A cellular Milnor-Witt (co)homology computation for the moduli space of stable, genus 0 curves with marked points

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Contents

Summary

We introduce a version of cellular homology for non-strictly cellular schemes that builds on ideas of Morel-Sawant. By allowing cohomologically trivial cells, instead of affine spaces, we ca fit $\overline{M_{0,n}}$, the moduli space of stable, genus 0 curves with n marked points, in this framework. The major benefit of our approach is the computability of this cellular complex. Additionally, we show that the (co)homology of this complex computes the (co)homology for strictly $\mathbb{A}^1$-invariant sheaves. Most computations are carried out for the Milnor-Witt K-theory sheaf. Classical invariants that can be deduced from our computations are Chow groups, singular cohomology of the complex points, and singular cohomology of the real points (with twisted coefficients).

Additionally, we study the range in which the real cycle class map

$$H^i_{\mathrm{Nis}}(X, I^j(\mathcal{L})) \to H^i_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$$

is an isomorphism for linear schemes. This extends the previous known results about strictly-cellular schemes of Hornbostel-Wendt-Xie-Zibrowius.

Introduction

Motivic homotopy theory introduces tools from algebraic topology to the realm of algebraic geometry. The main objective of this thesis lies in developing computational tools for schemes that are 'sufficiently close to affine spaces', paralleling classical computations in algebraic topology. The moduli space $\overline{M_{0,n}}$ of stable, genus 0, n-marked curves will be the example we are interested in.

The space $\overline{M_{0,n}}$ is the Deligne-Mumford compactification of the space of isomorphism classes of n distinct ordered points in $\mathbb{P}^1_k$, called $M_{0,n}$. Its complex points $\overline{M_{0,n}}(\mathbb{C})$ are well understood. For example the singular cohomology ring $H^*_{\text{sing}}(\overline{M_{0,n}}(\mathbb{C}), \mathbb{Z})$ was computed in [Kee92]. This was done by first finding a way to describe $\overline{M_{0,n}}$ as an iterated smooth blow-up of $(\mathbb{P}^1_k)^{\times(n-3)}$, describing the behavior of Chow rings under blow-ups (coming from the projective bundle formula) and using the isomorphism coming from the cycle class map $\mathrm{CH}^i(X) \to H^{2i}_{\text{sing}}(\overline{M_{0,n}}(\mathbb{C}), \mathbb{Z})$. This can be used to define Gromov-Witten invariants and furthermore to compute the number of degree d curves passing through $3d-1$ general points in $\mathbb{P}^2_{\mathbb{C}}$, see [FP97].

Lots of enumerative questions, like 'How many lines does a smooth cubic surface in $\mathbb{P}^3_k$ contain?', can be solved over algebraically closed fields using Chow rings, see [EH16]. In the case of curves on a cubic, the answer is 27. But over the reals this does not need to be true. There can be 3, 7, 15 or 27 lines. Quadratically enriched enumerative geometry allows us to get signed counts to enumerative questions for non-algebraically closed fields. One possible way to do this is by replacing Chow rings with Chow-Witt rings. The results are not integers anymore, but quadratic forms. In [KW21] this is used to show that there are $15\langle 1\rangle + 12\langle -1\rangle \in \mathrm{GW}(k)$ curves on a smooth cubic. By taking the rank, one recovers the 27 lines over $\mathbb{C}$ and using the signature, one gets a signed count of 3 curves over $\mathbb{R}$. In particular, this shows that there at least 3 curves over $\mathbb{R}$.

The invariants we are going to consider are (co)homology groups of strictly $\mathbb{A}^1$-invariant sheaves. One such example is the Chow-Witt group $\widetilde{\mathrm{CH}}^i(-, \mathcal{L}) = H^i_{\text{Nis}}(-, \underline{K}^{MW}_i(\mathcal{L}))$. It might seem that we just introduced completely new invariants, but Chow groups also fit in this formalism as $\mathrm{CH}^i(-) = H^i_{\text{Nis}}(-, \underline{K}^M_i)$. There is another group of classical origin, the cohomology of fundamental ideal powers $H^i_{\text{Nis}}(-, \underline{I}^j)$, sharing a close connection to the singular cohomology of the real points $H^*_{\text{sing}}(-(\mathbb{R}), \mathbb{Z})$. For $\overline{M_{0,n}}$ the singular cohomology ring $H^*_{\text{sing}}(\overline{M_{0,n}}(\mathbb{R}), \mathbb{Z})$ was computed in [EHKR10]. In cases where the scheme of interest is defined over $\mathbb{R}$ and the Chow ring contains no 2-torsion the Chow-Witt group can be seen as a non-trivial gluing of singular cohomology of the real and complex points. More precisely, this becomes

$$\widetilde{\mathrm{CH}}^i(X, \mathcal{L}) \cong H^i_{\mathrm{Nis}}(-, \underline{I}^i(\mathcal{L})) \times_{\mathrm{CH}(X)/2} \ker\left(\partial_{\mathcal{L}} \colon \mathrm{CH}(X) \to H^{i+1}_{\mathrm{Nis}}(-, \underline{I}^{i+1}(\mathcal{L}))\right),$$

which is the key diagram in [HW19].

The very first attempt to compute Chow-Witt groups of $\overline{M_{0,n}}$ would be to imitate Keel's computation of the Chow groups. This does not work so well, as Chow-Witt groups do not satisfy the projective bundle formula [Fas13]. One way to see the failure is to compute the cohomology of $\mathbb{P}^n_k$ and see that it is not free over the cohomology of $\mathrm{Spec}(k)$, see the computations in Chapter 5.

To do the computation, we consider a cellular structure on $\overline{M_{0,n}}$, following [MS23]. The difference to classically used strict cellular structures, where one stratifies the space by cells isomorphic to $\mathbb{A}^n_k$, is that we allow cells that are smooth, affine and cohomologically trivial, where the latter means cells for which $H^i_{\mathrm{Nis}}(-, \boldsymbol{M})$ vanishes for $i > 0$ and strictly $\mathbb{A}^1$-invariant sheaves of abelian groups $\boldsymbol{M}$. This extra flexibility allows one to use cells like $\mathbb{G}^d_m \times \mathbb{A}^n_k$ and this is necessary as there does not seem to be a known strict cellular structure, see Remark 4.1.12. More concretely, we stratify $\overline{M_{0,n}}$ by products of $M_{0,n'}$. These are smooth, affine and cohomologically trivial because they are complements of hyperplanes, see Lemma 4.1.9. With these cellular structures one can set up a version of cellular homology, very similar to the classical setting [Hat02], and perform computations. This complex is more suited for computations as it only requires knowledge about finitely many residues as opposed to the Rost-Schmid complex which, a priori, requires residues at all points, just like in ordinary topology. The other main ingredient is the computation of the differentials of the chain complex. This is done by restricting to smooth rational curves in $\overline{M_{0,n}}$ (meaning 1-dimensional subschemes of the space), which we have an abundance of, and computing individual residues for enough of those. This is the main part of the thesis.

Another class of schemes that is close enough to affine spaces, are linear schemes. These are schemes that can be iteratively constructed via localization sequences from affine spaces. We consider two notions of linear schemes: one following [Jan90] and a more restrictive notion of [Tot14]. By studying the behavior of the multiplication map $\langle\!\langle -1 \rangle\!\rangle \colon H^i(X, I^j(\mathcal{L})) \to H^i(X, I^{j+1}(\mathcal{L}))$ in parallel to what was done in [HWXZ21] for strictly cellular schemes, we are able to extend their results. The difference lies in weaker bounds depending on the non-cellularity appearing. This extension was already used in [MW24] to give new bounds of torsion exponents in the singular cohomology of oriented flag manifolds.

The thesis is structured in the following way. Chapter 1 starts with an introduction to Milnor-Witt K-theory for fields and its residue maps. Chapter 2 starts with an introduction to the Rost-Schmid complex in homological and cohomological notation together with the necessary functorialities. This complex provides a way to compute the cohomology of strictly $\mathbb{A}^1$-invariant sheaves, which are introduced in the second section of the chapter. A key tool in computations is Lemma 2.1.13, as it allows to compute differentials by restricting to curves. Chapter 3 introduces $\overline{M_{0,n}}$, the moduli space of stable, genus 0 curves with n marked points, alongside some standard facts. The second section collects two facts, that are used later. The first one is Lemma 3.2.2, which concretely give enough curves meeting boundary divisors in a prescribed way, and the second one is Proposition 3.2.6, which concretely computes the canonical bundle as an element of $\mathrm{Pic}(\overline{M_{0,n}})$ with respect to a specific basis. This presentation of the canonical bundle allows us to switch between homology and cohomology. Chapter 4 introduces cellular structures and the cellular homology. It is shown in Proposition 4.2.4 that this cellular complex actually computes the sheaf (co)homology. In Remark 4.1.12 we see that this added generality is necessary, as $\overline{M_{0,n}}$ does not come with a strictly-cellular structure. Chapter 5 starts with various computations for projective spaces. Section 2 is the main computation for $\overline{M_{0,n}}$. In chapter 6 we introduce two notions of linear schemes and study in which range the real cycle class map $H^i_{\mathrm{Nis}}(X, I^j(\mathcal{L})) \to H^i_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ is an isomorphism for those.

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1. Milnor-Witt K-theory

In this chapter the base field k is of characteristic not 2.

1.1. Preliminaries

Definition 1.1.1. Let k be a field. Define the Milnor-Witt K-theory $K_*^{MW}(k)$ to be the $\mathbb{Z}$-graded ring (not necessarily commutative) generated by symbols $[a]$ of degree 1 for $a \in k^\times$ and an additional symbol η in degree -1 satisfying the following relations:

(i) $[a][1-a] = 0 \in K_2^{MW}(k)$ for $a, 1-a \in k^\times$

(ii) $[ab] = [a] + [b] + \eta[a][b] \in K_1^{MW}(k)$ $a, b \in k^\times$

(iii) $\eta[a] = [a]\eta \in K_0^{MW}(k)$ for $a \in k^\times$

(iv) $(2 + \eta[-1])\eta = 0 \in K_{-1}^{MW}(k)$

Notation: $[a_1, \ldots, a_n] = [a_1] \ldots [a_n]$.

Remark 1.1.2. Taking the quotient by the ideal generated by η, immediately gives us the definition of Milnor K-theory, i.e. $K_*^{MW}(k)/(\eta) = K_*^M(k)$.

Remark 1.1.3. The Grothendieck-Witt ring $\mathrm{GW}(k)$ of a field k is the group completion of isomorphism classes of non-degenerate symmetric bilinear forms on k, where the addition is given by the direct sum and the multiplication by the tensor product. It is generated, as an abelian group, by symbols $\langle a \rangle$ for $a \in k^\times$ (corresponding to the form $k \times k \to k$, $(x, y) \mapsto axy$) satisfying the relations $\langle ab^2 \rangle = \langle a \rangle$ and $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$, see [Lam05, Theorem 4.1]. Using the isomorphism $\mathrm{GW}(k) \to K_0^{MW}(k)$ given by $\langle a \rangle \mapsto 1 + \eta[a]$ we will write $\langle a \rangle = 1 + \eta[a]$ in $K_0^{MW}(k)$. For ease of notation set $\langle a, b \rangle = \langle a \rangle + \langle b \rangle$ and $\varepsilon = -\langle -1 \rangle$.

More about symmetric bilinear forms can be found in [Lam05].

Remark 1.1.4. Another way to define $K_n^{MW}(k)$ is as the fiber product

$$\begin{tikzcd} K_n^{MW}(k) \arrow[rr] \arrow[dd] & & K_n^M(k) \arrow[d] \\ & & K_n^M(k)/2 \arrow[dl, "\cong"] \\ I^n(k) \arrow[r] & I^n(k)/I^{n+1}(k) & \end{tikzcd}$$

where $I^n(k)$ are the powers of the fundamental ideal $I(k) = \ker(\text{rank}\colon \mathrm{GW}(k) \to \mathbb{Z})$. We extend this to non-positive degrees by setting $I^n(k) = \mathrm{W}(k) = \mathrm{GW}(k)/(1 + \langle -1\rangle)$ for $n \leq 0$. The isomorphism $K_n^M(k)/2 \to I^n(k)/I^{n+1}(k)$ induced by $[u] \mapsto \langle -1, u\rangle$ comes from the resolution of the Milnor conjecture on quadratic forms, see [OVV07, Theorem 4.1] for a proof in characteristic 0 or [Dug04] for a survey. The left vertical map is

$$\eta^m[u_1, \dots, u_{n+m}] \mapsto \langle -1, u_1\rangle \dots \langle -1, u_{n+m}\rangle \in I^{n+m}(k) \subseteq I^n(k).$$

This way we have the following commutative diagram

$$\begin{tikzcd} K_{n+1}^{MW}(k) \arrow[r, "\cdot\eta"] \arrow[d] & K_n^{MW}(k) \arrow[d] \\ I^{n+1}(k) \arrow[r, hook] & I^n(k) \end{tikzcd}$$

and set $\bar{I}^n(k) = I^n(k)/I^{n+1}(k) \cong K_n^M(k)/2$, which is 2-torsion.

Lemma 1.1.5. *Let k be a field. The following relations hold in $K_*^{MW}(k)$:*

(i) $[ab] = [a] + \langle a\rangle[b]$,
(ii) $0 = [1]$, $1 = \langle 1\rangle$,
(iii) $\langle ab\rangle = \langle a\rangle\langle b\rangle$,
(iv) $\langle a^2\rangle = \langle 1\rangle$,
(v) $[a][b] = \varepsilon[b][a]$
(vi) $\langle a\rangle[b] = [b]\langle a\rangle$,
(vii) $\langle a^{-1}\rangle = \langle a\rangle^{-1}$,
(viii) $[\frac{a}{b}] = [a] - \langle \frac{a}{b}\rangle[b]$,
(ix) $[a][-a] = 0$,
(x) $\varepsilon[-1] = [-1]$,
(xi) $[a][a] = [a][-1] = [-1][a]$,
(xii) $[a^{-1}] = \varepsilon[a]$,
(xiii) $\eta = \varepsilon\eta$,
(xiv) $\langle a\rangle[a] = \langle -1\rangle[a]$.

Proof. See [Mor12, Section 3.1]. □

Lemma 1.1.6. *Let k be a field and $n \in \mathbb{Z}$. The abelian group $K_n^{MW}(k)$ is generated by symbols of the form $\eta^r[a_1, \dots, a_{n+r}]$ for $r \geq 0$ and $a_i \in k^\times$ modulo the relations:*

(i) $\eta^r[a_1, \dots, a_{n+r}] = 0$ *if $a_i + a_{i+1} = 1$ for some i*

(ii) $\eta^r[a_1, \dots, a_{i-1}, a_i b_i, a_{i+1}, \dots, a_{n+r}] = \eta^r[a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{n+r}]$
$$+ \eta^r[a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_{n+r}] + \eta^{r+1}[a_1, \dots, a_{i-1}, a_i, b_i, a_{i+1}, \dots, a_{n+r}]$$

(iii) $\eta^{r+2}[a_1, \dots, a_{i-1}, -1, a_{i+1}, \dots, a_{n+r+2}] = -2\eta^{r+1}[a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+r+2}]$.

Proof. By the defining relation, η is central. Therefore, any element can be written as a sum of these generators. The relations are verbatim the homogeneous relations for $K_*^{MW}(k)$. □

Remark 1.1.7. The set of generators of $K_n^{MW}(k)$ can be further reduced. One way to do this for $n \geq 1$ is to reduce the η-power to 0 using $\eta[a][b] = [ab] - [a] - [b]$, and for $n \leq 0$ to rewrite $\eta[a] = \langle a \rangle - \langle 1 \rangle$ noting that a product of these elements is a $\mathbb{Z}$-linear combination of forms $\langle a \rangle$. This results in the following set of generators:

(i) $n \geq 1$: the group $K_n^{MW}(k)$ is generated by the elements $[a_1, \dots, a_n]$ with $a_i \in k^\times$.
(ii) $n \leq 0$: the group $K_n^{MW}(k)$ is generated by the elements $\eta^{-n+1}[a]$ with $a \in k^\times$.

Another way to reduce the set of generators, in the case of a discretely valuated field, is to pick a uniformizer π and rewrite elements $a_i = \pi^{\nu(a_i)} u_i \in k^\times$ for some units $u_i \in \mathcal{O}_\nu^\times$. Start by using the product relation, separating everything into single uniformizers and units. Now, using $[a][a] = [a][-1]$ and graded-commutativity, reduce to at most one entry containing the uniformizer. This way every element can be written as a sum of symbols of the form $\eta^r[\pi, u_2, \dots, u_{n+r}]$ and $\eta^r[u_1, \dots, u_{n+r}]$ for $u_i \in \mathcal{O}_\nu^\times$.

Proposition 1.1.8. *Let k be a field with discrete valuation $\nu\colon k \to \mathbb{Z} \cup \{-\infty\}$ with valuation ring $\mathcal{O}_\nu$, maximal ideal $\mathfrak{m}_\nu$ and residue field $\kappa(\nu)$. Fix a uniformizer π. There is a unique homomorphism of abelian groups $\partial_\nu^\pi\colon K_*^{MW}(k) \to K_{*-1}^{MW}(\kappa(\nu))$ of degree -1 such that*

(i) $\partial_\nu^\pi(\eta\alpha) = \eta\partial_\nu^\pi(\alpha)$, *for* $\alpha \in K_*^{MW}(k)$,
(ii) $\partial_\nu^\pi([\pi, u_2, \dots, u_n]) = [\overline{u_2}, \dots, \overline{u_n}]$, *for* $u_2, \dots, u_n \in \mathcal{O}_\nu^\times$,
(iii) $\partial_\nu^\pi([u_1, \dots, u_n]) = 0$, *for* $u_1, \dots, u_n \in \mathcal{O}_\nu^\times$.

Moreover, there is a unique ring homomorphism $s_\nu^\pi\colon K_*^{MW}(k) \to K_*^{MW}(\kappa(\nu))$ given on generators by $s_\nu^\pi(\eta) = \eta$ and $s_\nu^\pi([\pi^n u]) = [\overline{u}]$ for $u \in \mathcal{O}_\nu^\times$.

Proof. This is [Mor12, Theorem 3.15]. □

Lemma 1.1.9. *Let $\alpha \in K^{MW}(k)$ and $u \in \mathcal{O}_\nu^\times$. Then:*

(i) $\partial_\nu^\pi([-\pi]\alpha) = \langle\overline{-1}\rangle s_\nu^\pi(\alpha)$,
(ii) $\partial_\nu^\pi([u]\alpha) = -\langle\overline{-1}\rangle[\overline{u}]\partial_\nu^\pi(\alpha)$,
(iii) $\partial_\nu^\pi(\langle u\rangle\alpha) = \langle\overline{u}\rangle\partial_\nu^\pi(\alpha)$,

(iv) $\partial_\nu^{u\pi}(\alpha) = \langle\overline{u^{-1}}\rangle\partial_\nu^\pi(\alpha)$.

Proof. It suffices to check this on generators $\eta^r[\pi, u_2, \dots, u_{n+r}]$ and $\eta^r[u_1, \dots, u_{n+r}]$ for $u_i \in \mathcal{O}_\nu^\times$ by Remark 1.1.7.

(i)
$$\begin{aligned}\partial_\nu^\pi([-\pi]\eta^r[u_1, \dots, u_{n+r}]) &= \eta^r\partial_\nu^\pi\left(([\pi] + [-1] + \eta[\pi][-1])[u_1, \dots, u_{n+r}]\right)\\ &= \eta^r(1 + \eta[\overline{-1}])[\overline{u_1}, \dots, \overline{u_{n+r}}] = \langle\overline{-1}\rangle\eta^r[\overline{u_1}, \dots, \overline{u_{n+r}}] = \langle\overline{-1}\rangle s_\nu^\pi(\alpha)\\ \partial_\nu^\pi([-\pi]\eta^r[\pi, u_2, \dots, u_{n+r}]) &= \eta^r\partial_\nu^\pi([-\pi][\pi, u_2, \dots, u_{n+r}]) = 0\\ &= \langle\overline{-1}\rangle\eta^r \underbrace{s_\nu^\pi([\pi])}_{=[1]=0} s_\nu^\pi([u_2, \dots, u_{n+r}]) = \langle\overline{-1}\rangle s_\nu^\pi(\eta^r[\pi, u_2, \dots, u_{n+r}])\end{aligned}$$

(ii)
$$\begin{aligned}\partial_\nu^\pi([u]\eta^r[u_1, \dots, u_{n+r}]) &= \eta^r\partial_\nu^\pi\left([u][u_1, \dots, u_{n+r}]\right) = 0 = \langle\overline{-1}\rangle[\overline{u}]\partial_\nu^\pi(\eta^r[u_1, \dots, u_{n+r}])\\ \partial_\nu^\pi([u]\eta^r[\pi, u_2, \dots, u_{n+r}]) &= \eta^r\partial_\nu^\pi(\underbrace{[u][\pi]}_{[\pi][u]\varepsilon}[u_2, \dots, u_{n+r}])\\ &= -\eta^r\partial_\nu^\pi([\pi][u][u_2, \dots, u_{n+r}]) - \eta^{r+1}\partial_\nu^\pi([\pi]\underbrace{[u][-1]}_{=[-1][u]}[u_2, \dots, u_{n+r}])\\ &= \eta^r(-1 - \eta[\overline{-1}])[\overline{u}][\overline{u_2}, \dots, \overline{u_{n+r}}] = -\langle\overline{-1}\rangle[\overline{u}]\partial_\nu^\pi(\eta^r[u_2, \dots, u_{n+r}])\end{aligned}$$

(iii)
$$\begin{aligned}\partial_\nu^\pi(\langle u\rangle\alpha) &= \partial_\nu^\pi((1 + \eta[u])\alpha) = \partial_\nu^\pi(\alpha) + \eta\partial_\nu^\pi([u]\alpha) = \partial_\nu^\pi(\alpha) + \eta(-\langle\overline{-1}\rangle[\overline{u}]\partial_\nu^\pi(\alpha))\\ &= (1 + \underbrace{\eta\varepsilon}_{=\eta}[\overline{u}])\partial_\nu^\pi(\alpha) = \langle\overline{u}\rangle\partial_\nu^\pi(\alpha)\end{aligned}$$

(iv)
$$\begin{aligned}\partial_\nu^{u\pi}(\eta^r[u_1, \dots, u_{n+r}]) &= 0 = \partial_\nu^\pi(\eta^r[u_1, \dots, u_{n+r}])\\ \partial_\nu^{u\pi}(\eta^r[\pi, u_2, \dots, u_{n+r}]) &= \partial_\nu^{u\pi}(\eta^r[u\pi \cdot u^{-1}, u_2, \dots, u_{n+r}])\\ &= \eta^r\left(\partial_\nu^{u\pi}\left([u\pi, u_2, \dots, u_{n+r}] + [u^{-1}, u_2, \dots, u_{n+r}] + \eta[u\pi, u^{-1}, u_2, \dots, u_{n+r}]\right)\right)\\ &= \eta^r\left([\overline{u_2}, \dots, \overline{u_{n+r}}] + \eta[\overline{u^{-1}}][\overline{u_2}, \dots, \overline{u_{n+r}}]\right) = \left(1 + \eta[\overline{u^{-1}}]\right)\eta^r[\overline{u_2}, \dots, \overline{u_{n+r}}]\\ &= \langle\overline{u^{-1}}\rangle\partial_\nu^\pi(\eta^r[\pi, u_2, \dots, u_{n+r}]).\end{aligned}$$

This finishes the proof. □

The additional term in the product relation for Milnor-Witt K-theory, in contrast to Milnor K-theory, causes the differential to depend on a choice of uniformizer, see Lemma 1.1.9(iv). To remedy this, we will introduce a twisted version of Milnor-Witt K-theory as a way to record the choice of uniformizer and get a choice-independent differential.

Remark 1.1.10. Let k be a field and L a one-dimensional k-vector space. The units $k^\times$ act on $K_*^{MW}(k)$ by multiplication with a form $\langle a\rangle$ and on $\mathbb{Z}[L \setminus \{0\}]$ by scalar multiplication. As the forms $\langle a\rangle$ lie in the center of $K_*^{MW}(k)$ it does not matter whether we multiply with them from the left or right.

Definition 1.1.11. Let k be a field and L a one-dimensional k-vector space. Define the Milnor-Witt K-theory of k twisted by L as

$$K_*^{MW}(k, L) = K_*^{MW}(k) \otimes_{\mathbb{Z}[k^\times]} \mathbb{Z}[L \setminus \{0\}].$$

Remark 1.1.12. For varying L, all the groups $K_*^{MW}(k, L)$ are non-canonically isomorphic by picking an element $\ell \in L \setminus \{0\}$, we get an isomorphism $K_*^{MW}(k) \to K_*^{MW}(k, L)$, $\alpha \mapsto \alpha \otimes \ell$. These choices of non-canonical isomorphisms play a crucial role in the upcoming computations.

If L is oriented (compare Definition 4.1.6), meaning we have chosen an isomorphism $N \otimes_k N \cong L$ for some N, then L admits a canonical choice of isomorphism $K_*^{MW}(k, L) \cong K_*^{MW}(k)$. Pick any element $n \in N \setminus \{0\}$ and consider $n \otimes_k n \in L$. This induces a canonical isomorphism because for any other element $n' \in N$ we have $n' \otimes_k n' = (un) \otimes_k (un) = u^2(n \otimes_k n) \in L$ for some $u \in k^\times$. As squares act trivially via $1 = \langle u^2 \rangle$ on $K_*^{MW}(k)$, this isomorphism is independent of the choice of n.

This is the reason why we are interested in the classes of line bundles in $\mathrm{Pic}/2$.

1.2. Residues

We would like to use the definition of the residues from the untwisted case to define a residue in the twisted case. The appearing reparametrization is necessary because a given non-zero section might vanish along divisors we want to compute the residue at.

Definition 1.2.1. Let k be a field with discrete valuation $\nu\colon k \to \mathbb{Z} \cup \{-\infty\}$ with valuation ring $\mathcal{O}_\nu$, maximal ideal $\mathfrak{m}_\nu$ and residue field $\kappa(\nu)$. Additionally let $\mathcal{L}$ be a line bundle on $\mathrm{Spec}(\mathcal{O}_\nu) = \{\mathrm{Spec}(k), \mathrm{Spec}(\kappa(\nu))\}$. For an $\ell \in \mathcal{L}_k \setminus \{0\} = \mathcal{L}|_{\mathrm{Spec}(k)} \setminus \{0\}$ there exist $\ell_0 \in \mathcal{L} \setminus \{0\}$ and $c \in k^\times$ such that $\ell = \ell_0 \otimes_{\mathcal{O}_\nu} c$. Define

$$\begin{aligned} \partial_\nu\colon K_*^{MW}(k, \mathcal{L}_k) &\to K_{*-1}^{MW}\left(\kappa(\nu), (\mathfrak{m}_\nu/\mathfrak{m}_\nu^2)^* \otimes_{\kappa(\nu)} \mathcal{L}_{\kappa(\nu)}\right), \\ \alpha \otimes \ell &\mapsto \partial_\nu^\pi(\langle c \rangle \alpha) \otimes \left(\overline{\pi}^* \otimes_{\kappa(\nu)} (\ell_0 \otimes_{\kappa(\nu)} 1_{\kappa(\nu)})\right) \end{aligned}$$

for some uniformizer π, where $\overline{\pi} \in \mathfrak{m}_\nu/\mathfrak{m}_\nu^2$ is the image of π and $\overline{\pi}^*$ is the dual of $\overline{\pi}$.

Lemma 1.2.2. *The homomorphism*

$$\partial_\nu\colon K_*^{MW}(k, \mathcal{L}_k) \to K_{*-1}^{MW}\left(\kappa(\nu), (\mathfrak{m}_\nu/\mathfrak{m}_\nu^2)^* \otimes_{\kappa(\nu)} \mathcal{L}_{\kappa(\nu)}\right)$$

defined above is well-defined, i.e. independent of the choice of uniformizer π and section ℓ_0.

Proof. Let $\pi' \in \mathcal{O}_\nu$ be another uniformizer. There is a unit $u \in \mathcal{O}_\nu^\times$ such that $\pi' = u\pi$. The element $\overline{\pi}^* \in (\mathfrak{m}_\nu/\mathfrak{m}_\nu^2)^* = \operatorname{Hom}_{\kappa(\nu)}(\mathfrak{m}_\nu/\mathfrak{m}_\nu^2, \kappa(\nu))$ is characterized by $\overline{\pi}^*(\overline{\pi}) = 1$. Therefore, we have $\overline{\pi'}^*(\overline{\pi}) = \overline{\pi'}^*(\overline{u^{-1}\pi'}) = \overline{u^{-1}}$ and thus $\overline{u^{-1}}\overline{\pi}^* = \overline{\pi'}^*$. Together with Lemma 1.1.9 this gives the desired independence of choice of uniformizer:

$$\begin{aligned}\partial_\nu^{\pi'}(\alpha) \otimes \overline{\pi'}^* = \partial_\nu^{u\pi}(\alpha) \otimes \overline{u\pi}^* &= \langle \overline{u^{-1}} \rangle \partial_\nu^\pi(\alpha) \otimes \left(\overline{u^{-1}}\overline{\pi}^*\right) \\ &= \langle \overline{u^{-1}} \rangle^2 \partial_\nu^\pi(\alpha) \otimes \overline{\pi}^* = \partial_\nu^\pi(\alpha) \otimes \overline{\pi}^*.\end{aligned}$$

To see the independence of the choice of section, let $\ell = \ell_0' \otimes_k c'$. The two sections ℓ_0 and ℓ_0' differ by a unit $u \in \mathcal{O}_\nu^\times$, i.e. $\ell_0 = u \cdot \ell_0'$. This satisfies $uc = c' \in k$ and hence $\overline{uc} = \overline{c'} \in \kappa(\nu)$. This shows the well-definedness by Lemma 1.1.9. □

Remark 1.2.3. The reparametrization of sections is necessary in computations. Consider the function field $k(t)$ together with the valuation associated to the uniformizer $-\frac{1}{t}$ (geometrically this describes the valuation at the point at infinity in $\mathbb{P}^1_k$). A line bundle over $\mathbb{P}^1_k$ can be trivialized over $\mathbb{P}^1_k \setminus \{\infty\}$ via some section u and over $\mathbb{P}^1_k \setminus \{0\}$ via some v. The transition map, for a line bundle $\mathcal{L}$, is given by $u = (\frac{1}{t})^{\deg(\mathcal{L})} v$. When computing the residue at t (i.e. $0 \in \mathbb{P}^1_k$), one has to use the section u and for the residue at $\frac{1}{t}$ (i.e. $\infty \in \mathbb{P}^1_k$) the section v. Note that depending on the degree of the line bundle, this introduces a symbol containing the uniformizer, influencing the residue.

The following computations will be used later as they will describe what residues at infinity look like.

Lemma 1.2.4. *Let k be a field. The residue $\partial^{\frac{1}{t}} \colon K_*^{MW}(k(t)) \to K_{*-1}^{MW}(k)$ associated with the uniformizer $\frac{1}{t}$ satisfies:*
(i) $\partial^{\frac{1}{t}}([t-a]) = \varepsilon \in K_0^{MW}(k) \quad \forall a \in k,$
(ii) $\partial^{\frac{1}{t}}\left(\langle \frac{1}{t} \rangle [t-a]\right) = -1 \in K_0^{MW}(k) \quad \forall a \in k,$
(iii) $\partial^{\frac{1}{t}}\left(\langle \frac{1}{t} \rangle\right) = \eta \in K_{-1}^{MW}(k) \quad \forall a \in k.$

Proof. Rewrite the symbol $[t-a]$, using $[xy] = [y] + [x]\langle y \rangle$, as:

$$\begin{aligned}[t-a] = \left[\left(\frac{1}{t}\right)^{-1}\left(\frac{t-a}{t}\right)\right] &= \left[\frac{t-a}{t}\right] + \left[\left(\frac{1}{t}\right)^{-1}\right]\left\langle \frac{t-a}{t} \right\rangle \\ &= \underbrace{\left[\frac{t-a}{t}\right]}_{\mapsto 0} + \underbrace{\varepsilon \left[\frac{1}{t}\right]\left\langle \frac{t-a}{t} \right\rangle}_{\mapsto \varepsilon\langle 1 \rangle = \varepsilon}\end{aligned}$$

(i) $\partial^{\frac{1}{t}}([t-a]) = \varepsilon$ since $\frac{t-a}{t} = 1 - \frac{a}{t} \equiv 1 \bmod \frac{1}{t}$ is a unit.

(ii) Rewriting

$$[t-a]\left\langle \frac{1}{t} \right\rangle = \left(\left[\frac{t-a}{t}\right] + \varepsilon \left[\frac{1}{t}\right] \left\langle \frac{t-a}{t} \right\rangle \right) \left(1 + \eta \left[\frac{1}{t}\right] \right)$$
$$= \underbrace{\left[\frac{t-a}{t}\right]}_{\mapsto 0} + \underbrace{\varepsilon \left[\frac{1}{t}\right] \left\langle \frac{t-a}{t} \right\rangle}_{\mapsto \varepsilon} + \underbrace{\eta \left[\frac{t-a}{t}\right] \left[\frac{1}{t}\right]}_{\mapsto \eta\varepsilon[1]=0} + \underbrace{\eta\varepsilon \left[\frac{1}{t}\right] \left[\frac{1}{t}\right] \left\langle \frac{t-a}{t} \right\rangle}_{=\eta\varepsilon\left[\frac{1}{t}\right][-1]\left\langle\frac{t-a}{t}\right\rangle \mapsto \eta\varepsilon[-1]}$$

gives $\partial^{\frac{1}{t}}\left([t-a]\left\langle \frac{1}{t}\right\rangle\right) = \varepsilon\,(1+\eta[-1]) = \varepsilon\,\langle -1\rangle = -1$.

(iii) Obvious by $\langle a\rangle = 1 + \eta[a]$.

This finishes the proof. □

Proposition 1.2.5. *Let F be a field. For $n \in \mathbb{Z}$ there is a split short exact sequence of $K_*^{MW}(F)$-modules*

$$0 \longrightarrow K_n^{MW}(F) \longrightarrow K_n^{MW}(F(x)) \xrightarrow{\sum_f \partial_f^f} \bigoplus_f K_{n-1}^{MW}(F[x]/f) \longrightarrow 0,$$

where the sum is taken over all monic irreducible polynomials $f \in F[x]$. The first homomorphism comes from the inclusion $F \subseteq F(x)$, and the second is the residue homomorphism associated to the f-adic valuation on $F(x)$ with uniformizer f.

Proof. This is [Mor12, Theorem 3.24]. □

Definition 1.2.6. Let $f \in F[x]$ be a monic irreducible polynomial. Define the geometric transfer as the composition

$$\tau_K^{F[x]/f}\colon K_n^{MW}(F[x]/f) \hookrightarrow \bigoplus_g K_n^{MW}(F[x]/g) \xrightarrow{s} K_{n+1}^{MW}(F(x)) \xrightarrow{-\partial_\infty^{-1/x}} K_n^{MW}(F),$$

where s is some section of $\sum_g \partial_g^g$ and the sum is taken over all monic irreducible polynomials $g \in F[x]$.

Proposition 1.2.7. *The geometric transfers $\tau_F^{F[x]/f}$ are well-defined, i.e. independent of the choice of section s.*

Proof. Suppose $s'\colon \bigoplus_g K_n^{MW}(F[x]/g) \to K_{n+1}^{MW}(F(x))$ is another section of $\sum_g \partial_g^g$. Then for any element $\alpha \in \bigoplus_g K_n^{MW}(F[x]/g)$ we have $\sum_g \partial_g^g(s(\alpha) - s'(\alpha)) = 0$. Therefore, $s(\alpha) - s'(\alpha) \in \ker(\sum_g \partial_g^g) = \operatorname{im}(K_n^{MW}(F) \hookrightarrow K_n^{MW}(F(x)))$ and we obtain the desired equality $(-\partial_\infty^{-1/x})(s(\alpha)) - (-\partial_\infty^{-1/x})(s'(\alpha)) = 0$, i.e. $\tau_F^{F[x]/f}$ is independent of the choice of section. □

Definition 1.2.8. Let $F \subseteq L$ be a finite field extension that is finitely generated over k. Denote by $\Omega_{L/k}$ and $\Omega_{F/k}$ the Kähler differentials over k. Pick a sequence of field extensions $F = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L$ such that L_{i+1}/L_i are simple extensions and define

$$\mathrm{Tr}_F^L = \mathrm{Tr}_{L_{n-1}}^{L_n} \circ \cdots \circ \mathrm{Tr}_{L_1}^{L_0} \colon K_n^{MW}(L, \det(\Omega_{L/k})) \to K_n^{MW}(F, \det(\Omega_{F/k})).$$

The transfer maps for simple extensions are defined by

$$\begin{aligned} \mathrm{Tr}_{L_{i+1}}^{L_i} \colon K_n^{MW}(L_{i+1}, \det(\Omega_{L_{i+1}/k})) &\to K_n^{MW}(L_i, \det(\Omega_{L_i/k})) \\ \alpha \otimes (l \otimes_{L_i} u) &\mapsto \tau_{L_i}^{L_{i+1}}(\langle u \rangle \alpha) \otimes l \end{aligned}$$

where we used $\det(\Omega_{L_{i+1}/k}) \cong \det(\Omega_{L_i/k}) \otimes_{L_i} L_{i+1}$.

Let $\mathcal{L}$ be a line bundle on $\mathrm{Spec}(F)$ and define by

$$\begin{aligned} \mathrm{Tr}_F^L \colon K_n^{MW}(L, \det(\Omega_{L/k}) \otimes \mathcal{L}|_L) &\to K_n^{MW}(F, \det(\Omega_{F/k}) \otimes \mathcal{L}) \\ \alpha \otimes \ell &\mapsto \mathrm{Tr}_F^L(\alpha) \otimes \ell \end{aligned}$$

the twisted transfer map.

Proposition 1.2.9. *The transfer maps Tr_F^L are well-defined, i.e. independent of the choice of sequence of simple extension and their generators.*

Proof. This is [Mor12, Section 4.2] in the case of trivial twists and $\mathrm{char}(k) \neq 2$, and [Mor12, Section 5.1] in the case of arbitrary twists and characteristic. □

2. The Rost-Schmid complex

General references for this chapter are [Mor12], [Fas07], [Fas08], and [Fas20]. In this chapter the base field k is of characteristic not 2 and perfect.

2.1. Preliminaries

Definition 2.1.1. Let X be a separated, finite type k-scheme and $\mathcal{L}$ a line bundle on X. Define the Rost-Schmid complex $C_{RS}(X, K_j^{MW}(\mathcal{L}))$ of weight $j \in \mathbb{Z}$ in degree i by

$$C_{RS}(X, K_j^{MW}(\mathcal{L}))_i := \bigoplus_{x \in X_{(i)}} K_{j+i}^{MW}(\kappa(x), \det(\Omega_{\kappa(x)/k}) \otimes \mathcal{L}_{\kappa(x)}),$$

where $X_{(i)}$ denotes the points of dimension i. For $x \in X_{(i)}$ and $y \in X_{(i-1)}$ the differential

$$d_y^x \colon K_{j+i}^{MW}(\kappa(x), \det(\Omega_{\kappa(x)/k}) \otimes \mathcal{L}_{\kappa(x)}) \to K_{j+i-1}^{MW}(\kappa(y), \det(\Omega_{\kappa(y)/k}) \otimes \mathcal{L}_{\kappa(y)})$$

is defined as follows. If $y \notin \overline{\{x\}}$, then set $d_y^x = 0$. If $y \in \overline{\{x\}}$, let Z be the normalization of $\overline{\{x\}}_y$. There are finitely many points $y_1, \dots, y_n \in Z$ above y. The differential d_y^x is defined to be the composition $\sum_{l=1}^n \mathrm{Tr}_{\kappa(y)}^{\kappa(y_l)} \circ \partial_{\nu_{y_l}}$.

Remark 2.1.2. Note that for fixed $x \in X_{(i)}$ and $\alpha \in K_{j+i}^{MW}(\kappa(x), \det(\Omega_{\kappa(x)/k}) \otimes \mathcal{L}_{\kappa(x)})$ there are only finitely many $y \in X_{(i-1)}$ with $d_y^x(\alpha) \neq 0$.

The normalization of $\overline{\{x\}}_y$ is necessary, because $\overline{\{x\}}$ might not be regular at y. So $\overline{\{x\}}_y$ might not be a regular local ring.

Alongside the points lying above $y \in \overline{\{x\}}$ there is a unique point $x_0 \in Z$ lying above x. As the normalization map is birational, we have an isomorphism $\kappa(x) \cong \kappa(x_0)$. The first map in the definition of the differential is

$$\bigoplus_{l=1}^n \partial_{\nu_{y_l}} : K_{j+i}^{MW}(\kappa(x_0), \det(\Omega_{\kappa(x_0)/k}) \otimes \mathcal{L}_{\kappa(x_0)}) \to \bigoplus_{l=1}^n K_{j+i-1}^{MW}(\kappa(y_l), \det(\Omega_{\kappa(y_l)/k}) \otimes \mathcal{L}_{\kappa(y_l)}),$$

where $\partial_{\nu_{y_l}}$ is the residue map coming from the valuation associated with the point y_l on the curve Z, see Definition 1.2.1. The twist bundle on the right $\det(\Omega_{\kappa(y_l)/k})$ is the same as $(\mathfrak{m}_{y_l}/\mathfrak{m}_{y_l}^2)^* \otimes_{\kappa(x)} \det(\Omega_{\kappa(x)/k})$ by the conormal sequence for smooth morphisms.

The second map is the sum of all the traces

$$\sum_{l=1}^{n} \mathrm{Tr}_{\kappa(y)}^{\kappa(y_l)} \colon \bigoplus_{l=1}^{n} K_{j+i-1}^{MW}(\kappa(y_l), \det(\Omega_{\kappa(y_l)/k}) \otimes \mathcal{L}_{\kappa(y_l)}) \to K_{j+i-1}^{MW}(\kappa(y), \det(\Omega_{\kappa(y)/k}) \otimes \mathcal{L}_{\kappa(y)}).$$

Theorem 2.1.3. *Let k be a field with $\mathrm{char}(k) \neq 2$, X be a separated finite type k-scheme, $j \in \mathbb{Z}$ and $\mathcal{L}$ be a line bundle on X. Then $(C_{RS}(X, K_j^{MW}(\mathcal{L})), d)$ is a chain complex.*

Notation 2.1.4. We will write $H_i^{RS}\left(Y, K_j^{MW}(\mathcal{L})\right)$ instead of $H_i\left(C_{RS}\left(Y, K_j^{MW}(\mathcal{L})\right)\right)$.

Definition 2.1.5. If X is a smooth equidimensional scheme with line bundle $\mathcal{L}$, we define

$$C^{RS}(X, K_j^{MW}(\mathcal{L}))^i = \bigoplus_{x \in X^{(i)}} K_{j-i}^{MW}(\kappa(x), \det(\mathfrak{m}_x/\mathfrak{m}_x^2)^{-1} \otimes \mathcal{L}_{\kappa(x)}).$$

and write $H^i\left(Y, K_j^{MW}(\mathcal{L})\right)$ instead of $H^i\left(C^{RS}\left(Y, K_j^{MW}(\mathcal{L})\right)\right)$.

Remark 2.1.6. Note that with this definition we have

$$C^{RS}(X, K_j^{MW}(\det(\Omega_{X/k}) \otimes \mathcal{L}))^i = C_{RS}(X, K_{j-\dim(X)}^{MW}(\mathcal{L}))_{\dim(X)-i}$$

which should be treated as an index rearranging. The change in twist bundle is there to make this definition compatible with other identifications and dualities. It does not carry any mathematical content. This definition results in

$$H^i(X, K_j^{MW}(\det(\Omega_{X/k}) \otimes \mathcal{L})) = H_{\dim(X)-i}^{RS}(X, K_{j-\dim(X)}^{MW}(\mathcal{L})).$$

There are multiple complexes appearing in the literature with slightly varying notations. The complex we call $C_{RS}(X, K_n^{MW})$ is called $C_{BM}(X, \mathbf{K}_{\dim(X)+n}^{MW})$ in [BCD+22]. For smooth schemes X the isomorphism [BCD+22, Theorem 4.2.11 in Chapter 6] gives $H_i(C_{BM}(X, \mathbf{K}_{\dim(X)-n}^{MW})) \cong H^{\dim(X)-i}(X, \mathbf{K}_{\dim(X)-n}^{MW}\{\omega_{X/k}\})$, where the latter is the sheaf cohomology. More on this will be in the next section. This explains why we add the twist by $\omega_{X/k} = \det(\Omega_{X/k})$ when defining the cohomological version C^{RS} above.

Remark 2.1.7. For $i \colon Z \hookrightarrow X$ a closed immersion with open complement $j \colon U \hookrightarrow X$ and a line bundle $\mathcal{L}$ on X, we have a short exact sequence of complexes

$$0 \longrightarrow C_{RS}(Z, K_m^{MW}(i^*\mathcal{L})) \longrightarrow C_{RS}(X, K_m^{MW}(\mathcal{L})) \longrightarrow C_{RS}(U, K_m^{MW}(j^*\mathcal{L})) \longrightarrow 0$$

and hence a long exact sequence in homology called the localization sequence. Concretely, this looks like:

$$H_l^{RS}(Z, K_m^{MW}(\mathcal{L})) \to H_l^{RS}(X, K_m^{MW}(\mathcal{L})) \to H_l^{RS}(U, K_m^{MW}(\mathcal{L})) \to H_{l-1}^{RS}(Z, K_m^{MW}(\mathcal{L})),$$

where we dropped the restriction of twists from the notation. Note that when looking at the analog for the cohomology there is an index shift in the cohomological degree and the K_m^{MW} sheaf degree, because the closed subscheme Z might have lower dimension. If we assume smoothness and set $c = \operatorname{codim}_X(Z)$, the same localization sequence looks like:

$$H^{q-c}(Z, K_{m-c}^{MW}(-)) \to H^q(X, K_m^{MW}(-)) \to H^q(U, K_m^{MW}(-)) \to H^{q+1-c}(Z, K_{m-c}^{MW}(-)),$$

where the twists at X and U are $\mathcal{L}$ and $j^*\mathcal{L}$, and at Z it is $\det(\Omega_{X/Z}) \otimes i^*\mathcal{L}$.

Definition 2.1.8. Let $f\colon X \to Y$ be a proper morphism and $\mathcal{L}$ a line bundle on Y. For $x \in X_{(i)}$ and $f(x) = y \in Y_{(i')}$ define

$$(f_*)_y^x\colon K_{j+i}^{MW}(\kappa(x), \det(\Omega_{\kappa(x)/k}) \otimes \mathcal{L}_{\kappa(x)}) \to K_{j+i'}^{MW}(\kappa(y), \det(\Omega_{\kappa(y)/k}) \otimes \mathcal{L}_{\kappa(y)})$$

to be trivial if the field extension $\kappa(y) \subseteq \kappa(x)$ is infinite and to be $\operatorname{Tr}_{\kappa(y)}^{\kappa(x)}$ if it is finite, in which case we have $i = i'$. Define

$$f_*\colon C_{RS}\left(X, K_j^{MW}(f^*(\mathcal{L}))\right) \to C_{RS}\left(Y, K_j^{MW}(\mathcal{L})\right)$$

to be the sum of all the $(f_*)_x^y$.

Theorem 2.1.9. *Let k be a field with $\operatorname{char}(k) \neq 2$, $f\colon X \to Y$ a proper morphism and $\mathcal{L}$ a line bundle on Y. Then*

$$f_*\colon C_{RS}\left(X, K_j^{MW}(f^*(\mathcal{L}))\right) \to C_{RS}\left(Y, K_j^{MW}(\mathcal{L})\right)$$

is a morphism of complexes.

Theorem 2.1.10. *Let k be a field with $\operatorname{char}(k) \neq 2$, $f\colon X \to Y$ a smooth morphism and $\mathcal{L}$ a line bundle on Y. Then there exists a pull-back*

$$f^*\colon C_{RS}\left(Y, K_j^{MW}(\mathcal{L})\right)_i \to C_{RS}\left(X, K_{j+r}^{MW}(\det(\Omega_{Y/X})^{-1} \otimes f^*(\mathcal{L}))\right)_{i-r},$$

where $r = \dim(Y) - \dim(X)$.

Proof. This is [BCD+22, Proposition 2.2.5. and 4.2.4 in Chapter 6]. □

Theorem 2.1.11. *Let k be a field with $\operatorname{char}(k) \neq 2$, $f\colon X \to Y$ a regular embedding and $\mathcal{L}$ a line bundle on Y. Then there exists a pull-back*

$$f^*\colon H_i^{RS}\left(Y, K_j^{MW}(\mathcal{L})\right) \to H_{i-r}^{RS}\left(X, K_{j+r}^{MW}(\det(\Omega_{Y/X})^{-1} \otimes f^*(\mathcal{L}))\right),$$

where $r = \dim(Y) - \dim(X)$.

Proof. This is [Fas20, Theorem 3.10 and Remark 3.12] or [BCD$^+$22, Chapter 6, Section 2.2] □

Remark 2.1.12. The pull-back for regular embeddings is constructed as usual by the deformation to the normal cone construction. Together with the pull-back for smooth morphism this gives a general pull-back morphism by pulling back along the projection $\mathrm{pr}_2\colon X \times Y \to Y$ and the graph $\Gamma_f\colon X \to X \times Y$.

In the case of X and Y smooth, and taking $i = \dim(Y)$, this recovers the sheaf pullback for $\underline{K}_j^{MW}(\mathcal{L})$, see [AF16, Theorem 2.11]. These sheaves will appear again in the next section.

The following technique I learned from Matthias Wendt.

Lemma 2.1.13. *Let $Z \hookrightarrow X$ be a regular embedding of codimension 1 and $d = \dim(X)$. Let $\iota\colon C \hookrightarrow X$ be a regularly embedded curve meeting Z transversally in point p. Then the following diagram $\langle -1\rangle$-commutes*

$$\begin{tikzcd}
H_d^{RS}(X \setminus Z, K_j^{MW}(\mathcal{L})) \arrow[r, "\partial"] \arrow[d, "\iota^*"'] & H_{d-1}^{RS}(Z, K_j^{MW}(\mathcal{L})) \arrow[d, "\iota^*"] \\
H_1^{RS}(C \setminus p, K_{j+d-1}^{MW}(\det(\mathcal{N}_{C/X})^{-1} \otimes \mathcal{L}|_C)) \arrow[r, "\partial"'] & H_0^{RS}(p, K_{j+d-1}^{MW}(\mathcal{L})),
\end{tikzcd}$$

where the horizontal morphisms come from the appropriate localization sequences.

Proof. This is done in [AF16, Section 2.3.1], compare [Fel20, Proposition 6.6]. □

Remark 2.1.14. The statement in the previous lemma is also true in the other degrees but we do not need that.

The sign comes from the fact that the pullback along regular embeddings is a composition of pullbacks along smooth morphisms, multiplication by $[t] \in K_1^{MW}(-)$ for a unit t and differentials from localization sequences. The differential commutes with pullbacks, $(-\langle -1\rangle)$-commutes with the multiplications, see Lemma 1.1.9 (ii), and (-1)-commutes with differentials from the localization sequence, see [Fel20, Proposition 6.6].

We can completely ignore the sign if all our computations are done by the previous lemma as this changes the differential by $\langle -1\rangle$, i.e. a unit.

Remark 2.1.15. Our goal is to compute differentials in localization sequences. Let $Z \hookrightarrow X$ be a closed embedding and U the open complement. A form defined on a $u \in U_{(i)}$ is first lifted to a form on $u \in X_{(i)}$, then one computes the Rost-Schmid differential d_x^u and restricts to Z. Therefore, we can treat the form on u as a form on $\overline{\{u\}}$ and compute the differential at a codimension 1 point. This can be done by restricting to curves due to Lemma 2.1.13. This way we can compute the whole form by computing its restriction to enough points. The amount of points needed depends on the concrete description of the cells. For affine spaces $\mathbb{A}^n_k$, forms are essentially constant so that a residue at a single point suffices to determine the differential.

2.2. $\mathbb{A}^1$-invariant sheaves

Remark 2.2.1. We defined Milnor-Witt K-theory K_n^{MW} for fields. By the framework of unramified sheaves in [Mor12] we obtain a sheaf $\underline{K}_n^{MW}$ on Sm_k, more precisely the category of essentially smooth k-schemes, whose evaluation on $\mathrm{Spec}(F)$ is $K_n^{MW}(F)$ for every field F with finite transcendence degree over k.

If X is irreducible with function field F, then $\underline{K}_n^{MW}(X) \subseteq K_n^{MW}(F)$ is the intersection of all kernels of residue maps at codimension 1 points of X. Also compare [Col95].

Analogously, this works for other constructions like $\underline{K}_n^M$, $\underline{I}^n$ and $\underline{\bar{I}}^n$, and their respective twists by line bundles.

Definition 2.2.2. Let $\boldsymbol{M}$ be a sheaf of abelian groups on Sm_k in the Nisnevich topology. It is called

(i) $\mathbb{A}^1$-invariant: if $\boldsymbol{M}(U) \xrightarrow{\cong} \boldsymbol{M}(U \times \mathbb{A}^1_k)$
(ii) strongly $\mathbb{A}^1$-invariant: if $H^i_{\mathrm{Nis}}(U, \boldsymbol{M}) \xrightarrow{\cong} H^i_{\mathrm{Nis}}(U \times \mathbb{A}^1_k, \boldsymbol{M})$ for $i = 0, 1$
(iii) strictly $\mathbb{A}^1$-invariant: if $H^i_{\mathrm{Nis}}(U, \boldsymbol{M}) \xrightarrow{\cong} H^i_{\mathrm{Nis}}(U \times \mathbb{A}^1_k, \boldsymbol{M})$ for $i \geq 0$

via the map induced by the projection $U \times \mathbb{A}^1_k \to U$ for all $U \in \mathrm{Sm}_k$.

Theorem 2.2.3. *Every strongly $\mathbb{A}^1$-invariant sheaf of abelian groups is already strictly $\mathbb{A}^1$-invariant.*

Proof. This is [Mor12, Corollary 5.45, Theorem 5.46]. □

Theorem 2.2.4. *The sheaf $\underline{K}_n^{MW}$ is the free strongly $\mathbb{A}^1$-invariant sheaf generated by the pointed scheme $(\mathbb{G}_m)^{\wedge n}$ for $n \geq 1$.*

In particular, $\underline{K}_n^{MW}$ is strictly $\mathbb{A}^1$-invariant.

Proof. This is [Mor12, Theorem 3.37] □

Definition 2.2.5. For any presheaf of groups $\boldsymbol{G}$ on Sm_k denote by $\boldsymbol{G}_{-1}$ the presheaf of groups defined by

$$U \mapsto \ker\left(\boldsymbol{G}(\mathbb{G}_m \times U) \xrightarrow{(\{1\}\times \mathrm{id}_U)^*} \boldsymbol{G}(U)\right)$$

called the contraction of $\boldsymbol{G}$. Iteratively define $\boldsymbol{G}_{-n} = (\boldsymbol{G}_{-n+1})_{-1}$.

Lemma 2.2.6. *If $\boldsymbol{G}$ is a strongly $\mathbb{A}^1$-invariant of abelian groups, then so is $\boldsymbol{G}_{-1}$.*

Proof. This is [Mor12, Lemma 2.32]. □

Remark 2.2.7. The sheaves we are considering are $\underline{K}_n^{MW}$, $\underline{K}_n^{M}$, $\underline{I}^n$ and $\underline{\bar{I}}^n$ which are all $\mathbb{Z}$-graded and strictly $\mathbb{A}^1$-invariant, by [Mor12, Theorem 2.46, Example 3.33, Example 3.34]. This grading is compatible with contractions by [Mor12, Section 2.3] and we analogously define Rost-Schmid complexes for these sheaves, see [Mor12, Chapter 5].

Theorem 2.2.8. *For any essentially smooth k-scheme X and any strongly $\mathbb{A}^1$-invariant sheaf of abelian groups $\boldsymbol{M}$ there are canonical isomorphisms*

$$H^i(C^{RS}(X, \boldsymbol{M})) \cong H^i_{\mathrm{Zar}}(X, \boldsymbol{M}) \cong H^i_{\mathrm{Nis}}(X, \boldsymbol{M}).$$

Proof. This is [Mor12, Corollary 5.43]. □

Remark 2.2.9. As all these notions coincide we will drop decorations and write $H^i(X, \mathbf{M})$. This is in line with Definition 2.1.5.

3. The moduli space $\overline{M_{0,n}}$

In this chapter the base field k is arbitrary. Later in Section 3.2 we will restrict to k being infinite.

3.1. Construction of the space

This section on the moduli space of stable, genus 0, n-marked curves can be extended indefinitely. There is so much one can write and we only need a fraction of the theory, that is why this section contains only the very basics needed for the later parts. The original construction was done in [Knu83a]. This is not the one we are going to use. In [Kee92] the space $\overline{M_{0,n}}$ is described via iterated blow-ups of $(\mathbb{P}^1_k)^{n-3}$ along smooth centers, which leads to a description of $\mathrm{CH}^*(\overline{M_{0,n}})$, see Lemma 3.2.2 and Theorem 3.1.12. More information along with a summary of this construction can be found in [ACG11].

Definition 3.1.1. A stable curve $(C; p_1, \dots, p_n)$ of genus 0 with n marked points is a curve C with closed points $p_1, \dots, p_n \in C$ such that:

(i) C has finitely many irreducible components C_i, which are all isomorphic to $\mathbb{P}^1_k$,
(ii) for $i \neq j$ the intersection $C_i \cap C_j$ is either empty or an ordinary double point,
(iii) the graph of C, which has the irreducible components as vertices and their intersections as edges, is a tree,
(iv) the points $p_1, \dots, p_n \in C$ are distinct,
(v) each marked point p_i lies on exactly one component C_j,
(vi) each component C_i contains at least 3 special points, i.e. markings or intersection points with other components.

An isomorphism of stable curves $f\colon (C; p_1, \dots, p_n) \to (C'; q_1, \dots, q_n)$ is an isomorphism of curves $f\colon C \to C'$ such that $f(p_i) = q_i$ for $i = 1, \dots, n$.

Remark 3.1.2. The first three conditions ensure that the curve C has arithmetic genus 0 and the last condition guarantees that it has trivial automorphism group.

Definition 3.1.3. The moduli problem for $\overline{M_{0,n}}$ is

$$S \mapsto \left\{ \left(\pi \colon \mathcal{C} \to S, (\sigma_i \colon S \to \mathcal{C})_{i=1,\dots,n}\right) \;\middle|\; \begin{array}{l} \pi \text{ is proper and flat}, \sigma_i \text{ are sections of } \pi, \\ \text{every geometric fiber of } \pi \text{ is a stable} \\ \text{genus 0 } n\text{-marked curve, where the} \\ \text{markings are defined by the sections} \end{array} \right\}$$

Notation 3.1.4. Denote by $\overline{M_{0,n}}$ the (fine) moduli space of stable, genus 0, n-marked curves.

When writing $\overline{M_{0,n}}(\mathbb{R})$ we means the set of $\mathbb{R}$-points endowed with the euclidean topology.

Theorem 3.1.5. *Let $n \geq 3$ be an integer, then*

(i) $\overline{M_{0,n}}$ is a smooth projective variety of dimension $n-3$.

(ii) $\overline{M_{0,n}}(\mathbb{R})$ is a connected, compact, smooth manifold of dimension $n-3$.

Proof. See [Knu83a, Theorem 2.7], [Knu83b], [ACG11, Chapter XIV, Theorem 5.1] and [Has03, Theorem 2.1]. □

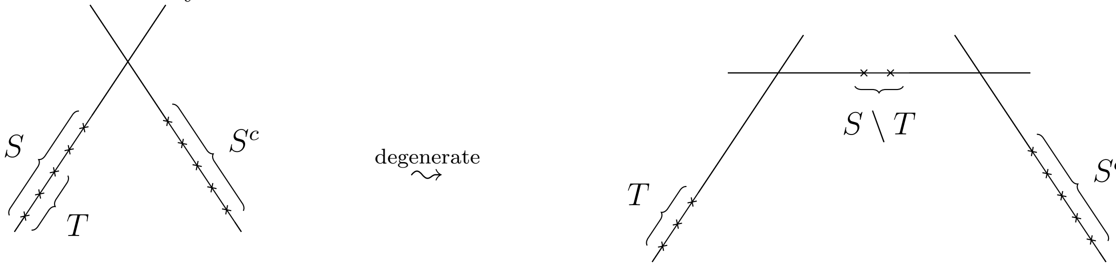
Remark 3.1.6. There are also the moduli spaces $M_{0,n}$ of smooth, genus 0, n-marked curves. These consists of isomorphism classes of $\mathbb{P}^1_k$ with n distinct points on it. As an isomorphism of marked curves respects the order of markings, we can send the first 3 points via Möbius transformation to $\{0, 1, \infty\}$ and use the rest as coordinates. This shows

$$M_{0,n} = \left(\mathbb{P}^1_k \setminus \{0, 1, \infty\}\right)^{n-3} \setminus \Delta^{\text{thick}}_{n-3},$$

where $\Delta^{\text{thick}}_{n-3} = \{(x_1, \dots, x_{n-3}) \in (\mathbb{P}^1_k \setminus \{0, 1, \infty\})^{n-3} \mid \exists i \neq j : x_i = x_j\}$ is the thick diagonal. Note that this isomorphism already depends on the choice of three markings (and their order).

Remark 3.1.7. The manifold $\overline{M_{0,5}}(\mathbb{R})$ is isomorphic to the connected sum of 5 real projective planes [Kol01, Proposition 86], and not orientable.

Remark 3.1.8. The points in $\overline{M_{0,n}}$ corresponding to smooth curves are exactly those stable curves having exactly one irreducible component. To better describe the curves in $\overline{M_{0,n}} \setminus M_{0,n}$ we introduce the following notation. Let $S \subseteq \{1, \dots, n\}$ be a subset such that $2 \leq |S| \leq n-2$. The boundary divisor D^S is the divisor whose generic element is a curve consisting of two components with all markings from S lying on one component and the ones from the complement S^c on the other. Clearly, we have $D^S = D^{S^c}$. The condition on the size of S ensures that this defines a stable curve. Each boundary divisor is smooth and itself isomorphic to a product $D^S \cong \overline{M_{0,|S|+1}} \times \overline{M_{0,|S^c|+1}}$ by treating each component individually and the node as an additional point on both components, see [Kee92, Fact 2]. This recursive structure allows to stratify and study $\overline{M_{0,n}}$ in a very combinatorial way.



generic element of D^S, $T \subseteq S$

generic element of $D^S \cap D^T$

For example we quickly see that for $D^S \cap D^T$ to be non-empty one of the following conditions has to be satisfied $S \subseteq T$, $S \subseteq T^c$, $T \subseteq S$ or $T^c \subseteq S$

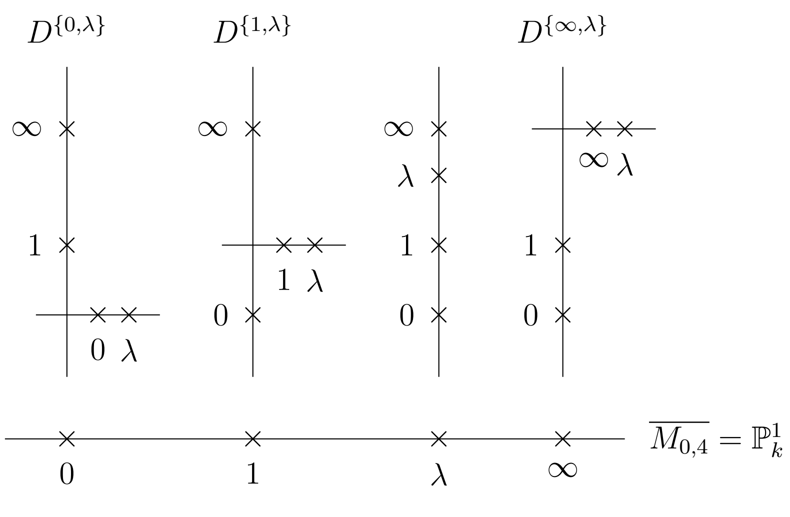
To avoid counting divisors twice, we occasionally assume $|S \cap \{1, 2, 3\}| \leq 1$. There are $2^n - 2n - 2$ subsets having at least 2 elements and at most $n-2$, resulting in each divisor D^S appearing twice. This way we see that there are $2^{n-1} - n - 1$ different boundary divisors in $\overline{M_{0,n}}$.

Example 3.1.9. A stable curve needs at least 3 markings, so the first cases are:

(i) $\overline{M_{0,3}} = M_{0,3} = \{*\}$ is a single point,

(ii) $\overline{M_{0,4}} = \mathbb{P}^1_k$,

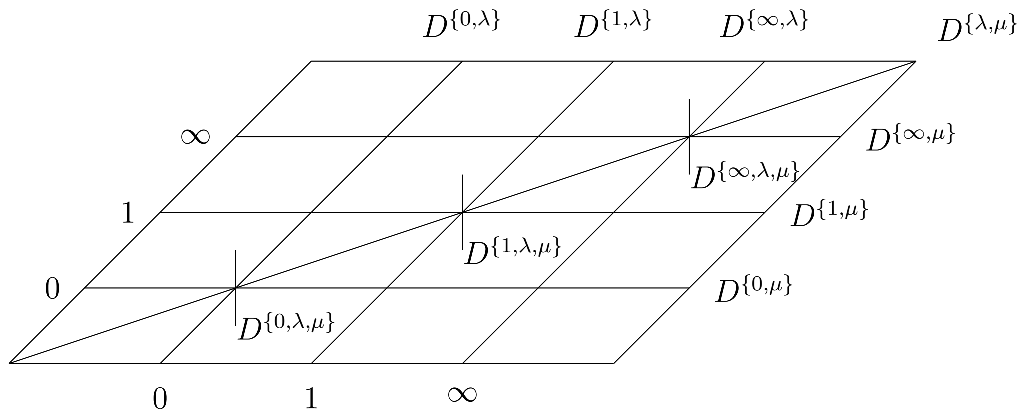
(iii) $\overline{M_{0,5}}$ is a degree 5 del Pezzo surface, i.e. the blow-up of $\mathbb{P}^2_k$ in 4 general points or equivalently, the blow-up of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ in the points $(0,0)$, $(1,1)$ and (∞, ∞).



To see that $\overline{M_{0,4}} = \mathbb{P}^1_k$, send the first three points via a Möbius transformation to $\{0, 1, \infty\}$ and use the third marking as a parameter. Calling the free parameter λ gives the previous picture.

The bottom $\mathbb{P}^1_k$ describes the moduli space and the stable curves above the points $\{0, 1, \lambda, \infty\}$ are the curves corresponding to the point in $\overline{M_{0,4}}$. Note that each boundary divisor is a single point, as each irreducible component contains exactly three special points and the curve corresponding to $\lambda \notin \{0, 1, \infty\}$ is a smooth 4-marked curve.

The following figure shows $\overline{M_{0,5}}$ as a blow-up of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with its 10 boundary divisors. The divisors $D^{\{0,\lambda,\mu\}}$, $D^{\{1,\lambda,\mu\}}$, and $D^{\{\infty,0,\lambda,\mu\}}$ are the exceptional divisors of the blow-up.



The picture is not an honest picture of $\overline{M_{0,5}}$. It is an irreducible surface and there is no point living in the intersection of more than two divisors. For example, $D^{\{0,\lambda\}}$, $D^{\{0,\mu\}}$, and $D^{\{0,\lambda,\mu\}}$ do not intersect in a single point, despite the picture suggesting they do.

Remark 3.1.10. In [Knu83a] it is shown that there are well-defined contraction morphisms $\overline{M_{0,n}} \to \overline{M_{0,n-1}}$. These morphism are given on geometric points by forgetting a chosen marking and contracting unstable components, i.e. components with less than three special points. Picking four markings $P = \{p_1, \dots, p_4\}$ and composing these morphisms results in a morphism $c_P\colon \overline{M_{0,n}} \to \overline{M_{0,4}} \cong \mathbb{P}^1$, where the isomorphism $\overline{M_{0,4}} \cong \mathbb{P}^1$ is given by taking cross-ratios, compare previous example.

Picking three markings q_0, q_1, q_∞ and using the others as coordinates results in a morphism $\prod_{p \notin \{q_0,q_1,q_\infty\}} c_{\{q_0,q_1,q_\infty,p\}}\colon \overline{M_{0,n}} \to (\mathbb{P}^1_k)^{n-3}$. This morphism precisely contracts the divisors D^S satisfying $3 \geq |S|$, assuming the convention $|\{q_0, q_1, q_\infty\} \cap S| \leq 1$.

Remark 3.1.11. We will use Keel's construction of $\overline{M_{0,n}}$, which we very briefly recall. Even though [Kee92] assumes the base field to be algebraically closed, this assumption is not necessary. The construction already works over $\mathrm{Spec}(\mathbb{Z})$ (e.g. compare [Has03]).

By universality of the moduli space there exists a morphism from Keel's construction to the moduli space, which becomes an isomorphism after base change to the algebraic closure [Kee92] and hence has to be an isomorphism over every field, by fpqc descent.

Consider $S \subseteq \{1, \dots, n\}$ with $|S \cap \{1, 2, 3\}| \leq 1$ and write $D_n^S \in \overline{M_{0,n}}$ to avoid confusion where divisors live. The space $\overline{M_{0,n}}$ can be inductively constructed, by blow-ups $B^{(j)}$, in the following way. Note that this again depends on a choice (and order) of three special markings, here $\{1, 2, 3\}$.

$$\begin{tikzcd}
\overline{M_{0,\{1,\dots,n+1\}}} \arrow[r] \arrow[rrrr, bend left, "\eta"] & B^{(k)} \arrow[r, "\pi^{(k-1)}"] & \dots \arrow[r] & B^{(2)} \arrow[r, "\pi^{(1)}"] & B^{(1)} = \overline{M_{0,\{1,\dots,n\}}} \times \overline{M_{0,\{1,2,3,n+1\}}} \arrow[d, "\mathrm{pr}_1"] \\
& & & & \overline{M_{0,\{1,\dots,n\}}} \arrow[llll, dashed, "\sigma_i"] \arrow[ul, dashed, "\sigma_i^{(2)}"] \arrow[u, dashed, bend left, "\sigma_i^{(1)}"]
\end{tikzcd}$$

The map η forgets the last marking for the first factor, and all but the first three and the last for the second factor, contracting components with less than three special points if necessary. The map σ_i replaces the i-th marking by a new component with markings i and $n+1$. Define $\sigma_i^{(1)} = \eta \circ \sigma_i$. This is a section for $\mathrm{pr}_1 \colon B^{(1)} \to \overline{M_{0,\{1,\dots,n\}}}$. Let $\pi^{(1)} \colon B^{(2)} \to B^{(1)}$ be the blow-up of $B^{(1)}$ along all $\sigma_i^{(1)}(D_n^S)$ for some $i \in S$ and $|S^c| = 2$. The map η factors through $\pi^{(1)}$. Use this to define $\sigma_i^{(2)}$ and repeat the process by blowing up along $\sigma_i^{(2)}(D_n^S)$ for some $i \in S$ and $|S^c| = 3$. Repeat this until $k = n - 2$.

To prove that this construction works one needs to prove:

(i) the definition of $\sigma_i^{(k)}(D_n^S)$ is independent of $i \in S$, and smooth of codimension 2,

(ii) η factors through all the blow-ups $\pi^{(k)}$ (this is the hard part),

(iii) $\eta^{(k)} \colon \overline{M_{0,\{1,\dots,n+1\}}} \to B^{(k)}$ contracts only $D_{n+1}^{S \cup \{n+1\}}$ for $|S^c| \geq k + 1$.

This describes $\overline{M_{0,n}}$ as an iterated blow-up of $(\mathbb{P}^1_k)^{n-3}$ along smooth codimension 2 subschemes by factoring the contraction map $\prod_{p \notin \{1,2,3\}} c_{\{1,2,3,p\}} \colon \overline{M_{0,n}} \to (\mathbb{P}^1_k)^{n-3}$.

Using this description, Keel computed the Chow ring of $\overline{M_{0,n}}$.

Theorem 3.1.12. *The Chow ring* $\mathrm{CH}^*(\overline{M_{0,n}})$ *has the following presentation in terms of boundary divisors:*

$$\mathrm{CH}^*(\overline{M_{0,n}}) = \mathbb{Z}[D^S | S \subseteq \{1, \dots, n\},\ 2 \leq |S| \leq n-2] \,/\, \textit{relations}$$

with each D^S *in degree* 1 *and subject to the following relations:*

(i) $D^S = D^{S^c}$,

(ii) $\sum\limits_{\substack{i,j \in S \\ k,l \notin S}} D^S = \sum\limits_{\substack{i,k \in S \\ j,l \notin S}} D^S = \sum\limits_{\substack{i,l \in S \\ j,k \notin S}} D^S$,

(iii) $D^S D^T = 0$ *unless* $S \subseteq T$, $T \subseteq S$, $S \subseteq T^C$ *or* $T^C \subseteq S$.

Proof. This is [Kee92, Theorem 4.1]. □

Remark 3.1.13. These are exactly the expected relations. The first and third are clear from the geometry of the D^S, as the third captures when exactly two different divisors intersect. The second one comes from the pullbacks along the maps $\overline{M_{0,n}} \to \overline{M_{0,4}} \cong \mathbb{P}^1_k$ forgetting all labels except i, j, k, l and the fact that any two points on $\mathbb{P}^1_k$ are rationally equivalent.

3.2. Canonical bundles and curves

The goal of this section is twofold. We would like to have a description of the canonical divisor $K_{\overline{M_{0,n}}} \in \mathrm{CH}^1(\overline{M_{0,n}})$ in the given presentation using boundary divisors, because this twist allows us to move from homology to cohomology, see Remark 2.1.6. The second one is that we will use rational curves in $\overline{M_{0,n}}$, meaning one dimensional subschemes, for the computation later on, see Remark 2.1.15. Writing down such curves is not a problem as we have an abundance of rational curves in $(\mathbb{P}^1_k)^{n-3}$ and can take their strict transforms. Due to the concrete description of the blow-up this is doable as there are no higher tangency condition to consider.

In this section we will assume that the field k we are working over is infinite. We could weaken this by requiring it to have enough elements, but decided to not do that.

Lemma 3.2.1. *Let* $C \cong \mathbb{P}^1_k$ *be a smooth rational curve in a smooth* k*-scheme* X*, then*

$$\det(\mathcal{N}_{C/X}) = (\omega_{X/k})_{|C} \in \mathrm{Pic}(\mathbb{P}^1_k)/2.$$

Proof. Follows immediately from the adjunction formula and $\omega_{\mathbb{P}^1_k} = 0 \in \mathrm{Pic}(\mathbb{P}^1_k)/2$. □

By a "smooth" point of D^S we mean a point in $D^S \setminus \left(\bigcup_{T \neq S} D^T\right)$, i.e. a smooth point of $\bigcup_T D^T$.

Lemma 3.2.2. *Let* $S \subseteq \{1, \ldots, n\}$ *with* $2 \leq |S| \leq n-2$*. Then there exists a rational curve* C *in* $\overline{M_{0,n}}$ *such that* $|C \cap D^S| = 1$ *and* $C \cap D^R \cap D^T = \emptyset$ *for all* R, T *with* $D^R \neq D^T$*.*

Moreover, the point $C \cap D^S$ *can be chosen to be any smooth point of* D^S *except for finitely many exceptions.*

Proof. The statement is obvious for $\overline{M_{0,4}} \cong \mathbb{P}^1_k = C$, so assume $n \geq 5$ from now on. We will construct $C \subseteq \overline{M_{0,n}}$ as the strict transform of a curve in $(\mathbb{P}^1)^{n-3}$. More precisely, we will use the strict transform of a curve in $(\mathbb{P}^1)^{n-2}$ and then forget the marking q_∞. Consider $\mathbb{P}^1_k$ with points $[0:1]=0$, $[1:1]=1$ and $[1:0]=\infty$. To simplify the notation consider the markings to be $\{q_0, q_1, q_\infty, p_1, \ldots, p_{n-2}\}$. At least one of S and S^C contains three elements, so by replacing S by S^C if necessary and reordering the other labels, we can assume $S = \{p_1, \ldots, p_r\}$ without changing the actual divisor D^S.

Consider the following rational curve D in $(\mathbb{P}^1_k)^{n-2}$:

$$[t:s] \mapsto \Big([t+\beta_1 s : s], \ldots, [t+\beta_r s : s], [\gamma_{r+1}:1], \ldots, [\gamma_{n-2}:1]\Big) \in (\mathbb{P}^1_k)^{n-2},$$

for pairwise distinct $\beta_i \in k$ and pairwise distinct $\gamma_j \in k \setminus \{0,1\}$ such that the collection consisting of

(i) $\gamma_j - \beta_i \in k$ for all (i,j),
(ii) $-\beta_i$ for all i,
(iii) $1-\beta_i$ for all i,

does not contain any duplicates. By assumption, k is infinite. Therefore such a collection of β_i and γ_j in k always exists as each condition only imposes finitely many exceptions.

Define C as the strict transform of D under the blow-up $\overline{M_{0,n+1}} \to (\mathbb{P}^1_k)^{n-2}$. To check the desired properties, we need to understand the points where D meets $(\mathbb{P}^1_k)^{n-2} \setminus M_{0,n+1}$, i.e. where a coordinate is 0, 1 or ∞, or two coordinates coincide. The conditions above correspond to the following cases:

(i) $[t+\beta_i s : s] = [\gamma_j : 1]$,
(ii) $[t+\beta_i s : s] = [0:1] = 0$,
(iii) $[t+\beta_i s : s] = [1:1] = 1$.

The only other point where coordinates agree or are equal to 0, 1 or ∞ is $[t:s]=[1:0]$. There the first r coordinates are all equal to $\infty = [1:0]$. With the exception of $[t:s]=[1:0]$, the assumptions guarantee that if one coordinate is in $\{0,1,\infty\}$ no other is and all coordinates are distinct, and that at most two coordinates agree at a time.

For all points except $[t:s]=[1:0]$ the point of D meeting $(\mathbb{P}^1_k)^{n-2} \setminus M_{0,n+1}$ lies away from the blow-up centers, i.e. inside the locus where the blow-up $\overline{M_{0,n+1}} \to (\mathbb{P}^1_k)^{n-2}$ is an isomorphism. Therefore the curve C meets the following divisors in a smooth point:

(i) $D^{\{p_i,p_j\}}$ over $[t:s] = [\gamma_j - \beta_i : 1]$,
(ii) $D^{\{q_0,p_i\}}$ over $[t:s] = [-\beta_i : 1]$,
(iii) $D^{\{q_1,p_i\}}$ over $[t:s] = [1-\beta_i : 1]$.

The last thing we need to see is which stable curve lies over the point $[t:s]=[1:0]$. To determine the curve we are going to use the following two observations. Firstly, if the images of a point under the contractions $c_{\{q_0,q_1,q_\infty,p\}} \colon \overline{M_{0,n}} \to (M_{0,4}) \cong \mathbb{P}^1_k$ for $p \notin \{q_0, q_1, q_\infty\}$ are all distinct and not 0, 1 or ∞, the point corresponds to a smooth stable curve, i.e. an element of $M_{0,n}$. Secondly, studying one side of a node can be done by considering the following map

$$\prod_{p\in S\setminus\{s_1,s_2\}} c_{\{s_1,s_2,s^c,p\}}\colon \overline{M_{0,n}} \supseteq D^S \cong \overline{M_{|S|+1}} \times \overline{M_{0,|S^c|+1}} \to \overline{M_{|S|+1}} \to (\mathbb{P}^1_k)^{|S|-2}$$

for fixed $s_1, s_2 \in S$ and $s^c \in S^c$.

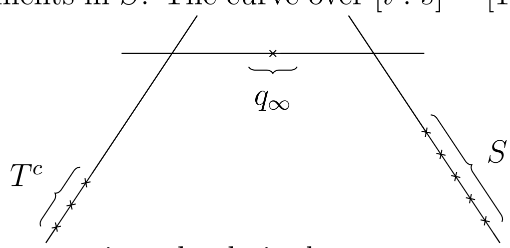
By the projection to $(\mathbb{P}^1_k)^{n-2}$, we know the curve meets the divisor $D^{\{q_\infty,p_1,\dots,p_r\}}$. This means that there exists a node separating the labels $T = \{q_\infty, p_1, \dots, p_r\} = S \cup \{q_\infty\}$ and $T^c = \{q_0, q_1, p_{r+1}, \dots, p_{n-2}\}$.

We start with the easier case of showing that there is no further node on the side of $T^c = \{q_0, q_1, p_{r+1}, \dots, p_{n-2}\}$. Consider the contraction map given by the product of the maps forgetting all markings except q_∞, q_0, q_1 and one of $p_{r+1}, \dots, p_{n-2}$, i.e. the product of $c_{\{q_\infty,q_0,q_1,p_i\}}$ for $i \in \{r+1, \dots, n-2\}$. Each of these maps is the cross-ratio of the four points. This results in $[\gamma_i : 1]$ for p_i, which is constant in the family, distinct and not in $\{0, 1, \infty\}$. Therefore there is no further node on the T^c side.

Next, we will treat the T side. Similarly, we consider the contraction map given by the product of the maps $c_{\{q_\infty,q_0,p_i,p_j\}}$ forgetting all markings except q_∞, q_0, p_i, p_j. Computing the cross-ratio over the point $[t : s] = [1 : 0]$ gives 1. The cross-ratio we chose for $\overline{M_{0,4}} \cong \mathbb{P}^1$ here sends the first marking to infinity, the second to 0 and the third to 1, i.e. $\frac{(x_4-x_2)(x_3-x_1)}{(x_4-x_1)(x_3-x_2)}$ thinking of it as a function in x_4. This shows that there is a node separating q_∞ from all other $p_1, \dots, p_r$. Considering the contraction maps $c_{\{q_\infty,p_1,p_2,p_i\}}$, give the cross-ratios

$$\frac{(t+\beta_i)-(t+\beta_1)}{(t+\beta_3)-(t+\beta_1)} = \frac{\beta_i-\beta_1}{\beta_3-\beta_1}.$$

As all of those are constant in the family, distinct and not in $\{0, 1, \infty\}$. There is no further node separating elements in S. The curve over $[t : s] = [1 : 0]$ therefore looks like



and forgetting the marking q_∞ gives the desired curve. □

Remark 3.2.3. The above construction allows one to construct a curve through any given smooth point of D^S, except finitely many. The finitely many exceptions come from the condition that $\gamma_j - \beta_i \in k$ are distinct. If this condition is omitted the construction above gives a curve through any given smooth point in D^S but meeting some exceptional divisor elsewhere.

The marking q_∞ might look special in the construction, but it is not. The only reason we do it this way is to simplify the cross-ratios appearing. One could do the same argument for any intersection of the markings $p_1, \dots, p_r$ which is not one of $0, 1, \infty$. Temporarily adding this point as a marking in the family, to see which curve lies over the intersection point, works too.

The following is [ACG11, Proposition 7.5]. As we will need part of the proof, we will recall it here.

Proposition 3.2.4. *Let $n \geq 4$ and fix distinct elements $i, j, k \in \{1, \dots, n\}$. A basis of* $\mathrm{Pic}(\overline{M_{0,n}})$ *is then given by*

$$D^S \text{ with } \qquad (i \in S \quad \text{and} \quad 2 \leq |S| \leq n-3) \quad \text{or} \quad S = \{j, k\}.$$

Proof. Consider D^S with $i \notin S$ and $|S| \geq 3$, then $D^{S^c} = D^S$ satisfies the first condition for S^c. Therefore, the boundary divisors not satisfying the conditions above are exactly the divisors $D^{\{s,t\}}$ with $i \notin \{s,t\} \neq \{j,k\}$. The following two relations, obtained from the presentation of $\mathrm{CH}^*(\overline{M_{0,n}})$ in Theorem 3.1.12, can be used to rewrite $D^{\{s,t\}}$.

$$D^{\{s,t\}} + D^{\{i,k\}} + \sum_{\substack{i,k \in S \\ s,t \notin S \\ |S|,|S^c| \geq 3}} D^S = D^{\{i,t\}} + D^{\{s,k\}} + \sum_{\substack{i,t \in S \\ s,k \notin S \\ |S|,|S^c| \geq 3}} D^S$$

$$D^{\{s,k\}} + D^{\{i,j\}} + \sum_{\substack{i,j \in S \\ s,k \notin S \\ |S|,|S^c| \geq 3}} D^S = D^{\{i,s\}} + D^{\{j,k\}} + \sum_{\substack{i,s \in S \\ j,k \notin S \\ |S|,|S^c| \geq 3}} D^S.$$

This immediately shows that this collection is a generating set for $\mathrm{Pic}(\overline{M_{0,n}})$.

For a proof of their linear independence see [ACG11, Proposition 7.5]. □

Remark 3.2.5. Rewriting $D^{\{s,t\}}$ in the above way results in an expression that is not symmetric in s, t and in j, k. Symmetry can easily be achieved.

$$\begin{aligned}
D^{\{s,t\}} &= D^{\{i,t\}} + D^{\{i,s\}} + D^{\{j,k\}} - D^{\{i,k\}} - D^{\{i,j\}} \\
&\quad + \sum_{\substack{i,s\in S\\ j,k\notin S\\ |S|,|S^c|\geq 3}} D^S + \sum_{\substack{i,t\in S\\ s,k\notin S\\ |S|,|S^c|\geq 3}} D^S - \sum_{\substack{i,k\in S\\ s,t\notin S\\ |S|,|S^c|\geq 3}} D^S - \sum_{\substack{i,j\in S\\ s,k\notin S\\ |S|,|S^c|\geq 3}} D^S \\
&= D^{\{i,t\}} + D^{\{i,s\}} + D^{\{j,k\}} - D^{\{i,k\}} - D^{\{i,j\}} \\
&\quad + \sum_{\substack{i,s,t\in S\\ j,k\notin S\\ |S^c|\geq 3}} D^S + \sum_{\substack{i,s\in S\\ j,k,t\notin S\\ |S|\geq 3}} D^S + \sum_{\substack{i,t\in S\\ j,k,s\notin S\\ |S|\geq 3}} D^S - \sum_{\substack{i,j,k\in S\\ s,t\notin S\\ |S^c|\geq 3}} D^S - \sum_{\substack{i,k\in S\\ j,s,t\notin S\\ |S|\geq 3}} D^S - \sum_{\substack{i,j\in S\\ k,s,t\notin S\\ |S|\geq 3}} D^S
\end{aligned}$$

Start by splitting the first sum into parts containing t and not containing t. Similarly, split the the third sum into into parts containing j and not containing j. Removing summands that appear both in the second and fourth sum, gives the remaining summands three and six.

Proposition 3.2.6. *Fix distinct elements $i, j, k \in \{1, \dots, n\}$ and therefore, a basis of* $\mathrm{Pic}(\overline{M_{0,n}})$. *The canonical divisor $K_{\overline{M_{0,n}}}$ can be written in that basis as $\sum_S c_s D^S$ with*

$$c_{\{j,k\}} = 2 - n, \qquad c_{\{i,j\}} = c_{\{i,k\}} = n - 4, \qquad c_{\{i,s\}} = -2 \text{ with } s \in \{1, \dots, n\} \setminus \{i, j, k\},$$

$$i \in S,\ 3 \leq |S| \leq n - 3: \qquad c_S = \begin{cases} n - 2 - |S|, & |\{j, k\} \cap S| \geq 1 \\ -|S|, & |\{j, k\} \cap S| = 0 \end{cases}$$

In particular, writing every divisor as D^S for some S such that $|S \cap \{i, j, k\}| \leq 1$ and replacing c_S by c_{S^c} accordingly, we have $c_S \equiv |S| \in \mathbb{Z}/2\mathbb{Z}$.

Proof. By [Pan97, Proposition 1] we have $K_{\overline{M_{0,n}}} = \sum_j^{\lfloor \frac{n}{2} \rfloor} \left(\frac{j(n-j)}{n-1} - 2 \right) \sum_{|S|=j} D^S$. To express this element in the basis coming from Proposition 3.2.4 we need to rewrite some D^S with $|S| = 2$ and $i \notin S \neq \{j, k\}$:

$$\begin{aligned}
c_{\{j,k\}} &= \left(\frac{2(n-2)}{n-1} - 2 \right) \left(2(n-3) + \binom{n-3}{2} + 1 \right) = 2 - n \\
c_{\{i,j\}} = c_{\{i,k\}} &= \left(\frac{2(n-2)}{n-1} - 2 \right) \left(-(n-3) - \binom{n-3}{2} + 1 \right) = n - 4 \\
c_{\{i,s\}} &= \left(\frac{2(n-2)}{n-1} - 2 \right) (2 + n - 4 + 1) = -2
\end{aligned}$$

To ease notation set $d_S = \left(\frac{|S|(n-|S|)}{n-1} - 2\right)$. For $i \in S$ and $3 \leq |S| \leq \left\lfloor \frac{n}{2} \right\rfloor$ we have

$$
\begin{aligned}
|\{j,k\} \cap S| = 2 : c_S &= d_S + \left(\frac{2(n-2)}{n-1} - 2\right)\left(-\binom{|S^c|}{2}\right) = n-2-|S| \\
|\{j,k\} \cap S| = 1 : c_S &= d_S + \left(\frac{2(n-2)}{n-1} - 2\right)\left(-\binom{|S^c|-1}{2} - (|S^c|-1)\right) = n-2-|S| \\
|\{j,k\} \cap S| = 0 : c_S &= d_S + \left(\frac{2(n-2)}{n-1} - 2\right)\left(\binom{|S|-1}{2} + (|S|-1)|S^c|\right) = -|S|
\end{aligned}
$$

For $i \in S$ and $\left\lfloor \frac{n}{2} \right\rfloor < |S| \leq n-3$ we get the same results, because $d_S = d_{S^c}$ and the counting is the same. $\square$

Remark 3.2.7. Consider the markings $\{1,\ldots,n\}$ and fix $(i,j,k) = (3,1,2)$. We perform a quick check in low dimensions:

$$
\begin{aligned}
K_{\overline{M_{0,4}}} &= -2D^{\{1,2\}} = -2D^{\{3,4\}}, \\
K_{\overline{M_{0,5}}} &= -3D^{\{1,2\}} + D^{\{1,3\}} + D^{\{2,3\}} - 2D^{\{3,4\}} - 2D^{\{3,5\}} \\
&= -3D^{\{3,4,5\}} + D^{\{2,4,5\}} + D^{\{1,4,5\}} - 2D^{\{3,4\}} - 2D^{\{3,5\}}.
\end{aligned}
$$

In the $n = 4$ case one needs to be careful to not count divisors too many times, as $D^{\{1,2\}} = D^{\{3,4\}}$. This is exactly what it should be, because $\overline{M_{0,4}} \cong \mathbb{P}^1_k$.

In the $n = 5$ case, the summands of $D^{\{3,4,5\}} + D^{\{2,4,5\}} + D^{\{1,4,5\}}$ are the three exceptional divisors of the blown up points. The $-4D^{\{3,4,5\}} - 2D^{\{3,4\}} - 2D^{\{3,5\}}$ comes from the pullback of the canonical divisor on $\mathbb{P}^1_k \times \mathbb{P}^1_k$. The coefficient -4 for exceptional divisor $D^{\{3,4,5\}}$ appears, because the divisors $\{\infty\} \times \mathbb{P}^1_k$ and $\mathbb{P}^1_k \times \{\infty\}$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ meet the corresponding blow-up center and each appear with a coefficient -2 in $K_{\mathbb{P}^1 \times \mathbb{P}^1}$.

4. Cellular structures and cellular homology

The following definition comes from [MS23]. Despite following their ideas, the cellular homology defined below differs from their $\mathbb{A}^1$-cellular homology. In [MS23], the differential is defined using cofiber sequences $\Omega_i \to \Omega_{i+1} \to \Omega_{i+1}/\Omega_i$ and the motivic homotopy purity theorem to express the latter in terms of spheres and $\Omega_{i+1} \setminus \Omega_i$, [MS23, Formula (2.2)]. Instead, we are using a description via localization sequences, using the knowledge of the residue maps to our advantage. Both variants can be used to compute $H^*_{\mathrm{Nis}}(-, \boldsymbol{M})$ for strictly $\mathbb{A}^1$-invariant sheaves. The main benefit of our definition is the computability as demonstrated in Chapter 5. We will directly compute the differentials for multiple cellular structures on $\mathbb{P}^n_k$. In [MS23], the homology of $\mathbb{P}^n_k$ is computed by considering $\mathbb{A}^{n+1}_k \setminus \{0\}$ as a $\mathbb{G}_m$-torsor on $\mathbb{P}^n_k$, computing the differential for $\mathbb{A}^{n+1}_k \setminus \{0\}$ and identifying the $\mathbb{G}_m$-action, [MS23, Section 2.5].

In this chapter the base field k is of characteristic not 2 and perfect.

4.1. Cellular structures

Definition 4.1.1. Let $X \in \mathrm{Sm}_k$ be an irreducible smooth k-scheme of dimension n. A strict cellular structure on X is an increasing filtration

$$\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_n = X,$$

with open subschemes Ω_i such that the reduced closed subschemes $X_i := \Omega_i \setminus \Omega_{i-1}$ are k-smooth and each irreducible component of X_i is isomorphic to $\mathbb{A}^{n-i}_k$.

Definition 4.1.2. A smooth k-scheme $X \in \mathrm{Sm}_k$ is called cohomologically trivial if $H^n_{\mathrm{Nis}}(X, \boldsymbol{M}) = 0$ for every strictly $\mathbb{A}^1$-invariant sheaf of abelian groups $\boldsymbol{M}$ and every $n \geq 1$.

Definition 4.1.3. Let $X \in \mathrm{Sm}_k$ be an irreducible smooth k-scheme of dimension n. A cellular structure on X is an increasing filtration

$$\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_s = X,$$

with open subschemes Ω_i such that the reduced closed subschemes $X_i := \Omega_i \setminus \Omega_{i-1}$ are k-smooth, affine, everywhere of codimension i and cohomologically trivial.

Remark 4.1.4. The idea is that all arguments for strict cellular structures that only use the cohomological vanishing of the 'cells' $\mathbb{A}_k^m$ will also work for cohomologically trivial cells. It might still be complicated to compute $H^0_{\mathrm{Nis}}(-, \boldsymbol{M})$ for the cells.

Lemma 4.1.5. *Let X be a cohomologically trivial smooth affine scheme over a perfect field, then every vector bundle on X is trivial.*

Proof. This is [MS23, Lemma 2.13]. □

Definition 4.1.6. Let $\mathcal{L}$ a line bundle on a scheme X. Then $\mathcal{L}$ is called orientable, if there exists an isomorphism $\mathcal{N} \otimes \mathcal{N} \cong \mathcal{L}$ for some line bundle $\mathcal{N}$ on X.

A fixed choice of such an isomorphism is called an orientation for $\mathcal{L}$.

Remark 4.1.7. The previous lemma shows that every vector bundle over a cohomologically trivial smooth affine scheme is orientable.

Lemma 4.1.8. *Let $U = \mathbb{A}_k^n \setminus (\bigcup_{i=1}^r V_i)$ be the complement of affine hyperplanes V_i, then U is cohomologically trivial.*

Proof. Proceed by induction on the number of hyperplanes r.

$\boldsymbol{r = 0}$**:** Strict $\mathbb{A}^1$-invariance immediately shows that $\mathbb{A}_k^n$ is cohomologically trivial.

$\boldsymbol{r = 1}$**:** Consider the localization pair $\mathbb{A}_k^{n-1} \cong V_1 \hookrightarrow \mathbb{A}_k^n$ with complement $U = \mathbb{A}_k^n \setminus V_1$. As V_1 is of codimension 1 the localization sequence from Remark 2.1.7 reads

$$H^q_{\mathrm{Nis}}(\mathbb{A}_k^n, \boldsymbol{M}) \longrightarrow H^q_{\mathrm{Nis}}(U, \boldsymbol{M}) \xrightarrow{\partial} H^q_{\mathrm{Nis}}(V_1, \boldsymbol{M}_{-1}),$$

where both outer terms vanish by strict $\mathbb{A}^1$-invariance of $\boldsymbol{M}$ and $\boldsymbol{M}_{-1}$. So the middle term vanishes, showing that $U = \mathbb{A}_k^n \setminus V_1$ is cohomologically trivial.

$\boldsymbol{r \rightsquigarrow r+1}$**:** Consider the localization pair $V_{r+1} \setminus (\bigcup_{i=1}^r V_i) \hookrightarrow \mathbb{A}_k^n \setminus (\bigcup_{i=1}^r V_i)$ with complement $U = \mathbb{A}_k^n \setminus \left(\bigcup_{i=1}^{r+1} V_i\right)$. Writing $V_{r+1} \setminus (\bigcup_{i=1}^r V_i) = V_{r+1} \setminus (\bigcup_{i=1}^r V_i \cap V_{r+1})$ and using that $V_i \cap V_{r+1}$ is either empty or isomorphic to $\mathbb{A}_k^{n-2}$, shows that this is a complement of at most r hyperplanes in $V_{r+1} \cong \mathbb{A}_k^{n-1}$ and therefore, cohomologically trivial by induction hypotheses. The localization sequence reads

$$H^q_{\mathrm{Nis}}\left(\mathbb{A}_k^n \setminus \left(\bigcup_{i=1}^r V_i\right), \boldsymbol{M}\right) \longrightarrow H^q_{\mathrm{Nis}}\left(U, \boldsymbol{M}\right) \xrightarrow{\partial} H^q_{\mathrm{Nis}}\left(V_{r+1} \setminus \left(\bigcup_{i=1}^r V_i\right), \boldsymbol{M}_{-1}\right),$$

where both ends vanish by induction hypothesis, showing that $U = \mathbb{A}_k^n \setminus \left(\bigcup_{i=1}^{r+1} V_i\right)$ is cohomologically trivial.

This finishes the proof. $\square$

Lemma 4.1.9. *Let $U = \mathbb{A}_k^n \setminus (\bigcup_{i=1}^r V_i)$ be the complement of affine hyperplanes V_i, then*

$$H^0_{\mathrm{Nis}}(U, K_q^{MW}) \cong \bigoplus_{i=0}^n \left(K_{q-i}^{MW}(k)\right)^{m_i},$$

where $m_0 = 1$ and $m_i = \left|\left\{J \subseteq \{1, \dots, r\} \mid |J| = i,\ \bigcap_{j \in J} V_j \neq \emptyset\right\}\right|$ is the number of non-empty intersections of i hyperplanes.

The isomorphism, $\bigoplus_{i=0}^n \left(K_{q-i}^{MW}(k)\right)^{m_i} \to H^0_{\mathrm{Nis}}(U, K_q^{MW}) \subseteq K_q^{MW}(k(t_1, \dots, t_r))$, is given by picking defining equations $\{f_i = 0\} = V_i$ and multiplying a summand $K^{MW}_{q-|J|}(k)$ corresponding to $J = \{j_1 < \dots < j_{|J|}\} \subseteq \{1, \dots, r\}$ by $[f_{j_{|J|}}] \dots [f_{j_1}]$.

Proof. Proceed by induction on r.

$\boldsymbol{r = 0}$**:** Strict $\mathbb{A}^1$-invariance of K_q^{MW} shows that $H^0_{\mathrm{Nis}}(\mathbb{A}_k^n, K_q^{MW}) \cong K_q^{MW}(k)$.

$\boldsymbol{r = 1}$**:** Consider the localization pair $\mathbb{A}_k^{n-1} \cong V_1 \hookrightarrow \mathbb{A}_k^n$ with complement $U = \mathbb{A}_k^n \setminus V_1$. Then the localization sequence reads

$$\begin{tikzcd} 0 \arrow[r] & H^0_{\mathrm{Nis}}(\mathbb{A}_k^n, K_q^{MW}) \arrow[r] & H^0_{\mathrm{Nis}}(U, K_q^{MW}) \arrow[r, "\partial"] & H^0_{\mathrm{Nis}}(V_1, K_{q-1}^{MW}) \arrow[l, bend left, dashed] \arrow[r] & 0, \end{tikzcd}$$

where the right zero comes from the cohomological triviality $H^1_{\mathrm{Nis}}(\mathbb{A}_k^n, K_q^{MW}) = 0$, by Lemma 4.1.8. The second term is isomorphic to $K_q^{MW}(k)$ and the fourth term to $K_{q-1}^{MW}(k)$. Let $f \in k[x_1, \dots, x_n]$ be a linear polynomial defining V_1, then a splitting of the sequence is given by multiplying with the symbol $[f]$, which shows the claim.

$\boldsymbol{r \rightsquigarrow r+1}$**:** Consider the localization pair $V_{r+1} \setminus (\bigcup_{i=1}^r V_i) \hookrightarrow \mathbb{A}_k^n \setminus (\bigcup_{i=1}^r V_i)$ with complement $U = \mathbb{A}_k^n \setminus \left(\bigcup_{i=1}^{r+1} V_i\right)$. Writing $V_{r+1} \setminus (\bigcup_{i=1}^r V_i) = V_{r+1} \setminus (\bigcup_{i=1}^r V_i \cap V_{r+1})$ and using that $V_i \cap V_{r+1}$ is either empty or isomorphic to $\mathbb{A}_k^{n-2}$, shows that this is a complement of at most r hyperplanes in $V_{r+1} \cong \mathbb{A}_k^{n-1}$. Therefore,

$$H^0_{\mathrm{Nis}}\left(V_{r+1} \setminus \left(\bigcup_{i=1}^r V_i\right), K_{q-1}^{MW}\right) \cong \bigoplus_{i=0}^n \left(K_{q-1-i}^{MW}(k)\right)^{m_i},$$

for $m_0 = 1$ and $m_i = \left|\left\{J \subseteq \{1,\dots,r\} \mid |J| = i,\ V_{r+1} \cap (\bigcap_{j\in J} V_j) \neq \emptyset\right\}\right|$. The localization sequence gives a short exact sequence

$$H^0_{\mathrm{Nis}}\left(\mathbb{A}^n_k \setminus (\bigcup_{i=1}^r V_i), K^{MW}_q\right) \hookrightarrow H^0_{\mathrm{Nis}}\left(U, K^{MW}_q\right) \overset{\partial}{\twoheadrightarrow} H^0_{\mathrm{Nis}}\left(V_{r+1} \setminus (\bigcup_{i=1}^r V_i), K^{MW}_{q-1}\right),$$

and the splitting is again given by multiplying with a defining equation of V_{r+1}. Adding the corresponding exponents $m_i + m_{i-1}$ for $i = 1,\dots,r+1$ gives exactly the stated exponents.

This finishes the proof. □

Example 4.1.10. Consider $\mathbb{A}^2_k \setminus \{xy = 0\}$ with order $V_1 = \{x = 0\}$ and $V_2 = \{y = 0\}$, then

$$H^0_{\mathrm{Nis}}\left(\mathbb{A}^2_k \setminus \{xy = 0\}, K^{MW}_q\right) = K^{MW}_q(k) \oplus [x]K^{MW}_{q-1}(k) \oplus [y]K^{MW}_{q-1}(k) \oplus [y][x]K^{MW}_{q-2}(k),$$

as a subgroup of $K^{MW}_q(k(x,y))$. For the other order, the last summand is $[x][y]K^{MW}_{q-2}(k)$ and thus differs from $[y][x]K^{MW}_{q-2}(k)$ by a factor of ε, i.e. up to a sign on the $K^{MW}_{q-2}(k)$ summand, the two isomorphisms agree.

Remark 4.1.11. Note that the isomorphism above depends not only on the choice of defining equations, but also on the order of hyperplanes. The ordering of hyperplanes is used to get an unordered basis. All of this does not concern the order of basis elements, which we do not care too much about unless we want to express the differential as a matrix.

Another way to get the above isomorphism is to look at the corresponding position in the Rost-Schmid complex for $\mathbb{A}^n_k$ and realize that this is exactly the group one gets if one omits the codimension 1 points corresponding to the r hyperplanes. The different number of summands depending on how the hyperplanes intersect comes from the observation, that $[f][g] \in [f]K^{MW}_1(k) \oplus [g]K^{MW}_1(k)$ if and only if they do not intersect, i.e. $f = g + a$ for $a \in k^\times$. By expanding $[\frac{af}{a}][\frac{-a(f-a)}{-a}]$ one sees $[f][f-a] = [f][-a] + [f-a]\varepsilon[a]$ and the other direction by computing residues at f.

Remark 4.1.12. We need to deal with non-strictly cellular schemes because we do not know a strictly cellular structure for $\overline{M_{0,n}}$, except in very low dimensions.

The basis coming from Proposition 3.2.4 does, in general, not correspond to a strictly cellular structure. This can already be seen for $\overline{M_{0,5}}$. The closure of the five 1-cells has three connected components, but this contradicts the fact that $\mathrm{CH}_0(\overline{M_{0,5}}) = \mathbb{Z}$.

4.2. Cellular homology

As mentioned in the beginning of this section, the cellular homology used here is different from the one used in [MS23].

Lemma 4.2.1. *Let $\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_s = X$ be a cellular structure, then*

$$H_k(\Omega_i, \boldsymbol{M}) = 0, \quad i \leq \dim(X) - k - 1,$$

for any strictly $\mathbb{A}^1$-invariant sheaf $\boldsymbol{M}$.

Proof. Proceed by induction on i.

$\boldsymbol{i = 0:}$ $\Omega_0 = \Omega_0 \setminus \Omega_{-1}$ is cohomologically trivial, so the only non-trivial homology group can appear in degree $k = \dim(\Omega_0) = \dim(X)$.

$\boldsymbol{i - 1 \rightsquigarrow i:}$ Consider the localization pair $\Omega_i \setminus \Omega_{i-1} \hookrightarrow \Omega_i$ with open complement Ω_{i-1}. The localization sequence from Remark 2.1.7 reads

$$H_k(\Omega_i \setminus \Omega_{i-1}, \boldsymbol{M}) \longrightarrow H_k(\Omega_i, \boldsymbol{M}) \longrightarrow H_k(\Omega_{i-1}, \boldsymbol{M}).$$

The left group vanishes by cohomological triviality for $k \neq \dim(X) - i$ and the right one vanishes for $k \leq \dim(X) - i$ by induction. Therefore, the middle term vanishes for $k \leq \dim(X) - i - 1$.

This finishes the proof. □

Lemma 4.2.2. *Let $\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_s = X$ be a cellular structure, then*

$$H_k(\Omega_i, \boldsymbol{M}) \cong H_k(X, \boldsymbol{M}), \quad i \geq \dim(X) - k + 1$$

for any strictly $\mathbb{A}^1$-invariant sheaf $\boldsymbol{M}$.

Proof. Consider the localization pair $\Omega_i \setminus \Omega_{i-1} \hookrightarrow \Omega_i$ with open complement Ω_{i-1}. The localization sequence from Remark 2.1.7 reads

$$H_k(\Omega_i \setminus \Omega_{i-1}, \boldsymbol{M}) \longrightarrow H_k(\Omega_i, \boldsymbol{M}) \longrightarrow H_k(\Omega_{i-1}, \boldsymbol{M}) \longrightarrow H_{k-1}(\Omega_i \setminus \Omega_{i-1}, \boldsymbol{M}).$$

The left term vanishes for $k \neq \dim(X) - i$ and the right term for $k - 1 \neq \dim(X) - i$, both by cohomological triviality. Therefore,

$$H_k(\Omega_{i-1}, \boldsymbol{M}) \cong H_k(\Omega_i, \boldsymbol{M}), \quad k \geq \dim(X) - i + 2,$$

which shows $H_k(\Omega_j, \boldsymbol{M}) \cong H_k(X, \boldsymbol{M})$ for $j \geq \dim(X) - k + 1$. □

Definition 4.2.3. Let $\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_s = X$ be a cellular structure and $\boldsymbol{M}$ a strictly $\mathbb{A}^1$-invariant sheaf. Define the cellular complex

$$C_i^{\mathrm{cell}}(X, \boldsymbol{M}) = H_{\dim(X)-i}(\Omega_i \setminus \Omega_{i-1}, \boldsymbol{M})$$

with differential $d_{\dim(X)-i}$ given by the composition

$$H_{\dim(X)-i}(\Omega_i \setminus \Omega_{i-1}, \boldsymbol{M}) \xrightarrow{\iota_*} H_{\dim(X)-i}(\Omega_i, \boldsymbol{M}) \xrightarrow{\partial} H_{\dim(X)-i-1}(\Omega_{i+1} \setminus \Omega_i, \boldsymbol{M}),$$

coming from the two appropriate localization sequences.

Proposition 4.2.4. *Let $\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \cdots \subseteq \Omega_s = X$ be a cellular structure and $\boldsymbol{M}$ a strictly $\mathbb{A}^1$-invariant sheaf. The cellular complex $C_*^{\mathrm{cell}}(X, \boldsymbol{M})$ is a complex and for all k:*

$$H_k(X, \boldsymbol{M}) \cong H_k(C_*^{\mathrm{cell}}(X, \boldsymbol{M})).$$

Proof. Set $n = \dim(X)$. The two localization sequences fit into the following diagram

$$\begin{tikzcd}[column sep=small]
 & & H_{n-i+1}(\Omega_{i-2}, \boldsymbol{M}) = 0 & & \\
 & H_{n-i+1}(\Omega_{i-1}, \boldsymbol{M}) \arrow[ur] \arrow[dr, "\partial_{n-i+1}"] & & & \\
H_{n-i+1}(\Omega_{i-1} \setminus \Omega_{i-2}, \boldsymbol{M}) \arrow[ur, two heads, "\iota_*"] \arrow[rr, dashed, "d_{n-i+1}"] & & H_{n-i}(\Omega_i \setminus \Omega_{i-1}, \boldsymbol{M}) \arrow[rr, dashed, "d_{n-i}"] \arrow[dr, two heads, "\iota_*"'] & & H_{n-i-1}(\Omega_{i+1} \setminus \Omega_i, \boldsymbol{M}) \\
 & & & H_{n-i}(\Omega_i, \boldsymbol{M}) \arrow[ur, "\partial_{n-i}"'] \arrow[dr] & \\
H_{n-i}(X, \boldsymbol{M}) \arrow[rr, equal, "\text{Lemma 4.2.2}"] & & H_{n-i}(\Omega_{i+1}, \boldsymbol{M}) \arrow[ur, hook, "j^*"] & & H_{n-i}(\Omega_{i-1}, \boldsymbol{M}) = 0 \\
 & 0 = H_{n-i}(\Omega_{i+1} \setminus \Omega_i, \boldsymbol{M}) \arrow[ur] & & &
\end{tikzcd}$$

where the vanishing is due to Lemma 4.2.1 or the dimension of the respective scheme. The cellular complex is in fact a complex because $\iota_* \circ \partial = 0$, as those are two consecutive maps in a localization sequence.

By injectivity of j^*, we have $H_{d-i}(X, \boldsymbol{M}) \cong \mathrm{im}(j^*) = \ker(\partial_{n-i})$ and by surjectivity of ι_*, we have $\iota_*(\ker(d_{n-i})) = \ker(\partial_{n-i})$ and $\mathrm{im}(d_{n-i+1}) = \mathrm{im}(\partial_{n-i+1})$. This means ι_* gives an isomorphism

$$\begin{aligned} H_k(C_*^{\mathrm{cell}}(X, \boldsymbol{M})) &= \ker(d_{n-i}) / \mathrm{im}(d_{n-i+1}) = \ker(d_{n-i}) / \ker(\iota_*) \\ &\cong \iota_*(\ker(d_{n-i})) = \ker(\partial_{n-i}) \cong \mathrm{im}(j^*) \cong H_{d-i}(X, \boldsymbol{M}). \end{aligned}$$

This finishes the proof. □

Remark 4.2.5. This is exactly the dual argument to the standard proof that cellular homology computes singular homology, as for example found in [Hat02, Theorem 2.35].

5. Computations

In this chapter the base field k is of characteristic not 2, infinite and perfect.

5.1. Computations for projective spaces

Example 5.1.1. To start computations there are quite a few choices we need to fix. Start by writing $\mathbb{P}^1_k = \mathrm{Proj}(k[x,y])$ with $\infty = [1:0]$ and complement $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$. Consider the standard (strict) cellular structure given by $\emptyset \subseteq \mathbb{A}^1_k \subseteq \mathbb{P}^1_k$, where the two differences are $\mathbb{A}^1_k$ and $\{\infty\}$. Denote the localization pair by $i^\infty \colon \{\infty\} \hookrightarrow \mathbb{P}^1_k \hookleftarrow \mathbb{A}^1_k : j$, and the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(d)$ by $\mathcal{L}$. Moreover, set $U_0 = \mathrm{Spec}(k[y])$, the open complement of 0 and $U_\infty = \mathrm{Spec}(k[x])$, the open complement of ∞. The gluing map on $\mathbb{P}^1_k$ is given by $U_0 \cap U_\infty \to U_\infty \cap U_0, x \mapsto y^{-1}$ and the one on $\mathcal{L}$ is given by $\mathcal{L}|_{U_\infty} = k[x]\cdot u$, $\mathcal{L}|_{U_0} = k[y]\cdot v$ gluing $u \mapsto y^{-d}v$.

The differential we want to compute is

$$H_1(\mathbb{A}^1_k, K^{MW}_q) \cong K^{MW}_{q+1}(k) \xrightarrow{d_1} K^{MW}_q(k) \cong H_0(\mathrm{Spec}(k), K^{MW}_q).$$

Picking orientations in 0 and ∞ gives $\mathcal{L}|_{\{0\}} \cong_u k$ and $\mathcal{L}|_{\{\infty\}} \cong_v k$. From the localization sequence we get the differential as (note $t = x$ and $t^{-1} = y$):

$$\begin{tikzcd}
K^{MW}_{q+1}(k) \cong_{u_*^{-1}} K^{MW}_{q+1}(k, \mathcal{L}|_{\{0\}}) \arrow[rr, bend left, "d_1"] \arrow[d] \arrow[r, "\partial_{\mathbb{A}^1_k/\mathbb{P}^1_k}"] & K^{MW}_q(k, \omega_\infty \otimes \mathcal{L}|_{\{\infty\}}) \cong_{y_* \otimes v_*} K^{MW}_q(k) \\
K^{MW}_{q+1}(k(t), \mathcal{L}|_{\{0\}} \otimes_k k(t)) \arrow[ur, "\partial^y_\infty"'] &
\end{tikzcd}$$

Note that here there is no pushforward along the inclusion $\iota_* \colon \Omega_0 \backslash \Omega_{-1} \to \Omega_0$ appearing, because $\Omega_{-1} = \emptyset$. The differential d_1 can be decomposed into four parts:

(i) $K^{MW}_{q+1}(k) \cong_{u_*^{-1}} K^{MW}_{q+1}(k, \mathcal{L}|_{\{0\}}) \cong H_1(\mathbb{A}^1_k, K^{MW}_q(\mathcal{L}|_{U_\infty}))$ via choice of orientation and homotopy invariance,

(ii) lifting the element along the inclusion $j \colon \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$,

(iii) the boundary map ∂^y_∞ giving the differential at the 0-cell $\infty \in \mathbb{P}^1_k$,

(iv) the choice of orientation for the 0-cell, as in (i).

In general, steps (i) and (iv) are the homological description of the cells depending on the choices made. Steps (ii) and (iii) describe the differential in the localization sequence and depend on the geometry and positioning of the cells relative to one another (see Definition 4.2.3).

In this case the differential $d_1\colon K^{MW}_{q+1}(k) \to K^{MW}_q(k)$, which depends on $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_k}(d)$, can be written explicitly as

$$\begin{aligned}
\alpha &\overset{(i)}{\longmapsto} \alpha \otimes u \\
&\overset{(ii)}{\longmapsto} \alpha \otimes (u \otimes 1) = \alpha \otimes \big(y^{-d} v \otimes 1\big) = \big(\langle y^{-d}\rangle\, \alpha\big) \otimes (v \otimes 1) \\
&\overset{(iii)}{\longmapsto} \partial^y_\infty \left(\langle y^{-d}\rangle\, \alpha\right) \otimes (y \otimes v) \\
&\overset{(iv)}{\longmapsto} \partial^y_\infty \left(\langle y^{-d}\rangle\, \alpha\right) = \partial^y_\infty \left(\langle y^{d \bmod 2}\rangle\, \alpha\right) = \begin{cases} \eta\alpha, & d \text{ odd}, \\ 0, & d \text{ even}. \end{cases}
\end{aligned}$$

This computes $H_i(\mathbb{P}^1_k, K^{MW}_q(\mathcal{L}))$:

$$H_1\left(\mathbb{P}^1_k, K^{MW}_q(\mathcal{O}(d))\right) = \begin{cases} {}_\eta K^{MW}_{q+1}(k), & d \text{ odd}, \\ K^{MW}_{q+1}(k), & d \text{ even}, \end{cases}$$

$$H_0\left(\mathbb{P}^1_k, K^{MW}_q(\mathcal{O}(d))\right) = \begin{cases} K^{MW}_q(k)/\eta, & d \text{ odd}, \\ K^{MW}_q(k), & d \text{ even}, \end{cases}$$

where ${}_\eta K^{MW}_{q+1}(k) = \ker(K^{MW}_{q+1}(k) \xrightarrow{\cdot\eta} K^{MW}_q(k))$. As the canonical bundle of $\mathbb{P}^1_k$ is $\omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2) = 0 \in \operatorname{Pic}(\mathbb{P}^1_k)/2$, the difference between homology and cohomology is only present in the change of degree and not in the twists, see Remark 2.1.6:

$$H^0\left(\mathbb{P}^1_k, K^{MW}_q(\mathcal{O}(d))\right) = \begin{cases} {}_\eta K^{MW}_q(k), & d \text{ odd}, \\ K^{MW}_q(k), & d \text{ even}, \end{cases}$$

$$H^1\left(\mathbb{P}^1_k, K^{MW}_q(\mathcal{O}(d))\right) = \begin{cases} K^{MW}_{q-1}(k)/\eta, & d \text{ odd}, \\ K^{MW}_{q-1}(k), & d \text{ even}. \end{cases}$$

Remark 5.1.2. This computation for $\mathbb{P}^1_k$ encapsulates multiple classical invariants.

The singular cohomology of the complex points, $\mathbb{CP}^1 \simeq S^2$, can be computed by considering the quotient map $K_q^{MW}(\mathcal{O}(d)) \to K_q^M$:

$$H^{2q}_{\text{sing}}(\mathbb{CP}^1_k, \mathbb{Z}) \cong \mathrm{CH}^q(\mathbb{P}^1_k) \cong H^q(\mathbb{P}^1_k, K_q^M) = \begin{cases} \mathbb{Z}, & q = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

The first isomorphism uses that $\mathbb{P}^1_k$ is strictly cellular and the second isomorphism holds in general, by definition. By considering the quotient map $K_q^{MW}(\mathcal{O}(d)) \to I^q(\mathcal{O}(d))$ we similarly get

$$H^0(\mathbb{P}^1_k, I^0(\mathcal{O}(d))) = \begin{cases} 0, & d \text{ odd}, \\ I^0(k), & d \text{ even}, \end{cases}$$

$$H^1(\mathbb{P}^1_k, I^1(\mathcal{O}(d))) = \begin{cases} I^0(k)/2I^0(k), & d \text{ odd}, \\ I^0(k), & d \text{ even}. \end{cases}$$

and further specializing to $k = \mathbb{R}$, $\mathbb{RP}^1 \simeq S^1$, we obtain:

$$H^q_{\text{sing}}(\mathbb{RP}^1, \mathbb{Z}) \cong H^q(\mathbb{P}^1_{\mathbb{R}}, I^q) = \begin{cases} \mathbb{Z}, & q = 0, 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$H^q_{\text{sing}}(\mathbb{RP}^1, \mathbb{Z}(\mathcal{O}(-1))) \cong H^q(\mathbb{P}^1_{\mathbb{R}}, I^q(\mathcal{O}(-1))) = \begin{cases} 0, & q = 0, \\ \mathbb{Z}/2\mathbb{Z}, & q = 1, \\ 0, & \text{otherwise.} \end{cases}$$

where $H^q_{\text{sing}}(-, \mathbb{Z}(\mathcal{L}))$ denotes singular cohomology with twisted coefficients, see e.g. [BT82, Chapter II.10 and Exercise II.10.7], and the real cycle class maps, see [HWXZ21] or Theorem 6.2.14, provide the isomorphisms $H^q(X, I^q(\mathcal{L})) \cong H^q_{\text{sing}}(X(\mathbb{R}), I^q(\mathcal{L}))$ for strictly cellular smooth $\mathbb{R}$-schemes X.

Example 5.1.3. Now consider the cellular structure on $\mathbb{P}^1_k$ given by removing more k-rational points, i.e. $\emptyset \subseteq \mathbb{P}^1_k \setminus \{\infty, p_2, \ldots, p_n\} \subset \mathbb{P}^1_k$. The chain complex for coefficients in $K_q^{MW}(\mathcal{L})$ is then given by

$$K_{q+1}^{MW}(k) \oplus \left(K_q^{MW}(k)\right)^{\oplus n-1} \xrightarrow{d_1} K_q^{MW}(k) \oplus \left(K_q^{MW}(k)\right)^{\oplus n-1}.$$

Note that to describe the first group we fixed an ordering of the points and used Lemma 4.1.9. Also compare with Remark 4.1.11.

The upper left entry in the $(n \times n)$-matrix associated to d_1 is the differential from the previous computation. To get the rest of the first column we need to compute the differentials of $\alpha \in K^{MW}_{q+1}(k)$ at the other points. Pick orientations in p_i for $i = 2, \ldots, n$ by $\mathcal{L}|_{p_i} \cong_u k$. The differential can then again be described by (note $t = x = y^{-1}$):

$$\begin{tikzcd}
K^{MW}_{q+1}(k) \cong_{u_*^{-1}} K^{MW}_{q+1}(k, \mathcal{L}|_{\{0\}}) \arrow[r, "\partial_{\mathbb{A}^1_k/\mathbb{P}^1_k}"] \arrow[d] & K^{MW}_{q}(k, \omega_{p_i} \otimes \mathcal{L}|_{\{p_i\}}) \cong_{[t-p_i]_* \otimes u_*} K^{MW}_q(k) \\
K^{MW}_{q+1}(k(t), \mathcal{L}|_{\{0\}} \otimes_k k(t)) \arrow[ur, "\partial^{[t-p_i]}_{p_i}"'] &
\end{tikzcd}$$

but this time the map vanishes because $\partial^{[t-p_i]}_{p_i}(\alpha) = 0$, since there is no $\langle y^{-d} \rangle$ factor appearing as no change of charts is necessary, and α has trivial valuation at $\{p_2, \ldots, p_n\}$. Geometrically this means that the constant form has no residues except at ∞.

To compute the upper row of the differential consider:

$$\begin{tikzcd}
K^{MW}_{q}(k) \cong_{u_*^{-1}} K^{MW}_{q}(k, \mathcal{L}|_{\{p_i\}}) \arrow[r, "\partial_{\mathbb{A}^1_k/\mathbb{P}^1_k}"] \arrow[d] & K^{MW}_{q}(k, \omega_{\infty} \otimes \mathcal{L}|_{\{\infty\}}) \cong_{y_* \otimes u_*} K^{MW}_q(k) \\
K^{MW}_{q+1}(k(t), \mathcal{L}|_{\{p_i\}} \otimes_k k(t)) \arrow[ur, "\partial^{y}_{\infty}"'] &
\end{tikzcd}$$

given by

$$\begin{aligned}
\alpha &\mapsto \alpha \otimes u \\
&\mapsto [t-p_i]\alpha \otimes (u \otimes 1) = [t-p_i]\alpha \otimes \left(y^{-d} v \otimes 1\right) = \left(\langle y^{-d} \rangle [t-p_i]\alpha\right) \otimes (v \otimes 1) \\
&\mapsto \partial^y_\infty \left(\langle y^{-d} \rangle [t-p_i]\alpha\right) \otimes (y \otimes v) \\
&\mapsto \partial^y_\infty \left(\langle y^{-d} \rangle [t-p_i]\alpha\right) = \partial^y_\infty \left(\langle y^{d \bmod 2} \rangle [t-p_i]\alpha\right) = \begin{cases} -\alpha, & d \text{ odd}, \\ \varepsilon\alpha, & d \text{ even}, \end{cases}
\end{aligned}$$

where the last equality comes from the computation of the differential in Lemma 1.2.4. Geometrically this corresponds to the fact that a function with a zero on $\mathbb{A}^1_k$ has a pole at ∞.

To complete the description of the differentials, we need to deal with the summands coming from a point p_i and their residues at some p_j, i.e. the $(n-1) \times (n-1)$ lower right subblock of the differential d_1. This is the identity matrix because $\partial^{[t-p_j]}([t-p_i]) = \delta_{i,j}$ and no coordinate changes are necessary.

The complete description of the differential for $\mathbb{P}^1_k$ and $\mathcal{L} \cong \mathcal{O}(d)$ is hence:

$$d_1\colon K^{MW}_{q+1}(k) \oplus \left(K^{MW}_q(k)\right)^{\oplus n-1} \longrightarrow K^{MW}_q(k) \oplus \left(K^{MW}_q(k)\right)^{\oplus n-1}$$
$$\begin{pmatrix} \eta & -1 & \dots & -1 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \ d \text{ odd} \qquad \begin{pmatrix} 0 & \varepsilon & \dots & \varepsilon \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \ d \text{ even}$$

This gives an alternative way to compute $H_i(\mathbb{P}^1_k, K^{MW}_q(\mathcal{L}))$. Because we did not use a strict cellular structure, the 0-th cohomology groups of the cells were more complicated, adding more $K^{MW}_q(k)$ summands. The additional relations we see encode the linear equivalence of any two points on $\mathbb{P}^1_k$, now with $\mathrm{GW}(k)$ coefficients.

Next we will do the computation for $\mathbb{P}^n_k$. Instead of directly computing the differential by pushforward along the inclusion and boundary map, we are going to restrict to curves, as described in Remark 2.1.15. As those curves are $\mathbb{P}^1_k$'s, we can directly apply the computation from the previous example. Note that for d_1 it is exactly the same situation as above.

Example 5.1.4. Consider the standard strict cellular structure on $\mathbb{P}^n_k$ given by fixing a complete flag of subspaces $\emptyset \subseteq \mathbb{P}^0_k \subseteq \mathbb{P}^1_k \subseteq \cdots \subseteq \mathbb{P}^n_k$. The chain complex for coefficients in $K^{MW}_q(\mathcal{L})$ is then given by

$$K^{MW}_{q+n}(k) \xrightarrow{d_n} K^{MW}_{q+n-1}(k) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} K^{MW}_{q+1}(k) \xrightarrow{d_1} K^{MW}_q(k).$$

To compute the differential $d_i\colon K^{MW}_{q+i}(k) \to K^{MW}_{q+i-1}(k)$ we consider the cell of dimension i, whose closure is a $\mathbb{P}^i_k$. There exists a rational curve meeting the boundary stratum in a single point. The determinant of the normal bundle of such a curve is $\mathcal{O}_{\mathbb{P}^1_k}(-i+1)$. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_k}(j)$ be the twist and observe that restricting to cells keeps the parity, i.e. $\mathcal{O}_{\mathbb{P}^n_k}(j)|_{\mathbb{P}^{n-1}_k} = \mathcal{O}_{\mathbb{P}^{n-1}_k}(j)$. Therefore, the differential d_i is given by multiplication with:

$$\begin{cases} \eta, & \text{for } j-i-1 \text{ odd,} \\ 0, & \text{for } j-i-1 \text{ even.} \end{cases}$$

This results in

$$H_i\left(\mathbb{P}^n_k, K^{MW}_q\right) = \begin{cases} K^{MW}_q(k), & i = 0, \\ {}_\eta K^{MW}_{q+i}(k), & 2 \leq i \leq n \text{ and } i \text{ even}, \\ K^{MW}_{q+i}(k)/\eta, & 1 \leq i < n \text{ and } i \text{ odd}, \\ K^{MW}_{q+n}(k), & i = n \text{ odd}, \end{cases}$$

$$H_i\left(\mathbb{P}^n_k, K^{MW}_q(\mathcal{O}(-1))\right) = \begin{cases} {}_\eta K^{MW}_{q+i}(k), & 1 \leq i \leq n \text{ and } i \text{ odd}, \\ K^{MW}_{q+i}(k)/\eta, & 0 \leq i < n \text{ and } i \text{ even} \\ K^{MW}_{q+n}(k), & i = n \text{ even}. \end{cases}$$

Writing this in terms of cohomology requires twists by the canonical bundle of $\mathbb{P}^n_k$, which is $\mathcal{O}_{\mathbb{P}^n_k}(-n-1)$. Therefore the parity of n plays a role in reading off the cohomology, resulting in

$$H^i\left(\mathbb{P}^n_k, K^{MW}_q\right) = \begin{cases} K^{MW}_q(k), & i = 0, \\ K^{MW}_{q-i}(k)/\eta, & 2 \leq i \leq n \text{ and } i \text{ even}, \\ {}_\eta K^{MW}_{q-i}(k), & 1 \leq i < n \text{ and } i \text{ odd}, \\ K^{MW}_{q-n}(k), & i = n \text{ odd}, \end{cases}$$

$$H^i\left(\mathbb{P}^n_k, K^{MW}_q(\mathcal{O}(-1))\right) = \begin{cases} K^{MW}_{q-i}(k)/\eta, & 1 \leq i \leq n \text{ and } i \text{ odd} \\ {}_\eta K^{MW}_{q-i}(k), & 0 \leq i < n \text{ and } i \text{ even}, \\ K^{MW}_{q-n}(k), & i = n \text{ even}. \end{cases}$$

This was previously computed in [Fas13, Theorem 11.7].

Remark 5.1.5. As for $\mathbb{P}^1_k$, we get similar descriptions for the classical invariants of $\mathbb{P}^n_k$.

The singular cohomology of the complex points is:

$$H^{2q}_{\text{sing}}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathrm{CH}^q(\mathbb{P}^n_{\mathbb{C}}) \cong H^q(\mathbb{P}^n_{\mathbb{C}}, K^M_q) = \begin{cases} \mathbb{Z}, & q = 0, \dots, n, \\ 0, & \text{otherwise}. \end{cases}$$

The I^q-cohomology is given by:

$$H^q(\mathbb{P}^n_k, I^q) = \begin{cases} I^0(k), & q = 0, \\ I^0(k)/2I^0(k), & 2 \leq q \leq n \text{ and } q \text{ even}, \\ 0, & 1 \leq q < n \text{ and } q \text{ odd}, \\ I^0(k), & q = n \text{ odd}, \end{cases}$$

and further specializing to $k = \mathbb{R}$ we see:

$$H^q_{\text{sing}}(\mathbb{RP}^n, \mathbb{Z}) \cong H^q(\mathbb{P}^n_{\mathbb{R}}, I^q) = \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/2\mathbb{Z}, & 2 \le q \le n \text{ and } q \text{ even}, \\ 0, & 1 \le q < n \text{ and } q \text{ odd}, \\ \mathbb{Z}, & q = n \text{ odd}, \end{cases}$$

and similarly for $I^q(\mathcal{O}(-1))$-cohomology and singular cohomology with twisted coefficients.

Example 5.1.6. To showcase the choice of orientations involved and the effects of non-strictly cellular structures, start with the cellular structure coming from the four lines $\{p\} \times \mathbb{P}^1_k, \mathbb{P}^1_k \times \{p\} \subseteq \mathbb{P}^1_k \times \mathbb{P}^1_k$ for $p \in \{0, \infty\}$. Cover $\mathbb{P}^1_k \times \mathbb{P}^1_k$ by the four open sets $U_{a,b} = (\mathbb{P}^1_k \setminus \{a\}) \times (\mathbb{P}^1_k \setminus \{b\}) \cong \mathbb{A}^2_k$ for $a, b \in \{0, \infty\}$. Pick orientations on those four, following the notation we used for $\mathbb{P}^1_k$, by u_1, u_2 on $U_{0,0}$, v_1, u_2 on $U_{\infty,0}$, u_1, v_2 on $U_{0,\infty}$ and v_1, v_2 on $U_{\infty,\infty}$. As before note $u_i = t_i$ and $v_i = t_i^{-1}$. Fix orientations on the four intersection points by $t_1 \otimes t_2$ on $(0,0)$, $t_1^{-1} \otimes t_2$ on $(\infty, 0)$, $t_1 \otimes t_2^{-1}$ on $(0, \infty)$ and $t_1^{-1} \otimes t_2^{-1}$ on (∞, ∞). A change of order of generators, such as for example picking $t_2 \otimes t_1$ on $(0,0)$ instead, differs from the previous choice by a -1 on the section side or equivalently a $\langle -1 \rangle$ on the form side.

The chain complex for this cellular structure and coefficients in $K^{MW}_{q-2}(\mathcal{L})$ is

$$K^{MW}_q(k) \oplus \left(K^{MW}_{q-1}(k)\right)^2 \oplus K^{MW}_{q-2}(k) \xrightarrow{d_2} \left(K^{MW}_{q-1}(k) \oplus K^{MW}_{q-2}(k)\right)^4 \xrightarrow{d_1} \left(K^{MW}_{q-2}(k)\right)^4,$$

where the 2-cell summands come from $\mathbb{A}^2_k \setminus (\mathbb{A}^1_k \times \{0\} \cup \{0\} \times \mathbb{A}^1_k)$, i.e. it is a hyperplane complement so $\mathbb{A}^2_k$ contributes a $K^{MW}_q(k)$ summand, the axes each contribute one $K^{MW}_{q-1}(k)$, and their intersection point $(0,0)$ contributes the $K^{MW}_{q-2}(k)$. The 1-cell summands all come from the four $\mathbb{A}^1_k \setminus \{*\} \cong \mathbb{G}_m$ and the 0-cell summands are the four intersection points of the four lines. Again this representation of the groups depends on a fixed ordering of the four lines.

Now we start with the computation of the differentials for the twists by $\mathcal{L} \cong \mathcal{O}(e_1, e_2)$ for $e_1, e_2 \in \mathbb{Z}$. Beginning with d_2 and the $K^{MW}_q(k)$ summand, i.e. the first column of d_2, we have:

$$K^{MW}_q(k) \longrightarrow K^{MW}_q(k(t_1, t_2), \mathcal{L} \otimes_k k(t_1, t_2)) \xrightarrow{\partial} K^{MW}_{q-1}(k) \oplus K^{MW}_{q-2}(k),$$

where the first map sends $\alpha \in K^{MW}_q(k)$ to

$$\alpha \otimes (u_1 u_2 \otimes 1) = \left\langle \frac{1}{t_1} \right\rangle^{e_1} \alpha \otimes (v_1 u_2 \otimes 1) = \left\langle \frac{1}{t_2} \right\rangle^{e_2} \alpha \otimes (u_1 v_2 \otimes 1)$$

Together with the residue map this can be summarized as follows:

$$d_2(\alpha) = \begin{cases} 0, & \text{for all differentials at } t_i = 0, \\ 0, & \text{for } t_i = \infty \text{ and } e_i \text{ even}, \\ \eta\alpha, & \text{for } t_i = \infty \text{ and } e_i \text{ odd}. \end{cases}$$

Continuing with the summand from $t_1 = 0$ (analogously for $t_2 = 0$) we get:

$$K_{q-1}^{MW}(k) \longrightarrow K_q^{MW}(k(t_1,t_2), \mathcal{L} \otimes_k k(t_1,t_2)) \xrightarrow{\partial} K_{q-1}^{MW}(k) \oplus K_{q-2}^{MW}(k),$$

where the first map sends $\alpha \in K_{q-1}^{MW}(k)$ to $[t_1]\alpha \otimes (u_1u_2 \otimes 1)$ and the composition becomes multiplication by

$$\begin{cases} 1, & \text{for all differentials at } t_1 = 0, \\ 0, & \text{for all differentials at } t_2 = 0, \\ \varepsilon, & \text{for } t_1 = \infty \text{ and } e_1 \text{ even}, \\ -1, & \text{for } t_1 = \infty \text{ and } e_1 \text{ odd}, \\ 0, & \text{for } t_2 = \infty \text{ and } e_2 \text{ even}, \\ \eta[t_1], & \text{for } t_2 = \infty \text{ and } e_2 \text{ odd}. \end{cases}$$

Geometrically, the last one means that the element is mapped to the K_{q-2}^{MW} summand of $t_2 = \infty$ having a simple zero at $(0, \infty)$. To finish the computation, consider the $K_{q-2}^{MW}(k)$ summand corresponding to $(0, 0)$:

$$K_{q-2}^{MW}(k) \longrightarrow K_q^{MW}(k(t_1,t_2), \mathcal{L} \otimes_k k(t_1,t_2)) \xrightarrow{\partial} K_{q-1}^{MW}(k) \oplus K_{q-1}^{MW}(k).$$

The first map sends $\alpha \in K_q^{MW}(k)$ to

$$\begin{aligned} [t_1][t_2]\alpha \otimes (u_1u_2 \otimes 1) &= \langle -1 \rangle^{e_1} [t_1][t_2]\alpha \otimes (v_1u_2 \otimes 1) = \langle -1 \rangle^{e_2} [t_1][t_2]\alpha \otimes (u_1v_2 \otimes 1) \\ &= \langle -1 \rangle^{e_1+e_2} [t_1][t_2]\alpha \otimes (v_1v_2 \otimes 1), \end{aligned}$$

where we use the equality $[a]\langle a \rangle = [a]\langle -1 \rangle$. The composition becomes

$$\begin{cases} [t_2], & \text{for all differentials at } t_1 = 0, \\ \varepsilon[t_1], & \text{for all differentials at } t_2 = 0, \\ \varepsilon[t_2], & \text{for } t_1 = \infty \text{ and } e_1 \text{ even}, \\ -[t_2], & \text{for } t_1 = \infty \text{ and } e_1 \text{ odd}, \\ [t_1], & \text{for } t_2 = \infty \text{ and } e_2 \text{ even}, \\ \langle -1 \rangle [t_1], & \text{for } t_2 = \infty \text{ and } e_2 \text{ odd}. \end{cases}$$

The differential d_1 is almost a straightforward copy of the $\mathbb{P}^1_k \setminus \{\infty, 0\}$ computation four times, one for each of $\{t_1 = 0\}$, $\{t_1 = \infty\}$, $\{t_2 = 0\}$ and $\{t_2 = \infty\}$. There is an additional $\langle -1 \rangle$ factor for all differentials coming from $\{t_1 = 0\}$ and $\{t_1 = \infty\}$ as the section part looks like $t_2^{\pm 1} \otimes t_1^{\pm 1}$ which is not the chosen orientation from the start of the example.

Fix an order of summands in the chain complex to represent the differentials by matrices in the following way. The summands corresponding to the 2-cell are in this order $\mathbb{A}^2_k$, $\{t_1 = 0\}$, $\{t_2 = 0\}$ and $(0, 0)$, the 1-cell summands are $\{t_1 = \infty\}$, $\{t_2 = \infty\}$, $\{t_1 = 0\}$ and $\{t_2 = 0\}$, and the 0-cell summands are (∞, ∞), $(0, \infty)$, $(\infty, 0)$ and $(0, 0)$. This gives the matrices in Appendix A.

To see that those maps define differentials use the equality $(1 + \langle -1 \rangle)\, \eta = 0$.

Remark 5.1.7. Note that the complexity of this computation only arises because we picked an unnecessarily complicated cellular structure. For products of projective spaces $\mathbb{P}^n_k \times \mathbb{P}^m_k$ one can take the standard strict cellular structure coming from the factors and the resulting computation for twists by the canonical bundle $\omega_{\mathbb{P}^n_k \times \mathbb{P}^m_k} = \mathcal{O}(-n-1, -m-1)$ looks identical to the computation of $H^*_{\mathrm{sing}}(\mathbb{RP}^n_k \times \mathbb{RP}^m_k, \mathbb{Z})$, replacing the multiplication by ± 2 with the multiplication by $\langle \pm 1 \rangle \eta$, see [Hat02, 3.B.Exercise 1]. The twists change whether all these multiplications appear in even or odd degrees.

5.2. Computations for $\overline{M_{0,n}}$

Convention 5.2.1. Fix the cellular structure $\emptyset = \Omega_{-1} \subseteq \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_{n-3} = \overline{M_{0,n}}$ given by

$$\Omega_i = \overline{M_{0,n}} \setminus \bigcup_{|I|=i+1} \Big(\bigcap_{i \in I} D^{S_i} \Big),$$

i.e. the points lying in at most i boundary divisors. This way we have $\Omega_0 = M_{0,n}$ and $\Omega_{i+1} \setminus \Omega_i$ are the points lying in exactly i boundary divisors.

This is the only cellular structure on $\overline{M_{0,n}}$ that we will consider.

Remark 5.2.2. Note that the intersection of m divisors, if non-empty, is isomorphic to a product of the form $\overline{M_{0,n_1}} \times \cdots \times \overline{M_{0,n_{m+1}}}$, see [Kee92, Fact 2]. If one removes all points lying in other boundary divisors one gets $M_{0,n_1} \times \cdots \times M_{0,n_{m+1}}$, which is a hyperplane complement.

Convention 5.2.3. Consider the space $\overline{M_{0,n}}$ and write the boundary divisors D^S according the usual convention $|S \cap \{1, 2, 3\}| \leq 1$. Define an order $\preccurlyeq$ on the set of boundary divisors by first ordering by size and then reverse lexicographically. This means for $S \neq T$:

$$\begin{aligned} D^S \preccurlyeq D^T \quad \Leftrightarrow \qquad & |S| < |T| \\ & \text{or } (|S| = |T| \text{ and } \max(S) < \max(T)) \\ & \text{or } (|S| = |T| \text{ and } \max(S) = \max(T) \text{ and } D^{S \setminus \{\max(S)\}} \preccurlyeq D^{T \setminus \{\max(T)\}}). \end{aligned}$$

For example $D^{\{1,7,8,9\}} \preccurlyeq D^{\{3,4,5,7,8\}} \preccurlyeq D^{\{1,2,6,7,8\}} \preccurlyeq D^{\{1,2,3,4,5,6\}}$.

This also induces an order on m-fold intersections of boundary divisors by reverse lexicographically ordering those subsets, i.e. $D^{S_1} \cap D^{T_1} \preccurlyeq D^{S_2} \cap D^{T_2}$ for $D^{S_i} \preccurlyeq D^{T_i}$ means $D^{T_1} \preccurlyeq D^{T_2}$ or $\big(D^{T_1} = D^{T_2}$ and $D^{S_1} \preccurlyeq D^{S_2}\big)$.

Remark 5.2.4. There are multiple different ways of viewing $M_{0,n}$ as a complement of hyperplanes in $\mathbb{A}_k^{n-3}$. We want to fix such an isomorphism.

Picking three markings, e.g. $\{q_0, q_1, q_\infty\}$, fixes a blow-up description $\varphi\colon \overline{M_{0,n}} \to (\mathbb{P}_k^1)^{n-1}$, see Remark 3.1.10 and Remark 3.1.11. The contraction φ restricts to an open embedding to $\mathbb{A}_k^{n-3}$ after removing all exceptional divisors, i.e. D^S with $|S| \geq 3$ assuming $|S \cap \{q_0, q_1, q_\infty\}| \leq 1$, and lines at infinity, i.e. $D^{\{q_\infty, \lambda\}}$ for $\lambda \notin \{q_0, q_1, q_\infty\}$. The remaining boundary divisors $D^{\{q_0,\lambda\}}$, $D^{\{q_1,\lambda\}}$ and $D^{\{\lambda,\mu\}}$ for $\lambda, \mu \notin \{q_0, q_1, q_\infty\}$ are mapped to the hyperplanes $\{x_\lambda = 0\}$, $\{x_\lambda = 1\}$ and $\{x_\lambda = x_\mu\}$ respectively. Therefore φ restricts to an isomorphism

$$\varphi\colon M_{0,n} \longrightarrow (\mathbb{A}_k^1 \setminus \{0,1\})^{n-3} \setminus \{(x_1, \dots, x_{n-3}) \mid \exists i \neq j : x_i = x_j\}.$$

We order the hyperplanes $\varphi(D^S)$ removed from $\mathbb{A}_k^{n-3}$ in terms of S according to Convention 5.2.3.

In the same way we can write $\bigcup_S D^S \setminus \bigcup_{S \neq T} D^S \cap D^T$ as a disjoint union of hyperplane complements in $\mathbb{A}_k^{n-4}$. For this we will write $D^S \setminus \bigcup_{S \neq T} D^S \cap D^T$ as such a hyperplane complement. Write $S = \{s_1 < s_2 < \dots < s_{|S|}\}$ and $S^c = \{s_1^c < \dots < s_{|S^c|}^c\}$. As before, fix a description of $D^S \cong \overline{M_{0,|S|+1}} \times \overline{M_{0,|S^c|+1}}$ as a blow-up of $(\mathbb{P}_k^1)^{|S|-2} \times (\mathbb{P}_k^1)^{|S^c|-2}$ corresponding to the markings $\{*, s_1, s_2\}$ and $\{*, s_1^c, s_2^c\}$ respectively, where $*$ is the node separating S and S^c. In this case the hyperplanes removed from $(\mathbb{A}_k^1)^{|S|-2} \times (\mathbb{A}_k^1)^{|S^c|-2}$ correspond to $D^S \cap D^T$ for T one of $\{s_1, \lambda\}$, $\{s_2, \lambda\}$, $\{\lambda, \mu\}$, $\{s_1^c, \lambda^c\}$, $\{s_2^c, \lambda^c\}$, or $\{\lambda^c, \mu^c\}$, where $\lambda, \mu \in \{s_3, \dots, s_{|S|}\}$ and $\lambda^c, \mu^c \in \{s_3^c, \dots, s_{|S^c|}^c\}$. They are again ordered according to Convention 5.2.3.

Definition 5.2.5. Let $\mathcal{L}$ be a line bundle on $\overline{M_{0,n}}$. Fix a basis of $\operatorname{Pic}(\overline{M_{0,n}})/2$ associated to $(3,1,2)$ by Proposition 3.2.4. Define

$$\tau_{\mathcal{L}}(S) = \operatorname{coeff}(D^S) \in \mathbb{Z}/2\mathbb{Z},$$

to be the coefficient of D^S in this representation.

Example 5.2.6. Consider the canonical bundle $\omega_{\overline{M_{0,5}}}$ of $\overline{M_{0,5}}$. From Remark 3.2.7 we know that it is represented by $D^{\{3,4,5\}} + D^{\{2,4,5\}} + D^{\{1,4,5\}} \in \mathrm{Pic}(\overline{M_{0,5}})/2$ with respect to the basis associated to $(3, 1, 2)$. Therefore,

$$\tau_{\omega_{\overline{M_{0,5}}}}(S) = \begin{cases} 1, & S = \{3,4,5\}, \{2,4,5\}, \{1,4,5\}, \\ 0, & \text{otherwise}, \end{cases}$$

in other words, we have $\tau_{\omega_{\overline{M_{0,5}}}}(S) = |S| \in \mathbb{Z}/2\mathbb{Z}$.

Remark 5.2.7. The choice of basis in the definition of $\tau_{\mathcal{L}}$ corresponds to fixing an inclusion $M_{0,n} \hookrightarrow \overline{M_{0,n}}$, hence a trivialization of the bundle $\mathcal{L}$ on that $M_{0,n}$.

Notation 5.2.8. Let $U = \mathbb{A}^n_k \setminus (\bigcup_{i=1}^r V_i)$ be the complement of affine hyperplanes V_i. By Lemma 4.1.9 we have an explicit description of

$$H^0_{\mathrm{Nis}}(U, K_q^{MW}) \cong \bigoplus_{i=0}^{n} \left(K_{q-i}^{MW}(k)\right)^{m_i},$$

where $m_0 = 1$ and $m_i = \left|\left\{J \subseteq \{1, \dots, r\} \mid |J| = i,\ \bigcap_{j\in J} V_j \neq \emptyset\right\}\right|$ is the number of non-empty intersections of i hyperplanes. As each summand corresponds to an intersection of hyperplanes, denote the respective generator by $[\bigcap_{j\in J} V_j]$ and choose $[\emptyset]$ for the m_0 summand.

As we will need to deal with multiple different spaces at the same time, we will use subscripts to which one the classes belong. For instance, $[D^S \cap D^T]$ will appear as a summand for D^S and D^T, so we write $[D^S \cap D^T]_{D^S}$ and $[D^S \cap D^T]_{D^T}$ for the respective summands.

Call the intersection of i divisors an i-intersection for short.

Remark 5.2.9. We think of the summand belonging to $[D^S \cap D^T]_{D^S}$ as forms defined on D^S that have a simple zero at $D^S \cap D^T$ or none at all if it is $[\emptyset]_{D^S}$.

All further computations are a task of bookkeeping all the non-canonical choices involved. The curves we restrict to come from strict transforms of curves C in $(\mathbb{P}^1)^{n-3}$ with prescribed residues along some divisors. These same curves C can be used for the computation of the (co)homology of $(\mathbb{P}^1)^{n-3}$ with respect to the cellular structure chosen on $M_{0,n}$. We can lift the $(\mathbb{P}^1)^{n-3}$ computation by understanding which boundary divisor the strict transform meets and its normal bundle. The behavior of the strict transform we understand from Lemma 3.2.2 and the normal bundle by Lemma 3.2.1 and Proposition 3.2.6. The higher codimension case is reduced to the codimension 0 case of a $\overline{M_{0,n'}}$ for smaller n', by considering the same curves in one of the $\overline{M_{0,n'}}$ factors of the intersection of divisors, see Remark 5.2.2.

Proposition 5.2.10. *Let $\mathcal{L}$ be a line bundle on $\overline{M_{0,n}}$. The top degree differential in the cellular complex for $\overline{M_{0,n}}$ is*

$$d_{n-3}\colon H_{n-3}\left(\overline{M_{0,n}} \setminus \left(\bigcup_T D^T\right), K_q^{MW}(\mathcal{L})\right) \to H_{n-4}\left(\bigcup_T D^T \setminus \bigcup_{S \neq T}\left(D^S \cap D^T\right), K_q^{MW}(\mathcal{L})\right)$$

and is given by

$$\begin{aligned} K_q^{MW}(k)[\emptyset]_{\overline{M_{0,n}}} &\longrightarrow \sum_T K_{q-1}^{MW}(k)[\emptyset]_{D^T} \\ \alpha &\longmapsto \begin{cases} \eta\alpha\ [\emptyset]_{D^T}, & \tau_{\mathcal{L}}(T) + |T| \equiv 1 \bmod 2,\ \text{for } T = \{3,\lambda\} \text{ or } |T| \geq 3, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

on the 0-intersections. For the 1-intersections consider some $S = \{1,\lambda\}, \{2,\lambda\}, \{\lambda,\mu\}$ and the differential becomes

$$K_{q-1}^{MW}(k)[D^S]_{\overline{M_{0,n}}} \longrightarrow \sum_T K_{q-1}^{MW}(k)[\emptyset]_{D^T} + \sum_{S',T} K_{q-2}^{MW}(k)[D^{S'} \cap D^T]_{D^T}$$

$$\alpha \longmapsto \begin{cases} \langle(-1)^{\tau_{\mathcal{L}}(T)+|T|}\rangle\alpha\ [\emptyset]_{D^T}, & S \subseteq T \\ -\langle(-1)^{\tau_{\mathcal{L}}(T)+|T|+1}\rangle\alpha\ [\emptyset]_{D^T}, & (S = \{1,\lambda\} \text{ or } S = \{2,\lambda\}) \text{ and } \{3,\lambda\} \subseteq T, \\ & \text{or } S = \{\lambda,\mu\} \text{ and } (\{3,\lambda\} \subseteq T \text{ or } \{3,\mu\} \subseteq T) \\ \eta\alpha\ [D^{S'} \cap D^T]_{D^T}, & \tau_{\mathcal{L}}(T) + |T| \equiv 1 \bmod 2 \text{ and} \\ & D^T \text{ meets } D^{S'} \text{ over } D^S \\ 0, & \text{otherwise} \end{cases}$$

Proof. To compute the differential we can restrict to a curve and compute it on there, see Lemma 2.1.13 and Remark 2.1.15. The restriction map for smooth curves is the pullback for sheaves, i.e. evaluation of forms at that point, see Remark 2.1.12. By Lemma 3.2.2 we can pick a rational curve C meeting a boundary divisor D^T, where T is arbitrary. The twist bundle of the differential on those curves is $\mathcal{L} \otimes \omega_C$ by Lemma 2.1.13. All possible situations of the $\mathbb{P}_k^1$ case were carried out in Example 5.1.3.

A constant non-zero form on $\overline{M_{0,n}}$, i.e. the elements in $K_q^{MW}(k)[\emptyset]_{\overline{M_{0,n}}}$, can only have a residue at some divisor D^T if that divisor appears with an odd multiplicity in the expression of the twist bundle $\mathcal{L} \otimes \omega_C$, i.e. $\tau_{\mathcal{L}}(T) + |T| \equiv 1 \bmod 2$ by Lemma 3.2.1 and Proposition 3.2.6. Compare all computations for constant forms in Section 5.1.

A form with prescribed residues at D^S, i.e. the elements in $K_{q-1}^{MW}(k)[D^S]_{\overline{M_{0,n}}}$, can only have residues at points lying over the hyperplane corresponding to S, i.e. $S \subseteq T$ the first case in the description of the differential, points over the parallel lines at infinity, i.e. the second case, or over points where a divisor D^T appears with an odd multiplicity in the twist bundle, i.e. the third case. In the first two cases the form is constant along the divisor D^T, hence the differential has the form $\alpha[D^S]_{\overline{M_{0,n}}} \mapsto \pm\langle\pm 1\rangle\alpha[\emptyset]_{D^T}$ for $S \subseteq T$, having no further residues along D^T. In the third case, the form is not constant along D^T having further residues along the divisors $D^{S'}$ over D^S and only there.

To determine the signs $+1$, -1, $\langle -1\rangle$, $-\langle -1\rangle$, note that the curve we restrict to is the same curve that can be used to do the computation for $(\mathbb{P}^1_k)^{n-3}$ where the cellular structure is induced by $M_{0,n}$ and the i-intersection of the complementary hyperplanes, compare Example 5.1.6. □

Corollary 5.2.11. *For $n \geq 3$ and $q \in \mathbb{Z}$ we have*

$$H_{n-3}(\overline{M_{0,n}}, K_q^{MW}(\mathcal{L})) \cong \begin{cases} K_q^{MW}(k), & \mathcal{L} = \omega_{\overline{M_{0,n}}}, \\ {}_\eta K_q^{MW}(k), & \textit{otherwise}, \end{cases}$$

and in particular $\widetilde{\mathrm{CH}}^0(\overline{M_{0,n}}) \cong \mathrm{GW}(k)$.

Proof. Except for $K_q^{MW}(k)[\emptyset]_{\overline{M_{0,n}}}$, no element can be in the kernel of the differential as each summand $K_q^{MW}(k)[D^{S_1}\cap\cdots\cap D^{S_m}]_{\overline{M_{0,n}}}$ is the only one mapping to $K_{q-m}^{MW}(k)[\emptyset]_{D^{S_1}\cap\cdots\cap D^{S_m}}$ and does so by a unit.

The computation of $\omega_{\overline{M_{0,n}}}$ from Proposition 3.2.6 shows that $\tau_{\omega_{\overline{M_{0,n}}}}(S) = |S|$. Therefore, the differential is trivial on the summand $K_q^{MW}(k)[\emptyset]_{\overline{M_{0,n}}} \to \sum_S K_{q-1}^{MW}(k)[\emptyset]_{D^S}$ when twisted by $\omega_{\overline{M_{0,n}}}$ and otherwise multiplication by η to at least one summand, showing the claim.

The additional statement follows since

$$\begin{aligned} \widetilde{\mathrm{CH}}^0(\overline{M_{0,n}}) &= H^0_{\mathrm{Nis}}(\overline{M_{0,n}}, \underline{K}_0^{MW}) \cong H_{n-3}(\overline{M_{0,n}}, K_0^{MW}(\omega_{\overline{M_{0,n}}})) \\ &\cong K_0^{MW}(k) = \mathrm{GW}(k). \end{aligned}$$

This finishes the proof. □

Proposition 5.2.12. *Let $\mathcal{L}$ be a line bundle on $\overline{M_{0,n}}$. The degree $n-4$ differential in the cellular complex for $\overline{M_{0,n}}$ is*

$$d_{n-4}\colon H_{n-4}\left(\Omega_1 \setminus \Omega_0, K_q^{MW}(\mathcal{L})\right) \to H_{n-5}\left(\Omega_2 \setminus \Omega_1, K_{q-1}^{MW}(\mathcal{L})\right).$$

On D^S with $S = \{s_1 < s_2 < \cdots < s_{|S|}\}$ it is given by

$$K_q^{MW}(k)[\emptyset]_{D^S} \longrightarrow \sum_T K_{q-1}^{MW}(k)[\emptyset]_{D^S \cap D^T}$$

$$\alpha \mapsto \begin{cases} \langle \pm 1 \rangle \eta\alpha\, [\emptyset]_{D^S \cap D^T}, & \{s_1, s_2\} \subseteq T \subseteq S \text{ and } \tau_{\mathcal{L}}(T) + \tau_{\mathcal{L}}(S) + |S \setminus T| + 1 \equiv 1 \bmod 2 \\ & \text{or } \{s_1, s_2\} \nsubseteq T \subseteq S \text{ and } \tau_{\mathcal{L}}(T) + |T| \equiv 1 \bmod 2 \\ & \text{or } T \cap S = \emptyset \text{ and } \tau_{\mathcal{L}}(T) + |T| \equiv 1 \bmod 2 \\ & \text{or } \{s_1^c, s_2^c\} \nsubseteq T \supseteq S \text{ and } |T \setminus S| + \tau_{\mathcal{L}}(S) + \tau_{\mathcal{L}}(T) \equiv 0 \bmod 2 \\ 0, & \text{otherwise} \end{cases}$$

on the 0-intersections within D^S. Where the sign is given by $\langle +1 \rangle$ if $D^S \preccurlyeq D^T$ and $\langle -1 \rangle$ if $D^T \preccurlyeq D^S$. For the 1-intersections within D^S consider a $T = \{s_1, \lambda\}, \{s_2, \lambda\}, \{\lambda, \mu\}$ or $T = \{s_1^c, \lambda^c\}, \{s_2^c, \lambda^c\}, \{\lambda^c, \mu^c\}$ if the intersection happens in the $\overline{M_{0,|S^c|+1}}$ factor of D^S

$$K_{q-1}^{MW}(k)[D^S \cap D^T]_{D^S} \longrightarrow \sum_U K_{q-1}^{MW}(k)[\emptyset]_{D^S \cap D^U} + \sum_{T', U} K_{q-2}^{MW}(k)[D^S \cap D^{T'} \cap D^U]_{D^S \cap D^U}$$

$$\alpha \longmapsto \begin{cases} \langle (-1)^{\tau_{\mathcal{L}}(U) + |U|} \rangle \alpha [\emptyset]_{D^S \cap D^U}, & T \subseteq U \subseteq S \text{ or } T \subseteq U \subseteq S^c \\ -\langle (-1)^{\tau_{\mathcal{L}}(U) + |U| + 1} \rangle \alpha [\emptyset]_{D^S \cap D^U}, & (T = \{s_1, \lambda\} \text{ or } \{s_2, \lambda\}) \text{ and } S^c \cup \{\lambda\} \subseteq U \\ & \text{or } T = \{\lambda, \mu\}, S^c \subseteq U \text{and } (\lambda \in U \text{ or } \mu \in U) \\ & \text{or } (T = \{s_1^c, \lambda^c\} \text{ or } \{s_2^c, \lambda^c\}) \text{ and } S \cup \{\lambda^c\} \subseteq U \\ & \text{or } T = \{\lambda^c, \mu^c\}, S \subseteq U \text{and } (\lambda^c \in U \text{ or } \mu^c \in U) \\ \eta\alpha\, [D^S \cap D^{T'} \cap D^U]_{D^S \cap D^U}, & \tau_{\mathcal{L}}(T) + |T| \equiv 1 \bmod 2 \text{ and} \\ & D^U \text{ meets } D^{T'} \text{ over } D^T \cap D^S \\ 0, & \text{otherwise} \end{cases}$$

all of which are multiplied by $\langle -1 \rangle$ if $D^U \preccurlyeq D^S$, as above.

Proof. The proof is basically identical to the proof of Proposition 5.2.10 because each computation happens solely within one of the factors of $D^S \cong \overline{M_{0,|S|+1}} \times \overline{M_{0,|S^c|+1}}$. Although, it requires slightly more care to treat the two factors, because the condition $|T \cap \{s_1, s_2, s_3\}| \leq 1$ determines whether one collapses the points of T or sends $S \setminus T$ to the additional marking when making the transition from D^S to $D^S \cap D^T$. Additionally one needs to determine the sign coming from the fact that there are two ways to arrive at $D^S \cap D^T$, namely one via first going to D^S and one via first going to D^T. The difference is a sign on the section side, so a $\langle -1 \rangle$ on the form side, depending on the order of D^S and D^T. This is the same situation as in Example 5.1.6.

As before, taking a constant form on one D^S, i.e. an element of $K_q^{MW}(k)[\emptyset]_{D^S}$, it can only have residues at divisors $D^S \cap D^T$ if that appears with an odd multiplicity in the twist bundle. The possibilities of T such that $D^S \cap D^T$ is not empty are $T \subseteq S$, $T \subseteq S^c$, $S \subseteq T^c$, and $S \subseteq T$. The middle two together form the case $S \cap T = \emptyset$. In the case of $\{s_1, s_2\} \subseteq T \subseteq S$ the curve we pick, to degenerate from D^S to $D^S \cap D^T$, moves all markings of $S \backslash T$ to the node. Therefore the twist bundle is $\tau_{\mathcal{L}}(T)+\tau_{\mathcal{L}}(S)+|S\backslash T|+1$. The second and third case collapses all markings of T, and the fourth moves all markings of $T \setminus S$ to the node just as the first one did. This explains all the twist conditions appearing in the statement.

The case of prescribed residues on $D^S \cap D^T$ for the D^S summand, i.e. elements in $K_{q-1}^{MW}(k)[D^S \cap D^T]_{D^S}$, behaves the same as for the codimension 0 situation. The difference is that there are now two possibilities whether the additional node is on the S part or the S^c part, resulting in exactly the same conditions as before once for S and once for S^c. □

Corollary 5.2.13. *For $n \geq 3$ we get the known description of $\mathrm{CH}^1(\overline{M_{0,n}})$ from Theorem 3.1.12.*

Proof. We know that $\mathrm{CH}^1(\overline{M_{0,n}}) = H_{n-2}(\overline{M_{0,n}}, K_{n-2}^M)$ and as $K_n^{MW}(\mathcal{L})/\eta = K_n^M$, there are no line bundles to consider. The markings are $\{1, 2, 3, \ldots, n\}$ and fix $(i, j, k) = (3, 1, 2)$. The chain complex here looks as follows.

$$\begin{tikzcd}
K_1^M(k)[\emptyset]_{\overline{M_{0,n}}} \arrow[dr, bend left, "\cdot\eta = 0"] & & \\
\bigoplus_{3\notin S,\ |S|=2} K_0^M(k)[D^S]_{\overline{M_{0,n}}} \arrow[r, "\cdot(\pm 1)"] & \bigoplus_S K_0^M(k)[\emptyset]_{D^S} \arrow[dr, bend left, "\cdot\eta = 0"] & \\
\underbrace{\bigoplus_{S,T\ldots} K_{-1}^M(k)[D^S \cap D^T]_{\overline{M_{0,n}}}}_{=0} \arrow[r] & \underbrace{\bigoplus_{S,T} K_{-1}^M(k)[D^S \cap D^T]_{D^S}}_{=0} \arrow[r] & \underbrace{\bigoplus_{S,T} K_{-1}^M(k)[\emptyset]_{D^S \cap D^T}}_{=0}
\end{tikzcd}$$

There are more rows appearing but all groups are $K_i^M(k)$ for $i < 0$ and thus vanish.

So the homology group in the middle is the cokernel of

$$\bigoplus_{3\notin S,\ |S|=2} K_0^M(k)[D^S]_{\overline{M_{0,n}}} \xrightarrow{\cdot(\pm 1)} \bigoplus_S K_0^M(k)[\emptyset]_{D^S}.$$

By Proposition 5.2.10 this map is given by

$$K^{MW}_{q-1}(k)[D^S]_{\overline{M_{0,n}}} \longrightarrow \sum_T K^{MW}_{q-1}(k)[\emptyset]_{D^T}$$

$$\alpha \longmapsto \begin{cases} \alpha\,[\emptyset]_{D^T}, & S \subseteq T \\ -\alpha\,[\emptyset]_{D^T}, & (S=\{1,\lambda\} \text{ or } S=\{2,\lambda\}) \text{ and } \{3,\lambda\} \subseteq T, \\ & \text{or } S=\{\lambda,\mu\} \text{ and } (\{3,\lambda\}\subseteq T \text{ or } \{3,\mu\}\subseteq T) \\ 0, & \text{otherwise} \end{cases}$$

which yield exactly the relations between the boundary divisors. □

Remark 5.2.14. The relations in the Chow ring of $\overline{M_{0,n}}$ all arise from pullbacks along projections to $\mathbb{P}^1_k$ coming from forgetting all but four of the markings. Restricting to curves parallels this behavior.

Computing the Chow groups this way is somewhat silly. Strictly speaking, our argument used a presentation of the Picard group from Proposition 3.2.4 to get the description of the canonical class as in Proposition 3.2.6. This is not really necessary, as one could have counted how many blow-up centers a curve in $(\mathbb{P}^1_k)^{n-3}$ meets during the iterated blow-up towards $\overline{M_{0,n}}$. Another way to avoid this is to note, that the description of the canonical and normal bundles does not matter at all for the (co)homology with coefficients in K^M, as the sheaf is independent of twists. So we could have done the computation for Milnor K-theory first, get the Picard group this way and continue from there.

The silliness comes from the fact that we definitely used Keel's construction from which the Chow ring description follows easily from the projective bundle formula [Kee92, Theorem 4.1].

Remark 5.2.15. Specializing the computation for $\overline{M_{0,5}}$ and twist by $\omega_{\overline{M_{0,5}}}$ we get:

$$\begin{aligned} H^0(\overline{M_{0,5}}, K^{MW}_q) &= K^{MW}_q(k), \\ H^1(\overline{M_{0,5}}, K^{MW}_q) &= {}_\eta K^{MW}_{q-1}(k) \oplus \left(K^{MW}_{q-1}(k)\right)^4, \\ H^2(\overline{M_{0,5}}, K^{MW}_q) &= K^{MW}_{q-2}(k)/\eta. \end{aligned}$$

This already shows the different behavior of the singular cohomology of the complex points $H^1_{\text{sing}}(\overline{M_{0,5}}(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}^5$, and of the real points $H^1_{\text{sing}}(\overline{M_{0,5}}(\mathbb{R}),\mathbb{Z}) = \mathbb{Z}^4$. We can also see the non-orientability $H^2_{\text{sing}}(\overline{M_{0,5}},\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. The more interesting 2-torsion phenomena are not yet visible for $\overline{M_{0,5}}$, see [EHKR10]. They completely determine the 2-torsion, which grows exponentially in n, whereas the rank grows polynomially.

We decided to not print the (30×12)-matrix for d_2 and the (15×30)-matrix for d_1.

Remark 5.2.16. The two propositions, Proposition 5.2.10 and Proposition 5.2.12, completely describe how to compute all the differentials in the cellular complex. For any necessary choice of orientation in the computation, it is easy to find one and to express the elements with respect to that choice. The problem is that currently there are too many choices to give a nice uniform answer. Given any integer $n \geq 3$ the computation for $\overline{M_{0,n}}$ is, from this point on, purely combinatorical and does not requires any additional geometric input.

6. Linear schemes and the real cycle class map

A different class of schemes that is well-suited for arguments using localization sequences is the class of linear schemes. The goal of this chapter is an extension of the real cycle class isomorphism for strictly cellular schemes, see [HWXZ21], to the more general class of linear schemes.

6.1. Linear schemes

The following definition of linearity comes from [Jan90]. As we will later deal with another version of linearity we will call these ones J-linear (for Jannsen) and the other, more restrictive notion, T-linear (for Totaro).

Definition 6.1.1. Let Y be a scheme. Then Y is called:

0-J-linear if $Y = \emptyset$ or $Y \cong \mathbb{A}_k^N$ for some $N \geq 0$.

n-J-linear if there exists a triple (Z, X, U) of k-schemes with closed immersion $Z \hookrightarrow X$ and open complement $U = X \setminus Z$ such that:

$Y = U$: Z and X are $(n-1)$-J-linear or

$Y = X$: Z and U are $(n-1)$-J-linear.

A scheme Y is called J-linear, if it is n-J-linear for some $n \geq 0$.

Remark 6.1.2. Any closed immersion is allowed in the definition of J-linear, so in particular non-regular ones.

Remark 6.1.3. The first construction option means that open complements of J-linear schemes in J-linear schemes are again J-linear. The second option will be used to show that stratifications by J-linear schemes are J-linear.

Definition 6.1.4. A stratification of a scheme X is a partition $X = \bigcup_{i\in I} U_i$, with $U_i \subseteq X$ locally closed, for which the following 'boundary condition' holds:

$$U_i \cap \overline{U_j} \neq \emptyset \quad \Rightarrow \quad U_i \subseteq \overline{U_j}.$$

A finite stratification is a stratification with finite index set I.

Remark 6.1.5. Two observations will help in the following proof. The boundary condition ensures that $\overline{U_j} = \bigcup_{U_i \subseteq \overline{U_j}} U_i$ and that there is at most one stratum U_j with $X = \overline{U_j}$.

To see that the second assertion holds assume there are two strata U and V whose closure is X. Then U and V are open in X, as they are locally closed. So $X \setminus U \subsetneq X$ is a closed set containing V. This is a contradiction as $X = \overline{V} \subseteq X \setminus U \subsetneq X$. Hence there can only be one.

Lemma 6.1.6. *Let X be a scheme admitting a finite stratification by locally closed J-linear schemes, then X is J-linear.*

Proof. Write the stratification as $X = \bigcup_{i=1}^{n} U_i$ with $n = |I|$ and U_i locally closed J-linear. Proceed by induction on the number of strata n.

$\boldsymbol{n = 1}$**:** $X = U_1$ is J-linear by assumption.

$\boldsymbol{n \rightsquigarrow n+1}$**:** In the stratification $X = \bigcup_{i=1}^{n+1} U_i$ pick one U_j with $X \setminus \overline{U_j} \neq \emptyset$ (exists by previous observation and $n+1 \geq 2$). Using the other observation write

$$X = \overline{U_j} \cup (X \setminus \overline{U_j}) = \left(\bigcup_{U_i \subseteq \overline{U_j}} U_i \right) \cup \left(\bigcup_{U_i \not\subseteq \overline{U_j}} U_i \right).$$

Now both parts are J-linear by induction hypothesis, since they are stratified by at most n strata. Thus X is J-linear as the disjoint union of a closed J-linear scheme and its J-linear complement.

This concludes the proof. □

Remark 6.1.7. Instead of picking an arbitrary U_j with $X \setminus \overline{U_j} \neq \emptyset$ in the previous proof, it is possible to pick a closed stratum. For this start with an arbitrary U_j and consider the closure $\overline{U_j} = \bigcup_{U_i \subseteq \overline{U_j}} U_i$. Repeating the process with any $U_i \subseteq \overline{U_j} \setminus U_j$ will terminate after finitely many steps, because in each step there are strictly less strata available and there are only finitely many to begin with. For the last stratum we necessarily have $\overline{U_i} = U_i$.

Lemma 6.1.8. *Let X be a scheme and $A_1, \dots, A_n$ closed irreducible J-linear subschemes of X with $\bigcap_{i\in I} A_i$ irreducible and J-linear for every subset $I \subseteq \{1,\dots,n\}$, then $\bigcup_{i=1}^n A_i$ is J-linear.*

Proof. Proceed by induction on n. The idea is to find an appropriate stratification of the union $\bigcup_{i=1}^n A_i$ by splitting it into $2^n - 1$ parts, according to which of the A_i an element belongs to.

$n = 1$: A_1 is J-linear by assumption.

$n \rightsquigarrow n+1$: Write

$$\bigcup_{i=1}^{n+1} A_i = \bigcup_{i=1}^{n+1} \bigcup_{\substack{J\subseteq\{1,\dots,n+1\},\\ |J|=i}} \left(\bigcap_{j\in J} A_j\right) \setminus \left(\bigcup_{j\notin J} A_j \cap \bigcap_{j\in J} A_j\right).$$

Using the J-linearity of the closed subschemes $\bigcup_{j\notin J}\left(A_j \cap \bigcap_{j'\in J} A_{j'}\right)$ (by induction hypotheses, since $|J^c| \le n$) and $\bigcap_{j\in J} A_j$ (by assumption) shows that the strata $\left(\bigcap_{j\in J} A_j\right) \setminus \left(\bigcup_{j\notin J} A_j \cap \bigcap_{j\in J} A_j\right)$ are all J-linear.

To see that this actually is a stratification the boundary condition needs to be checked. Picking a stratum means picking $U = \left(\bigcap_{j\in J} A_j\right) \setminus \left(\bigcup_{j\notin J} A_j \cap \bigcap_{j\in J} A_j\right)$ for some non-empty $J \subseteq \{1,\dots,n\}$. The stratum U is open in the closed and irreducible $\bigcap_{j\in J} A_j$, hence this is the closure of U.

Claim: Strata U' (corresponding to a J') meeting $\overline{U} = \bigcap_{j\in J} A_j$ satisfy $J \subseteq J'$. Suppose this is not the case and there is a $k \in J \setminus J'$, then there cannot be an element both in $\overline{U} = \bigcap_{j\in J} A_j \subseteq A_k$ and $U' = \left(\bigcap_{j\in J'} A_j\right) \setminus \left(\bigcup_{j\notin J'} A_j \cap \bigcap_{j\in J} A_j\right) \subseteq A_k^c$.

But $J \subseteq J'$ implies the wanted boundary condition $U' \subseteq \bigcap_{j\in J'} A_j \subseteq \bigcap_{j\in J} A_j = \overline{U}$. □

Remark 6.1.9. Taking $n = 3$ in the previous lemma gives the decomposition of the classical Venn diagram in its 7 parts. Explicitly the stratification will be:
$A_1 \cap A_2 \cap A_3$,
$(A_1 \cap A_2) \setminus A_3$, $(A_1 \cap A_3) \setminus A_2$, $(A_2 \cap A_3) \setminus A_1$,
$A_1 \setminus (A_2 \cup A_3)$, $A_2 \setminus (A_1 \cup A_3)$, $A_3 \setminus (A_1 \cup A_2)$.
The need for induction comes from the fact that for J-linearity the set differences of strata are written slightly different as (e.g. for $A_1 \setminus (A_2 \cup A_3)$):

$$A_1 \setminus \Big(((A_1 \cap A_2) \setminus (A_1 \cap A_2 \cap A_3)) \bigcup ((A_1 \cap A_3) \setminus (A_1 \cap A_2 \cap A_3)) \bigcup (A_1 \cap A_2 \cap A_3)\Big).$$

Lemma 6.1.10. *Let X be n-J-linear and Y be m-J-linear, then $X \times Y$ is $(n+m)$-J-linear.*

Proof. Let X be n-J-linear and Y be m-J-linear. The idea is to iteratively build the product, since the construction steps for J-linear schemes are stable under taking a product with a fixed scheme. For $(n,m)=(0,0)$ we have $X \cong \mathbb{A}_k^{\dim(X)}$ and $Y \cong \mathbb{A}_k^{\dim(Y)}$, so their product is $X \times Y \cong \mathbb{A}_k^{\dim(X)+\dim(Y)}$ and hence 0-J-linear.

The claim is that $X \times Y$ is $(n+m)$-J-linear. For X there exists a triple (Z, X', U) with a closed immersion $Z \hookrightarrow X'$ and open complement $U = X' \setminus Z$, such that:

$X = U$: Z and X' are $(n-1)$-J-linear or

$X = X'$: Z and U are $(n-1)$-J-linear,

by definition of n-J-linearity. By taking products with Y we get a triple with a closed immersion $Z \times Y \hookrightarrow X' \times Y$ and open complement $U \times Y = (X' \times Y) \setminus (Z \times Y)$, such that:

$X \times Y = U \times Y$: $Z \times Y$ and $X' \times Y$ are $(n-1+m)$-J-linear or

$X \times Y = X' \times Y$: $Z \times Y$ and $U \times Y$ are $(n-1+m)$-J-linear.

This shows the induction step (the respective statement for Y follows by symmetry). $\square$

Corollary 6.1.11. *The schemes $\mathbb{A}_k^n \times \mathbb{G}_m^d$ are d-J-linear.*

Proof. The scheme $\mathbb{A}_k^n$ is 0-J-linear by definition and $\mathbb{G}_m = \mathbb{A}_k^1 \setminus \{0\}$ is 1-J-linear. $\square$

6.2. The real cycle class map for linear schemes

All statements in this section are direct analogs of the corresponding statements in [HWXZ21, Chapter 5], with changed bounds. Only the more detailed look at T-linear schemes in Lemma 6.2.10 has no analog.

Lemma 6.2.1. *Let X be a n-J-linear scheme over $\mathbb{R}$, then the multiplication*

$$\langle\langle -1 \rangle\rangle \colon H_i^{RS}(X, \bar{I}^j) \to H_i^{RS}(X, \bar{I}^{j+1})$$

is an isomorphism for all $i \geq -j+n$ and injective for $i = -j+n-1$.

Proof. Proceed by induction on n. For $n=0$ we have $X \cong \mathbb{A}_{\mathbb{R}}^{\dim(X)}$. By $\mathbb{A}^1$-invariance both sides vanish for $i>0$ and for $i=0$ the map becomes

$$2 = \langle\langle -1 \rangle\rangle \colon \quad 2^j\mathbb{Z} = \bar{I}^j(\mathbb{R}) \cong H_0^{RS}(X, \bar{I}^j) \to H_0^{RS}(X, \bar{I}^{j+1}) \cong \bar{I}^{j+1}(\mathbb{R}) = 2^{j+1}\mathbb{Z},$$

which is an isomorphism.

Suppose we have a closed immersion $Z \hookrightarrow X'$ with open complement $U = X' \setminus Z$ then the localization sequence becomes

$$\begin{tikzcd}
H_{i+1}^{RS}(U,\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_i^{RS}(Z,\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_i^{RS}(X',\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_i^{RS}(U,\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_{i-1}^{RS}(Z,\bar{I}^j) \arrow[d, "\langle\langle -1\rangle\rangle"] \\
H_{i+1}^{RS}(U,\bar{I}^{j+1}) \arrow[r] & H_i^{RS}(Z,\bar{I}^{j+1}) \arrow[r] & H_i^{RS}(X',\bar{I}^{j+1}) \arrow[r] & H_i^{RS}(U,\bar{I}^{j+1}) \arrow[r] & H_{i-1}^{RS}(Z,\bar{I}^{j+1}),
\end{tikzcd}$$

where the diagram commutes as a diagram of W($\mathbb{R}$)-modules.

If $X = X'$, and Z,U are $(n-1)$-J-linear the first, second and fourth vertical morphisms are isomorphisms ($i \geq -j+n-1$) and the fifth is injective ($i-1 \geq -j+n-2$) by induction. Therefore, the morphism in the middle is an isomorphism by the 5-lemma for all $i \geq -j+n-1$.

For the other case of $X = U$, and Z,X' are $(n-1)$-J-linear the part of the localization sequence is

$$\begin{tikzcd}
H_i^{RS}(Z,\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_i^{RS}(X',\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_i^{RS}(U,\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_{i-1}^{RS}(Z,\bar{I}^j) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_{i-1}^{RS}(X',\bar{I}^j) \arrow[d, "\langle\langle -1\rangle\rangle"] \\
H_i^{RS}(Z,\bar{I}^{j+1}) \arrow[r] & H_i^{RS}(X',\bar{I}^{j+1}) \arrow[r] & H_i^{RS}(U,\bar{I}^{j+1}) \arrow[r] & H_{i-1}^{RS}(Z,\bar{I}^{j+1}) \arrow[r] & H_{i-1}^{RS}(X',\bar{I}^{j+1}).
\end{tikzcd}$$

For $i \geq -j+n$ the first, second, fourth and fifth vertical morphisms are isomorphisms and the 5-lemma gives the middle isomorphism again. For $i = -j+n-1$ the first and second are still isomorphisms, but the fourth is in general just injective ($i-1 = -j+n-1$) so the 4-lemma shows, that the middle morphism is injective. $\square$

Remark 6.2.2. The notation $H_i^{RS}(X,\bar{I}^j) = H_i(C_{RS}(X,\bar{I}^j))$ can unfortunately be misleading. The sheaves $\bar{I}^j$ vanish for $j < 0$, but the groups $H_i^{RS}(X,\bar{I}^j)$ do not, as the fundamental ideal power appearing in degree i is $\bar{I}^{i+j}$. The vanishing in this degree happens therefore, for $i+j < 0$.

Compare this with Remark 2.1.6. The reason we choose this indexing is to not clutter our notation even more. The Borel-Moore and cohomological notation both have index shifts in the localization sequences due to the dimensions involved.

Remark 6.2.3. The possible failure of surjectivity can already be seen at the end of the localization sequence. The upper row vanishes one degree before the lower one.

$$\begin{tikzcd}
H_i^{RS}(Z,\bar{I}^{-i}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_i^{RS}(X',\bar{I}^{-i}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_i^{RS}(U,\bar{I}^{-i}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & 0 \arrow[d, "\langle\langle -1\rangle\rangle"] \\
H_i^{RS}(Z,\bar{I}^{-i+1}) \arrow[r] & H_i^{RS}(X',\bar{I}^{-i+1}) \arrow[r] & H_i^{RS}(U,\bar{I}^{-i+1}) \arrow[r] & H_{i-1}^{RS}(Z,\bar{I}^{-i+1}),
\end{tikzcd}$$

For the use in a concrete example the bounds could be improved by keeping track of which construction steps are used in making the n-J-linear scheme. For example schemes with stratifications by affine spaces have the full range $i \geq -j$ where the multiplication $\langle\langle -1 \rangle\rangle \colon H_i^{RS}(X, \bar{I}^j) \to H_i^{RS}(X, \bar{I}^{j+1})$ is an isomorphism.

Remark 6.2.4. The requirement to be defined over $\mathbb{R}$ is somewhat necessary, because for general fields the $i = 0$ case does not hold. For finite fields $\mathbb{F}_q$ (or more generally fields in which every binary quadratic form is universal) we have $I(\mathbb{F}_q) = \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \cong \mathbb{Z}/2\mathbb{Z}$ and $I^2(\mathbb{F}_q) = 0$, so there cannot be an isomorphism from $\bar{I}^1(\mathbb{F}_q) \cong \mathbb{Z}/2\mathbb{Z}$ to $\bar{I}^2(\mathbb{F}_q) = 0$.

The proof works for any field where the $i = 0$ case holds.

Corollary 6.2.5. *Let X be a smooth, n-J-linear scheme over $\mathbb{R}$, then the multiplication*

$$\langle\langle -1 \rangle\rangle \colon H^i(X, \bar{I}^j) \to H^i(X, \bar{I}^{j+1})$$

is an isomorphism for all $j \geq i + n$ and injective for $j = i + n - 1$.

Proof. For X smooth we have $H^{RS}_{\dim(X)-i}(X, \bar{I}^{j-\dim(X)}) = H^i(C^{RS}(X, \bar{I}^j)) \cong H^i(X, \bar{I}^j)$ and therefore,

$$H^i(X, \bar{I}^j) = H^{RS}_{\dim(X)-i}(X, \bar{I}^{j-\dim(X)}) \xrightarrow{\cong} H^{RS}_{\dim(X)-i}(X, \bar{I}^{j-\dim(X)+1}) = H^i(X, \bar{I}^{j+1}),$$

since $\dim(X) - i \geq -j + \dim(X) + n$, which holds by assumption $j \geq i + n$. The injectivity part holds for $\dim(X) - i = -j + \dim(X) + n - 1$. $\square$

Corollary 6.2.6. *Let X be a smooth scheme over $\mathbb{R}$ which is stratified by at worst d-J-linear schemes, then the multiplication*

$$\langle\langle -1 \rangle\rangle \colon H^i(X, \bar{I}^j) \to H^i(X, \bar{I}^{j+1})$$

is an isomorphism for all $j \geq i + d$ and injective for $j = i + d - 1$.

Proof. In the proof of Lemma 6.2.1 the stratification step keeps the bounds, where the multiplication is an isomorphism, unchanged. $\square$

Remark 6.2.7. Using Corollary 6.1.11 together with Corollary 6.2.6 shows that smooth schemes stratified by cells of the form $\mathbb{A}^n_k \times \mathbb{G}^{d'}_m$ with all $d' \leq d$ satisfy the above property.

The following variant of linear schemes comes from [Tot14].

Definition 6.2.8. Let Y be a scheme. Then Y is called:

0-T-linear if $Y = \emptyset$ or $Y \cong \mathbb{A}^n_k$ for some $n \geq 0$.

n-T-linear if one of the following holds:

(i) there exists a triple $(Z, \mathbb{A}^n_k, Y)$ of k-schemes with closed immersion $Z \hookrightarrow \mathbb{A}^n_k$ and open complement $Y = \mathbb{A}^N_k \setminus Z$, where Z is $(n-1)$-T-linear

(ii) Y is stratified by $(n-1)$-T-linear schemes.

A scheme Y is called T-linear, if it is n-T-linear for some $n \geq 0$.

Remark 6.2.9. The difference between J-linear and T-linear schemes is that J-linearity is stable under open complements of linear schemes in arbitrary linear schemes, not just affine spaces $\mathbb{A}^n_k$ as it is for T-linearity. Lemma 6.1.6 shows that all T-linear schemes are J-linear. The author does not know any J-linear scheme that is not already T-linear.

In the more restrictive notion of T-linearity one can get a precise description of the influence the non-strictly cellular part has.

Lemma 6.2.10. *Let $(Z, \mathbb{A}^n_{\mathbb{R}}, X)$ be a triple of $\mathbb{R}$-schemes with closed immersion $Z \hookrightarrow \mathbb{A}^n_{\mathbb{R}}$ and open complement $X = \mathbb{A}^n_{\mathbb{R}} \setminus Z \neq \emptyset$, then the multiplication*

$$\langle\!\langle -1 \rangle\!\rangle \colon H^{RS}_i(X, \bar{I}^j) \to H^{RS}_i(X, \bar{I}^{j+1})$$

is an isomorphism for (i,j) in the following cases:

$\boldsymbol{i+j=0}$**:** *if and only if $H^{RS}_{i-1}(Z, \bar{I}^{-i+1}) = 0$*

$\boldsymbol{i+j \geq 1}$**:** *if and only if this holds for $(i-1, j)$ on Z*

Proof. The case distinction is necessary because of the previously mentioned vanishing in the localization sequence. Note that $\dim(Z) \leq n-1$ (otherwise $Z = \mathbb{A}^n_{\mathbb{R}}$ and $X = \emptyset$). The localization sequence here is:

$$\begin{tikzcd}
H^{RS}_i(Z,\bar{I}^j) \arrow[r] \arrow[d, "\langle\!\langle -1 \rangle\!\rangle"] & H^{RS}_i(\mathbb{A}^n_{\mathbb{R}},\bar{I}^j) \arrow[r] \arrow[d, "\langle\!\langle -1 \rangle\!\rangle"] & H^{RS}_i(X,\bar{I}^j) \arrow[r] \arrow[d, "\langle\!\langle -1 \rangle\!\rangle"] & H^{RS}_{i-1}(Z,\bar{I}^j) \arrow[r] \arrow[d, "\langle\!\langle -1 \rangle\!\rangle"] & H^{RS}_{i-1}(\mathbb{A}^n_{\mathbb{R}},\bar{I}^j) \arrow[d, "\langle\!\langle -1 \rangle\!\rangle"] \\
H^{RS}_i(Z,\bar{I}^{j+1}) \arrow[r] & H^{RS}_i(\mathbb{A}^n_{\mathbb{R}},\bar{I}^{j+1}) \arrow[r] & H^{RS}_i(X,\bar{I}^{j+1}) \arrow[r] & H^{RS}_{i-1}(Z,\bar{I}^{j+1}) \arrow[r] & H^{RS}_{i-1}(\mathbb{A}^n_{\mathbb{R}},\bar{I}^{j+1}).
\end{tikzcd}$$

$\boldsymbol{n = i = -j}$**:** The diagram becomes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \bar{I}^0(\mathbb{R}) \arrow[r] \arrow[d, "\langle\!\langle -1 \rangle\!\rangle"] & H^{RS}_n(X,\bar{I}^{-n}) \arrow[r] \arrow[d, "\langle\!\langle -1 \rangle\!\rangle"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \bar{I}^1(\mathbb{R}) \arrow[r] & H^{RS}_n(X,\bar{I}^{-n+1}) \arrow[r] & H^{RS}_{n-1}(Z,\bar{I}^{n+1}) \arrow[r] & 0,
\end{tikzcd}$$

by $\dim(Z) < n$, homotopy invariance and the vanishing of H^{RS} in that range. The second vertical map is an isomorphism, so by the five lemma the middle morphism is an isomorphism if and only if the fourth one is.

$\boldsymbol{n-1 \geq i = -j}$**:** The diagram becomes

$$\begin{tikzcd} * \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H_i^{RS}(X, \bar{I}^{-i}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ * \arrow[r] & 0 \arrow[r] & H_i^{RS}(X, \bar{I}^{-i+1}) \arrow[r] & H_{i-1}^{RS}(Z, \bar{I}^{i+1}) \arrow[r] & 0, \end{tikzcd}$$

by the same reasoning. Now we have $H_i^{RS}(X, \bar{I}^{-i}) \cong 0$ from the upper exact sequence and $H_i^{RS}(X, \bar{I}^{-i+1}) \cong H_{i-1}^{RS}(Z, \bar{I}^{i+1})$ from the lower one. Therefore, the middle map is an isomorphism if and only if $H_{i-1}^{RS}(Z, \bar{I}^{i+1}) \cong 0$.

$\boldsymbol{i+j \geq 1,\ i = n}$**:** The diagram becomes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \bar{I}^{n+j}(\mathbb{R}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_n^{RS}(X, \bar{I}^{j}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_{n-1}^{RS}(Z, \bar{I}^{j}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & 0 \arrow[d] \\ 0 \arrow[r] & \bar{I}^{n+j+1}(\mathbb{R}) \arrow[r] & H_n^{RS}(X, \bar{I}^{j+1}) \arrow[r] & H_{n-1}^{RS}(Z, \bar{I}^{j+1}) \arrow[r] & 0, \end{tikzcd}$$

by the same reasoning. The second map is an isomorphism, so the third map is an isomorphism if and only if the fourth is.

$\boldsymbol{i+j \geq 1,\ i \leq n-1}$**:** The diagram becomes

$$\begin{tikzcd} * \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H_i^{RS}(X, \bar{I}^{j}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & H_{i-1}^{RS}(Z, \bar{I}^{j}) \arrow[r] \arrow[d, "\langle\langle -1\rangle\rangle"] & 0 \arrow[d] \\ * \arrow[r] & 0 \arrow[r] & H_i^{RS}(X, \bar{I}^{j+1}) \arrow[r] & H_{i-1}^{RS}(Z, \bar{I}^{j+1}) \arrow[r] & 0, \end{tikzcd}$$

again by the same reasoning. Now we have isomorphisms $H_i^{RS}(X, \bar{I}^{j}) \cong H_{i-1}^{RS}(Z, \bar{I}^{j})$ and $H_i^{RS}(X, \bar{I}^{j+1}) \cong H_{i-1}^{RS}(Z, \bar{I}^{j+1})$, showing the assertion in the last case.

This finishes the proof. □

Remark 6.2.11. Assuming X and Z to be smooth in the previous lemma means for the $i+j=0$ case: $\langle\langle -1\rangle\rangle \colon H^i(X, \bar{I}^i) \to H^i(X, \bar{I}^{i+1})$ is an isomorphism if and only if $H^{i-c-1}(Z, \bar{I}^{i-c+1}) = 0$ for $c = \operatorname{codim}_{\mathbb{A}^n_{\mathbb{R}}}(Z) = n - \dim(Z)$.

Proposition 6.2.12. *Let X be a smooth scheme over $\mathbb{R}$ for which the multiplication*

$$\langle\langle -1\rangle\rangle \colon H^i(X, \bar{I}^j) \to H^i(X, \bar{I}^{j+1})$$

is an isomorphism for all $j \geq i+n$, then

$$\langle\langle -1\rangle\rangle \colon H^i(X, I^j(\mathcal{L})) \to H^i(X, I^{j+1}(\mathcal{L}))$$

is an isomorphism for all $j \geq i+n$ and all line bundles $\mathcal{L}$ on X.

Proof. The $n = 0$ case is treated in [HWXZ21, Proposition 5.5] and this proof is verbatim the same, with exactly one difference. The proof goes by downward induction, starting with the case $j \geq \dim(X)+1$ which holds by [Jac17, Corollary 8.11]. Here the downward induction terminates n steps earlier. □

When writing $X(\mathbb{R})$ we means the set of $\mathbb{R}$-points endowed with the euclidean topology.

Remark 6.2.13. Let X be a smooth scheme over $\mathbb{R}$. For non-negative integers $i, j \in \mathbb{Z}$, and a line bundle $\mathcal{L}$, there is a real cycle class map $H^i(X, I^j(\mathcal{L})) \to H^i_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$. This was defined in [Jac17] and further studied in [HWXZ21].

Theorem 6.2.14. *Let X be a smooth scheme over $\mathbb{R}$ and $\mathcal{L}$ a line bundle such that*

$$\langle\!\langle -1 \rangle\!\rangle \colon H^i(X, I^j(\mathcal{L})) \to H^i(X, I^{j+1}(\mathcal{L}))$$

is an isomorphism for all $j \geq i + n$, then the real cycle class map

$$H^i(X, I^j(\mathcal{L})) \to H^i_{sing}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$$

gives:

$j \geq i + n$: *a group isomorphism $H^i(X, I^j(\mathcal{L})) \cong H^i_{sing}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$.*

$j = i + n - 1$: *a group isomorphism to a subgroup of the singular cohomology*

$$H^i(X, I^j(\mathcal{L})) \cong 2 \cdot H^i_{sing}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \subseteq H^i_{sing}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

$j < i + n - 1$: *a map to the singular cohomology $H^i_{sing}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ with image*

$$\mathrm{im}\left(H^i(X, I^j(\mathcal{L})) \to H^i_{sing}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))\right) = 2^{i-j} \cdot H^i_{sing}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

Proof. The proof is again verbatim the same downward induction as in [HWXZ21, Theorem 5.7] just terminating a little bit earlier. □

Remark 6.2.15. In particular the statement of Theorem 6.2.14 holds for smooth J-linear varieties over $\mathbb{R}$, by Corollary 6.2.5 and Proposition 6.2.12.

A. Differentials in the case of $\mathbb{P}^1_{\boldsymbol{k}} \times \mathbb{P}^1_{\boldsymbol{k}}$

	d_1	d_2
e_1 even e_2 even	$\begin{pmatrix} 0 & -1 & 0 & \varepsilon & & & & \\ & & & 1 & 0 & -1 & & \\ & \langle -1\rangle & & & & & 0 & \varepsilon \\ & & & & & \langle -1\rangle & & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \varepsilon & & \\ & & & \varepsilon \\ 0 & & \varepsilon & \\ & & & 1 \\ & 1 & & \\ & & & 1 \\ & & 1 & \\ & & & \varepsilon \end{pmatrix}$
e_1 odd e_2 even	$\begin{pmatrix} 0 & -1 & \eta & -1 & & & & \\ & & & 1 & 0 & -1 & & \\ & \langle -1\rangle & & & & & \eta & -1 \\ & & & & & \langle -1\rangle & & 1 \end{pmatrix}$	$\begin{pmatrix} \eta & -1 & & \\ & & \eta & -1 \\ 0 & & \varepsilon & \\ & & & 1 \\ & 1 & & \\ & & & 1 \\ & & 1 & \\ & & & \varepsilon \end{pmatrix}$
e_1 even e_2 odd	$\begin{pmatrix} \langle -1\rangle\eta & \varepsilon & 0 & \varepsilon & & & & \\ & & & 1 & \langle -1\rangle\eta & \varepsilon & & \\ & \langle -1\rangle & & & & & 0 & \varepsilon \\ & & & & & \langle -1\rangle & & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \varepsilon & & \\ & & & \varepsilon \\ \eta & & -1 & \\ & \eta & & \langle -1\rangle \\ & 1 & & \\ & & & 1 \\ & & 1 & \\ & & & \varepsilon \end{pmatrix}$
e_1 odd e_2 odd	$\begin{pmatrix} \langle -1\rangle\eta & \varepsilon & \eta & -1 & & & & \\ & & & 1 & \langle -1\rangle\eta & \varepsilon & & \\ & \langle -1\rangle & & & & & \eta & -1 \\ & & & & & \langle -1\rangle & & 1 \end{pmatrix}$	$\begin{pmatrix} \eta & -1 & & \\ & & \eta & -1 \\ \eta & & -1 & \\ & \eta & & \langle -1\rangle \\ & 1 & & \\ & & & 1 \\ & & 1 & \\ & & & \varepsilon \end{pmatrix}$

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