Hom-Tensor Adjunctions for Quasi-Hopf Algebras

Inaugural-Dissertation

zur
Erlangung des Doktorgrades der
Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

Saeid Bagheri
aus Malayer, Iran

Düsseldorf
2008

Aus dem Institut für Mathematik
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit der Genehmigung der
Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

1. Referent: Prof. Dr. Robert Wisbauer
2. Referent: Prof. Dr. Fritz Grunewald
3. Referent: Prof. Viacheslav A. Artamonov

Tag der mündlichen Prüfung: **30.10.2008**

Hom-Tensor Adjunctions for Quasi-Hopf Algebras

Thesis for a Ph.D. degree submitted to the
Faculty of Mathematics and Natural Sciences
Heinrich-Heine University Düsseldorf

Saeid Bagheri
from Malayer, Iran

Düsseldorf
2008

Contents

Introduction

For a commutative ring k, the category $\mathbb{M}_k$ of k-modules is monoidal: the tensor product of two k-modules has again a natural k-module structure and for k-modules V, M, N, the canonical map

$$a_{V,M,N} : (V \otimes M) \otimes N \longrightarrow V \otimes (M \otimes N), \quad (v \otimes m) \otimes n \mapsto v \otimes (m \otimes n),$$

is an isomorphism. This means in particular that the composition of the endofunctors $V \otimes_k -, M \otimes_k - : \mathbb{M}_k \to \mathbb{M}_k$ is the same as the functor induced by the tensor product of k-modules V and M, that is, $(V \otimes_k M) \otimes_k -$. It is known from linear algebra that the endofunctors $V \otimes_k -, \mathrm{Hom}_k(V, -) : \mathbb{M}_k \to \mathbb{M}_k$, form an adjoint pair of functors with unit and counit

$$\eta_M : M \longrightarrow \mathrm{Hom}_k(V, V \otimes_k M), \quad m \longmapsto [v \mapsto v \otimes m],$$

$$\varepsilon_M : V \otimes \mathrm{Hom}_k(V, M) \longrightarrow M, \quad v \otimes f \mapsto f(v).$$

A *k-bialgebra* $(H, \mu, \iota, \Delta, \varepsilon)$ is a k-module H with k-linear maps

$\mu : H \otimes_k H \to H$, $\iota : k \to H$, for an associative algebra structure and
$\Delta : H \to H \otimes_k H$, $\varepsilon : H \to k$, for a coassociative coalgebra structure,

such that Δ and ε are algebra maps (equivalently μ and ι are coalgebra maps).

Denote the category of right H-modules by $\mathbb{M}_H$ and the category of right H-comodules by $\mathbb{M}^H$. For two modules $M, N \in \mathbb{M}_H$, the tensor product $M \otimes_k N$ is again a right H-module by the action $(m \otimes n) \cdot h = (m \otimes n)\Delta h$ (componentwise action). This turns $\mathbb{M}_H$ into a monoidal category. To make this work, coassociativity of the coproduct Δ is needed, since it is to show that for V, M and $N \in \mathbb{M}_H$, the k-linear isomorphism

$$a_{V,M,N} : (V \otimes_k M) \otimes_k N \to V \otimes_k (M \otimes_k N)$$

is also H-linear, that is - using the Sweedler notation -

$$((v \otimes m) \otimes n) \cdot h = (vh_{11} \otimes mh_{12}) \otimes nh_2 = vh_1 \otimes (mh_{21} \otimes nh_{22}) = (v \otimes (m \otimes n)) \cdot h.$$

Here the middle identity is just the coassociativity condition. In this case, it is easy to see that the composition of the functors $H \otimes_k (H \otimes_k -)$ can be identified with the functor induced by the the tensor product of the objects, namely $(H \otimes_k H) \otimes_k -$. This is an essential property in the theory of bialgebras and Hopf algebras.

A *right Hopf module* M is a right H-module $\rho_M : M \otimes_k H \to M$ as well as a right H-comodule $\rho^M : M \to M \otimes_k H$ such that $\rho^M(mh) = \rho^M(m)\Delta(h)$ for $m \in M$, $h \in H$.

For a bialgebra H, the endomorphisms ring $\mathrm{End}_k(H)$ has a second k-algebra structure with the convolution product $*$ and an $S \in \mathrm{End}_k(H)$ is an *antipode* if it is an inverse of the identity map with respect to the convolution product, that is, $id * S = \iota \circ \varepsilon = S * id$. A *Hopf algebra* is a bialgebra which has an antipode and the latter condition is equivalent to the fact that

$$- \otimes_k H : \mathbb{M}_k \to \mathbb{M}_H^H, \quad M \mapsto (M \otimes_k H, id \otimes \mu, id \otimes \Delta)$$

is an equivalence of categories (*Fundamental Theorem for Hopf algebras*) (see e.g. [7, 15.5]). The adjoint (inverse) to this functor was initially defined by *coinvariants* (see

[20, Proposition 1]) and it is shown in [7, 14.8] that it can be seen as the functor $\mathrm{Hom}_H^H(H, -)$.

The thesis is concerned with *quasi-bialgebras* as defined in Drinfeld [13] by requiring the same axioms as for bialgebras except for the coassociativity condition of the coproduct which is modified by a normalized 3-cocycle $\phi \in H \otimes H \otimes H$. Thus the map $a_{V,M,N}$ considered above is no longer H-linear and the subsequent theory of Hopf algebras cannot be transferred to the new situation immediately. For example, the convolution algebra $(\mathrm{End}_k(H), *)$ is no longer associative. However, the $a_{V,M,N}$ may be replaced by non-trivial associativity constraints in the monoidal category $\mathbb{M}_H$ and this leads the way to the necessary modification of the classical notions. The notion of an antipode was adapted to a *quasi-antipode* in Drinfeld [13]. The Fundamental Theorem corresponds to the comparison functor

$$- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H, \quad N \mapsto (N \otimes_k H, \varrho_{N\otimes H}, \varrho^{N\otimes H})$$

being an equivalence (see 12.4, 13.3 and 15.10). This was first shown by Hausser and Nill [17] by defining a projection $E : M \to M$ which leads to a *coinvariant functor* $(-)^{coH}$. Another projection $\bar{E} : M \to M$ was defined by Bulacu and Caenepeel [8] leading to a distinct (but isomorphic) coinvariant functor $(-)^{\overline{coH}}$.

The purpose of this thesis is to study various functors induced by the tensor product $- \otimes_k V$. They may go from ${}_H\mathbb{M} \to {}_H\mathbb{M}$, $\mathbb{M}_H \to \mathbb{M}_H$, ${}_H\mathbb{M}_H \to {}_H\mathbb{M}_H$, ${}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$, ${}_H\mathbb{M}_H \to {}_H\mathbb{M}_H^H$, etc. depending on V being a left or right H-module, a bimodule or a quasi-Hopf H-bimodule. In all these cases, we obtain the right adjoints as a variation of the Hom-functor and we give the intrinsic units and counits explicitely. Of particular interest is the observation that for any quasi-bialgebra H, the functor ${}_H\mathrm{Hom}_H^H(H \otimes_k H, -)$ is right adjoint to the comparison functor mentioned above.

In the first chapter, we state some facts about modules and Hopf algebra theory. In the second chapter, we recall notions from (monoidal) category theory needed to understand the general background of (quasi-) Hopf algebra theory.

In the third chapter, we generalize the Hom-tensor adjunction from the Hopf algebra case to the quasi-Hopf setting and describe the adjunctions between the functors $\mathrm{Hom}_k(V, -)$ and $- \otimes_k V$ (resp. $V \otimes_k -$) as endofunctors of ${}_H\mathbb{M}$, $\mathbb{M}_H$ and ${}_H\mathbb{M}_H$. The units and counits of these adjunctions are not the same as in the Hopf algebra case. We have to modify the adjunctions in such a way that the units and counits are morphisms in the corresponding categories.

For example, for a Hom-tensor adjunction on ${}_H\mathbb{M}$, the units and counits in ${}_H\mathbb{M}$, come out as (see 9.2)

$$\eta_M : M \longrightarrow {}^s\mathrm{Hom}_k(V, M \otimes_k V), \quad m \longmapsto [v \mapsto p_R\,(m \otimes v)],$$

$$\varepsilon_M : {}^s\mathrm{Hom}_k(V, M) \otimes V \longrightarrow M, \quad f \otimes v \mapsto \sum q_R^1\,[f(S(q_R^2)\,v)].$$

The corresponding results for $\mathbb{M}_H$ and ${}_H\mathbb{M}_H$ are considered in 9.11, 9.13, 9.15 and 9.16.

As a special case, if $V = A$ is a left H-module algebra, the functor $A \otimes_k -$ (resp. $- \otimes_k A$) is a monad on ${}_H\mathbb{M}$. In this case, we describe the isomorphism between the Eilenberg-Moore *module* category over this monad (that is in fact isomorphic to the module category over the associative algebra $A\#H$), and the Eilenberg-Moore *comodule* category $({}_H\mathbb{M})^{\mathrm{Hom}_k(A,-)}$ (see 9.7 and 4.9).

In [27], Schauenburg described the adjoint pair $(- \otimes_k V, - \otimes_k V^*)$ of endofunctors of ${}_H\mathbb{M}$ for a finite dimensional left H-module V over a base field. In 9.4 we give explicitely a functorial isomorphism $- \otimes_k V^* \to {}^s\mathrm{Hom}_k(V, -)$ for a finitely generated and projective k-module V. This yields Schauenburg's adjunction as a particular case of our adjunction in 9.2.

In section 11, we generalize the Hom-tensor relations from the module category over a quasi-Hopf algebra H to the module category over an H-comodule algebra (in the sense of Hausser and Nill [15]). The main idea that makes this generalization possible is that the coaction of a quasi-Hopf algebra H on a comodule algebra $\mathcal{B}$ gives rise to an action of the monoidal category ${}_H\mathbb{M}$ on the module category over $\mathcal{B}$. This yields an endofunctor $V \otimes_k - : {}_{\mathcal{B}}\mathbb{M} \to {}_{\mathcal{B}}\mathbb{M}$, for any left H-module V, which is left adjoint to some suitable Hom-functor.

In the fourth chapter, we show that with a suitable left H-module structure on Hom, the functor ${}_H\mathrm{Hom}_H^H(H \otimes H, -) : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$ is right adjoint to the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ (see 12.7).

For a quasi-Hopf algebra H, we consider the Hausser-Nill coinvariants functor $(-)^{coH} : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$ and the Bulacu-Caenepeel coinvariants functor $(-)^{\overline{coH}} : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$ which both are right adjoints to the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$. As one of the main results of this thesis, we obtain that these functors are isomorphic to the Hom-functor ${}_H\mathrm{Hom}_H^H(H \otimes H, -) : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$ (see 13.8). This gives a new description of the concept of coinvariants in terms of a Hom-functor and provides alternative techniques to handle modules over quasi-Hopf algebras.

These constructions are generalized to the category of left two-sided Hopf modules ${}_{\mathcal{A}}\mathbb{M}_H^H$, where $(\mathcal{A}, \rho, \phi_\rho)$ is a right H-comodule algebra. This category can be considered as the Eilenberg-Moore comodule category $({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H}$ over the comonad $- \otimes H : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H$ (see 4.8). Adopting the arguments of Hausser-Nill [17] and Bulacu-Caenepeel [8], we define two (isomorphic) types of coinvariants functors. Each of them gives rise to a version of the Fundamental Theorem. In 15.11 we describe both types of coinvariants in terms of the Hom-functor ${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, -) : {}_{\mathcal{A}}\mathbb{M}_H^H \to {}_{\mathcal{A}}\mathbb{M}$.

In the fifth chapter, we consider the category of right two-sided Hopf modules ${}_H\mathbb{M}_{\mathcal{A}}^H$ and define similar concepts in this category. In section 16, we introduce two versions of *coinvariants functors* $(-)^{coH}$ and $(-)^{\underline{coH}}$ for this category and describe them in terms of the Hom-functor ${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, -)$ (see 17.11). From the categorical point of view, all computations seem to be similar to the left case ${}_{\mathcal{A}}\mathbb{M}_H^H$ but they can not be derived from that results just by symmetry arguments.

***Acknowledgment ***

Firstly, I would like to express my sincere thanks to my advisor, Prof. Dr. Robert Wisbauer, for all his help, support and encouragement during my stay in Düsseldorf. He is an exceptionally caring advisor and a very nice teacher who helped me through many difficulties I had during my stay at Düsseldorf University. It was Prof. Wisbauer who introduced me into the theory of coalgebras, comodules and the (quasi)-Hopf algebra theory. In the final stages of the preparation of my thesis, he was a patient reader of the material and gave me many useful suggestions that improved the quality of the text. Throughout my study, his advice, guidance, and understanding were irreplaceable for my progress. I am truly grateful to him for everything.

I extend my thanks to Prof. Dr. Fritz Grunewald and Prof. Dr. Otto Kerner who kindly helped me during my stay in Düsseldorf in various ways.

Prof. Viacheslav A. Artamonov from MGU Moskau has patiently followed the presentations of my thesis, read the draft version of it and made useful suggestions. I am sincerely grateful to him for his discussions and advice.

I would like to thank Prof. Daniel Bulacu for reading a draft of my thesis and kindly answering some of my questions. Also, I am very grateful to Dr. Adriana Balan for helpful discussions and kind suggestions.

In my M. Sc. program, I was fortunate to have the chance to finish my work under the supervision of Prof. A. Haghany in Isfahan University of Technology. I take this opportunity to express my sincere thanks to him for many things that he taught me and for his kind encouragemants.

I am grateful to my friends and colleagues at the University of Düsseldorf, Mr. Carlos Preisser and Dr. Indah E. Wijayanti for helpful discussions. Specially I am grateful to Mr. Saeid Hamzehzarghani for his continuous help in many situations. I drew great profit from his knowledge in various branches of mathematics and computer science and talking with him was always rewarding for me.

Many thanks to the “Iranian Ministery of Science, Research and Technology” and also to the “universities of Malayer and Bu-Ali Sina” (Hamedan) for kind financial and official support that made my stay in Germany for more than 4 years possible.
Also, I am grateful to “Geselschft von Freunden und Förderern“ and ”Mathematisches Institut der HHU“ for their unbureaucratic financial support in the last months of my program.

Without the help and understanding of my situation by my family, the completion of this work and my studies was not possible at all. In particular, I am most indebted to my wife Mozhdeh, who has come with me thousands of kilometers away from her parents. Because of my concentration on Mathematics, she was most of the time alone responsible for taking care of the family and our two children. I would like to express my deepest thanks for her support, understanding and love.

For our daughters Bahar and Robina living in Germany was a nice opportunity to learn many new things and to grow up with two languages and cultures. However, my engagements with the studies did not permit me to pay as much attention to them as they needed. Hopefully, in future we will have more time to spend together.

Lastly and mostly, I thank my parents who provided me with so much love and optimism during my childhood that I will have a supply for life time. They taught me to face difficulties with humor and not to give up easily. Without learning that I would not be completing this thesis today.

Chapter 1

Preliminaries

Throughout this text, unless explicitely stated, we always suppose that k is a commutative ring with identity. All (co)algebras, bialgebras, Hopf algebras etc. will be over k; unadorned $\otimes$ and Hom mean $\otimes_k$ and Hom_k, respectively. For k-modules M, N, we denote by $\mathrm{Hom}_k(M,N)$ all k-module homomorphisms from M to N, $M^* := \mathrm{Hom}_k(M,k)$ and $\mathrm{End}_k(M) := \mathrm{Hom}_k(M,M)$. By $\tau_{M,N} : M \otimes N \to N \otimes M$ we denote the twist map which carries $m \otimes n$ to $n \otimes m$.

In this chapter we present some definitions and lemmas to be referred to later in this text. For more details about module theory we refer to [32] and about Hopf algebras, to [1], [7], [18], [22] and [29].

1 Algebras and coalgebras

1.1. Algebras and modules. A **k-algebra** is a k-module A together with k-linear maps $\mu_A : A \otimes A \to A$ and $\iota_A : k \to A$ and commutative diagrams

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{id\otimes\mu_A} & A\otimes A \\ {\scriptstyle \mu_A\otimes id}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ A\otimes A & \xrightarrow[\mu_A]{} & A, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{id\otimes\iota_A} & A\otimes A \\ {\scriptstyle \iota_A\otimes id}\downarrow & \searrow^{id_A} & \downarrow{\scriptstyle \mu_A} \\ A\otimes A & \xrightarrow[\mu_A]{} & A. \end{array}$$

Right **A-modules** are defined as k-modules M with an action $\varrho_M : M \otimes_k A \to M$ inducing the commutative diagrams

$$\begin{array}{ccc} M\otimes A\otimes A & \xrightarrow{id_M\otimes\mu_A} & M\otimes A \\ {\scriptstyle \varrho_M\otimes id_A}\downarrow & & \downarrow{\scriptstyle \varrho_M} \\ M\otimes A & \xrightarrow[\varrho_M]{} & M, \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{id\otimes\iota_A} & M\otimes A \\ & \searrow_{id} & \downarrow{\scriptstyle \varrho_M} \\ & & M. \end{array}$$

For the category of right A-modules we write $\mathbb{M}_A$ and denote the set of all A-module morphisms between $M, N \in \mathbb{M}_A$ by $\mathrm{Hom}_A(M,N)$. It is well known that A is a *projective generator* in ${}_A\mathbb{M}$.

1.2. Hom-tensor relations in ${}_k\mathbb{M}$. For any k-module V, the functors

$$-\otimes_k V : {}_k\mathbb{M} \longrightarrow {}_k\mathbb{M} \quad \text{and} \quad \mathrm{Hom}_k(V,-) : {}_k\mathbb{M} \longrightarrow {}_k\mathbb{M},$$

form an adjoint pair of functors.
For any k-module M, we have the k-linear morphism

$$\psi_M : M \otimes V^* \longrightarrow \mathrm{Hom}_k(V, M), \quad m \otimes f \longmapsto [v \mapsto f(v)m].$$

This induces a natural transformation $\psi : - \otimes V^* \to \mathrm{Hom}_k(V, -)$.
If ${}_kV$ is finitely generated and projective, there is a dual basis $\{v_i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$ for V with $v_i \in V$ and $v^i \in V^*$ such that for any $x \in V$ we have $x = \sum_{i=1}^n v^i(x)\, v_i$. In this case, for any k-module M, ψ_M is an isomorphism with inverse map $g \mapsto \sum_{i=1}^n g(v_i) \otimes v^i$, i.e. ψ is a natural isomorphism and the right adjoint of the tensor-functor $- \otimes_k V$ is again a tensor functor, namely $- \otimes_k V^* : {}_k\mathbb{M} \to {}_k\mathbb{M}$.

1.3. The category $\sigma[M]$. Let A be a k-algebra. A left A-module N is called **M-generated** if there exists an epimorphism $M^{(\Lambda)} \to N$ for some set Λ. The class of all M-generated modules is denoted by $\mathrm{Gen}(M)$. An A-module N is called **M-subgenerated** if it is (isomorphic to) a submodule of an M-generated module. A subcategory $\mathcal{C}$ of ${}_A\mathbb{M}$ is **subgenerated by** M, or M is a **subgenerator for** $\mathcal{C}$ if every object of $\mathcal{C}$ is subgenerated by M. By $\sigma[M]$ we denote the full subcategory of ${}_A\mathbb{M}$ whose objects are all M-subgenerated modules. This is the smallest full Grothendieck subcategory of ${}_A\mathbb{M}$ containing M. $\sigma[M]$ coincides with ${}_A\mathbb{M}$ if and only if A embeds into some (finite) coproduct of copies of M (see [32, 15.4]).

The **trace functor** $\tau^M : {}_A\mathbb{M} \to \sigma[M]$, which sends any $X \in {}_A\mathbb{M}$ to

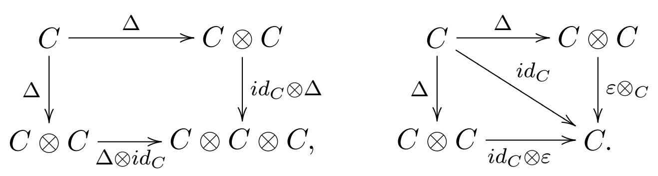
$$\tau^M(X) := \sum \{f(N) | N \in \sigma[M], f \in {}_A\mathrm{Hom}(N, X)\},$$

is right adjoint to the inclusion functor $\sigma[M] \to {}_A\mathbb{M}$ (e.g. [32, 45.11]).

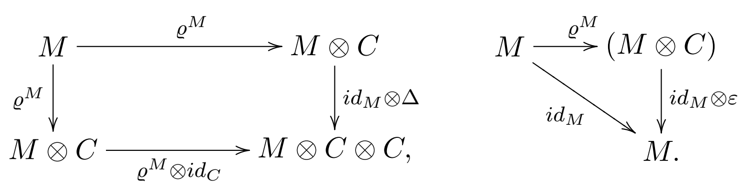
By definition, $\sigma[M]$ is closed under direct sums, factor modules and submodules in ${}_A\mathbb{M}$. Subcategories with these properties are said to be **closed subcategories** (of ${}_A\mathbb{M}$ or $\sigma[M]$). It is straightforward to see that any closed subcategory of ${}_A\mathbb{M}$ is of type $\sigma[N]$ for some N in ${}_A\mathbb{M}$. $N \in \sigma[M]$ is said to be a **generator** in $\sigma[M]$ if it generates all modules in $\sigma[M]$.

Reversing the arrows in the defining diagrams for algebras and their modules leads to the concepts

1.4. Coalgebras and Comodules. A **k-coalgebra** is a k-module C together with k-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ with commutative diagrams



A right C**-comodule** is a k-module M with a coaction $\varrho^M : M \to M \otimes_k C$ inducing the commutative diagrams



A k-linear map $f : M \to N$ between right C-comodules M and N is called a **C-comodule morphism** if it induces commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \varrho^M}\downarrow & & \downarrow{\scriptstyle \varrho^N} \\ M \otimes C & \xrightarrow[f \otimes id_C]{} & N \otimes C. \end{array}$$

The category of right C-comodules and C-comodule morphisms is denoted by $\mathbb{M}^C$ and the set of all morphisms between $M, N \in \mathbb{M}^C$ is written as $\mathrm{Hom}^C(M, N)$.
As a right comodule, C is a subgenerator in $\mathbb{M}^C$, that is, every right C-comodule is a subcomodule of a C-generated comodule (see 1.3). Note that $\mathbb{M}^C$ need not have projectives even if k is a field.

If C is a flat k-module, the category $\mathbb{M}^C$ is a Grothendieck category (see [7], 3.13). For a coalgebra (C, Δ, ε), the dual module $C^* = \mathrm{Hom}_k(C, k)$ is an associative k-algebra with unit element ε. The multiplication in C^* is the **convolution product**

$$\mu : C^* \otimes C^* \longrightarrow (C \otimes C)^* \xrightarrow{\Delta^*} C^*,$$

where $\Delta^* = \mathrm{Hom}_k(\Delta, k)$. Explicitely,

$$\forall f, g \in C^* \quad f * g = (f \otimes g) \circ \Delta : C \to k \otimes k \simeq k. \tag{1.1}$$

On the other hand, if (A, μ, ι) is a k-algebra, the transpose map $\mu^* : A^* \to (A \otimes A)^*$ does not in general carry A^* into $A^* \otimes A^*$. This is the case if A is finitely generated and projective as a k-module.

2 Bialgebras and Hopf algebras

2.1. Bialgebras. A k-module B that is a k-algebra (B, μ, ι) and a k-coalgebra (B, Δ, ε) is called a **k-bialgebra** if Δ and ε are algebra-maps, equivalently, if μ and ι are coalgebra maps. This means commutativity of the diagrams

$$\begin{array}{ccc} B \otimes B & \xrightarrow{\qquad \mu \qquad} & B \\ {\scriptstyle \Delta\otimes\Delta}\downarrow & & \\ (B \otimes B) \otimes (B \otimes B) & & \downarrow{\scriptstyle \Delta} \\ {\scriptstyle id\otimes\tau_{B,B}\otimes id}\downarrow & & \\ (B \otimes B) \otimes (B \otimes B) & \xrightarrow[\mu\otimes\mu]{} & B \otimes B, \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\ \iota\ } & B \\ {\scriptstyle \approx}\downarrow & & \downarrow{\scriptstyle \Delta} \\ k \otimes k & \xrightarrow[\iota\otimes\iota]{} & B \otimes B. \end{array}$$

Over any bialgebra $(B, \mu, \iota, \Delta, \varepsilon)$, the base ring k itself is a left and right B-module through the algebra map $\varepsilon : B \to k$ and also a left and right B-comodule through the coalgebra map $\iota : k \to B$. Also, $(\mathrm{End}_k(B), *)$ is an associative k-algebra with unit element $\iota \circ \varepsilon$ and the convolution product

$$\forall f, g \in \mathrm{End}_k(B) \qquad f * g = \mu \circ (f \otimes g) \circ \Delta. \tag{2.1}$$

.

2.2. Antipodes and Hopf algebras. An element $S \in \mathrm{End}_k(B)$ is called **left** (resp. **right**) **antipode** if it is left (resp. right) inverse to id_B with respect to the convolution product $*$ on $\mathrm{End}_k(B)$. In case S is a left and right antipode, it is called an **antipode**. A bialgebra H with an antipode is called a **Hopf algebra**.
The antipode S satisfies

$$S * id_B = id_B * S = \iota \circ \varepsilon \,,$$

which means explicitely

$$\mu \circ (S \otimes id_B) \circ \Delta = \mu \circ (id_B \otimes S) \circ \Delta = \iota \circ \varepsilon,$$

and, for $c \in B$ and $\Delta(c) = \sum c_1 \otimes c_2$,

$$\sum S(c_1)c_2 = \sum c_1 S(c_2) = \varepsilon(c) 1_B \,.$$

Notice that for any $f \in \mathrm{End}_k(B)$, being invertible with respect to $*$ does not mean that f is a bijective map.
A map $f : H_1 \to H_2$ of Hopf algebras with antipodes S_1 and S_2 is called a **Hopf algebra morphism** if it is an algebra as well as a coalgebra morphism satisfying

$$f\left(S_1(c)\right) = S_2(f(c)) \quad \forall c \in H.$$

2.3. Properties of the antipode. [7, 15.4]

(1) S is an anti-algebra-morphism, i.e.
   i) $S(ab) = S(b)S(a)$, for all $a, b \in H$.
   ii) $S \circ \iota = \iota$, which means $S(1_H) = 1_H$.
(2) S is an anti-coalgebra-morphism, i.e.
   i) $\varepsilon \circ S = \varepsilon$.
   ii) $\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S$, i.e. for $\Delta(c) = \sum c_1 \otimes c_2$,

$$\Delta S(c) = \sum S(c_2) \otimes S(c_1).$$

2.4. Group algebras and their duals. Let G be a group and $k[G]$ its group algebra, that is, $k[G]$ is a free k-module with basis G, and the product given by the group multiplication. Furthermore, $k[G]$ is a k-coalgebra with coproduct induced by $\Delta(g) = g \otimes g$ and counit $\varepsilon(g) = 1_k$, for $g \in G$. With these structures, $k[G]$ is a k-bialgebra and even a Hopf algebra with antipode S induced by $S(g) = g^{-1}$ for $g \in G$.
If G is a finite group of order $n \in \mathbb{N}$ with elements $\{g_1, \ldots, g_n\}$, the k-dual $k[G]^* = \mathrm{Hom}_k(k[G], k)$ is also a Hopf algebra. The multiplication of $f, g \in k[G]^*$ is given by $(f * g)(x) = f(x)g(x)$ for $x \in G$. To describe the coalgebra structure, let $\{g\}_{g \in G}$ and $\{e_g\}_{g \in G} \subset k[G]^*$ be a dual basis for $k[G]$. The coproduct and counit are defined by

$$\Delta(e_g) = \sum_{kh=g} e_k \otimes e_h, \quad \varepsilon(e_g) = \delta_{1,g}. \tag{2.2}$$

The antipode S of $k[G]^*$ is induced by $S(e_g) = e_{g^{-1}}$ for $g \in G$.

2.5. Modules over bialgebras.

i) Given a k-bialgebra $(H, \mu, \iota, \Delta, \varepsilon)$, for any pair M, N of left H-modules the algebra morphism $\Delta : H \to H \otimes H$ enables us to equip $M \otimes_k N$ with an H-module structure, given by

$$a \cdot (m \otimes n) = \Delta(a)(m \otimes n) = \sum_{(a)} a_1 m \otimes a_2 n. \tag{2.3}$$

Following [7, 13.4] we denote $M \otimes_k N$ with this diagonal H-module structure by $M \otimes_k^b N$.

ii) For any morphisms $f : M \to N$, $g : M' \to N'$ in ${}_H\mathbb{M}$, the k-linear map $f \otimes g : M \otimes^b M' \to N \otimes^b N'$ is a morphism in ${}_H\mathbb{M}$.

iii) For left H-modules M, N and L, the k-linear isomorphisms

$$(M \otimes^b N) \otimes^b L \simeq M \otimes^b (N \otimes^b L), \qquad k \otimes^b M \simeq M \simeq M \otimes^b k$$

are isomorphisms of H-modules. If H is a cocommutative k-bialgebra, then the twist map $\tau_{M,N} : M \otimes N \to N \otimes M$ is also an isomorphism of H-modules (see [18, III.5.1]).

2.6. Module structure on Hom. For any algebra H and left H-modules M, N, we have a left $H \otimes H^{op}$-module structure on $\mathrm{Hom}_k(M, N)$ by

$$((a \otimes a') \cdot f)(m) = (a\, f(a'\, m)),$$

for $a, a' \in H$, $f \in \mathrm{Hom}_k(M, N)$ and $m \in M$.

For a *Hopf algebra* with antipode S, the map

$$(id_H \otimes S) \circ \Delta : \; H \xrightarrow{\Delta} H \otimes H \xrightarrow{id_H \otimes S} H \otimes H^{op}$$

is an algebra morphism from H to $H \otimes H^{op}$. Through this morphism, we get an H-module structure on $\mathrm{Hom}_k(M, N)$ given by

$$(a \cdot g)(m) = \sum a_1(g(S(a_2)\, m)), \tag{2.4}$$

for $g \in \mathrm{Hom}_k(M, N)$, $a \in H$, and $m \in M$. In particular, for $N = k$, the above equality induces an H-module structure on $M^* = \mathrm{Hom}_k(M, k)$ which becomes

$$(a \cdot f)(m) = f(S(a)\, m) \tag{2.5}$$

for all $a \in H$, $f \in M^*$ and $m \in M$.

If the antipode S is bijective, another H-module structure can be defined on $\mathrm{Hom}_k(M, N)$ by

$$(a \cdot g)(m) = \sum a_2(g(S^{-1}(a_1)\, m)), \tag{2.6}$$

for $g \in \mathrm{Hom}_k(M, N)$, $a \in H$, and $m \in M$. In particular, for the trivial H-module $N = k$,

$$(a \cdot f)(m) = f(S^{-1}(a)\, m) \tag{2.7}$$

for all $a \in H$, $f \in M^*$ and $m \in M$.

2.7. Comodules over bialgebras. Let H be a k-bialgebra and M, N and L left H-comodules. Then $M \otimes_k N$ has a left H-comodule structure by the map

$$^{M\otimes N}\varrho : M \otimes_k N \xrightarrow{^M\varrho\otimes ^N\varrho} H \otimes M \otimes H \otimes N \xrightarrow{id\otimes\tau\otimes id} H \otimes H \otimes M \otimes N \xrightarrow{\mu\otimes id\otimes id} H \otimes M \otimes N$$

explicitly, for $m \in M$ and $n \in N$,

$$^{M\otimes N}\varrho(m \otimes n) = \sum m_{(-1)}n_{(-1)} \otimes m_0 \otimes n_0.$$

Following [7, 13.5], we denote $M \otimes_k N$ with this (diagonal) H-comodule structure by $M \otimes_k^c N$.

For any morphisms $f : M \to N$, $g : M' \to N'$ in $^H\mathbb{M}$, the tensor product map $f \otimes g : M \otimes^c M' \to N \otimes^c N'$, is a morphism in $^H\mathbb{M}$. The canonical isomorphisms

$$(M \otimes^c N) \otimes^c L \simeq M \otimes^c (N \otimes^c L), \quad k \otimes^c M \simeq M \simeq M \otimes^c k$$

are isomorphisms of H-comodules.
If H is a cocommutative k-bialgebra, the twist map $\tau_{M,N} : M \otimes N \to N \otimes M$ is also an isomorphism of H-comodules (see [18, III.6.2]).

Similar concepts for modules and comodules over a bialgebra or Hopf algebra can be considered also on the right side. We denote these categories with $\mathbb{M}_H$ and $\mathbb{M}^H$ respectively.

2.8. Hopf modules. Let H be a k-bialgebra. A k-module M is called a **right H-Hopf module** if M is

i) a right H-module by $\varrho_M : M \otimes_k H \to M$,

ii) a right H-comodule by $\varrho^M : M \to M \otimes_k H$,

iii) for all $m \in M$ and $h \in H$, $\varrho^M(mh) = \varrho^M(m)\Delta(h)$, for $m \in M, h \in H$.

The last condition means that $\varrho^M : M \to M \otimes_k^b H$ is H-linear and it is also equivalent to require $\varrho_M : M \otimes_k^c H \to M$ to be H-colinear.

2.9. Trivial Hopf modules. Let H be a k-bialgebra. For any k-module L,

i) $L \otimes_k H$ is a right H-Hopf module with the canonical stuctures

$$\begin{array}{rlll} \varrho^{L\otimes H} = id_L \otimes \Delta : & L \otimes_k H & \longrightarrow L \otimes_k H \otimes_k H, & l \otimes h \mapsto l \otimes \Delta(h), \\ \varrho_{L\otimes H} = id_L \otimes \mu : & L \otimes_k H \otimes_k H & \longrightarrow L \otimes_k H, & l \otimes h \otimes a \mapsto l \otimes ha. \end{array}$$

ii) For every k-module morphism $f : L \to L'$, the map

$$f \otimes id : L \otimes_k H \to L' \otimes_k H$$

is an H-Hopf module morphism.

In particular, $H \otimes_k H$ is a trivial right H-Hopf module.

2.10. H-modules and Hopf modules. Let H be a k-bialgebra. For a right H-module N, the right H-module $N \otimes_k^b H$ is a right H-Hopf module with the canonical comodule stucture

$$\varrho^{N\otimes H} = id_N \otimes \Delta : \quad N \otimes_k^b H \quad \to \quad N \otimes_k^b H \otimes_k H, \quad n \otimes h \mapsto n \otimes \Delta h.$$

For every H-module morphism $f : N \to N'$, the map

$$f \otimes id : \ N \otimes_k^b H \to N' \otimes_k^b H$$

is an H-Hopf module morphism.

In particular, $H \otimes_k^b H$ is a right H-Hopf modules. For any right H-module N, there is a Hopf module map

$$\gamma_N : N \otimes_k H \to N \otimes_k^b H, \quad n \otimes h \mapsto (n \otimes 1_H)\Delta(h). \tag{2.8}$$

γ_N is an isomorphism for all $N \in \mathbb{M}_H$ if and only if H is a Hopf algebra (see [7, 14.2, 14.3 and 15.8]).

2.11. The category $\mathbb{M}_H^H$. Let H be a k-bialgebra. The right H-Hopf modules, together with the maps which are both right H-comodule and right H-module morphisms, form a category that is denoted by $\mathbb{M}_H^H$. For objects M, M' in $\mathbb{M}_H^H$, we denote by $\mathrm{Hom}_H^H(M, M')$ the set of morphisms from M to M'. There is a faithful functor

$$\mathbb{M}_H^H \longrightarrow {}_{H^{op}\#H^*}\mathbb{M},$$

from the category $\mathbb{M}_H^H$ to the module category over the *smash product* $H^{op}\#H^*$. $\mathbb{M}_H^H$ can be considered as a full subcategory of ${}_{H^{op}\#H^*}\mathbb{M}$ if ${}_kH$ is locally projective.

2.12. Properties of $\mathbb{M}_H^H$. Let H be a k-bialgebra (see [7, 14.5, 14.6 and 14.15]).

i) $\mathbb{M}_H^H$ is closed under direct sums and factor modules.

ii) The right H-Hopf module $H \otimes_k^b H$ is a subgenerator in $\mathbb{M}_H^H$.

iii) The right H-Hopf module $H \otimes_k^c H$ is a subgenerator in $\mathbb{M}_H^H$.

iv) For any $M \in \mathbb{M}_H^H$, $N \in \mathbb{M}_H$,

$$\mathrm{Hom}_H^H(M, N \otimes_k H) \longrightarrow \mathrm{Hom}_H(M, N), \ f \mapsto (id \otimes \varepsilon) \circ f,$$

is a k-module isomorphism with inverse map $h \mapsto (h \otimes id) \circ \varrho^M$.

v) For any $K, L \in \mathbb{M}_k$,

$$\mathrm{Hom}_H^H(K \otimes_k H, L \otimes_k H) \longrightarrow \mathrm{Hom}_k(K, L), \ f \mapsto (id_L \otimes \varepsilon) \circ f(- \otimes 1_H),$$

is a k-module isomorphism with inverse map $h \mapsto (h \otimes id_H)$.

vi) If ${}_kH$ is flat, then $\mathbb{M}_H^H$ is a Grothendieck category and for Hopf modules $M, N \in \mathbb{M}_H^H$, the functors $\mathrm{Hom}_H^H(M, -) : \mathbb{M}_H^H \to \mathbb{M}_k$ and $\mathrm{Hom}_H^H(-, N) : \mathbb{M}_H^H \to \mathbb{M}_k$ are both left exact.

2.13. Coinvariants and Hopf modules. Let M be a right H-Hopf module. The **coinvariants** of H in M are defined as

$$M^{coH} = \{m \in M \mid \varrho^M(m) = m \otimes_k 1_H\} = \mathrm{Ke}\,(\varrho^M - (- \otimes 1_H)).$$

(1) The map

$$\nu_M : \mathrm{Hom}_H^H(H, M) \longrightarrow M^{coH}, \quad f \mapsto f(1_H),$$

is a k-module isomorphism with inverse map

$$\omega_M : M^{coH} \longrightarrow \mathrm{Hom}_H^H(H, M), \quad m \mapsto [h \mapsto m\,h].$$

In particular, $\mathrm{Hom}_H^H(H, H) \to H^{coH} = k\,1_H$ is a ring isomorphism.
We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_H^H(H,M) \otimes_k H & \longrightarrow & M \\ \downarrow{\scriptstyle \nu_M \otimes id_H} & & \downarrow{\scriptstyle id_M} \\ M^{coH} \otimes_k H & \longrightarrow & M, \end{array} \qquad \begin{array}{ccc} f \otimes h & \longrightarrow & f(h) \\ \downarrow & & \downarrow \\ f(1_H) \otimes h & \longrightarrow & f(1_H)h. \end{array}$$

(2) For any right H-module N, there is a k-module isomorphism

$$\nu'_{N \otimes H} : \mathrm{Hom}_H^H(H, N \otimes_k^b H) \longrightarrow N, \quad f \mapsto (id \otimes \varepsilon) \circ f(1_H)\,,$$

with inverse map $n \mapsto [h \mapsto \sum n h_1 \otimes h_2]$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_H^H(H, N \otimes_k^b H) \otimes_k H & \longrightarrow & N \otimes_k^b H \\ \downarrow{\scriptstyle \nu'_{N\otimes H} \otimes id} & & \downarrow{\scriptstyle id} \\ N \otimes_k H & \xrightarrow[\gamma_N]{} & N \otimes_k^b H, \end{array} \qquad \begin{array}{ccc} g \otimes h & \longrightarrow & g(h) \\ \downarrow & & \downarrow \\ (id_N \otimes \varepsilon) g(1) \otimes h & \longrightarrow & g(1)\Delta(h), \end{array}$$

where $\gamma_N : N \otimes H \to N \otimes^b H$ is the H-Hopf module morphism described in (2.8) (see also [7, 14.3]). In particular,

$$(H \otimes_k H)^{coH} \simeq \mathrm{Hom}_H^H(H, H \otimes_k H) \simeq H.$$

3 Co-chains and co-cycles

The (co)homology theory for Hopf algebras has been studied by Sweedler and others. V. G. Drinfeld has obtained new examples of (quasi-triangular) Hopf algebras from the old ones by "twisting" the structures by 2-cocycles. In Majid [22], the cochains and cocycles are defined for bialgebras and Hopf algebras.
In this section we follow the approach in [22], with some weaker conditions. In fact we have a unital multiplication and a counital comultiplication which are compatible, but we do not assume the comultiplication to be coassociative. This is done in view of the application of this theory to the quasi-Hopf algebra setting.

3.1. Cochains and cocycles without coassociativity condition. Let H be an associative algebra with a comultiplication $\Delta : H \to H \otimes H$ and a counit $\varepsilon : H \to k$ which both are algebra maps. For any $n \in \mathbb{N}$ and $i = 1, 2, \cdots, n$, we set

$$\Delta_i : H^{\otimes n} \to H^{\otimes n+1}, \quad \Delta_i = id \otimes id \otimes \cdots \otimes \underbrace{\Delta}_{i-th} \otimes \cdots \otimes id,$$

and

$$\Delta_0 = 1 \otimes (-), \quad \Delta_{n+1} = (-) \otimes 1.$$

We define an **n-cochain** ω as an invertible element in $H^{\otimes n}$, and its **coboundary** as the $(n+1)$-cochain

$$\partial^n \omega = (\prod_{i=0}^{i \text{ even}} \Delta_i \omega)(\prod_{i=1}^{i \text{ odd}} \Delta_i \omega^{-1}) \qquad 0 \le i \le n+1. \tag{3.1}$$

(The products are taken in increasing order).
An n-**cocycle** for H is an invertible element $\omega \in H^{\otimes n}$, such that $\partial^n \omega = 1$.
A cochain or cocycle ω is said to be **counital** or **normalized** if $\varepsilon_i(\omega) = 1$ for all $i = 1, 2, \cdots, n$, where $\varepsilon_i = id \otimes \cdots \otimes \underbrace{\varepsilon}_{i-th} \otimes \cdots \otimes id$.

Case $n = 1$**.** For $n = 1$, $\Delta_i : H \to H \otimes H, \quad i = 0, 1, 2,$

$$\Delta_0(h) = 1 \otimes h, \quad \Delta_1(h^{-1}) = \Delta(h^{-1}), \quad \text{and} \quad \Delta_2(h) = h \otimes 1,$$

then an element $h \in H$ is a 1-cocycle if and only if

$$1 = \partial^1 h = \Delta_0(h)\Delta_2(h)\Delta_1(h^{-1}) = (1 \otimes h)(h \otimes 1)\Delta(h)^{-1} \Leftrightarrow \Delta(h) = h \otimes h.$$

i.e. h is a semi-grouplike element. $h \in H$ is a **counital** 1-**cocycle** if and only if it is an invertible grouplike element.

Case $n = 2$**.** For $n = 2$, $\Delta_i : H \otimes H \to H^{\otimes 3}, \quad i = 0, 1, 2, 3,$ for an invertible element $R = \sum R^1 \otimes R^2$,

$$\Delta_0(R) = 1 \otimes R, \quad \Delta_1(R^{-1}) = (\Delta \otimes id)(R-1),$$

$$\Delta_2(R) = (id \otimes \Delta)(R), \quad \text{and} \quad \Delta_3(R^{-1}) = R^{-1} \otimes 1,$$

Thus, R is a 2-cocycle if and only if

$$1 = \partial^2 R = (1 \otimes R)(id \otimes \Delta)(R)(\Delta \otimes id)(R^{-1})(R^{-1} \otimes 1)$$

This corresponds to the equality

$$(1 \otimes R)(id \otimes \Delta)(R) = (R \otimes 1)(\Delta \otimes id)(R). \tag{3.2}$$

R is counital (normalized) if and only if

$$(\varepsilon \otimes id)(R) = 1 = (id \otimes \varepsilon)(R).$$

For example, let H be a braided bialgebra with universal R-matrix $\mathfrak{R}$. Then the universal R-matrix $\mathfrak{R}$ satisfies the equalities

$$(\Delta \otimes id_H)(\mathfrak{R}) = \mathfrak{R}_{13}\mathfrak{R}_{23} = \sum (\mathfrak{R}^1 \otimes 1 \otimes \mathfrak{R}^2)(1 \otimes \mathfrak{R}), \tag{3.3}$$

$$(id_H \otimes \Delta)(\mathfrak{R}) = \mathfrak{R}_{13}\mathfrak{R}_{12} = \sum (\mathfrak{R}^1 \otimes 1 \otimes \mathfrak{R}^2)(\mathfrak{R} \otimes 1), \tag{3.4}$$

$$\text{and} \quad \mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12}, \tag{3.5}$$

in other words

$$\sum(\mathfrak{R}\otimes 1)(\mathfrak{R}^1\otimes 1\otimes\mathfrak{R}^2)(1\otimes\mathfrak{R}) = \sum(1\otimes\mathfrak{R})(\mathfrak{R}^1\otimes 1\otimes\mathfrak{R}^2)(\mathfrak{R}\otimes 1).$$

and

$$(\varepsilon\otimes id)(\mathfrak{R}) = 1 = (id\otimes\varepsilon)(\mathfrak{R}), \tag{3.6}$$

(see [18, pp. 173-175]). Thus, $\mathfrak{R}$ is a normalized 2-cocycle.

Case $n = 3$. For $n = 3$, $\Delta_i : H^{\otimes 3} \to H^{\otimes 4}, \quad i = 0, 1, 2, 3, 4$, an invertible element $\phi \in H^{\otimes 3}$ is a 3-cocycle if and only if

$$\begin{aligned} 1 = \partial^3\phi &= \Delta_0(\phi)\Delta_2(\phi)\Delta_4(\phi)\Delta_1(\phi^{-1})\Delta_3(\phi^{-1}) \\ &= (1\otimes\phi)(id\otimes\Delta\otimes id)(\phi)(\phi\otimes 1)(\Delta\otimes id\otimes id)(\phi^{-1})(id\otimes id\otimes\Delta)(\phi^{-1}) \end{aligned}$$

so ϕ is a 3-cocycle if and only if

$$(1\otimes\phi)(id\otimes\Delta\otimes id)(\phi)(\phi\otimes 1) = (id\otimes id\otimes\Delta)(\phi)(\Delta\otimes id\otimes id)(\phi) \tag{3.7}$$

ϕ is counital (normalized) if and only if

$$(\varepsilon\otimes id\otimes id)(\phi) = 1\otimes 1 = (id\otimes\varepsilon\otimes id)(\phi) = (id\otimes id\otimes\varepsilon)(\phi).$$

We will use the concept of normalized 3-cocycles in the definition of quasi-bialgebras (see (7.3) and (7.4)).

Chapter 2

Tools from category theory

In this chapter we present some ingredients from category theory to be referred to later in this text. More details can be found in [3], [4], [5], [21], [24] and [28].

4 Monads and comonads

4.1. Adjoint Functors. A pair (L, R) of functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between categories $\mathbb{A}$ and $\mathbb{B}$ is called an **adjoint pair** if there exists a natural isomorphism

$$\Omega : \mathbb{B}(L(-), -) \longrightarrow \mathbb{A}(-, R(-)),$$

which can be described by natural transformations

$$\text{the unit} \quad \eta : id_{\mathbb{A}} \to RL, \qquad \text{and the counit} \quad \varepsilon : LR \longrightarrow id_{\mathbb{B}},$$

satisfying the triangular identities

$$\begin{array}{ccc} L & \xrightarrow{L\eta} & LRL \\ & {\scriptstyle 1_L} \searrow & \downarrow {\scriptstyle \varepsilon L} \\ & & L, \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\eta R} & RLR \\ & {\scriptstyle 1_R} \searrow & \downarrow {\scriptstyle R\varepsilon} \\ & & R. \end{array}$$

For any object A in $\mathbb{A}$ and B in $\mathbb{B}$,

$$\varepsilon_B = \Omega^{-1}_{A,B}(id_{R(B)}) \quad \text{and} \quad \eta_A = \Omega_{A,B}(id_{L(A)}).$$

Conversely, having a unit $\eta : id_{\mathbb{A}} \to RL$ and a counit $\varepsilon : LR \to id_{\mathbb{B}}$ satisfying the triangular identities, for any object A in $\mathbb{A}$ and B in $\mathbb{B}$, we obtain the natural isomorphism

$$\Omega_{A,B} : \mathbb{B}(L(A), B) \longrightarrow \mathbb{A}(A, R(B)),$$

given by

$$\Omega_{A,B}(f) = R(f) \circ \eta_A, \qquad \text{for} \quad f : L(A) \longrightarrow B, \tag{4.1}$$

with inverse map

$$\Omega^{-1}_{A,B}(g) = \varepsilon_B \circ L(g), \qquad \text{for} \quad g : A \longrightarrow R(B). \tag{4.2}$$

Let (L, R) be an adjoint pair of functors, then (e.g. [6])

i) R is full and faithful if and only if $\varepsilon : LR \to id_{\mathbb{B}}$ is an isomorphism.

ii) L is full and faithful if and only if $\eta : id_{\mathbb{A}} \to RL$ is an isomorphism.

iii) L is an equivalence if and only if η and ε are isomorphisms.

4.2. Natural transformations for adjoint pairs. Let $\mathbb{A} \xrightarrow{L} \mathbb{B} \xrightarrow{R} \mathbb{A}$ be an adjoint pair of functors with unit η and counit ε and $\mathbb{A} \xrightarrow{L'} \mathbb{B} \xrightarrow{R'} \mathbb{A}$ be another adjoint pair with unit η' and counit ε' between categories $\mathbb{A}$ and $\mathbb{B}$. Then there is a bijection between natural transformations

$$\mathrm{Nat}(L', L) \longrightarrow \mathrm{Nat}(R, R'), \quad \varphi \mapsto \bar{\varphi} := R'\varepsilon \circ R'\varphi R \circ \eta' R,$$

with inverse

$$\mathrm{Nat}(R, R') \longrightarrow \mathrm{Nat}(L', L), \quad \bar{\varphi} \mapsto \varphi := \varepsilon' R \circ L'\bar{\varphi} L \circ L'\eta.$$

In this case, following Kelly and Street [19], we say that φ and $\bar{\varphi}$ are **mates** under the given adjunctions (see also [23] and [6]).

4.3. F-modules. Given an endofunctor $F : \mathbb{A} \to \mathbb{A}$, an **$F$-module** (A, ϱ_A) consists of an object $A \in \mathbb{A}$ together with a morphism $\varrho_A : F(A) \to A$ in $\mathbb{A}$.
A morphism $f : A \to A'$ in $\mathbb{A}$ between F-modules is an **F-module morphism** provided it induces a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \varrho_A}\downarrow & & \downarrow{\scriptstyle \varrho_{A'}} \\ A & \xrightarrow[f]{} & A'. \end{array}$$

With these morphisms, the F-modules form a category which is denoted by $\mathbb{A}_F$. There is the faithful forgetful functor

$$U_F : \mathbb{A}_F \longrightarrow \mathbb{A}, \quad (A, \varrho_A) \mapsto A.$$

The relations between $\mathbb{A}_F$ and $\mathbb{A}$ are even stronger if additional conditions are imposed on the endofunctor F (see e.g. [31, 2.5.]).

4.4. Monads. A **monad** $\mathbb{F} = (F, \mu, \eta)$ on a category $\mathbb{A}$ consists of an endofunctor $F : \mathbb{A} \to \mathbb{A}$ and two natural transformations, the multiplication $\mu : F^2 \to F$ and the unit $\eta : id_{\mathbb{A}} \to F$, and commutative diagrams

$$\begin{array}{ccc} F^3 & \xrightarrow{F\mu} & F^2 \\ {\scriptstyle \mu F}\downarrow & & \downarrow{\scriptstyle \mu} \\ F^2 & \xrightarrow[\mu]{} & F, \end{array} \qquad \begin{array}{ccccc} F & \xrightarrow{\eta F} & F^2 & \xleftarrow{F\eta} & F. \\ & {\scriptstyle id_F}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle id_F} & \\ & & F & & \end{array}$$

Given monads $\mathbb{F} = (F, \mu, \eta)$ and $\mathbb{F}' = (F', \mu', \eta')$, a natural transformation $\alpha : F \to F'$ is called a **monad morphism** from $\mathbb{F}$ to $\mathbb{F}'$, if it induces commutativity of the diagrams

$$\begin{array}{ccccc} F^2 & \xrightarrow{F\alpha} & FF' & \xrightarrow{\alpha F'} & F'^2 \\ {\scriptstyle \mu}\downarrow & & & & \downarrow{\scriptstyle \mu'} \\ F & & \xrightarrow[\alpha]{} & & F', \end{array} \qquad \begin{array}{ccc} id_{\mathbb{A}} & & \\ {\scriptstyle \eta}\downarrow & \searrow^{\eta'} & \\ F & \xrightarrow[\alpha]{} & F'. \end{array}$$

4.5. Monads and their modules. Given a monad $\mathbb{F} = (F, \mu, \eta)$ on a category $\mathbb{A}$, an **$\mathbb{F}$-module** (A, ρ_A) consists of an object $A \in \mathbb{A}$ and an arrow $\rho_A : F(A) \to A$ in $\mathbb{A}$, with commutative diagrams

$$\begin{array}{ccc} F^2(A) & \xrightarrow{F(\rho_A)} & F(A) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \rho_A} \\ F(A) & \xrightarrow[\rho_A]{} & A, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & F(A) \\ & \searrow_{id_A} & \downarrow{\scriptstyle \rho_A} \\ & & A. \end{array}$$

$\mathbb{F}$-module morphisms are defined as in 4.3. The class all $\mathbb{F}$-modules together with $\mathbb{F}$-module morphisms form a category which is called the **Eilenberg-Moore module category** over the monad $\mathbb{F}$ and denoted by $\mathbb{A}_{\mathbb{F}}$.
As shown in Eilenberg-Moore [14], for a monad $\mathbb{F}$, the forgetful functor $U_F : \mathbb{A}_{\mathbb{F}} \to \mathbb{A}$ is right adjoint to the (free) functor

$$\phi_{\mathbb{F}} : \mathbb{A} \longrightarrow \mathbb{A}_{\mathbb{F}}, \quad A \longmapsto [(F(A),\ FF(A) \xrightarrow{\mu_A} F(A))\ ],$$

$$[\ A \xrightarrow{f} A'\ ] \mapsto [\ F(A) \xrightarrow{F(f)} F(A')\ ],$$

by the isomorphism

$$\mathbb{A}_{\mathbb{F}}(F(A), B)) \longrightarrow \mathbb{A}(A, U_{\mathbb{F}}(B)), \quad f \mapsto f \circ \eta_A,$$

for any $A \in \mathbb{A}$ and $B \in \mathbb{A}_{\mathbb{F}}$. Notice that $U_{\mathbb{F}} \circ \phi_{\mathbb{F}} = F$.

Dual to the preceding notions there is a theory of comodules which we sketch in the next paragraphs.

4.6. G-comodules. For a functor $G : \mathbb{A} \to \mathbb{A}$, a **$G$-comodule** (A, ϱ^A) is an $A \in \mathbb{A}$ with a morphism $\varrho^A : A \longrightarrow G(A)$ in $\mathbb{A}$.
A **G-comodule morphism** is a morphism $f : A \to A'$ in $\mathbb{A}$ between G-comodules A and A' inducing a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle \rho^A}\downarrow & & \downarrow{\scriptstyle \rho^{A'}} \\ G(A) & \xrightarrow[G(f)]{} & G(A'). \end{array}$$

The G-comodules together with G-comodule morphisms form a category which we denote by $\mathbb{A}^G$. The forgetful functor is faithful,

$$U^G : \mathbb{A}^G \longrightarrow A, \quad (A, \varrho^A) \mapsto A.$$

4.7. Comonads. A **comonad** $\mathbb{G} = (G, \delta, \varepsilon)$ on a category $\mathbb{A}$ consists of an endofunctor $G : \mathbb{A} \to \mathbb{A}$ and two natural transformations, the comultiplication $\delta : G \to G^2$ and the counit $\varepsilon : G \to id_{\mathbb{A}}$, such that the following diagrams commute

$$\begin{array}{ccc} G & \xrightarrow{\ \delta\ } & G^2 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta G} \\ G^2 & \xrightarrow[G\delta]{} & G^3, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\ \delta\ } & G^2 \\ {\scriptstyle\delta}\downarrow & \searrow^{id_G} & \downarrow{\scriptstyle\varepsilon G} \\ G^2 & \xrightarrow[G\varepsilon]{} & G. \end{array}$$

Comonad morphisms are defined in the same way as monad morphisms (see 4.6).

Given two comonads $\mathbb{G} = (G, \delta, \varepsilon)$ and $\mathbb{G}' = (G', \delta', \varepsilon')$, a natural transformation $\beta : G \to G'$ is called a **morphism of comonads** if the following diagrams commute

$$\begin{array}{ccc} G & \xrightarrow{\ \beta\ } & G' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ GG & \xrightarrow[\beta\star\beta]{} & G'G', \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\ \varepsilon\ } & id_{\mathbb{A}} \\ {\scriptstyle\beta}\downarrow & \nearrow_{\varepsilon'} & \\ G'. & & \end{array}$$

4.8. Comonads and their comodules. Given a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on a category $\mathbb{A}$, a $\mathbb{G}$**-comodule** (A, ρ^A) consists of an object $A \in \mathbb{A}$ and an arrow $\rho^A : A \to G(A)$ in $\mathbb{A}$, with commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \rho^A\ } & G(A) \\ {\scriptstyle\rho^A}\downarrow & & \downarrow{\scriptstyle\delta_A} \\ G(A) & \xrightarrow[G(\rho^A)]{} & GG(A), \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\ \rho^A\ } & G(A) \\ & \searrow^{id_A} & \downarrow{\scriptstyle\varepsilon_A} \\ & & A. \end{array}$$

The class all $\mathbb{G}$-comodules together with $\mathbb{G}$-comodule maps form a category which is called the **Eilenberg-Moore comodule category** over comonad $\mathbb{G}$ and denoted by $\mathbb{A}^{\mathbb{G}}$. The forgetful functor $U^G : \mathbb{A}^{\mathbb{G}} \to \mathbb{A}$ is left adjoint to the (free) functor

$$\phi^{\mathbb{G}} : \mathbb{A} \longrightarrow \mathbb{A}^{\mathbb{G}}, \quad A \longmapsto [(G(A),\ G(A) \xrightarrow{\delta_A} GG(A))\ ],$$

$$[\ A \xrightarrow{f} A'\ ] \mapsto [\ G(A) \xrightarrow{G(f)} G(A')\ ],$$

by the isomorphism

$$\mathbb{A}^{\mathbb{G}}(B, G(A)) \longrightarrow \mathbb{A}(U^{\mathbb{G}}(B), A), \quad f \mapsto \varepsilon_A \circ f,$$

for any $A \in \mathbb{A}$ and $B \in \mathbb{A}^{\mathbb{G}}$. Notice that $U^{\mathbb{G}} \circ \phi^{\mathbb{G}} = G$.

Monads and comonads are closely related to adjoint pairs of functors.

4.9. (Co)monads related to adjoints. Let $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors with unit $\eta : id_{\mathbb{A}} \to RL$ and counit $\varepsilon : LR \to id_{\mathbb{B}}$. Then

$$\mathbb{F} := (RL, R\varepsilon L, \eta), \quad RLRL \xrightarrow{R\varepsilon L} RL, \quad \eta : id_{\mathbb{A}} \longrightarrow RL,$$

is a monad on $\mathbb{A}$. Similarly, a comonad on $\mathbb{B}$ is defined by

$$\mathbb{G} := (LR, L\eta R, \varepsilon), \quad LR \xrightarrow{L\eta R} LRLR, \quad \varepsilon : LR \longrightarrow id_{\mathbb{B}}.$$

As observed by Eilenberg and Moore in [14], the monad structure of an endofunctor induces a comonad structure on its adjoint endofunctor. More precisely, as outlined in [6], for an adjoint pair $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ of functors:

(1) The following are equivalent:

  (a) L is a monad,

  (b) R is a comonad.

  In this case, the Eilenberg-Moore categories $\mathbb{A}_L$ and $\mathbb{A}^R$ are equivalent.

(2) The following are equivalent:

  (a) L is a comonad,

  (b) R is a monad.

  In this case, the Kleisli categories $\tilde{\mathbb{A}}_R$ and $\tilde{\mathbb{A}}^L$ are equivalent, where the Kleisli category $\tilde{\mathbb{A}}_R$ (resp. $\tilde{\mathbb{A}}^L$) is a subcategory of the Eilenberg-Moore category $\mathbb{A}_R$ (resp. $\mathbb{A}^L$) with objects $R(A)$ (resp. $L(A)$) for all $A \in \mathbb{A}$.

5 Monoidal categories

5.1. Monoidal categories. A category $\mathbb{A}$ is called a **monoidal (or tensor) category** if there exist a bifunctor $- \otimes - : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, a distinguished neutral object E, and natural isomorphisms

$$a : (- \otimes -) \otimes - \longrightarrow - \otimes (- \otimes -) \quad (\text{ associativity constraint })$$

$$\lambda : E \otimes - \longrightarrow id_{\mathbb{A}} \quad and \quad \rho : - \otimes E \longrightarrow id_{\mathbb{A}}$$

(left and right unit constraints) such that for all objects W, X, Y, Z in $\mathbb{A}$ the following two diagrams commute

$$\begin{array}{ccc}
[(W \otimes X) \otimes Y] \otimes Z & \xrightarrow{a_{W,X,Y} \otimes id_Z} & [W \otimes (X \otimes Y)] \otimes Z \\
 & & \downarrow{\scriptstyle a_{W,(X \otimes Y),Z}} \\
\downarrow{\scriptstyle a_{W \otimes X,Y,Z}} & & W \otimes [(X \otimes Y) \otimes Z] \\
 & & \downarrow{\scriptstyle id_W \otimes a_{X,Y,Z}} \\
(W \otimes X) \otimes (Y \otimes Z) & \xrightarrow[a_{W,X,Y \otimes Z}]{} & W \otimes [X \otimes (Y \otimes Z)],
\end{array}$$

$$\begin{array}{ccc}
(X \otimes E) \otimes Y & \xrightarrow{a_{A,E,Y}} & X \otimes (E \otimes Y) \\
 & \searrow{\scriptstyle \rho_X \otimes id_Y} & \downarrow{\scriptstyle id_X \otimes \lambda_Y} \\
 & & X \otimes Y.
\end{array}$$

A monoidal category $(\mathbb{A}, \otimes, E, a, \lambda, \rho)$ is said to be **strict** if the isomorphisms a, λ, and ρ are the identity morphisms. For a monoidal category $(\mathbb{A}, \otimes, E, a, \lambda, \rho)$, we shortly write $(\mathbb{A}, \otimes, E)$ or just $\mathbb{A}$ if no confusion arises.

5.2. Monoidal functors and natural transformations. Let $\mathbb{A}$ and $\mathbb{A}'$ be two monoidal categories. A **monoidal functor** from $\mathbb{A}$ to $\mathbb{A}'$ is a triple (F, φ_0, φ') where $F : \mathbb{A} \to \mathbb{A}'$ is a functor, $\varphi_0 : E' \to F(E)$ is an isomorphism and

$$\varphi' : F(-) \otimes' F(-) \longrightarrow F(- \otimes -)$$

is a natural isomorphism such that for all objects $U, V, W \in \mathbb{A}$, the diagrams

$$\begin{array}{ccc} E' \otimes' F(U) & \xrightarrow{\lambda'_{F(U)}} & F(U) \\ {\scriptstyle \varphi_0 \otimes' id_{F(U)}} \downarrow & & \uparrow {\scriptstyle F(\lambda_U)} \\ F(E) \otimes' F(U) & \xrightarrow[\varphi'_{E,U}]{} & F(E \otimes U), \end{array} \qquad \begin{array}{ccc} F(U) \otimes' E' & \xrightarrow{\rho'_{F(U)}} & F(U) \\ {\scriptstyle id_{F(U)} \otimes' \varphi_0} \downarrow & & \uparrow {\scriptstyle F(\rho_U)} \\ F(U) \otimes' F(E) & \xrightarrow[\varphi'_{U,E}]{} & F(U \otimes E), \end{array}$$

commute and also the following coherence diagram is commutative.

$$\begin{array}{ccc} (F(U) \otimes' F(V)) \otimes' F(W) & \xrightarrow{a'} & F(U) \otimes' (F(V) \otimes' F(W)) \\ {\scriptstyle \varphi'_{U,V} \otimes' id} \downarrow & & \downarrow {\scriptstyle id \otimes' \varphi'_{V,W}} \\ F(U \otimes V) \otimes' F(W) & & F(U) \otimes' F(V \otimes W) \\ {\scriptstyle \varphi'_{U \otimes V, W}} \downarrow & & \downarrow {\scriptstyle \varphi'_{U, V \otimes W}} \\ F((U \otimes V) \otimes W) & \xrightarrow[F(a)]{} & F(U \otimes (V \otimes W)). \end{array}$$

A monoidal functor (F, φ_0, φ') is said to be **strict** if φ_0 and φ' are the identity morphisms.

A **natural monoidal transformation**

$$\gamma : (F, \varphi_0, \varphi') \longrightarrow (G, \psi_0, \psi')$$

between monoidal functors $F, G : \mathbb{A} \to \mathbb{A}'$ is a natural transformation $\gamma : F \to G$ such that for each pair (U, V) of objects in $\mathbb{A}$, we get the commutative diagrams

$$\begin{array}{ccc} F(U) \otimes' F(V) & \xrightarrow{\varphi'_{U,V}} & F(U \otimes V) \\ {\scriptstyle \gamma_U \otimes' \gamma_V} \downarrow & & \downarrow {\scriptstyle \gamma_{(U \otimes V)}} \\ G(U) \otimes' G(V) & \xrightarrow[\psi'_{U,V}]{} & G(U \otimes V), \end{array} \qquad \begin{array}{ccc} & & F(E) \\ & {\scriptstyle \varphi_0} \nearrow & \downarrow {\scriptstyle \gamma_E} \\ E' & \xrightarrow[\psi_0]{} & G(E). \end{array}$$

A **natural monoidal isomorphism** is a natural monoidal transformation that is also a natural isomorphism. A **monoidal equivalence** between two monoidal categories is a monoidal functor $F : \mathbb{A} \to \mathbb{A}'$ such that there exists a monoidal functor $G : \mathbb{A}' \to \mathbb{A}$ and natural isomorphisms $\gamma : id_{\mathbb{A}'} \to FG$ and $\theta : GF \to id_{\mathbb{A}}$. For more details see [18].

5.3. Duality in a monoidal category. The concepts of evaluation and coevaluation morphisms, introduced for module categories over bialgebras and Hopf algebras, (see 5.4 below), can be generalized to monoidal categories in order to find left (and right) dual objects. We will encounter these concepts again in module categories over quasi-Hopf algebras (see 7.3).

A monoidal category $\mathbb{A}$ is called a monoidal category **with left duality** if, for each object V in $\mathbb{A}$, there exist an object V^* and morphisms

$$b_V : E \longrightarrow V \otimes V^* \quad and \quad d_V : V^* \otimes V \longrightarrow E$$

in the category $\mathbb{A}$ such that

$$(id_V \otimes d_V) \circ a_{V,V^*,V} \circ (b_V \otimes id_V) = id_V \quad (d_V \otimes id_{V^*}) \circ a^{-1}_{V^*,V,V^*} \circ (id_{V^*} \otimes b_V) = id_{V^*}.$$

Similarly, a monoidal category $\mathbb{A}$ has **right duality** if, for each object V in $\mathbb{A}$, there exist an object ${}^*V$ and morphisms

$$b'_V : E \longrightarrow {}^*V \otimes V \quad and \quad d'_V : V \otimes {}^*V \longrightarrow E$$

in the category $\mathbb{A}$ such that

$$(d'_V \otimes id_V) \circ a^{-1}_{V,{}^*V,V} \circ (id_V \otimes b'_V) = id_V, \quad (id_{{}^*V} \otimes d'_V) \circ a_{{}^*V,V,{}^*V} \circ (b'_V \otimes id_{{}^*V}) = id_{{}^*V}.$$

A category $\mathbb{A}$ is called **rigid** (or **autonomous**) if it has left and right duality.

5.4. Duality in module categories over a Hopf algebra.

i) Let H be a bialgebra over a commutative ring k. Then the category ${}_H\mathbb{M}$ of left H-modules is a monoidal category (see 2.5).

ii) If H is a Hopf algebra with an antipode S, the category $({}_H\mathbb{M})_{\rm fgp}$ of left H-modules that are finitely generated and projective over k, is a monoidal full subcategory of ${}_H\mathbb{M}$. As seen in 2.6, for any left H-module M, we can endow the dual module $M^* = \mathrm{Hom}_k(M,k)$ with the left H-action $(h \cdot f)(m) = f(S(h)\, m)$ and we have the evaluation map

$$d_M : M^* \otimes M \longrightarrow k, \quad f \otimes m \mapsto f(m),$$

for all $m \in M$ and $f \in M^*$. Now, if M is finitely generated and projective as a k-module, with dual basis $\{m_i\}_{i=1}^n$ and $\{m^i\}_{i=1}^n$, we have the coevaluation map

$$b_M : k \longrightarrow M \otimes M^*, \qquad 1 \mapsto \sum_i m_i \otimes m^i.$$

Both d_M and b_M are H-linear and satisfy

$$(id_M \otimes d_M) \circ (b_M \otimes id_M) = id_M \quad \text{and} \quad (d_M \otimes id_{M^*}) \circ (id_{M^*} \otimes b_M) = id_{M^*},$$

endowing $({}_H\mathbb{M})_{\rm fgp}$ with the structure of a monoidal category with left duality.

iii) If the antipode S is invertible, then for any left H-module M denote by ${}^*M$ the dual k-module $\mathrm{Hom}_k(M,k)$ equipped with the left H-action $(h \cdot f)(m) = f(S^{-1}(h)\, m)$ (see 2.6).
For any finitely generated projective k-module M, define

$$b'_M : k \longrightarrow {}^*M \otimes M, \quad 1 \mapsto \sum_i m^i \otimes m_i, \tag{5.1}$$

$$d'_M : M \otimes {}^*M \longrightarrow k, \quad m \otimes f \mapsto f(m), \tag{5.2}$$

using the same conventions as above. Then b'_M and d'_M are H-linear satisfying

$$(d'_M \otimes id_M) \circ (id_M \otimes b'_M) = id_M \quad and \quad (id_{{}^*M} \otimes d'_M) \circ (b'_M \otimes id_{{}^*M}) = id_{{}^*M}.$$

That is, $({}_H\mathbb{M})_{\rm fgp}$ has right duality, i.e. $({}_H\mathbb{M})_{\rm fgp}$ is autonomous (rigid).

6 Monoidal categories acting on categories

In the following, we present some necessary information about action of monoidal categories on categories to be refered to in this text. One can find more details in [24], [25] and [28].

6.1. Action of a monoidal category. Let $(\mathbb{A}, \otimes, E, a, \lambda, \rho)$ be a monoidal category. A **right $\mathbb{A}$-category** is a four-tuple $(\mathcal{D}, \diamond, \Psi, \mathbf{r})$, where $\mathcal{D}$ is a category, $\diamond : \mathcal{D} \times \mathbb{A} \to \mathcal{D}$ is a functor, and

$$\Psi : (- \diamond -) \diamond - \longrightarrow - \diamond (- \otimes -) \quad \text{and} \quad \mathbf{r} : - \diamond E \longrightarrow id$$

are natural isomorphisms such that for all objects $M \in \mathcal{D}$ and $X, Y, Z \in \mathbb{A}$,

$$(id \diamond a_{X,Y,Z}) \circ \Psi_{M,X\otimes Y,Z} \circ (\Psi_{M,X,Y} \diamond id) = \Psi_{M,X,Y\otimes Z} \circ \Psi_{M\diamond X,Y,Z}, \tag{6.1}$$

$$(id \diamond \lambda_X) \circ \Psi_{M,E,X} = \mathbf{r}_M \diamond id. \tag{6.2}$$

This means commutativity of the diagrams

$$\begin{array}{ccccc}
[(M \diamond X) \diamond Y] \diamond Z & \xrightarrow{\Psi_{M\diamond X,Y,Z}} & (M \diamond X) \diamond (Y \otimes Z) & & \\
 & & & \searrow^{\Psi_{M,X,Y\otimes Z}} & \\
\Big\downarrow{\scriptstyle \Psi_{M,X,Y}\diamond id} & & & & M \diamond [X \otimes (Y \otimes Z)] \\
 & & & \nearrow_{id\diamond a_{X,Y,Z}} & \\
[M \diamond (X \otimes Y)] \diamond Z & \xrightarrow[\Psi_{M,(X\otimes Y),Z}]{} & M \diamond [(X \otimes Y) \otimes Z], & &
\end{array}$$

$$\begin{array}{ccc}
(M \diamond E) \diamond X] & \xrightarrow{\Psi_{M,E,X}} & M \diamond (E \otimes X) \\
\Big\downarrow{\scriptstyle \mathbf{r}_M\diamond id} & & \Big\downarrow{\scriptstyle id\diamond\lambda_X} \\
M \diamond X & \xrightarrow[id_{M\diamond X}]{} & M \diamond X.
\end{array}$$

The natural isomorphism Ψ can be considered as a "mixed associativity constraint" of $\mathcal{D}$.

For any right $\mathbb{A}$-category $(\mathcal{D}, \diamond, \Psi, \mathbf{r})$ and $X \in \mathbb{A}$, we obtain an endofunctor

$$- \diamond X : \mathcal{D} \longrightarrow \mathcal{D}.$$

A **left $\mathbb{A}$-category** $(\mathcal{D}, \diamond', \Psi', \mathbf{l})$ consists of a category $\mathcal{D}$ together with natural isomorphisms

$$\Psi' : (- \otimes -) \diamond' - \longrightarrow - \diamond' (- \diamond' -) \quad \text{and} \quad \mathbf{l} : E \diamond' - \longrightarrow id,$$

with commutative diagrams

$$\begin{array}{ccccc}
[(X \otimes Y) \otimes Z] \diamond' M & \xrightarrow{a_{X,Y,Z}\diamond' id_M} & [X \otimes (Y \otimes Z)] \diamond' M & & \\
 & & & \searrow^{\Psi_{X,(Y\otimes Z),M}} & \\
\Big\downarrow{\scriptstyle \Psi_{(X\otimes Y),Z,M}} & & & & X \diamond' [(Y \otimes Z) \diamond' M] \\
 & & & \swarrow_{id_X\diamond'\psi} & \\
(X \otimes Y) \diamond' (Z \diamond' M) & \xrightarrow[\Psi_{X,Y,(Z\diamond' M)}]{} & X \diamond' [Y \diamond' (Z \diamond' M)], & &
\end{array}$$

$$\begin{array}{ccc}
(X \otimes E) \diamond' M & \xrightarrow{\Psi_{X,E,M}} & X \diamond' (E \diamond' M) \\
{\scriptstyle \rho_X \diamond' id}\big\downarrow & & \big\downarrow{\scriptstyle id \diamond' \mathbf{l}_M} \\
X \diamond' M & \xrightarrow[id_{X \diamond' M}]{} & X \diamond' M.
\end{array}$$

For any left $\mathbb{A}$-category $(\mathcal{D}, \diamond, \Psi, \mathbf{l})$ and $X \in \mathbb{A}$, we obtain an endofunctor

$$X \diamond - : \mathcal{D} \longrightarrow \mathcal{D}.$$

Let $(\mathbb{A}, \otimes, E, a, \lambda, \rho)$ be a monoidal category. Then we have a second monoidal structure $\bar{\mathbb{A}} := (\mathbb{A}, \bar{\otimes} := \otimes \circ \tau, E, \bar{a}, \rho, \lambda)$, where $\tau : \mathbb{A} \times \mathbb{A} \to \mathbb{A} \times \mathbb{A}$ is defined by $\tau(X, Y) = (Y, X)$, and $\bar{a}$ is defined by $\bar{a}_{X,Y,Z} := a^{-1}_{Z,Y,X}$. Now, if $(\mathcal{D}, \diamond', \Psi', \mathbf{l})$ is a **left** $\mathbb{A}$-category, then it becomes a **right** $\bar{\mathbb{A}}$-category, with $\diamond$ defined by $M \diamond X = X \diamond' M$, Ψ is defined by $\Psi_{M,X,Y} := \Psi'^{-1}_{Y,X,M}$, and with $\mathbf{r} = \mathbf{l}$. In this way, we have a bijective correspondence between **left** and **right** $\mathbb{A}$-category structures on a category $\mathcal{D}$, (and results on left (resp. right) $\mathbb{A}$-categories can be translated into results about right (resp. left) $\mathbb{A}$-categories).

6.2. A monoidal category acting on itself. As the first example, we can see that any monoidal category $(\mathbb{A}, \otimes, E, a, \lambda, \rho)$ is itself a left and right $\mathbb{A}$-category with $\diamond = \diamond' = \otimes$, $\Psi = \Psi' = a$ and $\mathbf{r} = \rho$ and $\mathbf{l} = \lambda$. Therefore, it can be considered as a left $\mathbb{A} \times \bar{\mathbb{A}}$-category.

6.3. Modules over an algebra in $\mathbb{A}$. Let $\mathbb{A}$ be a monoidal category and (A, μ, ι) be an algebra in this monoidal category. Then $\mathcal{D} = \mathbb{A}_A$, the category of all right A-modules in $\mathbb{A}$, is a left $\mathbb{A}$-category since for all $X \in \mathbb{A}$ and $(M, \varrho_M) \in \mathcal{D}$, $X \otimes M$ carries the structure of a right A-module by

$$(X \otimes M) \otimes A \simeq X \otimes (M \otimes A) \overset{id \otimes \varrho_M}{\longrightarrow} X \otimes M.$$

In this way, for any object $X \in \mathbb{A}$, we have an endofunctor

$$X \diamond - : \mathbb{A}_A \longrightarrow \mathbb{A}_A.$$

In the special case $\mathbb{A} = \mathbb{M}^H$, for a bialgebra H, let A be an algebra in $\mathbb{A}$ (a right H-comodule algebra). Then the category $\mathbb{B} = (\mathbb{M}^H)_A$ of all right (H, A)-Hopf modules is a left $\mathbb{M}^H$-category. So for any right H-comodule M, we have the endofunctor $M \otimes_k - : (\mathbb{M}^H)_A \to (\mathbb{M}^H)_A$. In particular, the endofunctor $A \otimes -$ is a monad on the category $(\mathbb{M}^H)_A$ of right (H, A)-Hopf modules.

Now let $\mathbb{A} = {}_H\mathbb{M}$, and A be an algebra in ${}_H\mathbb{M}$ (a left H-module algebra). Then the category $\mathcal{D} = {}_A({}_H\mathbb{M}) \simeq {}_{A\#H}\mathbb{M}$ is a right ${}_H\mathbb{M}$-category. Furthermore, if $\mathbb{A} = \mathbb{M}_H$ and A is an algebra in $\mathbb{A}$ (a right H-module algebra), then the category $\mathbb{B}' = (\mathbb{M}_H)_A$ of all right A-modules in $\mathbb{M}_H$, is a left $\mathbb{M}_H$-category.

6.4. Comodules over a coalgebra in $\mathbb{A}$. Let again $\mathbb{A}$ be a monoidal category and (C, Δ, ε) be a coalgebra in this monoidal category. Then $\mathcal{D} = \mathbb{A}^C$, the category of all right C-comodules in $\mathbb{A}$, is a left $\mathbb{A}$-category: for all $X \in \mathbb{A}$ and $(M, \varrho^M) \in \mathcal{D}$, $X \otimes M$ carries the structure of a right C-comodule by

$$X \otimes M \overset{id \otimes \varrho^M}{\longrightarrow} X \otimes (M \otimes C) \simeq (X \otimes M) \otimes C.$$

In the case $\mathbb{A} = \mathbb{M}^H$, for a k-bialgebra H, if C is a coalgebra in $\mathbb{A}$ (a right H-comodule coalgebra), let $\mathbb{B} = (\mathbb{M}^H)^C$. Then a k-module M is in $\mathbb{B} = (\mathbb{M}^H)^C$ if and only if M is a right H-comodule and a right C-comodule with commutative diagram

$$\begin{array}{ccccc} M & & \xrightarrow{\quad\rho_H^M\quad} & & M \otimes H \\ {\scriptstyle \rho_C^M}\downarrow & & & & \downarrow{\scriptstyle \rho_C^M \otimes id_H} \\ M \otimes C & \xrightarrow[id \otimes \rho_H^C]{} & M \otimes (C \otimes H) & \xrightarrow[\cong]{} & (M \otimes C) \otimes H. \end{array}$$

In this case, the category $\mathcal{D} = (\mathbb{M}^H)^C$ is a left $\mathbb{M}^H$-category.

If $\mathbb{A} = \mathbb{M}_H$ and C is a coalgebra in $\mathbb{A}$ (a right H-module coalgebra), then a k-module M is in $\mathcal{D}' = (\mathbb{M}_H)^C$ if and only if M is a right H-module and a right C-comodule such that

$$\rho_C^M(m\,h) = \sum m_{(0)}\, h_1 \otimes m_{(1)^C}\, h_2 = \rho_C^M(m)\,\Delta(h).$$

In this case, the category $\mathcal{D}' = (\mathbb{M}_H)^C$ is a left $\mathbb{M}_H$-category.

6.5. $\mathbb{A}$-functors and natural transformations. Let $\mathbb{A}$ be a monoidal category, $(\mathcal{D}, \diamond, \Psi, \mathbf{l})$ and $(\mathcal{D}', \diamond', \Psi', \mathbf{l}')$ be two left $\mathbb{A}$-categories. A (left) **$\mathbb{A}$-functor** (F, ξ) consists of a functor $F : \mathcal{D} \to \mathcal{D}'$ and a natural isomorphism

$$\xi : - \diamond' F(-) \longrightarrow F(- \diamond -)$$

satisfying $\xi_{E,M} = id_{F(M)}$, for all $M \in \mathcal{D}$, and the coherence condition (for all $X, Y \in \mathbb{A}, M \in \mathcal{D}$)

$$\xi_{X,Y\diamond M} \circ (X \diamond' \xi_{Y,M}) \circ \Psi'_{X,Y,F(M)} = F(\Psi_{X,Y,M}) \circ \xi_{X\otimes Y,M},$$

this means commutativity of the diagram

$$\begin{array}{ccccc} (X \otimes Y) \diamond' F(M) & \xrightarrow{\xi_{X\otimes Y,M}} & F((X \otimes Y) \diamond M) & & \\ & & & \searrow^{F(\Psi_{X,Y,M})} & \\ {\scriptstyle \Psi'_{X,Y,F(M)}}\downarrow & & & & F(X \diamond (Y \diamond M)) \\ & & & \nearrow_{\xi_{X,Y\diamond M}} & \\ X \diamond' (Y \diamond' F(M)) & \xrightarrow[id_X \diamond' \xi_{Y,M}]{} & X \diamond' F(Y \diamond M). & & \end{array}$$

Let (F, ξ) and $(F', \xi') : \mathcal{D} \to \mathcal{D}'$ be two $\mathbb{A}$-functors between $\mathbb{A}$-categories $\mathcal{D}$ and $\mathcal{D}'$. An **$\mathbb{A}$-natural transformation** between (F, ξ) and (F', ξ') is a natural transformation $\varphi : F \to F'$ such that for all $X \in \mathbb{A}, M \in \mathcal{D}$

$$\varphi_{X\diamond M} \circ \xi_{X,M} = \xi'_{X,M} \circ (id \diamond' \varphi_M),$$

this means commutativity of the diagram

$$\begin{array}{ccc} X \diamond' F(M) & \xrightarrow{\xi_{X,M}} & F(X \diamond M) \\ {\scriptstyle id_X \diamond' \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_{X\diamond M}} \\ X \diamond' F'(M) & \xrightarrow[\xi'_{X,M}]{} & F'(X \diamond M). \end{array}$$

6.6. (Co)modules over (co)algebras in $\mathbb{A}$-categories. Let $\mathbb{A}$ be a monoidal category and A an algebra in this category. If $(\mathcal{D}, \diamond', \Psi', \mathbf{l})$ is a right $\mathbb{A}$-category, then we can define right modules in $\mathcal{D}$ over A as follows. A **right module in $\mathcal{D}$ over A** or **right A-module in $\mathcal{D}$** is an object M of $\mathcal{D}$ together with a morphism $\varrho_M : M \diamond A \to M$ and the commutative diagrams

$$\begin{array}{ccccc} (M\diamond A)\diamond A & & \xrightarrow{\varrho_M\diamond id} & & M\diamond A \\ \downarrow{\scriptstyle \Psi_{M,A,A}} & & & & \downarrow{\scriptstyle \varrho_M} \\ M\diamond(A\otimes A) & \xrightarrow[id\diamond\mu_A]{} & M\diamond A & \xrightarrow[\varrho_M]{} & M, \end{array} \qquad \begin{array}{ccc} M\diamond E & \xrightarrow{\mathbf{r}_M} & M \\ & {\scriptstyle id\diamond\iota_A}\searrow \quad \nearrow{\scriptstyle \varrho_M} & \\ & M\diamond A. & \end{array}$$

We denote the category of right A-modules in $\mathcal{D}$ by $\mathcal{D}_A$.

Similarly, for a left $\mathbb{A}$-category $\mathcal{D}$ one can define left modules in $\mathcal{D}$ over A.

Furthermore, one can also define a right (resp. left) comodule in a right (resp. left) $\mathbb{A}$-category $\mathcal{D}$ over a coalgebra C in the monoidal category $\mathbb{A}$ as follows:

Let $(C, \underline{\Delta}, \underline{\varepsilon})$ be a coalgebra in the monoidal category $(\mathbb{A}, \otimes, E, a, \lambda, \rho)$ and $\mathcal{D}$ be a right $\mathbb{A}$-category. A right **comodule** (M, ϱ^M) **in $\mathcal{D}$ over C** is an object $M \in \mathcal{D}$ with a morphism $\varrho^M : M \to M \diamond C$ which is coassociative in the sense that

$$\Psi_{M,C,C} \circ (\varrho^M \diamond id_C) \circ \varrho^M = (id \diamond \underline{\Delta}) \circ \varrho^M, \quad \text{and} \quad (id \diamond \underline{\varepsilon}) \circ \varrho^M = \mathbf{r}_M.$$

i.e. the following diagrams are commutative.

$$\begin{array}{ccc} M & \xrightarrow{\varrho^M} & M\diamond C \\ & & \downarrow{\scriptstyle \varrho^M\diamond id_C} \\ \downarrow{\scriptstyle \varrho^M} & & (M\diamond C)\diamond C \\ & & \downarrow{\scriptstyle \Psi_{M,C,C}} \\ M\diamond C & \xrightarrow[id_M\diamond\underline{\Delta}]{} & M\diamond(C\otimes C), \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\varrho^M} & M\diamond C \\ & {\scriptstyle \mathbf{r}_M}\searrow & \downarrow{\scriptstyle id_M\diamond\underline{\varepsilon}} \\ & & M\diamond C. \end{array}$$

In short, a comodule in $\mathcal{D}$ over a coalgebra C in $\mathbb{A}$, is a module over an algebra C in $\mathbb{A}^{op}$.

Chapter 3

Hom-tensor relations for quasi-Hopf algebras

7 Quasi-bialgebras and quasi-Hopf algebras

Quasi-bialgebras and quasi-Hopf algebras were defined by Drinfeld in [13]. These are generalizations of the concepts of bialgebras and Hopf algebras in such a way that their module categories are still monoidal (even rigid monoidal, in the finite case).
The most important aspect of this generalization comes from the non-coassociativity of the comultiplication. However, this non-coassociativity is controlled by a 3-cocycle ϕ. In this way, we have a monoidal structure on the module category similar to the Hopf algebra case but with non-trivial associativity constraints of the tensor product. Thus, the forgetful functor will not be (coherent) monoidal. But it still preserves the tensor product.

Quasi-bialgebras and quasi-Hopf algebras were mainly considered over fields and in the finite dimensional case. However, many parts of the formalism still work over a commutative base ring. In this chapter, we outline these notions without any finiteness conditions. We recall some Hom-tensor relations in module categories over Hopf algebras and generalize them for modules over quasi-Hopf algebras and their comodule algebras. Our results imply the results for the finite dimensional (finitely generated and projective) cases.

7.1. Quasi-bialgebras. A four tuple $(H, \Delta, \varepsilon, \phi)$ is called a **quasi-bialgebra** if H is an associative k-algebra with unit, ϕ an invertible element in $H \otimes H \otimes H$, the comultiplication $\Delta : H \to H \otimes H$ and the counit $\varepsilon : H \to k$ are algebra maps, satisfying the identities for $h \in H$

$$(id \otimes \varepsilon) \circ \Delta(h) = h \otimes 1, \quad (id \otimes \varepsilon) \circ \Delta(h) = 1 \otimes h, \tag{7.1}$$

$$(id \otimes \Delta) \circ \Delta(h) = \phi \cdot (\Delta \otimes id) \circ \Delta(h) \cdot \phi^{-1}, \tag{7.2}$$

$$(id \otimes id \otimes \Delta)(\phi)(\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi)(id \otimes \Delta \otimes id)(\phi)(\phi \otimes 1), \tag{7.3}$$

$$(id \otimes \varepsilon \otimes id)(\phi) = 1 \otimes 1. \tag{7.4}$$

The identities (7.1), (7.3) and (7.4) imply also,

$$(\varepsilon \otimes id \otimes id)(\phi) = (id \otimes id \otimes \varepsilon)(\phi) = 1 \otimes 1. \tag{7.5}$$

ϕ is called the **Drinfeld reassociator**. The equation (7.3) is a 3-cocycle condition on ϕ (see section 3.1). We use the Sweedler type notations $\Delta(h) = \sum h_1 \otimes h_2$, and

$$(\Delta \otimes id) \circ \Delta(h) = \sum h_{11} \otimes h_{12} \otimes h_2, \quad (id \otimes \Delta) \circ \Delta(h) = \sum h_1 \otimes h_{21} \otimes h_{22}.$$

We denote the tensor components of ϕ by capital letters and those of ϕ^{-1} by small letters, namely

$$\phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum Y^1 \otimes Y^2 \otimes Y^3 = \sum T^1 \otimes T^2 \otimes T^3 = ...etc.$$

$$\phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum y^1 \otimes y^2 \otimes y^3 = \sum t^1 \otimes t^2 \otimes t^3 = ...etc.$$

As in the Hopf algebra case, (c.f. 2.5, see also section 5), we have:

For a quasi-bialgebra $(H, \Delta, \varepsilon, \phi)$, the categories ${}_H\mathbb{M}$, $\mathbb{M}_H$ and ${}_H\mathbb{M}_H$, with the tensor product $\otimes_k$, are monoidal categories.

Proof. We sketch a proof of this basic facts just for left H-modules. The proofs for right and bi-modules are similar (see remark below).

The associativity constraint for objects $M, N, L \in {}_H\mathbb{M}$ is given by

$$a_{M,N,L} : (M \otimes_k N) \otimes_k L \longrightarrow M \otimes_k (N \otimes_k L),$$

$$a_{M,N,L}((m \otimes n) \otimes l) = \phi \cdot (m \otimes (n \otimes l)).$$

For this, first we have to show that this is an H-linear map, i.e. for any $h \in H$,

$$a_{M,N,L}(h \cdot ((m \otimes n) \otimes l)) = h \cdot a_{M,N,L}((m \otimes n) \otimes l).$$

$$\begin{array}{rcl} L.H.S & = & a_{M,N,L}(h_1 \cdot (m \otimes n) \otimes h_2 l)) \\ & = & a_{M,N,L}(((\Delta \otimes id) \circ \Delta(h)) \cdot (m \otimes n \otimes l)) \\ & = & a_{M,N,L}((\phi^{-1} \cdot (id \otimes \Delta) \circ \Delta(h) \cdot \phi) \cdot (m \otimes n \otimes l)) \\ & = & \phi\, \phi^{-1} \cdot ((id \otimes \Delta) \circ \Delta(h)) \cdot \phi) \cdot (m \otimes (n \otimes l)) \\ & = & h \cdot a_{M,N,L}((m \otimes n) \otimes l) = R.H.S. \end{array}$$

Next, the associativity constraint a has to satisfy the pentagon diagram

$$\begin{array}{ccc} [V \otimes (M \otimes N)] \otimes L & \xrightarrow{a_{V,(M \otimes N),L}} & V \otimes [(M \otimes N) \otimes L] \\ \uparrow{\scriptstyle a_{V,M,N} \otimes id_L} & & \\ [(V \otimes M) \otimes N] \otimes L & & \downarrow{\scriptstyle id_V \otimes a_{M,N,L}} \\ \downarrow{\scriptstyle a_{(V \otimes M),N,L}} & & \\ (V \otimes M) \otimes (N \otimes L) & \xrightarrow[a_{V,M,(N \otimes L)}]{} & V \otimes [M \otimes (N \otimes L)], \end{array}$$

suppressing the symbol $\sum$, this means the commutativity of the diagram

$$\begin{array}{ccc} [X^1 v \otimes (X^2 m \otimes X^3 n)] \otimes l & \xrightarrow{a_{V,(M \otimes N),L}} & Y^1 X^1 v \otimes [(Y^2_1 X^2 m \otimes Y^2_2 X^3 n) \otimes Y^3 l] \\ \uparrow{\scriptstyle a_{V,M,N} \otimes id_L} & & \downarrow{\scriptstyle id_V \otimes a_{M,N,L}} \\ [(v \otimes m) \otimes n] \otimes l & & Y^1 X^1 v \otimes T^1 Y^2_1 X^2 m \otimes T^2 Y^2_2 X^3 n \otimes T^3 Y^3 l \\ \downarrow{\scriptstyle a_{(V \otimes M),N,L}} & & \downarrow{\scriptstyle id} \\ (X^1_1 v \otimes X^1_2 m) \otimes (X^2 n \otimes X^3 l) & \xrightarrow[a_{V,M,(N \otimes L)}]{} & Y^1 X^1_1 v \otimes [Y^2 X^1_2 m \otimes (Y^3_1 X^2 n \otimes Y^3_2 X^3 l)], \end{array}$$

that is,

$$a_{V,M,(N\otimes L)} \circ a_{(V\otimes M),N,L}(v\otimes m\otimes n\otimes l) = (id_V \otimes a_{M,N,L})\circ(a_{V,M\otimes N,L})\circ(a_{V,M,N}\otimes id_L)(v\otimes m\otimes n\otimes l).$$

$$\begin{aligned} L.H.S &= Y^1X^1_1 v\otimes [Y^2X^1_2 m\otimes (Y^3_1X^2 n\otimes Y^3_2X^3 l)] \\ &= (id\otimes id\otimes \Delta)(\phi)\cdot(\Delta\otimes id\otimes id)(\phi)\cdot(v\otimes m\otimes n\otimes l), \end{aligned}$$

$$\begin{aligned} R.H.S &= (Y^1X^1 v\otimes T^1Y^2_1X^2 m\otimes T^2Y^2_2X^3 n\otimes T^3Y^3 l) \\ &= (1\otimes\phi)\cdot(id\otimes\Delta\otimes id)(\phi)\cdot(\phi\otimes 1)\cdot(v\otimes m\otimes n\otimes l), \end{aligned}$$

and these expressions are equal by the axiom (7.3). Considering the trivial isomorphisms $M\otimes k \overset{\rho_M}{\simeq} M \overset{\lambda_M}{\simeq} k\otimes M$, the commutativity of the triangular diagram

$$\begin{array}{ccc} (M\otimes k)\otimes N & \xrightarrow{a_{M,k,N}} & M\otimes(k\otimes N) \\ & \searrow^{\rho_M\otimes id_N} & \downarrow{\scriptstyle id_M\otimes\lambda_N} \\ & & M\otimes N, \end{array}$$

follows from the axiom (7.4). □

Remark. For the category $\mathbb{M}_H$, the associativity constraint, for $M, N, L \in \mathbb{M}_H$, is

$$a'_{M,N,L} : (M\otimes_k N)\otimes_k L \longrightarrow M\otimes_k (N\otimes_k L),$$

$$a'_{M,N,L}((m\otimes n)\otimes l) = (m\otimes(n\otimes l))\cdot\phi^{-1},$$

Combining the left and right cases, we obtain the associativity constraint for (H,H)-bimodules as

$$a''_{M,N,L} : (M\otimes_k N)\otimes_k L \longrightarrow M\otimes_k (N\otimes_k L),$$

$$a''_{M,N,L}((m\otimes n)\otimes l) = \phi\cdot(m\otimes(n\otimes l))\cdot\phi^{-1}.$$

If $(H,\Delta,\varepsilon,\phi)$ and $(H',\Delta',\varepsilon',\phi')$ are quasi-bialgebras, then the tensor product $(H\otimes H', (I\otimes\tau_{H,H}\otimes I)\circ(\Delta\otimes\Delta'), \varepsilon\otimes\varepsilon', \sum X^1\otimes X'^1\otimes X^2\otimes X'^2\otimes X^3\otimes X'^3)$ is also a quasi-bialgebra.

7.2. Quasi-Hopf algebras. ([13] and [18]) A **quasi-antipode** (S,α,β) for a quasi-bialgebra H consists of an invertible algebra anti-automorphism $S : H \to H$ and elements $\alpha,\beta\in H$ with the identities, for $h\in H$,

$$\sum\nolimits_h S(h_1)\alpha h_2 = \varepsilon(h)\alpha, \quad \sum\nolimits_h h_1\beta S(h_2) = \varepsilon(h)\beta \tag{7.6}$$

$$\sum X^1\beta S(X^2)\alpha X^3 = 1, \qquad \sum S(x^1)\alpha x^2\beta x^3 = 1. \tag{7.7}$$

A **quasi-Hopf algebra** is a quasi-bialgebra H together with a quasi-antipode (S,α,β). The axioms for a quasi-Hopf algebra imply that $\varepsilon(\alpha)\varepsilon(\beta) = 1$, and $\varepsilon\circ S = \varepsilon$.
$(H,\Delta,\varepsilon,\phi,S,\alpha,\beta)$ expresses the complete data of a quasi-Hopf algebra.

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \phi, S, \alpha, \beta)$, we also have H^{op}, H^{cop}, and $H^{op,cop}$ as quasi-Hopf algebras, where "op" means opposite multiplication and "cop" means opposite comultiplication. The quasi-Hopf structures are obtained by putting $\phi_{op} = \phi^{-1}$, $\phi_{cop} = (\phi^{-1})^{321} = \sum x^3 \otimes x^2 \otimes x^1$, $\phi_{op,cop} = \phi^{321} = \sum X^3 \otimes X^2 \otimes X^1$, $S_{op} = S_{cop} = (S_{op,cop})^{-1} = S^{-1}$, $\alpha_{op} = S^{-1}(\beta)$, $\alpha_{cop} = S^{-1}(\alpha)$, $\alpha_{op,cop} = \beta$, $\beta_{op} = S^{-1}(\alpha)$, $\beta_{cop} = S^{-1}(\beta)$ and $\beta_{op,cop} = \alpha$.
This means that if $H = (H, \Delta, \varepsilon, \phi, S, \alpha, \beta)$ is a quasi-Hopf algebra, then we have new quasi-Hopf algebras given by

$$\begin{aligned} H^{op} &:= (H, \mu^{op}, \iota, \Delta, \varepsilon, \phi^{-1}, S^{-1}, S^{-1}(\beta), S^{-1}(\alpha)), \\ H^{cop} &:= (H, \mu, \iota, \Delta^{op}, \varepsilon, (\phi^{-1})^{321}, S^{-1}, S^{-1}(\alpha), S^{-1}(\beta)), \\ H^{op,cop} &:= (H, \mu^{op}, \iota, \Delta^{op}, \varepsilon, \phi^{321}, S, \beta, \alpha). \end{aligned}$$

7.3. Rigidity of $({}_H\mathbb{M})_{\mathbf{fgp}}$. Let H be a quasi-Hopf algebra. We consider the category $({}_H\mathbb{M})_{\mathrm{fgp}}$ of left H-modules which are finitely generated and projective as k-modules, and equip it with the monoidal structure induced by Δ and ϕ. Now for any object V in this category, with a dual basis $\{v_i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$, consider the k-module $V^* = \mathrm{Hom}_k(V, k)$ with the left H-action $(h \cdot f)(v) = f(S(h)\, v)$ (see (2.5)), and define the maps

$$b_V : k \longrightarrow V \otimes V^*, \quad 1 \mapsto \sum \beta v_i \otimes v^i, \tag{7.8}$$

$$d_V : V^* \otimes V \longrightarrow k, \quad f \otimes v \mapsto f(\alpha\, v). \tag{7.9}$$

Furthermore, we consider ${}^*V$ to be the same dual k-module, equipped with the left H-action given for all $h \in H$ and $f \in {}^*V$ by $(h \cdot f)(v) = f(S^{-1}(h)\, v)$ (see (2.7)) and define the maps

$$b'_V : k \longrightarrow {}^*V \otimes V, \quad 1 \mapsto \sum v^i \otimes S^{-1}(\beta)\, v_i, \tag{7.10}$$

$$d'_V : V \otimes {}^*V \longrightarrow k, \quad v \otimes g \mapsto g(S^{-1}(\alpha)\, v). \tag{7.11}$$

The maps b_V, d_V, b'_V and d'_V (defined above) are H-linear and all the following composites are identity maps:

(1) $V \cong k \otimes V \xrightarrow{b \otimes id} (V \otimes V^*) \otimes V \xrightarrow{a_{V,V^*,V}} V \otimes (V^* \otimes V) \xrightarrow{id \otimes d} V \otimes k \cong V$,

(2) $V^* \cong V^* \otimes k \xrightarrow{id \otimes b} V^* \otimes (V \otimes V^*) \xrightarrow{a^{-1}_{V^*,V,V^*}} (V^* \otimes V) \otimes V^* \xrightarrow{d \otimes id} k \otimes V^* \cong V^*$,

(3) $V \cong V \otimes k \xrightarrow{id \otimes b'} V \otimes ({}^*V \otimes V) \xrightarrow{a^{-1}_{V,{}^*V,V}} (V \otimes {}^*V) \otimes V \xrightarrow{d' \otimes id} k \otimes V \cong V$,

(4) ${}^*V \cong k \otimes {}^*V \xrightarrow{b' \otimes id} ({}^*V \otimes V) \otimes {}^*V \xrightarrow{a_{{}^*V,V,{}^*V}} {}^*V \otimes (V \otimes {}^*V) \xrightarrow{id \otimes d'} {}^*V \otimes k \cong {}^*V$.

This shows that the category $({}_H\mathbb{M})_{\mathrm{fgp}}$ of left H-modules that are finitely generated and projective as k-modules, is a *rigid* category.

7.4. Rigidity of $(\mathbb{M}_H)_{\mathbf{fgp}}$. Proceding in a similar way as above, we get that the category $(\mathbb{M}_H)_{\mathrm{fgp}}$ of right H-modules, that are finitely generated and projective as k-modules, is also a *rigid* category. In this case, the *left dual* of an object V is again $V^* = \mathrm{Hom}_k(V,k)$ as a k-module with the right H-module structure

$$(f \cdot h)(v) = f(v\, S^{-1}(h)) \quad \text{for } h \in H, v \in V \text{ and } f \in V^*,$$

and we have the evaluation and coevaluation

$$d_V : V^* \otimes V \longrightarrow k, \quad f \otimes v \mapsto f(v\, S^{-1}(\beta)).$$

$$b_V : k \longrightarrow V \otimes V^*, \quad 1 \mapsto \sum v_i\, S^{-1}(\alpha) \otimes v^i.$$

The *right dual* ${}^*V$ will be the same dual k-module $\mathrm{Hom}_k(V,k)$, equipped with the right H-action given for all $h \in H$ and $f \in {}^*V$ by

$$(f \cdot h)(v) = f(v\, S(h)),$$

and the evaluation and the coevaluation defined as

$$d'_V : V \otimes {}^*V \longrightarrow k, \quad v \otimes g \mapsto g(v\, \beta),$$

$$b'_V : k \longrightarrow {}^*V \otimes V, \quad 1 \mapsto \sum v^i \otimes v_i\, \alpha,$$

where $\{v_i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$ form a dual basis for the finitely generated projective k-modules V (see 1.2). Again, the maps b_V, d_V, b'_V and d'_V are H-linear and the corresponding compositions of maps (similar to the left H-module case) are identity maps, that is, the category $(\mathbb{M}_H)_{\mathrm{fgp}}$ is a *rigid* category.

7.5. Rigidity of $({}_H\mathbb{M}_H)_{\mathbf{fgp}}$. Combining the results about the rigidity of left and right H-modules, we can see with a similar argument, that the category $({}_H\mathbb{M}_H)_{\mathrm{fgp}}$ of (H,H)-bimodules which are finitely generated and projective as k-modules, is a *rigid* category. In this case, the *left dual* of an object $V \in ({}_H\mathbb{M}_H)_{\mathrm{fgp}}$ is again $V^* = \mathrm{Hom}_k(V,k)$ as a k-module with the (H,H)-bimodule structure

$$(h \cdot f \cdot h')(v) = f(S(h)\, v\, S^{-1}(h')) \quad \text{for } h, h' \in H, v \in V \text{ and } f \in V^*,$$

and the evaluation and coevaluation come out as

$$d_V : V^* \otimes V \longrightarrow k, \quad f \otimes v \mapsto f(\alpha\, v\, S^{-1}(\beta)),$$

$$b_V : k \longrightarrow V \otimes V^*, \quad 1 \mapsto \sum \beta\, v_i\, S^{-1}(\alpha) \otimes v^i.$$

Similarly, the *right dual* is ${}^*V = \mathrm{Hom}_k(V,k)$, equipped with the (H,H)-bimodule structure given for $h, h' \in H$ and $f \in {}^*V$ by

$$(h \cdot f \cdot h')(v) = f(S^{-1}(h)\, v\, S(h')),$$

with the evaluation and the coevaluation

$$d'_V : V \otimes {}^*V \longrightarrow k, \quad v \otimes g \mapsto g(S^{-1}(\alpha)\, v\, \beta),$$

$$b'_V : k \longrightarrow {}^*V \otimes V, \quad 1 \mapsto \sum v^i \otimes S^{-1}(\beta)\, v_i\, \alpha.$$

7.6. Gauge transformations. The definition of the quasi-bialgebras and quasi-Hopf algebra is "twist covariant" in the following sense.

Let $H = (H, \Delta, \varepsilon, \phi)$ be a quasi-bialgebra (resp. a quasi-Hopf algebra). A **gauge transformation** on H is an invertible element $F \in H \otimes H$ such that

$$(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1. \tag{7.12}$$

Using a gauge transformation F on H, one can build a new quasi-bialgebra (resp. a quasi-Hopf algebra) H_F by keeping the multiplication, unit and counit (and antipode S) of H and replacing the comultiplication of H by

$$\Delta_F : H \longrightarrow H \otimes H, \quad h \longmapsto F\,\Delta(h)\,F^{-1}, \tag{7.13}$$

(for $h \in H$), and with a new Drinfeld reassociator ϕ_F given by

$$\phi_F := (1 \otimes F)(id \otimes \Delta)(F)\,\phi\,(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1) \in H \otimes H \otimes H. \tag{7.14}$$

In case of a quasi-Hopf algebra, α and β must be replaced by

$$\alpha_F := \sum S(G^1)\alpha G^2, \quad \beta_F := \sum F^1 \beta S(F^2), \tag{7.15}$$

where we write by $F^{-1} = \sum G^1 \otimes G^2 \in H \otimes H$ (see [18, P. 373]).
Observe that if H happens to be a bialgebra, then H_F in general is not a bialgebra. This procedure provides non-trivial examples of quasi-bialgebras. However, we get again a bialgebra by twisting a bialgebra with a 2-cocycle F (see section 3.1).

In the Hopf algebra case, the antipode is an anti-coalgebra map, i.e.

$$(S \otimes S) \circ \Delta^{cop} = \Delta \circ S.$$

In this case, we have the identities like

$$\sum h_1 \otimes h_2 S(h_3) = h \otimes 1, \qquad \text{for } h \in H, \tag{7.16}$$

7.7. Some properties of quasi-antipode. For the quasi-Hopf algebra H, Drinfeld ([13]) defined a *gauge element* as an $f \in H \otimes H$, satisfying for $h \in H$,

$$\begin{aligned} f\Delta(h)f^{-1} &= (S \otimes S)\Delta^{cop}S^{-1}(h), & (7.17)\\ (S \otimes S \otimes S)(\phi^{321}) &= (1 \otimes f)(id \otimes \Delta)(f)\phi(\Delta \otimes id)(f^{-1})(f^{-1} \otimes 1), & (7.18)\\ (id \otimes \varepsilon)(f) &= (\varepsilon \otimes id)(f) = 1. & (7.19)\end{aligned}$$

Such an f can be computed explicitly as follows. First set

$$\sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (1 \otimes \phi^{-1})(id \otimes id \otimes \Delta)(\phi), \tag{7.20}$$

$$\sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\phi)(\phi^{-1} \otimes 1), \tag{7.21}$$

and then define γ and δ in $H \otimes H$ by

$$\gamma = \sum S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4, \tag{7.22}$$

$$\delta = \sum B^1 \beta S(B^4) \otimes B^2 \beta S(B^3). \tag{7.23}$$

f and f^{-1} are then given by the formula

$$f = \sum (S \otimes S)(\Delta^{op}(x^1))\gamma \Delta(x^2 \beta S(x^3)), \tag{7.24}$$

$$f^{-1} = \sum \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{op}(x^3)), \tag{7.25}$$

and f satisfies the relations

$$f\Delta(\alpha) = \gamma, \quad \Delta(\beta) f^{-1} = \delta. \tag{7.26}$$

Taking $f = \sum f^1 \otimes f^2$ and $f^{-1} = \sum g^1 \otimes g^2$ as in (7.24) and (7.25), it can be easily seen that

$$\sum f^1 \beta S(f^2) = S(\alpha), \quad \sum S(\beta f^1) f^2 = \alpha, \quad \sum g^1 S(g^2 \alpha) = \beta. \tag{7.27}$$

Formulas similar to (7.16) can be obtained for quasi-Hopf algebras. Following Hausser and Nill [15], [16], [17], define the elements

$$p_L = p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}(X^1 \beta) \otimes X^3 = (\mu^{op} \otimes id) \circ (S^{-1} \circ R_\beta \otimes id \otimes id)(\phi) \tag{7.28}$$

$$q_L = q_L^1 \otimes q_L^2 = \sum S(x^1)\alpha x^2 \otimes x^3 = (\mu \otimes id) \circ (S \otimes L_\alpha \otimes id)(\phi^{-1}) \tag{7.29}$$

$$p_R = p_R^1 \otimes p_R^2 = \sum x^1 \otimes x^2 \beta S(x^3) = (id \otimes \mu) \circ (id \otimes R_\beta \otimes S)(\phi^{-1}) \tag{7.30}$$

$$q_R = q_R^1 \otimes q_R^2 = \sum X^1 \otimes S^{-1}(\alpha X^3) X^2 = (id \otimes \mu^{op}) \circ (id \otimes id \otimes S^{-1} \circ L_\alpha)(\phi) \tag{7.31}$$

As showed in [15], for $h \in H$, they satisfy the equations

$$\sum \Delta(h_2) p_L (S^{-1}(h_1) \otimes 1) = p_L (1 \otimes h) \tag{7.32}$$

$$\sum (S(h_1) \otimes 1) q_L \Delta(h_2) = (1 \otimes h) q_L \tag{7.33}$$

$$\sum \Delta(h_1) p_R (1 \otimes S(h_2)) = p_R (h \otimes 1) \tag{7.34}$$

$$\sum (1 \otimes S^{-1}(h_2)) q_R \Delta(h_1) = (h \otimes 1) q_R \tag{7.35}$$

and

$$\sum \Delta(q_L^2) p_L (S^{-1}(q_L^1) \otimes 1) = 1 \otimes 1 \tag{7.36}$$

$$\sum (S(p_L^1) \otimes 1) q_L \Delta(p_L^2) = 1 \otimes 1 \tag{7.37}$$

$$\sum \Delta(q_R^1) p_R (1 \otimes S(q_R^2)) = 1 \otimes 1 \tag{7.38}$$

$$\sum (1 \otimes S^{-1}(p_R^2)) q_R \Delta(p_R^1) = 1 \otimes 1 \tag{7.39}$$

These identities will be used freely in the sequel. For example, the elements p_L, p_R, q_L and q_R and their relating identities are essential for stating the Hom-tensor relations in section 9 and defining the concept of *coinvarints* in section 13 and showing their properties.

8 Module algebras and smash products for quasi-bialgebras

Let $(H, \Delta, \varepsilon, \phi)$ be a quasi-bialgebra. Then the category of left (resp. right) H-modules $({}_H\mathbb{M}, \otimes_k, a)$ [resp. $(\mathbb{M}_H, \otimes_k, a')$] is a monoidal category, where

$$a_{M,N,L},\, b_{M,N,L} : (M \otimes_k N) \otimes_k L \longrightarrow M \otimes_k (N \otimes_k L),$$

$$a_{M,N,L}((m \otimes n) \otimes l) = \phi \cdot (m \otimes (n \otimes l)),$$

$$[resp. \quad a'_{M,N,L}((m \otimes n) \otimes l) = (m \otimes (n \otimes l)) \cdot \phi^{-1}],$$

are the corresponding associativity constraints, respectively (see 7.1).

8.1. Module algebras over quasi-bialgebras. Let $(H, \Delta, \varepsilon, \phi)$ be a quasi-bialgebra. A k-module A is called a **left H-module algebra** if A is an algebra in the monoidal category ${}_H\mathbb{M}$, i.e.

i) A has a multiplication $\mu_A : A \otimes A \longrightarrow A$ and a unit 1_A,

ii) $(ab)c = \sum (X^1\, a)[(X^2\, b)(X^3\, c)]$,

iii) $h \cdot (ab) = \sum (h_1\, a)(h_2\, b)$,

iv) $h \cdot 1_A = \varepsilon(h) 1_A$,

for $a, b, c \in A$, and $h \in H$, where $h \otimes a \mapsto h\, a$ is the left H-module structure of A, and $\phi = \sum X^1 \otimes X^2 \otimes X^3$ is the Drinfeld reassociator of H.

Let H be a quasi-bialgebra and A be a left H-module with an associative algebra structure defined by $\mu_A : A \otimes A \to A$, and $\iota : k \to A$. Then (unlike to the bialgebra case, even for $A = H$) A need not be an H-module algebra. However, if we have an algebra map $f : H \to A$, we can define the following multiplication on A making it an algebra in ${}_H\mathbb{M}$ which is denoted by A^f (see [11, Proposition 2.2]),

$$a * b = \sum f(X^1) a f(S(x^1 X^2) \alpha x^2 X^3_1) b f(S(x^3 X^3_2)). \tag{8.1}$$

A^f is a left H-module algebra with unit $f(\beta)$ and with the left adjoint action induced by f, that is,

$$h \triangleright_f a = \sum f(h_1) a f(S(h_2)),$$

for $a \in A$, and $h \in H$.
In particular, it induces an H-module algebra structure on the quasi-Hopf algebra H by left adjoint action and the new multiplication

$$a * b = \sum X^1 a S(x^1 X^2) \alpha x^2 X^3_1 b S(x^3 X^3_2). \tag{8.2}$$

This H-module algebra is denoted by H_0.

For a quasi-bialgebra $(H, \Delta, \varepsilon, \phi)$ and a left H-module algebra A, following Bulacu, Panaite and Van Oystaeyen [11], an H-module M is called a **left (A, H)-module** if it is an A-module in the monoidal category ${}_H\mathbb{M}$, i.e. if there exists a left weak action $\triangleright : A \otimes M \to M$, $a \otimes m \mapsto a \triangleright m$, such that for $m \in M$, and $a, b \in A$,

$$(ab) \triangleright m = \sum (X^1\, a) \triangleright [(X^2\, b) \triangleright (X^3\, m)], \tag{8.3}$$

$$h \cdot (a \triangleright m) = \sum (h_1 a) \triangleright (h_2 m), \tag{8.4}$$

$$1_A \triangleright m = m, \tag{8.5}$$

where $h \otimes m \mapsto hm$ is the left H-module structure of M. The category of left (A, H)-modules with morphisms that are left H-linear and preserve the weak A-action will be denoted by ${}_A({}_H\mathbb{M})$.

Symmetrically, for a left H-module algebra A, we define a right A-module M in the monoidal category ${}_H\mathbb{M}$, as a left H-module M with a right (weak) action $\triangleleft : M \otimes A \to M$, such that for $m \in M$, and $a, b \in A$,

$$m \triangleleft (ab) = \sum [(X^1 m) \triangleleft (X^2 a)] \triangleleft (X^3 b),$$

$$h \cdot (m \triangleleft a) = \sum (h_1 m) \triangleleft (h_2 a), \quad \text{and}$$

$$m \triangleleft 1_A = m.$$

Moreover, an H-module M is an (A, A)**-bimodule in** ${}_H\mathbb{M}$, if M is a left and a right A-module in ${}_H\mathbb{M}$, and

$$(a \triangleright m) \triangleleft b = \sum (X^1 a) \triangleright [(X^2 m) \triangleleft (X^3 b)], \tag{8.6}$$

for all $m \in M, a, b \in A$, that is, we have a commutative diagram

$$\begin{array}{ccc}
(A \otimes M) \otimes A & \xrightarrow{\triangleright \otimes id} & M \otimes A \\
\downarrow{\scriptstyle a_{A,M,A}} & & \\
A \otimes (M \otimes A) & & \downarrow{\scriptstyle \triangleleft} \\
\downarrow{\scriptstyle id \otimes \triangleleft} & & \\
A \otimes M & \xrightarrow[\triangleright]{} & M.
\end{array}$$

8.2. Smash products for bialgebras. For a bialgebra H and a left H-module algebra A, the **smash product** of A and H, denoted by $A \# H$, is the k-module $A \otimes H$, together with the multiplication

$$(a \# h)(b \# g) = \sum a(h_1 b) \# h_2 g, \tag{8.7}$$

for $a, b \in A$ and $g, h \in H$. In this case, a k-module M is a left $A\#H$-module if and only if M is a left A-module as well as a left H-module and the left A-module structure map $A \otimes M \longrightarrow M$ is an H-module morphism (see [29]).

Similar to the bialgebra case, one can define the concept of *smash product* for quasi-bialgebras (see [11]).

8.3. Smash products for quasi-bialgebras. Let H be a quasi-bialgebra and A a left H-module algebra. The **smash product** of A and H, denoted by $A\#H$, is the k-module $A\#H = A \otimes H$, together with the multiplication

$$(a \# h)(b \# g) = \sum (x^1 a)(x^2 h_1 b) \# x^3 h_2 g, \tag{8.8}$$

for $a, b \in A$ and $g, h \in H$. With this multiplication, $A\#H$ is an associative algebra with identity $1_A \# 1_H$. The canonical map $j : H \longrightarrow A\#H$ is an algebra map (see [11, Proposition 2.7]).

8.4. Module categories over smash products for quasi-bialgebras. Let H be a quasi-bialgebra and A a left H-module algebra. Then a k-module M is a left $A\#H$-module if and only if

(1) M has a left H-module stucture with left H-action $h \otimes m \mapsto h\,m$,

(2) A acts weakly on M, $\triangleright : A \otimes M \to M$, $a \otimes m \mapsto a \triangleright m$, such that (8.3), (8.4), and (8.5) are satisfied. (M must be a left A-module in ${}_H\mathbb{M}$).

This means that the categories ${}_A({}_H\mathbb{M})$ and ${}_{A\#H}\mathbb{M}$ are isomorphic. In fact, for any $M \in {}_A({}_H\mathbb{M})$ with left (weak) A-action $\triangleright$, there is an $A\#H$-module structure given by $(a\#h)m = a \triangleright (hm)$, thus $M \in {}_{A\#H}\mathbb{M}$. Conversely, if $M \in {}_{A\#H}\mathbb{M}$, then also $M \in {}_A({}_H\mathbb{M})$ by $a \triangleright m = (a\#1)m$ and $hm = (1_A\#h)m$, where $h \in H$, $a \in A$ and $m \in M$. Thus, ${}_{A\#H}\mathbb{M} \cong {}_A({}_H\mathbb{M})$ (see also [11, 2.15 and 2.16]).

9 Hom-tensor relations for quasi-Hopf algebras and their module algebras

In this section we will prove that there are adjunctions between tensor-functors and Hom-functors as endofunctors of the categories ${}_H\mathbb{M}$, $\mathbb{M}_H$ and ${}_H\mathbb{M}_H$ over a quasi-Hopf algebra H. First we recall that the classical Hom-tensor relation in the category ${}_k\mathbb{M}$ can be restricted to the category ${}_H\mathbb{M}$, where H is a Hopf algebra. After that we obtain some similar adjunction isomorphisms in case H is a quasi-Hopf algebra. For an H-module algebra A, we know that $A \otimes_k -$ (resp. $- \otimes_k A$) defines a monad on the categories ${}_H\mathbb{M}$, $\mathbb{M}_H$ and ${}_H\mathbb{M}_H$, with suitable structures on the tensor products. We can consider the Eilenberg-Moore *module* categories over these monads, and they are precisely the categories of A-modules in ${}_H\mathbb{M}$, $\mathbb{M}_H$ and ${}_H\mathbb{M}_H$ respectively. On the other hand, we will see that these categories are isomorphic to the Eilenberg-Moore *comodule* categories over the corresponding comonads that are right adjoint to the monads $A \otimes_k -$ (resp. $- \otimes_k A$).

For any k-bialgebra H, we know that $({}_H\mathbb{M}, \otimes_k^b, k)$ is a monoidal category and for any $M, N \in {}_H\mathbb{M}$, we have $M \otimes_k^b N \in {}_H\mathbb{M}$ (see 2.5). Furthermore, if H is a Hopf algebra with antipode S, then $\mathrm{Hom}_k(M, N)$ is a left H-module by $(h \cdot f)(m) = \sum h_1 f(S(h_2)\,m)$ (see 2.6), which we denote by ${}^s\mathrm{Hom}_k(M, N)$.

In case the antipode S is invertible, for any left H-modules M, N the k-module $\mathrm{Hom}_k(M, N)$ is a left H-module with another H-module structure given by $(h \cdot f)(m) = h_2 f(S^{-1}(h_1)\,m)$ (see 2.6), which is denoted by ${}^t\mathrm{Hom}_k(M, N)$.

Using the above notations, we recall from [7, 15.9] the

9.1. Hom-tensor relations for Hopf algebras. Let H be a Hopf algebra, and $M, N, V \in {}_H\mathbb{M}$.

(1) There is a functorial isomorphism

$${}_H\mathrm{Hom}(M \otimes_k^b V, N) \xrightarrow{\psi} {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}_k(V, N)), \quad f \longmapsto [m \mapsto f(m \otimes_k -)],$$

with inverse $g \longmapsto [m \otimes v \mapsto g(m)(v)]$, i.e. the functors

$$- \otimes^b V : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M}, \quad \text{and} \quad {}^s\mathrm{Hom}(V, -) : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M},$$

form an adjoint pair with unit and counit, in ${}_H\mathbb{M}$,

$$\eta_M : M \longrightarrow \mathrm{Hom}_k(V, M \otimes_k V), \quad m \longmapsto [v \mapsto m \otimes v],$$

$$\varepsilon_M : \mathrm{Hom}_k(V, M) \otimes V \longrightarrow M, \quad f \otimes v \mapsto f(v).$$

(2) If H has a bijective antipode S, then there is a functorial isomorphism

$${}_H\mathrm{Hom}(V \otimes_k^b M, N) \xrightarrow{\psi} {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}_k(V, N)), \quad f \longmapsto [m \mapsto f(- \otimes m)],$$

with inverse $g \mapsto [v \otimes m \mapsto g(m)(v)]$, i.e. the functors

$$V \otimes^b - : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M}, \quad {}^t\mathrm{Hom}(V, -) : {}_H\mathbb{M} \to {}_H\mathbb{M},$$

form an adjoint pair on ${}_H\mathbb{M}$ with unit and counit

$$\eta_M : M \longrightarrow {}^t\mathrm{Hom}_k(V, V \otimes_k M), \quad m \longmapsto [v \mapsto v \otimes m],$$

$$\varepsilon_M : V \otimes \mathrm{Hom}_k(V, M) \longrightarrow M \quad v \otimes f \mapsto f(v).$$

Now we state similar Hom-tensor relations in the module category ${}_H\mathbb{M}$ over a *quasi-Hopf algebra* H. For any quasi-bialgebra H, we know that $({}_H\mathbb{M}, \otimes_k^b, k, a)$ is a monoidal category (with non-trivial associativity constraint, see section 7).

If H is a quasi-Hopf algebra with quasi-antipode (S, α, β), then for any $M, N \in {}_H\mathbb{M}$ we have $\mathrm{Hom}_k(M, N) \in {}_H\mathbb{M}$ with the same left H-action as in the Hopf algebra case. Using the same notation as in the Hopf algebra case, we denote by ${}^s\mathrm{Hom}_k(M, N)$ the k-module $\mathrm{Hom}_k(M, N)$ with the left H-module structure $(h \cdot f)(m) = \sum h_1 f(S(h_2)\, m)$.

9.2. Theorem. (Adjunction $(- \otimes_k^b V, {}^s\mathrm{Hom}_k(V, -))$ on ${}_H\mathbb{M}$). *Let H be a quasi-Hopf algebra with quasi-antipode (S, α, β), and $M, N, V \in {}_H\mathbb{M}$. Then there is a functorial isomorphism*

$$\psi : {}_H\mathrm{Hom}(M \otimes_k^b V, N) \longrightarrow {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}_k(V, N)),$$

$$f \longmapsto \{m \mapsto [v \mapsto f(p_R\, (m \otimes v))]\},$$

with inverse map ψ' given by

$$g \longmapsto \{m \otimes v \mapsto \sum q_R^1\, [g(m)(S(q_R^2)\, v)]\},$$

where $p_R = \sum p_R^1 \otimes p_R^2$ and $q_R = \sum q_R^1 \otimes q_R^2$ are defined in (7.30) and (7.31), respectively. This means that the functors

$$- \otimes^b V : {}_H\mathbb{M} \to {}_H\mathbb{M}, \quad {}^s\mathrm{Hom}(V, -) : {}_H\mathbb{M} \to {}_H\mathbb{M},$$

form an adjoint pair with unit and counit, in ${}_H\mathbb{M}$,

$$\eta_M : M \longrightarrow {}^s\mathrm{Hom}_k(V, M \otimes_k V), \quad m \longmapsto [v \mapsto p_R\, (m \otimes v)],$$

$$\varepsilon_M : {}^s\mathrm{Hom}_k(V, M) \otimes V \longrightarrow M, \quad f \otimes v \mapsto \sum q_R^1\, [f(S(q_R^2)\, v)].$$

Proof. First we show that for any $f \in {}_H\mathrm{Hom}(M \otimes_k^b V, N)$, the image $\psi(f) \in {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}_k(V,N))$, i.e. $\psi(f)$ is H-linear. For any $h \in H$, $m \in M$ and $v \in V$ we have

$$\begin{aligned}
[\psi(f)(h\,m)](v) &= f(\sum p_R^1 h\, m \otimes p_R^2 v) \\
&= f(p_R\,(h \otimes 1)(m \otimes v)) \\
\text{by (7.34)} \quad &= \sum f(\Delta(h_1)\, p_R\,(1 \otimes S(h_2))(m \otimes v)) \\
\text{by } H\text{-linearity of } f \quad &= \sum h_1(f(p_R^1 m \otimes p_R^2 S(h_2)\, v).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[h \cdot (\psi(f)(m))](v) &= \sum h_1[(\psi(f)(m))(S(h_2)\, v)] \\
&= \sum h_1 f(p_R^1 m \otimes p_R^2 S(h_2)\, v).
\end{aligned}$$

This shows the H-linearity of $\psi(f)$.
Conversely, for any $h \in H$, $m \in M$, $v \in V$ and $g \in {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}_k(V,N))$,

$$\begin{aligned}
[\psi'(g)](h \cdot (m \otimes v)) &= [\psi'(g)](\sum h_1\, m \otimes h_2\, v) \\
&= \sum q_R^1(g(h_1\, m)(S(q_R^2) h_2\, v)) \\
g \text{ is } H\text{-linear} \quad &= \sum q_R^1[(h_1 \cdot (g(m)))(S(q_R^2) h_2\, v)) \\
&= \sum q_R^1 h_{11}[g(m)(S(q_R^2 h_{12}) h_2\, v)] \\
\text{by (7.35)} \quad &= \sum h q_R^1[g(m)(S(q_R^2)\, v)] = h\,[\psi'(g)](m \otimes v).
\end{aligned}$$

So $\psi'(g)$ is also left H-linear.
To show that ψ and ψ' are inverse to each other, take $m \in M, v \in V$ and $f \in {}_H\mathrm{Hom}(M \otimes V, N)$. Then

$$\begin{aligned}
[(\psi' \circ \psi)(f)](m \otimes v) &= \sum q_R^1[(\psi(f)(m))(S(q_R^2)\, v)] \\
&= \sum q_R^1[f(p_R^1\, m \otimes p_R^2 S(q_R^2)\, v)] \\
&= \sum q_R^1[f(p_R\,(1 \otimes S(q_R^2)) \cdot (m \otimes v)] \\
f \text{ is } H\text{-linear} \quad &= \sum [f(\Delta(q_R^1)\, p_R\,(1 \otimes S(q_R^2)) \cdot (m \otimes v)] \\
\text{by (7.38)} \quad &= f((1 \otimes 1)\,(m \otimes v)) = f(m \otimes v).
\end{aligned}$$

On the other hand, for any $m \in M, v \in V$ and $g \in {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}_k(V,N))$,

$$\begin{aligned}
[(\psi \circ \psi')(g)](m)(v) &= \psi'(g)(\sum p_R^1\, m \otimes p_R^2\, v)] \\
&= \sum q_R^1[g(p_R^1\, m)(S(q_R^2) p_R^2\, v)] \\
g \text{ is } H\text{-linear} \quad &= \sum q_R^1[p_R^1 \cdot (g(m))](S(q_R^2) p_R^2\, v) \\
&= \sum q_R^1 (p_R^1)_1\,(g(m))(S((p_R^1)_2) S(q_R^2) p_R^2\, v) \\
&= \sum q_R^1 (p_R^1)_1\,(g(m))(S(q_R^2 (p_R^1)_2) p_R^2\, v) \\
\text{by (7.39)} \quad &= g(m)(v).
\end{aligned}$$

This shows that ψ and ψ' are inverse to each other. □

9.3. Proposition. (Adjunction $(-\otimes_k^b A, {}^s\mathrm{Hom}_k(A,-))$ for a monad on ${}_H\mathbb{M}$). *Let H be a quasi-Hopf algebra and A be a left H-module algebra. Then the right adjoint functor ${}^s\mathrm{Hom}_k(A,-) : {}_H\mathbb{M} \to {}_H\mathbb{M}$ of the tensor functor $-\otimes_k^b A$ is a comonad on ${}_H\mathbb{M}$ with comultiplication given for any $N \in {}_H\mathbb{M}$ by*

$$\delta_N : {}^s\mathrm{Hom}_k(A,N) \longrightarrow {}^s\mathrm{Hom}_k(A, {}^s\mathrm{Hom}_k(A,N)),$$

$$f \longmapsto \{a \mapsto [b \mapsto \sum q_R^1 \{(X^1(p_R^1)_1 p_R^1 \cdot f)(S(q_R^2) \cdot [(X^2(p_R^1)_2 p_R^2\, a)(X^3 p_R^2\, b)])\}]\},$$

and counit

$$\epsilon_N : {}^s\mathrm{Hom}_k(A,N) \longrightarrow N, \quad f \mapsto f(1_A).$$

The Eilenberg-Moore comodule *category $({}_H\mathbb{M})^{{}^s\mathrm{Hom}_k(A,-)}$ is isomorphic to the category of right A-modules in ${}_H\mathbb{M}$, sending any right A-module $\triangleleft : N \otimes A \to N$ in ${}_H\mathbb{M}$ to N as a left H-module with the ${}^s\mathrm{Hom}_k(A,-)$-comodule structure*

$$\varrho^N : N \longrightarrow {}^s\mathrm{Hom}_k(A,N), \quad n \longmapsto [a \mapsto \sum (p_R^1\, n) \triangleleft (p_R^2\, a)].$$

Proof. Taking $V = A$, to be a left H-module algebra in 9.2 means that A is an algebra in ${}_H\mathbb{M}$, i.e. the functor $-\otimes_k A : {}_H\mathbb{M} \to {}_H\mathbb{M}$ is a monad (see 4.4 for the definition). Thus, its right adjoint, ${}^s\mathrm{Hom}_k(A,-) : {}_H\mathbb{M} \to {}_H\mathbb{M}$, is a comonad on ${}_H\mathbb{M}$ (by 4.9). The multiplication $\mu_A : A \otimes A \to A$ yields the commutative diagram

$$\begin{array}{ccc}
{}_H\mathrm{Hom}(M \otimes A, N) & \xrightarrow[\cong]{\psi_{M,N}} & {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}(A,N)) \\
\downarrow {\scriptstyle [id_M \otimes \mu_A, N]} & & \downarrow {\scriptstyle \tilde{\delta}_N} \\
{}_H\mathrm{Hom}(M \otimes (A \otimes A), N) & & {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}(A, {}^s\mathrm{Hom}(A,N))) \\
\downarrow {\scriptstyle [a_{M,A,A}, N]} & & \uparrow {\scriptstyle \psi_{M, {}^s\mathrm{Hom}(A,N)}} \\
{}_H\mathrm{Hom}((M \otimes A) \otimes A, N) & \xrightarrow[\psi_{M\otimes A,N}]{} & {}_H\mathrm{Hom}(M \otimes A, {}^s\mathrm{Hom}(A,N)),
\end{array}$$

and for $a, b \in A$, $m \in M$ and $g \in {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}(A,N))$, the map

$$\tilde{\delta}_N : {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}(A,N)) \longrightarrow {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}(A, {}^s\mathrm{Hom}(A,N))),$$

is given by

$$\begin{aligned}
& \tilde{\delta}_N(g)(m)(a)(b) \\
= \; & \psi_{M,{}^s\mathrm{Hom}(A,N)} \circ \psi_{M\otimes A,N} \circ [a_{M,A,A}, N] \circ [id_M \otimes \mu_A, N] \circ \psi'_{M,N}(g)(m)(a)(b) \\
= \; & \{\sum (\psi_{M\otimes A,N} \circ [a_{M,A,A}, N] \circ [id_M \otimes \mu_A, N] \circ \psi'_{M,N}(g))(p_R^1\, m \otimes p_R^2\, a)\}(b) \\
= \; & \sum \{[a_{M,A,A}, N] \circ [id_M \otimes \mu_A, N] \circ \psi'_{M,N}(g)\}([p_{R1}^1\, p_R^1\, m \otimes p_{R2}^1 p_R^2\, a] \otimes p_R^2\, b) \\
= \; & \sum \{[id_M \otimes \mu_A, N] \circ \psi'_{M,N}(g)\}(X^1(p_R^1)_1\, p_R^1\, m \otimes [X^2(p_R^1)_2 p_R^2\, a \otimes X^3 p_R^2\, b]) \\
= \; & \sum \{\psi'_{M,N}(g) \circ [id_M \otimes \mu_A]\}(X^1(p_R^1)_1\, p_R^1\, m \otimes [X^2(p_R^1)_2 p_R^2\, a \otimes X^3 p_R^2\, b]) \\
= \; & \sum \psi'_{M,N}(g)(X^1(p_R^1)_1\, p_R^1\, m \otimes (X^2(p_R^1)_2 p_R^2\, a)(X^3 p_R^2\, b)) \\
= \; & \sum q_R^1 \{g(X^1(p_R^1)_1 p_R^1\, m))(S(q_R^2) \cdot [(X^2(p_R^1)_2 p_R^2\, a)(X^3 p_R^2\, b)])\}.
\end{aligned}$$

By the left H-linearity of g, this yields the map $\tilde{\delta}_N : {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}(A,N)) \to {}_H\mathrm{Hom}(M, {}^s\mathrm{Hom}(A, {}^s\mathrm{Hom}(A,N)))$, explicitely given by

$$\tilde{\delta}_N(g)(m)(a)(b) = \sum q_R^1 \{(X^1(p_R^1)_1 p_R^1 \cdot g(m))(S(q_R^2) \cdot [(X^2(p_R^1)_2 p_R^2 a)(X^3 p_R^2 b)])\}.$$

By the Yoneda Lemma, this yields the comultiplication δ of this comonad as stated in the proposition.
The counit ϵ for δ of this comonad is obtained, by using the unit map $\iota : k \to A$ and the counit of adjunction, as the composition

$$\epsilon_N : {}^s\mathrm{Hom}_k(A,N) \xrightarrow{\iota_{\mathrm{Hom}_k(A,N)}} {}^s\mathrm{Hom}_k(A,N) \otimes A \xrightarrow{\varepsilon_N} N,$$

$$\begin{aligned} f \longmapsto f \otimes 1_A \longmapsto & \sum q_R^1 f(S(q_R^2)\, 1_A) \\ = & \sum q_R^1 f(\varepsilon(S(q_R^2)) 1_A) = \sum q_R^1 \varepsilon(q_R^2)\, f(1_A) \\ = & f(1_A). \end{aligned}$$

The Eilenberg-Moore *module* category $({}_H\mathbb{M})_{-\otimes A}$ over the monad $- \otimes_k A : {}_H\mathbb{M} \to {}_H\mathbb{M}$ is the category $({}_H\mathbb{M})_A$ of right A-modules in $({}_H\mathbb{M})$. We know from 4.9 that this category is isomorphic to the Eilenberg-Moore *comodule* category $({}_H\mathbb{M})^{{}^s\mathrm{Hom}_k(A,-)}$ over the comonad ${}^s\mathrm{Hom}_k(A,-)$. We describe this isomorphism explicitly.
For any $N \in ({}_H\mathbb{M})_A$ with the right (weak) A-action $n \otimes a \mapsto n \triangleleft a$ for $a \in A$ and $n \in N$, the ${}^s\mathrm{Hom}_k(A,-)$-comodule structure of N is given by the composition

$$\varrho^N : N \xrightarrow{\eta_N} {}^s\mathrm{Hom}_k(A, N \otimes A) \xrightarrow{[A,\varrho_N]} {}^s\mathrm{Hom}_k(A,N).$$

Explicitly, for any $n \in N$, and $a \in A$,

$$\varrho^N(n)(a) = \sum (p_R^1\, n) \triangleleft (p_R^2\, a).$$

In order to show that ϵ_N is a counit for ϱ_N, compute

$$\epsilon_N \circ \varrho^N(n) = \varrho^N(n)(1_A) = \sum (p_R^1\, n) \triangleleft (p_R^2\, 1_A) = \sum p_R^1 \varepsilon(p_R^2)\, n = n.$$

In this way, we obtain a functor $F : ({}_H\mathbb{M})_A \to ({}_H\mathbb{M})^{{}^s\mathrm{Hom}(A,-)}$.
Conversely, given a ${}^s\mathrm{Hom}_k(A,-)$-comodule structure map

$$\varrho^M : M \longrightarrow {}^s\mathrm{Hom}_k(A,M), \quad m \mapsto \varrho^M(m),$$

on a left H-module M, we define a right (weak) A-action on M as composition

$$\triangleleft' : M \otimes A \xrightarrow{\varrho^M \otimes A} {}^s\mathrm{Hom}_k(A,M) \otimes A \xrightarrow{\varepsilon_M} M.$$

Explicitly, for $a \in A$ and $m \in M$,

$$m \triangleleft' a = \sum q_R^1\, [\varrho^M(m)(S(q_R^2)\, a)].$$

This (weak) A-action is (by construction) a morphism in ${}_H\mathbb{M}$, and defines a right A-module structure on M in ${}_H\mathbb{M}$. This yields a functor $G : ({}_H\mathbb{M})^{{}^s\mathrm{Hom}(A,-)} \to ({}_H\mathbb{M})_A$, which is inverse to F by 4.9 (see also [6, 2.6]). □

9.4. Adjunction for f.g. projective V. From 9.2 we have the adjoint pair ($L := -\otimes_k^b V, R := {}^s\mathrm{Hom}_k(V,-)$) of endofunctors of ${}_H\mathbb{M}$ with unit and counit

$$\eta_M : M \longrightarrow {}^s\mathrm{Hom}_k(V, M\otimes_k V), \quad m \longmapsto [v \mapsto p_R\,(m\otimes v)],$$

$$\varepsilon_M : {}^s\mathrm{Hom}_k(V, M)\otimes V \longrightarrow M, \quad f\otimes v \mapsto \sum q_R^1\,[f(S(q_R^2)\,v)],$$

We know from 1.2 that there is a natural transformation $\psi : -\otimes_k V^* \to \mathrm{Hom}_k(V,-)$ between endofunctors of ${}_k\mathbb{M}$. For $V, M \in {}_H\mathbb{M}$, considering the diagonal left H-module structure on $M\otimes_k V^*$ and taking ${}^s\mathrm{Hom}_k(V,M)$ with the left H-structure given for $h\in H$, $v\in V$ and $f\in {}^s\mathrm{Hom}_k(V,M)$ by $(h\cdot f)(v) = f(S(h)\,v)$, we show that the k-linear morphism

$$\psi_M : M\otimes_k^b V^* \longrightarrow {}^s\mathrm{Hom}_k(V,M), \quad m\otimes f \longmapsto [v\mapsto f(v)m],$$

is left H-linear. To see this, for $h\in H$, $m\in M$, $v\in V$ and $f\in V^*$, we compute

$$\begin{aligned}[\psi_M(h\cdot(m\otimes f))](v) &= [\psi_M(\sum h_1\,m\otimes h_2\cdot f)](v)\\ &= \sum (h_2\cdot f)(v)[h_1\,m] = \sum f(S(h_2)\,v)h_1\,m).\end{aligned}$$

On the other hand,

$$\begin{aligned}[h\cdot\psi_M(m\otimes f)](v) &= \sum h_1\cdot[\psi_M(m\otimes f)(S(h_2)\,v)]\\ &= \sum h_1\,[f(S(h_2)\,v)m] = \sum f(S(h_2)\,v)h_1\,m.\end{aligned}$$

Over a base field, Schauenburg showed in [27] that for a quasi-Hopf algebra H and any finite dimensional left H-module V, we have an adjoint pair $(-\otimes V, -\otimes V^*)$ of endofunctors of ${}_H\mathbb{M}$. Referring to a dual basis for $V\in {}_H\mathbb{M}$, the computations of Schauenburg can be transferred to a commutative base ring k. For this, let $V\in {}_H\mathbb{M}$ with ${}_kV$ finitely generated and projective with a dual basis $\{v_i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$. Then we have an adjoint pair

$$L' := -\otimes_k^b V : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M}, \quad R' := -\otimes_k V^* : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M},$$

with unit

$$\eta'_M : M \xrightarrow{M\otimes b_V} M\otimes(V\otimes V^*) \xrightarrow{a^{-1}} (M\otimes V)\otimes V^*,$$
$$m \mapsto \sum m\otimes(\beta\,v_i\otimes v^i) \mapsto \sum x^1\,m\otimes x^2\beta\,v_i\otimes x^3\cdot v^i,$$

and counit

$$\varepsilon'_M : (M\otimes V^*)\otimes V \xrightarrow{a} M\otimes(V^*\otimes V) \xrightarrow{M\otimes d_V} M,$$
$$m\otimes g\otimes v \mapsto \sum X^1\,m\otimes(X^2\cdot g\otimes X^3\,v) \mapsto \sum g(S(q_R^2)\,v)q_R^1\,m.$$

By 4.2, there is a bijection

$$\mathrm{Nat}(-\otimes_k V, -\otimes_k V) \longrightarrow \mathrm{Nat}({}^s\mathrm{Hom}_k(V,-), -\otimes_k V^*).$$

Applying this bijection to the identity on the left side, we obtain the composition

$$\psi'_M : {}^s\mathrm{Hom}(V,M) \xrightarrow{\eta' R(M)} ({}^s\mathrm{Hom}(V,M)\otimes_k V)\otimes_k V^* \xrightarrow{R'(\varepsilon_M)} M\otimes_k V^*,$$

for $f \in {}^s\mathrm{Hom}_k(V,M)$ as

$$\begin{aligned}
\psi'_M(f) &= R'(\varepsilon_M)\circ\eta' R(M)(f) = R'(\varepsilon_M)(\sum x^1\cdot f\otimes x^2\beta\, v_i\otimes x^3\cdot v^i)\\
&= \sum q_R^1\,[(x^1\cdot f)(S(q_R^2)x^2\beta\, v_i)]\otimes x^3\cdot v^i\\
&= \sum X^1x_1^1\,[f(S(X^2x_2^1)\alpha X^3x^2\beta\, v_i)]\otimes x^3\cdot v^i.
\end{aligned}$$

It is straightforward to see directly that ψ' is natural in M, i.e. we obtain a natural transformation $\psi' : {}^s\mathrm{Hom}_k(V,-)\to -\otimes_k V^*$, given for any component $M\in {}_H\mathbb{M}$ as above. Now we show that ψ_M and ψ'_M are inverse to each other. For $v\in V$ and $f\in {}^s\mathrm{Hom}_k(V,M)$ we compute

$$\begin{aligned}
[\psi_M\circ\psi'_M(f)](v) &= [\psi_M(\sum X^1x_1^1\,[f(S(X^2x_2^1)\alpha X^3x^2\beta\, v_i)]\otimes x^3\cdot v^i)](v)\\
&= \sum (x^3\cdot v^i)(v)X^1x_1^1\,[f(S(X^2x_2^1)\alpha X^3x^2\beta\, v_i)]\\
&= \sum v^i(S(x^3)\,v)X^1x_1^1\,[f(S(X^2x_2^1)\alpha X^3x^2\beta\, v_i)]\\
&= \sum X^1x_1^1\,[f(S(X^2x_2^1)\alpha X^3x^2\beta v^i(S(x^3)\,v)v_i)]\\
&= \sum X^1x_1^1\,[f(S(X^2x_2^1)\alpha X^3x^2\beta S(x^3)\,v]\\
&= \sum X^1x_1^1\,[f(S(x_2^1)S(X^2)\alpha X^3x^2\beta S(x^3)\,v]\\
&= \sum q_R^1(p_R^1)_1\,[f(S(q_R^2(p_R^1)_2)p_R^2\,v]\\
\text{by (7.39)}\quad &= f(v).
\end{aligned}$$

Conversely,

$$\begin{aligned}
[\psi'_M\circ\psi_M](m\otimes f) &= \psi'_M(\sum X^1x_1^1\,[\psi_M(m\otimes f)(S(X^2x_2^1)\alpha X^3x^2\beta\, v_i)]\otimes x^3\cdot v^i)\\
&= \sum f(S(X^2x_2^1)\alpha X^3x^2\beta\, v_i)X^1x_1^1\,m\otimes x^3\cdot v^i\\
&= \sum f(S(q_R^2(p_R^1)_2)x^2\beta\, v_i)q_R^1(p_R^1)_1\,m\otimes x^3\cdot v^i\\
&= \sum q_R^1(p_R^1)_1\,m\otimes f(S(q_R^2(p_R^1)_2)x^2\beta\, v_i)x^3\cdot v^i\\
\text{by (7.39)}\quad &= m\otimes f.
\end{aligned}$$

This means for $V\in {}_H\mathbb{M}$ which is finitely generated projective as k-module, the natural transformation $\psi : -\otimes_k V^*\to {}^s\mathrm{Hom}_k(V,-)$, defined above, is a functorial isomorphism.

Thus we have shown

9.5. Theorem. (Adjunction $(-\otimes_k V, -\otimes_k V^*)$ on ${}_H\mathbb{M}$). *Let H be a quasi-Hopf algebra, $M,V\in {}_H\mathbb{M}$. Then*

(1) The map $\psi_M : M\otimes_k V^*\to {}^s\mathrm{Hom}_k(V,M)$, $m\otimes g\longmapsto [v\mapsto g(v)\,m]$, is a natural homomorphism in ${}_H\mathbb{M}$.

(2) If ${}_kV$ is finitely generated and projective with dual basis $\{v_i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$, then the map ψ_M give rise to a natural isomorphism *with inverse map*

$$\psi'_M : {}^s\mathrm{Hom}(V,M)\longrightarrow M\otimes_k V^*,\quad f\mapsto \sum X^1x_1^1\,f(S(X^2x_2^1)\alpha X^3x^2\beta v_i)\otimes x^3\cdot v^i,$$

For a quasi-Hopf algebra H, we assume that the quasi-antipode S is bijective, thus $\mathrm{Hom}_k(M,N)$ is also a left H-module by

$$(h\cdot f)(m)=\sum h_2\, f(S^{-1}(h_1)\, m)$$

which we denote by ${}^t\mathrm{Hom}_k(M,N)$ (see 2.6 and the text before of 9.1).

9.6. Theorem. (Adjunction $(V\otimes^b -, {}^t\mathrm{Hom}_k(V,-))$ on ${}_H\mathbb{M}$). *Let H be a quasi-Hopf algebra with quasi-antipode (S,α,β), and $M,N,V\in {}_H\mathbb{M}$. Then there is a functorial isomorphism*

$$\begin{array}{c} {}_H\mathrm{Hom}(V\otimes_k^b M,N)\xrightarrow{\theta} {}_H\mathrm{Hom}(M,{}^t\mathrm{Hom}_k(V,N)),\\ f\longmapsto \{m\mapsto [v\mapsto f(p_L\,(v\otimes m))]\}, \end{array}$$

with inverse map θ':

$$g\longmapsto \{v\otimes m\mapsto \sum q_L^2(g(m)(S^{-1}(q_L^1)\,v))\},$$

where $p_L=\sum p_L^1\otimes p_L^2$ and $q_L=\sum q_L^1\otimes q_L^2$ are defined in (7.28) and (7.29), respectively. Thus we have an adjoint pair of functors

$$V\otimes^b - : {}_H\mathbb{M}\longrightarrow {}_H\mathbb{M},\quad {}^t\mathrm{Hom}(V,-): {}_H\mathbb{M}\longrightarrow {}_H\mathbb{M},$$

with unit and counit, in ${}_H\mathbb{M}$, given by

$$\eta_M: M\longrightarrow {}^t\mathrm{Hom}_k(V,V\otimes_k M),\quad m\longmapsto \{v\mapsto \sum p_L\,(v\otimes m)\},$$

$$\varepsilon_M: V\otimes {}^t\mathrm{Hom}_k(V,M)\longrightarrow M,\quad v\otimes f\mapsto \sum q_L^2[f(S^{-1}(q_L^1)\,v)].$$

Proof. For any $f\in {}_H\mathrm{Hom}(V\otimes_k^b M,N)$, we show that $\theta(f)$ is H-linear. For any $h\in H, m\in M, v\in V$ we have

$$\begin{array}{rcl} [\theta(f)(h\,m)](v) &=& f(\sum p_L^1\,v\otimes p_L^2 h\,m)\\ &=& f(p_L\,(1\otimes h)(v\otimes m))\\ \text{by (7.32)} &=& f(\sum \Delta(h_2)\,p_L\,(S^{-1}(h_1)\otimes 1)(v\otimes m))\\ f \text{ is } H\text{-linear} &=& \sum h_2(f(\sum p_L^1 S^{-1}(h_1)\,v\otimes p_L^2\,m). \end{array}$$

On the other hand,

$$\begin{array}{rcl} [h\cdot(\theta(f)(m))](v) &=& \sum h_2[(\theta(f)(m))(S^{-1}(h_1)\,v)]\\ &=& \sum h_2 f(p_L^1 S^{-1}(h_1)\,v\otimes p_L^2\,m), \end{array}$$

showing H-linearity of $\theta(f)$.
Conversely, for any $h\in H, m\in M, v\in V$ and $g\in {}_H\mathrm{Hom}(M,{}^t\mathrm{Hom}_k(V,N))$,

$$\begin{array}{rcl} [\theta'(g)](h\cdot(v\otimes m)) &=& [\theta'(g)](\sum h_1\,v\otimes h_2\,m)\\ &=& \sum q_L^2(g(h_2\,m)(S^{-1}(q_L^1)h_1\,v))\\ &=& \sum q_L^2[(h_2\,(g(m)))(S^{-1}(q_L^1)h_1\,v))\\ &=& \sum q_L^2 h_{22}[g(m)(S^{-1}(q_L^1 h_{21})h_1\,v)]\\ \text{by (7.33)} &=& \sum h q_L^1[g(m)(S(q_L^2)\,v)] = h\cdot[\psi'(g)](m\otimes v). \end{array}$$

Thus, $\theta'(g)$ is also left H-linear. To show that θ and θ' are inverse to each other, let $m \in M, v \in V$ and $f \in {}_H\mathrm{Hom}(M \otimes V, N)$. Then

$$\begin{array}{rcl}[(\theta' \circ \theta)(f)](v \otimes m) & = & \sum q_L^2[(\theta(f)(m))(S^{-1}(q_L^1)\, v)] \\ & = & \sum q_L^2[f(p_L^1 S^{-1}(q_L^1)\, v \otimes p_L^2\, m)] \\ {\scriptstyle f \text{ is } H\text{-linear.}} & = & \sum [f(\Delta(q_L^2)\, p_L\, (S^{-1}(q_L^1) \otimes 1)\, (v \otimes m)] \\ {\scriptstyle \text{by } (7.36)} & = & f((1 \otimes 1)\, (v \otimes m)) = f(v \otimes m).\end{array}$$

On the other hand, for any $m \in M, v \in V$ and $g \in {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}_k(V, N))$,

$$\begin{array}{rcl}[(\theta \circ \theta')(g)](m)(v) & = & \theta'(g)(\sum p_L^1\, v \otimes p_L^2\, m)] \\ & = & \sum q_L^2[g(p_L^2\, m)(S^{-1}(q_L^1)p_L^1\, v)] \\ & = & \sum q_L^2[p_L^2 \cdot (g(m))](S^{-1}(q_L^1)p_L^1\, v) \\ & = & \sum q_L^2(p_L^2)_2\, (g(m))(S^{-1}((p_L^2)_1)S^{-1}(q_L^1)p_L^1\, v) \\ & = & \sum q_L^2(p_L^2)_2\, (g(m))(S^{-1}(q_L^1(p_L^2)_1)p_L^1\, v) \\ {\scriptstyle \text{by}(7.37)} & = & g(m)(v).\end{array}$$

This means that θ and θ' are inverse to each other. □

9.7. Proposition. (Adjunction $(A \otimes_k -, {}^t\mathrm{Hom}_k(A, -))$ **for algebras in** ${}_H\mathbb{M}$**).** *Let H be a quasi-Hopf algebra and A be a left H-module algebra. Then the right adjoint functor ${}^t\mathrm{Hom}_k(A, -) : {}_H\mathbb{M} \to {}_H\mathbb{M}$ of the tensor functor $A \otimes_k^b -$ is a comonad on ${}_H\mathbb{M}$ with comultiplication given, for any $N \in {}_H\mathbb{M}$, by*

$$\delta_N : {}^t\mathrm{Hom}_k(A, N) \longrightarrow {}^t\mathrm{Hom}_k(A, {}^t\mathrm{Hom}_k(A, N)),$$

$$f \longmapsto \{a \mapsto [b \mapsto \sum q_L^2\, \{(x^3(p_L^2)_2 p_L^2 \cdot f)(S^{-1}(q_L^1) \cdot [(x^1 p_L^1\, b)(x^2(p_L^2)_1\, a)])\}]\},$$

and counit

$$\epsilon_N : {}^t\mathrm{Hom}_k(A, N) \longrightarrow N, \quad f \mapsto f(1_A).$$

The Eilenberg-Moore comodule *category $({}_H\mathbb{M})^{{}^t\mathrm{Hom}_k(A,-)}$ is isomorphic to the category of left A-modules in ${}_H\mathbb{M}$, sending any left A-module $\triangleright : A \otimes_k N \to N$ in ${}_H\mathbb{M}$ to N itself with a ${}^t\mathrm{Hom}_k(A, -)$-comodule structure given by*

$$\varrho^N : N \longrightarrow {}^t\mathrm{Hom}_k(A, N), \quad n \mapsto [a \mapsto \sum (p_L^1\, a) \triangleright (p_L^2\, n)].$$

Proof. Let $V = A$ be a left H-module algebra, that is, the functor $A \otimes_k - : {}_H\mathbb{M} \to {}_H\mathbb{M}$ is a monad. Thus, its right adjoint, ${}^t\mathrm{Hom}_k(A, -) : {}_H\mathbb{M} \to {}_H\mathbb{M}$, is a comonad on ${}_H\mathbb{M}$ (see 4.9).

From the multiplication $\mu_A : A \otimes A \to A$ we get the commutative diagram

$$\begin{array}{ccc} {}_H\mathrm{Hom}(A \otimes M, N) & \xrightarrow[\cong]{\theta_{M,N}} & {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}(A, N)) \\ \downarrow {\scriptstyle [\mu_A \otimes id_M, N]} & & \downarrow {\scriptstyle \tilde{\delta}_N} \\ {}_H\mathrm{Hom}((A \otimes A) \otimes M, N) & & {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}(A, {}^t\mathrm{Hom}(A, N))) \\ \downarrow {\scriptstyle [a_{A,A,M}, N]} & & \uparrow {\scriptstyle \theta_{M, {}^t\mathrm{Hom}(A,N)}} \\ {}_H\mathrm{Hom}((A \otimes A) \otimes M, N) & \xrightarrow[\theta_{A \otimes M, N}]{} & {}_H\mathrm{Hom}(A \otimes M, {}^t\mathrm{Hom}(A, N)). \end{array}$$

Explicitely, for $a, b \in A$, $m \in M$ and $g \in {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}(A, N))$, the map

$$\tilde{\delta}_N : {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}(A, N)) \longrightarrow {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}(A, {}^t\mathrm{Hom}(A, N))),$$

is computed as

$$\begin{aligned}
& \tilde{\delta}_N(g)(m)(a)(b) \\
= \;& \theta_{M, {}^t\mathrm{Hom}(A,N)} \circ \theta_{A\otimes M, N} \circ [a^{-1}_{A,A,M}, N] \circ [\mu_A \otimes id_M, N] \circ \theta'_{M,N}(g)(m)(a)(b) \\
= \;& \{\sum (\theta_{A\otimes M,N} \circ [a^{-1}_{A,A,M}, N] \circ [\mu_A \otimes id_M, N] \circ \theta'_{M,N}(g))(p^1_L\, a \otimes p^2_L\, m)\}(b) \\
= \;& \sum \{[a^{-1}_{A,A,M}, N] \circ [\mu_A \otimes id, N] \circ \theta'_{M,N}(g)\}(p^1_L\, b \otimes [p^2_{L1}\, p^1_L\, a \otimes p^2_{L2} p^2_L\, m]) \\
= \;& \sum \{\theta'_{M,N}(g) \circ (\mu_A \otimes id) \circ a^{-1}_{A,A,M}\}(p^1_L\, b \otimes [(p^2_L)_1\, p^1_L\, a \otimes (p^2_L)_2 p^2_L\, m]) \\
= \;& \sum \{\theta'_{M,N}(g) \circ (\mu_A \otimes id_M)\}([x^1 p^1_L\, b \otimes x^2 (p^2_L)_1\, p^1_L\, a] \otimes x^3 (p^2_L)_2 p^2_L\, m) \\
= \;& \sum \{\theta'_{M,N}(g)\}((x^1 p^1_L\, b)(x^2 (p^2_L)_1\, p^1_L\, a) \otimes x^3 (p^2_L)_2 p^2_L\, m) \\
= \;& \sum q^2_L\, \{[g(x^3 (p^2_L)_2 p^2_L\, m)](S^{-1}(q^1_L) \cdot [(x^1 p^1_L\, b)(x^2 (p^2_L)_1\, p^1_L\, a)])\} \\
\text{H-linearity of g} \quad = \;& \sum q^2_L\, \{[x^3 (p^2_L)_2 p^2_L \cdot g(m)](S^{-1}(q^1_L) \cdot [(x^1 p^1_L\, b)(x^2 (p^2_L)_1\, p^1_L\, a)])\}.
\end{aligned}$$

This yields the map

$$\tilde{\delta}_N : {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}(A, N)) \longrightarrow {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}(A, {}^t\mathrm{Hom}(A, N))),$$

defined for $a, b \in A$ and $g \in {}_H\mathrm{Hom}(M, {}^t\mathrm{Hom}_k(A, N))$ by

$$\tilde{\delta}_N(g)(m)(a)(b) = \sum q^1_R\, \{(X^1 (p^1_R)_1 p^1_R \cdot g(m))(S(q^2_R) \cdot [(X^2 (p^1_R)_2 p^2_R\, a)(X^3 p^2_R\, b)])\}.$$

By the Yoneda Lemma, this gives the comultiplication δ of this comonad as

$$\delta_N : {}^t\mathrm{Hom}_k(A, N) \longrightarrow {}^t\mathrm{Hom}_k(A, {}^t\mathrm{Hom}_k(A, N)),$$

$$[\delta_N(f)(a)](b) = \sum q^2_L\, \{[x^3 (p^2_L)_2 p^2_L \cdot f](S^{-1}(q^1_L) \cdot [(x^1 p^1_L\, b)(x^2 (p^2_L)_1\, p^1_L\, a)])\},$$

for $a, b \in A$ and $f \in {}^t\mathrm{Hom}_k(A, N)$.

With a similar argument as in 9.3 and using the unit map $\iota : k \to A$ and the counit of adjunction, it can be shown that the counit ϵ for δ of this comonad is given by

$$\epsilon_N : \mathrm{Hom}_k(A, N) \longrightarrow N, \quad f \mapsto f(1_A).$$

The Eilenberg-Moore *module* category $({}_H\mathbb{M})_{A\otimes -}$ over the monad $A \otimes_k - : {}_H\mathbb{M} \to {}_H\mathbb{M}$, is the category of left (A, H)-modules ${}_A({}_H\mathbb{M})$. By 4.9, this is also isomorphic to the Eilenberg-Moore *comodule* category $({}_H\mathbb{M})^{{}^t\mathrm{Hom}_k(A,-)}$ over the comonad ${}^t\mathrm{Hom}_k(A, -)$. Explicitly, for any $N \in {}_A({}_H\mathbb{M})$ (with the left (weak) A-action $a \otimes n \mapsto a \triangleright n$), the ${}^t\mathrm{Hom}_k(A, -)$-comodule structure of N is given by the composition

$$\varrho^N : N \xrightarrow{\eta_N} {}^t\mathrm{Hom}_k(A, A \otimes N) \longrightarrow {}^t\mathrm{Hom}_k(A, N).$$

For any $n \in N$, and $a \in A$,

$$\varrho^N(n)(a) = \sum (p^1_L\, a) \triangleright (p^2_L\, n).$$

In this way, we obtain a functor $F : {}_A({}_H\mathbb{M}) \to ({}_H\mathbb{M})^{{}^t\mathrm{Hom}(A,-)}$.
Conversely, given a ${}^t\mathrm{Hom}_k(A,-)$-comodule structure

$$\varrho^M : M \longrightarrow {}^t\mathrm{Hom}_k(A, M), \quad m \mapsto \varrho^M(m),$$

on left H-module M, we define a left (weak) A-action $\triangleright'$ on M as composition

$$A \otimes M \xrightarrow{A\otimes\varrho^M} A \otimes {}^t\mathrm{Hom}_k(A, M) \xrightarrow{\varepsilon_M} M.$$

Explicitly, for $a \in A$ and $m \in M$,

$$a \triangleright' m = \sum q_L^2 \cdot [\varrho^M(m)(S^{-1}(q_L^1)\, a)].$$

This (weak) A-action is a morphism in ${}_H\mathbb{M}$, and defines a left A-module structure on M in ${}_H\mathbb{M}$. □

9.8. Corollary. *From 8.4 we know that the category ${}_A({}_H\mathbb{M})$ of left A-modules in ${}_H\mathbb{M}$ is isomorphic to the category of left modules over the associative algebra $A\#H$. So the composition*

$${}_{A\#H}\mathbb{M} \cong {}_A({}_H\mathbb{M}) \xrightarrow{F} ({}_H\mathbb{M})^{{}^t\mathrm{Hom}(A,-)},$$

of isomorphisms yields an isomorphism between ${}_{A\#H}\mathbb{M}$ and $({}_H\mathbb{M})^{{}^t\mathrm{Hom}(A,-)}$.
On the other hand, $A\#H$ is an associative algebra. Thus, $A\#H \otimes_k -$ is a monad and $\mathrm{Hom}_k(A\#H, -)$ *a comonad on $\mathbb{M}_k$. The Eilenberg-Moore* module *category* $(\mathbb{M}_k)_{A\#H\otimes_k -}$ *is nothing but the category ${}_{A\#H}\mathbb{M}$ and it is isomorphic to the Eilenberg-Moore* comodule *category* $(\mathbb{M}_k)^{\mathrm{Hom}_k(A\#H\otimes_k -)}$. *So we have*

$$(\mathbb{M}_k)^{\mathrm{Hom}_k(A\#H\otimes_k -)} \cong (\mathbb{M}_k)_{A\#H\otimes_k -} \cong {}_{A\#H}\mathbb{M} \cong {}_A({}_H\mathbb{M}) \xrightarrow{F} ({}_H\mathbb{M})^{{}^t\mathrm{Hom}(A,-)}.$$

9.9. Adjunction $(V\otimes_k^b -, {}^*V\otimes_k^b -)$ on ${}_H\mathbb{M}$. Let $V \in {}_H\mathbb{M}$ and ${}_kV$ be finitely generated and projective with a dual basis $\{v_i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$. Then we have the adjoint pair of functors

$$V \otimes_k - : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M}, \quad {}^*V \otimes_k - : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M}.$$

Computations similar to those of Schauenburg in [27] show that unit and counit of this adjunction are

$$u_M : M \xrightarrow{b'_V\otimes M} ({}^*V \otimes V) \otimes M \xrightarrow{a} {}^*V \otimes (V \otimes M),$$

$$m \mapsto \sum (v^i \otimes S^{-1}(\beta)\, v_i) \otimes m \mapsto \sum X^1 v^i \otimes (X^2 S^{-1}(\beta)\, v_i \otimes X^3 m),$$

$$c_M : V \otimes ({}^*V \otimes M) \xrightarrow{a^{-1}} (V \otimes {}^*V) \otimes M \xrightarrow{d'_V\otimes M} M,$$

$$v \otimes g \otimes m \mapsto \sum (x^1 v \otimes x^2 \cdot g) \otimes x^3 m) \mapsto \sum g(S^{-1}(q_L^1)\, v) q_L^2\, m.$$

Now, comparing the above right adjoint ${}^*V \otimes_k -$ with the right adjoint ${}^t\mathrm{Hom}(V, -)$ for the tensor functor $V \otimes_k - : {}_H\mathbb{M} \to {}_H\mathbb{M}$, introduced in 9.6, and using 4.2, with similar arguments as in 9.10, we obtain a functorial isomorphism between this two right adjoints.

9.10. Theorem. (Adjunction $(-\otimes_k V, -\otimes_k V^*)$ on ${}_H\mathbb{M}$). *Let H be a quasi-Hopf algebra, $M, V \in {}_H\mathbb{M}$. Then*

(1) *The map $\psi_M : {}^*V \otimes_k M \to {}^t\mathrm{Hom}_k(V, M)$, $\quad g \otimes m \longmapsto [v \mapsto g(v)\, m]$, is a natural homomorphism in ${}_H\mathbb{M}$.*

(2) *If ${}_kV$ is finitely generated and projective with dual basis $\{v_i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$, then the map ψ_M gives rise to a natural* isomorphism *with inverse map*

$$\psi'_M : {}^t\mathrm{Hom}(V, M) \longrightarrow {}^*V \otimes_k M,$$

$$f \mapsto \sum X^1 v^i \otimes x^3 X^3_2\, [f(S^{-1}(\alpha x^2 X^3_1) x^1 X^2 S^{-1}(\beta)\, v_i)].$$

By symmetry, similar adjunctions as 9.2 can be stated for right modules as well. For any $M, N \in \mathbb{M}_H$ we have that $\mathrm{Hom}_k(M, N)$ is a right H-module by

$$(f \cdot h)(m) = \sum f(m\, S(h_1))\, h_2,$$

which is denoted by $\mathrm{Hom}^s_k(M, N)$ (e.g. [7, 15.9]).

9.11. Theorem. (Adjunction $(V\otimes^b_k -, \mathrm{Hom}^s_k(V, -))$ on $\mathbb{M}_H$). *Let H be a quasi-Hopf algebra with quasi-antipode (S, α, β), and $M, N, V \in \mathbb{M}_H$. Then there is a functorial isomorphism*

$$\mathrm{Hom}_{-H}(V \otimes^b_k M, N) \xrightarrow{\psi} \mathrm{Hom}_{-H}(M, \mathrm{Hom}^s_k(V, N)),$$

$$f \longmapsto \{m \mapsto [v \mapsto f((v \otimes m)\, q_L)]\},$$

with inverse map ψ':

$$g \longmapsto \{v \otimes m \mapsto \sum [g(m)(v\, S(p^1_L))]\, p^2_L\},$$

where $p_L = \sum p^1_L \otimes p^2_L$ and $q_L = \sum q^1_L \otimes q^2_L$ as introduced in (7.28) and (7.29). Thus there is an adjunction between the functors

$$V \otimes^b_k - : \mathbb{M}_H \longrightarrow \mathbb{M}_H, \quad \mathrm{Hom}^s_k(V, -) : \mathbb{M}_H \longrightarrow \mathbb{M}_H,$$

with unit and counit given by

$$\eta_M : M \longrightarrow \mathrm{Hom}^s_k(V, V \otimes_k M), \quad m \longmapsto [v \mapsto (v \otimes m)\, q_L],$$

$$\varepsilon_M : V \otimes \mathrm{Hom}^s_k(V, M) \longrightarrow M, \quad v \otimes f \mapsto \sum [f(v\, S(p^1_L))]\, p^2_L.$$

The proof can be given with similar arguments as in 9.2.

9.12. Proposition. (Adjunction $(A \otimes^b_k -, \mathrm{Hom}^s_k(A, -))$ for algebras in $\mathbb{M}_H$). *Let H be a quasi-Hopf algebra and A be a right H-module algebra. Then the right adjoint functor $\mathrm{Hom}^s_k(A, -) : {}_H\mathbb{M} \to {}_H\mathbb{M}$ of the tensor functor $A \otimes^b_k -$ is a comonad on ${}_H\mathbb{M}$ with comultiplication given for any $N \in {}_H\mathbb{M}$ by*

$$\delta_N : \mathrm{Hom}^s_k(A, N) \longrightarrow \mathrm{Hom}^s_k(A, \mathrm{Hom}^s_k(A, N)),$$

$$f \longmapsto \{a \mapsto [b \mapsto \sum \{[f \cdot (q_L^2 (q_L^1)_2 X^3)]([(b\, q_L^2 X^1)(a\, q_L^1 (q_L^1)_1 X^2)] \cdot S(p_L^1))\}\, p_L^2\},$$

and counit

$$\epsilon_N : {}^t\mathrm{Hom}_k(A, N) \longrightarrow N, \quad f \mapsto f(1_A).$$

The Eilenberg-Moore comodule *category* $(\mathbb{M}_H)^{\mathrm{Hom}_k^s(A,-)}$ *is isomorphic to the category of left A-modules in $\mathbb{M}_H$, sending any left A-module $\triangleright : A \otimes N \to N$ in $\mathbb{M}_H$, to N itself with a $\mathrm{Hom}_k^s(A,-)$-comodule structure given by*

$$\varrho^N : N \longrightarrow \mathrm{Hom}_k^s(A, N), \quad n \mapsto [a \mapsto \sum (a\, q_L^1) \triangleright (n\, q_L^2)].$$

Proof. Let $V = A$ be a right H-module algebra, i.e. the functor $A \otimes_k^b - : \mathbb{M}_H \to \mathbb{M}_H$ is a monad. Thus, its right adjoint $\mathrm{Hom}_k^s(A,-) : \mathbb{M}_H \to \mathbb{M}_H$ is a comonad (see 4.9). The multiplication $\mu_A : A \otimes A \to A$ yields the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{-H}(A \otimes M, N) & \xrightarrow[\cong]{\psi_{M,N}} & \mathrm{Hom}_{-H}(M, \mathrm{Hom}^s(A, N)) \\
{\scriptstyle [\mu_A \otimes id_M, N]}\downarrow & & \downarrow{\scriptstyle \tilde{\delta}_N} \\
\mathrm{Hom}_{-H}((A \otimes A) \otimes M, N) & & \mathrm{Hom}_{-H}(M, \mathrm{Hom}^s(A, \mathrm{Hom}^s(A, N))) \\
{\scriptstyle [a_{A,A,M}, N]}\downarrow & & \uparrow{\scriptstyle \psi_{M, \mathrm{Hom}^s(A,N)}} \\
\mathrm{Hom}_{-H}((A \otimes A) \otimes M, N) & \xrightarrow[\psi_{A \otimes M, N}]{} & \mathrm{Hom}_{-H}(A \otimes M, \mathrm{Hom}^s(A, N)).
\end{array}$$

With similar arguments as in 9.7, we can see that for $a, b \in A$, $m \in M$ and $g \in \mathrm{Hom}_{-H}(M, {}^t\mathrm{Hom}(A, N))$, the morphism

$$\tilde{\delta}_N : {}_H\mathrm{Hom}(M, \mathrm{Hom}^s(A, N)) \longrightarrow \mathrm{Hom}_{-H}(M, \mathrm{Hom}^s(A, \mathrm{Hom}^s(A, N))),$$

is given by

$$\tilde{\delta}_N(g)(m)(a)(b) = \sum \{[g(m) \cdot (q_L^2 (q_L^1)_2 X^3)]([(b\, q_L^2 X^1)(a\, q_L^1 (q_L^1)_1 X^2)] \cdot S(p_L^1))\}\, p_L^2,$$

and by the Yoneda Lemma, we obtain the comultiplication δ of this comonad as stated above in the proposition.

With similar arguments as in 9.3, (see also 9.7), it can be shown that the counit ϵ for δ of this comonad is given by

$$\epsilon_N : \mathrm{Hom}_k(A, N) \longrightarrow N, \quad f \mapsto f(1_A).$$

The Eilenberg-Moore *module* category $(\mathbb{M}_H)_{A \otimes -}$ over the monad $A \otimes_k - : \mathbb{M}_H \to \mathbb{M}_H$, is the category ${}_A(\mathbb{M}_H)$ of left A-modules in $\mathbb{M}_H$. By 4.9, it is isomorphic to the Eilenberg-Moore *comodule* category $(\mathbb{M}_H)^{\mathrm{Hom}_k^s(A,-)}$ over the comonad $\mathrm{Hom}_k^s(A,-)$. We describe this isomorphism explicitly.

For any $N \in {}_A(\mathbb{M}_H)$ (with the left (weak) A-action $a \otimes n \mapsto a \triangleright n$), the $\mathrm{Hom}_k^s(A,-)$-comodule structure of N is given by the composition

$$\varrho^N : N \xrightarrow{\eta_N} \mathrm{Hom}_k^s(A, A \otimes N) \xrightarrow{[A, {}_N\varrho]} \mathrm{Hom}_k^s(A, N).$$

Explicitly, for any $n \in N$, and $a \in A$,

$$\varrho^N(n)(a) = \sum (a\, q_L^1) \triangleright (n\, q_L^2).$$

In this way, we obtain a functor

$$F : {}_A(\mathbb{M}_H) \longrightarrow (\mathbb{M}_H)^{\mathrm{Hom}^s(A,-)}.$$

Conversely, given a $\mathrm{Hom}_k^s(A,-)$-comodule structure

$$\varrho^M : M \longrightarrow \mathrm{Hom}_k^s(A, M), \quad m \mapsto \varrho^M(m),$$

on $M \in \mathbb{M}_H$, we define a left (weak) A-action $\triangleright'$ on M as composition

$$\triangleright' : A \otimes M \xrightarrow{A \otimes \varrho^M} A \otimes \mathrm{Hom}_k^s(A, M) \xrightarrow{\varepsilon_M} M,$$

explicitly, for $a \in A$ and $m \in M$,

$$a \triangleright' m = \sum [\varrho^M(m)(a\, S(p_L^1)\, p_L^2)].$$

□

Since the quasi-antipode S is bijective, $\mathrm{Hom}_k(M, N)$ is a right H-module by

$$(f \cdot h)(m) = \sum f(m\, S^{-1}(h_2))\, h_1,$$

which we denote by $\mathrm{Hom}_k^t(M, N)$. With a similar proof as in 9.6, we obtain

9.13. Theorem. (Adjunction $(- \otimes_k^b V, \mathrm{Hom}_k^t(V,-))$ on $\mathbb{M}_H$). *Let H be a quasi-Hopf algebra with quasi-antipode (S, α, β), and $M, N, V \in \mathbb{M}_H$. Then there is a functorial isomorphism*

$$\mathrm{Hom}_{-H}(M \otimes_k^b V, N) \xrightarrow{\theta} \mathrm{Hom}_{-H}(M, \mathrm{Hom}_k^t(V, N)),$$
$$f \longmapsto \{m \mapsto [v \mapsto f((m \otimes v)\, q_R)]\},$$

with inverse map θ':

$$g \longmapsto \{m \otimes v \mapsto \sum [g(m)(v\, S^{-1}(p_R^2))]\, p_R^1\},$$

where $p_R = \sum p_R^1 \otimes p_R^2$ and $q_R = \sum q_R^1 \otimes q_R^2$ as introduced in (7.30) and (7.31). Thus, there is an adjunction between the functors

$$- \otimes^b V : \mathbb{M}_H \to \mathbb{M}_H, \quad \mathrm{Hom}^t(V, -) : \mathbb{M}_H \to \mathbb{M}_H,$$

with unit and counit given by

$$\eta_M : M \longrightarrow \mathrm{Hom}_k^t(V, M \otimes_k^b V), \quad m \longmapsto [v \mapsto (m \otimes v)\, q_R],$$

$$\varepsilon_M : \mathrm{Hom}_k^t(V, M) \otimes V \longrightarrow M, \quad f \otimes v \mapsto \sum [f(v\, S^{-1}(p_R^2))]\, p_R^1.$$

9.14. Proposition. (Adjunction $(-\otimes_k^b A, \mathrm{Hom}_k^t(A,-)$ for algebras in $\mathbb{M}_H$). *Let H be a quasi-Hopf algebra and A be a right H-module algebra. Then the right adjoint functor $\mathrm{Hom}_k^t(A,-) : \mathbb{M}_H \to \mathbb{M}_H$ of the tensor functor $-\otimes_k^b A$ is a comonad on $\mathbb{M}_H$ with comultiplication given for any $N \in {}_H\mathbb{M}$ by*

$$\delta_N : \mathrm{Hom}_k^t(A,N) \longrightarrow \mathrm{Hom}_k^t(A, \mathrm{Hom}_k^t(A,N)),$$

$$f \longmapsto \{a \mapsto [b \mapsto \sum\{[f \cdot (q_R^1(q_R^1)_1 x^1)]([(a\, q_R^2(q_R^1)_2 x^2)(b\, q_R^2 x^3)] \cdot S^{-1}(p_R^2))\}\, p_R^1\},$$

and counit

$$\epsilon_N : \mathrm{Hom}_k^t(A,N) \longrightarrow N, \quad f \mapsto f(1_A).$$

The Eilenberg-Moore comodule *category $(\mathbb{M}_H)^{\mathrm{Hom}_k^t(A,-)}$ is isomorphic to the category of right A-modules in $\mathbb{M}_H$, corresponding with any right A-module $\triangleleft : N \otimes A \to N$ in $\mathbb{M}_H$, the right H-module N itself with a $\mathrm{Hom}_k^t(A,-)$-comodule structure given by*

$$\varrho^N : N \longrightarrow \mathrm{Hom}_k^t(A,N), \quad n \mapsto [a \mapsto \sum (n\, q_R^1) \triangleleft (a\, q_R^2)].$$

Proof. Take $V = A$ to be a right H-module algebra. The functor $-\otimes_k A : \mathbb{M}_H \to \mathbb{M}_H$ is a monad. Thus, its right adjoint, $\mathrm{Hom}_k^t(A,-) : \mathbb{M}_H \to \mathbb{M}_H$, is comonad on $\mathbb{M}_H$ (see 4.9).

By adjunction 9.13, the multiplication $\mu_A : A \otimes A \to A$ yields the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{-H}(M \otimes A, N) & \xrightarrow[\simeq]{\theta_{M,N}} & \mathrm{Hom}_{-H}(M, \mathrm{Hom}^t(A,N)) \\
\Big\downarrow {\scriptstyle [id_M \otimes \mu_A, N]} & & \Big\downarrow {\scriptstyle \tilde{\delta}_N} \\
\mathrm{Hom}_{-H}(M \otimes (A \otimes A), N) & & \mathrm{Hom}_{-H}(M, \mathrm{Hom}^t(A, \mathrm{Hom}^t(A,N))) \\
\Big\downarrow {\scriptstyle [a_{M,A,A}, N]} & & \Big\uparrow {\scriptstyle \theta_{M, \mathrm{Hom}^t(A,N)}} \\
\mathrm{Hom}_{-H}(M \otimes (A \otimes A), N) & \xrightarrow[\theta_{M \otimes A, N}]{\simeq} & \mathrm{Hom}_{-H}(M \otimes A, \mathrm{Hom}^t(A,N)).
\end{array}$$

With similar arguments as in 9.3, we can see that for $a, b \in A$, $m \in M$ and $g \in \mathrm{Hom}_{-H}(M, \mathrm{Hom}^t(A,N))$, the map

$$\tilde{\delta}_N : \mathrm{Hom}_{-H}(M, \mathrm{Hom}^t(A,N)) \longrightarrow \mathrm{Hom}_{-H}(M, \mathrm{Hom}^t(A, \mathrm{Hom}^t(A,N))),$$

is computed as

$$\tilde{\delta}_N(g)(m)(a)(b) = \sum\{[g(m) \cdot (q_R^1(q_R^1)_1 x^1)]([(a\, q_R^2(q_R^1)_2 x^2)(b\, q_R^2 x^3)] \cdot S^{-1}(p_R^2))\}\, p_R^1,$$

and using the Yoneda Lemma, we obtain the comultiplication δ of this comonad as stated.

With similar arguments as in 9.3 and 9.7, it can be shown that the counit ϵ for δ of this comonad is given by

$$\epsilon_N : \mathrm{Hom}_k(A,N) \longrightarrow N, \quad f \mapsto f(1_A).$$

The Eilenberg-Moore *module* category $(\mathbb{M}_H)_{-\otimes^b A}$ over the monad $-\otimes^b_k A : \mathbb{M}_H \to \mathbb{M}_H$ is the category $(\mathbb{M}_H)_A$ of right A-modules in $\mathbb{M}_H$.
By 4.9, this category is isomorphic to the Eilenberg-Moore *comodule* category $(\mathbb{M}_H)^{\mathrm{Hom}^t_k(A,-)}$ over the comonad $\mathrm{Hom}^t_k(A,-)$:
For any $N \in (\mathbb{M}_H)_A$ with right (weak) A-action $n \otimes a \mapsto n \triangleleft a$ for $a \in A$ and $n \in N$, the $\mathrm{Hom}^t_k(A,-)$-comodule structure of N is given by the composition

$$\varrho^N : N \xrightarrow{\eta_N} \mathrm{Hom}^t_k(A, N \otimes A) \xrightarrow{[A,\varrho_N]} \mathrm{Hom}^t_k(A,N),$$

i.e. for any $n \in N$, and $a \in A$,

$$\varrho^N(n)(a) = \sum (n\, q^1_R) \triangleleft (a\, q^2_R).$$

In this way, we obtain a functor $F : (\mathbb{M}_H)_A \to (\mathbb{M}_H)^{\mathrm{Hom}^t(A,-)}$.
Conversely, given any $\mathrm{Hom}^t_k(A,-)$-comodule structure

$$\varrho^M : M \longrightarrow \mathrm{Hom}^t_k(A,M), \quad m \mapsto \varrho^M(m),$$

on $M \in \mathbb{M}_H$, we define a right (weak) A-action on M as composition

$$\triangleleft' : M \otimes A \xrightarrow{\varrho^M \otimes A} \mathrm{Hom}^t_k(A,M) \otimes A \xrightarrow{\varepsilon_M} M,$$

i.e. for $a \in A$ and $m \in M$,

$$m \triangleleft' a = \sum [\varrho^M(m)(a\, S^{-1}(p^2_R))]\, p^1_R.$$

This (weak) A-action is (by construction) a morphism in $\mathbb{M}_H$, and defines a right A-module structure on M in $\mathbb{M}_H$. □

Combining the Hom-tensor adjunctions in the categories ${}_H\mathbb{M}$ and $\mathbb{M}_H$, we prove Hom-tensor relations for the bimodule category ${}_H\mathbb{M}_H$ over a *quasi-Hopf algebra* H. For any quasi-bialgebra H, we know that ${}_H\mathbb{M}_H$ is a monoidal category (see section 7) and for any $M, N \in {}_H\mathbb{M}_H$, $M \otimes_k N \in {}_H\mathbb{M}_H$ (with the diagonal left and right H-module structure).
If H is a quasi-Hopf algebra with quasi-antipode (S,α,β), then for any $M, N \in {}_H\mathbb{M}_H$ we have an (H,H)-bimodule structure on $\mathrm{Hom}_k(M,N)$ given by

$$(h \cdot f \cdot h')(m) = \sum h_1\, [f(S(h_2)\, m\, S^{-1}(h'_2))]\, h'_1, \tag{9.1}$$

which is denoted by ${}^s\mathrm{Hom}^t_k(M,N)$.

9.15. Theorem. **(Adjoint pair $(-\otimes^b_k V, {}^s\mathrm{Hom}^t(V,-))$ on ${}_H\mathbb{M}_H$).** *Let H be a quasi-Hopf algebra with quasi-antipode (S,α,β), and $M, N, V \in {}_H\mathbb{M}_H$. Then there is a functorial isomorphism*

$$ {}_H\mathrm{Hom}_H(M \otimes^b_k V, N) \xrightarrow{\psi} {}_H\mathrm{Hom}_H(M, {}^s\mathrm{Hom}^t_k(V,N)),$$

$$f \longmapsto \{m \mapsto [v \mapsto f(p_R\, (m \otimes v)\, q_R]\},$$

with inverse map ψ'*:*

$$g \longmapsto \{m \otimes v \mapsto \sum q_R^1[g(m)(S(q_R^2)\, v\, S^{-1}(p_R^2))]\, p_R^1\},$$

where $p_R = \sum p_R^1 \otimes p_R^2$ *and* $q_R = \sum q_R^1 \otimes q_R^2$ *are defined in (7.30) and (7.31). This means that the functors*

$$- \otimes^b V : {}_H\mathbb{M}_H \longrightarrow {}_H\mathbb{M}_H, \quad \textit{and} \quad {}^s\mathrm{Hom}^t(V, -) : {}_H\mathbb{M}_H \longrightarrow {}_H\mathbb{M}_H,$$

form an adjoint pair with unit and counit, (in ${}_H\mathbb{M}_H$*),*

$$\eta_M : M \longrightarrow {}^s\mathrm{Hom}_k^t(V, M \otimes_k V), \quad m \longmapsto [v \mapsto p_R\, (m \otimes v)\, q_R],$$

$$\varepsilon_M : {}^s\mathrm{Hom}_k^t(V, M) \otimes V \longrightarrow M, \quad f \otimes v \mapsto \sum q_R^1\, [f(S(q_R^2)\, v\, S^{-1}(p_R^2))]\, p_R^1.$$

Proof. First we show that for any $f \in {}_H\mathrm{Hom}_H(M \otimes_k^b V, N)$, we have $\psi(f) \in {}_H\mathrm{Hom}_H(M, {}^s\mathrm{Hom}_k^t(V, N))$. i.e. $\psi(f)$ is H-bilinear. for any $h \in H, m \in M, v \in V$,

$$\begin{aligned}
[\psi(f)(h\, m\, h')](v) &= f(p_R\, (h\, m\, h' \otimes v)\, q_R) \\
&= f(p_R\, (h \otimes 1)(m \otimes v)\, (h' \otimes 1)\, q_R) \\
\text{by (7.34) and (7.35)} \quad &= \sum f(\Delta(h_1)\, p_R\, (1 \otimes S(h_2))(m \otimes v)\, (1 \otimes S^{-1}(h'_2))\, q_R\, \Delta(h'_1)) \\
f \text{ is } H\text{-bilinear} \quad &= \sum h_1\, [f(p_R^1\, m\, q_R^1 \otimes p_R^2 S(h_2)\, v\, S^{-1}(h'_2) q_R^2)]\, h'_1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[h \cdot (\psi(f)(m)) \cdot h'](v) &= \sum h_1[(\psi(f)(m))(S(h_2)\, v\, S^{-1}(h'_2))]\, h'_1 \\
&= \sum h_1\, [f(p_R^1\, m\, q_R^1 \otimes p_R^2 S(h_2)\, v\, S^{-1}(h'_2) q_R^2)]\, h'_1,
\end{aligned}$$

showing the H-bilinearity of $\psi(f)$.
Conversely, for any $h, h' \in H, m \in M, v \in V$ and $g \in {}_H\mathrm{Hom}_H(M, {}^s\mathrm{Hom}_k^t(V, N))$,

$$\begin{aligned}
[\psi'(g)](h \cdot (m \otimes v) \cdot h') &= [\psi'(g)](\sum h_1\, m\, h'_1 \otimes h_2\, v\, h'_2) \\
&= \sum q_R^1\, [g(h_1\, m\, h'_1)(S(q_R^2) h_2\, v\, h'_2 S^{-1}(p2_R))]\, p_R^1 \\
&= \sum q_R^1[(h_1 \cdot (g(m)) \cdot h'_1)(S(q_R^2) h_2\, v\, h'_2 S^{-1}(p_R^2))]\, p_R^1 \\
&= \sum q_R^1 h_{11}\, \{g(m)[S(q_R^2 h_{12}) h_2\, v\, h'_2 S^{-1}(h'_{12} p_R^2))]\, h'_{11} p_R^1\} \\
\text{by (7.35) and (7.34)} \quad &= \sum h q_R^1[g(m)(S(q_R^2)\, v\, S^{-1}(p_R^2))]\, p_R^1 h' \\
&= h\, [\psi'(g)(m \otimes v)]\, h'.
\end{aligned}$$

Thus, $\psi'(g)$ is also H-bilinear.
Now we show that ψ and ψ' are inverse to each other. For any $m \in M, v \in V$ and $f \in {}_H\mathrm{Hom}_H(M \otimes V, N)$,

$$\begin{aligned}
[(\psi' \circ \psi)(f)](m \otimes v) &= \sum q_R^1[(\psi(f)(m))(S(q_R^2)\, v\, S^{-1}(p_R^2))]\, p_R^1 \\
&= \sum q_R^1[f(p_R^1\, m\, q_R^1 \otimes p_R^2 S(q_R^2)\, v\, S^{-1}(p_R^2) q_R^2)]\, p_R^1 \\
f \text{ is } H\text{-bilinear} \quad &= \sum [f(\Delta(q_R^1)\, p_R\, (1 \otimes S(q_R^2))\, (m \otimes v)\, (1 \otimes S^{-1}(p_R^2))\, q_R\, \Delta(p_R^1)] \\
\text{by (7.38) and (7.39)} \quad &= f((1 \otimes 1)\, (m \otimes v)\, (1 \otimes 1)) = f(m \otimes v).
\end{aligned}$$

On the other hand, for any $m \in M, v \in V$ and $g \in {}_H\mathrm{Hom}_H(M, {}^s\mathrm{Hom}_k^t(V,N))$,

$$
\begin{aligned}
[(\psi \circ \psi')(g)](m)(v) &= \psi'(g)(\sum p_R^1\, m\, q_R^1 \otimes p_R^2\, v\, q_R^2)] \\
&= \sum q_R^1[g(p_R^1\, m\, q_R^1)(S(q^2)p_R^2\, v\, q_R^2 S^{-1}(p_R^2))]\, p_R^1 \\
&= \sum q_R^1\, \{[p_R^1 \cdot (g(m)) \cdot q_R^1](S(q_R^2)p_R^2\, v\, q_R^2 S^{-1}(p_R^2))\}\, p_R^1 \\
&= \sum q_R^1 (p_R^1)_1\, \{(g(m))(S((p_R^1)_2)S(q_R^2)p_R^2\, v\, q_R^2 S^{-1}(p_R^2))\}\, p_R^1 \\
&= \sum q_R^1 (p_R^1)_1\, \{(g(m))(S(q_R^2(p_{R2}^1)p_R^2\, v\, q_R^2 S^{-1}((q_R^1)_2 p_R^2))\}\, (q_R^1)_1 p_R^1 \\
\text{by(7.38) and (7.39)} &= g(m)(v).
\end{aligned}
$$

This shows that ψ and ψ' are inverse to each other. □

As seen in the one-sided cases, for $M, N \in {}_H\mathbb{M}_H$, there are two possibilities to consider $\mathrm{Hom}_k(M,N)$ as a left (or right) H-module. (We denoted them by ${}^s\mathrm{Hom}_k(M,N)$, ${}^t\mathrm{Hom}_k(M,N)$, $\mathrm{Hom}_k^s(M,N)$, or $\mathrm{Hom}_k^t(M,N)$ respectively.) This can also be done for (H,H)-bimodules. Namely, for $M, N \in {}_H\mathbb{M}_H$, we have an alternative (H,H)-bimodule stucture on $\mathrm{Hom}_k(M,N)$

$$(h \cdot f \cdot h')(m) := \sum h_2\, [f(S^{-1}(h_1)\, m\, S(h_1'))]\, h_2', \tag{9.2}$$

for all $h, h' \in H$, $m \in M$ and $f \in \mathrm{Hom}_k(M,N)$. We denote by ${}^t\mathrm{Hom}_k^s(M,N)$ the k-module $\mathrm{Hom}_k(M,N)$ with the above left and right H-module structure. In this case, we have

9.16. Theorem. **($(V \otimes^b -, {}^t\mathrm{Hom}^s(V,-))$ as adjoint pair on ${}_H\mathbb{M}_H$).** *Let H be a quasi-Hopf algebra with quasi-antipode (S,α,β), and $M, N, V \in {}_H\mathbb{M}_H$. Then there is a functorial isomorphism*

$$ {}_H\mathrm{Hom}_H(V \otimes_k^b M, N) \xrightarrow{\psi} {}_H\mathrm{Hom}_H(M, {}^t\mathrm{Hom}_k^s(V,N)),$$

$$ f \longmapsto \{m \mapsto [v \mapsto f(p_L\,(v \otimes m)\, q_L)]\},$$

with inverse map ψ':

$$ g \longmapsto \{v \otimes m \mapsto \sum q_L^2[g(m)(S^{-1}(q_L^1)\, v\, S(p_L^1))]\, p_L^2\}$$

where $p_L = \sum p_L^1 \otimes p_L^2$ and $q_L = \sum q_L^1 \otimes q_L^2$ are defined in (7.28) and (7.29). This means that the functors

$$ V \otimes^b - : {}_H\mathbb{M}_H \longrightarrow {}_H\mathbb{M}_H, \quad {}^t\mathrm{Hom}^s(V,-) : {}_H\mathbb{M}_H \longrightarrow {}_H\mathbb{M}_H,$$

form an adjoint pair with unit and counit, in ${}_H\mathbb{M}_H$

$$ \eta_M : M \longrightarrow {}^t\mathrm{Hom}_k^s(V, V \otimes_k M), \quad m \longmapsto [v \mapsto p_L\,(v \otimes m)\, q_L],$$

$$ \varepsilon_M : V \otimes {}^t\mathrm{Hom}_k^s(V,M) \longrightarrow M, \quad v \otimes g \mapsto \sum q_L^2\, [g(S^{-1}(q_L^1)\, v\, S(p_L^1))]\, p_L^2.$$

10 Comodule algebras and quasi-smash products

For a quasi-bialgebra H, the category ${}_H\mathbb{M}$ is monoidal and an H-**module (co)algebra** is (by definition) a **(co)algebra** in this category. Because of non-coassociativity of H, this categorical definition cannot be used to introduce H-**comodule (co)algebras**. Hausser and Nill [15] gave a formal definition of an H-comodule algebra as a generalization of the definition of quasi-bialgebra.

10.1. Comodule algebras. Let $(H, \Delta, \varepsilon, \phi)$ be a quasi-bialgebra. A unital associative algebra $\mathcal{A}$ is called a **right** H-**comodule algebra** if there exist an algebra morphism $\rho : \mathcal{A} \to \mathcal{A} \otimes H$ and an invertible element $\phi_\rho \in \mathcal{A} \otimes H \otimes H$ such that

(R1) $\phi_\rho(\rho \otimes id_H) \circ \rho(a) = (id_H \otimes \Delta) \circ \rho(a) \cdot \phi_\rho \qquad \forall a \in \mathcal{A}.$

(R2) $(1_\mathcal{A} \otimes \phi)(id \otimes \Delta \otimes id)(\phi_\rho)(\phi_\rho \otimes 1_H) = (id \otimes id \otimes \Delta)(\phi_\rho)(\rho \otimes id \otimes id)(\phi_\rho)$

(R3) $(id_\mathcal{A} \otimes \varepsilon) \circ \rho = id_\mathcal{A}$

(R4) $(id_\mathcal{A} \otimes \varepsilon \otimes id_H)(\phi_\rho) = 1_\mathcal{A} \otimes 1_H$

Similarly, a unital associative algebra $\mathcal{B}$ is called a **left** H-**comodule algebra** if there exist an algebra morphism $\lambda : \mathcal{B} \to H \otimes \mathcal{B}$ and an invertible element $\phi_\lambda \in H \otimes H \otimes \mathcal{B}$ such that

(L1) $(id \otimes \lambda)(\lambda(b))\phi_\lambda = \phi_\lambda(\Delta \otimes id)(\lambda(b)) \quad \forall b \in \mathcal{B}$

(L2) $(1_H \otimes \phi_\lambda)(id \otimes \Delta \otimes id)(\phi_\lambda)(\phi \otimes 1_B) = (id \otimes id \otimes \lambda)(\phi_\lambda)(\Delta \otimes id \otimes id)(\phi_\lambda)$

(L3) $(\varepsilon \otimes id) \circ \lambda = id_\mathcal{B}$

(L4) $(id \otimes \varepsilon \otimes id)(\phi_\lambda) = 1_H \otimes 1_\mathcal{B}$

If $(\mathcal{A}, \rho, \phi_\rho)$ is a right H-comodule algebra, we also have

$$(id \otimes id \otimes \varepsilon)(\phi_\rho) = 1_\mathcal{A} \otimes 1_H.$$

If $(\mathcal{B}, \lambda, \phi_\lambda)$ is a left H-comodule algebra, then

$$(\varepsilon \otimes id \otimes id)(\phi_\lambda) = 1_H \otimes 1_\mathcal{B}.$$

Any quasi-bialgebra H is a particular example of a left and a right H-comodule algebra for $\mathcal{A} = \mathcal{B} = H$, $\rho = \lambda = \Delta$ and $\phi_\rho = \phi_\lambda = \phi$.
In analogy with the notation for the reassociator ϕ of a quasi-bialgebra H, we use capital letters for showing the components of ϕ_ρ and small letters for the components of ϕ_ρ^{-1}. Namely, we write

$$\phi_\rho = \sum \tilde{X}_\rho^1 \otimes \tilde{X}_\rho^2 \otimes \tilde{X}_\rho^3 = \sum \tilde{Y}_\rho^1 \otimes \tilde{Y}_\rho^2 \otimes \tilde{Y}_\rho^3 = \cdots$$

$$\phi_\rho^{-1} = \sum \tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 \otimes \tilde{x}_\rho^3 = \sum \tilde{y}_\rho^1 \otimes \tilde{y}_\rho^2 \otimes \tilde{y}_\rho^3 = \cdots$$

A similar notation is used for the element ϕ_λ of a left H-comodule algebra $\mathcal{B}$. If no confusion is possible, we will omit the subscripts ρ or λ in the tensor components of

$\phi_\rho, \phi_\lambda, \phi_\rho^{-1}, \phi_\lambda^{-1}$.

For a right H-comodule algebra $\mathcal{A}$, following Hausser and Nill [15], we define elements $\tilde{p}_\rho, \tilde{q}_\rho \in \mathcal{A} \otimes H$,

$$\tilde{p}_\rho = \tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2 = \sum \tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \tag{10.1}$$

$$\tilde{q}_\rho = \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2 = \tilde{X}_\rho^1 \otimes S^{-1}(\alpha \tilde{X}_\rho^3) \tilde{X}_\rho^2 \tag{10.2}$$

By [15, Lemma 9.1], we have the relations, for all $a \in \mathcal{A}$,

$$\rho(a_{(0)}) \tilde{p}_\rho [1_\mathcal{A} \otimes S(a_{(1)})] = \tilde{p}_\rho [a \otimes 1_H] \tag{10.3}$$

$$[1_\mathcal{A} \otimes S^{-1}(a_{(1)})] \tilde{q}_\rho \rho(a_{(0)}) = [a \otimes 1_H] \tilde{q}_\rho \tag{10.4}$$

$$\rho(\tilde{q}_\rho^1) \tilde{p}_\rho [1_\mathcal{A} \otimes S(\tilde{q}_\rho^2)] = 1_\mathcal{A} \otimes 1_H \tag{10.5}$$

$$[1_\mathcal{A} \otimes S^{-1}(\tilde{p}_\rho^2)] \tilde{q}_\rho \rho(\tilde{p}_\rho^1) = 1_\mathcal{A} \otimes 1_H \tag{10.6}$$

$$\phi_\rho (\rho \otimes id_H)(\tilde{p}_\rho)(\tilde{p}_\rho \otimes 1_H) = \sum (id_\mathcal{A} \otimes \Delta)(\rho(\tilde{x}_\rho^1) \tilde{p}_\rho)(1_\mathcal{A} \otimes g^1 S(\tilde{x}_\rho^3) \otimes g^2 S(\tilde{x}_\rho^2)) \tag{10.7}$$

$$(\tilde{q}_\rho \otimes 1_H)(\rho \otimes id_H)(\tilde{q}_\rho) \phi_\rho^{-1} = \sum [1_\mathcal{A} \otimes S^{-1}(f^2 \tilde{X}_\rho^3) \otimes S^{-1}(f^1 \tilde{X}_\rho^2)](id_\mathcal{A} \otimes \Delta)(\tilde{q}_\rho \rho(\tilde{X}_\rho^1)) \tag{10.8}$$

For a left H-comodule algebra $\mathcal{B}$ we define elements $\tilde{p}_\lambda, \tilde{q}_\lambda \in H \otimes \mathcal{B}$ as

$$\tilde{p}_\lambda = \tilde{p}_\lambda^1 \otimes \tilde{p}_\lambda^2 = \sum \tilde{X}_\lambda^2 S^{-1}(\tilde{X}_\lambda^1 \beta) \otimes \tilde{X}_\lambda^3 \tag{10.9}$$

$$\tilde{q}_\lambda = \tilde{q}_\lambda^1 \otimes \tilde{q}_\lambda^2 = \sum S(\tilde{x}_\lambda^1) \alpha \tilde{x}_\lambda^2 \otimes \tilde{x}_\lambda^3 \tag{10.10}$$

As shown in [15], they satisfy the equations (for $b \in \mathcal{B}$),

$$\sum \lambda(b_{(0)}) \tilde{p}_\lambda (S^{-1}(b_{(-1)}) \otimes 1_\mathcal{B}) = \tilde{p}_\lambda (1_H \otimes b) \tag{10.11}$$

$$\sum (S(b_{(-1)}) \otimes 1_\mathcal{B}) \tilde{q}_\lambda \lambda(b_{(0)}) = (1_H \otimes b) \tilde{q}_\lambda \tag{10.12}$$

$$\sum \lambda(\tilde{q}_\lambda^2) \tilde{p}_\lambda (S^{-1}(\tilde{q}_\lambda^1) \otimes 1_\mathcal{B}) = 1_H \otimes 1_\mathcal{B} \tag{10.13}$$

$$\sum (S(\tilde{p}_\lambda^1) \otimes 1_\mathcal{B}) \tilde{q}_\lambda \lambda(\tilde{p}_\lambda^2) = 1_H \otimes 1_\mathcal{B} \tag{10.14}$$

$$\phi_\lambda^{-1} (id_H \otimes \lambda)(\tilde{p}_\lambda)(1_H \otimes \tilde{p}_\lambda) = \sum (\Delta \otimes id_\mathcal{B})(\lambda(X_\lambda^3) \tilde{p}_\lambda)[S^{-1}(X_\lambda^2 g^2) \otimes S^{-1}(\tilde{X}_\lambda^1 g^1) \otimes 1_\mathcal{B}] \tag{10.15}$$

$$(1_H \otimes \tilde{q}_\lambda)(id_H \otimes \lambda)(\tilde{q}_\lambda) \phi_\lambda = \sum [S(\tilde{x}_\lambda^2) \otimes S(\tilde{x}_\lambda^1) \otimes 1_\mathcal{B}][f \otimes 1_\mathcal{B}][\Delta \otimes id_\mathcal{B}]([\tilde{q}_\lambda \lambda(\tilde{x}_\lambda^3)]). \tag{10.16}$$

10.2. Quasi-Smash products. Recall that for any k-algebra H, $H^* = \mathrm{Hom}_k(H, k)$ is an (H, H)-bimodule with left and right actions given by

$$h \rightharpoonup \varphi \leftharpoonup h' : H \longrightarrow k, \quad h'' \mapsto \varphi(h' h'' h) \qquad \forall h, h', h'' \in H \quad \forall \varphi \in H^*.$$

If H is a k-bialgebra and is finitely generated and projective as a k-module, then H^* is also a bialgebra. Now let $\mathcal{A}$ be a right H-comodule algebra. $\mathcal{A}$ can be considered as a left H^*-module algebra, and then the smash product $\mathcal{A}\#H^*$ is defined with multiplication

$$(a\#\varphi)(a'\#\psi) = \sum aa'_0\#(\varphi \leftharpoonup a'_1) * \psi.$$

For a quasi-bialgebra H, the convolution product in H^* is only associative up to conjugation with ϕ, namely,

$$[\varphi * \psi] * \theta = \sum (X^1 \rightharpoonup \varphi \leftharpoonup x^1) * [(X^2 \rightharpoonup \psi \leftharpoonup x^2) * (X^3 \rightharpoonup \theta \leftharpoonup x^3)],$$

for all $\varphi, \psi, \theta \in H^*$. In addition, for all $h \in H, \varphi, \psi \in H^*$,

$$h \rightharpoonup (\varphi * \psi) = \sum (h_1 \rightharpoonup \varphi) * (h_2 \rightharpoonup \psi),$$

$$(\varphi * \psi) \leftharpoonup h = \sum (\varphi \leftharpoonup h_1) * (\psi \leftharpoonup h_2).$$

This means that H^* **is an algebra in the monoidal category** ${}_H\mathbb{M}_H$.

Although H^* is not associative, we still keep the notation of the Hopf case for the action of the algebra H^* (in ${}_H\mathbb{M}_H$) on H

$$\varphi \rightharpoondown h = \sum \varphi(h_2)h_1, \quad h \leftharpoondown \varphi = \sum \varphi(h_1)h_2,$$

for all $\varphi \in H^*$, $h \in H$.

Let $(\mathcal{A}, \rho, \phi_\rho)$ be a right H-comodule algebra. Following Bulacu and Caenepeel [8], we define a multiplication on the k-module $\mathcal{A} \otimes H^*$ by

$$(a\bar{\#}\varphi)(a'\bar{\#}\psi) = \sum aa'_0\tilde{x_\rho}^1\bar{\#}(\varphi \leftharpoonup a'_1\tilde{x_\rho}^2) * (\psi \leftharpoonup \tilde{x_\rho}^3),$$

for all $a, a' \in \mathcal{A}$, and $\varphi, \psi \in H^*$, where we write $a\bar{\#}\varphi$ for $a \otimes \varphi$ and

$$\rho(a) = \sum a_{(0)} \otimes a_{(1)} \quad \textit{and} \quad \phi_\rho^{-1} = \sum \tilde{x_\rho}^1 \otimes \tilde{x_\rho}^2 \otimes \tilde{x_\rho}^3,$$

and denote $\mathcal{A} \otimes H^*$ with this structure by $\mathcal{A}\bar{\#}H^*$. In [9], it is proven that $\mathcal{A}\bar{\#}H^*$ is an algebra in the category ${}_H\mathbb{M}$ with unit $1_\mathcal{A}\#1_H$ and the left H-action

$$h \cdot (a\bar{\#}\varphi) = a\bar{\#}h \rightharpoonup \varphi \qquad \forall h \in H, a \in \mathcal{A}, \varphi \in H^*.$$

This is called the **quasi-smash product** of $\mathcal{A}$ and H^*.

11 Hom-tensor relations for comodule algebras

Let H be a quasi-Hopf algebra and $\mathcal{A}$ be an H-comodule algebra. We know that the coaction of H on $\mathcal{A}$ induces an action of the category of H-modules on the category of $\mathcal{A}$-modules (see section 6). In this section we display Hom-tensor relations for the module category over $\mathcal{A}$. Although this module category is not monoidal, we still have the tensor product $\otimes_k$ in it, and the action of $\mathbb{M}_H$ (resp. ${}_H\mathbb{M}$) on it makes some versions of Hom-tensor relations possible.

Let H be a quasi-Hopf algebra with quasi-antipode (S, α, β) and $(\mathcal{A}, \rho, \phi_\rho)$ be a right H-comodule algebra. Then the coaction

$$\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes_k H, \quad \rho(a) = \sum a_{(0)} \otimes a_{(1)},$$

gives a right action of $\mathbb{M}_H$ on the category $\mathbb{M}_\mathcal{A}$ of right $\mathcal{A}$-modules,

$$- \diamond - : \mathbb{M}_\mathcal{A} \times \mathbb{M}_H \longrightarrow \mathbb{M}_\mathcal{A}, \quad (N, V) \longmapsto N \otimes_k V,$$

where the right $\mathcal{A}$-module structure of $N \otimes_k V$ is given by

$$(n \otimes v) \cdot a = \sum n\, a_{(0)} \otimes v\, a_{(1)} = (n \otimes v)\, \rho(a),$$

for all $a \in \mathcal{A}, v \in V$, and $n \in N$. We denote by $N \otimes_k^{b'} V$ the k-module $N \otimes_k V$ with the above right $\mathcal{A}$-module structure.

There is a natural isomorphism

$$\psi : (- \diamond -) \diamond - \longrightarrow - \diamond (- \otimes -),$$

given by

$$\psi_{N,V,W} : (N \diamond V) \diamond W \longrightarrow N \diamond (V \otimes W), \quad (n \otimes v) \otimes w \mapsto [n \otimes (v \otimes w)] \cdot \phi_\rho^{-1},$$

for all $V, W \in \mathbb{M}_H$ and $N \in \mathbb{M}_\mathcal{A}$, and for $v \in V, w \in W$, and $n \in N$. The commutativity of the diagram

$$\begin{array}{ccccc}
[(N \diamond V) \diamond W] \diamond Z & \xrightarrow{\Psi_{N\diamond V,W,Z}} & (N \diamond V) \diamond (W \otimes Z) & & \\
 & & & \searrow^{\Psi_{N,V,W\otimes Z}} & \\
\downarrow{\scriptstyle \Psi_{N,V,W}\diamond id} & & & & N \diamond [V \otimes (W \otimes Z)] \\
 & & & \nearrow_{id\diamond a_{V,W,Z}} & \\
[N \diamond (V \otimes W)] \diamond Z & \xrightarrow[\Psi_{N,(V\otimes W),Z}]{} & N \diamond [(V \otimes W) \otimes Z] & &
\end{array}$$

is a consequence of the pentagon identity in the definition of the comodule algebra. In this way, for any right H-module V, we have an endofunctor

$$- \otimes_k^{b'} V : \mathbb{M}_\mathcal{A} \longrightarrow \mathbb{M}_\mathcal{A}, \quad N \mapsto N \otimes_k^{b'} V,$$

with the right $\mathcal{A}$-module structure on $N \otimes_k^{b'} V$ as given above. Generalizing the presentation in 9.13, we define a right $\mathcal{A}$-module structure on $\mathrm{Hom}_k(V, N)$ by

$$(f \cdot a)(v) = \sum f(v\, S^{-1}(a_{(1)}))\, a_{(0)},$$

for $a \in \mathcal{A}$, $v \in V$ and $f \in \mathrm{Hom}_k(V, N)$. We denote by $\mathrm{Hom}_k^{t'}(V, N)$ the k-module $\mathrm{Hom}_k(V, N)$ with the above right $\mathcal{A}$-module structure.

11.1. Theorem. (Adjunction $(-\otimes^{b'} V, \mathrm{Hom}^{t'}(V,-))$ on $\mathbb{M}_{\mathcal{A}}$). *Let H be a quasi-Hopf algebra with quasi-antipode (S,α,β) and $(\mathcal{A},\rho,\phi_\rho)$ be a right H-comodule algebra. For $V\in\mathbb{M}_H$ and $M,N\in\mathbb{M}_{\mathcal{A}}$, there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(M\otimes_k^{b'} V,N) \xrightarrow{\psi} \mathrm{Hom}_{\mathcal{A}}(M,\, \mathrm{Hom}_k^{t'}(V,N)),$$
$$f\longmapsto \{m\mapsto [v\mapsto f((m\otimes v)\,\tilde{q}_\rho)]\},$$

with inverse map ψ' given by

$$g\longmapsto \{m\otimes v\mapsto \sum [g(m)(v\,S^{-1}(\tilde{p}^2_\rho))]\,\tilde{p}^1_\rho\},$$

where

$$\tilde{p}_\rho=\sum \tilde{p}^1_\rho\otimes\tilde{p}^2_\rho=\sum \tilde{x}^1_\rho\otimes\tilde{x}^2_\rho\beta S(\tilde{x}^3_\rho),\quad \tilde{q}_\rho=\sum\tilde{q}^1_\rho\otimes\tilde{q}^2_\rho=\sum\tilde{X}^1_\rho\otimes S^{-1}(\alpha\tilde{X}^3_\rho)\tilde{X}^2_\rho.$$

This means that the functors

$$-\otimes_k^{b'} V:\,\mathbb{M}_{\mathcal{A}}\longrightarrow\mathbb{M}_{\mathcal{A}},\quad \mathrm{Hom}_k^{t'}(V,-):\,\mathbb{M}_{\mathcal{A}}\longrightarrow\mathbb{M}_{\mathcal{A}},$$

form an adjoint pair with unit and counit (in $\mathbb{M}_{\mathcal{A}}$)

$$\eta_M: M\longrightarrow \mathrm{Hom}_k^{t'}(V, M\otimes_k V),\quad m\longmapsto [v\mapsto (m\otimes v)\,\tilde{q}_\rho],$$
$$\varepsilon_M:\mathrm{Hom}_k^{t'}(V,M)\otimes V\longrightarrow M,\quad f\otimes v\mapsto\sum [f(v\,S^{-1}(\tilde{p}^2_\rho))]\tilde{p}^1_\rho.$$

Proof. First we show that for any $f\in\mathrm{Hom}_{\mathcal{A}}(M\otimes_k^{b'} V,N)$, $\psi(f)\in\mathrm{Hom}_{\mathcal{A}}(M,\,\mathrm{Hom}_k^{t'}(V,N))$, i.e. $\psi(f)$ is right $\mathcal{A}$-linear. For any $a\in\mathcal{A}, m\in M, v\in V$,

$$\begin{aligned}
[\psi(f)(m\,a)](v) &= f(\sum m\,a\tilde{q}^1_\rho\otimes v\,\tilde{q}^2_\rho)\\
&= f(\sum (m\otimes v)\,(a\otimes 1_H)\,\tilde{q}_\rho)\\
\text{by (10.4)}\quad &= \sum f((m\otimes v)\,(1_{\mathcal{A}}\otimes S^{-1}(a_{(1)}))\,\tilde{q}_\rho\,\rho(a_{(0)}))\\
f\text{ is right }\mathcal{A}\text{-linear}\quad &= \sum [f((m\otimes v)\,(1_{\mathcal{A}}\otimes S^{-1}(a_{(1)}))\,\tilde{q}_\rho)]\,a_{(0)}\\
&= \sum [f(m\,\tilde{q}^1_\rho\otimes v\,S^{-1}(a_{(1)})\tilde{q}^2_\rho)]\,a_{(0)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[(\psi(f)(m))\cdot a](v) &= \sum [(\psi(f)(m))(v\,S^{-1}(a_{(1)}))]\,a_{(0)}\\
&= \sum [f(m\,\tilde{q}^1_\rho\otimes v\,S^{-1}(a_{(1)})\tilde{q}^2_\rho)]\,a_{(0)},
\end{aligned}$$

showing the right $\mathcal{A}$-linearity of $\psi(f)$.
Conversely, for any $a\in\mathcal{A}, m\in M, v\in V$ and $g\in\mathrm{Hom}_{\mathcal{A}}(M,\,\mathrm{Hom}_k^{t'}(V,N))$,

$$\begin{aligned}
[\psi'(g)]((m\otimes v)\cdot a) &= [\psi'(g)](\sum m\,a_{(0)}\otimes v\,a_{(1)})\\
&= \sum\{g(m\,a_{(0)})(v\,a_{(1)}S^{-1}(\tilde{p}^2_\rho))\}\,\tilde{p}^1_\rho\\
g\text{ is right }\mathcal{A}\text{-linear}\quad &= \sum\{(g(m)\cdot a_{(0)})(v\,a_{(1)}S^{-1}(\tilde{p}^2_\rho))\}\,\tilde{p}^1_\rho\\
&= \sum\{g(m)(v\,a_{(1)}S^{-1}(a_{(0,1)}\tilde{p}^2_\rho))\}\,a_{(0,0)}\tilde{p}^1_\rho\\
\text{by (10.3)}\quad &= \sum\{g(m)(v\,S^{-1}(\tilde{p}^2_\rho))\}\,\tilde{p}^1_\rho a\\
&= [\psi'(g)(m\otimes v)]\,a.
\end{aligned}$$

So, $\psi'(g)$ is also right $\mathcal{A}$-linear. Now we show that ψ and ψ' are inverse to each other. For any $m \in M, v \in V$ and $f \in \mathrm{Hom}_{\mathcal{A}}(M \otimes V, N)$,

$$\begin{aligned}
[(\psi' \circ \psi)(f)](m \otimes v) &= \sum [(\psi(f)(m))(v\, S^{-1}(\tilde{p}^2_\rho))]\, \tilde{p}^1_\rho \\
&= \sum [f(m\, \tilde{q}^1_\rho \otimes v\, S^{-1}(\tilde{p}^2_\rho)\tilde{q}^2_\rho)]\, \tilde{p}^1_\rho \\
\text{\small f is right $\mathcal{A}$-linear} \quad &= \sum [f((m \otimes v)\,(1_{\mathcal{A}} \otimes S^{-1}(\tilde{p}^2_\rho))\, \tilde{q}_\rho\, \rho(\tilde{p}^1_\rho)] \\
\text{\small by(10.6)} \quad &= f(m \otimes v).
\end{aligned}$$

On the other hand, for any $m \in M, v \in V$ and $g \in \mathrm{Hom}_{\mathcal{A}}(M, \mathrm{Hom}^{t'}_k(V, N))$,

$$\begin{aligned}
[(\psi \circ \psi')(g)](m)(v) &= \psi'(g)(\sum m\, \tilde{q}^1_\rho \otimes v\, \tilde{q}^2_\rho) \\
&= \sum [g(m\, \tilde{q}^1_\rho)(v\, \tilde{q}^2_\rho S^{-1}(\tilde{p}^2_\rho))]\, \tilde{p}^1_\rho \\
\text{\small g is right $\mathcal{A}$-linear} \quad &= \sum \{[g(m) \cdot \tilde{q}^1_\rho](v\, \tilde{q}^2_\rho S^{-1}(\tilde{p}^2_\rho))\}\, \tilde{p}^1_\rho \\
&= \sum \{(g(m))(v\, \tilde{q}^2_\rho S^{-1}(\tilde{p}^2_\rho) S^{-1}((\tilde{q}^1_\rho)_{(1)}))\}\, (\tilde{q}^1_\rho)_{(0)} \tilde{p}^1_\rho \\
&= \sum \{(g(m))(v\, \tilde{q}^2_\rho S^{-1}((\tilde{q}^1_\rho)_{(1)} \tilde{p}^2_\rho))\}\, (\tilde{q}^1_\rho)_{(0)} \tilde{p}^1_\rho \\
\text{\small by (10.5)} \quad &= g(m)(v).
\end{aligned}$$

This means that ψ and ψ' are inverse to each other. $\square$

Now let $(\mathcal{B}, \lambda, \phi_\lambda)$ be a left H-comodule algebra. The coaction $\lambda : \mathcal{B} \to H \otimes \mathcal{B}$ induces a left action of ${}_H\mathbb{M}$ on the category ${}_{\mathcal{B}}\mathbb{M}$ of left $\mathcal{B}$-modules,

$$- \diamond - : {}_H\mathbb{M} \times {}_{\mathcal{B}}\mathbb{M} \longrightarrow {}_{\mathcal{B}}\mathbb{M}, \quad (V, N) \longmapsto V \otimes N,$$

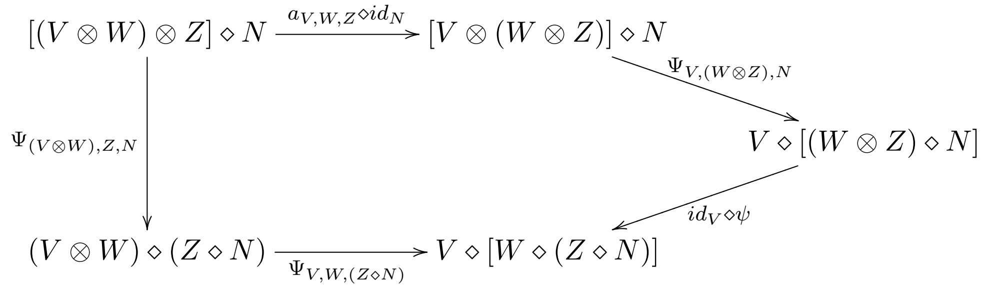
for all $N \in {}_{\mathcal{B}}\mathbb{M}$ and $V \in {}_H\mathbb{M}$, where the left $\mathcal{B}$-module structure of $V \otimes N$ is given by

$$b \cdot (v \otimes n) = \sum b_{(-1)}\, v \otimes b_{(0)}\, n = \lambda(b)(v \otimes n),$$

for all $b \in \mathcal{B}, v \in V$, and $n \in N$. We denote by $V \otimes^{b'}_k N$ the k-module $V \otimes_k N$ with the above left $\mathcal{B}$-module structure.
There is a natural isomorphism $\psi : (- \otimes -) \diamond - \to - \diamond (- \diamond -)$ given by

$$\psi_{V,W,N}(v \otimes w \otimes n) = \phi_\lambda \cdot (v \otimes w \otimes n)$$

for all $V, W \in {}_H\mathbb{M}$ $N \in {}_{\mathcal{B}}\mathbb{M}$, $v \in V, w \in W$ and $n \in N$. The commutativity of the diagram



is a consequence of the pentagon identity in the definition of the comodule algebra. In this way, for any left H-module V, we have an endofunctor

$$V \otimes_k^{b'} - : {}_{\mathcal{B}}\mathbb{M} \longrightarrow {}_{\mathcal{B}}\mathbb{M}, \quad N \mapsto V \otimes_k^{b'} N,$$

with the left $\mathcal{B}$-module structure on $V \otimes_k N$ given above. Generalizing the process defined for modules over quasi-Hopf algebras, we define a left $\mathcal{B}$-module structure on $\mathrm{Hom}_k(V, N)$ by

$$(b \cdot f)(v) = \sum b_{(0)}\, f(S^{-1}(b_{(-1)})\, v),$$

for all $b \in \mathcal{B}$, and $v \in V$ and $f \in \mathrm{Hom}_k(V, N)$. We denote by ${}^{t'}\mathrm{Hom}_k(V, N)$ the k-module $\mathrm{Hom}_k(V, N)$ with the above left $\mathcal{B}$-module structure.

11.2. Theorem. (Adjunction $(V \otimes_k^{b'} -, {}^{t'}\mathrm{Hom}(V, -))$ on ${}_{\mathcal{B}}\mathbb{M}$). *Let H be a quasi-Hopf algebra with quasi-antipode (S, α, β), $(\mathcal{B}, \lambda)$ a left H-comodule algebra, $V \in {}_H\mathbb{M}$, and $M, N \in {}_{\mathcal{B}}\mathbb{M}$. Then there is a functorial isomorphism*

$$ {}_{\mathcal{B}}\mathrm{Hom}(V \otimes_k^{b'} M, N) \xrightarrow{\theta} {}_{\mathcal{B}}\mathrm{Hom}(M, {}^{t'}\mathrm{Hom}_k(V, N)),$$

$$f \longmapsto \{m \mapsto [v \mapsto f(\tilde{p}_\lambda\, (v \otimes m))]\},$$

with inverse map θ':

$$g \longmapsto \{v \otimes m \mapsto \sum \tilde{q}_\lambda^2[g(m)(S^{-1}(\tilde{q}_\lambda^1)\, v)]\},$$

where

$$\tilde{p}_\lambda = \sum \tilde{p}_\lambda^1 \otimes \tilde{p}_\lambda^2 = \sum \tilde{X}_\lambda^2 S^{-1}(\tilde{X}_\lambda^1 \beta) \otimes \tilde{X}_\lambda^3, \quad \tilde{q}_\lambda = \sum \tilde{q}_\lambda^1 \otimes \tilde{q}_\lambda^2 = \sum S(\tilde{x}_\lambda^1)\alpha \tilde{x}_\lambda^2 \otimes \tilde{x}_\lambda^3.$$

This yields an adjunction between the functors

$$V \otimes^{b'} - : {}_{\mathcal{B}}\mathbb{M} \to {}_{\mathcal{B}}\mathbb{M}, \quad {}^{t'}\mathrm{Hom}_k(V, -) : {}_{\mathcal{B}}\mathbb{M} \longrightarrow {}_{\mathcal{B}}\mathbb{M},$$

with unit and counit given by

$$\eta'_M : M \longrightarrow {}^{t'}\mathrm{Hom}_k(V, V \otimes_k M), \quad m \longmapsto [v \mapsto \tilde{p}_\lambda\, (v \otimes m)],$$

$$\varepsilon'_M : V \otimes {}^{t'}\mathrm{Hom}_k(V, M) \longrightarrow M, \quad v \otimes f \mapsto \sum \tilde{q}_\lambda^2[f(S^{-1}(\tilde{q}_\lambda^1)\, v)].$$

Proof. For any $f \in {}_{\mathcal{B}}\mathrm{Hom}(V \otimes_k^b M, N)$, we show that $\theta(f)$ is left $\mathcal{B}$-linear. For any $b \in \mathcal{B}$, $m \in M$ and $v \in V$,

$$\begin{aligned}
[\theta(f)(b\, m)](v) &= f(\sum \tilde{p}_\lambda^1\, v \otimes \tilde{p}_\lambda^2 b\, m) \\
&= f(\tilde{p}_\lambda\, (1_H \otimes b)(v \otimes m)) \\
\text{by (10.11)} \quad &= f(\sum \lambda(b_{(0)})\, \tilde{p}_\lambda\, (S^{-1}(b_{(-1)}) \otimes 1_{\mathcal{B}})(v \otimes m)) \\
f \text{ is left } \mathcal{B}\text{-linear} \quad &= \sum b_{(0)}\, [f(\tilde{p}_\lambda^1 S^{-1}(b_{(-1)})\, v \otimes \tilde{p}_\lambda^2\, m)].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[b \cdot (\theta(f)(m))](v) &= \sum b_{(0)}\, [(\theta(f)(m))(S^{-1}(b_{(-1)})\, v)] \\
&= \sum b_{(0)}\, f(\tilde{p}_\lambda^1 S^{-1}(b_{(-1)})\, v \otimes \tilde{p}_\lambda^2\, m).
\end{aligned}$$

This shows the left $\mathcal{B}$-linearity of $\theta(f)$.
Conversely, for any $b \in \mathcal{B}, m \in M, v \in V$ and $g \in {}_{\mathcal{B}}\mathrm{Hom}(M, {}^{t'}\mathrm{Hom}_k(V, N))$,

$$\begin{aligned}
[\theta'(g)](b \cdot (v \otimes m)) &= [\theta'(g)](\sum b_{(-1)}\, v \otimes b_{(0)}\, m) \\
&= \sum \tilde{q}^2_\lambda\, [g(b_{(0)}\, m)(S^{-1}(\tilde{q}^1_\lambda) b_{(-1)}\, v)] \\
\text{g is left } \mathcal{B}\text{-linear} &= \sum \tilde{q}^2_\lambda\, [(b_{(0)} \cdot (g(m)))(S^{-1}(\tilde{q}^1_\lambda) b_{(-1)}\, v)] \\
&= \sum \tilde{q}^2_\lambda b_{(0,0)}\, [g(m)(S^{-1}(\tilde{q}^1_\lambda b_{(0,-1)}) b_{(-1)}\, v)] \\
\text{by (10.12)} &= \sum b \tilde{q}^2_\lambda [g(m)(S^{-1}(\tilde{q}^1_\lambda)\, v)] = b\, \{[\psi'(g)](m \otimes v)\}.
\end{aligned}$$

So, $\theta'(g)$ is also left $\mathcal{B}$-linear.
Now we show that θ and θ' are inverse to each other. For any $m \in M, v \in V$ and $f \in {}_{\mathcal{B}}\mathrm{Hom}(M \otimes V, N)$,

$$\begin{aligned}
[(\theta' \circ \theta)(f)](v \otimes m) &= \sum \tilde{q}^2_\lambda [(\theta(f)(m))(S^{-1}(\tilde{q}^1_\lambda)\, v)] \\
&= \sum \tilde{q}^2_\lambda\, [f(\tilde{p}^1_\lambda S^{-1}(\tilde{q}^1_\lambda)\, v \otimes \tilde{p}^2_\lambda\, m)] \\
\text{f is left } \mathcal{B}\text{-linear} &= \sum [f(\lambda(\tilde{q}^2_\lambda)\, \tilde{p}_\lambda\, (S^{-1}(\tilde{q}^1_\lambda) \otimes 1_{\mathcal{B}})\, (v \otimes m)] \\
\text{by (10.13)} &= f((1_H \otimes 1_{\mathcal{B}})\, (v \otimes m)) = f(v \otimes m).
\end{aligned}$$

On the other hand, for any $m \in M, v \in V$ and $g \in {}_{\mathcal{B}}\mathrm{Hom}(M, {}^{t'}\mathrm{Hom}_k(V, N))$,

$$\begin{aligned}
[(\theta \circ \theta')(g)](m)(v) &= \theta'(g)(\sum \tilde{p}^1_\lambda\, v \otimes \tilde{p}^2_\lambda\, m)] \\
&= \sum \tilde{q}^2_\lambda [g(\tilde{p}^2_\lambda\, m)(S^{-1}(\tilde{q}^1_\lambda) \tilde{p}^1_\lambda\, v)] \\
\text{g is left } \mathcal{B}\text{-linear} &= \sum \tilde{q}^2_\lambda [\tilde{p}^2_\lambda \cdot (g(m))](S^{-1}(\tilde{q}^1_\lambda) \tilde{p}^1_\lambda\, v) \\
&= \sum \tilde{q_\lambda}^2 \tilde{p_\lambda}^2_1\, \{(g(m))(S^{-1}(\tilde{p}^2_1) S^{-1}(\tilde{q}^1_\lambda) \tilde{p}^1_\lambda\, v)\} \\
&= \sum \tilde{q_\lambda}^2 \tilde{p}^2_1\, \{(g(m))(S^{-1}(\tilde{q_\lambda}^1 \tilde{p}^2_1) \tilde{p_\lambda}^1\, v)\} \\
\text{by (10.14)} &= g(m)(v).
\end{aligned}$$

This shows that θ and θ' are inverse to each other. □

Chapter 4

Hom-tensor relations for quasi-Hopf bimodules

12 Quasi-Hopf bimodules

Although a quasi-bialgebra H is not a coassociative coalgebra, it can be considered as a coalgebra in the bimodule category ${}_H\mathbb{M}_H$, so it makes sense to define comodules over this coalgebra in the monoidal category ${}_H\mathbb{M}_H$. This notion has been considered by Hausser and Nill in [17] under the name **quasi-Hopf H-bimodules** as a generalization of the concept of Hopf bimodules over Hopf algebras.

12.1. The Category ${}_H\mathbb{M}_H^H$. Let $(H, \Delta, \varepsilon, \phi)$ be a quasi-bialgebra, M an (H, H)-bimodule and

$$\varrho^M : M \longrightarrow M \otimes H, \quad \varrho^M(m) = \sum m_0 \otimes m_1,$$

an (H, H)-bimodule homomorphism. Then (M, ϱ^M) is called a right **quasi-Hopf H-bimodule** if for all $m \in M$,

$$(id_M \otimes \varepsilon) \circ \varrho^M = id_M, \tag{12.1}$$

$$\phi \cdot (\varrho^M \otimes id_H)(\varrho^M(m)) = (id_M \otimes \Delta)(\varrho^M(m)) \cdot \phi, \tag{12.2}$$

where we consider the diagonal left and right H-module structure on $M \otimes H$.

A **morphism of right quasi-Hopf H-bimodules** is an (H, H)-bimodule morphism $f : M \to L$ satisfying $\varrho^L \circ f = (f \otimes id) \circ \varrho^M$. The category of right quasi-Hopf H-bimodules with the above morphisms is denoted by ${}_H\mathbb{M}_H^H$.
By definition of a quasi-bialgebra, taking $M = H$ and $\varrho^M = \Delta$ gives an example of a quasi-Hopf H-bimodule.

12.2. (H, H)**-bimodules and quasi-Hopf bimodules.** Let H be a quasi-bialgebra and N an (H, H)-bimodule. With the following structures, $N \otimes_k H$ becomes a right quasi-Hopf H-bimodule. For all $a, b, h \in H, n \in N$, define

$$a \cdot (n \otimes h) \cdot b := \sum a_1 n b_1 \otimes a_2 h b_2 = \Delta(a)(n \otimes h)\Delta(b) \tag{12.3}$$

and a coaction $\varrho^{N \otimes H} : N \otimes H \to (N \otimes H) \otimes H$ is defined by

$$\varrho^{N \otimes H}(n \otimes h) := \phi^{-1} \cdot (id \otimes \Delta)(n \otimes h) \cdot \phi = \sum x^1\, n\, X^1 \otimes x^2 h_1 X^2 \otimes x^3 h_2 X^3. \tag{12.4}$$

For any (epi-)morphism $g : N_1 \to N_2$ in ${}_H\mathbb{M}_H$,

$$g \otimes id_H : N_1 \otimes H \longrightarrow N_2 \otimes H$$

is an (epi-)morphism in ${}_H\mathbb{M}_H^H$. This gives rise to a covariant functor

$$- \otimes_k H : {}_H\mathbb{M}_H \longrightarrow {}_H\mathbb{M}_H^H, \quad N \longmapsto (N \otimes H, \varrho_{N\otimes H}, \varrho^{N\otimes H}),$$

where, $\varrho_{N\otimes H}$ is our notation for the diagonal (H, H)-bimodule structure map given in (12.3) and $\varrho^{N\otimes H}$ is the coaction of $N \otimes H$ defined in (12.4).
In particular, $H \otimes H \in {}_H\mathbb{M}_H^H$ with the structure for all $h, a, b \in H$,

$$\begin{aligned} h \cdot (a \otimes b) &= \Delta(h)(a \otimes b) = \sum h_1 a \otimes h_2 b, \\ (a \otimes b) \cdot h &= (a \otimes b)\Delta(h) = \sum a h_1 \otimes b h_2, \\ \varrho^{H\otimes H}(a \otimes b) &= \sum x^1 a X^1 \otimes x^2 b_1 X^2 \otimes x^3 b_2 X^3 = \phi^{-1} \cdot (id \otimes \Delta)(a \otimes b) \cdot \phi. \end{aligned}$$

Next, we consider a special case of 12.2 taking the right H-module structure of N as the trivial one.

12.3. Left H-modules and quasi-Hopf bimodules. Let H be a quasi-bialgebra and N be a left H-module.

(1) $N \otimes H$ becomes a right quasi-Hopf H-bimodule with the bimodule structure, for all $a, b, h \in H$, and $n \in N$,

$$a \cdot (n \otimes h) \cdot b := \sum a_1 n \otimes a_2 h b = \Delta(a)(n \otimes hb), \tag{12.5}$$

and the coaction $\varrho^{N\otimes H} : N \otimes H \to (N \otimes H) \otimes H$ given by

$$\varrho^{N\otimes H}(n \otimes h) := \phi^{-1} \cdot (id \otimes \Delta)(n \otimes h) = \sum x^1 n \otimes x^2 h_1 \otimes x^3 h_2. \tag{12.6}$$

(2) If $g : N_1 \to N_2$ is an (epi-)morphism in ${}_H\mathbb{M}$, then

$$g \otimes id_H : N_1 \otimes H \longrightarrow N_2 \otimes H,$$

is an (epi-)morphism in ${}_H\mathbb{M}_H^H$.

12.4. Comparison functor. Let H be a quasi-bialgebra. We have seen that for any $N \in {}_H\mathbb{M}$, $N \otimes H \in {}_H\mathbb{M}_H^H$ with (H, H)-bimodule structure given in (12.5) and H-comodule structure map given in (12.6). This gives rise to the **comparison functor**

$$- \otimes_k H : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M}_H^H, \quad N \longmapsto (N \otimes H, \varrho_{N\otimes H}, \varrho^{N\otimes H}),$$

where, $\varrho_{N\otimes H}$ denotes the (H, H)-bimodule structure map given in (12.5) and $\varrho^{N\otimes H}$ is the right H-comodule structure of $N \otimes H$ defined in (12.6).

In [26, Proposition 3.6], Schauenburg showed that considering a convenient monoidal structure on ${}_H\mathbb{M}_H^H$, the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ is monoidal.

12.5. Subgenerator in ${}_H\mathbb{M}_H^H$. Let H be a quasi-bialgebra. Using a similar approach as in [7, 3.5 and 3.7], it can be shown that the category ${}_H\mathbb{M}_H^H$ is closed under direct sums and quotients. In the following we find a *subgenerator* for this category.
Since the algebra H is a generator in ${}_H\mathbb{M}$, for any quasi-Hopf H-bimodule $M \in {}_H\mathbb{M}_H^H$, the left H-module M can be considered as a homomorphic image of $H^{(\Lambda)}$, for some cardinal number Λ. Therefore $M \otimes H$ is a homomorphic image of

$$H^{(\Lambda)} \otimes H \cong (H \otimes H)^{(\Lambda)},$$

in ${}_H\mathbb{M}_H^H$. By definition, for any quasi-Hopf H-bimodule $M \in {}_H\mathbb{M}_H^H$, the coaction $\varrho^M : M \to M \otimes H$ is a morphism in the category ${}_H\mathbb{M}_H^H$. Then by 12.3, we can consider M as a subobject of $M \otimes H$ and this is generated by $H \otimes H \in {}_H\mathbb{M}_H^H$. Here, the structures of $H \otimes H$ given for $h, a, b \in H$ by

$$h \cdot (a \otimes b) \cdot h' = \Delta(h)(a \otimes b)(1 \otimes h') = \sum h_1 a \otimes h_2 b h' \qquad (12.7)$$

$$\varrho^{H\otimes H}(a \otimes b) = \sum x^1 a \otimes x^2 b_1 \otimes x^3 b_2 = \phi^{-1} \cdot (id \otimes \Delta)(a \otimes b). \qquad (12.8)$$

This means
Proposition. *For any quasi-bialgebra H, with the structures given above, $H \otimes H$ is a* subgenerator *for the category ${}_H\mathbb{M}_H^H$ of quasi-Hopf H-bimodules.*

The following Lemma helps us to find a right adjoint to the comparison functor $- \otimes_k H$ given in 12.4.

12.6. Lemma. (The functor ${}_H\mathrm{Hom}_H^H(V \otimes_k H, -)$). *Let H be a quasi-bialgebra and $V \in {}_H\mathbb{M}_H$.*

(1) *For $M \in {}_H\mathbb{M}_H$, ${}_H\mathrm{Hom}_H(V \otimes H, M) \in {}_H\mathbb{M}$ with the left H-module structure given for $h, h' \in H$ and $v \in V$, by*

$$(h' \cdot f)(v \otimes h) = f(v\, h' \otimes h).$$

In this way, we get a functor ${}_H\mathrm{Hom}_H(V \otimes H, -) : {}_H\mathbb{M}_H \to {}_H\mathbb{M}$.
In particular, if $M \in {}_H\mathbb{M}_H^H$, then ${}_H\mathrm{Hom}_H^H(V \otimes H, M) \in {}_H\mathbb{M}$ with left H-module structure given above, and we obtain the Hom-functor

$${}_H\mathrm{Hom}_H^H(V \otimes H, -) : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}.$$

(2) *Let $V \in {}_H\mathbb{M}_H$ and $N \in {}_H\mathbb{M}$. Consider N as an (H, H)-bimodule with the trivial right H-module structure. Then*

(i) $\psi : {}_H\mathrm{Hom}_H^H(V \otimes H, N \otimes H) \longrightarrow {}_H\mathrm{Hom}_H(V \otimes H, N), \quad f \mapsto (id \otimes \varepsilon) \circ f,$
is an isomorphism in ${}_H\mathbb{M}$ with inverse map ψ' given by

$$g \mapsto (g \otimes id_H) \circ \varrho^{V\otimes H}.$$

(ii) $\theta : {}_H\mathrm{Hom}_H(V \otimes H, N) \longrightarrow {}_H\mathrm{Hom}(V, N), \quad f \mapsto f(- \otimes 1_H),$
is an isomorphism in ${}_H\mathbb{M}$ with inverse map θ' given by

$$g \mapsto [v \otimes h \mapsto \varepsilon(h) g(v)],$$

(iii) We have the left H-module isomorphism

$$ {}_H\mathrm{Hom}(V,N) \longrightarrow {}_H\mathrm{Hom}_H^H(V\otimes H, N\otimes H), \quad g \longmapsto g\otimes id_H, $$

with the inverse map given, for $f \in {}_H\mathrm{Hom}_H^H(V\otimes H, N\otimes H)$, by

$$ f \mapsto (id\otimes\varepsilon)\circ f(-\otimes 1_H). $$

This means that the comparison functor $-\otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ is full and faithful.

Proof. (1) For all $h \in H$ and $f \in {}_H\mathrm{Hom}_H(V\otimes H, M)$, it is easy to see that $h\cdot f$ is an (H,H)-bilinear map. In this way, we have ${}_H\mathrm{Hom}_H(V\otimes H, M) \in {}_H\mathbb{M}$ and we obtain a functor

$$ {}_H\mathrm{Hom}_H(V\otimes H, -) : {}_H\mathbb{M}_H \longrightarrow {}_H\mathbb{M}. $$

In the particular case $M \in {}_H\mathbb{M}_H^H$ and $f \in {}_H\mathrm{Hom}_H^H(V\otimes H, M)$, the H-colinearity of $h\cdot f$ follows from the H-colinearity of f itself. Thus, ${}_H\mathrm{Hom}_H^H(V\otimes H, M) \in {}_H\mathbb{M}$ and we obtain a functor

$$ {}_H\mathrm{Hom}_H^H(V\otimes H, -) : {}_H\mathbb{M}_H^H \longrightarrow {}_H\mathbb{M}. $$

(2) (i) The quasi-bialgebra H is (by definition), a coalgebra in ${}_H\mathbb{M}_H$, i.e. the endofunctor

$$ G := -\otimes_k H : {}_H\mathbb{M}_H \longrightarrow {}_H\mathbb{M}_H, \quad N \mapsto N\otimes_k H, $$

is a *comonad.* Here, the (H,H)-bimodule structure on $N\otimes_k H$ is given for all $a, b, h \in H$, and $n \in N$ by

$$ a\cdot(n\otimes h)\cdot b = \sum a_1\, n\, b_1 \otimes a_2\, h\, b_2 = \Delta(a)\,(n\otimes h)\,\Delta(b). $$

The comultiplication δ of this comonad defined for $N \in {}_H\mathbb{M}_H$ by

$$ \begin{array}{rcl} \delta_N : N\otimes H & \longrightarrow & (N\otimes H)\otimes H, \\ n\otimes h & \mapsto & \sum x^1\, n\, X^1 \otimes x^2\, h_1\, X^2 \otimes x^3\, h_2\, X^3 = \phi^{-1}\cdot(id\otimes\Delta)(n\otimes h)\cdot\phi, \end{array} $$

and the counit ϵ of this δ is defined by $\epsilon_N = id_N\otimes\varepsilon : N\otimes H \to N$.
Furthermore, the category of two-sided Hopf modules ${}_H\mathbb{M}_H^H$ is isomorphic to the Eilenberg-Moore *comodule* category $({}_H\mathbb{M}_H)^{-\otimes H}$. Now, considering the comparison functor $-\otimes H : {}_H\mathbb{M}_H \to {}_H\mathbb{M}_H^H$ as the free functor which is right adjoint to the forgetful functor, by 4.8, we obtain the isomorphism of part (i).

(ii) First we note that for $f \in {}_H\mathrm{Hom}_H(V\otimes H, N)$, $h \in H$ and $v \in V$

$$ \begin{array}{rcl} h\,[\theta(f)(v)] & = & h\,[f(v\otimes 1_H)] \\ {\scriptstyle f \text{ is left } H\text{-linear}} & = & f(\sum h_1\, v\otimes h_2) \\ {\scriptstyle f \text{ is right } H\text{-linear}} & = & \sum f(h_1\, v\otimes 1_H)\, h_2 \\ & = & \sum f(h_1\, v\otimes 1_H)\,\varepsilon(h_2) \\ & = & f(h\, v\otimes 1_H) = \theta(f)(h\, v). \end{array} $$

This means that $\theta(f) \in {}_H\mathrm{Hom}(V,N)$. Also, it is straightforward to show that for $f \in {}_H\mathrm{Hom}(V,N)$, we have $\theta'(g) \in {}_H\mathrm{Hom}_H(V\otimes H, N)$. The H-linearity and the bijectivity of θ can be seen by direct computations.

(iii) Follows by combining the isomorphisms in parts (i) and (ii). □

Taking $V = H$ and considering the trivial right H-module structure on N, for $M \in {}_H\mathbb{M}_H^H$ we have a left H-module structure on ${}_H\mathrm{Hom}_H^H(H \otimes H, M)$ given for $h, a, b \in H$ and $f \in {}_H\mathrm{Hom}_H^H(H \otimes H, M)$ by

$$(h \cdot f)(a \otimes b) = f(a\,h \otimes b).$$

(The structures on $H \otimes_k H$ are given in (12.7) and (12.8)). Considering the left H-module structure on ${}_H\mathrm{Hom}_H^H(H \otimes H, M)$ given above, we show that the functor

$${}_H\mathrm{Hom}_H^H(H \otimes H, -) : {}_H\mathbb{M}_H^H \longrightarrow {}_H\mathbb{M},$$

is right adjoint to the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ defined in 12.4 (see also [7, 18.10]).

12.7. Theorem. (${}_H\mathrm{Hom}_H^H(H \otimes H, -)$ as right adjoint to the comparison-functor).
Let H be a quasi-bialgebra, $M \in {}_H\mathbb{M}_H^H$ and $N \in {}_H\mathbb{M}$. Then there is a functorial isomorphism

$${}_H\mathrm{Hom}_H^H(N \otimes H, M) \xrightarrow{\Omega} {}_H\mathrm{Hom}(N, {}_H\mathrm{Hom}_H^H(H \otimes H, M)),$$

$$f \longmapsto \{n \mapsto [a \otimes b \mapsto f(a\,n \otimes b)]\},$$

with inverse map Ω':

$$g \longmapsto [n \otimes h \mapsto g(n)(1_H \otimes h)].$$

This means that the comparison functor

$$- \otimes_k H : {}_H\mathbb{M} \longrightarrow {}_H\mathbb{M}_H^H, \quad N \mapsto (N \otimes H, \varrho_{N\otimes H}, \varrho^{N\otimes H}),$$

is left adjoint to the Hom*-functor*

$${}_H\mathrm{Hom}_H^H(H \otimes H, -) : {}_H\mathbb{M}_H^H \longrightarrow {}_H\mathbb{M},$$

with unit and counit

$$\eta_N : N \longrightarrow {}_H\mathrm{Hom}_H^H(H \otimes H, N \otimes H), \quad n \mapsto [a \otimes b \mapsto a\,n \otimes b],$$

$$\varepsilon_M : {}_H\mathrm{Hom}_H^H(H \otimes H, M) \otimes H \longrightarrow M, \quad f \otimes h \mapsto f(1 \otimes h).$$

Furthermore, the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ is full and faithful.

Proof. First we show that for any $f \in {}_H\mathrm{Hom}_H^H(N \otimes H, M)$, $\Omega(f)$ is left H-linear. For $h, a, b \in H$ and $n \in N$ we compute

$$[h \cdot (\Omega(f)(n))](a \otimes b) = \Omega(f)(n)(ah \otimes b) = f(ah\,n \otimes b) = [\Omega(f)(h\,n))](a \otimes b).$$

Thus, we have $\Omega(f) \in {}_H\mathrm{Hom}(N, {}_H\mathrm{Hom}_H^H(H \otimes H, M))$.
For $g \in {}_H\mathrm{Hom}(N, {}_H\mathrm{Hom}_H^H(H \otimes H, M))$, we show $\Omega'(g) \in {}_H\mathrm{Hom}_H^H(N \otimes H, M)$.

i) $\Omega'(g)$ is left H-linear. For $h, h' \in H$ and $n \in N$,

$$\begin{aligned}\Omega'(g)(h' \cdot (n \otimes h)) &= \Omega'(g)(h'_1 n \otimes h'_2 h) = g(h'_1 n)(1_H \otimes h'_2 h) \\ &= (h'_1 \cdot g(n))(1_H \otimes h'_2 h) = g(n)(h'_1 \otimes h'_2 h) \\ &= g(n)(\Delta(h')\,(1 \otimes h)) = h'\,[g(n)(1 \otimes h)] \\ &= h'\,[\Omega'(g)(n \otimes h)]\end{aligned}$$

ii) It can be easily seen that $\Omega'(g)$ is also right H-linear. For $h, h' \in H$ and $n \in N$,

$$\begin{aligned}\Omega'(g)((n \otimes h)\,h') &= \Omega'(g)(n \otimes hh') = g(n)(1_H \otimes hh') \\ &= g(n)((1 \otimes h)\,h') = [g(n)(1 \otimes h)]\,h' \\ &= [\Omega'(g)(n \otimes h)]\,h'\end{aligned}$$

iii) For the right H-colinearity of $\Omega'(g)$, we have to show that

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = (\Omega'(g) \otimes id)(x^1\, n \otimes x^2 h_1 \otimes x^3 h_2).$$

By the colinearity of $g(n)$,

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = \varrho^M(g(n)(1 \otimes h)) = g(n)(x^1 \otimes x^2 h_1) \otimes x^3 h_2.$$

On the other hand,

$$\begin{aligned}(\Omega'(g) \otimes id)(x^1\, n \otimes x^2 h_1 \otimes x^3 h_2) &= g(x^1\, n)(1 \otimes x^2 h_1) \otimes x^3 h_2 \\ &= [x^1 \cdot g(n)](1 \otimes x^2 h_1) \otimes x^3 h_2 \\ &= g(n)(x^1 \otimes x^2 h_1) \otimes x^3 h_2.\end{aligned}$$

This shows the H-colinearity of $\Omega'(g)$.

Now we show that Ω and Ω' are inverse to each other. For all $n \in N, h \in H$ and $f \in {}_H\mathrm{Hom}_H^H(H \otimes H, M)$,

$$\begin{aligned}(\Omega' \circ \Omega(f))(n \otimes h) &= (\Omega(f))(n)(1 \otimes h) \\ &= f(1_H\, n \otimes h) = f(n \otimes h).\end{aligned}$$

Conversely, for all $a, b \in H$, $n \in N$ and $g \in {}_H\mathrm{Hom}(N, {}_H\mathrm{Hom}_H^H(H \otimes H, M))$,

$$\begin{aligned}\{[(\Omega \circ \Omega')(g)](n)\}(a \otimes b) &= (\Omega'(g))(a\, n \otimes b) = g(a\, n)(1 \otimes b) \\ &= [a \cdot g(n)](1_H \otimes b) = g(n)(a \otimes b).\end{aligned}$$

It is straightforward to see that Ω is functorial in both components M and N.

The fully faithfulness of the comparison functor follows from Lemma 12.6. □

13 Fundamental Theorem for quasi-Hopf H-bimodules

Throughout this section, we consider H to be a quasi-Hopf algebra with a quasi-antipode (S, α, β). We have seen in 12.2 that for any left H-module N, the tensor product $N \otimes H$ is a right quasi-Hopf H-bimodule. Following Hausser and Nill [17], we observe that any quasi-Hopf H-bimodule M is isomorphic to such a tensor product $N \otimes H$, where N is a left H-module (the *coinvariants* of M). This is a generalization of *the Fundamental Theorem of Hopf modules* over a Hopf algebra by Larson and Sweedler [20], to quasi-Hopf algebras.

13.1. Hausser-Nill coinvariants in ${}_H\mathbb{M}_H^H$. For $M \in {}_H\mathbb{M}_H^H$, define a projection $E : M \to M$, for $m \in M, a \in H$, by

$$E(m) := \sum q_R^1 m_0 \, \beta S(q_R^2 m_1), \tag{13.1}$$

and put

$$a \blacktriangleright m := E(a\, m) \tag{13.2}$$

where $q_R = \sum q_R^1 \otimes q_R^2$ is defined as in (7.31).

For $M \in {}_H\mathbb{M}_H^H$, define the **HN-coinvariants** of M as $M^{coH} := E(M)$.

We have the following properties for $a, b \in H$, $m \in M$ (see [17, Proposition 3.4]).

(i) $E(m\, a) = \varepsilon(a) E(m)$,

(ii) $E^2 = E$,

(iii) $a \blacktriangleright E(m) = E(a\, m) = a \blacktriangleright m$,

(iv) $(ab) \blacktriangleright m = a \blacktriangleright (b \blacktriangleright m)$,

(v) $a\, E(m) = \sum [a_1 \blacktriangleright E(m)]\, a_2$,

(vi) $\sum E(m_0)\, m_1 = m$,

(vii) $\sum E(E(m)_0) \otimes E(m)_1 = E(m) \otimes 1$.

Due to (ii), (vi) and (vii), the following characterizations of *Hausser-Nill coinvariants* are equivalent:

$$\begin{aligned} M^{coH} := E(M) &= \{n \in M | E(n) = n\} \\ &= \{n \in M | \sum E(n_0) \otimes n_1 = E(n) \otimes 1\} \\ &= Ke((E \otimes id) \circ [\varrho^M - (- \otimes 1_H)]). \end{aligned}$$

M^{coH} with the left H-action $\blacktriangleright$ is a left H-module and for any morphism $f : M \to L$ in ${}_H\mathbb{M}_H^H$, it is not hard to show that

$$f(M^{coH}) \subseteq L^{coH}.$$

This gives rise to a functor $(-)^{coH} : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$ which - as we will see - is right adjoint to the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$.

13.2. Proposition. The adjoint pair $(- \otimes_k H, (-)^{coH})$ for HN-coinvariants. *Let H be a quasi-Hopf algebra, $N \in {}_H\mathbb{M}$ and $M \in {}_H\mathbb{M}_H^H$. Then there is a functorial isomorphism*

$$\psi_{N,M} : {}_H\mathrm{Hom}_H^H(N \otimes_k H, M) \longrightarrow {}_H\mathrm{Hom}(N, M^{coH}), \quad f \longmapsto [n \mapsto f(n \otimes 1)],$$

with inverse map $\psi'_{N,M}$ given by

$$g \longmapsto [n \otimes h \mapsto g(n)\, h)].$$

Thus, the functors

$$-\otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H, \quad (-)^{coH} : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$$

form an adjoint pair with unit and counit

$$\eta_N : N \longrightarrow (N \otimes H)^{coH}, \quad n \mapsto n \otimes 1,$$

$$\varepsilon_M : M^{coH} \otimes_k H \longrightarrow M, \quad m \otimes h \mapsto m\, h.$$

Proof. First, we show that $f(n \otimes 1) \in M^{coH}$: Since f is H-colinear,

$$\varrho^M(f(n \otimes 1)) = f(x^1 n \otimes x^2) \otimes x^3,$$

so we have

$$\begin{array}{rcl} E(f(n \otimes 1)) & = & \sum q_R^1 f(x^1 n \otimes x^2)\, \beta S(q_R^2 x^3) \\ {\scriptstyle f \text{ is } H\text{-linear}} & = & \sum f(\Delta(q_R^1)(x^1 n \otimes x^2 \beta S(x^3) S(q_R^2))) \\ & = & f(\sum \Delta(q_R^1) p_R (1 \otimes S(q_R^2))\, (n \otimes 1)) \\ {\scriptstyle \text{by } (7.38)} & = & f(n \otimes 1). \end{array}$$

Now, we show that $\psi := \psi_{N,M}$ and $\psi' := \psi'_{N,M}$ are inverse to each other. For $n \in N, h \in H$ and $f \in_H \mathrm{Hom}_H^H(N \otimes_k H, M)$,

$$[(\psi' \circ \psi)(f)](n \otimes h) = \psi(f)(n)\, h = f(n \otimes 1)\, h = f(n \otimes h).$$

Conversely, for $n \in N$ and $g \in {}_H\mathrm{Hom}(N, M^{coH})$,

$$[(\psi \circ \psi')(g)](n) = \psi'(g)(n \otimes 1) = g(n)\, 1 = g(n).$$

□

It is shown in [17, Lemma 3.6] that, for $N \in {}_H\mathbb{M}$, the coinvariants of the quasi-Hopf H-bimodule $N \otimes H$, come out as

$$(N \otimes H)^{coH} \simeq N,$$

and for $n \in N, h \in H$,

$$E(n \otimes h) = n \otimes \varepsilon(h) 1_H.$$

This means that the unit $\eta_N : N \longrightarrow (N \otimes H)^{coH}$ of the adjunction in 13.2 is an isomorphism. This gives another proof for the fully faithfulness of the comparison functor $-\otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ in the quasi-Hopf case (see 4.1, 12.6 and 12.7).

13.3. Fundamental Theorem of quasi-Hopf bimodules (I). (see [17, Theorem 3.8]) Let H be a quasi-Hopf algebra and $M \in {}_H\mathbb{M}_H^H$. Consider M^{coH} as a left H-module with left H-action ▶ defined above. Then the map

$$\varepsilon_M : M^{coH} \otimes H \longrightarrow M, \qquad m \otimes h \mapsto m\, h,$$

is an isomorphism of quasi-Hopf H-bimodules with inverse map

$$\varepsilon_M^{-1}(m) = \sum E(m_0) \otimes m_1 = (E \otimes id) \circ \varrho^M(m).$$

This yields an additional characterization of coinvariants for any quasi-Hopf H-bimodule M (see [17, Corollary 3.9]),

$$\begin{aligned} M^{coH} &= \{n \in M | \varrho^M(n) = \sum (x^1 \blacktriangleright n)\, x^2 \otimes x^3\} \\ &= Ke(\varrho^M - [(\varrho_M \otimes id) \circ (E \otimes id \otimes id)(\phi^{-1} \cdot (- \otimes 1_H \otimes 1_H))]). \end{aligned}$$

13.4. Bulacu-Caenepeel coinvariants. For a right quasi-Hopf H-bimodule (M, ϱ^M), Bulacu and Caenepeel in [8], gave an alternative definition for the coinvariants, denoted by $M^{\overline{coH}}$. For this, they used a different projection map

$$\bar{E} : M \longrightarrow M, \quad \bar{E}(m) = \sum m_0 \beta S(m_1).$$

This version of coinvariants is defined as $M^{\overline{coH}} = \bar{E}(M)$. We call them shortly as **BC-coinvariants**. They can be characterized as

$$\begin{aligned} M^{\overline{coH}} &= \{m \in M | \bar{E}(m) = m\} \\ &= \{m \in M | \varrho^M(m) = \sum x^1\, m\, S(x^3_2 X^3) f^1 \otimes x^2 X^1 \beta S(x^3_1 X^2) f^2\}, \\ &= Ke(\{\varrho^M - [(x^1 \otimes x^2)\,(- \otimes 1_H)\,(S(x^3_2 p^2_L) f^1 \otimes S(x^3_1 p^1_L) f^2)]\}) \end{aligned}$$

where $f = \sum f^1 \otimes f^2 \in H \otimes H$ is the gauge element, given in (7.24) (see the text before Lemma 3.6 in [8]).

$M^{\overline{coH}} = \bar{E}(M)$ forms a left H-module with respect to *the left adjoint action* of $h \in H$ on $m \in M$ (see [8, Lemma 3.6]),

$$h \triangleright m = \sum h_1\, m\, S(h_2).$$

For any morphism $f : M \to M'$ in ${}_H\mathbb{M}^H_H$, it is straightforward to see that

$$f(M^{\overline{coH}}) \subseteq M'^{\overline{coH}}.$$

This gives rise to a functor $(-)^{\overline{coH}} : {}_H\mathbb{M}^H_H \to {}_H\mathbb{M}$ which is also right adjoint to the comparison functor (see 13.6).

13.5. The relation between the projections E and $\bar{E}$. Let $M \in {}_H\mathbb{M}^H_H$ and $E, \bar{E} : M \to M$ be defined by

$$E(m) = \sum q^1_R\, m_0\, \beta S(q^2_R m_1), \qquad \bar{E}(m) = \sum m_0\, \beta S(m_1),$$

for all $m \in M$. Then it is shown in [8] that

i) $\bar{E}(m) = \sum E(p^1_R\, m)\, p^2_R$,

ii) $E(m) = \sum q^1_R\, \bar{E}(m)\, S(q^2_R)$,

iii) $\bar{E} : M^{coH} \to M^{\overline{coH}}$ and $E : M^{\overline{coH}} \to M^{coH}$ are inverse to each other, where

$p_R = \sum p_R^1 \otimes p_R^2$ is defined in (7.30), and $q_R = \sum q_R^1 \otimes q_R^2$ is defined in (7.31). In fact M^{coH} and $M^{\overline{coH}}$ are isomorphic as left H-modules. To see the H-linearity of E, take $h \in H, m \in M^{\overline{coH}}$ and compute

$$
\begin{aligned}
E(h \triangleright m) &= \sum E(h_1\, m_0\, \beta S(h_2 m_1)) \\
&= \sum q_R^1 h_{11}\, m_{00}\, \beta_1 S(h_2 m_1)_1 \beta S(q_R^2 h_{12} m_{01} \beta_2 S(h_2 m_1)_2) \\
&= \sum q_R^1 h_{11}\, m_{00}\, \beta_1 S(h_2 m_1)_1 \beta S(\beta_2 S(h_2 m_1)_2) S(q_R^2 h_{12} m_{01}) \\
&= \sum q_R^1 h_{11}\, m_{00}\, \varepsilon(\beta) \varepsilon(S(h_2 m_1)) \beta S(q_R^2 h_{12} m_{01}) \\
&= \sum \varepsilon(h_2 m_1) q_R^1 h_{11}\, m_{00}\, \beta S(h_{12} m_{01}) S(q_R^2) \\
&= \sum \varepsilon(h_2 m_1) q_R^1\, \bar{E}(h_1\, m_0)\, S(q_R^2) = E(h\, m) \\
&= h \blacktriangleright E(m).
\end{aligned}
$$

13.6. Proposition. (The adjoint pair $(- \otimes_k H, (-)^{\overline{coH}})$ for BC-coinvariants). *Let H be a quasi-Hopf algebra, $N \in {}_H\mathbb{M}$ and $M \in {}_H\mathbb{M}_H^H$. Then there is a functorial isomorphism*

$$
{}_H\mathrm{Hom}_H^H(N \otimes_k H, M) \xrightarrow{\psi_{N,M}} {}_H\mathrm{Hom}(N, M^{\overline{coH}}), \quad f \longmapsto [n \mapsto \bar{E}(f(n \otimes 1))],
$$

with inverse map $\psi'_{N,M}$ given by

$$
g \longmapsto [n \otimes h \mapsto \sum q_R^1\, g(n)\, S(q_R^2) h].
$$

This means that the functors

$$
{}_H\mathbb{M} \xrightarrow{- \otimes_k H} {}_H\mathbb{M}_H^H \xrightarrow{(-)^{\overline{coH}}} {}_H\mathbb{M},
$$

form an adjoint pair with unit and counit

$$
\eta_N : N \longrightarrow (N \otimes H)^{\overline{coH}}, \quad n \mapsto p_R\,(n \otimes 1),
$$

$$
\varepsilon_M : M^{\overline{coH}} \otimes_k H \longrightarrow M, \quad m \otimes h \mapsto \sum q_R^1\, m\, S(q_R^2) h.
$$

Proof. We show that ψ and ψ' are inverse to each other. For $n \in N, h \in H$ and $f \in {}_H\mathrm{Hom}_H^H(N \otimes_k H, M)$,

$$
\begin{aligned}
[(\psi' \circ \psi)(f)](n \otimes h) &= \sum q_R^1\, \psi(f)(n)\, S(q_R^2) h \\
&= \sum q_R^1\, \bar{E}(f(n \otimes 1))\, S(q_R^2) h \\
&= \sum q_R^1\, f(x^1\, n \otimes x^2)\, \beta S(q_R^2 x^3) h \\
&= \sum f((q_R^1)_1 x^1\, n \otimes (q_R^1)_2 x^2 \beta S(x^3) S(q_R^2) h) \\
&= \sum f((q_R^1)_1 p_R^1\, n \otimes (q_R^1)_2 p_R^2 S(q_R^2) h) \\
\text{by (7.38)} \quad &= f(n \otimes h).
\end{aligned}
$$

Conversely, for $n \in N$ and $g \in {}_H\mathrm{Hom}(N, M^{\overline{coH}})$,

$$\begin{aligned}
[(\psi \circ \psi')(g)](n) &= \bar{E}(\psi'(g)(n \otimes 1) = \bar{E}(q_R^1\, g(n)\, S(q_R^2)) \\
&= \sum (q_R^1)_1\, g(n)_0\, (S(q_R^2))_1 \beta S((q_R^1)_2\, g(n)_1\, (S(q_R^2))_2) \\
&= \sum (q_R^1)_1\, g(n)_0\, \varepsilon(S(q_R^2)) \beta S((q_R^1)_2 g(n)_1) \\
&= \sum (q_R^1)_1\, g(n)_0\, \beta S(g(n)_1) S((q_R^1)_2) \varepsilon(q_R^2) \\
&= \sum (q_R^1)_1\, \bar{E}(g(n))\, S((q_R^1)_2) \varepsilon(q_R^2) \\
&= \sum \varepsilon(q_R^2) q_R^1 \triangleright \bar{E}(g(n)) \\
\underset{g(n) \in M^{\overline{coH}}}{} &= g(n).
\end{aligned}$$

□

By the left H-module isomorphism between $M^{\overline{coH}}$ and M^{coH} (see 13.5), we have that $M^{\overline{coH}} \otimes H \cong M^{coH} \otimes H$ as quasi-Hopf H-bimodules. Thus, using $(-)^{\overline{coH}}$, we can restate:

13.7. The Fundamental Theorem of quasi-Hopf bimodules (II). *Let H be a quasi-Hopf algebra and M a right quasi-Hopf H-bimodule. Consider $M^{\overline{coH}} \otimes H$ as a right quasi-Hopf H-bimodule with the structures*

$$a \cdot (n \otimes h) \cdot b = \sum a_1 \triangleright n \otimes a_2 h b, \quad \varrho'(n \otimes h) = \sum x^1 \triangleright n \otimes x^2 h_1 \otimes x^3 h_2,$$

for $h, a, b \in H$ and $n \in M^{\overline{coH}}$. Then the map

$$\bar{\nu} : M^{\overline{coH}} \otimes H \longrightarrow M, \quad \bar{\nu}(n \otimes h) = \sum q_R^1\, n\, S(q_R^2) h$$

is an isomorphism of quasi-Hopf H-bimodules with the inverse map given by

$$\bar{\nu}^{-1}(m) = \sum \bar{E}(m_0) \otimes m_1.$$

The isomorphism $M^{\overline{coH}} \cong M^{coH}$ (see 13.5), implies $(N \otimes H)^{coH} \cong (N \otimes H)^{\overline{coH}}$ as left H-modules. The two versions of the Fundamental Theorem of quasi-Hopf bimodules show that both $(-)^{coH}$ and $(-)^{\overline{coH}}$ are right adjoints to the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ which is an equivalence of categories.

Remark. In case H is a Hopf algebra,

$$E(m) = \bar{E}(m) = \sum m_0\, S(m_1),$$

and both projections are equal to the identity map on $M^{coH} = M^{\overline{coH}}$. In this case, M^{coH} is invariant under the left adjoint action $h \triangleright m = \sum h_1 m S(h_2)$ in the sense that for all $h \in H$, $m \in M$,

$$E(h \triangleright m) = h \triangleright E(m)$$

and the fundamental theorem of quasi-Hopf bimodules reduces to the fundamental theorem of Hopf modules, stated by Larson and Sweedler. In this case, we have $M^{coH} \cong \mathrm{Hom}_H^H(H, M)$ (in ${}_k\mathbb{M}$).

For a *quasi-Hopf algebra* H, we have three different right adjoints for the comparison functor $-\otimes_k H$ given in 12.4, namely, the HN-coinvarints $(-)^{coH}$ (see 13.1), the BC-coinvarints $(-)^{\overline{coH}}$, (see 13.4) and the Hom-functor ${}_H\mathrm{Hom}_H^H(H\otimes H,-): {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$ (see 12.7). These three functors must be isomorphic and we describe these isomorphisms explicitly.

13.8. Theorem. (Coinvariants as Hom**-functor).** *Let H be a quasi-Hopf algebra and M a right quasi-Hopf H-bimodule.*

(1) *There is a functorial isomorphism in ${}_H\mathbb{M}$*

$$\bar{\psi}_M : {}_H\mathrm{Hom}_H^H(H\otimes_k H, M) \longrightarrow M^{coH}, \quad f \longmapsto f(1\otimes 1),$$

with inverse map $\bar{\psi}'_M$ given by

$$m \longmapsto [a\otimes b \mapsto E(a\,m)\,b],$$

for $a,b\in H$ and $m\in M^{coH}$.

(2) *There is a functorial isomorphism in ${}_H\mathbb{M}$,*

$$\bar{\theta}_M : {}_H\mathrm{Hom}_H^H(H\otimes_k H, M) \longrightarrow M^{\overline{coH}}, \quad f \longmapsto f(p_R),$$

with inverse map $\bar{\theta}'_M$ given by

$$m \longmapsto [a\otimes b \mapsto E(a\,m)\,b].$$

for $a,b\in H$ and $m\in M^{\overline{coH}}$.

Proof. (1) Using the isomorphism in 13.2 for $N=H$, we obtain the isomorphisms

$$\bar{\psi}_M : {}_H\mathrm{Hom}_H^H(H\otimes_k H, M) \overset{\psi_{H,M}}{\longrightarrow} {}_H\mathrm{Hom}(H, M^{coH}) \cong M^{coH},$$
$$f \longmapsto [a\mapsto f(a\otimes 1)] \mapsto f(1\otimes 1).$$

Here, $\psi_{H,M}$ is the isomorphism in 13.2 for $N=H$. The inverse map $\bar{\psi}'_M$ is obtained as the composition

$$M^{coH} \cong {}_H\mathrm{Hom}(H, M^{coH}) \overset{\psi'_{H,M}}{\longrightarrow} {}_H\mathrm{Hom}_H^H(H\otimes_k H, M),$$
$$m \longmapsto [a\mapsto a \blacktriangleright m = E(a\,m)] \longmapsto [a\otimes b \mapsto E(a\,m)\,b],$$

for all $a,b\in H$ and $m\in M^{coH}$.
Sofar, we have shown that $\bar{\psi}_M$ is a k-module isomorphism. To show the left H-linearity of $\bar{\psi}_M$, we compute for $h\in H$ and $f\in {}_H\mathrm{Hom}_H^H(H\otimes H, M)$,

$$\begin{aligned}
h\blacktriangleright \bar{\psi}_M(f) &= E(h\,f(1\otimes 1)) = \sum E(f(h_1\otimes h_2)) \\
&= \sum q_R^1\, f(h_1\otimes h_2)_0\, \beta S(q_R^2 f(h_1\otimes h_2)_1) \\
\text{\small f is H-colinear} \quad &= \sum q_R^1\, f(x^1h_1\otimes x^2h_{21})\beta S(q_R^2x^3h_{22}) \\
\text{\small f is H-linear} \quad &= \sum f(\Delta(q_R^1)\cdot(x^1h_1\otimes x^2h_{21})\beta S(h_{22})S(x^3)S(q_R^2)) \\
&= f(\sum \Delta(q_R^1)\cdot(x^1h\otimes x^2\beta S(x^3)S(q_R^2)) \\
&= f(\sum \Delta(q_R^1)p_R(1\otimes S(q_R^2))(h\otimes 1)) \\
\text{\small by (7.38)} \quad &= f(h\otimes 1) = (h\cdot f)(1\otimes 1) = \bar{\psi}_M(h\cdot f).
\end{aligned}$$

(2) For the isomorphism in 13.6, we set $N = H$ to obtain the isomorphism

$$\bar{\theta}_M : {}_H\mathrm{Hom}_H^H(H \otimes_k H, M) \overset{\psi_{H,M}}{\longrightarrow} {}_H\mathrm{Hom}(H, M^{\overline{coH}}) \cong M^{\overline{coH}},$$

$$f \longmapsto [a \mapsto \bar{E}(f(a \otimes 1))] \mapsto \bar{E}(f(1 \otimes 1)).$$

This means that for $f \in {}_H\mathrm{Hom}_H^H(H \otimes_k H, M)$,

$$\bar{\theta}_M(f) = \sum f(1 \otimes 1)_0 \, \beta S(f(1 \otimes 1)_1).$$

Now, similar to the proof of 13.2 (for $N = H$ and $n = 1_H$), using the H-colinearity of f, we obtain

$$\bar{\theta}_M(f) = \sum f(x^1 \otimes x^2) \, \beta S(x^3) = \sum f(x^1 \otimes x^2 \, \beta S(x^3)) = f(p_R).$$

The inverse map $\bar{\theta}'_M$ is obtained as the composition

$$M^{\overline{coH}} \cong {}_H\mathrm{Hom}(H, M^{\overline{coH}}) \overset{\psi'_{H,M}}{\longrightarrow} {}_H\mathrm{Hom}_H^H(H \otimes_k H, M),$$

$$\begin{aligned} m \longmapsto [a \mapsto a \triangleright m = \bar{E}(a\,m)] \longmapsto \{a \otimes b \ \ &\mapsto \ \ \sum q_R^1 \, \bar{E}(a\,m) \, S(q_R^2) b \\ &= \ \ E(a\,m) \, b\}, \end{aligned}$$

for all $a, b \in H$ and $m \in M^{\overline{coH}}$.
In a similar way as in the part (1), for the H-linearity of $\bar{\theta}_M$, we compute for $h \in H$ and $f \in {}_H\mathrm{Hom}_H^H(H \otimes H, M)$ we have:

$$\begin{aligned} h \triangleright \bar{\theta}_M(f) &= \bar{E}(h\, f(p_R)) = \sum \bar{E}(f(h_1 p^1 \otimes h_2 p^2)) \\ &= \sum f(h_1 p_R^1 \otimes h_2 p_R^2)_0 \, \beta S(f(h_1 p_R^1 \otimes h_2 p_R^2)_1) \\ \text{(f is H-colinear.)} \quad &= \sum f(x^1 h_1 p_R^1 \otimes x^2 h_{21} (p_R^2)_1) \beta S(x^3 h_{22} (p_R^2)_2) \\ &= \sum f(x^1 h_1 p_R^1 \otimes x^2 h_{21} (p_R^2)_1 \beta S(h_{22} (p_R^2)_2) S(x^3)) \\ &= f(\sum x^1 h \otimes x^2 \beta S(x^3)) = f(p_R \, (h \otimes 1)) \\ &= f(\sum p_R^1 h \otimes p_R^2) = (h \cdot f)(p_R) = \theta(h \cdot f). \end{aligned}$$

□

Remark. Another way to prove part (2) is to combine the isomorphism

$${}_H\mathrm{Hom}_H^H(H \otimes_k H, M) \overset{\bar{\psi}_M}{\longrightarrow} M^{coH}, \quad f \longmapsto f(1 \otimes 1),$$

in the part (1), with the isomorphism $\bar{E} : M^{coH} \to M^{\overline{coH}}$, to obtain the following composed isomorphism

$${}_H\mathrm{Hom}_H^H(H \otimes_k H, M) \overset{\bar{\psi}_M}{\longrightarrow} M^{coH} \overset{\bar{E}}{\longrightarrow} M^{\overline{coH}},$$

given for $f \in {}_H\mathrm{Hom}_H^H(H \otimes_k H, M)$ by

$$\begin{aligned} f \longmapsto f(1 \otimes 1) \longmapsto \bar{E}(f(1 \otimes 1)) &= \sum f(1 \otimes 1)_0\, \beta S(f(1 \otimes 1)_1) \\ \text{by } H\text{-colinearity of } f &= \sum f(x^1 \otimes x^2)\, \beta S(x^3) \\ \text{by } H\text{-linearity of } f &= \sum f(x^1 \otimes x^2 \beta S(x^3)) = f(p_R). \end{aligned}$$

The inverse map can be computed, for $a, b \in H$ and $m \in M^{\overline{coH}}$, as

$$\begin{aligned} m \overset{\bar{\theta}'_M}{\longmapsto} \{a \otimes b \mapsto E(a\, E(m))\, b &= \sum E([a_1 \blacktriangleright E(m)]\, a_2)\, b \\ &= [\sum E(a_1 \blacktriangleright E(m)) \varepsilon(a_2)]\, b \\ &= [E(a \blacktriangleright E(m))]\, b \\ &= [E(E(a\, m))]\, b = E(a\, m)\, b\}. \end{aligned}$$

Chapter 5

Hom-tensor relations for two-sided Hopf modules

14 The category ${}_{\mathcal{A}}\mathbb{M}_H^H$

The fact that a quasi-bialgebra is not coassociative entails that there is no trivial way to define a comodule category like in the bialgebra case. Nevertheless, we can associate monoidal categories to quasi-bialgebras, in which we can consider coalgebras and comodules over these coalgebras. This point of view has been used in [10], [17], [26], and [8] in order to define relative Hopf modules, quasi-Hopf bimodules, and two-sided two-cosided Hopf modules. For a quasi-bialgebra H, we know that the category of (H, H)-bimodules is monoidal and H itself is a coalgebra in this category.

For a quasi-bialgebra H and a right H-comodule algebra $(\mathcal{A}, \rho, \phi_\rho)$, we show that the tensor functor $- \otimes_k H$ is a comonad on the category ${}_{\mathcal{A}}\mathbb{M}_H$ and we consider the category of two-sided Hopf modules ${}_{\mathcal{A}}\mathbb{M}_H^H$ as the Eilenberg-Moore comodule category over this comonad. Furthermore, we show that the Hom-functor ${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, -)$ is a right adjoint to the comparison functor $- \otimes_k H$.

If H is a quasi-Hopf algebra, following Bulacu-Caenepeel [9, 8] and Bulacu-Torrecillas [12], we study the category of two-sided Hopf modules and state a generalized version of the Fundamental Theorem of Hopf modules by defining Hausser-Nill and Bulacu-Caenepeel type coinvariants for this category. Finally, we describe these versions of coinvariants in terms of a Hom-functor.

14.1. Category ${}_{\mathcal{A}}\mathbb{M}_H^H$ of two-sided Hopf modules. Let H be a quasi-bialgebra and $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra. A left **two-sided $(\mathcal{A}, H)$-Hopf module** is an $(\mathcal{A}, H)$-bimodule M, together with a k-linear map

$$\varrho^M : M \longrightarrow M \otimes H, \quad \varrho^M(m) = \sum m_0 \otimes m_1,$$

satisfying the relations

$$(id_M \otimes \varepsilon) \circ \varrho^M = id_M, \tag{14.1}$$

$$(id_M \otimes \Delta) \circ \varrho^M(m) = \phi_\rho \cdot (\varrho^M \otimes id_H) \circ \varrho^M(m) \cdot \phi^{-1}, \tag{14.2}$$

$$\varrho^M(a\, m) = \sum a_{(0)}\, m_0 \otimes a_{(1)} m_1, \tag{14.3}$$

$$\varrho^M(m\, h) = \sum m_0\, h_1 \otimes m_1 h_2, \tag{14.4}$$

for $m \in M$, $h \in H$ and $a \in \mathcal{A}$, where $\rho(a) = \sum a_{(0)} \otimes a_{(1)}$.
The category of left two-sided $(\mathcal{A}, H)$-Hopf modules and right H-linear, left $\mathcal{A}$-linear, and right H-colinear maps is denoted by ${}_{\mathcal{A}}\mathbb{M}_H^H$.

For the special case $\mathcal{A} = H$, the category of two-sided (H, H)-Hopf modules is nothing but the category of right quasi-Hopf H-bimodules (see section 12).

14.2. Proposition. (Subgenerator for ${}_{\mathcal{A}}\mathbb{M}_H^H$). *Let H be a quasi-bialgebra and $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra. Then*

(1) For any $N \in {}_{\mathcal{A}}\mathbb{M}$, we have $N \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$ with structure maps

$$a \cdot (n \otimes h) = \sum a_{(0)}\, n \otimes a_{(1)} h, \qquad (n \otimes h) \cdot h' = n \otimes hh', \tag{14.5}$$

$$\varrho^{N \otimes H}(n \otimes h) = \sum \tilde{x}_\rho^1\, n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2 = \phi_\rho^{-1} \cdot (id \otimes \Delta)(n \otimes h), \tag{14.6}$$

for all $h, h' \in H$, $n \in N$ and $a \in \mathcal{A}$.

(2) If $g : N_1 \to N_2$ is an (epi-)morphism in ${}_{\mathcal{A}}\mathbb{M}$, then

$$g \otimes id_H : N_1 \otimes H \longrightarrow N_2 \otimes H$$

is an (epi-)morphism in ${}_{\mathcal{A}}\mathbb{M}_H^H$.

(3) Endowed with the structure maps given, for $h, h' \in H$ and $a, a' \in \mathcal{A}$, by

$$a' \cdot (a \otimes h') = \sum a'_{(0)} a \otimes a'_{(1)} h, \quad (a \otimes h)\, h' = a \otimes hh',$$

$$\varrho^{\mathcal{A} \otimes H}(a \otimes h) = \sum \tilde{x}_\rho^1 a \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2,$$

$\mathcal{A} \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$ and it is a subgenerator *for this category, where*

$$\phi_\rho = \sum \tilde{X}_\rho^1 \otimes \tilde{X}_\rho^2 \otimes \tilde{X}_\rho^3, \quad \phi_\rho^{-1} = \sum \tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 \otimes \tilde{x}_\rho^3.$$

Proof. The parts (1) and (2) are straightforward to see.

(3) Using a similar approach as in section 12, we see that for any $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, the left $\mathcal{A}$-module M is a homomorphic image of $\mathcal{A}^{(\Lambda)}$, for some cardinal Λ. Therefore $M \otimes H$ is a homomorphic image of

$$\mathcal{A}^{(\Lambda)} \otimes H \cong (\mathcal{A} \otimes H)^{(\Lambda)}.$$

For any $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, the coaction $\varrho^M : M \to M \otimes H$ is a (mono-)morphism in the category ${}_{\mathcal{A}}\mathbb{M}_H^H$, so we can consider M as a subobject of $M \otimes H$, which is generated by the object $\mathcal{A} \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$. □

The parts (1) and (2) in the above proposition give rise to

14.3. The comparison functor $-\otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}_H^H$. Let H be a quasi-bialgebra and $(\mathcal{A}, \varrho, \phi_\varrho)$ a right H-comodule algebra. We have seen that for any $N \in {}_{\mathcal{A}}\mathbb{M}$, $N \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$ with the $(\mathcal{A}, H)$-bimodule structure given in 14.5 and the H-comodule structure map given in 14.6. This gives rise to the **comparison functor**

$$- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \longrightarrow {}_{\mathcal{A}}\mathbb{M}_H^H, \quad N \mapsto (N \otimes H, \varrho_{N\otimes H}, \varrho^{N\otimes H}),$$

where $\varrho_{N\otimes H}$ denotes the $(\mathcal{A}, H)$-bimodule and $\varrho^{N\otimes H}$ the right H-comodule structure of $N \otimes_k H$.

14.4. $-\otimes_k V$ as endofunctor of ${}_{\mathcal{A}}\mathbb{M}_H$. Let H be a quasi-bialgebra and $(\mathcal{A}, \rho, \phi_\rho)$ be a right H-comodule algebra, $N \in {}_{\mathcal{A}}\mathbb{M}_H$ and $V \in {}_H\mathbb{M}_H$. Then the coaction

$$\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes_k H, \quad \rho(a) = \sum a_{(0)} \otimes a_{(1)},$$

induces an $(\mathcal{A}, H)$-bimodule structure on $N \otimes_k V$, given for $h \in H$, $a \in \mathcal{A}$, $v \in V$, and $n \in N$ by

$$a \cdot (n \otimes v) \cdot h = \sum a_{(0)}\, n\, h_1 \otimes a_{(1)}\, v\, h_2 = \rho(a)\,(n \otimes v)\,\Delta(h).$$

In this way, for any $V \in {}_H\mathbb{M}_H$, we obtain an endofunctor

$$- \otimes_k V : {}_{\mathcal{A}}\mathbb{M}_H \longrightarrow {}_{\mathcal{A}}\mathbb{M}_H, \quad N \mapsto N \otimes_k V,$$

with the $(\mathcal{A}, H)$-bimodule structure on $N \otimes_k V$ given above.

Considering the special case that $V = H$, we obtain the endofunctor

$$G := - \otimes_k H : {}_{\mathcal{A}}\mathbb{M}_H \longrightarrow {}_{\mathcal{A}}\mathbb{M}_H, \quad N \mapsto N \otimes_k H,$$

with the $(\mathcal{A}, H)$-bimodule structure on $N \otimes_k H$ given for $h, h' \in H$, $a \in \mathcal{A}$, and $n \in N$ by

$$a \cdot (n \otimes h) \cdot h' = \sum a_{(0)}\, n\, h'_1 \otimes a_{(1)} h h'_2 = \rho(a)\,(n \otimes h)\,\Delta(h').$$

In this case, we show that $- \otimes_k H : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H$ is a *comonad.*

14.5. Theorem. ($-\otimes_k H$ as a comonad on ${}_{\mathcal{A}}\mathbb{M}_H$). *Let $(H, \Delta, \varepsilon, \phi)$ be a quasi-bialgebra and $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra. Then*

(1) The endofunctor $- \otimes_k H : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H$ is a comonad on ${}_{\mathcal{A}}\mathbb{M}_H$ with the comultiplication δ defined, for $N \in {}_{\mathcal{A}}\mathbb{M}_H$, by

$$\begin{array}{rcl} \delta_N : N \otimes H & \longrightarrow & (N \otimes H) \otimes H, \\ n \otimes h & \mapsto & \sum \tilde{x}_\rho^1\, n\, X^1 \otimes \tilde{x}_\rho^2\, h_1\, X^2 \otimes \tilde{x}_\rho^3\, h_2\, X^3 \\ & = & \phi_\rho^{-1} \cdot (id \otimes \Delta)(n \otimes h) \cdot \phi, \end{array}$$

and counit ϵ defined by $\epsilon_N = id_N \otimes \varepsilon : N \otimes H \to N$.

(2) The category of two-sided Hopf modules ${}_{\mathcal{A}}\mathbb{M}_H^H$ is isomorphic to the Eilenberg-Moore comodule category $({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H}$.

Proof. (1) To show that $(- \otimes H, \delta, \epsilon)$ is a comonad on ${}_{\mathcal{A}}\mathbb{M}_H$, first we show the coassociativity of δ, i.e. we show that for $N \in {}_{\mathcal{A}}\mathbb{M}_H$, $n \in N$ and $h \in H$,

$$\delta_{N\otimes H} \circ \delta_N(n \otimes h) = (\delta_N \otimes id_H) \circ \delta_N(n \otimes h). \tag{14.7}$$

For this, using the definition of δ_N, we compute

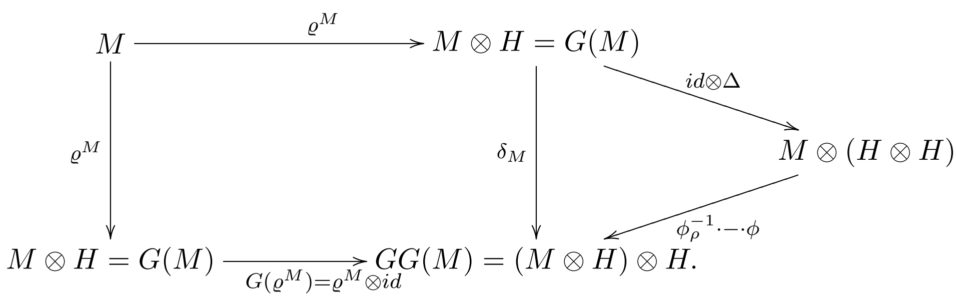
$$\begin{aligned}
\text{L.H.S} &= (\phi_\rho^{-1} \otimes 1) \cdot \{(id \otimes \Delta \otimes id)(\phi_\rho^{-1} \cdot [(id \otimes \Delta)(n \otimes h)] \cdot \phi)\} \cdot (\phi \otimes 1) \\
&= (\phi_\rho^{-1} \otimes 1) \cdot (id \otimes \Delta \otimes id)(\phi_\rho^{-1}) \cdot [(id \otimes \Delta \otimes id) \circ (id \otimes \Delta)(n \otimes h)] \\
&\quad \cdot (id \otimes \Delta \otimes id)(\phi) \cdot (\phi \otimes 1) \\
\text{by (7.2)} &= (\phi_\rho^{-1} \otimes 1) \cdot (id \otimes \Delta \otimes id)(\phi_\rho^{-1}) \cdot (1_{\mathcal{A}} \otimes \phi^{-1}) \cdot [(id \otimes id \otimes \Delta) \circ (id \otimes \Delta)(n \otimes h)] \\
&\quad \cdot (1_H \otimes \phi) \cdot (id \otimes \Delta \otimes id)(\phi) \cdot (\phi \otimes 1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\text{R.H.S} &= (\rho \otimes id \otimes id)(\phi_\rho^{-1}) \cdot \{(id \otimes \otimes id \Delta)(\phi_\rho^{-1} \cdot [(id_N \otimes \Delta)(n \otimes h)] \cdot \phi)\} \cdot (\Delta \otimes id \otimes id)(\phi) \\
&= (\rho \otimes id \otimes id)(\phi_\rho^{-1}) \cdot (id_N \otimes id_H \otimes \Delta)(\phi_\rho^{-1}) \cdot [(id \otimes id \otimes \Delta) \circ (id \otimes \Delta)(n \otimes h)] \\
&\quad \cdot (id \otimes id \otimes \Delta)(\phi) \cdot (\Delta \otimes id \otimes id)(\phi).
\end{aligned}$$

By (7.3) and (10.1) the both sides of (14.7) are equal to each other. Thus, δ is coassociative. It can be easily seen that ϵ, defined by $\epsilon_N = id_N \otimes \varepsilon : N \otimes H \to N$, is a counit for δ.

(2) To prove the isomorphism $({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H} \cong {}_{\mathcal{A}}\mathbb{M}_H^H$, we take an object $M \in ({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H}$ and note that we have a G-comodule structure morphism $\varrho^M : M \to M \otimes H = G(M)$ in ${}_{\mathcal{A}}\mathbb{M}_H$ making the following diagram commutative.



The commutativity of outer diagram is precisely the condition (14.2) on M to be a two-sided Hopf module. It is easy to see that the condition (14.1) is equivalent to the counitality of ϵ. □

The following Lemma helps to find a right adjoint to the comparison functor (see 14.3).

14.6. Lemma. **(${}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, -)$ as a functor into ${}_{\mathcal{A}}\mathbb{M}$).** *Let H be a quasi-bialgebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra and $V \in {}_{\mathcal{A}}\mathbb{M}_{\mathcal{A}}$.*

(1) If $M \in {}_{\mathcal{A}}\mathbb{M}_H$, then ${}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, M) \in {}_{\mathcal{A}}\mathbb{M}$ with the left $\mathcal{A}$-module structure given, for $h \in H$, $a \in \mathcal{A}$ and $v \in V$, by

$$(a \cdot f)(v \otimes h) = f(v\, a \otimes h).$$

In this way, we get the Hom-functor ${}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, -) : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}$.
In particular, if $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$ *then* ${}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, M) \in {}_{\mathcal{A}}\mathbb{M}$ *with left* $\mathcal{A}$*-module structure given above, and we obtain the Hom-functor*

$$ {}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, -) : {}_{\mathcal{A}}\mathbb{M}_H^H \to {}_{\mathcal{A}}\mathbb{M}. $$

(2) Let $V \in {}_{\mathcal{A}}\mathbb{M}_{\mathcal{A}}$ *and* $N \in {}_{\mathcal{A}}\mathbb{M}$. *Then*

(i) $\psi : {}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, N \otimes H) \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, N), \quad f \mapsto (id \otimes \varepsilon) \circ f$, *is an isomorphism in* ${}_{\mathcal{A}}\mathbb{M}$ *with inverse map* ψ' *given by*

$$ g \mapsto (g \otimes id_H) \circ \varrho^{V \otimes H}. $$

(ii) $\theta : {}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, N) \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}(V, N), \quad f \mapsto f(- \otimes 1_H)$, *is an isomorphism in* ${}_{\mathcal{A}}\mathbb{M}$ *with inverse map* θ' *given by*

$$ g \mapsto [v \otimes h \mapsto \varepsilon(h) g(v)], $$

(iii) There is a left $\mathcal{A}$*-module isomorphism*

$$ {}_{\mathcal{A}}\mathrm{Hom}(V, N) \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, N \otimes H), \quad g \longmapsto g \otimes id_H, $$

with inverse map given, for $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, N \otimes H)$, *by*

$$ f \mapsto (id \otimes \varepsilon) \circ f(- \otimes 1_H). $$

Thus the comparison functor $- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}_H^H$ *is full and faithful.*

Here, we consider the right H*-module structure of* N *to be the trivial one.*

Proof. (1) For all $a \in \mathcal{A}$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, M)$, it is easy to see that $a \cdot f$ is an $(\mathcal{A}, H)$-bilinear map. In this way, we have ${}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, M) \in {}_{\mathcal{A}}\mathbb{M}$ and we obtain a functor

$$ {}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, -) : {}_{\mathcal{A}}\mathbb{M}_H \longrightarrow {}_{\mathcal{A}}\mathbb{M}, $$

In case $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, M)$, the H-colinearity of of $a \cdot f$ follows from the H-colinearity of f itself. Thus, ${}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, M) \in {}_{\mathcal{A}}\mathbb{M}$ and we obtain a functor

$$ {}_{\mathcal{A}}\mathrm{Hom}_H^H(V \otimes H, -) : {}_{\mathcal{A}}\mathbb{M}_H^H \longrightarrow {}_{\mathcal{A}}\mathbb{M}. $$

(2) (i) As seen in 14.5, the functor $- \otimes_k H : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H$ is a comonad and the category ${}_{\mathcal{A}}\mathbb{M}_H^H$ of two-sided Hopf modules is just the Eilenberg-Moore comodule category $({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H}$ over this comonad. Now, considering the functor $- \otimes H : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H^H$ as the free functor which is right adjoint to the forgetful functor (by 4.8), we obtain the isomorphism of part (i).

(ii) First we note that for $f \in {}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, N)$, $h \in H$, $a \in \mathcal{A}$ and $v \in V$,

$$ \begin{aligned} a\,[\theta(f)(v)] &= a\,[f(v \otimes 1_H)] \\ \text{f is left } \mathcal{A}\text{-linear} \quad &= f(\sum a_{(0)}\, v \otimes a_{(1)}) \\ \text{f is right } H\text{-linear} \quad &= \sum f(a_{(0)}\, v \otimes 1_H)\, a_{(1)} \\ N \text{ is trivial right } H\text{-module} \quad &= \sum f(a_{(0)}\, v \otimes 1_H)\, \varepsilon(a_{(1)}) \\ &= f(a\, v \otimes 1_H) = \theta(f)(a\, v). \end{aligned} $$

This means that $\theta(f) \in {}_{\mathcal{A}}\mathrm{Hom}(V,N)$. It is straightforward to show that, for $g \in {}_{\mathcal{A}}\mathrm{Hom}(V,N)$, we have $\theta'(g) \in {}_{\mathcal{A}}\mathrm{Hom}_H(V \otimes H, N)$. Bijectivity and left $\mathcal{A}$-linearity of θ follow from direct computations.

(iii)This follows from the composition of the isomorphisms in parts (i) and (ii). □

14.7. Corollary. *Let H be a quasi-bialgebra and $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra.*

(1) For $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, we have a left $\mathcal{A}$-module structure on ${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M)$ given, for $h \in H$, $a, a' \in \mathcal{A}$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M)$, by

$$(a' \cdot f)(a \otimes h) = f(aa' \otimes h).$$

(2) For $N \in {}_{\mathcal{A}}\mathbb{M}$, the morphism

$$\eta_N : N \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, N \otimes H), \quad n \longmapsto [a \otimes h \mapsto a\,n \otimes h],$$

is an isomorphism with inverse map η'_N, given for $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, N \otimes H)$, by

$$\eta'_N(f) = (id \otimes \varepsilon) \circ f(1_{\mathcal{A}} \otimes 1_H).$$

Proof. (1) Follows directly from Lemma 14.6 by taking $V = \mathcal{A}$.

(2) Composition of the isomorphisms ψ' and θ' gives rise to the isomorphism

$$N \cong {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, N) \cong {}_{\mathcal{A}}\mathrm{Hom}_H(\mathcal{A} \otimes H, N) \cong {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, N \otimes H).$$

Using the above Lemma, we see that this composition gives precisely the isomorphism $\eta_N : N \to {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, N \otimes H)$ given above with the given inverse map η'_N. □

Now we show that this Hom-functor is a right adjoint to the comparison functor $- \otimes_k H$, described in 14.3.

14.8. Theorem. (Hom-tensor adjunction for ${}_{\mathcal{A}}\mathbb{M}_H^H$). *Let H be a quasi-bialgebra, $(\mathcal{A}, \varrho, \phi_\varrho)$ a right H-comodule algebra, $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, and $N \in {}_{\mathcal{A}}\mathbb{M}$. Then there is a functorial isomorphism*

$${}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M) \xrightarrow{\ \Omega\ } {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M)),$$
$$f \longmapsto \{n \mapsto [a \otimes h \mapsto f(a\,n \otimes h)]\},$$

with inverse map Ω' given by

$$g \longmapsto \{n \otimes h \mapsto g(n)(1_{\mathcal{A}} \otimes h)\}.$$

This means that the comparison functor

$$- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \longrightarrow {}_{\mathcal{A}}\mathbb{M}_H^H, \quad N \mapsto (N \otimes H, \varrho_{N \otimes H}, \varrho^{N \otimes H}),$$

is left adjoint to the Hom-*functor*

$${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, -) : {}_{\mathcal{A}}\mathbb{M}_H^H \longrightarrow {}_{\mathcal{A}}\mathbb{M},$$

with unit and counit given by

$$\eta_N : N \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, N \otimes H), \quad n \mapsto [a \otimes h \mapsto a\,n \otimes h],$$
$$\varepsilon_M : {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M) \otimes H \longrightarrow M, \quad f \otimes h \mapsto f(1_{\mathcal{A}} \otimes h).$$

Furthermore, the comparison functor $- \otimes H : {}_{\mathcal{A}}\mathbb{M}_H^H \to {}_{\mathcal{A}}\mathbb{M}$ is full and faithful.

Proof. First we show that for any $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M)$, $\Omega(f)$ is left $\mathcal{A}$-linear. For $h \in H, a, a' \in \mathcal{A}$ and $n \in N$,

$$[a' \cdot (\Omega(f)(n))](a \otimes h) = \Omega(f)(n)(aa' \otimes h) = f(n\, aa' \otimes h) = [\Omega(f)(a'\, n))](a \otimes h).$$

Thus, we have $\Omega(f) \in {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M))$.
For any $g \in {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M))$, we show that $\Omega'(g) \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M)$.

i) $\Omega'(g)$ is left $\mathcal{A}$-linear. For $a \in \mathcal{A}$ and $n \in N$,

$$\begin{aligned}
\Omega'(g)((n \otimes h) \cdot a) &= \sum \Omega'(g)(a_{(0)}\, n \otimes a_{(1)} h) = \sum g(a_{(0)}\, n)(1_{\mathcal{A}} \otimes a_{(1)} h) \\
{\scriptstyle g \text{ is right } \mathcal{A}\text{-linear}} &= \sum (a_{(0)} \cdot g(n))(1_{\mathcal{A}} \otimes a_{(1)} h) = \sum g(n)(a_{(0)} \otimes a_{(1)} h) \\
&= g(n)(\rho(a)\,(1 \otimes h)) = a\,[g(n)(1 \otimes h)] \\
&= a\,[\Omega'(g)(n \otimes h)].
\end{aligned}$$

ii) It can be easily seen that $\Omega'(g)$ is right H-linear.

iii) For the right H-colinearity of $\Omega'(g)$ we show that

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = \sum (\Omega'(g) \otimes id)(\tilde{x}_\rho^1\, n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2).$$

By the colinearity of $g(n)$,

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = \varrho^M(g(n)(1_{\mathcal{A}} \otimes h)) = g(n)(\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2.$$

On the other hand,

$$\begin{aligned}
(\Omega'(g) \otimes id)(\sum \tilde{x}_\rho^1\, n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2) &= \sum g(\tilde{x}_\rho^1\, n)(1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2 \\
{\scriptstyle g \text{ is } \mathcal{A}\text{-linear}} &= \sum [\tilde{x}_\rho^1 \cdot g(n)](1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2 \\
&= \sum g(n)(\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2.
\end{aligned}$$

This shows the H-colinearity of $\Omega'(g)$.

Now we show that Ω and Ω' are inverse to each other. For all $n \in N, h \in H$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M)$,

$$\begin{aligned}
(\Omega' \circ \Omega(f))(n \otimes h) &= (\Omega(f))(n)(1_{\mathcal{A}} \otimes h) \\
&= f(1_{\mathcal{A}}\, n \otimes h) = f(n \otimes h)
\end{aligned}$$

Conversely, for all $h \in H, n \in N, a \in \mathcal{A}$ and $g \in {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M))$,

$$\begin{aligned}
\{[(\Omega \circ \Omega')(g)](n)\}(a \otimes h) &= (\Omega'(g))(a\, n \otimes h) = g(a\, n)(1_{\mathcal{A}} \otimes h) \\
{\scriptstyle g \text{ is } \mathcal{A}\text{-linear}} &= [a \cdot g(n)](1_{\mathcal{A}} \otimes h) = g(n)(a \otimes h).
\end{aligned}$$

i.e. $\Omega \circ \Omega'(g) = g$. It is straightforward to see that Ω is functorial in both components M and N.
The fully faithfulness of the comparison functor follows by Lemma 14.6. $\square$

15 Coinvariants for ${}_{\mathcal{A}}\mathbb{M}_H^H$

15.1. Hausser-Nill-type coinvariants for ${}_{\mathcal{A}}\mathbb{M}_H^H$. Let H be a quasi-Hopf algebra and $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra. For $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, define a projection $E' : M \to M$, for $m \in M$, by

$$E'(m) := \sum \tilde{q}_\rho^1 \, m_0 \, \beta S(\tilde{q}_\rho^2 m_1), \tag{15.1}$$

and for $m \in M, a \in \mathcal{A}$ put

$$a \blacktriangleright m := E'(a\, m) \tag{15.2}$$

where $\tilde{q}_\rho = \sum \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2$ is defined as in (10.2).

For $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, define the **HN-type coinvariants** of M as $M^{coH} := E'(M)$.

Similar to 13.1 (see also [17, Proposition 3.4]), we have the following properties of the projection map $E' : M \to M$:

15.2. Proposition. (Properties of HN-type coinvariants). For $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, $a \in \mathcal{A}$, $h \in H$ and $m \in M$ and with above notations we have

(i) $E'(m\, h) = \varepsilon(h) E'(m)$,

(ii) $E'^2 = E'$,

(iii) $a \blacktriangleright E'(m) = E'(a\, m) = a \blacktriangleright m$,

(iv) $(ab) \blacktriangleright m = a \blacktriangleright (b \blacktriangleright m)$,

(v) $a\, E'(m) = \sum [a_{(0)} \blacktriangleright E'(m)]\, a_{(1)}$,

(vi) $\sum E'(m_0)\, m_1 = m$,

(vii) $\sum E'(E'(m)_0) \otimes E'(m)_1 = E'(m) \otimes 1$.

Proof. (i)

$$\begin{aligned} E'(m\, h) &= \sum \tilde{q}_\rho^1 \, (m\, h)_0 \, \beta S(\tilde{q}_\rho^2 (m\, h)_1) = \sum \tilde{q}_\rho^1 \, m_0 \, h_1 \beta S(\tilde{q}_\rho^2 m_1 h_2) \\ &= \varepsilon(h) \sum \tilde{q}_\rho^1 \, m_0 \, \beta S(\tilde{q}_\rho^2 m_1) = \varepsilon(h) E'(m). \end{aligned}$$

(ii) We use part (i) to compute

$$\begin{aligned} E'^2(m) &= E'(\sum \tilde{q}_\rho^1 \, m_0 \, \beta S(\tilde{q}_\rho^2 m_1)) \\ \text{by (i)} \quad &= \sum E'(\tilde{q}_\rho^1 \, m_0) \varepsilon(\beta S(\tilde{q}_\rho^2 m_1)) = \sum E'(\tilde{q}_\rho^1 \, m_0) \varepsilon(\beta) \varepsilon(\tilde{q}_\rho^2) \varepsilon(m_1) \\ &= \sum E'(\tilde{q}_\rho^1 \varepsilon(\tilde{q}_\rho^2) \, m_0 \varepsilon(m_1)) = E'(m). \end{aligned}$$

(iii)

$$\begin{aligned} a \blacktriangleright E'(m) &= E'(a\, E'(m)) = \sum E'(a\tilde{q}_\rho^1 \, m_0 \, \beta S(\tilde{q}_\rho^2 m_1)) \\ &= \sum E'(a\tilde{q}_\rho^1 \, m_0) \varepsilon(\beta S(\tilde{q}_\rho^2 m_1)) \\ &= \sum E'(a\tilde{q}_\rho^1 \, m_0) \varepsilon(\beta) \varepsilon \circ S(m_1) \varepsilon \circ S(\tilde{q}_\rho^2) \\ &= \sum E'(a\tilde{q}_\rho^1 \varepsilon(m_1) m_0) \varepsilon(\beta) \varepsilon(\tilde{q}_\rho^2) = \sum E'(a\tilde{q}_\rho^1 \, m) \varepsilon(\beta) \varepsilon(\tilde{q}_\rho^2) \\ &= \sum E'(a\tilde{q}_\rho^1 \varepsilon(\tilde{q}_\rho^2) \, m) = \sum E'(a\, m) = a \blacktriangleright m. \end{aligned}$$

(iv) follows immediately from part (iii).

$$\begin{array}{rrcl}
\text{(v)} & a\,E'(m) & = & a\sum \tilde{q}^1_\rho\, m_0\, \beta S(\tilde{q}^2_\rho m_1) \\
& \text{by (10.4)} & = & \sum \tilde{q}^1_\rho a_{(0)_{(0)}}\, m_0\, \beta S(m_1) S(a_{(0)_{(1)}}) S(\tilde{q}^2_\rho) a_{(1)} \\
& & = & \sum \tilde{q}^1_\rho a_{(0)_{(0)}}\, m_0\, \beta S(\tilde{q}^2_\rho a_{(0)_{(1)}} m_1) a_{(1)} \\
& & = & \sum \tilde{q}^1_\rho\, (a_{(0)}\, m)_0\, \beta S(\tilde{q}^2_\rho (a_{(0)}\, m)_1) a_{(1)} = \sum E'(a_{(0)}\, m)\, a_{(1)} \\
& \text{by (iii)} & = & \sum [a_{(0)} \blacktriangleright E'(m)]\, a_{(1)}.
\end{array}$$

$$\begin{array}{rrcl}
\text{(vi)} & E'(m_0)m_1 & = & \sum \tilde{q}^1_\rho m_{00} \beta S(\tilde{q}^2_\rho m_{01}) m_1 = \sum \tilde{X}^1_\rho m_{00} \beta S(\tilde{X}^2_\rho m_{01}) \alpha \tilde{X}^3_\rho m_1 \\
& & = & \sum m_0\, X^1 \beta S(m_{11} X^2) \alpha m_{12} X^3 \\
& & = & \sum m_0\, X^1 \beta S(X^2) S(m_{11}) \alpha m_{12} X^3 \\
& & = & \sum \varepsilon(m_1) m_0\, (X^1 \beta S(X^2) \alpha X^3) = m\, 1_H = m.
\end{array}$$

(vii)

$$\begin{array}{rcl}
\sum E'(E'(m)_0) \otimes E'(m)_1 & = & \sum E'([(\tilde{q}^1_\rho)\, m_0\, \beta S((\tilde{q}^2_\rho) m_1)]_0) \otimes [(\tilde{q}^1_\rho)\, m_0\, \beta S((\tilde{q}^2_\rho) m_1)]_1 \\
& = & \sum E'((\tilde{q}^1_\rho)_{(0)}\, m_{00}\, \beta_1 S((\tilde{q}^2_\rho) m_1)_1) \otimes (\tilde{q}^1_\rho)_{(1)} m_{01} \beta_2 S(\tilde{q}^2_\rho m_1))_2 \\
\text{by (i)} & = & \sum E'((\tilde{q}^1_\rho)_{(0)}\, m_{00}) \otimes \varepsilon(\beta_1) \varepsilon(S(\tilde{q}^2_\rho m_1)_1) (\tilde{q}^1_\rho)_{(1)} m_{01} \beta_2 S(\tilde{q}^2_\rho m_1)_2 \\
& = & \sum E'((\tilde{q}^1_\rho)_{(0)}\, m_{00}) \otimes (\tilde{q}^1_\rho)_{(1)} m_{01} \beta S(\tilde{q}^2_\rho m_1) \\
\text{by (14.2)} & = & \sum E'((\tilde{q}^1_\rho)_{(0)} \tilde{x}^1_\rho\, m_0\, X^1) \otimes (\tilde{q}^1_\rho)_{(1)} \tilde{x}^2_\rho m_{11} X^2 \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho m_{12} X^3) \\
\text{by (i)} & = & \sum E'((\tilde{q}^1_\rho)_{(0)} \tilde{x}^1_\rho\, m_0) \varepsilon(X^1) \otimes (\tilde{q}^1_\rho)_{(1)} \tilde{x}^2_\rho m_{11} X^2 \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho m_{12} X^3) \\
& = & \sum E'((\tilde{q}^1_\rho)_{(0)} \tilde{x}^1_\rho\, m_0) \otimes (\tilde{q}^1_\rho)_{(1)} \tilde{x}^2_\rho m_{11} \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho m_{12}) \\
\text{by (7.6)} & = & \sum E'((\tilde{q}^1_\rho)_{(0)} \tilde{x}^1_\rho\, m_0) \otimes (\tilde{q}^1_\rho)_{(1)} \tilde{x}^2_\rho \varepsilon(m_1) \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho) \\
& = & \sum E'((\tilde{q}^1_\rho)_{(0)} \tilde{x}^1_\rho\, m) \otimes (\tilde{q}^1_\rho)_{(1)} \tilde{x}^2_\rho \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho) \\
& = & \sum E'((\tilde{q}^1_\rho)_{(0)} \tilde{p}^1_\rho\, m) \otimes (\tilde{q}^1_\rho)_{(1)} p^2_\rho S(\tilde{q}^2_\rho) \\
\text{by (10.6)} & = & (E' \otimes id)([(1_{\mathcal{A}} \otimes 1_H)\, (m \otimes 1_H)]) = E'(m) \otimes 1_H.
\end{array}$$

□

By (ii), (vi) and (vii), the following characterizations of *HN-type coinvariants* are equivalent:

$$\begin{array}{rcl}
M^{coH} := E'(M) & = & \{n \in M | E'(n) = n\} \\
& = & \{n \in M | \sum E'(n_0) \otimes n_1 = E'(n) \otimes 1\} \\
& = & Ke((E' \otimes id) \circ [\varrho^M - (- \otimes 1_H)]).
\end{array}$$

M^{coH} with the left $\mathcal{A}$-action $\blacktriangleright$ is a left $\mathcal{A}$-module and for any morphism $f : M \to L$ in

${}_{\mathcal{A}}\mathbb{M}_H^H$, it is not hard to show that $f(M^{coH}) \subseteq L^{coH}$.
This gives rise to a functor $(-)^{coH} : {}_{\mathcal{A}}\mathbb{M}_H^H \to {}_{\mathcal{A}}\mathbb{M}$ which we will show to be right adjoint to the comparison functor.

15.3. Proposition. The adjoint pair $(-\otimes_k H, (-)^{coH})$ for HN-type coinvariants. *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ be a right H-comodule algebra, $N \in {}_{\mathcal{A}}\mathbb{M}$ and $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$. Then there is a functorial isomorphism*

$$\psi_{N,M} : {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes_k H, M) \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}(N, M^{coH}), \quad f \longmapsto [n \mapsto f(n \otimes 1)],$$

with inverse map $\psi'_{N,M}$ given by

$$g \longmapsto [n \otimes h \mapsto g(n)\, h)].$$

Thus, the functors

$$- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}_H^H, \quad (-)^{coH} : {}_{\mathcal{A}}\mathbb{M}_H^H \to {}_{\mathcal{A}}\mathbb{M},$$

form an adjoint pair with unit and counit

$$\eta_N : N \longrightarrow (N \otimes H)^{coH}, \quad n \mapsto n \otimes 1,$$

$$\varepsilon_M : M^{coH} \otimes_k H \longrightarrow M, \quad m \otimes h \mapsto m\, h.$$

Proof. First, we show that $f(n \otimes 1) \in M^{coH}$: Since f is H-colinear,

$$\varrho^M(f(n \otimes 1)) = f(\tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2) \otimes \tilde{x}_\rho^3,$$

so we have

$$\begin{aligned} E'(f(n \otimes 1)) &= \sum \tilde{q}_\rho^1 f(\tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2)\, \beta S(\tilde{q}_\rho^2 \tilde{x}_\rho^3) \\ {\scriptstyle f \text{ is } \mathcal{A}\text{-linear}} &= \sum f(\rho(\tilde{q}_\rho^1)\, (\tilde{x}^1 n \otimes \tilde{x}^2 \beta S(\tilde{x}^3) S(\tilde{q}_\rho^2) \\ &= f(\sum \rho(\tilde{q}_\rho^1)\, \tilde{p}_\rho\, (1 \otimes S(\tilde{q}_\rho^2))\, (n \otimes 1)) \\ {\scriptstyle \text{by (10.6)}} &= f(n \otimes 1). \end{aligned}$$

We show that $\psi := \psi_{N,M}$ and $\psi' := \psi'_{N,M}$ are inverse to each other. For $n \in N, h \in H$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes_k H, M)$,

$$\begin{aligned} [(\psi' \circ \psi)(f)](n \otimes h) &= \psi(f)(n)\, h = f(n \otimes 1)\, h \\ {\scriptstyle \text{by } H\text{-linearity of } f} &= f(n \otimes h). \end{aligned}$$

Conversely, for $n \in N$ and $g \in {}_{\mathcal{A}}\mathrm{Hom}(N, M^{coH})$,

$$[(\psi \circ \psi')(g)](n) = \psi'(g)(n \otimes 1) = g(n)\, 1 = g(n).$$

□

15.4. HN-type coinvariants of $N \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$. *For any $N \in {}_{\mathcal{A}}\mathbb{M}$, the HN-type coinvariants of the two-sided Hopf module $N \otimes H$, come out as*

$$(N \otimes H)^{coH} \simeq N,$$

and for $n \in N$ and $h \in H$, we have $E'(n \otimes h) = n \otimes \varepsilon(h) 1_H$.

Proof. The definition of the right H-module structure of $N \otimes H$ implies that $(n \otimes h) = (n \otimes 1)\, h$. Now by part (i) of the above proposition, we have:

$$E'(n \otimes h) = E'((n \otimes 1)\, h) = E'(n \otimes 1)\varepsilon(h),$$

thus we are left to show that

$$E'(n \otimes 1) = n \otimes 1.$$

$$\begin{aligned}
L.H.S &= E'(n \otimes 1) = \sum \tilde{q}^1_\rho\, (n \otimes 1)_0\, \beta S(\tilde{q}^2_\rho (n \otimes 1)_1) \\
&= \sum \tilde{q}^1_\rho \cdot (\tilde{x}^1_\rho n \otimes \tilde{x}^2_\rho)\, \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho) \\
&= \sum (\tilde{q}^1_\rho)_{(0)} \tilde{x}^1_\rho n \otimes (\tilde{q}^1_\rho)_{(1)} \tilde{x}^2_\rho \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho) \\
&= \sum \varrho(\tilde{q}^1_\rho) \tilde{p}_\rho [1 \otimes S(\tilde{q}^2_\rho)]\, (n \otimes 1) \\
\text{by (10.5)} &= (1 \otimes 1)\, (n \otimes 1) = n \otimes 1,
\end{aligned}$$

where $\tilde{p}_\rho = \sum \tilde{p}^1_\rho \otimes \tilde{p}^2_\rho$ and $\tilde{q}_\rho = \sum \tilde{q}^1_\rho \otimes \tilde{q}^2_\rho$ are defined in (10.1) and (10.2). □

This means that that the unit $\eta_N : N \longrightarrow (N \otimes H)^{coH}$ of the adjunction in 15.3 is an isomorphism with inverse map $n \otimes h \mapsto n\varepsilon(h)$. This gives another proof for fully faithfulness of the comparison functor $- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}^H_H$ in this case (see 4.1, 14.6 and 14.8).

15.5. Fundamental Theorem for ${}_{\mathcal{A}}\mathbb{M}^H_H$ with HN-type coinvariants. *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra and $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$. Consider $M^{coH} = E'(M)$ as a left $\mathcal{A}$-module with left $\mathcal{A}$-action $\blacktriangleright$, defined by*

$$a \blacktriangleright m := E'(a\, m) = \sum \tilde{q}^1_\rho a_{(0)}\, m_0\, \beta S(\tilde{q}^2_\rho a_{(1)} m_1).$$

Then the map

$$\varepsilon_M : M^{coH} \otimes H \longrightarrow M, \qquad m \otimes h \mapsto m\, h,$$

is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}^H_H$ with inverse map ε'_M given by

$$\varepsilon'_M(m) = \sum E'(m_0) \otimes m_1 = (E' \otimes id) \circ \varrho^M(m).$$

Proof. For $h \in H$ and $n \in N$,

$$\begin{aligned}
\varepsilon'_M \circ \varepsilon_M(n \otimes h) &= \varepsilon'_M(n\, h) = \sum E'(n_0\, h_1) \otimes n_1 h_2 \\
\text{by (i)} &= \sum E'(n_0)\varepsilon(h_1) \otimes n_1 h_2 \\
&= \sum E'(n_0) \otimes n_1 h = \sum (E'(n_0) \otimes n_1)(1 \otimes h) \\
\text{by (vii)} &= (n \otimes 1)(1 \otimes h) = n \otimes h.
\end{aligned}$$

Conversely, for $m \in M$,

$$\varepsilon_M \circ \varepsilon'_M(m) = \varepsilon_M(\sum E'(m_0) \otimes m_1) = \sum E'(m_0)\, m_1 = m.$$

Thus ε_M is indeed an isomorphism of k-modules.
We are left to show that ε_M is a morphism in ${}_{\mathcal{A}}\mathbb{M}^H_H$. By definition of the $(\mathcal{A}, H)$-bimodule structure of $M^{coH} \otimes H$, for $h \in H$, $a \in \mathcal{A}$ and $n \in M^{coH}$,

$$a \cdot (n \otimes h) \cdot h' = \sum a_{(0)} \blacktriangleright n \otimes a_{(1)} h h' = \sum E'(a_{(0)}\, n) \otimes a_{(1)} h h'.$$

Therefore, we have

$$\begin{aligned}\varepsilon_M(a\cdot(n\otimes h)\cdot h') &= \sum E'(a_{(0)}\,n)\,a_{(1)}hh' \\ \text{by (iii)}\quad &= \sum[a_{(0)}\blacktriangleright E'(n)]\,a_{(1)}hh' \\ &= a\,E'(n)\,hh' = a\,n\,hh' \\ &= a\,\varepsilon_M(n\otimes h)\,h'.\end{aligned}$$

Finally, we show that ε'_M (and therefore ε_M) is H-colinear: for $m\in M$,

$$\begin{aligned}\varrho^{M^{coH}\otimes H}(\varepsilon'_M(m)) &= \sum E'(\tilde{x}^1_\rho\,m_0)\otimes\tilde{x}^2_\rho m_{11}\otimes\tilde{x}^3_\rho m_{12} \\ &= \sum E'(m_{00}\,X^1)\otimes m_{01}X^2\otimes m_1X^3 \\ &= \sum E'(m_{00})\varepsilon(X^1)\otimes m_{01}X^2\otimes m_1X^3 \\ &= \sum E'(m_{00})\otimes m_{01}\otimes m_1 \\ &= (E'\otimes id)\varrho^M(m_0) = (\varepsilon'_M\otimes id)\varrho^M(m).\end{aligned}$$

□

The above form of the Fundamental Theorem yields an additional characterization of coinvariants, for any $M\in{}_{\mathcal{A}}\mathbb{M}^H_H$, as

$$\begin{aligned}M^{coH} &= \{n\in M|\varrho^M(n)=\sum(\tilde{x}^1_\rho\blacktriangleright n)\,\tilde{x}^2_\rho\otimes\tilde{x}^3_\rho\} \\ &= Ke(\varrho^M-[(\varrho_M\otimes id)\circ(E'\otimes id\otimes id)(\phi^{-1}_\rho(-\otimes 1_{\mathcal{A}}\otimes 1_H))])\end{aligned}$$

15.6. Bulacu-Caenepeel-type coinvariants for ${}_{\mathcal{A}}\mathbb{M}^H_H$. Let H be quasi-bialgebra and $\mathcal{A}$ a right H-comodule algebra. With similar arguments as in (13.4) (see also Bulacu-Caenepeel [8]), for any $M\in{}_{\mathcal{A}}\mathbb{M}^H_H$, we define a projection

$$\bar{E}': M\longrightarrow M,\quad m\mapsto\sum m_0\,\beta S(m_1),$$

and define **BC-type coinvariants** for $M\in{}_{\mathcal{A}}\mathbb{M}^H_H$ as

$$M^{\overline{coH}} := \bar{E}'(M) = \{m\in M\mid\bar{E}'(m)=m\},$$

generalizing the concept of *coinvariants* of quasi-Hopf bimodules $M\in{}_H\mathbb{M}^H_H$.

15.7. HN versus BC-type projections. *Let $M\in{}_{\mathcal{A}}\mathbb{M}^H_H$ and $E',\bar{E}': M\to M$ be defined by*

$$E'(m)=\sum\tilde{q}^1_\rho\,m_0\,\beta S(\tilde{q}^2_\rho m_1),\qquad \bar{E}'(m)=\sum m_0\,\beta S(m_1),$$

for all $m\in M$. Then

(i) $\bar{E}'(m)=\sum E'(\tilde{p}^1_\rho\,m)\,\tilde{p}^2_\rho,\quad E'(m)=\sum\tilde{q}^1_\rho\,\bar{E}'(m)\,S(\tilde{q}^2_\rho),$

(ii) $\bar{E}': M^{coH}\to M^{\overline{coH}}$ *is an isomorphism in* ${}_{\mathcal{A}}\mathbb{M}$ *with inverse* $E': M^{\overline{coH}}\to M^{coH}$,

where $\tilde{p}_\rho=\sum\tilde{p}^1_\rho\otimes\tilde{p}^2_\rho$ *and* $\tilde{q}_\rho=\sum\tilde{q}^1_\rho\otimes\tilde{q}^2_\rho$ *are defined in (10.1) and (10.2) respectively.*

Proof. (i)

$$\begin{aligned}
\sum E'(\tilde{p}_\rho^1 m)\, \tilde{p}^2 &= \sum \tilde{q}_\rho^1 (\tilde{p}_\rho^1)_{(0)}\, m_0\, \beta S(\tilde{q}_\rho^2 (\tilde{p}_\rho^1)_{(1)} m_1) \tilde{p}_\rho^2 \\
&= \sum \tilde{q}_\rho^1 (\tilde{p}_\rho^1)_{(0)}\, m_0\, \beta S(m_1) S(\tilde{q}_\rho^2 (\tilde{p}_\rho^1)_{(1)}) \tilde{p}_\rho^2 \\
\text{by (10.5)} \quad &= \sum \tilde{q}_\rho^1 (\tilde{p}_\rho^1)_{(0)}\, \bar{E}(m)\, S(\tilde{q}_\rho^2 (\tilde{p}_\rho^1)_{(1)}) \tilde{p}_\rho^2 = \bar{E}(m).
\end{aligned}$$

The other equality is an easy substitution of $\bar{E}'(m)$.

(ii) For any $m \in M^{coH}$,

$$\begin{aligned}
E'(\bar{E}'(m)) &= E'(\sum m_0\, \beta S(m_1)) \\
&= \sum \tilde{q}_\rho^1\, m_{00}\, \beta_1 (S(m_1))_1 \beta S(\tilde{q}_\rho^2 m_{01} \beta_2 S(m_1)_2) \\
&= \sum \tilde{q}_\rho^1\, m_{00}\, \beta_1 S(m_1)_1 \beta S(m_1)_2 S(\beta_2) S(\tilde{q}_\rho^2 m_{01}) \\
&= (\sum \tilde{q}_\rho^1\, m_{00}\, \beta S(\tilde{q}_\rho^2 m_{01})) \varepsilon(m_1) \varepsilon(\beta) \\
&= E'(m_0) \varepsilon(m_1) \varepsilon(\beta) = E'(m) = m.
\end{aligned}$$

On the other hand, for any $m \in M^{\overline{coH}}$,

$$\begin{aligned}
\bar{E}'(E'(m)) &= \bar{E}'(\sum \tilde{q}_\rho^1\, m_0\, \beta S(\tilde{q}_\rho^2 m_1)) \\
&= \bar{E}'(\sum \tilde{q}_\rho^1\, m_0\, \beta S(m_1) S(\tilde{q}_\rho^2)) \\
&= \bar{E}'(\tilde{q}_\rho^1\, \bar{E}'(m)\, S(\tilde{q}_\rho^2)) = \bar{E}'(\sum \tilde{q}_\rho^1\, m\, S(\tilde{q}_\rho^2)) \\
&= \sum (\tilde{q}_\rho^1\, m\, S(\tilde{q}_\rho^2))_0\, \beta S([\tilde{q}_\rho^1\, m\, S(\tilde{q}_\rho^2)]_1) \\
&= \sum (\tilde{q}_\rho^1\, m)_0\, S(\tilde{q}_\rho^2)_1 \beta S(S(\tilde{q}_\rho^2)_2) S((\tilde{q}_\rho^1\, m)_1) \\
&= \sum \varepsilon(\tilde{q}_\rho^2) (\tilde{q}_\rho^1\, m)_0\, \beta S((\tilde{q}_\rho^1\, m)_1) \\
&= \sum \varepsilon(\tilde{q}_\rho^2) \bar{E}'(\tilde{q}_\rho^1\, m) = \bar{E}'(\sum \tilde{q}_\rho^1 \varepsilon(\tilde{q}_\rho^2)\, m) \\
&= \bar{E}'(\varepsilon(\alpha) 1_H\, m) = \bar{E}'(m) = m.
\end{aligned}$$

For left $\mathcal{A}$-linearity of E' we compute

$$\begin{aligned}
E'(a \triangleright m) &= \sum E'(a_{(0)}\, m_0\, \beta S(a_{(1)} m_1)) \\
&= \sum \tilde{q}_\rho^1 a_{(0)_{(0)}}\, m_{00}\, \beta_1 S(a_{(1)} m_1)_1 \beta S(\tilde{q}_\rho^2 a_{(0)_{(1)}} m_{01} \beta_2 S(a_{(1)} m_1)_2) \\
&= \sum \tilde{q}_\rho^1 a_{(0)_{(0)}}\, m_{00}\, \beta_1 S(a_{(1)} m_1)_1 \beta S(\beta_2 S(a_{(1)} m_1)_2) S(\tilde{q}_\rho^2 a_{(0)_{(1)}} m_{01}) \\
&= \sum \tilde{q}_\rho^1 a_{(0)_{(0)}}\, m_{00}\, \varepsilon(\beta) \varepsilon(S(a_{(1)} m_1)) \beta S(\tilde{q}_\rho^2 a_{(0)_{(1)}} m_{01}) \\
&= \sum \varepsilon(a_{(1)} m_1) \tilde{q}_\rho^1 a_{(0)_{(0)}}\, m_{00}\, \beta S(a_{(0)_{(1)}} m_{01}) S(\tilde{q}_\rho^2) \\
&= \sum \varepsilon(a_{(1)} m_1) \tilde{q}_\rho^1\, \bar{E}'(a_{(0)}\, m_0)\, S(\tilde{q}_\rho^2) \\
&= E'(a\, m) = a \blacktriangleright E'(m).
\end{aligned}$$

□

With similar arguments as in [8, Lemma 3.6], we show

15.8. Proposition. (Characterization of BC-type coinvariants in ${}_{\mathcal{A}}\mathbb{M}_H^H$). *For a quasi-Hopf algebra H, a right H-comodule algebra $(\mathcal{A}, \rho, \phi_\rho)$ and $m \in M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, we have $m \in M^{\overline{coH}}$ if and only if*

$$\varrho^M(m) = \sum \tilde{x}^1_\rho\, m\, S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes \tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2\}. \tag{15.3}$$

Proof. Let $m \in M^{\overline{coH}}$. Then

$$\begin{aligned}
\varrho^M(m) &= \varrho^M(\bar{E}'(m)) = \sum m_{00}\, \beta_1 S(m_1)_1 \otimes m_{01}\beta_2 S(m_1)_2 \\
\text{by (7.26)} \quad &= \sum m_{00}\, \delta^1 f^1 S(m_1)_1 \otimes m_{01}\delta^2 f^2 S(m_1)_2 \\
\text{by (7.17)} \quad &= \sum m_{00}\, \delta^1 S(m_{12}) f^1 \otimes m_{01}\delta^2 S(m_{11}) f^2 \\
\text{by (7.23)} \quad &= \sum m_{00}\, x^1 Y^1 \beta S((m_{12} x^3_2 X^3 Y^3) f^1 \otimes m_{01} x^2 X^1 Y^2_1 \beta S((m_{11} x^3_1 X^2 Y^2_2) f^2 \\
\text{by (7.6)} \quad &= \sum m_{00}\, x^1 Y^1 \beta S((m_{12} x^3_2 X^3 Y^3) f^1 \otimes m_{01} x^2 X^1 \varepsilon(Y^2) \beta S((m_{11} x^3_1 X^2) f^2 \\
\text{by (7.4)} \quad &= \sum m_{00}\, x^1 \beta S((m_1 x^3)_2 X^3) f^1 \otimes m_{01} x^2 X^1 \beta S((m_1 x^3)_1 X^2) f^2 \\
\text{by (14.2)} \quad &= \sum \tilde{x}^1_\rho\, m_0\, \beta S((\tilde{x}^3_\rho m_{12})_2 X^3) f^1 \otimes \tilde{x}^2_\rho m_{11} X^1 \beta S((\tilde{x}^3_\rho m_{12})_1 X^2) f^2 \\
\text{by (7.2)} \quad &= \sum \tilde{x}^1_\rho\, m_0\, \beta S((\tilde{x}^3_\rho)_2 X^3 m_{12})) f^1 \otimes \tilde{x}^2_\rho X^1 m_{111} \beta S((\tilde{x}^3_\rho)_1 X^2 m_{112})) f^2 \\
\text{by (7.6)} \quad &= \sum \tilde{x}^1_\rho\, m_0\, \beta S((\tilde{x}^3_\rho)_2 X^3 m_{12})) f^1 \otimes \tilde{x}^2_\rho X^1 \varepsilon(m_{11}) \beta S((\tilde{x}^3_\rho)_1 X^2) f^2 \\
&= \sum \tilde{x}^1_\rho\, m_0\, \beta S((\tilde{x}^3_\rho)_2 X^3 m_1) f^1 \otimes \tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2 \\
&= \sum \tilde{x}^1_\rho\, \bar{E}'(m)\, S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes \tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2 \\
(m \in M^{\overline{coH}}) \quad &= \sum \tilde{x}^1_\rho\, m\, S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes \tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2.
\end{aligned}$$

Conversely, if we have

$$\varrho^M(m) = \sum \tilde{x}^1_\rho\, m\, S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes \tilde{x}^2 X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2,$$

then

$$\begin{aligned}
\bar{E}'(m) &= \sum m_0\, \beta S(m_1) \\
&= \sum \tilde{x}^1_\rho\, m\, S((\tilde{x}^3_\rho)_2 X^3) f^1 \beta S(\tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2) \\
&= \sum \tilde{x}^1_\rho\, m\, S((\tilde{x}^3_\rho)_2 X^3) f^1 \beta S(f^2) S(\tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2)) \\
\text{by (7.27)} \quad &= \sum \tilde{x}^1_\rho\, m\, S((\tilde{x}^3_\rho)_2 X^3) S(\alpha) S(\tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2)) \\
&= \sum \tilde{x}^1_\rho\, m\, S(\tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) \alpha (\tilde{x}^3_\rho)_2 X^3) \\
&= \sum \tilde{x}^1_\rho\, m\, S(\tilde{x}^2_\rho X^1 \beta S(X^2) S((\tilde{x}^3_\rho)_1) \alpha (\tilde{x}^3_\rho)_2 X^3) \\
&= \sum \tilde{x}^1_\rho\, m\, S(\tilde{x}^2_\rho X^1 \beta S(X^2) \varepsilon(\tilde{x}^3_\rho) \alpha X^3) \\
&= \sum m\, S(X^1 \beta S(X^2) \alpha X^3) = m.
\end{aligned}$$

□

Similar to the BC-coinvariats in (13.4), the above proposition gives a characterization of coinvariants $M^{\overline{coH}}$ for $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$ as

$$M^{\overline{coH}} = \{m \in M | \varrho^M(m) = \sum \tilde{x}^1_\rho\, m\, S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes \tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2\}, \tag{15.4}$$

These coinvariants can be also expressed as

$$\begin{aligned} M^{\overline{coH}} &= Ke(\varrho^M - \{\sum(\tilde{x}^1_\rho \otimes \tilde{x}^2_\rho)\,(- \otimes 1_H)\,[S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2]\}) \\ &= Ke(\varrho^M - \{\sum(\tilde{x}^1_\rho \otimes \tilde{x}^2_\rho)\,(- \otimes 1_H)\,[(S \otimes S) \circ \tau \circ (\Delta(\tilde{x}^3_\rho)\, p_L)\, f]\}) \end{aligned}$$

where $p_L = \sum p^1_L \otimes p^2_L$ is defined in (7.28), $f = \sum f^1 \otimes f^2$ is the Drinfeld gauge element given in equation (7.24), and τ is the twist map $a \otimes b \mapsto b \otimes a$.

One can define a new left $\mathcal{A}$-module structure on $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$ by

$$a \triangleright m := \sum a_{(0)}\, m\, S(a_{(1)}), \tag{15.5}$$

for $a \in \mathcal{A}$, and $m \in M$, where $\rho(a) = \sum a_{(0)} \otimes a_{(1)}$. With this left $\mathcal{A}$-action, $M^{\overline{coH}}$ can be considered as a left $\mathcal{A}$-submodule of M. It is straightforward to see that for any morphism $g : M \to L$ in ${}_{\mathcal{A}}\mathbb{M}^H_H$, we have $g(M^{\overline{coH}}) \subseteq L^{\overline{coH}}$.
In this way, we obtain an alternative *coinvariants functor*

$$(-)^{\overline{coH}} : {}_{\mathcal{A}}\mathbb{M}^H_H \longrightarrow {}_{\mathcal{A}}\mathbb{M},$$

which we will show to be right adjoint to the comparison functor (see 14.3)

$$- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}^H_H, \quad N \mapsto (N \otimes_k H, \varrho_{N \otimes_k H}, \varrho^{N \otimes_k H}).$$

15.9. Proposition. (The adjoint pair $(- \otimes_k H, (-)^{\overline{coH}})$ for BC-type coinvariants). *Let H be quasi-Hopf algebra and $\mathcal{A}$ a right H-comodule algebra, $N \in {}_{\mathcal{A}}\mathbb{M}$ and $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$. Then there is a functorial isomorphism*

$${}_{\mathcal{A}}\mathrm{Hom}^H_H(N \otimes_k H, M) \xrightarrow{\psi_{N,M}} {}_{\mathcal{A}}\mathrm{Hom}(N, M^{\overline{coH}}), \quad f \longmapsto [n \mapsto f(\tilde{p}_\rho\,(n \otimes 1))],$$

with inverse map $\psi'_{N,M}$ given by

$$g \longmapsto [n \otimes h \mapsto \sum \tilde{q}^1_\rho\, g(n)\, S(\tilde{q}^2_\rho) h)].$$

This means that the functors

$${}_{\mathcal{A}}\mathbb{M} \xrightarrow{-\otimes_k H} {}_{\mathcal{A}}\mathbb{M}^H_H \xrightarrow{(-)^{\overline{coH}}} {}_{\mathcal{A}}\mathbb{M},$$

form an adjoint pair with unit and counit

$$\bar{\eta}_N : N \longrightarrow (N \otimes H)^{\overline{coH}}, \quad n \mapsto \tilde{p}_\rho\,(n \otimes 1),$$

$$\bar{\varepsilon}_M : M^{\overline{coH}} \otimes_k H \longrightarrow M, \quad m \otimes h \mapsto \sum \tilde{q}^1_\rho\, m\, S(\tilde{q}^2_\rho) h.$$

Proof. We show that ψ and ψ' are inverse to each other. For $n \in N, h \in H$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}^H_H(N \otimes_k H, M)$,

$$\begin{aligned} [(\psi' \circ \psi)(f)](n \otimes h) &= \sum \tilde{q}^1_\rho\, \psi(f)(n)\, S(\tilde{q}^2_\rho) h = \sum \tilde{q}^1_\rho\, f(\tilde{p}_\rho\,(n \otimes 1))\, S(\tilde{q}^2_\rho) h \\ {\scriptstyle f \text{ is } (\mathcal{A},H)\text{-bilinear}} &= f(\sum \rho(\tilde{q}^1_\rho) \tilde{p}_\rho\,(n \otimes 1))\, S(\tilde{q}^2_\rho) h \\ &= f(\sum \rho(\tilde{q}^1_\rho) \tilde{p}_\rho (1_{\mathcal{A}} \otimes S(\tilde{q}^2_\rho))\,(n \otimes h)) \\ {\scriptstyle \text{by } (10.6)} &= f(n \otimes h). \end{aligned}$$

Conversely, for $n \in N$ and $g \in {}_{\mathcal{A}}\mathrm{Hom}(N, M^{\overline{coH}})$,

$$\begin{aligned}
[(\psi \circ \psi')(g)](n) &= \psi'(g)(\tilde{p}_\rho\,(n \otimes 1)) = \sum \tilde{q}^1_\rho\, g(\tilde{p}^1_\rho\, n)\, S(\tilde{q}^2_\rho))\tilde{p}^2_\rho \\
{\scriptstyle g \text{ is left } \mathcal{A}\text{-linear}} \quad &= \sum \tilde{q}^1_\rho\, (\tilde{p}^1_\rho \triangleright g(n))\, S(\tilde{q}^2_\rho)\tilde{p}^2_\rho \\
&= \sum \tilde{q}^1_\rho (\tilde{p}^1_\rho)_{(0)}\, g(n)\, S((\tilde{p}^1_\rho)_{(1)}) S(\tilde{q}^2_\rho)\tilde{p}^2_\rho \\
&= \sum \tilde{q}^1_\rho (\tilde{p}^1_\rho)_{(0)} \cdot g(n) \cdot S(\tilde{q}^2_\rho (\tilde{p}^1_\rho)_{(1)})\tilde{p}^2_\rho \\
{\scriptstyle \text{by (10.5)}} \quad &= g(n).
\end{aligned}$$

□

In order to state the *Fundamental Theorem* for the category ${}_{\mathcal{A}}\mathbb{M}^H_H$ with BC-type coinvariants, we first show that the unit map $\bar{\eta}_N$ is an isomorphism. For this, we show that for any $N \in {}_{\mathcal{A}}\mathbb{M}$,

$$(N \otimes H)^{\overline{coH}} = \{\tilde{p}^1_\rho\, n \otimes \tilde{p}^2_\rho | n \in N\}.$$

For $n \otimes h \in (N \otimes H)^{\overline{coH}}$,

$$\begin{aligned}
\varrho^{N \otimes H}(n \otimes h) &= \sum \tilde{x}^1_\rho \cdot (n \otimes h) \cdot S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes \tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2 \\
&= \sum (\tilde{x}^1_\rho)_{(0)}\, n \otimes (\tilde{x}^1_\rho)_{(1)} h S((\tilde{x}^3_\rho)_2 X^3) f^1 \otimes \tilde{x}^2_\rho X^1 \beta S((\tilde{x}^3_\rho)_1 X^2) f^2.
\end{aligned}$$

On the other hand, $\varrho^{N \otimes H}(n \otimes h) = \sum \tilde{x}^1_\rho\, n \otimes \tilde{x}^2_\rho h_1 \otimes \tilde{x}^3_\rho h_2$.
Comparing this two values for $\varrho^{N \otimes H}(n \otimes h)$ and applying $id \otimes \varepsilon \otimes id$ on both sides, we obtain

$$n \otimes h = \sum \varepsilon(h)(\tilde{p}^1_\rho\, n \otimes \tilde{p}^2_\rho).$$

This shows that the unit map

$$\bar{\eta}_N : N \longrightarrow (N \otimes H)^{\overline{coH}}, \quad n \mapsto \tilde{p}_\rho\,(n \otimes 1),$$

is an isomorphism with inverse map $n \otimes h \mapsto n\varepsilon(h)$. This gives another proof fpr fully faithfulness of the comparison functor $- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}^H_H$ in this case (see 14.6).

15.10. Fundamental Theorem for ${}_{\mathcal{A}}\mathbb{M}^H_H$ with BC-type coinvariants. *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right comodule algebra and $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$. Consider $M^{\overline{coH}} \otimes H$ as an object in ${}_{\mathcal{A}}\mathbb{M}^H_H$ with the structures*

$$a \cdot (n \otimes h) \cdot h' = \sum a_1 \triangleright n \otimes a_2 h h', \quad \varrho'(n \otimes h) = \sum \tilde{x}^1_\rho \triangleright n \otimes \tilde{x}^2_\rho h_1 \otimes \tilde{x}^3_\rho h_2,$$

for $h, h' \in H$, $a \in \mathcal{A}$ and $n \in M^{\overline{coH}}$. Then the map

$$\bar{\varepsilon}_M : M^{\overline{coH}} \otimes H \longrightarrow M, \quad \bar{\varepsilon}_M(n \otimes h) = \sum \tilde{q}^1_\rho\, n\, S(\tilde{q}^2_\rho) h$$

is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}^H_H$ with inverse map $\bar{\varepsilon}'_M$ given by

$$\bar{\varepsilon}'_M(m) = \sum \bar{E}(m_0) \otimes m_1.$$

Proof. By 15.7, we have the isomorphism $E' : M^{\overline{coH}} \to M^{coH}$ in ${}_{\mathcal{A}}\mathbb{M}$ and tensoring it with H, we obtain

$$E' \otimes id_H : M^{\overline{coH}} \otimes H \to M^{coH} \otimes H,$$

as an isomorphism in ${}_{\mathcal{A}}\mathbb{M}_H^H$. By the Hausser-Nill version of the Fundamental Theorem for ${}_{\mathcal{A}}\mathbb{M}_H^H$ 13.3, there is an isomorphism

$$\varepsilon_M : M^{coH} \otimes H \to M, \quad m \otimes h \mapsto m\,h.$$

in ${}_{\mathcal{A}}\mathbb{M}_H^H$. Combining these two isomorphisms, we have the isomorphism

$$\bar{\varepsilon}_M = \varepsilon_M \circ (E' \otimes id) : M^{\overline{coH}} \otimes H \longrightarrow M^{coH} \otimes H \longrightarrow M,$$

$$\begin{aligned}
m \otimes h \mapsto E'(m) \otimes h \quad &\mapsto \quad E'(m)\,h = \sum \tilde{q}^1_\rho\, m_0\, \beta S(\tilde{q}^2_\rho m_1) h \\
&= \quad \sum \tilde{q}^1_\rho\, m_0\, \beta S(m_1) S(\tilde{q}^2_\rho) h = \sum \tilde{q}^1_\rho\, \bar{E}'(m)\, S(\tilde{q}^2_\rho) h \\
{\scriptstyle m \in M^{\overline{coH}}} \quad &= \quad \sum \tilde{q}^1_\rho\, m\, S(\tilde{q}^2_\rho) h.
\end{aligned}$$

The inverse map $\bar{\varepsilon}'_M$ can be also computed directly as

$$\begin{aligned}
\bar{\varepsilon}'_M(m) \quad &= \quad (\bar{E}' \otimes id)(\sum E'(m_0) \otimes m_1) = \sum \bar{E}'(E'(m_0)) \otimes m_1 \\
&= \quad \sum \bar{E}'(\tilde{q}^1_\rho\, m_{00}\, \beta S(\tilde{q}^2_\rho m_{01}) \otimes m_1 \\
&= \quad \sum \bar{E}'(\tilde{q}^1_\rho\, m_{00}) \varepsilon(\beta) \varepsilon(\tilde{q}^2_\rho m_{01}) \otimes m_1 = \sum \bar{E}'(m_0) \otimes m_1.
\end{aligned}$$

□

As shown in the proceding sections, for any comodule algebra over a quasi-Hopf algebra H, the comparison functor $- \otimes_k H$, given in 14.3 has three right adjoint functors, namely

$${}_{\mathcal{A}}\mathrm{Hom}_H^H(H \otimes H, -), \quad (-)^{coH} \text{ and } (-)^{\overline{coH}} : {}_{\mathcal{A}}\mathbb{M}_H^H \longrightarrow {}_{\mathcal{A}}\mathbb{M}.$$

These have to be isomorphic and we describe the isomorphisms explicitly.

15.11. Theorem. (Coinvariants for ${}_{\mathcal{A}}\mathbb{M}_H^H$ as Hom-functor). *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra, and $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$.*

(1) There is a functorial isomorphism in ${}_{\mathcal{A}}\mathbb{M}$,

$$\bar{\psi}_M : {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M) \longrightarrow M^{coH}, \quad f \longmapsto f(1_{\mathcal{A}} \otimes 1_H),$$

with inverse map $\bar{\psi}'_M$ given by

$$m \longmapsto [a \otimes h \mapsto E(a\,m)\,h],$$

for $a \in \mathcal{A}, h \in H$ and $m \in M^{coH}$.

(2) There is a functorial isomorphism in ${}_{\mathcal{A}}\mathbb{M}$,

$$\bar{\theta}_M : {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M) \longrightarrow M^{\overline{coH}}, \quad f \longmapsto f(\tilde{p}_\rho),$$

with inverse map $\bar{\theta}'_M$ *given by*

$$m \longmapsto [a \otimes h \mapsto E(a\, m)\, h],$$

for $a \in \mathcal{A}, h \in H$ *and* $m \in M^{\overline{coH}}$.

Proof. (1) If we substitute $N = \mathcal{A}$ in the isomorphism in 15.3, we obtain for $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$ the isomorphisms

$$\bar{\psi}_M : {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M) \overset{\psi_{\mathcal{A},M}}{\longrightarrow} {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, M^{coH}) \cong M^{coH},$$

$$f \longmapsto [a \mapsto f(a \otimes 1_H)] \mapsto f(1_{\mathcal{A}} \otimes 1_H),$$

for $a \in \mathcal{A}$. The inverse map $\bar{\psi}'_M$ is obtained as the composition

$$M^{coH} \cong {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, M^{coH}) \overset{\psi'_{\mathcal{A},M}}{\longrightarrow} {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M),$$

$$m \mapsto [a \mapsto a \blacktriangleright m = E'(a\, m)] \mapsto [a \otimes h \mapsto E'(a\, m)\, h],$$

for $a \in \mathcal{A}$, $h \in H$ and $m \in M^{coH}$. Here, $\psi_{\mathcal{A},M}$ is the isomorphism given in 15.3 and $\psi'_{\mathcal{A},M}$ is its inverse.
It remains to show that $\bar{\psi}_M$ is right $\mathcal{A}$-linear: For $a \in \mathcal{A}$ and $f \in {}_H\mathrm{Hom}_{\mathcal{A}}^H(\mathcal{A} \otimes H, M)$,

$$\begin{aligned}
a \blacktriangleright \bar{\psi}_M(f) &= E(a\, f(1_{\mathcal{A}} \otimes 1_H)) = \sum E(f(a_{(0)} \otimes a_{(1)})) \\
&= \sum \tilde{q}^1_\rho\, f(a_{(0)} \otimes a_{(1)})_0\, \beta S(\tilde{q}^2_\rho f(a_{(0)} \otimes a_{(1)})_1) \\
\text{\scriptsize f is H-colinear} \quad &= \sum \tilde{q}^1_\rho\, f(\tilde{x}^1_\rho a_{(0)} \otimes \tilde{x}^2_\rho a_{(1)_1} \beta S(\tilde{q}^2_\rho \tilde{x}^3_\rho a_{(1)_2})) \\
\text{\scriptsize f is $\mathcal{A}$-linear} \quad &= \sum f(\rho(\tilde{q}^1_\rho)\, (\tilde{x}^1_\rho a_{(0)} \otimes \tilde{x}^2_\rho a_{(1)_1}) \beta S(a_{(1)_2}) S(\tilde{x}^3_\rho) S(\tilde{q}^2_\rho)) \\
\text{\scriptsize by (7.6)} \quad &= f(\sum \rho(\tilde{q}^1_\rho)\, (\tilde{x}^1_\rho a \otimes \tilde{x}^2_\rho \beta S(\tilde{x}^3_\rho) S(\tilde{q}^2_\rho)) \\
&= f(\sum \rho(\tilde{q}^1_\rho) \tilde{p}_\rho (1_{\mathcal{A}} \otimes S(\tilde{q}^2_\rho))(a \otimes 1)) \\
\text{\scriptsize by (10.6)} \quad &= f(a \otimes 1) = (a \cdot f)(1_{\mathcal{A}} \otimes 1_H) = \bar{\psi}_M(a \cdot f).
\end{aligned}$$

(2) If we set $N = \mathcal{A}$ in the isomorphism given in 15.9, we obtain the isomorphisms

$$\bar{\theta}_M : {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M) \overset{\psi_{\mathcal{A},M}}{\longrightarrow} {}_{\mathcal{A}}\mathrm{Hom}(H, M^{\overline{coH}}) \cong M^{\overline{coH}},$$

$$f \longmapsto [a \mapsto \bar{E}'(f(a \otimes 1)) = f(\tilde{p}_\rho\, (a \otimes 1))] \mapsto \bar{E}'(f(1_{\mathcal{A}} \otimes 1_H)) = f(\tilde{p}_\rho),$$

for all $a \in \mathcal{A}$. The inverse map $\bar{\theta}'_M$ is obtained as the composition

$$\bar{\theta}'_M : M^{\overline{coH}} \cong {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, M^{\overline{coH}}) \overset{\psi'_{\mathcal{A},M}}{\longrightarrow} {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M),$$

$$\begin{aligned}
m \longmapsto [a \mapsto a \triangleright m = \bar{E}(a\, m)] \longmapsto \{a \otimes b \ &\mapsto \ \sum \tilde{q}^1_\rho\, \bar{E}'(a\, m)\, S(\tilde{q}^2_\rho) h \\
&= \ E'(a\, m)\, h\},
\end{aligned}$$

for $a \in \mathcal{A}$, $h \in H$ and $m \in M^{\overline{coH}}$. Here, $\psi_{\mathcal{A},M}$ is the isomorphism given in 15.9 and $\psi'_{\mathcal{A},M}$ is its inverse.
Similar to the part (1) and considering the left $\mathcal{A}$-action $\triangleright$ on $M^{\overline{coH}}$, we must show that $\bar{\theta}_M$ is left $\mathcal{A}$-linear: For $a \in \mathcal{A}$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M)$,

$$\begin{aligned}
a \triangleright \bar{\theta}_M(f) &= \bar{E}(a\, f(1_{\mathcal{A}} \otimes 1_H)) = \sum \bar{E}(f(a_{(0)} \otimes a_{(1)})) \\
&= \sum f(a_{(0)} \otimes a_{(1)})_0\, \beta S(f(a_{(0)} \otimes a_{(1)})_1) \\
\text{\small f is H-colinear} \quad &= \sum f(\tilde{x}_\rho^1 a_{(0)} \otimes \tilde{x}_\rho^2 a_{(1)_1} \beta S(\tilde{x}_\rho^3 a_{(1)_2})) \\
&= f(\sum \tilde{x}_\rho^1\, a \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3)) = f(\sum \tilde{p}_\rho^1\, a \otimes \tilde{p}_\rho^2) \\
&= (a \cdot f)(\tilde{p}_\rho) = \bar{\theta}_M(a \cdot f).
\end{aligned}$$

□

Remark. Part (2) can be proved also by composing the isomorphism

$${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M) \xrightarrow{\bar{\psi}_M} M^{coH}, \quad f \longmapsto f(1_{\mathcal{A}} \otimes 1_H),$$

in the part (1), with the isomorphism $\bar{E}' : M^{coH} \to M^{\overline{coH}}$. This induces the isomorphism

$${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, M) \xrightarrow{\bar{\psi}_M} M^{coH} \xrightarrow{\bar{E}} M^{\overline{coH}},$$

given by

$$\begin{aligned}
f \longmapsto f(1 \otimes 1) \quad &\longmapsto \quad \bar{E}(f(1 \otimes 1)) \\
&= \quad \sum f(1 \otimes 1)_0\, \beta S(f(1 \otimes 1)_1) \\
\text{\small by H-colinearity of f} \quad &= \quad \sum f(\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2)\, \beta S(\tilde{x}_\rho^3) \\
\text{\small by H-linearity of f} \quad &= \quad \sum f(\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3)) = f(\tilde{p}_\rho).
\end{aligned}$$

The inverse map can be computed as

$$\begin{aligned}
m \xmapsto{\theta'} \{a \otimes h \quad &\mapsto \quad E(a\, E(m))\, h \\
&= \quad \sum E([a_{(0)} \blacktriangleright E(m)]\, a_{(1)})\, h \\
&= \quad [\sum E(a_{(0)} \blacktriangleright E(m)) \varepsilon(a_{(1)})]\, h \\
&= \quad [E(a \blacktriangleright E(m))]\, h \\
&= \quad [E(E(a\, m))]\, h = E(a\, m)\, h\},
\end{aligned}$$

for $a \in \mathcal{A}$, $h \in H$ and $m \in M^{\overline{coH}}$.

16 The category ${}_H\mathbb{M}_{\mathcal{A}}^H$

By symmetry and following Bulacu-Caenepeel [8] and Bulacu-Torrecillas [12], we can consider the category of two-sided Hopf modules from the right hand side.
As mentioned for the left hand version in section 14, for a quasi-bialgebra H and a right H-comodule algebra $(\mathcal{A}, \rho, \phi_\rho)$, the monoidal category ${}_H\mathbb{M}_H$ acts from the right

side on the category ${}_H\mathbb{M}_{\mathcal{A}}$ of $(H,\mathcal{A})$-bimodules. Thus we can consider the category of right comodules in ${}_H\mathbb{M}_{\mathcal{A}}$ over the coalgebra H in ${}_H\mathbb{M}_H$. This comodule category has been defined by Bulacu and Caenepeel in [9] as follows.

16.1. (Right) Two-sided Hopf modules. Let H be a quasi-bialgebra and $(\mathcal{A},\rho,\phi_\rho)$ a right H-comodule algebra. A **two-sided $(H,\mathcal{A})$-Hopf module** is an $(H,\mathcal{A})$-bimodule M, together with a k-linear map

$$\varrho^M : M \longrightarrow M\otimes H, \quad \varrho^M(m)=\sum m_0\otimes m_1,$$

satisfying the relations

$$\begin{aligned} (id_M\otimes\varepsilon)\circ\varrho^M &= id_M, \\ \phi\cdot(\varrho^M\otimes id_H)(\varrho^M(m)) &= (id_M\otimes\Delta)(\varrho^M(m))\cdot\phi_\rho, \\ \varrho^M(h\,m) &= \sum h_1\,m_0\otimes h_2 m_1, \\ \varrho^M(m\,a) &= \sum m_0\,a_{(0)}\otimes m_1 a_{(1)}, \end{aligned}$$

for all $m\in M$, $h\in H$, and $a\in\mathcal{A}$. The category of two-sided $(H,\mathcal{A})$-Hopf modules and left H-linear, right $\mathcal{A}$-linear, and right H-colinear maps is denoted by ${}_H\mathbb{M}^H_{\mathcal{A}}$.

As in 14.2, we find a subgenerator for ${}_H\mathbb{M}^H_{\mathcal{A}}$.

16.2. Proposition. (Subgenerator for ${}_H\mathbb{M}^H_{\mathcal{A}}$). *Let H be a quasi-bialgebra and $(\mathcal{A},\rho,\phi_\rho)$ a right H-comodule algebra. Then*

(1) For any $N\in\mathbb{M}_{\mathcal{A}}$, we have $N\otimes H\in{}_H\mathbb{M}^H_{\mathcal{A}}$ with structure maps defined for $h,h'\in H$, $n\in N$ and $a\in\mathcal{A}$, by

$$h'\cdot(n\otimes h)=n\otimes h'h, \qquad (n\otimes h)\cdot a=\sum na_{(0)}\otimes ha_{(1)}, \tag{16.1}$$

and

$$\varrho^{N\otimes H}(n\otimes h)=\sum n\tilde{X}^1_\rho\otimes h_1\tilde{X}^2_\rho\otimes h_2\tilde{X}^3_\rho=(id\otimes\Delta)(n\otimes h)\cdot\phi_\rho. \tag{16.2}$$

(2) If $g:N_1\to N_2$ is an (epi-)morphism in $\mathbb{M}_{\mathcal{A}}$, then

$$g\otimes id_H : N_1\otimes H\longrightarrow N_2\otimes H$$

is an (epi-)morphism in ${}_H\mathbb{M}^H_{\mathcal{A}}$.

(3) Endowed with the structure given, for $h,h'\in H$ and $a,a'\in\mathcal{A}$, by

$$h\cdot(a\otimes h')=a\otimes hh', \quad (a\otimes h)\cdot a'=\sum aa'_{(0)}\otimes ha'_{(1)},$$

$$\varrho^{\mathcal{A}\otimes H}(a\otimes h)=\sum a\tilde{X}^1\otimes h_1\tilde{X}^2\otimes h_2\tilde{X}^3,$$

$\mathcal{A}\otimes H\in{}_H\mathbb{M}^H_{\mathcal{A}}$ and it is a subgenerator *for the category ${}_H\mathbb{M}^H_{\mathcal{A}}$.*

Proof. The parts (1) and (2) are straightforward to see.
(3) Using a similar approach as in sections 12 and 14, we can see that for any $M \in {}_H\mathbb{M}_{\mathcal{A}}^H$, the right $\mathcal{A}$-module M can be considered as a homomorphic image of $\mathcal{A}^{(\Lambda)}$, for some cardinal number Λ. Therefore $M \otimes H$ is a homomorphic image of

$$\mathcal{A}^{(\Lambda)} \otimes H \cong (\mathcal{A} \otimes H)^{(\Lambda)}.$$

For any $M \in {}_H\mathbb{M}_{\mathcal{A}}^H$, the coaction $\varrho^M : M \to M \otimes H$ is a morphism in the category ${}_H\mathbb{M}_{\mathcal{A}}^H$, so we can consider M as a subobject of $M \otimes H$, which is generated by the object $\mathcal{A} \otimes H \in {}_H\mathbb{M}_{\mathcal{A}}^H$. □

The above proposition give rise to

16.3. The comparison functor $- \otimes_k H : \mathbb{M}_{\mathcal{A}} \to {}_H\mathbb{M}_{\mathcal{A}}^H$. Let H be a quasi-bialgebra and $(\mathcal{A}, \varrho, \phi_\varrho)$ a right H-comodule algebra. By 16.2, for any $N \in \mathbb{M}_{\mathcal{A}}$, $N \otimes H \in {}_H\mathbb{M}_{\mathcal{A}}^H$ with $(H, \mathcal{A})$-bimodule structure given in (16.1) and H-comodule structure map given in (16.2). This gives rise to the **comparison functor**

$$- \otimes_k H : \mathbb{M}_{\mathcal{A}} \longrightarrow {}_H\mathbb{M}_{\mathcal{A}}^H, \quad N \mapsto (N \otimes H, \varrho_{N \otimes H}, \varrho^{N \otimes H}).$$

16.4. Two-sided Hopf modules as comodules over a comonad. Let H be a quasi-bialgebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra, $N \in {}_H\mathbb{M}_{\mathcal{A}}$, and $V \in {}_H\mathbb{M}_H$. Then the coaction

$$\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes_k H, \quad \rho(a) = \sum a_{(0)} \otimes a_{(1)},$$

induces an $(H, \mathcal{A})$-bimodule structure on $N \otimes_k V$ given, for $h \in H$, $a \in \mathcal{A}$, $v \in V$, and $n \in N$ by

$$h \cdot (n \otimes v) \cdot a = \sum h_1 \, n \, a_{(0)} \otimes h_2 \, v \, a_{(1)} = \Delta(h) \, (n \otimes v) \, \rho(a).$$

In this way, for any $V \in {}_H\mathbb{M}_H$, we get an endofunctor

$$- \otimes_k V : {}_H\mathbb{M}_{\mathcal{A}} \longrightarrow \mathbb{M}_{\mathcal{A}}, \quad N \mapsto N \otimes_k V,$$

with the $(H, \mathcal{A})$-bimodule structure on $N \otimes_k V$ given above. In particular, for $V = H$, we obtain the endofunctor

$$G := - \otimes_k H : {}_{\mathcal{A}}\mathbb{M}_H \longrightarrow {}_{\mathcal{A}}\mathbb{M}_H, \quad N \mapsto N \otimes_k H,$$

with the $(H, \mathcal{A})$-bimodule structure on $N \otimes_k H$ given by

$$h' \cdot (n \otimes h) \cdot a = \sum h'_1 \, n \, a_{(0)} \otimes h'_2 \, v \, a_{(1)} = \Delta(h) \, (n \otimes v) \, \rho(a).$$

for all $h, h' \in H$, $a \in \mathcal{A}$, and $n \in N$. Similar to the case ${}_{\mathcal{A}}\mathbb{M}_H^H$ (see 14.5), we have

16.5. Theorem. ($- \otimes_k H$ as a comonad on ${}_H\mathbb{M}_{\mathcal{A}}$). *Let $(H, \Delta, \varepsilon, \phi)$ be a quasi-bialgebra, $(\mathcal{A}, \rho, \phi_\rho)$ be a right H-comodule algebra. Then*

(1) The endofunctor $- \otimes_k H : {}_H\mathbb{M}_{\mathcal{A}} \to {}_H\mathbb{M}_{\mathcal{A}}$ is a comonad on ${}_H\mathbb{M}_{\mathcal{A}}$ with the comultiplication δ defined for $N \in {}_H\mathbb{M}_{\mathcal{A}}$ by

$$\begin{aligned} \delta_N : N \otimes H &\longrightarrow (N \otimes H) \otimes H, \\ n \otimes h &\mapsto \sum x^1 \, n \, \tilde{X}_\rho^1 \otimes x^2 h_1 \tilde{X}_\rho^2 \otimes x^3 h_2 \tilde{X}_\rho^3 \\ &= \phi^{-1} \cdot (id \otimes \Delta)(n \otimes h) \cdot \phi_\rho, \end{aligned}$$

and counit ϵ defined by $\epsilon_N = id_N \otimes \varepsilon : N \otimes H \to N$.

(2) *The category of two-sided Hopf modules* ${}_H\mathbb{M}_{\mathcal{A}}^H$ *is isomorphic to the Eilenberg-Moore comodule category* $({}_H\mathbb{M}_{\mathcal{A}})^{-\otimes H}$.

The following lemma helps to find a right adjoint to the comparison functor.

16.6. Lemma. (The functor ${}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, -)$). *Let* H *be a quasi-bialgebra,* $(\mathcal{A}, \rho, \phi_\rho)$ *a right* H*-comodule algebra, and* $V \in {}_{\mathcal{A}}\mathbb{M}_{\mathcal{A}}$.

(1) *If* $M \in {}_H\mathbb{M}_{\mathcal{A}}$, *then* ${}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, M) \in \mathbb{M}_{\mathcal{A}}$ *with the right* $\mathcal{A}$*-module structure given, for* $h \in H$, $a \in \mathcal{A}$ *and* $v \in V$, *by*

$$(f \cdot a)(v \otimes h) = f(a\, v \otimes h).$$

In this way, we get the Hom-functor ${}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, -) : {}_H\mathbb{M}_{\mathcal{A}} \to \mathbb{M}_{\mathcal{A}}$.
In particular, if $M \in {}_H\mathbb{M}_{\mathcal{A}}^H$ *then* ${}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, M) \in \mathbb{M}_{\mathcal{A}}$ *with the right* $\mathcal{A}$*-module structure given above, and we obtain the Hom-functor*

$${}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, -) : {}_H\mathbb{M}_{\mathcal{A}}^H \to \mathbb{M}_{\mathcal{A}}.$$

(2) *Let* $V \in {}_{\mathcal{A}}\mathbb{M}_{\mathcal{A}}$ *and* $N \in \mathbb{M}_{\mathcal{A}}$. *Then*

(i) $\psi : {}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, N \otimes H) \longrightarrow {}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, N), \quad f \mapsto (id \otimes \varepsilon) \circ f$, *is an isomorphism in* $\mathbb{M}_{\mathcal{A}}$ *with inverse map* ψ' *given by*

$$g \mapsto (g \otimes id_H) \circ \varrho^{V \otimes H}.$$

(ii) $\theta : {}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, N) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(V, N), \quad f \mapsto f(- \otimes 1_H)$, *is an isomorphism in* $\mathbb{M}_{\mathcal{A}}$ *with inverse map* θ' *given by*

$$g \mapsto [v \otimes h \mapsto \varepsilon(h) g(v)].$$

(iii) *There is a right* $\mathcal{A}$*-module isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(V, N) \longrightarrow {}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, N \otimes H), \quad g \longmapsto g \otimes id,$$

with inverse map given for $f \in {}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, N \otimes H)$ *by*

$$f \mapsto (id \otimes \varepsilon) \circ f(- \otimes 1_H).$$

This means that the comparison functor $- \otimes_k H$ *is full and faithful.*

Here, we consider the left H*-module structure of* N *to be the trivial one.*

Proof. (1) For all $a \in \mathcal{A}$ and $f \in {}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, M)$, it is easy to see that $f \cdot a$ is an $(H, \mathcal{A})$-bilinear map. In this way, we have ${}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, M) \in \mathbb{M}_{\mathcal{A}}$, and we obtain a functor

$${}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, -) : {}_H\mathbb{M}_{\mathcal{A}} \longrightarrow \mathbb{M}_{\mathcal{A}}.$$

In case $M \in {}_H\mathbb{M}_{\mathcal{A}}^H$ and $f \in {}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, M)$, the H-colinearity of $f \cdot a$ follows from the H-colinearity of f itself, and ${}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, M) \in \mathbb{M}_{\mathcal{A}}$. Thus, we obtain the functor

$${}_H\mathrm{Hom}_{\mathcal{A}}^H(V \otimes H, -) : {}_H\mathbb{M}_{\mathcal{A}}^H \longrightarrow \mathbb{M}_{\mathcal{A}}.$$

(2) (i) As seen in 16.5, the functor $-\otimes_k H : {}_H\mathbb{M}_{\mathcal{A}} \to {}_H\mathbb{M}_{\mathcal{A}}$ is a comonad and the category ${}_H\mathbb{M}^H_{\mathcal{A}}$ of two-sided Hopf modules is just the Eilenberg-Moore comodule category $({}_H\mathbb{M}_{\mathcal{A}})^{-\otimes H}$ over this comonad. Now, considering the comparison functor $-\otimes H : {}_H\mathbb{M}_{\mathcal{A}} \to {}_H\mathbb{M}^H_{\mathcal{A}}$ as the free functor which is right adjoint to the forgetful functor, by 4.8, we obtain the isomorphism of part (i).

(ii) For $f \in {}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, N)$, $h, a \in H$ and $v \in V$,

$$\begin{aligned}
[\theta(f)(v)]\, a &= [f(v \otimes 1_H)]\, a \\
\text{f is right $\mathcal{A}$-linear} \quad &= f(\sum v\, a_{(0)} \otimes a_{(1)}) \\
\text{f is left H-linear} \quad &= \sum a_{(1)}\, f(v\, a_{(0)} \otimes 1_H) \\
&= \sum \varepsilon(a_{(1)}) f(v\, a_{(0)} \otimes 1_H) \\
&= f(v\, a \otimes 1_H) = \theta(f)(v\, a).
\end{aligned}$$

This means that $\theta(f) \in \mathrm{Hom}_{\mathcal{A}}(V, N)$. Also, it is straightforward to show that for $f \in \mathrm{Hom}_{\mathcal{A}}(V, N)$, we have $\theta'(g) \in {}_H\mathrm{Hom}_{\mathcal{A}}(V \otimes H, N)$. Right $\mathcal{A}$-linearity and bijectivity of θ are easy to see.

(iii) This is just the composition of the isomorphisms ψ' and θ' given in (i) and (ii). □

16.7. Corollary. *Let H be a quasi-bialgebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra.*

(1) For $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$ we have a right $\mathcal{A}$-module structure on ${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, M)$ given for $h \in H$, $a, a' \in \mathcal{A}$ and $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, M)$ by

$$(f \cdot a')(a \otimes h) = f(a'\, a \otimes h).$$

(2) For $N \in \mathbb{M}_{\mathcal{A}}$ the morphism

$$\eta_N : N \longrightarrow {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, N \otimes H), \quad n \longmapsto [a \otimes h \mapsto n\, a \otimes h],$$

is an isomorphism with inverse map η'_N given, for $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, N \otimes H)$, by

$$\eta'_N(f) = (id \otimes \varepsilon) \circ f(1_{\mathcal{A}} \otimes 1_H).$$

Proof. (1) Follows directly from the Lemma 16.6 by taking $V = \mathcal{A}$.

(2) Composition of the isomorphisms ψ' and θ' for $V = \mathcal{A}$ gives rise to the isomorphism

$$N \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, N) \cong {}_H\mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes H, N) \cong {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, N \otimes H).$$

Using part (1), this composition yields the isomorphism η_N with the given inverse map η'_N. □

Now we show that the Hom-functor ${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, -) : {}_H\mathbb{M}^H_{\mathcal{A}} \longrightarrow \mathbb{M}_{\mathcal{A}}$ is a right adjoint to the comparison functor in 16.3.

16.8. Theorem. (Hom-tensor adjunction for ${}_H\mathbb{M}^H_{\mathcal{A}}$). *Let H be a quasi-bialgebra, $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$, and $N \in \mathbb{M}_{\mathcal{A}}$. Then there is a functorial isomorphism*

$$\Omega := \Omega_{N,M} : {}_H\mathrm{Hom}^H_{\mathcal{A}}(N \otimes H, M) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(N, {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, M)),$$

$$f \longmapsto \{n \mapsto [a \otimes h \mapsto f(n\, a \otimes h)]\},$$

with inverse map $\Omega'_{N,M}$ given by

$$g \longmapsto \{n \otimes h \mapsto g(n)(1_{\mathcal{A}} \otimes h)\}.$$

This means that the comparison functor

$$- \otimes_k H : \mathbb{M}_{\mathcal{A}} \longrightarrow {}_H\mathbb{M}^H_{\mathcal{A}}, \quad N \mapsto (N \otimes H, \varrho_{N \otimes H}, \varrho^{N \otimes H}),$$

is left adjoint to the Hom-*functor*

$${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, -) : {}_H\mathbb{M}^H_{\mathcal{A}} \longrightarrow \mathbb{M}_{\mathcal{A}},$$

with unit and counit given by

$$\eta_N : N \longrightarrow {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, N \otimes H) \quad n \mapsto [a \otimes h \mapsto n\, a \otimes h],$$

$$\varepsilon_M : {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, M) \otimes H \longrightarrow M, \quad f \otimes h \mapsto f(1_{\mathcal{A}} \otimes h).$$

Furthermore, the comparison functor $- \otimes_k H : \mathbb{M}_{\mathcal{A}} \to {}_H\mathbb{M}^H_{\mathcal{A}}$ is full and faithful.

Proof. First we show that for any $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(N \otimes H, M)$, $\Omega(f)$ is right $\mathcal{A}$-linear. For $h \in H, a, a' \in \mathcal{A}$ and $n \in N$,

$$[(\Omega(f)(n)) \cdot a'](a \otimes h) = \Omega(f)(n)(a'a \otimes h) = f(n\, a'a \otimes h) = [\Omega(f)(n\, a')](a \otimes h).$$

So we have $\Omega(f) \in \mathrm{Hom}_{\mathcal{A}}(N, {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, M))$.
For any $g \in \mathrm{Hom}_{\mathcal{A}}(N, {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, M))$, we show that $\Omega'(g) \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(N \otimes H, M)$.

i) $\Omega'(g)$ is right $\mathcal{A}$-linear. For $a \in \mathcal{A}$ and $n \in N$,

$$\begin{aligned} \Omega'(g)((n \otimes h)\, a) &= \sum \Omega'(g)(n\, a_{(0)} \otimes h a_{(1)}) = \sum g(n\, a_{(0)})(1_{\mathcal{A}} \otimes h a_{(1)}) \\ \underset{g \text{ is right } \mathcal{A}\text{-linear}}{} &= \sum (g(n) \cdot a_{(0)})(1_{\mathcal{A}} \otimes h a_{(1)}) = \sum g(n)(a_{(0)} \otimes h a_{(1)}) \\ &= g(n)((1 \otimes h)\, \rho(a)) = [g(n)(1 \otimes h)]\, a \\ &= [\Omega'(g)(n \otimes h)]\, a. \end{aligned}$$

ii) It can be easily seen that $\Omega'(g)$ is also left H-linear.

iii) For the right H-colinearity of $\Omega'(g)$ we show that

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = \sum (\Omega'(g) \otimes id)(n\, \tilde{X}^1_\rho \otimes h_1 \tilde{X}^2_\rho \otimes h_2 \tilde{X}^3_\rho).$$

By colinearity of $g(n)$,

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = \varrho^M(g(n)(1 \otimes h)) = g(n)(\tilde{X}^1_\rho \otimes h_1 \tilde{X}^2_\rho) \otimes h_2 \tilde{X}^3_\rho.$$

On the other hand,

$$\begin{aligned}(\Omega'(g)\otimes id)(\sum n\,\tilde{X}^1_\rho\otimes h_1\tilde{X}^2_\rho\otimes h_2\tilde{X}^3_\rho) &= \sum g(n\,\tilde{X}^1_\rho)(1\otimes h_1\tilde{X}^2_\rho)\otimes h_2\tilde{X}^3_\rho\\ {\scriptstyle g\text{ is }\mathcal{A}\text{-linear}} &= \sum [g(n)\cdot\tilde{X}^1_\rho](1\otimes h_1\tilde{X}^2_\rho)\otimes h_2\tilde{X}^3_\rho\\ &= \sum g(n)(\tilde{X}^1_\rho\otimes h_1\tilde{X}^2_\rho)\otimes h_2\tilde{X}^3_\rho.\end{aligned}$$

This shows the H-colinearity of $\Omega'(g)$.

Now we show that Ω and Ω' are inverse to each other. For all $n\in N, h\in H$ and $f\in {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A}\otimes H, M)$,

$$(\Omega'\circ\Omega(f))(n\otimes h)=(\Omega(f))(n)(1\otimes h)=f(1_{\mathcal{A}}\,n\otimes h)=f(n\otimes h).$$

Conversely, for all $h\in H, n\in N, a\in\mathcal{A}$ and $g\in\mathrm{Hom}_{\mathcal{A}}(N, {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A}\otimes H, M))$,

$$\begin{aligned}\{[(\Omega\circ\Omega')(g)](n)\}(a\otimes h) &= (\Omega'(g))(n\,a\otimes h)=g(n\,a)(1_{\mathcal{A}}\otimes h)\\ {\scriptstyle g\text{ is }\mathcal{A}\text{-linear}} &= [g(n)\,a](1_{\mathcal{A}}\otimes h)=g(n)(a\otimes h).\end{aligned}$$

i.e. $\Omega\circ\Omega'(g)=g$. It is also straightforward to see that Ω is functorial in both components M and N.

The fully faithfulness of the comparison functor follows from Lemma 16.6. □

17 Coinvariants for ${}_H\mathbb{M}^H_{\mathcal{A}}$

17.1. Hausser-Nill-type coinvariants for ${}_H\mathbb{M}^H_{\mathcal{A}}$. Let H be a quasi-Hopf algebra, $(\mathcal{A},\rho,\phi_\rho)$ a right H-comodule algebra, and $M\in {}_H\mathbb{M}^H_{\mathcal{A}}$. We define a projection $E: M\to M$ for any $m\in M$ by

$$E(m):=\sum S^{-1}(\alpha m_1\tilde{p}^2_\rho)\,m_0\,\tilde{p}^1_\rho, \tag{17.1}$$

where $\tilde{p}_\rho=\sum p^1_\rho\otimes p^2_\rho$ is defined in (10.1). We define a new right action ◂ of $\mathcal{A}$ on M given for elements $a\in\mathcal{A}$ and $m\in M$ by

$$m ◂ a := E(m\,a)=\sum S^{-1}(\alpha m_1 a_{(1)}\tilde{p}^2_\rho)\,m_0\,a_{(0)}\tilde{p}^1_\rho. \tag{17.2}$$

The projection E and the action ◂ have the following properties:
Proposition. *For a quasi-Hopf algebra H and a right H-comodule algebra $(\mathcal{A},\rho,\phi_\rho)$, let $M\in {}_H\mathbb{M}^H_{\mathcal{A}}$, $m\in M$, $a\in\mathcal{A}$ and $h\in H$. Then with the above notations we have*

(i) $E(h\,m)=\varepsilon(h)E(m)$,

(ii) $E^2=E$,

(iii) $E(m) ◂ a=E(m\,a)=m ◂ a$,

(iv) $m ◂ (ab)=(m ◂ a) ◂ b$,

(v) $E(m)\,a=\sum a_{(1)}\,[E(m) ◂ a_{(0)}]$,

(vi) $\sum m_1\, E(m_0) = m$,

(vii) $\sum E(E(m)_0) \otimes E(m)_1 = E(m) \otimes 1$.

Proof.

$$\begin{aligned} \text{(i)} \quad E(h\,m) &= \sum S^{-1}(\alpha (h\,m)_1 \tilde{p}^2_\rho)(h\,m)_0\, \tilde{p}^1_\rho \\ &= \sum S^{-1}(\alpha h_2 m_1 \tilde{p}^2_\rho) h_1\, m_0\, \tilde{p}^1_\rho \\ &= \sum S^{-1}(m_1 \tilde{p}^2_\rho) S^{-1}(\alpha h_2) h_1\, m_0\, \tilde{p}^1_\rho \\ &= \sum S^{-1}(m_1 \tilde{p}^2_\rho) \varepsilon(h) S^{-1}(\alpha)\, m_0\, \tilde{p}^1_\rho \\ &= \varepsilon(h) \sum S^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho = \varepsilon(h) E(m). \end{aligned}$$

(ii) We use part (i) to compute

$$\begin{aligned} E^2(m) &= E(\sum S^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho) \\ \text{by (i)} \quad &= \sum \varepsilon(S^{-1}(\alpha m_1 \tilde{p}^2_\rho)) E(m_0\, \tilde{p}^1_\rho) \\ &= \sum \varepsilon(\alpha)\varepsilon(m_1)\varepsilon(\tilde{p}^2_\rho) E(m_0\, \tilde{p}^1_\rho) = E(m). \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad E(m) \blacktriangleleft a &= E(E(m)\, a) = \sum E(\sum S^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho a) \\ &= \sum \varepsilon(S^{-1}(\alpha m_1 \tilde{p}^2_\rho)) E(m_0\, \tilde{p}^1_\rho a) \\ &= \sum \varepsilon(\alpha)\varepsilon(m_1)\varepsilon(\tilde{p}^2_\rho) E(m_0\, \tilde{p}^1_\rho a) = E(m\,a) = m \blacktriangleleft a. \end{aligned}$$

$$\begin{aligned} \textit{(iv)} \quad m \blacktriangleleft (ab) = E(m\,(ab)) = E((m\,a)\,b) &= E(m\,a) \blacktriangleleft b \\ &= (m \blacktriangleleft a) \blacktriangleleft b. \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad E(m)\,a &= \sum S^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho a \\ \text{by (10.3)} \quad &= \sum S^{-1}(\alpha m_1 a_{(0)_{(1)}} \tilde{p}^2_\rho S(a_{(1)})\, m_0\, a_{(0)_{(0)}} \tilde{p}^1_\rho \\ &= \sum a_{(1)} S^{-1}(\alpha (m\, a_{(0)})_1 \tilde{p}^2_\rho)\, (m\, a_{(0)})_{(0)} \tilde{p}^1_\rho \\ &= \sum a_{(1)}\, E(m\, a_{(0)}) \\ \text{by (iii)} \quad &= \sum a_{(1)}\, [E(m) \blacktriangleleft a_{(0)}]. \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad m_1\, E(m_0) &= \sum m_1 S^{-1}(\alpha m_{01} \tilde{p}^2_\rho)\, m_{00}\, \tilde{p}^1_\rho \\ &= \sum m_1 \tilde{x}^3_\rho S^{-1}(\tilde{x}^2_\rho \beta) S^{-1}(\alpha m_{01})\, m_{00}\, \tilde{x}^1_\rho \\ &= \sum m_1 \tilde{x}^3_\rho S^{-1}(\alpha m_{01} \tilde{x}^2_\rho \beta)\, m_{00}\, \tilde{x}^1_\rho \\ &= \sum X^3 m_{12} S^{-1}(\alpha X^2 m_{11} \beta) X^1\, m_0 \\ &= \sum X^3 m_{12} S^{-1}(m_{11} \beta) S^{-1}(\alpha X^2) X^1\, m_0 \\ &= \sum X^3 \varepsilon(m_1) S^{-1}(\beta) S^{-1}(\alpha X^2) X^1\, m_0 \\ &= \sum X^3 S^{-1}(\alpha X^2 \beta) X^1\, m = m. \end{aligned}$$

$$
\begin{aligned}
\text{(vii)} \sum E(E(m)_0) \otimes E(m)_1 &= \sum E((S^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho)_0) \otimes (S^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho)_1 \\
&= \sum E(S^{-1}(\alpha m_1 \tilde{p}^2_\rho)_1\, m_{00}\, (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\alpha m_1 \tilde{p}^2_\rho)_2 m_{01} (\tilde{p}^1_\rho)_{(1)} \\
\text{by part (i)} &= \sum \varepsilon(S^{-1}(\alpha m_1 \tilde{p}^2_\rho)_1)\, E(m_{00}\, \tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\alpha m_1 \tilde{p}^2_\rho)_2 m_{01} (\tilde{p}^1_\rho)_{(1)} \\
&= \sum E(m_{00}\, (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\alpha m_1 \tilde{p}^2_\rho) m_{01} (\tilde{p}^1_\rho)_{(1)} \\
&= \sum E(x^1\, m_0\, \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\alpha x^3 m_{12} \tilde{X}^3_\rho \tilde{p}^2_\rho) x^2 m_{11} \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)} \\
\text{by part (i)} &= \sum \varepsilon(x^1) E(m_0\, \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\alpha x^3 m_{12} \tilde{X}^3_\rho \tilde{p}^2_\rho) x^2 m_{11} \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)} \\
&= \sum E(m_0\, \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\alpha m_{12} \tilde{X}^3_\rho \tilde{p}^2_\rho) m_{11} \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)} \\
&= \sum E(m_0\, \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\tilde{X}^3_\rho \tilde{p}^2_\rho) S^{-1}(\alpha m_{12} m_{11} \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)} \\
&= \sum E(m_0\, \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\tilde{X}^3_\rho \tilde{p}^2_\rho) S^{-1}(\alpha) \varepsilon(m_1) \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)} \\
&= \sum E(m\, \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\alpha \tilde{X}^3_\rho \tilde{p}^2_\rho) \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)} \\
&= \sum E(m\, \tilde{q}^1_\rho (\tilde{p}^1_\rho)_{(0)}) \otimes S^{-1}(\tilde{p}^2_\rho) \tilde{q}^2_\rho (\tilde{p}^1_\rho)_{(1)} \\
\text{by (10.5)} &= E(m) \otimes 1_H.
\end{aligned}
$$

□

Using (ii), (vi) and (vii), we obtain the following characterizations of *HN-type coinvariants* as

$$
\begin{aligned}
M^{coH} := E(M) &= \{n \in M | E(n) = n\} \\
&= \{n \in M | \sum E(n_0) \otimes n_1 = E(n) \otimes 1\} \\
&= Ke((E \otimes id) \circ [\varrho^M - (- \otimes 1_H)]).
\end{aligned}
$$

M^{coH} with the right $\mathcal{A}$-action ◀ is a right $\mathcal{A}$-module and for any morphism $f : M \to L$ in ${}_H\mathbb{M}^H_\mathcal{A}$, it is not hard to show that $f(M^{coH}) \subseteq L^{coH}$.
This gives rise to a functor $(-)^{coH} : {}_H\mathbb{M}^H_\mathcal{A} \to \mathbb{M}_\mathcal{A}$ which we show to be right adjoint to the comparison functor $- \otimes_k H : \mathbb{M}_\mathcal{A} \to {}_H\mathbb{M}^H_\mathcal{A}$.

17.2. Proposition. (The adjoint pair $(-\otimes H, (-)^{coH})$ for HN-type coinvariants in ${}_H\mathbb{M}^H_\mathcal{A}$). *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra, $N \in \mathbb{M}_\mathcal{A}$, and $M \in {}_H\mathbb{M}^H_\mathcal{A}$. Then there is a functorial isomorphism*

$$\psi_{N,M} : {}_H\mathrm{Hom}^H_\mathcal{A}(N \otimes_k H, M) \longrightarrow \mathrm{Hom}_\mathcal{A}(N, M^{coH}), \quad f \longmapsto [n \mapsto f(n \otimes 1)],$$

with inverse map $\psi'_{N,M}$ given by

$$g \longmapsto [n \otimes h \mapsto h\, g(n))].$$

Thus, these functors

$$- \otimes_k H : \mathbb{M}_\mathcal{A} \to {}_H\mathbb{M}^H_\mathcal{A}, \quad (-)^{coH} : {}_H\mathbb{M}^H_\mathcal{A} \to \mathbb{M}_\mathcal{A},$$

form an adjoint pair with unit and counit

$$\eta_N : N \longrightarrow (N \otimes H)^{coH}, \quad n \mapsto n \otimes 1,$$

$$\varepsilon_M : M^{coH} \otimes_k H \longrightarrow M, \quad m \otimes h \mapsto h\, m.$$

Proof. Using the H-colinearity of f, it is easy to see that $f(n \otimes 1) \in M^{coH}$. We show that $\psi := \psi_{N,M}$ and $\psi' := \psi'_{N,M}$ are inverse to each other. For $n \in N, h \in H$ and $f \in {}_H\mathrm{Hom}_{\mathcal{A}}^H(N \otimes_k H, M)$,

$$\begin{aligned} [(\psi' \circ \psi)(f)](n \otimes h) &= h\,\psi(f)(n) = h\,f(n \otimes 1) \\ \text{by } H\text{-linearity of } f \quad &= f(n \otimes h). \end{aligned}$$

Conversely, for $n \in N$ and $g \in \mathrm{Hom}_{\mathcal{A}}(N, M^{coH})$,

$$[(\psi \circ \psi')(g)](n) = \psi'(g)(n \otimes 1) = 1\,g(n) = g(n).$$

□

17.3. Proposition. (HN-type coinvariants of $N \otimes H \in {}_H\mathbb{M}_{\mathcal{A}}^H$). *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra. Then for any $N \in \mathbb{M}_{\mathcal{A}}$ the coinvariants of the two-sided Hopf module $N \otimes H$, comes out as*

$$(N \otimes H)^{coH} \simeq N,$$

and for $n \in N$ and $h \in H$ we have $E(n \otimes h) = n \otimes \varepsilon(h)1_H$.

Proof. By definition of the left H-module structure of $N \otimes H$, $(n \otimes h) = h \cdot (n \otimes 1)$. Thus,

$$E(n \otimes h) = E(h \cdot (n \otimes 1)) = \varepsilon(h)E(n \otimes 1),$$

thus, it is enough to show that $E(n \otimes 1) = n \otimes 1$. For this, we compute

$$\begin{aligned} E(n \otimes 1) &= \sum S^{-1}(\alpha (n \otimes 1)_1 \tilde{p}_\rho^2) \cdot (n \otimes 1)_0 \tilde{p}_\rho^1 \\ &= \sum S^{-1}(\alpha \tilde{X}_\rho^3 \tilde{p}_\rho^2) \cdot (n\,\tilde{X}_\rho^1 \otimes \tilde{X}_\rho^2) \cdot \tilde{p}_\rho^1 \\ &= \sum n\,\tilde{X}_\rho^1 (\tilde{p}_\rho^1)_{(0)} \otimes S^{-1}(\tilde{p}_\rho^2) S^{-1}(\alpha \tilde{X}_\rho^3) \tilde{X}_\rho^2 (\tilde{p}_\rho^1)_{(1)} \\ &= \sum n\,\tilde{q}_\rho^1 (\tilde{p}_\rho^1)_{(0)} \otimes S^{-1}(\tilde{p}_\rho^2) \tilde{q}_\rho^2 (\tilde{p}_\rho^1)_{(1)} \\ \text{by (10.5)} \quad &= n \otimes 1. \end{aligned}$$

where $\tilde{p}_\rho = \sum \tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2$ and $\tilde{q}_\rho = \sum \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2$ are defined in (10.1) and (10.2) respectively. The above equality means that the unit map $\eta_N : N \longrightarrow (N \otimes H)^{coH}$ of adjunction in 17.2 is an isomorphism with inverse map $n \otimes h \mapsto \varepsilon(h)n$ and this finishes the proof (see also 4.1 and 16.6). □

17.4. Fundamental Theorem for ${}_H\mathbb{M}_{\mathcal{A}}^H$ with HN-coinvariants. Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra and $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$. Consider $M^{coH} = E(M)$ as a right $\mathcal{A}$-module with the $\mathcal{A}$-action ◀, defined by

$$m \blacktriangleleft a = E(m\,a) = \sum S^{-1}(\alpha m_1 a_{(1)} \tilde{p}_\rho^2)\, m_0\, a_{(0)} \tilde{p}_\rho^1.$$

Then the map

$$\varepsilon_M : M^{coH} \otimes H \longrightarrow M, \qquad m \otimes h \mapsto h\,m,$$

is an isomorphism in ${}_H\mathbb{M}_{\mathcal{A}}^H$ with inverse map

$$\varepsilon'_M(m) = \sum E(m_0) \otimes m_1 = (E \otimes id) \circ \varrho^M(m).$$

Proof. For $h \in H$ and $n \in N$,

$$\begin{aligned}
\varepsilon'_M \circ \varepsilon_M(n \otimes h) &= \varepsilon'_M(h\, n) = \sum E(h_1\, n_0) \otimes h_2 n_1 \\
\text{by (i)} \quad &= \sum \varepsilon(h_1) E(n_0) \otimes h_2 n_1 = \sum E(n_0) \otimes h n_1 \\
&= \sum (1 \otimes h)\, (E(n_0) \otimes n_1) \\
\text{by (vii)} \quad &= (1 \otimes h)\, (n \otimes 1) = n \otimes h.
\end{aligned}$$

Conversely, for $m \in M$,

$$\varepsilon_M \circ \varepsilon'_M(m) = \varepsilon_M(\sum E(m_0) \otimes m_1) = \sum m_1\, E(m_0) = m.$$

Thus ε_M is indeed an isomorphism of k-modules.
We show that ε_M is a morphism in ${}_H\mathbb{M}^H_{\mathcal{A}}$. By definition of the $(H, \mathcal{A})$-bimodule structure of $M^{coH} \otimes H$ for $h \in H$, $a \in \mathcal{A}$ and $n \in M^{coH}$,

$$h' \cdot (n \otimes h) \cdot a = \sum n \blacktriangleleft a_{(0)} \otimes h' h a_{(1)} = \sum E(n\, a_{(0)}) \otimes h' h a_{(1)},$$

therefore, we have

$$\begin{aligned}
\varepsilon_M(h' \cdot (n \otimes h) \cdot a) &= \sum h' h a_{(1)}\, E(n\, a_{(0)}) \\
\text{by (iii)} \quad &= \sum h' h a_{(1)}\, [E(n) \blacktriangleleft a_{(0)}] \\
&= h' h\, E(n)\, a = h' h\, n\, a = h'\, \varepsilon_M(n \otimes h)\, a.
\end{aligned}$$

Finally, we show that ε'_M (and therefore ε_M) is H-colinear: for $m \in M$,

$$\begin{aligned}
\varrho^{M^{coH} \otimes H}(\varepsilon'_M(m)) &= \sum E(m_0) \blacktriangleleft \tilde{X}^1_\rho \otimes m_{11} \tilde{X}^2_\rho \otimes m_{12} \tilde{X}^3_\rho \\
&= \sum E(m_0\, \tilde{X}^1_\rho) \otimes m_{11} \tilde{X}^2_\rho \otimes m_{12} \tilde{X}^3_\rho \\
\text{by (14.2)} \quad &= \sum E(x^1\, m_{00}) \otimes x^2 m_{01} \otimes x^3 m_1 \\
&= \sum \varepsilon(x^1) E(m_{00}) \otimes x^2 m_{01} \otimes x^3 m_1 \\
\text{by (7.4)} \quad &= \sum E(m_{00}) \otimes m_{01} \otimes m_1 \\
&= (E \otimes id) \varrho^M(m_0) \otimes m_1 = (\varepsilon'_M \otimes id) \varrho^M(m).
\end{aligned}$$

□

The above Fundamental Theorem yields an additional characterization of coinvariants for any $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$ as

$$\begin{aligned}
M^{coH} &= \{n \in M | \varrho^M(n) = \sum \tilde{X}^2_\rho\, (n \blacktriangleleft \tilde{X}^1_\rho) \otimes \tilde{X}^3_\rho\} \\
&= Ke(\varrho^M - [({}_M\varrho \otimes id) \circ (E \otimes id \otimes id)((- \otimes 1_{\mathcal{A}} \otimes 1_H) \cdot \phi_\rho)]),
\end{aligned}$$

where ${}_M\varrho$ is the left H-module structure of M.

17.5. Bulacu-Torrecillas coinvariants (BT-coinvariants) in ${}_H\mathbb{M}^H_{\mathcal{A}}$. Let H be quasi-bialgebra and $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra. We have seen in section 16.3 that there is a comparison functor

$$- \otimes_k H : \mathbb{M}_{\mathcal{A}} \longrightarrow {}_H\mathbb{M}^H_{\mathcal{A}}, \quad N \mapsto (N \otimes H, \varrho_{N\otimes H}, \varrho^{N\otimes H}),$$

We have shown in 16.8 that this functor $- \otimes H$ is a left adjoint to the Hom-functor ${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, -)$ and a left adjoint to the Hausser-Nill coinvariants functor $(-)^{coH}$. Following Bulacu and Torrecillas [12], we consider another version of *coinvariants* in ${}_H\mathbb{M}^H_{\mathcal{A}}$. For any two-sided Hopf module $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$ we define a projection map $\underline{E} : M \to M$, by

$$\underline{E}(m) = \sum S^{-1}(\alpha m_1)\, m_0, \qquad \text{for } m \in M, \tag{17.3}$$

and for $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$ we introduce *BT-coinvariants* $M^{\underline{coH}}$ as

$$M^{\underline{coH}} := \underline{E}(M) = \{m \in M | \underline{E}(m) = m\}.$$

17.6. Proposition. (HN versus BT-type projections). *Let $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$ and $E, \underline{E} : M \to M$ be defined by*

$$E(m) = \sum S^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho, \qquad \underline{E}(m) = \sum S^{-1}(\alpha m_1)\, m_0,$$

for $m \in M$. Then

(i) $\underline{E}(m) = \sum \tilde{q}^2_\rho\, E(m\, \tilde{q}^1_\rho)\, \tilde{p}^2_\rho, \quad E(m) = \sum S^{-1}(\tilde{p}^2_\rho)\, \underline{E}(m)\, \tilde{p}^1_\rho),$

(ii) $\underline{E} : M^{coH} \to M^{\underline{coH}}$ *is an isomorphism in* $\mathbb{M}_{\mathcal{A}}$ *with inverse* $E : M^{\underline{coH}} \to M^{coH}$,

where $\tilde{p}_\rho = \sum \tilde{p}^1_\rho \otimes \tilde{p}^2_\rho$ and $\tilde{q}_\rho = \sum \tilde{q}^1_\rho \otimes \tilde{q}^2_\rho$ are defined in (10.1) and (10.2) respectively.

Proof. (i)

$$\begin{aligned}
\sum \tilde{q}^2_\rho E(m\, \tilde{q}^1_\rho) &= \sum \tilde{q}^2_\rho\, [S^{-1}(\alpha m_1 (\tilde{q}^1_\rho)_{(1)} \tilde{p}^2_\rho)\, m_0\, \tilde{q}^1_\rho)_{(0)} \tilde{p}^1_\rho \\
&= \sum \tilde{q}^2_\rho S^{-1}((\tilde{q}^1_\rho)_{(1)} \tilde{p}^2_\rho) S^{-1}(S^{-1}(\alpha m_1)\, m_0\, \tilde{q}^1_\rho)_{(0)} \tilde{p}^1_\rho \\
\text{by (10.6)} \quad &= \underline{E}(m).
\end{aligned}$$

The other equality is clear.

(ii) For any $m \in M^{coH}$,

$$\begin{aligned}
E(\underline{E}(m)) &= E(\sum S^{-1}(\alpha m_1)\, m_0) \\
&= \sum S^{-1}(\alpha S^{-1}(\alpha m_1)_2 m_{01} \tilde{p}^2_\rho) S^{-1}(\alpha m_1)_1\, m_{00}\, \tilde{p}^1_\rho \\
&= \sum S^{-1}(m_{01} \tilde{p}^2_\rho) S^{-1}(\alpha S^{-1}(\alpha m_1)_2) S^{-1}(\alpha m_1)_1\, m_{00}\, \tilde{p}^1_\rho \\
&= \sum S^{-1}(m_{01} \tilde{p}^2_\rho) \varepsilon(\alpha m_1) S^{-1}(\alpha)\, m_{00}\, \tilde{p}^1_\rho \\
&= \sum \varepsilon(\alpha m_1) S^{-1}(\alpha m_{01} \tilde{p}^2_\rho)\, m_{00}\, \tilde{p}^1_\rho = \varepsilon(m_1) E(m_0) = E(m) \\
{}_{m \in M^{coH}} \quad &= m.
\end{aligned}$$

On the other hand, for any $m \in M^{\underline{coH}}$,

$$\begin{aligned}
\underline{E}(E(m)) &= \underline{E}(\sum S^{-1}(\tilde{p}^2_\rho)\,\underline{E}(m)\,\tilde{p}^1_\rho) \\
{}_{m\in M^{\underline{coH}}} &= \underline{E}(\sum S^{-1}(\tilde{p}^2_\rho)\,m\,\tilde{p}^1_\rho) \\
&= \sum S^{-1}(\alpha S^{-1}(\tilde{p}^2_\rho)_2 m_1 (\tilde{p}^1_\rho)_{(1)}) S^{-1}(\tilde{p}^2_\rho)_1\, m_0\, (\tilde{p}^1_\rho)_{(0)} \\
&= \sum S^{-1}(m_1(\tilde{p}^1_\rho)_{(1)})\varepsilon(\tilde{p}^2_\rho) S^{-1}(\alpha)\, m_0\, (\tilde{p}^1_\rho)_{(0)} \\
&= \sum \varepsilon(\tilde{p}^2_\rho) S^{-1}(\alpha m_1 (\tilde{p}^1_\rho)_{(1)})\, m_0\, (\tilde{p}^1_\rho)_{(0)} \\
&= \sum \varepsilon(\tilde{p}^2_\rho)\underline{E}(m\,\tilde{p}^1_\rho) = \underline{E}(m) \\
{}_{m\in M^{\underline{coH}}} &= m.
\end{aligned}$$

For right $\mathcal{A}$-linearity of E we compute

$$\begin{aligned}
E(m \triangleleft a) &= \sum E(\underline{E}(m\,a)) = \sum E(S^{-1}(\alpha m_1 a_{(1)})\, m_0\, a_{(0)}) \\
&= \sum S^{-1}(\alpha S^{-1}(\alpha m_1 a_{(1)})_2 m_{01} a_{(0)_{(1)}} \tilde{p}^2_\rho) S^{-1}(\alpha m_1 a_{(1)})_1\, m_{00}\, a_{(0)_{(0)}} \tilde{p}^1_\rho \\
&= \sum S^{-1}(\alpha S^{-1}(m_{01} a_{(0)_{(1)}} \tilde{p}^2_\rho) S^{-1}(\alpha S^{-1}(\alpha m_1 a_{(1)})_2) S^{-1}(\alpha m_1 a_{(1)})_1\, m_{00}\, a_{(0)_{(0)}} \tilde{p}^1_\rho \\
&= \sum S^{-1}(\alpha S^{-1}(m_{01} a_{(0)_{(1)}} \tilde{p}^2_\rho) \varepsilon(\alpha m_1 a_{(1)}) S^{-1}(\alpha)\, m_{00}\, a_{(0)_{(0)}} \tilde{p}^1_\rho \\
&= \sum \varepsilon(\alpha m_1 a_{(1)})(\alpha m_{01} a_{(0)_{(1)}} \tilde{p}^2_\rho)\, m_{00}\, a_{(0)_{(0)}} \tilde{p}^1_\rho \\
&= \sum \varepsilon(m_1)\varepsilon(a_{(1)}) E(m_0\, a_{(0)}) = E(m\,a) = E(m) \triangleleft a.
\end{aligned}$$

□

Similar to (15.3), we show

17.7. Proposition. (Characterization of $M^{\underline{coH}}$). For a quasi-Hopf algebra H, a right H-comodule algebra $(\mathcal{A}, \rho, \phi_\rho)$ and $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$, we have

$$M^{\underline{coH}} = \{m \in M | \varrho^M(m) = \sum S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 g^2)\, m\, \tilde{X}^1_\rho \otimes S^{-1}(q^1_L (\tilde{X}^3_\rho)_1 g^1) \tilde{X}^2_\rho\}, \quad (17.4)$$

where $q_L = \sum q^1_L \otimes q^2_L \in H \otimes H$ is given by equation (7.29) and $f^{-1} = \sum g^1 \otimes g^2$ is given by equation (7.25).

Proof. Let $m \in M^{\underline{coH}}$, i.e. $\underline{E}(m) = m$. Then

$$\begin{aligned}
\varrho^M(m) &= \varrho^M(\underline{E}(m)) = \varrho^M(\sum S^{-1}(\alpha m_1)\, m_0) \\
&= \sum S^{-1}(\alpha m_1)_1\, m_{00} \otimes S^{-1}(\alpha m_1)_2 m_{01} \\
\text{by (7.17)} &= \sum S^{-1}(f^2\alpha_2 m_{12} g^2)\, m_{00} \otimes S^{-1}(f^1\alpha_1 m_{11} g^1) m_{01} \\
&= \sum S^{-1}(m_{12}g^2) S^{-1}(f^2\alpha_2)\, m_{00} \otimes S^{-1}(m_{11}g^1) S^{-1}(f^1\alpha_1) m_{01} \\
&= \sum [S^{-1}(m_{12}g^2) \otimes S^{-1}(m_{11}g^1)]\, [(S^{-1}\otimes S^{-1}) \circ \tau(f\Delta(\alpha))\, (m_{00}\otimes m_{01}) \\
\text{by (7.26)} &= \sum S^{-1}(m_{12}g^2) S^{-1}(\gamma^2)\, m_{00} \otimes S^{-1}(m_{11}g^1) S^{-1}(\gamma^1) m_{01} \\
&= \sum S^{-1}(\gamma^2 m_{12} g^2)\, m_{00} \otimes S^{-1}(\gamma^1 m_{11} g^1) m_{01} \\
\text{by (7.22)} &= \sum S^{-1}(S(X^1)\alpha x^3 X^3_2 m_{12} g^2)\, m_{00} \otimes S^{-1}(S(x^1X^2)\alpha x^2 X^3_1 m_{11} g^1) m_{01} \\
&= \sum S^{-1}(\alpha x^3 (X^3 m_1)_2 g^2) X^1\, m_{00} \otimes S^{-1}(\alpha x^2 (X^3 m_1)_1 g^1) x^1 X^2 m_{01} \\
\text{by (14.2)} &= \sum S^{-1}(\alpha x^3 (m_{12}\tilde{X}^3_\rho)_2 g^2) m_0\, \tilde{X}^1_\rho \otimes S^{-1}(\alpha x^2 (m_{12}\tilde{X}^3_\rho)_1 g^1) x^1 m_{01} \tilde{X}^2_\rho \\
&= \sum S^{-1}(\alpha x^3 m_{122} (\tilde{X}^3_\rho)_2 g^2) m_0\, \tilde{X}^1_\rho \otimes S^{-1}(\alpha x^2 m_{121} (\tilde{X}^3_\rho)_1 g^1) x^1 m_{11} \tilde{X}^2_\rho \\
\text{by (7.2)} &= \sum S^{-1}(\alpha m_{12} x^3 (\tilde{X}^3_\rho)_2 g^2) m_0\, \tilde{X}^1_\rho \otimes S^{-1}(\alpha m_{112} x^2 (\tilde{X}^3_\rho)_1 g^1) m_{111} x^1 \tilde{X}^2_\rho \\
&= \sum S^{-1}(\alpha m_{12} x^3 (\tilde{X}^3_\rho)_2 g^2) m_0\, \tilde{X}^1_\rho \otimes S^{-1}(x^2 (\tilde{X}^3_\rho)_1 g^1) S^{-1}(\alpha m_{112}) m_{111} x^1 \tilde{X}^2_\rho \\
\text{by (7.6)} &= \sum S^{-1}(\alpha m_{12} x^3 (\tilde{X}^3_\rho)_2 g^2) m_0\, \tilde{X}^1_\rho \otimes S^{-1}(x^2 (\tilde{X}^3_\rho)_1 g^1) \varepsilon(m_{11}) S^{-1}(\alpha) x^1 \tilde{X}^2_\rho \\
&= \sum S^{-1}(\alpha m_1 x^3 (\tilde{X}^3_\rho)_2 g^2) m_0\, \tilde{X}^1_\rho \otimes S^{-1}(\alpha x^2 (\tilde{X}^3_\rho)_1 g^1) x^1 \tilde{X}^2_\rho \\
\text{by (7.29)} &= \sum S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 g^2)\, \underline{E}(m)\, \tilde{X}^1_\rho \otimes S^{-1}(q^1_L (\tilde{X}^3_\rho)_1 g^1) \tilde{X}^2_\rho \\
{\scriptstyle m\in M^{\underline{coH}}} &= \sum S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 g^2)\, m\, \tilde{X}^1_\rho \otimes S^{-1}(q^1_L (\tilde{X}^3_\rho)_1 g^1) \tilde{X}^2_\rho.
\end{aligned}$$

Conversely, if we have

$$\varrho^M(m) = \sum S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 g^2)\, m\, \tilde{X}^1_\rho \otimes S^{-1}(q^1_L (\tilde{X}^3_\rho)_1 g^1) \tilde{X}^2_\rho,$$

then

$$\begin{aligned}
\underline{E}(m) &= \sum S^{-1}(\alpha m_1)\, m_0 \\
&= \sum S^{-1}(\alpha S^{-1}(q^1_L (\tilde{X}^3_\rho)_1 g^1) \tilde{X}^2_\rho)\, S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 g^2)\, m\, \tilde{X}^1_\rho \\
&= \sum S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 g^2 \alpha S^{-1}(g^1) S^{-1}(q^1_L (\tilde{X}^3_\rho)_1) \tilde{X}^2_\rho)\, m\, \tilde{X}^1_\rho \\
&= \sum S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 S^{-1}(\beta) S^{-1}(q^1_L (\tilde{X}^3_\rho)_1) \tilde{X}^2_\rho)\, m\, \tilde{X}^1_\rho \\
&= \sum S^{-1}(q^2_L (\tilde{X}^3_\rho)_2 S^{-1}(q^1_L (\tilde{X}^3_\rho)_1 \beta) \tilde{X}^2_\rho)\, m\, \tilde{X}^1_\rho \\
&= \sum S^{-1}(q^2_L \varepsilon(\tilde{X}^3_\rho) S^{-1}(\beta) S^{-1}(q^1_L (\tilde{X}^2_\rho)\, m\, \tilde{X}^1_\rho \\
&= \sum S^{-1}(x^3 S^{-1}(\alpha x^2 \beta) x^1)\, m \\
\text{by (7.6)} &= m.
\end{aligned}$$

□

The above proposition gives a characterization of coinvariants $M^{\underline{coH}}$ for two-sided Hopf modules $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$ as

$$\begin{aligned} M^{\underline{coH}} &= \{m \in M | \varrho^M(m) = \sum S^{-1}(q^2_L(\tilde{X}^3_\rho)_2 g^2)\, m\, \tilde{X}^1_\rho \otimes S^{-1}(q^1_L(\tilde{X}^3_\rho)_1 g^1)\tilde{X}^2_\rho, \} \\ &= Ke(\varrho^M - [(S^{-1}(q^2_L(\tilde{X}^3_\rho)_2 g^2) \otimes S^{-1}(q^1_L(\tilde{X}^3_\rho)_1 g^1))\,(-\otimes 1_H)\,(\tilde{X}^1_\rho \otimes \tilde{X}^2_\rho)] \end{aligned}$$

where $q_L = \sum q^1_L \otimes q^2_L$ is given by (7.29) and $f = \sum f^1 \otimes f^2$ and $f^{-1} = \sum g^1 \otimes g^2$ are the Drinfeld gauge element and its inverse given in equations (7.24) and (7.25) and τ is the twist map $a \otimes b \mapsto b \otimes a$.

We consider the right $\mathcal{A}$-module structure on $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$,

$$m \triangleleft a := \sum S^{-1}(a_{(1)})\, m\, a_{(0)}, \tag{17.5}$$

for $a \in \mathcal{A}$, and $m \in M$, where $\rho(a) = \sum a_{(0)} \otimes a_{(1)}$. If we restrict this $\mathcal{A}$-action to $M^{\underline{coH}}$, it can be considered as a right $\mathcal{A}$-submodule of M and it is straightforward to see that for any morphism $g : M \to L$ in ${}_H\mathbb{M}^H_{\mathcal{A}}$, we have $g(M^{\underline{coH}}) \subseteq L^{\underline{coH}}$.
In this way, we obtain another *coinvariants functor*, called **BT-coinvariants functor**.

$$(-)^{\underline{coH}} : {}_H\mathbb{M}^H_{\mathcal{A}} \longrightarrow \mathbb{M}_{\mathcal{A}},$$

which we will show to be right adjoint to the comparison functor from 16.3.

17.8. Proposition. (The adjoint pair $(- \otimes_k H, (-)^{\underline{coH}})$ for Bulacu-Torrecillas coinvariants in ${}_H\mathbb{M}^H_{\mathcal{A}}$). *Let H be quasi-Hopf algebra and $\mathcal{A}$ a right H-comodule algebra, $N \in \mathbb{M}_{\mathcal{A}}$ and $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$. Then there is a functorial isomorphism*

$${}_H\mathrm{Hom}^H_{\mathcal{A}}(N \otimes_k H, M) \xrightarrow{\psi_{N,M}} \mathrm{Hom}_{\mathcal{A}}(N, M^{\underline{coH}}), \quad f \longmapsto [n \mapsto f((n \otimes 1)\, \tilde{q}_\rho)],$$

with inverse map $\psi'_{N,M}$ given by

$$g \longmapsto [n \otimes h \mapsto \sum hS^{-1}(\tilde{p}^2_\rho)\, g(n)\, \tilde{p}^1_\rho].$$

This means that the functors

$$\mathbb{M}_{\mathcal{A}} \xrightarrow{-\otimes_k H} {}_H\mathbb{M}^H_{\mathcal{A}} \xrightarrow{(-)^{\underline{coH}}} \mathbb{M}_{\mathcal{A}},$$

form an adjoint pair with unit and counit

$$\underline{\eta}_N : N \longrightarrow (N \otimes H)^{\underline{coH}}, \quad n \mapsto (n \otimes 1)\, \tilde{q}_\rho,$$

$$\underline{\varepsilon}_M : M^{\underline{coH}} \otimes_k H \longrightarrow M, \quad m \otimes h \mapsto \sum hS^{-1}(\tilde{p}^2_\rho)\, m\, \tilde{p}^1_\rho.$$

Proof. We show that ψ and ψ' are inverse to each other. For $n \in N, h \in H$ and $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(N \otimes_k H, M)$,

$$\begin{aligned} [(\psi' \circ \psi)(f)](n \otimes h) &= \sum hS^{-1}(\tilde{p}^2_\rho)\, \psi(f)(n)\, \tilde{p}^1_\rho \\ &= \sum hS^{-1}(\tilde{p}^2_\rho)\, f((n \otimes 1)\, \tilde{q}_\rho)\, \tilde{p}^1_\rho \\ {\scriptstyle f \text{ is } (H,\mathcal{A})\text{-bilinear}} &= f(\sum hS^{-1}(\tilde{p}^2_\rho)\, (n \otimes 1)\, \tilde{q}_\rho\, \rho(\tilde{p}^1_\rho)) \\ &= f(\sum (n \otimes h)\, (1_{\mathcal{A}} \otimes S^{-1}(\tilde{p}^2_\rho))\tilde{q}_\rho \rho(\tilde{p}^1_\rho)) \\ {\scriptstyle \text{by } (10.5)} &= f(n \otimes h). \end{aligned}$$

Conversely, for $n \in N$ and $g \in \mathrm{Hom}_{\mathcal{A}}(N, M^{\underline{coH}})$,

$$\begin{aligned}
[(\psi \circ \psi')(g)](n) &= \psi'(g)((n \otimes 1)\,\tilde{q}_\rho) = \sum \tilde{q}^2_\rho S^{-1}(\tilde{p}^2_\rho)\, g(n\,\tilde{q}^1_\rho)\,\tilde{p}^1_\rho \\
{\scriptstyle g \text{ is right } \mathcal{A}\text{-linear}} \quad &= \sum \tilde{q}^2_\rho S^{-1}(\tilde{p}^2_\rho)\,(g(n) \triangleleft \tilde{q}^1_\rho)\,\tilde{p}^1_\rho \\
&= \sum \tilde{q}^2_\rho S^{-1}(\tilde{p}^2_\rho) S^{-1}((\tilde{q}^1_\rho)_{(1)})\, g(n)\,(\tilde{q}^1_\rho)_{(0)} \tilde{p}^1_\rho \\
{\scriptstyle \text{by } (10.6)} \quad &= g(n).
\end{aligned}$$

□

In order to state the *Fundamental Theorem* for the category ${}_H\mathbb{M}^H_{\mathcal{A}}$ in terms of Bulacu-Torrecillas coinvariants (BT-coinvariants), we first show that the unit map $\underline{\eta}_N$ is an isomorphism. For this, we show that for any $N \in \mathbb{M}_{\mathcal{A}}$,

$$(N \otimes H)^{\underline{coH}} = \{n\,\tilde{q}^1_\rho \otimes \tilde{q}^2_\rho | n \in N\}.$$

For element $n \otimes h \in (N \otimes H)^{\underline{coH}}$,

$$\begin{aligned}
\varrho^{N \otimes H}(n \otimes h) &= \sum S^{-1}(q^2_L(\tilde{X}^3_\rho)_2 g^2) \cdot (n \otimes h) \cdot \tilde{X}^1_\rho \otimes S^{-1}(q^1_L(\tilde{X}^3_\rho)_1 g^1)\tilde{X}^2_\rho \\
&= \sum n\,(\tilde{X}^1_\rho)_{(0)} \otimes S^{-1}(q^2_L(\tilde{X}^3_\rho)_2 g^2) h (\tilde{X}^1_\rho)_{(0)} \otimes S^{-1}(q^1_L(\tilde{X}^3_\rho)_1 g^1)\tilde{X}^2_\rho.
\end{aligned}$$

On the other hand, $\varrho^{N \otimes H}(n \otimes h) = \sum n\,\tilde{X}^1_\rho \otimes h_1 \tilde{X}^2_\rho \otimes h_2 \tilde{X}^3_\rho$.
Comparing this two values for $\varrho^{N \otimes H}(n \otimes h)$ and applying $id \otimes \varepsilon \otimes id$ on both sides, we obtain

$$n \otimes h = \sum \varepsilon(h)(n\,\tilde{p}^1_\rho \otimes \tilde{q}^2_\rho).$$

This shows that the unit map

$$\underline{\eta}_N : N \longrightarrow (N \otimes H)^{\underline{coH}}, \quad n \mapsto (n \otimes 1)\,\tilde{q}_\rho,$$

is an isomorphism with inverse map $n \otimes h \mapsto \varepsilon(h)n$.

17.9. Theorem. (The Fundamental Theorem for ${}_H\mathbb{M}^H_{\mathcal{A}}$ using BT-coinvariants). *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right comodule algebra and $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$. Consider $M^{\underline{coH}} \otimes H$ as an object in ${}_H\mathbb{M}^H_{\mathcal{A}}$ with the structures*

$$h' \cdot (n \otimes h) \cdot a = \sum n \triangleleft a_{(0)} \otimes h' h a_{(1)}, \quad \varrho^{N \otimes H}(n \otimes h) = \sum n \triangleleft \tilde{X}^1_\rho \otimes h_1 \tilde{X}^2_\rho \otimes h_2 \tilde{X}^3_\rho,$$

for $h, h' \in H$, $a \in \mathcal{A}$ and $n \in M^{\underline{coH}}$. Then the map

$$\underline{\varepsilon}_M : M^{\underline{coH}} \otimes H \longrightarrow M, \quad \underline{\varepsilon}_M(n \otimes h) = \sum h S^{-1}(\tilde{p}^2_\rho)\, n\, p^1_\rho,$$

is an isomorphism in ${}_H\mathbb{M}^H_{\mathcal{A}}$ with inverse map $\underline{\varepsilon}'_M$ given by

$$\underline{\varepsilon}'_M(m) = \sum \underline{E}(m_0) \otimes m_1,$$

where $\tilde{p}_\rho = \sum \tilde{p}^1_\rho \otimes \tilde{p}^2_\rho$ is the element defined in equation (10.1).

Proof. By 17.6, we have the isomorphism $E : M^{\underline{coH}} \to M^{coH}$ in $\mathbb{M}_{\mathcal{A}}$ and tensoring it with H, we obtain

$$E \otimes id_H : M^{\underline{coH}} \otimes H \longrightarrow M^{coH} \otimes H,$$

as an isomorphism in ${}_H\mathbb{M}^H_{\mathcal{A}}$. By the Hausser-Nill version of the Fundamental Theorem for ${}_H\mathbb{M}^H_{\mathcal{A}}$, there is an isomorphism

$$\varepsilon_M : M^{coH} \otimes H \longrightarrow M, \quad m \otimes h \mapsto h\,m.$$

in ${}_H\mathbb{M}^H_{\mathcal{A}}$. Combining these two isomorphisms, we have the isomorphism

$$\underline{\varepsilon}_M = \varepsilon_M \circ (E \otimes id) : M^{\underline{coH}} \otimes H \longrightarrow M^{coH} \otimes H \longrightarrow M,$$

$$\begin{aligned}
m \otimes h \mapsto E(m) \otimes h \quad &\mapsto \quad \sum h\,E(m) = \sum hS^{-1}(\alpha m_1 \tilde{p}^2_\rho)\, m_0\, \tilde{p}^1_\rho \\
&= \quad \sum hS^{-1}(\tilde{p}^2_\rho) S^{-1}(\alpha m_1)\, m_0\, \tilde{p}^1_\rho = \sum hS^{-1}(\tilde{p}^2_\rho)\, \underline{E}(m)\, \tilde{p}^1_\rho \\
{}_{m \in M^{\underline{coH}}} &= \quad \sum hS^{-1}(\tilde{p}^2_\rho)\, m\, \tilde{p}^1_\rho
\end{aligned}$$

The inverse map $\underline{\varepsilon}'_M$ can be computed as

$$\begin{aligned}
\underline{\varepsilon}'_M(m) &= (\underline{E} \otimes id)(\sum E(m_0) \otimes m_1) = \sum \underline{E}(E(m_0)) \otimes m_1 \\
&= \sum \underline{E}(S^{-1}(\alpha m_{01} \tilde{p}^2_\rho)\, m_{00}\, \tilde{p}^1_\rho) \otimes m_1 \\
&= \sum \varepsilon(S^{-1}(\alpha m_{01} \tilde{p}^2_\rho)) \underline{E}(m_{00}\, \tilde{p}^1_\rho) \otimes m_1 \\
&= \sum \varepsilon(\alpha)\varepsilon(m_{01})\varepsilon(\tilde{p}^2_\rho) \underline{E}(m_{00}\, \tilde{p}^1_\rho) \otimes m_1 = \sum \underline{E}(m_0) \otimes m_1.
\end{aligned}$$

□

17.10. Comparing the coinvariants for ${}_H\mathbb{M}^H_{\mathcal{A}}$ with Hom-functor. As seen in 16.3, we have the comparison functor $- \otimes_k H : \mathbb{M}_{\mathcal{A}} \to {}_H\mathbb{M}^H_{\mathcal{A}}$.
We have seen in 16.8 that the functor ${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, -)$ is right adjoint to the comparison functor and in 15.11 we observed that different definitions of coinvariants for $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$ lead to three different right adjoints for the comparison functor

$$- \otimes_k H : \mathbb{M}_{\mathcal{A}} \longrightarrow {}_H\mathbb{M}^H_{\mathcal{A}}, \quad N \mapsto (N \otimes H, \varrho_{N \otimes H}, \varrho^{N \otimes H}).$$

They are Hom-functor ${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, -) : {}_H\mathbb{M}^H_{\mathcal{A}} \to \mathbb{M}_{\mathcal{A}}$, the HN-type coinvariants functor $(-)^{coH} : {}_H\mathbb{M}^H_{\mathcal{A}} \to \mathbb{M}_{\mathcal{A}}$ and BT-coinvariant functor $(-)^{\underline{coH}} : {}_H\mathbb{M}^H_{\mathcal{A}} \to \mathbb{M}_{\mathcal{A}}$. Comparing these right adjoints, we can find a functorial isomorphisms between functors

$${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, -), (-)^{coH} \text{ and } (-)^{\underline{coH}} : {}_H\mathbb{M}^H_{\mathcal{A}} \longrightarrow \mathbb{M}_{\mathcal{A}}$$

We obtain this isomorphisms explicitly as follows:

17.11. Theorem. **(M^{coH} and $M^{\underline{coH}}$ for ${}_H\mathbb{M}^H_{\mathcal{A}}$ as Hom-Functor).** *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra and M a right two-sided $(H, \mathcal{A})$-Hopf module. Then*

(1) There is a functorial isomorphism in $\mathbb{M}_{\mathcal{A}}$

$$\bar{\psi}_M : {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes_k H, M) \longrightarrow M^{coH}, \quad f \longmapsto f(1_{\mathcal{A}} \otimes 1_H),$$

with inverse map $\bar{\psi}_M$ *given by*

$$m \longmapsto [a \otimes h \mapsto \sum h \, \underline{E}(m\, a)],$$

for $a \in \mathcal{A}, h \in H$*,* $m \in M$ *and* $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes_k H, M)$*.*

(2) There is a functorial isomorphism in $\mathbb{M}_{\mathcal{A}}$

$$\bar{\theta}_M : {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes_k H, M) \longrightarrow M^{\underline{coH}}, \quad f \longmapsto f(\tilde{q}_\rho),$$

with inverse map $\bar{\theta}'_M$ *given by*

$$m \longmapsto \{a \otimes h \mapsto \sum h S^{-1}(\tilde{p}^2_\rho) \, \underline{E}(m\, a) \, \tilde{p}^1_\rho = h \, E(m\, a)\},$$

for $a \in \mathcal{A}, h \in H$*,* $m \in M$ *and* $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes_k H, M)$*.*

Proof. (1) If we substitute $N = \mathcal{A}$ in the isomorphism in 17.2, we obtain for $M \in {}_H\mathbb{M}^H_{\mathcal{A}}$ the isomorphisms

$$\bar{\psi}_M : {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes_k H, M) \xrightarrow{\psi_{\mathcal{A},M}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M^{coH}) \cong M^{coH},$$

$$f \longmapsto [a \mapsto f(a \otimes 1_H)] \mapsto f(1_{\mathcal{A}} \otimes 1_H),$$

for $a \in \mathcal{A}$. The inverse map $\bar{\psi}'_M$ is obtained as the composition

$$M^{coH} \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M^{coH}) \xrightarrow{\psi'_{\mathcal{A},M}} {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes_k H, M),$$

$$m \mapsto [m \mapsto m \blacktriangleleft a = E(m\, a)] \mapsto [a \otimes h \mapsto h\, E(m\, a),$$

for $a \in \mathcal{A}$, $h \in H$ and $m \in M^{coH}$. Here, $\psi_{\mathcal{A},M}$ is the isomorphism given in 17.2 (for $N = \mathcal{A}$) and $\psi'_{\mathcal{A},M}$ is its inverse.
Considering the right $\mathcal{A}$-action $\blacktriangleleft$ on M^{coH}, we must show that $\bar{\psi}_M$ is right $\mathcal{A}$-linear: For $a \in \mathcal{A}$ and $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes H, M)$,

$$\begin{aligned}
\bar{\psi}_M(f) \blacktriangleleft a &= E(f(1 \otimes 1)\, a) = \sum E(f(a_{(0)} \otimes a_{(1)})) \\
&= \sum S^{-1}(\alpha f(a_{(0)} \otimes a_{(1)})_1 \tilde{p}^2_\rho) \, f(a_{(0)} \otimes a_{(1)})_0 \, \tilde{p}^1_\rho \\
\text{f is H-colinear} \quad &= \sum S^{-1}(\alpha a_{(1)_2} \tilde{X}^3_\rho \tilde{p}^2_\rho) \, f(a_{(0)} \tilde{X}^1_\rho \otimes a_{(1)_1} \tilde{X}^2_\rho) \, \tilde{p}^1_\rho \\
\text{f is $(H,\mathcal{A})$-bilinear} \quad &= \sum f(a_{(0)} \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)} \otimes S^{-1}(\alpha a_{(1)_2} \tilde{X}^3_\rho \tilde{p}^2_\rho) a_{(1)_1} \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)}) \\
\text{by (7.6)} \quad &= f(\sum a_{(0)} \tilde{X}^1_\rho (\tilde{p}^1_\rho)_{(0)} \otimes S^{-1}(\tilde{p}^2_\rho) \varepsilon(a_{(1)}) S^{-1}(\alpha \tilde{X}^3_\rho) \tilde{X}^2_\rho (\tilde{p}^1_\rho)_{(1)}) \\
&= f(\sum a \tilde{q}^1_\rho (\tilde{p}^1_\rho)_{(0)} \otimes S^{-1}(\tilde{p}^2_\rho) \tilde{q}^2_\rho (\tilde{p}^1_\rho)_{(1)}) \\
\text{by (10.6)} \quad &= f(a \otimes 1_H) = (f \cdot a)(1_{\mathcal{A}} \otimes 1_H) = \bar{\psi}_M(f \cdot a).
\end{aligned}$$

(2) If we set $N = \mathcal{A}$ in the isomorphism in 17.8, we obtain the isomorphisms

$$\bar{\theta}_M : {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A} \otimes_k H, M) \xrightarrow{\psi_{\mathcal{A},M}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M^{\underline{coH}}) \cong M^{\underline{coH}},$$

$$f \longmapsto [a \mapsto \underline{E}(f(a\otimes 1)) = f((a\otimes 1)\,\tilde{q}_\rho)] \mapsto \underline{E}(f((1_{\mathcal{A}}\otimes 1_H)\,\tilde{q}_\rho)) = f(\tilde{q}_\rho),$$

for all $a \in \mathcal{A}$. The inverse map θ' is obtained as the composition

$$\bar{\theta}'_M : M^{\underline{coH}} \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M^{\underline{coH}}) \overset{\psi'_{\mathcal{A},M}}{\longrightarrow} {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A}\otimes_k H, M),$$

$$\begin{aligned} m \longmapsto [a \mapsto m \triangleleft a = \underline{E}(m\,a)] \longmapsto \{a\otimes h \ &\mapsto \ \sum hS^{-1}(\tilde{p}^2_\rho)\,\underline{E}(m\,a)\,\tilde{p}^1_\rho) \\ &= \ h\,E(m\,a)\}, \end{aligned}$$

for $a \in \mathcal{A}$, $h \in H$ and $m \in M^{\overline{coH}}$. Here, $\psi_{\mathcal{A},M}$ is the isomorphism given in 17.8 (for $N = \mathcal{A}$) and $\psi'_{\mathcal{A},M}$ is its inverse.
$\bar{\theta}_M$ is right $\mathcal{A}$-linear. For $a \in \mathcal{A}$ and $f \in {}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A}\otimes H, M)$,

$$\begin{aligned} \bar{\theta}_M(f) \blacktriangleleft a &= \sum S^{-1}(a_{(1)})\,\bar{\theta}(f)\,a_{(0)} = \sum S^{-1}(a_{(1)})\,f(\tilde{q}_\rho)\,a_{(0)} \\ \text{f is $(H,\mathcal{A})$-bilinear} \quad &= \sum f(S^{-1}(a_{(1)})\cdot(\tilde{q}_\rho)\cdot a_{(0)}) \\ &= \sum f(\tilde{q}^1 a_{(0)_{(0)}} \otimes S^{-1}(a_{(1)})\tilde{q}^2_\rho a_{(0)_{(1)}}) \\ &= f(\sum(1\otimes S^{-1}(a_{(1)}))\,\tilde{q}_\rho\,\rho(a_{(0)})) \\ \text{by (10.4)} \quad &= f((a\otimes 1_H)\,\tilde{q}_\rho) = \sum f(a\tilde{q}^1_\rho \otimes \tilde{q}^2_\rho) \\ &= (f\cdot a)(\tilde{q}_\rho) = \bar{\theta}_M(f\cdot a). \end{aligned}$$

□

Remark. Part (2) can be proved also by composing the isomorphism

$${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A}\otimes_k H, M) \overset{\psi}{\longrightarrow} M^{coH}, \quad f \longmapsto f(1\otimes 1),$$

in part (1) with the isomorphism $\underline{E} : M^{coH} \to M^{\underline{coH}}$. We obtain the isomorphism

$${}_H\mathrm{Hom}^H_{\mathcal{A}}(\mathcal{A}\otimes_k H, M) \overset{\psi}{\longrightarrow} M^{coH} \overset{\underline{E}}{\longrightarrow} M^{\underline{coH}},$$

given by

$$\begin{aligned} f \longmapsto f(1\otimes 1) \quad &\longmapsto \quad \underline{E}(f(1\otimes 1)) \\ &= \quad \sum S^{-1}(\alpha f(1\otimes 1)_1)\,f(1\otimes 1)_0 \\ \text{by H-colinearity of f} \quad &= \quad \sum S^{-1}(\alpha\tilde{X}^3_\rho)\,f(\tilde{X}^1_\rho\otimes\tilde{X}^2_\rho) \\ \text{by H-linearity of f} \quad &= \quad \sum f(\tilde{X}^1_\rho\otimes S^{-1}(\alpha\tilde{X}^3_\rho)\tilde{X}^2_\rho) = f(\tilde{q}). \end{aligned}$$

The inverse map can be computed as

$$\begin{aligned} m \overset{\theta'}{\longmapsto} \{a\otimes h \mapsto h\,E(E(m))\,a \quad &= \quad \sum h\,E(a_{(1)}\,[E(m) \blacktriangleleft a_{(0)}]) \\ &= \quad \sum h\,\varepsilon(a_{(1)})\,E(E(m) \blacktriangleleft a_{(0)}) \\ &= \quad h\,E(E(m) \blacktriangleleft a) \\ &= \quad h\,E(E(m\,a)) = h\,E(m\,a)\}, \end{aligned}$$

for $a \in \mathcal{A}$, $h \in H$ and $m \in M^{\underline{coH}}$.

Bibliography

[1] Abe, E., *Hopf Algebras*, Cambridge Univ. Press, Cambridge (1977).

[2] Balan, A., *A Morita context and Galois extensions for quasi-Hopf Algebras*, to appear in Comm. Alg.

[3] Barr, M., Wells, C., *Toposes, Triples and Theories*, Springer(1985).

[4] Borceux, F., *Handbook of categorical Algebra 1*, Categories and Structures, Encyclopedia of Mathematics and its Applications **51**, Cambridge Univ. Press. (1994).

[5] Borceux, F., *Handbook of categorical Algebra 2*, Categories and Structures, Encyclopedia of Mathematics and its Applications **51**, Cambridge Univ. Press. (1994).

[6] Böhm, G., Brzeziński, T., and Wisbauer, R., *Monads and Comonads in Module categories*, arXiv:0804.1460 (2008).

[7] Brzeziński, T., Wisbauer, R., *Corings and Comodules*, London Math. Soc. LNS **309**, Cambridge University Press (2003).

[8] Bulacu, D., Caenepeel, S., *Integrals for (dual) Quasi-Hopf Algebras. Applications*, J. Alg. **266(2)**, 552-583 (2003).

[9] Bulacu, D., Caenepeel, S., *Two-sided two-cosided Hopf modules and Doi-Hopf Modules for Quasi-Hopf Algebras*, J. Alg. **270**, 55-95 (2003).

[10] Bulacu, D., Nauwelaerts, E., *Relative Hopf Modules for (dual) Quasi-Hopf Algebras*, J. Alg. **229** 632-659 (2000).

[11] Bulacu, D., Panaite, F., Van Oystayen, F., *Quasi-Hopf Algebra Actions and Smash Products*, Comm. Alg. **28(2)**, 631-651 (2000).

[12] Bulacu, D., Torrecillas, B., *Two-sided two-cosided Hopf modules and Yetter-Drinfeld Modules for Quasi-Hopf Algebras*, Appl. Categor. Str. **14**, 503-530 (2006).

[13] Drinfeld, V.G., *Quasi-Hopf algebras*, Leningrad Math. J. **1**, 1419-1457 (1990).

[14] Eilenberg, S., Moore, J.C., *Adjoint functors and Triples*, Ill. J. Math. **9**, 381-398 (1965).

[15] Hausser, F., Nill, F., *Diagonal crossed products by duals of quasi-quantum groups*, Rev. Math. Phys. **11**, 553-629 (1999).

[16] Hausser, F., Nill, F., *Doubles of quasi-quantum groups*, Comm. Math. Phys. **199**, 547-589 (1999).

[17] Hausser, F., Nill, F., *Integral theory for quasi-Hopf Algebras*, arXiv:Math. QA/9904164 (1999).

[18] Kassel, K., *Quantum Groups*, Springer, Berlin (1995).

[19] Kelly, G.M., and Street, R., *Review of the elements of* 2*-categories*, Category Sem. Proc. Sydney 1972/1973, LNM **420**, 75-103 (1974).

[20] Larson, R.G., Sweedler, M.E., *An associative orthogonal bilinear form for Hopf algebras*, Am. J. Math. **91** 75-94 (1969).

[21] Mac Lane, S., *Category for the working mathematician*, Springer (1971).

[22] Majid, S., *Foundations of Quantum Group Theory*, Cambridge Univ. Press, Cambridge (1995).

[23] Mesablishvili, B., Wisbauer, R., *Bimonads and Hopf monads on categories*, arXiv:0710.1163 (2007).

[24] Pareigis, B., *Non-additive ring and module theory II. $\mathcal{C}$-categories, $\mathcal{C}$-functors and $\mathcal{C}$-morphisms*, Publ. Math. Debrecen **24** 351-361 (1977).

[25] Pareigis, B., *Reconstruction of hidden symmetries*, J. Alg. **183**, 90-154 (1996).

[26] Schauenburg, P., *Hopf modules and the Double of a Quasi-Hopf Algebra*, Trans. Amer. Math. Soc. **354**, 3349-3378 (2002).

[27] Schauenburg, P., *Quotients of finite Quasi-Hopf Algebras*, arXiv:math/0204337v1 (2002).

[28] Schauenburg, P., *Actions of monoidal categories and generalized Hopf smash products*, J. Alg. **270**, 521-563 (2003).

[29] Sweedler, M.E., *Hopf Algebras*, Benjamin, New York (1969).

[30] Wisbauer, R., *From Galois field extensions to Galois comodules*, Chen, J.L. (ed.) et al. Advances in ring theory. Proc. of the 4th China-Japan-Korea Conference, Nanjing, China, June 24-28 2004. Nackensack, NJ: World Scientific. 263-281 (2005).

[31] Wisbauer, R., *Algebras versus Coalgebras*, Appl. Categor. Struct. **16** (1-2), 255-295 (2008).

[32] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading (1991).

Index