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Complement Reducible Uniform Hypergraphs

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Article

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Abstract

We investigate a generalization of complement reducible graphs, called co-graphs, for r-uniform hypergraphs. The operations of r-co-hypergraphs are the disjoint union of two given r-co-hypergraphs and the join operation, which inserts all hyperedges of cardinality r between the non-empty vertex subsets of two given r-co-hypergraphs. We show that the primal graphs of r-co-hypergraphs are special co-graphs and that r-co-hypergraphs are closed under complementation of r-uniform hypergraphs. This leads to a method that can determine whether an input hypergraph H is an r-co-hypergraph. If the answer is positive, we find a decomposition tree for H in polynomial time. We give specific formulas for how to compute several hypergraph parameters for r-uniform hypergraphs defined by the disjoint union of two r-uniform hypergraphs and the join of two r-uniform hypergraphs. The considered parameters are the size of a largest stable set, the size of a largest co-stable set, the size of a largest independent set, the size of a largest co-independent set, the size of a smallest vertex cover, the size of a smallest 2-transversal, the size of a smallest dominating set, the strong chromatic number, and the upper chromatic number. This leads to $\mathcal{O}(n)$ time algorithms to compute these values on r-co-hypergraphs on n vertices given by a decomposition tree. Further, we conclude relations for the considered parameters restricted to r-co-hypergraphs. Our methods generalize and re-prove several results known for co-graphs.

Keywords: hypergraphs; uniform hypergraphs; co-graphs; r-co-hypergraphs; algorithms; special graph classes

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1. Introduction

Hypergraphs are essential data structures that are often used in advanced research for representing and analyzing complex and large datasets. A hypergraph is a straightforward extension of a graph in which an edge can connect every number of vertices, which implies that it is a data structure that offers more modelling possibilities than an ordinary graph. Hypergraphs are useful to describe situations in the applied sciences, such as chemistry [1,2], epidemics [3], protein–protein interaction [4], and hypergraph learning [5,6]. Further applications can be found in [7] (Chapter 7). Hypergraphs are also very interesting from a theoretical point of view. There are numerous results on various coloring problems on hypergraphs, see [8] (Chapter 11) and [9].

The class of complement reducible graphs, called co-graphs for short, has been studied since the 1970s in [10–12] as graphs that can be constructed from the single-vertex graph by complementation and disjoint union. One of the first comprehensive studies of this class can be found in [13]. Co-graphs are exactly those graphs that do not contain the four-vertex path as an induced subgraph [13]. Co-graphs lead to a class of graphs arising in a large number of applications, see [13–16]. A lot of hard graph problems can be solved in linear time when restricted to co-graphs [13,17]. Co-graphs have been generalized to directed co-graphs [18,19], which also have applications in the field of genetics, see [20,21]. Many hard digraph problems can be solved in linear time if they are restricted to directed co-graphs [22,23].

We generalize co-graphs to r-uniform hypergraphs. The operations of complement reducible r-uniform hypergraphs, called r-co-hypergraphs, are the disjoint union of two given r-co-hypergraphs and the join operation, which inserts all hyperedges of cardinality r between the non-empty vertex subsets of two given r-co-hypergraphs, as suggested in [24]. We show that r-co-hypergraphs are closed under complementation of r-uniform hypergraphs and that r-co-hypergraphs can be defined by disjoint union and complementation as well as by join and complementation. This allows us to give a method which decides whether an input hypergraph H is an r-co-hypergraph. If the answer is positive, we find a decomposition tree for H in polynomial time. Further we characterize primal graphs of r-co-hypergraphs as special co-graphs. We present $\mathcal{O}(n)$ time algorithms to compute the size of a largest stable set, the size of a largest co-stable set, the size of a largest independent set, the size of a largest co-independent set, the size of a smallest vertex cover, the size of a smallest 2-transversal, the size of a smallest dominating set, the strong chromatic number, and the upper chromatic number on r-co-hypergraphs on n vertices given by a decomposition tree. Our solutions extend and re-prove several results which are known for co-graphs.

2. Preliminaries

2.1. Graphs

We use finite undirected *graphs* $G = (V(G), E(G))$, where $V(G)$ is a non-empty finite set of *vertices* and $E(G) \subseteq \{\{u,v\} \mid u,v \in V(G),\ u \neq v\}$ is a finite set of *edges* (we therefore do not consider graphs with loops, i.e., edges with a single vertex, or repeated edges, i.e., two or more edges containing identical set of vertices). Two vertices $v, w \in V(G)$ are *adjacent* in G, if $\{v,w\} \in E(G)$. For a vertex $v \in V(G)$ we denote by $N_G(v)$ the set of all vertices which are adjacent to v in G, i.e., $N_G(v) = \{w \in V(G) \mid \{v,w\} \in E(G)\}$. Vertex set $N_G(v)$ is called the set of all *neighbors* of v in G.

A graph J is a *subgraph* of G if $V(J) \subseteq V(G)$ and $E(J) \subseteq E(G)$. Graph J is an *induced subgraph* of G if additionally $E(J) = \{\{u,v\} \in E(G) \mid u,v \in V(J)\}$. For $U \subseteq V(G)$, we define by $G[U]$ the subgraph of G induced by the vertices of U. For a graph G its *complement graph* is defined by $\overline{G} = (V(G), \{\{u,v\} \mid u,v \in V(G), u \neq v, \{u,v\} \notin E(G)\})$.

We make use of the following indexed graphs. As usual, we denote by P_n the path on n vertices, by C_n the cycle on n vertices, by K_n the complete graph on n vertices, by $I_n = \overline{K_n}$ the empty graph (also called edgeless graph or null graph) on n vertices, and by $K_{n,m}$ the complete bipartite graph on $n + m$ vertices.

2.2. Co-Graphs

A *co-graph*, or *complement reducible graph* is a graph that can be constructed from the single-vertex graph by complementation and disjoint union, see [13]. There are several equivalent characterizations for co-graphs. From an algorithmic point of view we recall that co-graphs can be defined from the single-vertex graph by the following two operations. Let G_1 and G_2 be two vertex-disjoint graphs.

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- The *disjoint union* of G_1 and G_2, referred to as $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.
- The *join* of G_1 and G_2, referred to as $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\} \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$.

Definition 1 (co-graph; [13])**.** *Every graph $G = (\{v\}, \emptyset)$ on a single vertex, denoted by v, is a* co-graph*. If G_1 and G_2 are vertex-disjoint co-graphs, then $G_1 \cup G_2$ and $G_1 + G_2$ are co-graphs, too.*

Example 1.

1. *Every empty graph I_n with $n \in \mathbb{N}$ vertices is a co-graph by the expression $(\ldots((v_1 \cup v_2) \cup v_3) \cup \ldots) \cup v_n$.*
2. *Every complete graph K_n with $n \in \mathbb{N}$ vertices is a co-graph by the expression $(\ldots((v_1 + v_2) + v_3) + \ldots) + v_n$.*

Theorem 1 ([13])**.** *Co-graphs are (i) closed under complements and (ii) closed under induced subgraphs, but (iii) not closed under subgraphs.*

By the given definition, every co-graph can be described by a tree structure, which is referred to as a *co-tree*. The leaves of the co-tree correspond to the vertices of the co-graph and the inner nodes of the co-tree correspond to the operations performed on the subexpressions defined by the subtrees.

Theorem 2 ([13])**.** *For every graph G, in time $\mathcal{O}(|V(G)| + |E(G)|)$ we can decide whether G is a co-graph and, if so, provide a corresponding co-tree.*

Using the co-tree many hard problems can be solved in polynomial time if they are restricted to co-graphs [13,25]. For example, the following graph parameters can be calculated in linear time for co-graphs:

- the *independence number* α, i.e., the size of a maximum independent set—a subset of the vertices such that no two of them are connected by an edge;
- the *clique number* ω, i.e., the size of a maximum clique—a subset of the vertices such that every two of them are connected by an edge;
- the *clique covering number* θ, i.e., the minimum number of vertex-disjoint cliques needed to cover the vertex set;
- the *chromatic number* χ, i.e., the minimum number of vertex-disjoint independent sets needed to cover the vertex set, or equivalently, the minimum number of colors needed to color the vertices of a graph so that no two adjacent vertices have the same color;
- the *vertex cover number* τ, i.e., the size of a minimum vertex cover—a subset of the vertices that contains at least one vertex of each edge;
- the *domination number* γ, i.e., the size of a maximum dominating set—a subset of the vertices such that every vertex or one of its neighbors is contained in it;
- the *matching number* ν, i.e., the size of a maximum matching—a subset of the edges such that no two of them share a common vertex.

Please note that some of the given abbreviations for graph parameters such as $\alpha, \beta, \ldots$ will also be used to define special vertex sets in hypergraphs in Section 4. The context always makes it clear which parameter is meant.

The previous theorem can be used to efficiently determine the chromatic number of a co-graph, see [13]. Let G be a co-graph and T the corresponding co-tree. For a node w of T, let $G[w]$ denote the graph induced by the subtree of T with root w. To every leaf v of T we add the label $\chi(G[v]) = 1$. For every inner node w of T, depending on its type, we add the following label:

1. If w is a $\cup$-node with children v_1 and v_2, $\chi(G[w]) = \max\{\chi(G[v_1]), \chi(G[v_2])\}$;
2. if w is a $+$-node with children v_1 and v_2, $\chi(G[w]) = \chi(G[v_1]) + \chi(G[v_2])$.

Finally, for the root r of T it holds that $\chi(G[r]) = \chi(G)$.

For all graph parameters listed above, similar results are known from [13,25]. In Table 1 we summarize these results. Note that the formulas which are stated in Table 1 are not limited to co-graphs but also apply to general graphs.

Table 1. Graph parameters for single-vertex graphs, the disjoint union, and join operation for graphs and thus for co-graphs.

Parameter	v	$G_1 \cup G_2$	$G_1 + G_2$
α	1	$\alpha(G_1) + \alpha(G_2)$	$\max(\alpha(G_1), \alpha(G_2))$
ω	1	$\max(\omega(G_1), \omega(G_2))$	$\omega(G_1) + \omega(G_2)$
θ	1	$\theta(G_1) + \theta(G_2)$	$\max(\theta(G_1), \theta(G_2))$
χ	1	$\max(\chi(G_1), \chi(G_2))$	$\chi(G_1) + \chi(G_2)$
τ	0	$\tau(G_1) + \tau(G_2)$	$\min(\lvert V(G_1)\rvert + \tau(G_2), \lvert V(G_2)\rvert + \tau(G_1))$
γ	1	$\gamma(G_1) + \gamma(G_2)$	$\min(\gamma(G_1), \gamma(G_2), 2)$
ν	0	$\nu(G_1) + \nu(G_2)$	$\min(\lvert V(G_1)\rvert + \nu(G_2), \lvert V(G_2)\rvert + \nu(G_1), \lfloor \frac{\lvert V(G_1)\rvert + \lvert V(G_2)\rvert}{2} \rfloor$

2.3. Hypergraphs

We work with finite undirected *hypergraphs* $H = (V(H), E(H))$, where $V(H)$ is a non-empty finite set of *vertices* and $E(H)$ is a set of subsets of $V(H)$, the *hyperedges*. Edges with fewer than two elements are generally permitted, but will be disregarded here (thus we do not consider hypergraphs with loops, i.e., hyperedges with a single vertex, or repeated hyperedges, i.e., two or more hyperedges containing the same set of vertices). This is a generalization of the notion of graphs since the hyperedges can have every size instead of size two. In a hypergraph, two vertices are said to be *adjacent* if there is a hyperedge containing both of them.

Hypergraph H' is a *subhypergraph* of hypergraph H if $V(H') \subseteq V(H)$ and for every $e' \in E(H')$ there is an $e \in E(H)$ such that $e' = e \cap V(H')$ and $|e'| \geq 2$. For every subset $F \subseteq E(H)$ we call the hypergraph $H_F = (V(H), F)$ the *partial subhypergraph* of H. For every subset $W \subseteq V(H)$ we call $H_W = (W, E')$ with $E' = \{e \in E(H) \mid e \subseteq W\}$ the *induced subhypergraph* of H. That is, E' consists of all hyperedges of $E(H)$ that lie entirely in W.

Let I_n be the *empty hypergraph* (edgeless hypergraph or null hypergraph) on n vertices and $E(I_n) = \emptyset$.

The *rank* $r(H) = \max_{e \in E(H)} |e|$ is the maximum cardinality of a hyperedge in hypergraph H and the *co-rank* $cr(H) = \min_{e \in E(H)} |e|$ is the minimum cardinality of a hyperedge in H. If $r(H) = cr(H) = r$ the hypergraph H is *r-uniform* or *uniform*. For a set X we denote the set of all r-element subsets of X by $\binom{X}{r}$. For $2 \leq r \leq n$ we define the *complete r-uniform hypergraph* to be the hypergraph K_n^r for $|V(K_n^r)| = n$ and $E(K_n^r) = \binom{V(K_n^r)}{r}$.

Remark 1. *Let H be some r-uniform hypergraph, then $|E(H)| \leq |E(K^r_{|V(H)|})| = \binom{|V(H)|}{r}$.*

For an r-uniform hypergraph H the complement $\overline{H}$ is the r-uniform hypergraph with vertex set $V(H)$ and hyperedge set $\binom{V(H)}{r} \setminus E(H)$.

In a hypergraph H a *path P from v_0 to v_q* is a vertex–hyperedge alternating sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_q, v_q$ such that (i) $v_0, v_1, \ldots, v_q \in V(H)$, (ii) $v_0, v_1, \ldots, v_{q-1}$ are distinct, (iii) $v_1, \ldots, v_q$ are distinct, (iv) $e_1, \ldots, e_q \in E(H)$ are distinct, and (v) $v_{i-1} \in e_i$ and $v_i \in e_i$ for $i = 1, 2, \ldots, q$. The integer q is the *length* of the path P. Note that if there is a path from v_0 to v_q, then there is also a path from v_q to v_0. In this case, we say that P *connects* v_0 and v_q. A hypergraph is *connected* if for every pair of vertices, there is a path connecting them; otherwise, it is *not connected* or *disconnected*. In a hypergraph, a *connected component* is defined as a maximal set of vertices that are pairwise connected by a (possibly trivial) path.

In the analysis of hypergraphs, one of the most useful concepts is the *primal graph* (which is also called the 2-section, clique graph, representing graph, or Gaifman graph) of a hypergraph. The *primal graph* of a hypergraph H is defined as the graph

$$P(H) = (V(H), \{\{v_1, v_2\} \mid v_1 \neq v_2, \exists e \in E(H) : \{v_1, v_2\} \subseteq e\}).$$

Therefore, the hypergraph H is said to be connected or to have r components if the associated primal graph $P(H)$ is connected or has r components, respectively.

3. *r*-Co-Hypergraphs

We introduce a generalization of co-graphs to r-uniform hypergraphs. Therefore we use two operations. The disjoint union operation of two r-uniform hypergraphs is straight forward, and the join of two r-uniform hypergraphs H_1 and H_2 is motivated by [24]. Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs.

- The *disjoint union* of H_1 and H_2, denoted by $H_1 \oplus H_2$, is the hypergraph with vertex set $V(H_1) \cup V(H_2)$ and hyperedge set $E(H_1) \cup E(H_2)$.
- The *join* of H_1 and H_2, denoted by $H_1 \otimes H_2$, is the hypergraph with vertex set $V(H_1) \cup V(H_2)$ and hyperedge set $E(H) = E(H_1) \cup E(H_2) \cup \{e \subseteq V(H) \mid |e| = r, e \cap V(H_1) \neq \emptyset, e \cap V(H_2) \neq \emptyset\}$.

Please note that the disjoint union for r-uniform hypergraphs is identical to the disjoint union for graphs given in Section 2.2. We use different notations, since especially when we work with r-uniform hypergraphs and their primal graphs it is useful to know whether we combine hypergraphs or graphs.

Remark 2. *Every join operation $H_1 \otimes H_2$ inserts $\sum_{i=1}^{r-1} \binom{|V(H_1)|}{i}\binom{|V(H_2)|}{r-i}$ hyperedges of cardinality r. The related sum $\sum_{i=0}^{r} \binom{|V(H_1)|}{i}\binom{|V(H_2)|}{r-i} = \binom{|V(H_1)|+|V(H_2)|}{r}$ is known as the* Vandermonde's identity*. Thus, the number of hyperedges created by the join operation $H_1 \otimes H_2$ is $\binom{|V(H_1)|+|V(H_2)|}{r} - \binom{|V(H_1)|}{r} - \binom{|V(H_2)|}{r}$.*

Remark 3. *The join operation $H = H_1 \otimes H_2$ needs at least r vertices in both combined graphs. Thus, for $r > |V(H_1)| + |V(H_2)|$ it does not insert a hyperedge and also every join operation in H_1 and H_2 does not insert a hyperedge, which implies that $|E(H)| = 0$. Further, for $r = |V(H_1)| + |V(H_2)|$ the join inserts exactly one hyperedge, which implies that $|E(H)| = 1$.*

Next we consider lower bounds for the number of hyperedges in $H = H_1 \otimes H_2$ for $|V(H_1)| + |V(H_2)| > r$.

Lemma 1. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs, $r \geq 2$, and $|V(H_1)| + |V(H_2)| > r$, then $|E(H_1 \otimes H_2)| \geq |V(H_1)| + |V(H_2)| - 1$.*

Proof. First we show the result for $|V(H_1)| + |V(H_2)| = r + 1$.

- If $|V(H_1)| < r$ and $|V(H_2)| < r$, then by Remark 2 it holds $|E(H_1 \otimes H_2)| \geq \binom{r+1}{r} - \binom{|V(H_1)|}{r} - \binom{|V(H_2)|}{r} = \binom{r+1}{r} - 0 - 0 = r + 1 = |V(H_1)| + |V(H_2)|$.
- If either $|V(H_1)| = r$ and $|V(H_2)| < r$ or $|V(H_1)| < r$ and $|V(H_2)| = r$, then either $|V(H_2)| = 1$ or $|V(H_1)| = 1$ and by Remark 2 it holds $|E(H_1 \otimes H_2)| \geq \binom{r+1}{r} - \binom{|V(H_1)|}{r} - \binom{|V(H_2)|}{r} = \binom{r+1}{r} - 1 - 0 = r = |V(H_1)| + |V(H_2)| - 1$.

For $|V(H_1)| + |V(H_2)| \geq r + 2$ we remove vertices from $V(H_1)$ and $V(H_2)$ in order to have exactly $r + 1$ vertices. Formally we choose $W \subseteq V(H_1) \cup V(H_2)$ such that $W \neq V(H_1)$,

$W \neq V(H_2)$, $|W| = |V(H_1)| + |V(H_2)| - r - 1$ and consider the induced subhypergraphs $H_{1,V(H_1)\setminus W}$ of H_1 induced by $V(H_1) \setminus W$ and $H_{2,V(H_2)\setminus W}$ of H_2 induced by $V(H_2) \setminus W$. Since the removal of every vertex $w \in W$ and all hyperedges containing w reduces the number of hyperedges at least by one we have:

$$\begin{aligned} |E(H_1 \otimes H_2)| &\geq |E(H_{1,V(H_1)\setminus W}) \otimes H_{2,V(H_2)\setminus W})| + |W| \\ &\geq |V(H_1) \setminus W| + |V(H_2) \setminus W| - 1 + |W| \quad \text{first case} \\ &= |V(H_1)| + |V(H_2)| - |W| - 1 + |W| \\ &= |V(H_1)| + |V(H_2)| - 1 \end{aligned}$$

This shows the statement of the lemma. □

Corollary 1. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs, $r \geq 2$, and $|V(H_1)| + |V(H_2)| > r$, then $|E(H_1 \otimes H_2)| \geq r$.*

The class of *r-co-hypergraphs* is recursively defined for $r \geq 2$ as follows.

Definition 2 (r-co-hypergraphs)**.** *Every hypergraph $(\{v\}, \emptyset)$ on a single vertex, denoted by v, is an* r-co-hypergraph. *If H_1 and H_2 are vertex-disjoint r-co-hypergraphs, then $H_1 \oplus H_2$ and $H_1 \otimes H_2$ are* r-co-hypergraphs, *too.*

Example 2.

1. *The expression $(a \oplus b) \otimes c$ defines the 3-co-hypergraph shown in Figure 1a, which is the complete* 3*-uniform hypergraph on* 3 *vertices K_3^3.*
2. *The expression $(a \oplus b) \otimes (c \oplus d)$ defines the 3-co-hypergraph shown in Figure 1b, which is the complete* 3*-uniform hypergraph on* 4 *vertices K_4^3. Obviously, the expression $((a \oplus b) \otimes c) \otimes d$ also defines the 3-co-hypergraph K_4^3 shown in Figure 1b.*

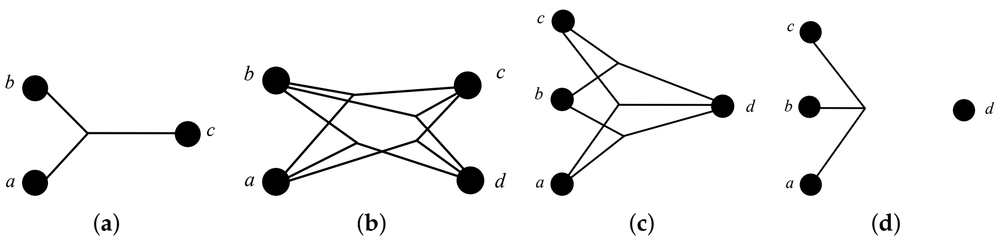

Figure 1. Four 3-co-hypergraphs considered in Examples 2 and 4. (**a**) The 3-co-hypergraph K_3^3 defined by expression $(a \oplus b) \otimes c$, (**b**) the 3-co-hypergraph K_4^3 defined by expression $(a \oplus b) \otimes (c \oplus d)$ and also by expression $((a \oplus b) \otimes c) \otimes d$, (**c**) the 3-co-hypergraph H defined by expression $((a \oplus b) \oplus c) \otimes d$, and (**d**) the 3-co-hypergraph $\overline{H}$ defined by expression $((a \otimes b) \otimes c) \oplus d$. We represent the vertices of hypergraphs graphically by black dots and the hyperedges are depicted as stars that have the vertices as their leaves.

Example 3.

1. *Every empty hypergraph with $n \in \mathbb{N}$ vertices I_n is an r-co-hypergraph for every $r \geq 2$ by the expression $(\ldots((v_1 \oplus v_2) \oplus v_3) \oplus \ldots) \oplus v_n$.*
2. *Every complete r-uniform hypergraph K_r^r on r vertices and one hyperedge of cardinality r is an r-co-hypergraph by the expression $((\ldots(v_1 \oplus v_2) \oplus \ldots) \oplus v_{r-1}) \otimes v_r$.*
3. *Every complete r-uniform hypergraph K_n^r on $n \geq r$ vertices is an r-co-hypergraph by the expression $(\ldots(((\ldots(v_1 \oplus v_2) \oplus \ldots) \oplus v_{r-1}) \otimes v_r) \otimes v_{r+1}) \ldots) \otimes v_n$.*

Remark 4. *The set of 2-co-hypergraphs is exactly the set of co-graphs.*

Next we consider some properties of the complement of r-uniform hypergraphs.

Example 4. *The expression* $((a \oplus b) \oplus c) \otimes d$ *defines the 3-co-hypergraph* H *shown in Figure 1c. It holds that vertices* a *and* b *are adjacent in* H *and also in the complement hypergraph* $\overline{H}$ *defined by expression* $((a \otimes b) \otimes c) \oplus d$ *shown in Figure 1d. Thus, sets of adjacent vertices (co-independent sets, see Section 4.4) do not correspond to sets of non-adjacent vertices (independent sets, see Section 4.3) in the complement graph (w.r.t.* r*-uniform hypergraphs), see Remark 10. A positive result in this context is obtained by considering sets including all hyperedges (co-stable sets, see Section 4.2) or none hyperedge (stable sets, see Section 4.1), see Remark 7.*

Using Lemma 1 we can establish lower bounds on the number of hyperedges in connected r-co-hypergraphs.

Corollary 2. *Let* $r \geq 2$ *and* H *be a connected* r*-co-hypergraph with* $|V(H)| > r$*, then* $|E(H)| \geq |V(H)| - 1$.

Corollary 3. *Let* $r \geq 2$ *and* H *be a connected* r*-co-hypergraph with* $|V(H)| > r$*, then* $|E(H)| \geq r$.

Closure properties for several graph transformations on co-graphs are given in Theorem 1. Next, we consider such properties on r-co-hypergraphs for $r \geq 3$.

Theorem 3. *Let* $r \geq 3$*. The set of all* r*-co-hypergraphs is (i) closed under complementation for* r*-uniform hypergraphs (ii) closed under induced subhypergraphs, (iii) not closed under subhypergraphs, and (iv) not closed under partial subhypergraphs.*

Proof.

(i) Let H be some r-co-hypergraph. We show by induction on the recursive structure of H that $\overline{H}$ is an r-co-hypergraph.

- If H is a single-vertex r-co-hypergraph, then $\overline{H}$ is an r-co-hypergraph.
- If $H = H_1 \oplus H_2$, then $\overline{H}$ is an r-co-hypergraph by the expression $\overline{H} = \overline{H_1} \otimes \overline{H_2}$.
- If $H = H_1 \otimes H_2$, then $\overline{H}$ is an r-co-hypergraph by the expression $\overline{H} = \overline{H_1} \oplus \overline{H_2}$.

(ii) Let H be some r-co-hypergraph and $W \subseteq V(H)$. An expression for the induced subhypergraph of H defined by the vertices of W can be obtained from an expression for H by deleting operations corresponding to vertices from $V(H) \setminus W$ and some obvious cleaning transformations.

(iii) The counterexample $H_r = I_{r-1} \otimes I_1$, $r \geq 3$, defines an r-co-hypergraph. Every subset $V' \subseteq V(H_r)$ on $r-1$ vertices defines a subhypergraph of H_r which is no r-co-hypergraph.

(iv) The counterexample $H_r = I_1 \otimes I_r$, $r \geq 3$, defines an r-co-hypergraph. By removing one hyperedge from $E(H_r)$ we obtain a partial subhypergraph of H_r on $r+1$ vertices and $r-1$ hyperedges, which is no r-co-hypergraph by Corollary 3.

This shows the statements of the theorem. □

For $r = 2$, it is well known that co-graphs can be defined by disjoint union and complementation, which also holds for $r \geq 3$ using the complement of r-uniform hypergraphs.

Lemma 2. *Let* H *be some* r*-uniform hypergraph. The following statements are equivalent.*

1. *H is an r-co-hypergraph.*
2. *H can be generated from the single-vertex hypergraph by complementation and disjoint union.*
3. *H can be generated from the single-vertex hypergraph by complementation and join.*

Proof. (1) $\Rightarrow$ (2) The join operation $H_1 \otimes H_2$ can be obtained by $\overline{\overline{H_1} \oplus \overline{H_2}}$. (2) $\Rightarrow$ (3) The disjoint union operation $H_1 \oplus H_2$ can be obtained by $\overline{\overline{H_1} \otimes \overline{H_2}}$. (3) $\Rightarrow$ (1) The complementation operation $\overline{H_1 \otimes H_2}$ can be obtained by $\overline{H_1} \oplus \overline{H_2}$. Both $\overline{H_1}$ and $\overline{H_2}$ can be handled in the same way until we encounter single-vertex hypergraphs. □

Lemma 3. *Let H be some r-co-hypergraph with $|V(H)| > 1$. Then either H or $\overline{H}$ is not connected.*

Proof. If H is the disjoint union of two r-co-hypergraphs H_1 and H_2, then H is obviously not connected. By Lemma 2 we have $\overline{H} = \overline{H_1 \oplus H_2} = \overline{H_1} \otimes \overline{H_2}$ and thus $\overline{H}$ is connected. If H is the join of two r-co-hypergraphs H_1 and H_2, then H is obviously connected. By Lemma 2 we have $\overline{H} = \overline{H_1 \otimes H_2} = \overline{H_1} \oplus \overline{H_2}$ and thus $\overline{H}$ is not connected. □

Every r-co-hypergraph can be represented by a binary decomposition tree (aka expression-tree) that reflects its construction starting with its vertices and using disjoint union and join operations.

Definition 3 (decomposition tree of r-co-hypergraphs)**.** *The* decomposition tree $T(H)$ *for an r-co-hypergraph H that contains only a single vertex v consists of a single node r (the root of T) labeled by v. The* decomposition tree $T(H)$ *for an r-co-hypergraph $H = H_1 \oplus H_2$ ($H = H_1 \otimes H_2$) is comprised of a copy T_1 of the decomposition tree for H_1, a copy T_2 of the decomposition tree for H_2, an additional node w (the root of T) labeled by $\oplus$ ($\otimes$), and two additional edges between the roots of T_1 and T_2 and node w. The root of T_1 represents the* left child *of w and the root of T_2 represents the* right child *of w.*

For some given hypergraph H we next show how to decide whether H is an r-co-hypergraph, and if the answer is positive we find a decomposition tree $T(H)$ in polynomial time by iterating the following steps until the hypergraphs consist of one vertex.

- If H is not connected, then let $H_1, \dots, H_k$ be its connected components. Recursively create the decomposition tree for each component H_i and create a $\oplus$-node as the predecessor node.
- Otherwise, consider the complement hypergraph $\overline{H}$ and its connected components $H_1, \dots, H_k$.
  - If $k = 1$ it holds that H and $\overline{H}$ are connected, thus H is no r-co-hypergraph by Lemma 3, we reject the input and stop our method.
  - If $k \geq 2$ we recursively create the decomposition tree for each complement hypergraph $\overline{H_i}$ and create a $\otimes$-node as the predecessor.

The obtained tree has to be modified at every node with more than two successors in order to make tree binary.

To characterize the primal graphs of r-co-hypergraphs, we consider the primal graphs $P(H)$ of the disjoint union $H = H_1 \oplus H_2$ and the join $H = H_1 \otimes H_2$ of two r-uniform hypergraphs H_1 and H_2.

Lemma 4. *Let $r \geq 3$ and H_1, H_2 be two r-uniform hypergraphs. The primal graph of $H_1 \oplus H_2$ is the graph $P(H_1) \cup P(H_2)$. The primal graph of $H_1 \otimes H_2$ is for $r > |V(H_1)| + |V(H_2)|$ the empty graph $I_{|V(H_1)|+|V(H_2)|}$ and for $r \leq |V(H_1)| + |V(H_2)|$ the clique $K_{|V(H_1)|+|V(H_2)|}$.*

Proof. Let H_1 and H_2 be two r-uniform hypergraphs and $r \geq 3$. For the primal graph of $H_1 \oplus H_2$ we know that the disjoint union for r-uniform hypergraphs does not change the adjacent vertices in $V(H_1) \cup V(H_2)$, which implies that $P(H_1 \oplus H_2)$ is the graph $P(H_1) \cup P(H_2)$.

For the primal graph of $H_1 \otimes H_2$ for $r > |V(H_1)| + |V(H_2)|$ we know by Remark 3 that $H_1 \otimes H_2$ has no hyperedges, which implies that $P(H_1 \otimes H_2)$ has no edges and thus $P(H_1 \otimes H_2)$ is isomorphic to the empty graph $I_{|V(H_1)|+|V(H_2)|}$.

For the primal graph of $H_1 \otimes H_2$ for $r \leq |V(H_1)| + |V(H_2)|$ it is easy to verify that every two vertices $u, v \in V(H_1 \otimes H_2) = V(H_1) \cup V(H_2)$ are adjacent in $H_1 \otimes H_2$, which implies that $\{u, v\} \in E(P(H_1 \otimes H_2))$. Thus, $P(H_1 \otimes H_2)$ is isomorphic to the complete graph $K_{|V(H_1)|+|V(H_2)|}$. □

This leads to the following properties of the primal graphs of r-co-hypergraphs.

Corollary 4. *Let H be some r-co-hypergraph. For $r = 2$ the primal graph $P(H)$ is H and thus a co-graph. For $r \geq 3$ and $H = H_1 \oplus H_2$ the primal graph $P(H)$ is the co-graph $P(H_1) \cup P(H_2)$. For $r \geq 3$ and $H = H_1 \otimes H_2$ for $r > |V(H_1)| + |V(H_2)|$ the primal graph $P(H)$ is the co-graph $I_{|V(H_1)|+|V(H_2)|}$ and for $r \leq |V(H_1)| + |V(H_2)|$ the primal graph $P(H)$ is the co-graph $K_{|V(H_1)|+|V(H_2)|}$.*

Corollary 5. *Let H be some r-uniform hypergraph and $r \geq 3$. Then, H is an r-co-hypergraph if and only if there is some $\ell \leq |V(H)|$ such that $P(H)$ is the disjoint union of ℓ cliques K_{n_i}, $1 \leq i \leq \ell$, where $n_i = 1$ or $n_i \geq r$ and $\sum_{i=1}^{\ell} n_i = |V(H)|$.*

Since the disjoint union of cliques is the set of P_3-free graphs, we know that for r-co-hypergraphs H the primal graphs $P(H)$ are P_3-free graphs.

4. Algorithms on *r*-Co-Hypergraphs

In the subsequent subsections (Sections 4.1–4.9) we show specific formulas how to compute several graph parameters for r-uniform hypergraphs defined by the binary disjoint union and join of two r-uniform hypergraphs. In the final subsection (Section 4.10) we apply our formulas to compare the considered parameters and conclude that they can be computed in polynomial time when restricted to r-uniform hypergraphs defined by disjoint union and join operation and thus on r-co-hypergraphs.

4.1. Max Stable Set

A set $V \subseteq V(H)$ that does not contain all vertices of a hyperedge of $E(H)$, i.e., $e \cap V \neq e$ for all $e \in E(H)$, is called a *stable set*. The *stability number* of a hypergraph H, denoted by $\alpha(H)$, is defined as the maximum cardinality of a stable set in H.

Remark 5. *For every r-uniform hypergraph H with $|E(H)| \geq 1$ we know $r - 1 \leq \alpha(H) \leq |V(H)| - 1$.*

Lemma 5. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 2$.*

1. $\alpha((\{v\}, \emptyset)) = 1$
2. $\alpha(H_1 \oplus H_2) = \alpha(H_1) + \alpha(H_2)$
3. $\alpha(H_1 \otimes H_2) = \max(\alpha(H_1), \alpha(H_2), r - 1)$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $\alpha(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\alpha((\{v\}, \emptyset)) = 1$ holds by definition.
2. The result follows since V is a stable set in $H_1 \oplus H_2$ if and only if $V \cap V(H_1)$ is a stable set in H_1 and $V \cap V(H_2)$ is a stable set in H_2.
3. The result follows since V is a stable set in $H_1 \otimes H_2$ if and only if V is a stable set in H_1 or V is a stable set in H_2, or $|V| \leq r - 1$.

4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3).

This shows the statements of the lemma. □

For $r = 2$ a stable set in an r-uniform hypergraph is an independent set in a graph. The results of Lemma 5 generalize the results of [13] (see Table 1) for computing the size of a largest independent set in a co-graph.

4.2. Max Co-Stable Set

For r-uniform hypergraphs H a *co-stable set* C is a set of vertices such that every subset of C of order r forms a hyperedge of H. Every set of up to $r-1$ vertices in an r-uniform hypergraph is a *(trivial) co-stable set* in H (The given definition of a co-stable set in r-uniform hypergraphs extends the definition of a clique in a graph. While a single vertex in a graph is a clique, in r-uniform hypergraphs every set of up to $r-1$ vertices is a (trivial) co-stable set, since every possible hyperedge is present). The *co-stability number* of an r-uniform hypergraph H, denoted by $\omega(H)$, is the maximum cardinality of a co-stable set in H.

Remark 6. *For every r-uniform hypergraph H with $|E(H)| \geq 1$ we know $r \leq \omega(H) \leq |V(H)|$.*

Since every co-stable set in some r-uniform hypergraph H corresponds to a stable set in $\overline{H}$ we can conclude the following remark.

Remark 7. *For every r-uniform hypergraph H it holds: $\omega(H) = \alpha(\overline{H})$.*

Lemma 6. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 2$.*

1. $\omega((\{v\}, \emptyset)) = 1$
2. $\omega(H_1 \oplus H_2) = \max(\omega(H_1), \omega(H_2), \min(|V(H_1)| + |V(H_2)|, r-1))$
3. $\omega(H_1 \otimes H_2) = \omega(H_1) + \omega(H_2)$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $\omega(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\omega((\{v\}, \emptyset)) = 1$ holds by definition.
2. By Remark 7 and Lemma 5(3.) for $r \leq |V(H_1)| + |V(H_2)|$ it holds

$$\begin{aligned} \omega(H_1 \oplus H_2) &= \alpha(\overline{H_1 \oplus H_2}) \\ &= \alpha(\overline{H_1} \otimes \overline{H_2}) \\ &= \max(\alpha(\overline{H_1}), \alpha(\overline{H_2}), r-1) \\ &= \max(\omega(H_1), \omega(H_2), \min(|V(H_1)| + |V(H_2)|, r-1)). \end{aligned}$$

For $r > |V(H_1)| + |V(H_2)|$ it holds $\omega(H_1 \oplus H_2) = |V(H_1)| + |V(H_2)|$, which is equal to $\max(\omega(H_1), \omega(H_2), \min(|V(H_1)| + |V(H_2)|, r-1))$.

3. By Remark 7 and Lemma 5(2.) it holds

$$\omega(H_1 \otimes H_2) = \alpha(\overline{H_1 \otimes H_2}) = \alpha(\overline{H_1} \oplus \overline{H_2}) = \alpha(\overline{H_1}) + \alpha(\overline{H_2}) = \omega(H_1) + \omega(H_2).$$

4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3). Thus, $\omega(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$ follows by the definition of the co-stability number.

This shows the statements of the lemma. □

For $r = 2$ a co-stable set in an r-uniform hypergraph is a clique in a graph. The results of Lemma 6 generalize the results of [13] (see Table 1) for computing the size of a largest clique in a co-graph.

4.3. Max Independent Set

A set $V \subseteq V(H)$ in which no two vertices are allowed to be adjacent, i.e., are not contained in a common hyperedge, is called an *independent set* or *strongly stable set*. The *independence number* of a hypergraph H, denoted by $\beta(H)$, is defined as the maximum cardinality of an independent set in H. (By using $\beta(H)$ we follow the notations of [26]. In [27] the maximum cardinality of a strongly stable set in H is denoted by $\overline{\alpha}(H)$ and in [7] the maximum cardinality of a strongly stable set in H is denoted by $\alpha'(H)$.)

The following relation has been mentioned in [26].

Remark 8 ([26])**.** *For every hypergraph H it holds that $\alpha(H) \geq \beta(H)$. If H is a graph then equality holds.*

Remark 9. *For every r-uniform hypergraph H with $|E(H)| \geq 1$ we know $1 \leq \beta(H) \leq |V(H)| - r + 1$.*

There is a close relation between the independence number of a hypergraph and the independence number of its primal graph.

Lemma 7. *For every hypergraph H it holds that $\alpha(P(H)) = \beta(H)$.*

Proof. Let $V \subseteq V(H)$ for a hypergraph H. Set V is an independent set in H. $\Leftrightarrow$ For every two vertices $u, v \in V$ there is no $e \in E(H)$ such that $\{u, v\} \subseteq e$. $\Leftrightarrow$ For every two vertices $u, v \in V$ it holds $\{u, v\} \notin E(P(H))$. $\Leftrightarrow$ Set V is an independent set in graph $P(H)$. □

For $r = 2$ computing the independence number of an r-co-hypergraph can be done by computing the independence number α of a co-graph (see Table 1). Next we assume $r \geq 3$. The following result for the join operation differs significantly to the case $r = 2$.

Lemma 8. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 3$.*

1. $\beta((\{v\}, \emptyset)) = 1$
2. $\beta(H_1 \oplus H_2) = \beta(H_1) + \beta(H_2)$
3. $\beta(H_1 \otimes H_2) = 1$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $\beta(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\beta((\{v\}, \emptyset)) = 1$ holds by definition.
2. By Lemma 7 and Corollary 4 it holds $\beta(H_1 \oplus H_2) = \alpha(P(H_1 \oplus H_2)) = \alpha(P(H_1) \cup P(H_2)) = \alpha(P(H_1)) + \alpha(P(H_2)) = \beta(H_1) + \beta(H_2)$.
3. If $r \leq |V(H_1)| + |V(H_2)|$, then for $r \geq 3$ the join operation implies that in $H_1 \otimes H_2$ every two vertices are adjacent.
4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3).

This shows the statements of the lemma. □

Corollary 6. *Let $r \geq 3$ and H be a connected r-co-hypergraph, then $\beta(H) = 1$.*

4.4. Max Co-Independent Set

A set $V \subseteq V(H)$ in which every two vertices are adjacent, i.e., are contained in a common hyperedge, is called a *co-independent set*. The *co-independence number* of a hypergraph H, denoted by $\delta(H)$, is defined as the maximum cardinality of a co-independent set in H.

Remark 10. *A relation similar to that given in Remark 7 on $\delta(H)$ and $\beta(\overline{H})$ is not possible, since two vertices in hypergraphs can be adjacent in the hypergraph and in its edge complement as shown in Example 4. For the hypergraph H shown in Figure 1c and its complement hypergraph $\overline{H}$ shown in Figure 1d we obtain the values $\beta(H) = 1$, $\delta(H) = 4$, $\beta(\overline{H}) = 2$, and $\delta(\overline{H}) = 3$.*

Remark 11. *For every r-uniform hypergraph H with $|E(H)| \geq 1$ we know $r \leq \delta(H) \leq |V(H)|$.*

There is a close relation between the co-independence number of a hypergraph and the clique number of its primal graph.

Lemma 9. *For every hypergraph H it holds that $\omega(P(H)) = \delta(H)$.*

Proof. Let $V \subseteq V(H)$ for a hypergraph H. Set V is a co-independent set in hypergraph H. $\Leftrightarrow$ For every two vertices $u, v \in V$ there is some $e \in E(H)$ such that $\{u, v\} \subseteq e$. $\Leftrightarrow$ For every two vertices $u, v \in V$ it holds $\{u, v\} \in E(P(H))$. $\Leftrightarrow$ Set V is a clique in graph $P(H)$. □

For $r = 2$ computing the co-independence number of an r-co-hypergraph can be done by computing the clique number ω of a co-graph (see Table 1). Next we assume $r \geq 3$. The following result for the join operation differs significantly to the case $r = 2$.

Lemma 10. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 3$.*

1. $\delta((\{v\}, \emptyset)) = 1$
2. $\delta(H_1 \oplus H_2) = \max(\delta(H_1), \delta(H_2))$
3. $\delta(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $\delta(H_1 \otimes H_2) = 1$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\delta((\{v\}, \emptyset)) = 1$ holds by definition.
2. By Lemma 9 and Corollary 4 it holds $\delta(H_1 \oplus H_2) = \omega(P(H_1 \oplus H_2)) = \omega(P(H_1) \cup P(H_2)) = \max(\omega(P(H_1)), \omega(P(H_2))) = \max(\delta(H_1), \delta(H_2))$.
3. If $r \leq |V(H_1)| + |V(H_2)|$, then for $r \geq 3$ the join operation implies that in $H_1 \otimes H_2$ every two vertices are adjacent.
4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3).

This shows the statements of the lemma. □

Corollary 7. *Let $r \geq 3$ and H be a connected r-co-hypergraph, then $\delta(H) = |V(H)|$.*

4.5. Min Vertex Cover

In a hypergraph, a *vertex cover* is a set of vertices such that every hyperedge contains at least one vertex from that set. The *vertex cover number (aka transversal number)* of a hypergraph H, denoted by $\tau(H)$, is the smallest size of a vertex cover in H.

In [9] it was mentioned (without a proof) that Gallai's Theorem [28] on the close relation between the vertex cover number and the independence number of graphs also holds for hypergraphs, if we consider the stability number.

Lemma 11. *For every hypergraph H it holds $\alpha(H) + \tau(H) = |V(H)|$.*

Proof. We show the equivalent equation $\alpha(H) = |V(H)| - \tau(H)$. Let $V \subseteq V(H)$ for a hypergraph H. Set V is a vertex cover in H. $\Leftrightarrow$ For every hyperedge $e \in E(H)$ it holds $e \cap V \neq \emptyset$. $\Leftrightarrow$ For every hyperedge $e \in E(H)$ it holds $e \cap (V(H) \setminus V) \neq e$. $\Leftrightarrow$ Set $V(H) \setminus V$ is a stable set in H. □

Remark 12. *For every r-uniform hypergraph H with $|E(H)| \geq 1$ we know $1 \leq \tau(H) \leq |V(H)| - r + 1$.*

We apply Lemma 11 to give formulas for the calculation of the vertex cover number of an r-co-hypergraph, which will be modified for computing a 2-transversal in Section 4.6.

Lemma 12. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 2$.*

1. $\tau((\{v\}, \emptyset)) = 0$
2. $\tau(H_1 \oplus H_2) = \tau(H_1) + \tau(H_2)$
3. $\tau(H_1 \otimes H_2) = \min(|V(H_1)| + \tau(H_2), |V(H_2)| + \tau(H_1), |V(H_1)| + |V(H_2)| - r + 1)$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $\tau(H_1 \otimes H_2) = 0$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\tau((\{v\}, \emptyset)) = 0$ holds by definition.
2. By Lemmas 11 and 5(2.) it holds $\tau(H_1 \oplus H_2) = |V(H_1 \oplus H_2)| - \alpha(H_1 \oplus H_2) = |V(H_1)| + |V(H_2)| - \alpha(H_1) - \alpha(H_2) = \tau(H_1) + \tau(H_2)$.
3. By Lemmas 11 and 5(3.) it holds

$$
\begin{aligned}
\tau(H_1 \otimes H_2) &= |V(H_1 \otimes H_2)| - \alpha(H_1 \otimes H_2) \\
&= |V(H_1)| + |V(H_2)| - \max(\alpha(H_1), \alpha(H_2), r - 1) \\
&= \min(|V(H_1)| + |V(H_2)| - \alpha(H_1), |V(H_1)| + |V(H_2)| - \alpha(H_2), \\
&\quad |V(H_1)| + |V(H_2)| - r + 1) \\
&= \min(|V(H_2)| + \tau(H_1), |V(H_1)| + \tau(H_2), |V(H_1)| + |V(H_2)| - r + 1).
\end{aligned}
$$

4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3).

This shows the statements of the lemma. □

For $r = 2$ a vertex cover in an r-uniform hypergraph is a vertex cover in a graph. The results of Lemma 12 generalize the known results of [13] (see Table 1) for computing the size of a smallest vertex cover in a co-graph.

4.6. Min 2-Transversal

In a hypergraph, a *2-transversal* is a set of vertices such that every hyperedge contains at least two vertices from that set. The *2-transversal number* of a hypergraph H, denoted by $\tau_2(H)$, is the smallest size of a 2-transversal in H.

Remark 13. *For every r-uniform hypergraph H with $|E(H)| \geq 1$ we know $2 \leq \tau_2(H) \leq |V(H)| - r + 2$.*

Lemma 13. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 2$.*

1. $\tau_2((\{v\}, \emptyset)) = 0$
2. $\tau_2(H_1 \oplus H_2) = \tau_2(H_1) + \tau_2(H_2)$
3. $\tau_2(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)| - r + 2$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $\tau_2(H_1 \otimes H_2) = 0$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\tau_2((\{v\}, \emptyset)) = 0$ holds by definition.
2. The result follows since V is a 2-transversal in $H_1 \oplus H_2$ if and only if $V \cap V(H_1)$ is a 2-transversal in H_1 and $V \cap V(H_2)$ is a 2-transversal in H_2.
3. Let $k = |V(H_1)| + |V(H_2)| - r + 2$ be the claimed 2-transversal number.

To show the upper bound $\tau_2(H_1 \otimes H_2) \le k$ we choose $|V(H_1)| + |V(H_2)| - r + 2$ arbitrary vertices from $V(H_1) \cup V(H_2)$ to obtain a 2-transversal V' of $H_1 \otimes H_2$. This implies that every hyperedge of $H_1 \otimes H_2$ has at most $|V(H_1)| + |V(H_2)| - (|V(H_1)| + |V(H_2)| - r + 2) = r - 2$ vertices which are not in V'. Thus, every hyperedge of $H_1 \otimes H_2$ has at least $r - (r - 2) = 2$ vertices from V', which implies that V' is a 2-transversal of hypergraph $H_1 \otimes H_2$.
In order to show the lower bound $\tau_2(H_1 \otimes H_2) \ge k$, we assume that there is a 2-transversal V' of $H_1 \otimes H_2$ such that $|V'| \le |V(H_1)| + |V(H_2)| - r + 1$. Thus, $W = V(H_1) \cup V(H_2) \setminus V'$ contains at least $r - 1$ vertices that are not in the 2-transversal V'.

- If $W \subseteq V(H_1)$, then we choose $x \in V(H_2)$ and $r - 1$ vertices $x_1, \dots, x_{r-1}$ from W. The join operation defines among others the hyperedge $\{x, x_1, \dots, x_{r-1}\}$ which has only the vertex x of V'.
- If $W \subseteq V(H_2)$, then we choose $x \in V(H_1)$ and $r - 1$ vertices $x_1, \dots, x_{r-1}$ from W. The join operation defines among others the hyperedge $\{x, x_1, \dots, x_{r-1}\}$ which has only the vertex x of V'.
- If $W \subseteq V(H_1) \cup V(H_2)$ and $W \not\subseteq V(H_1)$, $W \not\subseteq V(H_2)$, then we choose $x \in (V(H_1) \cup V(H_2)) \setminus W = V'$ and $r - 1$ vertices $x_1, \dots, x_{r-1}$ from W. The join operation defines among others the hyperedge $\{x, x_1, \dots, x_{r-1}\}$, which has only the vertex x of V'.

Thus V' is no 2-transversal, the assumption is not true, and the lower bound is shown.

4. This follows from the same arguments as given in the proof of Lemma 12(4.).

This shows the statements of the lemma. □

Lemma 13 shows that connected r-co-hypergraphs have the largest possible 2-transversal number for r-uniform hypergraphs mentioned in Remark 13.

Remark 14. *A k-*transversal *in a hypergraph is a set of vertices, such that every hyperedge contains at least k vertices from that set. The k-*transversal number *of a hypergraph H, denoted by $\tau_k(H)$, is the smallest size of a k-transversal in H. Our solution can be generalized to find k-transversals in r-co-hypergraphs for every $k \ge 2$ and $r \ge k$. Obviously for $r = k$ a k-transversal of a connected r-uniform hypergraph is the entire vertex set. The solution in Lemma 13(3.) can be generalized to $\tau_k(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)| - r + k$, if $r \le |V(H_1)| + |V(H_2)|$.*

4.7. Min Dominating Set

A set $D \subseteq V(H)$ is a *dominating set* of the hypergraph H if for every $v \in V(H) \setminus D$ there exists $u \in D$ such that u and v are adjacent in H; that is, there exists $e \in E(H)$ such that $u, v \in e$. The *domination number* of a hypergraph H, denoted by $\gamma(H)$, is defined as the minimum cardinality of a dominating set in H.

Remark 15. *For every r-uniform hypergraph H with $|E(H)| \ge 1$ we know $1 \le \gamma(H) \le |V(H)| - r + 1$.*

Lemma 14. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \ge 2$.*

1. $\gamma((\{v\}, \emptyset)) = 1$
2. $\gamma(H_1 \oplus H_2) = \gamma(H_1) + \gamma(H_2)$
3. $\gamma(H_1 \otimes H_2) = \min(\gamma(H_1), \gamma(H_2), 2)$, *if* $r \le |V(H_1)| + |V(H_2)|$
4. $\gamma(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\gamma((\{v\}, \emptyset)) = 1$ holds by definition.

2. Follows by the same arguments as given in the proof of Lemma 13(2.).
3. For the upper bound $\gamma(H_1 \otimes H_2) \leq \min(\gamma(H_1), \gamma(H_2), 2)$ we observe that every dominating set of H_1, every dominating set of H_2, and every set $D = \{x, y\}$ with $x \in V(H_1)$ and $y \in V(H_2)$ is a dominating set of $H_1 \oplus H_2$.
   In order to show the lower bound $\gamma(H_1 \otimes H_2) \geq \min(\gamma(H_1), \gamma(H_2), 2)$, we assume on the contrary that there is a dominating set of size $k < \min(\gamma(H_1), \gamma(H_2), 2)$ for $H_1 \otimes H_2$. We distinguish between the following two cases.
   - If $\min(\gamma(H_1), \gamma(H_2)) < 2$, then $\min(\gamma(H_1), \gamma(H_2), 2) = \min(\gamma(H_1), \gamma(H_2)) \leq 1$. But a dominating set of size $k < 1$ for $H_1 \otimes H_2$ is not possible since $|V(H_1)| \geq 1$ and $|V(H_2)| \geq 1$.
   - If $\min(\gamma(H_1), \gamma(H_2)) \geq 2$, then $\min(\gamma(H_1), \gamma(H_2), 2) = 2$. A dominating set of size $k < 2$ for $H_1 \otimes H_2$ has exactly one vertex $v \in V(H_j)$ for $j = 1$ or $j = 2$. But then v has to be adjacent to all vertices in $V(H_j) \setminus \{v\}$ which implies that $\min(\gamma(H_1), \gamma(H_2)) = 1$. Thus, a dominating set of size $k < 2$ for $H_1 \otimes H_2$ is not possible.

   Which leads to a contradiction in both cases and the lower bound follows.
4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3).

This shows the statements of the lemma. □

Corollary 8. *Let $r \geq 2$ and H be a connected r-co-hypergraph, then $1 \leq \gamma(H) \leq 2$.*

For $r = 2$ a dominating set in an r-uniform hypergraph is a dominating set in a graph. Thus, the results of Lemma 14 generalize the results of Table 1 for computing the size of a smallest dominating set in a co-graph.

4.8. Min Strong Coloring

A *strong k-coloring* of a hypergraph H is a function $c : V(H) \to \{1, \ldots, k\}$ such that if $u, v \in e$ for some $e \in E(H)$ and $u \neq v$, then we have that $c(u) \neq c(v)$. Thus, we assign distinct colors to vertices that are contained in a common hyperedge. The *strong chromatic number* of H, denoted by $s\chi(H)$, is the minimum value of k such that H has a strong k-coloring. A hypergraph H is *strongly k-colorable*, if $s\chi(H) \leq k$.

The strong chromatic number was independently introduced by Fellows [29] and Alon [30]. There is a close relation between the strong chromatic number of a hypergraph and the chromatic number of its primal graph [27].

Lemma 15 ([27]). *For every hypergraph H it holds that $\chi(P(H)) = s\chi(H)$.*

Remark 16. *For graphs H, i.e., 2-uniform hypergraphs, it holds $\chi(H) = s\chi(H)$.*

For $r = 2$ by Remark 16 we can compute the strong chromatic number by the chromatic number of a 2-co-hypergraph. By Remark 4 this can be done by the results on the chromatic number for co-graphs of [13] which are also given in Table 1. Next we assume $r \geq 3$. The following result for the join operation differs significantly to the case $r = 2$.

Lemma 16. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 3$.*

1. $s\chi((\{v\}, \emptyset)) = 1$
2. $s\chi(H_1 \oplus H_2) = \max(s\chi(H_1), s\chi(H_2))$
3. $s\chi(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $s\chi(H_1 \otimes H_2) = 1$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $s\chi((\{v\},\emptyset)) = 1$ holds by definition.
2. The result follows from the following equations.

$$\begin{array}{rcll} s\chi(H_1 \oplus H_2) & = & \chi(P(H_1 \oplus H_2)) & \text{by Lemma 15} \\ & = & \chi(P(H_1) \cup P(H_2)) & \text{by Lemma 4} \\ & = & \max(\chi(P(H_1)), \chi(P(H_2))) & \text{by Table 1} \\ & = & \max(s\chi(H_1), s\chi(H_2)) & \text{by Lemma 15} \end{array}$$

3. If $r \leq |V(H_1)| + |V(H_2)|$, the result follows from the following equations.

$$\begin{array}{rcll} s\chi(H_1 \otimes H_2) & = & \chi(P(H_1 \otimes H_2)) & \text{by Lemma 15} \\ & = & \chi(K_{|V(H_1)|+|V(H_2)|}) & \text{by Lemma 4} \\ & = & |V(H_1)| + |V(H_2)| & \end{array}$$

4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3).

This shows the statements of the lemma. □

Corollary 9. *Let $r \geq 3$ and H be a connected r-co-hypergraph, then $s\chi(H) = |V(H)|$.*

4.9. Max $\mathcal{C}$-Coloring

A $\mathcal{C}$-coloring of a hypergraph is defined as a coloring that satisfies an opposite coloring constraint. This constraint dictates the presence of two vertices with a *common* color inside each hyperedge. A *proper k-$\mathcal{C}$-coloring* of a hypergraph H is a function $c : V(H) \to \{1, \ldots, k\}$ for which every hyperedge $e \in E(H)$ has at least two vertices of the same color, which means that we have to avoid *polychromatic* hyperedges (we use the notation *polychromatic hyperedge* in the sense of [8] (Chapter 11). The same notation is also used to ensure that all k colors of a function $c : V(H) \to \{1, \ldots, k\}$ are used for every hyperedge, see e.g., [31]). The *upper chromatic number* of H, denoted by $\overline{\chi}(H)$, is defined as the maximum value of k such that H has a proper k-$\mathcal{C}$-coloring. A hypergraph H is said to be *k-$\mathcal{C}$-colorable*, if $\overline{\chi}(H) \geq k$.

The vertex sets $V_i = \{v \in V \mid c(v) = i\}$, with $1 \leq i \leq k$, serve to subdivide $V(H)$ into a partition of so called *color classes*.

Remark 17. *For $k \geq 2$ a proper k-$\mathcal{C}$-coloring always can be modified to a proper $(k-1)$-$\mathcal{C}$-coloring, e.g., by merging two color classes into one.*

Remark 18. *If we restrict to r-uniform hypergraphs and $r = 2$ (graphs) the $\mathcal{C}$-coloring problem is simple. Every connected graph G has $\overline{\chi}(G) = 1$. Further every graph G on k connected components has $\overline{\chi}(G) = k$.*

Remark 19. *By [9] every r-uniform hypergraph with at least one hyperedge admits a proper $(r-1)$-$\mathcal{C}$-coloring, because the pigeon-hole principle implies that each hyperedge contains a repeated color in every assignment of $r-1$ colors to the vertices. Therefore, for r-uniform hypergraphs H with at least one hyperedge we have $\overline{\chi}(H) \geq r-1$ and for r-uniform hypergraphs H with no hyperedges we have $\overline{\chi}(H) = |V(H)|$.*

Lemma 17. *Let H_1 and H_2 be two vertex-disjoint r-uniform hypergraphs and $r \geq 2$.*

1. $\overline{\chi}((\{v\},\emptyset)) = 1$
2. $\overline{\chi}(H_1 \oplus H_2) = \overline{\chi}(H_1) + \overline{\chi}(H_2)$
3. $\overline{\chi}(H_1 \otimes H_2) = r-1$, *if* $r \leq |V(H_1)| + |V(H_2)|$
4. $\overline{\chi}(H_1 \otimes H_2) = |V(H_1)| + |V(H_2)|$, *if* $r > |V(H_1)| + |V(H_2)|$

Proof.

1. $\overline{\chi}((\{v\},\emptyset)) = 1$ holds by definition.
2. The lower bound $\overline{\chi}(H_1 \oplus H_2) \geq \overline{\chi}(H_1) + \overline{\chi}(H_2)$ holds, since there obviously is a proper $\mathcal{C}$-coloring for $H_1 \oplus H_2$ using $\overline{\chi}(H_1) + \overline{\chi}(H_2)$ colors.
   The upper bound $\overline{\chi}(H_1 \oplus H_2) \leq \overline{\chi}(H_1) + \overline{\chi}(H_2)$ holds, since otherwise H_1 or H_2 has a proper $\mathcal{C}$-coloring using more than $\overline{\chi}(H_1)$ or $\overline{\chi}(H_2)$ colors, respectively.
3. To show the upper bound $\overline{\chi}(H_1 \otimes H_2) \leq r - 1$, we assume that there is a proper k-$\mathcal{C}$-coloring $c' : V(H_1 \otimes H_2) \to \{1, \ldots, k\}$ such that $k \geq r$. By Remark 17 we can assume that $k = r$. We distinguish between the following cases:
   - If c' uses r different colors $1, \ldots, r$ for the vertices in $V(H_1)$, then by choosing an arbitrary vertex v in $V(H_2)$ using color $c'(v)$, the considered join operation defines a hyperedge e using $r - 1$ vertices from H_1 using pairwise different colors from $\{1, \ldots, r\} \setminus \{c'(v)\}$ and vertex v from H_2 colored by $c'(v)$. Thus, e uses r different colors for the r vertices of e which is not possible in a proper r-$\mathcal{C}$-coloring.
   - If c' uses ℓ, $1 \leq \ell < r$, different colors $c_1, \ldots, c_\ell$ for the vertices in $V(H_1)$, then the considered join operation defines a hyperedge e using ℓ vertices from H_1 using pairwise different colors from $\{c_1, \ldots, c_\ell\}$ and $r - \ell$ vertices from H_2 colored pairwise different colors from $\{1, \ldots, r\} \setminus \{c_1, \ldots, c_\ell\}$. Thus, e uses r different colors for the r vertices of e which is not possible in a proper r-$\mathcal{C}$-coloring.

   Thus, in both cases the assumption is not true and the upper bound is shown.
   In order to show the lower bound $\overline{\chi}(H_1 \otimes H_2) \geq r - 1$ we can apply Remark 19.
4. If $r > |V(H_1)| + |V(H_2)|$, then $E(H_1 \otimes H_2) = \emptyset$ (Remark 3).

This shows the statements of the lemma. □

By Lemma 17 and Remark 19 connected r-co-hypergraphs H have the minimal possible upper chromatic number of $r - 1$ which is possible for r-uniform hypergraphs. Every assignment of $r - 1$ colors to $V(H)$ is a proper $(r-1)$-$\mathcal{C}$-coloring for H.

Voloshin [32] has mentioned the following relation.

Lemma 18 ([32]). *For every hypergraph H it holds that $\overline{\chi}(H) \leq \alpha(H)$.*

Even for r-co-hypergraphs H the difference $\alpha(H) - \overline{\chi}(H)$ can be arbitrarily large by the following examples.

Example 5. *Let $H_{k,r} = I_k \otimes I_k$ be the join inserting hyperedges of cardinality $r \geq 3$ between two empty hypergraphs on k vertices. Then, by Lemmas 5 and 17 it holds $\overline{\chi}(H_{k,r}) = r - 1 < \max(k, r - 1) = \alpha(H_{k,r})$. Thus, for $k > r - 1$ we get an arbitrary large difference $\alpha(H_{k,r}) - \overline{\chi}(H_{k,r})$.*

For $r = 2$ the remark on the difference of the stability number and upper chromatic number also holds true but the upper chromatic number is even constant for connected graphs (Remark 18).

Example 6. *For $r = 2$ we consider the 2-co-hypergraphs (co-graphs) $H = I_k \otimes I_k$. Then it holds $\overline{\chi}(H_{k,r}) = 1 < k = \alpha(H_{k,r})$.*

A 2-transversal can be used to give a proper k-$\mathcal{C}$-coloring of a hypergraph H as follows, as shown in [8] (Chapter 11) and [9].

Lemma 19. *For every hypergraph H it holds that $\overline{\chi}(H) \geq |V(H)| - \tau_2(H) + 1$.*

4.10. Results

Next we apply our formulas to compute and compare the considered parameters.

Theorem 4. *For every $r \geq 2$ and every r-co-hypergraph H on n vertices given by a decomposition tree $T(H)$ the size of a largest stable set, the size of a largest co-stable set, the size of a largest independent set, the size of a largest co-independent set, the size of a smallest vertex cover, the size of a smallest 2-transversal, the size of a smallest dominating set, the strong chromatic number, and the upper chromatic number can be computed in $\mathcal{O}(n)$ time.*

Proof. The result holds by the formulas shown in Sections 4.1–4.9. □

In addition to determining the sizes of the special vertex sets, it is also possible to compute a vertex set of this size using the results of Sections 4.1–4.7.

Theorem 5. *Let H be some r-co-hypergraph for $r \geq 3$, then it holds*

$$\beta(H) \leq \gamma(H) \leq \overline{\chi}(H) \leq \alpha(H) \quad \text{and} \quad s\chi(H) = \delta(H).$$

Proof. Let H be some r-co-hypergraph for $r \geq 3$. The relation $\beta(H) \leq \gamma(H)$ holds by Lemmas 8 and 14, $\gamma(H) \leq \overline{\chi}(H)$ holds by Lemmas 14 and 17, and $\overline{\chi}(H) \leq \alpha(H)$ holds by Lemmas 17 and 5. The equality $\delta(H) = s\chi(H)$ holds by Lemmas 10 and 16. □

5. Conclusions and Outlook

We introduced the class of r-co-hypergraphs. The operations of r-co-hypergraphs are the disjoint union of two given r-co-hypergraphs and the join operation, which inserts all hyperedges of cardinality r between the non-empty vertex subsets of two given r-co-hypergraphs. For $r = 2$ the set of r-co-hypergraphs corresponds to the well-known class of co-graphs which leads to a generalization of co-graphs for r-uniform hypergraphs. Similar to co-graphs also for $r \geq 3$ the set of r-co-hypergraphs can be defined by the disjoint union and complementation (w.r.t. r-uniform hypergraphs). The recursive structure is employed to formulate dynamic programming algorithms, thereby demonstrating that a substantial number of fundamental hypergraph problems can be solved in linear time for every r-co-hypergraph derived from a decomposition tree. In Table 2, we summarize our shown results for r-co-hypergraphs. Several results for r-co-hypergraphs re-prove and generalize the results for co-graphs summarized in Table 1.

Table 2. Hypergraph parameters for single-vertex hypergraphs, the disjoint union, and join operation for r-uniform hypergraphs and thus r-co-hypergraphs.

Parameter	Section	v	$H_1 \oplus H_2$	$H_1 \otimes H_2$ $r \leq \lvert V(H_1)\rvert + \lvert V(H_2)\rvert$	$H_1 \otimes H_2$ $r > \lvert V(H_1)\rvert + \lvert V(H_2)\rvert$ *
α	Section 4.1	1	$\alpha(H_1) + \alpha(H_2)$	$\max(\alpha(H_1), \alpha(H_2), r-1)$	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert$
ω	Section 4.2	1	$\max(\omega(H_1), \omega(H_2),$ $\min(\lvert V(H_1)\rvert + \lvert V(H_2)\rvert, r-1))$	$\omega(H_1) + \omega(H_2)$	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert$
β	Section 4.3	1	$\beta(H_1) + \beta(H_2)$	1 *	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert$
δ	Section 4.4	1	$\max(\delta(H_1), \delta(H_2))$	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert$ *	1
τ	Section 4.5	0	$\tau(H_1) + \tau(H_2)$	$\min(\lvert V(H_1)\rvert + \tau(H_2),$ $\lvert V(H_2)\rvert + \tau(H_1),$ $\lvert V(H_1)\rvert + \lvert V(H_2)\rvert - r + 1)$	0
τ_2	Section 4.6	0	$\tau_2(H_1) + \tau_2(H_2)$	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert - r + 2$	0
γ	Section 4.7	1	$\gamma(H_1) + \gamma(H_2)$	$\min(\gamma(H_1), \gamma(H_2), 2)$	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert$
$s\chi$	Section 4.8	1	$\max(s\chi(H_1), s\chi(H_2))$	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert$ *	1
$\overline{\chi}$	Section 4.9	1	$\overline{\chi}(H_1) + \overline{\chi}(H_2)$	$r-1$	$\lvert V(H_1)\rvert + \lvert V(H_2)\rvert$

* Applies only for $r \geq 3$.

Beside the considered graph parameters, there a number of so-called *structural parameters*, which measure the structure of a graph when it is decomposed into a tree structure. For graphs, the tree-width [33] and clique-width [34] are among the most important structural parameters.

The notion of tree-width and path-width for graphs can also be used for hypergraphs by considering the tree-width or path-width of the primal graph, as already done in [35]. By Lemma 4, and the results of [36] for graphs, we can get formulas for how to compute the tree-width and path-width of the disjoint union and join of two hypergraphs by the width of the involved hypergraphs. Since these formulas for path-width and tree-width are equal, the equality of path-width and tree-width for co-graphs known from [36] even holds for r-co-hypergraphs for every $r \geq 2$.

A further approach for the tree-width of hypergraphs is the notion of hypertree-width for hypergraphs, which was defined by Gottlob, Leone, and Scarello in [37] motivated by algorithmic problems from artificial intelligence and database theory. It remains to consider the hypertree-width for r-co-hypergraphs.

Following the idea of [35] for tree-width, the clique-width of hypergraphs can be defined by the clique-width of the primal graph using the definition in [34]. Since by Corollary 4 the primal graphs of r-co-hypergraphs are co-graphs, the clique-width of r-co-hypergraphs is at most 2.

In [38] a generalization of clique-width for r-uniform hypergraphs was suggested. The operations of hyper-r-clique-width are the creation of a new labeled vertex, the vertex-disjoint union of already constructed hypergraphs, the insertion of all possible hyperedges of cardinality r between vertices of r different labels of a defined hypergraph, and the uniform relabeling of vertices. It also remains to consider the hyper-r-clique-width for r-co-hypergraphs.

There are a lot of further interesting hypergraph problems which can be considered on r-co-hypergraphs. Some examples are rainbow coloring [39], harmonious coloring [40], non-monochromatic coloring [8] (Chapter 11), spanning hypertree [41], spanning hypergraph [42], matching [43], Hamiltonian cycle [44], Hamiltonian path [44], and Hamiltonian chain [45]. The objective of this paper is not to provide a comprehensive enumeration of hypergraph problems that can be solved through dynamic programming on r-co-hypergraphs. Rather, its aim is to demonstrate that a significant number of these problems can be addressed by dynamic programming solutions employing a similar methodology.

In Section 3, we sketched a method to decide whether an input hypergraph H is an r-co-hypergraph, and in the case of a positive answer we find a decomposition tree $T(H)$ in polynomial time. Since our method does not run in linear time, it remains to find such a method.

A graph class or graph property is *hereditary* if it is closed under induced subgraphs. A graph class is hereditary if and only if it admits a characterization in terms of minimal forbidden induced subgraphs, i.e., minimal graphs that do not belong to the class, see [46] (Chapter 2). Since this also holds for classes of r-uniform hypergraphs the closure property shown in Theorem 3(ii) implies the existence of a characterization by forbidden induced subhypergraphs for r-join-hypergraphs for $r \geq 3$. It remains to find such characterizations.

In [32] the notion of a *$\mathcal{C}$-perfect hypergraph* was introduced. A hypergraph H is *$\mathcal{C}$-perfect*, if $\overline{\chi}(H') = \alpha(H')$ for every induced subhypergraph H' of H. Examples 5 and 6 give for every $r \geq 2$ a set of r-co-hypergraphs H, such that $\overline{\chi}(H) < \alpha(H)$, which implies that for $r \geq 2$ the set of r-co-hypergraphs is not $\mathcal{C}$-perfect.

For co-graphs, and also for directed co-graphs, a number of sub-classes have been defined and investigated, see [13,47–50] and [51,52]. Such restrictions are also possible for r-co-hypergraphs by limiting the size or structure of one of the combined hypergraphs.

Furthermore, co-graphs and also directed co-graphs have a large number of real applications, see [13–16] and [20,21]. It remains to identify practical applications for r-co-hypergraphs.

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