# Profinite Invariance and Bounded Cohomology

Inaugural-Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

**Daniel Echtler**
aus Kaufbeuren

Düsseldorf, April 2026

aus dem Mathematischen Institut
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit der Genehmigung der
Mathematisch-Naturwissenschaftlichen Fakultät der
Heinrich-Heine-Universität Düsseldorf

Berichterstatter:

1. Jun.-Prof. Dr. Holger Kammeyer

2. Prof. Dr. Steffen Kionke

Tag der mündlichen Prüfung: 16. Juli 2026

In liebevoller Erinnerung an meine
Mutter, die mich stets unterstützt hat.

Claudia Echtler
1965–2022

## Abstract

We study to what extent the second real bounded cohomology is a profinite invariant. First, we construct pairs of residually finite groups with isomorphic profinite completions where one has non-vanishing second real bounded cohomology while the other one does not. These examples are lattices in higher rank simple Lie groups.

Using Galois cohomology, we actually classify all higher rank Lie groups admitting such examples. The complete list consists of those groups isogenous to $\mathrm{SO}^0(n,2)$ for $n \geq 6$ and the exceptional groups $\mathrm{E}_6^{(-14)}$ and $\mathrm{E}_7^{(-25)}$. For the classification, we use results from number theory and class field theory, Margulis arithmeticity and adelic superrigidity, as well as the classifications of semisimple algebraic groups and semisimple Lie groups.

Additionally, we introduce and study the notion of natural profinite invariants. These are functorial group invariants sending every Grothendieck pair to an isomorphism. We prove that there is no logical connection to the usual notion of profinite invariants. For this purpose we give examples of functorial invariants that are profinite, but not naturally profinite, as well as the other way around. Moreover, we also prove that the second (exact) bounded cohomology is not a natural profinite invariant.

# Contents

# Introduction

When studying groups, it proved to be fruitful to not only consider them as isolated objects, but to also study their actions on certain other mathematical objects. Representation theory for example is the study of groups via actions on vector spaces by linear maps. Lie groups can be studied via the action on the associated symmetric space. Similarly, $p$-adic Lie groups can be studied using the action on the Bruhat–Tits building. In geometric group theory, one examines the relationship of algebraic properties of groups and the properties of the geometric spaces they act on.

Taking a step back, the simplest objects a group can act on are finite sets via permutations. As the permutation groups of finite sets are finite themselves, this leads to the following, very broad, question.

**Question.** What can be inferred about a group from studying its finite quotients?

In order to ensure the existence of "enough" finite quotients, we usually assume groups are *residually finite*, i.e. every non-trivial element stays non-trivial in some finite quotient.

One way of systematically studying all finite quotients of a group $G$ simultaneously uses the *profinite completion* $\widehat{G}$. By a theorem of Dixon, Formanek, Poland, and Ribes, the profinite completions of two finitely generated groups $G$ and $H$ are isomorphic as topological groups if and only if the set of isomorphism classes of finite quotients of $G$ and $H$ are equal [23]. In fact, one can even drop the condition that $\widehat{G}$ and $\widehat{H}$ need to be isomorphic as *topological* groups and just ask for them to be isomorphic as abstract groups. This is due to an important result of Nikolov and Segal proving that every homomorphism of finitely generated profinite groups is automatically continuous [71].

Hence, we can make our question above precise as follows.

**Question** (Profinite properties)**.** Which properties or invariants of finitely generated residually finite groups only depend on the profinite completion?

This question can be asked not only in absolute terms for *all* finitely generated residually finite groups, but also among smaller classes of groups, such as finitely *presented* residually finite ones, for example.

There have been some results establishing the profiniteness of certain properties. For example, it is easy to see that the abelianization is a profinite invariant. In combination with Lück's approximation theorem [63], this proves that the first $\ell^2$-Betti number is a profinite invariant of *finitely presented* groups. Moreover, Sabbagh and Wilson proved that being polycyclic is a profinite property [92], Kionke and Schesler proved that being *uniformly amenable* is profinite [55, Theorem 1.2], and Kammeyer, Kionke, Raimbault, and Sauer proved that the sign of the Euler characteristic is a profinite invariant for arithmetic groups [50]. The last result has recently been improved by Kammeyer and Serafini to include most $S$-arithmetic groups [52].

However, more often than not, it turns out that properties are *not* profinite. For example Kazhdan's property (T) [3], property FA [19] and other fixed-point properties [12], finiteness properties [61], higher $\ell^2$-Betti numbers [51], and being amenable [55] are all not profinite.

Besides properties of groups, we can also ask whether a group is already determined by its finite quotients.

**Question** (Profinite rigidity)**.** Which finitely generated residually finite groups are determined by the profinite completion?

Again, this question can be asked not only in absolute terms, but also for smaller classes of groups. Many results on profinite rigidity often only give us that the *profinite genus* is finite, i.e. there are only finitely many groups (in the given class) with isomorphic profinite completion. For example, in the class of $S$-arithmetic subgroups of higher rank, the groups with the *congruence subgroup property* have finite genus [2], in the class of finitely generated 3-manifold groups, the finite volume hyperbolic 3-manifold groups have finite genus [60] and in the class of Coxeter groups, the right-angled Coxeter groups are profinitely rigid [21].

One of the guiding questions in profinite rigidity is due to Remeslennikov.

**Question** (Remeslennikov)**.** Are non-abelian free groups profinitely rigid?

So far we have only discussed the case where the profinite completions are isomorphic, without any control on the isomorphism. But, as taking the profinite completion is functorial, we could also consider group homomorphisms inducing an isomorphism of profinite completions. This leads to the notion of *Grothendieck pairs.* These were first studied by Grothendieck in a paper where he proved that the two groups in a Grothendieck pair have the same representation theory [37]. In the same paper, the question of the existence of non-trivial Grothendieck pairs was raised. The first such examples of finitely generated groups were constructed by Platonov and Tavgen' [79]. With similar techniques, Bridson and Grunewald later constructed the first finitely presented examples [13]. Since then, Grothendieck pairs have often been used to prove that properties are *not* profinite. For example the proof of Bridson that property FA and other fixed point properties are not profinite [12] and the proof of Kionke and Schesler showing that being amenable is not profinite [55] both constructed Grothendieck pairs exhibiting these behaviors.

## The content of this thesis

In this thesis, we will study the profiniteness of *bounded cohomology*, a functional analytical variation of group cohomology. Whereas in group cohomology one considers *all* cocycles, in bounded cohomology only the uniformly bounded ones are considered. This invariant was already considered by Johnson in order to study Banach algebras [48]. Only later on, when Gromov extended the theory and proved many of its fundamental properties [36], bounded cohomology became an independent field of research with applications in geometry and group theory.

The first counterexample to the profiniteness of bounded cohomology can be found as arithmetic subgroups in certain Spin-groups. It is similar in spirit to Aka's counterexamples to the profiniteness of property (T) [3].

**Proposition.** *There are arithmetic subgroups* $\Gamma \leq \mathrm{Spin}(7,2)(\mathbb{Z})$ *and* $\Lambda \leq \mathrm{Spin}(3,6)(\mathbb{Z})$ *such that* $\Gamma$ *and* $\Lambda$ *have isomorphic profinite completions* $\widehat{\Gamma} \cong \widehat{\Lambda}$*, and* $H_b^2(\Gamma;\mathbb{R}) \cong \mathbb{R}$ *while* $H_b^2(\Lambda;\mathbb{R}) \cong 0$*.*

This example led us to investigate in detail how often, or how rarely, such pairs of groups occur among higher rank lattices. For our purpose, we agree that by a *higher rank Lie group*, we mean the group of real points $G = \mathbf{G}(\mathbb{R})$ of a connected almost $\mathbb{R}$-simple affine algebraic $\mathbb{R}$-group $\mathbf{G}$ with $\mathrm{rank}_{\mathbb{R}}\, \mathbf{G} \geq 2$.

The main result of the thesis is a complete classification.

**Theorem.** *Let* $G$ *be a higher rank Lie group. There exist lattices* $\Gamma \leq G$ *and* $\Lambda \leq H$ *in another higher rank Lie group* $H$*, with* $\widehat{\Gamma} \cong \widehat{\Lambda}$*,* $H_b^2(\Gamma;\mathbb{R}) \ncong 0$ *and* $H_b^2(\Lambda;\mathbb{R}) \cong 0$ *if and only if* $G$ *is isogenous to*

$$\mathrm{SO}^0(n,2) \text{ with } n \geq 6, \qquad \mathrm{E}_6^{(-14)}, \quad \text{or} \quad \mathrm{E}_7^{(-25)}.$$

Due to a theorem of Monod and Shalom, the only lattices in higher rank Lie groups with non-trivial second bounded cohomology are the ones where the surrounding Lie group has a Hermitian symmetric space [69]. Thus, we can immediately restrict to the case of $G$ being one of these groups.

In order to prove the theorem, our task is twofold. On the one hand, for a higher rank Lie group $G$ isogenous to one listed in the theorem, we need to construct another higher rank Lie group $H$, which does *not* have a Hermitian symmetric space, as well as lattices $\Gamma \leq G, \Lambda \leq H$ with isomorphic profinite completions. On the other hand, for every remaining higher rank Lie group $G$ with Hermitian symmetric space, we need to show that every possible higher rank Lie group $H$, which admits a lattice profinitely isomorphic to one in $G$, also has to define a Hermitian symmetric space.

In both cases, by passing to finite index subgroups, we only need to consider one Lie group for every isogeny class.

For the first part, we use the following. Given two affine algebraic $\mathbb{Q}$-groups $\mathbf{G}$ and $\mathbf{H}$, both satisfying the *congruence subgroup property* and *strong approximation*, the existence of arithmetic subgroups $\Gamma \leq \mathbf{G}(\mathbb{Q})$ and $\Lambda \leq \mathbf{H}(\mathbb{Q})$ with isomorphic profinite completions is ensured by having isomorphisms $\mathbf{G} \cong_{\mathbb{Q}_p} \mathbf{H}$ over every local completion $\mathbb{Q}_p$. This, in turn, is a problem of classifying forms of algebraic groups, which can be solved using Galois cohomology. Starting with a suitable algebraic group $\mathbf{G}_0$ defined over $\mathbb{Q}$, we need to find elements in $H^1(\mathbb{Q}; \operatorname{Aut} \mathbf{G}_0)$, corresponding to two forms of $\mathbf{G}_0$, with the same image under the diagonal embedding

$$H^1(\mathbb{Q}; \operatorname{Aut} \mathbf{G}_0) \longrightarrow \bigoplus_{p \text{ prime}} H^1(\mathbb{Q}_p; \operatorname{Aut} \mathbf{G}_0).$$

If moreover, over the reals one of these forms corresponds to a Lie group with Hermitian symmetric space, while the other one does not, this gives the example we desire.

Conversely, for the remaining Lie groups with Hermitian symmetric space, we first apply Margulis arithmeticity to get that the lattices $\Gamma$ and $\Lambda$ are already arithmetic subgroups of algebraic groups $\mathbf{G}$ and $\mathbf{H}$, respectively. Moreover, these groups are anisotropic at all real places except for one, where they are isomorphic to the Lie groups $G$ and $H$. However, these algebraic groups might be defined over different number fields $K$ and $L$. Yet, adelic superrigidity proves that the two number fields are *locally isomorphic*, and that $\mathbf{G}$ and $\mathbf{H}$ are isomorphic as algebraic groups over these local isomorphisms [49]. Now in order to obtain that, over the one relevant real place, $\mathbf{H}$ defines a Lie group *with* Hermitian symmetric space as well, we again resort to Galois cohomology. This time, we need to infer enough information on the image of $\mathbf{H}$ under the map

$$H^1(L; \operatorname{Aut} \mathbf{G}_0) \longrightarrow H^1(\mathbb{R}; \operatorname{Aut} \mathbf{G}_0),$$

induced by the inclusion $L \to \mathbb{R}$. Here, $\mathbf{G}_0$ is the split group over $\mathbb{Q}$ with the same Dynkin type as $\mathbf{G}$ and $\mathbf{H}$.

Both of these proofs rely on various tools from number theory and class field theory, as well as the classification of semisimple algebraic and Lie groups.

When studying bounded cohomology, the kernel of the comparison map

$$H_b^n(\Gamma; \mathbb{R}) \to H^n(\Gamma; \mathbb{R}),$$

induced by the inclusion of bounded cocycles, is of interest. This kernel is usually called the *exact bounded cohomology* and is denoted by $EH_b^n(\Gamma; \mathbb{R})$. In degree two, the exact bounded cohomology detects non-trivial quasi-morphisms. These are maps $f\colon \Gamma \to \mathbb{R}$ with

$$\sup_{g,h\in\Gamma} |f(gh) - f(g) - f(h)| < \infty,$$

that are not of bounded distance from an honest homomorphism.

For our examples of lattices in higher rank Lie groups, Burger and Monod showed that the comparison map in degree two is always injective [17, Theorem 21]. Hence, our above examples do not prove that the second exact bounded cohomology is not a profinite invariant.

However, for lattices in rank one groups, the situation is quite different. A result of Fujiwara shows that, the space of quasi-morphisms is infinite dimensional [33]. But in this case, an important ingredient for constructing profinitely isomorphic lattices is missing: the congruence subgroup property. Serre conjectured that such lattices should *not* have the congruence subgroup property. Yet, lattices in the rank one groups $\mathrm{Sp}(n,1)$ and $\mathrm{F}_4^{(-20)}$ share many properties with higher rank lattices, like property (T), arithmeticity, and superrigidity, which might suggest that they also have the congruence subgroup property [62, Section 4]. If this were the case for $\mathrm{F}_4^{(-20)}$, we could once again use our toolbox from Galois cohomology in order to construct profinitely isomorphic lattices where one group has trivial second exact bounded cohomology and the other one does not.

In a joint paper with the author's PhD advisor on the above result [26], we raised the question of whether the second exact bounded cohomology is a profinite invariant or not. Very recently, this question was answered by Fournier-Facio by constructing a Grothendieck pair $N \to G$ where $EH_b^2(G;\mathbb{R})$ is infinite dimensional, while $EH_b^2(N;\mathbb{R})$ is trivial [31, Theorem B]. However, this construction does not give new counterexamples to the profiniteness of the second bounded cohomology.

Closely related to quasi-morphisms is the notion of *uniformly stable groups.* For a discrete group $\Gamma$ and a metric group $(G, d_G)$, an *$\varepsilon$-homomorphism* is a map $\varphi\colon \Gamma \to G$ such that the *defect*

$$D(\varphi) := \sup_{g,h\in\Gamma} d_G(\varphi(gh), \varphi(g)\varphi(h))$$

is less than $\varepsilon$. Given a class of metric groups $\mathcal{G}$, a group $\Gamma$ is *uniformly $\mathcal{G}$-stable* if for every $\varepsilon \geq 0$ all $\varepsilon$-homomorphisms $\Gamma \to G$, with $G \in \mathcal{G}$, are at uniformly bounded distance from actual group homomorphisms $\Gamma \to G$. Taking $\mathcal{G}$ to be the class of unitary operators on finite-dimensional Hilbert spaces, equipped with the operator norm, one obtains the notion of *Ulam stability.*

In a recent paper, Glebsky, Lubotzky, Monod, and Rangarajan study the stronger condition of uniform stability with respect to the class $\mathfrak{U}$ of groups of unitary operators on finite-dimensional Hilbert spaces, with metrics induced from sub-multiplicative norms on matrices [35]. For non-compact semisimple Lie groups, they define the technical condition "Property-$G(\mathcal{Q}_1, \mathcal{Q}_2)$" and prove that it implies uniform $\mathfrak{U}$-stability, and thus Ulam stability, of lattices in these Lie groups [35, Theorem 0.0.5]. They actually entertain the idea that Property-$G(\mathcal{Q}_1, \mathcal{Q}_2)$ might be necessary for Ulam stability [35, Section 7].

Interestingly, the Lie groups $\mathrm{SO}^0(n,2)$ with $n \geq 6$, $\mathrm{E}_6^{(-14)}$, and $\mathrm{E}_7^{(-25)}$ in our theorem all fail to have Property-$G(\mathcal{Q}_1, \mathcal{Q}_2)$. If it actually turned out to be true that Property-$G(\mathcal{Q}_1, \mathcal{Q}_2)$ is necessary for Ulam stability, our theorem would have the corollary that Ulam stability is not a profinite property.

In addition to studying bounded cohomology, we will also investigate Grothendieck pairs more closely. As a variation of profinite invariants, we introduce *natural profinite invariants*, functorial group invariants that induce isomorphisms for every Grothendieck pair. One such example can already be found in the original paper of Grothendieck. If $A$ is a commutative ring, sending a group $G$ to the category of finitely presented $A$-modules with $G$-action is a natural profinite invariant [37, Theoreme 1.2].

For adjoint functors, we give a useful characterization of being a natural profinite invariant. Using this we will then be able to prove that there is no logical connection between the notions of profinite invariants and natural profinite invariants.

| Functor | profinite invariant | natural profinite invariant |
|---|---|---|
| $(\,\cdot\,)_{\mathrm{ab}}\colon \mathrm{Group} \to \mathrm{Ab}$ | ✓ | ✓ |
| $H_{n\geq 2}(\,\cdot\,;\mathbb{Z})\colon \mathrm{Group} \to \mathrm{Ab}$ | ✗ | ✗ |
| $H_b^2(\,\cdot\,;\mathbb{R})\colon \mathrm{Group} \to \mathrm{Vect}_{\mathbb{R}}$ | ✗ | ✗ |
| $EH_b^2(\,\cdot\,;\mathbb{R})\colon \mathrm{Group} \to \mathrm{Vect}_{\mathbb{R}}$ | ✗ | ✗ |
| $\mathrm{Forget}\colon \mathrm{Group} \to \mathrm{Set}$ | ✓ | ✗ |
| $\mathrm{Bohr}\colon \mathrm{Group} \to \mathrm{CHGroup}$ | ✗ | ✓ |
| $L_S\colon \mathrm{Group} \to \mathrm{Group}[S^{-1}]$ | ✗ | ✓ |

In this table, Forget is the forgetful functor, Bohr the *Bohr compactification*, and $L_S$ the localization at the collection of all Grothendieck pairs.

For showing that the higher homology groups and the second (exact) bounded cohomology are *not* profinite invariants, we use the construction of Grothendieck pairs by Bridson and Grunewald. Choosing the correct "seed group" $Q$, a Rips construction gives us a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

of groups, where $N \to G$ is a Grothendieck pair. We then use the Hochschild–Serre spectral sequence in (bounded co)homology, in order to deduce that $N \to G$ does not induce an isomorphism in (co)homology.

Conversely, in order to prove that both the abelianization and the Bohr compactification *are* natural profinite invariants we use our characterization for adjoint functors being natural profinite invariants. Finally, the example showing that the Bohr compactification is *not* a profinite invariant is again similar to Aka's counterexample for the profiniteness of property (T).

While not part of this thesis, it would be interesting to identify cases in which being a natural profinite invariant implies being a profinite invariant or the other way around.

## The structure of this thesis

In Chapter 1, we recall some notions from the theory of algebraic groups and group schemes. In particular, we will discuss how Galois cohomology can be used to classify forms of algebraic groups. This allows us to take a schematic approach to the theory of algebraic groups, and thus in particular to consider the adèle points of an algebraic group.

Chapter 2 introduces some basics from number theory as well as the notion of $S$-arithmetic subgroups. For this purpose, we also construct integral models of algebraic groups. This enables us to give a more modern definition of the integral points of an algebraic group, in the language of schemes. Moreover, this chapter discusses the congruence completion as well as the congruence subgroup property.

In Chapter 3 we will finally come back to the question of profinite invariants. In particular, we will give the construction of how to get profinitely isomorphic arithmetic groups from locally isomorphic algebraic groups. Additionally, we also include the construction of Platonov and Tavgen' for non-trivial Grothendieck pairs, which becomes useful when discussing natural profinite invariants.

The heart of this thesis are Chapter 4 and Chapter 5. In Chapter 4, we prove the first of our main results. As already alluded to before, this uses the whole theory introduced in the previous chapters: the congruence subgroup property, the construction of profinitely isomorphic arithmetic groups, and the classification of forms of algebraic groups via Galois cohomology.

Chapter 5 introduces the notion of natural profinite invariants and discusses some examples. This chapter also contains our second main result on the connection between profinite invariants and natural profinite invariants and the necessary arguments involving spectral sequences. For the most part, this chapter is independent of the previous chapters on algebraic and arithmetic groups, it mostly builds on Chapter 3.

In Appendix A, the reader can find a brief introduction to the various kinds of cohomology appearing in this thesis. Besides the definition of group cohomology and bounded cohomology, this also contains some of the properties of bounded, non-abelian and Galois cohomology.

## Some general assumptions

All rings appearing in this thesis are assumed to be commutative and unital. The natural numbers include 0. Algebraic groups are usually denoted by bold face letters $(\mathbf{G}, \mathbf{H}, \dots)$, while we use capital Roman letters $(G, H, \dots)$ for groups and group schemes. In case we want to emphasize that a group is countable and discrete, e.g. for arithmetic subgroups or lattices, we use capital Greek letters $(\Gamma, \Lambda, \dots)$.

## Statement on personal contribution

Chapter 4 is based on the article “Bounded cohomology is not a profinite invariant” previously published in *Canadian Mathematical Bulletin* 67.2 (2024), co-authored with my advisor Holger Kammeyer. The results and original proofs were developed in close collaboration with equal contributions by both authors. Due to our scientific backgrounds, the necessary parts related to bounded cohomology were contributed by myself, while Holger Kammeyer gave the input on the Galois cohomological methods. The presentation in Chapter 4, however, was fully revised and rewritten.

## Acknowledgements

First and foremost, I would like to thank my advisor Holger Kammeyer. Not only for giving me the opportunity to pursue a PhD in mathematics, but also for his continued support. No matter if I had mathematical questions or personal problems to discuss, I always knew his door was open, and I could talk to him all the time. The various (sometimes very spontaneous) discussions we had at the office or on the way to lunch were indispensable to me.

I would also like to thank my "academic siblings" Giada Serafini and Adrian Baumann. I benefited greatly from all the exchanges we had. Whether I wanted to go into the details of a proof, discuss a vague idea, complain about notation, or talk about anything else, I could always count on you.

The supportive and welcoming environment created by my colleagues has been invaluable, and I am truly grateful. The (occasionally way too long) coffee breaks, the various fun (but sometimes also frustrating) trips to Wuppertal and Bochum, the Christmas parties, as well as the extra-professional activities we did together were a welcome change to the everyday work as a mathematician. While this list is not exhaustive, I would like to thank Fabian R., Clotilde, Mikel, Luca, Christian, Iker, Alena, Giada, Jan, Matthias, Christiane, Martina, Fabian K., Margherita, Doris, Moritz, and Adrian for the fantastic time I had here at the institute. (The list is in random order and does not reflect any favoritism I might or might not have. Or does it...?)

Special thanks go to Alena Ramona Meyer, for proofreading this thesis. Thank you, both for taking the time and your valuable input.

I would also like to thank Francesco Fournier-Facio, for making a preliminary version of his result on the non-profiniteness of the second exact bounded cohomology available to me.

This research was conducted in the framework of the research training group *GRK 2240: Algebro-Geometric Methods in Algebra, Arithmetic and Topology*, funded by the Deutsche Forschungsgemeinschaft. I am thankful for the financial support, which, among other things, enabled me to join many interesting international conferences.

# Chapter 1

## Algebraic groups

The theory of algebraic groups lives at the intersection of group theory and algebraic geometry. It has applications in many fields like algebraic topology, number theory, cryptography as well as theoretical physics. Classically linear algebraic groups are subgroups of $\mathrm{GL}_n(K)$ defined as the zero sets of polynomial. In a more modern setup algebraic groups are schemes which carry a group structure.

In this chapter we discuss some basic theory of algebraic groups, while ignoring most of the subtleties. Mostly we will focus on the theory of *affine algebraic groups* and will not discuss topics like elliptic curves or abelian varieties, which are in themselves huge fields of research.

The goal of this chapter is to give a rough overview of the theory of affine algebraic groups, which we will use in the following chapter discussing *arithmetic subgroups*. By no means is this an exhaustive treatment of the theory. There is a large collection of excellent literature doing so in much more detail, for example the books of Borel [8], Milne [67], Platonov and Rapinchuk [77], or the expository article by Tits on the classification of semisimple algebraic groups [100] and the references contained therein.

### Overview of this chapter

## 1.1 Group schemes and algebraic groups

In this section, we introduce some terminology on algebraic groups. We begin with a rather abstract definition but then give a more hands on description for the *affine* algebraic groups we are interested in.

**Definition 1.1.1** (Group object)**.** Let $\mathcal{C}$ be a category with finite products and terminal object 1. A *group object in* $\mathcal{C}$ is an object $G \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $e \in \mathrm{Hom}_{\mathcal{C}}(1, G)$, $m \in \mathrm{Hom}_{\mathcal{C}}(G \times G, G)$ and $\iota \in \mathrm{Hom}_{\mathcal{C}}(G, G)$ such that the diagrams

$$\begin{tikzcd} G \times G \times G \arrow[r, "{(\mathrm{id}, m)}"] \arrow[d, "{(m, \mathrm{id})}"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathrm{id} \times e"] \arrow[d, "e \times \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\mathrm{id} \times \iota"] \arrow[d, "\iota \times \mathrm{id}"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

commute. Here we also write $e$ for the composition $G \to 1 \xrightarrow{e} G$.

For two group objects $G, H \in \mathcal{C}$, a *morphism of group objects* $G \to H$ is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(G, H)$ such that

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "{(\varphi, \varphi)}"'] & G \arrow[d, "\varphi"] \\ H \times H \arrow[r, "m_H"'] & H \end{tikzcd}$$

commutes.

This definition abstracts the definition of a group by reformulating the defining properties in the language of category theory. Similarly, we can also reformulate what it means for a group to be commutative.

**Definition 1.1.2** (Commutative group object)**.** Let $\mathcal{C}$ be a category with finite products and terminal object 1. A group object $G$ in $\mathcal{C}$ is said to be commutative, if the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mathrm{swap}"] \arrow[d, "\mathrm{id}"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

commutes. Here $\mathrm{swap}\colon G \times G \to G \times G$ is the canonical map swapping the two factors.

As the definitions came from the defining properties of groups, it is not surprising that groups are just one example of group objects.

**Example 1.1.3** (Group objects)**.**

1. A group object in the category of sets is just a group. Morphisms between groups are just group homomorphisms.

2. A commutative group object in the category of sets is an abelian group.

3. A group object in the category of groups is an abelian group. This is due to the Eckmann–Hilton argument [64, Chapter III.6, Exercise 4]. Morphisms in this case are again just group homomorphisms.

4. A group object in the category of topological spaces is a topological group. In this case the morphisms are continuous group homomorphisms.

5. In the category of smooth manifolds, a group object is a Lie group. Morphisms between Lie groups are smooth group homomorphisms.

The following lemma will often be useful.

**Lemma 1.1.4** (Group objects as functors)**.** *Let $G$ be a group object in a category $\mathcal{C}$. Then there is an induced group structure on* $\mathrm{Hom}_{\mathcal{C}}(A, G)$ *for each object* $A \in \mathrm{Ob}(\mathcal{C})$*, which turns* $\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, G)$ *into a functor* $\mathcal{C} \to \mathrm{Group}$*. If $G$ is commutative, every induced group structure is in fact abelian and thus* $\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, G)$ *turns into a functor* $\mathcal{C} \to \mathrm{Ab}$*.*

*Moreover, if $\varphi\colon G \to H$ is a morphism of group objects, then*

$$\begin{aligned} \varphi^*\colon\ \mathrm{Hom}_{\mathcal{C}}(A, G) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, H), \\ f &\longmapsto \varphi \circ f \end{aligned}$$

*for all $A \in \mathcal{C}$, gives a natural transformation* $\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, G) \to \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, H)$ *of functors* $\mathcal{C} \to \mathrm{Group}$*.*

*Proof.* The contravariant functor $\mathrm{Hom}_{\mathcal{C}}(A, \cdot)$ preserves finite products. Thus, the defining diagrams of a group object give us that $\mathrm{Hom}_{\mathcal{C}}(A, G)$ is a group object in the category of sets. Obviously this is functorial. Similarly, the commutativity of $G$ translates to these groups being abelian.

For the second part, we only need to see that $\varphi^*\colon\ \mathrm{Hom}_{\mathcal{C}}(A, G) \to \mathrm{Hom}_{\mathcal{C}}(A, H)$ is a group homomorphism. But this is due to $\varphi$ being a morphism of group objects. $\square$

Conversely, one can show that an object $G \in \mathrm{Ob}(\mathcal{C})$, together with morphisms $m, e$ and $\iota$, is a (commutative) group object if for every $A \in \mathrm{Ob}(\mathcal{C})$ the induced maps endow $\mathrm{Hom}_{\mathcal{C}}(A, G)$ with a (commutative) group structure [64, Chapter III.6, Proposition 1, Exercise 3.].

**Definition 1.1.5** (Group scheme)**.** Let $R$ be a ring. A *group scheme over $R$* is a group object in the category of schemes over $R$. More explicitly, a group scheme over $R$ is a scheme $X$ over $R$ together with morphisms $e\colon \operatorname{Spec} R \to X$, $m\colon X \times_R X \to X$ and $\iota\colon X \to X$ of schemes over $R$ such that the diagrams

$$\begin{tikzcd} X \times_R X \times_R X \arrow[r, "(\mathrm{id}, m)"] \arrow[d, "(m, \mathrm{id})"'] & X \times_R X \arrow[d, "m"] \\ X \times_R X \arrow[r, "m"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{id} \times_R e"] \arrow[d, "e \times \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & X \times_R X \arrow[d, "m"] \\ X \times_R X \arrow[r, "m"'] & X \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\mathrm{id} \times_R \iota"] \arrow[d, "\iota \times \mathrm{id}"'] \arrow[dr, "e"] & X \times_R X \arrow[d, "m"] \\ X \times_R X \arrow[r, "m"'] & X \end{tikzcd}$$

commute.

A group scheme is called affine (respectively of finite type) if the underlying scheme is affine (respectively of finite type over $R$).

A morphism of group schemes is a morphism of group objects in the category of schemes over $R$.

More generally, one could also define group schemes over some base scheme. However, we will not make use of this more general definition, as later on we will mostly be interested in (affine) algebraic groups.

**Definition 1.1.6** (Algebraic group)**.** Let $K$ be a field. An *algebraic group over $K$* is a group scheme of finite type over $K$. Such an algebraic group is called *affine* if the underlying group scheme is affine.

A *morphism of algebraic groups* is a morphism of the underlying group schemes.

**Example 1.1.7** (Trivial group scheme)**.** Let $R$ be a ring. Since $\operatorname{Spec} R$ is (by definition) terminal in the category of schemes over $R$, there is a unique way of turning $\operatorname{Spec} R$ into a group scheme over $R$. This group scheme is called the *trivial group scheme over $R$*, and we denote it by $\{e\}$ or simply 1.

We will give two more, useful descriptions of group schemes in terms of the associated functor of points and the Hopf algebra. Both of these viewpoints can be useful to specify the group object structure on a given affine scheme.

**Remark 1.1.8** (Group schemes as functors)**.** Let $R$ be a ring. By the Yoneda lemma, every scheme $X$ over $R$ can be identified with the *functor of points*

$$X(\,\cdot\,) = \operatorname{Hom}_R(\,\cdot\,, X)\colon \mathrm{Aff}_R \longrightarrow \mathrm{Set},$$

from the category of affine schemes over $R$ to the category of sets [28, Proposition VI-2]. Moreover, using the duality of categories

$$\begin{aligned} \mathrm{Alg}_R &\longleftrightarrow \mathrm{Aff}_R \\ A &\longmapsto \operatorname{Spec} A \\ \Gamma(Y, \mathcal{O}_Y) &\longleftarrow\!\shortmid\, Y \end{aligned}$$

we can further identify $X$ with the covariant functor

$$\begin{aligned} X(\,\cdot\,)\colon \mathrm{Alg}_R &\longrightarrow \mathrm{Set} \\ A &\longmapsto \mathrm{Hom}_R(\operatorname{Spec} A, X). \end{aligned}$$

If now $X = G$ is a group scheme over $R$, we get by Lemma 1.1.4 that the functor of points is in fact a covariant functor

$$G(\,\cdot\,)\colon \mathrm{Alg}_R \longrightarrow \mathrm{Group}.$$

Finally, if $G = \operatorname{Spec} S$ is an *affine* group scheme, we get that this functor is given by

$$G(\,\cdot\,) = \mathrm{Hom}_{\mathrm{Alg}_R}(S, \,\cdot\,)\colon \mathrm{Alg}_R \longrightarrow \mathrm{Group}.$$

So equivalently, we could have defined a group scheme (respectively an affine group scheme) over $R$ to be a functor $G\colon \mathrm{Alg}_R \to \mathrm{Group}$ represented by an $R$-scheme (respectively an $R$-algebra).

From this point of view, a morphism $\varphi\colon G \to H$ of group schemes over $R$ is a natural transformation $\varphi\colon G(\,\cdot\,) \to H(\,\cdot\,)$ of functors $\mathrm{Alg}_R \to \mathrm{Group}$.

By the Yoneda Lemma, every such natural transformation, between the functor of points of two affine group schemes $G = \operatorname{Spec} S$ and $H = \operatorname{Spec} T$, is given by an $R$-algebra homomorphism $T \to S$.

Note that while every such algebra homomorphism $T \to S$ induces a natural transformation of $G(\,\cdot\,) \to H(\,\cdot\,)$ as functors $\mathrm{Alg}_R \to \mathrm{Set}$, not every such transformation gives a natural transformation as functors $\mathrm{Alg}_R \to \mathrm{Group}$. A necessary and sufficient condition for an algebra homomorphism to induce a morphism of the associated affine group schemes can be given as follows.

**Remark 1.1.9** (Group schemes as Hopf algebras)**.** Let $R$ be a ring and let $G = \operatorname{Spec} S$ be an affine group scheme over $R$. Using the duality of categories

$$\begin{aligned} \mathrm{Alg}_R &\longleftrightarrow \mathrm{Aff}_R \\ A &\longmapsto \operatorname{Spec} A \\ \Gamma(X, \mathcal{O}_X) &\longleftarrow\!\shortmid X \end{aligned}$$

we get that the $R$-algebra $S$ satisfies axioms dual to the ones in the definition of a group object. There are maps $m^*\colon S \to S \otimes_R S$, $e^*\colon S \to R$ and $\iota^*\colon S \to S$, called *comultiplication*, *counit* and *antipode* respectively, making the diagrams

$$\begin{tikzcd} S \otimes_R S \otimes_R S & S \otimes_R S \arrow[l, "(\mathrm{id}, m^*)"'] \\ S \otimes_R S \arrow[u, "(m^*, \mathrm{id})"] & S \arrow[l, "m^*"] \arrow[u, "m^*"'] \end{tikzcd} \qquad \begin{tikzcd} S & S \otimes_R S \arrow[l, "\mathrm{id} \otimes_R e^*"'] \\ S \otimes_R S \arrow[u, "e^* \otimes_R \mathrm{id}"] & S \arrow[l, "m^*"] \arrow[u, "m^*"'] \arrow[ul, "\mathrm{id}"'] \end{tikzcd}$$

$$\begin{tikzcd} S & S \otimes_R S \arrow[l, "\mathrm{id} \otimes_R \iota^*"'] \\ S \otimes_R S \arrow[u, "\iota^* \otimes_R \mathrm{id}"] & S, \arrow[l, "m^*"] \arrow[u, "m^*"'] \arrow[ul, "e^*"'] \end{tikzcd}$$

where we also write $e^*$ for the composition $S \xrightarrow{e^*} R \to S$, commute. This turns $S$ into a so-called *Hopf algebra*.

Conversely, every Hopf algebra $S$ over $R$ gives $\operatorname{Spec} S$ the structure of an affine group scheme over $R$.

From this point of view, if $G = \operatorname{Spec} S$ and $H = \operatorname{Spec} T$ are affine group schemes over $R$, then the morphism $\varphi\colon G \to H$ of affine group schemes correspond precisely to the morphism $T \to S$ of Hopf algebras over $R$, i.e. an $R$-algebra morphism that is compatible with the comultiplications.

Obviously, an affine group scheme over $R$ is of finite type if and only if the associated Hopf algebra is finitely generated as $R$-algebra.

Using the interpretation as functor of points, we finally give examples of group schemes.

**Example 1.1.10** (Group schemes)**.** Let $R$ be a ring and let $n \in \mathbb{N}$.

0. The trivial group scheme $\{e\}$ we introduced in Example 1.1.7 corresponds to the functor sending every $R$-algebra to the trivial group. By construction, it is represented by the ring $R$.

1. The additive group scheme is given by the functor

$$\begin{aligned} \mathbf{G}_a \colon \mathrm{Alg}_R &\longrightarrow \mathrm{Group} \\ A &\longmapsto (A, +). \end{aligned}$$

It is represented by the polynomial ring $R[T]$ and is, moreover, commutative.

2. The multiplicative group scheme is given by the functor

$$\begin{aligned} \mathbf{G}_m \colon \mathrm{Alg}_R &\longrightarrow \mathrm{Group} \\ A &\longmapsto (A^\times, \,\cdot\,). \end{aligned}$$

It is represented by $R[T, T^{-1}] = {R[T, T']}\big/{(TT' - 1)}$ and is again commutative.

3. The group scheme of $n$-th roots of unity is given by the functor

$$\begin{aligned} \mu_n \colon \mathrm{Alg}_R &\longrightarrow \mathrm{Group} \\ A &\longmapsto \{x \in A \mid x^n = 1\}. \end{aligned}$$

It is represented by ${R[T]}\big/{(T^n - 1)}$.

4. The general linear group of degree $n$ is given by the functor

$$\begin{aligned} \mathrm{GL}_n\colon \mathrm{Alg}_R &\longrightarrow \mathrm{Group} \\ A &\longmapsto \mathrm{GL}_n(A). \end{aligned}$$

   It is represented by

$$R[T_{1,1}, \dots, T_{n,n}, \det(T_{i,j})^{-1}] = R[T_{1,1}, \dots, T_{n,n}, d'] \big/ (\det(T_{i,j})d' - 1).$$

   Notice that $\mathrm{GL}_1 = \mathbf{G}_m$.

5. The special linear group of degree $n$ is given by the functor

$$\begin{aligned} \mathrm{SL}_n\colon \mathrm{Alg}_R &\longrightarrow \mathrm{Group} \\ A &\longmapsto \mathrm{SL}_n(A). \end{aligned}$$

   It is represented by $R[T_{1,1}, \dots, T_{n,n}] \big/ (\det(T_{i,j}) - 1)$.

Notice that all of these group schemes are both affine and of finite type, as the corresponding representing algebras are finitely generated. In particular, over a field they also give the first examples of (affine) algebraic groups.

**Example 1.1.11** (Morphisms of group schemes)**.** Let $R$ be a ring, let $n \in \mathbb{N}$ and let $A$ be an $R$-algebra.

1. By construction, we have the natural inclusion $\mathrm{SL}_n(A) \hookrightarrow \mathrm{GL}_n(A)$. This gives a natural transformation, i.e. a morphism of group schemes, $\mathrm{SL}_n \to \mathrm{GL}_n$.

2. The determinant

$$\begin{aligned} \det\colon\ \mathrm{GL}_n(A) &\longrightarrow A^\times = \mathbf{G}_m(A) \\ M &\longmapsto \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \prod_{i=1}^{n} M_{i,\sigma(i)} \end{aligned}$$

   is a group homomorphism natural in $A$. Thus, it induces a morphism of group schemes $\det\colon\ \mathrm{GL}_n \to \mathbf{G}_m$.

### 1.1.1 Subgroup schemes

The example that $\mathrm{SL}_n(A)$ is a subgroup of $\mathrm{GL}_n(A)$ might suggest saying that $\mathrm{SL}_n$ is a *subgroup scheme* of $\mathrm{GL}_n$. For this to make sense in our general setup of group schemes, we agree on the following definition.

**Definition 1.1.12** (Subgroup scheme)**.** Let $R$ be a ring and $G$ a group scheme over $R$. A *subgroup scheme of $G$* is a group scheme $H$ over $R$ such that $H$ is a closed subscheme of $G$ and the inclusion $H \to G$ is a group homomorphism.

This also gives the definition of an algebraic subgroup.

**Definition 1.1.13** (Algebraic subgroup)**.** Let $K$ be a field and let $\mathbf{G}$ be an algebraic group over $K$. An *algebraic subgroup of* $\mathbf{G}$ is a subgroup scheme of $\mathbf{G}$.

Note that some authors would use "closed subgroup scheme" for what we call a subgroup scheme. However, this does not change the definition of algebraic subgroups, as these are automatically closed [67, Proposition 1.41].

For an affine scheme $X = \operatorname{Spec} A$, there is a bijective correspondence between the ideals $I$ of $A$ and the closed subschemes of $X$ given by $I \mapsto \operatorname{Spec}(A/I)$ [40, Chapter II, Corollary 5.10]. Thus, we immediately get the following corollary.

**Corollary 1.1.14** (Subgroups of affine group schemes)**.** *Let $R$ be a ring, $G = \operatorname{Spec} S$ an affine group scheme over $R$, and $H$ a subgroup scheme of $G$. Then there is an ideal $I \subseteq S$ such that $H = \operatorname{Spec}(S/I)$.*

*In particular, algebraic subgroups of affine algebraic groups are again affine.*

**Remark 1.1.15** (Subgroup schemes as functors)**.** Let $R$ be a ring, let $G = \operatorname{Spec} S$ be an affine group scheme over $R$ and let $H$ be a subgroup scheme of $G$. Then $H$ is of the form $H = \operatorname{Spec}(S/I)$ for some ideal $I \subseteq S$ and the closed immersion $H \to G$ is induced by the quotient map $p\colon S \to S/I$. Thus for any $R$-algebra $A$, we get that the canonical map

$$\begin{aligned} H(A) = \operatorname{Hom}_{\mathrm{Alg}_R}\left({}^{S}\!\big/_{I}, A\right) &\longrightarrow \operatorname{Hom}_{\mathrm{Alg}_R}(S, A) = G(A) \\ f &\longmapsto f \circ p \end{aligned}$$

is an injective group homomorphism.

Conversely, let $G = \operatorname{Spec} S$ and $H = \operatorname{Spec} T$ be two affine group schemes over $R$, and let $H \to G$ be a morphism inducing injective group homomorphisms $H(A) \to G(A)$ for every $R$-algebra $A$. By the above correspondence of ideals and closed subschemes, the induced map $S \to T$ needs to be surjective for $H$ to become a closed subscheme of $G$.

Using the functorial point of view, we can also define what a *normal* subgroup scheme is supposed to be.

**Definition 1.1.16** (Normal subgroup scheme)**.** Let $R$ be a ring, let $G$ be a group scheme over $R$ and let $H \leq G$ be a subgroup scheme. Then $H$ is said to be *normal* if for every $R$-algebra $A$ the group $H(A)$ is a normal subgroup of $G(A)$.

**Remark 1.1.17** (Subgroup schemes as Hopf ideals)**.** Let $R$ be a ring, let $G = \operatorname{Spec} S$ be an affine group scheme over $R$ and let $H = \operatorname{Spec}(S/I)$ be a subgroup scheme of $G$. Then both $S$ and $S/I$ are Hopf algebras over $R$ and, as $H$ is a subgroup scheme of $G$, the Hopf algebra structure on $S$ induces the one on $S/I$. Thus, the ideal $I \subseteq S$ has to satisfy

$$\begin{aligned} m^*(I) &\subseteq S \otimes_R I + I \otimes_R S, \\ e^*(I) &= 0, \\ \text{and} \qquad i^*(I) &\subseteq I. \end{aligned}$$

An ideal satisfying these three properties is called a *Hopf ideal.* It is easy to see that Hopf ideals $I \subseteq S$ are precisely those ideals such that the quotient $S/I$ inherits a Hopf algebra structure. So subgroup schemes of affine group schemes are in 1:1-correspondence to the Hopf ideals of the associated Hopf algebra.

**Example 1.1.18** ($\mathrm{SL}_n \leq \mathrm{GL}_n$)**.** Let $R$ be a ring and let $n \in \mathbb{N}$. For every $R$-algebra $A$ we have the natural inclusion

$$\mathrm{SL}_n(A) = \{M \in \mathrm{Mat}_n(A) \mid \det A = 1\} \subseteq \mathrm{GL}_n(A),$$

given by the additional equation $\det = 1$. Thus, it is easy to see that these group homomorphisms are induced by the quotient map

$$\begin{aligned} R[T_{1,1}, \dots, T_{n,n}, d'] \big/ (\det(T_{i,j})d' - 1) &\longrightarrow R[T_{1,1}, \dots, T_{n,n}] \big/ (\det(T_{i,j}) - 1) \\ T_{i,j} &\longmapsto T_{i,j} \\ d' &\longmapsto 1. \end{aligned}$$

Hence, $\mathrm{SL}_n$ is indeed a subgroup scheme of $\mathrm{GL}_n$.

**Example 1.1.19** ($\mu_n \leq \mathbf{G}_m$)**.** Let $R$ be a ring and let $n \in \mathbb{N}$. For every $R$-algebra $A$ we have the natural inclusion

$$\mu_n(A) = \{x \in A^\times \mid x^n = 1\} \subseteq \mathbf{G}_m(A),$$

given by the additional equation $x^n = 1$. Thus, we obtain that these natural homomorphisms are induced by the quotient map

$$\begin{aligned} R[T, T'] \big/ (TT' - 1) &\longrightarrow R[T] \big/ (T^n - 1), \\ T &\longmapsto T \\ T' &\longmapsto T^{n-1} \end{aligned}$$

and so $\mu_n$ is a subgroup scheme of $\mathbf{G}_m = \mathrm{GL}_1$.

**Example 1.1.20** ($\mathbf{G}_a \leq \mathrm{GL}_2$)**.** Let $R$ be a ring. For every $R$-algebra $A$, we can consider $\mathbf{G}_a(A)$ as a subgroup of $\mathrm{GL}_2$, via the natural map

$$\begin{aligned} \varphi_A \colon \mathbf{G}_a(A) &\longrightarrow \mathrm{GL}_2(A) \\ a &\longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Here the additional equations needed to be satisfied by elements in $\varphi_A(\mathbf{G}_a(A)) \subseteq \mathrm{GL}_2(A)$ are $M_{1,1} = M_{2,2} = 1$ and $M_{2,1} = 0$. Thus, we get that the above natural group homomorphisms are induced by

$$\begin{aligned} R[T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}, d'] \big/ (\det(T_{i,j})d' - 1) &\longrightarrow R[T], \\ T_{1,1}, T_{2,2} &\longmapsto 1 \\ T_{2,1} &\longmapsto 0 \\ T_{1,2} &\longmapsto T \end{aligned}$$

implying that $\mathbf{G}_a$ is a subgroup scheme of $\mathrm{GL}_2$.

The examples above show that every affine group scheme we have seen so far can be realized as a subgroup scheme of some $\mathrm{GL}_n$. For affine algebraic groups this is in fact always the case.

**Theorem 1.1.21** ([67, Corollary 4.10])**.** *Every affine algebraic group is isomorphic to an algebraic subgroup of* $\mathrm{GL}_n$ *for some* $n \in \mathbb{N}$*.*

This result is the reason why affine algebraic groups are often referred to as *linear* algebraic groups.

### Kernels

Given a group homomorphism $f\colon \Lambda \to \Gamma$, its kernel $\ker f = \{x \in \Lambda \mid f(x) = e\}$ will be a normal subgroup of $\Lambda$. Now given a morphism $\varphi\colon H \to G$ of group schemes over $R$ we obtain for each $R$-algebra $A$ a group homomorphism $\varphi(A)\colon H(A) \to G(A)$. By taking kernels we get a functor $\mathrm{Alg}_R \to \mathrm{Group}$, $A \mapsto \ker(\varphi(A))$. The question is whether this functor is representable by a scheme, i.e. if it gives a group scheme.

**Proposition 1.1.22** (Kernel of morphisms of group schemes)**.** *Let $R$ be a ring, let $G$ and $H$ be group schemes over $R$ and let $\varphi\colon H \to G$ be a morphism of group schemes over $R$. Defining* $\ker(\varphi)$ *by the pullback square*

$$\begin{tikzcd} \ker(\varphi) = H \times_G \{e\} \arrow[r] \arrow[d] & H \arrow[d, "\varphi"] \\ \{e\} \arrow[r, "e"'] & G \end{tikzcd}$$

*gives a normal subgroup scheme of $H$, such that there is a natural isomorphism*

$$\ker(\varphi)(A) \cong \ker(\varphi(A))$$

*for every $R$-algebra $A$.*

*Proof.* Let $A$ be an $R$-algebra. As $\{e\} = \operatorname{Spec} R$ is terminal in the category of schemes over $R$, we obtain from the universal property of the pullback a natural isomorphism

$$\begin{aligned} \ker(\varphi)(A) &= \mathrm{Hom}_R(\operatorname{Spec} A, \ker(\varphi)) \\ &\cong \{g \in H(A) \mid \varphi \circ g = e\} \\ &= \{g \in H(A) \mid \varphi(A)(g) = e\} \\ &= \ker(\varphi(A)), \end{aligned}$$

where we also write $e$ for the composition $\operatorname{Spec} A \to \operatorname{Spec} R \xrightarrow{e} G$. This shows that $\ker(\varphi)$ is a group scheme over $R$ with the desired property. By construction, $\ker(\varphi)$ is a closed subscheme of $H$, so that it is indeed a subgroup scheme of $G$. Finally, as $\ker(\varphi)(A)$ is a normal subgroup in $H(A)$ for every $R$-algebra $A$, we have that $\ker(\varphi)$ is also a normal subgroup scheme. $\square$

**Example 1.1.23.** Let $R$ be a ring and let $n \in \mathbb{N}$. We consider the morphism of group schemes $\det\colon \mathrm{GL}_n \to \mathbf{G}_m$ given by

$$\begin{aligned} \det\colon \mathrm{GL}_n(A) &\longrightarrow \mathbf{G}_m(A) \\ M &\longmapsto \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} M_{i,\sigma(i)} \end{aligned}$$

for every $R$-algebra $A$. Then $\ker(\det) = \mathrm{SL}_n$, as this holds on the functor of points.

### 1.1.2 Exact sequences and quotients

As usual in algebraic geometry, defining images and quotients requires more effort than defining kernels. From the functorial point of view, for a group scheme $G$ and a normal subgroup scheme $N$ one would like to define the quotient $G/N$ to be given by the functor $A \mapsto G(A)/N(A)$. However, in general this is not possible as this functor might not be representable [80, Example 5.1.16]. This can for example be resolved by considering *fppf*[1] group schemes and defining the quotient to be the *fppf sheafification* of the above functor [80, Section 5.2.2].

However, as we are mostly interested in the theory of algebraic groups, we make an ad hoc definition of exactness for sequences of algebraic groups.

**Definition 1.1.24** (Short exact sequence of algebraic groups)**.** Let $K$ be a field. A sequence

$$1 \longrightarrow \mathbf{N} \xrightarrow{\ i\ } \mathbf{G} \xrightarrow{\ q\ } \mathbf{Q} \longrightarrow 1$$

of algebraic groups over $K$ is called a *short exact sequence*, if $i$ is an isomorphism from $\mathbf{N}$ to $\ker(q)$ and $q$ is faithfully flat, i.e. flat and surjective.

We note that, as in the case of classical group theory, every normal subgroup arises as the kernel of a quotient (i.e. a faithfully flat) morphism.

**Theorem 1.1.25** ([67, Theorem 5.14])**.** *Let $K$ be a field and $\mathbf{G}$ an algebraic group over $K$. Every normal algebraic subgroup $\mathbf{N} \trianglelefteq \mathbf{G}$ arises as the kernel of a quotient map $\mathbf{G} \to \mathbf{Q}$.*

In case $\mathbf{G}$ (and thus also $\mathbf{N}$) is affine, $\mathbf{Q}$ will be affine as well [67, Proposition 5.18]. Moreover, as quotients are unique (up to isomorphism) [67, Proposition 5.13], we can define the quotient of $\mathbf{G}$ by $\mathbf{N}$ as $\mathbf{G}/\mathbf{N} = \mathbf{Q}$.

**Remark 1.1.26** (Checking exactness)**.** In the above definition, surjectivity can simply be checked on the points of the algebraic closure. In general, showing flatness is more involved. However, in case the algebraic group $\mathbf{Q}$ is reduced, the assumption of flatness on $q$ is automatic by the surjectivity [67, Proposition 1.70].

This is in particular useful if $K$ is of characteristic zero as we only need to show surjectivity.

[1] fppf is an abbreviation of the French “fidèlement plat de présentation finie” meaning “faithfully flat of finite presentation”.

**Theorem 1.1.27** ([67, Corollary 8.39])**.** *Every algebraic group over a field of characteristic zero is smooth, and thus in particular reduced.*

The notion of quotients also allows us to define when a normal algebraic subgroup is of finite index. First let us define what a finite algebraic group is.

**Definition 1.1.28** (Finite algebraic group)**.** Let $K$ be a field. An algebraic group $\mathbf{G}$ over $K$ is said to be *finite* if for every field extension $K'|K$ the group $\mathbf{G}(K')$ is finite.

Note that it is not sufficient for $\mathbf{G}(K)$ to be finite in order for $\mathbf{G}$ to be finite. For example, consider the additive group $\mathbf{G}_a$ over the finite field $\mathbb{F}_2$. Here $\mathbf{G}_a(\mathbb{F}_2)$ is finite, but $\mathbf{G}_a$ is *not*, as $\mathbf{G}_a({\mathbb{F}_2}^{\mathrm{sep}})$ is infinite.

**Definition 1.1.29** (Normal subgroup of finite index)**.** Let $K$ be a field, $\mathbf{G}$ an algebraic group over $K$, and $\mathbf{N} \trianglelefteq \mathbf{G}$ a normal algebraic subgroup. Then $\mathbf{N}$ *has finite index in* $\mathbf{G}$ if the quotient $\mathbf{G}/\mathbf{N}$ is a finite algebraic group.

### 1.1.3 Extension and restriction of scalars

Given a group scheme $G$ over a ring $R$, we would like to turn it into other group schemes over possibly different rings. Here we want to discuss two such constructions: *extension of scalars* and *restriction of scalars*.

**Example 1.1.30** (Extension of scalars for modules)**.** Let $R$ be a ring, let $S$ be an $R$-algebra and let $M$ be an $R$-module. Then the *extension of scalars of $M$ to $S$* is the $S$-module

$$M_S := M \otimes_R S.$$

Now let $K|k$ be a field extension and let $V$ be a $k$-vector space. Given a $k$-basis $(b_i)_{i\in I}$ of $V$ we have a canonical $k$-linear isomorphism $V \cong_k k^{(I)}$. Thus, for the extension of scalars we get

$$V_K = V \otimes_k K \cong_K k^{(I)} \otimes_k K \cong_K (k \otimes_k K)^{(I)} \cong_K K^{(I)}.$$

So in other words, the extension of scalars $V_K$ is a $K$-vector space with the same basis as the $k$-vectors space $V$.

**Example 1.1.31** (Restriction of scalars for modules)**.** Let $R$ be a ring, let $S$ be an $R$-algebra via a ring homomorphism $f\colon R \to S$ and let $M$ be an $S$-module. Then the *restriction of scalars of $M$ to $R$* is the $R$-module $\operatorname{Res}_{S|R} M$, with the same underlying abelian group as $M$ and the $R$-module structure given by

$$\begin{aligned} M \times R &\longrightarrow M \\ (m, r) &\longmapsto m \cdot f(r). \end{aligned}$$

Let us again consider the case of a field extension $K|k$ and a $K$-vector space $V$. Then the restriction of scalars $\operatorname{Res}_{K|k} V$ is just the vector space $V$ considered as a $k$-vector space. In terms of bases we have the following description. Let $(b_i)_{i\in I}$ be a $K$-basis of $V$ and $(a_j)_{j\in J}$ be a $k$-basis of $K$. Then $\operatorname{Res}_{K|k} V$ is the $k$-vector space with basis $(a_j b_i)_{j\in J, i\in I}$.

Both of these constructions are functorial in the modules.

**Remark 1.1.32** (Adjointness of extension and restriction of scalars)**.** Let $R$ be a ring and let $S$ be an $R$-algebra via $f\colon R \to S$. Let $M$ be an $R$-module and let $N$ be an $S$-module. For an $R$-module homomorphism $\varphi\colon M \to \operatorname{Res}_{S|R} N$, we can define an $S$-module homomorphism $M_S \to N$ as the composition

$$\begin{aligned} M_S = M \otimes_R S \xrightarrow{\varphi \otimes \mathrm{id}_S} \operatorname{Res}_{S|R} N \otimes_R S &\longrightarrow N. \\ n \otimes s &\longmapsto n \cdot s \end{aligned}$$

This gives a natural (in both $M$ and $N$) map $\operatorname{Hom}_R(M, \operatorname{Res}_{S|R} N) \to \operatorname{Hom}_S(M_S, N)$.

Conversely, given an $S$-module homomorphisms $\psi\colon M_S \to N$, we can construct an $R$-module homomorphism $M \to \operatorname{Res}_{S|R} N$ as

$$\begin{aligned} M &\longrightarrow M \otimes_R R \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_R S = M_S \xrightarrow{\psi} N. \\ m &\longmapsto m \otimes 1 \end{aligned}$$

Again, we get a natural map $\operatorname{Hom}_S(M_S, N) \to \operatorname{Hom}_R(M, \operatorname{Res}_{S|R} N)$. It is easy to see that these two maps are mutually inverse and thus give an adjunction

$$\operatorname{Hom}_S((\,\cdot\,)_S, \,\cdot\,) \cong \operatorname{Hom}_R(\,\cdot\,, \operatorname{Res}_{S|R} \,\cdot\,).$$

Let us now discuss these two constructions in the setting of group schemes.

### Extension of scalars

**Lemma 1.1.33** (Base change of group schemes)**.** *Let $R$ be a ring, let $G$ be a group scheme over $R$ and let $S$ be an $R$-algebra. Then*

$$G_S := G \times_R S = G \times_R \operatorname{Spec} S$$

*is a group scheme over $S$.*

*If $G$ is affine (respectively of finite type over $R$), then $G_S$ will be affine (respectively of finite type over $S$) as well.*

*Proof.* As base change preserves products, it is easy to see that $G_S$ is again a group scheme over $S$. Moreover, base change preserves being of finite type, so if $G$ is of finite type over $R$, then $G_S$ will also be of finite type over $S$. Finally, if $G = \operatorname{Spec} A$ is affine we get that

$$G_S = \operatorname{Spec} A \times_R \operatorname{Spec} S = \operatorname{Spec}(A \otimes_R S)$$

is affine as well. □

**Example 1.1.34.** We consider the real algebraic group $\mathbf{G}\colon \mathrm{Alg}_{\mathbb{R}} \to \mathrm{Group}$ given by

$$\mathbf{G}(A) = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \;\middle|\; x, y \in A, x^2 + y^2 = 1 \right\} \leq \mathrm{SL}_2(A).$$

The base change to $\mathbb{C}$ is isomorphic to $\mathbf{G}_m$, via the mutually inverse natural isomorphisms

$$\begin{aligned} \varphi_A\colon \mathbf{G}_{\mathbb{C}}(A) &\longrightarrow \mathbf{G}_m(A) = A^\times & \quad\text{and}\quad & & \psi_A\colon \mathbf{G}_m(A) &\longrightarrow \mathbf{G}_{\mathbb{C}}(A), \\ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} &\longmapsto x - iy & & & a &\longmapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{aligned}$$

where in the second map $x = \frac{1}{2}(a + a^{-1})$ and $y = \frac{1}{2}(a - a^{-1})$.

However, $\mathbf{G} \ncong_{\mathbb{R}} \mathbf{G}_m$ as for the groups of real points we have

$$\mathbf{G}(\mathbb{R}) \cong_{\text{Group}} \{z \in \mathbb{C} \mid |z| = 1\} = \mathbb{S}^1 \ncong \mathbb{R}^\times = \mathbf{G}_m(\mathbb{R}).$$

Viewing affine group schemes as functors, applying base change is just pre-composition with the inclusion functor $\mathrm{Alg}_S \hookrightarrow \mathrm{Alg}_R$,

$$\begin{tikzcd} \mathrm{Alg}_S \arrow[r, hook] \arrow[rr, bend right, "G_S"'] & \mathrm{Alg}_R \arrow[r, "G"] & \text{Group}. \end{tikzcd}$$

This is formalized by the following corollary.

**Corollary 1.1.35** (Base change as functors)**.** *Let $R$ be a ring, let $S$ be an $R$-algebra and let $G$ be a group scheme over $R$. Restricting $G$ to the subcategory of $S$-algebras and considering it as a functor* $\mathrm{Alg}_S \to \mathrm{Group}$*, we have a natural isomorphism $G(\,\cdot\,) \cong G_S(\,\cdot\,)$.*

*Proof.* By the universal property of the pullback, we have a natural isomorphism

$$\mathrm{Hom}_S(\,\cdot\,, G_S) = \mathrm{Hom}_S(\,\cdot\,, G \times_R S) \cong \mathrm{Hom}_R(\,\cdot\,, G).$$

Now precomposing with Spec gives the claim. $\square$

### Restriction of scalars

Let $R$ be a ring, $S$ an $R$-algebra, and $G$ a group scheme over $S$. Using the structure map $R \to S$ of $S$ as an $R$-algebra, we can, via the composition

$$G \longrightarrow \operatorname{Spec} S \longrightarrow \operatorname{Spec} R,$$

consider $G$ as a scheme over $R$. However, this construction is in general *not* compatible with products and thus will in general not give $G$ the structure of a group scheme over $R$. So for the restriction of scalars, we have to use a different construction.

Using the adjointness of extension and restriction of scalars in the case of modules as motivation, we can ask for the restriction of scalars to be right-adjoint to extension of scalars. Thus, we ask for

$$\operatorname{Res}_{S|R} G(\,\cdot\,) = \mathrm{Hom}_R(\,\cdot\,, \operatorname{Res}_{S|R} G) \cong \mathrm{Hom}_S(\,\cdot\, \times_R S, G) = G(\,\cdot\, \times_R S),$$

or, from the viewpoint of functors $\mathrm{Alg}_R \to \mathrm{Group}$

$$\operatorname{Res}_{S|R} G(A) = G(A \otimes_R S),$$

for all $R$-algebras $A$. Using this as a definition, we always obtain a functor $\mathrm{Alg}_R \to \mathrm{Group}$ and in order to obtain a group scheme over $R$, only the question of representability remains.

**Definition 1.1.36** (Restriction of scalars)**.** Let $R$ be a ring, $S$ an $R$-algebra, and $G$ a group scheme over $S$. A scheme over $R$ is called *restriction of scalars of $G$ to $R$* if it represents the functor

$$\mathrm{Alg}_R \xrightarrow{\cdot \otimes_R S} \mathrm{Alg}_S \xrightarrow{G} \mathrm{Group}.$$

If it exists, we denote such a scheme by $\mathrm{Res}_{S|R}\, G$. It is unique up to isomorphisms by the Yoneda lemma.

In the case of algebraic groups and finite Galois extension, the restriction of scalars always exits and is known as *Weil restriction.*

**Theorem 1.1.37** (Weil restriction [67, Section 2.i])**.** *Let $K|k$ be a finite Galois extension and let $\mathbf{G}$ be an algebraic group over $K$. Then the restriction of scalars $\mathrm{Res}_{K|k}\, \mathbf{G}$ exists.*

**Remark 1.1.38** (Base change of restriction)**.** Let $K|k$ be a finite Galois extension and let $\mathbf{G}$ be an algebraic group over $K$. The restriction of scalars $\mathrm{Res}_{K|k}\, \mathbf{G}$ is an algebraic group over $k$, so we can apply base change back to $K$. For a $K$-algebra $A$ we then have

$$\left(\mathrm{Res}_{K|k}\, \mathbf{G}\right)_K(A) \cong \mathrm{Res}_{K|k}\, \mathbf{G}(A) = \mathbf{G}(A \otimes_k K),$$

where we have used Corollary 1.1.35. Now as a $K$-algebra

$$A \otimes_k K \cong_K \prod_{\sigma \in \mathrm{Gal}(K|k)} A^\sigma,$$

giving

$$\left(\mathrm{Res}_{K|k}\, \mathbf{G}\right)_K(A) \cong \prod_{\sigma \in \mathrm{Gal}(K|k)} \mathbf{G}(A^\sigma).$$

So as an algebraic group over $K$ we have $\mathrm{Res}_{K|k}\, \mathbf{G} \cong \prod_{\sigma \in \mathrm{Gal}(K|k)} \mathbf{G}^\sigma$, where $\mathbf{G}^\sigma$ is the algebraic group $A \mapsto \mathbf{G}(A^\sigma)$.

### 1.1.4 Models of group schemes

In Example 1.1.34 we have seen that different group schemes can have isomorphic base changes. This explains the following definition.

**Definition 1.1.39** (Model of a group scheme)**.** Let $R$ be a ring, let $S$ be an $R$-algebra and let $G$ be a group scheme over $S$. An *$R$-model of $G$* is a group scheme $H$ over $R$, such that there is an isomorphism $H_S \cong_S G$ of group schemes over $S$.

From the viewpoint of affine group schemes as representable functors, we thus have that $R$-models of an affine group scheme $G$ over $S$ are just possible extensions of $G\colon \mathrm{Alg}_S \to \mathrm{Group}$ to $\mathrm{Alg}_R$,

$$\begin{tikzcd} \mathrm{Alg}_S \arrow[r, "G"] \arrow[d, hook] & \mathrm{Group}. \\ \mathrm{Alg}_R \arrow[ur, dashed] & \end{tikzcd}$$

**Example 1.1.40.** The algebraic group given in Example 1.1.34 is a real model of the complex algebraic group $\mathbf{G}_m$. More generally, let $K$ be a field, let $a \in K^\times$ and consider the algebraic group $\mathbf{G}_a \colon \mathrm{Alg}_K \to \mathrm{Group}$ given by

$$\mathbf{G}_a(A) = \left\{ \begin{pmatrix} x & ay \\ y & x \end{pmatrix} \;\middle|\; x, y \in A, x^2 - ay^2 = 1 \right\} \leq \mathrm{SL}_2(A).$$

If $a$ is square free, then $\mathbf{G}_a$ is *not* isomorphic to $\mathbf{G}_m$. However, its base change to $K(\sqrt{a})$ *is* isomorphic to $\mathbf{G}_m$.

In fact, one can show for $a, b \in K^\times$, that the algebraic $K$-groups $\mathbf{G}_a$ and $\mathbf{G}_b$ are isomorphic if and only if $ab^{-1}$ is a square in $K$. In particular, over the algebraic (and even the separable) closure of $K$, all these groups are isomorphic to $\mathbf{G}_m \cong \mathbf{G}_1$.

As noticed in Lemma 1.1.33, being affine and of finite type is preserved under base change. We can now ask whether the same is true for models. Let $G$ be a group scheme over $S$ and let $H$ be an $R$-model of $G$, where $R \subseteq S$. If $G$ is affine/of finite type, is then $H$ also affine/of finite type? Moreover, if $G_0 \leq G$ is a subgroup scheme and $H_0$ is an $R$-model of $G_0$, is $H_0$ a subgroup scheme of $H$?

**Proposition 1.1.41** (Group schemes and descent)**.** *Let $R$ be a ring, let $S$ be a faithfully flat $R$-algebra, let $G$ be a group scheme over $S$ and let $H$ be an $R$-model of $G$.*

1. *If $G$ is affine, then $H$ is affine as well.*
2. *If $G$ is of finite type over $S$, then $H$ is also of finite type over $R$.*
3. *If $G_0 \leq G$ is a subgroup scheme, and $H_0$ is an $R$-model of $G_0$, then $H_0$ is a subgroup scheme of $H$.*

*In particular, this holds for field extension and algebraic groups.*

*Proof.* By assumption, we have that $\operatorname{Spec} S \to \operatorname{Spec} R$ is faithfully flat. As morphisms between affine schemes are always quasi-compact, this gives that $\operatorname{Spec} S \to \operatorname{Spec} R$ is a fpqc[2]-morphism. Since being affine, being of finite type, and being a closed immersion are stable under fpqc-descent [80, Section C.2, Table 1], we get the claim. □

### 1.1.5 Galois cohomology and forms of algebraic groups

Above we discussed *models* of group schemes, i.e. group schemes that give a certain other group scheme when base changed to some extension. For algebraic groups, it is in particular useful to study algebraic groups that become isomorphic over some field extension.

**Definition 1.1.42** (Forms of algebraic groups)**.** Let $K|k$ be a field extension and let $\mathbf{G}$ be an algebraic $k$-group. A $K|k$-form of $\mathbf{G}$ is an algebraic $k$-group $\mathbf{H}$ such that $\mathbf{G}_K \cong_K \mathbf{H}_K$.

In case $K = k^{\mathrm{sep}}$ is a separable closure of $k$, we simply talk about $k$-forms instead of $k^{\mathrm{sep}}|k$-forms.

[2] fpqc is an abbreviation for the French "fidèlement plate et quasi-compacte" meaning "faithfully flat and quasi-compact".

Using our previous definition of models, we could alternatively say a $K|k$-form of an algebraic $k$-group $\mathbf{G}$ is a $k$-model of the algebraic group $\mathbf{G}_K$.

The goal of this section is to present the classification of $K|k$-forms of affine algebraic groups in terms of *Galois cohomology.*

**Theorem 1.1.43** (Classification of $K|k$-forms)**.** *Let $K|k$ be a Galois extension and let $\mathbf{G}$ be an affine algebraic group over $k$. Denote by $E(K|k, \mathbf{G})$ the set of equivalence classes of $K|k$-forms of $\mathbf{G}$ up to $k$-isomorphism. Then there exists a natural bijection*

$$E(K|k, \mathbf{G}) \longrightarrow H^1(K|k, \mathrm{Aut}(\mathbf{G})),$$

*sending the equivalence class of $\mathbf{G}$ to the class represented by the trivial cocycle.*

Galois cohomology is a special case of non-abelian cohomology, in this case

$$H^*(K|k; \mathrm{Aut}(\mathbf{G})) = H^*(\mathrm{Gal}(K|k); \mathrm{Aut}_K(\mathbf{G}_K)),$$

for some affine algebraic group $\mathbf{G}$ over $k$. For the definition and some basic properties of non-abelian cohomology we refer the reader either to Serre's book on the topic [95, Chapter I, §5] or Section A.3, where all the lengthy computations omitted by Serre are spelled out.

**Remark 1.1.44** (Inner forms)**.** Let $K|k$ be a Galois extension and let $\mathbf{G}$ be an affine algebraic group over $k$. The above theorem associates to a $K|k$-form $\mathbf{H}$ of $\mathbf{G}$, a cocycle $\mathrm{Gal}(K|k) \to \mathrm{Aut}_K(\mathbf{G}_K)$. This cocycle might only take values in $\mathrm{Inn}(\mathbf{G}_K)$, or, more generally, might be equivalent to one with values in $\mathrm{Inn}(\mathbf{G}_K)$. In this case we call $\mathbf{H}$ an *inner form of* $\mathbf{G}$. Otherwise, we say $\mathbf{H}$ is an *outer form of* $\mathbf{G}$.

It is not hard to see that inner forms of $\mathbf{G}$ are precisely the ones in the image of

$$H^1(K|k; \mathrm{Inn}(\mathbf{G})) \longrightarrow H^1(K|k; \mathrm{Aut}(\mathbf{G})).$$

### The Galois action

In order to make sense of the above Galois cohomology, we first need to describe the action of the Galois group $\mathrm{Gal}(K|k)$ on the automorphism group $\mathrm{Aut}_K(\mathbf{G}_K)$.

**Remark 1.1.45** (Abstract description of the action)**.** Let $K|k$ be a separable extension and let $X$ be a scheme over $k$. For $\sigma \in \mathrm{Gal}(K|k)$ we consider the commutative diagram

$$\begin{tikzcd}
X_K \arrow[r, "\pi"] \arrow[d, "\sigma^*"', "\cong"] & \operatorname{Spec} K \arrow[d, "\operatorname{Spec}\sigma"] \\
X_K \arrow[r, "\pi"] \arrow[d] & \operatorname{Spec} K \arrow[d] \\
X \arrow[r] & \operatorname{Spec} k,
\end{tikzcd}$$

where both the upper and lower square are pullbacks. By the pasting law for pullbacks, the outer square is also a pullback. Moreover, since $\sigma$ is an automorphism over $k$, we in fact have that the outer and lower pullbacks are isomorphic over $k$. So, in particular, we can indeed assume that the upper morphism is given by the same structure morphism $\pi\colon X_K \to \operatorname{Spec} K$. Note that, although $\sigma^*\colon X_K \to X_K$ is an isomorphism of $k$-schemes, it is in general *not* an isomorphism of $K$-schemes.

By naturality of pullbacks, we get that this construction is compatible with composition, i.e. that for $\sigma, \tau \in \operatorname{Gal}(K|k)$ we have $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$. So $\operatorname{Gal}(K|k)$ acts on $X_K$ via $k$-automorphisms.

Now let $Y$ be another scheme over $k$. Similarly as above, we obtain for $\sigma \in \operatorname{Gal}(K|k)$ a $k$-isomorphism $\sigma^*\colon Y_K \to Y_K$. Hence, for every morphism of schemes $\varphi \in \operatorname{Hom}_K(X_K, Y_K)$, we have a commutative diagram

$$\begin{tikzcd}
 & Y_K \arrow[rr] \arrow[dd, "\sigma^*" near end] & & \operatorname{Spec} K \arrow[dd, "\operatorname{Spec}\sigma"] \\
X_K \arrow[ur, "(\sigma^*)^{-1}\circ\varphi\circ\sigma^*"] \arrow[rr] \arrow[dd, "\sigma^*"'] & & \operatorname{Spec} K \arrow[ur, equal] \arrow[dd, "\operatorname{Spec}\sigma" near start] & \\
 & Y_K \arrow[rr] & & \operatorname{Spec} K. \\
X_K \arrow[ur, "\varphi"] \arrow[rr] & & \operatorname{Spec} K \arrow[ur, equal] &
\end{tikzcd}$$

Here the horizontal maps are the structure morphisms of $X_K$ and $Y_K$, respectively. The commutativity of the bottom square encodes that $\varphi$ is a morphism over $K$. Similarly, the commutativity of the top square gives that

$$^{\sigma}\varphi := (\sigma^*)^{-1} \circ \varphi \circ \sigma^*\colon X_K \longrightarrow Y_K$$

is a morphism over $K$. As before, it is easy to see that this is compatible with composition, such that we obtain a group action $\operatorname{Gal}(K|k) \curvearrowright \operatorname{Hom}_K(X_K, Y_K)$.

Moreover, by construction it is also easy to see that if $\psi\colon Y \to Z$ is another morphisms of $k$-schemes, we get $^{\sigma}(\psi \circ \varphi) = {}^{\sigma}\psi \circ {}^{\sigma}\varphi$ for all $\sigma \in \operatorname{Gal}(K|k)$, and thus the action restricts to one on $\operatorname{Aut}_K(X_K)$.

Finally, as pullbacks preserve products, it is also easy to see that this is compatible with the structure of an algebraic group, giving a group action of the Galois group on the (auto)morphisms of algebraic groups.

Let us also give a more hands-on description of this action in the case of affine algebraic groups.

For affine schemes $X, Y$ over $k$, a morphism $X_K \to Y_K$ is essentially defined by finitely many polynomials with coefficients in $K$. The action of $\sigma \in \operatorname{Gal}(K|k)$ on such a morphism is then simply given by applying $\sigma$ to all the coefficients of the polynomial, to get another morphism $X_K \to Y_K$. Let us see that this is actually the correct description of the action above.

**Remark 1.1.46** (Explicit description of the Galois action)**.** Let $K|k$ be a separable extension and let $X = \operatorname{Spec} S$ be an affine scheme of finite type over $k$. Then we can write

$$S = {k[T_1, \dots, T_n]}\big/{(f_1, \dots, f_r)},$$

for some $n, r \in \mathbb{N}$ and $f_1, \dots, f_r \in k[T_1, \dots, T_n]$.

For $\sigma \in \operatorname{Gal}(K|k)$, the morphism $\alpha_\sigma \colon X_K \to X_K$ is induced by the morphism

$$\mathrm{id}_S \otimes_k \sigma \colon S \otimes_k K \to S \otimes_k K.$$

Under the canonical identification

$$S \otimes_k K \cong_K {K[T_1, \dots, T_n]}\big/{(f_1, \dots, f_r)},$$

this morphism is given by

$$\begin{aligned} \beta_\sigma \colon {K[T_1, \dots, T_n]}\big/{(f_1, \dots, f_r)} &\longrightarrow {K[T_1, \dots, T_n]}\big/{(f_1, \dots, f_r)}. \\ \left[\sum_I a_I T^I\right] &\longmapsto \left[\sum_I \sigma(a_I) T^I\right] \end{aligned}$$

Now let $Y = \operatorname{Spec} S'$ be another affine scheme of finite type over $k$. Again we can write

$$S' = {k[T_1, \dots, T_m]}\big/{(g_1, \dots, g_s)}$$

for some $m, s \in \mathbb{N}$ and $g_1, \dots, g_s \in k[T_1, \dots, T_m]$. Every morphism $\varphi \colon X_K \to Y_K$ of schemes over $K$ corresponds to a $K$-algebra homomorphism

$$\begin{aligned} \psi \colon {K[T_1, \dots, T_m]}\big/{(g_1, \dots, g_s)} &\longrightarrow {K[T_1, \dots, T_n]}\big/{(f_1, \dots, f_r)}, \\ [T_i] &\longmapsto [h_i] \end{aligned}$$

for polynomials $h_1, \dots, h_m \in K[T_1, \dots, T_n]$ with $g_j(h_1, \dots, h_m) = 0$ for all $j \in \{1, \dots, s\}$.

So we obtain that ${}^\sigma\varphi \colon X_K \to Y_K$ corresponds to the composition

$$\beta_\sigma \circ \psi \circ (\beta_\sigma)^{-1} = \beta_\sigma \circ \psi \circ \beta_{\sigma^{-1}}.$$

This composition is just given by

$$\begin{aligned} \psi \colon {K[T_1, \dots, T_m]}\big/{(g_1, \dots, g_s)} &\longrightarrow {K[T_1, \dots, T_n]}\big/{(f_1, \dots, f_r)}, \\ [T_i] &\longmapsto [{}^\sigma h_i] \end{aligned}$$

where ${}^\sigma h_i$ is the polynomial obtained from $h_i$ by applying $\sigma$ to all coefficients, as for $i \in \{1, \dots, m\}$ we have

$$\begin{aligned} \beta_\sigma \circ \psi \circ \beta_{\sigma^{-1}}([T_i]) &= \beta_\sigma \circ \psi([T_i]) \\ &= \beta_\sigma([h_i]) \\ &= [{}^\sigma h_i]. \end{aligned}$$

As in Section A.4 we will now work with a functor $A$ from the category of field extensions of $k$ to the category of groups with the following properties:

(1) For an extension $K|k$ we have $A(K) = \varinjlim_i A(K_i)$, where $K_i$ are all finitely generated field extensions of $k$ with $k \subseteq K_i \subseteq K$.

(2) If $K \to K'$ is a morphism of field extensions, the induced morphism $A(K) \to A(K')$ is injective.

(3) If $K'|K$ is a Galois extension, we have $A(K) = H^0(\mathrm{Gal}(K'|K); A(K'))$.

Let us verify that $K \mapsto \mathrm{Aut}_K(\mathbf{G}_K)$ actually satisfies these properties. We start with the second property.

**Proposition 1.1.47.** *Let $k$ be a field, let $\mathbf{G} = \operatorname{Spec} S$ be an affine algebraic group over $k$ and let $\sigma\colon K \to K'$ be a morphism of field extensions of $k$. Then the induced morphism $\mathrm{Aut}_K(\mathbf{G}_K) \to \mathrm{Aut}_{K'}(\mathbf{G}_{K'})$, given by pullback, is injective.*

*Proof.* Since field extensions are faithfully flat, we have that

$$\cdot \otimes_K \mathrm{id}_{K'} \colon \operatorname{Hom}_{\mathrm{Alg}_K}(A, B) \longrightarrow \operatorname{Hom}_{\mathrm{Alg}_{K'}}(A \otimes_K K', B \otimes_K K')$$

is injective for all $K$-algebras $A, B$. In particular, the induced map

$$\begin{aligned} \mathrm{Aut}_K(S \otimes_k K) &\longrightarrow \mathrm{Aut}_{K'}((S \otimes_k K) \otimes_K K') \cong \mathrm{Aut}_{K'}(S \otimes_k K') \\ \varphi &\longmapsto \varphi \otimes_K \mathrm{id}_{K'} \end{aligned}$$

of Hopf algebra automorphisms is injective as well. $\square$

**Proposition 1.1.48.** *Let $K|k$ be a field extension, let $\mathbf{G}$ be an affine algebraic group over $k$ and let $(K_i)_i$ be all finitely generated subextensions $k \subseteq K_i \subseteq K$. Then there is a canonical isomorphism*

$$\mathrm{Aut}_K(\mathbf{G}_K) \cong \varinjlim_i \mathrm{Aut}_{K_i}(\mathbf{G}_{K_i}).$$

*Proof.* First we have that $K$ can be written as the ascending union of the subextensions $K_i$. Moreover, considering the groups $\mathrm{Aut}_{K_i}(\mathbf{G}_{K_i})$ as subgroups of $\mathrm{Aut}_K(\mathbf{G}_K)$ we get that the direct limit of the groups $\mathrm{Aut}_{K_i}(\mathbf{G}_{K_i})$ is an ascending union

$$\varinjlim_i \mathrm{Aut}_{K_i}(\mathbf{G}_{K_i}) = \bigcup_i \mathrm{Aut}_{K_i}(\mathbf{G}_{K_i}) \subseteq \mathrm{Aut}_K(\mathbf{G}_K)$$

as well.

Now let $\varphi \in \mathrm{Aut}_K(\mathbf{G}_K)$. As already used in Remark 1.1.46, $\varphi$ is defined by finitely many polynomials with coefficients in $K$. If we take $K_0$ to be the field generated by these finitely many coefficients, we obtain a finitely generated subextension $k \subseteq K_0 \subseteq K$. By construction, $\varphi$ is already defined over $K_0$, showing that $\varphi \in \bigcup_i \mathrm{Aut}_{K_i}(\mathbf{G}_{K_i})$. $\square$

Finally, let us also verify the last property.

**Proposition 1.1.49.** *Let $k$ be a field, let $\mathbf{G}$ be an affine algebraic group over $k$, let $K|k$ be an extension of $k$ and let $K'|K$ be a Galois extension. Then we have*

$$H^0(\mathrm{Gal}(K'|K); \mathrm{Aut}_{K'}(\mathbf{G}_{K'})) = \mathrm{Aut}_K(\mathbf{G}_K).$$

*Proof.* By considering $\mathbf{G}_K$ instead of $\mathbf{G}$, we can without loss of generality assume that $K = k$. Now, by definition, we have

$$H^0(\mathrm{Gal}(K'|k); \mathrm{Aut}_{K'}(\mathbf{G}_{K'})) = \mathrm{Aut}_{K'}(\mathbf{G}_{K'})^{\mathrm{Gal}(K'|k)},$$

so we only need to show $\mathrm{Aut}_{K'}(\mathbf{G}_{K'})^{\mathrm{Gal}(K'|k)} = \mathrm{Aut}_k(\mathbf{G})$, where we consider $\mathrm{Aut}_k(\mathbf{G})$ as a subgroup of $\mathrm{Aut}_{K'}(\mathbf{G}_{K'})$ using Proposition 1.1.47.

"$\supseteq$": Let $\varphi \in \mathrm{Aut}_k(\mathbf{G})$ and $\sigma \in \mathrm{Gal}(K'|k)$. As described in Remark 1.1.46, ${}^\sigma\varphi$ is given by applying $\sigma$ to the coefficients of the polynomials defining $\varphi$. Since $\varphi$ is defined over $k$, all of these coefficients are $\mathrm{Gal}(K'|k)$-invariant. So ${}^\sigma\varphi = \varphi$.

"$\subseteq$": Now let $\varphi \in \mathrm{Aut}_{K'}(\mathbf{G}_{K'})^{\mathrm{Gal}(K'|k)}$. Again $\varphi$ is defined by polynomials over $K'$. Since $\varphi$ is $\mathrm{Gal}(K'|k)$-invariant, we get that the defining polynomials are $\mathrm{Gal}(K'|k)$-invariant as well. But then all the coefficients are already in $K'^{\mathrm{Gal}(K'|k)} = k$. So $\varphi$ is defined over $k$, i.e. $\varphi \in \mathrm{Aut}_k(\mathbf{G})$. □

### The proof of Theorem 1.1.43

**Construction 1.1.50.** Let $K|k$ be a separable field extension, let $\mathbf{G}$ be an algebraic group over $k$ and let $\mathbf{G}'$ be a $K|k$-form of $\mathbf{G}$, i.e. another algebraic group over $k$ such that there is a $K$-isomorphism $\Theta\colon \mathbf{G}_K \to \mathbf{G}'_K$. Then the map

$$\begin{aligned} \varphi\colon\ \mathrm{Gal}(K|k) &\longrightarrow \mathrm{Aut}_K(\mathbf{G}) \\ \sigma &\longmapsto \Theta^{-1} \circ \sigma \cdot \Theta \end{aligned}$$

is a cocycle. For $\sigma, \tau \in \mathrm{Gal}(K|k)$ we have

$$\begin{aligned} \varphi(\sigma\tau) &= \Theta^{-1} \circ (\sigma\tau) \cdot \Theta \\ &= \Theta^{-1} \circ \sigma \cdot (\tau \cdot \Theta) \\ &= \Theta^{-1} \circ \sigma \cdot (\Theta \circ \Theta^{-1} \circ \tau \cdot \Theta) \\ &= \Theta^{-1} \circ \sigma \cdot \Theta \circ \sigma \cdot (\Theta^{-1} \circ \tau \cdot \Theta) \\ &= \varphi(\sigma) \circ \sigma \cdot \varphi(\tau). \end{aligned}$$

Moreover, the class represented by this cocycle in $H^1(K|k; \mathrm{Aut}_K(\mathbf{G}))$ is independent of the chosen isomorphism $\mathbf{G}_K \cong_K \mathbf{G}'_K$. If $\Phi\colon \mathbf{G}_K \to \mathbf{G}'_K$ is another $K$-isomorphism, we have

$$\Phi^{-1} \circ {}^\sigma\Phi = (\Theta^{-1} \circ \Phi)^{-1} \circ \Theta^{-1} \circ {}^\sigma\Theta \circ {}^\sigma(\Theta^{-1} \circ \Phi)$$

for all $\sigma \in \operatorname{Gal}(K|k)$ where $\Theta^{-1} \circ \Phi \in \operatorname{Aut}_K(\mathbf{G}_K)$. Thus, the two cocycles defined by $\Theta$ and $\Phi$ are cohomologous, and we obtain a well-defined map

$$\{\mathbf{H} \text{ algebraic group over } k \mid \mathbf{G}_K \cong_K \mathbf{H}_K\} \longrightarrow H^1(K|k; \operatorname{Aut}_K(\mathbf{G}_K)).$$

Finally, if $\mathbf{G}''$ is another algebraic group over $k$, isomorphic to $\mathbf{G}'$ via a $k$-isomorphism $\Phi\colon \mathbf{G}' \to \mathbf{G}''$, then $\mathbf{G}''$ is also a $K|k$-form of $\mathbf{G}$ by the $K$-isomorphism

$$\Phi_K \circ \Theta\colon \mathbf{G}_K \longrightarrow \mathbf{G}'_K \longrightarrow \mathbf{G}''_K.$$

Since $\Phi_K$ is $\operatorname{Gal}(K|k)$-invariant, we get for all $\sigma \in \operatorname{Gal}(K|k)$ that

$$\begin{aligned}(\Phi_K \circ \Theta)^{-1} \circ {}^\sigma(\Phi_K \circ \Theta) &= \Theta^{-1} \circ \Phi_K^{-1} \circ {}^\sigma\Phi_K \circ {}^\sigma\Theta \\ &= \Theta^{-1} \circ \Phi_K^{-1} \circ \Phi_K \circ {}^\sigma\Theta \\ &= \Theta^{-1} \circ {}^\sigma\Theta.\end{aligned}$$

So the cocycles defined by $\mathbf{G}'$ and $\mathbf{G}''$ are the same. Hence, we in fact obtain a map

$$\Omega\colon E(K|k, \mathbf{G}) \longrightarrow H^1(K|k; \operatorname{Aut}_K(\mathbf{G}_K)).$$

Notice that by construction the image of $\mathbf{G}$ under $\Omega$ is the trivial cocycle.

**Proposition 1.1.51.** *Let $K|k$ be a Galois extension and let $\mathbf{G}$ be an algebraic group over $k$. The map $\Omega$ constructed above is injective.*

*Proof.* Let $\mathbf{G}', \mathbf{G}''$ be two $K|k$-forms of $\mathbf{G}$ whose associated cocycles are cohomologous. This means that for $K$-isomorphisms $\Theta\colon \mathbf{G}_K \to \mathbf{G}'_K$ and $\Phi\colon \mathbf{G}_K \to \mathbf{G}''_K$, there exists $a \in \operatorname{Aut}_K(\mathbf{G}_K)$ such that

$$\Phi^{-1} \circ {}^\sigma\Phi = a^{-1} \circ \Theta^{-1} \circ {}^\sigma\Theta \circ {}^\sigma a,$$

for all $\sigma \in \operatorname{Gal}(K|k)$. Equivalently, we have for all $\sigma \in \operatorname{Gal}(K|k)$ that

$$\Theta \circ a \circ \Phi^{-1} = {}^\sigma\Theta \circ {}^\sigma a \circ {}^\sigma\Phi^{-1} = {}^\sigma(\Theta \circ a \circ \Phi^{-1}).$$

So $\Theta \circ a \circ \Phi^{-1}\colon \mathbf{G}''_K \to \mathbf{G}'_K$ is a $\operatorname{Gal}(K|k)$-invariant $K$-isomorphism and thus already given by a $k$-isomorphism $\mathbf{G}'' \cong_k \mathbf{G}'$. ☐

**Proposition 1.1.52.** *Let $K|k$ be a Galois extension and let $\mathbf{G}$ be an affine algebraic group over $k$. The map $\Omega$ constructed above is surjective.*

*Sketch of proof.* Since every affine algebraic group and every automorphism of such an algebraic group is defined by finitely many polynomials, we may assume without loss of generality that we are working over a finite Galois extension $K|k$.

Let

$$\varphi\colon \operatorname{Gal}(K|k) \to \operatorname{Aut}_K(\mathbf{G}_K)$$

be a cocycle representing an element in $H^1(\mathrm{Gal}(K|k), \mathrm{Aut}_K(\mathbf{G}_K))$. Using this cocycle, we define

$$\begin{aligned}\widetilde{\varphi}\colon \mathrm{Gal}(K|k) &\longrightarrow \mathrm{Aut}_k(\mathbf{G}_K)\\ \sigma &\longmapsto \varphi(\sigma)\circ(\sigma^*)^{-1},\end{aligned}$$

where $\sigma^*\colon X_K \to X_K$ is the $k$-isomorphism constructed in Remark 1.1.45. This is in fact a group homomorphism, as for $\sigma, \tau \in \mathrm{Gal}(K|k)$ we have

$$\begin{aligned}\widetilde{\varphi}(\sigma\circ\tau) &= \varphi(\sigma\circ\tau)\circ\big((\sigma\circ\tau)^*\big)^{-1}\\ &= \varphi(\sigma\circ\tau)\circ(\tau^*\circ\sigma^*)^{-1}\\ &= \varphi(\sigma\circ\tau)\circ(\sigma^*)^{-1}\circ(\tau^*)^{-1}\\ &= \varphi(\sigma)\circ{}^{\sigma}\varphi(\tau)\circ(\sigma^*)^{-1}\circ(\tau^*)^{-1} && \text{(by the cocycle condition)}\\ &= \varphi(\sigma)\circ(\sigma^*)^{-1}\circ\varphi(\tau)\circ\sigma^*\circ(\sigma^*)^{-1}\circ(\tau^*)^{-1}\\ &= \widetilde{\varphi}(\sigma)\circ\widetilde{\varphi}(\tau).\end{aligned}$$

Here we have used the description of the $\mathrm{Gal}(K|k)$-action given in Remark 1.1.45.

Since $\mathrm{Gal}(K|k)$ is a finite group, we can now form the quotient

$$_{\sigma}\mathbf{G} = {}_{\mathrm{Gal}(K|k)}\backslash^{\mathbf{G}_K}.$$

In our setting of *affine* algebraic groups, it is actually feasible to describe this quotient. Write $\mathbf{G} = \operatorname{Spec} S$, where $S$ is the Hopf algebra of $\mathbf{G}$. Then $\mathbf{G}_K = \operatorname{Spec}(S \otimes_k K)$ and the Galois action on $\mathbf{G}_K$ (as a $k$-scheme) gives an action on $S \otimes_k K$ by $k$-algebra homomorphisms. Then, as a scheme, we have $_{\sigma}\mathbf{G} = \operatorname{Spec}\big((S\otimes_k K)^{\mathrm{Gal}(K|k)}\big)$. Since $\mathrm{Gal}(K|k)$ is finite, the $k$-algebra $(S\otimes_k K)^{\mathrm{Gal}(K|k)}$ is again of finite type [72][27, Theorem 13.17] and thus $_{\sigma}\mathbf{G}$ is a $k$-scheme of finite type.

Moreover, one can check that the action of $\mathrm{Gal}(K|k)$ is compatible with the $K$-Hopf algebra structure of $S \otimes_k K$, which is due to the fact that $K^{\mathrm{Gal}(K|k)} = k$. This gives a $k$-Hopf algebra structure on $(S\otimes_k K)^{\mathrm{Gal}(K|k)}$ and thus the structure of an algebraic group over $k$ on $_{\sigma}\mathbf{G}$.

By construction, one can show that $(_{\sigma}\mathbf{G})_K \cong_K \mathbf{G}_K$, i.e. that $_{\sigma}\mathbf{G}$ is a $K|k$-form of $\mathbf{G}$. Finally, we also have by construction that the cocycle associated with $_{\sigma}\mathbf{G}$ is cohomologous to $\varphi$, which shows the desired surjectivity. "□"

We note that while we only gave an argument for the case of *affine* algebraic groups this is more generally true for all algebraic groups [95, Chapter III, §1.3].

## 1.2 Affine algebraic groups

In the following, we will introduce some notions from the theory of affine algebraic groups. This will by no means be a comprehensive treatment of the theory, but is just intended to introduce the required terminology we will use later on.

From now on we assume that all algebraic groups are affine. Moreover, in order to avoid some subtleties one would need to care for in positive characteristic, we will also assume that all algebraic groups are smooth and thus in particular reduced. In characteristic zero this condition is automatically satisfied.

### Connected algebraic groups

Recall that a topological space $X$ is said to be *connected* if it cannot be decomposed into a disjoint union of non-empty open subsets. This definition can in particular be applied to algebraic groups. In fact, there are multiple equivalent conditions for an algebraic group $\mathbf{G} = \operatorname{Spec} S$ over a field $K$ to be connected [67, Summary 1.36]:

1. $\mathbf{G}$ is connected.
2. $\mathbf{G}$ is geometrically connected.
3. $\mathbf{G}$ is irreducible.
4. $S$ is an integral domain.

In case $\mathbf{G}$ is *not* connected, one can always consider the connected component $\mathbf{G}^0$ of $\mathbf{G}$ containing the identity. Let us gather some properties of this identity component.

**Proposition 1.2.1** ([67, Proposition 1.34, Proposition 1.52]). *Let $\mathbf{G}$ be an algebraic group over a field $K$. The identity component $\mathbf{G}^0$ of $\mathbf{G}$ is a normal algebraic subgroup of $\mathbf{G}$. Its formation commutes with extension of the base field, i.e. for every field extension $K'|K$ we have $(\mathbf{G}^0)_{K'} \cong (\mathbf{G}_{K'})^0$.*

Moreover, a subgroup of $\mathbf{G}$ has finite index if and only if it contains $\mathbf{G}^0$ [67, Section 6.b]. In particular, connected algebraic groups have no proper algebraic subgroups of finite index.

### Lie algebra of an algebraic group

Let us recall the definition of a Lie algebra.

**Definition 1.2.2** (Lie algebra). A *Lie algebra* over a field $K$ is a (finite dimensional) $K$-vector space $\mathfrak{g}$ together with a $K$-bilinear map

$$[\,\cdot\,,\,\cdot\,] \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

called the *Lie bracket*, such that

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$ and
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

The second relation satisfied by the Lie bracket is called the *Jacobi identity.*

To every algebraic group one can always associate a Lie algebra.

**Definition 1.2.3** (Tangent space)**.** Let $K$ be a field and $\mathbf{G}$ an algebraic group over $K$. The *tangent space of* $\mathbf{G}$ *(at the neutral element)* is

$$T(\mathbf{G}) = \ker\Big(\mathbf{G}(K[\varepsilon]) \longrightarrow \mathbf{G}(K)\Big),$$

where $\varepsilon^2 = 0$ and the map $K[\varepsilon] \to K$ is given by $\varepsilon \mapsto 0$.

It is easy to see that there is an isomorphism

$$T(\mathbf{G}) \cong \mathrm{Hom}_K\Big({}^{I_{\mathbf{G}}}\!/_{I_{\mathbf{G}}^2}, K\Big),$$

where $I_{\mathbf{G}}$ is the *augmentation ideal*, the kernel of the counit $K[\mathbf{G}] \to K$ [67, Section 10.b, 10.6].

**Definition 1.2.4** (Lie algebra of an algebraic group)**.** For an algebraic group $\mathbf{G}$ over a field $K$, we define the *Lie algebra of* $\mathbf{G}$ to be

$$\mathrm{Lie}(\mathbf{G}) = \mathrm{Hom}_K\Big({}^{I_{\mathbf{G}}}\!/_{I_{\mathbf{G}}^2}, K\Big).$$

While we did not specify the Lie bracket, there is actually a unique functorial way of defining a bracket on the Lie algebra of an algebraic group [67, Section 10.d].

This associated Lie algebra is a pretty good invariant of the algebraic group. For example, we have the following properties.

**Proposition 1.2.5** ([67, Proposition 10.15])**.** *Let $K$ be a field, $\mathbf{G}$ a connected algebraic group of $K$, and $\mathbf{H} \subseteq \mathbf{G}$ a (smooth) algebraic subgroup. If* $\mathrm{Lie}(\mathbf{H}) = \mathrm{Lie}(\mathbf{G})$*, then* $\mathbf{H} = \mathbf{G}$.

**Proposition 1.2.6** ([67, Corollary 10.17])**.** *Let $K$ be a field, $\mathbf{G}$ a connected algebraic group over $K$, and $\mathbf{H}_1, \dots, \mathbf{H}_n$ (smooth) algebraic subgroups of $\mathbf{G}$. If the Lie algebras of the $\mathbf{H}_i$ generate* $\mathrm{Lie}(\mathbf{G})$ *as a Lie algebra, then the $\mathbf{H}_i$ generate $\mathbf{G}$.*

The connectedness assumption is essential as $\mathrm{Lie}(\mathbf{G}^0) = \mathrm{Lie}(\mathbf{G})$.

### 1.2.1 Semisimple and almost simple groups

We now come to an important definition in the theory of affine algebraic groups: the notion of a *semisimple* algebraic group.

As in classical group theory, one can define the notion of a *solvable* algebraic group. This is done by copying the classical definition.

**Definition 1.2.7** (Solvable algebraic group)**.** Let $K$ be a field. An algebraic group $\mathbf{G}$ over $K$ is *solvable* if it admits a subnormal series

$$\{e\} = \mathbf{G}_r \trianglelefteq \mathbf{G}_{r-1} \trianglelefteq \ldots \trianglelefteq \mathbf{G}_0 = \mathbf{G},$$

such that each quotient $\mathbf{G}_i/\mathbf{G}_{i+1}$ is commutative.

Similar as in the classical theory, subgroups, quotients, and extensions of solvable groups are again solvable [67, Proposition 6.27]. Using this, one can show that every algebraic group contains a unique largest connected solvable normal subgroup [67, Proposition 6.42]. This will be relevant in the following definition.

**Definition 1.2.8** (Semisimple algebraic group)**.** Let $K$ be a field and let $\mathbf{G}$ be a connected algebraic group over $K$. The largest connected solvable normal subgroup $R(\mathbf{G})$ of $\mathbf{G}$ is called the *radical* of $\mathbf{G}$. We call the algebraic group $\mathbf{G}$ *semisimple* if the radical over the algebraic closure is trivial.

Similar to solvability and semisimplicity of algebraic groups, one can also define solvability and semisimplicity of Lie algebras. Over a field of characteristic zero, a connected algebraic group is semisimple if and only the associated Lie algebra is [67, Lemma 23.68].

**Example 1.2.9** ($\mathrm{GL}_n$ is not semisimple)**.** Let $K$ be a field and $n \in \mathbb{N}$. We consider the algebraic group $\mathrm{GL}_n$. Since scalar multiples of the identity matrix form a connected solvable normal subgroup isomorphic to $\mathbf{G}_m$, we see that the radical $R(\mathrm{GL}_n)$ has to be non-trivial. In fact, it is easy to see that the radical is indeed isomorphic to $\mathbf{G}_m$. Hence, $\mathrm{GL}_n$ is *not* semisimple.

In classical group theory there are also simple groups: groups that do not contain any non-trivial normal subgroups. This definition can also be made for algebraic groups. However, it is often more useful to only restrict to *almost simple groups.*

**Definition 1.2.10** (Almost simple algebraic group)**.** Let $K$ be a field. An algebraic group $\mathbf{G}$ over $K$ is *almost simple*, if it is semisimple, non-commutative and every proper normal algebraic subgroup is finite. If moreover, $\mathbf{G}$ remains almost simple over $K^{\mathrm{sep}}$ it is called *absolutely almost simple* or *geometrically almost simple.*

### Isogenies

First let us define the *center* of an algebraic group.

**Definition 1.2.11** (Center of an algebraic group)**.** Let $K$ be a field and $\mathbf{G}$ an algebraic group over $K$. The *center* $Z(\mathbf{G})$ *of* $\mathbf{G}$ is the algebraic group given by

$$Z(\mathbf{G})(A) = \Big\{ g \in \mathbf{G}(A) \;\Big|\; g \in Z(\mathbf{G}(R)) \text{ for all } A\text{-algebras } R \Big\},$$

for all $K$-algebras $A$. Here $Z(\mathbf{G}(R))$ is the center of the group $\mathbf{G}(R)$, i.e. the subgroup of elements commuting with every other element.

One actually needs to verify that the above indeed defines an algebraic group. A proof can for example be found in the book by Milne [67, Proposition 1.92]. Note that in general, the center of a smooth algebraic group does not need to be smooth again. For example, over $\mathbb{F}_p$ the center of $\mathrm{SL}_p$ is the non-reduced group $\mu_p$. This will not be relevant for us, but should be kept in mind when working with algebraic groups in positive characteristic.

Since the center of an algebraic group is always commutative, centers of semisimple algebraic groups are finite.

**Definition 1.2.12** (Isogeny)**.** Let $K$ be a field. A surjective homomorphism $\mathbf{G} \to \mathbf{H}$ of connected algebraic groups over $K$ is an *isogeny* if it has finite kernel. Moreover, an isogeny is called *central* if the kernel is contained in the center of $\mathbf{G}$. If $\mathbf{G} \to \mathbf{H}$ is an isogeny we also say that $\mathbf{G}$ *covers* $\mathbf{H}$.

For fields of characteristic zero, every isogeny is central [80, Proposition 5.6.15]. In case the algebraic group $\mathbf{H}$ is semisimple, one can show that $\mathbf{G}$ is also semisimple [67, Proposition 19.14]. Moreover, it is easy to see that an isogeny of algebraic groups induces an isomorphism of the corresponding Lie algebras.

By definition, one can think of isogenies as finite sheeted coverings. With this picture in mind we give the following definition.

**Definition 1.2.13** (Simply connected and adjoint groups)**.** Let $K$ be a field. A semisimple algebraic group $\mathbf{G}$ over $K$ is *simply connected* if every central isogeny $\mathbf{H} \to \mathbf{G}$ is an isomorphism. Conversely, a semisimple algebraic group $\mathbf{G}$ over $K$ is *adjoint* if every central isogeny $\mathbf{G} \to \mathbf{H}$ is an isomorphism. Equivalently an adjoint group is one with trivial center [80, Proposition 5.6.19]

So in some sense simply connected groups are precisely those that do not admit non-trivial finite sheeted coverings and adjoint groups are the ones not covering any other group.

**Proposition 1.2.14** ([80, Proposition 5.6.21][67, Proposition 18.8, Remark 18.27])**.** *Let $K$ be a field and $\mathbf{G}$ a semisimple group over $K$.*

1. *Among all semisimple groups that centrally cover $\mathbf{G}$, there is a maximal one $\widetilde{\mathbf{G}}$ that centrally covers all others. This $\widetilde{\mathbf{G}}$ is unique up to isomorphism (as a group with central isogeny to $\mathbf{G}$) and is simply connected. It is called* the simply connected covering *or* universal covering of $\mathbf{G}$*.*

2. *Among all semisimple groups that $\mathbf{G}$ centrally covers, there is a minimal one* $\operatorname{Ad}\mathbf{G}$ *that is centrally covered by all others. This* $\operatorname{Ad}\mathbf{G}$ *is unique up to isomorphism (as a group with a central isogeny from $\mathbf{G}$ to it) and is adjoint. It is called* the adjoint group of $\mathbf{G}$*.*

*Both the formation of $\widetilde{\mathbf{G}}$ and* $\operatorname{Ad}\mathbf{G}$ *commute with extending the base field.*

It is easy to see that $\mathbf{G}/Z(\mathbf{G})$ has trivial center and thus $\operatorname{Ad}\mathbf{G} \cong \mathbf{G}/Z(\mathbf{G})$.

**Corollary 1.2.15.** *For a semisimple group, being simply connected or adjoint is stable under extension of the base field.*

A special case of an isogeny is an *almost direct product.*

**Definition 1.2.16** (Almost direct product)**.** Let $K$ be a field and $\mathbf{G}$ an algebraic group over $K$. For algebraic subgroups $\mathbf{G}_1, \dots, \mathbf{G}_r \leq \mathbf{G}$ we say that $\mathbf{G}$ *is the almost direct product of the subgroups* $\mathbf{G}_1, \dots, \mathbf{G}_r$ if the multiplication map $\mathbf{G}_1 \times \dots \times \mathbf{G}_r \to \mathbf{G}$ is an isogeny.

Semisimple groups can always be expressed as an almost direct product of almost simple normal subgroups.

**Theorem 1.2.17** ([67, Theorem 21.51])**.** *Let* $\mathbf{G}$ *be a semisimple algebraic group over a field* $K$*. Then* $\mathbf{G}$ *has only finitely many almost simple normal algebraic subgroups* $\mathbf{G}_1, \dots, \mathbf{G}_r$*, and it is the almost direct product of them. Every connected normal subgroup of* $\mathbf{G}$ *is the product of those* $\mathbf{G}_i$ *that are contained in it and is centralized by the remaining ones.*

The $\mathbf{G}_i$ in the above theorem are called the *almost simple factors of* $\mathbf{G}$.

**Corollary 1.2.18.** *Let* $K$ *be a field of characteristic zero and let* $\mathbf{G}$ *be a simply connected semisimple algebraic group over* $K$*. Then* $\mathbf{G}$ *is the* direct *product of its almost simple factors.*

*Proof.* By the above theorem we have an isogeny

$$\mathbf{G}_1 \times \dots \times \mathbf{G}_r \longrightarrow \mathbf{G}.$$

As we are in characteristic zero, this isogeny is automatically central. Thus, $\mathbf{G}$ being simply connected gives that this isogeny is an isomorphism. □

### 1.2.2 Algebraic tori and rank

**Definition 1.2.19** (Split torus, algebraic torus)**.** Let $K$ be a field. A *split torus* over $K$ is an algebraic group over $K$ isomorphic to $\mathbf{G}_m^n$ for some $n \in \mathbb{N}$. An *(algebraic) torus* over $K$ is an algebraic group that is a $K$-form of a split torus, i.e. isomorphic to $\mathbf{G}_m^n$ over $K^{\mathrm{sep}}$ for some $n \in \mathbb{N}$. In both cases we call $n$ the *rank* of the torus.

**Example 1.2.20.** Let $K$ be a field. In Example 1.1.40, we have seen that the algebraic groups $\mathbf{G}_a \colon \mathrm{Alg}_K \to \mathrm{Group}$ given by

$$\mathbf{G}_a(A) = \left\{ \begin{pmatrix} x & ay \\ y & x \end{pmatrix} \;\middle|\; x, y \in A, x^2 - ay^2 = 1 \right\} \leq \mathrm{SL}_2(A),$$

for $a \in K^\times$, are $K$-forms of $\mathbf{G}_m$. Hence, these give first examples of algebraic tori of rank 1 over $K$.

**Example 1.2.21.** Let $K|k$ be a finite Galois extension. As seen in Remark 1.1.38, the base changed restriction $(\operatorname{Res}_{K|k} \mathbf{G}_m)_K$ is isomorphic to $\prod_{\sigma \in \mathrm{Gal}(K|k)} \mathbf{G}_m^\sigma$. Now over the separable closure every factor is again isomorphic to $\mathbf{G}_m$, thus showing that $\operatorname{Res}_{K|k} \mathbf{G}_m$ is an algebraic tori.

In a given algebraic group, we can consider subgroups that are algebraic tori.

**Definition 1.2.22** (Rank of an algebraic group)**.** Let $K$ be a field and $\mathbf{G}$ an algebraic group over $K$. The *(absolute) rank of* $\mathbf{G}$, denoted by $\operatorname{rank} \mathbf{G}$, is the rank of a maximal torus in $\mathbf{G}_{K^{\mathrm{sep}}}$. The *$K$-rank of* $\mathbf{G}$, denoted by $\operatorname{rank}_K \mathbf{G}$, is the rank of a maximal split torus of $\mathbf{G}$.

For the notion of rank to be well-defined we need the rank of maximal tori to always be the same. In fact, any two maximal tori are conjugate to one another [67, Theorem 17.10]. Obviously, the $K$-rank is always less than or equal to the absolute rank. If the two are equal the algebraic group is called *split*.

**Definition 1.2.23** (Split group)**.** Let $K$ be a field and $\mathbf{G}$ a semisimple algebraic group over $K$. We call $\mathbf{G}$ *split* if it contains a split maximal torus, i.e. if $\operatorname{rank} \mathbf{G} = \operatorname{rank}_K \mathbf{G}$.

If one would like to be more precise, we should actually call such groups *splittable* and only call them *split* once we chose a split maximal torus. This is also the convention some authors, like Milne [67, Definition 19.22] for example, use.

As the "opposite" of split groups, with maximal $K$-rank, there are also anisotropic groups of minimal $K$-rank.

**Definition 1.2.24** (Anisotropic group)**.** Let $K$ be a field. A semisimple algebraic group $\mathbf{G}$ over $K$ is called *anisotropic* if it has $K$-rank $\operatorname{rank}_K \mathbf{G} = 0$.

Over the reals every semisimple algebraic group has a unique anisotropic form.

**Theorem 1.2.25** (Cartan [67, Theorem 25.56])**.** *Every semisimple algebraic group over the reals has an anisotropic form, which is unique up to isomorphism.*

### 1.2.3 On the classification of semisimple algebraic groups

Above we have seen that every semisimple algebraic group admits a central covering by a simply connected semisimple algebraic group. Thus, our classification will only be up to *isogeny.*

**Definition 1.2.26** (Isogenous)**.** Let $K$ be a field and $\mathbf{G}, \mathbf{G}'$ be two semisimple algebraic groups over $K$. We say $\mathbf{G}$ *and* $\mathbf{G}'$ *are (strictly) isogenous* if there are central isogenies $\mathbf{H} \to \mathbf{G}$ and $\mathbf{H} \to \mathbf{G}'$, i.e. if they have the same simply connected covering.

### The case of algebraically closed fields of characteristic zero

Let us first assume that we are working over a field of characteristic zero. Here we can use that a connected algebraic group is semisimple if and only if its Lie algebra is. Moreover, for every semisimple Lie algebra $\mathfrak{g}$ one can construct a simply connected semisimple algebraic group $\mathbf{G}(\mathfrak{g})$, having $\mathfrak{g}$ as Lie algebra [67, Theorem 23.70]. Hence, classifying simply connected semisimple algebraic groups is precisely the same as classifying semisimple Lie algebras.

For algebraically closed fields of characteristic zero there is a well-known classification of semisimple Lie algebras in terms of their *root systems* and thus *Dynkin diagrams*. This is for example extensively discussed in the book of Humphreys [44, Chapter II–V]. So here simply connected semisimple groups can also be classified in terms of Dynkin diagrams.

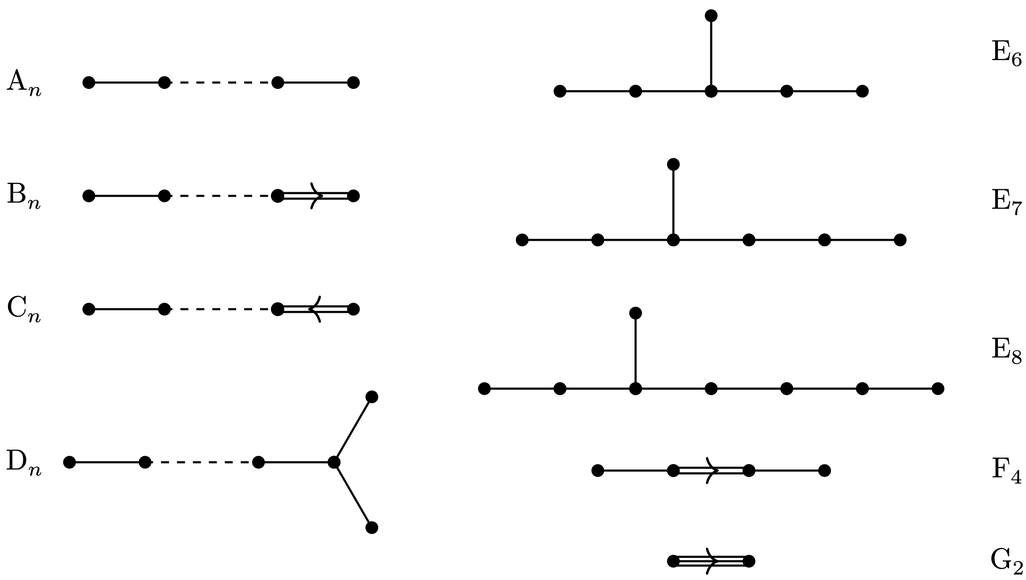


Figure 1.1: The connected Dynkin diagrams

### The case of algebraically closed fields

Now let us assume that the field $K$ we are working with is algebraically closed. In contrast to the characteristic zero case, algebraic groups do not need to be semisimple if and only if the associated Lie algebra is semisimple. Even if this were the case, there is not such a nice classification of semisimple Lie algebras in terms of their Dynkin diagrams as in the characteristic zero situation.

However, surprisingly, one can still classify semisimple algebraic groups in terms of root systems and Dynkin diagrams. For that purpose let $\mathbf{G}$ be a semisimple algebraic group over $K$, let $\mathbf{T} \leq \mathbf{G}$ be a maximal (split) torus and let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. Then one can consider the so-called *adjoint representation* [67, Section 10.d]

$$\mathrm{Ad}\colon \mathbf{T} \longrightarrow \mathrm{GL}_{\mathfrak{g}}\,.$$

As $\mathbf{T}$ is isomorphic to $\mathbf{G}_m^n$, this representation can be diagonalized to give a finite decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(\mathbf{G},\mathbf{T})} \mathfrak{g}_\alpha,$$

which can be used to define a root system of $(\mathbf{G}, \mathbf{T})$ [67, Section 21.a, Section 21.c]. These root systems completely classify simply connected semisimple groups [67, Theorem 23.58, Proposition 23.59] and can in turn again be classified by Dynkin diagrams.

#### The general case

Finally, let us consider the case where $K$ does not need to be algebraically closed. The constructions used in the case of algebraically closed fields only relied on the maximal torus being split. Thus, the same arguments can also be applied for split groups with a chosen split maximal torus.

**Theorem 1.2.27** ([67, Theorem 23.62, Proposition 24.1])**.** *Let $K$ be a field. Two split semisimple groups over $K$ are strictly isogenous if and only if they have the same Dynkin diagram. Every Dynkin diagram arises from a split simply connected semisimple group over $K$. Such a group is absolutely almost simple if and only if its Dynkin diagram is connected.*

This theorem only deals with the classification of *split* semisimple groups. For a general simply connected semisimple group $\mathbf{G}$ defined over $K$, the base change to the separable closure $\mathbf{G}_{K^{\mathrm{sep}}}$ is always split. Thus, we obtain a corresponding Dynkin diagram associated to $\mathbf{G}$. We call the type of the Dynkin diagram $(\mathrm{A}_n, \mathrm{B}_n, \dots)$ the *Cartan–Killing type of* $\mathbf{G}$. By the classification, there is a unique split simply connected semisimple group $\mathbf{H}$ over $K$ with the same Cartan–Killing type as $\mathbf{G}$. By uniqueness, we get that $\mathbf{G}_{K^{\mathrm{sep}}}$ and $\mathbf{H}_{K^{\mathrm{sep}}}$ have to be isomorphic, so $\mathbf{G}$ and $\mathbf{H}$ are $k$-forms of one another. We call $\mathbf{H}$ the *split form of* $\mathbf{G}$.

**Remark 1.2.28** (Studying forms of split simply connected semisimple groups)**.** Let $K$ be a field and let $\mathbf{G}$ be a split simply connected semisimple group over $K$. Here one can identify the short exact sequence

$$1 \longrightarrow \mathrm{Inn}(\mathbf{G}) \longrightarrow \mathrm{Aut}(\mathbf{G}) \longrightarrow \mathrm{Out}(\mathbf{G}) \longrightarrow 1,$$

where $\mathrm{Aut}(\mathbf{G})$ is the group of $K$-automorphisms of $\mathbf{G}$ and $\mathrm{Inn}(\mathbf{G})$ is the subgroup of automorphisms that become inner over $K^{\mathrm{sep}}$, with the *split* short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Ad}\,\mathbf{G}(K) \arrow[r] & \mathrm{Aut}(\mathbf{G}) \arrow[r] & \mathrm{Sym}\,\Delta \arrow[r] \arrow[l, bend left, dashed] & 1, \end{tikzcd}$$

where $\operatorname{Sym}\Delta$ is the group of Dynkin diagram symmetries of the Dynkin diagram associated to $\mathbf{G}$ [67, Section 24.b]. In particular, over the separable closure $K^{\mathrm{sep}}$ this gives a short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \operatorname{Ad}\mathbf{G}(K^{\mathrm{sep}}) \arrow[r] & \operatorname{Aut}(\mathbf{G}_{K^{\mathrm{sep}}}) \arrow[r] & \operatorname{Sym}\Delta \arrow[r] \arrow[l, bend left, dashed] & 1, \end{tikzcd}$$

which is additionally $\operatorname{Gal}(K)$-equivariant with respect to the trivial action on $\operatorname{Sym}\Delta$ [67, Section 24.b, 24.7].

Thus, applying Galois cohomology to this exact sequence, gives a short exact sequence (of pointed sets)

$$\begin{tikzcd} 1 \arrow[r] & H^1(K;\operatorname{Ad}\mathbf{G}) \arrow[r] & H^1(K;\operatorname{Aut}\mathbf{G}) \arrow[r] & H^1(K;\operatorname{Sym}\Delta) \arrow[r] \arrow[l, bend left, dashed] & 1, \end{tikzcd}$$

which again splits. Here, the inner forms of $\mathbf{G}$ (Remark 1.1.44) are precisely the ones in the image of the first map.

For our purpose, we say that a form of $\mathbf{G}$ is *quasi-split* if its corresponding element of $H^1(K;\operatorname{Aut}\mathbf{G})$ lies in the image of the splitting. To any form $\mathbf{H}$ of $\mathbf{G}$ we can associate a quasi-split form $\mathbf{H}'$, called *the quasi-split form of* $\mathbf{H}$, by first mapping to $H^1(K;\operatorname{Sym}\Delta)$ and then going back using the splitting. By twisting the above sequence with the cocycle representing $\mathbf{H}'$ one can see that $\mathbf{H}$ is an inner form of its quasi-split group [67, Theorem 23.51, Corollary 23.53]. Two forms of $\mathbf{G}$ are *inner twists of one another* if they have the same quasi-split form.

The inner forms of a quasi-split form $\mathbf{G}'$ can in turn be studied by applying Galois cohomology to the short exact sequence

$$1 \longrightarrow Z(\mathbf{G}') \longrightarrow \mathbf{G}' \longrightarrow \operatorname{Ad}\mathbf{G}' \longrightarrow 1$$

of algebraic $K$-groups. On the $K^{\mathrm{sep}}$-points this sequence stays exact and as $Z(\mathbf{G}')(K^{\mathrm{sep}})$ is central in $\mathbf{G}'(K^{\mathrm{sep}})$, we obtain a "long" exact sequence of pointed sets

$$\begin{tikzcd} 1 \arrow[r] & Z(\mathbf{G}')(K) \arrow[r] & \mathbf{G}'(K) \arrow[r] & \operatorname{Ad}\mathbf{G}'(K) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (\tikztotarget) }] & \\ & H^1(K;Z(\mathbf{G}')) \arrow[r] & H^1(K;\mathbf{G}') \arrow[r] & H^1(K;\operatorname{Ad}\mathbf{G}') \arrow[r] & H^2(K;Z(\mathbf{G}')) \end{tikzcd}$$

in Galois cohomology from Proposition A.3.10.

# Chapter 2

# $S$-arithmetic groups

Given an algebraic group over $\mathbb{Q}$, we would like to be able to talk about its "integral points". In case the algebraic group is already defined over $\mathbb{Z}$, like $\mathrm{GL}_n$ or $\mathrm{SL}_n$ for example, this notion should coincide with the integral points of the corresponding group scheme, i.e. with $\mathrm{GL}_n(\mathbb{Z})$ or $\mathrm{SL}_n(\mathbb{Z})$ respectively. However, in general it is not obvious how one can define such a notion. With Section 1.1.4 on models of group schemes in mind, we would like to define the integral points of an algebraic group to be the $\mathbb{Z}$-points of an integral model of this group. This immediately raises two questions:

1. Do integral models of algebraic groups always exist?
2. How do the integral points differ for different integral models?

For affine algebraic groups, we will answer both of these question in this chapter. Moreover, in the end we will also discuss the *congruence subgroup problem* which asks about the relation between *arithmetic* and *congruence subgroups*. As we will do all of this not only over $\mathbb{Q}$, but more generally over any *number field*, we first include a primer on some of the necessary notions.

From now on, we will assume that all algebraic groups are affine. Moreover, we also assume, unless explicitly stated otherwise, that all group schemes are affine and of finite type over their base.

### Overview of this chapter

## 2.1 Number fields and adèles

### 2.1.1 Normed and valued fields

**Definition 2.1.1** (Normed field)**.** Let $K$ be a field. A *norm on $K$* is a map $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ such that:

1) For all $x \in K$, we have $|x| = 0$ if and only if $x = 0$.
2) For all $x, y \in K$, we have $|x \cdot y| = |x| \cdot |y|$.
3) For all $x, y \in K$, we have $|x + y| \leq |x| + |y|$.

A field together with a norm is called a *normed field.*

We usually denote the norm of a normed field $K$ by $|\cdot|_K$. If the field is clear from the context we might also just write $|\cdot|$.

**Example 2.1.2.** The usual absolute value on $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ is a norm. For reasons that will become apparent later on, we denote this norm on $\mathbb{Q}$ by $|\cdot|_\infty$.

One way of obtaining normed fields is by using valuations.

**Definition 2.1.3** (Valuation)**.** Let $K$ be a field. A *(discrete) valuation on $K$* is a surjective group homomorphism $v\colon K^\times \to \mathbb{Z}$ such that

$$v(x + y) \geq \min(v(x), v(y))$$

for all $x, y \in K$.

By convention, we usually write $v(0) = \infty$ and consider a valuation to be a map $K \to \mathbb{Z} \cup \{\infty\}$.

**Example 2.1.4** ($p$-adic valuation)**.** Let $p$ be a prime. We define the *$p$-adic valuation $\nu_p$* on $\mathbb{Q}$ as follows. For $x \in \mathbb{Q}^\times$ we can decompose its denominator and numerator uniquely into prime factors to obtain a unique description

$$x = \varepsilon(x) \prod_{q \text{ prime}} q^{\nu_q(x)},$$

where $\varepsilon(x) \in \{\pm 1\}$ and $\nu_q(x) \in \mathbb{Z}$ is non-zero for only finitely many primes $q$.

It is easy to see that $\nu_p$ defines a surjective group homomorphism. To verify the defining property let $x, y \in \mathbb{Q}^\times$ and assume without loss of generality that $\nu_p(x) \leq \nu_p(y)$. Then

$$\begin{aligned} x + y &= \varepsilon(x) \prod_q q^{\nu_q(x)} + \varepsilon(y) \prod_q q^{\nu_q(y)} \\ &= p^{\nu_q(x)} \cdot \left( \varepsilon(x) \prod_q q^{\nu_q(x)} + \varepsilon(y) p^{\nu_p(y) - \nu_p(x)} \prod_{q \neq p} q^{\nu_q(y)} \right), \end{aligned}$$

where the last factor can only contain non-negative powers of $p$. Thus, we have

$$\nu_p(x + y) \geq \nu_p(x) = \min(\nu_p(x), \nu_p(y)).$$

**Example 2.1.5** ($\mathfrak{p}$-adic valuation)**.** As a generalization of the above example, let $R$ be a Dedekind domain with field of fractions $K$ and let $\mathfrak{p} \subseteq R$ be a maximal ideal. The $\mathfrak{p}$-*adic valuation on* $K$ is given as follows. For a non-zero element $a \in R$, we can decompose the ideal $(a)$ uniquely as a product of maximal ideals

$$(a) = \prod_{\mathfrak{q} \in \mathrm{mSpec}(R)} \mathfrak{q}^{\nu_{\mathfrak{q}}(a)}.$$

Then writing $x \in K^\times$ as a fraction $x = a/b$ with $a, b \in R \setminus \{0\}$ we define

$$\nu_{\mathfrak{p}}(x) = \nu_{\mathfrak{p}}(a) - \nu_{\mathfrak{p}}(b).$$

Similar as above, one can show that this defines a valuation on $K$ [53, Proposition 6.2].

**Example 2.1.6** (Normed fields using valuations)**.** Let $K$ be a field with a valuation $v$ and let $0 < a < 1$. Using the valuation, one can define a norm on $K$ as $|x|_v = a^{v(x)}$, where we use the convention $a^\infty = 0$.

From the defining property of a valuation, we get that $|\cdot|_v$ not only satisfies the triangle inequality, but also the *strong triangle inequality*

$$|x + y| \leq \max(|x|, |y|) \qquad \text{for all } x, y \in K.$$

**Example 2.1.7** ($p$-adic norm)**.** Using the $p$-adic valuation, for some prime $p$, we can define the $p$-*adic norm* $|\cdot|_p$ on $\mathbb{Q}$. Here one usually uses the base $p^{-1}$, such that $|x|_p = p^{-\nu_p(x)}$ for all $x \in \mathbb{Q}$. With this definition it is easy to see that

$$|x|_\infty \cdot \prod_{p \text{ prime}} |x|_p = 1 \qquad \text{for all } x \in \mathbb{Q}^\times.$$

**Definition 2.1.8** ((Non-)Archimedean norm)**.** A norm $|\cdot|$ on a field $K$ is called *non-Archimedean* if it satisfies the strong triangle inequality

$$|x + y| \leq \max(|x|, |y|)$$

for all $x, y \in K$. Otherwise, it is called an *Archimedean norm.*

In many cases, the strong triangle inequality is very useful.

**Proposition 2.1.9** (Norm estimation for zeros of polynomials)**.** *Let $K$ be a non-Archimedean normed field. Any zero $x \in K$ of a monic polynomial*

$$p = T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 \in K[T]$$

*satisfies*

$$|x| \leq \max\{1, |a_{n-1}|, \dots, |a_0|\}.$$

*Proof.* If $x \leq 1$ there is nothing to show, so let us assume $|x| > 1$. Then $x$ is in particular non-zero, so we can write

$$\begin{aligned} x &= x^n x^{-n+1} \\ &= (-1)(a_{n-1}x^{n-1} + \dots + a_1 x + a_0)x^{-n+1} \\ &= (-1)(a_{n-1} + a_{n-2}x^{-1} + \dots + a_2 x^{2-n} + a_0 x^{1-n}). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} |x| &= |a_{n-1} + a_{n-2}x^{-1} + \dots + a_2 x^{2-n} + a_0 x^{1-n}| \\ &\leq \max\{|a_{n-1}|, |a_{n-2}||x|^{-1}, \dots, |a_2||x|^{2-n}, |a_0||x|^{1-n}\} \\ &\leq \max\{|a_{n-1}|, |a_{n-2}|, \dots, |a_2|, |a_0|\}, \end{aligned}$$

as $|x| > 1$. □

**Lemma 2.1.10** (Valuation ring)**.** *Let $K$ be a non-Archimedean normed field. Then*

$$\mathcal{O} = \{x \in K \mid |x| \leq 1\} \subseteq K$$

*is a local ring with maximal ideal*

$$\mathcal{m} = \{x \in K \mid |x| < 1\}.$$

*It is called the* valuation ring of *$K$.*

*Proof.* Obviously, we have $0, 1 \in \mathcal{O}$. Moreover, for $x, y \in K$ with $|x|, |y| \leq 1$ we have

$$|x \cdot y| = |x| \cdot |y| \leq 1 \qquad \text{and} \qquad |x + y| \leq \max(|x|, |y|) \leq 1,$$

proving that $\mathcal{O}$ is indeed a ring. A similar argument, where we replace "$\leq$" by "$<$", shows that $\mathcal{m}$ is an ideal.

In order to see that $\mathcal{O}$ is local with maximal ideal $\mathcal{m}$, we prove that every element not in $\mathcal{m}$ is a unit in $\mathcal{O}$. Let $x \in \mathcal{O}$ with $|x| = 1$, by multiplicativity

$$|x^{-1}| = |x|^{-1} = 1.$$

So $x^{-1} \in \mathcal{O}$, showing that $x$ is a unit. □

Using Proposition 2.1.9, we immediately see that the valuation ring is always integrally closed.

**Example 2.1.11** (Valuation ring of $p$-adic valuation)**.** Let $p$ be a prime. Then the valuation ring of $\mathbb{Q}$ with respect to the $p$-adic norm is

$$\mathcal{O} = \{x \in \mathbb{Q} \mid |x|_p \leq 1\} = \{x \in \mathbb{Q} \mid \nu_p(x) \geq 1\}.$$

Thus, $\mathcal{O}$ consists of only those reduced fractions whose denominators do not contain any powers of $p$. So $\mathcal{O}$ equals the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the ideal generated by $p$.

More generally, let $R$ be a Dedekind domain with field of fractions $K$ and let $\mathfrak{p} \subseteq R$ be a maximal ideal. Then the valuation ring of $K$ with respect to the $\mathfrak{p}$-adic norm is

$$\mathcal{O} = \{x \in K \mid |x|_{\mathfrak{p}} \leq 1\} = \{x \in K \mid \nu_{\mathfrak{p}}(x) \geq 1\}.$$

So $\mathcal{O}$ consists of all elements whose denominators do not contain elements in $\mathfrak{p}$, i.e. $\mathcal{O} = R_{\mathfrak{p}}$, the localization of $R$ at $\mathfrak{p}$ [53, Proposition 6.14].

#### Complete normed fields

Similarly to how the real numbers can be obtained from $\mathbb{Q}$ by completing with respect to the absolute value, one can complete any field $K$ with respect to any norm $|\cdot|$. This is achieved by considering the ring of Cauchy sequences (with respect to $|\cdot|$) and taking the residue field $\widehat{K}$ of the maximal ideal given by all null sequences. Then $K$ can be embedded densely into $\widehat{K}$ as the constant sequences. Moreover, the given norm can be uniquely extended to a norm on $\widehat{K}$ by defining

$$\left|[(a_n)_{n\in\mathbb{N}}]\right| = \lim_{n\to\infty} |a_n|.$$

Finally, one can show that $(\widehat{K}, |\cdot|)$ is the (up to isomorphism) unique complete normed field that contains $(K, |\cdot|)$ as a dense subfield. The details of this process can for example be found in the book of Keune [53, Section 10.2].

In case the given norm is Archimedean, we have the following surprising result.

**Theorem 2.1.12** (Ostrowski [70, Kapitel II, Theorem 4.2])**.** *Let $K$ be an Archimedean normed field. Then, as a normed field, the completion $\widehat{K}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.*

**Remark 2.1.13** (Archimedean norms as embeddings)**.** As a normed field $K$ can always be embedded into the completion $\widehat{K}$, the above theorem gives that every Archimedean norm on a field $K$ comes from an embedding $K \hookrightarrow \mathbb{C}$ by restricting the norm on $\mathbb{C}$ to one on $K$.

If the image of such an embedding lies in $\mathbb{R}$, the completion with respect to this norm will be isomorphic to $\mathbb{R}$. If the image does not lie in $\mathbb{R}$, we can apply complex conjugation to obtain another embedding $K \hookrightarrow \mathbb{C}$ which induces the same norm on $K$, and thus will give the same completion, isomorphic to $\mathbb{C}$.

**Example 2.1.14** (Completions of $\mathbb{Q}$)**.** Let us take a look at the different completions of $\mathbb{Q}$. Completing with respect to the absolute value $|\cdot|_\infty$ gives the real numbers. For a prime $p$, the completion of $\mathbb{Q}$ with respect to the $p$-adic norm $|\cdot|_p$ is given by the $p$-adic numbers

$$\mathbb{Q}_p = \left\{ \sum_{i=k}^{\infty} a_i p^i \;\middle|\; k \in \mathbb{Z}, a_i \in \{0, \dots, p-1\} \text{ for all } i \geq k \right\}.$$

The norm of a (non-zero) formal power series $s = \sum a_i p^i$ is $|s|_p = p^{-k}$, where $k$ is the smallest integer $i$ with $a_i \neq 0$. Thus, the valuation ring is given by the *$p$-adic integers*

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i \;\middle|\; a_i \in \{0, \dots, p-1\} \text{ for all } i \in \mathbb{N} \right\}.$$

#### Equivalence of norms and places

Note that, both the notion of a Cauchy sequence and a null sequence do not directly depend on the given norm, but only on the *equivalence class* of the norm.

**Definition 2.1.15** (Equivalence of norms)**.** Let $K$ be a field. Two norms $|\cdot|_1, |\cdot|_2$ on $K$ are called *equivalent* if they define the same topology on $K$.

Equivalently, there exists an $s \in \mathbb{R}_{>0}$ such that

$$|x|_1 = {|x|_2}^s$$

for all $x \in K$ [53, Proposition 10.5].

Note that by using the second characterization, the valuation ring (Lemma 2.1.10) does not change when using an equivalent norm.

In fact, we have already seen equivalent norms. Namely, the ones coming from valuations.

**Example 2.1.16** (Equivalent norms from valuations)**.** Let $K$ be a field, $v$ a valuation on $K$ and $0 < a < 1$. In Example 2.1.6 we have seen that $|x|_{v,a} = a^{v(x)}$ defines a norm on $K$. This norm will depend on the chosen base $a$, but changing the base always gives an equivalent norm. For $0 < b < 1$ and $x \in K$ we have

$$|x|_{v,b} = b^{v(x)} = a^{\log_a(b)v(x)} = {|x|_{v,a}}^{\log_a(b)}.$$

Over the rational numbers, it is not too hard to classify all norms up to equivalence.

**Theorem 2.1.17** (Ostrowski [70, Kapitel II, Satz 3.7])**.** *Every norm on $\mathbb{Q}$ is either equivalent to the absolute value $|\cdot|_\infty$ or a p-adic norm $|\cdot|_p$ for some prime p.*

As equivalent norms define the same topology and the same completion, we are usually only interested in norms up to equivalence.

**Definition 2.1.18** (Place of a field)**.** Let $K$ be a field. A *place of $K$* is an equivalence class of a norm with respect to equivalence of norms. We call a place *finite* if some (and thus every) norm representing it is non-Archimedean. Otherwise, i.e. if some (and thus every) representing norm is Archimedean, we will call the place *infinite*.

The set of all places is denoted by $V(K)$, the set of finite places by $V_f(K)$ and the set of infinite places by $V_\infty(K)$.

By Example 2.1.16, all norms coming from a valuation are equivalent. Thus, for a valuation $v$ on a field $K$ we will usually just write $v \in V(K)$ to denote the equivalence class given by the norms induced from $v$. Moreover, we will also simply write $K_v$ to denote the completion of $K$ with respect to any of these norms.

**Remark 2.1.19** (Real and complex places)**.** By Theorem 2.1.12, we know that the completion of a field $K$ with respect to an Archimedean norm will be isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. Using this, we call an infinite place *real* if the completion is isomorphic to $\mathbb{R}$ and *complex* if the completion is isomorphic to $\mathbb{C}$.

By further using our observation in Remark 2.1.13, we can then identify real places with embeddings $K \hookrightarrow \mathbb{R}$ and complex places with pairs of complex conjugate embeddings $K \hookrightarrow \mathbb{C}$.

### 2.1.2 Number fields

**Definition 2.1.20** (Number field, ring of integers)**.** A *number field* is a finite field extension of the rational numbers $\mathbb{Q}$.

For a number field $K$ we define its *ring of integers* to be

$$\mathcal{O}_K := \{x \in K \mid \text{there exists } p \in \mathbb{Z}[T] \text{ monic such that } p(x) = 0\}.$$

It is well-known that any number field admits an *integral basis*, i.e. a $\mathbb{Z}$-basis $(\alpha_1, \dots, \alpha_n)$ of $\mathcal{O}_K$ where $n = [K : \mathbb{Q}]$ [70, Kapitel I, Satz 2.10][53, Corollary 1.38], and that the ring of integers is a Dedekind domain [70, Kapitel I, Theorem 3.1][53, Theorem 2.46].

**Proposition 2.1.21** (Field of fractions of $\mathcal{O}_K$)**.** *Let $K$ be a number field with ring of integers $\mathcal{O}_K$. Then $K$ is the field of fractions of $\mathcal{O}_K$.*

*Proof.* Denote by $L \subseteq K$ the field of fractions of $\mathcal{O}_K$. Now let $x \in K$. Since $K|\mathbb{Q}$ is a finite, and thus algebraic, field extension, $x \in K$ is algebraic over $\mathbb{Q}$. Thus, there exists a polynomial

$$p = a_n T^n + a_{n-1} T^{n-1} + \dots + a_1 T + a_0 \in \mathbb{Q}[T],$$

such that $p(x) = 0$. By clearing the denominators, we can assume without loss of generality that $p \in \mathbb{Z}[T]$. Then

$$\begin{aligned} 0 &= a_n^{n-1} p(x) \\ &= a_n^{n-1} \left( a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \right) \\ &= a_n^n x^n + a_{n-1} a_n^{n-1} x^{n-1} + \dots + a_1 a_n^{n-1} x + a_n^{n-1} a_0 \\ &= (a_n x)^n + a_{n-1} (a_n x)^{n-1} + \dots + a_1 a_n^{n-2} (a_n x) + a_n^{n-1} a_0. \end{aligned}$$

So $a_n x$ is a zero of a monic integral polynomial, i.e. an element in $\mathcal{O}_K$. Using $a_n \in \mathbb{Z} \subseteq \mathcal{O}_K$ we thus get $x = \frac{a_n x}{a_n} \in L$. □

**Remark 2.1.22.** In fact, the above proof shows that already the localization of $\mathcal{O}_K$ at the subring $\mathbb{Z}$ gives $K$.

**Proposition 2.1.23** ([53, Lemma 3.17])**.** *Let $K$ be a number field. Every non-trivial ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ is of finite index, i.e. $\mathcal{O}_K/\mathfrak{a}$ is finite.*

*Proof.* Let $(\alpha_1, \dots, \alpha_n)$ be an integral basis of $K$ and let $a \in \mathfrak{a} \setminus \{0\}$. Then $a\alpha_1, \dots, a\alpha_n$ is a $\mathbb{Z}$-basis of $a\mathcal{O}_K$ and $a\mathcal{O}_K \subseteq \mathfrak{a} \subseteq \mathcal{O}_K$. Hence, $\mathfrak{a}$ has to be a $\mathbb{Z}$-module of rank $n$ as well. Thus, we can choose $a \in \mathfrak{a} \cap \mathbb{N}_{>0}$, giving us that the index of $a\mathcal{O}_K$ in $\mathcal{O}_K$ is $a^n$. Finally, the index of $\mathfrak{a}$ in $\mathcal{O}_K$ has to divide this number. □

#### Places of number fields

For number fields, the set of finite places is in bijection with the maximal ideals of the ring of integers. This is also the reason why places of number fields are sometimes called *primes*. We will *not* do so and still refer to them as places. Whenever we speak of primes, we mean *prime numbers*.

**Proposition 2.1.24** ([53, Theorem 10.12])**.** *For every number field $K$, the map*

$$\begin{aligned} \mathrm{mSpec}(\mathcal{O}_K) &\longmapsto V_f(K) \\ \mathfrak{p} &\longmapsto |\cdot|_{\mathfrak{p}} \end{aligned}$$

*is a bijection.*

Using this proposition, we will usually assume that a finite place of a number field is given by a maximal ideal, and thus a valuation on $K$. Moreover, by abuse of notation we will often simply write $\mathfrak{p} \in V(K)$ for both finite and infinite places. However, we will always denote the norm representing the place by $|\cdot|_{\mathfrak{p}}$.

In combination with Remark 2.1.19, we can thus describe all the places of a number field.

**Remark 2.1.25** (Product formula for number fields)**.** In Example 2.1.7, we have seen that for every non-trivial rational number, the product of all $p$-adic valuations and the absolute value will always give one. For number fields, we can obtain a similar result by choosing the correct basis to get a norm from the $\mathfrak{p}$-adic valuation:

For a maximal ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, we define the *norm* $N(\mathfrak{p})$ to be the cardinality of $\mathcal{O}_K/\mathfrak{p}$. As the basis defining the $\mathfrak{p}$-adic norm on $K$, we choose $N(\mathfrak{p})^{-1}$, so that $|x|_{\mathfrak{p}} = N(\mathfrak{p})^{-\nu_{\mathfrak{p}}(x)}$. For a real place $\mathfrak{p}$, given by an embedding $\sigma\colon K \to \mathbb{R}$, we define $|x|_{\mathfrak{p}} = |\sigma(x)|$ and for a complex place $\mathfrak{p}$, given by an embedding $\sigma\colon K \to \mathbb{C}$, we define $|x|_{\mathfrak{p}} = |\sigma(x)|^2 = \sigma(x)\overline{\sigma(x)}$.

Then for every $x \in K^*$ we have

$$\prod_{\mathfrak{p} \in V(K)} |x|_{\mathfrak{p}} = 1.$$

A proof can for example be found in the book of Neukirch [70, Kapitel III, Satz 1.2] or the book of Keune [53, Proposition 10.24].

In the following, we will always assume that the $\mathfrak{p}$-adic norms are given in this way, so that the product formula holds.

**Definition 2.1.26** (Saturated set of places)**.** A set $S$ of places of a number field $K$ is called *saturated* if it contains all infinite places, i.e. if $V_\infty(K) \subseteq S$.

**Definition 2.1.27** ($S$-integers)**.** Let $K$ be a number field and let $S \subseteq V(K)$ be a finite saturated set of places. Then we define the *ring of $S$-integers* to be

$$\begin{aligned} \mathcal{O}_S &:= \{x \in K \mid \forall_{\mathfrak{p} \in V(K) \setminus S} \colon |x|_{\mathfrak{p}} \leq 1\} \\ &= \{x \in K \mid \forall_{\mathfrak{p} \in V(K) \setminus S} \colon \nu_{\mathfrak{p}}(x) \geq 0\}. \end{aligned}$$

By definition, we have that $\mathcal{O}_S$ is the intersection of the valuation rings of the finite places not contained in $S$. This easily proves that $\mathcal{O}_S$ is indeed a ring and, by using Proposition 2.1.9, $\mathcal{O}_K \subseteq \mathcal{O}_S$.

**Example 2.1.28.** Consider $K = \mathbb{Q}$. We identify the set of all finite valuation on $\mathbb{Q}$ with the set of all prime numbers and write $\infty$ for the infinite place. We have already seen that the valuation ring of the $p$-adic norm is the local ring $\mathbb{Z}_{(p)}$. Thus for $S = \{\infty, p_1, \dots, p_n\}$ we obtain

$$\mathcal{O}_S = \bigcap_{p \notin S} \mathbb{Z}_{(p)} = \mathbb{Z}[p_1^{-1}, \dots p_n^{-1}],$$

as the only primes allowed to show up in the denominators are the ones in $S$.

**Example 2.1.29** (Ring of integers)**.** Let $K$ be a number field. By identifying the set of finite places with the maximal ideals of $\mathcal{O}_K$, and the valuation rings with the corresponding localizations, we obtain for $S = V_\infty(K)$ that

$$\mathcal{O}_S = \bigcap_{\mathfrak{p} \in \mathrm{mSpec}\, \mathcal{O}_K} (\mathcal{O}_K)_{\mathfrak{p}} = \mathcal{O}_K.$$

**Remark 2.1.30** (Non-Archimedean completions of number fields)**.** Let $K$ be a number field and $\mathfrak{p} \subseteq \mathcal{O}_K$ a maximal ideal. We have already seen how to construct the completion $K_{\mathfrak{p}}$ with respect to the $\mathfrak{p}$-adic valuation. A more algebraic approach would be the following. First define the valuation ring $\mathcal{O}_{\mathfrak{p}}$ to be the inverse limit

$$\mathcal{O}_{\mathfrak{p}} = \varprojlim_{n \in \mathbb{N}_{\geq 1}} \mathcal{O}_K / \mathfrak{p}^n = \left\{ (x_n \mathfrak{p}^n)_n \in \prod_{n \in \mathbb{N}_{\geq 1}} \mathcal{O}_K / \mathfrak{p}^n \;\middle|\; x_{n+1}\mathfrak{p}^n = x_n \mathfrak{p}^n \text{ for all } n \in \mathbb{N}_{\geq 1} \right\}.$$

An element $x$ in this inverse limit can be interpreted as a sequence $(x_n)_{n \in \mathbb{N}_{\geq 1}}$ in $\mathcal{O}_K$. As $x_n - x_{n+1} \in \mathfrak{p}^n$, this is a Cauchy sequence and $x$ can be considered as the limit, giving the connection to our previous construction. Then $K_{\mathfrak{p}}$ can be defined to be the field of fractions of $\mathcal{O}_{\mathfrak{p}}$ [53, Alternative construction 10.37].

Formalizing this identification shows that the valuation ring of $K_{\mathfrak{p}}$ is, as a topological ring, isomorphic to the above inverse limit (where every $\mathcal{O}_K/\mathfrak{p}^n$ carries the discrete topology) [70, Kapitel II, Satz 4.5]. In particular, as every quotient $\mathcal{O}_K/\mathfrak{p}^n$ is finite (Proposition 2.1.23), using Tychonoff's theorem, we see that $\mathcal{O}_{\mathfrak{p}}$ is a compact topological ring. By the definition of $\mathcal{O}_{\mathfrak{p}}$, as the closed unit ball in $K_{\mathfrak{p}}$, this additionally shows that $K_{\mathfrak{p}}$ is locally compact [70, Kapitel II, Satz 5.1].

**Remark 2.1.31** (The various kinds of $\mathcal{O}$'s)**.** To a number field $K$ we can associate various different rings:

- The valuation ring of $K$ with respect to a $\mathfrak{p}$-adic norm. This turned out to be the localization $(\mathcal{O}_K)_{\mathfrak{p}}$.
- The ring of $S$-integers $\mathcal{O}_S$ for a finite saturated set of places $S \subseteq V(K)$. For $S = V_\infty(K)$, this recovered the ring of integers $\mathcal{O}_K$.
- The valuation ring of the completion $K_{\mathfrak{p}}$, usually denoted by $\mathcal{O}_{\mathfrak{p}}$.

The reader should be aware that these are in general all different, and that they *should not* be confused with one another. However, in the following we will rarely use the valuation ring of $K$ with respect to a $\mathfrak{p}$-adic norm, thus avoiding possible confusion between $(\mathcal{O}_K)_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{p}}$.

### 2.1.3 Adèle rings

It is often useful to study not only a single completion of a number field, but all of them simultaneously. This leads to the notion of the *adèle ring*.

**Definition 2.1.32** (Restricted product)**.** Let $(X_i)_{i\in I}$ be a family of locally compact topological spaces and for all but finitely many $i \in I$, let $Y_i \subseteq X_i$ be open compact subsets. We define the *restricted product of* $(X_i)_{i\in I}$ *with respect to* $(Y_i)_{i\in I}$ to be

$$\prod_{i\in I}(X_i, Y_i) = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; x_i \in Y_i \text{ for almost all } i \in I \right\} \subseteq \prod_{i\in I} X_i.$$

**Remark 2.1.33** (Topology on the restricted product)**.** Let $(X_i)_{i\in I}$ be a family of locally compact spaces and for all but finitely many $i \in I$, let $Y_i \subseteq X_i$ be open compact subsets. For $S \subseteq I$ finite and containing all $i \in I$ for which $Y_i$ is *not* defined, we consider

$$X_S := \prod_{i\in S} X_i \times \prod_{i\notin S} Y_i$$

with the product topology. As a product of finitely many locally compact spaces and compact spaces, $X_S$ is itself locally compact [53, Lemma 20.2].

By defining each $X_S$ to be open in $\prod_{i\in I}(X_i, Y_i)$, we obtain a locally compact topology on the restricted product. This topology is called the *restricted product topology*.

In more categorical terms, we can write $\prod_{i\in I}(X_i, Y_i)$ as the direct limit

$$\prod_{i\in I}(X_i, Y_i) = \varinjlim_{S\subseteq I \text{ finite}} X_S = \varinjlim_{S\subseteq I \text{ finite}} \prod_{i\in S} X_i \times \prod_{i\notin S} Y_i,$$

and the restricted product topology agrees with the direct limit topology. Note that, in general the restricted product topology *does not* agree with the subspace topology induced by the topology on $\prod_{i\in I} X_i$.

**Remark 2.1.34** (Group/ring structures)**.** It is easy to see that, if every $X_i$ is a locally compact group (respectively ring) and every $Y_i$ is a compact open subgroup (respectively subring), then the componentwise multiplication (and addition) gives the restricted product $\prod_{i\in I}(X_i, Y_i)$ the structure of a locally compact group (respectively ring).

**Remark 2.1.35.** Let $(G_i)_{i\in I}$ be a family of locally compact groups and for almost all $i \in I$, let $H_i \leq G_i$ be compact open subgroups. By construction, we have

$$\bigoplus_{i\in I} G_i \subseteq \prod_{i\in I}(G_i, H_i) \subseteq \prod_{i\in I} G_i.$$

This is why Tate suggested using the symbol "$\prod$" for the restricted product. It lies in between the coproduct (direct sum) "$\coprod$" and the product "$\prod$". Some authors use the symbol $\prod'$ instead.

**Definition 2.1.36** (Adèle ring)**.** Let $K$ be a number field and $S \subseteq V(K)$ a finite set of places. We define the *$S$-adèle ring of $K$* (or *ring of $S$-adèles*) to be

$$\mathbb{A}_{K,S} = \prod_{\mathfrak{p} \in V(K) \setminus S} (K_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}}),$$

where $\mathcal{O}_{\mathfrak{p}}$ is the valuation ring of $K_{\mathfrak{p}}$.

We call $\mathbb{A}_K := \mathbb{A}_{K,\emptyset}$ the *adèle ring of $K$* and $\mathbb{A}_K^f := \mathbb{A}_{K,V_\infty(K)}$ the *finite adèle ring of $K$*.

As every element of $K$ is in $\mathcal{O}_{\mathfrak{p}}$ for almost all $\mathfrak{p} \in V_f(K)$, there are diagonal embeddings of $K$ into the various adèle rings of $K$. Let us record some properties of these embeddings.

**Proposition 2.1.37** ([77, Proposition 1.5])**.** *Let $K$ be a number field. The image of the diagonal embedding $K \to \mathbb{A}_K$ is discrete.*

**Theorem 2.1.38** (Strong approximation [77, Theorem 1.5])**.** *Let $K$ be a number field. For every* non-empty *finite set of places $S \subseteq V(K)$, the image of the diagonal embedding $K \to \mathbb{A}_{K,S}$ is dense.*

### Interlude: locally isomorphic fields and isomorphic adèle rings

We now want to briefly discuss the connection between two number fields having isomorphic adèle rings and other concepts like being *arithmetically equivalent* and being *locally isomorphic.*

Arithmetically equivalent fields are a well-known concept in algebraic number theory.

**Definition 2.1.39** (Arithmetically equivalent fields [56, Chapter III, Theorem 1.2])**.** Two number fields $K$ and $K'$ are *arithmetically equivalent* if they have the same Dedekind zeta function $\zeta_K = \zeta_{K'}$.

While arithmetically equivalent fields do not need to be isomorphic, they share many properties, like for example the degree, roots of unity, and the number of real and complex places [56, Chapter III, Theorem 1.4].

Arithmetically equivalent fields can also be characterized in terms of the local completions.

**Proposition 2.1.40** ([56, Chapter VI, Remark 2.1 iv)])**.** *Two number fields $K$ and $K'$ are arithmetically equivalent if and only if there exists a bijection $V_f(K) \to V_f(K'), \mathfrak{p} \mapsto \mathfrak{p}'$, such that there are isomorphisms of topological fields*

$$K_{\mathfrak{p}} \cong_{\mathrm{Top}} K'_{\mathfrak{p}'} \qquad \textit{for almost all } \mathfrak{p} \in V_f(K).$$

Dropping the assumption of having this isomorphism for only *almost* all places, gives the stronger property of being *locally isomorphic.*

**Definition 2.1.41** (Locally isomorphic fields [56, Chapter VI, Definition 2.2])**.** Two number fields $K$ and $K'$ are *locally isomorphic* if there is a bijection $V_f(K) \to V_f(K'), \mathfrak{p} \mapsto \mathfrak{p}'$, such that

$$K_{\mathfrak{p}} \cong_{\mathrm{Top}} K'_{\mathfrak{p}'} \qquad \text{for all } \mathfrak{p} \in V_f(K).$$

Such a bijection and isomorphisms can immediately be combined to give a topological isomorphism of finite adèle rings $\mathbb{A}_K^f \cong_{\mathrm{Top}} \mathbb{A}_{K'}^f$. Moreover, as such fields have the same number of real and complex places, the bijection can be extended to the set of *all* places, and thus gives a topological isomorphism of the full adèle rings $\mathbb{A}_K \cong_{\mathrm{Top}} \mathbb{A}_{K'}$.

In fact, the converse also holds true.

**Theorem 2.1.42** (Characterization using (finite) adèle rings [56, Chapter VI, Proposition 2.5])**.** *Let $K$ and $K'$ be two number fields. A topological isomorphism of the adèle rings $\mathbb{A}_K \cong_{\mathrm{Top}} \mathbb{A}_{K'}$ corresponds uniquely to a bijection $V(K) \to V(K'), \mathfrak{p} \mapsto \mathfrak{p}'$ and a family of isomorphisms $K_\mathfrak{p} \cong_{\mathrm{Top}} K'_{\mathfrak{p}'}$.*

*Similarly, a topological isomorphism of the* finite *adèle rings $\mathbb{A}_K^f \cong_{\mathrm{Top}} \mathbb{A}_{K'}^f$ corresponds uniquely to a bijection of all* finite *places and a family of topological isomorphisms of the corresponding local fields.*

*Proof.* As already discussed above, such a family of isomorphism can easily be turned into a topological isomorphism of adèle rings.

For the converse implications, we use that every closed maximal ideal $\mathscr{m} \subseteq \mathbb{A}_K$ is given by $\mathscr{m}_\mathfrak{p} = \ker(\mathbb{A}_K \to K_\mathfrak{p})$, the kernel of the projection, for some $\mathfrak{p} \in V(K)$ [56, Chapter VI, Lemma 2.4]. Thus, for $\mathfrak{p} \in V(K)$, the isomorphism $\mathbb{A}_K \cong_{\mathrm{Top}} \mathbb{A}_{K'}$ maps the closed maximal ideal $\mathscr{m}_\mathfrak{p} \subseteq \mathbb{A}_K$ to another closed maximal ideal $\mathscr{m}_{\mathfrak{p}'} \subseteq \mathbb{A}_{K'}$ for some $\mathfrak{p}' \in V(K')$. Moreover, it induces an isomorphism

$$K_\mathfrak{p} \cong_{\mathrm{Top}} \mathbb{A}_K/\mathscr{m}_\mathfrak{p} \cong_{\mathrm{Top}} \mathbb{A}_{K'}/\mathscr{m}_{\mathfrak{p}'} \cong_{\mathrm{Top}} K'_{\mathfrak{p}'}.$$

For the analogous statement in the finite case, we observe that the statement for closed maximal ideals also holds true for the finite adèle rings. Thus, the same proof as above applies. □

## 2.2 $S$-arithmetic subgroups

### 2.2.1 Integral models

One way of constructing integral models for semi-simple algebraic groups by using their root systems is due to Chevalley [20]. However, we will use a different construction. We first learned of this approach, as well as of the proof of the commensurability of the resulting points (Section 2.2.2), in a minicourse given by Steffen Kionke [54]. Although this idea can occasionally be found in the literature [30, Section §1], there does not seem to be a complete proof for our setting, hence we include one here.

**Example 2.2.1** (Integral model of $\mathrm{GL}_n$)**.** For any number field $K$, the algebraic group $\mathrm{GL}_n$ has an $\mathcal{O}_K$-model given by

$$\mathrm{Spec}\left(\frac{\mathcal{O}_K[T_{1,1}, \dots, T_{n,n}, d']}{(\det(T_{i,j}) \cdot d' - 1)}\right).$$

**Theorem 2.2.2** (Pullback of model)**.** *Let $K$ be a number field, let $\Phi\colon \mathbf{H} \hookrightarrow \mathbf{G}$ be an embedding of algebraic groups over $K$ and assume that $G$ is an $\mathcal{O}_K$-model of $\mathbf{G}$. Then there exists an $\mathcal{O}_K$-model $H$ of $\mathbf{H}$ and an embedding $\varphi\colon H \to G$ inducing the given embedding $\Phi$.*

*Proof.* We prove the claim by passing to the corresponding Hopf algebras. For ease of notation, let us assume that $G_K$ is not only isomorphic but actually equal to $\mathbf{G}$, so that $K[\mathbf{G}] = \mathcal{O}_K[G] \otimes_{\mathcal{O}_K} K$.

First, we define the $\mathcal{O}_K$-subalgebra

$$A \coloneqq \operatorname{im}\begin{pmatrix} \mathcal{O}_K[G] \to K[\mathbf{G}] \overset{\Phi^*}{\twoheadrightarrow} K[\mathbf{H}] \\ x \mapsto x \otimes 1 \end{pmatrix} \subseteq K[\mathbf{H}].$$

By construction, we can factorize $\Phi^*|_{\mathcal{O}_K[G]}\colon \mathcal{O}_K[G] \to K[\mathbf{H}]$ as

$$\mathcal{O}_K[G] \xrightarrow{\varphi^*} A \xhookrightarrow{i} K[\mathbf{H}].$$

As $K$ is a localization of $\mathcal{O}_K$, and thus a flat $\mathcal{O}_K$-algebra, we get that the canonical map

$$A \otimes_{\mathcal{O}_K} K \to K[\mathbf{H}] \otimes_{\mathcal{O}_K} K = K[\mathbf{H}] \otimes_K K \cong K[\mathbf{H}],$$

induced by the inclusion $A \subseteq K[\mathbf{H}]$, is still injective. Moreover, by definition we also have the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_K[G] \otimes_{\mathcal{O}_K} K \arrow[r, two heads, "\varphi^* \otimes \mathrm{id}_K"] \arrow[d, equal] & A \otimes_{\mathcal{O}_K} K \arrow[d] \\
K[\mathbf{G}] \arrow[r, "\Phi^*"'] & K[\mathbf{H}],
\end{tikzcd}$$

which shows that this canonical map is also surjective and that $\varphi^*$ in fact induces $\Phi^*$.

It remains to show that $A$ has a Hopf algebra structure over $\mathcal{O}_K$, inducing the given Hopf algebra structure of $K[\mathbf{H}]$ over $K$. For the construction of the counit, antipode and comultiplication, consider the commutative diagrams

$$\begin{tikzcd}
& K[\mathbf{G}] \arrow[rr, two heads, "\Phi^*"] \arrow[dd, "e^*"' near start] & & K[\mathbf{H}] \arrow[dd, "e^*"] \\
\mathcal{O}_K[G] \arrow[ur] \arrow[rr, two heads, crossing over, "\varphi^*" near end] \arrow[dd, "e^*"'] & & A \arrow[ur, hook, "i"'] & \\
& K \arrow[rr, equal] & & K \\
\mathcal{O}_K \arrow[ur, hook] \arrow[rr, equal] & & \mathcal{O}_K \arrow[ur, hook] \arrow[from=uu, dashed, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& K[\mathbf{G}] \arrow[rr, two heads, "\Phi^*"] \arrow[dd, "\iota^*"' near start] & & K[\mathbf{H}] \arrow[dd, "\iota^*"] \\
\mathcal{O}_K[G] \arrow[ur] \arrow[rr, two heads, crossing over, "\varphi^*" near end] \arrow[dd, "\iota^*"'] & & A \arrow[ur, hook, "i"'] & \\
& K[\mathbf{G}] \arrow[rr, two heads] & & K[\mathbf{H}] \\
\mathcal{O}_K[G] \arrow[ur] \arrow[rr, two heads] & & A \arrow[ur, hook, "i"'] \arrow[from=uu, dashed, crossing over] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & K[\mathbf{G}] \arrow[rr, two heads, "\Phi^*"] \arrow[dd, "m^*" near start] & & K[\mathbf{H}] \arrow[dd, "m^*"] \\
\mathcal{O}_K[G] \arrow[ur] \arrow[rr, two heads, "\varphi^*" near end] \arrow[dd, "m^*"'] & & A \arrow[ur, hook, "i"'] \arrow[dd, dashed] & \\
 & K[\mathbf{G}] \otimes_{\mathcal{O}_K} K[\mathbf{G}] \arrow[rr, two heads] & & K[\mathbf{H}] \otimes_{\mathcal{O}_K} K[\mathbf{H}]. \\
\mathcal{O}_K[G] \otimes_{\mathcal{O}_K} \mathcal{O}_K[G] \arrow[ur] \arrow[rr, two heads] & & A \otimes_{\mathcal{O}_K} A \arrow[ur, hook, "i\otimes i"'] &
\end{tikzcd}$$

In order to see that in the last diagram, $i \otimes i$ is injective, we first note that the canonical map

$$A \longrightarrow A \otimes_{\mathcal{O}_K} K \cong K[\mathbf{H}]$$

coincides with the inclusion $A \subseteq K[\mathbf{H}]$. So $A$ is a torsion-free $\mathcal{O}_K$-module and, as $\mathcal{O}_K$ is a Dedekind domain, thus a *flat* $\mathcal{O}_K$-module. Hence,

$$i \otimes \mathrm{id}_A \colon A \otimes_{\mathcal{O}_K} A \longrightarrow K[\mathbf{H}] \otimes_{\mathcal{O}_K} A$$

is injective. Moreover, it is easy to see that $\mathrm{id}_{K[\mathbf{H}]} \otimes i$ is an isomorphism, as

$$K[\mathbf{H}] \otimes_{\mathcal{O}_K} A \cong K[\mathbf{H}] \otimes_{\mathcal{O}_K} K \otimes_{\mathcal{O}_K} A \cong K[\mathbf{H}] \otimes_{\mathcal{O}_K} K[\mathbf{H}].$$

Thus, $i \otimes i = (\mathrm{id}_{K[\mathbf{H}]} \otimes i) \circ (i \otimes \mathrm{id}_A)$ is indeed injective.

Now for each diagram, we can use the surjectivity of the front upper morphism and the injectivity of the lower right morphism, in order to construct a unique morphism completing the cube. We define these morphisms to be the counit, antipode and comultiplication on $H$.

By construction, it is easy to see that they indeed satisfy the defining relations of a Hopf algebra. We can replace the vertical morphisms by the corresponding relations in $\mathcal{O}_K[G], K[\mathbf{G}]$, and $K[\mathbf{H}]$ and then use the same uniqueness of the constructed morphisms. Moreover, as the “front-to-back” morphisms are all given by tensoring with $K$, we also see that this Hopf algebra structure on $A$ induces the one given on $K[\mathbf{H}]$. Hence, we obtain that $H = \operatorname{Spec} A$ is an $\mathcal{O}_K$-form of $\mathbf{H}$.

Finally, as $G$ is of finite type, we have that $\mathcal{O}_K[G]$ is a finitely generated $\mathcal{O}_K$-algebra, and thus $A$ is also finitely generated. So $H$ is indeed an affine $\mathcal{O}_K$-group scheme of finite type. □

In Proposition 1.1.41, we have seen that being affine, being of finite type, and being a subgroup scheme descends for faithfully flat ring extension. While we cannot apply this proposition to the extension of $K$ over $\mathcal{O}_K$, the above theorem provides us with a way of constructing $\mathcal{O}_K$-models that still satisfy these properties.

**Remark 2.2.3** (Generalization)**.** We only used that $K$ is the field of fractions of $\mathcal{O}_K$ and that torsion-free modules over $\mathcal{O}_K$ are already flat. Thus, we can apply the same proof for any so-called *Prüfer domain* $P$ with field of fractions $K$, in order to find $P$-models for algebraic groups over $K$.

**Corollary 2.2.4** (Integral models for affine algebraic groups). *Let $K$ be a number field. For every algebraic group over $K$ there exists an $\mathcal{O}_K$-model.*

*Proof.* By Theorem 1.1.21, every (affine) algebraic group can be realized as an algebraic subgroup of some $\mathrm{GL}_n$. Using the canonical $\mathcal{O}_K$-model of $\mathrm{GL}_n$ given in Example 2.2.1, together with the above theorem, we obtain an $\mathcal{O}_K$-model of the given algebraic group. □

Let us investigate the points of the $\mathcal{O}_K$-model constructed above.

**Remark 2.2.5** (Points of the $\mathcal{O}_K$-model). Let $K$ be a number field, let $\Phi\colon \mathbf{H} \to \mathbf{G}$ be an embedding of algebraic groups over $K$, let $G$ be an $\mathcal{O}_K$-model of $\mathbf{G}$ and let $H$ be the affine $\mathcal{O}_K$-model of $\mathbf{H}$ constructed above. For simplicity, let us again assume that $\mathbf{G} = G_K$, i.e. $K[\mathbf{G}] = \mathcal{O}_K[G] \otimes_{\mathcal{O}_K} K$.

Now let $B$ be a torsion-free $\mathcal{O}_K$-algebra. Then $B \to B \otimes_{\mathcal{O}_K} K$ is injective and hence $H(B)$ fits into a commutative square

$$\begin{tikzcd} H(B) \arrow[r, hook] \arrow[d, hook] & H(B \otimes_{\mathcal{O}_K} K) \cong \mathbf{H}(B \otimes_{\mathcal{O}_K} K) \arrow[d, hook] \\ G(B) \arrow[r, hook] & G(B \otimes_{\mathcal{O}_K} K) = \mathbf{G}(B \otimes_{\mathcal{O}_K} K). \end{tikzcd}$$

So, as subgroups of $\mathbf{G}(B \otimes_{\mathcal{O}_K} K)$, we have

$$H(B) \subseteq G(B) \cap \mathbf{H}(B \otimes_{\mathcal{O}_K} K).$$

We claim that this is in fact an equality. Considering $x \in G(B) \cap \mathbf{H}(B \otimes_{\mathcal{O}_K} K)$ as algebra homomorphisms $x\colon \mathcal{O}_K[G] \to B$ and $x\colon K[\mathbf{H}] \to B \otimes_{\mathcal{O}_K} K$, respectively, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_K[G] \arrow[r, two heads] \arrow[d, "x"'] & \mathcal{O}_K[H] \arrow[r] \arrow[d, dashed] & K[\mathbf{H}] \arrow[d, "x"] \\ B \arrow[r, equal] & B \arrow[r, hook] & B \otimes_{\mathcal{O}_K} K. \end{tikzcd}$$

Here, we can uniquely fill in the morphism $\mathcal{O}_K[H] \to B$, showing that $x$ represents an element in $H(B) = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[H], B)$.

In particular, for an algebraic group $\mathbf{G}$ with a given embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_n$, we can identify the $\mathcal{O}_K$-points of the constructed $\mathcal{O}_K$-model $G$ with the subgroup

$$G(\mathcal{O}_K) = \mathrm{GL}_n(\mathcal{O}_K) \cap \mathbf{G}(K),$$

where $\mathbf{G}(K)$ is considered as subgroup of $\mathrm{GL}_n(K)$ using the given embedding.

The above already gives a nice description of the points of the integral model. However, the question of how these points differ depending on the choice of the integral model still remains. In the next section, we will prove the following, which is sufficient for many applications.

**Theorem 2.2.6.** *Let $K$ be a number field, let $\mathbf{G}$ be an algebraic group over $K$ and let $G_1, G_2$ be two $\mathcal{O}_K$-models of $\mathbf{G}$. Then, as subgroups of $\mathbf{G}(K)$, the two groups $G_1(\mathcal{O}_K)$ and $G_2(\mathcal{O}_K)$ are commensurable, i.e. the intersection of the two groups is of finite index in both of them.*

In order to prove this theorem, we first need to introduce the notion of *(principal) congruence subgroups.*

### 2.2.2 Congruence subgroups

**Definition 2.2.7** (Principal congruence subgroup)**.** Let $R$ be a ring, let $G$ be a group scheme over $R$, let $A$ be an $R$-algebra and let $\mathfrak{a} \subseteq A$ be a non-trivial ideal. Then we call

$$G(A, \mathfrak{a}) = \ker\left( G(A) \to G\left( {}^{A}\!/\!{}_{\mathfrak{a}} \right) \right)$$

a *principal congruence subgroup of* $G(A)$.

**Theorem 2.2.8.** *Let $K$ be a number field, let $\mathbf{G}$ be an algebraic group over $K$ and let $G_1, G_2$ be two $\mathcal{O}_K$-models of $\mathbf{G}$. There exists an $\ell \in \mathcal{O}_K$ such that for every torsion-free $\mathcal{O}_K$-algebra $A$ and every non-trivial ideal $\mathfrak{a} \subseteq A$ we have*

$$G_1(A, \ell \cdot \mathfrak{a}) \subseteq G_2(A, \mathfrak{a})$$

*as subgroups of $\mathbf{G}(A \otimes_{\mathcal{O}_K} K)$.*

*Proof.* Let $f_1, \dots, f_n$ and $g_1, \dots, g_m$ be generators of $\mathcal{O}_K[G_1]$ and $\mathcal{O}_K[G_2]$ respectively. By replacing $f$ with $f - e^*(f)$, we can assume, without loss of generality, that applying the counit $e^*$ sends all of these generators to zero.

As $G_1$ and $G_2$ are $\mathcal{O}_K$-models of $\mathbf{G}$, we can map the $f_i$'s and $g_j$'s to $K[\mathbf{G}]$ by using the maps $\mathcal{O}_K[G_i] \to \mathcal{O}_K[G_i] \otimes_{\mathcal{O}_K} K \cong K[\mathbf{G}]$. We denote these images by $\overline{f}_i$ and $\overline{g}_j$, respectively. Now $\overline{f}_1, \dots, \overline{f}_n$ generate $K[\mathbf{G}]$ as $K$-algebra. Thus, we can find polynomials $p_j \in K[T_1, \dots, T_n]$, such that

$$\overline{g}_j = p_j(\overline{f}_1, \dots, \overline{f}_n).$$

By applying the counit to this equality we obtain

$$\begin{aligned} 0 = e^*(\overline{g}_j) &= e^*(p_j(\overline{f}_1, \dots, \overline{f}_n) \\ &= p_j(e^*(\overline{f}_1), \dots, e^*(\overline{f}_n) \\ &= p_j(\overline{e^*(f_1)}, \dots, \overline{e^*(f_n)}) \\ &= p_j(0, \dots, 0), \end{aligned}$$

i.e. all the constant terms of the polynomials $p_j$ are zero.

Using Proposition 2.1.21, we can clear out the denominators and find $0 \neq \ell \in \mathcal{O}_K$ such that $\ell \cdot p_1, \dots, \ell \cdot p_m \in \mathcal{O}_K[T_1, \dots, T_n]$.

Now, let $0 \neq \mathfrak{a} \subseteq A$ be a non-trivial ideal and define $\mathfrak{b} = \ell \cdot \mathfrak{a}$. As $A$ is torsion-free, we again have $\mathfrak{b} \neq 0$. Identify $x \in G_1(A, \mathfrak{b})$ with an algebra homomorphism $x\colon \mathcal{O}_K[G_1] \to A$ such that $x(f_i) \equiv 0 \mod \mathfrak{b}$. Under the inclusion $G_1(A) \subseteq \mathbf{G}(A \otimes_{\mathcal{O}_K} K)$, $x$ corresponds to a morphism $x\colon K[\mathbf{G}] \to A \otimes_{\mathcal{O}_K} K$, such that $x(\overline{f}_i) \in \mathfrak{b} \subseteq A \subseteq A \otimes_{\mathcal{O}_K} K$.

In order to see that $x$ also defines an element in $G_2(A, \mathfrak{a})$, we need to see that $x(\overline{g}_j) \in \mathfrak{a}$ for all $j$. For this, we first use

$$x(\overline{g}_j) = x(p_j(\overline{f}_1, \dots, \overline{f}_n)) = p_j(\underbrace{x(\overline{f}_1)}_{\in \mathfrak{b}}, \dots, \underbrace{x(\overline{f}_n)}_{\in \mathfrak{b}}).$$

As $\mathfrak{b} = \ell \cdot \mathfrak{a}$, we can write every $x(\overline{f}_i)$ as $\ell \cdot h_i$ for some $h_i \in \mathfrak{a}$. Thus, with $p_j = \sum_I \lambda_I T_*^I$, we get

$$x(\overline{g}_j) = p_j(\ell \cdot h_1, \dots, \ell \cdot h_n) = \sum_I \lambda_I (\ell \cdot h_*)^I = \sum_I \lambda_I \ell^{|I|} h_*^I.$$

Now, the constant term of $p_j$ is zero, so we always have $|I| > 0$. Hence, $\lambda_I \ell^{|I|} \in \mathcal{O}_K$, as $\ell \cdot p_j \in \mathcal{O}_K[T_1, \dots, T_n]$, and thus $x(\overline{g}_j) \in \mathfrak{a}$. Since this holds for every $j$, we obtain

$$G_1(A, \mathfrak{b}) \subseteq G_2(A, \mathfrak{a}). \qquad \square$$

**Remark 2.2.9** (Generalization)**.** As before, this is again not the most general formulation of this proposition. We only used that $K$ is the field of fractions of $\mathcal{O}_K$ and that the canonical map $A \to A \otimes_{\mathcal{O}_K} K$ is injective, i.e. that $A$ is a torsion-free $\mathcal{O}_K$-module.

**Corollary 2.2.10.** *Let $K$ be a number field, let $\mathbf{G}$ be an algebraic group over $K$ and let $S \subseteq V(K)$ be a finite saturated set of places. Then for any two $\mathcal{O}_K$-models $G_1$, $G_2$ of $\mathbf{G}$, we have that $G_1(\mathcal{O}_S)$ and $G_2(\mathcal{O}_S)$ are commensurable as subgroups of $\mathbf{G}(K)$.*

*Proof.* We need to prove that $G_1(\mathcal{O}_S) \cap G_2(\mathcal{O}_S)$ is of finite index in both $G_1(\mathcal{O}_S)$ and $G_2(\mathcal{O}_S)$. For this, we apply Theorem 2.2.8 twice to obtain $\ell_1, \ell_2 \subseteq \mathcal{O}_K$ with

$$G_2(\mathcal{O}_S, \ell_2\mathcal{O}_S) \subseteq G_1(\mathcal{O}_S, \ell_1\mathcal{O}_S) \subseteq G_2(\mathcal{O}_S, \mathcal{O}_S) = G_2(\mathcal{O}_S).$$

In particular, we obtain $G_2(\mathcal{O}_S, \ell_2\mathcal{O}_S) \subseteq G_1(\mathcal{O}_S, \ell_1\mathcal{O}_S) \subseteq G_1(\mathcal{O}_S) \cap G_2(\mathcal{O}_S)$.

Now every quotient of $\mathcal{O}_S$ is finite, so $G_1(\mathcal{O}_S, \ell_1\mathcal{O}_S)$ and $G_2(\mathcal{O}_S, \ell_2\mathcal{O}_S)$ are of finite index in $G_1(\mathcal{O}_S)$ and $G_2(\mathcal{O}_S)$, respectively. Hence, $G_1(\mathcal{O}_S) \cap G_2(\mathcal{O}_S)$ is of finite index in both $G_1(\mathcal{O}_S)$ and $G_2(\mathcal{O}_S)$. $\square$

**Definition 2.2.11** ($S$-congruence subgroup)**.** Let $K$ be a number field, $\mathbf{G}$ an algebraic group over $K$, and $S \subseteq V(K)$ a finite saturated set of places. A subgroup of $\mathbf{G}(K)$ is called an *$S$-congruence subgroup* if it contains a principal congruence subgroup $G(\mathcal{O}_S, \mathfrak{a})$, for some $\mathcal{O}_K$-model $G$ of $\mathbf{G}$ and some non-trivial ideal $\mathfrak{a} \subseteq \mathcal{O}_S$.

In case $S = V_\infty(K)$, and thus $\mathcal{O}_S = \mathcal{O}_K$, we also speak of a *congruence subgroup*.

Theorem 2.2.8 shows that this definition is independent of the chosen $\mathcal{O}_K$-model of $\mathbf{G}$. Finally, we can define the notion of an *$S$-arithmetic subgroup*.

**Definition 2.2.12** ($S$-arithmetic subgroup)**.** Let $K$ be a number field, $\mathbf{G}$ an algebraic group over $K$, and $S \subseteq V(K)$ a finite saturated set of places. An *$S$-arithmetic subgroup* of $\mathbf{G}(K)$ is a subgroup commensurable with $G(\mathcal{O}_S)$ for some $\mathcal{O}_K$-model $G$ of $\mathbf{G}$.

Again, if $S = V_\infty(K)$ we just speak of an *arithmetic subgroup.*

Although $S$-arithmetic subgroups are only defined up to commensurability, we will often use the notation $\mathbf{G}(\mathcal{O}_S)$ to denote an $S$-arithmetic subgroup.

**Remark 2.2.13** (Classical definition of arithmetic groups)**.** In a more classical setup, where an algebraic group is defined to be a subgroup $\mathbf{G} \leq \mathrm{GL}_n(\mathbb{Q})$ defined by polynomial, the integral points $\mathbf{G}(\mathbb{Z})$ are defined to be $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathrm{GL}_n(\mathbb{Z})$. For two embeddings $\mathbf{G} \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ the two groups might differ, however they will always be commensurable. Thus, an arithmetic subgroup is defined to be commensurable with $\mathbf{G}(\mathbb{Z})$ for some embedding of $\mathbf{G}$ into $\mathrm{GL}_n(\mathbb{Q})$.

This is also what is mirrored in the above construction of integral models: We used an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_n$ of algebraic groups in order to get an integral model of $\mathbf{G}$. In Remark 2.2.5, we have seen that the integral points of this model are the same as in the classical viewpoint. Finally, arithmetic subgroups are again only defined up to commensurability.

By a theorem of Borel and Harish-Chandra, arithmetic subgroups in a semisimple algebraic group $\mathbf{G}$ over $\mathbb{Q}$ are always lattices in the real points $\mathbf{G}(\mathbb{R})$. This hold more generally for $S$-arithemtic subgroups.

**Theorem 2.2.14** (Borel–Harish-Chandra [65, Chapter I, Theorem 3.2.4])**.** *Let $K$ be a number field, let $\mathbf{G}$ be a semisimple algebraic group over $K$, let $S' \subseteq V(K)$ be a finite set of places containing all infinite places where $\mathbf{G}$ is* not *anisotropic and let $S = S' \cup V_\infty(K)$.*

- *The diagonal embedding turns $\mathbf{G}(\mathcal{O}_S)$ into a lattice in $\prod_{\mathfrak{p} \in S'} \mathbf{G}(K_\mathfrak{p})$.*
- *This lattice is cocompact if and only if $\mathbf{G}$ is anisotropic over $K$.*

### 2.2.3 The congruence subgroup problem

For now, let us fix a number field $K$ and an algebraic group $\mathbf{G}$ over $K$ with an $\mathcal{O}_K$-model $G$. By construction the principal congruence subgroups $G(\mathcal{O}_S, \mathfrak{a}) \subseteq G(\mathcal{O}_S)$ are of finite index. Thus, also every $S$-congruence subgroup of $G(\mathcal{O}_S)$ is of finite index. Now the *congruence subgroup problem* asks for the converse.

**Question 2.2.15** (Congruence subgroup problem, classical)**.** Is every subgroup of finite index of $G(\mathcal{O}_S)$ an $S$-congruence subgroup?

A more modern approach, first introduced by Serre [96], uses the so-called *congruence kernel.* For this, we introduce two topologies on $G(\mathcal{O}_S)$. The *$S$-arithmetic topology* is given by defining finite index subgroups to form an open neighbourhood basis of the identity. Similarly, the *$S$-congruence topology* is given by defining principal congruence subgroups $G(\mathcal{O}_S, \mathfrak{a})$ to be an open neighbourhood basis of the identity. As every $S$-congruence subgroup is also an $S$-arithmetic subgroup, a priori the $S$-arithmetic topology is finer than the $S$-congruence topology, and we can reformulate the congruence subgroup problem as follows.

**Question 2.2.16** (Congruence subgroup problem, topological)**.** Do the $S$-arithmetic and $S$-congruence topology on $G(\mathcal{O}_S)$ agree?

In order to define the congruence kernel, we need to briefly discuss the *Hausdorff completion of a topological group*. We will not go into detail, but refer the interested reader to Bourbaki's book on topology [10, Chapter II, Chapter III].

**Remark 2.2.17** (Hausdorff completion of a topological group)**.** In Section 2.1.1, we saw that a normed field can be completed with respect to the given norm. Similarly, one can define the completion of a metric space. The most general setup in which this works is the theory of "uniform spaces" [10, Chapter II]. This theory allows one to speak of uniform continuity, Cauchy *filters* (in some sense a generalization of sequences) and convergence of filters. Thus, one also gets a notion of a *complete* uniform space: a uniform space in which every Cauchy filter converges.

For every uniform space $X$, there exists a complete Hausdorff uniform space $\widehat{X}$ together with a uniformly continuous map $i\colon X \to \widehat{X}$, satisfying the following universal property:

For every uniformly continuous map $f\colon X \to Y$ to a complete Hausdorff uniform space $Y$, there exists a unique uniformly continuous map $\hat{f}\colon \widehat{X} \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \\ \widehat{X} \arrow[ur, dashed, "\exists! \hat{f}"'] & \end{tikzcd}$$

commute [10, Section II.7.3, Theorem 3]. The image $i(X)$ is always dense in $\widehat{X}$ and, if $X$ was already Hausdorff, $i\colon X \to i(X)$ is an isomorphism of uniform spaces.

Now for a topological group $G$, one can define the *two-sided uniformity* compatible with the group structure. This gives that the Hausdorff completion $\widehat{G}$ carries again a group structure [10, Section III.3, Theorem 1, Exercise 6]. Here the universal property reads as follows:

Every continuous group homomorphism $f\colon G \to H$ into a complete Hausdorff group, can be uniquely factorized as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "i"'] & H, \\ \widehat{G} \arrow[ur, dashed, "\exists! \hat{f}"'] & \end{tikzcd}$$

with $\hat{f}\colon \widehat{G} \to H$ a continuous group homomorphism [10, Section III.3, Proposition 8].

Coming back to the $S$-arithmetic and $S$-congruence topology on $G(\mathcal{O}_K)$. Since the intersection of all $S$-congruence subgroups only contains the identity, we get that both the $S$-arithmetic and the $S$-congruence topologies are Hausdorff. Now, denoting by

$$\widehat{G(\mathcal{O}_S)}^S \qquad \text{and} \qquad \overline{G(\mathcal{O}_S)}^S$$

the $S$-arithmetic and $S$-congruence completion, respectively, the universal property gives a continuous group homomorphism

$$\pi_G^S \colon \widehat{G(\mathcal{O}_S)}^S \longrightarrow \overline{G(\mathcal{O}_S)}^S.$$

**Lemma 2.2.18** (Identifications of completions)**.** *As topological groups we have isomorphisms*

$$\widehat{G(\mathcal{O}_S)}^S \cong \varprojlim_{N \trianglelefteq_{\text{f.i.}} G(\mathcal{O}_S)} G(\mathcal{O}_S)/N,$$

*where $N$ runs through all finite index normal subgroups, and similarly*

$$\overline{G(\mathcal{O}_S)}^S \cong \varprojlim_{\mathfrak{a} \trianglelefteq \mathcal{O}_S} G(\mathcal{O}_S)/G(\mathcal{O}_S, \mathfrak{a}),$$

*where $\mathfrak{a}$ runs through all non-trivial ideals of $\mathcal{O}_S$.*

*Proof.* We only give the argument for the first isomorphism. The argument for the second isomorphism is essentially the same.

By construction, the inverse limit is a closed subgroup of the product

$$\prod_{N \trianglelefteq_{\text{f.i.}} G(\mathcal{O}_S)} G(\mathcal{O}_S)/N$$

of discrete finite groups. Hence, it is a compact Hausdorff group and thus, as compact groups are already complete [10, Section I.4.1, Theorem 1], a complete Hausdorff group.

Moreover, the intersection of all $S$-congruence subgroups is trivial, so we can identify $G(\mathcal{O}_S)$ with a dense subgroup of this inverse limit. Thus, the universal property of the Hausdorff completion gives us that the inclusion of $G(\mathcal{O}_S)$ into the inverse limit extends to an isomorphism [10, Section II.3.7, Proposition 13]. □

**Corollary 2.2.19.** *The canonical map*

$$\pi_G^S \colon \widehat{G(\mathcal{O}_S)}^S \longrightarrow \overline{G(\mathcal{O}_S)}^S$$

*is surjective.*

*Proof.* First, we note that $G(\mathcal{O}_S)$ is dense in $\overline{G(\mathcal{O}_S)}^S$ and lies, by construction, in the image of $\pi_G^S$. Now, by the above proposition, $\widehat{G(\mathcal{O}_S)}^S$ is compact, so the image of $\pi_G^S$ is compact as well and hence, as a subgroup of a Hausdorff group, in particular closed. But then the image must already contain the closure of $G(\mathcal{O}_S)$, i.e. all of $\overline{G(\mathcal{O}_S)}^S$. □

**Definition 2.2.20** ($S$-congruence kernel)**.** Let $K$ be a number field, $\mathbf{G}$ an algebraic group over $K$ and $G$ an $\mathcal{O}_K$-model of $\mathbf{G}$. For $S \subseteq V(K)$ finite and saturated, we define

$$C^S(G) = \ker \pi_G^S$$

to be the *$S$-congruence kernel of $G$*.

By definition, the $S$-congruence kernel lies in a short exact sequence

$$1 \longrightarrow C^S(G) \longrightarrow \widehat{G(\mathcal{O}_S)}^S \xrightarrow{\pi_G^S} \overline{G(\mathcal{O}_S)}^S \longrightarrow 0,$$

which gives an alternative formulation of the congruence subgroup problem.

**Question 2.2.21** (Congruence subgroup problem, congruence kernel)**.** Is the $S$-congruence kernel $C^S(G)$ trivial?

### The congruence kernel as an invariant of algebraic groups

The formulations of the congruence subgroup problem we have seen so far were always dependent on the chosen $\mathcal{O}_K$-model $G$. However, the congruence kernel is actually an invariant of the algebraic group $\mathbf{G}$, which we want to see in the following.

Similarly to the $S$-arithmetic and $S$-congruence topology on $G(\mathcal{O}_K)$, we can define an $S$-arithmetic and $S$-congruence topology on $\mathbf{G}(K)$ by defining $S$-arithmetic and $S$-congruence subgroups, respectively to form an open neighborhood basis of the identity. Again, these give $\mathbf{G}(K)$ two different structures as topological Hausdorff groups, and we can form the Hausdorff completion to obtain complete Hausdorff groups. As before, we denote by

$$\widehat{\mathbf{G}(K)}^S \qquad \text{and} \qquad \overline{\mathbf{G}(K)}^S$$

the $S$-arithmetic and $S$-congruence completion of $\mathbf{G}(K)$, respectively.

Using again the universal property we obtain a continuous group homomorphism

$$\pi^S \colon \widehat{\mathbf{G}(K)}^S \longrightarrow \overline{\mathbf{G}(K)}^S.$$

**Lemma 2.2.22.** *The $S$-arithmetic (respectively $S$-congruence) completion of $G(\mathcal{O}_S)$ can be identified with the closure of $G(\mathcal{O}_S)$ in the $S$-arithmetic (respectively $S$-congruence) completion of $\mathbf{G}(K)$. Moreover, this closure is an open subgroup as well.*

*Proof.* The closure of $G(\mathcal{O}_S)$ in $\widehat{\mathbf{G}(K)}^S$ is again a complete Hausdorff group [10, Section II.3.4, Proposition 8]. Thus, using the uniqueness of Hausdorff completions [10, Section II.3.7, Proposition 13] gives the claim.

Since $G(\mathcal{O}_S)$ is open in $\mathbf{G}(K)$ with respect to the $S$-arithmetic topology, there is an open subgroup $V \leq \widehat{\mathbf{G}(K)}^S$ with $G(\mathcal{O}_S) = \mathbf{G}(K) \cap V$. As $\mathbf{G}(K)$ is dense and $V$ is also closed, we obtain that the closure of $G(\mathcal{O}_S)$ is $V$.

The proof for the $S$-congruence completion is the same. $\square$

**Corollary 2.2.23.** *Both the S-arithmetic completion $\widehat{\mathbf{G}(K)}^S$ and the S-congruence completion $\overline{\mathbf{G}(K)}^S$ are locally compact.*

*Proof.* By the lemma above, the compact group $\widehat{G(\mathcal{O}_K)}^S$ can be identified with an open subgroup of $\widehat{\mathbf{G}(K)}^S$, proving local compactness.

Again, the proof for the $S$-congruence completion is the same. □

**Corollary 2.2.24.** *The canonical continuous group homomorphism*

$$\pi^S \colon \widehat{\mathbf{G}(K)}^S \longrightarrow \overline{\mathbf{G}(K)}^S$$

*is surjective.*

*Proof.* We already know that this holds for the corresponding completions of $G(\mathcal{O}_S)$. Hence, the image of $\pi^S$ contains the open subgroup $\overline{G(\mathcal{O}_S)}^S$, and is thus itself open. Now, as a subgroup, it then also has to be closed. Since it, moreover, contains the dense subgroup $\mathbf{G}(K)$, the image has to be all of $\overline{\mathbf{G}(K)}^S$. □

Everything we have discussed so far shows that we have a commutative diagram of complete topological Hausdorff groups

$$\begin{tikzcd}
1 \arrow[r] & C^S(G) \arrow[r] \arrow[d, hook] & \widehat{G(\mathcal{O}_K)}^S \arrow[r, "\pi_G^S"] \arrow[d, hook] & \overline{G(\mathcal{O}_K)}^S \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \ker \pi^S \arrow[r] & \widehat{\mathbf{G}(K)}^S \arrow[r, "\pi^S"'] & \overline{\mathbf{G}(K)}^S \arrow[r] & 1,
\end{tikzcd}$$

with exact rows. Going into the definition of the Hausdorff completion, one can show equality for the kernels.

**Proposition 2.2.25.** *In the above diagram we have $C^S(G) = \ker \pi^S$.*

We will not prove this statement and instead refer the reader to either the book of Humphreys [43, Chapter VI, §17] or the book of Sury [99, Lemma 4-5.1]. Both discuss only the case of $\mathbf{G} = \mathrm{SL}_n$, but one easily sees that the proofs generalize to all algebraic groups.

**Definition 2.2.26** ($S$-congruence kernel)**.** Let $K$ be a number field, $\mathbf{G}$ an algebraic group over $K$ and $S \subseteq V(K)$ a finite saturated set of places. The *$S$-congruence kernel of* $\mathbf{G}$ is

$$C^S(\mathbf{G}) = \ker \pi^S.$$

#### An identification of $\overline{\mathrm{G}(K)}^S$

Before we continue our discussion on the congruence subgroup problem, we want to give a description of the $S$-congruence completion. In Lemma 2.2.18, we have seen that we can make the identification

$$\overline{G(\mathcal{O}_S)}^S \cong \varprojlim_{\mathfrak{a} \trianglelefteq \mathcal{O}_S} G(\mathcal{O}_S) / G(\mathcal{O}_S, \mathfrak{a}).$$

Since, by definition, $G(\mathcal{O}_S, \mathfrak{a}) = \ker\big(G(\mathcal{O}_S) \to G(\mathcal{O}_S/\mathfrak{a})\big)$, we can identify $G(\mathcal{O}_S)/(\mathcal{O}_S, \mathfrak{a})$ with $G(\mathcal{O}_S/\mathfrak{a})$, provided that the map $G(\mathcal{O}_S) \to G(\mathcal{O}_S/\mathfrak{a})$ is surjective. Assuming this surjectivity for all non-trivial ideals $\mathfrak{a} \subseteq \mathcal{O}_S$ gives the following lemma.

**Lemma 2.2.27.** *Assume that the canonical morphism $G(\mathcal{O}_S) \to G(\mathcal{O}_S/\mathfrak{a})$ is surjective for every non-trivial ideal $\mathfrak{a} \subseteq \mathcal{O}_S$. Then for the $S$-congruence completion of $G(\mathcal{O}_S)$ we have*

$$\overline{G(\mathcal{O}_S)}^S \cong \prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_\mathfrak{p}).$$

*Proof.* From Lemma 2.2.18 in combination with the surjectivity assumption we obtain

$$\overline{G(\mathcal{O}_S)}^S \cong \varprojlim_{\mathfrak{a} \trianglelefteq \mathcal{O}_S} G(\mathcal{O}_S) / G(\mathcal{O}_S, \mathfrak{a}) \cong \varprojlim_{\mathfrak{a} \trianglelefteq \mathcal{O}_S} G\left( \mathcal{O}_S / \mathfrak{a} \right).$$

Then, using that group schemes are representable functors and thus compatible with limits, we get

$$\overline{G(\mathcal{O}_K)}^S \cong \varprojlim_{\mathfrak{a} \trianglelefteq \mathcal{O}_S} G\left( \mathcal{O}_S / \mathfrak{a} \right) \cong G\left( \varprojlim_{\mathfrak{a} \trianglelefteq \mathcal{O}_S} \mathcal{O}_S / \mathfrak{a} \right).$$

By the Chinese remainder theorem, the inverse limit can be decomposed as

$$\varprojlim_{\mathfrak{a} \trianglelefteq \mathcal{O}_S} \mathcal{O}_S / \mathfrak{a} \cong \prod_{\mathfrak{p} \in \mathrm{mSpec}(\mathcal{O}_S)} \varprojlim_{n \in \mathbb{N}_{\geq 1}} \mathcal{O}_S / \mathfrak{p}^n \cong \prod_{\mathfrak{p} \in V(K) \setminus S} \mathcal{O}_\mathfrak{p},$$

where $\mathcal{O}_\mathfrak{p}$ is the valuation ring of $K_\mathfrak{p}$. Thus, we finally obtain

$$\overline{G(\mathcal{O}_S)}^S \cong G\left( \prod_{\mathfrak{p} \in V(K) \setminus S} \mathcal{O}_\mathfrak{p} \right) \cong \prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_\mathfrak{p}),$$

proving the statement. □

**Remark 2.2.28** (On the surjectivity assumption)**.** Note that the surjectivity assumption is in general not satisfied. One counterexample is given by $\mathbf{G}_m$ as a group scheme over $\mathbb{Z}$. Here $\mathbf{G}_m(\mathbb{Z}) = \{\pm 1\}$, while for every prime $p$ we have $\mathbf{G}_m(\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^\times$.

However, one can easily see that the canonical embedding

$$G(\mathcal{O}_S) \longrightarrow \prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_p)$$

induces the $S$-congruence topology on $G(\mathcal{O}_S)$, such that the characterization of the Hausdorff completion [10, Section II.3.7, Proposition 13] allows us to identify $\overline{G(\mathcal{O}_S)}^S$ with the closure of $G(\mathcal{O}_S)$ in this product.

In the case of $\overline{\mathbf{G}(K)}^S$ we have a very similar result.

**Proposition 2.2.29** (Identification of adèle points)**.** *Let $K$ be a number field, let $\mathbf{G}$ be an algebraic group over $K$, let $G$ be an $\mathcal{O}_K$-model of $\mathbf{G}$, and let $S \subseteq V(K)$ be finite. The $\mathbb{A}_{K,S}$-points of $\mathbf{G}$ can be identified with the restricted product*

$$\mathbf{G}(\mathbb{A}_{K,S}) \cong \prod_{\mathfrak{p} \in V(K) \setminus S} (G(K_\mathfrak{p}), G(\mathcal{O}_\mathfrak{p})).$$

*Proof.* The proof does not depend on the index set, so for the sake of simplicity we only prove the statement for the adèle ring $\mathbb{A}_K$.

Let $\mathcal{O}_K[G]$ be the Hopf algebra of $G$. As $G$ is of finite type, $\mathcal{O}_K[G]$ is finitely generated. Moreover, being a Dedekind domain, $\mathcal{O}_K$ is in particular Noetherian, so $\mathcal{O}_K[G]$ is in fact finitely presented. Now the Hom-functor $\mathrm{Hom}_R(T, \,\cdot\,)$ commutes with direct limits for every ring $R$ and every finitely presented $R$-algebra $T$ [98, Tag 00QO]. In particular, this holds for $G = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[G], \,\cdot\,)$. Thus,

$$\begin{aligned} G(\mathbb{A}_K) &= G\left(\prod_{\mathfrak{p} \in V(K)} (K_\mathfrak{p}, \mathcal{O}_\mathfrak{p})\right) \\ &= G\left(\varinjlim_S \prod_{\mathfrak{p} \in S} K_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} \mathcal{O}_\mathfrak{p}\right) \\ &\cong \varinjlim_S G\left(\prod_{\mathfrak{p} \in S} K_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} \mathcal{O}_\mathfrak{p}\right) \\ &\cong \varinjlim_S \prod_{\mathfrak{p} \in S} G(K_\mathfrak{p}) \times \prod_{\mathfrak{p} \notin S} G(\mathcal{O}_\mathfrak{p}) \\ &= \prod_{\mathfrak{p} \in V(K)} (G(K_\mathfrak{p}), G(\mathcal{O}_\mathfrak{p})), \end{aligned}$$

where we used that, as a representable functor, $G$ commutes with products. Finally, since $G$ was a model of $\mathbf{G}$ we have $\mathbf{G}(\mathbb{A}_K) \cong G(\mathbb{A}_K)$, finishing the proof. □

**Remark 2.2.30** (The choice of $\mathcal{O}_K$-model)**.** In the above statement, we actually do not need to choose an $\mathcal{O}_K$-model of $\mathbf{G}$. The algebraic group $\mathbf{G}$ is defined by finitely many polynomials, with finitely many coefficients each. Now for almost all $\mathfrak{p} \in V_f(K)$, all of these coefficients are elements of $\mathcal{O}_\mathfrak{p}$. Thus, using those polynomials, we can still make sense of $\mathbf{G}(\mathcal{O}_\mathfrak{p})$ for almost all $\mathfrak{p} \in V_f(K)$ (which is enough for the restricted product).

**Remark 2.2.31** ($S$-congruence topology using adèles)**.** Under the above identification, the diagonal embedding $K \to \mathbb{A}_{K,S}$ corresponds to the diagonal embedding

$$\mathbf{G}(K) \longrightarrow \prod_{\mathfrak{p} \in V(K) \setminus S} (\mathbf{G}(K_\mathfrak{p}), \mathbf{G}(\mathcal{O}_\mathfrak{p})).$$

Moreover, if $S$ is saturated, one can check that the induced topology on $\mathbf{G}(K)$ is precisely the $S$-congruence topology.

By using, once again, the characterization of the Hausdorff completion, we get a description of the $S$-congruence completion.

**Corollary 2.2.32.** *Let $K$ be a field, $\mathbf{G}$ an algebraic group over $K$, and $S \subseteq V(K)$ a finite saturated set of places. The $S$-congruence completion $\overline{\mathbf{G}(K)}^S$ is isomorphic to the closure of $\mathbf{G}(K)$ in $\mathbf{G}(\mathbb{A}_{K,S})$.*

Knowing that $K$ is dense in $\mathbb{A}_{K,S}$ (Theorem 2.1.38), one might expect that the same should also be true for the points of $\mathbf{G}$, i.e. that $\mathbf{G}(K)$ is dense in $\mathbf{G}(\mathbb{A}_{K,S})$. However, this is in general not the case and is precisely what *strong approximation* states.

**Definition 2.2.33** (Strong approximation)**.** Let $K$ be a number field, $\mathbf{G}$ an algebraic group over $K$, and $S \subseteq V(K)$ a finite saturated set. We say that $\mathbf{G}$ satisfies *strong approximation with respect to $S$*, if the diagonal embedding $\mathbf{G}(K) \to \mathbf{G}(\mathbb{A}_{K,S})$ has dense image.

**Corollary 2.2.34.** *Let $\mathbf{G}$ satisfy strong approximation with respect to $S$. Then*

$$\overline{\mathbf{G}(K)}^S \cong \mathbf{G}(\mathbb{A}_{K,S}),$$

*as well as*

$$\overline{G(\mathcal{O}_S)}^S \cong \prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_\mathfrak{p}).$$

*Proof.* The first part follows directly from the definition of strong approximation. For the second part we note that, in $\mathbf{G}(\mathbb{A}_{K,S})$, we have

$$G(\mathcal{O}_S) = \mathbf{G}(K) \cap \prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_\mathfrak{p}).$$

Here $\prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_\mathfrak{p})$ is open in $\mathbf{G}(\mathbb{A}_{K,S})$, so that $G(\mathcal{O}_S)$ is dense in it. □

The following theorem is usually the one giving strong approximation.

**Theorem 2.2.35** (Strong approximation theorem [77, Theorem 7.12])**.** *Let $K$ be a number field, $\mathbf{G}$ a* semisimple *algebraic group over $K$, and $S \subseteq V(K)$ finite saturated. Then $\mathbf{G}$ has strong approximation with respect to $S$ if and only if*

1. *$\mathbf{G}$ is simply connected and*
2. *$\mathbf{G}$ does not contain any almost simple factor $\mathbf{H}$ with $\mathbf{H}\left(\prod_{\mathfrak{p} \in S} K_\mathfrak{p}\right)$ compact.*

### The congruence subgroup property

Coming back to the congruence subgroup problem, we have reformulated the problem as whether the congruence kernel is trivial. Moreover, we have also seen that the congruence kernel $C^S(\mathbf{G})$ is an invariant of the algebraic group $\mathbf{G}$. As a closed subgroup of a profinite group, the congruence kernel is itself a profinite group and thus either uncountable or finite. In case it is finite, $S$-arithmetic subgroups should still be "close to being $S$-congruence subgroups".

**Definition 2.2.36** (Congruence subgroup property)**.** Let $K$ be a number field, $\mathbf{G}$ an algebraic group over $K$, and $S \subseteq V(K)$ a finite saturated set of places. We say that $G$ has the *congruence subgroup property with respect to* $S$ if the congruence kernel $C^S(\mathbf{G})$ is finite.

Obviously, a positive answer to the congruence subgroup problem implies the congruence subgroup property. We note that in the literature the abbreviation "CSP" is common for both the congruence subgroup *problem* and the congruence subgroup *property.* So when reading about CSP one should always be aware what the author means precisely. We will, however, not use this abbreviation in order to avoid this ambiguity.

Due to previous results on the congruence subgroup property, Serre asked whether the following holds [97, Footnote on p. 489]. Nowadays, this question is usually formulated as a conjecture.

**Conjecture 2.2.37** (Serre)**.** Let $K$ be a number field, $\mathbf{G}$ an absolutely almost simple simply connected algebraic group over $K$, and $S \subseteq V(K)$ finite saturated. Then the congruence kernel $C^S(\mathbf{G})$ should be finite if $\operatorname{rank}_S \mathbf{G} := \sum_{\mathfrak{p} \in S} \operatorname{rank}_{K_\mathfrak{p}} \geq 2$ and $\operatorname{rank}_{K_\mathfrak{p}} \mathbf{G} \geq 1$ for all $\mathfrak{p} \in S \setminus V_\infty(K)$. Conversely, the congruence kernel $C^S(\mathbf{G})$ should be infinite if $\operatorname{rank}_S \mathbf{G} = 1$.

There are many results supporting this conjecture, however there are still some open cases. Most prominently it is still open whether $\mathbb{Q}$-groups whose group of real points is isomorphic to $\mathrm{Sp}(n, 1)$ or $\mathrm{F}_4^{(-20)}$ have the congruence subgroup property [62, Section 4].

Let us only give a single result establishing the congruence subgroup property for some algebraic groups. For a better overview of the current state, we instead refer the reader to various survey articles by Rapinchuk [86], Prasad and Rapinchuk [82] or Raghunathan [85] and the references contained in there.

**Theorem 2.2.38** (Raghunathan [84, Theorem D])**.** *Let $K$ be a number field, $\mathbf{G}$ an absolutely almost simple simply connected algebraic group over $K$ with $\operatorname{rank}_K \mathbf{G} \geq 2$. Then for any finite saturated $S \subseteq V(K)$, the congruence kernel $C^S(\mathbf{G})$ is finite.*

# Chapter 3

# Profinite invariance

The field of geometric group theory studies on which kinds of geometric spaces certain groups can act. One might ask what can be learned about a group when studying only the actions on spaces with *finite* automorphism group, most notably finite sets.

This inevitably leads to the question of *profinite rigidity.*

**Question.** Which groups are determined by the finite quotients?

In case there are two groups with the same finite quotients, one might still ask which properties they share, yielding the question of *profinite properties.*

**Question.** Which group properties are determined by the finite quotients of a group?

Obviously, not every group is determined by its finite quotients, as for example simple groups do not have any finite quotients at all. So for both of these questions, it is reasonable to restrict to a class of groups with "many" finite quotients. Usually one considers residually finite groups.

This chapter will introduce the notions of *profinite rigidity*, *profinite properties* and *profinite invariants.* Moreover, we provide some first examples of properties and invariants that are (not) profinite. Finally, we also briefly discuss *Grothendieck pairs*, which is another variation of the question of profinite rigidity, and one way of constructing them.

For a nice overview of the topic we recommend the survey articles by Reid [87] and Bridson [11].

### Overview of this chapter

## 3.1 On the question of profinite properties

### 3.1.1 Ensuring finite quotients: residual finiteness

As already stated in the introduction of this chapter, it is reasonable to focus on groups admitting many finite quotients. It turns out that *residually finite groups* are the natural class of groups to study profiniteness questions.

**Definition 3.1.1** (Residually finite group)**.** A group $G$ is called *residually finite* if for every $g \in G \setminus \{1\}$ there exists a group homomorphism $\varphi\colon G \to F$ to a finite group such that $\varphi(g) \neq 1$.

For finitely generated groups, or more generally countable groups, being residually finite is equivalent to the existence of a *residual chain*.

**Definition 3.1.2** (Residual chain)**.** Let $G$ be a group. A *residual chain in* $G$ is a family of finite index normal subgroups $(G_k)_{k\in\mathbb{N}}$ with $G_{k+1} \subseteq G_k$ for all $k \in \mathbb{N}$ and $\bigcap_{k\in\mathbb{N}} G_k = \{1\}$.

**Proposition 3.1.3** (Characterization of residually finite groups)**.** *For a group $G$ the following are equivalent:*

1. *The group $G$ is residually finite.*
2. *The intersection of all normal subgroups of finite index of $G$ is trivial.*

*If additionally $G$ is finitely generated this is equivalent to:*

3. *There is a residual chain in $G$.*

*Proof.* **1. $\Rightarrow$ 2.** First assume $G$ is residually finite. For $g \in G \setminus \{1\}$ there exists a group homomorphism $\varphi\colon G \to F$ to a finite group with $\varphi(g) \neq 1$. Thus, $g$ is not in the finite index normal subgroup $\ker\varphi$ and so $\bigcap_{N \trianglelefteq_{\text{f.i.}} G} N = \{1\}$.

**2. $\Rightarrow$ 1.** Conversely, assume $\bigcap_{N \trianglelefteq_{\text{f.i.}} G} N = \{1\}$. Then for every $g \in G \setminus \{1\}$ there exists a finite index normal subgroup $N \trianglelefteq_{\text{f.i.}} G$, such that $g \notin N$. Thus, the canonical projection $\pi\colon G \to G/N$ is a group homomorphism to a finite group with $\pi(g) \neq 1$.

Now assume that $G$ is finitely generated.

**2. $\Rightarrow$ 3.** For $k \in \mathbb{N}$ let
$$G_k = \bigcap_{\substack{N \trianglelefteq G \\ [G:N] \le k}} N.$$
Since there are only finitely many subgroups of $G$ of index $\le k$, this intersection is in fact a finite intersection of normal subgroups of finite index. This gives a residual chain as
$$\bigcap_{k\in\mathbb{N}} G_k = \bigcap_{N \trianglelefteq_{\text{f.i.}} G} N = 1$$
is trivial by assumption.

**3. $\Rightarrow$ 2.** Conversely, let $(G_k)_{k\in\mathbb{N}}$ be a residual chain. Since every $G_k$ is a normal subgroup of finite index we get
$$\bigcap_{N \trianglelefteq_{\text{f.i.}} G} N \subseteq \bigcap_{k\in\mathbb{N}} G_k = \{1\}.$$
□

### 3.1.2 Finite quotients and the profinite completion

The *profinite completion* combines all finite quotients of a group into a single object.

**Definition 3.1.4** (Profinite completion)**.** Let $G$ be a group. The collection of all finite index normal subgroups of $G$ form an inverse system. We define the *profinite completion* of $G$ to be the topological group

$$\widehat{G} = \varprojlim_{N \trianglelefteq_{\text{f.i.}} G} G/N = \left\{ (x_N N)_N \in \prod_{N \trianglelefteq_{\text{f.i.}} G} G/N \;\middle|\; x_N M = x_M M \text{ for every } N \subseteq M \right\}.$$

We have a canonical map $G \to \widehat{G}$ given by $g \mapsto (gN)_N$. This map has dense image by construction.

**Remark 3.1.5** (Universal property [88, Lemma 3.2.1])**.** Let $G$ be a group. The profinite completion $\widehat{G}$, together with the canonical map $\iota\colon G \to \widehat{G}$, has the following universal property:

For every profinite group $H$ and every group homomorphism $\varphi\colon G \to H$ there exists a *unique* continuous group homomorphism $\widehat{\varphi}\colon \widehat{G} \to H$ with $\varphi = \widehat{\varphi} \circ \iota$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\iota"'] & H \\ \widehat{G} \arrow[ur, dashed, "\exists!\widehat{\varphi}"'] & \end{tikzcd}$$

It suffices to check this universal property only for *finite* groups $H$.

**Example 3.1.6.** The profinite completion of a finite group is the group itself. This can easily be checked using the universal property.

**Example 3.1.7.** Consider the group of integers $\mathbb{Z}$. By the Chinese remainder theorem, every finite quotient of $\mathbb{Z}$ is a product of cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$, for some prime $p$ and $n \in \mathbb{N}_{\geq 1}$. From this, it is easy to see that the profinite completion of $\mathbb{Z}$ is given by the product of all $p$-adic completions

$$\widehat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p.$$

**Corollary 3.1.8** (Profinite completion and residual finiteness)**.** *Let $G$ be a group. Then $G$ is residually finite if and only if the canonical map $G \to \widehat{G}$ is injective. In particular, the profinite topology on $G$ coincides with the subspace topology induced by $\widehat{G}$.*

*Proof.* By construction, the kernel of the canonical map $G \to \widehat{G}$ is $\bigcap_{N \trianglelefteq_{\text{f.i.}} G} N$. According to Proposition 3.1.3, this intersection is trivial if and only if $G$ is residually finite. $\square$

For finitely generated groups, the profinite completion is in fact already determined by the finite quotients of the group.

**Theorem 3.1.9** ([23, Main Theorem]). *Let $G$ and $H$ be finitely generated groups. Then $\widehat{G}$ and $\widehat{H}$ are isomorphic as topological groups if and only if the set of isomorphism classes of finite quotients of $G$ is equal to the set of isomorphism classes of finite quotients of $H$.*

By the work of Nikolov and Segal [71, Theorem 1.1], one can moreover drop the condition that the profinite completions need to be isomorphic as *topological* groups and can instead only require that they are isomorphic as abstract groups.

This allows us to formulate the questions on profinite rigidity and profinite properties in terms of profinite completions.

**Definition 3.1.10** (Profinite rigidity). Let $G$ be a finitely generated residually finite group. We say *$G$ is profinitely rigid* if for every finitely generated residually finite group $H$ with $\widehat{G} \cong \widehat{H}$ we have $G \cong H$.

**Example 3.1.11.** Finite groups are profinitely rigid, as they are equal to their own profinite completions.

**Definition 3.1.12** (Profinite property, profinite invariant). Let $P$ be a property of finitely generated residually finite groups. We say that *$P$ is a profinite property* if for all finitely generated residually finite groups $G$ and $H$ with $\widehat{G} \cong \widehat{H}$ we have that $G$ has $P$ if and only if $H$ has $P$.

Similarly, an invariant $I$ of finitely generated residually finite groups is a *profinite invariant*, if for all finitely generated residually finite groups $G$ and $H$ with $\widehat{G} \cong \widehat{H}$ we have that $I(G)$ agrees with $I(H)$.

### 3.1.3 Profinite invariance: some (non-)examples

The first and easiest example of a profinite property is satisfying a law.

**Lemma 3.1.13** (Satisfying a law is profinite). *Let $n \in \mathbb{N}$ and let $w \in F_n$. If $G$ and $H$ are finitely generated residually finite groups with $\widehat{G} \cong \widehat{H}$ and $G$ satisfies $w$, i.e. for all $a_1, \dots, a_n \in G$ we have $w(a_1, \dots, a_n) = 1$, then $H$ satisfies $w$ as well.*

*Proof.* Assume that $H$ does not satisfy $w$. Then there are $b_1, \dots, b_n \in H$ such that $w(b_1, \dots, b_n) \neq 1$. Since $H$ is residually finite, this gives a finite quotient $F$ of $H$ such that $w(b_1, \dots, b_n)$ is non-trivial in this finite quotient. However, as $G$ and $H$ have isomorphic profinite completions, they have the same finite quotients by Theorem 3.1.9. So $F$ is also a quotient of $G$. But then $F$ satisfies $w$ as well, giving a contradiction. ☐

**Corollary 3.1.14** (Being abelian is a profinite property). *Let $G$ and $H$ be finitely generated residually finite groups with $\widehat{G} \cong \widehat{H}$. If $G$ is abelian, then $H$ is as well.*

*Proof.* Apply the above lemma to the word $[a, b] = aba^{-1}b^{-1} \in F_2$. ☐

Using the classification of finitely generated abelian groups, one can use this in order to prove that finitely generated abelian groups are profinitely rigid.

**Lemma 3.1.15** (Abelian groups are profinitely rigid)**.** *Let $G$ and $H$ be finitely generated residually finite groups with $\widehat{G} \cong \widehat{H}$. If $G$ is abelian, then $G \cong H$.*

*Proof.* We have already seen that in this case $H$ needs to be abelian as well. Thus, we can write $G \cong \mathbb{Z}^n \oplus T_1$ as well as $H \cong \mathbb{Z}^m \oplus T_2$, where $T_1$ and $T_2$ are finite abelian groups.

First we can prove that $n = m$. Assume $n > m$ and let $p$ be a prime large enough such that $p$ does not divide $|T_1|$ and $|T_2|$. Then $(\mathbb{Z}/p\mathbb{Z})^n$ is a quotient of $G$ which cannot be a quotient of $H$, giving a contradiction.

Now, if $T_1$ is not isomorphic to $T_2$, then without loss of generality $T_1$ is not a quotient of $T_2$. Take again $p$ a prime not dividing $|T_1|$ and $|T_2|$. Then $(\mathbb{Z}/p\mathbb{Z})^n \oplus T_1$ is a finite quotient of $G$ which cannot be a quotient of $H$. Again a contradiction. □

The proof in fact already shows that the abelianization is a profinite invariant.

**Corollary 3.1.16** (Abelianization is profinite)**.** *The abelianization is a profinite invariant.*

Moving away only slightly from abelian groups, one loses profinite rigidity.

**Proposition 3.1.17** ([4][87, Theorem 3.3])**.** *There are two non-isomorphic extensions of $\mathbb{Z}/25\mathbb{Z}$ by $\mathbb{Z}$ which have isomorphic profinite completions.*

*Sketch of proof.* Consider the two groups $G = \mathbb{Z}/25\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ and $H = \mathbb{Z}/25\mathbb{Z} \rtimes_\beta \mathbb{Z}$, where $\alpha$ is the automorphism $x \mapsto 6 \cdot x$ and $\beta$ is the automorphism $x \mapsto 11 \cdot x$. One can directly verify that these two groups are not isomorphic.

However, we have $G \times \mathbb{Z} \cong H \times \mathbb{Z}$, which gives us that $G$ and $H$ have isomorphic profinite completions [4, Section 2, Proposition]. "□"

Using Lück's approximation theorem [63], Corollary 3.1.16 shows that the first $\ell^2$-Betti number is a profinite invariant of *finitely presented* groups. First let us discuss a canonical choice of a residual chain and some properties.

**Definition 3.1.18** (Canonical residual chain)**.** Let $G$ be a finitely generated residually finite group. We call the residual chain

$$G_k = \bigcap_{\substack{N \trianglelefteq G \\ [G:N] \leq k}} N,$$

constructed in the proof of Proposition 3.1.3, the *canonical residual chain of $G$*.

**Proposition 3.1.19.** *Let $G, H$ be two finitely generated, residually finite groups with $\widehat{G} \cong \widehat{H}$. Then for the canonical residual chains $(G_k)_{k\in\mathbb{N}}$ and $(H_k)_{k\in\mathbb{N}}$ we have that*

$$G/G_k \cong H/H_k \qquad \textit{and} \qquad \widehat{G_k} \cong \widehat{H_k}$$

*for all $k \in \mathbb{N}$.*

*Proof.* It is well-know that for a residually finite group $\Gamma$ the assignment

$$\begin{aligned}\varphi_\Gamma\colon \{N \mid N \leq \Gamma, [\Gamma : N] < \infty\} &\longrightarrow \{U \mid U \leq \widehat{\Gamma} \text{ open}\}\\ N &\longmapsto \overline{N}\end{aligned}$$

is a bijection preserving both the index and being normal. In case $N \trianglelefteq \Gamma$ is a normal subgroup of finite index, we have $\Gamma/N \cong \widehat{\Gamma}/\overline{N}$ [88, Proposition 3.2.2]. Moreover, it is also well-known that, for each $N \leq \Gamma$ of finite index, $\varphi_\Gamma(N) = \overline{N}$ is (isomorphic to) the profinite completion of $N$.

Since $\widehat{G} \cong \widehat{H}$, using that both $\varphi_G$ and $\varphi_H$ preserve the index and being normal, it is easy to see that $\varphi_H^{-1} \circ \varphi_G$ maps $G_k$ to $H_k$, and so $\overline{G_k} = \overline{H_k}$. In particular, we obtain

$$G/G_k \cong \widehat{G}/\overline{G_k} = \widehat{H}/\overline{H_k} \cong H/H_k,$$

as well as

$$\widehat{G_k} \cong \overline{G_k} = \overline{H_k} \cong \widehat{H_k}. \qquad \square$$

**Corollary 3.1.20** (First $\ell^2$-Betti number is profinite)**.** *Let $G$ and $H$ be finitely* presented *residually finite groups. If $G$ and $H$ are profinitely isomorphic, then $b_1^{(2)}(G) = b_1^{(2)}(H)$.*

*Proof.* By Lück's approximation theorem, we have

$$b_1^{(2)}(G) = \lim_{k\to\infty} \frac{b_1(G_k)}{[G : G_k]},$$

where $(G_k)_{k\in\mathbb{N}}$ is the canonical residual chain of $G$. Similarly, this holds for $H$. Now as $G$ and $H$ are profinitely isomorphic, so are $G_k$ and $H_k$ for every $k \in \mathbb{N}$ by Proposition 3.1.19. As the abelianization is a profinite property (Corollary 3.1.16), so is the first Betti number $b_1(\,\cdot\,) = \dim_{\mathbb{Q}}((\,\cdot\,)_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q})$. Thus, making again use of Proposition 3.1.19, we get

$$\frac{b_1(G_k)}{[G : G_k]} = \frac{b_1(H_k)}{[H : H_k]}$$

for all $k \in \mathbb{N}$ and thus $b_1^{(2)}(G) = b_1^{(2)}(H)$. $\square$

In contrast to the first $\ell^2$-Betti number, Kammeyer and Sauer proved that none of the higher $\ell^2$-Betti numbers are profinite invariants [51]. Again using Lück's approximation theorem, one can use this to prove that the higher homology groups are not profinite invariants.

**Corollary 3.1.21** (Non-profiniteness of higher homology groups)**.** *Let $n \in \mathbb{N}_{\geq 2}$. The $n$-th homology $H_n(\,\cdot\,;\mathbb{Q})$ group with coefficients in $\mathbb{Q}$ is* not *a profinite invariant.*

*In particular, $H_n(\,\cdot\,;\mathbb{Z})$ is* not *a profinite invariant either.*

*Proof.* Let $G$ and $H$ be finitely generated residually finite groups of type $F$ with isomorphic profinite completions exhibiting the non-profiniteness of $b_n^{(2)}$, i.e. $b_n^{(2)}(G) \neq b_n^{(2)}(H)$. Such groups are given by the examples of Kammeyer and Sauer [51], after passing to finite index subgroups when necessary.

By Lück's approximation theorem [63] we can compute the $\ell^2$-Betti numbers using any residual chain. In particular, for the canonical residual chains $(G_k)_{k\in\mathbb{N}}$ and $(H_k)_{k\in\mathbb{N}}$ of $G$, respectively $H$, we get that

$$\lim_{k\to\infty} \frac{b_n(G_k)}{[G : G_k]} = b_n^{(2)}(G) \neq b_n^{(2)}(H) = \lim_{k\to\infty} \frac{b_n(H_k)}{[H : H_k]}.$$

Thus there is some $k \in \mathbb{N}$ with

$$\frac{b_n(G_k)}{[G : G_k]} \neq \frac{b_n(H_k)}{[H : H_k]}.$$

But, by Proposition 3.1.19, we have $[G : G_k] = [H : H_k]$, so

$$\dim_{\mathbb{Q}}(H_n(G_k; \mathbb{Q})) = b_n(G_k) \neq b_n(H_k) = \dim_{\mathbb{Q}}(H_n(H_k; \mathbb{Q}))$$

and thus $H_n(G_k; \mathbb{Q}) \ncong H_n(H_k; \mathbb{Q})$. Finally, again by Proposition 3.1.19, we have that $\widehat{G_k} \cong \widehat{H_k}$ which shows that $H_n(\,\cdot\,; \mathbb{Q})$ is not a profinite invariant.

For the statement concerning homology with integral coefficients, we use the natural isomorphism $H_n(\,\cdot\,; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(\,\cdot\,; \mathbb{Q})$ to see that also $H_n(G_k; \mathbb{Z}) \ncong H_n(H_k; \mathbb{Z})$. □

**Remark 3.1.22.** There are alternative proofs showing that the higher homology groups are not profinite. For example, Bridson and Reid constructed two groups that are profinitely isomorphic where one is finitely presented, and thus has finite dimensional $H_2(\,\cdot\,; \mathbb{Q})$, and the other has infinite dimensional second homology [14, Theorem B]. Then a kind of "dimension shifting" argument can be used to get similar statements for the other higher homology groups. We will see how this can be done in Section 5.2.1.

**Remark 3.1.23** (Some more (non-)examples)**.** There are many other examples of properties which are (not) profinite. Let us name just a few.

1. Pickel proved that nilpotent groups have *finite genus* among nilpotent groups, i.e., given a nilpotent group, there are only finitely many isomorphism classes of nilpotent groups with isomorphic profinite completion [76].

2. Sabbagh and Wilson proved that being polycyclic is a profinite property [92].

3. Aka constructed profinitely isomorphic arithmetic groups where one has Kazhdan's property (T) and the other does not [3].

4. As a strengthening of this result, Cheetham-West, Lubotzky, Reid, and Spitler proved that having property FA is not profinite [19]. Recently Fournier-Facio further improved the result by showing that property NL is not profinite [31].

5. Lubotzky proved that being of type $F_n$ is not a profinite property [61].
6. For arithmetic groups with the congruence subgroup property, Kammeyer, Kionke, Raimbault, and Sauer proved that the sign of the Euler characteristic is a profinite property [50]. This was generalized to $S$-arithmetic groups, except possibly those of type ${}^6\mathrm{D}_4$, by Kammeyer and Serafini [52].
7. Liu proved that finite-volume hyperbolic 3-manifold groups have finite genus among finitely generated 3-manifold groups [60].
8. Kionke and Schesler constructed groups acting on trees which show that being amenable is not profinite. In the same paper, they also proved that being uniformly amenable is a profinite property [55].
9. Corson, Hughes, Möller, and Varghese proved that affine Coxeter groups are profinitely rigid [22]. In a different paper the same authors proved that right-angled Coxeter groups are profinitely rigid among all Coxeter groups [21].

We conclude this section by stating one of the big open questions in the field of profinite rigidity.

**Question 3.1.24** (Remeslennikov)**.** Are finitely generated (non-abelian) free groups profinitely rigid?

## 3.2 Profinitely isomorphic arithmetic groups

In this section, we give a construction of how to obtain different arithmetic subgroups with isomorphic profinite completions.

**Remark 3.2.1** (Profinite vs. congruence completion)**.** Let $K$ be a number field, $S \subseteq V(K)$ a finite saturated set of places, and $G$ an $\mathcal{O}_K$-model of an algebraic $K$-group $\mathbf{G}$. The $S$-congruence kernel $C^S(\mathbf{G})$ measures precisely the difference of the profinite and $S$-congruence completion of $G(\mathcal{O}_S)$,

$$0 \longrightarrow C^S(\mathbf{G}) \longrightarrow \widehat{G(\mathcal{O}_S)} \longrightarrow \overline{G(\mathcal{O}_S)}^S \longrightarrow 0.$$

If the $S$-congruence kernel is trivial, the profinite completion and $S$-congruence completion agree. In Corollary 2.2.34 we have seen that the $S$-congruence completion is given by

$$\overline{G(\mathcal{O}_S)}^S \cong \prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_\mathfrak{p})$$

if $\mathbf{G}$ satisfies strong approximation with respect to $S$. Thus, given another $\mathcal{O}_K$-model $H$ of an algebraic group $\mathbf{H}$ over $K$ with trivial $S$-congruence kernel and strong approximation with respect to $S$, it suffices to prove

$$G(\mathcal{O}_\mathfrak{p}) \cong H(\mathcal{O}_\mathfrak{p}) \qquad \text{for all } \mathfrak{p} \in V(K) \setminus S,$$

in order to obtain $\widehat{G(\mathcal{O}_S)} \cong \widehat{H(\mathcal{O}_S)}$.

In the more general setting that the congruence kernels are only finite, this condition is still enough to find *finite index* subgroups of $G(\mathcal{O}_S)$ and $H(\mathcal{O}_S)$ with isomorphic profinite completions.

**Proposition 3.2.2** (Profinitely isomorphic arithmetic groups)**.** *Let $K$ be a number field, let $S \subseteq V(K)$ be a finite saturated set of places and let $G, H$ be $\mathcal{O}_K$-models of algebraic $K$-groups $\mathbf{G}$ and $\mathbf{H}$, respectively. Moreover, assume that both $\mathbf{G}$ and $\mathbf{H}$ have the congruence subgroup property, as well as strong approximation, with respect to $S$ and that for every $\mathfrak{p} \in V(K) \setminus S$ we have an isomorphism $G(\mathcal{O}_\mathfrak{p}) \cong H(\mathcal{O}_\mathfrak{p})$.*

*Then there are $S$-arithmetic subgroups $\Gamma \leq \mathbf{G}(K)$, $\Lambda \leq \mathbf{H}(K)$ with isomorphic profinite completions $\widehat{\Gamma} \cong \widehat{\Lambda}$.*

*Proof.* As the $S$-congruence kernel $C^S(\mathbf{G})$ is finite, we can find a finite index open subgroup $\widehat{\Gamma_0} \leq \widehat{G(\mathcal{O}_S)}$ which intersects the congruence kernel trivially. This subgroup is the profinite completion of a finite index subgroup $\Gamma_0 \leq G(\mathcal{O}_S)$ [88, Proposition 3.2.2]. As $\widehat{\Gamma_0}$ intersects the congruence kernel trivially, it agrees with the closure in the $S$-congruence completion $\overline{\Gamma_0}^S \leq \overline{G(\mathcal{O}_S)}^S$. Moreover, $\overline{\Gamma_0}^S$ is of finite index in $\overline{G(\mathcal{O}_S)}^S$, so as a finite index closed subgroup it is open as well.

Similarly, we can find a finite index subgroup $\Lambda_0 \leq H(\mathcal{O}_S)$ such that the profinite completion $\widehat{\Lambda_0}$ agrees with the closure in the $S$-congruence completion $\overline{\Lambda_0}^S \leq \overline{H(\mathcal{O}_S)}^S$.

Now, under the identification

$$\overline{G(\mathcal{O}_S)}^S \cong \prod_{\mathfrak{p} \in V(K) \setminus S} G(\mathcal{O}_\mathfrak{p}) \cong \prod_{\mathfrak{p} \in V(K) \setminus S} H(\mathcal{O}_\mathfrak{p}) \cong \overline{H(\mathcal{O}_S)}^S$$

the intersection $\overline{\Gamma_0}^S \cap \overline{\Lambda_0}^S$ is a finite index open subgroup in both $\overline{G(\mathcal{O}_S)}^S$ and $\overline{H(\mathcal{O}_S)}^S$. Hence, this intersection corresponds to finite index open subgroups $\widehat{\Gamma} \leq \widehat{\Gamma_0}$ and $\widehat{\Lambda} \leq \widehat{\Lambda_0}$. Again, these are the profinite completions of finite index subgroups $\Gamma \leq \Gamma_0$ and $\Lambda \leq \Lambda_0$.

So both $\Gamma$ and $\Lambda$ are of finite index in $G(\mathcal{O}_S)$ and $H(\mathcal{O}_S)$, respectively, and thus are $S$-arithmetic subgroups. Finally, we have

$$\widehat{\Gamma} \cong \overline{\Gamma}^S = \overline{\Gamma_0}^S \cap \overline{\Lambda_0}^S = \overline{\Lambda}^S \cong \widehat{\Lambda}$$

by construction of $\Gamma$ and $\Lambda$. □

Aka used this idea in order to construct two arithmetic subgroups of Spin-groups with isomorphic profinite completion, where one has Kazhdan's property (T) and the other one does not.

**Lemma 3.2.3** ([3, Lemma 2])**.** *For any prime $p$, the two quadratic forms*

$$q_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad \text{and} \qquad q_2 = -x_1^2 - x_2^2 - x_3^2 - x_4^2$$

*are isometric over $\mathbb{Z}_p$.*

*Proof.* In case $p \equiv 1 \mod 4$, there is $i \in \mathbb{Z}_p$ with $i^2 = -1$. Hence, already the quadratic forms $x_1^2$ and $-x_1^2$ are isometric.

If $p \equiv 3 \mod 4$, there are $a, b \in \mathbb{Z}_p$ such that $a^2 + b^2 = -1$. Thus, the quadratic forms

$$x_1^2 + x_2^2 \qquad \text{and} \qquad -x_1^2 - x_2^2$$

are isometric via the transformation matrix $\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$.

Finally, for $p = 2$ there is $a \in \mathbb{Z}_2$ with $a^2 = -7$. Using the orthogonal transformation matrix

$$\begin{pmatrix} 2 & 1 & 1 & a \\ -1 & 2 & -a & 1 \\ -1 & a & 2 & -1 \\ -a & -1 & 1 & 2 \end{pmatrix},$$

one gets the claimed isometry of $q_1$ and $q_2$. □

For standard quadratic forms of signature $(n, m)$, i.e.

$$x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - \cdots - x_{n+m}^2,$$

this lemma allows us to always flip four signs simultaneously over every $\mathbb{Z}_p$.

**Corollary 3.2.4.** *Let $n, m \in \mathbb{N}_{\geq 2}$. There are arithmetic subgroups*

$$\Gamma \leq \mathrm{Spin}(n+4, m)(\mathbb{Q}) \qquad \textit{and} \qquad \Lambda \leq \mathrm{Spin}(n, m+4)(\mathbb{Q})$$

*with isomorphic profinite completion $\widehat{\Gamma} \cong \widehat{\Lambda}$.*

*Proof.* The simply connected simple algebraic groups $\mathrm{Spin}(q)$, with $q$ a quadratic form over $\mathbb{Z}$, can already be defined over the integers. Moreover, this construction is functorial in the quadratic form $q$ [51, Section 3]. Thus, by using the above Lemma 3.2.3, we immediately see that we have isomorphisms

$$\mathrm{Spin}(n+4, m)(\mathbb{Z}_p) \cong \mathrm{Spin}(n, m+4)(\mathbb{Z}_p)$$

for every prime $p$.

Additionally, the $\mathbb{Q}$-ranks of $\mathrm{Spin}(n+4, m)$ and $\mathrm{Spin}(n, m+4)$ are $\min(n+4, m)$ and $\min(n, m+4)$, respectively. So our assumption $n, m \geq 2$ ensures the congruence subgroup property by Theorem 2.2.38. Thus, using Proposition 3.2.2, we obtain $\Gamma$ and $\Lambda$ as desired. □

**Remark 3.2.5** (Different fields of definition)**.** Let us note that the proof of Proposition 3.2.2 did not use that $\mathbf{G}$ and $\mathbf{H}$ were defined over the same number field $K$.

In fact, let $\mathbf{G}$ and $\mathbf{H}$ be defined over $K$ and $K'$ respectively and let $S \subseteq V(K)$ and $S' \subseteq V(K')$ be finite saturated sets of places. Assume that there is a bijection $V(K) \setminus S \to V(K') \setminus S', \mathfrak{p} \mapsto \mathfrak{p}'$, and that $\mathbf{G}$ and $\mathbf{H}$ have both the congruence subgroup property as well as strong approximation with respect to $S$ and $S'$, respectively. If there is an isomorphism

$$G(\mathcal{O}_{K,\mathfrak{p}}) \cong H(\mathcal{O}_{K',\mathfrak{p}'}) \qquad \text{for every } \mathfrak{p} \in V(K) \setminus S,$$

then the same arguments show that there is an $S$-arithmetic subgroup $\Gamma \leq \mathbf{G}(K)$ and an $S'$-arithmetic subgroup $\Lambda \leq \mathbf{H}(K')$ with $\widehat{\Gamma} \cong \widehat{\Lambda}$.

This can in particular be applied to arithmetically equivalent or locally isomorphic number fields, where $\mathbf{G}$ and $\mathbf{H}$ come from the same algebraic group over $\mathbb{Q}$.

**Example 3.2.6.** Let $K = \mathbb{Q}(\sqrt[8]{7})$ and $K' = \mathbb{Q}(\sqrt[8]{112})$. A result by Komatsu [59] shows that $K$ and $K'$ are locally isomorphic but *not* isomorphic. Hence, there is a bijection of places of $K$ and $K'$ such that the local completions are isomorphic (Theorem 2.1.42). In particular, the valuation rings at all finite places are also isomorphic. Thus, we can find arithmetic subgroups in $\mathrm{SL}_n(K)$ and $\mathrm{SL}_n(K')$, for $n \geq 3$, with isomorphic profinite completions. Here the condition of $n \geq 3$ ensures the congruence subgroup property using Theorem 2.2.38.

A slight variation of Proposition 3.2.2, which does not use $\mathcal{O}_K$-forms explicitly, is the following.

**Proposition 3.2.7** (Profinite isomorphism using congruence completions)**.** *Let $K$ be a number field, $S \subseteq V(K)$ a finite saturated set of places and $\mathbf{G}$, $\mathbf{H}$ two algebraic groups over $K$ satisfying the congruence subgroup property with respect to $S$. If there is an isomorphism $\overline{\mathbf{G}(K)}^S \cong \overline{\mathbf{H}(K)}^S$ of the $S$-congruence completions, then there are $S$-arithmetic subgroups of $\mathbf{G}(K)$ and $\mathbf{H}(K)$, respectively, with isomorphic profinite completions.*

*Proof.* After choosing $\mathcal{O}_K$-models $G, H$ for $\mathbf{G}$ and $\mathbf{H}$, respectively, the proof is essentially the same as the one of Proposition 3.2.2. The only difference is that we do not necessarily have $\overline{G(\mathcal{O}_S)}^S \cong \overline{H(\mathcal{O}_S)}^S$. However, the intersection of $\overline{\Gamma_0}^S$ and $\overline{\Lambda_0}^S$, as subgroups of $\overline{\mathbf{G}(K)}^S \cong \overline{\mathbf{H}(K)}^S$, still gives finite index subgroups of $\overline{G(\mathcal{O}_S)}^S$ and $\overline{H(\mathcal{O}_S)}^S$, respectively. □

Recall from Corollary 2.2.34, that in case $\mathbf{G}$ satisfies strong approximation with respect to $S$, we get a more hands-on description of the $S$-congruence completion as

$$\overline{\mathbf{G}(K)}^S \cong \mathbf{G}(\mathbb{A}^f_{K,S}) \cong \prod_{\mathfrak{p} \in V(K) \setminus S} (\mathbf{G}(K_\mathfrak{p}), \mathbf{G}(\mathcal{O}_\mathfrak{p})).$$

This allows us to give a criterion in terms of forms of algebraic groups, and thus in terms of Galois cohomology, when two algebraic groups have isomorphic $S$-congruence completions.

**Proposition 3.2.8** (Isomorphic adèle points)**.** *Let $K$ be a number field, let $S \subseteq V(K)$ be a finite saturated set of places and let $\mathbf{G}, \mathbf{H}$ be two algebraic groups over $K$. If $\mathbf{G}$ and $\mathbf{H}$ are isomorphic over $K_\mathfrak{p}$ for every $\mathfrak{p} \in V(K) \setminus S$, then there exists an isomorphism*

$$\mathbf{G}(\mathbb{A}_{K,S}) \cong \mathbf{H}(\mathbb{A}_{K,S}).$$

*Proof.* Since $\mathbf{G}$ and $\mathbf{H}$ are isomorphic as algebraic groups over $K_\mathfrak{p}$, we have in particular an isomorphism $\mathbf{G}(K_\mathfrak{p}) \cong \mathbf{H}(K_\mathfrak{p})$. Thus, in order to get the statement for the adèle points, we only need to see that for almost all $\mathfrak{p} \in V(K) \setminus S$, this isomorphism restricts to one of $\mathbf{G}(\mathcal{O}_\mathfrak{p}) \cong \mathbf{H}(\mathcal{O}_\mathfrak{p})$.

Let $G$ and $H$ be $\mathcal{O}_K$-models of $\mathbf{G}$ and $\mathbf{H}$, respectively. For almost all places $\mathfrak{p} \in V_f(K)$ both $G(\mathcal{O}_\mathfrak{p})$ and $H(\mathcal{O}_\mathfrak{p})$ are so-called *hyperspecial subgroups* of $\mathbf{G}(K_\mathfrak{p}) \cong \mathbf{H}(K_\mathfrak{p})$ [101, p. 55, 3.9.1]. Any two hyperspecial subgroups are isomorphic via a unique inner automorphism of $\mathbf{G}(K_\mathfrak{p}) \cong \mathbf{H}(K_\mathfrak{p})$ [81, Proposition 4.5]. Hence, we can modify the given isomorphism $\mathbf{G}(K_\mathfrak{p}) \cong \mathbf{H}(K_\mathfrak{p})$ by an inner automorphism in order to obtain one that maps $G(\mathcal{O}_\mathfrak{p})$ to $H(\mathcal{O}_\mathfrak{p})$. Doing so for almost all places suffices to get an isomorphism of the adèle points. $\square$

The author would like to thank Will Sawin for his MathOverflow answer providing us with the above proof [93].

**Remark 3.2.9** (Different fields of definition II)**.** As noted before in Remark 3.2.5, we did not use that $\mathbf{G}$ and $\mathbf{H}$ were defined over the same number field $K$ in any essential way.

Recall, that two number fields $K$ and $K'$ are called arithmetically equivalent if there is a bijection $V_f(K) \to V_f(K'), \mathfrak{p} \mapsto \mathfrak{p}'$ such that for almost all $\mathfrak{p} \in V_f(K)$, we have isomorphisms $K_\mathfrak{p} \cong K'_{\mathfrak{p}'}$. Choosing $S \subseteq V(K)$ finite saturated, such that we have these isomorphisms for every $\mathfrak{p} \notin S$, we have $\mathbb{A}_{K,S} \cong \mathbb{A}_{K',S'}$. Moreover, we can still ask for $\mathbf{G}$ and $\mathbf{H}$ to be isomorphic over every $\mathfrak{p} \in V(K) \setminus S$, in the sense that $\mathbf{G}_{K_\mathfrak{p}} \cong \mathbf{H}_{K'_{\mathfrak{p}'}}$ over the isomorphism $K_\mathfrak{p} \cong K'_{\mathfrak{p}'}$.

In this setup, the same proofs as above apply to give similar results.

For future reference, let us gather what we have seen in this section.

**Corollary 3.2.10** (Profinite isomorphism from local isomorphisms)**.** *Let $K$ be a number field, let $S \subseteq V(K)$ be a finite saturated set of places and let $\mathbf{G}, \mathbf{H}$ be two algebraic groups over $K$. Assume that both $\mathbf{G}$ and $\mathbf{H}$ have the congruence subgroup property as well as strong approximation with respect to $S$.*

*If for every $\mathfrak{p} \in V(K) \setminus S$, the algebraic groups $\mathbf{G}$ and $\mathbf{H}$ are isomorphic over $K_\mathfrak{p}$, then there exist $S$-arithmetic subgroups of $\mathbf{G}(K)$ and $\mathbf{H}(K)$ with isomorphic profinite completions.*

*Proof.* Using the local isomorphisms, Proposition 3.2.8 gives an isomorphism of the finite $S$-adèle points of $\mathbf{G}$ and $\mathbf{H}$. Since both $\mathbf{G}$ and $\mathbf{H}$ have strong approximation with respect to $S$, these finite adèle points are just the $S$-congruence completions of $\mathbf{G}(K)$ and $\mathbf{H}(K)$, respectively (Corollary 2.2.34). Finally, this allows us to apply Proposition 3.2.7, in order to obtain the desired $S$-arithmetic subgroups. $\square$

## 3.3 Grothendieck pairs

When studying profinite rigidity, one is looking at groups with the same profinite completion. Such groups are only known to have isomorphic profinite completions without control of this isomorphism. However, it is also interesting and relevant to study the situation where the isomorphism of profinite completions is induced by a morphism of the groups. Such morphisms were first studied by Grothendieck [37], hence they are now called *Grothendieck pairs*.

**Definition 3.3.1** (Grothendieck pair)**.** Let $G, H$ be finitely generated, residually finite groups. A group homomorphism $u\colon H \to G$ defines a *Grothendieck pair* if $\hat{u}\colon \widehat{H} \to \widehat{G}$ is an isomorphism. If $u$ is *not* an isomorphism, we say that $u$ is a *non-trivial* Grothendieck pair.

One first easy observation is that $u\colon H \to G$ always has to be injective. Using this, we will usually assume that $H$ is a subgroup of $G$.

**Lemma 3.3.2.** *Let $u\colon H \to G$ be a Grothendieck pair. Then $u$ is injective.*

*Proof.* Since both $G$ and $H$ are residually finite, we have a commutative diagram

$$\begin{tikzcd} H \arrow[r, "u"] \arrow[d, hook] & G \arrow[d, hook] \\ \widehat{H} \arrow[r, "\widehat{u}"'] & \widehat{G}, \end{tikzcd}$$

with injective vertical morphisms. Now, by assumption, $\hat{u}$ is an isomorphism, so the restriction $u$ has to be injective as well. ☐

In his original paper [37], Grothendieck asked whether there are non-trivial Grothendieck pairs consisting of finitely presented groups. The first non-trivial finitely *generated* Grothendieck pairs were given by Platonov and Tavgen' [79], using a construction we will discuss in Section 3.3.1. By a similar construction, Bridson and Grunewald were able to construct non-trivial finitely *presented* Grothendieck pairs [13].

Before we give examples of Grothendieck pairs, let us first discuss when a group homomorphism of finitely generated residually finite groups induces an injection, respectively surjection, on profinite completions.

**Lemma 3.3.3** ([78, Lemma 1])**.** *Let $u\colon H \to G$ be a group homomorphism of finitely generated residually finite groups. The induced map $\hat{u}\colon \widehat{H} \to \widehat{G}$ of profinite completions is*

1. *surjective if and only if the image $u(H)$ is, with respect to the profinite topology, dense in $G$, and*
2. *injective if and only if $u$ is injective and the profinite topology on $u(H)$ coincides with the one induced by the profinite topology of $G$.*

*Proof.* We consider the commutative diagram

$$\begin{tikzcd} H \arrow[r, "u"] \arrow[d, hook] & G \arrow[d, hook] \\ \widehat{H} \arrow[r, "\widehat{u}"'] & \widehat{G}. \end{tikzcd}$$

Here the two vertical arrows are injective as both $H$ and $G$ are assumed to be residually finite.

We begin by proving that, as subspaces of $\widehat{G}$, $u(H)$ is dense in $\hat{u}(\widehat{H})$. As the image of a compact space, $\hat{u}(\widehat{H})$ is compact in $\widehat{G}$. With $\widehat{G}$ being Hausdorff, we get in particular that $\hat{u}(\widehat{H})$ is closed. Denoting the by $\overline{(\,\cdot\,)}$ the closure in $\widehat{G}$ and $\widehat{H}$, we thus have

$$\hat{u}(\widehat{H}) = \overline{\hat{u}(\widehat{H})} = \overline{\hat{u}(\overline{H})} = \overline{\hat{u}(H)} = \overline{u(H)}$$

where we used that $H$ is dense in $\widehat{H}$. So $u(H)$ is actually dense in $\hat{u}(\widehat{H})$.

1. First assume that $u(H)$ is dense in $G$. Since $G$ is dense in $\widehat{G}$ and the profinite topology on $G$ agrees with the subspace topology, $u(H)$ is dense in $\widehat{G}$ as well. But the closure of $u(H)$ in $\widehat{G}$ is $\hat{u}(\widehat{H})$, so $\hat{u}(\widehat{H}) = \widehat{G}$, proving surjectivity.

   Conversely, let $\hat{u}$ be surjective. Since $u(H)$ is dense in $\hat{u}(\widehat{H}) = \widehat{G}$ and $u(H)$ is already a subspace of $G$, it already has to be dense in $G$.

2. Let $\hat{u}$ be injective. Since $\widehat{H}$ is compact and $\widehat{G}$ is Hausdorff, the usual compact–Hausdorff argument gives that $\hat{u}$ is a homeomorphism onto its image, i.e. an embedding. Restricting to the subspaces $H$ and $G$ gives the claim.

   Now let $u$ be an embedding, i.e. $u$ is injective and the profinite topology on $u(H)$ coincides with the subspace topology induced by the profinite topology on $G$. Both $\widehat{H}$ and $\hat{u}(\widehat{H})$ are compact and thus in particular complete [10, Section III.3.3, Corollary 1]. So the homeomorphism $u\colon H \to u(H)$ of dense subsets can be uniquely extended to a continuous homomorphism $\widehat{H} \to \hat{u}(\widehat{H})$, which will also be a homeomorphism [10, Section III.3.3, Proposition 5]. Obviously, $\hat{u}$ is such a continuous extension. So it is a homeomorphism onto its image, and thus in particular injective. □

Let us reformulate these conditions in terms of finite index subgroups. Since later on we will only be interested in the case where $H$ is a subgroup of $G$, we will already make this additional assumption to declutter the statement.

**Corollary 3.3.4** ([78, Corollary p. 91]). *Let $H \leq G$ be groups and denote by $u\colon H \to G$ the inclusion.*

1. *The map $\hat{u}$ is surjective if and only if $H$ is* not *contained in any proper finite index subgroup of $G$.*
2. *The map $\hat{u}$ is injective if and only if for every finite index subgroup $H_1 \leq H$, there exists a finite index subgroup $G_1 \leq G$ with $G_1 \cap H \subseteq H_1$.*

*Proof.*

1. Let $\hat{u}$ be surjective and let $G_1 \leq G$ be a finite index subgroup with $H \subseteq G_1$. As $\hat{u}$ is surjective, we know that $H$ is dense in $\widehat{G}$. Thus,

$$\widehat{G} = \overline{H} \subseteq \overline{G_1} \subseteq \widehat{G},$$

and so $\overline{G_1} = \widehat{G}$. But using the canonical bijection of finite index subgroups of $G$ and open subgroups of $\widehat{G}$, this already proves $G_1 = G$.

Now let us assume that $\hat{u}$ is *not* surjective, i.e. $\overline{H} \neq \widehat{G}$. Then there is $g \in \widehat{G}$ and an open normal subgroup $U \leq \widehat{G}$ such that $gU \cap \overline{H} = \emptyset$. As the union of open cosets, $U\overline{H}$ is an open subgroup of $\widehat{G}$. So by the subgroup correspondence, it corresponds to a finite index subgroup of $G$ containing $H$. Since $g^{-1} \notin U\overline{H}$, it has to be a *proper* subgroup of $G$.

2. This is just a reformulation that the profinite topology on $H$ coincides with the one induced by the one on $G$. □

### 3.3.1 The construction of Platonov–Tavgen

We now come to the construction of non-trivial Grothendieck pairs due to Platonov and Tavgen' [79].

**Definition 3.3.5** (Fiber product)**.** Let $\varphi\colon G \to Q$ be a surjective group homomorphism. Then the *fiber product of* $\varphi$ is defined to be

$$P = \big\{(g,h) \in G \times G \;\big|\; \varphi(g) = \varphi(h)\big\} \leq G \times G.$$

It is easy to see that if $G$ is finitely generated and $Q$ is finitely presented, then $P$ is finitely generated as well. The following theorem is more involved.

**Theorem 3.3.6** ([5, 1-2-3 Theorem])**.** *Let $1 \to N \to G \xrightarrow{\varphi} Q \to 1$ be a short exact sequence of groups and denote by $P$ the fiber product of $\varphi$. If $N$ is finitely generated (type $F_1$), $G$ is finitely presented (type $F_2$), and $Q$ is of type $F_3$, then $P$ is finitely presented.*

Platonov and Tavgen' used the characterization for surjectivity and injectivity in Corollary 3.3.4 and essentially proved the following.

**Theorem 3.3.7** ([13, Theorem 5.1])**.** *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of groups and let $P \leq G \times G$ be the associated fiber product. If $G$ is finitely generated, $Q$ has no finite quotients, and $H_2(Q;\mathbb{Z}) = 0$, then the inclusion $u\colon P \hookrightarrow G \times G$ induces an isomorphism $\hat{u}\colon \hat{P} \to \widehat{G} \times \widehat{G}$.*

The same arguments also prove this slight variation.

**Proposition 3.3.8** ([12, Lemma 4.4])**.** *Let $1 \to N \to G \to Q \to 1$ be an exact sequence of finitely generated groups. If $\widehat{Q} = 1$ and $H_2(Q;\mathbb{Z}) = 0$, then $N \hookrightarrow G$ induces and isomorphism of profinite completions.*

**Remark 3.3.9** (The construction of Platonov and Tavgen')**.** The original construction of non-trivial Grothendieck pairs of finitely generated groups by Platonov and Tavgen' worked as follows:

Let

$$Q = \langle a,b,c,d \mid a^{-1}ba = b^2,\ b^{-1}cb = c^2,\ c^{-1}dc = d^2,\ d^{-1}ad = a^2\rangle$$

be the Higman group. This group has no finite quotients [42] and is acyclic [24]. So, in particular, $H_2(Q;\mathbb{Z}) = 0$. Now, consider the short exact sequence

$$1 \longrightarrow N \longrightarrow F_4 \longrightarrow Q \longrightarrow 1,$$

where the map $F_4 \to Q$ sends the four generators of $F_4$ to the generators of $Q$. Then Theorem 3.3.7 shows that the inclusion of the fiber product $P \hookrightarrow F_4 \times F_4$ is a Grothendieck pair.

**Remark 3.3.10** (The refinement by Bridson and Grunewald)**.** Bridson and Grunewald used the same idea to construct the first examples of non-trivial Grothendieck pairs of finitely presented groups. However, to obtain a short exact sequence to start with, they used a variation of the Rips construction [89] due to Wise which produces residually finite groups:

**Theorem 3.3.11** (Wise–Rips construction [103])**.** *There is an algorithm that associates with every finite group presentation a short exact sequence of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

*where $Q$ is the group with the given presentation, $N$ is generated by three elements, and the group $G$ is a torsion-free, residually finite, hyperbolic (and thus finitely presented) group.*

Applying this variant of the Rips construction to the Higman group gives a short exact sequence with *finitely presented* fiber product $P$. Hence, this time Theorem 3.3.7 proves that $P \hookrightarrow G \times G$ is a Grothendieck pair of finitely presented groups.

# Chapter 4

# Bounded cohomology is not a profinite invariant

We will now study an example of an invariant which is *not* profinite: *bounded cohomology*. Bounded cohomology is a functional analytical variation of (group) cohomology, where, instead of *all* cochains, only the *uniformly bounded* ones are considered. This gives functors $H_b^n(\,\cdot\,;\mathbb{R})\colon \mathrm{Group} \to \mathrm{Vect}_{\mathbb{R}}$. For the definition and some basic results we refer the reader either to Section A.2 in the appendix or (for more details) to the books by Frigerio [32] and Monod [68].

As it turns out, there are lattices in Lie groups with isomorphic profinite completion but different bounded cohomology. We will in fact give a full classification of which higher rank Lie groups this phenomenon can occur in. The classification makes heavy use of the classification of forms of algebraic groups in terms of Galois cohomology, as discussed in Section 1.1.5.

This chapter is based on joint work with the author's PhD advisor Holger Kammeyer [26].

### Overview of this chapter

## 4.1 Bounded cohomology of lattices in Lie groups

The second bounded cohomology of a lattice in a simple Lie group can actually be computed in terms of the surrounding Lie group. This is due to the following theorem by Monod and Shalom, which extends a previous result of Burger and Monod [17, Corollary 1.6].

**Theorem 4.1.1** ([69, Theorem 1.4])**.** *Let $K$ be a local field, $\mathbf{G}$ be a connected almost simple algebraic group over $K$ with $\operatorname{rank}_K \mathbf{G} \geq 2$ and let $\Gamma \leq G = \mathbf{G}(K)$ be a lattice. Then we have*

$$H_b^2(\Gamma;\mathbb{R}) \cong_{\mathbb{R}} \begin{cases} \mathbb{R} & \text{if } K = \mathbb{R} \text{ and } \pi_1(G) \text{ is infinite} \\ 0 & \text{in all other cases.} \end{cases}$$

*An analogous statement holds for the* continuous *bounded cohomology of $G$.*

**Remark 4.1.2** (The rank one case)**.** The above theorem gives no information about lattices in rank one groups. In fact, here the situation is quite different. Lattices in Lie groups of rank one act on hyperbolic spaces and thus have infinite dimensional second bounded cohomology [33].

It is well-known that for simple Lie groups, the condition of having infinite fundamental group is equivalent to having a Hermitian symmetric space [41, Chapter VIII, Theorem 6.1]. By the classification of real simple Lie groups, these are isogenous to

$$\begin{array}{cc} \mathrm{SU}(n,m) \text{ for } n,m \geq 2, & \mathrm{SO}^0(n,2) \text{ for } n \geq 3, \\ \mathrm{SO}^*(2n) \text{ for } n \geq 4, & \mathrm{Sp}(n;\mathbb{R}) \text{ for } n \geq 2, \\ \mathrm{E}_6^{(-14)} \text{ or} & \mathrm{E}_7^{(-25)} \end{array}$$

[41, Chapter X, §6.3]. Note that we have the accidental isogenies

$$\mathrm{SO}^0(3,2) \approx \mathrm{Sp}(2;\mathbb{R}), \qquad \mathrm{SO}^0(4,2) \approx \mathrm{SU}(2,2) \quad \text{and} \qquad \mathrm{SO}^0(6,2) \approx \mathrm{SO}^*(8).$$

### 4.1.1 A first example: $\mathrm{Spin}(n,2)$

In Section 3.2 we have already seen some examples of profinitely isomorphic arithmetic groups.

**Lemma 4.1.3.** *Let $n \in \mathbb{N}_{\geq 7}$. There are arithmetic subgroups $\Gamma \leq \mathrm{Spin}(n,2)(\mathbb{Q})$ and $\Lambda \leq \mathrm{Spin}(n-4,6)(\mathbb{Q})$ with $\widehat{\Gamma} \cong \widehat{\Lambda}$ and $H_b^2(\Gamma;\mathbb{R}) \cong \mathbb{R}$ while $H_b^2(\Lambda;\mathbb{R}) \cong 0$.*

*Proof.* Making use of Corollary 3.2.4, we find arithmetic subgroups $\Gamma \leq \mathrm{Spin}(n,2)(\mathbb{Q})$ and $\Lambda \leq \mathrm{Spin}(n-4,6)(\mathbb{Q})$ with isomorphic profinite completions.

In order to compute the bounded cohomology, we use the theorem by Monod and Shalom. The two arithmetic subgroups $\Gamma$ and $\Lambda$ are lattices in $G = \mathrm{Spin}(n,2)(\mathbb{R})$ and $H = \mathrm{Spin}(n-4,6)(\mathbb{R})$, respectively. Both of these groups have higher $\mathbb{R}$-rank and $G$, being isogenous to $\mathrm{SO}^0(n,2)$, has a Hermitian symmetric space, while $H$, being isogenous to $\mathrm{SO}^0(n-4,6)$, does not. Thus, we get $H_b^2(\Gamma;\mathbb{R}) \cong \mathbb{R}$ and $H_b^2(\Lambda;\mathbb{R}) \cong 0$ as desired. □

This already provides the first counterexample to the second bounded cohomology being a profinite invariant.

## 4.2 The full classification of examples

Having seen the first examples of the non-profiniteness of the second bounded cohomology, we now want to give the complete classification of such examples in the class of lattices in higher rank Lie groups.

**Definition 4.2.1.** Let $G$ be a higher rank Lie group, i.e. the real points $G = \mathbf{G}(\mathbb{R})$ of a connected almost $\mathbb{R}$-simple linear algebraic $\mathbb{R}$-group $\mathbf{G}$ with $\operatorname{rank}_{\mathbb{R}} \mathbf{G} \geq 2$. We say that $G$ *exhibits non-profinite second bounded cohomology* if there exists a lattice $\Gamma \leq G$ as well as a lattice $\Lambda \leq H$ in another higher rank Lie group $H$, such that $\widehat{\Gamma} \cong \widehat{\Lambda}$ and $H_b^2(\Gamma;\mathbb{R}) \not\cong 0$ while $H_b^2(\Lambda;\mathbb{R}) \cong 0$.

**Theorem 4.2.2.** *Let $G$ be a higher rank Lie group. Then $G$ exhibits non-profinite second bounded cohomology if and only if it is isogenous to*

$$\mathrm{SO}^0(n,2) \textit{ for } n \geq 6, \qquad \mathrm{E}_6^{(-14)}, \quad \textit{or} \qquad \mathrm{E}_7^{(-25)}.$$

Due to Theorem 4.1.1, the only higher rank Lie groups we need to consider are the ones with Hermitian symmetric space. Moreover, by possibly passing to finite index subgroups, we also only need to consider one Lie group for every isogeny class.

The next sections provide the constructions of lattices exhibiting the non-profiniteness of second bounded cohomology in the cases of $\mathrm{E}_7^{(-25)}$, $\mathrm{E}_6^{(-14)}$, and $\mathrm{SO}^0(n,2)$.

Afterwards, we will show that the remaining Lie groups with Hermitian symmetric space do not exhibit non-profinite second bounded cohomology.

### 4.2.1 $\mathbf{E}_7^{(-25)}$

Let $\mathbf{E_7}$ be the unique simply connected absolutely almost simple $\mathbb{Q}$-split algebraic $\mathbb{Q}$-group of type $\mathrm{E}_7$. As the Dynkin diagram of type $\mathrm{E}_7$ does not have any symmetries, the short exact sequence

$$1 \longrightarrow \operatorname{Ad}\mathbf{E_7} \longrightarrow \operatorname{Aut}\mathbf{E_7} \longrightarrow \operatorname{Sym}\Delta \longrightarrow 1$$

gives $\operatorname{Ad}\mathbf{E_7} \cong \operatorname{Aut}\mathbf{E_7}$, and thus $H^1(\mathbb{Q};\operatorname{Ad}\mathbf{E_7})$ classifies all $\mathbb{Q}$-forms of type $\mathrm{E}_7$.

In order to study this Galois cohomology set, we consider the short exact sequence

$$1 \longrightarrow Z(\mathbf{E_7}) \longrightarrow \mathbf{E_7} \longrightarrow \operatorname{Ad}\mathbf{E_7} \longrightarrow 1$$

of $\operatorname{Gal}(\mathbb{Q})$-groups. Here the center $Z(\mathbf{E_7})$ is isomorphic to the group of second roots of unity $\mu_2$ [77, Table on p. 332]. For $p$ a prime or $p = \infty$, using the functoriality of Galois cohomology and the naturality of the "long" exact sequence, we get an induced diagram

$$\begin{tikzcd}
H^1(\mathbb{Q};\mathbf{E_7}) \arrow[r, "\pi"] \arrow[d] & H^1(\mathbb{Q};\operatorname{Ad}\mathbf{E_7}) \arrow[r, "\Delta"] \arrow[d, "f_p"] & H^2(\mathbb{Q};\mu_2) \arrow[d, "b_p"] \\
H^1(\mathbb{Q}_p;\mathbf{E_7}) \arrow[r, "\pi_p"'] & H^1(\mathbb{Q}_p;\operatorname{Ad}\mathbf{E_7}) \arrow[r, "\Delta_p"'] & H^2(\mathbb{Q}_p;\mu_2),
\end{tikzcd}$$

with exact rows. Here we agree on the convention $\mathbb{Q}_\infty = \mathbb{R}$. Let us collect some information about these diagrams.

**Lemma 4.2.3.** *Let $p$ be a prime. Then $\Delta_p$ has trivial kernel.*

*Proof.* By a result of Kneser [58] the Galois cohomology $H^1(\mathbb{Q}_p; \mathbf{E_7})$ is trivial. So $\pi_p$ has trivial image and thus, by exactness, $\Delta_p$ has trivial kernel. □

**Proposition 4.2.4.** *The image of $\pi_\infty \colon H^1(\mathbb{R}; \mathbf{E_7}) \to H^1(\mathbb{R}; \operatorname{Ad} \mathbf{E_7})$ consists of two elements, corresponding to the real forms $\mathrm{E}_7^{(7)}$ and $\mathrm{E}_7^{(-25)}$.*

*Proof.* The map $\pi_\infty$ is part of an exact sequence

$$\mathbf{E_7}(\mathbb{R}) \xrightarrow{\varphi} \operatorname{Ad} \mathbf{E_7}(\mathbb{R}) \xrightarrow{\delta_\infty} H^1(\mathbb{R}; \mu_2) \longrightarrow H^1(\mathbb{R}; \mathbf{E_7}) \xrightarrow{\pi_\infty} H^1(\mathbb{R}; \operatorname{Ad} \mathbf{E_7}) \xrightarrow{\Delta_\infty} H^2(\mathbb{R}; \mu_2).$$

Since $\mu_2$ is an abelian group, $\delta_\infty$ is a group homomorphism [95, Chapter I, §5.6, Corollary 2]. So using exactness we have

$$\operatorname{im} \delta_\infty \cong {}^{\operatorname{Ad} \mathbf{E_7}(\mathbb{R})}\!\big/\!{}_{\varphi(\mathbf{E_7}(\mathbb{R}))}.$$

Being defined by the real points of a simply connected semisimple $\mathbb{R}$-group, $\mathbf{E_7}(\mathbb{R})$ is a *connected* Lie group [67, Theorem 25.54]. However, $\operatorname{Ad} \mathbf{E_7}(\mathbb{R})$ has two connected components [1, Table 5], so $\operatorname{im} \delta_\infty$ has two connected components as well. Since $H^1(\mathbb{R}; \mu_2) \cong \mathbb{Z}/2$, this proves that $\delta_\infty$ is surjective, and thus exactness shows that $\pi_\infty$ has trivial kernel. As $H^1(\mathbb{R}; \mathbf{E_7})$ has only two elements [1, Table 3], we obtain that $\pi_\infty$ is actually injective.

Now let $[a] \in H^1(\mathbb{R}; \mathbf{E_7})$ be the non-trivial class. The action of the twisted group ${}_a \operatorname{Ad} \mathbf{E_7}(\mathbb{R})$ on $H^1(\mathbb{R}; \mu_2) \cong \mathbb{Z}/2$ needs to be transitive [95, Chapter I, §5.5, Corollary 2]. So ${}_a \operatorname{Ad} \mathbf{E_7}(\mathbb{R})$ has to be disconnected. However, the only non-split disconnected form of type $\mathrm{E}_7$ is the Hermitian form $\mathrm{E}_7^{(-25)}$ [1, Table 5]. This shows that $\pi_\infty([a]) \in H^1(\mathbb{R}; \operatorname{Ad} \mathbf{E_7})$ corresponds to the real from $\mathrm{E}_7^{(-25)}$.

Obviously, the unit class $1 \in H^1(\mathbb{R}; \mathbf{E_7})$ corresponds to the split form $\mathrm{E}_7^{(7)}$. □

Now fix a prime $p_0$. Taking the direct sum of the above diagram over all primes excluding $p_0$, while including $\infty$, we obtain the diagram

$$\begin{tikzcd}
H^1(\mathbb{R}; \mathbf{E_7}) \arrow[r, "\pi_\infty"] & \bigoplus_{p \neq p_0} H^1(\mathbb{Q}_p; \operatorname{Ad} \mathbf{E_7}) \arrow[r, "\oplus \Delta_p"] & \bigoplus_{p \neq p_0} H^2(\mathbb{Q}_p; \mu_2) \\
H^1(\mathbb{Q}; \mathbf{E_7}) \arrow[r, "\pi"] \arrow[u] \arrow[d] & H^1(\mathbb{Q}; \operatorname{Ad} \mathbf{E_7}) \arrow[r, "\Delta"] \arrow[u, "f"] \arrow[d, "f_{p_0}"] & H^2(\mathbb{Q}; \mu_2) \arrow[u, "b"] \arrow[d, "b_{p_0}"] \\
\{*\} \arrow[r] & H^1(\mathbb{Q}_{p_0}; \operatorname{Ad} \mathbf{E_7}) \arrow[r, "\Delta_{p_0}"'] & H^2(\mathbb{Q}_{p_0}; \mu_2),
\end{tikzcd}$$

with exact rows. Here only $H^1(\mathbb{R}; \mathrm{E}_7)$ remains in the left-hand column, as the sets $H^1(\mathbb{Q}_p; \mathrm{E}_7)$ are trivial for every prime $p$ by Kneser's theorem [57]. Moreover, the map

$f$ is surjective by the work of Prasad and Rapinchuk [83, Proposition 1]. So, using Proposition 4.2.4, we can find $\alpha \in H^1(\mathbb{Q}; \operatorname{Ad}\mathbf{E_7})$ which splits at every prime $p \neq p_0$ and corresponds to the real form $\mathrm{E}_7^{(-25)}$ at $p = \infty$. Moreover, we may assume that the algebraic $\mathbb{Q}$-group $\mathbf{G}$ defined by $\alpha$ has $\operatorname{rank}_{\mathbb{Q}} \mathbf{G} = 3$ [83, Theorem 1].

**Proposition 4.2.5.** *The class $\alpha$, and thus $\mathbf{G}$, splits at $p_0$ as well.*

*Proof.* As $\mathrm{E}_7^{(-25)}$ is in the image of $\pi_\infty$, using exactness we get $\oplus\Delta_p(f(\alpha)) = 1$. By commutativity, this gives $b(\Delta(\alpha)) = 1$.

The Albert–Brauer–Hasse–Noether Theorem from global class field theory gives a short exact sequence

$$1 \longrightarrow \operatorname{Br}(\mathbb{Q}) \longrightarrow \bigoplus_{p \leq \infty} \operatorname{Br}(\mathbb{Q}_p) \xrightarrow{s} \mathbb{Q}/\mathbb{Z} \longrightarrow 1,$$

where $s$ sums up *local invariants* [39, Theorem 14.11]. Now $H^2(\,\cdot\,; \mu_2)$ is the subgroup of $\operatorname{Br}(\,\cdot\,)$ consisting of order two elements [39, Proposition 6.12]. So, in particular, we get the injectivity of $b\colon H^2(\mathbb{Q}; \mu_2) \to \bigoplus_{p \neq p_0} H^2(\mathbb{Q}_p; \mu_2)$. Hence, $b(\Delta(\alpha)) = 1$ gives $\Delta(\alpha) = 1$ and thus also $\Delta_{p_0}(f_{p_0}(\alpha)) = b_{p_0}(\Delta(\alpha)) = 1$. As we have seen in Lemma 4.2.3, $\Delta_{p_0}$ has trivial kernel, so that $f_{p_0}(\alpha) = 1 \in H^1(\mathbb{Q}; \operatorname{Ad}\mathbf{E_7})$, i.e. $\alpha$ splits at $p_0$. $\square$

By the above proposition, $\mathbf{G}$ and $\mathbf{E_7}$ become isomorphic over $\mathbb{Q}_p$ for every prime $p$. Moreover, as $\operatorname{rank}_{\mathbb{Q}} \mathbf{G} = 3$ and $\operatorname{rank}_{\mathbb{Q}} \mathbf{E_7} = 7$, both $\mathbf{G}$ and $\mathbf{E_7}$ have the congruence subgroup property. Hence, applying Corollary 3.2.10 gives arithmetic subgroups $\Gamma \leq \mathbf{G}(\mathbb{Q})$, $\Lambda \leq \mathbf{E_7}(\mathbb{Q})$ with isomorphic profinite completions.

Both $\Gamma$ and $\Lambda$ are lattices in the corresponding real points. By construction, these are $\mathrm{E}_7^{(-25)}$ and $\mathrm{E}_7^{(7)}$. As only $\mathrm{E}_7^{(-25)}$ has a Hermitian symmetric space, the theorem of Monod–Shalom (Theorem 4.1.1) gives $H_b^2(\Gamma; \mathbb{R}) \cong \mathbb{R}$ and $H_b^2(\Lambda; \mathbb{R}) \cong 0$.

### 4.2.2 $\mathbf{E_6^{(-14)}}$

Fix a prime $p_0$ and denote by $\mathbf{{}^2E_6}$ a simply connected absolutely almost simple quasi-split $\mathbb{Q}$-group which splits neither at $p_0$ nor $\infty$. That such a group exists can either be seen by using the result of Prasad and Rapinchuk, which allows prescribing the local behaviour almost everywhere [83, Proposition 1], or already the previous work of Borel and Harder, which allows this prescription at only finitely many places [9, p. 58, Proposition].

Again we consider the short exact sequence

$$1 \longrightarrow Z(\mathbf{{}^2E_6}) \longrightarrow \mathbf{{}^2E_6} \longrightarrow \operatorname{Ad}\mathbf{{}^2E_6} \longrightarrow 1$$

of $\operatorname{Gal}(\mathbb{Q})$-groups. (In this case the center $Z(\mathbf{{}^2E_6})$ is a so-called *norm one torus* [77, Table on p. 332] however, this does not really matter for us.)

As before, we obtain for any prime $p$ an induced diagram

$$\begin{tikzcd} H^1(\mathbb{Q};{}^{\mathbf{2}}\mathbf{E_6}) \arrow[r, "\pi"] \arrow[d] & H^1(\mathbb{Q};\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6}) \arrow[r, "\Delta"] \arrow[d] & H^2(\mathbb{Q};Z({}^{\mathbf{2}}\mathbf{E_6})) \arrow[d] \\ H^1(\mathbb{Q}_p;{}^{\mathbf{2}}\mathbf{E_6}) \arrow[r, "\pi_p"'] & H^1(\mathbb{Q}_p;\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6}) \arrow[r, "\Delta_p"'] & H^2(\mathbb{Q}_p;Z({}^{\mathbf{2}}\mathbf{E_6})) \end{tikzcd}$$

with exact rows. Again by the result of Kneser [57], we have $H^1(\mathbb{Q}_p;{}^{\mathbf{2}}\mathbf{E_6}) = 1$ for every prime $p$. Moreover, in this case we also have that the Galois cohomology of the center $H^1(\mathbb{Q}_p;Z({}^{\mathbf{2}}\mathbf{E_6}))$ is trivial [77, Proposition 6.14]. Hence, by exactness, we also have $H^1(\mathbb{Q}_p;\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6}) = 1$ for every prime $p$.

According to the tables by Adams and Taïbi [1, Table 3], $H^1(\mathbb{R};\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6})$ consists of three elements, corresponding to the quasi-split form $\mathrm{E}_6^{(2)}$, the Hermitian form $\mathrm{E}_6^{(-14)}$, and the compact form $\mathrm{E}_6^{(-78)}$

So the results of Prasad and Rapinchuk [83, Proposition 1, Theorem 1] already prove that the natural map

$$H^1(\mathbb{Q};\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6}) \longrightarrow \bigoplus_{p\leq\infty} H^1(\mathbb{Q}_p;\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6})$$

is surjective, allowing us to find $\alpha, \beta \in H^1(\mathbb{Q};\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6})$, which become the same in $H^1(\mathbb{Q}_p;\mathrm{Ad}\,{}^{\mathbf{2}}\mathbf{E_6})$ for every prime $p$ and correspond to $\mathrm{E}_6^{(-14)}$ and $\mathrm{E}_6^{(2)}$, respectively, over the reals. Here we can assume that the corresponding algebraic groups $\mathbf{G}_1$ and $\mathbf{G}_2$ have $\mathrm{rank}_{\mathbb{Q}}\,\mathbf{G}_1 = 4$ and $\mathrm{rank}_{\mathbb{Q}}\,\mathbf{G}_2 = 2$, such that both have the congruence subgroup property.

As in the above case of $\mathrm{E}_7$, applying Corollary 3.2.10 gives lattices $\Gamma \leq \mathbf{G}_1(\mathbb{R}) = \mathrm{E}_6^{(-14)}$ and $\Lambda \leq \mathbf{G}_2(\mathbb{R}) = \mathrm{E}_6^{(2)}$ with isomorphic profinite completions. Again the theorem of Burger–Monod gives $H_b^2(\Gamma;\mathbb{R}) \cong \mathbb{R}$ while $H_b^2(\Lambda;\mathbb{R}) \cong 0$.

### 4.2.3 $\mathbf{SO^0(n,2)}$

As the spin group $\mathrm{Spin}(n,2)(\mathbb{R})$ is isogenous to $\mathrm{SO}^0(n,2)$, the examples of Lemma 4.1.3 already prove that the groups isogenous to $\mathrm{SO}^0(n,2)$ with $n \geq 7$ exhibit non-profinite second bounded cohomology. So only the case of $\mathrm{SO}^0(6,2)$ remains to be examined.

Let ${}^{\mathbf{3/6}}\mathbf{D}_4$ be a simply connected absolutely almost simple quasi-split $\mathbb{Q}$-group, which localizes to the quasi-split triality form ${}^3\mathrm{D}_4$ at a fixed prime $p_0$ and splits over the reals. As in the previous section, the existence of such a group can be seen using the result of Borel and Harder [9, p. 58, Proposition].

Again, we have that both $H^1(\mathbb{Q}_{p_0};{}^{\mathbf{3/6}}\mathbf{D}_4)$ [57] and $H^2(\mathbb{Q}_{p_0};{}^{\mathbf{3/6}}\mathbf{D}_4)$ [77, Proposition 6.14] are trivial, implying that $H^1(\mathbb{Q}_{p_0};\mathrm{Ad}\,{}^{\mathbf{3/6}}\mathbf{D}_4)$ is trivial as well. So, as before, the results of Prasad and Rapinchuk gives us that the canonical map

$$H^1(\mathbb{Q};\mathrm{Ad}\,{}^{\mathbf{3/6}}\mathbf{D}_4) \longrightarrow \bigoplus_{p\leq\infty} H^1(\mathbb{Q}_p;\mathrm{Ad}\,{}^{\mathbf{3/6}}\mathbf{D}_4)$$

is surjective. Hence, we can choose (inner) $\mathbb{Q}$-forms $\mathbf{G}_1$ and $\mathbf{G}_2$ of ${}^{3/6}\mathbf{D}_4$ that localize to the inner forms $\mathrm{SO}^0(6,2)$ and $\mathrm{SO}^0(4,4)$ over the reals and to the trivial inner twist of ${}^{3/6}\mathbf{D}_4$ at every finite place. By construction $\mathbf{G}_1$ and $\mathbf{G}_2$ become isomorphic over $\mathbb{Q}_p$ for every prime $p$.

Again, we can choose both groups in such a way that $\mathrm{rank}_{\mathbb{Q}}\, \mathbf{G}_1 = \mathrm{rank}_{\mathbb{Q}}\, \mathbf{G}_2 = 2$ [83, Theorem 1], ensuring the congruence subgroup property.

As before, Corollary 3.2.10 gives lattices $\Gamma \leq \mathbf{G}_1(\mathbb{R}) = \mathrm{SO}^0(6,2)$ and $\Lambda \leq \mathbf{G}_2(\mathbb{R}) = \mathrm{SO}^0(4,4)$ with isomorphic profinite completions and by the theorem of Monod–Shalom we have $H^2_b(\Gamma;\mathbb{R}) \cong \mathbb{R}$ while $H^2_b(\Lambda;\mathbb{R}) \cong 0$.

## 4.3 Excluding the remaining possibilities

We now want to see that the remaining higher rank Lie groups defining Hermitian symmetric spaces

$$\begin{array}{llll} \mathrm{SU}(n,m) & \text{for } n,m \geq 2, & \mathrm{Sp}(n,\mathbb{R}) & \text{for } n \geq 2, \\ \mathrm{SO}^*(2n) & \text{for } n \geq 5, \text{ and} & \mathrm{SO}^0(5,2) & \end{array}$$

do *not* exhibit non-profinite second bounded cohomology. As before, we only need to do so for one group in every isogeny class. So let $G$ be a group in the above list, and let $\Gamma \leq G$ be a lattice. By Margulis arithmeticity [65, Chapter IX, Theorem 1.11, pp. 293–294 ($**$), Remark 1.6 (i)], there is a totally real number field $K$ and a simply connected absolutely almost simple $K$-group $\mathbf{G}$, such that:

- $\mathbf{G}$ is anisotropic at all infinite places except one, say $v$,
- $\mathbf{G}(K_v)$ is isogenous to $G$ and thus $\mathrm{rank}_{K_v}\, \mathbf{G} \geq 2$, and
- $\mathbf{G}(\mathcal{O}_K)$ is commensurable to $\Gamma$.

Let $\Lambda \leq H$ be a lattice in another higher rank Lie group $H$ with $\widehat{\Gamma} \cong \widehat{\Lambda}$. We have to show that $H$ also defines a Hermitian symmetric space. Again by Margulis arithmeticity, there is a totally real number field $L$ and a simply connected absolutely almost simple $L$-group $\mathbf{H}$, such that $\mathbf{H}$ is anisotropic at all infinite places except one, say $w$, $\mathbf{H}(L_w)$ is isogenous to $H$, and $\mathbf{H}(\mathcal{O}_L)$ is commensurable to $\Lambda$.

**Proposition 4.3.1.** *There is a bijection of finite places $V_f(K) \to V_f(L), \mathfrak{p} \mapsto \mathfrak{p}'$ such that $K_{\mathfrak{p}} \cong L_{\mathfrak{p}'}$ and $\mathbf{G}_{K_{\mathfrak{p}}} \cong \mathbf{H}_{L_{\mathfrak{p}'}}$ for every finite place $\mathfrak{p} \in V_f(K)$.*

*In particular, the groups $\mathbf{G}$ and $\mathbf{H}$ have the same Cartan–Killing type.*

*Proof.* By adèlic superrigidity [49, Theorem 3.4], there is an isomorphism $j\colon \mathbb{A}^f_L \to \mathbb{A}^f_K$ of topological rings and a group scheme isomorphism $\eta\colon \mathbf{G} \times_K \mathbb{A}^f_K \to \mathbf{H} \times_L \mathbb{A}^f_L$ over $j$. Using Theorem 2.1.42, the isomorphism $j$ corresponds uniquely to a bijection $\mathfrak{p} \mapsto \mathfrak{p}'$ of finite places of $K$ and $L$ as well as a family of isomorphisms $K_{\mathfrak{p}} \cong_{\mathrm{Top}} L_{\mathfrak{p}'}$.

Moreover, for $\mathfrak{p} \in V_f(K)$ we have that $K_{\mathfrak{p}} \subseteq \mathbb{A}^f_K$ is a minimal ideal which is mapped to $L_{\mathfrak{p}'} \subseteq \mathbb{A}^f_L$ when applying $j$. Hence, pulling back the isomorphism $\eta\colon \mathbf{G} \times_K \mathbb{A}^f_K \to \mathbf{H} \times_L \mathbb{A}^f_L$ gives the desired isomorphisms $\mathbf{G}_{K_{\mathfrak{p}}} \cong \mathbf{H}_{L_{\mathfrak{p}'}}$. $\square$

**Remark 4.3.2.** The above proposition shows that $K$ and $L$ are locally isomorphic. Thus, we not only have a bijection of finite places, but also that $K$ and $L$ have the same number of infinite places (which are all real in our case). Thus, we can extend the bijection to all places, where we may assume without loss of generality that $v$ is mapped to $w$.

For ease of notation, in the following we will always use this bijection implicitly to identify the places of $K$ and the places of $L$. Moreover, we will also denote the algebraic groups $\mathbf{G}_{K_\mathfrak{p}}$ and $\mathbf{H}_{L_\mathfrak{p}}$ simply by $\mathbf{G}_\mathfrak{p}$ and $\mathbf{H}_\mathfrak{p}$ respectively.

Since over the reals, there is only one anisotropic form of every type (Theorem 1.2.25), we not only have that $\mathbf{G}$ and $\mathbf{H}$ are isomorphic over all finite places, but also over all real places, except $v$. So we only need to study to group $\mathbf{H}_v$.

Knowing that $\mathbf{G}$ and $\mathbf{H}$ have the same Cartan–Killing type is already enough to cover the case of $\mathrm{SO}^0(5,2)$.

**Lemma 4.3.3.** *The groups $G \approx \mathrm{SO}^0(5,2)$ do not exhibit non-profinite second bounded cohomology.*

*Proof.* As $G$ and $H$ have to have the same Cartan–Killing type, $H$ has to be isogenous to

$$\mathrm{SO}^0(7,0) = \mathrm{SO}(7), \quad \mathrm{SO}^0(6,1), \quad \mathrm{SO}^0(5,2), \text{ or } \quad \mathrm{SO}^0(4,3).$$

Since we assumed both $G$ and $H$ to be of higher rank, we can immediately exclude the cases of $\mathrm{SO}^0(7,0)$ and $\mathrm{SO}^0(6,1)$ as these have rank 0 and 1, respectively.

In order to also exclude $\mathrm{SO}^0(4,3)$, we make use of a profinite invariant: the dimension of the symmetric space modulo 4 [50, Theorem 2.1]. This result by Kammeyer, Kionke, Raimbault, and Sauer shows that $H$ cannot be isogenous to $\mathrm{SO}^0(4,3)$, as its symmetric space has dimension 12, whereas the dimension of the symmetric space of $\mathrm{SO}^0(5,2)$ is 10 [41, Chapter X, §6, Table V].

Hence, $H$ needs to be isogenous to $\mathrm{SO}^0(5,2)$ and thus also defines a Hermitian symmetric space. ☐

### 4.3.1 $\mathrm{Sp}(n;\mathbb{R})$

In the case of $\mathrm{Sp}(n;\mathbb{R})$ we again use our, by now, familiar tool of Galois cohomology.

**Proposition 4.3.4.** *The groups $G \approx \mathrm{Sp}(n;\mathbb{R})$ with $n \geq 2$ do not exhibit non-profinite second bounded cohomology.*

*Proof.* Let $\mathbf{Sp_n}$ be the unique simply connected $\mathbb{Q}$-split $\mathbb{Q}$-group of type $\mathrm{C}_n$. As the Dynkin diagram of type $\mathrm{C}_n$ does not have symmetries, we get $\mathrm{Aut}\,\mathbf{Sp_n} = \mathrm{Ad}\,\mathbf{Sp_n}$. So forms of this group are classified by $H^1(\,\cdot\,;\mathrm{Ad}\,\mathbf{Sp_n})$. Again, we use the short exact sequence

$$1 \longrightarrow Z(\mathbf{Sp_n}) \longrightarrow \mathbf{Sp_n} \longrightarrow \mathrm{Ad}\,\mathbf{Sp_n} \longrightarrow 1$$

of $\mathrm{Gal}(\mathbb{Q})$-groups. Here the center $Z(\mathbf{Sp_n})$ is isomorphic to the group of second roots of unity $\mu_2$ [77, Table on p. 332].

This short exact sequence gives an exact sequence

$$H^1(F; \mathbf{Sp_n}) \longrightarrow H^1(F; \operatorname{Ad}\mathbf{Sp_n}) \xrightarrow{\delta_F} H^2(F; \mu_2)$$

in Galois cohomology for every field extension $F|\mathbb{Q}$. By looking at the tables by Adams and Taïbi [1, Table 1], we get that $H^1(\mathbb{R}; \mathbf{Sp_n})$ has a single element. Using exactness, and the pointedness of the sequence, we can deduce that $\ker \delta_{\mathbb{R}}$ consist of only one element corresponding to the split form $\mathrm{Sp}(n; \mathbb{R})$.

Moreover, the above sequences combine to a commutative diagram

$$\begin{tikzcd}
H^1(K; \operatorname{Ad}\mathbf{Sp_n}) \arrow[r, "\delta_K"] \arrow[d] & H^2(K; \mu_2) \arrow[d] \\
\bigoplus_{\mathfrak{p}} H^1(K_{\mathfrak{p}}; \operatorname{Ad}\mathbf{Sp_n}) \arrow[r, "\oplus\delta_{K_{\mathfrak{p}}}"] \arrow[d, phantom, "\wr\Vert"] & \bigoplus_{\mathfrak{p}} H^2(K_{\mathfrak{p}}; \mu_2) \arrow[d, phantom, "\wr\Vert"] \\
\bigoplus_{\mathfrak{p}} H^1(L_{\mathfrak{p}}; \operatorname{Ad}\mathbf{Sp_n}) \arrow[r, "\oplus\delta_{L_{\mathfrak{p}}}"'] & \bigoplus_{\mathfrak{p}} H^2(L_{\mathfrak{p}}; \mu_2) \\
H^1(L; \operatorname{Ad}\mathbf{Sp_n}) \arrow[r, "\delta_L"'] \arrow[u] & H^2(L; \mu_2), \arrow[u]
\end{tikzcd}$$

where the sums run over the *finite* places of $K$ and $L$. Denoting the classes corresponding to $\mathbf{G}$ and $\mathbf{H}$ by $\alpha \in H^1(K; \operatorname{Ad}\mathbf{Sp_n})$ and $\beta \in H^1(L; \operatorname{Ad}\mathbf{Sp_n})$, respectively, we get that $\delta_K(\alpha)$ and $\delta_L(\beta)$ map diagonally to corresponding elements in

$$\bigoplus_{\mathfrak{p}} H^2(K_{\mathfrak{p}}; \mu_2) \cong \bigoplus_{\mathfrak{p}} H^2(L_{\mathfrak{p}}; \mu_2).$$

Additionally, since $\mathbf{G}$ and $\mathbf{H}$ are anisotropic at every real place except possibly $v$, this still holds true if we include all infinite places *except* $v$ in the direct sums.

Now we again make use of the Albert–Brauer–Hasse–Noether theorem, this time the number field version. It gives two short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Br}(K) \arrow[r] & \bigoplus_{\mathfrak{p}\in V(K)} \mathrm{Br}(K_{\mathfrak{p}}) \arrow[r, "s"] \arrow[d, "\cong"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Br}(L) \arrow[r] & \bigoplus_{\mathfrak{p}\in V(L)} \mathrm{Br}(L_{\mathfrak{p}}) \arrow[r, "s"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 1,
\end{tikzcd}$$

where, in both cases, $s$ is given by summing up local invariants [39, Theorem 14.11]. Using the bijection of places and the isomorphisms of local completions given by Proposition 4.3.1 and Remark 4.3.2, we obtain the isomorphism of the middle terms.

Again, $H^2(\,\cdot\,; \mu_2)$ corresponds to the 2-torsion of the Brauer group [39, Proposition 6.12]. So by the above, this isomorphism identifies $\delta_K(\alpha)$ and $\delta_L(\beta)$ at all places except $v$.

As both $\delta_K(\alpha)$ and $\delta_L(\beta)$ are global elements, the sum of the local invariants has to be zero. However, the local invariants already agree everywhere except possibly $v$. Thus, they have to agree at the remaining infinite place $v$ as well.

Finally, the local invariant over the reals is injective, so we conclude $\delta_{\mathbb{R}}(\alpha_v) = \delta_{\mathbb{R}}(\beta_v)$. Now by assumption, $\mathbf{G}_v(\mathbb{R}) \approx \mathrm{Sp}(n;\mathbb{R})$ is the split form, so $\alpha_v$ is trivial. Hence, $\delta_{\mathbb{R}}(\beta_v)$ is trivial. With the kernel of $\delta_{\mathbb{R}}$ being trivial, this shows that $\beta_v$ has to be trivial as well. Thus, $H \approx \mathrm{Sp}(n;\mathbb{R})$ defines a Hermitian symmetric space. □

### 4.3.2 $\mathrm{SU}(n,m)$

Above we have established that $\mathbf{G}$ and $\mathbf{H}$ have the same Cartan–Killing type. At the one real place $v$ where $\mathbf{G}$ and $\mathbf{H}$ might not be anisotropic we can actually say even more.

**Proposition 4.3.5.** *The real groups $\mathbf{G}_v$ and $\mathbf{H}_v$ are inner twists of one another.*

*Proof.* Let $\mathbf{G}_0$ be the up to $\mathbb{Q}$-isomorphism unique $\mathbb{Q}$-split simply connected absolutely almost simple $\mathbb{Q}$-group of the same Cartan–Killing type as $\mathbf{G}$ and $\mathbf{H}$. Then we get a split short exact sequence

$$1 \longrightarrow \mathrm{Ad}\,\mathbf{G}_0 \xrightarrow{i} \mathrm{Aut}\,\mathbf{G}_0 \xrightarrow{\pi} \mathrm{Sym}\,\Delta \longrightarrow 1$$

of $\mathrm{Gal}(\mathbb{Q})$-groups.

Being inner twists of one another means that the two elements in $H^1(\mathbb{R};\mathrm{Aut}\,\mathbf{G}_0)$ representing $\mathbf{G}_v$ and $\mathbf{H}_v$ have the same image in $H^1(\mathbb{R};\mathrm{Sym}\,\Delta)$. This holds obviously true if $\mathrm{Sym}\,\Delta$ is trivial. The only other symmetry group that can occur in our case is $\mathrm{Sym}\,\Delta \cong \mathbb{Z}/2\mathbb{Z}$, in which case

$$H^1(F;\mathrm{Sym}\,\Delta) \cong {}^{F^\times}\!\big/_{(F^\times)^2}.$$

Now, denote by $\alpha \in H^1(k;\mathrm{Aut}\,\mathbf{G}_0)$ and $\beta \in H^1(\ell;\mathrm{Aut}\,\mathbf{G}_0)$ the elements represented by $\mathbf{G}$ and $\mathbf{H}$, respectively. Using functoriality of Galois cohomology, we get a commutative diagram

$$\begin{tikzcd}
\alpha \in H^1(K;\mathrm{Aut}\,\mathbf{G}_0) \arrow[r, "\pi_*"] \arrow[d] & H^1(K;\mathbb{Z}/2\mathbb{Z}) \arrow[d] \\
\prod_{\mathfrak{p}\in V_f(K)} H^1(K_{\mathfrak{p}};\mathrm{Aut}\,\mathbf{G}_0) \arrow[r, "\pi_*"] \arrow[d, phantom, "\wr"] & \prod_{\mathfrak{p}\in V_f(K)} H^1(K_{\mathfrak{p}};\mathbb{Z}/2\mathbb{Z}) \arrow[d, phantom, "\wr"] \\
\prod_{\mathfrak{p}\in V_f(L)} H^1(L_{\mathfrak{p}};\mathrm{Aut}\,\mathbf{G}_0) \arrow[r, "\pi_*"'] & \prod_{\mathfrak{p}\in V_f(L)} H^1(L_{\mathfrak{p}};\mathbb{Z}/2\mathbb{Z}) \\
\beta \in H^1(L;\mathrm{Aut}\,\mathbf{G}_0) \arrow[r, "\pi_*"'] \arrow[u] & H^1(L;\mathbb{Z}/2\mathbb{Z}). \arrow[u]
\end{tikzcd}$$

Since we have isomorphisms $\mathbf{G}_\mathfrak{p} \cong \mathbf{H}_\mathfrak{p}$ for every $\mathfrak{p} \in V_f(K)$, we get that $\pi_*\alpha \in H^1(K;\mathbb{Z}/2\mathbb{Z})$ and $\pi_*\beta \in H^1(L;\mathbb{Z}/2\mathbb{Z})$ map diagonally to elements corresponding to one another under the induced isomorphism

$$\prod_{\mathfrak{p}\in V_f(K)} H^1(K_\mathfrak{p};\mathbb{Z}/2\mathbb{Z}) \cong \prod_{\mathfrak{p}\in V_f(L)} H^1(L_\mathfrak{p};\mathbb{Z}/2\mathbb{Z}).$$

Moreover, $\mathbf{G}$ and $\mathbf{H}$ localize to the anisotropic real form at all infinite places except $v$, so this can be extended to all places *except* $v$. In order to conclude that $\pi_*\alpha$ and $\pi_*\beta$ also have the same image in

$$H^1(K_v;\mathbb{Z}/2\mathbb{Z}) \cong H^1(L_v;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{R}^\times/(\mathbb{R}^\times)^2$$

we make use of Hilbert reciprocity. For $\mathfrak{p}$ a place of $K$, denote by

$$(\,\cdot\,,\,\cdot\,)_\mathfrak{p} \colon K_\mathfrak{p}^\times/(K_\mathfrak{p}^\times)^2 \times K_\mathfrak{p}^\times/(K_\mathfrak{p}^\times)^2 \to \{\pm 1\}$$

the corresponding Hilbert symbol and let $(a_\mathfrak{p})_{\mathfrak{p}\in V(K)}$ and $(b_\mathfrak{p})_{\mathfrak{p}\in V(L)}$ be the images of $\pi_*\alpha$ and $\pi_*\beta$ in

$$\prod_{\mathfrak{p}\in V(K)} K_\mathfrak{p}^\times/(K_\mathfrak{p}^\times)^2 \qquad \text{and} \qquad \prod_{\mathfrak{p}\in V(L)} L_\mathfrak{p}^\times/(L_\mathfrak{p}^\times)^2,$$

respectively. As they come from global elements, Hilbert reciprocity [70, Kapitel VI, Theorem 8.1] shows

$$\prod_{\mathfrak{p}\in V(K)} (-1, a_\mathfrak{p})_\mathfrak{p} = 1 = \prod_{\mathfrak{p}\in V(L)} (-1, b_\mathfrak{p})_\mathfrak{p}.$$

Moreover, as $\pi_*\alpha$ and $\pi_*\beta$ correspond over all places, except possibly $v$, we know that the Hilbert symbols at these places have to agree as well. This allows us to conclude that the remaining Hilbert symbols $(-1, a_v)_v$ and $(-1, b_v)_v$ also have to agree. But then $a_v$ and $b_v$ already have to be the same in

$$\mathbb{R}^\times/(\mathbb{R}^\times)^2 \cong \mathbb{Z}/2\mathbb{Z},$$

showing that $\mathbf{G}_v$ and $\mathbf{H}_v$ are inner twists of one another. □

**Remark 4.3.6.** We want to make two small remarks:

1. Although we restricted ourselves to the case of $\operatorname{Sym}\Delta \cong \mathbb{Z}/2\mathbb{Z}$, the statement is nonetheless true more generally for all types [26, Proposition 8]. Even more generally, if $\mathbf{G}$ and $\mathbf{H}$ are not anisotropic at only *one* infinite place, one can still show that they have to have the same number of places where $\mathbf{G}$ and $\mathbf{H}$ are inner (respectively outer) forms [52, Proposition 16].

2. Alternatively, we could have used Poitou–Tate duality in the above proof. The middle part of the Poitou–Tate sequence [95, Chapter II, §6.3, Theorem C]

$$\cdots \longrightarrow H^1(K;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \prod_{\mathfrak{p}\in V(K)} H^1(K_\mathfrak{p};\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(K;\mathbb{Z}/2\mathbb{Z}')^* \longrightarrow \cdots$$

can be used to make essentially the same argument we made above using Hilbert reciprocity. However, when using this approach, the difficulty lies in identifying the second map as the product of the Hilbert symbols.

Proposition 4.3.5 can immediately be applied to deal with one of our cases, this time the one of $\mathrm{SU}(n,m)$.

**Proposition 4.3.7.** *The groups $G \approx \mathrm{SU}(n,m)$, with $n, m \geq 2$, do not exhibit non-profinite second bounded cohomology.*

*Proof.* The above proposition shows that $\mathbf{G}_v$ and $\mathbf{H}_v$ are inner twists of one another. By the classification of real simple Lie groups we know that the generalized special unitary groups $\mathrm{SU}(n', m')$, with $n', m' \geq 2$ and $n' + m' = n + m$, are the only real outer forms of type $\mathrm{A}_{n+m-1}$. Hence, $H$ has to be isogenous to such a group and thus defines a Hermitian symmetric space. ☐

### 4.3.3 $\mathrm{SO}^*(2n)$

In the remaining case of $\mathrm{SO}^*(2n)$, $n \geq 5$, it is not as easy to apply the Galois cohomology machinery we used so far. Not only does the Dynkin diagram have $\mathbb{Z}/2\mathbb{Z}$ as its symmetry group, but also whether $\mathrm{SO}^*(2n)$ is an inner or outer form depends on $n$ being even or odd. Finally, the center of the quasi-split form is not just given by $\mu_2$ and also depends on the parity of $n$. [77, Table on p. 332]. Hence, to deal with this final case, we resort to using the theory of Lie algebras.

**Proposition 4.3.8.** *The groups $G \approx \mathrm{SO}^*(2n)$ with $n \geq 5$ do not exhibit non-profinite second bounded cohomology.*

*Proof.* Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$ defined over $K$. By a theorem of Jacobson and Weil [94, Theorem 4.5.10], there exists a central simple $K$-algebra $A$ with an involution $\sigma$ of the first kind and orthogonal type such that the Lie algebra

$$\mathrm{Skew}(A, \sigma) = \{x \in A \mid \sigma(x) = -x\},$$

together with the commutator Lie bracket $[x, y] = xy - yx$, is $K$-isomorphic to the Lie algebra $\mathfrak{g}$.

Similarly, there exists a central simple $L$-algebra $B$ with involution $\tau$ such that the Lie algebra $\mathfrak{h}$ of $\mathbf{H}$ is isomorphic to $\mathrm{Skew}(B, \tau)$.

Now, as $\mathbf{G}$ and $\mathbf{H}$ are isomorphic at every place $\mathfrak{p}$, except possibly $v$, the corresponding Lie algebras

$$\begin{aligned}\mathfrak{h}_{\mathfrak{p}} &\cong \mathrm{Skew}(A, \sigma)_{\mathfrak{p}} \cong \mathrm{Skew}(A_{\mathfrak{p}}, \sigma_{\mathfrak{p}}) \\ \mathfrak{g}_{\mathfrak{p}} &\cong \mathrm{Skew}(B, \tau)_{\mathfrak{p}} \cong \mathrm{Skew}(B_{\mathfrak{p}}, \tau_{\mathfrak{p}})\end{aligned}$$

are isomorphic as well. As $n \geq 5$, a theorem by Jacobson proves that such an isomorphism can be extended uniquely to an isomorphism of algebras with involution [46, Chapter X, Theorem 12]

$$(A_{\mathfrak{p}}, \sigma_{\mathfrak{p}}) \cong (B_{\mathfrak{p}}, \tau_{\mathfrak{p}}).$$

In particular, $A$ and $B$ represent the same elements in the Brauer group $\mathrm{Br}(K_{\mathfrak{p}}) \cong \mathrm{Br}(L_{\mathfrak{p}})$.
Making again use of the Albert–Brauer–Hasse–Noether theorem

$$1 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{\mathfrak{p}\in V(K)} \mathrm{Br}(K_{\mathfrak{p}}) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 1,$$

we see that the elements represented by $A$ and $B$ in $\mathrm{Br}(K_v)$ have to agree.

Now, as $G \approx \mathrm{SO}^*(2n)$, we know that $\mathfrak{g}_v \cong \mathfrak{so}^*(2n)$ is defined by a central simple algebra over the quaternions. So $A_v$ and $B_v$ are the non-trivial element in $\mathrm{Br}(K_v) = \mathrm{Br}(\mathbb{R})$. However, all other real Lie algebras $\mathfrak{so}(p,q)$ of type $\mathrm{D}_n$ are defined by split central simple algebras over the reals, and are thus trivial in the Brauer group. So, $\mathfrak{h}_v$ needs to be isomorphic to $\mathfrak{so}^*(2n)$ as well, giving us that $\mathbf{H}$ is also isogenous to $\mathrm{SO}^*(2n)$, finishing the proof. □

## 4.4 On the question of profinite invariance of quasi-morphisms

Finally, let us discuss whether having non-trivial quasi-morphism, or equivalently having non-trivial exact bounded cohomology $EH^2_b(\,\cdot\,;\mathbb{R})$ (Proposition A.2.10), is a profinite property.

First we note that, by a theorem of Burger and Monod [17, Corollary 1.6], the lattices we constructed above all have trivial exact bounded cohomology. Thus, our strategy above does not give any examples of exact bounded cohomology not being a profinite property.

The picture is quite different in the case of lattices in rank one groups. As already noted in Remark 4.1.2, these lattices have infinite dimensional second bounded cohomology. In fact, they have infinite dimensional second exact bounded cohomology [33]. However, an important ingredient for constructing lattices with isomorphic profinite completion is missing: the congruence subgroup property. Serre conjectured that all groups of rank one should *not* have the congruence subgroup property [97, Footnote 1]. This is still an open question for the rank one groups $\mathrm{Sp}(n,1)$ and $\mathrm{F}_4^{(-20)}$. As lattices in these groups exhibit other "higher rank properties" like Kazhdan's property (T), arithmeticity, and superrigidity, one might suspect that, contrary to Serre's conjecture, they also have the congruence subgroup property [62, Section 4].

As all groups of type $\mathrm{F}_4$ split over $p$-adic fields [100, p. 60], there is a unique simply connected absolutely almost simple $\mathbb{Q}$-group of type $\mathrm{F}_4$, giving the Lie group $\mathrm{F}_4^{(-20)}$ over the reals. Denote this algebraic group by $\mathbf{F}_4^{(-20)}$. Moreover, let $\mathbf{F}_4$ be the unique absolutely almost simple algebraic *split* $\mathbb{Q}$-group of type $\mathrm{F}_4$. As both $\mathbf{F}_4$ and $\mathbf{F}_4^{(-20)}$ split over every $\mathbb{Q}_p$, assuming the congruence subgroup property for $\mathbf{F}_4^{(-20)}$, Corollary 3.2.10 gives us lattices $\Gamma \leq \mathbf{F}_4(\mathbb{R})$ and $\Lambda \leq \mathbf{F}_4^{(-20)}(\mathbb{R})$ with isomorphic profinite completions. As a lattice in a higher rank group $EH^2_b(\Gamma;\mathbb{R}) \cong 0$ while $EH^2_b(\Lambda;\mathbb{R})$ is infinite dimensional.

So we have the following dichotomy.

**Proposition 4.4.1.** *At most one of the following statements is true:*

*1) The rank one group* $\mathbf{F}_4^{(-20)}$ *has the congruence subgroup property.*

*2) For lattices in simple Lie groups, the vanishing of the exact bounded cohomology* $EH_b^2(\,\cdot\,;\mathbb{R})$ *is a profinite property.*

This would allow us to prove Serre's conjecture by proving that exact bounded cohomology is a profinite invariant for lattices. However, as this conjecture has been open for more than 50 years this seems to be a hard problem.

Conversely, if one were to be able to disprove Serre's conjecture, this would not only give examples for the non-profiniteness of the exact bounded cohomology, but also, and admittedly more interestingly, the existences of a non-residually finite hyperbolic group [62, Remark 4.2].

**Remark 4.4.2** (Other counterexamples?)**.** We briefly want to discuss other possible sources of counterexamples to the profiniteness of having quasi-morphisms. For constructing finitely generated residually finite groups with isomorphic profinite completions, there are three main sources of examples.

- Arithmetic subgroups in algebraic groups with the congruence subgroup property. Here we just saw that they (most likely) won't produce any counterexamples.

- Grothendieck pairs coming from the Platonov–Tavgen construction, usually using some kind of Rips construction as input. In the Grothendieck pair $N \to G$ directly obtained from the Rips construction the group $G$ will be hyperbolic. As an infinite normal subgroup, $N$ will then be acylindrically hyperbolic [74, Corollary 1.5]. Thus, both $G$ and $N$ will have an infinite space of non-trivial quasi-morphisms [38, Theorem A]. Even when applying the fiber product construction, both of the groups will still have $G$ as a quotient, proving that they have many quasi-morphisms.

- Branch groups acting on rooted trees, as used by Kionke and Schesler [55]. These groups always contain a lot of torsion elements, so it seems unlikely that they have non-trivial quasi-morphisms.

However, very recently Fournier-Facio gave the first example of a Grothendieck pair $N \to G$, where $N$ has no non-trivial quasi-morphisms and $G$ has an infinite dimensional space of non-trivial quasi-morphisms [31, Main Theorem]. Starting with a Grothendieck pair, coming from applying the Rips construction to the Higman group, one uses *iterated group theoretic Dehn fillings* to produce Grothendieck pairs $N_i \to G_i$ where the stable commutator length of the $N_i$ gets smaller and smaller. In the limit this ensures that $N = N_\infty$ has vanishing stable commutator length, and thus no non-trivial quasi-morphisms (Corollary A.2.16). On the other hand, every $G_i$ and $G = G_\infty$ has the Higman group as quotient and thus lots of quasi-morphisms.

# Chapter 5
# Natural profinite invariants

In Section 3.3, we saw the definition of a Grothendieck pair and in Section 3.3.1 we have seen one way of constructing them. The research on Grothendieck pairs was initiated by Grothendieck himself, with a paper where he studied the linear representations of Grothendieck pairs [37]. In this paper he also raised the question for the existence of finitely presented non-trivial Grothendieck pairs.

Recently, Jaikin-Zapirain and Lubotzky revisited some of these results [47]. In both papers the authors consider the class $\mathcal{C}$ of groups $K$ such that

$$\begin{aligned} u^*\colon \mathrm{Hom}_{\mathrm{Group}}(G,K) &\longrightarrow \mathrm{Hom}_{\mathrm{Group}}(H,K) \\ f &\longmapsto f\circ u \end{aligned}$$

is a bijection for every Grothendieck pair $u\colon H\to G$. This might lead to the following questions.

1. Which other functors from the category of groups are *natural* profinite invariants, meaning they send all Grothendieck pairs to isomorphisms?
2. Are such invariants automatically profinite invariants?

In this chapter we will discuss both of these questions. As it turns out, the notions of "profinite invariants" and "natural profinite invariants" are logically independent.

### Overview of this chapter

## 5.1 The notion of natural profinite invariants

Let us briefly revisit the paper by Grothendieck. A remarkable result is the following theorem on the class $\mathcal{C}$.

**Theorem 5.1.1** ([37], [47, Theorem 1.1])**.**

*(1) The class $\mathcal{C}$ is closed under commensurability, inverse limits and direct products, and contains all nilpotent groups.*

*(2) If $A$ is a commutative ring and $G$ is an affine group scheme of finite type over $A$, then $G(A) \in \mathcal{C}$.*

*(3) Compact Hausdorff groups are in $\mathcal{C}$.*

*(4) Let $\Lambda$ be a finitely generated residually finite group. If $\Lambda \in \mathcal{C}$, then $\Lambda$ is* left Grothendieck rigid*, i.e. every Grothendieck pair $\Lambda \to \Gamma$ is trivial.*

Using this Theorem one sees that for every non-trivial Grothendieck pair $\Lambda \to \Gamma$ we have $\Lambda \notin \mathcal{C}$, i.e. that the functor $\mathrm{Hom}_{\mathrm{Group}}(\,\cdot\,, \Lambda)\colon \mathrm{Group} \to \mathrm{Set}$ does *not* map every Grothendieck pair to a bijection. Another, and far simpler, example is the forgetful functor $\mathrm{Group} \to \mathrm{Set}$.

**Definition 5.1.2** (Natural profinite invariant)**.** Let Group be the category of groups, let $\mathcal{D}$ be any category and let $F\colon \mathrm{Group} \to \mathcal{D}$ be a functor. Then $F$ is called a *natural profinite invariant*, if for every Grothendieck pair $u\colon H \to G$ the induced morphism $F(u)\colon P(H) \to P(G)$ is an isomorphism.

This definition seems to be a natural[1] variation of the one of *profinite invariants.* For an invariant $P$ to be called a profinite invariant, we require that whenever two finitely generated residually finite groups $\Lambda$ and $\Gamma$ have isomorphic profinite completions $\widehat{\Lambda} \cong \widehat{\Gamma}$, they also have the same (or in our context isomorphic) invariant $P(\Lambda) \cong P(\Gamma)$. Now, for *natural* profinite invariance we not only ask for *any* isomorphism between the profinite completions and the corresponding invariants, but instead these isomorphisms should be induced by the given group homomorphism $\Lambda \to \Gamma$.

**Example 5.1.3** (Trivial examples)**.**

1. Take the identity functor $\mathrm{id}\colon \mathrm{Group} \to \mathrm{Group}$. Since not all groups are profinitely rigid, this is not a profinite invariant. Moreover, not all groups are Grothendieck rigid, so this is also not a natural profinite invariant.

2. By definition, the profinite completion functor $\widehat{\cdot}\colon \mathrm{Group} \to \mathrm{Group}$ is both a profinite and a natural profinite invariant.

[1]This pun is somewhat intended.

**Remark 5.1.4** (Localization). Let $S$ be the collection of all Grothendieck pairs and let $\mathrm{Group}[S^{-1}]$ be the localization of the category of groups at this class of morphisms. By construction, the localization functor $L_S\colon \mathrm{Group} \to \mathrm{Group}[S^{-1}]$ is a natural profinite invariant. Moreover, it is *universal* in the sense that a functor $F\colon \mathrm{Group} \to \mathcal{D}$ is a natural profinite invariant if and only if it factors (up to natural isomorphism) through the localization functor.

$$\begin{tikzcd} \mathrm{Group} \arrow[r, "F"] \arrow[d, "L_S"'] & \mathcal{D} \\ \mathrm{Group}[S^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

The main result of Grothendieck's original paper gives an astounding example of a natural profinite invariant.

**Theorem 5.1.5** ([37, Theoreme 1.2]). *For a commutative ring $A$ and a group $G$, denote by $\mathrm{Rep}_A(G)$ the category of finitely presented $A$-modules together with a $G$-action. Then any Grothendieck pair $u\colon H \to G$ induces an equivalence of categories*

$$u^*\colon \mathrm{Rep}_A(G) \longrightarrow \mathrm{Rep}_A(H).$$

It suggests itself to ask about the relation between profinite invariants and natural profinite invariants. As already noted above, the forgetful functor is not a natural profinite invariant, however, it is easy to see that it is a profinite invariant.

**Lemma 5.1.6.** *The cardinality of a group (or in other words the forgetful functor sending a group to the underlying set) is a profinite invariant.*

*Proof.* Let $\Lambda$ and $\Gamma$ be two finitely generated residually finite groups such that $\widehat{\Lambda} \cong \widehat{\Gamma}$. As both groups are finitely generated, they are either finite or countably infinite. If one of them, say $\Lambda$, is finite we have that $\Lambda = \widehat{\Lambda} \cong \widehat{\Gamma}$. Thus, $\Gamma$ has to be the same finite group as $\Lambda$. Otherwise, both are countably infinite, and thus the underlying sets are in bijection. $\square$

In Section 5.3.2 we will see a natural profinite invariant that is *not* a profinite invariant.

## 5.2 Some non-examples

After seeing some first examples, we now want to discuss some more invariants for which we have already discussed the profinite invariance. For this we will study short exact sequences

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

with $N \to G$ a Grothendieck pair.

### Interlude: spectral sequences and five-term exact sequences

Studying the (co)homological invariants of short exact sequences often leads to arguments involving spectral sequences. For the basic terminology on spectral sequences and convergence, we recommend the books by McCleary [66, Part I], Weibel [102, Chapter 5], and Rotman [91, Chapter 10]. Here we collect some facts about spectral sequences which we will use later. All spectral sequences are assumed to be first-quadrant spectral sequences.

**Proposition 5.2.1** (General five-term exact sequence)**.** *Let $n \in \mathbb{N}$ and let $E_2^{p,q}$ be a cohomological spectral sequence converging towards $H^*$. Assume that $E_\infty^{p,q} \cong 0$ for all $p+q = n-1$ with $p, q > 0$. Then there exists a natural five-term exact sequence*

$$0 \longrightarrow E_n^{n-1,0} \longrightarrow H^{n-1} \longrightarrow E_n^{0,n-1} \xrightarrow{d_n^{0,n-1}} E_n^{n,0} \longrightarrow H^n.$$

*Similarly, if $E^2_{p,q}$ is a homological spectral sequence converging towards $H_*$, with $E^\infty_{p,q} = 0$ for all $p+q = n-1$ with $p, q > 0$, then there exists a natural five-term exact sequence*

$$H_n \longrightarrow E^n_{n,0} \xrightarrow{d^n_{n,0}} E^n_{0,n-1} \longrightarrow H_{n-1} \longrightarrow E^n_{n-1,0} \longrightarrow 0.$$

*Proof.* Since $d_n^{0,n-1}$ is the last non-trivial differential involving $E_*^{0,n-1}$ and $E_*^{n,0}$, we have an exact sequence

$$0 \longrightarrow E_\infty^{0,n-1} \longrightarrow E_n^{0,n-1} \xrightarrow{d_n^{0,n-1}} E_n^{n,0} \longrightarrow E_\infty^{n,0} \longrightarrow 0. \tag{1}$$

By the convergence towards $H^n$, we have extension problems

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & F^0H^{n-1} & \longrightarrow & E_\infty^{n-1,0} & \longrightarrow & 0 \\
0 & \longrightarrow & F^0H^{n-1} & \longrightarrow & F^1H^{n-1} & \longrightarrow & E_\infty^{n-2,1} & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & F^{n-2}H^{n-1} & \longrightarrow & H^{n-1} & \longrightarrow & E_\infty^{0,n-1} & \longrightarrow & 0
\end{array}$$

and the assumption on $E_\infty^{p,q}$ gives that most of these extension problems degenerate. Thus, we obtain another exact sequence

$$0 \longrightarrow E_\infty^{n-1,0} \longrightarrow H^{n-1} \longrightarrow E_\infty^{0,n-1} \longrightarrow 0, \tag{2}$$

where in fact $E_\infty^{n-1,0} = E_n^{n-1,0}$. Finally, the convergence again gives extension problems

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & F^0H^{n} & \longrightarrow & E_\infty^{n,0} & \longrightarrow & 0 \\
0 & \longrightarrow & F^0H^{n} & \longrightarrow & F^1H^{n} & \longrightarrow & E_\infty^{n-1,1} & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & F^{n-1}H^{n} & \longrightarrow & H^{n} & \longrightarrow & E_\infty^{0,n} & \longrightarrow & 0,
\end{array}$$

where $F^0H^n \subseteq H^n$. In particular, we get the exact sequence

$$0 \longrightarrow E_\infty^{n,0} \longrightarrow H^n. \tag{3}$$

Combining the three exact sequences (1), (2) and (3) gives the desired sequence

$$0 \longrightarrow E_n^{n-1,0} \longrightarrow H^{n-1} \longrightarrow E_n^{0,n-1} \xrightarrow{d_n^{0,n-1}} E_n^{n,0} \longrightarrow H^n.$$

The proof of the homological version works similarly. First we have the exact sequence

$$0 \longrightarrow E^\infty_{n,0} \longrightarrow E^n_{n,0} \xrightarrow{d^n_{n,0}} E^n_{0,n-1} \longrightarrow E^\infty_{0,n-1} \longrightarrow 0.$$

The convergence in total degree $n-1$ and the assumption on $E^\infty_{p,q}$ gives a short exact sequence

$$0 \longrightarrow E^\infty_{0,n-1} \longrightarrow H_{n-1} \longrightarrow E^\infty_{n-1,0} \longrightarrow 0,$$

where $E^\infty_{n-1,0} = E^n_{n-1,0}$. Finally, the convergence in total degree $n$ gives, as one of the extension problems,

$$0 \longrightarrow F_{n-1}H_n \longrightarrow H_n \longrightarrow E^\infty_{n,0} \longrightarrow 0.$$

Again, combining these three sequences, while truncating the last one, gives the desired five-term exact sequence. □

In the case $n = 2$ the assumption on $E_\infty^{p,q}$ is empty. Thus, we immediately get the following.

**Corollary 5.2.2** (Five-term exact sequence). *Let $E^2_{p,q}$ be a cohomological spectral sequence converging towards $H^n$. Then there exists a natural five-term exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow H^2$$

*Similarly, if $E_2^{p,q}$ is a homological spectral sequence converging towards $H_n$, then there exists a natural five-term exact sequence*

$$H_2 \longrightarrow E^2_{2,0} \longrightarrow E^2_{0,1} \longrightarrow H_1 \longrightarrow E^2_{1,0} \longrightarrow 0.$$

**Corollary 5.2.3** (Inflation-restriction sequence in group (co)homology). *Let $R$ be a ring, let*

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 0$$

*be a short exact sequence of groups and let $M$ be an $R[G]$-module. Then there are two natural five-term exact sequences*

$$0 \to H^1(Q; M^N) \to H^1(G; M) \to H^1(N; M)^Q \to H^2(Q; M^N) \to H^2(G; M)$$

*and*

$$H_2(G; M) \to H_2(Q; M_N) \to H_1(N; M)_Q \to H_1(G; M) \to H_1(Q; M_N) \to 0,$$

*where the maps in same degree are induced by the morphisms in the short exact sequence.*

*Proof.* The two sequences arise as five-term exact sequences of the cohomological and homological Hochschild–Serre spectral sequences [102, Section 6.8][91, Theorem 10.52]

$$E_2^{p,q} = H^p(Q; H^q(N; M)) \Longrightarrow H^{p+q}(G; M)$$

and

$$E^2_{p,q} = H_p(Q; H_q(N; M)) \Longrightarrow H_{p+q}(G; M).$$

For the additional statement that the maps in the exact sequence are induced by the morphisms in the short exact sequence, we use the naturality of both the spectral sequences and the five-term exact sequence. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, equal] \arrow[d, equal] & N \arrow[r] \arrow[d, "i"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, "i"] \arrow[d] & G \arrow[r, "p"] \arrow[d, "p"] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 0 \arrow[r] & Q \arrow[r, equal] & Q \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By naturality, we then obtain a diagram of five-term exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^1(Q; M^N) \arrow[r, equal] \arrow[d, equal] & H^1(Q; M^N) \arrow[r] \arrow[d, "p^*"] & 0 \arrow[r] \arrow[d] & H^2(Q; M^N) \arrow[r, equal] \arrow[d, equal] & H^2(Q; M^N) \arrow[d, "p^*"] \\
0 \arrow[r] & H^1(Q; M^N) \arrow[r] \arrow[d] & H^1(G; M) \arrow[r] \arrow[d, "i^*"] & H^1(N; M)^Q \arrow[r] \arrow[d, equal] & H^2(Q; M^N) \arrow[r] \arrow[d] & H^2(G; M) \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & H^1(N; M) \arrow[r, equal] & H^1(N; M) \arrow[r] & 0 \arrow[r] & H^2(N; M),
\end{tikzcd}$$

giving the desired identifications of the maps. The same proof also works for the homological statement. □

**Corollary 5.2.4** (Five-term exact sequence in bounded cohomology)**.** *Let*

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 0$$

*be a short exact sequence of groups. Then there is a natural five-term exact sequence*

$$0 \longrightarrow H_b^2(Q; \mathbb{R}) \longrightarrow H_b^2(G; \mathbb{R}) \longrightarrow H_b^2(N; \mathbb{R})^Q \longrightarrow H_b^3(Q; \mathbb{R}) \longrightarrow H_b^3(G; \mathbb{R})$$

*where the maps in equal degree are induced by the morphisms in the short exact sequence.*

*Proof.* The existence of this five-term exact sequence uses the Hochschild–Serre spectral sequence in bounded cohomology. For a short exact sequence of groups as given here, there exists a spectral sequence $E_2^{p,q} \Longrightarrow H_b^2(G; \mathbb{R})$ such that $E_2^{p,q} \cong H_b^p(Q; H_b^q(N; \mathbb{R}))$ whenever

$p = 0$, $q = 0$ or the induced semi-norm on $H_b^q(N;\mathbb{R})$ is a norm [68, Section IV.12][25, Theorem 3.1.17].

Since we have $H_b^1(N;\mathbb{R}) \cong 0$, we get $E_2^{p,1} \cong H_b^p(Q; H_b^1(N;\mathbb{R})) \cong 0$ for all $p \in \mathbb{N}$. In particular $E_\infty^{1,1} \cong 0$. Now the general five-term exact sequence construction in Proposition 5.2.1 with $n = 3$ gives an exact sequence

$$0 \longrightarrow E_3^{2,0} \longrightarrow H_b^2(G;\mathbb{R}) \longrightarrow E_3^{0,2} \longrightarrow E_3^{3,0} \longrightarrow H_b^3(G;\mathbb{R}).$$

Here all the terms on the third page of the spectral sequence are actually isomorphic to the terms on the second page, as $E_2^{p,1} \cong 0$. This, and the identification of the second page on the two axis, gives the desired sequence

$$0 \longrightarrow H_b^2(Q;\mathbb{R}) \longrightarrow H_b^2(G;\mathbb{R}) \longrightarrow H_b^2(N;\mathbb{R})^Q \longrightarrow H_b^3(Q;\mathbb{R}) \longrightarrow H_b^3(G;\mathbb{R}).$$

The identifications of the maps again uses the naturality as the proof of Corollary 5.2.3. □

### 5.2.1 Higher homology groups

In Corollary 3.1.21, we have seen that the higher homology groups are *not* a profinite invariant. This also holds for natural profinite invariance.

**Proposition 5.2.5.** *The second homology group with integer coefficients $H_2(\,\cdot\,;\mathbb{Z})$ is* not *a natural profinite invariant.*

*Proof.* Bridson and Reid constructed a group $Q$ that is finitely presented, super-perfect, i.e. $H_1(Q;\mathbb{Z}) \cong H_2(Q;\mathbb{Z}) \cong 1$, has no finite quotients and infinite dimensional $H_3(Q;\mathbb{Q})$ [14, Proposition 6.5].

Using this group as input for the Wise–Rips construction (Theorem 3.3.11) we obtain a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with $G$ residually finite, hyperbolic and $N$ finitely generated. As $Q$ is super-perfect and has no non-trivial finite quotients, we obtain from Proposition 3.3.8 that $N \to G$ is a Grothendieck pair.

Now, we want to prove that $H_2(N;\mathbb{Z}) \to H_2(G;\mathbb{Z})$ is not an isomorphism. For this we use the Hochschild–Serre spectral sequence

$$E_{p,q}^2 = H_p(Q; H_q(N;\mathbb{Z})) \Longrightarrow H_{p+q}(G;\mathbb{Z})$$

associated with the above short exact sequence. First, since $N$ is finitely generated, so is $H_1(N;\mathbb{Z}) \cong N_{\mathrm{ab}}$. Thus, $\mathrm{Aut}(H_1(N;\mathbb{Z}))$ is a residually finite group. Since $Q$ has no non-trivial finite quotients, the $Q$-action on $H_1(N;\mathbb{Z})$ is trivial. This allows us to use the universal coefficient theorem, in order to identify

$$E_{1,1}^2 = H_1(Q; H_1(N;\mathbb{Z})) \cong 0 \qquad \text{and} \qquad E_{2,1}^2 = H_2(Q; H_1(N;\mathbb{Z})) \cong 0.$$

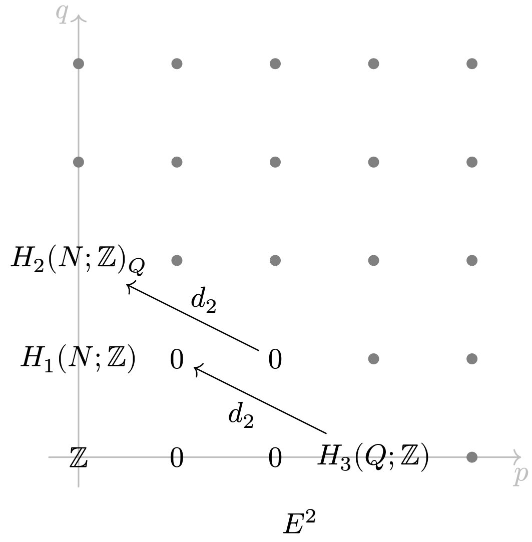


Figure 5.1: The second page of the spectral sequence

As this implies $E^\infty_{1,1} \cong 0$, Proposition 5.2.1 gives a natural five-term exact sequence

$$H_3(G;\mathbb{Z}) \longrightarrow E^3_{3,0} \longrightarrow E^3_{0,2} \longrightarrow H_2(G;\mathbb{Z}) \longrightarrow E^3_{2,0} \longrightarrow 0.$$

Using the above identifications of the second page, we can identify this exact sequence with

$$H_3(G;\mathbb{Z}) \longrightarrow H_3(Q;\mathbb{Z}) \longrightarrow H_2(N;\mathbb{Z})_Q \longrightarrow H_2(G;\mathbb{Z}) \longrightarrow 0.$$

Now $H_3(G;\mathbb{Z})$ is finitely generated as $G$ is hyperbolic, whereas $H_3(Q;\mathbb{Z})$ is *not* finitely generated. Thus, exactness of the sequence gives that $H_2(N;\mathbb{Z})_Q \to H_2(G;\mathbb{Z})$ is *not* injective. Thus, the composition $H_2(N;\mathbb{Z}) \twoheadrightarrow H_2(N;\mathbb{Z})_Q \to H_2(G;\mathbb{Z})$ is not injective either. □

**Remark 5.2.6.** The above proposition was basically already proven by Bridson and Reid. While their theorem does not explicitly state that they construct a Grothendieck pair, their proof is essentially the same as ours [14, Theorem 6.1, Section 6.3].

**Corollary 5.2.7.** *For every $n \in \mathbb{N}_{\geq 2}$, the $n$-th homology group with integer coefficients $H_n(\,\cdot\,;\mathbb{Z})$ is* not *a natural profinite invariant.*

*Proof.* For $n = 2$ this is the statement of Proposition 5.2.5. From here on we can proceed by induction on $n$:

Suppose $H_n(\,\cdot\,;\mathbb{Z})$ is not a natural profinite invariant, i.e. there exists a Grothendieck pair $\varphi\colon N \to H$ such that $H_n(\varphi;\mathbb{Z})\colon H_n(N;\mathbb{Z}) \to H_n(H;\mathbb{Z})$ is *not* an isomorphism. We now consider $\varphi \times \mathrm{id}_\mathbb{Z}\colon N \times \mathbb{Z} \to H \times \mathbb{Z}$. As taking profinite completion is compatible with products, this is still a Grothendieck pair. Moreover, as

$$H_q(\mathbb{Z};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } q \in \{0,1\} \\ 0 & \text{otherwise,} \end{cases}$$

the Künneth theorem [16, Section V.5, Corollary 5.8] gives a commutative square

$$\begin{tikzcd} H_{n+1}(N \times \mathbb{Z};\mathbb{Z}) \arrow[r] \arrow[d, "\cong"'] & H_{n+1}(H \times \mathbb{Z},\mathbb{Z}) \arrow[d, "\cong"] \\ H_{n+1}(N;\mathbb{Z}) \oplus H_n(N;\mathbb{Z}) \arrow[r] & H_{n+1}(H;\mathbb{Z}) \oplus H_n(H;\mathbb{Z}), \end{tikzcd}$$

where both of the horizontal maps are induced by $\varphi$. Now by assumption on $\varphi$ the lower map is not an isomorphism in the $H_n$-factor. Thus, the upper map is not an isomorphism either, showing that $H_{n+1}(\,\cdot\,;\mathbb{Z})$ is not a natural profinite invariant. $\square$

### 5.2.2 Bounded cohomology

One result of Chapter 4 was that the second bounded cohomology is *not* a profinite invariant. The answer to whether the second bounded cohomology is a *natural* profinite invariant turns out to be "no" as well.

**Proposition 5.2.8.** *Neither the second bounded cohomology $H_b^2(\,\cdot\,;\mathbb{R})$ nor the second exact bounded cohomology $EH_b^2(\,\cdot\,;\mathbb{R})$ are natural profinite invariants.*

*Proof.* We consider the Higman group

$$Q = \langle a,b,c,d \mid a^{-1}ba = b^2,\ b^{-1}cb = c^2,\ c^{-1}dc = d^2,\ d^{-1}ad = a^2 \rangle.$$

As in Remark 3.3.10, applying the Wise–Rips construction Theorem 3.3.11 gives a short exact sequence

$$1 \longrightarrow N \xrightarrow{\iota} G \longrightarrow Q \longrightarrow 1,$$

where $N$ is generated by three elements and $G$ is a torsion-free, residually finite, hyperbolic group.

Again, since $Q$ is acyclic [24] and has no finite quotients [42], we obtain from Proposition 3.3.8 that $\iota$ induces an isomorphism $\widehat{N} \to \widehat{G}$, i.e. that $\iota\colon N \to G$ is a Grothendieck pair.

Moreover, the above short exact sequence of groups gives a five-term exact sequence

$$\begin{tikzcd} 1 \arrow[r] & H_b^2(Q;\mathbb{R}) \arrow[r] & H_b^2(G;\mathbb{R}) \arrow[r] \arrow[dr, "H_b^2(\iota)"'] & H_b^2(N;\mathbb{R})^H \arrow[r] \arrow[d, hook] & H_b^3(Q;\mathbb{R}) \arrow[r] & H_b^3(G;\mathbb{R}) \\ & & & H_b^2(N;\mathbb{R}) & & \end{tikzcd}$$

in bounded cohomology (Corollary 5.2.4). Now, by construction, $Q$ is a non-trivial amalgam [42], so we have that $H_b^2(Q;\mathbb{R})$ is non-trivial (in fact, it is infinite dimensional)

[34]. Hence, by exactness, the map $H_b^2(G;\mathbb{R}) \to H_b^2(N;\mathbb{R})^H$, and thus also $H_b^2(G;\mathbb{R}) \to H_b^2(N;\mathbb{R})$, is *not* injective.

For the statement concerning the second exact bounded cohomology, we first use the acyclicity of $Q$ to obtain a similar five term (in fact, long) exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \overbrace{H^2(Q;\mathbb{R})}^{\cong 0} \arrow[r] & H^2(G;\mathbb{R}) \arrow[r] \arrow[dr, "H^2(\iota)"'] & H^2(N;\mathbb{R})^H \arrow[r] \arrow[d, hook] & \overbrace{H^3(Q;\mathbb{R})}^{\cong 0} \arrow[r] & H^3(G;\mathbb{R}). \\
 & & & H^2(N;\mathbb{R}) & &
\end{tikzcd}$$

By taking the kernels of the comparison maps between these two exact sequences, we get another exact sequence

$$\begin{tikzcd}
1 \arrow[r] & EH_b^2(Q;\mathbb{R}) \arrow[r] & EH_b^2(G;\mathbb{R}) \arrow[r] \arrow[dr, "H_b^2(\iota)"'] & EH_b^2(N;\mathbb{R})^H \arrow[d, hook] \\
 & & & EH_b^2(N;\mathbb{R}),
\end{tikzcd}$$

where $EH_b^2(Q;\mathbb{R}) \cong H_b^2(Q;\mathbb{R})$ due to the acyclicity of $Q$. Thus, the same arguments as above apply. $\square$

**Remark 5.2.9.** By construction, $G$ is a non-elementary hyperbolic group. Moreover, as $N \to G$ is a (non-trivial) Grothendieck pair, we have that $N$ is an infinite normal subgroup of $G$. Hence, both $G$ and $N$ are *acylindrically hyperbolic groups* [74, Corollary 1.5], and so both $EH_b^2(G;\mathbb{R})$ and $EH_b^2(N;\mathbb{R})$ are infinite dimensional [38, Theorem A].

Thus, the above proof provides neither a new example for the failure of profinite invariance of $H_b^2(\,\cdot\,;\mathbb{R})$ nor an example for the failure of profinite invariance of $EH_b^2(\,\cdot\,;\mathbb{R})$.

## 5.3 Using adjointness

As the question of natural profinite invariance only makes sense for functors, we can use tools from category theory in order to learn more about this property. We have already seen one of them in Remark 5.1.4, where we used localization in order to describe natural profinite invariants as those functors that factor through the localization at all Grothendieck pairs.

### 5.3.1 Adjoint functors and natural profinite invariance

A useful tool from category theory is the Yoneda lemma. It will give a characterization of the natural profinite invariance of *adjoint* functors in terms of the class $\mathcal{C}$ studied by Grothendieck.

**Theorem 5.3.1** (Natural profinite invariance from adjointness)**.** *Let $\mathcal{D}$ be a locally small category and let $F\colon$ Group $\to \mathcal{D}$ be left-adjoint to $G\colon \mathcal{D} \to$ Group. Then $F$ is a natural profinite invariant if and only if for every $X \in \mathcal{D}$ we have $G(X) \in \mathcal{C}$.*

*Proof.* Let $\varphi\colon \Lambda \to \Gamma$ be a Grothendieck pair. By the adjunction of $F$ and $G$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Group}}(\Gamma, G(\,\cdot\,)) \arrow[r, "\varphi^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{Group}}(\Lambda, G(\,\cdot\,)) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathcal{D}}(F(\Gamma),\,\cdot\,) \arrow[r, "F(\varphi)^*"'] & \mathrm{Hom}_{\mathcal{D}}(F(\Lambda),\,\cdot\,) \end{tikzcd}$$

whose morphisms are natural transformations.

First, assume that for each $X \in \mathcal{D}$ we have $G(X) \in \mathcal{C}$. Then we get in particular that

$$\varphi^*\colon\ \mathrm{Hom}_{\mathrm{Group}}(\Gamma, G(X)) \longrightarrow \mathrm{Hom}_{\mathrm{Group}}(\Gamma, G(X))$$

is an isomorphism for every $X \in \mathcal{D}$. So the upper natural transformation in our diagram is, in fact, a natural isomorphism. By commutativity, so is the lower natural transformation and using the Yoneda embedding, we obtain that $F(\varphi)\colon F(\Lambda) \to F(\Gamma)$ is an isomorphism. As this holds for every Grothendieck pair, we obtain that $F$ is a natural profinite invariant.

Conversely, assume that $F$ is a natural profinite invariant. Then $F(\varphi)$ is an isomorphism, and thus $F(\varphi)^*$ is a natural isomorphism. By commutativity, we get that then $\varphi^*$ is also a natural isomorphism, i.e. for every $X \in \mathcal{D}$

$$\varphi^*\colon\ \mathrm{Hom}_{\mathrm{Group}}(\Gamma, G(X)) \longrightarrow \mathrm{Hom}_{\mathrm{Group}}(\Lambda, G(X))$$

is an isomorphism. Again, as this holds for every Grothendieck pair $\Lambda \to \Gamma$, we get that $G(X) \in \mathcal{C}$. ☐

One first example we could apply this theorem to is the profinite completion. However, we already know that the profinite completion is, for trivial reasons, a natural profinite invariant (Example 5.1.3). But more generally, it can be applied to the pro-$\mathfrak{C}$ completion, for any formation $\mathfrak{C}$ of finite groups.

**Corollary 5.3.2** (Pro-$\mathfrak{C}$ completion is a natural profinite invariant)**.** *Let $\mathfrak{C}$ be a formation of finite groups and denote the category of pro-$\mathfrak{C}$ groups by* Pro-$\mathfrak{C}$*. Then the pro-$\mathfrak{C}$ completion* $(\,\cdot\,)_{\widehat{\mathfrak{C}}}\colon \mathrm{Group} \to \text{Pro-}\mathfrak{C}$ *is a natural profinite invariant.*

*Proof.* The pro-$\mathfrak{C}$ completion is left-adjoint to the forgetful functor Pro-$\mathfrak{C} \to$ Group [88, Lemma 3.2.1]. Moreover, pro-$\mathfrak{C}$ groups are compact Hausdorff groups [88, Theorem 2.1.3], and thus in $\mathcal{C}$ by Theorem 5.1.1. So we can apply the above theorem to get the natural profinite invariance. ☐

This result should not be too surprising, as the profinite completion "knows everything about finite quotients" and the pro-$\mathfrak{C}$ completion "knows everything about finite quotients in $\mathfrak{C}$".

One other example where the theorem can be applied is the abelianization.

**Corollary 5.3.3** (Abelianization is a natural profinite invariant)**.** *The abelianization* $(\,\cdot\,)_{\mathrm{ab}}\colon \mathrm{Group} \to \mathrm{Ab}$ *is a natural profinite invariant.*

*Proof.* The abelianization functor is left-adjoint to the forgetful functor Ab $\to$ Group. Moreover, by Theorem 5.1.1, every nilpotent group is in $\mathcal{C}$, so in particular every abelian group is in $\mathcal{C}$. Hence, Theorem 5.3.1 gives the claim. $\square$

**Corollary 5.3.4.** *Let $A$ be an abelian group. Then both $H_1(\,\cdot\,;A)$ and $H^1(\,\cdot\,;A)$ are natural profinite invariants.*

*Proof.* By the universal coefficient theorem for homology, we have a natural isomorphism

$$H_1(\,\cdot\,;A) \cong H_1(\,\cdot\,;\mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

Similarly, the universal coefficient theorem for cohomology gives a natural isomorphism

$$H^1(\,\cdot\,;A) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\,\cdot\,;\mathbb{Z}), A).$$

Since $H_1(\,\cdot\,;\mathbb{Z}) \cong (\,\cdot\,)_{\mathrm{ab}}$ is a natural profinite invariant we get the claim. $\square$

### 5.3.2 The Bohr compactification

Another example, where Theorem 5.3.1 can be applied, is the *Bohr compactification*[2].

To our knowledge, Jaikin-Zapirain and Lubotzky were the first to study the profiniteness of the Bohr compactification. They gave concrete proofs that it is a natural profinite invariant [47, Corollary 1.3] and that it is *not* a profinite invariant [47, Proposition 6.1]. In fact, for our proof of Theorem 5.3.1 we abstracted their concrete proof and generalized it to *all* adjoint functors.

**Definition 5.3.5** (Bohr compactification)**.** Let $G$ be a group. The *Bohr compactification of $G$* is a pair $(\mathrm{Bohr}(G), \beta)$, consisting of a compact Hausdorff group $\mathrm{Bohr}(G)$ and a group homomorphism $\beta\colon G \to \mathrm{Bohr}(G)$, satisfying the following universal property: for every compact Hausdorff group $K$ and every group homomorphism $\alpha\colon G \to K$, there exists a unique continuous group homomorphism $\alpha'\colon \mathrm{Bohr}(G) \to K$ with $\alpha = \alpha' \circ \beta$, i.e. such that the diagram

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & K \\ \mathrm{Bohr}(G) \arrow[ur, dashed, "\alpha'"'] & \end{tikzcd}$$

commutes.

**Theorem 5.3.6** (Existence of Bohr compactification [7, Theorem 4.C.3])**.** *Let $G$ be a group. Then the Bohr compactification* $\mathrm{Bohr}(G)$ *of $G$ exists and is unique up to unique isomorphism.*

[2]Named after mathematician Harald Bohr, who was part of the Danish football team winning the silver medal at the 1908 Olympic Games. (He was also the younger brother of physicist and Nobel Prize winner Niels Bohr.)

*Proof.* As usual, the uniqueness of the Bohr compactification follows from the universal property. Thus, we only need to prove existence.

Let $\big(\pi_i \colon G \to U(V_i)\big)_{i\in I}$ be a family of representatives of all equivalence classes of finite dimensional unitary representations of $G$. We consider

$$\begin{aligned} \beta\colon G &\longrightarrow \prod_{i\in I} U(V_i) \\ g &\longmapsto \big(\pi_i(g)\big)_{i\in I} \end{aligned}$$

and define $\mathrm{Bohr}(G) = \overline{\beta(G)}$. Since the product $\prod_{i\in I} U(V_i)$ of compact Hausdorff groups is again compact Hausdorff we have that $\mathrm{Bohr}(G)$ is a compact Hausdorff group as well.

For the universal property, let $\alpha\colon G \to K$ be a group homomorphism to a compact Hausdorff group $K$ and let $\big(\varrho_j \colon K \to U(U_j)\big)_{j\in J}$ be a family of representatives of all continuous finite dimensional unitary representations of $G$. As above, let us consider the continuous group homomorphism

$$\begin{aligned} K &\longrightarrow \prod_{j\in J} U(U_j) \\ k &\longmapsto \big(\varrho_j(k)\big)_{j\in J}. \end{aligned}$$

By the Peter–Weyl theorem [75], this homomorphism is injective. Since $K$ is compact and $\prod_{j\in J} U(U_i)$ is Hausdorff, we can thus identify $K$ with its image in $\prod_{j\in J} U(U_j)$.

Now let $j_0 \in J$, and denote by $\mathrm{pr}_{j_0} \colon \prod_{j\in J} U(U_j) \to U(U_{j_0})$ the canonical projection. Then the composition $\mathrm{pr}_{j_0} \circ \alpha\colon G \to U(U_{j_0})$ is a finite dimensional representation of $G$ and thus is equivalent to some $\pi_{i(j_0)}$ with $i(j_0) \in I$. Let $\varphi_j \colon V_{i(j_0)} \to U_{j_0}$ be a unitary operator exhibiting this equivalence, i.e. such that

$$\mathrm{pr}_{j_0} \circ \alpha(g) = \varphi_{j_0} \circ \pi_{i(j_0)}(g) \circ \varphi_{j_0}^{-1}$$

for all $g \in G$. We now define

$$\begin{aligned} \alpha' \colon \mathrm{Bohr}(G) &\longrightarrow K \\ (\psi_i)_{i\in I} &\longmapsto (\varphi_j \circ \psi_{i(j)} \circ \varphi_j^{-1})_{j\in J}. \end{aligned}$$

By construction this is a well-defined continuous group homomorphism. Moreover, for $g \in G$ we have

$$\alpha' \circ \beta(g) = \big(\varphi_j \circ \pi_{i(j)}(g) \circ \varphi_j^{-1}\big)_{j\in J} = \big(\mathrm{pr}_j \circ \alpha(g)\big)_{j\in J} = \alpha(g),$$

i.e. $\alpha' \circ \beta = \alpha$.

The uniqueness of $\alpha'$ follows as $\mathrm{im}(\beta)$ is dense in $\mathrm{Bohr}(G)$. □

As the Bohr compactification always exists, it is easy to see that we in fact get a functor $\mathrm{Bohr}\colon \mathrm{Group} \to \mathrm{CHGroup}$ to the category of compact Hausdorff groups. The universal property of the Bohr compactification simply states that this functor is left-adjoint to the forgetful functor $\mathrm{CHGroup} \to \mathrm{Group}$.

**Corollary 5.3.7** (Bohr compactification is a natural profinite invariant)**.** *Denote by* CHGroup *the category of compact Hausdorff groups. The Bohr compactification* Bohr: Group $\to$ CHGroup *is a natural profinite invariant.*

*Proof.* As noted above, Bohr is left-adjoint to the forgetful functor. Moreover, every compact Hausdorff group is in $\mathcal{C}$ by Theorem 5.1.1. So, by Theorem 5.3.1, we get that Bohr is a natural profinite invariant. ☐

In contrast, it turns out that the Bohr compactification is *not* a profinite invariant. In order to show this, we use the following theorem due to Bekka.

**Theorem 5.3.8** ([6, Theorem 3])**.** *Let* $\mathbf{G}$ *be a connected, simply connected, and almost simple* $\mathbb{Q}$*-group. Assume that the real semisimple Lie group* $\mathbf{G}(\mathbb{R})$ *is not locally isomorphic to a group of the form* $\mathrm{SO}(m,1)\times K$ *or* $\mathrm{SU}(m,1)\times K$ *for a compact Lie group* $K$*. Let* $\mathbf{G}_{\mathrm{nc}}$ *be the product of the almost* $\mathbb{R}$*-simple factors* $\mathbf{G}_i$ *of* $\mathbf{G}$ *for which* $\mathbf{G}_i(\mathbb{R})$ *is non-compact. Let* $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$ *be an arithmetic subgroup. We have a direct product decomposition*

$$\mathrm{Bohr}(\Gamma) \cong \mathrm{Bohr}(\Gamma)^0 \times \widehat{\Gamma}$$

*and an isomorphism*

$$\mathrm{Bohr}(\Gamma)^0 \cong {}^{\mathbf{G}(\mathbb{R})}\!/\!_{\mathbf{G}_{\mathrm{nc}}(\mathbb{R})},$$

*induced by the natural maps* $\Gamma \to \mathbf{G}(\mathbb{R})/\mathbf{G}_{\mathrm{nc}}(\mathbb{R})$ *and* $\Gamma \to \widehat{\Gamma}$*.*

**Proposition 5.3.9** ([47, Proposition 6.1])**.** *Consider the following two quadratic forms*

$$\begin{aligned} & q_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 - \sqrt{2}x_7^2 - \sqrt{2}x_8^2 \\ \text{and} \quad & q_2 = x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - \sqrt{2}x_7^2 - \sqrt{2}x_8^2. \end{aligned}$$

*Then there exist finite-index subgroups* $\Gamma \leq \mathrm{Spin}(q_1)(\mathbb{Z}[\sqrt{2}])$ *and* $\Lambda \leq \mathrm{Spin}(q_2)(\mathbb{Z}[\sqrt{2}])$*, with* $\widehat{\Gamma} \cong \widehat{\Lambda}$*, while* $\mathrm{Bohr}(\Gamma) \not\cong \mathrm{Bohr}(\Lambda)$*.*

*Proof.* By considering $\mathrm{Spin}(q_i)$ as algebraic groups over $\mathbb{Q}(\sqrt{2})$ and restricting scalars to $\mathbb{Q}$ we obtain almost simple algebraic $\mathbb{Q}$-groups $\mathbf{G}_i$.

Similarly to the proof of Corollary 3.2.4, we can use Lemma 3.2.3 to flip four signs simultaneously over every $\mathbb{Z}_p[\sqrt{2}]$ in order to see that $\mathbf{G}_1(\mathbb{Z}) = \mathrm{Spin}(q_1)(\mathbb{Z}[\sqrt{2}])$ and $\mathbf{G}_2(\mathbb{Z}) = \mathrm{Spin}(q_2)(\mathbb{Z}[\sqrt{2}])$ have isomorphic congruence completions. Since the $\mathbb{Q}$-rank of both $\mathbf{G}_1$ and $\mathbf{G}_2$ is greater than 2, we have the congruence subgroup property by Theorem 2.2.38. Thus, we can apply Proposition 3.2.2 in order to get finite-index subgroups $\Gamma \leq \mathrm{Spin}(q_1)(\mathbb{Z}[\sqrt{2}])$ and $\Lambda \leq \mathrm{Spin}(q_2)(\mathbb{Z}[\sqrt{2}])$, with $\widehat{\Gamma} \cong \widehat{\Lambda}$.

To compute the Bohr compactification of $\Gamma$ and $\Lambda$ we make use of Theorem 5.3.8. The real points of $\mathbf{G}_1$ and $\mathbf{G}_2$ are given by

$$\begin{aligned} \mathbf{G}_1(\mathbb{R}) &\cong \mathrm{Spin}(6,2)(\mathbb{R}) \times \mathrm{Spin}(8,0)(\mathbb{R}) \\ \mathbf{G}_2(\mathbb{R}) &\cong \mathrm{Spin}(2,6)(\mathbb{R}) \times \mathrm{Spin}(4,4)(\mathbb{R}), \end{aligned}$$

induced by the two embeddings $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{R}$, $\sqrt{2} \mapsto \sqrt{2}$ and $\sqrt{2} \mapsto -\sqrt{2}$. Here the only compact factor is $\mathrm{Spin}(8,0)$ and thus Theorem 5.3.8 gives

$$\mathrm{Bohr}(\Gamma)^0 \cong \mathrm{Spin}(8,0)(\mathbb{R}) \qquad \text{and} \qquad \mathrm{Bohr}(\Lambda)^0 \cong 0.$$

So $\mathrm{Bohr}(\Gamma) \not\cong \mathrm{Bohr}(\Lambda)$, proving the claim. □

**Remark 5.3.10.** The fact that there is a natural profinite invariant $F\colon \mathrm{Group} \to \mathcal{D}$ which is *not* a profinite invariant allows us to give another such example: the localization functor $L_S\colon \mathrm{Group} \to \mathrm{Group}[S^{-1}]$ at the class of Grothendieck pairs (Remark 5.1.4).

If $L_S$ was a profinite invariant, then the factorization of $F$ through $L_S$ would give that $F$ is also a profinite invariant. But this is not the case.

$$\begin{tikzcd} \mathrm{Group} \arrow[r, "F"] \arrow[d, "L_S"'] & \mathcal{D} \\ \mathrm{Group}[S^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

The list of examples we have seen can be summarized in the following table.

| Functor | profinite invariant | natural profinite invariant |
|---|---|---|
| $\widehat{\cdot}\colon \mathrm{Group} \to \mathrm{Group}$ | ✓ | ✓ |
| $(\,\cdot\,)_{\mathrm{ab}}\colon \mathrm{Group} \to \mathrm{Ab}$ | ✓ | ✓ |
| $\mathrm{id}\colon \mathrm{Group} \to \mathrm{Group}$ | ✗ | ✗ |
| $H^2_b(\,\cdot\,;\mathbb{R})\colon \mathrm{Group} \to \mathrm{Vect}_{\mathbb{R}}$ | ✗ | ✗ |
| $EH^2_b(\,\cdot\,;\mathbb{R})\colon \mathrm{Group} \to \mathrm{Vect}_{\mathbb{R}}$ | ✗ | ✗ |
| $H_{n\geq 2}(\,\cdot\,;\mathbb{Z})\colon \mathrm{Group} \to \mathrm{Ab}$ | ✗ | ✗ |
| $\mathrm{Forget}\colon \mathrm{Group} \to \mathrm{Set}$ | ✓ | ✗ |
| $\mathrm{Bohr}\colon \mathrm{Group} \to \mathrm{CHGroup}$ | ✗ | ✓ |
| $L_S\colon \mathrm{Group} \to \mathrm{Group}[S^{-1}]$ | ✗ | ✓ |

# Appendix A

# Cohomology

Both homology and cohomology proved to be invaluable tools in various fields of mathematics. From group cohomology, singular cohomology and cellular cohomology to $L^2$-cohomology, Čech cohomology, sheaf cohomology and various kinds of $K$-theory, without cohomology many parts of modern mathematics would not be possible in their current form.

This thesis is no exception. Large parts revolve around two kinds of cohomology: *bounded cohomology* and *Galois cohomology*. Despite neither being a cohomology theory in the classical sense, they are nonetheless interesting to study.

Bounded cohomology in its current form was introduced by Gromov in a paper where he studied minimal and simplicial volume [36]. Later on, the theory was generalized by Ivanov [45] and Noskov [73] to allow for twisted coefficients.

A priori, Galois cohomology is just group cohomology of the Galois group of a Galois extension. The first Galois cohomology $H^1(K|k; \operatorname{Aut}\mathbf{G})$ can be used to classify the $k$-forms of an algebraic $k$-group $\mathbf{G}$, as seen for example in Section 1.1.5. However, as the automorphism group of $\mathbf{G}$ is in general not abelian, we first need to define *non-abelian* cohomology.

Since the definition of both bounded and non-abelian cohomology is based on ordinary group cohomology, we start with a brief recollection thereof.

### Overview of this appendix

## A.1 Group cohomology

We will only give the definition of group cohomology and compute examples in degree zero and one. For a thorough treatment of both group homology and group cohomology, we refer the reader to the book of Brown [16].

**Definition A.1.1** (Bar resolution)**.** Let $G$ be a topological group and let $A$ be an abelian group with a continuous action $G \curvearrowright A$. The *bar resolution of $G$ with coefficients in $A$* is the following cochain complex:

- The cochain modules are
$$C^n(G;A) := \operatorname{Hom}_{\mathrm{Top}}(G^n, A).$$
- For $\varphi \in C^n(G;A)$ the coboundary $\delta^n\varphi$ is given by
$$\begin{aligned}\delta^n\varphi(g_1,\dots,g_{n+1}) = {} & g_1 \cdot \varphi(g_2,\dots,g_{n+1}) \\ & + \sum_{i=1}^{n}(-1)^i\varphi(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) \\ & + (-1)^{n+1}\varphi(g_1,\dots,g_n).\end{aligned}$$

**Definition A.1.2** (Group cohomology)**.** Let $G$ be a topological group acting continuously on an abelian group $A$. Then the *(continuous) group cohomology of $G$ with coefficients in $A$* is defined as the cohomology of the corresponding bar resolution

$$H^n(G;A) := \frac{\ker\left(\delta^n \colon C^n(G;A) \to C^{n+1}(G;A)\right)}{\operatorname{im}\left(\delta^{n-1} \colon C^{n-1}(G;A) \to C^n(G;A)\right)}.$$

Since $A$ is an abelian group, there is a canonical group structure on $H^n(G;A)$ inherited from the group structure on $C^n(G;A)$.

**Example A.1.3** (Cohomology in degree 0)**.** It is easy to see that we have a canonical isomorphism $C^0(G;A) \cong A$, such that for $a \in A$ the coboundary operator is given by $\delta^0 a(g) = g \cdot a - a$. Since $\delta^{-1}$ is trivial, this gives

$$H^0(G;A) = \ker\delta^0 = \{a \in A \mid \forall_{g\in G} \colon ga = a\} = A^G.$$

**Example A.1.4** (Cohomology in degree 1)**.** Let $\varphi \in C^1(G;A) = \operatorname{Hom}_{\mathrm{Top}}(G,A)$. Then we have
$$\delta^1\varphi(g,h) = g \cdot \varphi(h) - \varphi(gh) + \varphi(g).$$

So we get
$$\ker\delta^1 = \{\varphi \in \operatorname{Hom}_{\mathrm{Top}}(G,A) \mid \varphi(gh) = \varphi(g) + g \cdot \varphi(h)\},$$

and hence
$$H^1(G;A) = \frac{\{\varphi \in \operatorname{Hom}_{\mathrm{Top}}(G,A) \mid \varphi(gh) = \varphi(g) + g \cdot \varphi(h)\}}{\{\varphi \in \operatorname{Hom}_{\mathrm{Top}}(G,A) \mid \exists_{a\in A} \colon \varphi(g) = g \cdot a - a\}}.$$

Thus, two cocycles $\varphi, \psi \in \ker \delta^1$ represent the same element in $H^1(G; A)$ if and only if there exists some $a \in A$ with

$$\psi(g) - \varphi(g) = g \cdot a - a,$$

or equivalently

$$\psi(g) = -a + \varphi(g) + g \cdot a.$$

In particular, if the group action $G \curvearrowright A$ is trivial we obtain

$$H^1(G; A) = \{\varphi \in \mathrm{Hom}_{\mathrm{Top}}(G, A) \mid \varphi(gh) = \varphi(g) + \varphi(h)\} = \mathrm{Hom}_{\mathrm{TopGroup}}(G, A).$$

**Remark A.1.5** (Functoriality)**.** Using our definition of the bar resolution, it is easy to see that post-/precomposing yields covariant functoriality in the coefficients and contravariant functoriality in the group.

## A.2 Bounded cohomology

Bounded cohomology is a variant of group cohomology, where instead of *all* cocycles only *uniformly bounded* cocycles are considered. In order to talk about boundedness, the coefficients need to be endowed with a (semi-)norm. For the sake of simplicity, we will only consider bounded cohomology of discrete groups with trivial real coefficients. A more complete picture of the theory can be found in the books by Monod [68] and Frigerio [32].

**Definition A.2.1** (Bounded bar resolution)**.** Let $G$ be a discrete group. The *bounded bar resolution of $G$* is the following cochain complex:

- The cochain modules are

$$C_b^n(G; \mathbb{R}) = \ell^\infty(G^n, \mathbb{R}).$$

- For $\varphi \in C_b^n(G; \mathbb{R})$ the coboundary $\delta_b^n \varphi$ is given by the same formula as before:

$$\begin{aligned} \delta_b^n \varphi(g_1, \dots, g_{n+1}) &= \varphi(g_2, \dots, g_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i \varphi(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &\quad + (-1)^{n+1} \varphi(g_1, \dots, g_n). \end{aligned}$$

**Definition A.2.2** (Bounded cohomology)**.** Let $G$ be a group. The *bounded cohomology of $G$* is defined as the cohomology of the associated bounded bar resolution

$$H_b^n(G; A) = \frac{\ker\left(\delta_b^n : C_b^n(G; A) \to C_b^{n+1}(G; A)\right)}{\mathrm{im}\left(\delta_b^{n-1} : C_b^{n-1}(G; A) \to C_b^n(G; A)\right)}.$$

**Remark A.2.3** (Bounded cohomology as semi-normed vector spaces)**.** Using the normed structure of the coefficients, the bounded cohomology groups carry some more information.

Using the $\ell^\infty$-norm, the cochain modules $C_b^n(G;\mathbb{R}) = \ell^\infty(G^n, R)$ have a canonical structure as normed vector spaces. On the bounded cohomology, this induces the structure of a *semi-normed* vector space. Sometimes more insight can be obtained by not only studying the bounded cohomology groups, but also the semi-norm on them. However, we will ignore this additional structure for the most part.

**Example A.2.4** (Bounded cohomology in degree 0)**.** As in the unbounded case, there is a canonical isomorphism $C_b^0(G;\mathbb{R}) \cong \mathbb{R}$, such that the differential is given by $\delta_b^0 a(g) = a - a = 0$ for all $a \in \mathbb{R}$. Thus, we have

$$H_b^0(G;\mathbb{R}) = \ker \delta_b^0 \cong \mathbb{R}.$$

**Example A.2.5** (Bounded cohomology in degree 1)**.** The same computation as in Example A.1.4 shows that

$$H_b^1(G;\mathbb{R}) = \{\varphi \in \ell^\infty(G,\mathbb{R}) \mid \varphi(gh) = \varphi(g) + \varphi(h)\}.$$

As every group homomorphism to $\mathbb{R}$ is unbounded, we thus get $H_b^1(G;\mathbb{R}) = 0$.

**Remark A.2.6** (Functoriality)**.** As in the case of group cohomology, we have that bounded cohomology is a functor, which is contravariant in the group.

### A.2.1 Exact bounded cohomology and quasi-morphisms

By construction, the bounded bar resolution is a subcomplex of the usual bar resolution (where $G$ acts trivially on $\mathbb{R}$). Thus, the inclusion induces a map

$$c^n \colon H_b^n(G;\mathbb{R}) \longrightarrow H^n(G;\mathbb{R}),$$

from bounded cohomology to group cohomology. This map is called the *n-th comparison map*.

The computations of $H^1(G;\mathbb{R}) = \mathrm{Hom}_{\mathrm{Group}}(G,\mathbb{R})$ and $H_b^1(G;\mathbb{R}) = 0$ in Example A.1.4 and Example A.2.5, respectively, already show that the comparison map is in general *not* surjective. Below we will also see that in general it is *not* injective either.

**Definition A.2.7** (Exact bounded cohomology)**.** Let $G$ be a group. The *exact bounded cohomology of $G$* is defined to be the kernel of the comparison map,

$$EH_b^n(G;\mathbb{R}) = \ker(c^n \colon H_b^n(G;\mathbb{R}) \to H^n(G;\mathbb{R})).$$

**Definition A.2.8** (Quasi-morphism)**.** Let $G$ be a group. A *quasi-morphism of $G$* is a map $\varphi \colon G \to \mathbb{R}$ such that

$$D(\varphi) := \sup_{g,h\in G} \{\varphi(gh) - \varphi(g) - \varphi(h)\} < \infty.$$

The constant $D(\varphi)$ is called the *defect of $\varphi$*. We denote the space of quasi-morphisms of $G$ by $\mathrm{QM}(G)$.

A quasi-morphism $\varphi$ is called *homogeneous* if it is $\mathbb{Z}$-linear, i.e. $\varphi(g^n) = n\varphi(g)$ for all $g \in G$ and $n \in \mathbb{Z}$. We denote the subspace of homogeneous quasi-morphisms by $\widetilde{\mathrm{QM}}(G)$.

Using the triangle inequality, it is easy to see that the $\mathrm{QM}(G)$ and $\widetilde{\mathrm{QM}}(G)$ are actually real vector spaces. Moreover, we have the inclusions

$$\mathrm{Hom}_{\mathrm{Group}}(G, \mathbb{R}) \subseteq \widetilde{\mathrm{QM}}(G), \qquad \ell^\infty(G, \mathbb{R}) \subseteq \mathrm{QM}(G).$$

**Proposition A.2.9** (Homogenization of quasi-morphisms)**.** *Let $G$ be a group and let $\varphi\colon G \to \mathbb{R}$ be a quasi-morphism. Then there is a unique homogeneous quasi-morphism $\widetilde{\varphi}$ such that $\widetilde{\varphi} - \varphi \in \ell^\infty(G, \mathbb{R})$. This quasi-morphism is given by*

$$\widetilde{\varphi}(g) = \lim_{n\to\infty} \frac{\varphi(g^n)}{n}.$$

*Proof.* We begin by proving uniqueness. Let $\psi, \psi'$ be homogeneous quasi-morphisms such that $\psi - \varphi, \psi' - \varphi \in \ell^\infty(G, \mathbb{R})$. Then for $g \in G$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
|\psi(g) - \psi'(g)| &= \frac{1}{n}|\psi(g^n) - \psi'(g^n)| \\
&= \frac{1}{n}|\psi(g^n) - \varphi(g^n) + \varphi(g^n) - \psi'(g^n)| \\
&\le \frac{1}{n}\left(|\psi(g^n) - \varphi(g^n)| + |\psi'(g^n) - \varphi(g^n)|\right) \\
&\le \frac{1}{n}\left(|\psi - \varphi|_\infty + |\psi' - \varphi|_\infty\right).
\end{aligned}$$

As the right-hand side tends to 0 as $n \to \infty$, we obtain $\psi = \psi'$.

For the explicit description of $\widetilde{\varphi}$, we first prove the existence of the limit. Let $g \in G$ and $n \in \mathbb{N}$. Applying the quasi-morphism condition iteratively we have

$$|\varphi(g^n) - n\varphi(g)| \le (n-1)D(\varphi) \le nD(\varphi). \tag{$*$}$$

Now, for $n, m \in \mathbb{N}$ we get

$$\begin{aligned}
\left|\frac{\varphi(g^n)}{n} - \frac{\varphi(g^m)}{m}\right| &= \left|\frac{\varphi(g^n)}{n} - \frac{\varphi(g^{mn})}{mn} + \frac{\varphi(g^{mn})}{mn} = \frac{\varphi(g^m)}{m}\right| \\
&\le \frac{1}{n}\left|\varphi(g^n) - \frac{\varphi(g^{mn})}{m}\right| + \frac{1}{m}\left|\varphi(g^m) - \frac{\varphi(g^{mn})}{n}\right| \\
&\le \frac{1}{n}D(\varphi) + \frac{1}{m}D(\varphi).
\end{aligned}$$

So $\varphi(g^n)/n$ is a Cauchy sequence which converges due to the completeness of $\mathbb{R}$. Thus dividing $(*)$ by $n$ and letting $n$ tend to infinity shows $|\widetilde{\varphi} - \varphi|_\infty \le D(\varphi)$, giving that $\widetilde{\varphi} - \varphi \in \ell^\infty(G, \mathbb{R})$. This also proves that $\widetilde{\varphi}$ is the sum of a quasi-morphism and a uniformly bounded map, thus itself a quasi-morphism. Finally, by construction $\widetilde{\varphi}$ is $\mathbb{Z}$-linear as

$$\widetilde{\varphi}(g^n) = \lim_{m\to\infty} \frac{\varphi(g^{mn})}{m} = n \lim_{m\to\infty} \frac{\varphi(g^{mn})}{mn} = n\widetilde{\varphi}(g).$$

So $\widetilde{\varphi}$ is indeed a homogeneous quasi-morphism with $\widetilde{\varphi} - \varphi \in \ell^\infty(G, \mathbb{R})$. ∎

The definition of a quasi-morphism states precisely that $\delta^1\varphi\colon G^2 \to \mathbb{R}$ is uniformly bounded. Since the differentials used to define bounded cohomology and group cohomology agree, we obviously have $\delta_b^2\delta^1\varphi = 0$. So $\delta^1\varphi$ is, in fact, a bounded cocycle. Moreover, by construction, it is a coboundary (in group cohomology) and thus trivial as an element in $H^2(G;\mathbb{R})$. Hence, we obtain a well-defined linear map

$$\begin{aligned} \Theta\colon \operatorname{QM}(G) &\longrightarrow EH_b^2(G;\mathbb{R}) \\ \varphi &\longmapsto [\delta^1\varphi]. \end{aligned}$$

**Proposition A.2.10.** *Let $G$ be a group. The map $\Theta$ induces an isomorphism*

$$EH_b^2(G;\mathbb{R}) \cong \frac{\operatorname{QM}(G)}{\operatorname{Hom}_{\mathrm{Group}}(G,\mathbb{R}) \oplus \ell^\infty(G,\mathbb{R})}.$$

*Proof.* We first prove that $\Theta$ is surjective. Let $\varphi \in \ell^\infty(G^2,\mathbb{R})$ be a cocycle representing an element in $EH_b^2(G;\mathbb{R})$. As $\varphi$ becomes trivial in $H^2(G;\mathbb{R})$, there is $\psi \in \operatorname{Hom}_{\mathrm{Set}}(G,\mathbb{R})$ with $\varphi = \delta^1\psi$. Since $\varphi$ is uniformly bounded, this shows $\psi \in \operatorname{QM}(G)$ and thus $\varphi = \Theta(\psi)$. Hence, $\Theta$ is indeed surjective.

Using the isomorphism theorem, it now remains to show that

$$\ker\Theta = \operatorname{Hom}_{\mathrm{Group}}(G,\mathbb{R}) \oplus \ell^\infty(G,\mathbb{R}).$$

Note that, as there are no non-trivial uniformly bounded group homomorphisms the right-hand side is actually a direct sum of subspaces in $\operatorname{QM}(G)$.

Now for $\varphi \in \operatorname{Hom}_{\mathrm{Group}}(G,\mathbb{R})$ we have $\delta^1\varphi = 0$. For $\varphi \in \ell^\infty(G,\mathbb{R})$ we have that $\delta^1\varphi = \delta_b^1\varphi$ is already a *bounded* coboundary, and thus trivial in $H_b^2(G;\mathbb{R})$. So obviously the inclusion "$\supseteq$" holds.

Conversely, let $\varphi \in \ker\Theta$. As $[\delta^1\varphi]$ is trivial in $H_b^2(G;\mathbb{R})$, there is $\psi \in \ell^\infty(G,\mathbb{R})$ such that $\delta^1\varphi = \delta_b^1\psi$. However, then $\varphi - \psi$ is a 1-cocycle, i.e. a group homomorphism, and so we get $\varphi \in \operatorname{Hom}_{\mathrm{Group}}(G,\mathbb{R}) \oplus \ell^\infty(G,\mathbb{R})$. ☐

**Corollary A.2.11.** *Let $G$ be a group. Then there is a canonical isomorphism*

$$EH_b^2(G;\mathbb{R}) \cong \frac{\widetilde{\operatorname{QM}}(G)}{\operatorname{Hom}_{\mathrm{Group}}(G,\mathbb{R})}.$$

*Proof.* As every non-trivial homogeneous quasi-morphisms is unbounded, we get from Proposition A.2.9 that $\operatorname{QM}(G) = \widetilde{\operatorname{QM}}(G) \oplus \ell^\infty(G,\mathbb{R})$. Combining this with the previous proposition, we get the claim. ☐

For free groups, it is easy to construct quasi-morphisms. One way of construction are so-called *counting quasi-morphisms* due to Brooks [15, Section §3].

**Example A.2.12** (Counting quasi-morphisms [18, Section 2.3.2])**.** Let $F$ be a free group on a symmetric generating set $S$ and let $w$ be a reduced word in $S$. We define the *counting function* $C_w\colon F \to \mathbb{N}$ to be

$$C_w(g) = \text{ number of copies of } w \text{ in the reduced word representing } g$$

and the *counting quasi-morphism* $H_w \colon F \to \mathbb{R}$ to be

$$H_w(g) = C_w(g) - C_{w^{-1}}(g).$$

One can check that this is actually a quasi-morphism.

Another class of examples is given by so-called *split quasi-morphisms* due to Rolli [90].

**Example A.2.13** (Split quasi-morphisms [32, Section 2.6])**.** Let $F_2$ be a free group on two generators $\{a, b\}$. For

$$\alpha \in \ell^\infty_{\mathrm{odd}}(\mathbb{Z}, \mathbb{R}) = \{\varphi \in \ell^\infty(\mathbb{Z}, \mathbb{R}) \mid \varphi(-n) = -\varphi(n) \text{ for all } n \in \mathbb{Z}\}$$

we define $f_\alpha \colon F_2 \to \mathbb{R}$ by

$$f_\alpha(g) = f_\alpha(s_1^{n_1} \dots s_k^{n_k}) = \sum_{i=1}^{k} \alpha(n_k),$$

where $s_1^{n_1} \dots s_k^{n_k}$ is the (unique!) word in $\{a, b\}$ representing $g$.

Again, it can be shown that this is indeed a quasi-morphism. Moreover, one can prove that the map $\ell^\infty_{\mathrm{odd}}(\mathbb{Z}, \mathbb{R}) \to EH_b(F_2; \mathbb{R})$, given by $\alpha \mapsto [\delta^1(f_\alpha)]$, is injective.

The above shows, in particular, that $EH^2_b(F_2; \mathbb{R})$ is infinite dimensional. So we see that the comparison map is in general *not* injective.

More generally, the infinite dimensionality of $EH^2_b(G; \mathbb{R})$ is known for (non-elementary) hyperbolic groups [29], groups acting on hyperbolic space [33], amalgamated free products of groups [34], as well as for acylindrically hyperbolic groups [38, Theorem A].

### A.2.2 Stable commutator length

For the sake of completeness, we also want to mention the connection between quasi-morphisms and *stable commutator length*. We will, however, not go into details and instead refer the interested reader to the book of Calegari [18, Chapter 2].

**Definition A.2.14** ((Stable) commutator length)**.** Let $G$ be a group. We define the *commutator length* of $g \in [G, G]$ to be

$$\mathrm{cl}_G(g) = \min\{n \in \mathbb{N} \mid g = [a_1, b_1] \dots [a_n, b_n] \text{ for some } a_i, b_i \in G\}.$$

Moreover, we define the *stable commutator length* of $g \in [G, G]$ to be

$$\mathrm{scl}_G(g) = \lim_{n \to \infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

Often the convention $\mathrm{cl}_G(g) = \infty$ is used if $g \notin [G, G]$.

Computing commutator lengths is usually difficult. For computing *stable* commutator lengths, the following can be used.

**Theorem A.2.15** (Bavard duality [18, Theorm 2.70])**.** *Let $G$ be a group. Then for any $g \in [G,G]$ we have*

$$\mathrm{scl}_G(g) = \frac{1}{2} \sup_{[\varphi]\in\widetilde{\mathrm{QM}}(G)\setminus\mathrm{Hom}_{\mathrm{Group}}(G,\mathbb{R})} \frac{|\varphi(g)|}{D(\varphi)}.$$

**Corollary A.2.16.** *Let $G$ be a group. Then $G$ has no non-trivial quasi-morphisms, i.e. $EH_b^2(G;\mathbb{R}) = 0$ if and only if $\mathrm{scl}_G(g) = 0$ for all $g \in [G,G]$.*

## A.3 Non-abelian cohomology

In Section A.1, we defined the cohomology *groups* $H^n(G;A)$. This was only possible due to $A$ being an abelian group. Now, we want to weaken the assumptions on our coefficients $A$. Some more details on non-abelian cohomology, for example the approach via homogeneous spaces that we omit here, can be found in Serre's book on Galois cohomology [95, Chapter I, §5].

**Definition A.3.1** ($G$-set, $G$-group)**.** Let $G$ be a topological group. A *$G$-set* is a set $E$ together with a continuous $G$-action $G \curvearrowright E$ (where $E$ is endowed with the discrete topology).

A *$G$-group* is a group $A$ together with a continuous $G$-action $G \curvearrowright A$ by group homomorphisms (where, again, $A$ is endowed with the discrete topology).

We now use the computations in the above examples as definitions for non-abelian cohomology.

**Definition A.3.2** (Non-abelian cohomology)**.** Let $G$ be a topological group and let $E$ be a $G$-set. We define the *zeroth non-abelian cohomology of $G$ with coefficients in $E$* to be

$$H^0(G;E) = E^G.$$

For a $G$-group $A$, we define the space of cocycles to be

$$Z^1(G;A) = \{\varphi\colon G \to A \text{ continuous} \mid \varphi(gh) = \varphi(g)g \cdot \varphi(h)\}.$$

Two cocycles $\varphi, \psi \in Z^1(G;A)$ are called *cohomologous* if there exists an $a \in A$ with

$$\psi(g) = a^{-1}\varphi(g)g \cdot a.$$

Being cohomologous defines an equivalence relation $\sim$ on $Z^1(G;A)$, and we define the *first non-abelian cohomology of $G$ with coefficients in $A$* to be

$$H^1(G;A) = Z^1(G;A)/\sim.$$

If $E$ is a $G$-set, then $H^0(G;E)$ is in general just a set. If $A$ is a $G$-group, then both $H^0(G;A)$ and $H^1(G;A)$ are pointed sets, with base point $e$ and $\mathrm{const}_e$ respectively.

In case $A$ is an abelian group, by construction, the non-abelian cohomology agrees with usual group cohomology.

**Example A.3.3.** Let $\varphi \in Z^1(G; A)$ be a cocycle.

- We have
$$\varphi(e) = \varphi(ee) = \varphi(e)e \cdot \varphi(e) = \varphi(e)\varphi(e),$$
and thus $\varphi(e) = e$.

- Moreover, we have
$$e = \varphi(e) = \varphi(g^{-1}g) = \varphi(g^{-1})g^{-1} \cdot \varphi(g),$$
and thus $\varphi(g^{-1}) = \big(g^{-1} \cdot \varphi(g)\big)^{-1}$.

### A.3.1 Functoriality

Let $G$ and $H$ be groups, let $A$ be a $G$-group and $B$ be an $H$-group. Moreover, let $\alpha\colon H \to G$ and $\beta\colon A \to B$ be group homomorphisms that are compatible in the sense that
$$\beta\big(\alpha(h) \cdot a\big) = h \cdot \beta(a)$$
for all $h \in H$ and $a \in A$.

Then we also obtain maps in cohomology
$$\begin{aligned} H^0(\alpha; \beta)\colon H^0(G; A) = A^G &\longrightarrow B^H = H^0(H; B) \\ a &\longmapsto \beta(a) \end{aligned}$$
and
$$\begin{aligned} H^1(\alpha; \beta)\colon H^1(G; A) &\longrightarrow H^1(G'; A') \\ [\varphi] &\longmapsto [\beta \circ \varphi \circ \alpha]. \end{aligned}$$

Let us check that $\varphi \mapsto \beta \circ \varphi \circ \alpha$ preserves being a cocycle and being cohomologous. First we have
$$\begin{aligned} \beta \circ \varphi \circ \alpha(gh) &= \beta \circ \varphi\big(\alpha(g)\alpha(h)\big) && (\alpha \text{ is a group homomorphism}) \\ &= \beta\Big(\varphi\big(\alpha(g)\big)\alpha(g) \cdot \varphi\big(\alpha(h)\big)\Big) && (\varphi \text{ is a cocycle}) \\ &= \beta \circ \varphi \circ \alpha(g)\beta\Big(\alpha(g) \cdot \varphi\big(\alpha(h)\big)\Big) && (\beta \text{ is a group homomorphism}) \\ &= \beta \circ \varphi \circ \alpha(g)g \cdot \beta \circ \varphi \circ \alpha(h), && (\text{compatibility of } \alpha \text{ and } \beta) \end{aligned}$$
so being a cocycle is preserved. Now, for two cohomologous cocycles $\varphi, \psi \in Z^1(G; A)$, there exists $a \in A$ with
$$\psi(g) = a^{-1}\varphi(g)g \cdot a.$$

Thus,

$$\begin{aligned}\beta \circ \psi \circ \alpha(h) &= \beta \circ \psi(\alpha(h)) \\ &= \beta\Big(a^{-1}\varphi(\alpha(h))\alpha(h) \cdot a\Big) \\ &= \beta(a)^{-1}\beta \circ \varphi \circ \alpha(h)\beta(\alpha(h) \cdot a)) && (\beta \text{ is a group homomorphism}) \\ &= \beta(a)^{-1}\beta \circ \varphi \circ \alpha(h)h \cdot \beta(a), && (\text{compatibility of } \alpha \text{ and } \beta)\end{aligned}$$

and so being cohomologous is also preserved.

By construction, it is easy to see that this construction is functorial in both $\alpha$ and $\beta$.

**Example A.3.4.** Let $G$ be a group, let $A$ be a $G$-group and let $h \in G$. We consider the group homomorphisms

$$\begin{aligned} c_h \colon G &\longrightarrow G & \qquad t_h \colon A &\longrightarrow A \\ g &\longmapsto hgh^{-1} & a &\longmapsto h^{-1} \cdot a. \end{aligned}$$

These morphisms are compatible as

$$h^{-1} \cdot ((hgh^{-1}) \cdot a) = g \cdot (h^{-1} \cdot a),$$

and thus induce maps $H^n(c_h; t_h)$ in cohomology.

In degree 0 we have for $a \in H^0(G; A)$ that

$$H^0(c_h; t_h)(a) = t_h(a) = h^{-1}a = a$$

as $H^0(G; A) = A^G$, so $H^0(c_h; t_h) = \mathrm{id}$. Moreover, for a cocycle $\varphi \in Z^1(G; A)$ we have

$$\begin{aligned} t_h \circ \varphi \circ c_h(g) &= h^{-1} \cdot (\varphi(hgh^{-1}) \\ &= h^{-1} \cdot (\varphi(hg)(hg) \cdot \varphi(h^{-1})) && (\varphi \text{ is a cocycle}) \\ &= h^{-1} \cdot (\varphi(h)h \cdot \varphi(g)(hg) \cdot \varphi(h^{-1})) && (\varphi \text{ is a cocycle}) \\ &= h^{-1} \cdot \varphi(h)\varphi(g)g \cdot \varphi(h^{-1}) \\ &= (h^{-1} \cdot \varphi(h))\varphi(g)g \cdot (h^{-1} \cdot \varphi(h))^{-1}, && (\text{by Example A.3.3}) \end{aligned}$$

i.e. the image of $\varphi$ is cohomologous to $\varphi$, and so we also have $H^1(c_h; t_h) = \mathrm{id}$.

### A.3.2 Exact sequences

Let $B$ be a $G$-group and let $A \leq B$ be a $G$-subgroup, i.e. a subgroup of $B$ invariant under the $G$-action. Then the quotient $B/A$ is a pointed $G$-set, and thus $H^0(G; B/A) = (B/A)^G$ is a pointed space.

Now let $b \in B$ such that $bA \in (B/A)^G$. Then, for every $g \in G$, there exists $a \in A$ such that

$$g \cdot b = ba, \qquad \text{or written differently} \qquad b^{-1}g \cdot b = a \in A.$$

In fact, defining $\varphi(g) = b^{-1}g \cdot b$ gives a cocycle: for $g, h \in G$ we have

$$\varphi(gh) = b^{-1}(gh) \cdot b = b^{-1}g \cdot bg \cdot (b^{-1}h \cdot b) = \varphi(g)g \cdot \varphi(h).$$

Moreover, for two representatives of $bA$, the two cocycles are cohomologous. For $a \in A$ we have

$$(ba)^{-1}g \cdot (ba) = a^{-1}b^{-1}g \cdot bg \cdot a = a^{-1}\varphi(g)g \cdot a.$$

Thus, we obtain a well-defined map $\delta\colon (B/A)^G \to H^1(G; A)$.

**Proposition A.3.5.** *Let $B$ be a $G$-group and let $A \leq B$ be a $G$-subgroup. Then*

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow (B/A)^G \xrightarrow{\delta} H^1(G; A) \longrightarrow H^1(G; B)$$

*is an exact sequence of pointed sets, meaning that for every map the preimage of the base point agrees with the image of the previous map.*

*Proof.* Since $A^G \to B^G$ is simply the inclusion, exactness at $A^G$ is clear. Also, the exactness at $B^G$ is easy to see.

Now let $b \in B$ with $bA \in (B/A)^G$. Then $\varphi(g) = b^{-1}g \cdot b$ is cohomologous to $\mathrm{const}_e$ if and only if there exists $a \in A$ with

$$e = a^{-1}\varphi(g)g \cdot a = a^{-1}b^{-1}g \cdot bg \cdot a = (ba)^{-1}g \cdot (ba),$$

which is the case if and only if $ba \in B^G$. Hence, the sequence is exact at $(B/A)^G$.

Finally, a cocycle $\varphi \in Z^1(G; A)$ is cohomologous to $\mathrm{const}_e$ in $Z^1(G; B)$ if and only if there is $b \in B$ such that

$$\varphi(g) = b^{-1}g \cdot b, \qquad \text{or equivalently} \qquad g \cdot b = b \cdot \underbrace{\varphi(a)}_{\in A}.$$

This is the case if and only if $bA \in (B/A)^G$, and thus shows exactness at $H^1(G; A)$. □

If $A \leq B$ is moreover a *normal* $G$-subgroup, the quotient $B/A$ has a $G$-group structure, and we can extend the exact sequence by one more term.

**Proposition A.3.6.** *Let $B$ be a $G$-group and let $A \trianglelefteq B$ be a normal $G$-subgroup. Then*

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow (B/A)^G \xrightarrow{\delta} H^1(G; A) \longrightarrow H^1(G; B) \longrightarrow H^1(G; B/A)$$

*is an exact sequence of pointed sets.*

*Proof.* Let $\varphi \in Z^1(G; B)$ be a cocycle that is cohomologous to $\mathrm{const}_A$ in $Z^1(G; B/A)$. Then there exists $b \in B$ such that

$$b^{-1}\varphi(g)g \cdot bA = A,$$

i.e. $\varphi$ is cohomologous, in $Z^1(G; B)$ to a cocycle in $Z^1(G; A)$ so the sequence is exact at $H^1(G; B)$. □

Finally, suppose that $A \leq B$ is a $G$-subgroup contained in the center

$$Z(B) = \{a \in B \mid \forall_{b \in B} \colon ab = ba\}.$$

Then $A$ is in particular an abelian normal subgroup. So the usual group cohomology $H^2(G; A)$ is defined, and we can extend the sequence by yet another term.

Let $\varphi \in Z^1(G; B/A)$ be a cocycle. We choose a function $\widetilde{\varphi} \colon G \to B$ such that

$$\varphi(g) = \widetilde{\varphi}(g)A$$

for all $g \in G$. Since $\varphi$ was a cocycle we have

$$\widetilde{\varphi}(gh)A = \varphi(gh) = \varphi(g)g \cdot \varphi(h) = \widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(h)A,$$

i.e. we have $\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(h) \in A$.

**Lemma A.3.7.** *Let $\varphi$ and $\widetilde{\varphi}$ be as above. The map $f \colon G^2 \to A$ given by*

$$f(g,h) = \widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(h)$$

*is a 2-cocycle.*

*Proof.* We have

$$\begin{aligned}
\delta^2 f(g,h,k) &= g \cdot f(h,k)(f(gh,k))^{-1} f(g,hk)(f(g,h))^{-1} \\
&= g \cdot (\widetilde{\varphi}(hk)^{-1}\widetilde{\varphi}(h)h \cdot \widetilde{\varphi}(k))(\widetilde{\varphi}(ghk)^{-1}\widetilde{\varphi}(gh)(gh) \cdot \widetilde{\varphi}(k))^{-1} \\
&\quad \widetilde{\varphi}(ghk)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(hk)(\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(h))^{-1} \\
&= g \cdot \widetilde{\varphi}(hk)^{-1}g \cdot \widetilde{\varphi}(h)(gh) \cdot \widetilde{\varphi}(k)(gh) \cdot \widetilde{\varphi}(k)^{-1}\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(ghk) \\
&\quad \widetilde{\varphi}(ghk)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(hk)\underbrace{g \cdot \widetilde{\varphi}(h)^{-1}\widetilde{\varphi}(g)^{-1}\widetilde{\varphi}(gh)}_{\in A \subseteq Z(B)} \\
&= g \cdot \widetilde{\varphi}(hk)^{-1}g \cdot \widetilde{\varphi}(h)g \cdot \widetilde{\varphi}(h)^{-1}\widetilde{\varphi}(g)^{-1}\widetilde{\varphi}(gh)\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(hk) \\
&= g \cdot \widetilde{\varphi}(hk)^{-1}g \cdot \widetilde{\varphi}(h)g \cdot \widetilde{\varphi}(h)^{-1}\widetilde{\varphi}(g)^{-1}\widetilde{\varphi}(gh)\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(hk) \\
&= g \cdot \widetilde{\varphi}(hk)^{-1}\widetilde{\varphi}(g)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(hk) \\
&= g \cdot \widetilde{\varphi}(hk)^{-1}g \cdot \widetilde{\varphi}(hk) \\
&= e.
\end{aligned}$$

□

**Lemma A.3.8.** *The class represented by $f$ in $H^2(G; A)$ is independent of the chosen $\widetilde{\varphi}$.*

*Proof.* Let $\widetilde{\varphi}, \tilde{\psi} \colon G \to B$ be two maps with

$$\widetilde{\varphi}(g)A = \varphi(g) = \tilde{\psi}(g)A.$$

Taking

$$\theta(g) := \tilde{\psi}(g)\widetilde{\varphi}(g)^{-1} \in A$$

for each $g \in G$, defines a map $\theta\colon G \to A$. With this map we have

$$\begin{aligned}
&\tilde{\psi}(gh)^{-1}\tilde{\psi}(g)g \cdot \tilde{\psi}(h)(\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(h))^{-1} \\
&\qquad = \tilde{\psi}(gh)^{-1}\tilde{\psi}(g)\underbrace{g \cdot \tilde{\psi}(h)g \cdot \widetilde{\varphi}(h)^{-1}}_{= g \cdot \theta(h) \in A \subseteq Z(B)}\widetilde{\varphi}(g)^{-1}\widetilde{\varphi}(gh) \\
&\qquad = \tilde{\psi}(gh)^{-1}\underbrace{\tilde{\psi}(g)\widetilde{\varphi}(g)^{-1}}_{= \theta(g) \in A \subseteq Z(B)}\widetilde{\varphi}(gh)g \cdot \theta(h) \\
&\qquad = \tilde{\psi}(gh)^{-1}\widetilde{\varphi}(gh)g \cdot \theta(h)\theta(g) \\
&\qquad = (\underbrace{\tilde{\psi}(gh)\widetilde{\varphi}(gh)^{-1}}_{= \theta(gh)})^{-1}g \cdot \theta(h)\theta(g) && \text{(see below)} \\
&\qquad = \theta(gh)^{-1}\theta(g)g \cdot \theta(h) \\
&\qquad = \delta^2\theta(g,h).
\end{aligned}$$

Here we used that

$$\tilde{\psi}(gh)^{-1}\widetilde{\varphi}(gh) = (\tilde{\psi}(gh)\widetilde{\varphi}(gh)^{-1})^{-1} = (\theta(gh))^{-1},$$

as

$$\begin{aligned}
\tilde{\psi}(gh)^{-1}\widetilde{\varphi}(gh)\,\underbrace{\theta(gh)}_{\in A \subseteq Z(B)} &= \tilde{\psi}(gh)^{-1}\theta(gh)\widetilde{\varphi}(gh) \\
&= \tilde{\psi}(gh)^{-1}\tilde{\psi}(gh)\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(gh) \\
&= e.
\end{aligned}$$

$\square$

Thus, we obtain a well-defined map $Z^1(G; B/A) \to H^2(G; A)$. Finally, we also have that two cohomologous cocycles give the same class in $H^2(G; A)$.

**Lemma A.3.9.** *Let $\varphi, \psi \in Z^1(G; B/A)$ be cohomologous. Then their images in $H^2(G; A)$ agree.*

*Proof.* Since $\varphi$ and $\psi$ are cohomologous, there exists $bA \in B/A$ such that

$$\widetilde{\varphi}(g)A = \varphi(g) = (bA)^{-1}\psi(g)g \cdot bA = (b^{-1}\tilde{\psi}(g)g \cdot b)A.$$

As the image in $H^2(G; A)$ was independent of the choice of $\widetilde{\varphi}$, we can without loss of generality assume that $\widetilde{\varphi}(g) = b^{-1}\tilde{\psi}(g)g \cdot b$. Then we have

$$\begin{aligned}
\widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(h) &= (b^{-1}\tilde{\psi}(gh)(gh) \cdot b)^{-1}b^{-1}\tilde{\psi}(g)g \cdot bg \cdot (b^{-1}\tilde{\psi}(h)h \cdot b) \\
&= (gh) \cdot b^{-1}\tilde{\psi}(gh)^{-1}bb^{-1}\tilde{\psi}(g)g \cdot bg \cdot b^{-1}g \cdot \tilde{\psi}(h)(gh) \cdot b \\
&= (gh) \cdot b^{-1}\underbrace{\tilde{\psi}(gh)^{-1}\tilde{\psi}(g)g \cdot \tilde{\psi}(h)}_{\in A \subseteq Z(B)}(gh) \cdot b \\
&= \tilde{\psi}(gh)^{-1}\tilde{\psi}(g)g \cdot \tilde{\psi}(h)(gh) \cdot b^{-1}(gh) \cdot b \\
&= \tilde{\psi}(gh)^{-1}\tilde{\psi}(g)g \cdot \tilde{\psi}(h).
\end{aligned}$$

$\square$

This shows that the above construction gives a well-defined map

$$\Delta\colon H^1(G; B/A) \to H^2(G; A).$$

**Proposition A.3.10.** *Let $B$ be a $G$-group and let $A \leq B$ be a $G$-subgroup contained in the center of $B$. Then*

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow (B/A)^G \xrightarrow{\delta} H^1(G;A) \longrightarrow H^1(G;B) \longrightarrow H^1(G;B/A) \xrightarrow{\Delta} H^2(G;A)$$

*is an exact sequence of pointed sets.*

*Proof.* We only need to check exactness at $H^1(G;B/A)$. For $\varphi \in Z^1(G;B/A)$ we have that

$$\Delta(\varphi)(g,h) = \widetilde{\varphi}(gh)^{-1}\widetilde{\varphi}(g)g \cdot \widetilde{\varphi}(h)$$

is trivial in $H^2(G;A)$ if and only if any lift $\widetilde{\varphi}\colon G \to B$ of $\psi$ is a cocycle. This in turn is the case if and only if $\varphi$ is given by an element in $H^1(G;B)$. $\square$

## A.4 Galois cohomology

In the following, $k$ will always be a field and $A$ will denote a functor from the category of field extensions of $k$ to the category of groups with the following properties:

(1) For an extension $K|k$ we have $A(K) = \varinjlim_i A(K_i)$, where $K_i$ are all finitely generated field extension of $k$ with $k \subseteq K_i \subseteq K$.

(2) If $K \to K'$ is a morphism of field extensions, the induced morphism $A(K) \to A(K')$ is injective.

(3) If $K'|K$ is a Galois extension, we have $A(K) = H^0\big(\mathrm{Gal}(K'|K); A(K')\big)$.

One example of functors satisfying these assumptions are *algebraic groups* over $k$.
In the following, we will prove that we can define

$$H^n(k;A) = H^n\big(\mathrm{Gal}(k^{\mathrm{sep}}|k); A(k^{\mathrm{sep}})\big)$$

independent of the chosen separable closure of $k$. Moreover, we will also see that this definition is functorial in the field $k$. We will not prove anything more than these basic properties. The curious reader should again consult Serre's book [95].

**Proposition A.4.1.** *Let $K|k$ and $L|\ell$ be Galois extensions and let $\varphi\colon k \to \ell$ and $\Phi\colon K \to L$ be morphisms of fields such that*

$$\begin{array}{ccc} K & \xrightarrow{\Phi} & L \\ \uparrow & & \uparrow \\ k & \xrightarrow[\varphi]{} & \ell \end{array}$$

*commutes.*

*Then $\Phi$ induces compatible group homomorphisms*

$$\mathrm{Gal}(L|\ell) \longrightarrow \mathrm{Gal}(K|k) \qquad \textit{and} \qquad A(K) \longrightarrow A(L).$$

*The induced map in cohomology*

$$H^n\big(\mathrm{Gal}(K|k); A(K)\big) \longrightarrow H^n\big(\mathrm{Gal}(L|\ell); A(L)\big)$$

*does* not *depend on* $\Phi$*.*

*Proof.* We begin by constructing the map $\mathrm{Gal}(L|\ell) \to \mathrm{Gal}(K|k)$. For $\sigma \in \mathrm{Gal}(L|\ell)$ we have $\sigma\big(\operatorname{im}\Phi\big) \subseteq \operatorname{im}\Phi$:

Let $a \in K$, let $\mu_a \in k[X]$ be its minimal polynomial and $n = \deg \mu_a$. As $K|k$ is Galois, $\mu_x$ has $n$ distinct roots in $K$. The images of these roots under $\Phi$ are the zeros of $\varphi(\mu_x) \in \ell[X]$. Now, applying $\sigma$ only permutes the zeros of $\varphi(\mu_x)$, so $\sigma\big(\Phi(x)\big)$ is still a zero of $\varphi(\mu_x)$ and thus in the image of $\Phi$.

As this holds for all elements in $\mathrm{Gal}(L|\ell)$, we can define

$$\begin{aligned} \mathrm{Gal}(\Phi)\colon\ \mathrm{Gal}(L|\ell) &\longrightarrow \mathrm{Gal}(K|k) \\ \sigma &\longmapsto \Phi^{-1} \circ \sigma \circ \Phi, \end{aligned}$$

which is obviously a group homomorphism. Moreover, by the functoriality of $A$, it is easy to see that this morphism is compatible with $A(\Phi)\colon A(K) \to A(L)$.

Now let $\widetilde{\Phi}\colon K \to L$ be another extension of $\varphi$. A similar argument as above proves that $\tau = \Phi^{-1} \circ \widetilde{\Phi} \in \mathrm{Gal}(K|k)$ and thus

$$\mathrm{Gal}(\Phi)(\sigma) = \tau \circ \mathrm{Gal}(\widetilde{\Phi})(\sigma) \circ \tau^{-1},$$

i.e. $\mathrm{Gal}(\Phi) = c_\tau \circ \mathrm{Gal}(\widetilde{\Phi})$. By the functoriality of $A$, we moreover have

$$A(\Phi) = A(\widetilde{\Phi}) \circ A(\tau^{-1}) = A(\widetilde{\Phi}) \circ t_\tau.$$

Finally, using the functoriality of $H^n(\,\cdot\,;\,\cdot\,)$ this gives

$$\begin{aligned} H^n\big(\mathrm{Gal}(\Phi); A(\Phi)\big) &= H^n\big(c_\tau \circ \mathrm{Gal}(\widetilde{\Phi}); A(\widetilde{\Phi}) \circ t_\tau\big) \\ &= H^n\big(\mathrm{Gal}(\widetilde{\Phi}); A(\widetilde{\Phi})\big) \circ H^n\big(c_\tau; t_\tau\big) \\ &= H^n\big(\mathrm{Gal}(\widetilde{\Phi}); A(\Phi)\big) \circ \mathrm{id} && \text{(by Example A.3.4)} \\ &= H^n\big(\mathrm{Gal}(\widetilde{\Phi}); A(\widetilde{\Phi})\big), \end{aligned}$$

and thus the claimed independence of $\Phi$. □

**Corollary A.4.2.** *The cohomology*

$$H^n(k; A) := H^n(k^{\mathrm{sep}}|k; A) = H^n\big(\mathrm{Gal}(k^{\mathrm{sep}}|k); A(k^{\mathrm{sep}})\big)$$

*is independent of the chosen separable closure $k^{\mathrm{sep}}$ of $k$. Moreover, it is both functorial in the field $k$ and in $A$.*

*Proof.* Let $K$ be another separable closure of $k$. The identity $k \to k$ can always be extended (in a non-canonical way) to a map $k^{\mathrm{sep}} \to K$. This induces an isomorphism $H^n(k^{\mathrm{sep}}|k; A) \to H^n(K|k; A)$, which, by the proposition above, is independent of the chosen extension $k^{\mathrm{sep}} \to K$. Thus, we still obtain a *canonical* isomorphism $H^n(k^{\mathrm{sep}}|k; A) \cong H^n(K|k; A)$.

The functoriality in $A$ is clear from the functoriality of non-abelian cohomology in the coefficients. For the functoriality in the field we argue similar as above. Let $\varphi\colon k \to \ell$ be a morphism of fields. Every such morphism can be extended (in a non-canonical way) to a map $k^{\mathrm{sep}} \to \ell^{\mathrm{sep}}$. Due to Proposition A.4.1 this extension induces a canonical map $H^n(k; A) \to H^n(\ell; A)$. As the proposition also gives uniqueness, this is functorial. $\square$

# Bibliography


[1] J. Adams and O. Taïbi. "Galois and Cartan cohomology of real groups." In: *Duke Mathematical Journal* 167.6 (2018), pp. 1057–1097. DOI: 10.1215/00127094-2017-0052. Cited on pages 78, 80, 83

[2] M. Aka. "Arithmetic groups with isomorphic finite quotients." In: *Journal of Algebra* 352 (2012), pp. 322–340. DOI: 10.1016/j.jalgebra.2011.10.033. Cited on page x

[3] M. Aka. "Profinite completions and Kazhdan's property (T)." In: *Groups, Geometry, and Dynamics* 6.2 (2012), pp. 221–229. DOI: 10.4171/GGD/155. Cited on pages x, xi, xiv, 65, 67

[4] G. Baumslag. "Residually finite groups with the same finite images." In: *Compositio Mathematica* 29.3 (1974), pp. 249–252. EuDML: 89238. Cited on page 63

[5] G. Baumslag, M. R. Bridson, C. F. Miller III, and H. Short. "Fibre products, non-positive curvature, and decision problems." In: *Commentarii Mathematici Helvetici* 75 (2000), pp. 457–477. DOI: 10.1007/s000140050136. Cited on page 73

[6] B. Bekka. "The Bohr compactification of an arithmetic group." In: *Transactions of the American Mathematical Society* 378 (2025), pp. 3755–3777. DOI: 10.1090/tran/9422. Cited on page 102

[7] B. Bekka and P. de la Harpe. *Unitary representations of groups, duals, and characters.* Mahtematical Surveys and Monographs 250. American Mathematical Society, 2020. DOI: 10.1090/surv/250. Cited on page 100

[8] A. Borel. *Linear Algebraic Groups.* 2nd ed. Graduate Texts in Mathematics 126. Springer, 1991. DOI: 10.1007/978-1-4612-0941-6. Cited on page 1

[9] A. Borel and G. Harder. "Existence of discrete cocompact subgroups of reductive groups over local fields." In: *Journal für die reine und angewandte Mathematik* 298 (1978), pp. 53–64. DOI: 10.1515/crll.1978.298.53. Cited on pages 79, 80

[10] N. Bourbaki. *General Topology. Chapters 1–4.* Elements of Mathematics. Springer, 1995. DOI: 10.1007/978-3-642-61701-0. Cited on pages 51–53, 55, 72

[11] M. R. Bridson. *Chasing finite shadows of infinite groups through geometry.* To appear in the Proceeding of 9ECM, the European Congress of Mathematics in Sevilla. 2025. arXiv: 2504.11684v1 [math.GR]. Cited on page 59

[12] M. R. Bridson. “Profinite isomorphisms and fixed-point properties.” In: *Algebraic & Geometric Topology* 24 (2024), pp. 4103–4114. DOI: 10.2140/agt.2024.24.4103. Cited on pages x, 73

[13] M. R. Bridson and F. J. Grunewald. “Grothendieck’s problem concerning profinite completions and representations of groups.” In: *Annals of Mathematics* 160.2 (2004), pp. 359–373. DOI: annals.2004.160.359. Cited on pages x, xiv, 71, 73, 74

[14] M. R. Bridson and A. W. Reid. “Nilpotent Completions of Groups, Grothendieck Pairs, and Four Problems of Baumslag.” In: *International Mathematics Research Notices* 2015.8 (2015), pp. 2111–2140. DOI: 10.1093/imrn/rnt353. Cited on pages 65, 95, 96

[15] R. Brooks. “Some Remarks on Bounded Cohomology.” In: *Riemann Surfaces and Related Topics* (State University of New York, Stony Brook, June 5, 1978–June 9, 1987). Ed. by I. Kra and B. Maskit. Annals of Mathematics Studies 97. Princeton University Press, 1980, pp. 53–63. DOI: 10.1515/9781400881550-006. Cited on page 110

[16] K. S. Brown. *Cohomology of Groups*. Graduate Texts in Mathematics 87. Springer, 1982. DOI: 10.1007/978-1-4684-9327-6. Cited on pages 97, 106

[17] M. Burger and N. Monod. “Bounded cohomology of lattices in higher rank Lie groups.” In: *Journal of the European Mathematical Society* 1.2 (1999), pp. 199–235. DOI: 10.1007/S100970050007. Cited on pages xii, 76, 87. Corrected in “Erratum: Bounded cohomology of lattices in higher rank Lie groups.” In: *Journal of the European Mathematical Society* 1.3 (1999), p. 338. DOI: 10.1007/S100970050010

[18] D. Calegari. *scl*. MSJ Memoirs 20. Mathematical Society of Japan, 2009. DOI: 10.2969/msjmemoirs/020010000. Cited on pages 110–112

[19] T. Cheetham-West, A. Lubotzky, A. W. Reid, and R. Spitler. “Property FA is not a profinite property.” In: *Groups, Geometry and Dynamics* 19.3 (2025), pp. 1081–1087. DOI: 10.4171/GGD/802. Cited on pages x, 65

[20] C. Chevalley. “Certains schémas de groupes semi-simples.” In: *Séminaire Bourbaki*. Vol. 6. Société Mathématique de France, 1961, pp. 219–234. EuDML: 109612. Cited on page 44

[21] S. M. Corson, S. Hughes, P. Möller, and O. Varghese. “Higman–Thompson groups and profinite properties of right angled Coxeter groups.” In: *Selecta Mathematica New Series* 32.28 (2026). DOI: 10.1007/s00029-026-01130-4. Cited on pages x, 66

[22] S. M. Corson, S. Hughes, P. Möller, and O. Varghese. “Profinite rigidity of affine Coxeter groups.” In: *Mathematische Zeitschrift* 311.11 (2025). DOI: 10.1007/s00209-025-03799-3. Cited on page 66

[23] J. D. Dixon, E. W. Formanek, J. C. Poland, and L. Ribes. “Profinite completions and isomorphic finite quotients.” In: *Journal of Pure and Applied Algebra* 23.3 (1982), pp. 227–231. DOI: 10.1016/0022-4049(82)90098-6. Cited on pages ix, 62

[24] E. Dyer and A. T. Vasquez. “Some small aspherical spaces.” In: *Journal of the Australian Mathematical Society* 16.3 (1973). DOI: 10.1017/S1446788700015147. Cited on pages 74, 97

[25] D. Echtler. “Spectral sequences in bounded cohomology.” Master’s thesis. Universität Regensburg, 2021. Cited on page 95

[26] D. Echtler and H. Kammeyer. “Bounded cohomology is not a profinite invariant.” In: *Canadian Mathematical Bulletin* 67.2 (2024), pp. 379–390. DOI: 10.4153/S0008439523000826. Cited on pages xiii, xv, 75, 85

[27] D. Eisenbud. *Commutative Algebra. with a View Toward Algebraic Geometry.* Graduate Texts in Mathematics 150. Springer, 1995. DOI: 10.1007/978-1-4612-5350-1. Cited on page 23

[28] D. Eisenbud and J. Harris. *The Geometry of Schemes.* Graduate Texts in Mathematics 197. Springer, 2000. DOI: 10.1007/b97680. Cited on page 4

[29] D. B. A. Epstein and K. Fujiwara. “The second bounded cohomology of word-hyperbolic groups.” In: *Topology* 36.6 (1997), pp. 1275–1289. DOI: 10.1016/S00$0-9383(96)00046-8. Cited on page 111

[30] T. V. Fomina. “Integral forms of linear algebraic groups.” In: *Mathematical Notes* 61 (1997), pp. 346–351. DOI: 10.1007/BF02355417. Cited on page 44. Russian original in: *Matematicheskie Zametki* 61.3 (1997), pp. 424–430

[31] F. Fournier-Facio. *Profinite isomorphisms, stable commutator length, and fixed point properties.* preprint. 2026. arXiv: 2603.12095v1 [math.GR]. Cited on pages xiii, 65, 88

[32] R. Frigerio. *Bounded Cohomology of Discrete Groups.* Mathematical Surveys and Monographs 227. American Mathematical Society, 2017. DOI: 10.1090/surv/227. Cited on pages 75, 107, 111

[33] K. Fujiwara. “The Second Bounded Cohomology of a Group Acting on a Gromov-Hyperbolic Space.” In: *Proceedings of the London Mathematical Society* 76.1 (1998), pp. 70–94. DOI: 10.1112/S0024611598000033. Cited on pages xii, 76, 87, 111

[34] K. Fujiwara. “The Second Bounded Cohomology of an Amalgamated Free Product of Groups.” In: *Transactions of the American Mathematical Society* 352.3 (2000), pp. 1113–1129. DOI: 10.1090/s0002-9947-99-02282-5. Cited on pages 98, 111

[35] L. Glebsky, A. Lubotzky, N. Monod, and B. Rangarajan. *Asymptotic Cohomology and Uniform Stability for Lattices in Semisimple Groups.* preprint. 2023. arXiv: 2301.00476 [math.GR]. Cited on page xiii

[36] M. Gromov. “Volume and bounded cohomology.” In: *Publications Mathématiques de l’IHÉS* 56 (1982), pp. 5–99. DOI: 10.5802/pmihes.3. Cited on pages xi, 105

[37] A. Grothendieck. “Représentations linéaires et compactification profinie des groupes discrets.” In: *Manuscripta Mathematica* 2 (1970), pp. 375–396. DOI: 10.1007/BF01719593. Cited on pages x, xiii, 71, 89–91

[38] U. Hamenstädt. "Bounded cohomology and isometry groups of hyperbolic spaces." In: *Journal of the European Mathematical Society* 10.2 (2008), pp. 351–349. DOI: 10.4171/JEMS/112. Cited on pages 88, 98, 111

[39] D. Harari. *Galois Cohomology and Class Field Theory*. Universitext. Springer, 2020. DOI: 10.1007/978-3-030-43901-9. Cited on pages 79, 83

[40] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics 52. Springer, 1977. DOI: 10.1007/978-1-4757-3849-0. Cited on page 8

[41] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. Graduate Studies in Mathematics 34. American Mathematical Society, 2001. DOI: 10.1090/gsm/034. Cited on pages 76, 82

[42] G. Higman. "A Finitely Generated Infinite Simple Group." In: *Journal of the London Mathematical Society* 1 (1951), pp. 61–64. DOI: 10.1112/jlms/s1-26.1.61. Cited on pages 74, 97

[43] J. E. Humphreys. *Arithmetic Groups*. Lecture Notes in Mathematics 789. Springer, 1980. DOI: 10.1007/BFb0094567. Cited on page 54

[44] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics 9. Springer, 1972. DOI: 10.1007/978-1-4612-6398-2. Cited on page 30

[45] N. V. Ivanov. "Foundations of the theory of bounded cohomology." In: *Journal of Soviet Mathematics* 37 (1987), pp. 1090–1115. DOI: 10.1007/BF01086634. Cited on page 105. Russian original in: *Investigations in topology. Part V*. Zapiski Nauchnykh Seminarov LOMI 143. 1985, pp. 69–109

[46] N. Jacobson. *Lie Algebras*. Dover Books on Mathematics. Corrected reprint of the 1962 edition. Dover Publications, 1979. Cited on page 86

[47] A. Jaikin-Zapirain and A. Lubotzky. "Some remarks on Grothendieck pairs." In: *Groups, Geometry and Dynamics* 19.2 (2025), pp. 597–616. DOI: 10.4171/GGD/889. Cited on pages 89, 90, 100, 102

[48] B. E. Johnson. *Cohomology in Banach algebras*. Memoirs of the American Mathematical Society 127. American Mathematical Society, 1972. DOI: 10.1090/memo/0127. Cited on page xi

[49] H. Kammeyer and S. Kionke. "Adelic superrigidity and profinitely solitary lattices." In: *Pacific Journal of Mathematics* 313.1 (2021), pp. 137–158. DOI: 10.2140/pjm.2021.313.137. Cited on pages xii, 81

[50] H. Kammeyer, S. Kionke, J. Raimbault, and R. Sauer. "Profinite invariants of arithmetic groups." In: *Forum of Mathematics, Sigma* 8 (2020), p. 22. DOI: 10.1017/fsm.2020.43. Cited on pages x, 66, 82

[51] H. Kammeyer and R. Sauer. "$S$-arithmetic spinor groups with the same finite quotients and distinct $\ell^2$-cohomology." In: *Groups, Geometry and Dynamics* 14.3 (2020), pp. 857–869. DOI: 10.4171/GGD/566. Cited on pages x, 64, 65, 68

[52] H. Kammeyer and G. Serafini. "On the Euler characteristic of $S$-arithmetic groups." In: *Journal of the London Mathematical Society* 113.3 (2026). DOI: `10.1112/jlms.70505`. Cited on pages x, 66, 85

[53] F. Keune. *Number Fields*. Radboud University Press, 2023. DOI: `10.54195/IPVU4488`. Cited on pages 35–42

[54] S. Kionke. *An introduction to arithmetic groups*. Minicourse given at the summer school "Analytic and Arithmetic Groups". Heinrich Heine University Düsseldorf, Aug. 19–23, 2024. URL: `https://www.math.uni-duesseldorf.de/~internet/summer.school.2024/`. Cited on page 44

[55] S. Kionke and E. Schesler. "Amenability and profinite completions of finitely generated groups." In: *Groups, Geometry, and Dynamics* 17.4 (2023), pp. 1235–1258. DOI: `10.4171/GGD/732`. Cited on pages x, 66, 88

[56] N. Klingen. *Arithmetical Similarities. Prime Decomposition and Finite Group Theory*. Oxford Mathematical Monographs. Oxford Clarendon Press, 1998. DOI: `10.1093/oso/9780198535980.001.0001`. Cited on pages 43, 44

[57] M. Kneser. "Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern. I." In: *Mathematische Zeitschrift* 88.1 (1965), pp. 40–47. DOI: `10.1007/BF01112691`. Cited on pages 78, 80

[58] M. Kneser. "Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern. II." In: *Mathematische Zeitschrift* 89.3 (1965), pp. 250–272. DOI: `10.1007/BF02116869`. Cited on page 78

[59] K. Komatsu. "On the adele rings and zeta-functions of algebraic number fields." In: *Kodai Mathematical Journal* 1.3 (1978), pp. 394–400. DOI: `kmj/1138035648`. Cited on page 69

[60] Y. Liu. "Finite-volume hyperbolic 3-manifolds are almost determined by their finite quotient groups." In: *Inventiones mathematicae* 231 (2022), pp. 741–804. DOI: `10.1007/s00222-022-01155-4`. Cited on pages x, 66

[61] A. Lubotzky. "Finiteness properties and profinite completions." In: *Bulletin of the London Mathematical Society* 46.1 (2013), pp. 103–110. DOI: `10.1112/blms/bdt070`. Cited on pages x, 66

[62] A. Lubotzky. "Some more non-arithmetic rigid groups." In: *Geometry, spectral theory, groups, and dynamics* (Haifa, Israel, Dec. 29, 2003–Jan. 9, 2004). Ed. by M. Entov, Y. Pinchover, and M. Sageev. Contemporary Mathematics 387. American Mathematical Society, 2005, pp. 237–244. DOI: `10.1090/conm/387`. Cited on pages xiii, 58, 87, 88

[63] W. Lück. "Approximating $L^2$-invariants by their classical counterparts." In: *EMS Surveys in Mathematical Sciences* 3.2 (1994), pp. 269–344. DOI: `10.4171/EMSS/18`. Cited on pages x, 63, 65

[64] S. Mac Lane. *Categories for the Working Mathematician*. Second Edition. Graduate Texts in Mathematics 5. Springer, 1978. DOI: 10.1007/978-1-4757-4721-8. Cited on page 3

[65] G. A. Margulis. *Discrete Subgroups of Semisimple Lie Groups*. Vol. 17. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1991. Cited on pages 50, 81

[66] J. McCleary. *A User's Guide to Spectral Sequences*. 2nd ed. Cambridge Studies in Advanced Mathematics 58. Cambridge University Press, 2001. DOI: 10.1017/CBO9780511626289. Cited on page 92

[67] J. S. Milne. *Algebraic Groups. The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics 170. Cambridge University Press, 2017. DOI: 10.1017/9781316711736. Cited on pages 1, 8, 10–12, 15, 24–32, 78

[68] N. Monod. *Continuous bounded cohomology of locally compact groups*. Lecture Notes in Mathematics 1758. Springer, 2001. DOI: 10.1007/b80626. Cited on pages 75, 95, 107

[69] N. Monod and Y. Shalom. "Cocyle superrigidity and bounded cohomology for negatively curved space." In: *Journal of Differential Geometry* 67.3 (2004), pp. 395–455. DOI: 10.4310/jdg/1102091355. Cited on pages xi, 76

[70] J. Neukirch. *Algebraische Zahlentheorie*. Masterclass. Springer, 1992. DOI: 10.1007/978-3-540-37663-7. Cited on pages 37–41, 85

[71] N. Nikolov and D. Segal. "On finitely generated profinite groups, I. strong completeness and uniform bounds." In: *Annals of Mathematics* 165.1 (2007), pp. 171–238. DOI: 10.4007/annals.2007.165.171. Cited on pages ix, 62

[72] E. Noether. "Der Endlichkeitssatz der Invarianten endlicher Gruppen." In: *Mathematische Annalen* 77.1 (1915), pp. 89–92. DOI: 10.1007/BF01456821. Cited on page 23

[73] G. A. Noskov. "Bounded cohomology of discrete groups with coefficients." In: *Leningrad Mathematical Journal* 2.5 (1991), pp. 1067–1084. Cited on page 105. Russian original in: *Algebra i Analiz* 2.5 (1990), pp. 146–164

[74] D. V. Osin. "Acylindrically hyperbolic groups." In: *Transactions of the American Mathematical Society* 368.2 (2016), pp. 851–888. DOI: 10.1090/tran/6343. Cited on pages 88, 98

[75] F. Peter and H. Weyl. "Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe." In: *Mathematische Annalen* 97 (1927), pp. 737–755. DOI: 10.1007/BF01447892. Cited on page 101

[76] P. F. Pickel. "Finitely Generated Nilpotent Groups with Isomorphic Finite Quotients." In: *Transactions of the American Mathematical Society* 160 (1971), pp. 327–341. DOI: 10.2307/1995809. Cited on page 65

[77] V. Platonov and A. Rapinchuk. *Algebraic Groups and Number Theory*. Trans. from the Russian by R. Rowen. Pure and Applied Mathematics 139. Academic press, 1994. DOI: 10.1016/s0079-8169(08)62066-8. Cited on pages 1, 43, 57, 77, 79, 80, 82, 86

[78] V. P. Platonov and O. I. Tavgen'. "Grothendieck's problem on profinite completions and representations of groups." In: *K-Theory* 4 (1990), pp. 89–101. DOI: 10.1007/BF00534194. Cited on pages 71, 72

[79] V. P. Platonov and O. I. Tavgen'. "On Grothendieck's problem on profinite completions of groups." In: *Soviet Mathematics, Doklady* 33 (1986), pp. 822–825. Cited on pages x, xv, 71, 73. Russian original in: *Daklady Akademii Nauk SSSR* 228.5 (1986), pp. 1054–1058

[80] B. Poonen. *Rational points on varieties.* Graduate Studies in Mathematics 186. American Mathematical Society, 2017. DOI: 10.1090/gsm/186. Cited on pages 11, 16, 27

[81] G. Prasad. "A new approach to unramified descent in Bruhat-Tits theory." In: *American Journal of Mathematics* 142.1 (2020), pp. 215–253. DOI: 10.1353/ajm.2020.0005. Cited on page 70

[82] G. Prasad and A. Rapinchuk. "Developments on the congruence subgroup problem after the work of Bass, Milnor and Serre." In: *Collected papers of John Milnor. Algebra.* Ed. by H. Bass and L. T. Yuen. Vol. V. American Mathematical Society, 2010, pp. 307–324. Cited on page 58

[83] G. Prasad and A. S. Rapinchuk. "On the existence of isotropic forms of semi-simple algebraic groups over number fields with prescribed local behavior." In: *Advances in Mathematics* 207.2 (2006), pp. 646–660. DOI: 10.1016/j.aim.2006.01.001. Cited on pages 79–81

[84] M. S. Raghunathan. "On the congruence subgroup problem." In: *Publications Mathématiques de l'IHÉS* 46 (1976), pp. 107–161. DOI: 10.1007/BF02684320. Cited on page 58

[85] M. S. Raghunathan. "The congruence subgroup problem." In: *Proceedings of the Hyderabad Conference on Algebraic Groups.* Ed. by S. Ramanan, C. Musili, and N. Mohan Kumar. Manoj Prakashan, 1991, pp. 465–494. Cited on page 58

[86] A. Rapinchuk. "The Congruence Subgroup Problem." In: *Algebra, K-Theory, Groups, and Education. On the Occasion of Hyman Bass's 65th Birthday.* Ed. by T. Y. Lam and A. R. Magid. Contemporary Mathematics 243. American Mathematical Society, 1999, pp. 175–188. DOI: 10.1090/conm/243/03693. Cited on page 58

[87] A. W. Reid. "Profinite properties of discrete groups." In: *Groups St Andrews 2013* (University of St. Andrews, UK, Aug. 3–11, 2013). Ed. by C. M. Campbell, M. R. Quick, E. F. Robertson, and C. M. Roney-Dougal. London Mathematical Society Lecture Note Series 422. Cambridge University Press, 2015, pp. 73–104. DOI: 10.1017/CBO9781316227343.004. Cited on pages 59, 63

[88] L. Ribes and P. Zalesskii. *Profinite Groups*. Second Edition. Vol. 40. Ergebnise der Mathematik und ihrer Grenzgebiete. Springer, 2010. DOI: 10.1007/978-3-642-01642-4. Cited on pages 61, 64, 67, 99

[89] E. Rips. “Subgroups of small cancellation groups.” In: *Bulletin of the London Mathematical Society* 14.1 (1982), pp. 45–47. DOI: 10.1112/blms/14.1.45. Cited on page 74

[90] P. Rolli. *Quasi-morphisms on Free Groups*. preprint. 2009. arXiv: 0911.4234v2 [math.GR]. Cited on page 111

[91] J. J. Rotman. *An Introduction to Homological Algebra*. 2nd ed. Universitext. Springer, 2009. DOI: 10.1007/b98977. Cited on pages 92, 94

[92] G. Sabbagh and J. S. Wilson. “Polycyclic groups, finite images, and elementary equivalence.” In: *Archiv der Mathematik* 57 (1991), pp. 221–227. DOI: 10.1007/BF01196851. Cited on pages x, 65

[93] W. Sawin. *Do locally isomorphic algebraic groups have isomorphic adèle points?* MathOverflow. URL: https://mathoverflow.net/q/508318 (version 2026-02-19). Cited on page 70

[94] T. Schoeneberg. *Semisimple Lie algebras and their classification over* $\mathfrak{p}$*-adic fields*. Mémoires de la SMF 151. Société Mathématique de France, 2017. DOI: 10.24033/msmf.459. Cited on page 86

[95] J.-P. Serre. *Galois Cohomology*. Trans. from the French by P. Ion. Springer Monographs in Mathematics. Springer, 2002. DOI: 10.1007/978-3-642-59141-9. Cited on pages 17, 23, 78, 85, 112, 118

[96] J.-P. Serre. “Groupes de congruence.” In: *Séminaire Bourbaki*. Vol. 10. Société Mathématique de France, 1968, pp. 275–291. EuDML: 109731. Cited on page 50

[97] J.-P. Serre. “Le Problème des Groupes de Congruence Pour $\mathrm{SL}_2$.” In: *Annals of Mathematics* 92.3 (1970), pp. 489–527. DOI: 10.2307/1970630. Cited on pages 58, 87

[98] The Stacks Project Authors. *The Stacks Project*. 2018. URL: https://stacks.math.columbia.edu. Cited on page 56

[99] B. Sury. *The Congruence Subgroup Problem. An Elementary Approach Aimed at Applications*. Texts and Readings in Mathematics. Hindustan Book Agency, 2003. DOI: 10.1007/978-93-86279-19-4. Cited on page 54

[100] J. Tits. “Classification of Algebraic Semisimple Groups.” In: *Algebraic Groups and Discontinuous Subgroups* (Boulder, Colorado). Ed. by A. Borel and G. D. Mostow. American Mathematical Society, 1966, pp. 33–62. DOI: 10.1090/pspum/009. Cited on pages 1, 87

[101] J. Tits. "Reductive groups over local fields." In: *Automorphic forms, representations and L-functions, Part 1* (Oregon State University, July 11–Aug. 5, 1977). Ed. by A. Borel and W. Casselman. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 1979, pp. 29–69. DOI: 10.1090/pspum/033.1. Cited on page 70

[102] C. A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advances Mathematics 38. Cambridge University Press, 1994. DOI: 10.1017/CBO9781139644136. Cited on pages 92, 94

[103] D. T. Wise. "A residually finite version of Rips's construction." In: *Bulletin of the London Mathematical Society* 35.1 (2016), pp. 23–29. DOI: 10.1112/S0024609302001406. Cited on page 74