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Journal of Logic, Language and Information (2025) 34:515–555
https://doi.org/10.1007/s10849-025-09442-y

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Received: 25 January 2023 / Accepted: 6 October 2025 / Published online: 29 October 2025

Abstract
We investigate propositional logics which enrich classical logic with a binary connective '‖' representing ambiguity. Some of these logics have been established in the literature. We briefly present and compare the most interesting of these. We generalize all existing approaches, defining the family of full ambiguity logics by certain basic requirements they have to meet. We introduce further examples of ambiguity logics, investigate the structure of the family, and show how conceptual properties correspond to formal properties. The most important notion for ambiguity logics is what we call trust: either we trust that ambiguous terms are used consistently in one sense in an argument, or we do not. Formally, every reasonable ambiguity logic is either closed under uniform substitution (this corresponds to trust), or it is closed under substitution of equivalents – but closure under both results in (a specific type of) triviality. This correlation between conceptual and mathematical properties is not straightforward, but our results show that it is well-founded.

Keywords Ambiguity · Reasoning · Proof theory · Non-classical logic

1 Introduction

Intro and short summary We say a word/phrase/proposition is ambiguous, if it has two or more clearly distinct meanings, only one of which is usually intended in a single usage. In traditional philosophical literature, ambiguity is considered as detrimental and hostile to sound reasoning, and one of the main tenets of formal logic is to avoid ambiguity. However, as a matter of fact, in natural language ambiguity is uniquitous, and humans deal with this very successfully and efficiently (see e.g. Piantadosi et al., 2011). This has lead to the endeavor to formalize reasoning with ambiguity by means of logics. We can distinguish three different categories of these logics: 1. Logics which allow to explicitly represent ambiguity with an additional binary connective '‖'. There have been several more or less independent approaches, most prominently van Eijck and Jaspars (1995); Wurm (2021). 2. Logics where ambiguity is a semantic property

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of non-logical symbols; interesting approaches are van Deemter (1996); Vanackere (1997); Batens (2002). 3. Logics taking a different perspective altogether, as multi-agent settings (see Halpern and Kets,2014). In this article, we will scrutinize logics of the **first category** only, that is, logics with an ambiguity connective '$\|$' which has to satisfy a number of combinatorial and inferential properties.

The main result of this article can be stated as: there exists a FIELD OF AMBIGUITY LOGICS in the sense of 1., with its own systematicity, which puts the different logical approaches so far into context, and allows to relate and construct ambiguity logics with a set of general formal methods. For now, we restrict ourselves to *propositional* ambiguity logics, by which we mean logics in the language of propositional logic with the additional connective '$\|$'. Moreover, we restrict ourselves to logics which (conservatively) extend classical logic. The central notion for ambiguity logics is what we call **trust**. We use this term throughout the article as a technical term in a very specific sense, meaning the assumption that ambiguous propositions are intended *consistently in one sense* throughout an argument (more explanations and examples can be found in Section 2.3). Formally, trust corresponds to closure under uniform substitution of atoms, and distrust is roughly the lack thereof. Trust is central for the main results on ambiguity logics: the Basic Closure Theorem (presented already in Wurm, 2021) states that a non-trivial ambiguity logic cannot be closed both under uniform substitution and substitution of equivalents. Hence every **trustful logic** which is non-trivial lacks closure under substitution of equivalents (and hence transitivity). Dually, ambiguity logics which are non-trivial and closed under substitution of equivalents cannot be trustful; we call these **distrustful logics**. In trustful logics (closed under uniform substitution of atoms), each inference can be seen as a generally valid scheme regardless of content, but on the other side inferentially equivalent formulas are *not* generally interchangeable in all contexts; distrustful logics have exactly the opposite properties (see Definition 4). The In-Out-Lemma states that every trustful logic, where at least equivalent *classical* formulas are interchangeable, contains a distrustful logic as its **inner logic** (an important notion to be made precise). Finally, the Trust Theorem states that every non-trivial distrustful logic can be extended to a non-trivial trustful ambiguity logic, but never vice versa. This is the mathematical counterpart to the intuition that in a situation of trust, we accept more valid arguments than in a situation of distrust (see again Section 2.3 for examples).

In this general perspective, we present the existing logics TAL (see Wurm, 2021, trustful, with commutative version cDAL) and DAL (see van Eijck and Jaspars, 1995, distrustful, with commutative version cDAL), and provide some new results on them. In particular, we show that whereas TAL is a very reasonable logic of trust, cTAL is trivial (in a technical sense). We also present some new and rather exotic logic as $L_{\blacklozenge\blacksquare}$, the smallest ambiguity logic, and $\tau L_{\blacklozenge\blacksquare}$, the smallest trustful ambiguity logic. The list of methods we use to construct ambiguity logics are the following: 1. Proof theory, 2. Truth operators, 3. Trust closure, 4. Algebraic extensions (extending the congruence algebra) of existing logics.

To sum up, we aim to give a first overview of the field of propositional ambiguity logics in our sense, and provide a toolbox of methods and concepts to understand these logics. We will not deal with possible applications here, but the important point in this regard is: in treating ambiguity, logical pluralism is unavoidable.

Structure of the text Section 2 shortly provides some background knowledge and a list of core properties of ambiguity, some of which we think every logic for ambiguity has to satisfy, some of which we think might be desirable, but do not need to be satisfied (see Wurm, 2021, formoredetails). Moreover, we present our basic conventions on notation and some fundamental definitions, among which Definition 1 of a **logic of ambiguity**, which we will elaborate later to Definition 7 of a **full ambiguity logic**. We define, explain and illustrate the central notions of trust/distrust, and state the Basic Closure Theorem. We then explain why we call these logics **logics of trust** and **logics of distrust**.

In Section 3, we present the distrustful logic cDAL with non-commutative version DAL (see see vanEijck and Jaspars, 1995) and the trustful logic TAL with commutative version cTAL (Wurm, 2021). We will see in particular that contrary to TAL, cTAL is not very reasonable, and that trust does not seem to go well together with commutative ambiguity.

Section 4 introduces the concept of a congruence algebra of a logic. In a logic closed under substitution of equivalents, this corresponds to the Lindenbaum-Tarski algebra. For trustful ambiguity logics, it does not: we construct congruence classes, the elements of the algebra, based on $\equiv$, which means *interchangeability in all contexts* $\Gamma[-]$, not $\dashv\vdash$. We will then investigate the congruence algebra of TAL, cTAL and cDAL and show that cDAL is the inner logic of cTAL.

Section 5 presents some general results on the family of full ambiguity logics. We have a look at its lattice structure and present its minimal and maximal members. We present the In-Out Lemma: in the case of a reasonable (classically congruent) trustful logic $\vdash_L$, the relation $\leqq_L$ (based on substitutability) defines a distrustful ambiguity logic. We present additional methods for constructing ambiguity logics: one method is actually old, by means of truth operators (introduced in van Deemter, 1996). We introduce the concept of trust closure, and present the Trust Theorem, which both relate distrustful and trustful logics. We also present the minimal trustful ambiguity logic. Since logics resulting from trust closure seem to have rather odd properties, we introduce the notion of algebraic extension, where we enlarge an (outer) logic by adding new equivalences to its inner logic.

Section 6 firstly gives a list of prominent open questions and problems, and then a short summary of the main results and, finally, the lesson to be learned.

2 The Notion of Ambiguity

2.1 Conceptual and Formal Properties

The main field where ambiguity arises is in natural language, and in fact ambiguity is ubiquitous there. We will hardly find an utterance which is not ambiguous by the most rigid standards.[1] Nonetheless, there are some commonly shared prejudices about ambiguity, and even though they have been falsified many times, they maintain some

[1] For example, operationalizing “reading" as SynSet in WordNet.

vitality. Since we have treated the topic in (Wurm, 2021, 2023), we will just quickly state the facts (contradicting the prejudices) without further explanation:

1. Humans do not always disambiguate, and disambiguation is not a prerequisite to understanding (see for example Fornaciari et al., 2021; van Deemter, 1996; van Eijck and Jaspars, 1995)
2. Ambiguity is not (necessarily) syntactic in nature: it does not make sense to consider an ambiguous word/phrase/sentence simply as two distinct sentences which happen to look/sound identical.
3. Ambiguity is distinct from disjunction in its semantic and inferential properties (see also Saka,2007, and the above.).
4. Ambiguity cannot be adequately treated by meta-representations, such as underspecification languages (see Reyle, 1993; Egg, 2010, for overview), unless these are logics themselves.

Accepting this leads to what we can call the **fundamental assumption**, namely that humans can reason with ambiguity, and that ambiguity is a semantic relation between propositions (see Wurm, 2023, for an investigation into this semantic relation.). The fundamental assumption leads us to investigate propositional logics with a binary ambiguity connector '$\|$' satisfying some key properties. We will now give the list of these properties of ambiguity; they are separated into 1. MANDATORY PROPERTIES, which have to be satisfied, and 2. FACULTATIVE PROPERTIES, which might be desirable but are not mandatory. Note that all these properties only apply to ambiguity in semantics; if we can disambiguate from morphosyntactic context, there is no semantic ambiguity in our sense. Since we have discussed these properties for '$\|$' in (Wurm, 2021, 2023), we just mention them here, for ease of reference.

Mandatory The following properties are fundamental properties of ambiguity, in the sense that if a phenomenon does not satisfy one of those, we would not consider it to be ambiguity. Accordingly, every logic for reasoning with ambiguity has to satisfy them.

1. Universal Distribution (UD): all connectives and all semantic operations uniformly distribute over ambiguity, without altering it, that is:

$$\neg(\alpha||\beta) \text{ is equivalent to } \neg\alpha||\neg\beta,$$
$$(\alpha\|\beta) \wedge \gamma \text{ is equivalent to } (\alpha \wedge \gamma)\|(\beta \wedge \gamma) \text{ etc.}$$

2. Unambiguous Entailments (UE): $\alpha \wedge \beta$ entails $\alpha\|\beta$ entails $\alpha \vee \beta$.
3. Idempotence (id): $\alpha\|\alpha$ is equivalent to α.[2]
4. Conservative extension: if we extend a logic with $\|$, we should extend it conservatively.
5. Associativity ($\|$assoc): $\alpha\|(\beta\|\gamma)$ is equivalent to $(\alpha\|\beta)\|\gamma$.

[2] Note that if $\alpha \vee \alpha$ is equivalent to α is equivalent to $\alpha \wedge \alpha$, then this follows from (UE), but this need not actually hold in all logics of ambiguity.

Facultative Now comes a list of properties we want to consider, but which need not be necessarily satisfied. So these are important parameters of reasoning with ambiguity, rather than prerequisites.

1. Commutativity: $\alpha\|\beta$ is equivalent to $\beta\|\alpha$
2. Uniform Usage (UU): this means ambiguity terms are used consistently in one sense, and corresponds to closure under uniform substitution.
3. Monotonicity ($\|$mon): If α entails α', β entails β', then $\alpha\|\beta$ entails $\alpha'\|\beta'$. This is weaker than UU.
4. Strong Law of Disambiguation (LoD): $\alpha\|\beta\|\gamma$ and $\neg\beta$ entail $\alpha\|\gamma$ (here we subsume the case where α or γ is empty).
5. Weak Law of Disambiguation (wLoD): $\alpha\|\beta\|\gamma$ and $\neg\beta$ entail $\alpha\|\gamma$, provided β is unambiguous.[3]

2.2 Basic Conventions and Definitions

This subsection covers most of the notational conventions and some key definitions. We denote the classical sequent calculus by CL. All logics we consider here will be in the same formula language, which (conservatively) extends classical logic with an additional binary connective '$\|$'. More formally: we have a countable set *Var* of propositional variables, and define $Form(\mathsf{AL})$ by

1. if $p \in Var$, then $p \in Form(\mathsf{AL})$;
2. if $\phi, \chi \in Form(\mathsf{AL})$, then $(\phi \wedge \chi)$, $(\phi \vee \chi)$, $(\neg\phi) \in Form(\mathsf{AL})$;
3. if $\phi, \chi \in Form(\mathsf{AL})$, then $(\phi\|\chi) \in Form(\mathsf{AL})$;
4. nothing else is in $Form(\mathsf{AL})$.

If we can derive a formula without 3., we say it is in $Form(\mathsf{CL})$, or simply classical, and we sometimes use $Form(\|)$, $Form(\|, \vee)$ etc. with the obvious meaning. As usual, we will omit outermost parentheses of formulas. We will generally use lowercase Greek variables $\alpha, \beta, \gamma, ...$ for arbitrary formulas, $p, q, r, ...$ for propositional variables. Moreover, we will sometimes use $a, b, c, ...$ as variables for classical formulas (in $Form(\mathsf{CL})$). Hence $\alpha = a_1\|a_2$ intends that a_1, a_2 are classical formulas. On some occasions, we omit '$\|'$ and write pq for $p\|q$, but only for atoms.

For proof theory, we need to generalize classical contexts and sequents. We do this by distinguishing two different types of contexts: The classical context is denoted by $(\ldots, \ldots)$, which basically embeds classical logic, and the ambiguous context is denoted by $(\ldots, \ldots)$, which allows to introduce the new connective '$\|$'.[4] We denote contexts by uppercase letters (mostly Greek, but sometimes also Latin). Our presentation is somewhat sloppy, for more details, consider Wurm (2021). Formally:

1. ϵ, the empty sequence, is a well-formed context, which we also call the **empty context**.

[3] Note that ($\|$mon) entails wLoD, yet not LoD. This is, briefly, since $\beta \wedge \neg\beta$ is a contradiction for classical β, but not for arbitrary β.

[4] We have found this idea briefly mentioned as a way to approach substructural logic in Restall (2008), and structures similar to our contexts contexts are found in Dyckhoff et al. (2012). They are also used in the context of linear logic, see for example de Groote (1996).

2. If $\gamma \in Form(\mathsf{AL})$, then γ is a well-formed context.
3. If $\Gamma_1, ..., \Gamma_i$ are well-formed contexts, then $(\Gamma_1\textbf{,} ...\textbf{,} \Gamma_i)$ and $(\textbf{;}\Gamma_1\textbf{;} ...\textbf{;} \Gamma_i)$ are well-formed contexts.

We write $\Gamma[\alpha]$ to refer to a subformula α of a context Γ; same for $\Gamma[\Delta]$, where Δ is a context. We sometimes conceive of $\Gamma[-]$ as a function, which takes a formula as argument and yields a context. We have a number of **conventions for sequents**, to increase readability. These are important, as we make full use of them already in presenting the calculus.

– We omit the outermost brackets in contexts.
– ϵ is a neutral element for both**,** , **;** : $(\Gamma\textbf{,} \epsilon) = \Gamma = (\Gamma\textbf{;} \epsilon)$ etc.
– We let $(\Delta_1\textbf{,} (\Delta_2\textbf{,} ...\textbf{,} \Delta_i))$ and $((\Delta_1\textbf{,} ...\textbf{,} \Delta_{i-1})\textbf{,} \Delta_i)$ just be alternative notation for $(\Delta_1\textbf{,} ...\textbf{,} \Delta_i)$, same for $(\Delta_1\textbf{;} (\Delta_2\textbf{;} ...\textbf{;} \Delta_i))$

Another convention is the following: a **logic** is a pair $L = (Form, \vdash_L)$, with *Form* its set of well-formed formulas and $\vdash_L \subseteq Form \times Form$ its consequence relation. We write $\alpha \vdash_L \beta$ if the sequent is valid in L. However, sometimes it will be convenient to speak of a sequent $\alpha \vdash \beta$ independently of a logic. Then we also write $\Vdash_L \alpha \vdash \beta$ or $\nVdash_L \alpha \vdash \beta$, meaning that the sequent is (or is not) valid in L.[5]

Having said this, the relation $\vdash_L$ is *always* extended from formulas to contexts via functions l, r, which are simple maps from contexts to formulas:

$$l(\alpha) = \alpha = r(\alpha) \qquad r(\Gamma\textbf{,}\ \Delta) = r(\Gamma) \vee r(\Delta)$$
$$l(\Gamma\textbf{;}\ \Delta) = l(\Gamma)\|l(\Delta) = r(\Gamma\textbf{;}\ \Delta) \qquad l(\Gamma\textbf{,}\ \Delta) = l(\Gamma) \wedge l(\Delta)$$

For every logic L, we will always implicitly assume that

$$\Gamma \vdash_L \Delta \text{ iff } l(\Gamma) \vdash_L r(\Delta) \tag{1}$$

This makes sure that $\leqq_L, \equiv_L$ (see Definition 5) are well-defined for every logic L. An important notion is the one of **ambiguous normal form**, which results from distributing out all ambiguity, until we remain with a formula $a_1\|...\|a_i$, where $a_1, ..., a_i$ are classical. (UD) ensures that every formula is equivalent to all of its ambiguous normal forms. For example, $(p \wedge r)\|(q \wedge r)\|(p \wedge s)\|(q \wedge s)$ and $(p \wedge r)\|(p \wedge s)\|(q \wedge r)\|(q \wedge s)$ are ambiguous normal forms of $(p\|q) \wedge (r\|s)$.

2.3 The Basic Closure Theorem, Trust and Distrust, Inner and Outer Logics

We now give a preliminary working definition for logics of ambiguity, which we will elaborate later on (Definition 7).

Definition 1 A logic L is a **logic of ambiguity** (in the weak sense) if $L = (Form(\mathsf{AL}), \vdash_L)$, and

1. $\vdash_{\mathsf{CL}} \subseteq \vdash_L$ (extends classical logic)
2. L satisfies universal distribution and unambiguous entailments :

[5] This is usual in algebra, where both $t =_{\mathbf{BA}} t'$ and $\mathbf{BA} \models t = t'$ are synonym.

   (a) $\alpha \wedge (\beta \| \gamma) \dashv\vdash_L (\alpha \wedge \beta) \| (\alpha \wedge \gamma)$
   (b) $\alpha \vee (\beta \| \gamma) \dashv\vdash_L (\alpha \vee \beta) \| (\alpha \vee \gamma)$
   (c) $\neg(\alpha \| \beta) \dashv\vdash_L \neg\alpha \| \neg\beta$
   (d) $\alpha \wedge \beta \vdash_L \alpha \| \beta \vdash_L \alpha \vee \beta$

3. $(\alpha \| \beta) \| \gamma \dashv\vdash_L \alpha \| (\beta \| \gamma)$
4. $\alpha \| \alpha \dashv\vdash_L \alpha$

We will present some results for logics of ambiguity in this broader sense. However, this definition is slightly too liberal, and we will present a more restrictive definition of full ambiguity logic in Section 2.4. We first present the Basic Closure Theorem, which (we believe) is the most important result on logics of ambiguity. The proof of the Basic Closure Theorem can be found in Wurm (2021), following up to Wurm and Lichte (2016). The proof is algebraic in nature, but carries over to logics via completeness of algebraic semantics (see Wurm, 2017).

Theorem 2 *(Basic Closure Theorem)*

1. *Let $\mathcal{L} = (Form(\mathsf{AL}), \vdash_L)$ be a logic of ambiguity (in the weak sense) which is closed under uniform substitution of atoms and the rule (cut), where moreover $\alpha \| \beta \dashv\vdash_L \beta \| \alpha$. Then $\mathcal{L}$ is inconsistent.*
2. *Let $\mathcal{L} = (Form(\mathsf{AL}), \vdash_L)$ be a logic of ambiguity (in the weak sense) which is closed under uniform substitution of atoms and the rule (cut). Then for all $\alpha, \beta, \gamma \in Form(\mathsf{AL})$, we have $\alpha \| \gamma \| \beta \dashv\vdash_L \alpha \| \beta$.*

This is a key result, because it shows that algebra and “normal" logics, closed under (cut) and uniform substitution, are **fundamentally inappropriate** for reasoning with ambiguity. The property of Theorem 2.2 is slightly weaker than inconsistency, yet it is strong enough to exclude any logic which satisfies it as a reasonable logic. We now introduce a technical notion of triviality of a logic, which allows to uniquely refer to this property.

Definition 3 A logic of ambiguity L is **margin-trivial** (or **m-trivial**), if for all formulas $\alpha, \beta, \gamma, \alpha \| \beta \| \gamma \dashv\vdash_L \alpha \| \gamma$.

M-triviality serves as an important benchmark to decide when a logic of ambiguity is “too permissive" to be actually useful. The immediate consequence of the Basic Closure Theorem is the following:

Observation 1 *The only way to incorporate the basic features of ambiguity into a reasonable system for reasoning with ambiguity is to abandon one key feature of algebra itself.*

However, there are two key features which can be abandoned while still satisfying the mandatory properties of ambiguity:

1. Uniform substitution of atomic propositions by arbitrary formulas preserves the truth of sequents: $\alpha \vdash_L \beta$, $\sigma : Var \to Form(\mathsf{AL})$ a uniform substitution, entail $\sigma(\alpha) \vdash_L \sigma(\beta)$. We call this **closure under u-substitution**.

2. Substitution of arbitrary $\dashv\vdash_L$-equivalent formulas preserves the truth of sequents: $\alpha \dashv\vdash_L \beta$ and $\Gamma[\alpha] \vdash_L \Delta$ entail $\Gamma[\beta] \vdash_L \Delta$; same on the right. We call this **closure under e-substitution**.

Note that these two notions correspond to two (of three) properties of abstract logics in the sense of Tarski (1936)![6] Let us illustrate this with two examples.

Example 1 $p \vee \neg p$ is obviously a theorem in every logic of ambiguity. However, the uniform substitution $\sigma : p \mapsto p\|q$ need not preserve this: $(p\|q) \vee \neg(p\|q)$ need not be a theorem (see cDAL). However, assume $(p\|q) \vee \neg(p\|q)$ *is* a theorem in a logic of ambiguity L. Then

$$(p \vee \neg p)\|(p \vee \neg q)\|(\neg p \vee q)\|(q \vee \neg q) \tag{2}$$

is also a theorem (by the laws of universal distribution). However,

$$(r \vee \neg r)\|(p \vee \neg q)\|(\neg p \vee q)\|(q \vee \neg q) \tag{3}$$

need *not* be a theorem, even though $p \vee \neg p \dashv\vdash_L r \vee \neg r$ holds by conservative extension, because (3) is not equivalent to a substitution of a classical theorem (see e.g. $\tau L_{\blacklozenge\blacksquare}$, Section 5.4.2).

Hence one of the two closure properties has to be abandoned, if we do not want to end up with m-triviality. This leaves us with **four kinds** of logics of ambiguity:

1. logics of ambiguity which are closed under e- and u-substitution. These logics are m-trivial and do not seem to be particularly interesting.
2. logics of ambiguity which are not closed under e-substitution, but closed under u-substitution. We call these the (non-trivial) **logics of trust**. The reason for this is, in a nutshell, that every inference is valid as a *scheme*, regardless of the exact content of propositional variables (the “content" of a variable can be thought of as the result of the substitution).
3. logics of ambiguity which are closed under e-substitution, but not under u-substitution. We call these logics (non-trivial) **logics of distrust**. The reason is that inferences are not schemes: if we (uniformly) substitute an atom p in a formula with an ambiguous formula, a valid argument might become invalid.
4. logics of ambiguity which are closed neither under e- nor u-substitution. We cannot say these logics are generally uninteresting, but for now we have neither a motivation nor a relevant example for these logics, so we will not look at them here.

Hence we will mostly investigate logics of kind 2. and 3., logics of trust and logics of distrust. Note that there is an interesting correlation between (necessary) formal and conceptual properties of logics: trust is closure under u-substitution (“arguments are schemes"), and the lack thereof is partly compensated by closure under e-substitution. Let us illustrate this with some examples.

[6] In the presence of a lattice order, e-congruence is slightly stronger than transitivity of inference

Example 2 Every ambiguity logic should satisfy Modus Ponens with unambiguous propositions. But consider the following argument:

(4) `Peter loves plants.`
`If someone loves plants, he loves nature.`
$\therefore$ `Peter loves nature`

This is not necessarily correct, since `plant` is famously ambiguous. This illustrates how lack of closure under u-substitution corresponds to distrust: defining '$\rightarrow$' as usual, $p, p \rightarrow q \vdash q$ is valid in every logic of ambiguity, but

$$p \| r, (p \| r) \rightarrow q \vdash q \tag{5}$$

need not be valid under distrust (see cDAL below). On the other hand, in a trustful logic Modus Ponens is *always* valid, also for ambiguous premises, hence we accept (4)[7].

Put simply: Distrustful logics of ambiguity are useful to reason with the devil, or if statements come from different and independent sources[8], but maybe not appropriate to reason with your friends and family. Distrustful logics seem problematic in linguistic applications, as lexical word meanings are always somewhat opaque, and ambiguity can never be completely excluded (just think of metaphors and irony), and hence with distrust we can never be sure whether an argument holds or not.

Example 3 I can tell my kids on two distinct occasions, but each in a trustful setting, both (5-a) and (5-b):

(6) a. `Ice is water (in frozen form)`
b. `Water is liquid.`
c. `Ice is liquid.`

However, this does not entail (5-c), which I would never sustain! This illustrates that in trustful reasoning, we cannot assume transitivity of inference, and since (provided a lattice order) transitivity follows from substitution of equivalents, we cannot assume closure under e-substitution (see, Wurm, 2023, for further discussion of this example).

Definition 4 Assume $L = (\mathit{Form}(\mathsf{AL}), \vdash_L)$ is a logic (of ambiguity).

1. We say L is a **trustful logic**, if for every uniform substitution $\sigma : \mathit{Var} \rightarrow \mathit{Form}(\mathsf{AL})$, $\Gamma \vdash_L \Delta$ entails $\sigma(\Gamma) \vdash_L \sigma(\Delta)$
2. We say L is a **distrustful logic**, if $\Gamma[\alpha] \vdash_L \Delta$, $\alpha' \vdash_L \alpha$ entail $\Gamma[\alpha'] \vdash_L \Delta$, and $\Gamma \vdash_L \Delta[\beta]$, $\beta \vdash_L \beta'$ entail $\Gamma \vdash_L \Delta[\beta']$.

Note that the definition of distrustful logics is slightly more general than interchangeability of $\dashv\vdash$-equivalents. It is equivalent in case the logic is naturally ordered (Definition 30). However, we will see some logics where this is not the case (see $\tau L_{\blacklozenge\blacksquare}$ below), and our more general definition prevents some problems.

7 See Wurm (2023) for further discussion

8 This can be the case for entries in ontologies, see Arapinis and Vieu (2015).

This gives rise to an important distinction, namely between **inner logics** and **outer logics**. The relation $\vdash$ means *entailment*, but in a trustful logic L, $\alpha \vdash_L \beta$ does *not* entail that α is logically stronger than β in *all contexts*. This notion is the corresponding inner logic $\leqq_L$:

Definition 5 Let L be a logic. We write $\alpha \leqq_L \beta$ iff $\Gamma[\beta] \vdash_L \Delta$ implies $\Gamma[\alpha] \vdash_L \Delta$ and $\Delta \vdash_L \Gamma[\alpha]$ implies $\Delta \vdash_L \Gamma[\beta]$. We call $\leqq_L$ the **inner logic** of L, and we write $\alpha \equiv_L \beta$ iff $\alpha \leqq_L \beta$ and $\beta \leqq_L \alpha$.

By analogy, we sometimes call $\vdash_L$ the **outer logic**. $\equiv_L$ is the relation of *congruence* or interchangeability. It is by definition an equivalence relation. $\alpha \leqq_{\mathsf{TAL}} \beta$ can be conceived of as: "α is logically stronger than β in all contexts", and it is by definition a pre-order (reflexive, transitive), and hence a partial order up to $\equiv_{\mathsf{TAL}}$ congruence. In a calculus which is closed under e-congruence (corresponds to admissible (cut)), $\leqq_L$ usually coincides with $\vdash_L$, but not always.

Lemma 6 *Let L be a logic of ambiguity. Then* $\leqq_L \subseteq \vdash_L$ *iff for all* α, $\alpha \vdash_L \alpha$. *Moreover, if L is trustful and non-trivial, then* $\leqq_L \subsetneq \vdash_L$.

Proof *If* $\alpha \vdash_L \alpha$, $\alpha \leqq_L \beta$ obviously entails $\alpha \vdash_L \beta$.

Only if Contraposition: assume there is an α such that $\alpha \nvdash_L \alpha$. $\leqq_L$ by definition is reflexive, hence $\leqq_L \nsubseteq \vdash_L$.

Finally, assume L is trustful and non-trivial, hence it cannot be closed under (cut). $\leqq_L$ by definition is, hence the two cannot be identical. $\dashv$

We will later (Section 5.1) see the logic $L_{\blacklozenge\blacksquare}$, for which we cannot always derive $\alpha \vdash \alpha$, hence this is not a trivial requirement. For the investigation of trustful (outer) logics, the inner logic $\leqq$ is absolutely crucial. Its meaning for trustful logics of ambiguity can be maybe compared to the meaning of the Deduction Theorem for Hilbert calculi: it allows to get from one derivable sequent to another one quickly. It is however not straightforward to establish properties for $\leqq_L$, $\equiv_L$, given a relation $\vdash_L$. For example, $\vdash_L \subseteq \vdash_{L'}$ does not generally entail $\leqq_L \subseteq \leqq_{L'}$ (see the logics $L_{\blacklozenge\blacksquare}$ and $\tau L_{\blacklozenge\blacksquare}$ in Section 5).

2.4 Full Ambiguity Logics

We are now ready for the definition of a full ambiguity logic, which we believe is the most reasonable and natural one. We have decided that full ambiguity logics have to satisfy the mandatory properties of Section 2.1 for $\leqq$ and $\equiv$, because only congruence means that meanings are really identical. As regards classical connectives, we do not have any requirements except the conservative extension of classical logic. This opens the way to logics where classical operators behave non-classical in ambiguous contexts (see Section 5.4).

Definition 7 A logic $L = (\mathit{Form}(\mathsf{AL}), \vdash_L)$ is a **full ambiguity logic** if it satisfies the following:

1. $\vdash_{\mathsf{CL}} \subseteq \vdash_L$ (extends classical logic)

2. L satisfies universal distribution and unambiguous entailments for congruence:
   (a) $\alpha \wedge (\beta \| \gamma) \equiv_L (\alpha \wedge \beta) \| (\alpha \wedge \gamma)$
   (b) $\alpha \vee (\beta \| \gamma) \equiv_L (\alpha \vee \beta) \| (\alpha \vee \gamma)$
   (c) $\neg(\alpha \| \beta) \equiv_L \neg\alpha \| \neg\beta$
   (d) $\alpha \wedge \beta \leqq_L \alpha \| \beta \leqq_L \alpha \vee \beta$
3. $(\alpha \| \beta) \| \gamma \equiv_L \alpha \| (\beta \| \gamma)$
4. $\alpha \| \alpha \equiv_L \alpha$
5. $\alpha \vdash_L \beta, \sigma : \mathit{Var} \to \mathit{Form}(\mathsf{CL})$ entails $\sigma(\alpha) \vdash_L \sigma(\beta)$
6. $\alpha \vdash_L a, a \vdash_L \beta, a \in \mathit{Form}(\mathsf{CL})$ entails $\alpha \vdash_L \beta$

Since we are convinced that logics of ambiguity which are not full ambiguity logics are mostly uninteresting, we will simply write ambiguity logic to refer logics which satisfy Definition 7. In most cases, this will not make a difference with respect to Definition 1, but in some it will.

A note on the list of axioms: properties (1-4) are obvious and have been established as mandatory properties of ambiguity. We do not require the usual properties of abstract logics in the sense of Tarski (1936), in particular transitivity and closure under u-substitution. But importantly, both properties are only abandoned if ambiguous propositions are involved, classical propositions should behave classically. We refer to functions $\sigma : \mathit{Var} \to \mathit{Form}(\mathsf{CL})$ as classical substitutions, hence 5 states that logics are closed under uniform classical substitution (which entails closure under renaming of variables). Similarly, 6 states that classical propositions allow for transitive inferences. Note that 4 is derivable provided that the congruence algebra of L is lattice ordered, in particular if $\alpha \wedge \alpha \equiv_L \alpha \equiv_L \alpha \vee \alpha$, but this need not generally hold. Of course, this is a rather minimal choice for properties: we do *not* require that $\wedge$ be associative, idempotent, we do not require the distributive and DeMorgan laws to hold etc. (for an example see Section 5.4.2).

3 2 × 2 Logics of Ambiguity: Trust and Order

3.1 Trustful Reasoning with Ambiguity: TAL and cTAL

In this section, we will introduce 2 × 2 logics of ambiguity, according to the two features $(\pm)$trust and $(\pm)\|$-commutativity. We first present the Gentzen-style proof theory of TAL and cTAL, two logics of trust, which are closed under u-substitution, but not e-substitution. We formulate the calculus in a way to make structural rules, as far as they are desired, admissible (see Negri and Plato, 2001, on the topic). TAL has been investigated at length in Wurm (2021), so we state many properties without explicit proofs. Note that we slightly change the presentation with respect to Wurm (2021).[9]

$$(\mathrm{ax})\ \frac{}{\alpha\ \Gamma \vdash \alpha\ \Delta}$$

$$(\wedge \mathrm{I})\ \frac{\Gamma[\alpha\ \beta] \vdash \Theta}{\Gamma[\alpha \wedge \beta] \vdash \Theta} \qquad (\mathrm{I}\wedge)\ \frac{\Gamma \vdash \Theta[\alpha] \quad \Gamma \vdash \Theta[\beta]}{\Gamma \vdash \Theta[\alpha \wedge \beta]}$$

[9] The additional rules (I$\Diamond$) and ($\Diamond$I) from Wurm (2021) are actually admissible, see Lemma 23.

$$(\vee I)\ \frac{\Gamma[\alpha] \vdash \Theta \quad \Gamma[\beta] \vdash \Theta}{\Gamma[\alpha \vee \beta] \vdash \Theta} \qquad (I\vee)\ \frac{\Gamma \vdash \Theta[(\alpha\ \beta)]}{\Gamma \vdash \Theta[\alpha \vee \beta]}$$

$$(\neg I)\ \frac{\Gamma \vdash \Delta\ (\alpha_1; ...; \alpha_i)}{\Gamma\ (\neg\alpha_1; ...; \neg\alpha_i) \vdash \Delta} \qquad (I\neg)\ \frac{\Gamma\ (\alpha_1; ...; \alpha_i) \vdash \Delta}{\Gamma \vdash \Delta\ (\neg\alpha_1; ...; \neg\alpha_i)}$$

$$(\ ,\text{comm})\ \frac{\Gamma[\Psi\ \Theta]}{\Gamma[\Theta\ \Psi]} \qquad (\ ,\text{weak})\ \frac{\Gamma[\Delta]}{\Gamma[(\Delta\ \Psi)]} \qquad (\ ,\text{contr})\ \frac{\Gamma[(\Delta\ \Delta)]}{\Gamma[\Delta]}$$

The absence of $\vdash$ means that the rules can be applied on both sides of $\vdash$.

$$(I; I)\ \frac{\Gamma\ \Lambda \vdash \Delta\ \Psi \quad \Theta\ \Lambda \vdash \Phi\ \Psi}{(\Gamma; \Theta)\ \Lambda \vdash (\Delta; \Phi)\ \Psi} \qquad (I\|)\ \frac{\Theta[\alpha; \beta]}{\Theta[\alpha\|\beta]}$$

$$(;\ \text{assoc})\ \frac{\Psi[(\Gamma; (\Delta; \Theta))]}{\Psi[((\Gamma; \Delta); \Theta))]} \qquad (;\ \text{contr})\ \frac{\Gamma[\alpha; \alpha]}{\Gamma[\alpha]}$$

Here double lines indicate that the rule works in both directions.

$$(\text{inter1})\ \frac{\Gamma[(\Delta; \Psi), \Delta] \quad \Gamma[(\Delta; \Psi), \Psi]}{\Gamma[(\Delta; \Psi)]}$$

$$(\text{inter2})\ \frac{\Gamma[\Psi, (\Delta; \Psi; \Delta')] \quad \Gamma[(\Delta; (\beta, \Psi); \Delta')]}{\Gamma[(\Delta; \Psi; \Delta')]}$$

Proof trees are defined inductively as usual. We call the logic of all sequents derivable by the rules so far TAL, and denote its consequence relation with $\vdash_{\mathsf{TAL}}$. TAL is a trustful logic of ambiguity. Finally, there is ; -commutativity:

$$(;\ \text{comm})\ \frac{\Gamma[(\beta; \alpha)]}{\Gamma[(\alpha; \beta)]}$$

TAL with this additional rule is called cTAL. (inter1),(inter2) have alternative, equivalent formulations: (distr) is equivalent to (inter1), (subst) is equivalent to (inter2). (distr) and (subst) are easier to grasp intuitively, but they have a disadvantage that (i) they are not invertible, and (ii) they pose some problems in proving admissibility of structural rules.[10] We present them, though they are not strictly part of the calculus:

$$(\text{distr})\ \frac{\Gamma[(\Delta; \Psi), \Theta_1] \quad \Gamma[(\Delta; \Psi), \Theta_2]}{\Gamma[(\Delta, \Theta_1); (\Psi, \Theta_2)]} \qquad (\text{subst})\ \frac{\Gamma[\Psi] \quad \Gamma[(\Delta; \beta; \Delta')]}{\Gamma[(\Delta; \Psi; \Delta')]}$$

There is another important pair of rules of inverse distribution, which are admissible (and hence not part of the calculus, see Wurm, 2021), but will play a role in some proofs:

[10] This will play no role here, but in (Wurm, 2021) most structural rules have been proved to be admissible

$$\text{(invDistr1)}\ \frac{\Gamma[((\Delta,\Theta);\Psi)]}{\Gamma[((\Delta;\Psi),\Theta)]} \qquad \text{(invDistr2)}\ \frac{\Gamma[(\Delta;(\Psi,\Theta))]}{\Gamma[((\Delta;\Psi),\Theta)]}$$

Now consider the rules (cut),(classic cut). (cut) is **not** part neither of TAL nor cTAL, (classic cut) is admissible.

$$\text{(cut)}\ \frac{\Gamma[\alpha] \vdash \Delta \quad \Theta \vdash \alpha}{\Gamma[\Theta] \vdash \Delta} \qquad \text{(classic cut)}\ \frac{\Gamma, \alpha \vdash \Delta \quad \Theta \vdash \alpha}{\Gamma, \Theta \vdash \Delta}, \text{ where } \alpha \in \mathit{Form}(\mathsf{CL})$$

We call the logics with (cut) TAL^{cut}, cTAL^{cut}. Basic results are the following (which are partly obvious, partly proved in (Wurm, 2021)):

- TAL and cTAL conservatively extend classical logic,
- TAL, cTAL are consistent
- TAL and cTAL are closed under u-substitution
- (classic cut) is admissible in both TAL, cTAL (standard elimination proof)
- TAL and cTAL do not admit the rule (cut)
- TAL^{cut} is m-trivial, cTAL^{cut} is inconsistent (corollary of the Basic Closure Theorem).

This can be strengthened: define the transitive closure of a logic L as usual by just making $\vdash_L$ transitive on formulas (not contexts). The following is proved in (Wurm, 2021):

Lemma 8 *The transitive closure of* $\vdash_{\mathsf{TAL}}$ *is m-trivial, the transitive closure of* $\vdash_{\mathsf{cTAL}}$ *is inconsistent.*

Invertibility of a rule basically means that it can also be applied "upside down", in the indirect sense that from the derivability of the conclusion, we can infer the derivability of the premises. This depends not only on the logic, but also on the formulation of rules. For example, (inter1),(inter2) are invertible, whereas the equivalent (distr),(subst) are not. Classical logic has a fully invertible calculus. In our calculus, we can achieve *almost* full invertibility:

Lemma 9 *In* TAL *and* cTAL, *all rules are invertible except for (I; I) and (, weak).*

Proof This consists of two parts: invertibility of all rules except (I; I),(, weak) is an easy exercise (see Wurm, 2021). The non-invertibility of (I; I) in TAL is also straightforward, together with (; assoc): consider a sequent $p;(q;r) \vdash (p;q);r$: this is obviously not invertible. (, weak) is obviously not invertible, but admissible. ⊣

A more intricate example is the sequent $p;\neg p \vdash q;\neg q$, provable in TAL, cTAL, but not invertible. As we have said, TAL and cTAL are logics of trust. There is one particular pair of rules which makes the difference, namely (¬I),(I¬). These are the rules which ensure that $\alpha \vee \neg\alpha$ is a theorem for arbitrary α, and $\alpha \wedge \neg\alpha$ a contradiction. These are the rules which are only sound under the assumption of uniform usage, and these are the rules which do not preserve the identity of $\leqq_{\mathsf{TAL}}$ with $\vdash_{\mathsf{TAL}}$: actually, it is possible to prove that in the calculus without negation, (cut) is admissible. Hence the unrestricted negation rule is really what makes the logics TAL, cTAL trustful.

Lemma 10 TAL *and* cTAL *are full ambiguity logics.*

The proof for TAL can be found in Wurm (2021), proof for cTAL is basically identical.

3.2 Distrustful Reasoning with Ambiguity: DAL and cDAL

Whereas TAL (non-commutative) is in our view the most reasonable logic for trustful reasoning with ambiguity, DAL and cDAL (commutative) are probably the most reasonable logics for distrustful reasoning, first presented by van Eijck and Jaspars (1995). Their proof theory is characterized by

1. Using all rules of cTAL except negation rules (I¬),(¬I),
2. adding (cut), and
3. adding the following restricted negation rules (we only provide them for one side and skip duals, (contra) and (swap) are self-dual):

$$(at\neg)\ \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p}, \ p \text{ atomic} \qquad (contra)\ \frac{\Gamma \vdash \Delta}{\neg\Delta \vdash \neg\Gamma} \qquad (eDN)\ \frac{\Gamma \vdash \Delta, \neg\neg\alpha}{\Gamma \vdash \Delta, \alpha}$$

$$(swap)\ \frac{\Gamma \vdash \Delta, \alpha \quad \Gamma, \beta \vdash \Delta}{\Gamma, \neg\alpha \vdash \Delta, \neg\beta}\ .$$

Introducing double negation is admissible. $\neg\Gamma$ is an abbreviation for pointwise negation of all formulas in Γ. Since DAL, cDAL allow (cut), (inter1),(inter2) are admissible, so we do not need them. DAL is the non-commutative version (without (; comm)), cDAL the commutative version (with (; comm)).

We will focus on (commutative) cDAL in this article, since cDAL is much more easily accessible from the semantic side. Take a classical model $M \subseteq Var$. Classically, there are only two possibilities: $M \models \alpha$ or $M \not\models \alpha$, where α is *not true* means as much as α is *false*. This is no longer true with ambiguity. This is the starting point for van Eijck and Jaspars (1995). We will initially follow this presentation, but then show a presentation using operators ■, ♦ (introduced by van Deemter, 1996) which is easier to handle. Here is the double list of verifying ($\models$) and falsifying ($\rightsquigarrow$) conditions for formulas/connectives, as presented in (van Eijck & Jaspars, 1995).

$\models$1 $M \models p_i$ iff $p_i \in M$.
$\models$2 $M \models \alpha \wedge \beta$ iff $M \models \alpha$ and $M \models \beta$.
$\models$3 $M \models \alpha \vee \beta$ iff at least one holds, $M \models \alpha$ or $M \models \beta$.
$\models$4 $M \models \neg\alpha$ iff $M \rightsquigarrow \alpha$
$\models$5 $M \models \alpha\|\beta$ iff $M \models \alpha \wedge \beta$
$\rightsquigarrow$1 $M \rightsquigarrow p_i$ iff $p_i \notin M$
$\rightsquigarrow$2 $M \rightsquigarrow \alpha \wedge \beta$ iff at least one holds, $M \rightsquigarrow \alpha$ or $M \rightsquigarrow \beta$
$\rightsquigarrow$3 $M \rightsquigarrow \alpha \vee \beta$ iff $M \rightsquigarrow \alpha$ and $M \rightsquigarrow \beta$
$\rightsquigarrow$4 $M \rightsquigarrow \neg\alpha$ iff $M \models \alpha$
$\rightsquigarrow$5 $M \rightsquigarrow \alpha\|\beta$ iff $M \rightsquigarrow \alpha \vee \beta$

Note the central point: for verification, '$\|$' behaves like $\wedge$, for falsification, '$\|$' behaves like $\vee$. Hence it is a connective which switches the condition according to "truth modality". This ensures its intermediate position between $\wedge$ and $\vee$. This defines two relations between models and formulas. We write

- ▶ $\alpha \models \beta$ iff $M \models \alpha$ implies $M \models \beta$
- ▶ $\alpha \rightsquigarrow \beta$ iff $M \rightsquigarrow \alpha$ implies $M \rightsquigarrow \beta$.

Definition 11 We define the relation $\models_{\mathsf{cDAL}}$ by $\alpha \models_{\mathsf{cDAL}} \beta$ iff both $\alpha \models \beta$ and $\beta \rightsquigarrow \alpha$. We say $\models_{\mathsf{cDAL}} \alpha$, or α is a tautology of cDAL if for all M, $M \models \alpha$. We say $\alpha \models_{\mathsf{cDAL}}$, or α is a contradiction of cDAL, if for all M, $M \rightsquigarrow \alpha$.

Note that we consider classical models, hence $M \models \alpha$ implies $M \not\rightsquigarrow \alpha$ and conversely.

Theorem 12 *(van Eijck & Jaspars, 1995)* $\alpha \models_{\mathsf{cDAL}} \beta$ *iff* $\alpha \vdash_{\mathsf{cDAL}} \beta$.

This is the (semantic) definition of cDAL by van Eijck and Jaspars (1995), which coincides with the proof-theoretic definition, hence we will simply write $\vdash_{\mathsf{cDAL}}$. Obviously, for classical α, β we have $\alpha \models \beta$ iff $\beta \rightsquigarrow \alpha$ (contraposition) iff $\alpha \vdash_{\mathsf{cDAL}} \beta$. So $\vdash_{\mathsf{cDAL}}$ is a conservative extension of $\vdash_{\mathsf{CL}}$. We now give an alternative presentation for $\vdash_{\mathsf{cDAL}}$, using the two maps $\blacksquare, \blacklozenge : \mathit{Form}(\mathsf{CL}) \to \mathit{Form}(\mathsf{AL})$:

$$
\begin{array}{llll}
\blacksquare p & = p & \blacklozenge p & = p \\
\blacksquare(\alpha \wedge \beta) & = (\blacksquare\alpha) \wedge (\blacksquare\beta) & \blacklozenge(\alpha \wedge \beta) & = (\blacklozenge\alpha) \wedge (\blacklozenge\beta) \\
\blacksquare(\alpha \vee \beta) & = (\blacksquare\alpha) \vee (\blacksquare\beta) & \blacklozenge(\alpha \vee \beta) & = (\blacklozenge\alpha) \vee (\blacklozenge\beta) \\
\blacksquare(\neg\alpha) & = \neg(\blacklozenge\alpha) & \blacklozenge(\neg\alpha) & = \neg(\blacksquare\alpha) \\
\blacksquare(\alpha\|\beta) & = (\blacksquare\alpha) \wedge (\blacksquare\beta) & \blacklozenge(\alpha\|\beta) & = (\blacklozenge\alpha) \vee (\blacklozenge\beta)
\end{array}
$$

Lemma 13 *For all* $\alpha \in \mathit{Form}(\mathsf{AL})$,

1. $M \models \alpha$ *iff* $M \models \blacksquare\alpha$ *iff* $M \not\rightsquigarrow \blacksquare\alpha$
2. $M \rightsquigarrow \alpha$ *iff* $M \not\models \blacklozenge\alpha$ *iff* $M \rightsquigarrow \blacklozenge\alpha$

The first bi-implication can be proved by induction on formula complexity and straightforward checking of the conditions. $M \models \blacksquare\alpha$ iff $M \not\rightsquigarrow \blacksquare\alpha$ holds in virtue of $\blacksquare\alpha$ being classical. Same for $\blacklozenge$.

Lemma 14 $\alpha \vdash_{\mathsf{cDAL}} \beta$ *iff* $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ *and* $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$.

Proof *If* Assume $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ and $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$. Then by Lemma 13.1 (and classical completeness), if $M \models \alpha$, then $M \models \beta$. Moreover, by contraposition (valid for classical formulas) we obtain $M \not\models \blacklozenge\beta$ entails $M \not\models \blacklozenge\alpha$, where $M \not\models \blacklozenge\gamma$ is equivalent to $M \rightsquigarrow \gamma$ (Lemma 13.2). Hence $\alpha \vdash_{\mathsf{cDAL}} \beta$.

Only if Assume $\alpha \vdash_{\mathsf{cDAL}} \beta$. Hence $\alpha \models \beta$, so (Lemma 13.1) $M \models \blacksquare\alpha \Leftrightarrow M \models \alpha \Rightarrow M \models \beta \Leftrightarrow M \models \blacksquare\beta$. Thus $\blacksquare\alpha \models \blacksquare\beta$. Moreover, $\beta \rightsquigarrow \alpha$, so dually $M \rightsquigarrow \blacklozenge\beta \Leftrightarrow M \rightsquigarrow \beta \Rightarrow M \rightsquigarrow \alpha \Leftrightarrow M \rightsquigarrow \blacklozenge\alpha$. Since $\blacklozenge\alpha, \blacklozenge\beta$ are classical, $\blacklozenge\beta \rightsquigarrow \blacklozenge\alpha$ iff $\blacklozenge\alpha \models \blacklozenge\beta$. $\dashv$

This shows how $\vdash_{\mathsf{cDAL}}$ can be very neatly reduced to $\vdash_{\mathsf{CL}}$. The operators $\blacklozenge, \blacksquare$ will later on play other important roles for the analysis and construction of ambiguity logics. DAL and cDAL are obviously consistent and a conservative extensions of classical logic. Moreover, we have the following simple results which show that cDAL is a full ambiguity logic:

Lemma 15 *For all* $\alpha, \beta, \gamma \in Form(\mathsf{AL})$*,*

1. $\blacksquare(\alpha \wedge (\beta\|\gamma)) \dashv\vdash_{\mathsf{CL}} \blacksquare((\alpha \wedge \beta)\|(\alpha \wedge \gamma))$
2. $\blacksquare(\alpha \vee (\beta\|\gamma)) \dashv\vdash_{\mathsf{CL}} \blacksquare((\alpha \vee \beta)\|(\alpha \vee \gamma))$
3. $\blacksquare(\neg(\alpha\|\beta)) \dashv\vdash_{\mathsf{CL}} \blacksquare(\neg\alpha\|\neg\beta)$

Same for $\blacklozenge$.

Proof is an easy exercise using DeMorgan laws and $\wedge, \vee$-idempotence. Note that $\vdash_{\mathsf{DAL}}$ and $\leqq_{\mathsf{DAL}}$ coincide, and we have obviously $\blacksquare\alpha \vdash_{\mathsf{DAL}} \alpha \vdash_{\mathsf{DAL}} \blacklozenge\alpha$ and $\blacksquare\alpha \vdash_{\mathsf{cDAL}} \alpha \vdash_{\mathsf{cDAL}} \blacklozenge\alpha$. As a next result, we see the following:

Lemma 16 cDAL *satisfies (UD), (UE), and (‖mon), that is:*

1. $(\alpha\|\beta) \wedge \gamma \dashv\vdash_{\mathsf{cDAL}} (\alpha \wedge \gamma)\|(\beta \wedge \gamma)$
2. $(\alpha\|\beta) \vee \gamma \dashv\vdash_{\mathsf{cDAL}} (\alpha \vee \gamma)\|(\beta \vee \gamma)$
3. $\neg(\alpha\|\beta) \dashv\vdash_{\mathsf{cDAL}} (\neg\alpha)\|(\neg\beta)$
4. $\alpha \wedge \beta \vdash_{\mathsf{cDAL}} \alpha\|\beta \vdash_{\mathsf{cDAL}} \alpha \vee \beta$
5. $\alpha\|\beta \vdash_{\mathsf{cDAL}} (\alpha \vee \gamma)\|(\beta \vee \delta)$

These are all straightforward to verify with Lemma 14, same for (‖assoc). Hence:

Corollary 17 cDAL *is a full ambiguity logic.*

For DAL, these results hold as well, but not having a suitable semantics or reduction to CL, they are lengthy to prove. Now these results together with the Basic Closure Theorem already entail the following:

Lemma 18 *The logic* cDAL *is not closed under uniform substitution.*

For example, $p \vee \neg p$ is a theorem in cDAL. However, if we uniformly substitute p by $p\|q$, we obtain $(p\|q) \vee \neg(p\|q)$. We have

$$\blacksquare((p\|q) \vee \neg(p\|q)) = (\blacksquare p\|q) \vee (\neg\blacklozenge p\|q) = (p \wedge q) \vee \neg(p \vee q) \tag{7}$$

which is not a classical theorem, hence $\nvdash_{\mathsf{cDAL}} (p\|q) \vee \neg(p\|q)$. It is easy to see the trick: negation changes $\blacksquare$ to $\blacklozenge$. Also Modus Ponens (MP) is not generally valid in cDAL:

$$\nVdash_{\mathsf{cDAL}} (p\|q) \to r, p\|q \vdash r$$

It is an easy exercise to verify this; the reason is the same as above in Lemma 18: define, as usual, $(p\|q) \to r \equiv \neg(p\|q) \vee r$, hence

$$\blacklozenge((p\|q) \to r), \blacklozenge(p\|q) = (p \wedge q) \to r, p \vee q \tag{8}$$

which obviously does not imply r. Conceptually, this reflects distrust: maybe the formula $p\|q$ is *intended in a different sense* than the premise of $(p\|q) \to r$, hence MP is not applicable. In other words, we do not trust that ambiguous terms are used consistently in an argument. Even for a logic as simple and intuitive as cDAL, there are some slightly counterintuitive results. Lemma 14 shows for every formula α, we have

$$\alpha \dashv\vdash_{\mathsf{cDAL}} \blacklozenge\alpha \| \blacksquare\alpha \tag{9}$$

This means: every formula in cDAL is equivalent to a *binary* ambiguity. It also entails that all formulas of the form $a \| \neg a$, a an arbitrary classical formula, are equivalent. So we derive sequents like $p \| \neg p \vdash_{\mathsf{cDAL}} q \| \neg q$. Since equivalence entails congruence for DAL, we can also derive sequents like

$$p \| \neg p \| \neg p \dashv\vdash_{\mathsf{cDAL}} q \| \neg q \| \neg p \tag{10}$$

This can be generalized to the following observation:

Lemma 19 *Assume $\beta \in Form(\mathsf{CL})$, and $\blacksquare\alpha \vdash_{\mathsf{CL}} \beta \vdash_{\mathsf{CL}} \blacklozenge\alpha$. Then $\alpha \equiv_{\mathsf{cDAL}} \alpha \| \beta$.*

Proof $\alpha \equiv_{\mathsf{cDAL}} \blacksquare\alpha \| \blacklozenge\alpha$. By assumption, $\blacksquare\alpha \equiv_{\mathsf{CL}} (\blacksquare\alpha) \wedge \beta = \blacksquare(\alpha \| \beta)$, $\blacklozenge\alpha \equiv_{\mathsf{CL}} (\blacklozenge\alpha) \vee \beta = \blacklozenge(\alpha \| \beta)$. Hence $\alpha \equiv_{\mathsf{cDAL}} (\blacksquare(\alpha \| \beta)) \| (\blacklozenge(\alpha \| \beta)) \equiv_{\mathsf{cDAL}} \alpha \| \beta$. ⊣

For example, $p \| \neg p \equiv_{\mathsf{cDAL}} p \| \neg p \| q$, and so:

Lemma 20 *1.* cDAL *does not satisfy the general law of disambiguation.*

2. cDAL *satisfies the weak law of disambiguation, $\alpha \| b \| \gamma, \neg b \vdash \alpha \| \gamma$, where b is a formula of classical logic.*

Proof 1. Take the sequent $(p \| \neg p \| q) \wedge \neg(p \| \neg p) \vdash q$, which is an instance of the general law of disambiguation. This is not valid in cDAL, as can be easily checked.

2. cDAL satisfies ($\|$mon), and if b is classical, $b \wedge \neg b$ is a contradiction, so $(\alpha \| b \| \gamma) \wedge \neg b \equiv_{\mathsf{cDAL}} (\alpha \wedge \neg b) \| (b \wedge \neg b) \| (\gamma \wedge \neg b) \vdash_{\mathsf{cDAL}} \alpha \| \alpha \| \gamma \equiv_{\mathsf{cDAL}} \alpha \| \gamma$ ⊣

Lemma 20.1 is slightly surprising. It is an important result because it proves that the law of disambiguation is independent from other properties of ambiguity. We see how the law of disambiguation depends on whether 1. $\beta \wedge \neg\beta$ is a contradiction, and 2. whether ($\|$mon) is holds. Since in cDAL, $\beta \wedge \neg\beta$ is not generally a a contradiction, the general LoD does not hold. But $\beta \wedge \neg\beta$ is a contradiction in all trustful logics of ambiguity, hence they usually satisfy the general LoD. This should also illustrate how the general LoD cannot be a necessary criterion to define a "logic of ambiguity": it would in fact exclude most distrustful logics!

We present one more result that we will need later on. The operator $\blacklozenge$ obviously makes every proposition "more true", the operator $\blacksquare$ makes it "less true". Hence if $\blacklozenge\alpha$ is false in a classical model, then $\blacksquare\alpha$ is also false, and if $\blacksquare\alpha$ is true in a model, then $\blacklozenge\alpha$ is also true. What is less obvious is that $\blacklozenge$ preserves theoremhood, and dually $\blacksquare$ preserves contradictions, even when we close under u-substitution.

Lemma 21 *Assume $\alpha \in Form(\mathsf{CL})$, $\sigma : Var \to Form(\mathsf{AL})$ is a uniform substitution. Then there is a function $f : \wp(Var) \to \wp(Var)$ such that*

1. *if $M \not\models \blacklozenge\sigma(\alpha)$, then $f(M) \not\models \alpha$*
2. *if $M \models \blacksquare\sigma(\alpha)$, then $f(M) \models \alpha$*

Proof Define $f(M) := \{p : M \models \blacklozenge\sigma(p)\}$. We prove that this does the job, via simultaneous induction (for both 1. and 2.) on formula complexity of α.

Base case: $\alpha = p$.

1. Contraposition: assume $f(M) \models p$. Then by definition $M \models \blacklozenge\sigma(p)$.

2. Assume $M \models \blacksquare\sigma(p)$. Hence $M \models \blacklozenge\sigma(p)$. Hence $p \in f(M)$, so $f(M) \models p$.

Induction hypothesis (IH): Assume the claims 1.,2. hold for α, β.

Induction step:

$\alpha \wedge \beta$.

1. Assume there is an M such that $M \not\models \blacklozenge\sigma(\alpha \wedge \beta)$ $(= (\blacklozenge\sigma(\alpha)) \wedge (\blacklozenge\sigma(\beta)))$. Case i: $M \not\models \blacklozenge\sigma(\alpha)$. Then the claim follows by IH. Case ii: parallel.

2. Assume that $M \models \blacksquare\sigma(\alpha \wedge \beta)$ $(= \blacksquare\sigma(\alpha) \wedge \blacksquare\sigma(\beta))$. Hence $f(M) \models \alpha$, $f(M) \models \beta$, hence $f(M) \models \alpha \wedge \beta$.

$\alpha \vee \beta$ Dual.

$\neg\alpha$

1. Assume that $M \not\models \blacklozenge\sigma(\neg\alpha)$ $(= \neg\blacksquare\sigma(\alpha))$. Hence $M \not\models \neg\blacksquare\sigma(\alpha)$, so $M \models \blacksquare\sigma(\alpha)$. By IH2, $f(M) \models \alpha$, hence $f(M) \not\models \neg\alpha$.

2. Dual. $\dashv$

Corollary 22 *If* $\alpha \in$ *Form*(CL), σ : *Var* $\to$ *Form*(AL) *is a uniform substitution, then*

1. *if* α *is a tautology, then* $\blacklozenge\sigma(\alpha)$ *is a theorem in* cDAL.
2. *if* α *is a contradiction, then* $\blacksquare\sigma(\alpha)$ *is a contradiction in* cDAL.

Proof of this is a simple contraposition using the previous lemma. So $\blacklozenge\sigma(\alpha)$ does not preserve truth of α in a model (provided α is not a theorem), but it preserves theoremhood.

To sum up, cDAL is a very simple and natural logic of ambiguity, and many results can be obtained as easy exercise. Another big advantage is that it is straightforward to extend it to predicate logic (or fragments thereof, like description logic), which seems a prerequisite for real-world applications.

3.3 Trust without Order: cTAL and $L_{\blacksquare\blacklozenge}$

We have stated here (and argued in Wurm, 2021) that TAL is a very reasonable trustful logic of ambiguity. We will now show the same is *not* true for cTAL, since it is way too permissive. This makes a more general point: trustful reasoning does not go well together with $\|$-commutativity. If we trust in uniform usage, we should keep track of ordering (which roughly corresponds to plausibility). The argument is purely formal though, rather than conceptual. Consider the following rules.

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta \quad \Theta_1, \Theta_2 \vdash \Delta}{(\Gamma_1; \Theta_1), (\Gamma_2; \Theta_2) \vdash \Delta}\ (;I) \qquad \frac{\Delta \vdash \Gamma_1, \Gamma_2 \quad \Delta \vdash \Theta_1, \Theta_2}{\Delta \vdash (\Gamma_1; \Theta_1), (\Gamma_2; \Theta_2)}\ (I;)$$

Lemma 23 *In* TAL, cTAL, *the rules* $(;I)$, $(I;)$ *are admissible.*

Proof In TAL and cTAL, $\Gamma \vdash \Delta$ iff $l(\Gamma) \vdash r(\Delta)$ (Lemma 9). Hence $\Gamma_1, \Gamma_2 \vdash \Delta$ entails $\Gamma_1, l(\Gamma_2) \vdash \Delta$ entails $\Gamma_1 \vdash \Delta, \neg(l(\Gamma_2))$, and $\Theta_1, \Theta_2 \vdash \Delta$ entails $\Theta_1, l(\Theta_2) \vdash \Delta$ entails $\Theta_1 \vdash \Delta, \neg(l(\Theta_2))$.

We derive $\Gamma_1; \Theta_1 \vdash \Delta, (\neg l(\Gamma_2); \neg l(\Theta_2))$, so $(\Gamma_1; \Theta_1), (\neg\neg l(\Gamma_2); \neg\neg l(\Theta_2)) \vdash \Delta$, and by double negation congruence (see Wurm, 2021), the claim follows. ⊣

In cTAL we can then make the following proof:

$$\dfrac{\dfrac{\dfrac{\alpha, \beta \vdash \alpha \wedge \beta \quad \beta, \alpha \vdash \alpha \wedge \beta}{(\alpha; \beta), (\beta; \alpha) \vdash \alpha \wedge \beta}\,(\mathrm{I};\,)}{(\alpha; \beta), (\alpha; \beta) \vdash \alpha \wedge \beta}\,(;\,\mathrm{comm})}{\alpha; \beta \vdash \alpha \wedge \beta}\,(,\,\mathrm{contr})$$

Dually, we can derive $\alpha \vee \beta \vdash_{\mathsf{cTAL}} \alpha \| \beta$. In this sense, we can say that cTAL is **too trustful**: these sequents are clearly counterintuitive. Interestingly, even though we derive $\alpha \vee \beta \vdash_{\mathsf{cTAL}} \alpha \| \beta$ and $\alpha \| \beta \vdash_{\mathsf{cTAL}} \alpha \wedge \beta$, the logic is still consistent and conservatively extends classical logic, because of the absence of (cut). To see where cTAL "goes wrong", consider that in TAL, we can derive the weaker

$$(\alpha \| \beta) \wedge (\beta \| \alpha) \vdash_{\mathsf{TAL}} \alpha \wedge \beta$$

This does not strike us as necessarily incorrect: basically if the first reading can be α and β, and the second reading can be α and β, $\alpha \wedge \beta$ follows. Now it comes even worse for cTAL. Consider the following lemma, which holds only for cTAL, not for TAL. It is subtle, but it will make a big difference:

Lemma 24 *In* cTAL, $\Gamma[\alpha]; \Gamma[\beta] \vdash \Delta$ *iff* $\Gamma[\alpha; \beta] \vdash \Delta$. *Dually on the right.*

Proof By induction over complexity of $\Gamma[-]$. Base case (identity function) is clear, since then $\alpha; \beta = \Gamma[\alpha]; \Gamma[\beta] = \Gamma[\alpha; \beta]$. Now assume it holds for some $\Gamma[-]$.

, $(\Gamma[\alpha], \Gamma'); (\Gamma[\beta], \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ iff $(\Gamma[\alpha]; \Gamma[\beta]), \Gamma', \Gamma' \vdash_{\mathsf{cTAL}} \Delta$ (by 2× (distr/invDistr)) iff $(\Gamma[\alpha]; \Gamma[\beta]), \Gamma' \vdash_{\mathsf{cTAL}} \Delta$ (by (, contr)) iff $\Gamma[\alpha; \beta], \Gamma' \vdash_{\mathsf{cTAL}} \Delta$ (by IH). Same on the right.

; $(\Gamma[\alpha]; \Gamma'); (\Gamma[\beta]; \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ iff $(\Gamma[\alpha]; \Gamma[\beta]; \Gamma'; \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ (by (; comm) iff $(\Gamma[\alpha; \beta]; \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ (by (; contr)). Same on the right. ⊣

Now consider the following rules of ambiguous weakening:

$$(;\,\mathrm{weak})\ \dfrac{\Gamma[\Theta] \vdash \Delta}{\Gamma[(\Theta; \Psi)] \vdash \Delta} \qquad (\mathrm{weak};\,)\ \dfrac{\Gamma \vdash \Delta[\Theta]}{\Gamma \vdash \Delta[(\Theta; \Psi)]}$$

Lemma 25 *(; weak), (weak;) are admissible in* cTAL.

Proof Assume we have $\Gamma[\Theta] \vdash_{\mathsf{cTAL}} \Delta$. We can apply (, weak) and obtain:

$$\dfrac{\dfrac{\dfrac{\Gamma[\Psi], \Gamma[\Theta] \vdash \Delta \quad \Gamma[\Theta], \Gamma[\Psi] \vdash \Delta}{(\Gamma[\Psi]; \Gamma[\Theta]), (\Gamma[\Theta]; \Gamma[\Psi]) \vdash \Delta}\,(;\,\mathrm{I})}{(\Gamma[\Psi]; \Gamma[\Theta]), (\Gamma[\Psi]; \Gamma[\Theta]) \vdash \Delta}\,(;\,\mathrm{comm})}{(\Gamma[\Psi]; \Gamma[\Theta]) \vdash \Delta}\,(,\,\mathrm{contr})$$

Finally, we obtain $\Gamma[\Psi; \Theta] \vdash_{\mathsf{cTAL}} \Delta$ by the previous lemma. ⊣

Now this is of course *way* too trustful: it entails that $a_1 \| ... \| a_i \vdash_{\mathsf{cTAL}} b_1 \| ... \| b_j$ if there is at least one pair (n, m) such that $a_n \vdash_{\mathsf{CL}} b_m$ (this is a sufficient criterion, not necessary).

Lemma 26 $\alpha \vdash_{\mathsf{cTAL}} \beta$ *iff* $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$

Proof *Only if* Follows from $\blacksquare\alpha \leqq_{\mathsf{cTAL}} \alpha$, $\beta \leqq_{\mathsf{cTAL}} \blacklozenge\beta$ and conservative extension.

If: Assume $\alpha = a_1\|...\|a_i$, $\beta = b_1\|...\|b_j$ are in ambiguous normal form (all a, b classical). This comes without loss of generality from Lemma 15 and Lemma 10.

$\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ means then as much as $a_1, ..., a_i \vdash_{\mathsf{CL}} b_1, ..., b_j$. By conservative extension, we have $a_1,..., a_i \vdash_{\mathsf{cTAL}} b_1, ..., b_j$.

Now we can apply (**;** weak) to every formula of the sequent, until we obtain $\underbrace{\alpha, ..., \alpha}_{i \text{ times}} \vdash_{\mathsf{cTAL}} \underbrace{\beta, ..., \beta}_{j \text{ times}}$; then we can apply (**,** contr) to obtain $\alpha \vdash_{\mathsf{cTAL}} \beta$. ⊣

This means: our logic cTAL has a simple characterization in terms of classical logic! The same does not apply, however, to TAL. Note that (classic cut) and the fact that $p \vee \neg p \vdash_{\mathsf{cTAL}} p\|\neg p$ entail that $p\|\neg p$ is a *theorem* of cTAL, and, by a dual argument, at the same time a *contradiction*! Note also that the proof relies heavily on (**,** contr), put differently, the fact that $\alpha \wedge \alpha \equiv \alpha$, and one might start to wonder whether the rule should be taken for granted in the context of ambiguity.

Corollary 27 cDAL $\subseteq$ cTAL

Proof $\blacksquare\beta \vdash_{\mathsf{CL}} \blacklozenge\beta$, hence if $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$, then $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$, so $\alpha \vdash_{\mathsf{cTAL}} \beta$. ⊣

So in this case, trust does not only come with closure under substitution, but with more effectively valid inferences. We will generalize this observation in the Trust Theorem. Note also the following corollary:

Corollary 28 cTAL *is m-trivial.*

We have $\blacksquare(\alpha\|\beta) \vdash_{\mathsf{CL}} \blacklozenge(\alpha\|\gamma)$ for arbitrary α, β, γ, so the claim is obvious. To sum up this discussion, we can say the following: trust goes well together with an ordering of readings (which roughly corresponds to plausibility ordering). Maybe this is even the *essence of trust* (in ambiguity): that we have to assume a fixed ordering of plausibility, since otherwise, trust becomes ingenuousness.

Finally, let us make the following observation: both cDAL and cTAL have simple truth-theoretic characterizations, whereas their non-commutative counterparts DAL, TAL require a complex proof-theory. However, their proof-theories seem to have some “natural" properties, since adding commutativity makes them become so easily accessible in terms of truth (this relates to a property which we will call *classically congruent* later, see Definition 54).

4 Congruence Algebras and Inner Logics

4.1 Congruence Algebras and Generalized Boolean Algebras

The Basic Closure Theorem states that we cannot think algebraically about reasoning with ambiguity. This however does *not* prevent us from analyzing logics of ambiguity algebraically. In particular, we can construct, for every logic of ambiguity L, a **congruence algebra**, which is similar to the well-known Lindenbaum-Tarski algebra, but its elements are *not* $\dashv\vdash_L$- equivalence classes, but rather $\equiv_L$**-congruence classes** (the relations coincide for most logics of distrust, though).

For every logic of ambiguity L, $\equiv_L$ is obviously an equivalence relation, hence we can define the algebra of its congruence classes, or its **congruence algebra**:

$$Cong(L) := (Form(\mathsf{AL})_{\equiv_L}, \wedge, \vee, \neg, \|)$$

We know that logical connectives are well-defined operators on congruence classes, independently of representatives (whereas this is not satisfied for $\dashv\vdash_L$!). The Basic Closure Theorem in this sense obtains a different reading/meaning: for all non-trivial logics of ambiguity, their congruence algebra will *not be Boolean*.

Lemma 29 *Let L be a logic of ambiguity. Assume the Boolean reduct of* $Cong(L)$ *is a Boolean algebra. Then L is m-trivial.*

This is proved in (Wurm, 2021, Theorem 75). Obviously, $Cong(L)$ can be equipped with an order relation $\leq_{Cong(L)}$, where $\alpha \leq_{Cong(L)} \beta$ iff $\alpha \wedge \beta \equiv_L \alpha$ and $\alpha \vee \beta \equiv_L \beta$. The following notion is an important criterion for "plausibility" of logics, which basically states that $\wedge, \vee$ behave in a "logical way":

Definition 30 An ambiguity logic L is **naturally ordered** if for all $\alpha, \beta \in Form(\mathsf{AL})$, $\alpha \leq_{Cong(L)} \beta$ iff $\alpha \leqq_L \beta$.

This is a very basic requirement. Note that this is not equivalent with requiring that $Cong(L)$ is lattice ordered: $Cong(L)$ can be lattice ordered, where $\alpha \leq_{Cong(L)} \beta$. Hence $\Gamma[\beta] \vdash \Delta$ entails $\Gamma[\alpha \vee \beta] \vdash \Delta$. This need not entail $\Gamma[\alpha] \vdash \Delta$, because $(\vee I)$ need not be invertible (for example, TAL without (inter2)). Conversely, a logic can be naturally ordered, yet we might have $\alpha \lneqq_{Cong(L)} \alpha \wedge \alpha$ ($\wedge, \vee$ need not be idempotent). Every logic we have considered until now is both lattice ordered and naturally ordered, but we will see possible counterexamples in Section 5.4.2.

The algebraic study of naturally lattice ordered logics of ambiguity is basically (an application of) the study of generalized Boolean algebras, a well-established algebraic field which explores the space between distributive lattices with a unary operator and Boolean algebras. So let us have a look at some classes of generalized Boolean algebras.

Definition 31 A **generalized Boolean algebra** (GBA) $(B, \wedge, \vee, \sim, 0, 1)$ is an algebra where $(B, \wedge, \vee, 0, 1)$ is a bounded, distributive lattice, and $\sim$ a unary function.

Since every GBA is lattice ordered, the ordering $\leq$ is defined implicitly by $\wedge$ and $\vee$. $(B, \wedge, \vee, \sim, 0, 1)$ is

1. an **Ockham algebra**, if it satisfies $\sim (x \wedge y) = \sim x \vee \sim y$, $\sim (x \vee y) = \sim x \wedge \sim y$, and $\sim 0 = 1$, $\sim 1 = 0$.
2. a **DeMorgan algebra**, if in addition it satisfies $\sim\sim x = x$.
3. a **Kleene algebra**, if in addition it satisfies $x \wedge \sim x \leq y \vee \sim y$.

Importantly, this is to say that the equations are **valid**, meaning they are true whatever we substitute for the variables (u-substitution).

4.2 On the Congruence Algebra of TAL

We will see that the congruence algebra of TAL is a Kleene algebra. At the same time, for all known classes of generalized Boolean algebras, Kleene algebras seem to be the most specific class in which the congruence algebra of TAL is contained.

Lemma 32 *Cong*(TAL) *is naturally ordered.*

Proof Assume $\alpha \leq_{Cong(\mathsf{TAL})} \beta$. So $\alpha \wedge \beta \equiv_{\mathsf{TAL}} \alpha$, hence $\Gamma[\beta] \vdash_{\mathsf{TAL}} \Delta$ implies $\Gamma[\alpha \wedge \beta] \vdash_{\mathsf{TAL}} \Delta$ implies $\Gamma[\alpha] \vdash_{\mathsf{TAL}} \Delta$; similar on the right and dual for $\vee$, so $\alpha \leqq_{\mathsf{TAL}} \beta$

Conversely, assume $\alpha \leqq_{\mathsf{TAL}} \beta$. Hence $\Gamma[\alpha \wedge \beta] \vdash_{\mathsf{TAL}} \Delta$ iff $\Gamma[\alpha, \beta] \vdash_{\mathsf{TAL}} \Delta$ iff $\Gamma[\alpha, \alpha] \vdash_{\mathsf{TAL}} \Delta$ iff $\Gamma[\alpha] \vdash_{\mathsf{TAL}} \Delta$, so $\alpha \wedge \beta \equiv_{\mathsf{TAL}} \alpha$; similar on the right and dual for $\vee$. ⊣

This proof can be conceived of as a blueprint, as it works equally for cTAL, cDAL, DAL, so we will simple use $\leqq_{\mathsf{TAL}}$ instead of $\leq_{Cong(\mathsf{TAL})}$ etc. Next, we prove the Context Lemma, which allows us to easily establish congruence results. We conceive of contexts $\Gamma[-]$ as functions, and say $\Gamma[-]$ is **classic** if $\Gamma[\alpha] = (\Gamma_1, \alpha, \Gamma_2)$, and $\Gamma[-]$ is ambiguous if $\Gamma[\alpha] = (\Gamma_1; \alpha; \Gamma_2)$. Note that the identity function is both classical and ambiguous. Obviously, every context function $\Psi[-]$ can be brought into the form

$$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-] \circ \Delta_i[-] \circ \Gamma_{i+1}[-] \tag{11}$$

where each $\Gamma_j[-]$ is a classic context function, each $\Delta_j[-]$ an ambiguous context function, and $\circ$ denotes function composition.

Lemma 33 *(Context Lemma)*

1. *Assume for all contexts* $\Gamma, \Delta,\ \Gamma, \beta \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma, \alpha \vdash_{\mathsf{TAL}} \Delta$. *Then for all context-functions* $\Gamma[-]$, *contexts* $\Delta,\ \Gamma[\beta] \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma[\alpha] \vdash_{\mathsf{TAL}} \Delta$.
2. *Assume for all contexts* $\Gamma, \Delta,\ \Delta \vdash_{\mathsf{TAL}} \Gamma, \beta$ *entails* $\Delta \vdash_{\mathsf{TAL}} \Gamma, \alpha$. *Then for all contexts* Γ, *context-functions* $\Delta[-],\ \Gamma \vdash_{\mathsf{TAL}} \Delta[\beta]$ *entails* $\Gamma \vdash_{\mathsf{TAL}} \Delta[\alpha]$.

We only prove 1., since 2. is completely dual.

Proof We make an induction over the complexity of context functions, that is i in (11).

Base case is $i = 0$, which is trivial (identity function). Assume (IH) the claim holds for some i, and we have

$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-] \circ \Delta_{i+1}[-] \circ \Gamma_{i+1}[-](\beta)$	$\vdash_{\mathsf{TAL}} \Psi$.	Then
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](A; (\beta, B); C)$	$\vdash_{\mathsf{TAL}} \Psi$	(alt. notation)
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](A; (\beta, \alpha, B); C)$	$\vdash_{\mathsf{TAL}} \Psi$	(,weak)
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](\beta, (A; (\alpha, B); C))$	$\vdash_{\mathsf{TAL}} \Psi$	(invDistr)
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](\alpha, (A; (\alpha, B); C))$	$\vdash_{\mathsf{TAL}} \Psi$	(by IH)

Now we can apply (inter2) to obtain the desired result:

$$\frac{\Gamma_1[-] \circ ... \circ \Gamma_i[-](\alpha, (A; (\alpha, B); C)) \vdash \Psi \quad \Gamma_1[-] \circ ... \circ \Gamma_i[-](A; (\beta, \alpha, B); C)] \vdash \Psi}{\Gamma_1[-] \circ ... \circ \Gamma_i[-]((A; (\alpha, B); C)) \vdash \Psi}$$

which is an alternative notation for

$$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_{i-1}[-] \circ \Delta_{i+1}[-] \circ \Gamma_{i+1}[-](\alpha) \vdash \Psi \qquad \dashv$$

This lemma is very useful: many congruence proofs, which can be tedious inductions over contexts or proof lengths, become simple exercises. For example, from our general invertibility lemma, we know that $\Gamma, \neg\alpha \vdash_{\mathsf{TAL}} \Delta$ if and only if $\Gamma \vdash_{\mathsf{TAL}} \alpha, \Delta$. Hence we have $\Gamma, \neg\neg\alpha \vdash_{\mathsf{TAL}} \Delta$ iff $\Gamma, \alpha \vdash_{\mathsf{TAL}} \Delta$. It is also easy to see that $\Gamma, \alpha \vee \beta \vdash_{\mathsf{TAL}} \Delta$ entails $\Gamma, \alpha \| \beta \vdash_{\mathsf{TAL}} \Delta$ entails $\Gamma, \alpha \wedge \beta \vdash_{\mathsf{TAL}} \Delta$. The following results are easy exercises to prove (also proved in Wurm, 2021).

Lemma 34 *In* TAL*, for all* α, β, γ*, we have:*

1. $\alpha \equiv_{\mathsf{TAL}} \neg\neg\alpha$
2. $\neg(\alpha \wedge \beta) \equiv_{\mathsf{TAL}} \neg\alpha \vee \neg\beta$
3. $(\alpha \vee \beta) \wedge \gamma \equiv_{\mathsf{TAL}} (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$

Moreover, we obtain a useful result, which will have some importance later on. Recall that (classic cut) is admissible in TAL (for proof, see Wurm, 2021).

Lemma 35 *Assume* $\alpha \vdash_{\mathsf{CL}} \beta$*. Then* $\alpha \leqq_{\mathsf{TAL}} \beta$*.*

Proof From $\Gamma, \beta \vdash_{\mathsf{TAL}} \Delta$ and $\alpha \vdash_{\mathsf{CL}} \beta$ we can deduce $\Gamma, \alpha \vdash_{\mathsf{TAL}} \Delta$ by (classic cut). Same goes for $\Gamma \vdash_{\mathsf{TAL}} \Delta, \alpha$ and $\alpha \vdash_{\mathsf{CL}} \beta$, which entail $\Gamma \vdash_{\mathsf{TAL}} \Delta, \beta$. Hence we simply apply the Context Lemma, and $\alpha \leqq_{\mathsf{TAL}} \beta$ follows. $\dashv$

Recall the maps l, r from contexts to formulas. We define complexity of contexts as follows: for all formulas α, $|\alpha| = 1$; $|\Gamma, \Delta| = |\Gamma| + |\Delta|$ and $|\Gamma; \Delta| = |\Gamma| + |\Delta|$. We write $\Gamma \equiv^l_{\mathsf{TAL}} \Delta$ if $\Theta, \Gamma \vdash_{\mathsf{TAL}} \Psi \iff \Theta, \Delta \vdash_{\mathsf{TAL}} \Psi$; parallel for $\equiv^r_{\mathsf{TAL}}$.

Lemma 36 *For all contexts* Ξ*,*

1. $\neg\Xi \equiv^r_{\mathsf{TAL}} \neg l(\Xi)$
2. $\neg\Xi \equiv^l_{\mathsf{TAL}} \neg r(\Xi)$

Proof We only prove claim 1. To increase readability, we put $a := l(A)$, $b := l(B)$.

Induction over $|\Xi|$. If $\Xi = \xi$ is a formula, the claim is obvious. Assume it holds for all $\Xi : |\Xi| \leq n$ for a given n. We distinguish two cases

Case 1. $\Xi = (A, B)$. Note that $\neg l(A, B) = \neg(l(A) \wedge l(B)) =: \neg(a \wedge b)$.

Assume	$\Gamma \vdash \neg(a \wedge b),\ \Delta$	Assume	$\Gamma \vdash \Delta,\ \neg A,\ \neg B$
Hence	$\Gamma,\ (a \wedge b) \vdash \Delta$	Hence	$\Gamma \vdash \Delta,\ \neg a,\ \neg b$
Hence	$\Gamma,\ a,\ b \vdash \Delta$	Hence	$\Gamma,\ a\, b \vdash \Delta$
Hence	$\Gamma \vdash \Delta,\ \neg a,\ \neg b$	Hence	$\Gamma,\ (a \wedge b) \vdash \Delta$
Hence	$\Gamma \vdash \Delta,\ \neg A,\ \neg B$	Hence	$\Gamma,\ \vdash \neg(a \wedge b),\ \Delta$
Hence	$\Gamma \vdash \Delta,\ \neg(A,\ B)$	Hence	$\Gamma,\ \vdash \neg l(A,\ B)\, \Delta$

Case 2. $\Xi = (A; B)$. Note that $\neg l(A; B) = \neg(l(A) \| l(B)) =: \neg(a \| b)$.

Assume	$\Gamma \vdash \neg(a \| b)\, \Delta$	Assume	$\Gamma \vdash \neg(A;\ B)\, \Delta$
Hence	$\Gamma\ a \| b \vdash \Delta$	Hence	$\Gamma \vdash (\neg A; \neg B)\, \Delta$

Hence	$\Gamma\,(a; b) \vdash \Delta$	Hence	$\Gamma \vdash (\neg a; \neg b)\,\Delta$
Hence	$\Gamma \vdash (\neg a; \neg b)\,\Delta$	Hence	$\Gamma\,(\neg\neg a; \neg\neg b) \vdash \Delta$
Hence	$\Gamma \vdash (\neg A; \neg B)\,\Delta$	Hence	$\Gamma\,(a; b) \vdash \Delta$
Hence	$\Gamma \vdash \neg(A; B)\,\Delta$	Hence	$\Gamma\,(a\|b) \vdash \Delta$
		Hence	$\Gamma \vdash \neg(a\|b)\,\Delta$

$\dashv$

This gives us the following nice corollary:

Lemma 37 *(Generalized negation)*

1. $\Gamma, \Xi \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma \vdash_{\mathsf{TAL}} \neg\Xi, \Delta$
2. $\Gamma \vdash_{\mathsf{TAL}} \Xi, \Delta$ *entails* $\Gamma, \neg\Xi \vdash_{\mathsf{TAL}} \Delta$

Proof Assume $\Gamma, \Xi \vdash_{\mathsf{TAL}} \Delta$. Hence $\Gamma, l(\Xi) \vdash_{\mathsf{TAL}} \Delta$, so $\Gamma \vdash_{\mathsf{TAL}} \neg l(\Xi), \Delta$, hence $\Gamma \vdash_{\mathsf{TAL}} \neg\Xi, \Delta$. $\dashv$

We will negate not only contexts, but also context functions, as in $(\neg\Gamma)[-]$, with the obvious meaning. Note that $(\neg\Gamma)[\alpha] = (\neg\Gamma[-])(\alpha) \neq \neg(\Gamma[\alpha]) = (\neg\Gamma)[\neg\alpha]$. The law of contraposition holds in *Cong*(TAL):

Lemma 38 $\alpha \leqq_{\mathsf{TAL}} \beta$ *iff* $\neg\beta \leqq_{\mathsf{TAL}} \neg\alpha$

Proof $\Rightarrow$ Assume $\alpha \leqq_{\mathsf{TAL}} \beta$, and $\Gamma[\neg\alpha] \vdash_{\mathsf{TAL}} \Delta$. Then $\vdash_{\mathsf{TAL}} \neg(\Gamma[\neg\alpha]), \Delta$, hence by DN-congruence $\vdash_{\mathsf{TAL}} \neg(\Gamma)[\alpha], \Delta$. So by assumption $\vdash_{\mathsf{TAL}} (\neg\Gamma)[\beta], \Delta$, hence $(\neg\neg\Gamma)[\neg\beta] \vdash_{\mathsf{TAL}} \Delta$, so $\Gamma[\neg\beta] \vdash_{\mathsf{TAL}} \Delta$. Similar on the right hand side.
$\Leftarrow$ Immediate from double negation congruence. $\dashv$

Now we return to the question we have mentioned in the beginning: where can we locate *Cong*(TAL) in the family of generalized Boolean algebras? The following is already immediate:

Lemma 39 *The Boolean reduct of Cong*(TAL) *is a DeMorgan algebra.*

We now strengthen this result. Having double negation, DeMorgan law and the distributive laws for lattices, we are lacking the inequation

$$\alpha \wedge \sim \alpha \leq \beta \vee \sim \beta \tag{12}$$

which in our terms means: every contradiction of TAL is logically stronger ($\leqq_{\mathsf{TAL}}$-smaller) than every theorem of TAL. Again, Lemma 33 allows us to prove this in a simple fashion:

Lemma 40 *1. Assume* $\Gamma[\Theta] \vdash_{\mathsf{TAL}} \Delta$ *and* $\Psi \vdash_{\mathsf{TAL}}$ *(Ψ is a contradiction). Then* $\Gamma[\Psi] \vdash_{\mathsf{TAL}} \Delta$ *is derivable.*
2. Assume $\Gamma \vdash_{\mathsf{TAL}} \Delta[\Theta]$ *and* $\vdash_{\mathsf{TAL}} \Psi$ *(Ψ is a theorem). Then* $\Gamma \vdash_{\mathsf{TAL}} \Delta[\Psi]$ *is derivable.*

Proof We only prove 1. It is obvious that if $\Psi \vdash_{\mathsf{TAL}}$, we can derive $\Gamma, \Psi \vdash_{\mathsf{TAL}} \Delta$ (by weakening). So if $\Gamma, \Theta \vdash_{\mathsf{TAL}} \Delta$ and $\Psi \vdash_{\mathsf{TAL}}$, then $\Gamma, \Psi \vdash_{\mathsf{TAL}} \Delta$. From here, the claim follows via Lemma 33 (Context Lemma). $\dashv$

From this double result, the following follows easily:

Corollary 41 *Let α be an arbitrary contradiction of* TAL, *β and arbitrary theorem. Then $\alpha \leqq_{\mathsf{TAL}} \beta$. Hence the Boolean reduct of Cong*(TAL) *is a Kleene algebra.*

Proof $\Gamma[\beta] \vdash \Delta$ entails $\Gamma[\alpha] \vdash \Delta$ since α is a contradiction; $\Delta \vdash \Gamma[\alpha]$ entails $\Delta \vdash \Gamma[\beta]$ since β is a theorem. $\dashv$

Seemingly, the previous lemma even proves a stronger claim: theorems are maximal in the sense that one can substitute a theorem for any other formula on the right, and a contradiction for any other formula on the left. But we have to be careful: not all theorems/contradictions are $\equiv_{\mathsf{TAL}}$-equivalent! Actually, this would entail that the TAL-congruence algebra is Boolean (follows from results on generalized Boolean algebras), and hence in an additional step, that TAL is m-trivial. Where is the mistake in that line of thought? All theorems are interchangeable on the right of $\vdash_{\mathsf{TAL}}$, all contradictions are interchangeable on the left. But: different theorems might have different distributions on the *left* of $\vdash_{\mathsf{TAL}}$, and contradictions on the *right*. In fact, Corollary 41 does not even entail that for some theorem β and an arbitrary α which is *not* a theorem, we have $\alpha \leqq_{\mathsf{TAL}} \beta$; in Corollary 41 we need the additional premise that α is a contradiction. We illustrate this with an example. Let T_1, T_2, T_3 be arbitrary theorems of TAL, $\bot$ an arbitrary contradiction. Obviously,

$$(T_1 \| \bot) \vee \neg(T_1 \| \bot) \equiv_{\mathsf{TAL}} (T_1 \vee \neg T_1) \| (\bot \vee \neg T_1) \| (T_1 \vee \neg \bot) \| (\bot \vee \neg \bot) \quad (13)$$

are both theorems of TAL. Moreover, we have $(\bot \vee \neg T_1) \vdash_{\mathsf{TAL}}$. By two applications of (**I;** I) we can derive (abbreviated proof)

$$\frac{(T_1 \vee \neg T_1) \vdash T_2 \quad (\bot \vee \neg T_1) \vdash \alpha \quad (T_1 \vee \neg \bot) \| (\bot \vee \neg \bot) \vdash T_3}{(T_1 \vee \neg T_1) \| (\bot \vee \neg T_1) \| ((T_1 \vee \neg \bot) \| (\bot \vee \neg \bot)) \vdash_{\mathsf{TAL}} T_2 \| \alpha \| T_3} \quad (14)$$

(for arbitrary α). However, we *cannot* derive (for arbitrary α)

$$p \vdash T_2 \| \alpha \| T_3, \text{ where } p \text{ does not occur in } T_2 \| \alpha \| T_3 \quad (15)$$

Assume we can. Then we can substitute $p \mapsto p \vee \neg p$, apply (classic cut) to obtain $\vdash T_2 \| \alpha \| T_3$. Hence for all α, β, we would obtain $\beta \vdash T_2 \| \alpha \| T_3$ (weakening), which would result in m-triviality (use (**I;** I) and (inter2)). Hence

$$p \not\leqq_{\mathsf{TAL}} (T_1 \| \bot) \vee \neg(T_1 \| \bot) \quad (16)$$

whereas $p \leqq_{\mathsf{TAL}} p \vee \neg p$. So there are distinct congruence classes of theorems. We can however prove the following:

Lemma 42 *All classical theorems are $\equiv_{\mathsf{TAL}}$ equivalent and $\leqq_{\mathsf{TAL}}$ maximal. Dually for classical contradictions.*

Proof We have already proved that every formula can be substituted by a theorem on the right. Let $T \in Form(\mathsf{CL})$ be a classical theorem and assume $\Gamma[T] \vdash_{\mathsf{TAL}} \Delta$. We obtain $\Gamma[T\mathbf{,}\alpha] \vdash_{\mathsf{TAL}} \Delta$ (**,** weak), $\Gamma[\alpha]\mathbf{,}\, T \vdash_{\mathsf{TAL}} \Delta$ (invDistr), $\Gamma[\alpha] \vdash_{\mathsf{TAL}} \Delta$ (classic cut). Hence $\alpha \leqq_{\mathsf{TAL}} T$. Since α is arbitrary, this proves both claims. $\dashv$

Corollary 43 $\leqq_{\mathsf{TAL}}, \equiv_{\mathsf{TAL}}$ *are not closed under u-substitution.*

This might be surprising at first glimpse, but once we see the In-Out Lemma, it will already follow from the Basic Closure Theorem. Note also that some additional laws, like the law of semi-complementation or the orthomodular law (well-known from quantum logic), would entail that $Cong(\mathsf{TAL})$ is Boolean, hence they cannot hold.

4.3 Truth and Validity: On the Congruence Algebra of cDAL

In the congruence algebra of cDAL, at first glimpse the issue is straightforward: since cDAL is closed under e-substitution, we have

$$\alpha \leqq_{\mathsf{cDAL}} \beta \text{ if and only if } \alpha \vdash_{\mathsf{cDAL}} \beta$$

(only if because of Lemma 6, if because of the rule (cut)). In cDAL, and more generally in every distrustful logic L which contains the identity relation on formulas, the outer logic $\vdash_L$ and the inner logic $\leqq_L$ coincide.

There is another distinction which gains some importance, namely between **truth** and **validity** of (in)equations: an equation in $Cong(\mathsf{cDAL})$ is valid, if it is true for all uniform substitutions. For example: all *classical* theorems are equivalent in cDAL. But the equation

$$p \vee \neg p \leqq_{\mathsf{cDAL}} q \vee \neg q \tag{17}$$

is **true**, not **valid**, since its truth is not preserved by substitutions. Note that the situation is not different from $\leqq_{\mathsf{TAL}}$ (and $\leqq_{\mathsf{cTAL}}$). Lemma 21 entails that the ambiguous substitution of a classical theorem in the truth-theoretic approach can be either strictly true, or undefined, but it is never strictly false. This holds dually for contradictions, so for example $(p\|q) \wedge \neg(p\|q) \vdash_{\mathsf{cDAL}} (r\|s) \vee \neg(r\|s)$ holds, as is easy to verify. Hence the inequation

$$p \wedge \neg p \leqq_{\mathsf{cDAL}} q \vee \neg q \tag{18}$$

is **valid** in cDAL (as it is in TAL!). As another example, $\alpha \leqq_{\mathsf{cDAL}} T$ is true for an arbitrary α and classical theorem T, but $\alpha \leqq T$ is not generally valid in cDAL: $p \vdash (p\|q) \vee \neg(p\|q)$ does not hold in cDAL: assume p is true, q is false, this falsifies the sequent. Hence there is an even smaller *inner logic of validity*, which satisfies for example the DeMorgan laws, but not (17). That gives rise to a question: are there rules/techniques to make sure that an inequality is *valid* in cDAL (not only true)?

Lemma 44 *Assume* α, β *are positive Boolean formulas, and* $\alpha \vdash_{\mathsf{cDAL}} \beta$. *Then* $\alpha \leqq_{\mathsf{cDAL}} \beta$ *is valid.*

Proof If there is no ambiguity and no negative polarity involved, then $\blacksquare\sigma(\alpha) = \sigma'(\alpha)$, where for all $p \in Var$, $\sigma'(p) = \blacksquare\sigma(p)$. So assume $\alpha \vdash_{\mathsf{cDAL}} \beta$, this entails $\alpha \vdash_{\mathsf{CL}} \beta$, hence $\sigma'(\alpha) \vdash_{\mathsf{CL}} \sigma'(\beta)$, so $\blacksquare\sigma(\alpha) = \sigma'(\alpha) \vdash_{\mathsf{cDAL}} \sigma'(\beta) = \blacksquare\sigma(\beta)$. Same for $\blacklozenge$. $\dashv$

This entails a number of things, in particular that all distributive laws for $\wedge, \vee$ are valid in cDAL. Moreover, we have double negation and DeMorgan laws:

Lemma 45 *For all formulas α, β, the following equations are valid:*

1. $\alpha \equiv_{\mathsf{cDAL}} \neg\neg\alpha$
2. $\neg(\alpha \vee \beta) \equiv_{\mathsf{cDAL}} \neg\alpha \wedge \neg\beta$
3. $\neg(\alpha \wedge \beta) \equiv_{\mathsf{cDAL}} \neg\alpha \vee \neg\beta$

Proof 1. We have $\blacksquare(\neg\neg\alpha) \equiv_{\mathsf{CL}} \blacksquare(\alpha)$ for all α, hence also for $\sigma(\alpha)$ for arbitrary σ.
2. $\blacksquare\neg(\alpha \vee \beta) = \neg(\blacklozenge\alpha \vee \blacklozenge\beta) \equiv_{\mathsf{CL}} \neg(\blacklozenge\alpha) \wedge \neg(\blacklozenge\beta) = \blacksquare(\neg\alpha \wedge \neg\beta)$. Dual for $\blacklozenge$.
3. Parallel. $\dashv$

There is a pattern: the important thing is that all formulas occurring multiple times in the equation have the same polarity. Hence every equality, which 1. is valid in Boolean algebras, in which 2. we have the same variables on the left and the right side, and 3. all these variables have the same polarity is valid in the congruence algebra of cDAL. It is easy to prove that *Cong*(cDAL) satisfies all requirements for ambiguity like Unambiguous Entailments, Universal Distribution etc., and that *Cong*(cDAL) is a Kleene Algebra.

4.4 On the Congruence Algebra of cTAL

The Context Lemma 33 does obviously also hold for cTAL, and so do the congruences we have proved for TAL. We will not spell them out here, since the relation $\leqq_{\mathsf{cTAL}}$ has a pleasant surprise for us, as can be seen in the following lemma. Recall that $\alpha \vdash_{\mathsf{cTAL}} \beta$ iff $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$.

Lemma 46 $\alpha \leqq_{\mathsf{cTAL}} \beta$ *iff* $\alpha \vdash_{\mathsf{cDAL}} \beta$.

Proof *If* Assume $\alpha \vdash_{\mathsf{cDAL}} \beta$. Hence $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$, $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$. Assume moreover $\Gamma[\beta] \vdash_{\mathsf{cTAL}} \Delta$, which means $\blacksquare\Gamma[\beta] \vdash_{\mathsf{CL}} \blacklozenge\Delta$. By (cut) (admissible in classical logic), we then have $\blacksquare\Gamma[\alpha] \vdash_{\mathsf{CL}} \blacklozenge\Delta$, so $\blacksquare\Gamma[\alpha] \vdash_{\mathsf{cTAL}} \blacklozenge\Delta$. Dual argument for $\Gamma \vdash_{\mathsf{cTAL}} \Delta[\alpha]$.

Only if Contraposition: assume $\nVdash_{\mathsf{cDAL}} \alpha \vdash \beta$. Case 1: $\nVdash_{\mathsf{CL}} \blacksquare\alpha \vdash \blacksquare\beta$. This entails by Lemma 26 that $\nVdash_{\mathsf{cTAL}} \alpha \vdash \blacksquare\beta$. However, we obviously have $\beta \vdash_{\mathsf{cTAL}} \blacksquare\beta$, so $\alpha \nleqq_{\mathsf{cTAL}} \beta$. Case 2: $\nVdash_{\mathsf{CL}} \blacklozenge\alpha \vdash \blacklozenge\beta$. Dually. $\dashv$

This is a simple, yet surprising result: cDAL is the **inner logic** of cTAL. Hence $\leqq_{\mathsf{cTAL}}$ is *not* closed under u-substitution, but under e-substitution. This gives us the following interesting corollary: by definition, every outer logic $\vdash_L$ gives rise to a unique inner logic $\leqq_L$. But the converse is not true:

Corollary 47 *There are logics L, L' such that $\vdash_L \neq \vdash_{L'}$, yet $\leqq_L = \leqq_{L'}$*

5 The Family: the Bigger Picture

5.1 Lattice Structure

Until now, we have focused on prominent logics, now we will investigate further on the structure of the family of full ambiguity logics and its less prominent members.

Logics are usually ordered by (set-theoretic) inclusion of $\vdash_L$. We will do the same here, and write $L \subseteq L'$ iff $\vdash_L \subseteq \vdash_{L'}$.

Lemma 48 *Assume I is an arbitrary index set, and for all $i \in I$, L_i is a full ambiguity logic. Then $\bigwedge\{L_i : i \in I\} = (Form(\mathsf{AL}), \bigcap\{\vdash_i : i \in I\})$ is a full ambiguity logic.*

Proof 1. If $\vdash_{\mathsf{CL}} \subseteq \vdash_i$ for all $i \in I$, then $\vdash_{\mathsf{CL}} \subseteq \bigcap\{\vdash_i : i \in I\}$. 2.-4. The (in-)equations really have the form of implications (definition of $\leqq$), which are preserved under arbitrary intersections. 5.-6. These requirements also have the form of implications, which are preserved under arbitrary intersections. $\dashv$

Set-theoretic unions do not preserve these properties, hence infinite intersections are important to define joins. Now let us have a closer look at the upper and lower bounds of the lattice. We have already seen that $\mathsf{cTAL} = L_{\blacksquare\blacklozenge}$ is a very large, m-trivial ambiguity logic. In fact, it is easy to show that it is the largest:

Lemma 49 *$L_{\blacksquare\blacklozenge}$ is the largest ambiguity logic: if for some full ambiguity logic L, $\alpha \vdash_L \beta$, then $\alpha \vdash_{\blacksquare\blacklozenge} \beta$.*

Proof $\blacksquare\alpha \leqq_L \alpha \leqq_L \blacklozenge\alpha$ in every full ambiguity logic L. Hence $\alpha \vdash_L \beta$ implies $\blacksquare\alpha \vdash_L \blacklozenge\beta$. Since L conservatively extends CL, we have $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$, and hence $\alpha \vdash_{\blacksquare\blacklozenge} \beta$. $\dashv$

It is actually very easy to construct further full ambiguity logics with these operators: recall Lemma 15, which states that $\blacksquare(\alpha \wedge (\beta \| \gamma)) \equiv_{\mathsf{CL}} \blacksquare((\alpha \wedge \beta) \| (\alpha \wedge \gamma))$ etc., for both $\blacksquare$, $\blacklozenge$ and all connectives. We also have $\blacksquare(\alpha \wedge \beta) \vdash_{\mathsf{CL}} \blacksquare(\alpha \| \beta) \vdash_{\mathsf{CL}} \blacksquare(\alpha \vee \beta)$, same for $\blacklozenge$. These and some other simple results ensure that whenever we use the operators $\blacksquare$, $\blacklozenge$ to define a consequence relation in terms of classical logic for the language of ambiguity logics, we can be sure that it actually is a full ambiguity logic. Hence we can easily construct the logic $L_{\blacklozenge\blacksquare}$.

Definition 50 Define $L_{\blacklozenge\blacksquare} = (Form(\mathsf{AL}), \vdash_{\blacklozenge\blacksquare})$ by $\alpha \vdash_{\blacklozenge\blacksquare} \beta$ iff $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$.

$L_{\blacklozenge\blacksquare}$ is a very particular and restrictive logic. In particular, $\not\vdash_{\blacklozenge\blacksquare} p \| q \vdash p \| q$. This means that (I**;** I) is not sound in this logic. This provides us also with an example of a logic where ($\|$mon) does not hold. It is easy to see that $L_{\blacklozenge\blacksquare}$ is a logic of *extreme mistrust*:

$$a_1 \| \ldots \| a_i \vdash_{\blacklozenge\blacksquare} b_1 \| \ldots \| b_j$$

is derivable iff under an *arbitrary* left reading a_n, *all* right readings b_m follow. It is easy to see that $L_{\blacklozenge\blacksquare}$ is closed under e-substitution, yet not under u-substitution. We can actually prove the following:

Lemma 51 *$L_{\blacklozenge\blacksquare}$ is the smallest full ambiguity logic.*

Proof Assume without loss of generality that $\gamma = a_1 \| \ldots \| a_n$, $\delta = b_1 \| \ldots \| b_m$ are both in ambiguous normal form. Assume moreover $\blacklozenge\gamma \vdash_{\mathsf{CL}} \blacksquare\delta$. Hence for an arbitrary full ambiguity logic L, we have $\blacklozenge\gamma \vdash_L \blacksquare\delta$ (by conservative extension). Then, since $\gamma \leqq_L \blacklozenge\gamma$, we obtain $\gamma \vdash_L \blacksquare\delta$, and since $\blacksquare\delta \leqq_L \delta$, we obtain $\gamma \vdash_L \delta$. $\dashv$

Hence $L_{\blacklozenge\blacksquare}$ has a special position in the family of full ambiguity logics. Note a peculiarity: $L_{\blacklozenge\blacksquare}$ is *commutative*. Yet, it is not only minimal for commutative logics, but for all full ambiguity logics. Finally, note that $L_{\blacklozenge\blacksquare}$ does not even satisfy the weak law of disambiguation:

$$(p\|q\|r) \wedge \neg q \vdash p\|r \tag{19}$$

is *not* valid in $L_{\blacklozenge\blacksquare}$ (easy to check). This is directly connected to the fact that (**I;** I) is not sound in this logic. $L_{\blacklozenge\blacksquare}$ provides us also with an example of a logic where the outer logic does not contain the inner logic: we do not have $\alpha \vdash_{\blacklozenge\blacksquare} \alpha$ for all α, but by definition $\alpha \leqq_{\blacklozenge\blacksquare} \alpha$ (see also Lemma 6). The inner logic $\leqq_{\blacklozenge\blacksquare}$ is in this case actually *larger* than the outer logic, and it is monotonic (satisfies $\|$mon), so $\vdash_{\blacklozenge\blacksquare} \subsetneq \leqq_{\blacklozenge\blacksquare}$. We can make this more precise:

Lemma 52 *$\alpha \leqq_{\blacklozenge\blacksquare} \beta$ if and only if $\alpha \vdash_{\mathsf{cDAL}} \beta$.*

Proof *If*: Assume $\alpha \vdash_{\mathsf{cDAL}} \beta$. Hence $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$, $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$. Assume $\Gamma[\beta] \vdash_{\blacklozenge\blacksquare} \Delta$, so $\blacklozenge\Gamma[\beta] \vdash_{\mathsf{CL}} \blacksquare\Delta$. This entails that $\blacklozenge\Gamma[\alpha] \vdash_{\mathsf{CL}} \blacksquare\Delta$. Dually on the right.

Only if: Assume $\alpha \nvdash_{\mathsf{cDAL}} \beta$. Case 1: $\blacksquare\alpha \nvdash_{\mathsf{CL}} \blacksquare\beta$. We have $\blacksquare\alpha \vdash_{\blacklozenge\blacksquare} \alpha$, but by case assumption $\blacksquare\alpha \nvdash_{\blacklozenge\blacksquare} \beta$, and hence $\alpha \nleqq_{\blacklozenge\blacksquare} \beta$. Case 2: $\blacklozenge\alpha \nvdash_{\mathsf{CL}} \blacklozenge\beta$: dually. ⊣

Hence cTAL, the largest full ambiguity logic (m-trivial and trustful), and the smallest full ambiguity logic (which is distrustful) have the same identical inner logic, which is cDAL! Let us state the main result of this section:

Lemma 53 *The family of full ambiguity logics is a complete, bounded lattice.*

Proof We have arbitrary intersections, hence we can define $\bigvee\{L_i : i \in I\} = \bigcap\{L_j : \bigcup\{\vdash_i : i \in I\} \subseteq \vdash_j\}$ (unions do not preserve certain properties, in particular 6). $L_{\blacklozenge\blacksquare}$ is its minimal element, $L_{\blacksquare\blacklozenge}$ its maximal element. ⊣

Note that completeness is important for extensions: We can close a distrustful logic L for example under u-substitution. The result will not necessarily be a full ambiguity logic, but we can intersect all full ambiguity logics which contain this substitution closure, and the result is an ambiguity logic.

5.2 The In-Out Lemma

Let L be a full ambiguity logic. Its inner logic $\leqq_L$ is always closed under e-substitution. This leads to the following question: If L is a trustful ambiguity logic, is $\leqq_L$ then a distrustful ambiguity logic? For every full ambiguity logic L, it is obvious that $\leqq_L$ satisfies (UD),(UE),(id) etc. However, one condition is not necessarily satisfied, namely that $\leqq_L$ extends classical logic.

Definition 54 We say a trustful ambiguity logic L is **classically congruent** if 1. $\vdash_{\mathsf{CL}} \subseteq \leqq_L$, and 2. for all $\sigma : \mathit{Var} \to \mathit{Form}(\mathsf{CL})$, $\alpha \leqq_L \beta$ implies $\sigma(\alpha) \leqq_L \sigma(\beta)$.

This gives us another way of constructing distrustful ambiguity logics: namely as *inner logics* of trustful logics – presupposing that their inner logic contains $\vdash_{\mathsf{CL}}$ and is closed under classical substitution. We think that classical congruence is a central criterion for a trustful logic to be reasonable; we will see some strange logics not satisfying this in Section 5.4.2.

Lemma 55 *(In-Out Lemma) Assume L is a trustful classically congruent full ambiguity logic. Then $\leqq_L$ is the consequence relation of a distrustful ambiguity logic.*

Proof Check Definition 7. 1. of holds by assumption, 2.-4. are obvious, 5. holds by assumption and 6. holds because $\leqq_L$ is always transitive. ⊣

Corollary 56 $\leqq_{\mathsf{TAL}}$ *is a non-commutative distrustful ambiguity logic.*

This follows from Lemma 35. Moreover, since $\vdash_{\mathsf{TAL}}$ contains the identity relation of formulas (for all α, $\alpha \vdash_{\mathsf{TAL}} \alpha$), it is straightforward that $\leqq_{\mathsf{TAL}}$ is non-trivial (otherwise TAL would be also m-trivial). Since $\leqq_{\mathsf{TAL}}$ by definition is closed under e-substitution, it cannot be closed under u-substitution. Hence we have another full ambiguity logic basically "for free", with the only problem that we do not know very much about it yet. We conjecture the following:

Conjecture 1 $\alpha \leqq_{\mathsf{TAL}} \beta$ *iff* $\alpha \vdash_{\mathsf{DAL}} \beta$

This would result in a very nice symmetry, but given that both TAL and DAL for now only have proof-theoretic characterizations, there does not seem to be an easy way to show this.

5.3 Negation Congruence

We have seen three logics based on/related to the "truth operators" $\blacksquare$, $\blacklozenge$, namely cDAL, cTAL and $L_{\blacklozenge\blacksquare}$. There are two we have not yet considered, namely $L_{\blacklozenge\blacklozenge}$ and $L_{\blacksquare\blacksquare}$, which can already be found in van Deemter (1996).

1. $L_{\blacksquare\blacksquare}$: $\alpha \vdash_{\blacksquare\blacksquare} \beta$ iff $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$
2. $L_{\blacklozenge\blacklozenge}$: $\alpha \vdash_{\blacklozenge\blacklozenge} \beta$ iff $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$

We will quickly discuss these logics here. Our results on $\blacksquare$, $\blacklozenge$ already entail that these are full ambiguity logics. Obviously, $\vdash_{\mathsf{cDAL}} = \vdash_{\blacksquare\blacksquare} \cap \vdash_{\blacklozenge\blacklozenge}$. We have

$$p \vee q \equiv_{\blacklozenge\blacklozenge} p \| q \tag{20}$$

$$p \wedge q \equiv_{\blacksquare\blacksquare} p \| q \tag{21}$$

Hence in both logics '$\|$' coincides with a classical connective. Note however that the logics are not trivially equivalent to classical logic; we illustrate this for $L_{\blacklozenge\blacklozenge}$:

$$\neg(p\|q) \equiv_{\blacklozenge\blacklozenge} \blacklozenge\neg(p\|q) \equiv_{\blacklozenge\blacklozenge} \neg\blacksquare(p\|q) \equiv_{\blacklozenge\blacklozenge} \neg(p \wedge q) \equiv_{\blacklozenge\blacklozenge} \neg p \vee \neg q \tag{22}$$

We also obviously have $p \vee q \vdash_{\blacklozenge\blacklozenge} p\|q$, yet $\neg(p\|q) \nvdash_{\blacklozenge\blacklozenge} \neg(p \vee q)$. This means that the rule of contraposition is not sound in $L_{\blacklozenge\blacklozenge}$ (and $L_{\blacksquare\blacksquare}$), contrary to all logics we have observed until now. Also $L_{\blacklozenge\blacklozenge}$ and $L_{\blacksquare\blacksquare}$ are asymmetric in an obvious way. They have another more peculiar property which distinguishes them from all other logics we have considered so far: we have defined $\alpha \equiv_L \beta$ by $\Gamma[\alpha] \vdash_L \Delta$ iff $\Gamma[\beta] \vdash_L \Delta$ and $\Gamma \vdash_L \Delta[\alpha]$ iff $\Gamma \vdash_L \Delta[\beta]$: α and β are exchangeable in arbitrary contexts. For all logics up to this section, $\alpha \equiv_L \beta$ naturally entailed that α and β can be exchanged as *subformulas* as well:

Definition 57 A logic L is **negation congruent** if $\alpha \equiv_L \beta$ entails for all $\gamma[-]$, $\gamma[\alpha] \equiv_L \gamma[\beta]$.

This is not trivial, since as a subformula of $\gamma[-]$, α (or β) can be within the scope of *negation*. And in fact, we see the following:

Lemma 58 *$L_{\blacklozenge\blacklozenge}$, $L_{\blacksquare\blacksquare}$ are not negation congruent.*

Proof Counterexample: $p\|q \equiv_{\blacklozenge\blacklozenge} p \vee q$, yet $\neg(p\|q) \not\equiv_{\blacklozenge\blacklozenge} \neg(p \vee q)$ (both are easy to see). Dual for $L_{\blacksquare\blacksquare}$. ⊣

We think that it is an essential and useful property for an ambiguity logic to be negation congruent, hence we have a point in disfavor of $L_{\blacksquare\blacksquare}$ and $L_{\blacklozenge\blacklozenge}$. However, the logics nicely illustrate that the requirement is not trivial.

5.4 Trust Closure and the Trust Theorem

5.4.1 The Trust Theorem

We have seen logics of trust and logics of distrust. The Trust Theorem is crucial for understanding their relation. Assume L_t is a logic of trust, L_d a logic of distrust. This obviously does not entail that $L_d \subseteq L_t$: just consider that cDAL $\not\subseteq$ TAL, because of commutativity. But there is one important question: is there a trustful logic L_t, distrustful logic L_d, such that $L_t \subseteq L_d$? This would somehow contradict our intuitive conception of what trust means: if we trust each other, we tend to accept additional valid arguments and inferences, which we would refute if we distrust (see 2.3) – otherwise what would be the point of trust? We will prove that trust always increases the set of valid inferences. The underlying reason for this is that e-substitution preserves closure under u-substitution, but not vice versa. Assume L is a full ambiguity logic. We let eL, its closure under e-substitution, denote the smallest logic such that

1. $\Gamma \vdash_L \Delta$ entails $\Gamma \vdash_{eL} \Delta$
2. $\Gamma[\alpha] \vdash_{eL} \Delta[\beta], \alpha' \vdash_{eL} \alpha, \beta \vdash_{eL} \beta'$ entail $\Gamma[\alpha'] \vdash_{eL} \Delta[\beta']$

e is a closure operator: $L \subseteq eL$, if $L \subseteq L'$, then $eL \subseteq eL'$, and $e(eL) = eL$.

Lemma 59 *For every trustful ambiguity logic L, $\leqq_L \subseteq \leqq_{eL}$*

Proof In every trustful ambiguity logic L, we have $\gamma \vdash_L \gamma$ for all $\gamma \in Form(\mathsf{AL})$, and hence $\leqq_L \subseteq \vdash_L$ (Lemma 6). Assume $\alpha \leqq_L \beta$. Then $\alpha \vdash_L \beta$, hence $\Gamma[\beta] \vdash_{eL} \Delta$ entails $\Gamma[\alpha] \vdash_{eL} \Delta$. Dual on the right, so $\alpha \leqq_{eL} \beta$. ⊣

Lemma 60 *Assume L is a trustful ambiguity logic. Then eL is a trustful ambiguity logic as well.*

Proof eL is a full ambiguity logic, since we firstly have $\vdash_L \subseteq \vdash_{eL}$ and secondly $\leqq_L \subseteq \leqq_{eL}$, so we satisfy conditions 1.-4., 6. is immediate. To see that eL is closed under u-substitution: L is trustful, so $\Gamma[\alpha] \vdash_L \Delta[\beta]$, $\alpha' \vdash_L \alpha$, $\beta \vdash_L \beta'$ entail $\sigma(\Gamma[\alpha]) \vdash_L \sigma(\Delta[\beta])$, $\sigma(\alpha') \vdash_L \sigma(\alpha)$ etc. Hence $\sigma(\Gamma[\alpha']) \vdash_{eL} \sigma(\Delta[\beta'])$. ⊣

More generally, closure under σ is preserved over e-substitution steps. Note also that if L is distrustful, then $L = eL$. This entails that a full ambiguity logic of trust cannot be extended to a full ambiguity logic of distrust:

Lemma 61 *Assume L_d is a non-trivial distrustful ambiguity logic, L_t a trustful ambiguity logic. Then $\vdash_{L_t} \not\subseteq \vdash_{L_d}$.*

Proof Assume $\vdash_{L_t} \subseteq \vdash_{L_d}$. Then we have $\vdash_{L_t} \subseteq \vdash_{eL_t} \subseteq \vdash_{eL_d} = \vdash_{L_d}$. However, eL_t is by the previous lemma closed under both e- and u-substitution and hence m-trivial (Basic Closure Theorem). So L_d has to be m-trivial too, contradiction. ⊣

This result confirms our intuitive approach: there is an asymmetry between e-substitution and u-substitution, which entails that under trustful reasoning, we have more valid inferences than under distrustful reasoning.

Now we consider the other direction: we will show that every distrustful logic can be extended to a trustful logic, preserving non-triviality. This will require some effort though. We firstly define the relations $\leqq^1_{\mathsf{AL}}$, $\leqq_{\mathsf{AL}}$ as follows (we use $\Gamma \equiv^1_{\mathsf{AL}} \Delta$ as abbreviation for $\Gamma \leqq^1_{\mathsf{AL}} \Delta$ and $\Gamma \geqq^1_{\mathsf{AL}} \Delta$, $\circ$ to denote relation composition):

Definition 62 We let $\leqq^1_{\mathsf{AL}}$ be the smallest relation such that for all $\Gamma[-], \alpha, \beta, \gamma \in$ *Form*(AL),

$$\begin{array}{ll}\Gamma[\alpha \wedge (\beta \| \gamma)] \equiv^1_{\mathsf{AL}} \Gamma[(\alpha \wedge \beta) \| (\alpha \wedge \gamma)] & \Gamma[\alpha \vee (\beta \| \gamma)] \equiv^1_{\mathsf{AL}} \Gamma[(\alpha \vee \beta) \| (\alpha \vee \gamma)] \\ \Gamma[\neg(\alpha \| \beta)] \equiv^1_{\mathsf{AL}} \Gamma[\neg\alpha \| \neg\beta] & \Gamma[\alpha \wedge \beta] \leqq^1_{\mathsf{AL}} \Gamma[\alpha \| \beta] \leqq_L \Gamma[\alpha \vee \beta] \\ \Gamma[(\alpha \| \beta) \| \gamma] \equiv^1_{\mathsf{AL}} \Gamma[\alpha \| (\beta \| \gamma)] & \Gamma[\alpha \| \alpha] \equiv^1_{\mathsf{AL}} \Gamma[\alpha]\end{array}$$

Put $\leqq^{n+1}_{\mathsf{AL}} := \leqq^n_{\mathsf{AL}} \circ \leqq^1_{\mathsf{AL}}$, and let $\leqq_{\mathsf{AL}} = \bigcup_{n \in \mathbb{N}_0} \leqq^n_{\mathsf{AL}}$ denote the reflexive transitive closure of $\leqq^1_{\mathsf{AL}}$.

Definition 63 Assume $L = (Form(\mathsf{AL}), \vdash_L)$ is a logic (not necessarily a full ambiguity logic).

1. RL is the smallest logic such that i. $\vdash_L \subseteq \vdash_{RL}$ and ii. if $\alpha' \leqq_{\mathsf{AL}} \alpha$ or $\alpha' \vdash_{\mathsf{CL}} \alpha$, and $\beta \leqq_{\mathsf{AL}} \beta'$ or $\beta \vdash_{\mathsf{CL}} \beta'$, then $\alpha \vdash_{RL} \beta$ implies $\alpha' \vdash_{RL} \beta'$. Hence RL is the smallest logic containing L which is closed under Definition 7.2.-4. and composition with $\vdash_{\mathsf{CL}}$.
2. σL is the smallest logic such that $\vdash_L \subseteq \vdash_{\sigma L}$, and $\vdash_{\sigma L}$ is closed under uniform substitution into *Form*(AL).
3. SL is the smallest logic such that $\vdash_L \subseteq \vdash_{SL}$, and $\vdash_{SL}$ is closed under uniform substitution into *Form*(CL) (Definition 7.5.).
4. TL is the smallest logic such that i. $\vdash_L \subseteq \vdash_{TL}$, and ii. if $\alpha \vdash_{TL} a \vdash_{TL} \beta$, $a \in$ *Form*(CL), then $\alpha \vdash_{TL} \beta$. Hence TL is the closure of L under classic transitivity)(Definition 7.6.).
5. τL, the **trust closure** of L, is the smallest full ambiguity logic containing σL.

If $L = \sigma L$, we say L is closed under σ; same for R, S, T. Note that every full ambiguity logic is closed under R, S, T. Completeness of the lattice of full ambiguity logics already entails that every logic L is included in a unique smallest full ambiguity logic τL, hence the concept is well-defined. It is easy to see that

$$\tau L = (TR\sigma)^* L \tag{23}$$

where $(TR\sigma)^* L = \bigcup_{n \in \mathbb{N}} (TR\sigma)^n L$. Note also:

- All operators R, S, σ, T preserve the property of being a conservative extension of classical logic.
- Since substitutions are closed under compositions, we have $\alpha \vdash_{\sigma L} \beta$ iff there are α', β' and a u-substitution σ such that $\alpha' \vdash_L \beta', \sigma(\alpha') = \alpha, \sigma(\beta') = \beta$.
- Similarly, if L is closed under R, then $\alpha \vdash_{TL} \beta$ iff either $\alpha \vdash_L \beta$, or there is an $a \in Form(\mathsf{CL})$ such that $\alpha \vdash_L a \vdash_L \beta$.

There are a number of questions around these operators, but the main question is: does trust-closure preserve non-triviality? We will settle this question positively.

Lemma 64 *Assume L is closed under R, S. Then*

1. *$R\sigma L$ is closed under R, S*
2. *TL is closed under R, S.*

Proof 1. Closure under R holds by definition for $R\sigma L$. Closure under S: $\leqq_{\mathsf{AL}}$ and $\vdash_{\mathsf{CL}}$ are both closed under S, $\vdash_{\sigma L}$ is also closed under S, hence so is their composition (see the argument in Lemma 60).

2. Closure under R: Assume $a \in Form(\mathsf{CL})$. L is closed under R, S. Hence if $\alpha \leqq_{\mathsf{AL}} \alpha' \vdash_L a$ or $\alpha \vdash_{\mathsf{CL}} \alpha' \vdash_L a$, then $\alpha \vdash_L a$, same for $a \vdash_L \beta$. Hence TL is obviously closed under R. Closure under S: $\alpha \vdash_L a \vdash_L \beta$ implies $S(\alpha) \vdash_L S(a) \vdash_L S(\beta)$, where S is a classical substitution, hence $S(a)$ is classical. Hence $\alpha \vdash_{TL} \beta$ implies $S(\alpha) \vdash_{TL} S(\beta)$. ⊣

To prove this, we first simplify the problem of m-triviality. We can actually think of a formula $w \in Form(\|)$ as a word in Var^*. For ease of reading, in the following lemmas and proofs we will make use of simple language-theoretic conventions, and write pq for $p\|q$, and p^* for the set $\{\epsilon, p, pp, ...\}$ etc.

Lemma 65 *A full ambiguity logic L is m-trivial iff $pqr \vdash_L pq'r$, for some pairwise distinct p, q, q', r.*

Proof *Only if* holds by definition. *If*: assume $pqr \vdash_L pq'r$, and pick some arbitrary $\alpha, \gamma \in Form(\mathsf{CL})$, $\beta, \beta' \in Form(\mathsf{AL})$. By classical substitution, we have $\alpha\|\blacklozenge\beta\|\gamma \vdash_L \alpha\|\blacksquare\beta'\|\gamma$, and by ambiguity laws, $\alpha\|\beta\|\gamma \vdash_L \alpha\|\beta'\|\gamma$. ⊣

Lemma 66 *Assume L is closed under R, S, and for some pairwise distinct p, q, q', r, we have $pqr \vdash_{R\sigma L} pq'r$. Then $pqr \vdash_L pq'r$.*

Proof Assume $pqr \leqq_{\mathsf{AL}} w \vdash_{\sigma L} v \leqq_{\mathsf{AL}} pq'r$, where $w' \vdash_L v'$ and $\sigma(w') = w$, $\sigma(v') = v$. We can distinguish two kinds of letters, namely *common* letters which occur in both w', v', and *separate* letters which occur in only one of them. Note that for a common letter a, we necessarily have $\sigma(a) \equiv_{\mathsf{AL}} x \in \{p, r, rp\}$, because any other formula cannot be reduced to a substring occurring both in a word congruent to pqr and $pq'r$ (no $\wedge$, no $\vee$, no $\neg$, and no other sequence of letters can be reduced on both sides of $\vdash$). We now construct a classical σ' from σ which yields the sequent.

Separate letters For all separate letters a in w', if $\sigma(a)$ contains q, put $\sigma'(a) = p \vee q \vee r$. Otherwise, simply put $\sigma'(a) = \blacklozenge\sigma(a)$. For all separate letters a in v', if $\sigma(a)$ contains q', put $\sigma'(a) = p \wedge q' \wedge r$. Otherwise, simply put $\sigma'(a) = \blacksquare\sigma(a)$.

Common letters Assume a occurs in both w', v'. We distinguish two cases:

a) Assume $\sigma(a) \equiv_{\mathsf{AL}} b \in \{p, r\}$. Then simply put $\sigma'(a) = b$, and the substitution is classical, and does not change the image modulo $\equiv_{\mathsf{AL}}$.

b) Assume $\sigma(a) \equiv_{\mathsf{AL}} rp$. This entails that a occurs between separate letters both in w' and v' (because in w or resp. v, rp can only occur between two occurrences of q or resp. q'). We simply put $\sigma'(a) = r$ (but p would work just as well).

This completes the construction of σ', which is classical. We now show that $pqr \leqq_{\mathsf{AL}} \sigma'(w')$, the proof that $\sigma'(v') \leqq_{\mathsf{AL}} pq'r$ is exactly dual.

We have $w' = x_1 y_1 x_2 ... x_i y_i x_{i+1}$, where each x_j consists of letters whose image does not contain q, each y_j of separate letters whose image contains q.

For all j, $\sigma'(y_j) = (p \vee q \vee r)^n$ for some n. $\sigma'(x_1) = p^m$, $\sigma'(x_{i+1}) = r^o$. Now consider $x_2, ..., x_i$.

We necessarily have, for $2 \leq j \leq i$, $\sigma(x_j) \in (r^* p^*)^*$. This is necessary (not sufficient) for σ to be appropriate, but it is sufficient to make σ' appropriate. We can easily show that there is $z \in (r^* p^*)^*$ such that $z \leqq_{\mathsf{AL}} \sigma'(x_j)$, by induction over $|x_j|$. The base $|x_j| = 1$ is clear by definition of σ'.

Assume $\sigma'(x_j) \leqq_{\mathsf{AL}} z \in (r^* p^*)^*$. Now consider $x_j a$, $a \in Var$. Cases: $\sigma(a) \equiv_{\mathsf{AL}} p$, then $\sigma'(a) = p$, and the claim follows. $\sigma(a) \equiv_{\mathsf{AL}} r$ – parallel. $\sigma(a) \equiv_{\mathsf{AL}} rp$, then $\sigma'(a) \equiv_{\mathsf{AL}} p$, and the claim follows. None of the three: then a is separate, $\sigma(a)$ does not contain q, $\sigma'(a) = \blacklozenge\sigma(a)$, and the claim follows. $\dashv$

Now we need some auxiliary lemmas:

Lemma 67 *Assume $\alpha \in Form(\mathsf{AL})$, $a \in Form(\mathsf{CL})$. Then*

1. *$\alpha \leqq_{\mathsf{AL}} a$ entails $\blacklozenge\alpha \vdash_{\mathsf{CL}} a$*
2. *$a \leqq_{\mathsf{AL}} \alpha$ entails $a \vdash_{\mathsf{CL}} \blacksquare\alpha$*

Proof 1. Induction over $\leqq^n$. Base case is $\alpha = \blacklozenge\alpha = a$. Now it is easy to check that $\alpha' \leqq^1 \alpha$ entails $\blacklozenge\alpha' \vdash_{\mathsf{CL}} \blacklozenge\alpha$ (see Lemma 15), and the claim follows from transitivity of $\vdash_{\mathsf{CL}}$. Parallel for 2. $\dashv$

Definition 68 Assume $\sigma : Var \to Form(\mathsf{AL})$. We define $\sigma^{\blacklozenge}, \sigma^{\blacksquare} : Var \to Form(\mathsf{CL})$ by $\sigma^{\blacklozenge}(p) = \blacklozenge\sigma(p)$, $\sigma^{\blacksquare}(p) = \blacksquare\sigma(p)$ for all $p \in Var$.

We extend $\sigma^{\blacklozenge}, \sigma^{\blacksquare}$ to canonically to arbitrary formulas. Note that $\sigma^{\blacklozenge}, \sigma^{\blacksquare}$ are *uniform* classical substitutions, since $\sigma^{\blacklozenge}(\neg\alpha) = \neg\sigma^{\blacklozenge}(\alpha)$, same for $\sigma^{\blacksquare}$, whereas $\blacklozenge\sigma(\neg\alpha) = \neg\blacksquare\sigma(\alpha)$! We have the following consequence: for $a, b \in Form(\mathsf{CL})$, $\sigma : Var \to Form(\mathsf{AL})$

- $a \equiv_{\mathsf{CL}} b$ implies $\sigma^{\blacklozenge}(a) \equiv_{\mathsf{CL}} \sigma^{\blacklozenge}(b)$ (closure under classical substitution).
- $a \equiv_{\mathsf{CL}} b$ does *not* imply $\blacklozenge\sigma(a) \equiv_{\mathsf{CL}} \blacklozenge\sigma(b)$

It is easy to find counterexamples involving $p \vee \neg p$ etc. Therefore, we write $a \equiv_{DM} b$, if b can be obtained from a via the DeMorgan and double negation laws. These two are sufficient to bring any classical formula into negation normal form (only atoms negated).

Lemma 69 *Assume $a, b \in Form(\mathsf{CL})$ and $a \equiv_{DM} b$. Then for arbitrary $\sigma : Var \to Form(\mathsf{AL})$, $\blacklozenge\sigma(a) \equiv_{\mathsf{CL}} \blacklozenge\sigma(b)$ and $\blacksquare\sigma(a) \equiv_{\mathsf{CL}} \blacksquare\sigma(b)$.*

Proof We have (by way of example)

$$\begin{aligned}\blacklozenge\sigma(\neg(a \wedge b)) &= \neg((\blacksquare\sigma(a)) \wedge (\blacksquare\sigma(b)))\\ &\equiv_{\mathsf{CL}} (\neg(\blacksquare\sigma(a))) \vee (\neg(\blacksquare\sigma(b)))\\ &= \blacklozenge\sigma((\neg a) \vee (\neg b))\end{aligned}$$

Since $\vdash_{\mathsf{CL}}$ is congruent (equivalents are substitutable), this is sufficient. ⊣

Lemma 70 *Assume $a \in Form(\mathsf{CL})$, $\sigma : Var \to Form(\mathsf{AL})$. Then*

1. $\sigma^{\blacklozenge}(a) \vdash_{\mathsf{CL}} \blacklozenge(\sigma(a))$
2. $\blacksquare(\sigma(a)) \vdash_{\mathsf{CL}} \sigma^{\blacksquare}(a)$

Proof We only prove 1., since 2. is dual. Note that by Lemma 69, we can without loss of generality assume that a is in negation normal form, that is, only atoms are negated. We make an induction over the complexity of a.

Base case $a = p$ is clear since by definition, $\blacklozenge\sigma(p) = \sigma^{\blacklozenge}(p)$. For $\neg p$, the claim is also straightforward: $\blacklozenge\sigma(\neg p) = \neg\blacksquare\sigma(p)$, and $\sigma^{\blacklozenge}(\neg p) = \neg\blacklozenge\sigma(p)$. Now since $\blacksquare\sigma(p) \vdash_{\mathsf{CL}} \blacklozenge\sigma(p)$, by contraposition we have $\neg\blacklozenge\sigma(p) \vdash_{\mathsf{CL}} \neg\blacksquare\sigma(p)$.

Now we make an induction for $\wedge, \vee$.

$\wedge$ Assume $a = b \wedge c$, by induction hypothesis $\sigma^{\blacklozenge}(b) \vdash_{\mathsf{CL}} \blacklozenge\sigma(b), \sigma^{\blacklozenge}(c) \vdash_{\mathsf{CL}} \blacklozenge\sigma(c)$, hence $\sigma^{\blacklozenge}(b \wedge c) = \sigma^{\blacklozenge}(b) \wedge \sigma^{\blacklozenge}(c) \vdash_{\mathsf{CL}} \blacklozenge\sigma(b) \wedge \blacklozenge\sigma(c) = \blacklozenge\sigma(b \wedge c)$.

$\vee$ Dual. ⊣

Lemma 71 *Assume L is a conservative extension of* CL *and closed under R, S. Assume moreover $w \in Form(\|)$, $a \in Form(\mathsf{CL})$. Then*

1. *$w \vdash_{R\sigma L} a$ implies $w \vdash_L a$.*
2. *$a \vdash_{R\sigma L} w$ implies $a \vdash_L w$.*

Proof We only prove 1., since 2. is dual. Note that if $w \in Form(\mathsf{CL})$, the claim is obvious by conservative extension. Hence assume $w \leqq_{\mathsf{AL}} \sigma(\alpha) \vdash_{\sigma L} \sigma(\beta) \leqq_{\mathsf{AL}} a$. Since $w \in Form(\|)$, we must have $\alpha \in Form(\vee, \|)$. Since $\vdash_L$ is closed under $\leqq_{\mathsf{AL}}$, we can without loss of generality assume that $\beta \in Form(\mathsf{CL})$ (substitute $\|$ by $\vee$, see Lemma 67). We must also have $\sigma : Var \to Form(\vee, \|)$, since any other connective occurring in $\sigma(\alpha)$ would exclude $w \leqq_{\mathsf{AL}} \sigma(\alpha)$, for $w \in Form(\|)$. We have

1. $w \leqq_{\mathsf{AL}} \sigma(\alpha) \leqq_{\mathsf{AL}} \sigma^{\blacklozenge}(\alpha)$ (no negation in α, hence $\sigma^{\blacklozenge}(\alpha) = \blacklozenge\sigma(\alpha)$)
2. $\sigma^{\blacklozenge}(\alpha) \vdash_L \sigma^{\blacklozenge}\beta$ ($\sigma^{\blacklozenge}$ is classical, L closed under S)
3. $\sigma^{\blacklozenge}(\beta) \vdash_{\mathsf{CL}} \blacklozenge\sigma(\beta)$ (Lemma 70, $\beta \in (\mathsf{CL})$)
4. $\blacklozenge\sigma(\beta) \vdash_{\mathsf{CL}} a$ (Lemma 67)

Hence $w \leqq_{\mathsf{AL}} \sigma(\alpha) \leqq_{\mathsf{AL}} \sigma^{\blacklozenge}(\alpha) \vdash_L \sigma^{\blacklozenge}(\beta) \vdash_{\mathsf{CL}} \blacklozenge\sigma(\beta) \vdash_{\mathsf{CL}} a$. By assumption L is closed under composition with both $\leqq_{\mathsf{AL}}$ and $\vdash_{\mathsf{CL}}$, so $w \vdash_L a$. ⊣

Note that this property relies in particular on the fact that w does not contain any negation symbol!

Corollary 72 *Let L be a full ambiguity logic, and assume L is non-trivial. Then τL is non-trivial.*

Proof Assume τL is m-trivial, hence for pairwise distinct p, q, q', r, we have $pqr \vdash_{(TR\sigma)^* L} pq'r$, hence there is an n such that $pqr \vdash_{(TR\sigma)^{n+1} L} pq'r$. Since L is closed under R, S, so is $(TR\sigma)^n L$. We distinguish cases as to the outermost T:

Case 1: $pqr \vdash_{R\sigma(TR\sigma)^n L} pq'r$. Then $pqr \vdash_{(TR\sigma)^n L} pq'r$ (Lemma 66).

Case 2: $pqr \vdash_{R\sigma(TR\sigma)^n L} a \vdash_{R\sigma(TR\sigma)^n L} pq'r$. This entails $pqr \vdash_{(TR\sigma)^n L} a \vdash_{(TR\sigma)^n L} pq'r$ (Lemma 71), and so $pqr \vdash_{(TR\sigma)^n L} pq'r$, since $(TR\sigma)^n L$ is closed under T.

This way, we can iteratively reduce n, yielding finally $pqr \vdash_L pq'r$ (L is by assumption closed under R, S, T). ⊣

Corollary 73 *(Trust Theorem)*

1. *Every distrustful ambiguity logic L can be extended to a unique smallest trustful ambiguity logic τL, where $\vdash_L \subseteq \vdash_{\tau L}$. Moreover, if L is non-trivial, then τL is non-trivial.*
2. *No trustful ambiguity logic can be extended to a non-trivial distrustful ambiguity logic: if L_t is a trustful ambiguity logic, L_d is a distrustful ambiguity logic, and $\vdash_{L_t} \subseteq \vdash_{L_d}$, then L_d is m-trivial.*

In the remainder of this subsection, we will see that trust closure results in rather odd logics. This will lead to the last method for constructing full ambiguity logics, namely the one of algebraic extensions of congruence algebras.

5.4.2 The Smallest Trustful Ambiguity Logic and Algebraic Extensions

A logic worth looking at is the *smallest trustful ambiguity logic*, which is the trust closure of $L_{\blacklozenge\blacksquare}$, denoted by $\tau L_{\blacklozenge\blacksquare}$, where $L_{\blacklozenge\blacksquare}$ is the minimal full ambiguity logic.

$\tau L_{\blacklozenge\blacksquare}$ is considerably stronger than $L_{\blacklozenge\blacksquare}$. In particular, for all α, we have $\alpha \vdash_{\tau L_{\blacklozenge\blacksquare}} \alpha$, since $p \vdash_{\blacklozenge\blacksquare} p$. This allows us to show that whereas $L_{\blacklozenge\blacksquare}$ is commutative, $\tau L_{\blacklozenge\blacksquare}$ is not: $p\|q \vdash_{\tau L_{\blacklozenge\blacksquare}} p\|q$, yet the sequent $p\|q \vdash q\|p$ is not derivable: $\tau L_{\blacklozenge\blacksquare}$ derives only sequents which hold in *all* trustful ambiguity logics! Similarly, $(p\|q) \vee \neg(p\|q)$ is a theorem in $\tau L_{\blacklozenge\blacksquare}$, $(p\|q) \vee \neg(q\|p)$ is *not*. (I**;** I) is still not sound in this logic: we can derive $p \vdash_{\tau L_{\blacklozenge\blacksquare}} p \vee q$, $r \vdash_{\tau L_{\blacklozenge\blacksquare}} r \vee s$, but

$$p\|r \vdash (p \vee q)\|(r \vee s) \tag{24}$$

is *not* derivable in $\tau L_{\blacklozenge\blacksquare}$: it cannot be a substitution of any formula derivable in $L_{\blacklozenge\blacksquare}$, and it cannot be obtained by means R, S, T either, as is simple, but lengthy to check. This entails that (**,** weak) is not admissible in $\tau L_{\blacklozenge\blacksquare}$, and so $Cong(L_{\blacklozenge\blacksquare})$ is not lattice ordered, and we conjecture it is not naturally ordered either. Note that $\tau L_{\blacklozenge\blacksquare} \not\subseteq$ cDAL: this follows as a corollary from the Trust Theorem. A sequent which illustrates this is $\vdash (p\|q) \vee \neg(p\|q)$, valid in $\tau L_{\blacklozenge\blacksquare}$. We also have inverse results:

Lemma 74 DAL $\not\subseteq \tau L_{\blacklozenge\blacksquare}$

This follows from (24), which is derivable in DAL, yet not in $\tau L_{\blacklozenge\blacksquare}$. Hence $\tau L_{\blacklozenge\blacksquare}$ is incomparable with both DAL and cDAL. What is particularly striking about $\tau L_{\blacklozenge\blacksquare}$, or more generally about logics of the form τL, is that their inner logics seem to be very small: considering $\leqq_{\tau L_{\blacklozenge\blacksquare}}$, not only does it not contain $\vdash_{\mathsf{CL}}$, but it does not seem to satisfy any of the typical (generalized) Boolean inequations, such as DeMorgan law, $\wedge\vee$-distribution etc. Hence whereas the inner logic of $L_{\blacklozenge\blacksquare}$ is rather large (recall that $\leqq_{\blacklozenge\blacksquare} = \vdash_{\mathsf{cDAL}}$), here we obtain examples of logics with really restrictive inner logics (proofs seem to be very lengthy though).

In this context, it is interesting to work with **algebraic extensions**. For example, take the logic $\tau L_{\blacklozenge\blacksquare}$ and a set of equations $\{E_1, ..., E_n\}$. Then $\tau L_{\blacklozenge\blacksquare}(E_1, ..., E_n)$ is the smallest trustful ambiguity logic which 1. contains $\tau L_{\blacklozenge\blacksquare}$ and 2. whose congruence algebra satisfies $E_1, ..., E_n$ (meaning the equations are *valid*). Note that algebraic extensions are meaningful mostly for trustful logics, since in distrustful logics the inner logic and the outer logic usually coincide, so we can implement extensions by proof rules. So we define the following:

Definition 75 Assume L is a full ambiguity logic. Then $\tau L(E_1, ..., E_n)$ is the smallest trustful ambiguity logic which contains L and whose congruence algebra satisfies $E_1, ..., E_n$.

For example in $\tau L_{\blacklozenge\blacksquare}$ basically none of the usual Boolean axioms hold up to congruence. From here, we can construct

- $\tau L_{\blacklozenge\blacksquare}(DN)$, which is $\tau L_{\blacklozenge\blacksquare}$ with double negation congruence. So the smallest full ambiguity logic containing $L_{\blacklozenge\blacksquare}$, closed under uniform substitution and satisfying $\neg\neg\alpha \equiv \alpha$
- There is a logic $\tau L_{\blacklozenge\blacksquare}(DN, DM)$, which is $\tau L_{\blacklozenge\blacksquare}(DN)$ with DeMorgan congruence, etc.

Hence we can add all sorts of congruence-axioms, resulting in a lattice order of logics. Now this opens an interesting question for TAL, cTAL: can we characterize cTAL as axiomatic extension of τcDAL, and TAL as extension of τDAL? We have already proved that the congruence algebra *Cong*(cTAL) is a Kleene algebra, so the question is whether conversely τcDAL(Kleene) (τcDAL with all axioms for Kleene algebras) is equal to cTAL. The answer is probably negative, the argument is as follows: The rules ($\neg$I),(I$\neg$) preserve $\leqq$ if and only if the negated formula is classical. Since validity of equations means truth under arbitrary substitutions, there is no hope of bringing this into an algebraic form.

Conjecture 2 *The rules (¬I),(I¬) are not sound in* τcDAL(*Kleene*)

More generally, the correspondence of proof rules and (in)equations for congruence algebras is problematic for a number of reasons. We will not go into detail here, just state the following final conjecture:

Conjecture 3 *There is no set of equalities* $\boldsymbol{E}$ *such that* cTAL $= \tau$cDAL($\boldsymbol{E}$)*, and no set of equalities* $\boldsymbol{E}$ *such that* TAL $= \tau$DAL($\boldsymbol{E}$).

6 Conclusion

6.1 Open Questions, Loose Ends

We start the conclusion with a list of prominent open problems and questions.

Commutative Logics of Trust So far, we have seen several commutative logics of trust. However, none of them was really convincing: cTAL turned out to be equivalent to $L_{\blacksquare\blacklozenge}$, an m-trivial logic. Logics like $\tau L_{\blacklozenge\blacksquare}$ (or τcDAL, which we have not looked at) are non-trivial commutative logics. However, they seem to fail to satisfy fundamental properties like the laws for distributive lattices, and $\tau L_{\blacklozenge\blacksquare}$ does not even admit the fundamental ambiguation rule (I**;** I). The question is: is there a reasonable commutative logic of trust? So far, we have not seen one, and we leave this question open here. Still, it is striking that trust and commutativity do not go well together. We conjecture that trust requires keeping track of plausibility.

The issue of reducibility We have seen sequents of the form $p\|\neg p \vdash q\|\neg q$, which are valid even in a conservative logic as cDAL, but also in TAL. Such a sequent can be called *irreducible*, since it is in ambiguous normal form, yet the unambiguous “submeanings" on the left/right are not in any classical entailment relation. Having a logic with irreducibles is paramount (modulo commutativity and associativity) to the rule (I**;** I) not being invertible. The only logic without irreducibles we have considered here is $L_{\blacklozenge\blacksquare}$, which however does not derive many reducible sequents either, like $p\|q \vdash p\|q$. Is there a full ambiguity logic which derives all and only reducible sequents?

Logics without e- and u-substitution? The Basic Closure Theorem entails that there are four types of logics: 1. closed under u-substitution, not e-substitution (trustful), 2. closed under e-substitution, not u-substitution (distrustful), 3. closed under both (m-trivial), 4. closed under none of the two. It is this fourth type we have not observed yet. Given that we require closure under classical substitution, it does not seem to be straightforward to construct logics of this type. So do these logics exist, and are they interesting?

Which logic is most adequate? In this article, we have made a plea for logical pluralism, or rather: we have shown that it is necessary. There simply is not one most plausible logic of ambiguity. But then, of course, some are more reasonable than others: for example, cDAL seems to be a very reasonable logic of distrust, TAL a very reasonable logic of trust. Most of the other logics we presented have some properties which are strongly counterintuitive (even though also cDAL and TAL have properties which are mildly counterintuitive). So the big conceptual (not mathematical) question is: which logic is adequate for which purpose? This is a question which can be approached from at least three different perspectives:

1. Cognitive science: how do humans actually reason? A very good case study (though in a slightly different setting) can be found in (Frost-Arnold and Beebe, 2020).
2. Computer science: Arapinis and Vieu (2015) show that ambiguity almost inevitably enters into formal ontologies. Provided ontologies are based on fragments of first-order logic, which logic is the most apt for ambiguous ontologies?
3. Philosophy: It is also a philosophical question for (at least) argumentation theory. Of course this overlaps with 1., but this cannot be reduced to cognitive science (otherwise all logic were cognitive science).

6.2 The Lesson to be Learned

What do we learn from this article? First and foremost, there is a field of (full) ambiguity logics, with the Basic Closure Theorem and the Trust Theorem as their main cornerstones. What they show is the following: there *cannot* be “the" logic of ambiguity: every non-trivial ambiguity logic necessarily lacks at least one fundamental closure property in the sense of Tarski (1936), closure under substitution of equivalents or closure under uniform substitution. Moreover, this mathematical choice corresponds closely to a conceptual choice: whether we are in a situation of **trust** or **distrust**. Trust corresponds to closure under uniform substitution, distrust to its lack. We showed that every (non-trivial) distrustful logic can be extended to a (non-trivial) trustful logic, but *never* the other way around: hence trust (ceteris paribus) allows for more valid arguments than distrust, which confirms our intuition. This correlation between conceptual and mathematical properties is a neat finding.

In other cases, even if intuitions might be clear in the assumptions we make, they can leave us puzzled as we proceed to discover the consequences which follow from them. For example, the fact that there do not seem to be reasonable commutative logics of trust is surprising and somewhat puzzling. The fact that every classically congruent logics of trust contains, as an *inner logic*, a distrustful logic, is a pleasant surprise. Even more surprisingly, the prominent distrustful logic cDAL is the inner logic of the commutative version of the most prominent trustful logic cTAL, as well as the inner logic of the minimal ambiguity logic $L_{\blacklozenge\blacksquare}$.

Ambiguity is a field where we need logical pluralism, or, as van Deemter (1996) put it: logical consequence itself is ambiguous. This means that even beyond the most prominent logics DAL, cDAL, TAL, cTAL, there are interesting logics, which however have strange properties: they might not satisfy properties as the DeMorgan laws (as presumably $\tau L_{\blacklozenge\blacksquare}$, τcDAL), or not even satisfy $\alpha \vdash \alpha$ for all α (as $L_{\blacklozenge\blacksquare}$). We have to decide which logical properties we consider desirable in which circumstances.

Acknowledgements Thanks to the anonymous reviewers, especially to one, who helped improve this article a lot.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflicts of Interest Author A declares that he/she has no conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

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