Bielliptic surfaces over arbitrary ground fields

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Summary

Bielliptic surfaces form one of the four classes of minimal surfaces of Kodaira dimension zero. Over algebraically closed fields every bielliptic surface arises as a quotient of a product of two genus-one curves by a finite commutative group scheme. We study the classification of bielliptic surfaces in an arithmetic setting, i.e. over arbitrary ground fields. Our main result states that in this context not every bielliptic surface is a quotient of a product of two curves. This can be attributed to an obstruction in a second cohomology group. We furthermore construct concrete examples of bielliptic surfaces that are not quotients of the above form.

Contents

Introduction

The Enriques classification of minimal *complex* smooth, proper, minimal algebraic surfaces is a particularly classical topic, whose study was initiated in the early 20th century in the larger framework of the Italian school of algebraic geometry; for an exposition see e.g. [Băd01; Bea96; Sha96]. It was studied by prominent mathematicians of the time, including Enriques, Severi and Castelnuovo. Their classification is an elaborate collection of statements concerning many classes of surfaces, each determined by specific numerical invariants. Only later did the concept of the *Kodaira dimension* provide a more natural framework for the classification of surfaces. This invariant measures in some sense the 'size' of the dualising sheaf. For surfaces it takes values in the set $\{-\infty, 0, 1, 2\}$. Fixing the Kodaira dimension, we obtain four classes of minimal surfaces. They are called: the *rational and ruled surfaces*, the *surfaces of Kodaira dimension* 0, the *honestly elliptic surfaces*, and *surfaces of general type*, corresponding to Kodaira dimensions $-\infty$, 0, 1 and 2, respectively.

Our main interest lies with surfaces whose Kodaira dimension equals zero. Since elliptic curves also have Kodaira dimension zero, these surfaces may be seen as their two-dimensional counterparts. One of the results in the Enriques classification states that there are four subclasses of minimal surfaces, distinguished by the second Betti number, namely *K3 surfaces* with $b_2 = 22$, *Enriques surfaces* with $b_2 = 10$, *abelian surfaces* with $b_2 = 6$, and *bielliptic surfaces* (also called *hyperelliptic surfaces*) with $b_2 = 2$; see Theorem 3.1.23. Each family has its own rich geometric theory.

Although this characterisation gives a natural and intrinsic definition of each of the classes, it does not give much immediate insight into the geometric properties, nor does it unveil the interrelationships between the four classes. For example, the Kummer construction produces a K3 surface from an abelian surface. Additionally, the first two classes are inextricably linked: every complex Enriques surface is the quotient of a K3 surface by a free involution.

The last family of bielliptic surfaces is often overshadowed by its three brothers, despite displaying interesting properties that are not shared by the other classes. For example, fundamental groups of bielliptic surfaces are noncommutative, related to the so-called *crystallographic groups*; see [Iit69]. The other three families have an abelian fundamental group. Furthermore, bielliptic surfaces may possess numerically trivial invertible sheaves that are not algebraically trivial.

Complex bielliptic surfaces are characterised as quotients of abelian surfaces by free actions of finite commutative groups. In that sense they are to abelian surfaces what Enriques surfaces are to K3 surfaces. Not every abelian variety arises as a cover of a complex bielliptic surface: a structure theorem states that every bielliptic surface is the quotient of a product $\widetilde{C} \times \widetilde{D}$ of two elliptic curves, where the finite commutative group G acts diagonally on the two factors, such that the action is by translation on the first factor $\widetilde{C}$, but has fixed points on the second factor $\widetilde{D}$; see Theorem 3.4.1. We note the similarities to *ruled* surfaces, which arise if we ask instead that $\widetilde{D} \cong \mathbb{P}^1$, and abelian surfaces, which correspond to $G = 0$, respectively.

In contrast to Enriques surfaces, where no concrete description of all free involutions on K3 surfaces is known, the *Bagnera–de Franchis classification* gives a concrete and complete description of all bielliptic surfaces. The assumptions on the actions on the two factors impose severe restrictions on the isomorphism class of the group G. It turns out that there are seven isomorphism classes for the group G, whose actions are listed in Theorem 3.4.4, but see also List VI.20 of [Bea96] or p. 339 of [BF08]. This partitions the set of bielliptic surfaces into seven *types*, as follows.

Type	(a1)	(a2)	(b1)	(b2)	(c1)	(c2)	(d)
G	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$

The above structure result implies that a complex bielliptic surface admits two elliptic fibrations, to be defined in Definition 3.1.30, given by the two projections $f\colon (\widetilde{C} \times \widetilde{D})/G \to \widetilde{C}/G = P$ and $g\colon (\widetilde{C} \times \widetilde{D})/G \to \widetilde{D}/G = B$; in a more general context see Sections 3.2 and 3.3. The two fibrations justify the nomenclature *bielliptic*. Note that the apparent symmetry in the two factors $\widetilde{C} \leftrightarrow \widetilde{D}$ is broken by the different postulates in the action of the group scheme G. The curve $P = \widetilde{C}/G$ is an elliptic curve, being the quotient of an elliptic curve by a free action, whereas the existence of fixed points for the action on $\widetilde{D}$ is the cause for ramification in the quotient map $\widetilde{D} \to B$, whence $B \cong \mathbb{P}^1$ follows from the Riemann–Hurwitz formula. Each of the two elliptic fibrations hence has its own characteristics: e.g. all fibres of f are smooth, whereas for g has degenerate fibres.

The Enriques classification of surfaces was extended to algebraically closed fields of arbitrary characteristic by Bombieri and Mumford in a series of articles [Mum69; BM77; BM76]. It turns out that the classification remains intact in characteristic 0 and characteristic $p \geq 5$. In order to include the small characteristics 2 and 3, their marvellous insight is to allow actions by *group schemes* that may be non-smooth. For example, an Enriques surface in characteristic 2 need not necessarily be the quotient of a K3 surface by a $\mathbb{Z}/2\mathbb{Z}$-action: it may also be the quotient by μ_2 or α_2 of a so-called *K3-like surface*. The K3-like surface should be thought of as a 'non-smooth version of a K3 surface'. The non-smoothness of the surface is, in some sense, compensated by the non-smoothness of the group scheme.

A similar adjustment is necessary for bielliptic surfaces. In the small characteristics $p = 2$ and $p = 3$, the curve $\widetilde{D}$ occurring in the structure theorem may cease to be smooth, instead being isomorphic to the rational cuspidal curve. In this case, the product $\widetilde{C} \times \widetilde{D}$ should be thought of as a 'non-smooth version of an abelian surface'. Bielliptic surfaces for which $\widetilde{D}$ is not smooth are referred to as *quasi-bielliptic surfaces*. Note that we include quasi-bielliptic surfaces within the class of bielliptic surfaces, which is natural in light of the Enriques classification. Nevertheless it is a non-standard convention; see the discussion in Remark 3.2.18. Bombieri and Mumford extended the Bagnera–de Franchis classification include quasi-bielliptic surfaces, giving a list of non-reduced finite commutative group schemes G and actions such that the quotient $(\widetilde{C} \times \widetilde{D})/G$ is smooth; see Theorem 3.4.12.

The extension of the Enriques classification to positive characteristic of Bombieri and Mumford provides a reasonable context for certain counterexamples in small characteristic. For example, the *Igusa surface* is a notable 'pathological' surface in characteristic 2, being the first known example of a smooth, proper scheme with a non-reduced Picard scheme; see [Igu55]. It also satisfies the curious property that its tangent sheaf is trivial, even though it is not a twisted form of an abelian variety, which is impossible in characteristic 0; see e.g. [MS87]. Both properties are elucidated by an Igusa surface being an bielliptic surface in critical characteristic: in our terminology, an Igusa surface is nothing more than an example of a bielliptic surface of type (a1).

Bielliptic surfaces also exhibit interesting behaviour from an arithmetic point of view. For instance, the first known example of a smooth, proper scheme of finite type for which the Brauer–Manin obstruction is insufficient to explain the failure of the Hasse principle for the existence of a rational point is a bielliptic surface of type (a1) over $\mathbb{Q}$, as given by Skorobogatov in [Sko99]; see also §8 of [Sko01]. The crucial feature of bielliptic surfaces used in this construction is the noncommutativity of their geometric fundamental group. A refinement of the Brauer–Manin obstruction, the so-called *descent obstruction*, is sufficient to explain the failure of the Hasse principle; see Cor. 3.1 of [Sko09]. Related to this is the existence a bielliptic surface X over $\mathbb{Q}$ with a curious property, as constructed in [CSS97]. The set of real points $X(\mathbb{R})$ equipped with the Euclidean topology has two connected components, and the set of rational points $X(\mathbb{Q})$ is dense in one component only. This provides a counterexample to a conjecture by Mazur; see [Maz92; Maz95]

Bielliptic surfaces also form a natural class of *examples*. As a consequence of their similarity to abelian surfaces, they often display non-trivial and interesting behaviour in various aspects of algebraic geometry. On the other hand, the concrete description of the Bagnera–de Franchis classification regularly allows for concrete insight. See for example the articles [Pot17; Nue25; Ree23; Tak20; CF03; Mar22].

This dissertation is concerned with the classification of bielliptic surfaces over arbitrary ground fields. In *critical characteristic*, which can be either $p = 2$ or $p = 3$, we propose to partition the collection of bielliptic surfaces into three classes: *ordinary*, *classical* and *supersingular*. This trichotomy aids in the clarification of the critical behaviour that occurs in these low characteristics, for example non-smoothness of the Picard scheme. This terminology is analogous to the trichotomy of Enriques surfaces in characteristic 2, as introduced by Bombieri and Mumford. Aided by the trichotomy, we propose an extension of the definition of *type* to quasi-bielliptic surfaces; see Table 3.3.

The main aim is to study bielliptic surfaces in an arithmetic setting, i.e. over arbitrary ground fields. The ground field is in particular not assumed to be algebraically closed or even perfect. A central question is to what extent the Bagnera–de Franchis classification holds over arbitrary ground fields. If the Bagnera–de Franchis cover $\widetilde{C}_{k^{\text{alg}}} \times \widetilde{D}_{k^{\text{alg}}} \to X^{\text{alg}}$ descends to a morphism over k, we say that there exists a *Bagnera–de Franchis cover* $Z \to X$. Provided it exists, it shares many properties with the Bagnera–de Franchis cover over an algebraically closed field: for example, Z decomposes as the product $Z = \widetilde{C} \times \widetilde{D}$ of two genus-one curves.

The finite commutative group scheme G arising from the Bagnera–de Franchis classification is intrinsic to the bielliptic surface: we show that its Cartier dual can be recovered as a certain subgroup scheme of the Picard scheme; see Proposition 4.1.28. If X admits a Bagnera–de Franchis cover, then a Bagnera–de Franchis cover Z is canonically equipped with a G-action such that $X \cong Z/G$. Even though the group scheme G descends to an arbitrary ground field, our first main result states that the Bagnera–de Franchis cover may not descend along with it.

Theorem A (See Theorem 4.1.30 and Corollary 4.3.13)**.** *Let X be a bielliptic surface over an arbitrary ground field k. There is a cohomological obstruction $\alpha \in \mathrm{H}^2(k, G)$ in the second fppf-cohomology to the existence of a Bagnera–de Franchis cover. If $p \neq 2, 3$ and $\widetilde{C}_{k^{\text{alg}}}$ and $\widetilde{D}_{k^{\text{alg}}}$ are not isogenous, then the non-vanishing of the obstruction furthermore implies that X is not isomorphic to the quotient of a product of smooth genus-one curves by the free action of a finite group scheme.*

We study the cohomological obstruction closely. Our second main result consequently states a number of unrelated criteria under which the obstruction necessarily vanishes. Some depend only on arithmetic properties of the ground field, whereas others take geometric properties of the bielliptic surface into account.

Theorem B (See Sections 4.2 and 4.3)**.** *Let X be a bielliptic surface over an arbitrary ground field k. The obstruction to the existence of a Bagnera–de Franchis cover vanishes if one of the following conditions is met:*

(i) $\mathrm{Br}(k)[2] = 0$ *and* $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z}) = 0$*;*

(ii) *k is a global field, the short exact sequence* (4.1.4) *splits, and for every place v of k the base-change to the completion $X_v = X \otimes k_v$ admits a Bagnera–de Franchis cover;*

(iii) *the Albanese P has a rational point;*

(iv) *f is smooth and there exists a canonical cover $Y \to X$ such that the Stein factor D of the composition $Y \to X \to B$ has a rational point.*

(v) *X is a quasi-bielliptic surface;*

Condition (ii) is a local to global principle, cf. Corollary 4.2.14. Considering (i), we show that the cohomology group $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z})$ vanishes if $\mathrm{Br}(k(\zeta_3)) = 0$ (see Theorem 4.2.18). Both Brauer groups vanish if e.g. k is quasi-algebraically closed (see Corollary 4.2.19). We also note that (iii) holds if X has a rational point; in this case the original proof of Bombieri and Mumford generalises verbatim, as verified in detail in [Tak20].

Our third main result concerns the construction of a bielliptic surface that does not admit a Bagnera–de Franchis cover, ensuring that the above results are not vacuous. The construction requires a suitable assumption on the Brauer group of the ground field, so that it can be seen as a partial converse to (i) of the above result.

Theorem C. *Let k be a ground field with $p \neq 2$ and $\mathrm{Br}(k)[2] \neq 0$. Then the bielliptic surface constructed in Chapter 5 does not admit a Bagnera–de Franchis cover.*

In light of the cohomological obstruction, considering Bagnera–de Franchis covers is not an intrinsic approach to classifying bielliptic surfaces in an arithmetic setting. Instead we propose that it is more natural to regard a bielliptic surface as a quotient of its *canonical cover*, which over an algebraically closed ground field is a certain intermediate cover of its Bagnera–de Franchis cover; see Section 3.5. In non-critical characteristic the canonical cover is the minimal cover by an abelian surface. If $X = (\widetilde{C} \times \widetilde{D})/G$ and $\widetilde{D}$ is smooth, the canonical cover is described concretely as the quotient $Y = (\widetilde{C} \times \widetilde{D})/H$, where $H \subset G$ is the subgroup scheme that acts on $\widetilde{D}$ by translation. For non-smooth $\widetilde{D}$ we instead require that H act by translation on the smooth locus of $\widetilde{D}$, which is isomorphic to a copy of $\mathbb{A}^1$; see Notation 3.5.1.

If the quotient map $Y \to X$ descends to the ground field k, we say that X *admits a canonical cover.* In this case it turns out that there is no cohomological obstruction, which is well-known in tame characteristic.

Theorem D (See Theorem 4.1.21)**.** *Every bielliptic surface admits a canonical cover.*

Within the above study of covers of a bielliptic surface, the Picard scheme of X plays a central role. We also study the orthogonally related *Néron–Severi group scheme* $\mathrm{NS}_{X/k}$ of a bielliptic surface, whose torsion subgroup scheme is closely related to other cohomological invariants of X. For example, over the ground field of complex numbers there are isomorphisms $\mathrm{NS}(X)_{\mathrm{tors}} \cong \mathrm{H}^1(X, \mathbb{Z})_{\mathrm{tors}} \cong \mathrm{Br}(X)$. By the central result of [Suw83], the torsion group scheme $\mathrm{NS}^{\tau}_{X/k}$ encodes the Hodge numbers and de Rham numbers of the bielliptic surface; see Theorem 7.2.2. This in particular allows us to compute these cohomological invariants for Jacobian quasi-bielliptic surfaces in characteristic 2.

Theorem E (See Table 7.2)**.** *Let X be a quasi-bielliptic surface of type (d). Then $h^1_{\mathrm{dR}} = h^2_{\mathrm{dR}} = 2$, and its Hodge diamond*

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

coincides with the Hodge diamond of bielliptic surfaces over the complex numbers. Let X be a quasi-bielliptic surface of type (c1). Then $h^1_{\mathrm{dR}} = 3$ and $h^2_{\mathrm{dR}} = 4$, and its Hodge diamond is as follows.

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 1 & \\ 0 & & 4 & & 0 \\ & 1 & & 2 & \\ & & 1 & & \end{array}$$

Even in the cases where we do not determine the Hodge and de Rham numbers definitively, we limit the possibilities drastically. The result of Suwa also implies that degeneration of the Hodge-to-de Rham spectral sequence may be read off from the Hodge diamond and the de Rham numbers. The possibilities are limited sufficiently to classify the bielliptic surfaces whose Hodge-to-de Rham spectral sequence does not degenerate.

Theorem F (See Corollary 7.2.7)**.** *Let X be a bielliptic surface. The Hodge-to-de Rham spectral sequence does not degenerate if and only if*

- *X is supersingular of type (a1) or (a2); or*
- *X is ordinary of type (b1) or (c1).*

Structure of the thesis

In Chapter 1 we briefly recall the theory of torsors, which arise throughout the remainder of the text, and formulate a number of results for later reference. We specialise in Chapter 2 to torsors under abelian varieties, that we call *para-abelian varieties.* From Chapter 3 onwards we study bielliptic surfaces. After proving a number of fundamental properties of bielliptic surfaces such as the existence of the two fibrations, we treat the theory of bielliptic surfaces over an algebraically closed ground field of arbitrary characteristic. Chapter 4 is then concerned with generalising this theory to an arbitrary ground field, with a focus on the Bagnera–de Franchis cover and canonical cover: we define and study the cohomological obstruction to the existence of a BdF-cover. A bielliptic surface of type (a2) that does not admit a BdF-cover is constructed in Chapter 5. It is assembled out of suitable para-elliptic curves that are constructed using the arithmetic of para-elliptic curves as studied in Chapter 6. Finally Chapter 7 studies the torsion subgroup scheme of the Néron–Severi group scheme. It consequently leads to the computation of most of the Hodge and de Rham numbers of quasi-bielliptic surfaces and the classification of bielliptic surfaces for which the Hodge-to-de Rham spectral sequence does not degenerate.

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>
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Voor To Swets

Chapter 1

Torsors

Torsors arise ubiquitously throughout this dissertation. For example, we study bielliptic surfaces through certain covers in Chapter 4. These covers turn out to be torsors. We also frequently encounter para-abelian varieties, which are torsors under abelian varieties; cf. Chapters 2 and 6. We briefly treat the important parts of the theory of torsors, to be referenced throughout the text. For a more thorough treatment of the subject we refer to the literature, e.g. [Sko01; Mil80; Gir71].

1.1 Torsors and the first cohomology group

In this section we define and study torsors. We follow §2 of [Sko01]. Let S be a base scheme and let G be a group scheme. Throughout this dissertation we work in the fppf-topology unless specified otherwise. Here 'fppf' is the standard French abbreviation for *fidèlement plate et de présentation finie*, meaning *faithfully flat and of finite presentation.*

A torsor is, roughly speaking, a twisted form of the action of a group scheme on itself by left multiplication. We mostly consider commutative group schemes, in which case the distinction between left and right multiplication is not important. We consider instead the following equivalent definition.

Definition 1.1.1. Let X be a faithfully flat scheme of finite presentation over S and let $\alpha\colon X \times G \to X$ be a left G-action on X. We say that $X \to S$ is a *G-torsor* (or: *torsor under* G) if the map $\mathrm{pr}_X \times \alpha\colon X \times G \to X \times X$ is an isomorphism. A homomorphism of torsors is an equivariant morphism of schemes.

Example 1.1.2. The most basic example of a G-torsor is $X = G$, with action given by left multiplication. This is called the *trivial* G-torsor.

It follows directly from the definition that every torsor is a twisted form of the trivial torsor. Indeed, since $X \to S$ is itself an fppf-cover, the defining assumption states that the pullback of a torsor along itself $X \times X$ is isomorphic to the trivial torsor $X \times G$. The converse turns out to be true as well: every G-scheme which is locally isomorphic to the trivial torsor is a torsor. Indeed, $G \times X \to X \times X$ is an isomorphism locally on an fppf-cover, hence an isomorphism; see Prop. 2.7.1.viii of [EGA IV$_2$].

Example 1.1.3 (Galois field extensions are torsors). Suppose $S = \mathrm{Spec}(k)$ is the spectrum of a ground field k. Let k'/k be a finite Galois extension, whose Galois group $G = \mathrm{Gal}(k'/k)$ we consider as constant group scheme. Let $X = \mathrm{Spec}(k')$ equipped with the natural G-action.. Then the fibred product is

$$X \times_S X = \mathrm{Spec}(k' \otimes_k k') \xleftarrow{\mathrm{pr}_{k'} \times \alpha} \mathrm{Spec}(k') \times G,$$

so that X is indeed a G-torsor.

There is a partial converse to the above. Let G be a finite abstract group, considered as constant group scheme. A G-torsor is a finite étale cover of k, hence decomposes as a union of spectra of separable field extensions. A *connected* G-torsor is hence isomorphic to $\mathrm{Spec}(k')$, with k'/k a separable field extension of degree $|G|$. Since G acts on $\mathrm{Spec}(k')$, the cardinality of the automorphism group $\mathrm{Aut}(k'/k)$ is at least its degree and therefore k'/k is Galois with Galois group G.

Example 1.1.4 (Line bundles). Let X be a $\mathbb{G}_m$-torsor and consider the quotient $L = (X \times \mathbb{A}^1)/\mathbb{G}_m$, where $\mathbb{G}_m$ acts diagonally on the product, and the action of $\mathbb{G}_m$ on $\mathbb{A}$ is given by scaling by the inverse

$(\lambda, x) \mapsto \lambda^{-1}x$, cf. Notation 1.1.13. Since $\mathrm{H}^1(S, \mathbb{G}_m)$ may equivalently be calculated in the Zariski topology, we verify that L is locally isomorphic to $(\mathbb{G}_m \times \mathbb{A}^1)/\mathbb{G}_m \cong \mathbb{A}^1$, hence L is a line bundle. Conversely let L be a line bundle on S with zero section $0 \in L(S)$. Then the open subscheme $X = L \setminus \{0\}$ is stable under the natural $\mathbb{G}_m$-action on L. Restricting to a Zariski open cover, we see similarly that X is a $\mathbb{G}_m$-torsor.

Example 1.1.5. Torsors also arise as quotient maps of free group scheme actions. Let G be a group scheme acting freely on a scheme X, such that the quotient $S = X/G$ exists as a scheme: then the quotient map $X \to S$ is a G-torsor.

Proposition 1.1.6. *Let X and Y be two G-torsor. If there exists a G-equivariant morphism $f\colon X \to Y$, then X and Y are isomorphic.*

Proof. To verify that f is an isomorphism, we may by the theory of descent restrict to an fppf-cover that trivialises both X and Y. Thus let $f\colon G \to G$ be a G-equivariant morphism of schemes. The identity element $e \in G(S)$ is mapped to a section $f(e) \in G(S)$. By equivariance f is given on T-points by $g \mapsto g \cdot f(e)$, which is an isomorphism. □

Remark 1.1.7. The above result essentially states that every morphism in the category of G-torsors is an isomorphism. This is the defining property for a category to be a *groupoid*.

Corollary 1.1.8. *Let X be a G-torsor. If $X(S)$ is nonempty, then X is isomorphic to the trivial torsor.*

Proof. The choice of rational point $x \in X(S)$ determines an equivariant morphism $G \to X$ by $g \mapsto g{\cdot}x$. □

By varying the base scheme, we obtain another useful result. Let $p\colon S' \to S$ be a morphism of base schemes and let $p^*G = G \times S'$ denote the pullback of G, considered as group scheme over S'; denote the projection map by $f\colon p^*G \to G$. Let X' be a p^*G-torsor and let X be a G-torsor.

Definition 1.1.9. We say that $F\colon X' \to X$ is *equivariant* if

$$\begin{tikzcd}
p^*G \times X' \arrow[r, "\alpha_{X'}"] \arrow[d, "f \times F"] & X' \arrow[d, "F"] \\
G \times X \arrow[r, "\alpha_X"] & X
\end{tikzcd}$$

commutes; cf. Definition 1.2.1.

Lemma 1.1.10. *Let G be a group scheme over S and let $p\colon S' \to S$ be a morphism of base schemes. Let $X \to S$ be a G-torsor and let $X' \to S'$ be a p^*G-torsor. Let $X' \to X$ be an equivariant morphism such that the square*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] \\
S' \arrow[r, "p"] & S
\end{tikzcd} \tag{1.1.1}$$

is commutative. Then (1.1.1) *is Cartesian.*

Proof. Let $Y = S' \times_S X$, then $Y \to S'$ is a G-torsor by its action on X. The natural map $X' \to Y$ is G-equivariant, hence a morphism of torsors and thus an isomorphism. □

Isomorphism classes of G-torsors, being twisted forms of the trivial G-torsors, define cohomology classes in a first non-abelian cohomology. Roughly speaking, it is a rule of thumb that $\mathrm{H}^1(S, \mathrm{Aut}_X)$ classifies isomorphism classes of twisted forms of the object X. Here the first cohomology is computed in the fppf-topology. We furthermore remark that it only has the structure of a pointed set instead of a group, unless $\mathrm{Aut}_{X/S}$ is a *commutative* group scheme.

In our case the automorphism group scheme of the trivial torsor G is isomorphic to a copy of G by left-multiplication. By general theory a G-torsor X defines a cohomology class $[X] \in \mathrm{H}^1(S, G)$, as follows, following Prop. III.4.6 of [Mil80]. Let $(U_i \to S)_i$ be a fppf-covering such that $X \times U_i$ is the trivial torsor and pick a global section $x_i \in \Gamma(U_i, X)$. On the overlaps $U_{ij} = U_i \cap U_j$ the choices of x_i differ up to a unique element $g_{ij} \in \Gamma(U_{ij}, G)$. It is not difficult to show that g_{ij} satisfies the cocycle condition.

The converse is in general not true: not every cocycle class in $\mathrm{H}^1(S, G)$ defines a torsor. That is because the first cohomology $\mathrm{H}^1(S, G)$ actually only classifies *sheaves of G-torsors* on the fppf-site of S, which may or may not be representable by a scheme. Our group schemes will always fall into one of the following two classes for which representability always holds; see Thm. III.4.3 of op. cit.

Theorem 1.1.11. *Let G be a commutative group scheme such that either*

(i) *G is affine over S; or*

(ii) *G is regular and smooth over S.*

Then any sheaf of G-torsors is representable by a scheme. Therefore $\mathrm{H}^1(S, G)$ classifies isomorphism classes of G-torsors.

Example 1.1.12. Suppose $S = \mathrm{Spec}(k)$ where k is a field. Let G be a smooth commutative group scheme over k. Then the fppf-cohomology $\mathrm{H}^1(k, G)$ may be computed in the étale topology, which coincides with the Galois cohomology $\mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), G(k^{\mathrm{sep}}))$ for the Galois action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $G(k^{\mathrm{sep}})$. Let X be a G-torsor. The 1-cocycle $\varphi\colon \mathrm{Gal}(k^{\mathrm{sep}}/k) \to G(k^{\mathrm{sep}})$ can be described explicitly as follows. Pick a geometric point $x \in X(k^{\mathrm{sep}})$, then for each $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$ there is a unique $\varphi(\sigma) \in G(k^{\mathrm{sep}})$ such that $\sigma \cdot x = \alpha(\varphi(\sigma), x)$. A short computation shows that $\varphi\colon \mathrm{Gal}(k^{\mathrm{sep}}/k) \to G(k^{\mathrm{sep}})$ satisfies the cocycle condition.

We now consider functoriality of cohomology. It is contravariant in the base and covariant in the group scheme. Let $p\colon S' \to S$ be a morphism of base scheme and let $G' = G \times_S S'$, then the natural map $p^*\colon \mathrm{H}^1(S', G') \to \mathrm{H}^1(S, G)$ should correspond to a certain construction on the level of torsors directly. It is not difficult to see that the image of $[X]$ is the cohomology class $[X']$, where X' is the base change $X \times S'$. Similarly, let $f\colon G_1 \to G_2$ be a morphism of group schemes, and consider

$$(X \times G_2)/G_1,$$

where the G_1-action is anti-diagonal on $X \times G_2$. Letting G_2 act on $X \wedge^{G_1} G_2$ through the second coordinate, it gets the structure of a torsor under G_2. It turns out that $f_*([X_1]) = [X \wedge^{G_1} G_2]$, where $f_*\colon \mathrm{H}^1(S, G_1) \to \mathrm{H}^1(S, G_2)$ is the natural map.

Suppose now that G is an affine commutative group scheme, so that $\mathrm{H}^1(S, G)$ is naturally equipped with the structure of a group. We describe the group operation explicitly on the level of torsors.

Notation 1.1.13. Let G be an affine commutative group scheme and let X and Y be G-torsors. Then the *contracted product* $X \wedge_S^G Y$ is the G-torsor

$$X \wedge_S^G Y = (X \times_S Y)/G,$$

where the G-action on $X \times Y$ is anti-diagonal: the action on X is as usual, but the action on Y is the *inverse* action $(g, y) \mapsto g^{-1}y$ on Y.

Theorem 1.1.14. *Let G be an affine commutative group scheme and let X and Y be two G-torsors. Then $[X] + [Y] = [X \wedge^G Y]$.*

1.2 Morphisms of torsors and the long exact sequence

Let S be a base scheme. Having introduced torsors as *objects*, we now study morphisms between torsors Recall that Proposition 1.1.6 states that any morphism of G-torsors is an isomorphism. The category of G-torsors is hence a groupoid, and for our purposes not very interesting. We should instead allow the group scheme G to change.

Definition 1.2.1. The *category of torsors* (Torsor) is defined as follows. Its objects are pairs (G, X), where G is a group scheme and X is a G-torsor. A morphism $(G_1, X_1) \to (G_2, X_2)$ is a pair of morphisms $f\colon G_1 \to G_2$ and $F\colon X_1 \to X_2$, such that the following diagram is commutative:

$$\begin{tikzcd}
G_1 \times X_1 \arrow[r, "\alpha_1"] \arrow[d, "f \times F"] & X_1 \arrow[d, "F"] \\
G_2 \times X_2 \arrow[r, "\alpha_2"] & X_2
\end{tikzcd}$$

In this case, we say that F is *equivariant* (with respect to f).

There is a cohomological criterion for the existence of a morphism of torsors $(G_1, X_1) \to (G_2, X_2)$ extending a *surjective* underlying homomorphism of group schemes $f\colon G_1 \to G_2$.

Lemma 1.2.2. *Let $f\colon G_1 \to G_2$ be a surjective morphism of group schemes with kernel K. Let X_1 and X_2 be torsors for G_1 and G_2 respectively. Then there is an equivariant morphism $F\colon X_1 \to X_2$ if and only if $f_*([X_1]) = [X_2]$, which is unique up to the action of G_1.*

Proof. Recall that $f_*([X_1])$ denotes the isomorphism class of the contracted product

$$X_1 \wedge^{G_1} G_2 = (X_1 \times G_2)/G_1 \cong X_1/K,$$

where the isomorphism follows from the surjectivity of f. It remains to verify that the existence of a G_1-equivariant map $X_1/K \xrightarrow{\sim} X_2$ is equivalent to the existence of a G_1-equivariant map $F\colon X_1 \to X_2$.

If $X_1/K \xrightarrow{\sim} X_2$ is such an isomorphism, then let F be the composition $X_1 \to X_1/K \xrightarrow{\sim} X_2$. Conversely the action of K on X_2 is trivial, hence by the universal property of quotients the map F induces a map $X_1/K \to X_2$. This is an isomorphism by descent: a trivialising cover of X_1 also trivialises X_2, and restricting to this cover we reduce to the isomorphism $G_1/K \cong G_2$ induced by f.

Last of all, the map $F\colon X_1 \to X_2$ depends only on the choice of isomorphism $X_1/K \xrightarrow{\sim} X_2$, which is unique up to the G_1-action. □

In the setting of Lemma 1.2.2, there is a short exact sequence

$$0 \longrightarrow K \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 0. \tag{1.2.1}$$

This induces a long exact sequence in cohomology.

$$0 \longrightarrow \Gamma(X, K) \longrightarrow \Gamma(X, G_1) \longrightarrow \Gamma(X, G_2) \longrightarrow \mathrm{H}^1(X, K) \longrightarrow \mathrm{H}^1(X, G_1) \longrightarrow \mathrm{H}^1(X, G_2) \xrightarrow{\delta} \mathrm{H}^2(X, K). \tag{1.2.2}$$

Proposition 1.2.3. *Suppose that $f\colon G_1 \to G_2$ is a surjective morphism of group schemes with kernel K. Let X_2 be a G_2-torsor. The following are equivalent:*

(i) *there exists a G_1-torsor X_1 and an equivariant morphism $F\colon X_1 \to X_2$;*

(ii) *there exists a G_1-torsor X_1 such that $X_1/K \cong X_2$;*

(iii) *there exists a cohomology class $[X_1] \in \mathrm{H}^1(S, G_1)$ such that $f_*([X_1]) = [X_2]$;*

(iv) *$\delta([X_2]) = 0$, with δ as in (1.2.2).*

Remark 1.2.4. Although the existence of the stated objects are equivalent, some contain more data than others. For example a $F\colon X_1 \to X_2$ as in case (i) induces an isomorphism $X_1/K \xrightarrow{\sim} X_2$, whereas in cases (ii), (iii) and (iv) such an isomorphism exists but cannot be chosen canonically.

In that sense, the image $\delta([X_2])$ is an *obstruction* for X_2 to arise from a G_1-torsor.

1.3 Lifting property and Leray–Serre five-term exact sequence

Let S be a base scheme, and let $p\colon X \to S$ be a scheme. Let G be a *commutative* group scheme.. A useful cohomological tool is the *Leray–Serre spectral sequence* $E_2^{r,s} = \mathrm{H}^r(S, R^s p_*(p^*G)) \Rightarrow \mathrm{H}^{r+s}(X, p^*G)$. Most of the useful information about the low degrees of the spectral sequence can be obtained through the five-term exact sequence

$$0 \longrightarrow \mathrm{H}^1(S, p_*p^*G) \xrightarrow{p^*} \mathrm{H}^1(X, p^*G) \xrightarrow{t} \mathrm{H}^0(S, R^1p_*(p^*G)) \xrightarrow{\partial} \mathrm{H}^2(S, p_*p^*G) \longrightarrow \mathrm{H}^2(X, p^*G). \tag{1.3.1}$$

Remark 1.3.1. If G is noncommutative, then there is an analogue of the above five-term exact sequence in non-abelian cohomology, excluding the last term; see §V of [Gir71]. Recall that the ith cohomology only has the structure of a pointed set for $i = 1$, 2.

Since $R^1p_*(p^*G)$ is the sheafification of the functor $S' \mapsto \mathrm{H}^1(p^{-1}(S'), p^*G)$ in the fppf-topology, there are canonical maps $t_{S'} \colon \mathrm{H}^1(X_{S'}, p^*G) \to \Gamma(S', R^1p_*p_*(p^*G))$ for every fppf $S' \to S$. Taking $S' = S$ gives the map t of the five-term exact sequence above. The description of the boundary map ∂, called *transgression* in op. cit., is more involved. In a sense, an element of $\mathrm{H}^0(S, R^1p_*(p^*G))$ describes a 'descent datum' for a G-torsor over X, which may or may not be obstructed to exist by an element of $\mathrm{H}^2(S, G)$.

The situation simplifies if the natural map $G \to p_*p^*G$ is an isomorphism. This happens in one of the following cases; see Prop. 2.2.4 of [Sko01].

Proposition 1.3.2. *Suppose that one of the following conditions is satisfied:*

- *the group scheme G is finite and S is geometrically connected;*
- *the group scheme G is of multiplicative type and $h^0(\mathcal{O}_X) = 1$;*
- *the group scheme G is affine and X is projective.*

Then the canonical map $G \to p_*p^*G$ is an isomorphism.

Suppose from now on that G satisfies $G = p_*p^*G$. Then the five term exact sequence becomes the simpler

$$0 \longrightarrow \mathrm{H}^1(S, G) \xrightarrow{p^*} \mathrm{H}^1(X, p^*G) \xrightarrow{t} \mathrm{H}^0(S, R^1p_*(p^*G)) \xrightarrow{\partial} \mathrm{H}^2(S, G) \longrightarrow \mathrm{H}^2(X, p^*G). \tag{1.3.2}$$

Return to the setting of Section 1.2: let $f \colon G_1 \to G_2$ be a surjective morphism of commutative group schemes with kernel $K = \mathrm{Ker}(f)$ and let X_1 be a torsor under G_1. Restricting the G_1-action to the subgroup scheme K, let us consider $X_2 = X_1/K$. The induced action of $G_1/K = G_2$ on X_2 gives it the structure of a G_2-torsor. Furthermore, the quotient map $F \colon X_1 \to X_2$ canonically has the structure of a K-torsor.

We are still interested in the converse question: given a G_2-torsor X_2 and a K-torsor $X \to X_2$, can one equip X with the structure of a G_1-torsor? A partial answer is given in Proposition 1.2.3. We use the above five-term exact sequence to give another criterion. Thus fix a G_2-torsor X_2 with structure morphism $p \colon X_2 \to S$. Our main source is §V.3.2.9.1 of [Gir71].

Following loc. cit., the cohomology $\mathrm{H}^0(T_2, R^1f_*G)$ contains a certain special cohomology class τ, corresponding to the central short exact sequence (1.2.1), which we will describe in more detail. An element of $\mathrm{H}^0(S, R^1p_*(p^*K))$ informally describes a 'descent datum' for a p^*K-torsor over X on some trivialising cover of X_2; more precisely, pick isomorphisms $X_2 \cong G_2$ on some trivialising cover, then on this cover τ describes the p^*K-torsors $G_1 \to G_1/K \cong X_2$. In this way, one can view $\partial(\tau)$ as an obstruction for the 'descent datum' of a p^*K-torsor to be effective.

We combine the exact sequences (1.2.2) and (1.3.2) into one diagram with exact rows.

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{H}^1(S,K) \arrow[r] \arrow[dd, "\mathrm{id}"] & \mathrm{H}^1(X_2, p^*K) \arrow[r, "t"] & \mathrm{H}^0(S, R^1p_*(p^*K)) \arrow[r, "\partial"] \arrow[d, phantom, "\cup"] & \mathrm{H}^2(S,K) \arrow[dd, "\mathrm{id}"] \\
 & & & & \tau \quad [X_2] \arrow[d, phantom, "\cap"] & \\
\cdots \arrow[r] & \Gamma(S, G_2) \arrow[r] & \mathrm{H}^1(S,K) \arrow[r] & \mathrm{H}^1(S, G_1) \arrow[r, "f_*"] & \mathrm{H}^1(S, G_2) \arrow[r, "\delta"] & \mathrm{H}^2(S,K)
\end{tikzcd} \tag{1.3.3}$$

Although it is true that the above diagram is commutative, this is not particularly interesting, since both compositions are 0 by exactness of the two rows. Nevertheless, the two exact sequences interact in a particularly nice way, resulting in the so-called *lifting property of torsors*, as coined in Prop. 3.2.3 of [Sko01]. We cite the following fundamental lemma from Prop. 3.2.9 of [Gir71].

Lemma 1.3.3 (Obstructions agree)**.** *There is an equality of obstructions $\delta([X_2]) = \partial(\tau)$.*

The above Lemma is used throughout the dissertation. We frequently apply both exact sequences (1.2.2) and (1.3.2). This result guarantees that the obstructions are equal.

We can interpret the above result as stating that the fibre of t over τ is empty if and only if the fibre of f_* over $[X_2]$ is empty. If the fibres are nonempty, we relate both of them by a map which is not quite canonical. It follows directly from Lemma 1.3.3 and (1.3.3).

Theorem 1.3.4 (Lifting property of torsors)**.** *The choice of elements of the fibres $t^{-1}(\tau)$ and $f_*^{-1}([X_2])$ determines a* $\mathrm{H}^1(S, K)$*-equivariant bijection*

$$t^{-1}(\tau)/\Gamma(S, G_2) \longrightarrow f_*^{-1}([X_2]), \tag{1.3.4}$$

where the $\Gamma(S, G_2)$*-action on* $\mathrm{H}^1(X_2, p^*K)$ *is given by the natural pullback maps. In particular, the existence of a* G_1*-torsor* X_1 *such that* $X_1/K \cong X_2$ *is equivalent to the existence of a* K*-torsor* $X_1 \to X_2$ *mapping to* τ *in* $\mathrm{H}^0(S, R^1p_*(p^*K))$*.*

A more down-to-earth way of phrasing the same result is as follows; cf. Prop. 3.2.3 of [Sko01], whose proof is more explicit in nature.

Theorem 1.3.5. *Let* X_2 *be a* G_2*-torsor and let* $F\colon X_1 \to X_2$ *be a* K*-torsor. Suppose that* F *maps to* $\tau \in \mathrm{H}^0(S, R^1p_*(p^*K))$*. Then we can equip* X_1 *canonically with the structure of a* G_1*-torsor which extends the* K*-action.*

Note in particular that the underlying scheme structure of X_1 is the same, regardless of whether we consider it as K-structure of G_1-torsor. This is not so clear from our cohomological treatment of Theorem 1.3.4.

Proposition 1.3.6. *Suppose that* $p_*p^*K = K$*. Then the statements of Proposition 1.2.3 are equivalent to*

(v) (Lifting Property of torsors) *there exists a* p^*K*-torsor* $[Y] \in \mathrm{H}^1(X_2, p^*K)$ *which maps to the element* $\tau \in \mathrm{H}^0(S, R^1p_*(p^*K))$*.*

Remark 1.3.7. Similarly to Remark 1.2.4, the data in the objects in the cases (iii) and (v) is not the equivalent. For the implication (iii)⇒(v), if $[X_1] \in \mathrm{H}^1(S, G_1)$, then the class $[X_1] \in \mathrm{H}^1(X_2, K)$ depends on a choice of isomorphism $X_1/K \cong G_2$. Conversely, suppose $[Y] \in \mathrm{H}^1(X_2, p^*K)$ is a cohomology class as in (v). Although Y can be equipped with the structure of a G_1-torsor, this also really depends on a choice.

The difficult term to describe explicitly in the five-term exact sequence (1.3.2) is $\mathrm{H}^0(S, R^1p_*(p^*G))$. A fundamental result of Raynaud helps to describe the first derived pushforward sheaf more explicitly under certain mild assumptions. Recall that a group scheme G is *of multiplicative type* if it is diagonalisable locally in the fpqc-topology; see Tome II, Exp. IX, §1 of [SGA 3]. The following result is Prop. 6.2.1 of [Ray70].

Theorem 1.3.8 (Raynaud Correspondence)**.** *Suppose* $p\colon X \to Y$ *is a proper, flat morphism of finite type. Let* G *be a finite, flat group scheme of finite type and let* $G^\vee = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ *be its Cartier dual. Then there is a canonical map*

$$R^1p_*(p^*G) \longrightarrow \underline{\mathrm{Hom}}(G^\vee, \mathrm{Pic}_{X/Y}),$$

which is an isomorphism if either

(i) $p_*\mathcal{O}_X = \mathcal{O}_Y$*; or*

(ii) G *is of multiplicative type.*

Under this canonical isomorphism, the five-term exact sequence becomes as follows.

$$0 \longrightarrow \mathrm{H}^1(Y, G) \longrightarrow \mathrm{H}^1(X, p^*G) \longrightarrow \mathrm{Hom}(G^\vee, \mathrm{Pic}_{X/Y}) \xrightarrow{\partial} \mathrm{H}^2(Y, G). \tag{1.3.5}$$

Let $Z \to X$ be a p^*G-torsor. If the morphism $G^\vee \to \mathrm{Pic}_{X/Y}$ is trivial, then $Z \to X$ is the pullback of a G-torsor over Y. The other extremity occurs when $G^\vee \to \mathrm{Pic}_{X/Y}$ is injective, in some sense indicating non-triviality of $Z \to X$ as p^*G-torsor.

By the Raynaud correspondence, we associate to every p^*G-torsor $Z \to X$ over X a morphism $G^\vee \to \mathrm{Pic}_{X/Y}$ of group schemes. By naturality we can locate the image of this morphism more precisely. We state a notable property of the map $G^\vee \to \mathrm{Pic}_{X/Y}$.

Proposition 1.3.9. *Let $p\colon X \to Y$ be a proper, flat morphism of finite type. Let $\pi\colon Z \to X$ be a p^*G-torsor, such that $p \circ \pi\colon Z \to Y$ satisfies the conditions of Theorem 1.3.8. Let $G^\vee \to \mathrm{Pic}_{X/Y}$ be the associated homomorphism. The image of $G^\vee$ in $\mathrm{Pic}_{X/Y}$ is contained in the kernel of the pullback map $\pi^*\colon \mathrm{Pic}_{X/Y} \to \mathrm{Pic}_{Z/Y}$.*

Proof. Naturality of the Leray–Serre spectral sequence (1.3.5) implies the following diagram is commutative:

$$\begin{tikzcd}
H^1(X, p^*G) \arrow[r] \arrow[d, "\pi^*"] & \mathrm{Hom}(G^\vee, \mathrm{Pic}_{X/Y}) \arrow[d, "\pi^* \circ -"] \\
\mathrm{H}^1(Z, \pi^*p^*G) \arrow[r] & \mathrm{Hom}(G^\vee, \mathrm{Pic}_{Z/Y})
\end{tikzcd}$$

Since $Z \to X$ is a torsor, the pullback $\pi^*Z = Z \times_X Z \cong Z \times_S G$ is the trivial torsor over Z, so the cohomology class $[Z]$ maps to the identity element of $\mathrm{H}^1(Z, \pi^*p^*G)$. By commutativity, the composition $G^\vee \to \mathrm{Pic}_{X/Y} \xrightarrow{\pi^*} \mathrm{Pic}_{Z/Y}$ is constant. □

Example 1.3.10. Let G be a finite subgroup scheme of $\mathbb{G}_m$. Let Y be a G-torsor on X. Let $\chi\colon G_T \to \mathbb{G}_{m,T}$ be a T-valued point of the Cartier dual $G^\vee$. The pushforward $\chi_*([Y]) = [Y \wedge^G \mathbb{G}_m]$ defines a cohomology class in $\mathrm{H}^1(X_T, \mathbb{G}_{m,T}) = \mathrm{Pic}(X_T)$. As such, this defines a morphism $G^\vee \to \mathrm{Pic}_{X/S}$. According to Thm. 2.3.6 of [Sko01] this is a concrete description of the map $\mathrm{H}^1(X, p^*G) \to \mathrm{Hom}(G^\vee, \mathrm{Pic}_{X/S})$ defined above.

Examples for the above in case G is an abelian variety are given in Section 2.2.

The above provides motivation for Picard schemes. We often use the main result of Jensen to determine the Picard scheme of a torsor. In order to state it, we introduce the following terminology, following [Fog73].

Notation 1.3.11. Let X be a locally noetherian G-scheme. Define the *fixed locus* X^G as the schematic image of the coproduct $\bigsqcup Z \to X$, taken over the inclusions $Z \to X$ of all closed and invariant subschemes of X.

Under mild assumptions, the fixed locus is well-behaved. The following is a special case Thm. 2.3 of op. cit., where it is stated over base schemes, though the separation assumption seems to be missing in loc. cit.

Proposition 1.3.12. *Let X be a locally noetherian and separated G-scheme. Then X^G is a invariant closed subscheme, whose functor of points equals $S \mapsto X(S)^G$ on the category of locally noetherian and separated schemes with quasi-compact morphisms.*

Example 1.3.13. The separation assumption is needed, as the following example shows. Let X be the affine line with double origin, obtained from gluing two copies of $\mathbb{A}^1$ along the open subschemes $\mathbb{A}^1 \setminus \{0\}$. Consider the $\mathbb{Z}/2\mathbb{Z}$-action by permuting the two affine charts. The closed point corresponding to any nonzero $x \in k$ is invariant under the G-action, so that X^G contains $\mathbb{A}^1 \setminus \{0\}$. But the schematic image is closed, from which it follows that $X^G = X$. The two origins are however permuted non-trivially by $\mathbb{Z}/2\mathbb{Z}$, and therefore X^G is not invariant.

We now state a useful result to compute the Picard scheme of quotients. See Thm. 2.1 of [Jen87] for a proof.

Theorem 1.3.14. *Let k be a ground field, and let Y be a proper, geometrically integral scheme with $h^0(\mathcal{O}_Y) = 1$, so that its Picard scheme is representable. Let G be a finite group scheme acting freely on Y, such that the quotient $X = Y/G$ exists. The G-action on Y determines a G-action on the Picard scheme by pullback of invertible sheaves. Then there is a left-exact sequence*

$$0 \longrightarrow G^\vee \longrightarrow \mathrm{Pic}_{X/k} \longrightarrow (\mathrm{Pic}_{Y/k})^G.$$

Suppose that either

(i) *G is a twisted form of the constant group scheme associated to a finite cyclic group;*

(ii) *G is reduced and $G^\vee$ is infinitesimal;*

(iii) *G is infinitesimal and $G^\vee$ is reduced;*

(iv) *G is local of height ≤ 1,*

then the natural map $\mathrm{Pic}_{Y/k} \to (\mathrm{Pic}_{X/k})^G$ is surjective.

1.4 Kummer theory

Let S be a base scheme. In this section we collect various perspectives on μ_n-torsors. The algebro-geometric version of Kummer theory relates μ_n-torsors to n-torsion invertible sheaves with an explicit choice of global non-vanishing section of their nth tensor power. We start however by considering more general μ_n-actions on schemes, which are related to certain $\mathbb{Z}/n\mathbb{Z}$-gradings since μ_n is *diagonalisable.* The relation to Kummer theory is made explicit by a result in Tome I, Exp. 1 of [SGA 3].

A *diagonalisable* group scheme is defined to be the Cartier dual of the constant group scheme of an abelian group M: typical examples include the multiplicative group scheme $\mathbb{G}_m = \underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{G}_m)$ and $\mu_n = \underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$; see Tome I, Exp. I, §4.4 of [SGA 3]. They have the remarkable property that group scheme actions can be equivalently seen through *M-gradings*; see §4.7 of op. cit., or §2.3 of [Ber23]. Let us state the main result. For a proof we refer the reader to ibid.

Proposition 1.4.1. *The relative spectrum functor* $\mathscr{A} \mapsto \operatorname{Spec} \mathscr{A}$ *is an anti-equivalence from the category of* $\mathbb{Z}/n\mathbb{Z}$*-graded* $\mathcal{O}_S$*-algebra's to the category of affine schemes over* S *with a* μ_n*-action.*

Let $\mathscr{A}$ be a $\mathbb{Z}/n\mathbb{Z}$-graded $\mathcal{O}_S$-algebra; we write $\mathscr{A} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \mathscr{A}_j$ for the decomposition into its homogeneous parts. There is a criterion when the μ_n-action induces a μ_n-torsor: we cite Prop. 4.1 of Tome II of Exp. VIII of [SGA 3].

Proposition 1.4.2. *The morphism* $\operatorname{Spec} \mathscr{A} \to S$ *is a* μ_n*-torsor if and only if the following two conditions hold:*

(i) *Every* $\mathscr{A}_j$ *is an invertible sheaf on* S*;*

(ii) *The natural morphisms* $\mathscr{A}_i \otimes \mathscr{A}_j \to \mathscr{A}_{i+j}$ *are isomorphisms.*

Suppose that $\operatorname{Spec} \mathscr{A}$ is a μ_n-torsor. Then (ii) implies that $\mathscr{A}_0 \otimes \mathscr{A}_0 \cong \mathscr{A}_0$, and since $\mathscr{A}_0$ is an invertible sheaf by (i) we conclude that $\mathscr{A}_0 \cong \mathcal{O}_S$. From (ii) it also follows by induction that $\mathscr{A}_1^{\otimes i} \cong \mathscr{A}_i$ for any $i \in \mathbb{Z}/n\mathbb{Z}$, where the isomorphism is given by i-fold multiplication. In other words, a μ_n-torsor is essentially determined by an invertible sheaf $\mathscr{L} = \mathscr{A}_1$, as long as we keep track of the identification $\mathscr{L}^{\otimes n} \cong \mathcal{O}_S$, amounting to the choice of a nonzero global section $s \in \Gamma(S, \mathscr{L}^{\otimes n})$.

This description of μ_n-torsors also arises from a cohomological viewpoint, namely through *Kummer theory.* Originally developed in the study of Galois field extensions with Galois group $\mathbb{Z}/n\mathbb{Z}$ of fields containing a primitive nth root of unity, we recap it from the perspective of algebraic geometry. Let $R = \mathrm{H}^0(\mathcal{O}_S)$ and fix a positive integer $n \geq 1$.

Proposition 1.4.3. *There is a short exact sequence*

$$1 \longrightarrow R^*/R^{*n} \longrightarrow \mathrm{H}^1(S, \mu_n) \longrightarrow \operatorname{Pic}(S)[n] \longrightarrow 0. \tag{1.4.1}$$

Proof. Consider the Kummer short exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 1, \tag{1.4.2}$$

which is exact in the fppf-topology in any characteristic. Its long exact sequence

$$\ldots \longrightarrow R^* \xrightarrow{n} R^* \longrightarrow \mathrm{H}^1(S, \mu_n) \longrightarrow \operatorname{Pic}(S) \xrightarrow{n} \operatorname{Pic}(S) \longrightarrow \ldots. \tag{1.4.3}$$

yields the above short exact sequence. □

Remark 1.4.4. Recall that we work by default in the fppf-topology. This is important, since if μ_n is non-smooth the sequence (1.4.2) fails to be short exact in the étale topology: there may not exist an nth root of an element after an arbitrary étale base extension if n is divisible by the characteristic exponent.

The natural map $\mathrm{H}^1(S, \mu_n) \to \operatorname{Pic}(S)[n]$ hence also maps a μ_n-torsor to an n-torsion invertible sheaf $\mathscr{L}$ on S. There is a clear converse to this statement; see p. 125 of [Mil80].

Theorem 1.4.5 (Kummer Theory)**.** *The map*

$$\{(\mathscr{L}, s) \mid \mathscr{L} \textit{ is an invertible sheaf on } S \textit{ and } s \colon \mathcal{O}_S \xrightarrow{\sim} \mathscr{L}^{\otimes n}\}/\cong \longrightarrow \mathrm{H}^1(S, \mu_n);$$

$$[(\mathscr{L}, s)] \longmapsto \operatorname{Spec}_S \left(\bigoplus_{j=0}^{n-1} \mathscr{L}^{\otimes -j} \right),$$

is a bijection. Here, two pairs $(\mathscr{L}_1, s_1)$ and $(\mathscr{L}_2, s_2)$ are considered to be equivalent if there is an isomorphism $\varphi\colon \mathscr{L}_1 \xrightarrow{\sim} \mathscr{L}_2$ such that $\varphi^* s_2 = s_1$. The $\mathcal{O}_S$-algebra structure on $\bigoplus_{j=0}^{n-1} \mathscr{L}^{\otimes -j}$, is induced by the natural maps $\mathscr{L}^{\otimes -i} \otimes \mathscr{L}^{\otimes -j} \to \mathscr{L}^{\otimes -(i+j)}$ if $i+j < n$ and $\mathscr{L}^{\otimes -i} \otimes \mathscr{L}^{\otimes -j} \to \mathscr{L}^{\otimes -(i+j)} \xrightarrow{s} \mathscr{L}^{\otimes -(i+j-n)}$ if $i + j \geq n$. The composition with the above map $\mathrm{H}^1(S, \mu_n)$ maps a pair $(\mathscr{L}, s)$ to $\mathscr{L}$.

Example 1.4.6. Suppose $S = \mathrm{Spec}(k)$, in which case $\mathscr{L}$ is isomorphic to the structure sheaf. A nonzero section of $\mathscr{L}^{\otimes n}$ is simply an element $\lambda \in k^{*n}$. Denote $1 \in \mathrm{H}^0(S, \mathscr{L})$ by x. Interpreting $\mathscr{L}$ as a k-module, there is then an isomorphism of k-algebra's

$$\bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \mathscr{L}^{\otimes -j} \cong \frac{k[x]}{(x^n - \lambda)},$$

whose isomorphism class fundamentally depends on the class of λ in k^*/k^{*n}. We conclude that the $\mathcal{O}_S$-algebra structure on $\bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \mathscr{L}^{\otimes -j}$ really depends on the choice of section $s\colon \mathcal{O}_S \xrightarrow{\sim} \mathscr{L}^{\otimes n}$. In fact, the isomorphism class depends *only* on the choice of section: since $\mathrm{Pic}(k) = 0$ it follows from the long exact sequence (1.4.3) that the natural map $k^*/k^{*n} \to \mathrm{H}^1(k, \mu_n)$ is an isomorphism. We use this natural isomorphism freely throughout the dissertation.

We also recover the original statement from field theory.

Corollary 1.4.7 (Classical Kummer Theory)**.** *Let k be a field which contains a primitive nth root of unity. Then any cyclic Galois extension k'/k of degree n is of the form $k' = k\big(\sqrt[n]{\lambda}\big)$, for some $\lambda \in k^*/k^{*n}$.*

Proof. A cyclic Galois extension k'/k of degree n torsor is nothing but a $\mathbb{Z}/n\mathbb{Z}$-torsor over $\mathrm{Spec}(k)$, but note that in our case there is an isomorphism $\mathbb{Z}/n\mathbb{Z} \cong \mu_n$. Since $\mathrm{Pic}(k) = \mathrm{H}^1(k, \mathbb{G}_m)$ by Hilbert 90 [Mil80, Prop. III.4.9], it follows that the natural map $k^*/k^{*n} \to \mathrm{H}^1(k, \mu_n)$ is an isomorphism. Thus class of the torsor $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ hence corresponds to some $\lambda \in k^*$, unique up to nth powers, which by Example 1.4.6 satisfies $k' = k[x]/(x^n - \lambda)$. The assumption that k' is a field implies that λ is not an nth power, so $k' = k\big(\sqrt[n]{\lambda}\big)$. ☐

Remark 1.4.8. The classic proof of this version of Kummer theory is constructive. Let σ be a generator for $\mathrm{Gal}(k'/k)$. Since k'/k is separable, it is a primitive field extension may write $k' = k(\alpha)$. Consider the *Lagrange resolvent*

$$\beta = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \zeta_n^{-i} \sigma^i(\alpha).$$

of α, which is nonzero by the independence of characters [Stacks, Tag 0CKL]. The crucial property is that $\sigma^i(\beta) = \zeta_n^i \cdot \beta$. From Galois theory it then follows that $\lambda = \beta^n$ is contained in k, but not in any intermediate fields of k'. It is therefore a primitive element.

Proposition 1.4.9. *Let $\pi\colon X \to S$ be a μ_n-torsor, mapping to the class of the invertible sheaf $\mathscr{L}$ of order n. The kernel of $\pi^*\colon \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ is cyclic of order n generated by the class of $\mathscr{L}$.*

Proof. By Theorem 1.3.14 there is a left-exact sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(X),$$

so that the kernel is indeed cyclic of order n. Since $\mathscr{L}$ has order n, it suffices to verify that $\pi^*\mathscr{L} \cong \mathcal{O}_X$. Let $U_i = \mathrm{Spec}\, A_i$ be an open affine cover of S trivialising $\mathscr{L}$, and let $f_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_S)$ be a cocycle representing $\mathscr{L}$. The pre-image of π over U_i is isomorphic to $\mathrm{Spec}\, A_i[x_i]/(x_i^n - a_i)$, where the gluing is given by $f_{ij} x_j = x_i$. The a_i hence satisfy $f_{ij}^n a_j = a_i$. The pullback $\pi^*\mathscr{L}$ is still represented by the cocycle $f_{ij} \in \Gamma(\pi^{-1}(U_i \cap U_j), \mathcal{O}_X^*)$, but it is a coboundary since it equals x_i/x_j, so its class in $\mathrm{Pic}(T)$ is trivial. ☐

Conversely, the above property is essentially a universal property for μ_n-torsors. The following result is essentially Prop. 0.2.14 of [CDL24]

Proposition 1.4.10. *Let $\pi\colon X \to S$ be a finite morphism of proper schemes over a ground field k. Let $\mathscr{L}$ be an invertible sheaf on S such that $\pi^*\mathscr{L} \cong \mathcal{O}_X$. Then there is a positive integer n and a section $s\colon \mathcal{O}_S \xrightarrow{\sim} \mathscr{L}^{\otimes n}$ of the nth tensor power, such that π factors over the μ_n-torsor determined by the pair $(\mathscr{L}, s)$.*

Proof. Pick a nonzero global section $t\colon \mathcal{O}_X \xrightarrow{\sim} \pi^*\mathscr{L}$. Any choice of s determines a non-vanishing global section $\pi^*s\colon \mathcal{O}_X \xrightarrow{\sim} \pi^*\mathscr{L}^{\otimes n}$. The sections π^*s and $t^{\otimes n}$ differ by a scalar $\lambda \in k^*$, i.e. $\lambda = t^{\otimes n}/\pi^*s$. A difference choice of t multiplies λ by an nth power, so that λ is well-defined as element of k^*/k^{*n}. We then have a chain of isomorphisms of $\mathcal{O}_X$-algebras

$$\pi^*\left(\bigoplus_{i=0}^{n-1} \mathscr{L}^{\otimes i}\right) = \bigoplus_{i=0}^{n-1} \pi^*\mathscr{L}^{\otimes i} \cong \bigoplus_{i=0}^{n-1} \mathcal{O}_X \cong \frac{\mathcal{O}_X[T]}{(T^n - \lambda)}.$$

The first isomorphism is determined in each degree by the choice of $t^{\otimes i}$, and the last isomorphism is determined by the choice of t as $\mathcal{O}_X$-algebra generator. Note that $t^{\otimes n} = \lambda\pi^*s$ indeed implies that t satisfies the polynomial equation $T^n - \lambda$. Replacing without loss of generality s by λs, we may set $\lambda = 1$. Since the polynomial $T^m - 1$ has a root, there is a surjection onto $\mathcal{O}_X$. By the functorial definition of the relative spectrum [Stacks, Tag 01LQ], there is hence a morphism

$$X \longrightarrow \operatorname{Spec}_S \bigoplus_{i=0}^{n-1} \mathscr{L}^{\otimes i}$$

commuting with the morphisms to S. ☐

We quantify the influence of the choice of section more precisely. Given $\lambda \in k^*$ let $[\lambda]$ be its image in $\mathrm{H}^1(k, \mu_n)$. By abuse of notation, we also use $[\lambda]$ to denote its image in $\mathrm{H}^1(S, \mu_n)$.

Proposition 1.4.11. *Let $\mathscr{L}$ be an invertible sheaf on S, and suppose that $s\colon \mathcal{O}_S \xrightarrow{\sim} \mathscr{L}^{\otimes n}$ is an isomorphism. Let $T \to S$ be the μ_n-torsor corresponding to the pair $(\mathscr{L}, s)$. Let $\lambda \in k^*$ and let T_λ be the μ_n-torsor corresponding to $(\mathscr{L}, \lambda s)$. Then there is an isomorphism*

$$T_\lambda = T \wedge^{\mu_n} [\lambda^{-1}].$$

Conceptually, the fibre of $\mathrm{H}^1(S, \mu_n) \to \operatorname{Pic}(S)[n]$ over $\mathscr{L}$ equals $[T] + \mathrm{H}^1(k, \mu_n)$ by the short exact sequence (1.4.1), which contains T_λ. To see that the translate element really equals $[\lambda^{-1}]$, we calculate the effects of choosing a different section through a concrete computation.

Proof. Let $U = \operatorname{Spec}(R) \subset S$ be an affine open subset on which $\mathscr{L}_S$ is trivial, so there exists a local section $t\colon \mathcal{O}_U \xrightarrow{\sim} \mathscr{L}|_U$. Since $t^{\otimes n}$ and s are local trivializations of $\mathscr{L}^{\otimes n}|_U$, there exists a $\mu \in R^*$ such that $t^{\otimes n} = \mu \cdot s$. Then the open subscheme of Y lying over U is isomorphic to $\operatorname{Spec}(\bigoplus_{j=0}^{n-1} \Gamma(U, \omega_S^{\otimes -j}) \cong \operatorname{Spec}(R[x]/(x^n - \mu))$. Similarly, the open subscheme of T_λ lying over U is $\operatorname{Spec}(R[x]/(x^n - \mu\lambda^{-1}))$, in view of the equality $t^{\otimes n} = \mu\lambda^{-1} \cdot \lambda s$. The μ_n-actions are inherited from the $\mathbb{Z}/n\mathbb{Z}$-gradings.

On the right hand side, we may calculate the contracted product locally over U, hence we consider the quotient of

$$\operatorname{Spec}\left(\frac{R[x_1]}{(x_1^n - \mu)}\right) \times \operatorname{Spec}\left(\frac{k[x_2^{-1}]}{(x_2^{-n} - \lambda^{-1})}\right) = \operatorname{Spec}\left(\frac{R[x_1, x_2^{-1}]}{(x_1^n - \mu, x_2^{-n} - \lambda^{-1})}\right)$$

by the μ_n-action induced by the $\mathbb{Z}/n\mathbb{Z}$-grading. Here, we consider T_1 and T_2^{-1} to be of degree $\overline{1}$ and $\overline{-1}$ respectively, since we consider the opposite action in one of the factors. By [Ber23, Lem. 2.3.8] the quotient is isomorphic to the spectrum of the homogeneous part of degree $\overline{0}$ in the $\mathbb{Z}/n\mathbb{Z}$-grading, which is clearly generated by $T_1T_2^{-1}$ and isomorphic to $\operatorname{Spec}(R[T_1T_2^{-1}]/((T_1T_2^{-1})^n - \mu\lambda^{-1})$ and thus by the previous paragraph isomorphic to the open subscheme of T_λ over U. These isomorphisms glue to an isomorphism $T_\lambda \cong T \wedge^{\mu_n} [\lambda^{-1}]$. ☐

We emphasise that T and its *twist* T_λ are geometrically isomorphic. After base-change to an algebraic closure, the element λ will be an nth power, so that the contracted product with the class of λ^{-1} yields an isomorphic torsor.

1.5 Cohomology of unipotent group schemes

Let k be a ground field. Where Kummer theory was concerned with torsors under the multiplicative group schemes μ_n, which form the typical examples of diagonalisable group schemes, this section is concerned with a certain statement on the cohomology of an 'orthogonal' class of group schemes, namely the *unipotent group schemes.* Most interesting examples arise in characteristic $p > 0$. Our main source is Tome II, Exp. XVII of [SGA 3].

Definition 1.5.1. A group scheme G is called *unipotent* if every finite dimensional representation $G \to \mathrm{GL}_V$ has a nonzero fixed vector, i.e. $V^G \neq 0$.

A list of equivalent characterisations of unipotence is given Thm. 3.5 of op. cit. We state the most important points.

Theorem 1.5.2. *Let G be a group scheme of finite type over a field. The following are equivalent:*

(i) *G is unipotent;*

(ii) *G is a subgroup scheme of the group scheme of strictly upper triangular matrices;*

(iii) *there is a filtration of G whose successive quotients are subgroup schemes of $\mathbb{G}_a$.*

Example 1.5.3. In light of (iii), typical examples of unipotent group schemes are $\mathbb{G}_a$, in characteristic $p > 0$ also α_p and $\mathbb{Z}/p\mathbb{Z}$. Successive extensions of α_p and $\mathbb{Z}/p\mathbb{Z}$ are also unipotent, e.g. $\mathbb{Z}/p^n\mathbb{Z}$, α_{p^n}, and the p-torsion subgroup scheme of a supersingular elliptic curve.

Example 1.5.4. Let k be a field of characteristic $p > 0$ and let V be a $\mathbb{Z}/p\mathbb{Z}$-representation. That is, there is an automorphism T of V such that $T^p = \mathrm{id}_V$. In characteristic p, the Frobenius isomorphism then implies that $(T - \mathrm{id}_V)^p = 0$, from which it follows that $T - \mathrm{id}_V$ has a non-zero kernel, so that T has a non-zero fixed point. This shows directly that $\mathbb{Z}/p\mathbb{Z}$ is unipotent in characteristic p. An example of a typical $\mathbb{Z}/p\mathbb{Z}$-representation is given by $\mathbb{Z}/p\mathbb{Z} \to \mathrm{GL}_2$; $\overline{1} \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, whose fixed locus is generated by the vector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

Lemma 1.5.5. *Let G be a subgroup scheme of $\mathbb{G}_a$. Then $\mathrm{H}^i(k, G) = 0$ for $i \geq 2$.*

Proof. The subgroup scheme G is either finite or $\mathbb{G}_a$. In the latter case $\mathrm{H}^i(k, \mathbb{G}_a) = 0$ for $i \geq 1$ is well-known, which essentially follows from the normal basis theorem; see Prop. X.1 of [Ser79]. Otherwise, the quotient $\mathbb{G}_a/G$ is isomorphic to $\mathbb{G}_a$; see Lem. 2.3 of Tome II, Exp. XVII of [SGA 3]. The short exact sequence

$$0 \longrightarrow G \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a \longrightarrow 0$$

then induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}(k, \mathbb{G}_a) \longrightarrow \mathrm{H}^i(k, G) \longrightarrow \mathrm{H}^i(k, \mathbb{G}_a) \longrightarrow \cdots .$$

Since the cohomology groups $\mathrm{H}^i(k, \mathbb{G}_a)$ vanish for $i \geq 1$, it follows that $\mathrm{H}^i(k, G) = 0$ for $i \geq 2$. □

Proposition 1.5.6. *Let G be a commutative unipotent group scheme. Then $\mathrm{H}^i(k, G) = 0$ for $i \geq 2$.*

Proof. The characterisation of Thm. 3.5 of Tome II, Exp. XVII of [SGA 3] states that a unipotent group admits a filtration where the successive quotients are isomorphic to finite subgroup schemes of $\mathbb{G}_a$, i.e. copies of $\mathbb{G}_a$, α_{p^n}, and twisted forms of $(\mathbb{Z}/p\mathbb{Z})^r$. The result follows now by induction, using Lemma 1.5.5. □

Chapter 2

Para-abelian varieties

Abelian varieties are defined as smooth, proper and geometrically integral group schemes. They arise naturally in many contexts of algebraic geometry. One-dimensional abelian varieties are referred to as *elliptic curves*, and the study of their rational points is a central topic in number theory. The theory of abelian varieties is well-studied in the literature. We point the reader to e.g. [EGM; Mum08].

Our focus is directed mainly towards *twisted forms of abelian varieties*, which we refer to as *para-abelian varieties*. This terminology was coined by Grothendieck in [Gro62] and has reappeared recently in a series of articles by Schröer et al. [LS23; Sch23a; BDS24]. Similarly, twisted forms of elliptic curves are referred to as *para-elliptic curves*. It turns out that any para-abelian variety can be equipped canonically with the structure of a torsor under an abelian variety. In that sense, we specialise the theory of Chapter 1 to abelian varieties. We study the so-called *associated abelian variety* of a para-abelian variety in Section 2.1 The functoriality of the associated abelian variety is the topic of Section 2.2. In Section 2.3 we study the para-abelian surfaces that admit elliptic fibrations.

2.1 Para-abelian varieties and their associated abelian varieties

In this section we give a brief introduction to the theory of para-abelian varieties. We closely follow §4 and §5 of [LS23], where the theory is developed in the broader generality of algebraic spaces. We instead summarise the theory over a ground field k within the category of schemes.

Recall that an *abelian variety* A is a proper, geometrically integral group scheme. The group scheme structure places a large restriction on the scheme structure: an abelian variety is necessarily smooth with trivial dualising sheaf. The group scheme axioms dictate that any abelian variety has a rational point $e \in A(k)$, which also poses an arithmetic restriction on the underlying scheme A. The class of para-abelian varieties capture exactly the geometry of abelian varieties, without demanding the existence of rational points.

Definition 2.1.1. A scheme X over a field k is called a *para-abelian variety* if there is a field extension k'/k such that the base-change $X' = X \otimes_k k'$ admits the structure of an abelian variety. A para-abelian variety of dimension 1 is called a *para-elliptic curve.*

Example 2.1.2. A para-elliptic curve C is simply a smooth genus-one curve. Let k'/k be a field extension and let $C' = C \otimes k'$. If C has a k'-valued point, the choice of $O \in C(k')$ induces an isomorphism $C' \to \mathrm{Pic}^0_{C'/k'}$ by $P \mapsto \mathcal{O}_{C'}(P - O)$. In this way the base-change C' inherits the structure of an elliptic curve.

Para-abelian varieties satisfy similar geometric properties: by fpqc-descent, a para-abelian variety is smooth and proper with trivial dualising sheaf. By fpqc-descent, a para-abelian variety X is smooth and proper with trivial dualising sheaf. From now on, let X denote a para-abelian variety. The following is Prop. 4.3 of loc. cit., generalising Thm. 6.14 of [MFK94].

Proposition 2.1.3. *For each $e \in X(k)$ there is a* unique *group scheme structure on X with e as identity section.*

A theorem of Matsumura and Oort states that for a proper scheme X the functor

$$(k\text{-}\mathsf{Alg}) \longrightarrow (\mathsf{Set}); \quad R \longmapsto \mathrm{Aut}(X \otimes R/R) = \{\phi\colon X_R \to X_R \mid \phi \text{ is an } R\text{-automorphism}\}$$

is representable; see Thm. 3.7 of [MO67]. We denote the scheme representing the above functor by $\mathrm{Aut}_{X/k}$, called the *automorphism group scheme.* Concretely, by identifying an automorphism ϕ with its graph in $X \times X$, it can be defined as an open subscheme of the Hilbert scheme $\mathrm{Hilb}_{X\times X/k}$. Also representable for proper schemes is the Picard scheme $\mathrm{Pic}_{X/k}$, due to a result of Murre, see [Mur64]. We denote by $\mathrm{Pic}^{\tau}_{X/k}$ the closed subscheme parametrising numerically trivial invertible sheaves.

Note that the automorphism group scheme $\mathrm{Aut}_{X/k}$ acts naturally on the Picard scheme $\mathrm{Pic}_{X/k}$ by pullback of invertible sheaves. The numerically trivial sheaves are stable under this action, so we consider the induced action of $\mathrm{Aut}_{X/k}$ on $\mathrm{Pic}^{\tau}_{X/k}$.

Definition 2.1.4. Let $A \subset \mathrm{Aut}_{X/k}$ be the *inertia subgroup sheaf*, which is the functor defined on R-valued points by

$$A(R) = \{\phi \in \mathrm{Aut}_{P/k}(R) \mid \phi^*\colon \mathrm{Pic}^{\tau}_{P/k}(R) \to \mathrm{Pic}^{\tau}_{P/k}(R) \text{ is the identity}\}.$$

We cite a number of critical results from our main source [LS23]. The following statements summarise Prop. 5.1, 5.2, 5.5 and Thm. 5.3 of op. cit.

Proposition 2.1.5. *Let X be a para-abelian variety. The functor A is representable by an abelian variety. Its action on X is free and transitive, so that X has the structure of an A-torsor. Furthermore, there is a canonical identification* $\mathrm{Pic}^{\tau}_{X/k} = \mathrm{Pic}^{\tau}_{A/k}$.

Definition 2.1.6. Let X be a para-abelian variety. We call A the *associated abelian variety of X.*

Since X is canonically equipped with the structure of an A-torsor, it corresponds to a cohomology class $[X]$ in $\mathrm{H}^1(k, A)$. The group $\mathrm{H}^1(k, A)$ is called the *Weil-Châtelet group.* It is an important arithmetic invariant associated to the abelian variety A. Since A is smooth, the cohomology may also be computed through Galois cohomology

$$\mathrm{H}^1(k, A) = \mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), A(k^{\mathrm{sep}})) = \varinjlim_{k'/k} \mathrm{H}^1(\mathrm{Gal}(k'/k), A(k')),$$

where the direct limit ranges over all *finite Galois* extensions k'/k. It follows that every cohomology class has a finite order, which in this context is usually given a different name.

Definition 2.1.7. The *period* $\mathrm{per}(X)$ of a para-abelian variety X is the order of $[X]$ in the Weil–Châtelet group $\mathrm{H}^1(k, A)$.

It turns out that the construction of the associated abelian variety is entirely functorial, see [LS23, Prop. 5.4]. We study functoriality further in the upcoming section.

Proposition 2.1.8. *Let $F\colon X_1 \to X_2$ be a morphism of para-abelian varieties. There is a unique homomorphism $f = F_*\colon A_1 \to A_2$ making F into an A_1-equivariant morphism.*

Remark 2.1.9. The functor induces a map on automorphism groups $\mathrm{Aut}(X) \to \mathrm{Aut}(A)$. In fact the previous proposition holds in families, as shown in op. cit., so for any scheme T there are compatible maps $\mathrm{Aut}(X_T) \to \mathrm{Aut}(A_T)$ which hence induce a map on automorphism group schemes $\mathrm{Aut}_{X/k} \to \mathrm{Aut}_{A/k}$.

Remark 2.1.10. The maps $F\colon X_1 \to X_2$ and $f\colon A_1 \to A_2$ are twisted forms, in the sense that there is a ground field extension k'/k and suitable isomorphisms $X_1 \otimes k' \cong A_1 \otimes k'$ and $X_2 \otimes k' \cong A_2 \otimes k'$ such that $F \otimes k'$ and $f \otimes k'$ agree.

2.2 Morphisms between para-abelian varieties

In the previous section, we defined a functor from the full subcategory of para-abelian varieties to the category of abelian varieties by mapping a para-abelian variety its associated abelian variety. In this section we study surjective morphisms between para-abelian varieties and deduce a number of important criteria. Throughout, let X_1, X_2 be para-abelian varieties with associated abelian variety A_1, A_2 respectively.

Since a para-abelian variety is canonically a torsor under its associated abelian variety, we may apply the results of Section 1.2 to classify surjective morphisms $F\colon X_1 \to X_2$ between para-abelian varieties. This gives a number of useful cohomological criteria for the existence and non-existence for certain morphisms between para-abelian varieties. For example, n-covers, sign involutions, or automorphisms with a geometric fixpoint.

Let $f\colon A_1 \to A_2$ be the morphism of associated abelian varieties and let $K = \mathrm{Ker}(f)$. In other words, we have a short exact sequence

$$0 \longrightarrow K \longrightarrow A_1 \longrightarrow A_2 \longrightarrow 0. \tag{2.2.1}$$

The action of A_1 on X_1 restricts to an action of K, which gives F the structure of a K-torsor. In this setting hence Section 1.3 applies. We note that conversely, any torsor over a para-abelian variety X under a finite commutative group scheme K can be thought of as coming from an isogeny. If G is finite étale this is the Serre–Lang theorem; see Thm. 10.36 of [EGM], §18 of [Mum08], or [LS57]. In fact, an analogous statement holds if G is possibly non-smooth and noncommutative; see [Nor83].

Recall that $f\colon A_1 \to A_2$ is an *isogeny* if f is finite and surjective, or equivalently if f is finite and $\dim(A_1) = \dim(A_2)$; see Prop. 5.2 of [EGM]. We extend this notion by descent to para-abelian varieties.

Definition 2.2.1. A morphism of para-abelian varieties $F\colon X_1 \to X_2$ is called an *isogeny* if one of the following equivalent conditions is fulfilled:

(i) F is finite and $\dim(X_1) = \dim(X_2)$;

(ii) after some base-change we can endow X_1 and X_2 with the structure of abelian varieties such that F is an isogeny of abelian varieties;

(iii) the induced morphism $f = F_*$ on the associated abelian varieties is an isogeny.

Note indeed that the notion of finiteness and equidimensionality descend by Prop. 2.7.1 of [EGA IV$_2$]. The equivalence of the last two points follows since f and F are twisted forms in the sense of Remark 2.1.10. Since isogenies are surjective, they satisfy the hypotheses of the cohomological criterion Lemma 1.2.2. As such, we may use the Weil–Châtelet group to study the existence of isogenies. In that light we also consider the expanded statements Propositions 1.2.3 and 1.3.6.

Proposition 2.2.2. *Let A_1, A_2 be abelian varieties, let $f\colon A_1 \to A_2$ be an isogeny with kernel K and let X_2 be an A_2-torsor. Let $p\colon X_2 \to \mathrm{Spec}(k)$ denote the structure morphism. The following are equivalent:*

(i) *there exists an A_1-torsor X_1 and a morphism $F\colon X_1 \to X_2$ inducing f;*

(ii) *there exists an A_1-torsor X_1 such that $X_1/K \cong X_2$;*

(iii) *there exists a cohomology class $[X_1] \in \mathrm{H}^1(k, A_1)$ such that $f_*([X_1]) = [X_2]$;*

(iv) $\delta([X_2]) = 0$ *in $\mathrm{H}^2(k, K)$, where δ is the boundary map in the long exact sequence associated to* (2.2.1)*;*

(v) (Lifting Property) *there exists a cohomology class $[X_1] \in \mathrm{H}^1(X_2, K)$ mapping to the element $\tau \in \mathrm{H}^0(k, R^1p_*(p^*K))$ of Section 1.3.*

If the isogeny F of (i) *exists, it is unique up to the action of $A_1(k)$.*

In this list of equivalences, property (v) remains the most mysterious. In the case of para-abelian varieties there is however a much more concrete description of the element τ using the Raynaud correspondence, which applies since para-abelian varieties are proper. Note that the Raynaud correspondence applies since abelian varieties are proper. We investigate what homomorphism corresponds to τ under the canonical isomorphism $\mathrm{H}^0(k, R^1p_*(p^*K)) \cong \mathrm{Hom}(K^\vee, \mathrm{Pic}^\tau_{X_2/k})$.

Note that $\mathrm{Pic}^\tau_{X_2/k} = \mathrm{Pic}^\tau_{A_2/k} = A_2^\vee$ is the *dual abelian variety.* The Néron–Severi group of an abelian variety is torsion-free, so the dual variety may equivalently be defined as $A_2^\vee = \mathrm{Pic}^0_{A_2/k}$. Duality is an important tool in the study of abelian varieties; see e.g. §§6-7 of [EGM]. Taking the dual of an abelian variety defines *contravariant* endofunctor on the category of abelian surfaces: an isogeny $f\colon A_1 \to A_2$ induces a *dual isogeny* denoted $f^\vee\colon A_2^\vee \to A_1^\vee$ by pullback of invertible sheaves. For any abelian variety A, the double dual $(A^\vee)^\vee$ is naturally identified with A.

If E is an elliptic curve with identity element ∞, then $E^\vee = \mathrm{Pic}^0_{E/k}$ is canonically isomorphic to E. The isomorphism $E \to \mathrm{Pic}^0_{E/k}$ is given on T-points by $P \mapsto \mathcal{O}(P - \infty)$. Since elliptic curves are *self-dual*, we may make the identification $E = E^\vee$. Let $f\colon E_1 \to E_2$ be an isogeny. Under this identification, the dual isogeny is an isogeny $f^\vee\colon E_2 \to E_1$ in the other direction. There are certain cases in which it may be helpful to not make this identification: for example, it aids in regulating the co- or contravariant functoriality.

We study the kernel of the dual isogeny. By Thm. 7.5 of [EGM] or §15 of [Mum08], there is a canonical isomorphism $\mathrm{Ker}(f^\vee) = K^\vee$, where $K^\vee = \underline{\mathrm{Hom}}(K, \mathbb{G}_m)$ denotes the *Cartier dual* of K. It hence sits in a short exact sequence.

$$0 \longrightarrow K^\vee \longrightarrow A_2^\vee \longrightarrow A_1^\vee \longrightarrow 0. \tag{2.2.2}$$

Conversely, the dual isogeny $f^\vee\colon A_2^\vee \to A_1^\vee$ is the quotient map by $K^\vee$ and may thus be recovered from the subgroup scheme $K^\vee \subset A_2^\vee$ only. This subgroup scheme also encodes the original isogeny $f\colon A_1 \to A_2$. In other words, the inclusion map $K^\vee \to A_2^\vee$ contains substantial information. The following correspondence is well known and is shown in Prop. 2.3.11 under the assumption that K is of multiplicative type.

Lemma 2.2.3. *Let $f\colon A_1 \to A_2$ be an isogeny of abelian varieties with kernel K. Under the canonical isomorphism $\mathrm{H}^0(k, R^1p_*(p^*K)) \cong \mathrm{Hom}(K^\vee, A_2^\vee)$ of the Raynaud correspondence Theorem 1.3.8, the element τ of Section 1.3 maps to the inclusion of the subgroup scheme $K^\vee \to A_2^\vee$.*

Proposition 2.2.4. *In the setting of Proposition 2.2.2, the equivalent statements hold if and only if*

(vi) (Lifting Property) *there exists a cohomology class $[X_1] \in \mathrm{H}^1(X_2, K)$ mapping to the canonical inclusion $K^\vee \to A_2^\vee = \mathrm{Pic}^0_{X_2/k} \subset \mathrm{Pic}_{X_2/k}$.*

Remark 2.2.5. Let $F\colon X_1 \to X_2$ be an isogeny and let $K = \mathrm{Ker}(F_*)$. We note that the implication (i)⇒(v) of Proposition 2.2.4 is natural, in the sense that the K-torsor F maps to the inclusion $K^\vee \subset A_2^\vee$.

In general, one needs to be careful with these equivalences, as the objects in some cases contain more data than others. For example, given an isogeny $F\colon X_1 \to X_2$ as in (i), this induces an isomorphism $X_1/K \cong X_2$. From the data in case (iii) however, one can only deduce that such an isomorphism must exist.

For later reference, let us state the relevant exact sequences of (1.3.3) in the context of abelian varieties and para-abelian varieties.

$$0 \longrightarrow \mathrm{H}^1(k, K) \longrightarrow \mathrm{H}^1(X_2, K) \longrightarrow \mathrm{Hom}(K^\vee, A_2^\vee) \xrightarrow{\partial} \mathrm{H}^2(k, K) \longrightarrow \mathrm{H}^2(X_2, K) \tag{2.2.3}$$

$$\cdots \longrightarrow A_2(k) \longrightarrow \mathrm{H}^1(k, K) \longrightarrow \mathrm{H}^1(k, A_1) \xrightarrow{f^*} \mathrm{H}^1(k, A_2) \longrightarrow \mathrm{H}^2(k, K) \longrightarrow \cdots \tag{2.2.4}$$

This cohomological characterisation gives us a good understanding of when an isogeny $F\colon X_1 \to X_2$ inducing a fixed $f = F_*$ exists. We specialise to different kinds of isogenies, but first note the following useful criterion.

Lemma 2.2.6. *In the setting of Proposition 2.2.4, if $\deg(f)$ and $\mathrm{per}(X_2)$ are coprime, then the equivalent conditions hold.*

Proof. Since the order of K is $\deg(f)$, the cohomology group $\mathrm{H}^2(k, K)$ is annihilated by $\deg(f)$. The order of $[X_2]$ in $\mathrm{H}^1(k, A_2)$ is coprime to this, so $\delta([X_2]) = 0$. □

First we investigate to what extent a 'dual isogeny' of para-abelian varieties exists, though in a sense different than described above. Given an isogeny $f\colon A_1 \to A_2$ of abelian varieties, there is an isogeny $g\colon A_2 \to A_1$ such that the compositions $g \circ f$ and $f \circ g$ are multiplication by $\deg(f)$ on A_1 and A_2, respectively; see Prop. 5.12 of [EGM]. This follows roughly from the fact that the kernel of f is finite, so it is contained in the n-torsion $A[n]$, where $n = |K| = h^0(\mathcal{O}_K)$ is the order of K. If $A_1 = E_1$ and $A_2 = E_2$ are elliptic curves, then by self-duality we may identify g with $f^\vee$.

Given an isogeny of para-abelian varieties $F\colon X_1 \to X_2$, the following results explore to what extent a $G\colon X_2 \to X_1$ exists, with 'similar' properties, to be clarified below. First, we suppose that the composition $G \circ F$ be a twisted form of multiplication by $\deg(F) = \deg(f)$.

Proposition 2.2.7 (Dual isogeny)**.** *Let $F\colon X_1 \to X_2$ be an isogeny inducing $f\colon A_1 \to A_2$. There is a map $G\colon X_2 \to X_1$ such that $(G \circ F)_*$ is multiplication by $\deg(f)$ if and only if $\deg(f) \equiv 1 \bmod \mathrm{per}(X_1)$.*

Proof. By the existence of F, we have that $f_*([X_1]) = [X_2]$. Let $g\colon A_2 \to A_1$ be the isogeny such that $g \circ f$ is multiplication by $\deg(f)$. There is a $G\colon X_2 \to X_1$ inducing g if and only if $[X_1] = g_*([X_2])$, where the right hand side equals $g_*(f_*([X_1])) = \deg(f) \cdot [X_1]$. ☐

The above notion is stronger than if we had required that the composition $G \circ F$ be a twisted form of multiplication by some (possibly larger) integer, although the two notions coincide for abelian surfaces.

Proposition 2.2.8 (A 'weak' dual isogeny)**.** *Let $F\colon X_1 \to X_2$ be an isogeny inducing $f\colon A_1 \to A_2$. There is a map $G\colon X_2 \to X_1$ such that $(G \circ F)_*$ is multiplication by some integer if and only if* $\deg(f)$ *and* $\mathrm{per}(X_1)$ *are coprime.*

Proof. Suppose first that $(G \circ F)_*$ is multiplication by $m \in \mathbb{N}$. Then m is divisible by $\deg(f)$ so set $n = m/\deg(f)$. Now $n \deg(f)[X_1] = [X_1]$, so $\deg(f)$ is coprime to $\mathrm{per}(X_1)$. The converse is similar: all steps are reversible. ☐

The statement of the following result seems quite unrelated, but can in fact be proven by a very similar method. Let E be an elliptic curve and let C be an E-torsor. For any field extension k'/k, the set of k'-valued points $C(k')$ is either empty, or in non-canonical bijection with $E(k')$ by Corollary 1.1.8. The bijection $E(k') \to C(k')$ depends on the choice of a point in $C(k')$. If $C(k')$ is nonempty, we call it a *splitting field* for C. We show that in some cases, the set of splitting fields characterises the underlying scheme of an E-torsor.

Proposition 2.2.9. *Suppose E is an elliptic curve with* $\mathrm{End}(E) = \mathbb{Z}$. *Let C and D be E-torsors such that for all field extensions k'/k the set of rational points $C(k')$ is empty if and only if $D(k')$ is empty. Under the assumption that the period of C is* 2, 3, 4 *or* 6*, then C and D are isomorphic as schemes.*

Proof. The case that C and D have k-points is trivial, so we suppose they have no k-points. Let $K = \kappa(C)$ be the function field of C. The inclusion $\mathrm{Spec}(K) \to C$ hence yields a K-rational point on D. Since C is a smooth curve, it extends to a map $F\colon C \to D$.

We show that F is non-constant. Suppose F maps the generic point $\mathrm{Spec}(K)$ to a closed point $p \in D$. This yields a tower of field extensions $k \subset \kappa(p) \subset K$, where $\kappa(p)/k$ is finite. Since $h^0(C) = 1$ we know that k is algebraically closed in K, so in fact $\kappa(p) = k$ and D has a k-point.

Therefore F is a surjective map between equidimensional para-elliptic curves and hence an isogeny of para-elliptic curves. By symmetry, there is also an isogeny $G\colon D \to C$. Let f and g be the induced endomorphisms of E, which by assumption can be interpreted as integers. Then $f_*([C]) = [D]$ and $g_*([D]) = [C]$, so $fg \equiv 1 \bmod \mathrm{per}(C)$. By assumption on the period, we have that $\varphi(\mathrm{per}(C)) \leq 2$, where φ denotes the Euler-phi function, hence it follows that $f \equiv \pm 1 \bmod \mathrm{per}(C)$. Then $[D] = f_*([C])$ is isomorphic as torsor C or its pullback along the sign involution of E. In either case $[D]$ is isomorphic to $[C]$ as a scheme. ☐

Remark 2.2.10. Let E be an elliptic curve with complex multiplication, so $\mathrm{End}(E)$ is an order in a imaginary quadratic number field. Given an isogeny ϕ of E which is not multiplication by some integer, its *norm* is the composition $\phi^\vee \circ \phi$, which is (multiplication by) an integer. If the norm is 1 modulo the period of C, then the cohomology class $[D] = \phi_*([C]) \in \mathrm{H}^1(k, E)$ satisfies $\phi^\vee_*([D]) = [C]$. As such, there are isogenies $C \to D$ and $D \to C$, which are twisted forms of ϕ and $\phi^\vee$, respectively. Then it is clear that C and D have the same splitting fields.

We return to studying particular kinds of isogenies between para-abelian varieties. Consider now the case where $A_1 = A_2 = A$ and where f is multiplication by an integer n. We call an $F\colon X_1 \to X_2$ such that F_* is multiplication by n an *n-covering*, generalising the notion of Dfn. 3.3.1 of [Sko01]. The following result follows immediately from the equivalence (i)⇔(iii) of Proposition 2.2.2.

Corollary 2.2.11 (n-coverings)**.** *Let $F\colon X_1 \to X_2$ be an n-covering, then $n \cdot [X_1] = [X_2]$ Conversely, for any cohomology class $[X_1]$ satisfying $n \cdot [X_1] = [X_2]$ there is an n-covering map $F\colon X_1 \to X_2$. In particular, if X_2 has a rational point, then* $\mathrm{per}(X_1)$ *divides n. And if $X_2 \cong X_1$, then* $\mathrm{per}(X_1)$ *divides $n - 1$.*

Fix an integer n and fix a para-abelian variety X_2 with associated abelian variety A. Consider the five-term exact sequence of the Leray–Serre spectral sequence with the Raynayd correspondence of (1.3.5).

$$0 \longrightarrow \mathrm{H}^1(k, A[n]) \longrightarrow \mathrm{H}^1(X_2, A[n]) \xrightarrow{t} \mathrm{Hom}(A[n]^\vee, A^\vee) \xrightarrow{\partial} \mathrm{H}^2(k, A[n]) \longrightarrow \cdots .$$

From Lemma 2.2.3 it follows that the set of isomorphism classes of para-abelian varieties X_1 that admit an n-covering $X_1 \to X_2$ is in bijection to the fibre of t over the inclusion $A[n]^\vee \subset A^\vee$. Recall that this bijection is not canonical and depends on the choice of n-covering $X_1 \to X_2$. The fibre of t is nonempty if and only if the image of the inclusion under ∂ is zero.

Setting $X_2 = A$, the n-coverings $X \to A$ are classified by the cohomology group $\mathrm{H}^1(k, A[n])$. Given an n-covering $X \to A$ the fibre over the identity element is a torsor under $A[n]$, which defines a cohomology class in $\mathrm{H}^1(k, A[n])$. Conversely, given an $A[n]$-torsor P we construct the contracted product $X = A \wedge^{A[n]} P$. The map $n \circ \mathrm{pr}_A \colon A \times P \to A$ induces to an n-covering on the quotient $X \to A$.

Heuristically, the cohomology group $\mathrm{H}^1(k, A[n])$ classifies twisted forms of objects with automorphism group scheme $A[n]$. The multiplication by n morphism $A \xrightarrow{n} A$ in some sense has $A[n]$ as 'automorphism group scheme', since any dashed arrow making the diagram

$$\begin{tikzcd}
A \arrow[r, dashed] \arrow[d, "n"] & A \arrow[d, "n"] \\
A \arrow[r, "\mathrm{id}"] & A
\end{tikzcd}$$

commute is translation by an n-torsion element.

We now explore endomorphisms of a given para-abelian variety, i.e. where $A_1 = A_2 = A$ and $X_1 = X_2 = X$ are fixed.

Theorem 2.2.12 (Endomorphisms of A)**.** *Let f be an endomorphism of A and let $H = \mathrm{Ker}(1 - f)$. The following are equivalent:*

(i) *there exists an endomorphism F of X such that $F_* = f$;*

(ii) *the cohomology class $[X]$ lies in the kernel of $(1 - f)_*$;*

(iii) *there is an H-torsor T such that $X \cong A \wedge^H T$.*

Proof. From the equivalence (i)⇔(iii) of Proposition 2.2.2 it follows that such an endomorphism F exists if and only if $f_*([X]) = [X]$. This happens if and only if $f_*([X]) - [X] = 0$. Consider now the short exact sequence

$$0 \longrightarrow H \longrightarrow A \xrightarrow{1-f} A \longrightarrow 0,$$

which yields the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^1(k, H) \longrightarrow \mathrm{H}^1(k, A) \xrightarrow{(1-f)_*} \mathrm{H}^1(k, A) \longrightarrow \cdots .$$

Then $[X]$ lies in the kernel of $(1 - f)_*$ if and only if it lies in the image of $\mathrm{H}^1(k, H)$, i.e. is of the form $X \cong A \wedge^H T$ for some $[T] \in \mathrm{H}^1(k, H)$. ☐

We recover a number of statements from the literature. For example, if $f = \pm n$ we recover Prop. 19 of [Cla06]. In the special case that $f = -1$ is the sign involution, we recover Thm. 1.2 of [BDS24], which states the following.

Corollary 2.2.13 (sign involutions)**.** *The following are equivalent:*

(i) *X admits a sign involution;*

(ii) $2[X] = 0$ *in* $\mathrm{H}^1(k, A)$*;*

(iii) *there is an $A[2]$-torsor T such that $X \cong A \wedge^{A[2]} T$.*

Combining Corollary 2.2.11 with above result, it follows that a para-abelian variety has a sign involution if and only if it has a 2-covering $X \to A$. This equivalences arises naturally for elliptic curves in the context of Section 6.1; see e.g. Remark 6.1.17 and Lemma 6.1.20.

There is also a concrete construction. If σ is a sign involution on X, then the composition

$$X \xrightarrow{\mathrm{id} \times \sigma} X \times X \xleftarrow{\sim} A \times X \xrightarrow{\mathrm{pr}_A} A$$

is a two-covering. Conversely, if $F\colon X \to A$ is a two-covering, then

$$\sigma\colon X \longrightarrow X; x \longmapsto \alpha(-F(x), x) \tag{2.2.5}$$

is a sign involution, where $\alpha\colon A \times X \to X$ denotes the action of A on X. Indeed, after base changing to a field k'/k such that $X \otimes k' \cong A'$ by choice of point making $F'\colon A \to A$ into multiplication by two, then σ states $x \mapsto$ '$x - 2x$'.

In [BDS24] the above result was shown using the so-called *scheme of sign involutions* $\mathrm{Inv}^{\mathrm{sgn}}_{X/k}$, which is a closed subscheme of $\mathrm{Aut}_{X/k}$ parametrising exactly the sign involutions. For existence we refer to op. cit. The proof of above Corollary in op. cit. revelas that $\mathrm{Inv}^{\mathrm{sgn}}_{X/k}$ is a twisted form of A. We strengthen Corollary 2.2.13 by describing the cohomology class in the Weil–Châtelet group in terms of the cohomology class $[X]$, following the spirit of the arguments in loc. cit.

Theorem 2.2.14. *The scheme of sign involutions* $\mathrm{Inv}^{\mathrm{sgn}}_{X/k}$ *is the torsor under* A *corresponding to the cohomology class* $2[X] \in \mathrm{H}^1(k, A)$.

Proof. Since A is smooth, we may compute the cohomology group $\mathrm{H}^1(k, A)$ in terms of Galois cohomology $\mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), A(k^{\mathrm{sep}}))$. The cohomology class $[X]$ then corresponds to a 1-cocycle

$$\varphi\colon\ \mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow A(k^{\mathrm{sep}}).$$

An explicit description of φ is given as follows: pick a point $x \in X(k^{\mathrm{sep}})$, then $\sigma \cdot x = \varphi(\sigma) + x$, where $+\colon A \times X \to X$ denotes the canonical action of A on X by translation. The automorphism group scheme $\mathrm{Aut}_{X/k}$ is a twisted form of $\mathrm{Aut}_{A/k}$. The Galois action on $\mathrm{Aut}(A \otimes k^{\mathrm{sep}})$ corresponding to $\mathrm{Aut}_{X/k}$ is, according to Lem. 3.1 of [ST23], induced by conjugation

$$\sigma \cdot \psi\colon x \longmapsto \sigma \cdot \psi(\sigma \cdot x) = \psi(x + \varphi(\sigma)) + \varphi(\sigma).$$

If ψ is a sign involution of $A \otimes k^{\mathrm{sep}}$, then it follows from (1) on p. 4 of [BDS24] that above expression equals

$$\psi(x + \varphi(\sigma)) + \varphi(\sigma) = \psi(x) + 2\varphi(\sigma).$$

Therefore 2φ is a 1-cocycle corresponding to $[\mathrm{Inv}^{\mathrm{sgn}}_{X/k}] \in \mathrm{H}^1(k, A)$. ☐

We now consider other automorphisms of the abelian variety A, which are not necessarily the sign involution. Recall that an abelian variety A is *simple* if it does not have non-trivial abelian subvarieties. If A is simple, then $\mathrm{End}(A)$ does not have zero-divisors, see [EGM, Cor. 12.7].

Corollary 2.2.15. *Let* X *be a para-abelian variety with an automorphism of order* 6 *with a geometric fixpoint. If* A *is simple, then* X *has a rational point.*

Proof. Let ω be the associated automorphism of order 6 on A. The subring $\mathbb{Z}[\omega] \subset \mathrm{End}(A)$ is an integral domain in which ω has order 6 and hence is a root of the polynomial $t^2 - t + 1 = 0$. We rewrite this to the equation $1 - \omega = \omega^{-1}$. The existence of a stated automorphism of order 6 on X is equivalent with $0 = (1 - \omega)_*([X]) = \omega_*^{-1}([X])$. Repeatedly applying ω_*^{-1}, it follows that $[X] = 0$, so X has a rational point. ☐

Remark 2.2.16. The assumption on the simplicity of A is necessary: consider for example the product of an elliptic curve with an automorphism of order 6 with any para-elliptic curve without rational point, the latter equipped with the identity map. The product map has order 6 and plenty of geometric fixpoints, but the factor without rational points obstructs the existence of rational points on the product.

Remark 2.2.17. This line of argument is special to (multiples of) 6. The argument relies on the fact that $1 - \zeta_6 = \zeta_6^5$, but could work more generality if there are positive integers n and $i < n$ such that $1 - \zeta_n = \zeta_n^i$. We will show that if such integers exist, that n must be divisible by 6. Then the polynomials $X^i + X - 1$ and $X^n - 1$ over $\mathbb{Z}$ (or equivalently over $\mathbb{Q}$, by primitivity) share an irreducible factor. The polynomial $X^i + X - 1$ has been factorised by Selmer in [Sel56, Thm. 1]: it is irreducible unless $i \equiv 5 \bmod 6$, in which case it is the product of $X^2 - X + 1$ and an (explicit) irreducible polynomial, which can be defined recursively through the identity

$$X^{i+6} - X + 1 = X^i - X + 1 + (X^2 - X + 1)(X^{i+4} + X^{i+3} - X^{i+1} - X^i).$$

First suppose that $i \equiv 5 \bmod 6$ and the irreducible factor $X^2 - X + 1$ shares a root with $X^n - 1$. Since $X^2 - X + 1$ is the sixth cyclotomic polynomial, it follows directly that n is divisible by 6. The other case is that the (other) irreducible factor $X^i - X + 1$ or $(X^i - X + 1)/(X^2 - X + 1)$ shares a root with $X^n - 1$. By irreducibility it must be a divisor, from which it follows that it is a cyclotomic polynomial and hence palindromic. This only happens for $i = 2$, which again yields the sixth cyclotomic polynomial $X^2 - X + 1$, so n is divisible by 6. (See also §3 of loc. cit.)

In fact, this last Corollary can be generalised to actions by étale group schemes of order 6 with a geometric fixpoint. First, we state a helpful result from [BS04]; see its Prop. 2.

Lemma 2.2.18. *Let G be an étale group scheme acting on X, which by Remark 2.1.9 naturally induces an action on A. Let X^G be the scheme of G-invariants of X, and let A^G be the subgroup scheme of G-invariants of A. If X^G is nonempty, then the natural map $\mathrm{H}^1(k, A^G) \to \mathrm{H}^1(k, A)$ maps $[X^G]$ to $[X]$.*

We note that this proposition is in some sense similar to Theorem 2.2.12, where A^G plays a similar role as H, since they both give a criterion for the cohomology class of a group scheme to come from $\mathrm{H}^1(k, A^G)$ or $\mathrm{H}^1(k, H)$. In fact, in the special case that G is the constant group scheme associated to the group $\mathbb{Z}$, which acts on X by some *automorphism* F and hence on A by some automorphism f, then $A^G = \mathrm{Ker}(1 - f) = H$ and both results state that $[X]$ is the image of a class in $\mathrm{H}^1(k, A^G) = \mathrm{H}^1(k, H)$. We can similarly draw the following corollary, which was noted in op. cit. under the assumption that A is an elliptic curve; cf. its p. 33.

Proposition 2.2.19. *Let X be a para-abelian variety with the action of an étale group scheme G of order 6 with a geometric fixpoint. If A is simple, then X has a rational point.*

Proof. This follows directly from Lemma 2.2.18 once we show that A^G consists of a single point. To prove this, we assume without loss of generality that k is algebraically closed. Let ω be a generator for $G(k)$, then by simplicity $\mathbb{Z}[\omega] \subset \mathrm{End}(A)$ is a domain and hence satisfies the relation $\omega^2 - \omega + 1 = 0$. This implies that $1 - \omega = \omega^{-1}$, so the ω-invariants are exactly the kernel of ω^{-1}, which is 0. ☐

2.3 Elliptic para-abelian surfaces

In this section we study so-called *elliptic para-abelian surfaces*, since we later encounter them naturally in the study of canonical covers of bielliptic surfaces. Before treating the para-abelian case, let us first explain the theory given the existence of rational points. Recall that an elliptic fibration is a proper morphism in Stein factorisation such that the generic fibre is a smooth genus-one curve.

Definition 2.3.1. An abelian surface A is called *elliptic* if there is an elliptic fibration $f\colon A \to E$.

Remark 2.3.2. We postpone a more detailed discussion of elliptic fibrations to Section 3.1; e.g. for a definition of *elliptic fibration* see Definition 3.1.30 below.

Remark 2.3.3. The elliptic fibration f is not part of the data. This is the natural definition in view of Proposition 2.3.10.

The notation suggests that the codomain E should be an elliptic curve. Of course it is not surprising that E is a curve, but it is perhaps not so clear why the genus should equal one. This remarkable fact turns out to be a consequence of the powerful Blanchard's Lemma. Let us first recall its statement: the following is Thm. 7.2.1 of [Bri17].

Theorem 2.3.4 (Blanchard's Lemma)**.** *Let G be a connected algebraic group. Let X be a scheme of finite type with a G-action. Let Y be a scheme of finite type and let $f\colon X \to Y$ a proper morphism in Stein factorisation. Then there exists a unique action of G on Y such that f is equivariant.*

By abuse of terminology, we also refer to the following corollary, which is Cor. 7.2.2 in op. cit., as Blanchard's lemma.

Corollary 2.3.5 (Blanchard's Lemma)**.** *Let $f\colon X \to Y$ be a morphism of proper schemes of finite type in Stein factorisation. Then f induces a homomorphism $f_*\colon \mathrm{Aut}^0_{X/k} \to \mathrm{Aut}^0_{Y/k}$, such that f is $\mathrm{Aut}^0_{X/k}$-equivariant.*

Proof. Apply Theorem 2.3.4 to the action of $\mathrm{Aut}^0_{X/k}$ on X. ☐

We specialise to abelian varieties. In this case, Blanchard's lemma yields a substantial restriction on the existence morphisms in Stein factorisation where the domain is an abelian variety.

Lemma 2.3.6. *Let A be an abelian variety and let X be a proper scheme, such that $f\colon A \to X$ is in Stein factorisation. If $\mathrm{Aut}^0_{X/k}$ is affine, then X is a point.*

Proof. Since $A = \mathrm{Aut}^0_{A/k}$ is proper and $\mathrm{Aut}^0_{X/k}$ is affine, the induced homomorphism $f_*\colon \mathrm{Aut}^0_{A/k} \to \mathrm{Aut}^0_{X/k}$ obtained through the latter version of Blanchard's lemma is constant. Since $\mathrm{Aut}^0_{A/k}$ acts transitively on A and trivially on X through f_*, the only possibility is that f is constant. But a morphism in Stein factorisation is surjective, hence X consists of a single point. ☐

Although this may seem to be a reasonably specific criterion, we state two immediate and remarkable consequences, in which we take advantage of the fact that the group scheme PGL_n is affine.

Proposition 2.3.7. *If $f\colon A \to \mathbb{P}^n$ is a morphism in Stein factorisation, then $n = 0$.*

Proof. This is a special case of Lemma 2.3.6, since $\mathrm{Aut}^0_{\mathbb{P}^n/k} = \mathrm{PGL}_n$ is affine. ☐

Proposition 2.3.8. *Let A be an abelian variety and let C be a proper curve of genus at least* 2*. There is no morphism $f\colon A \to C$ in Stein factorisation.*

Proof. For curves of genus at least two, the dualising sheaf ω_C is ample. Fix $n \geq 1$ such that the tensor power $\omega_C^{\otimes n}$ is very ample, inducing an embedding $i\colon C \to \mathbb{P}\Gamma(C, \omega_C^{\otimes n})$. The natural action of $\mathrm{Aut}_{C/k}$ on C extends to a compatible action on $\mathbb{P}\Gamma(C, \omega_C^{\otimes n})$ by the pullback of differentials. Thus there is an injective morphism $\mathrm{Aut}^0_{C/k} \to \mathrm{Aut}_{\mathbb{P}\Gamma(C,\omega_C^{\otimes n})} = \mathrm{PGL}_N$ with $N = h^0(\omega_C^{\otimes n}) - 1$. Now $\mathrm{Aut}^0_{C/k}$ is a subgroup scheme of an affine group scheme and hence affine. We conclude by Lemma 2.3.6. ☐

Combining the above propositions completely classifies morphisms in Stein factorisation from abelian varieties to curves.

Theorem 2.3.9. *Let A be an abelian variety, let E be a curve and let $f\colon A \to E$ be a proper morphism in Stein factorisation. Then E is a genus-one curve.*

Proof. Combine Propositions 2.3.7 and 2.3.8. ☐

Suppose f is furthermore assumed to be an elliptic fibration. Then by definition, the codomain is assumed to be smooth, hence can be given the structure of an elliptic curve. This justifies our choice of notation. Note that since we have chosen a rational point on A, there is a rational point on E making f into a homomorphism of abelian varieties. Since f is an elliptic fibration, the kernel $\widetilde{J} = \mathrm{Ker}(f)$ is an elliptic curve. It follows that *every* fibre of f is smooth, so no degenerate fibres occur. Due to the short exact sequence

$$0 \longrightarrow \widetilde{J} \longrightarrow A \longrightarrow E \longrightarrow 0, \tag{2.3.1}$$

an elliptic abelian surface is an extension of two elliptic curves. The converse is true as well: recall that an abelian variety is called *simple* if there are no non-trivial abelian subvarieties.

Proposition 2.3.10. *Let A be an abelian surface. The following are equivalent:*

(i) *the abelian surface A is elliptic;*

(ii) *the abelian surface A has a subgroup scheme $\widetilde{J}$ which is an elliptic curve;*

(iii) *the abelian surface A is not simple.*

Proof. The implications (i)⇒(ii)⇒(iii) are clear from the above discussion. For the converse, we note that for dimension reasons a non-trivial abelian subvariety $\widetilde{J}$ of an abelian surface is an elliptic curve, and that the quotient map $A \to A/\widetilde{J}$ is an elliptic fibration. ☐

Recall the Poincaré irreducibility theorem, which states that any abelian variety is isogenous to a product of simple abelian varieties. Since an elliptic abelian variety is not simple, it is isogenous to a product of elliptic curves. More precisely, it follows that there are elliptic curves $\widetilde{E} \subset A$ and $\widetilde{J} \subset A$ such that the intersection $H = \widetilde{E} \cap \widetilde{J}$ is finite and the addition map $\widetilde{E} \times \widetilde{J} \to A$ is an isogeny. It is for this reason that elliptic abelian surfaces are sometimes called *split* abelian surfaces in the literature.

There is a natural isomorphism $A = (\widetilde{E} \times \widetilde{J})/H$. Regarding H also as a subgroup scheme of $\widetilde{E}$, there is a further isomorphism $E = \widetilde{E}/H$. For the sake of symmetry, we set $J = \widetilde{J}/H$: indeed, we may also regard A as an elliptic surface through the fibration $A \to J$. These objects sit naturally in the following diagram:

$$\begin{tikzcd}
\widetilde{J} \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{E} \times \widetilde{J} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{E} \arrow[d] \\
J & A \arrow[l] \arrow[r] & E
\end{tikzcd} \tag{2.3.2}$$

The vertical maps are quotients by a free H-action, hence are H-torsors. By equivariance it follows from Lemma 1.1.10 that both squares are Cartesian. Therefore the fibration $F\colon A \to E$ is étale locally isomorphic to a product

Combining the two fibrations, we find a morphism of abelian varieties $A \to E \times J$. These morphisms are compatible, in the following sense.

Proposition 2.3.11. *The diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & H \arrow[r] \arrow[d, "\Delta^-"] & \widetilde{E} \times \widetilde{J} \arrow[r, "+"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2 \arrow[r] \arrow[d, "+"] & \widetilde{E} \times \widetilde{J} \arrow[r] \arrow[d, "+"] & E \times J \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[d] & A \arrow[r] \arrow[uur, "\mathrm{id}"'] & E \times J \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

commutes, with exact rows and exact first column.

From the above short exact sequences, it follows that the quotient maps $\widetilde{E} \times \widetilde{J} \to A$ and $A \to E \times J$ naturally have the structure of H-torsors. The quotient maps $\widetilde{E} \to E$ and $\widetilde{J} \to J$ are also naturally H-torsors. We describe these torsors in terms of the theory developed in Section 1.3, by considering the Leray–Serre spectral sequences attached to the structure morphisms of $E \times J$, and the factors E and J.

Proposition 2.3.12. *The diagram*

$$\begin{tikzcd}
0 \arrow[r] & \begin{array}{c} \mathrm{H}^1(k, H) \\ \oplus \\ \mathrm{H}^1(k, H) \end{array} \arrow[r] \arrow[d, "+"] & \begin{array}{c} \mathrm{H}^1(E, H) \\ \oplus \\ \mathrm{H}^1(J, H) \end{array} \arrow[r] \arrow[d, "\mathrm{pr}_E^* + \mathrm{pr}_J^*"] & \begin{array}{c} \mathrm{Hom}(H^\vee, E^\vee) \\ \oplus \\ \mathrm{Hom}(H^\vee, J^\vee) \end{array} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(k, H) \arrow[r] & \mathrm{H}^1(E \times J, H) \arrow[r] & \mathrm{Hom}(H^\vee, E^\vee \times J^\vee) \arrow[r] & 0
\end{tikzcd} \tag{2.3.3}$$

is a morphism of short exact sequences. In the top row, the pair of H-torsors $(\widetilde{E} \to E, \widetilde{J} \to J)$ maps to the pair of inclusions $(H^\vee \to E^\vee, H^\vee \to J^\vee)$. Consequently in the bottom row, the H-torsor $A \to E \times J$ maps to the inclusion $H^\vee \to E^\vee \times J^\vee$.

Proof. The rows arise from the five-term exact sequences (2.2.3), which reduce to short exact sequences due to the existence of rational points. Note that the boundary maps ∂_E, ∂_J and $\partial_{E \times J}$ vanish due to the existence of rational points, so that the five-term exact sequences restrict to short exact sequences. The morphisms between the rows follow from the naturality of the Leray–Serre spectral sequence.

The dual of the quotient map $E \to E/H$ is the $H^\vee$-torsor $\widetilde{E} \to E$, which by Lemma 2.2.3 maps to the inclusion $H^\vee \to E^\vee$. In a similar way, the H-torsor $\widetilde{J} \to J$ amps to the inclusion $H^\vee \to J^\vee$. The last statement follows by commutativity, since

$$A = (\widetilde{E} \times \widetilde{J})/H = ((\widetilde{E} \times J) \times_{E\times J} (E \times \widetilde{J}))/H = (\widetilde{E} \times J) \wedge^H_{E\times J} (E \times \widetilde{J}),$$

corresponding to the cohomology class $\mathrm{pr}_E^*([\widetilde{E} \to E]) + \mathrm{pr}_J^*([\widetilde{J} \to J])$. □

We now develop a similar theory for para-abelian surfaces. Throughout let X denote a para-abelian surface. Let A be the associated abelian surface, so that X is an A-torsor.

Definition 2.3.13. A para-abelian surface X is called *elliptic* if there is an elliptic fibration $F\colon X \to C$.

The base-change $X^{\mathrm{alg}} \to C^{\mathrm{alg}}$ is then an elliptic fibration on an abelian surface, and we have seen that hence C^{alg} is an elliptic curve, so C is para-elliptic. Note that although X being an elliptic para-abelian surface implies that X^{alg} is also an elliptic abelian surface, the converse does not hold.

Example 2.3.14. Examples over finite fields arise naturally in cryptography as Jacobians of certain hyperelliptic curves. See for example [Sat09].

Instead, the connection between para-abelian surfaces and abelian surfaces should be sought through the theory of the associated abelian variety.

Proposition 2.3.15. *A para-abelian surface X is elliptic if and only if its associated abelian surface A is elliptic.*

Proof. If $F\colon X \to C$ is an elliptic fibration then $f = F_*$ is an elliptic fibration $A \to C$. Conversely consider $f\colon A \to E$ and let $[C] = f_*([X])$. Unravelling definitions, this means that $C = X/\widetilde{J}$, where $\widetilde{J} = \mathrm{Ker}(f)$. The quotient map $X \to C$ is a twisted form of f, hence is an elliptic fibration. □

Let X be an elliptic para-abelian surface. Since the associated abelian surface A is elliptic, there are elliptic curves E and J and an isogeny $A \to E \times J$ with kernel H. A similar statement still holds without rational points. Note that H, as a subgroup scheme of A, acts freely on X.

Proposition 2.3.16. *The quotient X/H is isomorphic to a product $C \times D$ of para-elliptic curves.*

Proof. The abelian surface A has two elliptic fibrations $f\colon A \to E$ and $g\colon A \to J$. Define the para-elliptic curve D through the cohomology class $[D] = g_*([X])$, which defines an elliptic fibration $G\colon X \to D$. Note that similarly $[C] = f_*([X])$, corresponding to the elliptic fibration $F\colon X \to C$. The diagonal map $X \to C \times D$ is a twisted form of $A \to E \times J$. By the universal property of the quotient, there is an induced map $X/H \to C \times D$, which is a twisted form of the isomorphism $A/H \xrightarrow{\sim} E \times J$ and is hence an isomorphism. □

Although A is covered also by a product $\widetilde{E} \times \widetilde{J}$, this is a lot more subtle in the context of para-abelian surfaces. We show that there is a certain cohomological obstruction to the existence of a cover of an elliptic para-abelian surface by a product of para-elliptic curves of a certain form. We treat this question by considering H-torsors over C and D, in analogy to before. As before, we consider the Leray–Serre spectral sequences corresponding to the structure morphisms of $C \times D$, and the factors C and D, cf. (2.2.3).

Proposition 2.3.17. *The diagram*

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix} \mathrm{H}^1(k,H) \\ \oplus \\ \mathrm{H}^1(k,H) \end{matrix} \arrow[r] \arrow[d, "+"] & \begin{matrix} \mathrm{H}^1(C,H) \\ \oplus \\ \mathrm{H}^1(D,H) \end{matrix} \arrow[r] \arrow[d, "\mathrm{pr}_C^*+\mathrm{pr}_D^*"] & \begin{matrix} \mathrm{Hom}(H^\vee,E^\vee) \\ \oplus \\ \mathrm{Hom}(H^\vee,J^\vee) \end{matrix} \arrow[r, "\partial_C\oplus\partial_D"] \arrow[d, "\cong"] & \begin{matrix} \mathrm{H}^2(k,H) \\ \oplus \\ \mathrm{H}^2(k,H) \end{matrix} \arrow[r, "p_C^*\oplus p_D^*"] \arrow[d, "+"] & \begin{matrix} \mathrm{H}^2(C,H) \\ \oplus \\ \mathrm{H}^2(D,H) \end{matrix} \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(k,H) \arrow[r] & \mathrm{H}^1(C\times D,H) \arrow[r] & \mathrm{Hom}(H^\vee,E^\vee\times J^\vee) \arrow[r, "\partial_{C\times D}"] & \mathrm{H}^2(k,H) \arrow[r] & \mathrm{H}^2(C\times D,H)
\end{tikzcd} \tag{2.3.4}$$

commutes, with exact rows. In the bottom row, the H-torsor $X \to C \times D$ maps to the inclusion $H^\vee \to E^\vee \times J^\vee$.

The main difference with Proposition 2.3.12 is the potential non-vanishing of the boundary maps ∂_E and ∂_J. If the images of the inclusion $H^\vee \to E^\vee$ and $H^\vee \to J^\vee$ vanish in the cohomology group $\mathrm{H}^2(k, H)$, then there are H-torsors $\widetilde{C} \to C$ and $\widetilde{D} \to D$, which play the role of the covers $\widetilde{E} \to E$ and $\widetilde{J} \to J$. From commutativity of (2.3.4) it follows that there is an induced cover $\widetilde{C} \times \widetilde{D} \to X$, as a quotient by the anti-diagonal H-action. However in general the boundary map of a para-elliptic curve will be nonzero; we study the boundary map closely in Section 6.3 and refer to section for examples.

Let $\alpha = \partial_C(H^\vee \to E^\vee)$ be the obstruction in $\mathrm{H}^2(k, H)$ for Y to arise as a quotient of a product. Note that $\partial_J(H^\vee \to J^\vee) = -\alpha$, since the diagonal embedding $H^\vee \to E^\vee \times J^\vee$ arises from the cohomology class $[X] \in \mathrm{H}^1(C \times D, H)$. Commutativity and exactness of (2.3.4) direct implies the following result.

Theorem 2.3.18. *The following are equivalent:*

(i) *There are H-torsors $\widetilde{C} \to C$ and $\widetilde{D} \to D$ such that $Y \cong (\widetilde{C} \times \widetilde{D})/H$;*

(ii) *$\partial_C(H^\vee \to E^\vee) = 0$, where $H^\vee \to E^\vee$ denotes the inclusion;*

(iii) *$\partial_D(H^\vee \to J^\vee) = 0$, where $H^\vee \to J^\vee$ denotes the inclusion.*

The cohomology classes $[Y]$ in $\mathrm{H}^1(C \times D, H)$ which do not satisfy (i) forms a non-trivial element of the cokernel of $\mathrm{pr}_C^* + \mathrm{pr}_D^*$. Through a more accurate diagram chase we may also phrase the following more general form of the same result.

Proposition 2.3.19. *The diagram* (2.3.4)*, induces a natural isomorphism*

$$\mathrm{Coker}(\mathrm{pr}_C^* + \mathrm{pr}_D^*) = \mathrm{Ker}(p_C^* \oplus p_D^*) \cap \Delta^-(\mathrm{H}^2(k, H)),$$

where $\Delta^-(\mathrm{H}^2(k, H))$ denote the anti-diagonal in $\mathrm{H}^2(k, H) \oplus \mathrm{H}^2(k, H)$.

Proof. This is a diagram chase. From the above diagram it is clear that an element of $\mathrm{H}^1(C \times D, H)$ maps to $\mathrm{H}^2(k, H) \oplus \mathrm{H}^2(k, H)$. By commutativity and exactness it lands within $\mathrm{Ker}(p_C^* \oplus p_D^*)$ and $\mathrm{Ker}(+) = \Delta^-(\mathrm{H}^2(k, H))$.

Conversely, any element in the intersection of these two kernels comes from an element of $\mathrm{H}^1(C \times D, H)$, which is unique up to elements of $\mathrm{H}^1(C, H) \oplus \mathrm{H}^1(D, H)$ and $\mathrm{H}^1(k, H)$. Since $+$ is surjective, this latter element lifts to $\mathrm{H}^1(k, H) \oplus \mathrm{H}^1(k, H)$ and maps to $\mathrm{H}^1(C, H) \oplus \mathrm{H}^1(D, H)$. Whence this procedure defines a unique element of the cokernel of $\mathrm{pr}_C^* + \mathrm{pr}_D^*$. □

Theorem 2.3.20. *Suppose that there is an isogeny $C_1 \times D_1 \to X$, where C_1 and D_1 are para-elliptic curves, such that the composition $C_1 \times D_1 \to X \to C \times D$ is the product of two isogenies $C_1 \to C$ and $D_1 \to D$. Then there are H-torsors $\widetilde{C} \to C$ and $\widetilde{D} \to D$ such that $X = (\widetilde{C} \times \widetilde{D})/H$, where H acts anti-diagonally on the product $\widetilde{C} \times \widetilde{D}$. In other words, the cohomology class α vanishes in $\mathrm{H}^2(k, H)$.*

Proof. The isogenies $C_1 \times D_1 \to X \to C \times D$ induce isogenies of abelian varieties, whose kernels sit inside a short exact sequence

$$0 \longrightarrow K \longrightarrow G_C \times G_D \longrightarrow H \longrightarrow 0,$$

where $C_1 \to C$ is a G_C-torsor and $D_1 \to D$ is a G_D-torsor, such that X is isomorphic to the quotient $(C_1 \times C_2)/K$. In a cohomological language, this means that the natural map $\mathrm{H}^1(C \times D, G_C \times G_D) \to \mathrm{H}^1(C \times D, H)$ maps the cohomology class of the $G_C \times G_D$-torsor $C_1 \times D_1 \to C \times D$ to the cohomology class of the H-torsor $X \to C \times D$. The surjective map $G_C \times G_D \to H$ induces a pair of maps $G_C \to H$ and $G_D \to H$, and hence also the map $G_C \times G_D \to H \times H$, such that the following diagram is commutative.

$$\begin{tikzcd}
G_C \times G_D \arrow[rr] \arrow[dr] & & H \times H \arrow[dl, "+"] \\
& H &
\end{tikzcd}$$

By naturality, the following square commutes.

$$\begin{tikzcd}
\mathrm{H}^1(C, G_C) \times \mathrm{H}^1(D, G_D) \arrow[r] \arrow[d] & \mathrm{H}^1(C \times D, G_C \times G_D) \arrow[d] \\
\mathrm{H}^1(C, H) \times \mathrm{H}^1(D, H) \arrow[r, "\mathrm{pr}_C^* + \mathrm{pr}_D^*"] & \mathrm{H}^1(C \times D, H)
\end{tikzcd}$$

The pair of cohomology classes $([C_1 \to C], [D_1 \to D])$ in $\mathrm{H}^1(C, G_C) \times \mathrm{H}^1(D, G_D)$ maps to the cohomology class $[X \to C \times D]$. Letting $[\widetilde{C} \to C] \in \mathrm{H}^1(C, H)$ and $[\widetilde{D} \to D] \in \mathrm{H}^1(D, H)$ be the images of $[C_1 \to C] \in \mathrm{H}^1(C, G_C)$ and $[D_1 \to D] \in \mathrm{H}^1(D, G_D)$, respectively, then

$$X = (\widetilde{C} \times D) \wedge^H_{C \times D} (C \times \widetilde{D}) = (\widetilde{C} \times \widetilde{D})/H.$$

Since the cohomology class $[\widetilde{C} \to C]$ in $\mathrm{H}^1(C, H)$ maps to the inclusion $H^\vee \to E^\vee$, by exactness $\alpha = 0$. □

In this way, the cohomology class $\alpha \in \mathrm{H}^2(k, H)$ obstruct the existence of certain covers $C_1 \times D_1 \to X$. The main restriction is that the isogeny $C_1 \times D_1 \to C \times D$ is diagonal. In the generic case, we may remove this assumption. For the purposes of Theorem 2.3.20, one may without loss of generality swap the factors C_1 and D_1, in which case the following criterion is sufficient.

Proposition 2.3.21. *Suppose that the elliptic curves E and J associated to C and D are non-isogenous. Then an isogeny $C_1 \times D_1 \to C \times D$ is either the product of two isogenies $C_1 \to C$ and $D_1 \to D$, or the product of two isogenies $C_1 \to D$ and $D_1 \to C$.*

Proof. Let E_1 and J_1 be the associated elliptic curves of C and D. It suffices to treat the induced isogeny on associated abelian surfaces $E_1 \times J_1 \to E \times J$. It can be identified with a 2×2-matrix

$$\phi = \begin{pmatrix} \phi_{EE} & \phi_{JE} \\ \phi_{EJ} & \phi_{JJ} \end{pmatrix},$$

where each of the entries are homomorphisms of elliptic curves $\phi_{EE} \colon E_1 \to E$, $\phi_{EJ} \colon E_1 \to J$, $\phi_{JE} \colon J_1 \to E$ and $\phi_{JJ} \colon J_1 \to J$. If ϕ_{EE} is nonzero, then E and E_1 are isogenous, so that ϕ_{EJ} and ϕ_{JE} are zero and ϕ is diagonal. On the other hand if ϕ_{EE} is zero, then ϕ_{JE} must be nonzero, since ϕ is an isogeny, so J_1 and E are isogenous and by a similar argument it follows that ϕ is anti-diagonal. □

Remark 2.3.22. One can also argue conceptually, using an argument similar to what we see in Section 3.3. Under the assumption that E and J are non-isogenous, it follows that the Picard number ρ of the product $E \times J$ equals 2. Therefore the Picard number of $E_1 \times J_1$ is also 2. The Gram matrix of the intersection product on $\mathrm{Num}(E_1 \times J_1)$ is the hyperbolic plane $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so there are two rays of effective divisor classes satisfying $D^2 = 0$ and the only fibrations of $E_1 \times J_1$ are the projections onto the two factors. The Stein factorisation of $E_1 \times J_1 \to E$ is hence one of the two projections projection; the Stein factorisation of $E_1 \times J_1 \to J$ is the projection onto the other factor, since otherwise it violates surjectivity. It is now not difficult to see that $E_1 \times J_1 \to E \times J$ is either the product of two induced maps $E_1 \to E$ and $J_1 \to J$, or of two maps $E_1 \to J$ and $J_1 \to E$.

Corollary 2.3.23. *If E and J are non-isogenous and α does not vanish in $\mathrm{H}^2(k, H)$, then X is not covered by a product of para-elliptic curves.*

Chapter 3

Bielliptic surfaces

3.1 The Enriques classification of surfaces

Much of 19th- and 20th-century algebraic geometry was concerned with the theory of *algebraic surfaces*, at the time defined as smooth, two-dimensional complex algebraic varieties. The main collection of results of the time is the *Enriques classification of surfaces*. The classification was extended in the 1970s by Bombieri and Mumford in a series of papers [Mum69; BM77; BM76] to algebraically closed fields of arbitrary characteristic. Adapting this classification to higher dimensions is an active area of research relating to the *minimal model programme*.

Despite the added complexity, it is valuable to develop the theory in a general arithmetic context, namely that of an *arbitrary* ground field. In particular, we do not assume k to be algebraically closed, or even perfect, and no assumption on the characteristic is made.

We explain the relevance of imperfect ground fields. Consider a scheme X admitting a fibration to a curve. Its geometry is heavily reflected in the generic fibre, which naturally lives over the function field of the curve. Of course the function field may be non-perfect, even if the ground field is algebraically closed. The inseparable field extensions of the function field then influence the geometry of X in a substantial way. A prominent example of this phenomenon is constituted by schemes admitting a *quasi-elliptic fibration*, i.e. a fibration whose generic fibre is a regular but non-smooth genus-one curve; see Example 3.1.36 and the surrounding discussion. Quasi-elliptic fibrations play an important role in the adaptation of the Enriques classification of surfaces to the small characteristics $p = 2$ and $p = 3$. This occurs for example in the theory of bielliptic surfaces, where a certain otherwise elliptic fibration turns out to be quasi-elliptic; see Definition 3.1.30. This indicates the importance of algebraic geometry over imperfect ground fields, even for algebraic geometers who prefer to work in the context of an algebraically closed ground field of positive characteristic.

In order to phrase a coarse version of the Enriques classification of surfaces, we first treat a number of generalities about invertible sheaves on schemes that can be found in most standard references on the classification of surfaces, e.g. [Băd01; Sha96; Bea96; Laz04a]. We swiftly locate four interesting classes of surfaces that includes the main interest of this thesis: the so-called *bielliptic surfaces*. They are the focus of our study for the remainder of this chapter.

Lemma 3.1.1. *Let X be a proper equidimensional scheme of dimension d and let $\mathscr{L}$ be an invertible sheaf on X. Then the function $n \mapsto h^0(\mathscr{L}^{\otimes n})$ lies in $O(n^d)$ as $n \to \infty$.*

Proof. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be ample invertible sheaves such that $\mathscr{L} = \mathscr{L}_1 \otimes \mathscr{L}_2^\vee$. There is a natural inclusion $\mathrm{H}^0(\mathscr{L}) \subset \mathrm{H}^0(\mathscr{L}_1)$, so without loss of generality we may assume that $\mathscr{L}$ is ample. Since the higher cohomology of ample sheaves vanishes, it follows that $h^0(\mathscr{L}^{\otimes n}) = \chi(\mathscr{L}^{\otimes n})$. Now the result follows directly from the asymptotic Riemann–Roch formula [Laz04a, Thm. 1.1.24]. ☐

This bounds the asymptotic growth of $h^0(\mathscr{L}^{\otimes n})$, as a function of n. To quantify precisely how sharp this bound is, we introduce the following invariant.

Definition 3.1.2. Let $\mathscr{L}$ be an invertible sheaf on a proper scheme X. The *Iitaka dimension of $\mathscr{L}$* is defined as

$$\kappa(\mathscr{L}) = \kappa(X, \mathscr{L}) = \inf\{m \in \mathbb{Z} \mid n \mapsto h^0(\mathscr{L}^{\otimes n})/n^m \text{ is bounded as } n \to \infty\}.$$

The Iitaka dimension takes values in the finite set $\{-\infty, 0, 1, \cdots, \dim(X)\}$.

Example 3.1.3. Let X be a proper scheme and let $\mathscr{L}$ be an invertible sheaf on X.

- If $\mathscr{L}$ is anti-ample, it follows that $h^0(\mathscr{L}^{\otimes n}) = 0$ for all $n > 0$, so that the Iitaka dimension is given by $\kappa(\mathscr{L}) = -\infty$;
- If $\mathscr{L}$ is ample, then from the asymptotic Riemann–Roch formula [Laz04a, Ex. 1.2.19] it follows that the Iitaka dimension equals $\kappa(\mathscr{L}) = \dim(X)$;
- If $\mathscr{L} \cong \mathcal{O}_X$, then $h^0(\mathscr{L}^{\otimes n}) = 1$ so the Iitaka dimension is $\kappa(\mathscr{L}) = 0$.
- We generalise the previous example. Suppose $\mathscr{L}$ is torsion in the Picard group with order m, so that $\mathscr{L}^{\otimes m} \cong \mathcal{O}_X$. Then
$$h^0(\mathscr{L}^{\otimes n}) = \begin{cases} 1 & \text{if } n \text{ is divisible by } m; \\ 0 & \text{if } n \text{ is not divisible by } m, \end{cases}$$
is bounded with a (nonzero) constant subsequence, so the Iitaka dimension is $\kappa(\mathscr{L}) = 0$.

Under suitable assumptions, it turns out that the asymptotic growth of $h^0(\mathscr{L}^{\otimes n})$ measures the dimension of a certain scheme attached to X. Suppose that X is also normal. Consider the ring

$$R(\mathscr{L}) = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathscr{L}^{\otimes n}).$$

The following is well-known, see e.g. §14 of [Băd01] or Cor. 2.1.38 of [Laz04a].

Proposition 3.1.4. *Suppose $R(\mathscr{L})$ is finitely generated, so that the homogeneous spectrum* $\operatorname{Proj} R(\mathscr{L})$ *is a scheme of finite type. Then $\kappa(\mathscr{L}) = \dim \operatorname{Proj} R(\mathscr{L})$.*

Suppose that X is proper and Gorenstein, in which case there is a canonical choice of invertible sheaf on X by the dualising sheaf ω_X. The Iitaka dimension $\kappa(\omega_X)$ is a useful invariant, since it is stable under base-change. However, it is not a *birational invariant.*

Example 3.1.5. Let X be the rational cuspidal curve, defined by the homogeneous equation $y^2 z = x^3$ in $\mathbb{P}^2$. For a degree-three plane curve, the adjunction formula states that $\omega_X \cong \mathcal{O}_X$, whence $\kappa(\omega_X) = 0$. Note that X is birational to $\mathbb{P}^1$, as the normalisation map is $\mathbb{P}^1 \to X$. Since $\omega_{\mathbb{P}^1}$ is anti-ample, it follows that $\kappa(\omega_{\mathbb{P}^1}) = -\infty$.

Nevertheless, restricting ourselves to *smooth* and proper schemes, the Iitaka dimension $\kappa(\omega_X)$ does turn out to be a birational invariant. This is because the *plurigenera* $h^0(\omega_X^{\otimes n})$ are birational invariants for smooth schemes [Băd01, Prop. 5.7] and the Iitaka dimension $\kappa(\omega_X)$ simply captures their asymptotic growth. We hence define the Kodaira dimension as a birational invariant for proper schemes that admit a resolution of singularities.

Definition 3.1.6. Let X be a proper scheme. Assume that the base-change X^{alg} admits a resolution of singularities $\widetilde{X} \to X^{\mathrm{alg}}$. Then we define the *Kodaira dimension* as

$$\operatorname{kod}(X) = \kappa\big(\widetilde{X}, \omega_{\widetilde{X}}\big).$$

Remark 3.1.7. This definition does not depend on the choice of resolution of singularities $\widetilde{X} \to X^{\mathrm{alg}}$, since any two choices of $\widetilde{X}$ are smooth, proper and birational to each other.

Remark 3.1.8. A resolution of singularities always exists in characteristic 0 by a celebrated result of Hironaka; see [Hir64]. The existence of a resolution of singularities in positive characteristic is an open problem, though results in low-dimensional cases (curves, surfaces and threefolds) have been established; see the survey article [Hau10].

The Kodaira dimension forms a discrete invariant of the d-dimensional scheme X, which hence yields a coarse classification of proper Gorenstein schemes. Though not strictly necessary, let us also assume that X is smooth and geometrically integral. We will return to the case $d = 2$, but we can already investigate the simpler case $d = 1$, comprising *smooth curves.*

Example 3.1.9 (Classification of smooth curves). Let C be a smooth proper curve of genus $g = h^1(\mathcal{O}_C)$ with $h^0(\mathcal{O}_C) = 1$. Using Riemann–Roch and Serre duality we see that

$$\begin{aligned} \operatorname{kod}(C) &= -\infty && \iff g = 0, && (C \text{ is a } \textit{Brauer–Severi} \text{ curve, i.e. } C^{\mathrm{alg}} \cong \mathbb{P}^1;) \\ \operatorname{kod}(C) &= 0 && \iff g = 1, && (C \text{ is a } \textit{para-elliptic} \text{ curve, as in Chapter 2;}) \\ \operatorname{kod}(C) &= 1 && \iff g \geq 2, && (C \text{ is } \textit{of general type.}) \end{aligned}$$

Although this classification is coarser than the one obtained through the genus, the Kodaira dimension is already able to distinguish interesting classes of smooth curves, namely Brauer–Severi curves and para-elliptic curves. Note that curves of general type share the property that ω_C is ample.

We now pursue the classification of surfaces, mostly following the notes of Shafarevich [Sha66]. Since we do not assume the ground field k to be perfect, we have to make a distinction between regularity and smoothness. It turns out that some statements are more naturally phrased in terms of regular surfaces. Nevertheless, in most of the thesis we are interested in smooth surfaces specifically. For clarity, we recall the definition or a surface in the context of their classification.

Definition 3.1.10. A *surface* is a proper, geometrically integral scheme of dimension 2.

The classification of regular surfaces is handled much better up to *birational equivalence*. The central construction herein is that of the *blow-up*: let $Z \subset X$ be a closed subscheme corresponding to the sheaf of ideals $\mathcal{I}$. Then the blow-up $\widetilde{X}$ of X with centre Z is defined as the homogeneous spectrum of the $\mathcal{O}_X$-algebra $\bigoplus_{n=0}^{\infty} \mathcal{I}^n$. The blow-up of a regular surface in a closed point is again regular (see pp. 15–16 of op. cit.) and the structural morphism $\widetilde{X} \to X$ is a proper, birational morphism. Conversely, it is a deep fact in the classification of surfaces that any proper, birational morphism arises in this way: the following 'decomposition theorem' can be found on p. 55 of op. cit.

Theorem 3.1.11 (Decomposition Theorem). *A proper, birational morphism between regular surfaces decomposes as a sequence of blow-ups in closed points.*

Care is needed in the distinction between regular and smooth surfaces over imperfect fields. So far, the notion of regularity is the correct one: it is, for example, not generally true that the blow-up of a smooth surface in a closed point is smooth, as the following example illustrates.

Example 3.1.12. Suppose k has characteristic $p > 1$ and contains an element a that is not a pth power. The standard affine open $\mathbb{A}^1 \times \mathbb{A}^1 = \operatorname{Spec} k[S, T]$ of $\mathbb{P}^1 \times \mathbb{P}^1$ contains the closed point x corresponding to the maximal ideal $(S, T^p - a)$, which has purely inseparable residue field $k' = k\big(\sqrt[p]{a}\big)$. One chart of the blow-up of $\mathbb{A}^1 \times \mathbb{A}^1$ in x is given by

$$\operatorname{Spec} k[S, T, (T^p - a)/S] \cong \operatorname{Spec} k[S, T, U]/(T^p - SU - a).$$

Over an algebraic closure the defining equation becomes $(T - \sqrt[p]{a})^p - SU = 0$, which is the standard equation of an A_{p-1}-singularity for the geometric point with $T = \sqrt[p]{a}$ and $S = U = 0$.

Remark 3.1.13. Note in particular that an A_{p-1}-singularity arises through the base-change of a regular local ring. Some singularities cannot appear in this way. A complete list of types of rational double points (according to Artin's classification [Art77] including positive characteristic) that descend to regular schemes is given in [Sch08].

Above decomposition theorem is of main importance in the classification of surfaces: since blow-ups are reasonably well understood, one can approach many questions using a birational model. We hence investigate whether a regular surface X arises as a blow-up of another surface in a closed point x and, consequently, how to 'undo' the blow-up. It is key to consider the *exceptional locus* E, which is the subscheme of $\widetilde{X}$ on which $\widetilde{X} \to X$ is not an isomorphism, which is abstractly given by $E = \operatorname{Proj} \bigoplus_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}$. Here E is isomorphic to a copy of $\mathbb{P}^1$ over the residue field $\kappa(x)$ and satisfies the numerical condition $E \cdot \omega_X = -[\kappa(x) : k]$.

Definition 3.1.14. A (-1)-*curve* (also called an *exceptional curve*) is an integral curve $E \subset X$ abstractly isomorphic to $\mathbb{P}^1_R$ (where $R = \mathrm{H}^0(\mathcal{O}_E)$), that satisfies the numerical condition $E \cdot \omega_X = -\dim_k R$.

The terminology stems from the Italian school of algebraic geometry, where k is assumed to be an algebraically closed field, say of characteristic 0. Under this assumption we have $R = k$, so that $E \cdot \omega_X = -1$ and $E^2 = -1$ by the adjunction formula. Similarly, an integral curve C isomorphic to $\mathbb{P}^1$ satisfying $C^2 = -2$ is called a (-2)-*curve*.

There are a number of equivalent formulations to this numerical condition. Through the adjunction formula, one is that $E^2 = -[k' : k]$, hence that the normal bundle $\mathcal{N}_{E/X}$ has degree -1 when restricted to $E \cong \mathbb{P}^1_{k'}$. Let $\mathcal{I}$ denote the ideal sheaf associated to E. Then one may equivalently assume that the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ has degree 1 when restricted to E.

We have stated that the blow-up of a closed point produces a (-1)-curve. The converse is a classical statement due to Castelnuovo and can be found in a general context e.g. on p. 102 of [Sha66] or in [Stacks, Tag 0CEF], but see also Thm. V.5.7 of [Har13] for the statement over an algebraically closed ground field.

Theorem 3.1.15 (Castelnuovo's contraction theorem)**.** *Let X be a regular surface and let $E \subset X$ be a (-1)-curve. There exist a unique regular surface Y and a unique proper, birational morphism $f \colon X \to Y$, such that*

- *the image of E is a closed point;*
- *the morphism $f \colon X \to Y$ is the blow-up of Y at the closed point $f(E)$.*

We say that a morphism f satisfying the above properties is a *contraction* of the exceptional curve E. One can also say that the (-1)-curve E is *blown down*. Repeatedly blowing down yields a sequence of birational regular surfaces. Since the Picard rank $\rho = \operatorname{rk}\operatorname{Num}(X) \in \mathbb{N}$ is a decreasing invariant, this process must terminate and we arrive at a surface which cannot be blown down further.

Definition 3.1.16. A regular surface X is called *minimal* (over the ground field k) if any proper birational morphism $X \to Y$ to a regular surface is an isomorphism. A *minimal model* of a regular surface X is a minimal surface birational to X.

As a corollary of the Castelnuovo contraction theorem, we find the following criterion for minimality.

Corollary 3.1.17. *A regular surface is minimal if and only if it contains no (-1)-curves.*

The minimal model need not be unique: a regular surface may admit non-isomorphic minimal models. Nevertheless, it is shown in p. 141 of [Sha66] that if a regular, proper surface admits non-isomorphic minimal models, then its plurigenera vanish and hence the Kodaira dimension equals $-\infty$. This relates to the fact that two (-1)-curves E_1 and E_2 may intersect: one may choose whether to blow down E_1 or E_2, though the image of the other curve will cease to be exceptional.

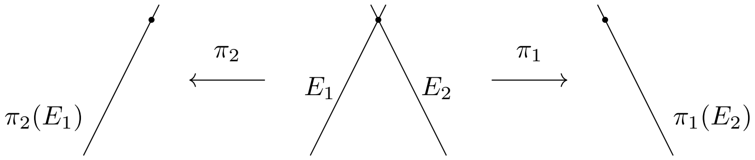

Figure 3.1: Intersecting exceptional curves can be blown down individually

Example 3.1.18. The rational surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are non-isomorphic minimal surfaces. It is well-known that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point admits a birational morphism to $\mathbb{P}^2$; see e.g. Exc. 3.3.9 of [CLS11].

We emphasise that the notion of minimality depends on the choice of ground field k. Fix a field extension k'/k. First of all, the notion of regularity is not stable under base-change, so that $X' = X \otimes k'$ may fail to be regular. But even if we sidestep this issue, perhaps by assuming that X is smooth, or even that k is perfect, the base change of a minimal surface may cease to remain minimal. Suppose the situation over k' is as in Figure 3.1, where two exceptional curves intersect non-trivially. Suppose also that k'/k is Galois and that the Galois group $\operatorname{Gal}(k'/k)$ permutes the curves E_1 and E_2. Over k, the subscheme $(E_1 \cup E_2)/\operatorname{Gal}(k'/k) \subset X$ is a curve, that is not exceptional since $(E_1 + E_2)^2 = E_1^2 + E_2^2 + 2E_1 \cdot E_2 = 0$.

We illustrate this behaviour over a perfect ground field through the following example: the classification of minimal geometrically rational surfaces. They form an interesting subclass of surfaces satisfying $\operatorname{kod}(X) = -\infty$. The following classification result can be found in §3.4 of [Has09].

Theorem 3.1.19 (Classification of minimal geometrically rational surfaces). *Suppose the ground field k is perfect. Let X be a smooth, minimal and geometrically rational surface. Then X is either*

- *isomorphic to $\mathbb{P}^2$;*
- *a smooth quadric in $\mathbb{P}^3$ with $\mathrm{Pic}(X) = \mathbb{Z}$;*
- *a smooth del Pezzo surface with $\mathrm{Pic}(X) = \mathbb{Z}\omega_X$;*
- *a conic bundle over a rational curve, with $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.*

If k is algebraically closed, then the classification of smooth, minimal, rational surfaces is treated in [Bǎd01], see Thm. 12.8, where it is shown that the only possibilities are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a so-called *Hirzebruch surface*, where the latter two cases are conic bundles over $\mathbb{P}^1$. We emphasise that del Pezzo surfaces with $\mathrm{Pic}(X) = \mathbb{Z}\omega_X$ do not occur within the classification over an algebraically closed field, since each such surface is geometrically isomorphic to the blow-up of $\mathbb{P}^2$ in at most nine points, or to the product $\mathbb{P}^1 \times \mathbb{P}^1$; see §13.3 of [Sha96]. In particular a smooth del Pezzo surface X over a perfect field with *degree* ω_X^2 at most 8 is not geometrically minimal, even though it may be minimal.

Let us also consider the converse direction: if the base-change $X' = X \otimes k'$ is minimal, then what can be said about the minimality of X? This behaves nicely for separable field extensions.

Proposition 3.1.20. *Let X be a geometrically regular surface and let k'/k be a separable field extension. If the base-change $X' = X \otimes k'$ is minimal, then X is minimal.*

Proof. Let $X \to Y$ be a birational morphism, where Y is a regular surface. Since k'/k is a separable field extension, it follows from Prop. 6.7.4 of [EGA IV$_2$] that the base-change $Y' = Y \otimes k'$ is regular as well. By minimality the morphism $X' \to Y'$ is an isomorphism, so that $X \to Y$ is an isomorphism by étale descent. □

It seems like the situation for inseparable field extensions is quite restricted as well, but we do not treat this case further. Instead let us state an argument which ensures that this subtlety is not be an issue in the class of surfaces in which we are interested most, namely smooth surfaces of Kodaira dimension 0. For this, we apply a main structural theorem in the classification over an algebraically closed field, conducted by Bombieri and Mumford in [BM77]. The following is Thm. 1 in op. cit.

Theorem 3.1.21. *Suppose k is algebraically closed and let X be a minimal surface with $\mathrm{kod}(X) = 0$. Then $\omega_X^{\otimes 4} \cong \mathcal{O}_X$ or $\omega_X^{\otimes 6} \cong \mathcal{O}_X$. In particular, ω_X is torsion in $\mathrm{Pic}(X)$ and hence is a numerically trivial invertible sheaf.*

Corollary 3.1.22. *Let X be a smooth surface.*

- *If $\mathrm{kod}(X) \geq 0$ and X^{alg} is minimal, then X is minimal.*
- *If $\mathrm{kod}(X) = 0$, then X^{alg} is minimal if and only if X is minimal.*

Proof. Suppose first that $\mathrm{kod}(X) \geq 0$ and that $X^{\mathrm{alg}} = X \otimes k^{\mathrm{alg}}$ is minimal. It is well-known that $\omega_{X^{\mathrm{alg}}}$ is nef; see e.g. [BM77]. It follows that ω_X is also nef, so that there are no (-1)-curves and indeed X is minimal.

Now suppose that X is minimal. Let $X^{\mathrm{alg}} \to X^{\mathrm{alg}}_{\mathrm{min}}$ be a birational morphism to a regular minimal model. Its exceptional locus R^{alg} is encoded within the dualising sheaf, since by Theorem 3.1.21 we have that

$$\omega_{X^{\mathrm{alg}}}^{\otimes 12} \cong \omega_{X^{\mathrm{alg}}/X^{\mathrm{alg}}_{\mathrm{min}}}^{\otimes 12} \cong \mathcal{O}(-12R^{\mathrm{alg}})$$

is anti-effective. Let R be the reduced subscheme of the Weil divisor attached to the invertible sheaf $\omega_X^{\otimes -12}$. In order to show that R is contractible, we invoke the criterion Cor. 5.3 of [Sch00], which is a generalisation of Castelnuovo's contraction theorem. In this case we verify that $R \cdot \omega_X = R^2 \leq 0$, since it is a negative-definite curve after base-change. It follows that we can contract R, hence descending $X^{\mathrm{alg}} \to X^{\mathrm{alg}}_{\mathrm{min}}$ to the ground field k. □

With the discussion of minimality and base-change out of the way, we continue with the Enriques classification of smooth surfaces performed by Bombieri and Mumford over an algebraically closed field in a series of articles [Mum69; BM77; BM76]. Let X be a smooth surface; there are four classes, according to whether the Kodaira dimension of X equals $-\infty$, 0, 1 or 2. We are mainly interested in the case where $\mathrm{kod}(X) = 0$, which in the light of Example 3.1.9 can be viewed as generalisations of elliptic curves. We cite the main structural result for the case $\mathrm{kod}(X) = 0$, as on p. 25 of [BM77].

Theorem 3.1.23 (Tetrachotomy of minimal surfaces of Kodaira dimension 0)**.** *Suppose k is algebraically closed. Let X be a smooth, minimal surface with* $\mathrm{kod}(X) = 0$*. Then X falls into one of four classes, as in Table 3.1.*

Surface	b_2	b_1	$\chi(\mathcal{O}_X)$	$h^2(\mathcal{O}_X)$	$h^1(\mathcal{O}_X)$	Δ	p
Bielliptic	2	2	0	$\begin{cases} 0 \\ 1 \end{cases}$	1 2	0 2	any 2 or 3
Abelian	6	4	0	1	2	0	any
Enriques	10	0	1	$\begin{cases} 0 \\ 1 \end{cases}$	0 1	0 2	any 2
K3	22	0	2	1	0	0	any

Table 3.1: Tetrachotomy of proper, smooth, minimal surfaces of Kodaira dimension $\mathrm{kod}(X) = 0$ over an algebraically closed field

Through the classification is stated over algebraically closed fields, we may without loss of generality base-change a smooth minimal surface of Kodaira dimension 0 to an algebraic closure. It then falls into one of the above four classes, distinguished by their second étale Betti number b_2. This leads to the following definition.

Definition 3.1.24. Let X be a smooth, minimal surface of Kodaira dimension 0. It is called

- a *bielliptic surface* if $b_2 = 2$;
- a *para-abelian surface* if $b_2 = 6$;
- an *Enriques surface* if $b_2 = 10$;
- a *K3 surface* if $b_2 = 22$.

Remark 3.1.25. The analysis performed in §5 of op. cit. shows that a smooth, minimal surface of Kodaira dimension 0 with $b_2 = 6$ with a rational point indeed admits the structure of an abelian surface. Then para-abelian surfaces defined as above are their twisted forms, hence are indeed two-dimensional para-abelian varieties in the sense of Definition 2.1.1.

Remark 3.1.26. A note on the terminology of 'bielliptic surfaces'. In some sources, mainly those which work in a complex analytic setting, bielliptic surfaces are referred to as *hyperelliptic surfaces*, which historically stems from the Italian school of algebraic geometry. We emphasise that there is no relation to *hyperelliptic curves*, i.e. the double covers over $\mathbb{P}^1$. We instead prefer the more modern terminology *bielliptic* due to the relation with the geometry of the surface, cf. Sections 3.2 and 3.3. We also note that in characteristics 2 and 3 some sources subdivide the class of bielliptic surfaces further, into bielliptic surfaces and so-called *quasi-bielliptic surfaces*; we postpone a discussion of terminology to Remark 3.2.18.

The invariant $\Delta = 2h^1(\mathcal{O}_X) - b_1$ of Table 3.1 is of importance in positive characteristic. It vanishes in characteristic 0 since then $b_1 = 2h^{0,1}$ by Hodge theory. An equivalent definition over an algebraically closed field is

$$\Delta = 2\big(\dim \text{of tangent space of } \mathrm{Pic}^0_{X/k} - \dim \text{of tangent space of } (\mathrm{Pic}^0_{X/k})_{\mathrm{red}}\big); \tag{3.1.1}$$

see pp. 24–25 of [BM77]. Note that the formuation of $(\mathrm{Pic}^0_{X/k})_{\mathrm{red}}$ commutes with base-field extensions by Lem. 3.3.7 of [Bri17], which applies since the Picard scheme $\mathrm{Pic}^0_{X/k}$ is proper, so that this expression makes sense over all ground fields. Since in characteristic 0 all group schemes are reduced, we again see that $\Delta = 0$ unless $p \geq 2$. The table in fact shows that $\Delta = 0$ for surfaces of Kodaira dimension zero, unless $p = 2$, 3. This is special to Kodaira dimension zero: it turns out that surfaces of larger Kodaira dimension may have an arbitrarily non-reduced Picard scheme in any positive characteristic; see Thm. 2.2 and Prop. 3.1 of [Lie09].

It turns out that most interesting characteristic p behaviour occurs whenever $\Delta > 0$. For later reference, we observe the following characterisation.

Proposition 3.1.27. *Let X be a bielliptic surface or an Enriques surface. The following are equivalent:*

(i) *X has a geometrically non-reduced Picard scheme;*

(ii) *X has a non-reduced Picard scheme;*

(iii) $\Delta > 0$*;*

(iv) $\Delta = 2$*;*

(v) $h^2(\mathcal{O}_X) = 1$*;*

(vi) $\omega_X \cong \mathcal{O}_X$*.*

Proof. The Picard scheme $\mathrm{Pic}^0_{X/k}$ of a smooth, integral scheme is proper, so the equivalence of the first two points follows from Lem. 3.3.7 of [Bri17]. The equivalence between the second and the third points follows from (3.1.1). The numerical conditions are clear from Table 3.1. For the last two conditions, we note that $h^2(\mathcal{O}_X) = h^0(\omega_X)$ by Serre duality and that ω_X is torsion due to Theorem 3.1.21. ☐

Remark 3.1.28. In fact, the equivalence of (i) to (iii) only requires that the Picard scheme $\mathrm{Pic}^0_{X/k}$ be proper. If $\mathrm{kod}(X) = 0$ then there are equivalences amongst (i) to (iv), and between (v) and (vi). Only the equivalence between (iv) and (v) requires the full assumption that X be either a bielliptic surface or an Enriques surface.

For the sake of exposition assume that the ground field is algebraically closed for the remainder of this discussion. The four classes of smooth, minimal surfaces of Kodaira dimension 0 are interrelated by a number of standard constructions, except in certain small *critical* characteristics, as displayed in Figure 3.2. In critical characteristics, we instead obtain 'singular versions' of the expected classes, although the Kodaira dimension may decrease.

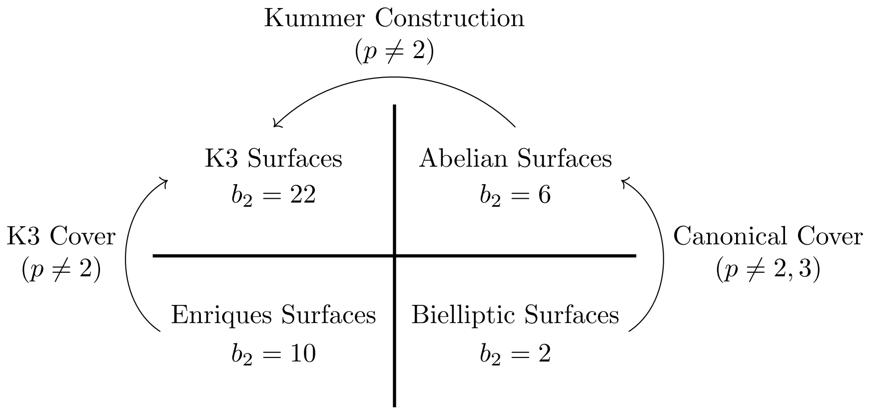

Figure 3.2: The relations between minimal surfaces of Kodaira dimension 0 with $k = k^{\mathrm{alg}}$

Given an abelian surface A over a field with $p \neq 2$, the *Kummer construction* creates a K3 surface as the minimal resolution of the 16 rational double points of type A_1 of the quotient $A/\{\pm 1\}$, where -1 acts on A as the sign involution. In characteristic $p = 2$ the situation changes: there are fewer but more complicated singularities on the quotient $A/\{\pm 1\}$. If A is a supersingular abelian variety, there is only a unique elliptic singularity. Then the minimal resolution turns out to be a rational surface, instead of a K3 surface. We briefly sketch the correct generalisation of the Kummer construction for $p = 2$: one replaces the abelian surface A by a non-smooth surface, namely the self-product of the rational cuspidal curve, which can be equipped with the suitable action of an infinitesimal group scheme μ_2 or α_2. Remarkably, the quotient has only mild singularities. A resolution of singularities then yields either a K3 surface or a rational surface. In some sense, the singularities of the surface are 'cancelled' by the singularities coming from the group scheme. A detailed explanation of the above, and more, can be found in e.g. [Sch07; KS21; Ber23].

Another example is the *K3 cover* of an Enriques surface. If $p \neq 2$, then any Enriques surface is a quotient of a K3 surface by a free $\mathbb{Z}/2\mathbb{Z}$-action. The K3 cover coincides with the universal cover, since a K3 surface is simply connected. Again, this construction breaks down in the critical characteristic $p = 2$. The K3 cover in the critical characteristic $p = 2$ as defined by Bombieri and Mumford in [BM76] exhibits an Enriques surface as the quotient of a so-called *K3-like surface* by a group scheme of order 2, which may be either $\mathbb{Z}/2\mathbb{Z}$, μ_2 or α_2. A K3-like surface has the cohomology of a K3 surface, but may be non-smooth. In some sense, the singularities on a K3-like surface are offset by the non-smoothness of the group schemes μ_2 and α_2. For a detailed description of the theory of K3 covers in characteristic 2, we refer to [CD89; Sch21a].

The last example is the *canonical cover* of bielliptic surfaces, which we discuss in more detail in Section 3.5. In this case there are two critical characteristics $p = 2$ and $p = 3$. Outside of these characteristics, any bielliptic surface is covered by an abelian surface. Analogously, it may be necessary in critical characteristic to cover a bielliptic surface by a non-smooth surface that has the cohomology of an abelian surface. The surfaces for which this is necessary are precisely the quasi-bielliptic surfaces. We expand further on the similarities and analogies between the K3 cover of an Enriques surface and the canonical cover of a bielliptic surface throughout Section 3.5.

We finish this section with a discussion about elliptic and quasi-elliptic fibrations, jointly referred to as *genus-one fibrations*, since we encounter them extensively throughout our study of bielliptic surfaces. In fact, they form a constant factor within the study of (subclasses of) smooth surfaces of Kodaira dimension ≤ 1. For example, within the class of surfaces with $\mathrm{kod}(X) = 0$, it is a fact that any bielliptic or Enriques surface admits a genus-one fibration; for Enriques surfaces see Thm. 5.7.1 of [CD89], for bielliptic surfaces this is shown in the upcoming sections. We also note our study of elliptic abelian surfaces in Section 2.3.

Definition 3.1.29. A *genus-one fibration* is a proper morphism $f\colon X \to Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and the generic fibre of f is a genus-one curve.

Note that in our definition we make no assumption on the smoothness of the generic fibre, but smoothness of X implies that the generic fibre X_η is at least a *regular* genus-one curve over the function field $\kappa(Y)$. Smoothness of X also implies that a geometric generic fibre X_η^{alg} is a reduced scheme over an algebraic closure of the function field $\kappa(Y)^{\mathrm{alg}}$, as follows from Cor. 7.3 of [Băd01] since f is in Stein factorisation.

Definition 3.1.30. Let X be a smooth surface and let $f\colon X \to Y$ be a genus-one fibration.

(i) If X_η is a *smooth* curve, then we say that f is an *elliptic fibration*;

(ii) If X_η is a *regular but non-smooth* curve, then we say that f is a *quasi-elliptic fibration.*

In the latter case, there is a field extension $K/\kappa(Y)$ such that the base-change $X_\eta \otimes_{\kappa(Y)} K$ is not regular. Since the base-change of a regular scheme by a separable field extension remains regular, the field extension $K/\kappa(Y)$ is necessarily inseparable and we see that regular but non-smooth curves exist only over imperfect ground fields. Note that the function field of Y may be imperfect, even if the ground field k itself is algebraically closed with $p > 0$. In this sense, regular but non-smooth curves arise naturally as generic fibres of suitable fibrations.

Definition 3.1.31. A *quasi-elliptic curve* is a proper, regular, geometrically reduced but non-smooth curve of genus one.

Though the existence of quasi-elliptic curves may seem pathological, in the context of genus-one fibrations they should actually be regarded as natural generalisations of para-elliptic curves. Indeed: many phenomena in positive characteristic may be explained by regular but non-smooth curves. For example, the existence of quasi-elliptic fibrations yields a natural counterexample to Bertini's theorem on the smoothness of a general member of a linear system Cor. 10.9 of [Har13].

Proposition 3.1.32. *Let C be a quasi-elliptic curve over a field K with $h^0(\mathcal{O}_C) = 1$. Then K is imperfect with $p = 2$ or 3. Furthermore C is a twisted form of the rational cuspidal curve, i.e. the base-change to an algebraic closure $C \otimes K^{\mathrm{alg}}$ is a rational cuspidal curve.*

Proof. To see that quasi-elliptic curves only exist over non-perfect fields of characteristics $p = 2$ and 3, one can also invoke Tate's genus change formula [Tat52]. If C is a regular curve over a field k and k'/k is a purely inseparable field extension, then $C \otimes k'$ may be non-regular. Let C' be the normalisation of $C \otimes k'$, then Tate's formula states that $h^1(\mathcal{O}_{C'}) - h^1(\mathcal{O}_C)$ is divisible by $(p-1)/2$. If $h^1(\mathcal{O}_C) = 1$, then $h^1(\mathcal{O}_{C'}) = 0$ and divisibility only holds for $p = 2$ and $p = 3$.

Since the base-change $C^{\mathrm{alg}} = C \otimes k^{\mathrm{alg}}$ is a non-smooth integral genus-one curve, there is a unique non-smooth point, which is either a nodal singularity or a cuspidal singularity. Since C is regular, the local ring of the singularity must be analytically irreducible by a theorem of Zariski [Zar45], which rules out the nodal singularity. □

Remark 3.1.33. If one drops the requirement that C is geometrically reduced, then there are a lot more possibilities for regular but non-smooth genus-one curves. They are classified in [Sch22].

Remark 3.1.34. The quasi-elliptic curves in characteristics 2 and 3 occur naturally within a larger family of curves of higher genus. In [HS24] the authors construct, for $n \geq 1$ and $p \geq 2$ prime, a collection of non-smooth curves $X_{p,n}$ of genus $h^1(\mathcal{O}_{X_{p,n}}) = (np^{n+1} - (n+2)p^n + 2)/2$, whose twisted forms can be regular. If $(p,n) = (2,0)$ or $(3,1)$ then the curve $X_{p,n}$ is indeed isomorphic to the rational cuspidal curve.

The above classification of quasi-elliptic curves aids our understanding of quasi-elliptic fibrations.

Corollary 3.1.35. *Suppose k is algebraically closed, and let $f\colon X \to Y$ be a quasi-elliptic fibration. Then for almost all $y \in Y$, the fibre $f^{-1}(y)$ is a rational cuspidal curve.*

We give a local example of a quasi-elliptic fibration in characteristic $p = 3$. Though this example is quite general, we refer to [Lan79] for a more complete description of quasi-elliptic fibrations in characteristic 3. Examples in characteristic 2 can be given through the deformation theory of the rational cuspidal curve. Explicit equations in characteristic $p = 2$ are given by Queen in [Que71; Que72].

Example 3.1.36. Consider the schemes $Y = \operatorname{Spec} k\llbracket t\rrbracket$, $X = \operatorname{Spec} k\llbracket t\rrbracket[x,y]/(y^2 - x^3 - t)$ and the natural map $f\colon X \to Y$. The generic fibre X_η is given by by the equation $y^2 = x^3 + t$ over the function field $k(\!(t)\!)$. This curve is not geometrically regular and hence not smooth: since $p = 3$, over the field extension $k(\!(t^{1/3})\!)$ the equation becomes $y^2 = (x + t^{1/3})^3$, which is a rational, cuspidal curve.

The ideal (y) in $k\llbracket t\rrbracket[x,y]/(y^2 - x^3 - t)$ defines a closed subscheme $Z \subset X$, $Z = \operatorname{Spec} k\llbracket t\rrbracket/(x^3 + t)$. The intersection with the closed fibre is the closed subscheme corresponding to the maximal ideal (x,y) of $k[x,y]/(y^2 - x^3)$, which is the unique non-regular point of the fibre.We first make the following simplification. The situation in the generic fibre is different: the intersection point is the closed point corresponding to the maximal ideal (y) of $k(\!(t)\!)[x,y]/(y^2 - x^3 - t)$, whose residue field is $k(\!(t)\!)[x]/(x^3 + t) \cong k(\!(t^{1/3})\!)$.

To verify that the curve $\operatorname{Spec} k(\!(t)\!)[x,y]/(y^2 - x^3 - t)$ is regular, we only have to check that the local ring of the unique non-smooth point is regular. Let $\mathfrak{m}$ be the maximal ideal (y). Then regularity follows since $\mathfrak{m}/\mathfrak{m}^2 = y \cdot k(\!(t^{1/3})\!)$. Underlying this phenomenon is the observation that the residue field of the non-smooth point $k(\!(t^{1/3})\!)$ is inseparable over the residue field $k(\!(t)\!)$.

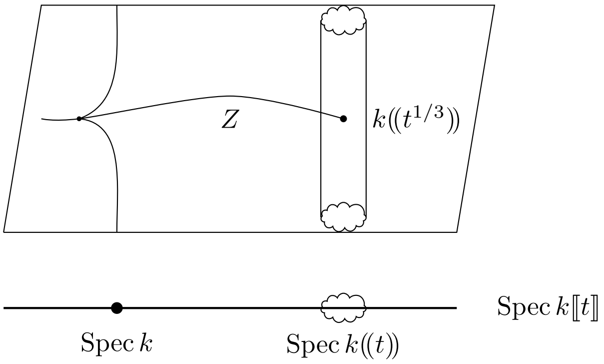

Figure 3.3: An artistic impression of the quasi-elliptic fibration of Example 3.1.36

3.2 The Albanese fibration

We now start with the study of bielliptic surfaces specifically. The word 'bielliptic' reflects a very important property of the geometry, namely that a bielliptic surface X comes equipped with two distinct genus-one fibrations, namely the *Albanese morphism* $f\colon X \to P$ to a para-elliptic curve and *another fibration* $g\colon X \to B$ to a Brauer–Severi curve. Though the Albanese morphism can be either an elliptic fibration or a quasi-elliptic fibration, with the latter of course only in characteristics 2 and 3, the other fibration is always an elliptic fibration. In this section we construct and study the Albanese map. We begin with a general background on the Albanese and then specialise to the case of bielliptic surfaces.

Classically, say over an algebraically closed ground field, the Albanese variety of a pointed scheme (X, x_0) is constructed as an abelian variety A equipped with a morphism $f\colon X \to A$, where x_0 maps to the identity element of A. Over non-algebraically closed fields, the possible absence of rational points suggests that we should instead work with para-abelian varieties, which has the additional advantage that the choice of base points is not necessary.

The Albanese variety has been constructed in a large generality. By work of Brochard it is defined for algebraic stacks; see §§7–8 of [Bro19]. In the context of open algebraic spaces it is constructed in [Sch23a], extending the work in [LS23] over a base scheme. For proper varieties and schemes over a ground field, the Albanese map was already defined by Matsusaka in [Mat52] and Grothendieck in [Gro62]. We refer also to the treatment of [Wit08, App. A] for proper, geometrically integral schemes. Although it is not strictly necessary, we assume for simplicity that X is a proper, geometrically integral and smooth scheme, prior to restricting ourselves to bielliptic surfaces.

Definition 3.2.1. An *Albanese morphism* is a pair (P, f), where P is a para-abelian variety and $f\colon X \to P$ is a morphism, such that for each pair (Q, g) satisfying the same assumptions, there is a unique morphism $h\colon P \to Q$ such that $g = h \circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & P \arrow[d, dashed, "\exists! h"] \\ & Q \end{tikzcd}$$

Remark 3.2.2. We refer to the scheme P as the *Albanese variety.* By abuse of terminology we may mention the *Albanese* only when referring to either the morphism or the scheme.

Since the Albanese morphism is defined by a universal property, uniqueness is immediate, but existence results can be rather subtle. In our context the existence of an Albanese morphism is well established. We state the formulation as in Cor. 10.5 of [LS23] without proof.

Theorem 3.2.3. *Over ground fields, an Albanese morphism $f\colon X \to P$ exists. It commutes with ground field extensions, is equivariant with respect to group scheme actions, and is functorial in X.*

Remark 3.2.4. The universal property essentially guarantees that the Albanese defines a left-adjoint functor to the full inclusion functor $(\mathsf{ParaAbVar}/k) \to (\mathsf{Sch}/k)$.

Example 3.2.5. Suppose $k = \mathbb{C}$ and let $n > 0$. The Albanese of projective n-space $\mathbb{P}^n$ is 0. Indeed, consider any morphism to a g-dimensional abelian variety $f\colon \mathbb{P}^n \to A$. Since $\mathbb{P}^n$ is simply connected, the analytification of f factors over the universal cover of a g-dimensional complex abelian variety, which is isomorphic to $\mathbb{C}^g$; but since $\mathbb{CP}^n$ is compact, this lift must be constant. We later give an algebraic proof that the Albanese of $\mathbb{P}^n$ is a point.

Although we do not expand upon the construction of the Albanese variety, we mention that it is intimately connected to the existence of the *maximal abelian subvariety* of the Picard scheme, i.e. an abelian subvariety containing all other subvarieties. The existence of maximal abelian subvarieties of group schemes of finite type was shown in §7 of [LS23], using a functorial three-step filtration. Let $\mathrm{Pic}^{\tau}_{X/k}$ be the subgroup scheme that parametrises numerically trivial invertible schemes.

Definition 3.2.6. Let $\mathrm{Pic}^{\alpha}_{X/k}$ be the *maximal abelian subvariety* of $\mathrm{Pic}^{\tau}_{X/k}$.

Connectivity of abelian varieties gives an inclusion $\mathrm{Pic}^{\alpha}_{X/k} \subset \mathrm{Pic}^{0}_{X/k}$. If X is smooth and geometrically integral, then the Picard scheme $\mathrm{Pic}^{0}_{X/k}$ is proper, although it may be non-reduced in positive characteristics. Over a perfect ground field the reduced subscheme inherits the structure of a group scheme, which implies that the maximal abelian subvariety is given by $\mathrm{Pic}^{\alpha}_{X/k} = (\mathrm{Pic}^{0}_{X/k})_{\mathrm{red}}$. One should take care over imperfect ground fields: here, properness of $\mathrm{Pic}^{\tau}_{X/k}$ implies that $(\mathrm{Pic}^{0}_{X/k})_{\mathrm{red}}$ is still naturally equipped with the structure of a group scheme; see Lem. 3.3.7 of [Bri17].

The concept of the maximal abelian subvariety allows us to state an equivalent definition for the Albanese morphism, which is in fact the one used in [LS23]. The equivalence is a main result of op. cit.; see §§8, 10 for a proof.

Theorem 3.2.7. *A morphism of schemes $f\colon X \to P$ is an Albanese morphism for X if and only if the induced map on Picard schemes induces an isomorphism $f^*\colon \mathrm{Pic}^{0}_{P/k} \to \mathrm{Pic}^{\alpha}_{X/k}$.*

Example 3.2.8. Let C be a smooth curve of genus $g = h^1(\mathcal{O}_X)$. It is well-known that $\mathrm{Pic}^{0}_{C/k}$ is an abelian variety of dimension g, so that the maximal abelian subvariety coincides with $\mathrm{Pic}^{0}_{C/k}$. If X is the Albanese of C, then it follows that $\mathrm{Pic}^{0}_{X/k} = \mathrm{Pic}^{0}_{C/k}$, so that X is a torsor under the dual abelian variety of $\mathrm{Pic}^{0}_{C/k}$. In particular the Albanese variety of C is g-dimensional. Note in particular that the Albanese map is not surjective, by considering dimensions.

There is in fact a general dimension formula for the Albanese variety; see [BM77] for a proof.

Proposition 3.2.9. *The dimension of the Albanese variety equals $b_1/2$, where b_1 denotes the first étale Betti number.*

Remark 3.2.10. If the Picard scheme is reduced, then $2h^1(\mathcal{O}_X) = b_1$, so that the Albanese dimension equals $h^1(\mathcal{O}_X)$. This follows from (3.1.1), since the invariant Δ vanishes. Alternatively, it is a consequence of the equivalent definition Theorem 3.2.7. Note that $\dim P = \dim \mathrm{Pic}^{0}_{P/k} = \dim \mathrm{Pic}^{\alpha}_{X/k} = \dim \mathrm{Pic}^{0}_{X/k}$ by smoothness of the Picard scheme, whose dimension coincides with the dimension of the tangent space at the identity element, which can be identified with $\mathrm{H}^1(X, \mathcal{O}_X)$.

Example 3.2.11. Since the first étale Betti number of $\mathbb{P}^n$ vanishes, we see that the dimension of its Albanese is 0. This generalises Example 3.2.5.

Example 3.2.12. According to Table 3.1, the first étale Betti number of K3 surfaces and Enriques surfaces vanishes. We conclude similarly that their Albanese varieties vanish, so that there are no non-constant morphisms from a K3 or Enriques surface to a para-abelian variety.

In preparation for the study of bielliptic surfaces, suppose X is a smooth, proper and geometrically integral scheme with $b_1 = 2$. Similarly to the above examples, this implies that the Albanese variety is one-dimensional, hence a para-elliptic curve. The following proof is similar to that of Prop. V.15 of [Bea96].

Proposition 3.2.13. *Let X be a proper scheme with $b_1 = 2$. Then the Albanese morphism $f\colon X \to P$ is in Stein factorisation.*

Proof. Let $Y = \mathrm{Spec}_P\, f_*\mathcal{O}_X$ be the Stein factorisation of f, so that f factors as a composition of $\widetilde{f}\colon X \to Y$ and $u\colon Y \to P$. The hypothesis that $b_1 = 2$ implies that P is a para-elliptic curve, so Y is a curve since u is finite. The Stein factorisation coincides with the relative normalisation of P in X. Thus Y is normal, whence smooth. Let $g\colon Y \to Q$ be the Albanese morphism of Y. We computed in Example 3.2.8 that the dimension of Q equals the genus of Y.

We claim that the composition $g \circ \widetilde{f}\colon X \to Q$ is an Albanese morphism for X. We verify the universal property: let $X \to R$ be a morphism to a para-abelian variety, which by the universal property of P factors as $h \circ f$ for a unique morphism $h\colon P \to R$, then we show that there exists a unique $\widetilde{h}\colon Q \to R$ such that $\widetilde{h} \circ g = h \circ u$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd, "\widetilde{f}"'] & P \arrow[dr, "h"] & \\
 & & R \\
Y \arrow[uur, "u"'] \arrow[r, "g"] & Q \arrow[ur, dashed, "\exists!\widetilde{h}"] &
\end{tikzcd}$$

By the universal property of the Albanese Q of Y, there is a unique $\widetilde{h}\colon Q \to R$ such that $\widetilde{h} \circ g = h \circ u$. Precomposing with $\widetilde{f}$ gives $\widetilde{h} \circ g \circ \widetilde{f} = h \circ u \circ \widetilde{f} = h \circ f$, which treats existence. For uniqueness of $\widetilde{h}$, suppose that there is a second morphism $\widetilde{h}'\colon Q \to R$ also satisfying $\widetilde{h}' \circ g \circ \widetilde{f} = h \circ f$. We thus find that $\widetilde{h}' \circ g \circ \widetilde{f} = h \circ u \circ \widetilde{f} = \widetilde{h} \circ g \circ \widetilde{f}$. Note that $\widetilde{f}$ is an epimorphism, because it is proper and in Stein factorisation, whence we conclude that $\widetilde{h} \circ g = h \circ u = \widetilde{h}' \circ g$. Then $\widetilde{h} = \widetilde{h}'$ follows from the uniqueness in the universal property of the Albanese morphism of Q.

Because $g \circ \widetilde{f} \colon X \to Q$ satisfies the universal property of the Albanese morphism, there is an isomorphism of Albanese varieties $P \cong Q$. Comparing dimensions we find that $h^1(\mathcal{O}_Y) = 1$, so Y is a para-elliptic curve. Again applying the universal property of the Albanese, we see that the map u is an isomorphism. Hence f is in Stein factorisation. ☐

Remark 3.2.14. The proof simplifies if $\Delta = 0$, which is automatic in characteristic 0, since then the assumption $b_1 = 2$ translates to $h^1(\mathcal{O}_X) = 1$. There is a chain of inequalities

$$1 = h^1(\mathcal{O}_P) \leq h^1(\mathcal{O}_Y) \leq h^1(\mathcal{O}_X) = 1,$$

where the last inequality holds because $\widetilde{f}$ is in Stein factorisation. This directly shows that $h^1(\mathcal{O}_Y) = 1$. See also p. 189 of [Sha96].

From now on let X be a bielliptic surface. Note that the defining assumption that $b_2 = 2$ implies through Table 3.1 that $b_1 = 2$, so that the above Proposition applies. Note that the Albanese is furthermore proper, because X is proper and P is separated. It is also flat, because f is surjective and P is regular; see Prop. III.9.7 of [Har13]. We now study the fibres of f through the *numerical group* $\mathrm{Num}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$, where $\mathrm{Pic}^\tau(X)$ comprises all numerically trivial invertible sheaves on X. Recall that $\mathrm{Num}(X)$ is a finitely generated free abelian group, see e.g. Prop. 1.1.16 of [Laz04a], and that its rank ρ is commonly referred to as the *Picard number.* The following result is very classical, tracing back to Bagnera and de Franchis; see §9 of [BF10].

Proposition 3.2.15. *Let X be a bielliptic surface. Its Picard number equals $\rho = 2$.*

Proof. The Igusa–Severi inequality states that $\rho \leq b_2 = 2$; see [Igu60]. Let F be a fibre of f and let $\mathscr{L} = \mathcal{O}(H)$ be an ample invertible sheaf on X. Then clearly $F^2 = 0$ and $H^2 > 0$ and $H \cdot F > 0$, so that F and H are linearly independent in $\mathrm{Num}(X)$. ☐

Theorem 3.2.16. *The closed fibres of f are irreducible genus-one curves. Therefore the Albanese morphism of a bielliptic surface is a genus-one fibration.*

Proof. Irreducible components of fibres would increase the Picard number ρ. Let F be any closed fibre. By the adjunction formula we get

$$0 = \omega_X \cdot F + F^2 = 2h^1(\mathcal{O}_F) - 2,$$

whence $h^1(\mathcal{O}_F) = 1$. Since the fibre F is arbitrary, it follows from flatness of f that the genus of the generic fibre also equals 1. ☐

The Albanese of a bielliptic surface can hence either be an elliptic fibration or a quasi-elliptic fibration, although the latter only occurs in characteristics $p = 2$ and $p = 3$. We introduce the following terminology.

Definition 3.2.17. A *quasi-bielliptic surface* is a bielliptic surface whose Albanese morphism is a quasi-elliptic fibration.

Remark 3.2.18. In our terminology, a quasi-bielliptic surface is also a bielliptic surface. We warn the reader that this seems to be non-standard: in the literature a *bielliptic surface* is often assumed to have an elliptic Albanese fibration. The author is of the opinion that this separation is unnecessary and that it is much more natural to view quasi-bielliptic surfaces as a subset of bielliptic surfaces. From the correct perspective, many statements about bielliptic surfaces hold regardless of whether the Albanese morphism is smooth.

Remark 3.2.19. We have now encountered critical behaviour in small characteristics $p = 2$ and $p = 3$ multiple times: first of all the Picard scheme $\mathrm{Pic}^0_{X/k}$ may be non-reduced and second of all the Albanese map f may be a quasi-elliptic fibration. These two phenomena are independent of one another: a quasi-bielliptic surface may or may not have a reduced Picard scheme and a bielliptic surface with a non-reduced Picard scheme may or may not be quasi-bielliptic. We will see examples over algebraically closed fields in Section 3.4, where in fact all possibilities of critical behaviour are tabulated; see Table 3.6.

The study of elliptic fibrations is often pursued through its degenerate fibres. Supposing that the Albanese map is an elliptic fibration, is extremely well-behaved in this regard.

Proposition 3.2.20.

(i) *If f is an elliptic fibration, then* all fibres *are smooth genus-one curves.*

(ii) *If f is a quasi-elliptic fibration, then* every fibre *is a quasi-elliptic curve or a rational cuspidal curve.*

In either case f has no wild fibres, so $R^1 f_* \mathcal{O}_X$ is an invertible sheaf.

Proof. The first part is Prop. 5 of [BM77]. For the second part, see p. 27 of op. cit. or §7.8 of [EGA III_2]. ☐

We state two immediate consequences.

Proposition 3.2.21. *A bielliptic surface contains no curves C with $C^2 = -2$.*

Proof. Suppose otherwise. By the adjunction formula

$$-2 = C^2 + \omega_X \cdot C = 2g(C) - 2,$$

so $g(C) = 0$. By the Riemann–Hurwitz formula there is no surjective map $C \to P$, so C is contained in a fibre of f. But this is impossible since the fibres are irreducible with $F^2 = 0$. ☐

Theorem 3.2.22. *The invertible sheaves ω_X and $f^*(R^1 f_* \mathcal{O}_X)^\vee$ are isomorphic.*

Proof. This is a direct consequence of the canonical bundle formula for genus-one fibrations; see Thm. 2 of [BM77], using the fact that there are no multiple fibres and that $\omega_P \cong \mathcal{O}_P$. ☐

Especially the latter observation is frequently useful, culminating in the main application in Section 4.2.

3.3 The numerical group and the other fibration

A frequently reoccurring strategy in the study of surfaces is to investigate the numerical group $\operatorname{Num}(X) = \operatorname{Pic}(X)/\operatorname{Pic}^\tau(X)$ equipped with extra structure, such as the class of the canonical divisor, the intersection pairing, and the cone of effective or even ample divisors. This often yields fundamental information about the nature of the surface in question. This is for example part of the philosophy of *Mori theory*, which provides a framework for generalising the Enriques classification of Section 3.1 to higher dimensions, by regarding the classes of exceptional curves as *extremal rays* inside the numerical group. The Enriques classification of surfaces from the perspective of Mori theory is treated in detail in [Cil20].

It sometimes possible to determine the isomorphism class of a surface from numerical invariants together with knowledge of certain configuration of curves, the latter of which can be regarded as computations of the intersection pairings in the numerical group. This strategy is for example used in the determination of explicit equations for certain *Hilbert modular surfaces*, as in §VIII of [Gee88]. In the case of bielliptic surfaces, the information encoded in the numerical group yields a second genus-one fibration, which is in a sense 'transversal' to the Albanese fibration.

To this end, we first discuss in a general setting how invertible sheaves produce rational maps and morphisms. In order to construct this morphism, we apply the theory of the homogeneous spectrum Proj; our main source is §3.7 of [EGA II]. Let X be a proper, geometrically integral scheme and fix an invertible sheaf $\mathscr{L}$, which we assume to be *semi-ample*, i.e. a tensor power $\mathscr{L}^{\otimes d}$ is *basepoint-free* for some $d > 0$. Let $R(\mathscr{L}) = \bigoplus_{n=0}^\infty \Gamma(X, \mathscr{L}^{\otimes n})$ denote its associated graded ring. Consider the rational map $\operatorname{Proj}_X \bigoplus_{n=0}^\infty \mathscr{L}^{\otimes n} \dashrightarrow \operatorname{Proj} R(\mathscr{L})$ as in (3.7.1) of op. cit., where $\operatorname{Proj}_X \bigoplus_{n=0}^\infty \mathscr{L}^{\otimes n}$ is naturally isomorphic to X by 3.1.7 and 3.1.8.iii of op. cit. The assumption that $\mathscr{L}$ is semi-ample guarantees that this rational map is defined everywhere, i.e. is a *morphism* of schemes, denoted by $r_{\mathscr{L}} \colon X \to \operatorname{Proj} R(\mathscr{L})$. This same assumption ensures also that $R(\mathscr{L})$ is a finitely generated algebra, allowing us to remain in the realm of schemes of finite type; see Thm. 9.14 of [Bǎd01].

Proposition 3.3.1. *Let $\mathscr{L}$ be a semi-ample invertible sheaf on X. The morphism $r_{\mathscr{L}}$ is proper and is in Stein factorisation.*

Proof. Properness follows since X is proper and a homogeneous spectrum is separated. Stein factorisation is a local property on the codomain and may be verified through a simple computation on the basis of standard affine opens $D_+(s)$, for $s \in \mathrm{H}^0(X, \mathscr{L}^{\otimes d})$. $\square$

Definition 3.3.2. A *fibration* is a proper morphism $f\colon X \to Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Remark 3.3.3. Note that this definition makes sense in light of Definition 3.1.29. Also note that a fibration is by definition in Stein factorisation, so that its fibres are connected due to Zariski's main theorem; see Cor. 11.3. of [Har13].

Fibrations have the following desirable property.

Proposition 3.3.4. *Let $f\colon X \to Y$ be a fibration. Then the induced map $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is injective.*

Proof. Let $\mathscr{L}$ be an invertible sheaf on Y such that $f^*(\mathscr{L}) \cong \mathcal{O}_X$. Then

$$\mathcal{O}_Y = f_*(\mathcal{O}_X \otimes f^*(\mathscr{L})) \cong f_*(\mathcal{O}_X) \otimes \mathscr{L} = \mathscr{L}$$

by the projection formula. $\square$

From a classical perspective $r_{\mathscr{L}}$ coincides with the morphism defined by the *linear system* $|\mathscr{L}^{\otimes d}|$ where $d > 0$ is sufficiently large; see Thm. 2.1.27 of [Laz04a]. This indicates that the morphism $r_{\mathscr{L}}$ depends only on to $\mathscr{L}$ up to its tensor powers: in fact, the natural map $\operatorname{Proj} R(\mathscr{L}^{\otimes d}) \to \operatorname{Proj} R(\mathscr{L})$ is an isomorphism for $d > 0$ and

$$\begin{tikzcd}
X \arrow[r, "r_{\mathscr{L}^{\otimes d}}"] \arrow[dr, "r_{\mathscr{L}}"'] & \operatorname{Proj} R(\mathscr{L}^{\otimes d}) \arrow[d, "\cong"] \\
 & \operatorname{Proj} R(\mathscr{L})
\end{tikzcd}$$

commutes, by Prop. 3.1.8.i of [EGA II].

We show that $r_{\mathscr{L}}$ also depends only on the numerical equivalence class of $\mathscr{L}$, hence on the *ray* $\mathbb{R}_{>0} \cdot \mathscr{L}$ in $\operatorname{Num}(X)_{\mathbb{R}} = \operatorname{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We have to take care here with our assumption that $\mathscr{L}$ is semi-ample, since semi-ampleness is not preserved by numerical equivalence; see e.g. Ex. 10.3.3 of [Laz04b]. This is in contrast to the notion of ampleness, which can be verified through the numerical conditions of the Nakai–Moishezon criterion, Thm. V.1.10 of [Har13].

Lemma 3.3.5. *Suppose that $\mathscr{L}$ is semi-ample and let $C \subset X$ be an integral proper curve. Then, the following are equivalent*

(i) *The image $r_{\mathscr{L}}(C) \subset \operatorname{Proj} R(\mathscr{L})$ is a point.*

(ii) *$\mathscr{L}|_C \in \operatorname{Pic}(C)$ is torsion;*

(iii) *The intersection product $\mathscr{L} \cdot C$ equals 0;*

(iv) *For any positive integer n such that $\mathrm{H}^0(C, \mathscr{L}|_C^{\otimes n}) \neq 0$, the nth tensor power $\mathscr{L}|_C^{\otimes n}$ is trivial on C.*

Proof. Recalling that the intersection product $\mathscr{L} \cdot C$ equals $\deg(\mathscr{L}|_C)$, the implications (ii)⇒(iii)⇒(iv) are clear. For the converse (iv)⇒(ii) we note that there always exists a positive integer n satisfying the condition, because $\mathscr{L}$ is semi-ample.

It remains to show the equivalence (i)⇔(ii). Suppose first that $C \subset X$ contracts to a point, and let $x \in \operatorname{Proj} R(\mathscr{L})$ be its image. Let $n \geq 1$ be an integer and let $s \in \Gamma(X, \mathscr{L}^{\otimes n})$ be a global section which does not vanish at the generic point η of C. Then $r_{\mathscr{L}}(\eta) = x$ lies in the distinguished open subset $D_+(s)$, so s vanishes nowhere along the pre-image $r_{\mathscr{L}}^{-1}(x) = C$. We conclude that multiplication by $s|_C$ induces an isomorphism of invertible sheaves $\mathcal{O}_C \xrightarrow{\sim} \mathscr{L}_C^{\otimes n}$.

Conversely, suppose that $\mathscr{L}|_C$ is torsion. Since the morphism $r_{\mathscr{L}}$ depends on $\mathscr{L}$ only up to tensor powers, we may assume without loss of generality that $\mathscr{L}|_C \cong \mathcal{O}_C$. The image $r_{\mathscr{L}}(C)$ can be identified with the homogeneous spectrum $\operatorname{Proj} R(\mathscr{L}|_C)$. But $R(\mathscr{L}|_C) = \bigoplus_{n=0}^{\infty} \Gamma(C, \mathcal{O}_C) = k'[T]$ where $k' = \Gamma(C, \mathcal{O}_C)$ is a finite field extension of k, which gives $r_{\mathscr{L}}(C) = \operatorname{Spec} k'$. $\square$

We emphasise the equivalence (i)$\Leftrightarrow$(iii), which states that the dimension of the image $r_{\mathscr{L}}(C)$ depends only on the vanishing of the intersection product $\mathscr{L} \cdot C$. If $r_{\mathscr{L}}(C)$ is a point, we say that $r_{\mathscr{L}}$ *contracts* the curve C. The contraction behaviour of curves is therefore a numerical property.

Lemma 3.3.6. *Let X be a proper, geometrically connected scheme. For every two points there is a connected curve containing both.*

Proof. By Chow's lemma we may assume that X is projective. Without loss of generality we may assume that X is irreducible. Pick a closed embedding $X \to \mathbb{P}^N$ and a hyperplane section $H \subset \mathbb{P}^N$ passing through x and y. The intersection $X \cap H$ is connected by the lemma of Enriques–Severi–Zariski; see Cor. 7.9 of [Har13]. Proceed by induction on $\dim(X)$ until $\dim(X \cap H) = 1$. ☐

Lemma 3.3.7. *Let X, Y, Z be proper, geometrically integral schemes and let $f\colon X \to Y$ and $g\colon X \to Z$ be two fibrations. If f and g contract the same integral curves, then there exists an isomorphism $\psi\colon Y \xrightarrow{\sim} Z$ such that $g = \psi \circ f$. In other words, the morphisms f and g can be identified.*

Proof. We claim that f is constant on the fibres of g. Indeed, let G be any fibre of g, and let x, y be any two closed points of G. By connectivity of G we may apply Lemma 3.3.6 to construct a curve $C \subset G$ containing x and y. Since $g(C)$ is a point, the curve C is also contracted by f, hence $f(x) = f(y)$. By symmetry of the hypotheses we also conclude that g is constant on the fibres of f.

By Lemma 3.3.6, the main hypothesis implies that f is constant on the fibres of g and that similarly g is constant on the fibres of f. The condition that f is a fibration implies by Lem. 8.11.1 of [EGA II] that the map

$$\operatorname{Hom}(Y, Z) \longrightarrow \operatorname{Hom}(X, Z); \quad \phi \longmapsto \phi \circ f$$

is a bijection onto the collection of functions which are constant on the fibres of f. By surjectivity, there is a morphism $\widetilde{g}\colon Y \to Z$ such that $\widetilde{g} \circ f = g$. Due to the symmetry among f and g, we find in a similar way that there is a morphism $\widetilde{f}\colon Z \to Y$ such that $\widetilde{f} \circ g = f$. But then $\widetilde{f} \circ \widetilde{g} \circ f = f$, and we conclude that $\widetilde{f} \circ \widetilde{g} = \mathrm{id}_Y$ since fibrations are epimorphisms in the category of schemes. The equality $\widetilde{g} \circ \widetilde{f} = \mathrm{id}_Z$ follows in a similar way. ☐

Combining Lemmata 3.3.5 and 3.3.7 grants the following result.

Theorem 3.3.8. *Let $\mathscr{L}$ and $\mathscr{L}'$ be numerically equivalent semi-ample invertible sheaves. Then there is an isomorphism $\psi\colon \operatorname{Proj} R(\mathscr{L}) \to \operatorname{Proj} R(\mathscr{L}')$ such that $r_{\mathscr{L}'} = \psi \circ r_{\mathscr{L}}$.*

In summary, the fibration $r_{\mathscr{L}}$ is determined by the *ray* $\mathbb{R}_{>0} \cdot \mathscr{L}$ in the numerical group $\operatorname{Num}(X)_{\mathbb{R}} = \operatorname{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, by choosing a semi-ample representative from the numerical equivalence class.

For the remainder of this section, let X be a bielliptic surface. Recall that in this case the numerical group $\operatorname{Num}(X)$ is a free abelian group of rank 2. We start by constructing a suitable $\mathbb{Q}$-basis of $\operatorname{Num}(X)_{\mathbb{Q}} = \operatorname{Num}(X) \otimes \mathbb{Q}$.

Lemma 3.3.9. *Let X be a bielliptic surface. Let $f\colon X \to P$ be the Albanese fibration and let F be a closed fibre of f. There is a divisor G on X such that $G^2 = 0$ and $F \cdot G > 0$.*

Proof. Let $\mathscr{L} = \mathcal{O}(H)$ be an ample invertible sheaf on X and define $G = 2(F \cdot H) \cdot H - H^2 \cdot F$. An easy computation shows that $G^2 = 0$ and that $G \cdot F = 2(F \cdot H)^2 > 0$. ☐

It is non-trivial to show that $\mathcal{O}(G)$ is numerically equivalent to a semi-ample invertible sheaf, so we briefly touch upon the relevant literature if k is algebraically closed. It is shown in the proof of Thm. 3 of [BM77] that G is numerically equivalent to an effective divisor, which we regard as curve $C \subset X$. From the results of [Mum69], Steps (II) and (III) it follows that C is semi-ample and furthermore that $\mathcal{O}(C)$ induces an elliptic fibration $g = r_{\mathcal{O}(C)}\colon X \to \mathbb{P}^1$; cf. Thm. 7.11 of [Băd01]. Lemma 3.3.5 implies that G is a fibre for g and is hence basepoint-free, so also semi-ample.

We conclude that G is also semi-ample over an arbitrary ground field. Lemmata 3.3.5 and 3.3.7 ensure that the numerical equivalence between G and C does not create further technical issues, so that G induces an elliptic fibration.

Theorem 3.3.10. *Let X be a bielliptic surface. There is an elliptic fibration $g\colon X \to B$, where B is a Brauer–Severi curve, whose closed fibres are numerically equivalent to multiples of G. It is unique in the sense of Lemma 3.3.7.*

Remark 3.3.11. The existence of g was also shown in a number theoretic setting in Prop. 5.6 of [CV18], where the authors construct (a multiple of) G by Galois-descent.

Remark 3.3.12. Recall that the fibres F and G provide a $\mathbb{Q}$-basis for $\operatorname{Num}(X)$, independently of the choice of ground field. In other words, the Galois action on $\operatorname{Num}(X^{\mathrm{alg}}) \cong \mathbb{Z}^2$ is trivial. In the language of [Sch23b] this means that the *numerical sheaf* $\operatorname{Num}_{X/k} = \operatorname{Pic}_{X/k} / \operatorname{Pic}^{\tau}_{X/k}$ is constant. Therefore we may also apply Thm. 1.3 of op. cit. to descend the elliptic fibration g from an algebraically closed ground field to an arbitrary ground field. The proof of lod. cit. essentially combines the Galois descent argument of [CV18] with a treatment of purely inseparable field extensions.

Notation 3.3.13. The fibration $g\colon X \to B$ is called the *other fibration* or the *second fibration.*

We emphasise that this fibration is always an *elliptic* fibration, i.e. that g is never quasi-elliptic. This is in clear contrast to the Albanese fibration, which can be quasi-elliptic in small characteristics. A further contrast is that g has degenerate fibres, as can be seen most easily from the Bagnera–de Franchis classification of Section 3.4. There is a notable restriction on the non-smooth fibres: the low Picard number $\rho = 2$ imposes that the fibres of g are irreducible, since any further irreducible components contribute to the rank of the numerical group. An explicit description of the degenerate fibres of the complex numbers is given in [Ser90], where they are used to construct elements generating the torsion subgroup of the Néron–Severi group $\operatorname{NS}(X)$.

Remark 3.3.14. In the small characteristics 2 and 3 the elliptic fibration may have *wildly ramified fibres.* In view of the canonical bundle formula of [BM77], the existence of wild fibres is related to the 'unexpected triviality of ω_X' discussed in Remark 3.5.4 below. A complete description of the wild fibres for bielliptic surfaces with smooth Albanese are listed on p. 38 of op. cit. There seems to be no explicit description for quasi-bielliptic surfaces, although Thm. 8.16.iv of [Zim19] limits the possibilities.

To see that B is a Brauer–Severi curve, we may base-change to an algebraic closure and argue that $B \otimes k^{\mathrm{alg}} \cong \mathbb{P}^1_{k^{\mathrm{alg}}}$, as follows from the results of [Mum69; BM77]. Alternatively, we can also argue directly.

Proposition 3.3.15. *The curve B has genus zero.*

Proof. By functoriality of the Albanese, there is an induced morphism $P \to \operatorname{Alb}(B)$, which sits in the following commutative square.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & P \arrow[d, "\operatorname{Alb}(g)"] \\ B \arrow[r] & \operatorname{Alb}(B) \end{tikzcd}$$

Since the composition $X \to \operatorname{Alb}(B)$ factors both over f and g, it must contract both of their fibres and hence be constant. This is only possible if $\operatorname{Alb}(B) = 0$. Then apply the dimension formula Proposition 3.2.9, combined with the fact that $b_1(B)/2 = h^1(\mathcal{O}_B)$ because the Picard scheme of a smooth curves is smooth. $\square$

The terminology of Notation 3.3.13 presupposes that there are only two fibrations on a bielliptic surface. This is indeed the case.

Proposition 3.3.16. *Let X be a bielliptic surface and let C be a curve. Let $h\colon X \to C$ be a morphism such that $h_*\mathcal{O}_X = \mathcal{O}_C$. Then there exists an isomorphism $\psi\colon C \to P$ or an isomorphism $\psi\colon C \to B$ such that $f = \psi \circ h$ or $g = \psi \circ h$ respectively. In other words: f and g are the only fibrations of X to a curve.*

Proof. The subset of $\operatorname{Num}(X) \otimes \mathbb{Q}$ of divisors satisfying $D^2 = 0$ consists of the two lines through F and G. If H is a closed fibre of h, then $H^2 = 0$ and therefore H is linearly equivalent to a multiple of F or G. Since the morphism defined by the linear system of a divisor only depends on the ray it generates in $\operatorname{Num}(X)$, the result follows. $\square$

Remark 3.3.17. We can see the above quite visually in the context of Mori theory: the fibres F and G span the *extremal rays* of the pseudo-effective cone $\overline{\operatorname{NE}}(X)$. In two dimensions, it is clear that a closed cone has at most two extremal rays.

Remark 3.3.18. We have treated the existence of the two genus-one fibrations f and g in the setting of a ground field k. In fact, much of the above holds over a Noetherian base scheme S. Let $\mathscr{X}$ be a family of bielliptic surfaces over S and suppose that 6 is invertible on S. Then the Albanese fibration exists by Thm. 10.2 of [LS23], considering that $\mathrm{Pic}^0_{\mathscr{X}/S} \subset \mathrm{Pic}^\tau_{\mathscr{X}/S}$ is a maximal abelian subvariety due to the absence of critical behaviour; cf. Ch. 2, Prop. 1.17 of [Boa21]. The existence of the other fibration is more subtle: as the proof of Lemma 3.3.9 indicates, one needs to assume that $\mathscr{X}$ admits an S-ample invertible sheaf, see Lem. 1.6 of [Sei87].

Note that the fibres F and G are linearly independent in $\mathrm{Num}(X)_\mathbb{Q}$, which is clear from the intersection pairing $F^2 = G^2 = 0$ while $F \cdot G > 0$. The Gram matrix is proportional to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; in fact, it is shown in Prop. 3.1 of [Boa21] that the lattice $\mathrm{Num}(X)$ is even and unimodular, hence isomorphic to the hyperbolic plane. It thus follows that the intersection number $F \cdot G$ measures how far F and G are removed from spanning $\mathrm{Num}(X)$. To remove the dependence on the choice of F and G within the ray, we normalise accordingly.

Notation 3.3.19. Let X be a bielliptic surface, with two fibrations $f \colon X \to P$ and $g \colon X \to B$. Let $x \in P$ and $y \in B$ be closed points, and let $F = f^{-1}(x)$ and $G = g^{-1}(y)$ be the corresponding fibres. We define the *intersection invariant* γ as

$$\gamma = \frac{F \cdot G}{[\kappa(x) : k] \cdot [\kappa(y) : k]}.$$

Remark 3.3.20. Note that the value of γ is independent of the choice of points x and y and is invariant under base-change.

The intersection invariant γ is a fundamental discrete invariant of bielliptic surfaces, for which we provide a suitable meaningful interpretation in the upcoming section. Consequently we see that $\gamma > 1$, so that the fibres F and G turn out to never form a $\mathbb{Z}$-basis for the numerical group $\mathrm{Num}(X)$. This can be partly explained by the multiple fibres of g. A $\mathbb{Z}$-basis for $\mathrm{Num}(X)$ in terms of F and G is given in Table 2 of [Ser90].

3.4 The Bagnera–de Franchis classification

The classification of *complex* bielliptic surfaces is a particularly classical topic, whose study was initiated in the early 20th century in the larger framework of the Italian school of algebraic geometry. The main object of this school was to classify complex algebraic surfaces, generalising the classification of curves by genus. Although the tetrachotomy of surfaces of Kodaira dimension 0 had not yet been discovered, bielliptic surfaces formed a notable subclass of surfaces with geometric genus $p_g = 0$ and irregularity $q = 1$. Valuing the importance of these surfaces, the Paris Academy of Science organised a competition for the reputable Bordin prize in 1907, with the goal of classifying so-called *hyperelliptic surfaces*, which at the time were defined to be quotients of abelian surfaces.

The first 'team' to participate in this competition consisted of the famous geometers Federigo Enriques and Francesco Severi. In their 1907 manuscript [ES07], published in final form in 1909 as [ES09], they treated the special case where the abelian surface is the Jacobian of a smooth genus-two curve. Their partial classification notably excludes the surfaces which we nowadays call *bielliptic*, which instead arise as a quotient of two elliptic curves by a finite group. Enriques and Severi addressed the general version in 1908 in their follow-up article [ES08]; however, by this time, the 'other team' in the competition had already independently listed the full classification.

The competing team consisted of the Italian geometers Giuseppe Bagnera and Michele de Franchis. They determined the full classification of complex bielliptic surfaces, called the *Bagnera–de Franchis classification* in their honour, in terms of the possible groups acting on a product of two complex elliptic curves. They announced their results in [BF07] in 1907 only after the deadline for the Bordin prize had passed; the full manuscript was published in [BF08], with the list of bielliptic surfaces in §6. Part of the delay was caused by a technical but superfluous assumption, which de Franchis only eliminated decades later in 1936 in the short note [Fra36a] followed by the article [Fra36b] (both as cited in [Cil98; Cat03]).

The Bordin prize of 1907 was hence awarded to Enriques and Severi for their partial results. De Franchis states in private correspondence with Guccia, the director of the at the time prestigious *Circolo Matematico di Palermo* (Mathematical Circle of Palermo), that he had pointed out several flaws in a draft that Enriques and Severi had submitted to the Paris Academy of Science. There was however enough time to resubmit a version in which a number (but not all) of the suggestions of de Franchis were adopted. According to de Franchis, the members of the Academy were unable to detect the remaining mistakes. In the end, the Paris Academy of Science awarded the Bordin prize in 1909 *on the same subject*, this time to Bagnera and de Franchis, for the considerable contributions made in e.g. [BF10].

The above history is part of the excellent historic article [Cil98], but see also [Cat03, p. 30], [OR]. As a final introductory remark, we note that the classification of bielliptic surfaces to algebraically closed fields of arbitrary characteristic is due to Bombieri and Mumford [BM77; BM76], as part of a series of articles extending the entire Enriques classification of surfaces to positive characteristic; see also the first part [Mum69].

In this section k denotes an *algebraically closed* ground field of arbitrary characteristic. We explore how this theory generalises to arbitrary ground fields in Chapter 4. First, we state the classification over algebraically closed fields in its modern form, subdividing bielliptic surfaces into seven different *types*. We also propose terminology to control the critical behaviour that occurs in small characteristics $p = 2$ and $p = 3$, mirroring the trichotomy of Enriques surfaces in characteristic 2 that was introduced in [BM76].

Let us start with a coarse version of the structure theorem for bielliptic surfaces, whose proof for bielliptic surfaces can be found on pp. 33–35 of [BM77], though the proof generalises verbatim to quasi-bielliptic surfaces; see Thm. 4 in loc. cit. and Thm. 1 of [BM76]. Our proposed notation is unfortunate due to the cumbersome tildes, but is necessary for later consistency with Section 3.5, where C and D should instead denote the Stein factors of the canonical cover.

Theorem 3.4.1 (Structure theorem for bielliptic surfaces)**.** *A bielliptic surface X over an algebraically closed field is isomorphic to a quotient $X \cong (\widetilde{C} \times \widetilde{D})/G$, where*

- *$\widetilde{C}$ is an elliptic curve;*
- *$\widetilde{D}$ is a smooth genus-one curve, or the rational cuspidal curve;*
- *G is a finite subgroup scheme of $\widetilde{C}$, acting by translation;*
- *G acts faithfully on $\widetilde{D}$;*
- *the Albanese fibration $f\colon X \to P$ equals the projection $(\widetilde{C} \times \widetilde{D})/G \to \widetilde{C}/G$;*
- *the second fibration $g\colon X \to \mathbb{P}^1$ equals the projection $(\widetilde{C} \times \widetilde{D})/G \to \widetilde{D}/G$.*

The curve $\widetilde{D}$ is the rational cuspidal curve if and only if X is a quasi-bielliptic surface.

The action of G on the product is free despite the fixed points of the G-action on $\widetilde{D}$, because the action on $\widetilde{C}$ is free. Therefore the quotient exists as a scheme and the quotient map $\widetilde{C} \times \widetilde{D} \to X$ is naturally a G-torsor. The proof in op. cit. shows that this description of X as a quotient only depends on the choice of (reduced) closed fibres of the two fibrations f and g. We propose the following terminology for the quotient map $\widetilde{C} \times \widetilde{D} \to X$.

Definition 3.4.2. Let X be a bielliptic surface. A G-torsor $\widetilde{C} \times \widetilde{D} \to X$ is called the *Bagnera–de Franchis cover* (or: *BdF-cover*) if it satisfies the conditions listed in Theorem 3.4.1.

Remark 3.4.3. We may abuse terminology and refer to the surface $\widetilde{C} \times \widetilde{D}$ as the Bagnera–de Franchis cover.

The natural follow-up question for refining the classification of bielliptic surfaces is to classify the possible subgroup schemes G of an elliptic curve acting on a genus-one curve, such that the conditions of Theorem 3.4.1 are satisfied. Bagnera and de Franchis showed that, over the field of complex numbers, the only possible group schemes are the constant group schemes $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ subject to the conditions that n is a non-trivial divisor of 12, $d \mid n$, and $nd \leq 9$. This amounts to seven isomorphism classes, each referred to by a different *type*, which is either (a1), (a2), (b1), (b2), (c1), (c2), or (d). We consider *type (a)* to indicate the union of types (a1) and (a2), and similarly for types (b) and (c).

Type	(a1)	(a2)	(b1)	(b2)	(c1)	(c2)	(d)
G	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$

Table 3.2: The groups occurring the Bagnera–de Franchis classification over $\mathbb{C}$

Remarkably, this list remains the same over algebraically closed fields of characteristics not equal to 2 or 3, while in characteristics 2 and 3 only small adjustments are needed to deal with bielliptic surfaces with smooth Albanese. A more elaborate treatment is necessary in order to include the quasi-bielliptic surfaces as well. First we treat the case where $\widetilde{D}$ is smooth, following pp. 35–37 of [BM77].

Theorem 3.4.4 (Bagnera–de Franchis classification; smooth Albanese)**.** *Let $G \subset \widetilde{C}$ be a finite subgroup scheme that acts on a* smooth *genus-one curve $\widetilde{D}$, as in Theorem 3.4.1. With a suitable choice of rational point on $\widetilde{D}$ giving it the structure of an elliptic curve $\widetilde{J}$, we have either*

- (a1) *$G \cong \mathbb{Z}/2\mathbb{Z}$ with action $x \mapsto -x$;*
- (a2) *$G \cong \mathbb{Z}/2\mathbb{Z} \times \mu_2$ where $\mathbb{Z}/2\mathbb{Z}$ acts by $x \mapsto -x$ and where μ_2 acts by translation;*
- (b1) *$G \cong \mathbb{Z}/3\mathbb{Z}$ with action $x \mapsto \omega x$, where $\omega\colon \widetilde{J} \to \widetilde{J}$ is an elliptic curve automorphism of order 3;*
- (b2) *$G \cong (\mathbb{Z}/3\mathbb{Z})^2$ (with $p \neq 3$) with action $x \mapsto \omega x$ and $x \mapsto x + a$ for some $a \in \widetilde{J}[3]$ satisfying $\omega a = a$;*
- (c1) *$G \cong \mathbb{Z}/4\mathbb{Z}$ with action $x \mapsto ix$, where $i\colon \widetilde{J} \to \widetilde{J}$ is an elliptic curve automorphism of order 4;*
- (c2) *$G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (with $p \neq 2$) with action $x \mapsto ix$ and $x \mapsto x + a$ for some $a \in \widetilde{J}[2]$ with $ia = a$;*
- (d) *$G \cong \mathbb{Z}/6\mathbb{Z}$ with action $x \mapsto -\omega x$, where ω is as in case (b1).*

Proof. The isomorphism group scheme of $\widetilde{D}$ decomposes as a semidirect product

$$\mathrm{Aut}_{\widetilde{D}/k} = \widetilde{J} \rtimes \mathrm{Aut}_{\widetilde{J}/k}, \tag{3.4.1}$$

where the latter factor parametrises the automorphisms of the elliptic curve, and is isomorphic to either $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, or $Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$, where Q_8 denotes the quaternion group of order 8. The latter two cases occur only in characteristics 3 and 2, respectively.

Denote the action of G by $\phi\colon G \to \mathrm{Aut}_{\widetilde{D}/k}$ and consider the projection $\pi\colon \mathrm{Aut}_{\widetilde{D}/k} \to \mathrm{Aut}_{\widetilde{J}/k}$. Since G is abelian, the image of $\pi \circ \phi$ is contained within an abelian subgroup of $\mathrm{Aut}_{\widetilde{J}/k}$, which is a cyclic group, even in the last two noncommutative cases. Furthermore $\mathrm{Im}\,\pi \circ \phi$ cannot be trivial, since then the image of ϕ would be contained entirely within the first factor $\widetilde{J}$, and the quotient $\widetilde{D}/G$ would have genus one.

Choose an element $g \in G$ such that $\pi(\phi(g))$ generates its image. The automorphism $\phi(g)$ has a fixed point since $\phi(g) \notin \widetilde{J}$. We replace the distinguished point of $\widetilde{J}$ by this fixed point, so that $\mathrm{Im}\,\phi$ is itself a direct product $G \cong \mathrm{Im}\,\phi = H \times N$, where $H \subset G$ is the maximal subgroup scheme that acts on $\widetilde{D}$ by translations and where $N \subset \mathrm{Aut}_{\widetilde{J}/k}$ is cyclic of order n. The possible values of n are $2, 3, 4$ or 6.

The subscheme $H \subset \widetilde{D}$ is invariant under the action of N by commutativity of G. The set of fixed points of $\phi(g) \in N$ is well-known for each possible value of n, and can be deduced from the explicit formulas of automorphisms of elliptic curves of e.g. App. A of [Sil09]. This leads to a case distinction.

(a) Suppose $n = 2$. The automorphism $\phi(g)$ is the sign involution on $\widetilde{J}$. Therefore $H \subset \widetilde{J}[2]$. Since $G \cong H \times \mathbb{Z}/2\mathbb{Z} \subset \widetilde{J}$ is also two-torsion, we see that either $H = 0$ or $H = \mu_2$, corresponding to cases (a1) or (a2), respectively.

(b) Suppose $n = 3$. The existence of an elliptic curve automorphism ω of order 3 implies that the j-invariant of $\widetilde{J}$ equals 0. If $p \neq 3$ then the fixed locus of ω is a subgroup scheme isomorphic to $\mathbb{Z}/3\mathbb{Z}$, so that either $H = 0$ or $H = \mathbb{Z}/3\mathbb{Z}$, leading to cases (b1) and (b2) respectively. If $p = 3$ then the fixed locus of ω is the subgroup scheme $\alpha_3 \subset \widetilde{J}$, so either $H = 0$ or $H = \alpha_3$. The latter case is not possible, since $G \cong \alpha_3 \times \mathbb{Z}/3\mathbb{Z}$ is not a subgroup scheme of an elliptic curve.

(c) Suppose $n = 2$. Similarly, the existence of an elliptic curve automorphism i of order 4 implies that the j-invariant of $\widetilde{J}$ is 1728. The fixed locus for i is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $p \neq 2$, and α_2 if $p = 2$. As before, the case of $H = \alpha_2$ does not occur, so $H = 0$, or $H = \mathbb{Z}/2\mathbb{Z}$ if $p \neq 2$, corresponding to cases (c1) and (c2) respectively.

(d) Suppose $n = 6$, so $\widetilde{J}$ is the elliptic curve with j-invariant 0 and automorphism ω of order 3. The automorphism $\phi(g)$ is given by $-\omega$, which has order 6. Its fixed locus consists of only the identity element of $\widetilde{J}$, so that $H = 0$ is the only possibility. □

Remark 3.4.5. The existence of an elliptic curve automorphism of order 3 or 4 puts a restriction on the j-invariant of $\widetilde{D}$, namely $j(\widetilde{D}) = 0$ and $j(\widetilde{D}) = 1728$, respectively.

Remark 3.4.6. Different proposals for the labels of the type can be found in the literature: for example, some sources use (1), (2), ..., (7) for (a1), (a2), ..., (d), respectively. Our labelling convention is more structural, since a bielliptic surface of type (a1) shares similarity with one of type (a2). It traces back to Bombieri and Mumford in [BM77], although we have made a slight adjustment. Many sources, including op. cit., separate the type (a2) into two types: one for $p \neq 2$ in which case G is the constant group scheme $(\mathbb{Z}/2\mathbb{Z})^2$, and an additional case denoted (a3) in characteristic $p = 2$, where the group scheme $G = \mathbb{Z}/2\mathbb{Z} \times \mu_2$ is non-smooth. The author is of the opinion that this distinction is unnatural and unnecessary, wherefore we include type (a3) within type (a2).

This treats the classification of bielliptic surfaces with a smooth Albanese, i.e. all except for the quasi-bielliptic surfaces. The case where $\widetilde{D}$ is a rational cuspidal curve requires a separate analysis, which rests upon the description of the automorphism group scheme of the rational cuspidal curve as a semidirect product of three factors

$$\mathrm{Aut}_{\widetilde{D}/k} = \mathbb{G}_a \rtimes A \rtimes \mathbb{G}_m. \tag{3.4.2}$$

Here A is an infinitesimal group scheme that vanishes outside of characteristics 2 and 3; see Prop. 6 of [BM76] and the natural generalisation Thm. 8.1 of [HS24]. The non-vanishing of A in some sense permits the existence of quasi-bielliptic surfaces, since the quotient of $\widetilde{C} \times \widetilde{D}$ by a subgroup scheme of $\mathbb{G}_a \rtimes \mathbb{G}_m$ is non-smooth; see p. 212 of [BM76]. Determining explicitly the list of group schemes G with action on $\widetilde{D}$ is, in the words of Bombieri and Mumford, 'a tedious problem', so we state the classification without proof.

To do this in a satisfactory manner, we first introduce helpful terminology. This is necessary, since the possibilities for G at first sight seem to be quite arbitrary. The author believes that they can nevertheless been seen as *degenerations* of one of the above seven classical types. Our terminology quantifies the possible kinds of degenerations by creating a trichotomy for bielliptic surfaces in critical characteristics. This is analogous to the established trichotomy of Enriques surfaces in the critical characteristic $p = 2$. For bielliptic surfaces there are two possible critical characteristics, namely $p = 2$ and $p = 3$.

Definition 3.4.7. Let X be a bielliptic surface over a field of characteristic exponent $p \geq 1$. The characteristic is *tame* if p is coprime to the intersection invariant γ of Notation 3.3.19; otherwise the characteristic is *critical.*

We show in Proposition 3.4.17 that the intersection invariant γ equals the order of the group scheme G. This motivates our choice of terminology: if the order of G is coprime to the characteristic exponent, we should expect tame behaviour. In case the Albanese is smooth, it follows from Theorem 3.4.4 that the characteristic $p = 2$ is critical for types (a), (c) and (d), and that the characteristic $p = 3$ is critical for types (b) and (d). There are no other critical characteristics. This should make sense, since a quotient by e.g. $\mathbb{Z}/p\mathbb{Z}$ should only be critical in characteristic p.

Remark 3.4.8. We phrase the above definition in terms of the intersection invariant γ, since it depends only on intrinsic data, i.e. it does not reference the objects introduced in Theorem 3.4.1. Therefore Definition 3.4.7 remains a valid definition over an arbitrary ground field.

Definition 3.4.9 (Trichotomy of bielliptic surfaces)**.** Let X be a bielliptic surface in critical characteristic. We say that

- X is *ordinary* if the Albanese map f is smooth, i.e. if X is not quasi-bielliptic;
- X is *classical* if X is quasi-bielliptic and P is ordinary;
- X is *supersingular* if P is supersingular.

Remark 3.4.10. By the structure theorem Theorem 3.4.1, the Albanese P is isogenous to $\widetilde{C}$, so that one is ordinary if and only if the other is ordinary. If X is an ordinary bielliptic surface, then the existence of a non-trivial étale subgroup scheme N whose order is divisible by the characteristic p implies that $\widetilde{C}$ and hence P is ordinary. We conclude that the three classes in the trichotomy are mutually disjoint.

Remark 3.4.11. The first half of the Bagnera–de Franchis classification Theorem 3.4.4 also handles some bielliptic surfaces in critical characteristic, namely the ordinary bielliptic surfaces. These surfaces arise as notable *counterexamples.* For example, an ordinary bielliptic surface of type (a1) is also referred to as an *Igusa surface*, cf. [Igu55]. It is the first known example of a smooth, proper scheme whose Picard scheme is non-smooth. This seemingly pathological behaviour is quite natural in the context of the other bielliptic surfaces; see Table 3.6 below.

At first sight it may seem to be more natural to swap the definitions of 'ordinary' and 'classical' bielliptic surfaces, but we reassure the reader that the above definition is correct in analogy with the trichotomy of Enriques surfaces in characteristic 2. Recall from Table 3.1 and Proposition 3.1.27 that Enriques surfaces exhibit similar critical behaviour if $p = 2$. It is well-established to separate them into three different classes, also called *ordinary*, *classical* and *supersingular.*

The trichotomy of bielliptic surfaces shares a lot of similarities to that of Enriques surfaces, except for notably the definition. For an Enriques surfaces Y, the trichotomy is defined by investigating the possible isomorphism classes for the group scheme $\mathrm{Pic}^{\tau}_{Y/k}$, which has order 2 by Thm. 1.2.1 of [CD89]. Then

$$\mathrm{Pic}^{\tau}_{Y/k} \cong \begin{cases} \mu_2 & \text{if } Y \text{ is ordinary;} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } Y \text{ is classical;} \\ \alpha_2 & \text{if } Y \text{ is supersingular.} \end{cases}$$

In case of bielliptic surfaces, the group scheme $\mathrm{Pic}^{\tau}_{X/k}$ contains an elliptic curve $\mathrm{Pic}^{\alpha}_{X/k}$, so this classification breaks down. That is the main reason that Definition 3.4.9 is much more involved compared to Enriques surfaces. However, there are still similarities to the Cartier dual $N^{\vee}$ of the group scheme N defined below; see Table 3.8.

Despite the name, classical Enriques surfaces are not necessarily better behaved than the other classes; in fact, in many cases the ordinary Enriques surfaces behave most like their counterparts in characteristic $p \neq 2$. It much depend on the properties of interest. In a similar way for bielliptic surfaces, some properties are well-behaved for ordinary surfaces, wheras others are better behaved for classical surfaces; see also Remark 3.4.22 below. In general, it seems like the class of supersingular surfaces satisfy most critical behaviour.

We are now in a position to list systematically the possible group schemes G and group scheme actions on the rational cuspidal curve $\widetilde{D}$ satisfying the properties of Theorem 3.4.1, as given on p. 214 of [BM76]. The detailed description of the action is not important for the moment, but we list it in full detail for later use. We simultaneously propose a new definition of the type of a quasi-bielliptic surface through our indexing, which naturally extends the types as given in Theorem 3.4.4.

Theorem 3.4.12 (Bagnera–de Franchis classification; quasi-bielliptic surfaces)**.** *Let $G \subset \widetilde{C}$ be a subgroup scheme, that acts on the* rational cuspidal *curve $\widetilde{D} = \operatorname{Spec} k[t] \cup \operatorname{Spec} k[t^{-2}, t^{-3}]$, as in Theorem 3.4.1. If X is classical, we have either*

(a1) $p = 2$ and $G \cong \mu_2$ acting by $t \mapsto at + \lambda(a+1)t^2 + (a+1)t^4$ where $a^2 = 1$, for some $\lambda \in k$;

(a2) $p = 2$ and $G \cong \mu_2 \times \mathbb{Z}/2\mathbb{Z}$ acting as in (a1), plus $t \mapsto t + \xi$, where ξ satisfies $\xi^4 + \lambda\xi^2 + \xi = 0$;

(b1) $p = 3$ and $G \cong \mu_3$ acting by $t \mapsto at + (1-a)t^3$ where $a^3 = 1$;

(b2) $p = 3$ and $G \cong \mu_3 \times \mathbb{Z}/3\mathbb{Z}$ acting as in (b1), plus $t \mapsto t + 1$;

(c1) $p = 2$ and $G \cong \mu_4$ acting by $t \mapsto at + (a + a^2)t^2 + (1 + a^2)t^4$ where $a^4 = 1$;

(c2) $p = 2$ and $G \cong \mu_4 \times \mathbb{Z}/2\mathbb{Z}$ acting as in (c1), plus $t \mapsto t + 1$;

(d) $p = 2$ and $G \cong \mu_6 = \mu_2 \times \mu_3$, where μ_2 acts as in (a1) with $\lambda = 0$, plus $t \mapsto \zeta t$ where $\zeta^3 = 1$;

(d) $p = 3$ and $G \cong \mu_6 = \mu_3 \times \mu_2$, where μ_3 acts as in (b1), plus $t \mapsto -t$.

If X is supersingular, we have either

(a1) $p = 2$ and $G \cong \alpha_2$ acting by $t \mapsto t + \lambda a t^2 + a t^4$ where $a^2 = 0$, with $\lambda = 0$ or 1;

(a2) $p = 2$ *and* $G \cong M_2$*, which is the two-torsion subgroup scheme of a supersingular elliptic curve, acting by* $t \mapsto t + a + \lambda a^2t^2 + a^2t^4$ *where* $a^4 = 0$*;*

(b1) $p = 3$ *and* $G \cong \alpha_3$ *acting by* $t \mapsto t + at^3$ *where* $a^3 = 0$*;*

(d) $p = 2$ *and* $G \cong \alpha_2 \times \mu_3$*, where* α_2 *acts as in (a1), plus* $t \mapsto \zeta t$ *for* $\zeta^3 = 1$*;*

(d) $p = 3$ *and* $G \cong \alpha_3 \times \mu_2$*, where* α_3 *acts as in (b1), plus* $t \mapsto -t$*.*

Aside from notational differences, our list still differs slightly from the one given by Bombieri and Mumford in the case of quasi-bielliptic surfaces. We explain the differences in the following remarks.

Remark 3.4.13. As was mentioned in p. 489 of [Lan79] and Rk. 5.12, p. 15 of [Mar22], the group scheme of order 9 of maps $t \mapsto t + a + a^3t^3$ (with $a^9 = 0$) mentioned by Bombieri and Mumford does not occur, as it is not isomorphic to a subgroup scheme of any elliptic curve. Indeed, from Dieudonné theory it follows that it is a twisted form of α_9, which is not a subgroup scheme of an elliptic curve, since its Cartier dual has embedding dimension 2; see (15.5) of [Oor66]. In our notation, this causes the non-existence of supersingular quasi-bielliptic surfaces of type (b2). Note that supersingular quasi-bielliptic surfaces of type (a2) do exist, where the group G is M_2, the two-torsion subgroup of a supersingular elliptic curve.

Remark 3.4.14. As explained in Rk. 5.13, p. 17 of [Mar22] we may take $\lambda = 0$ or 1 without loss of generality, since all actions with nonzero λ are conjugate.

In summary, Theorems 3.4.4 and 3.4.12 classify the possible group schemes and group scheme actions that can occur in the structure theorem of bielliptic surfaces Theorem 3.4.1. They are jointly referred to as the *Bagnera–de Franchis classification.* For clarity we tabulate the possible isomorphism classes of G according to type; see Table 3.3. The group schemes arising in the last three columns indicate occurrences in critical characteristic, as listed in the third column.

Type	Tame Char.	Crit. Char$^{\text{s}}$	Ordinary Bielliptic	Classical Q.-Biell.	Supersing. Q.-Biell.
(a1)	μ_2	2	$\mathbb{Z}/2\mathbb{Z}$	μ_2	α_2
(a2)	$\mu_2 \times \mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z} \times \mu_2$	$\mu_2 \times \mathbb{Z}/2\mathbb{Z}$	M_2
(b1)	μ_3	3	$\mathbb{Z}/3\mathbb{Z}$	μ_3	α_3
(b2)	$\mu_3 \times \mathbb{Z}/3\mathbb{Z}$	3	$\nexists$	$\mu_3 \times \mathbb{Z}/3\mathbb{Z}$	$\nexists$
(c1)	μ_4	2	$\mathbb{Z}/4\mathbb{Z}$	μ_4	$\nexists$
(c2)	$\mu_4 \times \mathbb{Z}/2\mathbb{Z}$	2	$\nexists$	$\mu_4 \times \mathbb{Z}/2\mathbb{Z}$	$\nexists$
(d)	μ_6	$2, 3$	$\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \times \mu_{6/p}$	μ_6	$\alpha_p \times \mu_{6/p}$

Table 3.3: The group scheme G in all cases of the Bagnera–de Franchis classification

Remark 3.4.15. Only after applying the isomorphism $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ in tame characteristic does Table 3.2 agree with the 'tame characteristic' column of Table 3.3. The use of multiplicative group schemes is preferred: first of all, because it turns out to be more accurate in the context of non-algebraically closed fields, c.f. Table 3.8 and Proposition 4.1.24; another reason comes from Remark 3.4.20.

Remark 3.4.16. Note that not all entries in the table are filled: there is no ordinary bielliptic surface of type (b2) or (c2), and there is no supersingular quasi-bielliptic surface of type (b2), (c1) or (c2). The author is not aware of conceptual justification of this fact; cf. the concluding remarks in the introduction of [Lan79].

There are a number of reasons why our proposed extension of the definition of 'type' for quasi-bielliptic surfaces is the natural one. First of all, note that the order of the group scheme G is constant among bielliptic surfaces of the same type. For example for type (a1), the group schemes $\mathbb{Z}/2\mathbb{Z}$, μ_2 and α_2 all have order 2. Looking row by row, the group schemes occurring for quasi-bielliptic surfaces indeed seem to be infinitesimal degenerations of the groups occurring over $\mathbb{C}$ as listed in Table 3.2: a copy of $\mathbb{Z}/p\mathbb{Z}$ may be replaced by the multiplicative group scheme μ_p in the classical case, or the unipotent group scheme α_p in the supersingular case.

This is analogous to the trichotomy of Enriques surfaces in characteristic 2. Enriques surfaces can similarly be described naturally as a quotient by a group scheme G of order 2. The isomorphism class of G distinguishes the three classes of Enriques surfaces in characteristic 2: ordinary entails $G = \mathbb{Z}/2\mathbb{Z}$, classical entails $G = \mu_2$, and supersingular entails $G = \alpha_2$.

A consequence of this explicit description of bielliptic surfaces is that we can compute a number of numerical invariants explicitly in terms of the group scheme G and the action. This adheres to the philosophy that a bielliptic surface is studied best through its covers, as discussed in Chapter 4. We first compute the invariant γ of Notation 3.3.19. Recall that the *order* of G is defined as $|G| = h^0(\mathcal{O}_G)$.

Proposition 3.4.17. *The intersection invariant γ equals the order $|G|$.*

Proof. Without loss of generality suppose that k is algebraically closed. Consider a closed fibre F of f and a closed fibre G of g, as elements of the numerical group $\mathrm{Num}(X)$. Consider the product map $f \times g\colon X \to P \times \mathbb{P}^1$. Then the fibres F and G are the pullbacks of $\mathrm{Spec}(k) \times \mathbb{P}^1$ and $P \times \mathrm{Spec}(k)$ respectively, so from the naturality of the intersection form it follows that

$$F \cdot G = \deg(f \times g)\big((\mathrm{Spec}(k) \times \mathbb{P}^1) \cdot (P \times \mathrm{Spec}(k)\big) = \deg(f \times g),$$

since the intersection on $P \times \mathbb{P}^1$ has multiplicity 1. The map $\widetilde{C} \times \widetilde{D} \to X$ has degree $|G|$ and the composition $\widetilde{C} \times \widetilde{D} \to X \to P \times \mathbb{P}^1$ has degree $|G|^2$, being quotients by the free action of G and G^2, respectively. This yields $\gamma = |G|$. ☐

This explains a posteriori our definition of tame and critical characteristic: if the order of the group scheme is coprime with the characteristic exponential, we can expect tame behaviour. As a corollary, we see from Table 3.3 that the only possible tame characteristics are 2 and 3.

Type	(a1)	(a2)	(b1)	(b2)	(c1)	(c2)	(d)
γ	2	4	3	9	4	8	6

Table 3.4: The intersection invariant γ by type

Recall from Theorem 3.1.21 that the canonical sheaf ω_X has finite order in the Picard group $\mathrm{Pic}(X)$. Its order is an important invariant intrinsic to the bielliptic surface, so let us fix notation for it.

Notation 3.4.18. The order of ω_X in the Picard group is denoted m.

Proposition 3.4.19. *The order m is tabulated according to the type of X in Table 3.5.*

Proof. The integer m is minimal such that G acts trivially on $\omega_{\widetilde{C}}^{\otimes m} \otimes \omega_{\widetilde{D}}^{\otimes m}$. Since G acts by translations on $\widetilde{C}$, we only consider the action of G on $\omega_{\widetilde{D}}^{\otimes m}$. This is a case-by-case computation using the actions listed in Theorems 3.4.4 and 3.4.12; cf. [BM77, p. 37], [BM76, p. 214]. Note however that the only group schemes that act non-trivially on $\omega_{\widetilde{D}}$ are the copies of μ_m that do not act by translations. ☐

Type	Tame Char.	Ordinary Biell.	Classical Q.-Biell.	Supersingular Q.-Biell.
(a)	2	1	2	1
(b)	3	1	3	1
(c)	4	1	4	$\nexists$
(d)	6	$6/p$	6	$6/p$

Table 3.5: The order m of the canonical sheaf ω_X in the Picard group of X, according to type

Remark 3.4.20. Comparing Tables 3.3 and 3.5 we note that m is the order of the maximal multiplicative subgroup scheme of G which does not act by translation. An explanation for this phenomenon is provided in Remark 3.5.4.

Recall from Remark 3.2.19 that there are two independent kinds of critical behaviour in characteristics 2 and 3: the quasi-ellipticity of the Albanese and the non-reducedness of the Picard scheme. We are now in a position to tabulate for which types of bielliptic surface which behaviour occurs. Recall from Proposition 3.1.27 that the Picard scheme of a bielliptic surface is non-reduced if and only if $\omega_X \cong \mathcal{O}_X$, i.e. if and only if $m = 1$, and that Table 3.5 indicates exactly for which types of bielliptic surface this happens. We tabulate all four possibilities for critical behaviour in Table 3.6. Note that a form of critical behaviour always occurs in critical characteristic, except for ordinary bielliptic surfaces of type (d).

	Reduced Picard scheme	Non-reduced Picard scheme
Smooth Albanese	Tame characteristic, Type (d) ordinary bielliptic	Type (a), (b), (c) ordinary bielliptic
Quasi-bielliptic	Classical quasi-bielliptic, Type (d) supersingular bielliptic	Type (a), (b) supersingular quasi-bielliptic

Table 3.6: The critical behaviour in characteristics 2 and 3

Example 3.4.21. Non-reducedness of the Picard scheme is explained by the canonical sheaf ω_X being 'unexpectedly' trivial. Although ω_X is non-trivial for a bielliptic surface of type (d), its dualising sheaf does have an 'unexpectedly small' order $m = 6/p$ if it is ordinary or supersingular compared to tame characteristics where $m = 6$. By the Bagnera–de Franchis classification, a bielliptic surface of type (d) can be described as a further quotient of a bielliptic surface, which is of type (a1) if $p \neq 3$, of type (b1) if $p \neq 2$, and both occur if the Albanese of X is smooth. This consequently produces examples of bielliptic surfaces with smooth Picard scheme that are canonically covered by bielliptic surfaces with non-smooth Picard scheme.

Remark 3.4.22. At first sight, ordinary bielliptic surfaces seem to be the best behaved among bielliptic surfaces in critical characteristic, since they admit an étale cover by an abelian surface and due to the absence of a quasi-elliptic fibration. On the other hand, the Picard scheme of an ordinary bielliptic surface is often non-smooth due to an 'unexpected triviality' of the canonical divisor. In that regard, the classical bielliptic surfaces seem to be better behaved: Table 3.5 shows that the order of the canonical sheaf agrees with tame characteristic, so that the Picard scheme is smooth. Supersingular bielliptic surfaces display bad behaviour in both aspects.

For the remainder of this section, we study geometric properties of the Bagnera–de Franchis cover $\widetilde{C} \times \widetilde{D}$. We continue to work in the context of an algebraically closed field. The following properties are of use in Chapter 4, where we generalise the theory to arbitrary ground fields. Recall that the quotient map $\widetilde{C} \times \widetilde{D} \to X$ naturally admits the structure of a G-torsor. This has a number of direct consequences, the first one pertaining to the Albanese map. Recall that the conditions of Theorem 3.4.1 impose that the fibrations $f\colon X \to P$ and $g\colon X \to \mathbb{P}^1$ coincide with the projections $(\widetilde{C} \times \widetilde{D})/G \to \widetilde{C}/G$ and $(\widetilde{C} \times \widetilde{D})/G \to \widetilde{D}/G$, respectively. In other words, the following diagram is commutative.

$$\begin{tikzcd}
\widetilde{D} \arrow[d] & \widetilde{C} \times \widetilde{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{C} \arrow[d] \\
\mathbb{P}^1 & X \arrow[l, "g"'] \arrow[r, "f"] & P
\end{tikzcd} \tag{3.4.3}$$

Here, all vertical arrows are quotients by the action of the group scheme G. Since the action of G on $\widetilde{C}$ is free, the middle and rightmost vertical arrows are in fact G-torsors. This contrasts with the leftmost vertical arrow, which is not a G-torsor since the action on $\widetilde{D}$ is not free. As a consequence of Lemma 1.1.10, we directly obtain the following result.

Proposition 3.4.23. *The square on the right in* (3.4.3) *is Cartesian.*

This Lemma does not apply to the left square since $\widetilde{D} \to \mathbb{P}^1$ is not a G-torsor. The left square is in fact never Cartesian: otherwise all closed fibres of g would be isomorphic to copies of $\widetilde{C}$, but g has singular fibres. In contrast, if $\widetilde{D}$ is smooth then all fibres of f are smooth by Proposition 3.2.20. Despite its simplicity and apparent innocence, it turns out that the above Proposition is quite fundamental to the theory we develop in Chapter 4.

The Bagnera–de Franchis cover can detect whether the bielliptic surface X is quasi-bielliptic or not.

Proposition 3.4.24. *The following are equivalent:*

(i) *the Albanese map $f\colon X \to P$ is smooth;*

(ii) *the curve $\widetilde{D}$ is an elliptic curve;*

(iii) *the Bagnera–de Franchis cover $\widetilde{C} \times \widetilde{D}$ is an abelian surface;*

(iv) *the Bagnera–de Franchis cover* $\widetilde{C} \times \widetilde{D}$ *is smooth;*

(v) *the group scheme* G *is smooth.*

Proof. The implication (i)⇒(ii) holds because a closed fibre of f is isomorphic to a copy of $\widetilde{D}$, which one can see either from the structure theorem Theorem 3.4.1 or from Proposition 3.4.23. The implications (ii)⇒(iii)⇒(iv) are trivial. Recall that $\widetilde{C} \times \widetilde{D} \to X$ is a G-torsor, so (iv)⇒(v) holds by fppf-descent; see [Stacks, Tag 02VL]. Last of all (v)⇒(i) follows from Table 3.3. □

We observe the following easy fact for later reference.

Proposition 3.4.25. *The projections* $\widetilde{C} \times \widetilde{D} \to \widetilde{C}$ *and* $\widetilde{C} \times \widetilde{D} \to \widetilde{D}$ *are in Stein factorisation.*

3.5 The canonical cover

The Bagnera–de Franchis classification is a central tool in the analysis of bielliptic surfaces, since it allows us to regard a bielliptic surface as a quotient of a product of two genus-one curves. These curves are furthermore smooth in tame characteristics, thereby rendering bielliptic surfaces as quotients of abelian surfaces; q.v. Proposition 3.4.24. As a consequence, many questions about bielliptic surfaces in tame characteristic can be answered using the theory of abelian surfaces by passing to the Bagnera–de Franchis cover. This approach is used in some sense in virtually all articles about bielliptic surfaces; see e.g. [Tak20; Mar22; Ser90; CF03], and it appears implicitly in many more.

Aside from the Bagnera–de Franchis cover, there is a second cover that fulfils a similar job, called the *canonical cover.* In tame characteristic it is the *minimal covering* of a bielliptic surface by an abelian surface, making it in some sense more intrinsic to the bielliptic surface. It occurs frequently in the literature; see e.g. [Pot17; HLT20; Nue25; BM90; Ree23; Fer+22]. In critical characteristic the Bagnera–de Franchis cover and canonical cover may cease to be abelian surfaces, but should instead be considered to be ‘non-smooth versions of abelian surfaces’.

In the reoccurring analogy between Enriques surfaces and bielliptic surfaces, the canonical cover should be thought of as the K3 cover; cf. Figure 3.2. Indeed, the K3 cover of an Enriques surface in characteristic 2 may be a non-smooth *K3-like* surface, similar to how the canonical cover may be a non-smooth *abelian-like* surface. This abelian-like surface may or may not then be isomorphic to a product of genus-one curves.

In this section k remains an algebraically closed field of arbitrary characteristic. Fix a bielliptic surface $X \cong (\widetilde{C} \times \widetilde{D})/G$ as in Theorem 3.4.1. Denote the action of G on $\widetilde{D}$ by $\phi\colon G \to \mathrm{Aut}_{\widetilde{D}/k}$. Our starting point is a certain decomposition of the group scheme G.

For simplicity let us discuss first the case where $\widetilde{D}$ is smooth. In the course of the proof of Theorem 3.4.4 we encountered a subgroup scheme $H \subset G$ that acts on $\widetilde{D}$ by translations. Under the semidirect product decomposition (3.4.1) of the automorphism group scheme of $\widetilde{D}$, it equals the pre-image $H = \phi^{-1}(\widetilde{J})$, where $\widetilde{J}$ is the associated elliptic curve. The possible isomorphism classes of H and its quotient $N = G/H$ are then used to distinguish the possible cases in the Bagnera–de Franchis classification.

If X is quasi-bielliptic, we consider in a similar way the threefold semidirect product decomposition (3.4.2) of the automorphism group scheme of the rational cuspidal curve. The role of $\widetilde{J}$ is taken over by the normal subgroup scheme $\mathbb{G}_a$, since it acts on the smooth locus $\mathbb{A}^1 = \operatorname{Spec} k[t]$ by translations. It is hence clear how to extend the definition of H to the quasi-bielliptic case.

Notation 3.5.1. Let $H \subset G$ be the inverse image

$$H = \begin{cases} \phi^{-1}(\widetilde{J}) & \text{if } \widetilde{D} \text{ is smooth;} \\ \phi^{-1}(\mathbb{G}_a) & \text{if } \widetilde{D} \text{ is the rational cuspidal curve,} \end{cases}$$

which acts freely on the smooth locus of $\widetilde{D}$. Also let $N = G/H$ denote the quotient group scheme.

Since the action of G on $\widetilde{D}$ is described explicitly in the Bagnera–de Franchis classification, it is not difficult to compute H and hence N by type. We tabulate the isomorphism classes of H and N in Tables 3.7 and 3.8, respectively.

There are a number of interesting patterns to observe. First of all note that the isomorphism class of N is the same for types (a1) and (a2), types (b1) and (b2) and types (c1) and (c2). We have taken

Type	Tame Char.	Crit. Char$^{\mathrm{s}}$	Ordinary Biell.	Classical Q.-Biell.	Supersing. Q.-Biell.
(a1)	0	$p=2$	0	0	0
(a2)	$\mathbb{Z}/2\mathbb{Z}$	$p=2$	μ_2	$\mathbb{Z}/2\mathbb{Z}$	α_2
(b1)	0	$p=3$	0	0	0
(b2)	$\mathbb{Z}/3\mathbb{Z}$	$p=3$	$\nexists$	$\mathbb{Z}/3\mathbb{Z}$	$\nexists$
(c1)	0	$p=2$	0	0	$\nexists$
(c2)	$\mathbb{Z}/2\mathbb{Z}$	$p=2$	$\nexists$	$\mathbb{Z}/2\mathbb{Z}$	$\nexists$
(d)	0	$p=2,3$	0	0	0

Table 3.7: The group scheme H in all cases of the Bagnera–de Franchis classification

Type	Tame Char.	Crit. Char$^{\mathrm{s}}$	Ordinary Biell.	Classical Q.-Biell.	Supersing. Q.-Biell.
(a)	μ_2	$p=2$	$\mathbb{Z}/2\mathbb{Z}$	μ_2	α_2
(b)	μ_3	$p=3$	$\mathbb{Z}/3\mathbb{Z}$	μ_3	α_3
(c)	μ_4	$p=2$	$\mathbb{Z}/4\mathbb{Z}$	μ_4	$\nexists$
(d)	μ_6	$p=2,3$	$\mathbb{Z}/6\mathbb{Z}$	μ_6	α_p

Table 3.8: The group scheme N in all cases of the Bagnera–de Franchis classification

advantage of this fact to save space in Table 3.8. On the other hand, note that $H = 0$ for types (a1), (b1), (c1) and (d). Also note that the orders of H and N only depend on the type. This produces new numerical invariants.

Notation 3.5.2. Let $n = h^0(\mathcal{O}_N)$ and $d = h^0(\mathcal{O}_H)$ denote the orders of N and H, respectively.

Type	(a1)	(a2)	(b1)	(b2)	(c1)	(c2)	(d)
γ	2	4	3	9	4	8	6
n	2	2	3	3	4	4	6
d	1	2	1	2	1	2	1

Table 3.9: The numerical invariants γ, n and d by type

Remark 3.5.3. Note that each type is determined uniquely by the pair of natural numbers (n, d). It is hence also possible to determine the type through Table 3.9, which one can argue is more intrinsic than the case-by-case definition given implicitly in the Bagnera–de Franchis classification Theorems 3.4.4 and 3.4.12.

By comparing Tables 3.5 and 3.9 we see that $n = m$ in tame characteristic: we hence refer to n as the *expected order* of ω_X. The actual order m may be strictly smaller than the expected order, so the quotient n/m is a measure of certain critical behaviour.

Remark 3.5.4. The discrepancy between n and m in critical characteristic can be interpreted in the context of the proof of Proposition 3.4.19. Recall that m is minimal such that G acts trivially on $\omega_{\widetilde{D}}^{\otimes m}$. Note that certain subgroups of G already act trivially on $\omega_{\widetilde{D}}$. In tame characteristic the kernel of this action is H, so that $N = G/H \cong \mu_n$ results in the order m being equal to n. In critical characteristic, however, the induced action N on $\omega_{\widetilde{D}}$ may have a non-trivial kernel and the further quotient is isomorphic to a copy of μ_m for $m|n$ a proper divisor.

The extreme case with $m = 1$ occurs if and only if the Picard scheme of X is reduced by Proposition 3.1.27; cf. Table 3.6. A more modest type of critical behaviour occurs for ordinary and supersingular bielliptic surfaces of type (d), since they satisfy $n \neq m$ even though $m \neq 1$; cf. Example 3.4.21. Note that n/m is always a power of p, so it makes sense to consider the following invariant, tabulated in Table 3.10.

Notation 3.5.5. We set $i = \log_p(n/m)$.

We return to Table 3.9. The identity $\gamma = nd$ follows easily from the short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow N \longrightarrow 0. \tag{3.5.1}$$

Type	Tame Char.	Ordinary Biell.	Classical Q.-Biell.	Supersingular Q.-Biell.
(a)	0	1	0	1
(b)	0	1	0	1
(c)	0	2	0	$\nexists$
(d)	0	1	0	1

Table 3.10: The invariant i in all cases of the Bagnera–de Franchis classification

We observe from the Bagnera–de Franchis classification that this short exact sequence is split, unless X is a supersingular quasi-bielliptic surface of type (a2), in which case $G = M_2$ is a non-split extension of $N = \alpha_2$ by $H = \alpha_2$. Recall that in case $\widetilde{D}$ is smooth, we already noted during the course of the proof of Theorem 3.4.4 that $G = H \times N$ if $\widetilde{D}$ is smooth.

Example 3.5.6. A splitting of (3.5.1) induces an N-action on the product $\widetilde{C} \times \widetilde{D}$. One can check that in each case the quotient $\widetilde{X} = (\widetilde{C} \times \widetilde{D})/N$ is a bielliptic surface lying over $X = (\widetilde{C} \times \widetilde{D})/G$. The invariants of $\widetilde{X}$ are $\gamma(\widetilde{X}) = n(\widetilde{X}) = n(X)$ and $d(\widetilde{X}) = 1$. This construction is non-trivial only if the type of X is (a2), (b2) or (c2), in which case the type of $\widetilde{X}$ is (a1), (b1), or (c1), respectively. In characteristic 0 this construction appears explicitly in Lem. 2.3.ii of [Fer+22] and Lem. 2.5 of [Nue25].

The class bielliptic surfaces with the properties of Example 3.5.6 is given a special name. The terminology is motivated by e.g. Cor. 3.5 of [Boa21], which states that over an algebraically closed field of tame characteristic, the Albanese fibration $f\colon X \to P$ admits a section if and only if X has type (a1), (b1), (c1) or (d). We do not give a proof, but one direction is easy: if X is a bielliptic surface with $d = 1$, then the action of $G = N$ on $\widetilde{D}$ has rational fixed points and the choice of rational point of $\widetilde{D}^G(k)$ defines a section for $f\colon (\widetilde{C} \times \widetilde{D})/G \to \widetilde{C}/G$. Since elliptic fibrations with a section are called *Jacobian*, we introduce the following terminology.

Definition 3.5.7. A bielliptic surface X is called *of Jacobian type* if $d = 1$, i.e. if its type is (a1), (b1), (c1) or (d).

We emphasise that above result only holds over algebraically closed ground fields: if k is not algebraically closed then the Albanese fibration $f\colon X \to P$ may only admit a section after a finite ground field extension. In light of Chapter 4 where we develop the theory of bielliptic surfaces over arbitrary ground fields, we elect to use this terminology instead of the more common *Jacobian bielliptic surfaces*, which suggests that f would have a section over the ground field.

In a similar but complementary fashion to Example 3.5.6 we consider the quotient of $\widetilde{C} \times \widetilde{D}$ by the group scheme H. This quotient is of fundamental importance.

Definition 3.5.8. The *canonical cover* of a bielliptic surface $X = (\widetilde{C} \times \widetilde{D})/G$ is the map $\pi\colon (\widetilde{C} \times \widetilde{D})/H \to (\widetilde{C} \times \widetilde{D})/G$. We denote the total space by $Y = (\widetilde{C} \times \widetilde{D})/H$.

Remark 3.5.9. As with the Bagnera–de Franchis cover, we may abuse terminology and refer to the surface Y as the canonical cover.

The induced N-action on the quotient Y is free, so it gives the canonical cover the structure of an N-torsor. Note that the canonical cover and BdF-cover coincide for a bielliptic surface of Jacobian type.

The remainder of this section is dedicated to the study of Y as scheme, i.e. without its canonical action by N Most of its properties are invariant under base-change and hence still applicable in the more general setting of Chapter 4, where the ground field is no longer assumed to be algebraically closed. We first investigate smoothness of Y, in analogy to Proposition 3.4.24.

Proposition 3.5.10. *The following are equivalent:*

(i) *the Albanese fibration f is smooth;*

(ii) *the canonical cover Y can be given the structure of an abelian surface;*

(iii) *the canonical cover Y is smooth;*

(iv) *the group scheme N is smooth.*

Proof. If f is smooth, then $\widetilde{D}$ is a smooth genus-one curve and $\widetilde{C} \times \widetilde{D}$ is an abelian surface. By definition H acts by translations, so the choice of rational point makes the quotient $(\widetilde{C} \times \widetilde{D})/H$ into an abelian surface. The implication (ii)⇒(iii) is trivial. It thus remains to show (iii)⇒(iv)⇒(i). Since $\pi\colon Y \to X$ is an N-torsor, by fppf-descent it then follows that the group scheme N is smooth, which by Table 3.8 only occurs if X is not quasi-bielliptic. ☐

Remark 3.5.11. Comparing Proposition 3.5.10 with Proposition 3.4.24, we see in particular that the canonical cover is smooth if and only if the BdF-cover is smooth. Therefore N is smooth if and only if G is smooth, which can also be observed by comparing Table 3.3 and Table 3.8.

In the quasi-bielliptic case, one can think of the surface Y as a 'non-smooth version of an abelian surface'. This fits with the analogy of the K3 cover of an Enriques surface: in the critical characteristic $p = 2$, the K3 cover of an Enriques surface can be a "non-smooth version of a K3 surface", whence it is referred to as a *K3-like surface*; see [BM76; CD89; Sch21a]. Instead of smoothness, we will have to content ourselves with the weaker property of being *Gorenstein*, which is also shared with non-smooth K3 covers, cf. [BM76, p. 221] or [CD89, Prop. 1.3.1].

Lemma 3.5.12. *The surface Y is Gorenstein.*

Proof. Although the group scheme N may not be smooth, it is always Gorenstein since any group scheme over a field is a complete intersection; see Exp. VII_B, Cor. 5.5.1 of [SGA 3]. Alternatively, we can simply observe that all group schemes occurring in Table 3.8 are Gorenstein. Since $\pi\colon Y \to X$ is an N-torsor, it follows from [Stacks, Tag 0C05] and [Stacks, Tag 0C03] that this morphism is Gorenstein. Because X is smooth, and hence Gorenstein, it follows that Y is Gorenstein as well. ☐

Therefore the dualising sheaf ω_Y exists and is an invertible sheaf. Again in analogy with the dualising sheaf of a K3 cover, this sheaf turns out not to be very complicated. See Prop. 1.3.1 of [CD89] for the corresponding statement in the context of Enriques surfaces.

Proposition 3.5.13. *The invertible sheaves ω_Y and $\mathcal{O}_Y$ are isomorphic.*

Proof. The group scheme G acts trivially on $\omega_{\widetilde{C}}$ and its subgroup scheme H acts trivially on $\omega_{\widetilde{D}}$ since it restricts to translations on the smooth locus. Since H acts trivially on the tensor product $\omega_{\widetilde{C}} \otimes \omega_{\widetilde{D}}$, it follows that $\omega_Y \cong \mathcal{O}_Y$. ☐

Corollary 3.5.14. *The surface Y has cohomology*

$$h^0(\mathcal{O}_Y) = 1, \qquad h^1(\mathcal{O}_Y) = 2, \qquad h^2(\mathcal{O}_Y) = 1.$$

Proof. The fact that $h^0(\mathcal{O}_Y) = 1$ is clear. By Serre duality we also see that $h^2(\mathcal{O}_Y) = h^0(\omega_Y) = h^0(\mathcal{O}_Y) = 1$. Since $\widetilde{C} \times \widetilde{D} \to Y$ is an H-torsor, we calculate its Euler characteristic using $0 = \chi(\mathcal{O}_{\widetilde{C}\times\widetilde{D}}) = d \cdot \chi(\mathcal{O}_Y)$ through Thm. 2 of §15 of [Mum08]. It follows that $h^1(\mathcal{O}_Y) = 2$. ☐

Remark 3.5.15. A weaker version of Thm. 2 of op. cit., in which we assume the group scheme to be étale, is also sufficient in the proof of Corollary 3.5.14. Note that Y sits in the middle of a composition $\widetilde{C} \times \widetilde{D} \to Y \to X$, where $Z \to Y$ is an H-torsor, $Y \to X$ is an N-torsor and where $\chi(\mathcal{O}_{\widetilde{C}\times\widetilde{D}}) = \chi(\mathcal{O}_X) = 0$. Observe from Tables 3.7 and 3.8 that in each case either N or H is étale. Depending on these two cases, we argue either by $\chi(\mathcal{O}_Y) = n \cdot \chi(\mathcal{O}_X) = 0$ or $0 = \chi(\mathcal{O}_{\widetilde{C}\times\widetilde{D}}) = d \cdot \chi(\mathcal{O}_Y)$.

In other words, the canonical cover Y has the cohomology of an abelian surface. This serves as justification for our earlier comments on regarding Y as a non-smooth version of an abelian surface. In the reoccurring analogy with Enriques surfaces, we note that the K3-like cover has the cohomology of a K3-surface; see Prop. 9 of [BM76], Prop. 5.2 of [Sch21a], or Prop. 1.3.1 of [CD89].

The projections of $\widetilde{C} \times \widetilde{D}$ onto the two factors induce maps $Y \to \widetilde{C}/H$ and $Y \to \widetilde{D}/H$.

Notation 3.5.16. We denote the quotients by $C = \widetilde{C}/H$ and $D = \widetilde{D}/H$.

Note that, as before, there is an induced action of $N = G/H$ on the quotient $C = \widetilde{C}/H$. Since the action of G on $\widetilde{C}$ is free, it follows that the natural map $C \to P$ is an N-torsor. The situation for D is different: although D does inherit an N-action, it *never* acts freely and ramification occurs in the map $D \to B$. One can see this by a comparison of genera: the curve B is isomorphic to $\mathbb{P}^1$, but we see next that D is a genus-one curve.

Proposition 3.5.17. *The curve C is smooth of genus one.*

Proof. This is immediate, since $\widetilde{C}$ is an elliptic curve and H is a subgroup of translations. ☐

Proposition 3.5.18. *The genus of the curve D equals one. Furthermore, the following are equivalent:*

(i) *the Albanese fibration f is smooth;*

(ii) *D is a smooth genus-one curve.*

Proof. Recall that f is smooth if and only if $\widetilde{D}$ is smooth. The only potentially difficult case is the one where $\widetilde{D}$ is the rational cuspidal curve. We show by computation that if $\widetilde{D}$ is a rational cuspidal curve and $H \cong \mathbb{Z}/p\mathbb{Z}$ or α_p acts as in the Bagnera–de Franchis classification, then $\widetilde{D}/H$ is also isomorphic to the rational cuspidal curve.

Suppose first that $H = \mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{A}^1 = \operatorname{Spec} k[t]$ by $t \mapsto t+1$. the ring of invariants equals $k[t^p - t]$: note first of all that $t^p - t = \prod_{i\in\mathbb{Z}/p\mathbb{Z}}(t-i)$ is invariant, and conversely if $f \in k[t]$ satisfies $f(t) = f(t+1)$, then its roots form orbits $\alpha, \alpha+1, \dots, \alpha+p-1$, so that f is a product of polynomials of the form

$$\prod_{i\in\mathbb{Z}/p\mathbb{Z}} (t-\alpha-i) = (t-\alpha)^p - t + \alpha = t^p - t + \alpha - \alpha^p,$$

which are all contained in $k[t^p - t]$. Denote $s = t^{-1}$ in $k(t)$. Consider the open subscheme $\mathbb{A}^1 \setminus (\mathbb{Z}/p\mathbb{Z}) = \operatorname{Spec} k[t, (t^p - t)^{-1}] = \operatorname{Spec} k[s, (s^{p-1} - 1)^{-1}]$, which forms a $\mathbb{Z}/p\mathbb{Z}$-stable affine subscheme. Concretely, the action is given by $t \mapsto t+1$ and hence by $s \mapsto (s^{-1} - 1)^{-1} = s/(1-s)$. This time, the subring of invariants is generated by $(t^p - t)^{-1} = s^p/(1 - s^{p-1})$. Since the rational cuspidal curve is the union of the $\mathbb{Z}/p\mathbb{Z}$-stable affine subschemes $\operatorname{Spec} k[t]$ and $\operatorname{Spec} k[s^2, s^3, (1-s^{p-1})^{-1}]$, the quotient map is the canonical map

$$\operatorname{Spec} k[t] \cup \operatorname{Spec} k[s^2, s^3, (1-s^{p-1})^{-1}] \longrightarrow \operatorname{Spec} k[t^p - t] \cup \operatorname{Spec} k[s^{2p}/(1-s^{p-1})^2, s^{3p}/(1-s^{p-1})^3].$$

The codomain is hence a rational cuspidal curve.

Now suppose that $H = \alpha_p$, in which case the action on $\mathbb{A}^1 = \operatorname{Spec} k[t]$ is given on R-points by either $t \mapsto t + a$ or $t \mapsto t + at^p$, where $a^p = 0$. In either case the ring of invariants $R[t]^{\alpha_p(R)}$ equals $R[t^p]$ if $\alpha_p(R) \neq 0$, so the quotient $\mathbb{A}^1/\alpha_p$ equals $\operatorname{Spec} k[t^p]$. Letting $s = t^{-1}$ again, we find similarly that $\operatorname{Spec} k[s^2, s^3]_{(s^2,s^3)}/\alpha_p = \operatorname{Spec} k[s^{2p}, s^{3p}]_{(s^{2p},s^{3p})}$. Considering that the rational cuspidal curve is the union $\operatorname{Spec} k[t] \cup \operatorname{Spec} k[s^2, s^3]_{(s^2,s^3)}$, we may glue the two quotient maps together on the two α_p-stable affine charts, resulting in the relative Frobenius map of the rational cuspidal curve to itself. ☐

Extending the diagram (3.4.3), we consider the following commutative diagram. We note that the horizontal maps in the top two rows are equivariant, where we equip the top row with the natural G-action and the middle row with the natural N-action.

$$\begin{tikzcd}
D \arrow[d] & C \times D \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\square"] & C \arrow[d] \\
D/H \arrow[d] & X = (C \times D)/H \arrow[l] \arrow[r] \arrow[d, "\pi"'] \arrow[r, phantom, shift right=4ex, "\square"] & C/H \arrow[d] \\
D/G & Y = (C \times D)/G \arrow[l, "g"'] \arrow[r, "f"'] & C/G
\end{tikzcd} \tag{3.5.2}$$

Proposition 3.5.19. *The squares on the right in* (3.5.2) *are Cartesian. The top left square is Cartesian if $\widetilde{D}$ is smooth.*

Proof. It suffices to treat the top squares, as then the bottom right square follows from Proposition 3.4.23. Note that the action of N on $\widetilde{C}$ is free, and the action of $\widetilde{D}$ is free if $\widetilde{D}$ is smooth. In these cases the diagrams are Cartesian by applying Lemma 1.1.10. ☐

We finish this section with a number of generally useful results for later reference.

Proposition 3.5.20. *The maps $Y \to C$ and $Y \to D$ are in Stein factorisation.*

Proof. This follows directly from Cor. 7.8.8 of [EGA III$_2$], since the maps are proper and flat and the geometric fibres are integral. □

In critical characteristics, the Picard scheme of the quotient $X = (\widetilde{C} \times \widetilde{D})/G$ may be non-reduced, which happens exactly if $\omega_X \cong \mathcal{O}_X$, see Proposition 3.1.27. The canonical cover does not exhibit this particular critical behaviour.

Proposition 3.5.21. *The Picard scheme of the canonical cover Y is reduced.*

Proof. The top right Cartesian square of (3.5.2) induces the commutative square on Picard schemes.

$$\begin{tikzcd}
\mathrm{Pic}_{C/k} \arrow[r] \arrow[d] & \mathrm{Pic}_{\widetilde{C}/k} \arrow[d] \\
\mathrm{Pic}_{Y/k} \arrow[r] & \mathrm{Pic}_{\widetilde{C}/k} \times \mathrm{Pic}_{\widetilde{D}/k}
\end{tikzcd}$$

By Theorem 1.3.14, which applies since the order of $H^\vee$ is at most 3, the kernels of the horizontal arrows are both isomorphic to $H^\vee$. This gives us a morphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^\vee \arrow[r] \arrow[d] & \mathrm{Pic}_{C/k} \arrow[r] \arrow[d] & \mathrm{Pic}_{\widetilde{C}/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^\vee \arrow[r] & \mathrm{Pic}_{Y/k} \arrow[r] & \mathrm{Pic}_{\widetilde{C}/k} \times \mathrm{Pic}_{\widetilde{D}/k} \arrow[r] & 0
\end{tikzcd}$$

Since $Y \to C$ is a fibration, the induced map on Picard schemes is injective, cf. Proposition 3.3.4. The induced map $H^\vee \to H^\vee$ is hence an isomorphism. The snake lemma now implies that there is a short exact sequence

$$0 \longrightarrow \mathrm{Pic}_{C/k} \longrightarrow \mathrm{Pic}_{Y/k} \longrightarrow \mathrm{Pic}_{\widetilde{D}/k} \longrightarrow 0. \tag{3.5.3}$$

The outer terms are smooth, hence the middle term is smooth as well. □

Remark 3.5.22. In almost all cases of the Bagnera–de Franchis classification, smoothness of the Picard scheme of Y can be observed directly. For example, if X is ordinary, then the Albanese f is smooth, so Y can be given the structure of an abelian surface, which has a smooth Picard scheme. Alternatively if $H = 0$ then the natural map $\widetilde{C} \times \widetilde{D} \to Y$ is an isomorphism, so $\mathrm{Pic}_{Y/k} = \mathrm{Pic}_{\widetilde{C}/k} \times \mathrm{Pic}_{\widetilde{D}/k}$, which is also smooth. Since classical bielliptic surfaces do not have a reduced Picard scheme, this covers all cases except for the supersingular bielliptic surfaces of type (a2), for which above proof seems to be necessary.

Chapter 4

The two covers of a bielliptic surface

A bielliptic surface over an algebraically closed field can be seen as a certain quotient of a product of two genus-one curves, by the Bagnera–de Franchis classification of Section 3.4. This provides a potent strategy to study bielliptic surfaces over algebraically closed ground fields: it reduces their study to products of genus-one curves and general results about quotients by finite, commutative group schemes. So far, our study of bielliptic surfaces in Sections 3.4 and 3.5 has relied mostly on this approach.

This approach breaks down over arbitrary ground fields, i.e. without the assumption that the ground field is algebraically closed, since the Bagnera–de Franchis classification is not applicable. We instead start from a slightly different philosophy, in which the bielliptic surface X remains the central object: instead of regarding X as a quotient, we study the various *torsors* over X. This leads to a more intrinsic description of the Bagnera–de Franchis covers and canonical covers of X – of course provided that they exist. Although we show that a canonical cover always exists, our main result states that there may be a cohomological obstruction to the existence of a Bagnera–de Franchis cover. An explicit example of a bielliptic surface of type (a2) with non-vanishing obstruction is constructed in characteristic not 2 in Chapter 5.

A note on the notation and terminology introduced in Sections 3.4 and 3.5: although the Bagnera–de Franchis classification is not applicable here, most of the notation and terminology naturally extend to the context of an arbitrary ground field. For example, we define invariants like γ, n, d and i simply as the corresponding invariants of the base-change $X^{\mathrm{alg}} = X \otimes k^{\mathrm{alg}}$. A number of their definitions as given in Sections 3.4 and 3.5 actually work verbatim in this more general setting. For example, the definition of the intersection invariant γ in terms of the intersection multiplicity of the two fibres does not require the ground field to be algebraically closed. Also the definition of critical characteristic (Definition 3.4.7) and the trichotomy of bielliptic surface (Definition 3.4.9) are phrased to extend verbatim over arbitrary ground fields. The *type* of X may be defined either as the type of X^{alg} or, as in Remark 3.5.3, through the invariants n and d using Table 3.9.

Fix an arbitrary ground field k. We emphasise that k is not assumed to be algebraically closed and may even be imperfect. Therefore fix an algebraic closure k^{alg}. Throughout, let X be a bielliptic surface.

4.1 The covers as torsors

The main non-trivial part in generalising the theory of Sections 3.4 and 3.5 concerns Bagnera–de Franchis covers and canonical covers. We first clarify our notion of these two covers over non-algebraically closed fields.

Definition 4.1.1.

- A cover $Z \to X$ is called a *Bagnera–de Franchis cover* (abbreviated: a *BdF-cover*) if the base-change $Z^{\mathrm{alg}} \to X^{\mathrm{alg}}$ is the Bagnera–de Franchis cover in the sense of Definition 3.4.2.
- A cover $Y \to X$ is called a *canonical cover* if the base-change $Y^{\mathrm{alg}} \to X^{\mathrm{alg}}$ is the canonical cover in the sense of Definition 3.5.8.

In other words, we define the coverings to be twisted forms of their analogues over an algebraically closed field.

Remark 4.1.2. A note about terminology: in Sections 3.4 and 3.5 we spoke about 'the' Bagnera–de Franchis cover and 'the' canonical cover. This is justified over an algebraically closed ground field, since the covers exist and are unique up to isomorphism. In the context of an arbitrary ground field however, it turns out that uniqueness usually fails: the different covers become isomorphic only after base-change to k^{alg} and can thus be *twisted forms* of one another. The question of existence is a delicate one and is studied in the upcoming sections. Due to failure of uniqueness, we instead speak about *a* Bagnera–de Franchis cover and *a* canonical cover.

Assuming the existence of a BdF-cover or a canonical cover, many of the properties described in Sections 3.4 and 3.5 descend directly to arbitrary ground fields. For example Propositions 3.4.24 and 3.5.10 are concerned with the smoothness of a BdF-cover and a canonical cover, respectively, which is a property that may be verified without loss of generality over an algebraic closure. It turns out that more structure descends than one may expect.

Proposition 4.1.3. *Suppose Z is a BdF-cover for X. Let $\widetilde{C}$ and $\widetilde{D}$ be the Stein factors of the compositions $Z \to X \to P$ and $Z \to X \to B$ respectively. The natural map $Z \to \widetilde{C} \times \widetilde{D}$ is an isomorphism.*

Proof. The natural map $Z \to \widetilde{C} \times \widetilde{D}$ is an isomorphism after base-change to k^{alg}; see Proposition 3.4.25. By descent, it is an isomorphism over k; see Prop. 2.7.1.viii of [EGA IV$_2$]. ☐

Proposition 4.1.4. *Suppose $Z = \widetilde{C} \times \widetilde{D}$ is a BdF-cover for X. Then the map $Z \to X$ is canonically a G-torsor, where G is the kernel of the associated map on elliptic curves of $\widetilde{C} \to P$, as in Section 2.1.*

Proof. The map of para-elliptic curves $\widetilde{C} \to P$ is canonically a G-torsor. Since $Z = X \times_P \widetilde{C}$ by Proposition 3.4.23, the pullback inherits the structure of a torsor. ☐

In other words, a BdF-cover retains canonically the structure of a product of two curves, as well as the structure of a G-torsor. A canonical cover does not always decompose as a product, even over an algebraically closed ground field. However, it retains the fibrations to the curves C and D by Proposition 3.5.20.

Notation 4.1.5. Suppose that $\pi\colon Y \to X$ is a canonical cover. We let C and D be the Stein factors of the compositions $f \circ \pi$ and $g \circ \pi$, respectively.

We emphasise that the curves C and D depend substantially on the choice of canonical cover, if it exists. Similarly to a BdF-cover, the N-torsor structure on a canonical cover descends similarly in light of Proposition 3.5.19.

Proposition 4.1.6. *If X has a canonical cover, then the map $Y \to X$ is canonically an N-torsor, where N is the kernel of the associated map on elliptic curves of $C \to P$.*

For context, it is not true in general that a G^{alg}-torsor $Z^{\mathrm{alg}} \to X^{\mathrm{alg}}$ descends in any sense to a G-torsor $Z \to X$, since the group scheme action may only be defined after a field extension. It may also occur that a variety becomes isomorphic to a product only after a base-change, for example due to a non-trivial Galois action permuting the two factors; cf. Example 2.3.14.

Over an algebraically closed field there is a clear quotient map $\widetilde{C} \times \widetilde{D} \to Y$ from the BdF-cover to the canonical cover. The situation over an arbitrary ground field is not much different, as the following result indicates.

Proposition 4.1.7. *If X admits a BdF-cover $Z = \widetilde{C} \times \widetilde{D}$ then it also admits a canonical cover Y. Concretely, Y is given by the quotient of Z by a subgroup scheme $H \subset G$ that acts freely on the smooth locus of $\widetilde{D}$.*

Proof. Recall that $\widetilde{C} \times \widetilde{D} \to X$ is canonically a G-torsor. We aim to define the subgroup scheme $H \subset G$ in a similar way to Notation 3.5.1. Since the ground field k is not assumed to be perfect, the curve $\widetilde{D}$ may be a regular twisted form of the rational cuspidal curve. Its automorphism group scheme $\mathrm{Aut}_{\widetilde{D}/k}$ is a twisted form of the automorphism group scheme of the rational cuspidal curve $\mathbb{G}_a \rtimes A \rtimes \mathbb{G}_m$, by a conjugation action, as follows from Lem. 3.1. of [ST23]. The subgroup scheme $\mathbb{G}_a$ is normal in $\mathrm{Aut}_{\widetilde{D}/k}$, hence stable under the conjugation action. Therefore it defines a twisted normal subgroup scheme $\widetilde{\mathbb{G}}_a$ in $\mathrm{Aut}_{\widetilde{D}/k}$. We define H to be pre-image of $\widetilde{\mathbb{G}}_a$ under the group scheme action, similar to Notation 3.5.1. It is hence clear that the base-change $H \otimes k^{\mathrm{alg}}$ coincides with this definition over an algebraically closed field. Setting $Y = (\widetilde{C} \times \widetilde{D})/H$, the further quotient map $Y \to X$ by $N = G/H$ is a canonical cover. ☐

Definition 4.1.8. A BdF-cover $Z \to X$ is said to be *a BdF-cover over Y* if $Z/H \cong Y$.

A substantial downside of Definition 4.1.1 is the rather extrinsic nature of the definitions of covers due to the reliance on a base-change to an algebraic closure. A consequence is that existence is not clear over arbitrary ground fields. A more intrinsic approach is to study the covers as torsors over X. We hence apply the theory covered in Chapter 1, most notably Section 1.3.

Let K be a finite commutative group scheme. Recall that the first cohomology group $\mathrm{H}^1(X, K)$ classifies isomorphism classes of K-torsors over X, since K is affine. We repeat the five-term exact sequence (1.3.5), obtained from the five-term exact sequence of the Leray–Serre spectral sequence by invoking the Raynaud correspondence Theorem 1.3.8:

$$0 \longrightarrow \mathrm{H}^1(k, K) \longrightarrow \mathrm{H}^1(X, K) \longrightarrow \mathrm{Hom}(K^\vee, \mathrm{Pic}_{X/k}) \longrightarrow \mathrm{H}^2(k, K) \longrightarrow \cdots . \tag{4.1.1}$$

Taking $K = G$ to be the group scheme described in Proposition 4.1.4, a BdF-cover $Z \to X$ defines a group scheme homomorphism $G^\vee \to \mathrm{Pic}_{X/k}$. Similarly, taking $K = N$ as in Proposition 4.1.6, a canonical cover defines a homomorphism $N^\vee \to \mathrm{Pic}_{X/k}$. Therefore a BdF-cover (resp. canonical cover) is nothing more than a G-torsor (resp. N-torsor) over X that maps to this distinguished homomorphism.

This perspective allows us to address the non-uniqueness of a BdF-cover and a canonical cover directly. Let $Z \to X$ be a BdF-cover and let P be a G-torsor over k. Then the contracted product $Z \wedge^G P \to X$ is a twisted form of $Z \to X$ and therefore again a BdF-cover. This is referred to as *twisting by the G-torsor P*. Since the cohomology classes of $Z \to X$ and its twist in $\mathrm{H}^1(X, G)$ are translates by an element of $\mathrm{H}^1(k, G)$, exactness of (4.1.1) implies that they map to the same homomorphism $G^\vee \to \mathrm{Pic}_{X/k}$. The converse also holds by exactness: a torsor that maps to a given homomorphism $G^\vee \to \mathrm{Pic}_{X/k}$ is unique up to twists by G-torsors over k. Note that in the context of an algebraically closed ground field all torsors over k are trivial since $\mathrm{H}^1(k, K) = 0$, from which we recover the uniqueness of the BdF-cover in this setting.

The above paragraph is of course also valid for canonical covers after replacing the group scheme G by N. For context, let us note that the twists of a canonical cover by N-torsors play a substantial role in [BS04] in the construction of a bielliptic surface over a number field for which the Brauer–Manin obstruction is not sufficient to explain the failure of the Hasse principle.

Remark 4.1.9. Setting $K = \mu_\ell$ brings us back to the setting of Kummer theory. There the choice of μ_ℓ-torsor depends on the choice of a section s, which is unique up to an element of $k^*/k^{*\ell} = \mathrm{H}^1(k, \mu_\ell)$; cf. Theorem 1.4.5.

The conclusion of above discussion is that all BdF-covers map to the same homomorphism $G^\vee \to \mathrm{Pic}_{X/k}$. Once we have an intrinsic definition of this homomorphism, we may redefine a BdF-cover to be any torsor that maps to this given homomorphism. For the canonical cover we similarly search an intrinsic definition of $N^\vee \to \mathrm{Pic}_{X/k}$.

Proposition 4.1.10. *Suppose that X has a BdF-cover, hence also a canonical cover. Let G and N be the finite commutative group schemes of Propositions 4.1.4 and 4.1.6. The corresponding maps to the Picard scheme $G^\vee \to \mathrm{Pic}_{X/k}$ and $N^\vee \to \mathrm{Pic}_{X/k}$ are injective with images contained in $\mathrm{Pic}^\alpha_{X/k}$. The latter map factors over the former via the inclusion $N^\vee \subset G^\vee$. In other words, we may regard $N^\vee$ and $G^\vee$ as subgroup schemes $N^\vee \subset G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$.*

Proof. Consider first the homomorphism $G^\vee \to \mathrm{Pic}_{X/k}$. Since $G^\vee$ is finite, its image lies inside the numerically trivial part $\mathrm{Pic}^\tau_{X/k}$. Recall that $f\colon X \to P$ denotes the Albanese fibration, and that $\mathrm{Pic}^\tau_{P/k} = \mathrm{Pic}^0_{P/k}$ since P is a para-elliptic curve. Consider the five-term exact sequences obtained from the Leray–Serre spectral sequence (1.3.5) applied to $P \to \mathrm{Spec}(k)$ and $X \to \mathrm{Spec}(k)$ by invoking the Raynaud correspondence Theorem 1.3.8. By naturality the following diagram is commutative with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(k, G) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^1(P, G) \arrow[r] \arrow[d, "f^*"] & \mathrm{Hom}(G^\vee, \mathrm{Pic}^0_{P/k}) \arrow[r] \arrow[d, "f^* \circ -"] & \mathrm{H}^2(k, G) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
0 \arrow[r] & \mathrm{H}^1(k, G) \arrow[r] & \mathrm{H}^1(X, G) \arrow[r] & \mathrm{Hom}(G^\vee, \mathrm{Pic}^\tau_{X/k}) \arrow[r] & \mathrm{H}^2(k, G) \arrow[r] & \cdots
\end{tikzcd} \tag{4.1.2}$$

Since a BdF-cover $Z \to X$ is the pullback of a G-torsor $\widetilde{C} \to P$ along the Albanese $f\colon X \to P$, since it holds after base change to an algebraic closure k^{alg} by Proposition 3.4.23, it follows that

$f^*([\widetilde{C} \to P]) = [Z \to X]$. Therefore, the commutativity of the second square implies that the map $G^\vee \to \mathrm{Pic}^\tau_{X/k}$ factors over $f^*\colon \mathrm{Pic}^0_{P/k} \to \mathrm{Pic}^\tau_{X/k}$. Recall that f^* is an isomorphism when restricting the codomain to the maximal abelian subvariety $\mathrm{Pic}^\alpha_{X/k}$.

The map $G^\vee \to \mathrm{Pic}^0_{P/k}$ corresponding to the G-torsor $\widetilde{C} \to P$ is injective, since the latter is an isogeny of para-elliptic curves. We conclude that $G^\vee \to \mathrm{Pic}^\alpha_{X/k}$ is injective; cf. Lemma 2.2.3. From now on, we regard it as the inclusion of a subgroup scheme. Arguing in a similar way, we conclude that $N^\vee \to \mathrm{Pic}^\alpha_{X/k}$ is also injective.

Let $H \subset G$ be the subgroup scheme that acts by translation on the smooth locus of $\widetilde{D}$, as in Proposition 4.1.7. Then $Y = Z/H$ is a canonical cover, from which we can conclude that $N \cong G/H$. Furthermore

$$Z \wedge^G N = (Z \times N)/G \cong Z/H = Y,$$

so the cohomology class of a BdF-cover $[Z \to X] \in \mathrm{H}^1(X, G)$ maps to the cohomology class $[Y \to X] \in \mathrm{H}^1(X, N)$ of a canonical cover under the quotient map $G \to N$. Note that by Cartier duality the quotient map induces an inclusion $i\colon N^\vee \to G^\vee$. By naturality of the Leray–Serre spectral sequence, the following diagram is commutative with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(k,G) \arrow[r] \arrow[d] & \mathrm{H}^1(X,G) \arrow[r] \arrow[d] & \mathrm{Hom}(G^\vee, \mathrm{Pic}^\tau_{X/k}) \arrow[r] \arrow[d, "-\circ i"] & \mathrm{H}^2(k,G) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{H}^1(k,N) \arrow[r] & \mathrm{H}^1(X,N) \arrow[r] & \mathrm{Hom}(N^\vee, \mathrm{Pic}^\tau_{X/k}) \arrow[r] & \mathrm{H}^2(k,N) \arrow[r] & \cdots
\end{tikzcd}$$

Commutativity of the second square implies that the inclusion $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ indeed factors over the inclusion $G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ via the inclusion $N^\vee \subset G^\vee$. In other words, it follows that $N^\vee \subset G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$. □

Remark 4.1.11. Restricting to subgroup schemes of $\mathrm{Pic}^\alpha_{X/k}$ instead of $\mathrm{Pic}^\tau_{X/k}$ is helpful in limiting the search for the subgroup schemes $N^\vee$ and $G^\vee$. We do not exclude many covers by considering onlysubgroup schemes of the maximal abelian subvariety $\mathrm{Pic}^\alpha_{X/k}$, in the following sense. We study the quotient $\mathrm{Pic}^\tau_{X/k} / \mathrm{Pic}^\alpha_{X/k} = \mathrm{NS}^\tau_{X/k}$ in $\mathrm{Pic}^\tau_{X/k} / \mathrm{Pic}^\alpha_{X/k} = \mathrm{NS}^\tau_{X/k}$ in Chapter 7. From Table 7.1 and Proposition 7.1.22 we read off that the quotient group scheme is quite small: its order is at most 4.

The above does not directly yield an intrinsic description of the canonical covers and BdF-covers. This is because the subgroup schemes $N^\vee \subset G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ are so far only defined for bielliptic surfaces admitting a canonical cover or a BdF-cover, respectively, and an intrinsic definition is lacking. We redefine these subgroup schemes for all bielliptic surfaces in the following subsections, without assumption on the existence of either of the covers. In order to achieve this in a consistent matter, we first study the subgroup schemes defined by the covers, if they exist, as explicitly as possible. These descriptions will then serve as redefinitions; see Notations 4.1.16 and 4.1.27.

Remark 4.1.12. Suppose X is an Enriques surface. In the reoccurring analogy between the canonical cover and the K3 cover of an Enriques surface, the issue of finding the correct subgroup scheme $G^\vee \subset \mathrm{Pic}^\tau_{X/k}$ does not arise for Enriques surfaces: since $\mathrm{Pic}^\tau_{X/k}$ has order 2, one should simply take the entire group scheme $G^\vee = \mathrm{Pic}^\tau_{X/k}$, which may be isomorphic to either μ_2, $\mathbb{Z}/2\mathbb{Z}$ or α_2. The situation is not as easy if X is a bielliptic surface, since then $\mathrm{Pic}^\alpha_{X/k}$ is an elliptic curve, which has infinitely many finite subgroup schemes.

We begin with canonical covers, followed by BdF-covers.

4.1.1 Canonical cover

Suppose X is a bielliptic surface that admits a canonical cover. Information about the subgroup scheme $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ of Proposition 4.1.10 can be obtained by using properties of the canonical cover over an algebraically closed ground field, e.g. those of Section 3.5. For example, the following is a consequence of Proposition 3.5.13, which states that $\omega_Y \cong \mathcal{O}_Y$.

Lemma 4.1.13. *Suppose X has a canonical cover and let $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ be the subgroup scheme of Proposition 4.1.10. Then $\omega_X \in N^\vee(k)$.*

Proof. In Proposition 1.3.9 we have seen that the image of the map $N^\vee \to \mathrm{Pic}^\tau_{X/k}$ is contained in the kernel of the pullback map $\pi^*\colon \mathrm{Pic}^\tau_{X/k} \to \mathrm{Pic}^\tau_{Y/k}$. By Theorem 1.3.14 this kernel is isomorphic to a copy of $N^\vee$, which is hence induced by the inclusion $N^\vee \to \mathrm{Pic}^\tau_{X/k}$. Now it suffices to note that $\pi^*\omega_X \cong \omega_Y \cong \mathcal{O}_Y$, where the first isomorphism is as in pp. 221–222 of [BM76], and the second isomorphism follows from Proposition 3.5.13. $\square$

It follows that the subgroup scheme isomorphic to $\mathbb{Z}/m\mathbb{Z}$ generated by ω_X is contained in $N^\vee$. It turns out to be quite an important subgroup scheme.

Notation 4.1.14. Let $M^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ be the subgroup scheme isomorphic to $\mathbb{Z}/m\mathbb{Z}$ generated by the class of the invertible sheaf ω_X.

Lemma 4.1.15. *Suppose X has a canonical cover. Then $M^\vee$ is the maximal étale subgroup scheme of $N^\vee$.*

Proof. It is clear that $M^\vee$ is an étale subgroup scheme of $N^\vee$. It suffices to check after base-change to an algebraic closure that $M^\vee \otimes k^{\mathrm{alg}}$ is the maximal étale subgroup scheme of $N^\vee \otimes k^{\mathrm{alg}}$. This can be verified in all cases of the Bagnera–de Franchis classification using Tables 3.5 and 3.8. $\square$

The inclusion $M^\vee \subseteq N^\vee$ is a group-scheme-theoretic analogue of the fact that the invariant m divides n. This latter fact follows from the former by comparing orders. Note in particular that $m = n$ if and only if $M^\vee = N^\vee$. The failure of m and n to coincide (cf. Remark 3.5.4) measures exactly the failure of the inclusion $M^\vee \subset N^\vee$ to be an equality. In tame characteristics we may hence characterise $N^\vee$ as $M^\vee = \langle \omega_X \rangle$. In this case, the canonical cover is determined by the canonical bundle. In critical characteristics, however, the dualising sheaf ω_X does not contain sufficient information to determine a canonical cover. This happens when the dualising sheaf is trivial, in which case the Picard scheme is non-reduced, or when the order is 'unexpectedly small', as in Example 3.4.21. Since the quotient $N^\vee/M^\vee$ is an infinitesimal group scheme, this suggests that $N^\vee$ is obtained from $M^\vee$ by 'enlarging by' an infinitesimal group scheme of order p^i, with the invariant i as in Notation 3.5.5. We introduce infinitesimal group schemes using Frobenius kernels, as follows.

Notation 4.1.16. Let $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ be the group scheme defined by $N^\vee = M^\vee + \mathrm{Pic}^\alpha_{X/k}[F^i]$ and let $N = \underline{\mathrm{Hom}}(N^\vee, \mathbb{G}_m)$ be its Cartier dual.

With this definition, the quotient $N^\vee/M^\vee$ is isomorphic to the ith iterated Frobenius kernel $\mathrm{Pic}^\alpha_{X/k}[F^i]$, which is infinitesimal. Over an algebraically closed field, the quotient is isomorphic to μ_{p^i} if P is ordinary and to α_p if P is supersingular, since in the latter case $i = 1$ by Table 3.10. In either case the description of $N^\vee/M^\vee$ matches that as expected from Table 3.8. To avoid further notational confusion, we immediately show that this description of $N^\vee$ coincides with the subgroup scheme $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ obtained from a canonical cover.

Proposition 4.1.17. *Let X be a bielliptic surface that admits a canonical cover. As N-torsor the canonical cover $Y \to X$ corresponds to the subgroup scheme $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ of Notation 4.1.16. In other words, the subgroup scheme $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ of Proposition 4.1.10 coincides with that of Notation 4.1.16.*

Proof. Let $N^\vee$ be the subgroup scheme in sense of Proposition 4.1.10. The quotient $N^\vee/M^\vee$ is infinitesimal by Lemma 4.1.15, so is annihilated by some power of the Frobenius. It follows that that $N^\vee = M^\vee + \mathrm{Pic}^\alpha_{X/k}[F^j]$ for some $j \geq 0$. The order equals $n = mp^j$, c.f. Lemma 4.1.19 below. This identity only holds for $j = i$. $\square$

The assumption on the existence of a canonical cover turns out to be vacuous: we may use the above description of $N^\vee = M^\vee + \mathrm{Pic}^\alpha_{X/k}[F^i]$ to show that a canonical cover exists for any bielliptic surface. We treat a slightly more general case: consider instead the following family of subgroup schemes.

Notation 4.1.18. For any $j \geq 0$, we define the subgroup scheme $N_j^\vee$ of $\mathrm{Pic}^\alpha_{X/k}$ by $N_j^\vee = M^\vee + \mathrm{Pic}^\alpha_{X/k}[F^j]$. Denote its Cartier dual by $N_j = \underline{\mathrm{Hom}}(N_j^\vee, \mathbb{G}_m)$.

Lemma 4.1.19. *For any $j \geq 0$, the addition map $M^\vee \times \mathrm{Pic}^\alpha_{X/k}[F^j] \to N_j^\vee$ is an isomorphism.*

Proof. Note that $M^\vee \cap \mathrm{Pic}^\alpha_{X/k}[F^j] = 0$, since $M^\vee$ is étale and the Frobenius kernel is infinitesimal. The result now follows from the short exact sequence

$$0 \longrightarrow G \cap H \xrightarrow{\Delta^-} G \times H \xrightarrow{+} G + H \longrightarrow 0$$

for any two subgroup schemes G and H of a common commutative group scheme. □

For $j \geq 0$ consider $K = N_j$ in the five-term exact sequence of the Leray–Serre spectral sequence (4.1.1).

$$0 \longrightarrow \mathrm{H}^1(k, N_j) \longrightarrow \mathrm{H}^1(X, N_j) \longrightarrow \mathrm{Hom}(N_j^\vee, \mathrm{Pic}_{X/k}) \longrightarrow \mathrm{H}^2(k, N_j) \longrightarrow \cdots$$

The inclusion $N_j^\vee \subset \mathrm{Pic}_{X/k}$ maps to a cohomology class in $\mathrm{H}^2(k, N_j)$, which is the obstruction to the existence of a corresponding N_j-torsor $Y_j \to X$.

Proposition 4.1.20. *For any $j \geq 0$, the image of the inclusion $N_j^\vee \to \mathrm{Pic}_{X/k}$ in $\mathrm{H}^2(k, N_j)$ is zero.*

Proof. Take Cartier duals of the group schemes in Lemma 4.1.19, so that $N_j = M \times \mathrm{Pic}^\alpha_{X/k}[V^j]$, where V denotes the Verschiebung. Whence $\mathrm{H}^2(k, N_j) = \mathrm{H}^2(k, M) \times \mathrm{H}^2(k, \mathrm{Pic}^\alpha_{X/k}[V^j])$ naturally. For the first factor, consider the boundary map $\mathrm{Hom}(M^\vee, \mathrm{Pic}^\tau_{X/k}) \to \mathrm{H}^2(k, M)$. Since $M \cong \mu_m$ we are in the realm of Kummer theory. The boundary map can be identified with the obstruction map $\mathrm{Pic}^\tau_{X/k}[m] \to \mathrm{Br}(k)[m]$. Since the class of $\omega_X \in \mathrm{Pic}_{X/k}(k)$ comes from an invertible sheaf, its image in the Brauer group vanishes. For the second factor we note that $\mathrm{Pic}^\alpha_{X/k}[V^j]$ is a unipotent group scheme, so that its second cohomology vanishes by Proposition 1.5.6. □

We conclude that for any $j \geq 0$, the inclusion of $N_j^\vee$ into $\mathrm{Pic}^\tau_{X/k}$ comes from an N_j-torsor, which is uniquely determined up to an element of $\mathrm{H}^1(k, N_j)$. The existence of these covers in the special case $j = i$ directly gives us the existence of a canonical cover.

Theorem 4.1.21. *Every bielliptic surface admits a canonical cover.*

For each $j \geq 0$ we pick a $N_j^\vee$-torsor $Y_j \to X$. Although this notation depends fundamentally on choices of elements in $\mathrm{H}^1(k, N_j)$, for each j the possible choices for Y_j are geometrically isomorphic. It thus makes sense to consider properties which are preserved by and descend under field extensions: for example, we may state that $\omega_{Y_j} \cong \mathcal{O}_{Y_j}$, which follows from $M^\vee \subset N_j^\vee$.

Proposition 4.1.22. *The surfaces Y_j have cohomology*

$$h^0(\mathcal{O}_{Y_j}) = 1, \qquad h^1(\mathcal{O}_{Y_j}) = 2, \qquad h^2(\mathcal{O}_{Y_j}) = 1.$$

Proof. The identity $h^0(\mathcal{O}_{Y_j}) = 1$ follows since the N_j-torsor $Y_j \to X$ is non-trivial and $h^2(\mathcal{O}_{Y_j}) = 1$ follows by Serre duality from $\omega_{Y_j} \cong \mathcal{O}_{Y_j}$. The Euler characteristic equals $\chi(\mathcal{O}_{Y_j}) = np^j \cdot \chi(\mathcal{O}_X) = 0$ by Thm. 2 of §15 of [Mum08]. We conclude that $h^1(\mathcal{O}_X) = 1$. □

In other words, the surfaces Y_j have the cohomology of an abelian surface. Note that it is a direct generalisation of Corollary 3.5.14. Compare their proofs: in the former proof we computed the Euler characteristic of the canonical cover $\chi(\mathcal{O}_Y)$ over an algebraically closed field using a BdF-cover $Z \to Y$, wheras over an arbitrary ground field we resort to using the more intrinsic N_i-torsor $Y_i \to X$.

We use the N_j-torsors Y_j to give an intrinsic description of the invariant $i \in \{0, 1, 2\}$, which is otherwise quite mysterious.

Proposition 4.1.23. *The integer i is minimal such that the cover Y_i has a reduced Picard scheme.*

Proof. We may suppose without loss of generality that k is algebraically closed: indeed, since $\mathrm{Pic}^0_{Y/k}$ is proper, the notions of reducedness and geometric reducedness coincide by Lem. 3.3.7 of [Bri17]; cf. Proposition 3.1.27. Let j be the minimal integer such that Y_j has a reduced Picard scheme. The inequality $j \leq i$ follows from Proposition 3.5.21. The surface Y_j is the quotient of Y by the Verschiebung kernel $\mathrm{Pic}^0_{P/k}[V^{i-j}]$, acting as a subgroup scheme of N. We observe that in all cases of the Bagnera–de Franchis classification Theorems 3.4.4 and 3.4.12 the quotient of $Y = (\widetilde{C} \times \widetilde{D})/H$ by a non-trivial subgroup scheme of N is a bielliptic surface. Therefore Y_j is a bielliptic surface with trivial dualising sheaf, whose Picard scheme is hence non-reduced. □

This completes the search for an intrinsic definition for all major invariants of the bielliptic surface that are introduced in Chapter 3; by 'intrinsic' we mean specifically without reference to the Bagnera–de Franchis classification Theorems 3.4.4 and 3.4.12. Indeed, combining this description of i with $m = \operatorname{ord}(\omega_X)$ gives us the value of $n = mp^i$. Since the intersection invariant γ is defined in Notation 3.3.19 as the normalised intersection of the two fibres, we can also compute $d = \gamma/n$.

The possibilities for the isomorphism class of the group scheme $N^\vee$ as defined in Notation 4.1.16 are quite limited.

Proposition 4.1.24. *The description of the isomorphism classes of the group scheme N as depicted in Table 3.8 remains accurate over arbitrary ground fields, except for the column indicating ordinary bielliptic surfaces, where twisted forms may occur.*

Proof. Suppose first that X is either a bielliptic surface in tame characteristic or a classical quasi-bielliptic surface, which share the property that $i = 0$. This implies that $N^\vee = M^\vee \cong \mathbb{Z}/m\mathbb{Z}$, so that $N \cong \mu_m$. If X is instead a supersingular quasi-bielliptic surface, then $N^\vee \cong \mu_m \times \operatorname{Pic}^\alpha_{X/k}[F]$. Since $\operatorname{Pic}^\alpha_{X/k}$ is supersingular, its Frobenius kernel is isomorphic to α_p and no twisted forms arise because $\mathrm{H}^1(k, \operatorname{Aut}_{\alpha_p/k}) = \mathrm{H}^1(k, \mathbb{G}_m) = 0$ by Hilbert 90. Therefore $N \cong \mu_m \times \alpha_p$ in this case. □

Remark 4.1.25. Suppose that X is an ordinary bielliptic surface and consider $N^\vee = \mathbb{Z}/m\mathbb{Z} \times \operatorname{Pic}^\alpha_{X/k}[F^i]$. Since $\operatorname{Pic}^\alpha_{X/k}$ is an ordinary elliptic curve in this case, the (iterated) Frobenius kernel is a twisted form of $\mathbb{Z}/p^i\mathbb{Z}$. We do not have much control over the twisted forms of $\mathbb{Z}/p^i\mathbb{Z}$: although the induced action of N on the smooth genus-one curve D implies that $N^\vee \subset \operatorname{Aut}_{D/k}$, and although it is possible to argue that in fact $N^\vee \subset \operatorname{Aut}_{J/k}$, this latter automorphism group scheme may contain twisted forms of $\mathbb{Z}/p^i\mathbb{Z}$ if the j-invariant of J is 0. Since there are no twisted forms of $\mathbb{Z}/2\mathbb{Z}$, we can only deduce that the tabulation remains accurate in characteristic 2 for types (a) and (d).

Let us consider the case $j = 0$, when the $\mathbb{Z}/m\mathbb{Z}$-torsor $Y_0 \to X$ is in some sense a 'naive canonical cover', since it ignores the infinitesimal group scheme at play. The following classification clarifies why it cannot serve as canonical cover in critical characteristics.

Proposition 4.1.26. *The surface Y_0 is*

(i) *a para-abelian surface if and only if the characteristic is tame;*

(ii) *a non-smooth surface if and only if X is a classical quasi-bielliptic surface;*

(iii) *a bielliptic surface with non-reduced Picard scheme if and only if X is an ordinary bielliptic surface or a supersingular quasi-bielliptic surface.*

Proof. Let $p \geq 1$ denote the characteristic exponent of k. Note from Table 3.5 that X is classical if and only if $p \geq 2$ divides m. On the other hand $i = 0$ and $\gcd(m, p)$ happens only in tame characteristics, whereas $i > 1$ occurs only for ordinary bielliptic surfaces and supersingular quasi-bielliptic surfaces.

If m is divisible by the characteristic exponent $p \geq 2$, then the group scheme $M \cong \mu_m$ is Gorenstein but non-smooth, therefore so is the surface Y_0, corresponding to case (ii). Thus suppose that m is coprime to p. Since μ_m is smooth we also see that Y_0 is smooth. We show that it is a geometrically minimal and hence minimal surface. The image of a (-1)-curve Y_0 is a curve on X must remain a curve, which must still be a rational curve by the Riemann–Hurwitz formula. But Proposition 3.2.21 indicates that there are no rational curves on a bielliptic surface; see its proof. Therefore Y_0 must fall into one of the four cases of the Enriques classification Theorem 3.1.23.

The vanishing of the Euler characteristic reveals that Y_0 is either a bielliptic surface or an abelian surface; this can alternatively be deduced from the non-vanishing of the Albanese due to the surjective map $Y_0 \to X \to P$. If $i > 0$ then the Picard scheme of Y_0 is non-reduced by Proposition 4.1.23, so that Y_0 is a bielliptic surface. If $i = 0$ then the reducedness of the Picard scheme similarly implies that Y_0 is para-abelian. Finally, we note that $i = 0$ and $\gcd(m, p) = 1$ happens only in tame characteristic, whereas $i > 1$ happens only for ordinary bielliptic surfaces and supersingular quasi-bielliptic surfaces. □

Especially case (iii) is undesirable. This case includes all bielliptic surfaces with $m = 1$, in which case the torsor $Y_0 \to X$ is simply the identity. As such, in general nothing is gained by considering X as a quotient of Y_0. We remark that Table 3.10 displays how the invariant i contributes only in case (iii).

4.1.2 Bagnera–de Franchis covers

Now that we have a description of the subgroup scheme $N^\vee \subset \mathrm{Pic}^\alpha_{X/k}$. We use its description to construct the group scheme $G^\vee \supset N^\vee$.

Notation 4.1.27. Let $G^\vee = N^\vee + \mathrm{Pic}^\alpha_{X/k}[d]$ as subgroup scheme of $\mathrm{Pic}^\alpha_{X/k}$.

In case X has a BdF-cover, this definition is consistent with our earlier treatment of the subgroup scheme $G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$.

Proposition 4.1.28. *Let X be a bielliptic surface that admits a BdF-cover. As G-torsor, a BdF-cover $Z \to X$ corresponds to the subgroup scheme $G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ of Notation 4.1.27. In other words, the subgroup scheme $G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ of Proposition 4.1.10 coincides with that of Notation 4.1.27.*

Proof. We first consider the G-torsor $\widetilde{C} \to P$ between para-elliptic curves, which induces an isogeny of associated elliptic curves. Since this is a quotient by $\widetilde{E}[V^i] + \widetilde{E}[d]$, as can be verified after base-change to an algebraic closure from the Bagnera–de Franchis classification Theorem 3.4.1 and Table 3.3, the cover corresponds to the Cartier dual subgroup scheme $\mathrm{Pic}^0_{P/k}[F^i] + \mathrm{Pic}^0_{P/k}[d]$ of $\mathrm{Pic}^0_{P/k}$. Because (3.4.3) is Cartesian, we see that a BdF-cover corresponds to the subgroup scheme

$$G^\vee = f^*(\mathrm{Pic}^0_{P/k}[F^i] + \mathrm{Pic}^0_{P/k}[d]) = \mathrm{Pic}^\alpha_{X/k}[F^i] + \mathrm{Pic}^\alpha_{X/k}[d] = N^\vee + \mathrm{Pic}^\alpha_{X/k}[d]$$

of $\mathrm{Pic}^\alpha_{X/k}$, considering that the Albanese f induces an isomorphism $f^* \colon \mathrm{Pic}^0_{P/k} \xrightarrow{\sim} \mathrm{Pic}^\alpha_{X/k}$. □

Recall that there is a five-term exact sequence

$$0 \longrightarrow \mathrm{H}^1(k, G) \longrightarrow \mathrm{H}^1(X, G) \longrightarrow \mathrm{Hom}(G^\vee, \mathrm{Pic}^\tau_{X/k}) \longrightarrow \mathrm{H}^2(k, G) \longrightarrow \cdots \tag{4.1.3}$$

by taking $K = G$ in (4.1.1). The analogue of Proposition 4.1.20 does not hold for the subgroup scheme $G^\vee$: it may occur that the inclusion $G^\vee \to \mathrm{Pic}^\tau_{X/k}$ maps to a nonzero element of $\mathrm{H}^2(k, G)$. Consequently, there is a cohomological obstruction for a bielliptic surface to admit a Bagnera–de Franchis cover. The vanishing of this obstruction is both necessary and sufficient for X to admit a Bagnera–de Franchis cover.

Notation 4.1.29. The *obstruction (for X to admit a BdF-cover) α* is the image of the inclusion $G^\vee \to \mathrm{Pic}^\tau_{X/k}$ in $\mathrm{H}^2(k, G)$ along map of the five-term exact sequence (4.1.3).

Theorem 4.1.30. *A bielliptic surface X admits a Bagnera–de Franchis cover if and only if its obstruction vanishes in $\mathrm{H}^2(k, G)$.*

We construct an example of a bielliptic surface where the obstruction is nonzero in Chapter 5. In a few cases the obstruction vanishes trivially.

Example 4.1.31. If k is algebraically closed then $\mathrm{H}^2(k, G) = 0$. The vanishing of the obstruction corresponds to the structure theorem of bielliptic surfaces Theorem 3.4.1 stating in essence that a Bagnera–de Franchis cover exists over an algebraically closed ground field.

Example 4.1.32. Suppose that X has a rational point. In this case the proof of Bombieri and Mumford [BM77; BM76] of the Bagnera–de Franchis classification generalises without issues. This is done in detail in [Tak20]; see its Lem. 2.2. Alternatively, the choice of rational point on X determines a section for the $\mathrm{H}^2(k, G) \to \mathrm{H}^2(X, G)$, so the boundary map $\mathrm{Hom}(G^\vee, \mathrm{Pic}^\tau_{X/k}) \to \mathrm{H}^2(k, G)$ is 0.

In Section 4.2 we study the obstruction further and formulate a number of more powerful criteria for the obstruction to vanish.

If k is algebraically closed, or more generally if X has a BdF-cover, we define the subgroup scheme $H \subset G$ as the maximal subgroup scheme which acts freely on the smooth locus of $\widetilde{D}$. It sits inside a short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow N \longrightarrow 0 \tag{4.1.4}$$

similar to the short exact sequence (3.5.1) of the previous chapter. We recall that it is almost always geometrically split, except for supersingular quasi-bielliptic surfaces of type (a2). There is also a dual short exact sequence

$$0 \longrightarrow N^\vee \longrightarrow G^\vee \longrightarrow H^\vee \longrightarrow 0. \tag{4.1.5}$$

Over an arbitrary ground field we may hence recover the group scheme H through its Cartier dual. It clearly coincides with the earlier definition by Proposition 4.1.7.

Notation 4.1.33. Define the group scheme $H^\vee$ as the quotient $H^\vee = G^\vee/N^\vee$ and denote its Cartier dual by $H = \underline{\mathrm{Hom}}(H^\vee, \mathbb{G}_m)$.

Theorem 4.1.34. *The description of the isomorphism classes of the group scheme H as depicted in Table 3.7 remains accurate over arbitrary ground fields.*

Proof. Since groups of order at most 2 do not admit twisted forms, the only interesting case consists of bielliptic surfaces of type (b2). In this case $G^\vee = \mathrm{Pic}^\alpha_{X/k}[3]$ contains the normal subgroup $N^\vee \cong \mathbb{Z}/3\mathbb{Z}$. By self-duality of elliptic curves, the quotient $H^\vee = G^\vee/N^\vee$ is hence isomorphic to the Cartier dual $(\mathbb{Z}/3\mathbb{Z})^\vee = \mu_3$, so that $H \cong \mathbb{Z}/3\mathbb{Z}$. □

Remark 4.1.35. Even if the group schemes H and N are exactly as described in Tables 3.7 and 3.8, the group scheme G may still be a twisted form of its entry listed in Table 3.3. From the perspective of the short exact sequence (4.1.4), there may be different extensions of N by H that are geometrically isomorphic.

The group schemes $N^\vee \subset G^\vee$ are canonically subgroup schemes of $\mathrm{Pic}^\alpha_{X/k}$. For any choice of canonical cover $\pi\colon Y \to X$, there is a short exact sequence

$$0 \longrightarrow N^\vee \longrightarrow \mathrm{Pic}^\tau_{X/k} \xrightarrow{\pi^*} \mathrm{Pic}^\tau_{Y/k} \longrightarrow 0,$$

by Theorem 1.3.14, where we use that N acts by translations on Y and hence trivially on the Picard scheme. The image of $G^\vee \subset \mathrm{Pic}^\tau_{X/k}$ along the pullback π^* is thus a subgroup scheme isomorphic to $G^\vee/N^\vee = H^\vee$. The Leray–Serre spectral sequence associated to $Y \to \mathrm{Spec}(k)$ yields the five-term exact sequence

$$0 \longrightarrow \mathrm{H}^1(k, H) \longrightarrow \mathrm{H}^1(Y, H) \longrightarrow \mathrm{Hom}(H^\vee, \mathrm{Pic}^\tau_{Y/k}) \longrightarrow \mathrm{H}^2(k, H) \longrightarrow \mathrm{H}^2(Y, H) \longrightarrow \cdots . \tag{4.1.6}$$

By naturality, the set of H-torsors over Y mapping to the above inclusion $H^\vee \subset \mathrm{Pic}^\tau_{Y/k}$ corresponds bijectively to the set of BdF-covers $Z \to X$ such that $Z/Y \cong H$, i.e. the set of BdF-covers over Y. The five-term exact sequence reveals that for each choice of canonical cover there is an obstruction in $\mathrm{H}^2(k, H)$ for the existence of a BdF-cover over Y. We emphasise that this obstruction depends on the choice of canonical covers: it is possible that some canonical cover admit a BdF-cover over it, whereas another canonical cover may not. We clarify the situation in the upcoming section.

4.2 Pullback along the Albanese

The starting point of this section is the observation that we may study canonical covers and BdF-coves as pullbacks of covers over the Albanese P, as follows from Proposition 3.5.19 stating that the right-hand squares of (3.5.2) are Cartesian. This has been used previously, e.g. in the proofs of Propositions 4.1.10 and 4.1.28: the above observation corresponds directly to the fact that $N^\vee$ and $G^\vee$ are subgroup schemes of the maximal abelian subvariety $\mathrm{Pic}^\alpha_{X/k}$, in light of the isomorphism $f^*\colon \mathrm{Pic}^0_{P/k} \xrightarrow{\sim} \mathrm{Pic}^\alpha_{X/k}$. A cohomological study of torsors over P reveals an equivalent description of the obstruction $\alpha \in \mathrm{H}^2(k, G)$ of a bielliptic surface to admit a Bagnera–de Franchis cover, purely in terms of the Albanese P. The study of torsors over a given para-elliptic curve is manageable in light of the theory developed in Sections 1.3 and 2.2; we crucially use the lifting property. We correspondingly state a number of criteria for the obstruction α to vanish, depending on properties of the ground field or the bielliptic surface. This cohomological perspective further clarifies the relations between canonical covers and BdF-covers.

Let us first return to the observation above that the Albanese map $f\colon X \to P$ induces an isomorphism $f^*\colon \mathrm{Pic}^0_{P/k} \xrightarrow{\sim} \mathrm{Pic}^\alpha_{X/k}$. We may hence recover the torsors of interest over X as certain torsors over P. By abuse of notation we also consider the group schemes $M^\vee$, $N^\vee$ and $G^\vee$ as subgroup schemes of $\mathrm{Pic}^0_{P/k}$.

Example 4.2.1. Recall that $M^\vee$ is the subgroup scheme of $\mathrm{Pic}^\alpha_{X/k}$ isomorphic to $\mathbb{Z}/m\mathbb{Z}$ that is generated by $\langle \omega_X \rangle$. Since $\omega_X = f^*(R^1 f_* \mathcal{O}_X)^\vee$ by Theorem 3.2.22, we identify $M^\vee$ with the subgroup scheme of $\mathrm{Pic}^\tau_{X/k}$ isomorphic to $\mathbb{Z}/m\mathbb{Z}$ generated by $R^1 f_* \mathcal{O}_X$.

This perspective allows us to improve upon Example 4.1.32.

Theorem 4.2.2. *Let X be a bielliptic surface whose Albanese variety has a rational point. Then X admits a Bagnera–de Franchis cover.*

Proof. By the commutativity of (4.1.2), the obstructions for the inclusions $G^\vee \to \mathrm{Pic}^\alpha_{X/k}$ and $G^\vee \to \mathrm{Pic}^0_{P/k}$ coincide. A rational point of P determines a section for the map $\mathrm{H}^2(k, G) \to \mathrm{H}^2(P, G)$, which is thus injective. This implies that the boundary map $\mathrm{Hom}(G^\vee, \mathrm{Pic}^0_{P/k}) \to \mathrm{H}^2(k, G)$ is zero. The obstruction α lies in the image, hence is 0. $\square$

Notation 4.2.3. We further let F be the associated elliptic curve to P and $F^\vee = \mathrm{Pic}^0_{F/k}$ its dual elliptic curve.

Although elliptic curves are self-dual (i.e. there is a canonical isomorphism $F \cong \mathrm{Pic}^0_{F/k} = F^\vee$), we refrain from making this identification. In this context, the author is of the opinion that the omission of this identification aids in keeping track of the direction of the natural quotient maps between various elliptic curves and morphisms between them. To that end, consider the following elliptic curves.

Notation 4.2.4. Denote $E^\vee = F^\vee/N^\vee$ and $\widetilde{E}^\vee = F^\vee/G^\vee$. Let E and $\widetilde{E}$ denote their respective dual elliptic curves.

The motivation for introducing the elliptic curve E, is that the Stein factor C is naturally an E-torsor. Indeed, the N-torsor $C \to P$ is a twisted form of the isogeny $E \to F$ with kernel N, since both correspond to the subgroup scheme $N^\vee \subset \mathrm{Pic}^0_{P/k} = \mathrm{Pic}^0_{E/k} = E^\vee$. In a similar way, the para-elliptic curve $\widetilde{C}$, if it exists, is a torsor under the elliptic curve $\widetilde{E}$. In some sense, the associated elliptic curve $\widetilde{E}$ is intrinsically defined, even though the para-elliptic curve $\widetilde{C}$ may not exist.

Again we refrain from making the canonical identifications $E = E^\vee$ and $\widetilde{E} = \widetilde{E}^\vee$. This helps to distinguish the canonical isogenies $F^\vee \to E^\vee \to \widetilde{E}^\vee$, from their dual isogenies $\widetilde{E} \to E \to F$. The quotient maps $F^\vee \to E^\vee \to \widetilde{E}^\vee$ fit in the following diagram, that has exact rows and exact first column:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & N^\vee \arrow[r] \arrow[d] & F^\vee \arrow[r] \arrow[d, "\mathrm{id}"] & E^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G^\vee \arrow[r] \arrow[d] & F^\vee \arrow[r] \arrow[d] & \widetilde{E}^\vee \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & H^\vee \arrow[r] \arrow[d] & E^\vee \arrow[r] \arrow[uur, "\mathrm{id}"'] & \widetilde{E}^\vee \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd} \tag{4.2.1}$$

We consider the dual diagram. That is to say: we consider the dual elliptic curves with dual isogenies. The kernels are subsequently given by their Cartier duals; see e.g. Thm. 7.5 of [EGM].

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & H \arrow[r] \arrow[d] & \widetilde{E} \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & \widetilde{E} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[uur, "\mathrm{id}"'] & F \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd} \tag{4.2.2}$$

In the following we apply the *lifting property* of Section 1.3 to the rows of (4.2.2). Viewing this diagram as a composition of morphisms between short exact sequences, we consider the induced maps between the long exact sequences. As such, the following diagram is commutative with exact rows and some exact columns.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathrm{H}^1(k,H) \arrow[r] \arrow[d] & \mathrm{H}^1(k,\widetilde{E}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^1(k,E) \arrow[r] \arrow[d] & \mathrm{H}^2(k,H) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(k,G) \arrow[r] \arrow[d] & \mathrm{H}^1(k,\widetilde{E}) \arrow[r] \arrow[d] & \mathrm{H}^1(k,F) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^2(k,G) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(k,N) \arrow[r] \arrow[d] & \mathrm{H}^1(k,E) \arrow[r] \arrow[uur, "\mathrm{id}"'] & \mathrm{H}^1(k,F) \arrow[r] & \mathrm{H}^2(k,N) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & & \vdots &
\end{tikzcd} \tag{4.2.3}$$

Of course, the only columns that are exact are those that are part of the long exact sequence of (4.1.4). The above diagram is clearly not quite complete: it is natural to extend it with the boundary map $\delta\colon \mathrm{H}^i(k,N) \to \mathrm{H}^{i+1}(k,H)$ induced by this short exact sequence.

Lemma 4.2.5. *The following diagram is commutative:*

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & & \\
\cdots \arrow[r] & \mathrm{H}^1(k,H) \arrow[r] \arrow[d] & \mathrm{H}^1(k,\widetilde{E}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^1(k,E) \arrow[r] \arrow[d] & \mathrm{H}^2(k,H) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(k,G) \arrow[r] \arrow[d] & \mathrm{H}^1(k,\widetilde{E}) \arrow[r] \arrow[d] & \mathrm{H}^1(k,F) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^2(k,G) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(k,N) \arrow[r] \arrow[uurrr, rounded corners, "\delta"] & \mathrm{H}^1(k,E) \arrow[r] \arrow[uur, "\mathrm{id}"'] & \mathrm{H}^1(k,F) \arrow[r] & \mathrm{H}^2(k,N) \arrow[r] \arrow[d] & \cdots \\
 & & & & \vdots &
\end{tikzcd} \tag{4.2.4}$$

Proof. We verify that δ equals the composition $\mathrm{H}^1(k,N) \to \mathrm{H}^1(k,E) \to \mathrm{H}^2(k,H)$. Consider the following morphism of short exact sequences induced by (4.2.2)

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] \arrow[d, "\mathrm{id}"] & G \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & \widetilde{E} \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

which yields a morphism of long exact sequences, part of which is the following commutative square

$$\begin{tikzcd}
\mathrm{H}^i(k,N) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^{i+1}(k,H) \arrow[d, "\mathrm{id}"] \\
\mathrm{H}^i(k,E) \arrow[r] & \mathrm{H}^{i+1}(k,H)
\end{tikzcd}$$

in which the horizontal maps are the boundary maps. □

The study of torsors over P is related to the above diagram by the *lifting property of torsors* of Theorem 1.3.4. Applied to the bottom two exact rows of (4.2.2), the choice of isomorphisms $P \cong C/N \cong C/G$ determines $\mathrm{H}^1(k, N)$-equivariant bijections:

$$\{C \to P \mid N\text{-torsor corr. to } N^\vee \subset F^\vee\}/F(k) \xrightarrow{\sim} \{C \mid E\text{-torsor, mapping to } [P] \in \mathrm{H}^1(k,F)\}; \quad (4.2.5)$$

$$\{\widetilde{C} \to P \mid G\text{-torsor corr. to } G^\vee \subset F^\vee\}/F(k) \xrightarrow{\sim} \{\widetilde{C} \mid \widetilde{E}\text{-torsor, mapping to } [P] \in \mathrm{H}^1(k,F)\}. \quad (4.2.6)$$

The action of the abstract group $F(k)$ on the set of isomorphism classes of torsors is induced by the translation action of F on P. The choice of isomorphisms $P \cong C/N \cong \widetilde{C}/G$ of F-torsors is not so important to us, so we mostly ignore the $F(k)$-action.

The bijections (4.2.5) and (4.2.6) allow us to study the existence of canonical covers and BdF-covers through the existence of certain torsors over P: we are essentially asking when (4.2.6) is the empty bijection. This question is then answered fully by the diagram (4.2.4), as it contains all information regarding these torsors, so also regarding the coverings up to the $F(k)$-action on P. We observe a number of immediate consequences: the following result is a result of Lemma 1.3.3.

Theorem 4.2.6. *The existence of a Bagnera–de Franchis cover is equivalent to the existence of a cohomology class* $[\widetilde{C}] \in \mathrm{H}^1(k, \widetilde{E})$ *mapping to* $[P] \in \mathrm{H}^1(k, F)$. *The obstruction to the existence of a Bagnera–de Franchis cover* α *equals the image of* $[P]$ *in* $\mathrm{H}^2(k, G)$ *along the boundary map.*

The obstruction is, in essence, an obstruction to the existence of a para-elliptic curve $\widetilde{C}$. This makes sense in view of Proposition 3.4.23. It is notable that its associated elliptic curve $\widetilde{E}$ is, however, intrinsically defined in Notation 4.2.4 due to the existence of a rational point on E.

Corollary 4.2.7. *If* $n = d$, *then a Bagnera–de Franchis cover exists if and only if the cohomology class* $[P] \in \mathrm{H}^1(k, F)$ *is divisible by* n.

Proof. In light of the definition of $G^\vee$, of Notation 4.1.27, the assumption implies that $G^\vee = \mathrm{Pic}^0_{P/k}[n]$ because $N^\vee \subset \mathrm{Pic}^0_{P/k}[n]$. Therefore, there is an isomorphism $\widetilde{E} \cong F$ such that the natural map $\widetilde{E} \to F$ is multiplication by n. A pre-image in $\mathrm{H}^1(k, \widetilde{E})$ hence corresponds to an element $\mathrm{H}^1(k, F)$ whose n-fold multiple equals $[P]$. Alternatively, the choice of a Bagnera–de Franchis cover is equivalent with the choice of an n-cover $\widetilde{C} \to P$, so we may apply Corollary 2.2.11. ☐

We locate the obstruction more precisely within $\mathrm{H}^2(k, G)$.

Proposition 4.2.8. *The obstruction* $\alpha \in \mathrm{H}^2(k, G)$ *is contained in the subgroup* $\mathrm{Ker}(\mathrm{H}^2(k, G) \to \mathrm{H}^2(k, N)) = \mathrm{Im}(\mathrm{H}^2(k, H) \to \mathrm{H}^2(k, G))$.

Proof. This follows from commutativity of (4.2.3), since the image of $[P]$ in $\mathrm{H}^2(k, N)$ vanishes by virtue of Proposition 4.1.20 and Lemma 1.3.3. ☐

The obstruction α arises as the image of an element of $\mathrm{H}^2(k, H)$. By commutativity of (4.2.4), such a lift is the image of a cohomology class $[C] \in \mathrm{H}^1(k, E)$ corresponding to a canonical cover of X. A lift of the obstruction has a geometric interpretation in terms of the chosen canonical cover Y. Recall that if Z is a BdF-cover, then $H \subset G$ acts naturally on Z and the quotient Z/H is a canonical cover.

Theorem 4.2.9. *Let* $Y \to X$ *be a fixed canonical cover of* X *and let* C *denote the Stein factor of the Albanese fibration. There exists a BdF-cover* $Z \to X$ *over* Y *if and only if* $[C]$ *maps to* 0 *in* $\mathrm{H}^2(k, H)$.

Proof. In a similar way to (4.2.5) and (4.2.6), the lifting property Theorem 1.3.4 yields a $\mathrm{H}^1(k, E)$-equivariant bijection

$$\{\widetilde{C} \to C \mid H\text{-torsor, corr. to } H^\vee \subset E^\vee\}/E(k) \xrightarrow{\sim} \{\widetilde{C} \mid \widetilde{E}\text{-torsor, mapping to } C \in \mathrm{H}^1(k,E)\}. \quad (4.2.7)$$

The domain is nonempty if and only if there is a BdF-cover factoring over Y, by Proposition 3.5.19, whereas the codomain is nonempty if and only if $[C]$ maps to 0 in $\mathrm{H}^2(k, H)$ in (4.2.4). ☐

Notation 4.2.10. Fix a canonical cover $Y \to X$ with Stein factors C and D. The image of $[C] \in \mathrm{H}^1(k, E)$ in $\mathrm{H}^2(k, H)$ along the boundary map is referred to as *the obstruction for* X *to admit a BdF-cover over* Y.

The obstruction for X to admit a BdF-cover over Y depends measurably on the choice of canonical cover. The latter is only unique up to an element of $\mathrm{H}^1(k, N)$. Let $Y_1 \to X$ and $Y_2 \to X$ be two canonical covers of X, that differ by an element λ of $\mathrm{H}^1(k, N)$. We describe how the obstructions of $[C_1]$ and $[C_2]$ in $\mathrm{H}^2(k, H)$ are related.

Proposition 4.2.11. *Let $\lambda \in \mathrm{H}^1(k, N)$ such that $[C_1] - [C_2] = \lambda$. Then the obstructions of $[C_1]$ and $[C_2]$ in $\mathrm{H}^2(k, H)$ differ by the boundary map $\delta(\lambda)$.*

Proof. Commutativity with the snake map directly states that the difference between the obstructions of $[C_1]$ and $[C_2]$ in $\mathrm{H}^2(k, H)$ equals the boundary $\delta(\lambda)$. ☐

According to Theorem 4.1.34, the isomorphism class of H over arbitrary ground fields is displayed in Table 3.7. The number of possibilities for H are limited; thus, so are the possibilities for the second cohomology group $\mathrm{H}^2(k, H)$. If H is unipotent, e.g. α_p or $\mathbb{Z}/p\mathbb{Z}$ in characteristic p, then $\mathrm{H}^2(k, H) = 0$, q.v. Proposition 1.5.6. We observe that H is always unipotent if X is a quasi-bielliptic surface. Therefore Proposition 4.2.8

Corollary 4.2.12. *A quasi-bielliptic surface admits a Bagnera–de Franchis cover.*

Corollary 4.2.13. *If $\mathrm{Br}(k)[2] = 0$ and $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z}) = 0$ then any bielliptic surface admits a Bagnera–de Franchis cover.*

Proof. By Theorem 4.1.34 we observe from Table 3.7 that H is isomorphic to 0 or μ_2 or $\mathbb{Z}/3\mathbb{Z}$ if X is not quasi-bielliptic. ☐

Above, we used in characteristic not 2 that $\mathrm{H}^2(k, \mathbb{Z}/\mathbb{Z}) = \mathrm{H}^2(k, \mu_2) \cong \mathrm{Br}(k)[2]$, which is non-zero for e.g. $k = \mathbb{R}$. A similar isomorphism $\mathbb{Z}/3\mathbb{Z} \cong \mu_3$ holds outside of characteristic 3 only if the ground field contains a primitive cube root of unity $\zeta_3 \in k^{\mathrm{alg}}$. This has an amusing consequence.

Corollary 4.2.14 (A local-to-global principle)**.** *Suppose that k is a global field containing a primitive cube root of unity. We denote a place of k by v, and its completion by k_v. Let X be a bielliptic surface and fix a canonical cover $Y \to X$. Suppose that $X \otimes k_v$ admits a BdF-cover over the canonical cover $Y \otimes k_v$ for every place. Then X also admits a BdF-cover over Y.*

In particular, if the cohomology class $[C] \in \mathrm{H}^1(k, E)$ of the Stein factor C of a canonical cover Y lies in the Tate–Shafarevich group $Ш(E/k) = \mathrm{Ker}(\mathrm{H}^1(k, E) \to \prod_v \mathrm{H}^1(k_v, E_v))$, then X admits a BdF-cover over Y.

Proof. Since $H = \mathbb{Z}/d\mathbb{Z} \cong \mu_d$, the obstruction for X to admit a BdF-cover over Y lies in the image of $\mathrm{H}^2(k, H) \cong \mathrm{Br}(k)[d]$. By the Albert–Brauer–Hasse–Noether theorem (originally [AH32; HBN32] over a number field; see [Hür92] for a proof in the general case) the natural map $\mathrm{Br}(k) \to \prod_v \mathrm{Br}(k_v)$ is injective. Therefore triviality of the obstruction in all $\mathrm{Br}(k_v)[d]$ implies that the obstruction vanishes in $\mathrm{Br}(k)[d]$. ☐

Remark 4.2.15. The assumption that k contains a primitive cube root of unity is necessary in case X is a bielliptic surface of type (b), since otherwise $H = \mathbb{Z}/3\mathbb{Z}$ may not be isomorphic to μ_3. As far as the author is aware, there is no similar local-to-global principle for the cohomology groups $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z})$, in the sense that the natural map $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z}) \to \prod_v \mathrm{H}^2(k_v, \mathbb{Z}/3\mathbb{Z})$ may not be injective.

From the above result it does not follow that the existence of a BdF-cover at every completion implies the existence of a BdF-cover over k. Indeed, there may not be a local-to-global principle for the group scheme G similar to the Albert–Brauer–Hasse–Noether theorem. The situation is elucidated by considering the following morphism of long exact sequences.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^1(k, N) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^2(k, H) \arrow[r] \arrow[d, tail] & \mathrm{H}^2(k, G) \arrow[r] \arrow[d] & \mathrm{H}^2(k, N) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod_v \mathrm{H}^1(k_v, N) \arrow[r] & \prod_v \mathrm{H}^2(k_v, H) \arrow[r] & \prod_v \mathrm{H}^2(k_v, G) \arrow[r] & \prod_v \mathrm{H}^2(k_v, N) \arrow[r] & \cdots
\end{tikzcd} \tag{4.2.8}$$

Without additional assumptions one cannot use the injectivity of $\mathrm{H}^2(k, H) \to \prod_v \mathrm{H}^2(k_v, H)$ to conclude anything about the injectivity of $\mathrm{H}^2(k, G) \to \mathrm{H}^2(k_v, G)$, even when restricting to elements that map to

0 in $\mathrm{H}^2(k, N)$ and $\prod_v \mathrm{H}^2(k_v, N)$, respectively. E.g. the four lemma requires injectivity of $\mathrm{H}^2(k, N) \to \prod_v \mathrm{H}^2(k_v, N_v)$ and surjectivity of $\mathrm{H}^1(k, N) \to \prod_v \mathrm{H}^1(k_v, N_v)$.

We consider the additional assumption that (4.1.4) is split. Recall that the it is usually geometrically split, except in the case of supersingular quasi-bielliptic surfaces of type (a2). But even if it split after a field extension, it may not be split over the ground field. For example, if X is an ordinary bielliptic surface then G is isomorphic to the two-torsion of an elliptic curve, .

Corollary 4.2.16. *Let k be a global field containing a cube root of unity. Let X be a bielliptic surface such that the short exact sequence (4.1.4) is split. If the base-change $X \otimes k_v$ admits a BdF-cover for every place, then X also admits a BdF-cover.*

Thus, if the cohomology class of the Albanese $[P]$ lies in the Tate–Shafarevich group $\text{Ш}(F/k)$ of F, then X admits a BdF-cover.

Proof. The consequence of the short exact sequence being split is that the boundary maps in (4.2.8) are zero. By Proposition 4.2.11 there is an element in $\mathrm{H}^2(k, H)$ mapping to the obstruction α in $\mathrm{H}^2(k, G)$, which is unique because of injectivity. Applying this to the completions, it follows that the obstructions in $\mathrm{H}^2(k_v, H)$ for $X \otimes k_v$ to admit a BdF-cover over some canonical cover are all zero. Now the Albert–Brauer–Hasse–Noether theorem implies that the obstruction in $\mathrm{H}^2(k, H)$ vanishes. ☐

Example 4.2.17. Suppose that the short exact sequence (4.1.4) is split. The choice of a section $N \to G$ determines a retraction $G \to H$ and dually yields a splitting $H^\vee \to G^\vee$ of the dual short exact sequence (4.1.5). In this way $H^\vee$ can be seen also as a subgroup scheme of $\mathrm{Pic}^\alpha_{X/k}$. Naturality yields the following commutative diagram with exact rows, in which the downward vertical arrows are injective due the dashed retractions.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^1(X, H) \arrow[r] \arrow[d] & \mathrm{Hom}(H^\vee, \mathrm{Pic}^\tau_{X/k}) \arrow[r] \arrow[d] & \mathrm{H}^2(k, H) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(X, G) \arrow[r] \arrow[u, dashed, bend right] & \mathrm{Hom}(G^\vee, \mathrm{Pic}^\tau_{X/k}) \arrow[r] \arrow[u, dashed, bend right] & \mathrm{H}^2(k, G) \arrow[r] \arrow[u, dashed, bend right] & \cdots
\end{tikzcd}$$

The H-torsor over X corresponding to the inclusion $H^\vee \subset G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ is hence also obstructed by $\alpha \in \mathrm{H}^2(k, H)$ to exist. Suppose the obstruction vanishes and choose a BdF-cover for X. Consider the quotient $\widetilde{X} = (\widetilde{C} \times \widetilde{D})/N$ where N acts on a Bagnera–de Franchis cover as a subgroup scheme of G by choice of section of (4.1.4), as in Example 3.5.6. The further quotient map $\widetilde{X} \to X$ is an H-torsor over X. Its cohomology class is the image of $[\widetilde{C} \times \widetilde{D} \to X]$ under the above dashed retraction. We conclude that the existence of $\widetilde{X}$ depends both on the short exact sequence (4.1.4) being split, as well as the obstruction to admit a BdF-cover.

Up to now we assumed the existence of a primitive cube root of unity in k, since $\mathbb{Z}/3\mathbb{Z}$ and μ_3 only become isomorphic after base-change to $k(\zeta_3)$. We may instead apply the theory of the Weil restriction of scalars to relate $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z})$ with $\mathrm{Br}(k)[3]$: we refer to Lemma 4.2.21 below. Let us first state its consequence.

Theorem 4.2.18. *Let k be a ground field for which* $\mathrm{Br}(k)[2] = 0$ *and* $\mathrm{Br}(k(\zeta_3))[3] = 0$. *Then any bielliptic surface admits a BdF-cover.*

Proof. Let X be a bielliptic surface. We only need to treat the case $H = \mathbb{Z}/d\mathbb{Z}$ for $d = 2$, 3. As before we may exclude the case where $d = p$, since then $\mathbb{Z}/d\mathbb{Z}$ is unipotent and $\mathrm{H}^2(k, H) = 0$. If $H = \mathbb{Z}/2\mathbb{Z}$ outside of characteristic 2, then the isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mu_2$ yields an isomorphism $\mathrm{H}^2(k, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Br}(k)[2]$, which vanishes by assumption. If $H = \mathbb{Z}/3\mathbb{Z}$ over a field with $p \neq 3$ containing a primitive cube root of unity then we similarly find $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z}) \cong \mathrm{Br}(k)[3]$, which is similarly 0 since $k = k(\zeta_3)$. If k does not contain a primitive cube root of unity then Lemma 4.2.21 below with $j = 2$ implies that $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z})$ is the kernel of the norm map $\mathrm{Br}(k(\zeta_3))[3] \to \mathrm{Br}(k)$, which is trivial by assumption. ☐

A number of examples of fields with trivial Brauer group are listed on p. 162 of [Ser79]. We mention in particular the *quasi-algebraically closed* fields, also called *C_1 fields*, whose defining property is that for every $N \geq 2$, any hypersurface in $\mathbb{P}^N$ of degree $N - 2$ needs to have a rational point. Quasi-algebraically

closed fields have trivial Brauer group by e.g.
Prop. 10 of op. cit. Examples of C_1-fields include algebraically closed fields, finite fields by the Chevalley–Warning theorem (Cor. 1 in §I.2 of [Ser73]), and function fields of curves over algebraically closed fields by Tsen's theorem (Thm. 1.2.14 of [SC21]). For finite fields one may also argue that the Brauer group is zero through Wedderburn's little theorem; see Thm. 1.2.13 of [CS21].

Corollary 4.2.19. *A bielliptic surface over a quasi-algebraically closed field admits a BdF-cover.*

Proof. If k is quasi-algebraically closed, then so is the finite extension $k(\zeta_3)$. Therefore, both Brauer groups vanish. ☐

Remark 4.2.20. There is an alternative argument for bielliptic surfaces over a finite field whose Albanese morphism is smooth. Since a canonical cover is a para-abelian surface, it has a rational point by Lang's Theorem; q.v. Thm. 2 of [Lan56]. It follows that X also has a rational point, so that Example 4.1.32 applies. Alternatively, the para-elliptic curve P has a rational point by Lang's theorem, so that Theorem 4.2.2 applies. See also Thm. 3.1 of [Ryb16], correcting the earlier version Thm. 2.3 of [Ryb08].

The proof of Theorem 4.2.18 relied on the fact that $\mathrm{H}^2(k, \mathbb{Z}/3\mathbb{Z})$ is isomorphic to the kernel of the norm map $\mathrm{Br}(k')[3] \to \mathrm{Br}(k)[3]$, where $k' = k[t]/(t^2+t+1)$, as shown in the following lemma. It is inspired by the results of §1 of [LS10] concerning twisted forms of μ_p in characteristic $p > 0$, though the philosophy mostly applies also outside of critical characteristic. In fact, it is possible to generalise to twisted forms of group schemes with constant and cyclic automorphism group scheme. For the sake of simplicity we state it for μ_3 only.

Lemma 4.2.21. *Let k be a ground field in which 3 is invertible. Let $k' = k[t]/(t^2+t+1)$, so that $k' = k(\zeta_3)$ if k contains no primitive cube root of unity, and $k' \cong k \times k$ otherwise. The choice of $t \in k'$ induces a morphism $\mathbb{Z}/3\mathbb{Z} \otimes k' \to \mu_3 \otimes k'$; hence, by the universal property of the restriction of scalars, a morphism $\mathbb{Z}/3\mathbb{Z} \to \mathrm{Res}_{k'/k}\, \mu_{3,k'}$. It sits inside a short exact sequence*

$$0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathrm{Res}_{k'/k}\, \mu_{3,k'} \longrightarrow \mu_3 \longrightarrow 1, \tag{4.2.9}$$

where $\mathrm{Res}_{k'/k}\, \mu_{3,k'} \to \mu_3$ denotes the norm map. This induces an isomorphism

$$\mathrm{H}^j(k, \mathbb{Z}/3\mathbb{Z}) \cong \mathrm{Ker}(\mathrm{Norm}_{k'/k} \colon \mathrm{H}^j(k', \mu_{3,k'}) \to \mathrm{H}^j(k, \mu_3)).$$

Proof. We may verify exactness after base-change to an algebraic closure. Assume without loss of generality that k contains a primitive cube root of unity ζ_3. The two choices of primitive root of unity induce an isomorphism $k' \cong k \times k$, so $\mathrm{Res}_{k'/k}\, \mu_{3,k'} \cong \mu_3 \times \mu_3$. The composition $\mathbb{Z}/3\mathbb{Z} \to \mathrm{Res}_{k'/k}\, \mu_{3,k'} \cong \mu_3 \times \mu_3$ is then given by sending a generator to the pair (ζ_3, ζ_3^2). Its norm indeed equals 1, showing that the image is contained within the kernel. Equality then follows by comparing orders.

The identity $\mu_3 \otimes k' \to \mu_3 \otimes k'$ induces a map $\mu_3 \to \mathrm{Res}_{k'/k}\, \mu_{3,k'}$, such that the composition $\mu_3 \to \mathrm{Res}_{k'/k}\, \mu_{3,k'} \to \mu_3$ is the automorphism $x \mapsto x^2$ of μ_3; cf. [Stacks, Tag 03SH]. Thus pre-composing $\mu_3 \to \mathrm{Res}_{k'/k}\, \mu_{3,k'}$ with the automorphism $x \mapsto x^2$ yields a splitting of the short exact sequence.

This yields short exact sequences in cohomology

$$0 \longrightarrow \mathrm{H}^j(k, \mathbb{Z}/3\mathbb{Z}) \longrightarrow \mathrm{H}^j(k, \mathrm{Res}_{k'/k}\, \mu_{3,k'}) \xrightarrow{\mathrm{Norm}_{k'/k}} \mathrm{H}^j(k, \mu_3) \longrightarrow 0$$

for each $j \geq 0$. Since μ_3 is smooth, we may compute the cohomology in the étale topology, in which case there is a natural isomorphism $\mathrm{H}^j(k, \mathrm{Res}_{k'/k}\, \mu_{3,k'}) = \mathrm{H}^j(k', \mu_{3,k'})$. Indeed, the restriction of scalars is the pushforward functor on sheaves along the structure morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, which is exact in the étale topology by [Stacks, Tag 03QP]. ☐

We note that for $j = 1$, the group $\mathrm{H}^1(k, \mathbb{Z}/3\mathbb{Z})$ classifies $\mathbb{Z}/3\mathbb{Z}$-torsors over the ground field. Non-trivial torsors correspond to Galois extensions with Galois group $\mathbb{Z}/3\mathbb{Z}$. We investigate in the following example.

Example 4.2.22. Let k be a ground field with $p \neq 3$. If it contains a primitive cube root of unity, then every $\mathbb{Z}/3\mathbb{Z}$ Galois-extension of k is of the form $k(\sqrt[3]{\lambda})$ for some $\lambda \in k$ by Kummer theory; see Corollary 1.4.7. Suppose instead that k contains no primitive cube root of unity and let $k(\alpha)/k$ be a cubic Galois extension and let $\sigma \in \mathrm{Gal}(k(\alpha)/k)$ be a generator of the Galois group. By Kummer theory, there is an element $\lambda \in k(\zeta_3)$, defined uniquely up to cubes, such that $k(\zeta_3, \alpha) = k(\zeta_3, \sqrt[3]{\lambda})$. Since the classic proof of Kummer theory is constructive, we can describe λ explicitly through

$$\sqrt[3]{\lambda} = \alpha + \zeta_3\sigma(\alpha) + \zeta_3^2\sigma^2(\alpha).$$

We then compute that

$$\begin{aligned}
\mathrm{Norm}_{k(\zeta_3)/k}(\lambda) &= \mathrm{Norm}_{k(\zeta_3)/k}(\mathrm{Norm}_{k(\zeta_3,\alpha)/k(\zeta_3)}(\sqrt[3]{\lambda})) \\
&= \mathrm{Norm}_{k(\zeta_3,\alpha)/k}(\sqrt[3]{\lambda}) \\
&= \mathrm{Norm}_{k(\alpha)/k}(\mathrm{Norm}_{k(\zeta_3,\alpha)/k(\alpha)}(\alpha + \zeta_3\sigma(\alpha) + \zeta_3^2\sigma^2(\alpha))) \\
&= \mathrm{Norm}_{k(\alpha)/k}((\alpha + \zeta_3\sigma(\alpha) + \zeta_3^2\sigma^2(\alpha))(\alpha + \zeta_3^2\sigma(\alpha) + \zeta_3\sigma^2(\alpha)) \\
&= \mathrm{Norm}_{k(\alpha)/k}(\alpha^2 + \sigma(\alpha^2) + \sigma^2(\alpha^2) - \alpha\sigma(\alpha) - \sigma(\alpha)\sigma^2(\alpha) - \sigma^2(\alpha)\alpha) \\
&= \mathrm{Norm}_{k(\alpha)/k}(\mathrm{Tr}_{k(\alpha)/k}(\alpha^2 - \alpha\sigma(\alpha))) \\
&= (\mathrm{Tr}_{k(\alpha)/k}(\alpha^2 - \alpha\sigma(\alpha)))^3
\end{aligned}$$

is a cube in k. Since λ, considered as element of $k(\zeta_3)^*/k(\zeta_3)^{*3}$, lies in the kernel of the norm map $k(\zeta_3)^*/k(\zeta_3)^{*3} \to k^*/k^{*3}$, it is likely that the class of λ corresponds to the field extension $k(\alpha)/k$ under the isomorphism of Lemma 4.2.21 with $j = 1$, although we do not verify that this agrees with the maps induced on the cohomological groups. To conclude, we note that $\mathrm{Norm}_{k(\zeta_3)/k}(a + b\zeta_3) = a^2 - ab + b^2$, so that elements of the kernel of the norm map $k(\zeta_3)^* \to k^*/k^{*3}$ correspond to non-trivial solutions of the equation

$$a^2 - ab + b^2 = c^3, \quad (a, b, c \in k).$$

In case $k = \mathbb{Q}$ one can classify the solutions to this equation using standard techniques in algebraic number theory, using the fact that $\mathbb{Q}(\zeta_3)$ has class number 1.

4.3 Bielliptic surfaces with smooth Albanese

Since any quasi-bielliptic surface admits a Bagnera–de Franchis cover, we restrict our study throughout this section to bielliptic surfaces with smooth Albanese. We fix a canonical cover $\pi\colon Y \to X$ with Stein factors C and D. Our assumption implies that Y is a para-abelian surface and that C and D are para-elliptic curves. Let A be the associated abelian surface of Y and let E and J be the associated elliptic curves of C and D, respectively.

The fibration $Y \to C$ is an elliptic fibration, making Y into an elliptic para-abelian surface, as studied in Section 2.3. One main result of that section is the existence of a cohomological obstruction to the existence of a certain isogeny $\widetilde{C} \times \widetilde{D} \to Y$. This closely mirrors the cohomological obstruction to the existence of a BdF-cover over Y. Our first aim is to show that these potential covers coincide and that their obstructions in $\mathrm{H}^2(k, H)$ are equal.

We first treat the situation for the associated abelian varieties, which is slightly simpler due to the existence of rational points. We use notation of the previous section: we define $\widetilde{E}$ as the dual elliptic curve of the quotient $\widetilde{E}^\vee = E^\vee/H^\vee$.

Consider that since the pullback map $\pi^*\colon \mathrm{Pic}^\tau_{X/k} \to A^\vee$ has kernel $N^\vee$ by Theorem 1.3.14. The image of $G^\vee \subset \mathrm{Pic}^\alpha_{X/k}$ maps to a subgroup scheme $H^\vee \subset A^\vee$, providing the data for a potential BdF-cover through the five-term exact sequence (4.1.6). In a similar way, the image of the subgroup scheme $G^\vee \subset \mathrm{Pic}^0_{P/k} \to E^\vee$ is isomorphic to a copy of $H^\vee \subset E^\vee$, which provides the data for the coverings $\widetilde{E} \to E$ and $\widetilde{C} \to C$, for the latter provided that it exists. Since $Y \to C$ is in Stein factorisation, the induced map $E^\vee \to A^\vee$ is injective. The two copies of $H^\vee \subset A^\vee$ coincide by commutativity of the following square.

$$\begin{tikzcd}
A^\vee & E^\vee \arrow[l] \\
\mathrm{Pic}^\alpha_{X/k} \arrow[u, "\pi^*"] & \mathrm{Pic}^0_{P/k} \arrow[l, "f^*"'] \arrow[u]
\end{tikzcd}$$

Proposition 4.3.1. *There exists an elliptic curve $\widetilde{J}$ with subgroup scheme H such that $J = \widetilde{J}/H$ and $A = (\widetilde{E} \times \widetilde{J})/H$, where H is considered as anti-diagonal subgroup scheme.*

Proof. The subgroup scheme $H^\vee \subset A^\vee = \mathrm{Pic}^0_{Y/k}$ determines the data of a BdF-cover over Y, which may or may not be obstructed to exist. If we instead consider $A^\vee$ as $\mathrm{Pic}^0_{A/k}$, it instead determines the data of a cover $B \to A$, which is unobstructed due to the existence of a rational point on A. Let $B^\vee = A^\vee/H^\vee$, then the dual abelian variety $B = \mathrm{Pic}^0_{B/k}$ has a subgroup scheme isomorphic to H such that $B/H \cong A$ and such that the quotient map $B \to A$ is an H-torsor corresponding to this data; see Lemma 2.2.3. The base change B^{alg} to an algebraic closure is a BdF-cover of X^{alg} over Y^{alg}.

Similarly, the subgroup scheme $H^\vee \subset E^\vee$ provides the data for the isogeny $\widetilde{E} \to E$ of elliptic curves. Since $H^\vee \subset E^\vee \subset A^\vee$, it follows from naturality that there is a morphism $B \to \widetilde{E}$. Let $\widetilde{J}$ be its kernel. Then $B = \widetilde{E} \times \widetilde{J}$, since the natural comparison map is an isomorphism after base change to an algebraic closure. By construction we have $(\widetilde{E} \times \widetilde{J})/H \cong A$. □

The H-torsor $\widetilde{J} \to J$ subsequently defines a subgroup scheme $H^\vee \subset J^\vee$, which arose from the subgroup scheme $H^\vee \subset A^\vee$.

Lemma 4.3.2. *The subgroup scheme $H^\vee \subset A^\vee$ is the image of a subgroup scheme $H^\vee \subset J^\vee$ under the pullback map $J^\vee \to A^\vee$ of the fibration $Y \to D$.*

Proof. From $(\widetilde{E} \times \widetilde{J})/H = A$ it follows dually that $A^\vee/H^\vee \cong \widetilde{E}^\vee \times \widetilde{J}^\vee$. The map $J^\vee \to \widetilde{J}^\vee$ has degree d, since $\widetilde{D}^{\mathrm{alg}} \to D^{\mathrm{alg}}$ is an H^{alg}-torsor after base-change to k^{alg}. The kernel of $A^\vee \to \widetilde{E}^\vee \times \widetilde{J}^\vee$ also has order d, so from the commutativity of the square

$$\begin{tikzcd}
J^\vee \arrow[r] \arrow[d] & A^\vee \arrow[d] \\
\widetilde{J}^\vee \arrow[r] & \widetilde{E}^\vee \times \widetilde{J}^\vee
\end{tikzcd}$$

it follows that $H^\vee$ is contained in the image of $J^\vee$ in $A^\vee$. □

Notation 4.3.3. Let $\widetilde{J}^\vee = \mathrm{Pic}^0_{\widetilde{J}/k} = J^\vee/H^\vee$, which are equal by Theorem 1.3.14.

The situation is much more symmetric than in the quasi-bielliptic case: both $E^\vee$ and $J^\vee$ have a subgroup isomorphic to $H^\vee$. If X is a bielliptic surface with smooth Albanese, then the description of $Y = (\widetilde{C} \times \widetilde{D})/H$ is indeed symmetric in the two factors, since both $\widetilde{C}$ and $\widetilde{D}$ are smooth curves acted upon by a finite subgroup of translations. Dually, there is a different way to recover the subgroup scheme $H^\vee \subset E^\vee \times J^\vee$.

Proposition 4.3.4. *There is a canonical isomorphism $A^\vee = (E^\vee \times J^\vee)/H^\vee$, where $H^\vee$ is embedded anti-diagonally. Dually, the kernel of $A \to E \times J$ is isomorphic to H.*

Proof. Consider the subgroup scheme $H^\vee \times H^\vee \subset E^\vee \times J^\vee$. Then the anti-diagonally embedded subgroup scheme $H^\vee \subset H^\vee \to H^\vee$ lies in the kernel of the map $E^\vee \times J^\vee \to A^\vee$. Since both have order $d = |H|$, it follows that $H^\vee$ is the kernel and hence that $A^\vee = (E^\vee \times J^\vee)/H^\vee$. The dual version states that the kernel of $A \to E \times J$ is isomorphic to H; see e.g. §15 of [Mum08]. □

The images of the subgroup schemes $H^\vee \subset E^\vee$ and $H^\vee \subset J^\vee$ are equal in $A^\vee$, as foreshadowed in Lemma 4.3.2. Similarly, the fact that $A^\vee = \widetilde{E}^\vee \times \widetilde{J}^\vee$ implies that $A = (\widetilde{E} \times \widetilde{J})/H$. It follows that $\widetilde{E}$ and $\widetilde{J}$ are the elliptic curves of Section 2.3 such that $A = (\widetilde{E} \times \widetilde{J})/H$.

Let us return to the elliptic para-abelian surface Y. Since the isogeny $Y \to C \times D$ is a twisted form of $A \to E \times J$, it is naturally equipped with the structure of an H-torsor.

Remark 4.3.5. From this it follows that the degree of $Y \to C \times D$ is d. This also follows easily from a direct argument, which does not use the assumption on the smoothness of the Albanese f: consider the following commutative square, in which the integers next to the arrows denote the degrees.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "n"] & C \times D \arrow[d, "n^2"] \\
X \arrow[r, "\gamma"] & P \times B
\end{tikzcd}$$

Then $\deg(Y \to C \times D) = \gamma/n$, which equals d in view of the short exact sequence (4.1.4).

Both Theorem 4.2.9 and Proposition 2.3.17 describe a cohomological obstruction in $\mathrm{H}^2(k, H)$ to the existence of a certain covering $\widetilde{C} \times \widetilde{D} \to Y$. The above indicates that the two covers coincide, if they exist. The obstructions for existence are equal in view of the lifting property of torsors (4.2.7), by Lemma 1.3.3. As a consequence, we find the following result extending Theorem 4.2.2. Recall the following commutative diagram with exact rows from (2.3.4).

$$\begin{tikzcd}
0 \arrow[r] & {\substack{\mathrm{H}^1(k,H)\\ \oplus\\ \mathrm{H}^1(k,H)}} \arrow[r] \arrow[d, "+"] & {\substack{\mathrm{H}^1(C,H)\\ \oplus\\ \mathrm{H}^1(D,H)}} \arrow[r] \arrow[d, "\mathrm{pr}_C^*+\mathrm{pr}_D^*"] & {\substack{\mathrm{Hom}(H^\vee,E^\vee)\\ \oplus\\ \mathrm{Hom}(H^\vee,J^\vee)}} \arrow[r, "\partial_C\oplus\partial_D"] \arrow[d, "\cong"] & {\substack{\mathrm{H}^2(k,H)\\ \oplus\\ \mathrm{H}^2(k,H)}} \arrow[r, "p_C^*\oplus p_D^*"] \arrow[d, "+"] & {\substack{\mathrm{H}^2(C,H)\\ \oplus\\ \mathrm{H}^2(D,H)}} \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(k,H) \arrow[r] & \mathrm{H}^1(C\times D,H) \arrow[r] & \mathrm{Hom}(H^\vee,E^\vee\times J^\vee) \arrow[r, "\partial_{C\times D}"] & \mathrm{H}^2(k,H) \arrow[r] & \mathrm{H}^2(C\times D,H)
\end{tikzcd}$$

Corollary 4.3.6. *Let X be a bielliptic surface with smooth Albanese and fix a canonical cover $Y \to X$. The obstruction for X to admit a Bagnera–de Franchis cover over Y is contained in* $\mathrm{Ker}(p_C^*) \cap \mathrm{Ker}(p_D^*)$. *Therefore C or D having a rational point, implies that X admits a BdF-cover over Y.*

Remark 4.3.7. With above methods, the existence of a rational point on B should not be enough to conclude the existence of a BdF-cover. This boils down to the failure of the bottom left square of (3.5.2) to be Cartesian. Recall that B is a Brauer–Severi curve, so the existence of a rational point implies that $B \cong \mathbb{P}^1$.

Example 4.3.8. In some cases, conversely, the existence of a BdF-cover implies that B has a rational point. Let X be a bielliptic surface of type (a2) with smooth Albanese that admits a BdF-cover such that the short exact sequence (4.1.4) is split. Carrying out the construction of Example 3.5.6, we find a bielliptic surface $\widetilde{X} = (\widetilde{C} \times \widetilde{D})/N$, such that the further quotient by $H \cong \mu_2$ is isomorphic to X. Let $\widetilde{g} \colon \widetilde{X} \to \widetilde{B}$ be the second fibration of $\widetilde{X}$. The quotient $\widetilde{B}/H$ being isomorphic to B. This results in double cover of Brauer–Severi curves, in which case the following lemma applies.

Lemma 4.3.9. *Let $f \colon B_2 \to B_1$ be a morphism of degree two between Brauer–Severi curves. Then $B_1 \cong \mathbb{P}^1$.*

Proof. By [Lie17] a morphism to a Brauer–Severi curve is determined by a point on the Picard scheme of B_2, corresponding to the pullback $\mathscr{L} = f^*(\mathcal{O}_{B_1}(1)) \in \mathrm{Pic}_{B_2/k}(k)$. Since the degree of $\mathscr{L}$ is 2, it is isomorphic to $\omega_{B_1}^\vee$ and is thus representable by an invertible sheaf. This implies that B_1 is the trivial Brauer–Severi curve, hence has a rational point. □

Remark 4.3.10. We sketch an alternative argument, omitting some details. The pushforward $\mathcal{E} = f_*\mathcal{O}_{B_2}$ is locally free of rank 2. By the Grothendieck decomposition theorem, the base-change $\mathcal{E} \otimes k^{\mathrm{alg}}$ is a direct sum of invertible sheaves. In the language of [Nov24], $\mathcal{E}$ is said to be *absolutely split.* By Thm. 5.1 of op. cit. it is the direct sum of locally free sheaves on B_1, where the individual summands are tensor products of copies of ω_{B_2} and, in the notation of loc. cit., the unique non-split extension $\mathcal{W}_1$ of ω_{B_1} by $\mathcal{O}_{B_1}$; compare to Cor. 1.54 and the more general Cor. 1.46 of [Nov14]. Note that if $B_1 \cong \mathbb{P}^1$ then $\mathcal{W}_1 \cong \mathcal{O}(-1)^{\oplus 2}$. The sheaf cohomology of $\mathcal{E}$ is

$$h^i(B_1, \mathcal{E}) = h^i(B_2, \mathcal{O}_{B_2}) = \begin{cases} 1 & \text{if } i = 0; \\ 0 & \text{if } i \neq 0. \end{cases}$$

This puts a restriction on the possible decompositions of $\mathcal{E}$: computing the cohomology of the individual summands, one finds that $\mathcal{E} \cong \mathcal{O}_{B_1} \oplus \mathcal{W}_1^{\otimes n}$ for some $n \geq 0$. We omit the details. But then the rank of $\mathcal{E}$ is odd, which is a contradiction.

Even if the bielliptic surface X may be obstructed to admit a BdF-cover, it could be possible to exhibit X as a quotient of a product of two para-elliptic curves, for example by allowing the degree of the cover to be increased. The non-existence of a BdF-cover nevertheless places a strong restriction on the existence of these more general covers, in some cases also leading to a non-existence result for them. To be more precise, we consider products of para-elliptic curves $C_1 \times D_1$ equipped with a free group scheme action, such that the quotient is our given bielliptic surface X.

In tame characteristics we can reduce the situation to a canonical cover, so that all surfaces involved are para-abelian. The following can also be seen as a corollary of Prop. 0.2.14 of [CDL24].

Proposition 4.3.11. *Suppose X is a bielliptic surface in tame characteristic. Then any torsor $\psi\colon C_1 \times D_1 \to X$ factors over some canonical cover $C_1 \times D_1 \to Y$.*

Proof. Since ψ is a torsor, the pullback $\psi^*\omega_X$ is trivial, cf. p. 222 of [BM76]. From the universal property of μ_n-torsors Proposition 1.4.10 it then follows that there is a choice of section $s\colon \mathcal{O}_X \xrightarrow{\sim} \omega_X^{\otimes n}$ such that ψ factors over the μ_n-torsor $\pi\colon Y \to X$ determined by the pair (ω_X, s). Since $i = 0$ this is simply the canonical cover. ☐

Remark 4.3.12. Suppose X is an ordinary bielliptic surface. In the notation of Section 4.1.1, the proof shows that any such ψ factors over the cover $Y_0 \to X$, but there seems to be no apparent reason for it to factor also over the torsor $Y \to Y_0$ under $\mathrm{Pic}^{\alpha}_{X/k}[V^i] = \mathbb{Z}/p^i\mathbb{Z}$.

Corollary 4.3.13. *Let X be a bielliptic surface in tame characteristic. Suppose that E and J are geometrically non-isogenous elliptic curves. If there is a torsor $C_1 \times D_1 \to X$, where C_1 and D_1 are para-elliptic curves, then X admits a BdF-cover.*

Proof. Let $C_1 \times D_1 \to X$ be a torsor. By Proposition 4.3.11 it factors over some torsor $C_1 \times D_1 \to Y$, where Y is a canonical cover. Let C and D be the Stein factors of Y, which are not isogenous because E and J are not geometrically isogenous. Then Corollary 2.3.23 implies that the obstruction for X to admit a BdF-cover over Y vanishes. In particular, X admits a BdF-cover. ☐

The contrapositive is a non-existence result for covers over X that is stronger than the non-existence of a BdF-cover. It states: let X be a bielliptic surface that does not admit a BdF-cover, and assume that E and J are geometrically non-isogenous elliptic curves. Then there are no torsors $C_1 \times D_1 \to X$, where C_1 and D_1 are para-elliptic curves. In this sense, the obstruction for X to admit a BdF-cover in fact also obstructs more general covers of X.

Remark 4.3.14. It does not suffice to assume only that E and J are not isogenous over the ground field. They are only defined up to twisted forms, since they depend on Y, which is unique only up to twisted forms. It is possible that for some choice of canonical cover Y, the curves E and J are not isogenous, whereas for another choice of canonical cover they are isogenous, as discussed below.

Consider elliptic curves with affine equation $E_d\colon y^2 = x^3 + d$, for $d \in k$. The j-invariant is 0 and hence they are geometrically isomorphic. We describe different approaches to show that they can be non-isogenous, at least over number fields.

Example 4.3.15. If $k = \mathbb{Q}$, then §X.6 of [Sil09] computes the Mordell–Weil rank of a number of those elliptic curves, which can be either 0, 1 or 2. Since the Mordell–Weil rank is an isogeny invariant of elliptic curves, this distinguishes at least three classes of non-isogenous but geometrically isomorphic elliptic curves.

Example 4.3.16. Let k be a number field. We give a nice non-constructive proof of the existence of non-isogenous, geometrically isomorphic elliptic curves. Note that the elliptic curves E_d and $E_{d'}$ are isomorphic if and only if d/d' is a square in k, hence the collection of E_d indexed by k^*/k^{*2} forms an infinite family of pairwise non-isomorphic elliptic curves. By a corollary of Shafarevich's Theorem, §1.4, p. IV-7 of [Ser97] only finitely many of them can be isogenous to a given E_d.

Example 4.3.17. Suppose again $k = \mathbb{Q}$. In this case, the argument of Example 4.3.16 can be refined. By §IV.1.3 of op. cit., the locus of bad reduction is an isogeny invariant of elliptic curves. For the elliptic curve E_d, this locus consists of the prime 2 as well as all the primes such that the valuation of d is odd.

Chapter 5

An obstructed bielliptic surface

In this section we construct bielliptic surfaces over fields of characteristic not 2 whose obstructions to admit Bagnera–de Franchis covers do not vanish. Since the obstruction vanishes for quasi-bielliptic surfaces, throughout this section we consider only bielliptic surfaces with a smooth Albanese map. We furthermore should not consider bielliptic surfaces of type (a1), (b1), (c1) or (d), since they also always admit a Bagnera–de Franchis cover. We focus on the simplest non-trivial type (a2). Then the group scheme H is isomorphic to μ_2, so that a potential obstruction lies in the two-torsion of the Brauer group $\mathrm{Br}(k)[2]$. Its non-vanishing is hence necessary for the existence of a bielliptic surface of type (a2) without Bagnera–de Franchis cover. Our construction can be seen as a converse statement: a non-split quaternion algebra determines a bielliptic surface of type (a2) without Bagnera–de Franchis cover. This is the main result of this chapter.

Theorem 5.0.1. *Let k be a ground field with $p \neq 2$. There exists a bielliptic surface of type (a2) which does not admit a Bagnera–de Franchis cover if and only if* $\mathrm{Br}(k)[2] \neq 0$.

We refrain from making an assumption on the characteristic of the ground field insofar as possible: ordinary bielliptic surfaces in characteristic $p = 2$ are included in the majority of the following analysis. We consequently distinguish the group schemes μ_2 and $\mathbb{Z}/2\mathbb{Z}$. The bottleneck for including the critical characteristic arises from an insufficient understanding of the theory of two-descent in characteristic 2, e.g. the construction of a para-elliptic curve of period 2 with a given obstruction in $\mathrm{Br}(k)[2]$ of Section 6.1.

How to construct a bielliptic surface? Any bielliptic surface X of type (a2) with smooth Albanese sits in a diagram of the following form.

$$\begin{tikzcd}
\widetilde{C} \times \widetilde{D} \arrow[d, dashed, "\mu_2"] & \\
Y \arrow[r, "\mu_2"] \arrow[d, "\mathbb{Z}/2\mathbb{Z}"] & C \times D \\
X &
\end{tikzcd}$$

Here Y is a canonical cover for X and C and D are the Stein factors of the two elliptic fibrations, as in Notation 4.1.5. Each morphism is a quotient by a certain finite group scheme, indicated alongside the arrow. The dashed arrow indicates that the scheme $\widetilde{C} \times \widetilde{D}$, and hence the morphism to Y, exists if and only if X admits a Bagnera–de Franchis cover. The usual way of constructing a bielliptic surface is by taking a suitable quotient of a product $\widetilde{C} \times \widetilde{D}$. With this method however the obstruction always vanishes. Instead, we should consider a quotient of a para-abelian surface Y directly, where Y is isogenous to a product $C \times D$ but not covered by a product $\widetilde{C} \times \widetilde{D}$, as in Section 2.3. We briefly outline the construction in more detail.

Step 1. Constructing the μ_2-cover $Y \to C \times D$. We first construct a para-abelian surface Y as an appropriate μ_2-cover of the product $C \times D$, with the auxilary property that a further cover by a product $\widetilde{C} \times \widetilde{D}$ should not exist. By Kummer theory a μ_2-cover of $C \times D$ is determined up to isomorphism by an invertible sheaf $\mathscr{L}$ of order 2 and a nowhere vanishing section $s\colon \mathscr{O}_{C\times D} \xrightarrow{\sim} \mathscr{L}^{\otimes 2}$. The invertible sheaf $\mathscr{L}$ defines a k-point of the Picard scheme $\mathrm{Pic}_{C\times D/k}[2]$, hence two classes of invertible sheaves $L_C \in \mathrm{Pic}_{C/k}[2](k)$ and $L_D \in \mathrm{Pic}_{D/k}[2](k)$. We require that L_C and L_D are obstructed to come from invertible sheaves, since otherwise they determine covers of the factors $\widetilde{C} \to C$ and $\widetilde{D} \to D$ such that $Y = (\widetilde{C} \times \widetilde{D})/\mu_2$. The obstruction for $\mathscr{L}$ coming from an invertible sheaf is the sum of the obstructions of L_C and L_D, causing the individual obstructions to be inverses. We note the similarities to Section 2.3.

Step 2. Constructing the involution on Y. We then construct a certain $\mathbb{Z}/2\mathbb{Z}$-action on Y, such that the quotient surface X is a bielliptic surface not admitting a BdF-cover over the canonical cover Y. Suppose for the sake of exposition that the obstruction α vanishes, so that the cover $\widetilde{C} \times \widetilde{D}$ exists. Then the $\mathbb{Z}/2\mathbb{Z}$-action on Y should be induced by a $\mathbb{Z}/2\mathbb{Z}$-action on $\widetilde{C} \times \widetilde{D}$ of the form $(x, y) \mapsto (x + a, \psi(y))$, where ψ is a sign involution on $\widetilde{D}$. Note that we may replace the function in the second coordinate $y \mapsto \psi(y)$ without loss of generality by $y \mapsto \psi(y) + b$, as the latter function still is an involution with a geometric fixpoint. We subdivide the construction into two steps. We construct:

- **Step 2.1.** a translation-involution on Y, corresponding to $(x, y) \mapsto (x + a, y + b)$ on $\widetilde{C} \times \widetilde{D}$; and
- **Step 2.2.** a family of sign involution of Y over C, corresponding to $(x, y) \mapsto (x, \psi(y))$ on $\widetilde{C} \times \widetilde{D}$.

The above description is of course meaningless if the obstruction α does not vanish. Nevertheless, we construct the automorphisms of Y which are 'twisted forms' of automorphisms of the above form, i.e. after base-change to an algebraic closure k^{alg}, when the cover $\widetilde{C}_{k^{\mathrm{alg}}} \times \widetilde{D}_{k^{\mathrm{alg}}}$ of Y^{alg} does exist, they are induced by an automorphisms of the above form; cf. Proposition 5.2.1.

Step 3. Verifying that X has no BdF-cover. A routine verification shows that the quotient $X = Y/(\mathbb{Z}/2\mathbb{Z})$ is indeed a bielliptic surface and that Y is a canonical cover of X. So far, this construction yields that X admits no BdF-cover over Y. It may still be possible for X to admit a BdF-cover over a different canonical cover. We show that X may be constructed so that it admits no Bagnera–de Franchis cover at all.

Step 4. = Step 0. Constructing the para-elliptic curves C and D. Each step in the construction has placed additional restraints on the para-elliptic curves C and D, and it is a-priori not clear that curves with the given properties exist. We construct explicit examples using the arithmetic theory of *descent* in the upcoming section; see Chapter 6.

For the sake of clarity, let us state the data to be used throughout the construction. It consists of

- a (nonzero) Brauer class $\alpha \in \mathrm{Br}(k)[2]$;
- an elliptic curve E, together with a torsor $[C] \in \mathrm{H}^1(k, E)$ and a rational point $P \in E[2](k)$;
- an elliptic curve J, together with a torsor $[D] \in \mathrm{H}^1(k, J)$ and a rational point $Q \in J[2](k)$.

In order to state the necessary assumptions, we first interpose the following definitions. Let $\mathbb{Z}/2\mathbb{Z}$ be the subgroup of E generated by P and set $\widetilde{E} = E/(\mathbb{Z}/2\mathbb{Z})$. The dual isogeny $\widetilde{E} \to E$ is a μ_2-cover corresponding to the invertible sheaf $\mathcal{O}_E(P - \infty) \in \mathrm{Pic}^0(E)$. Similarly, let $\mathbb{Z}/2\mathbb{Z}$ be the subgroup scheme of J generated by Q and set $\widetilde{J} = J/(\mathbb{Z}/2\mathbb{Z})$, which is both a quotient and a cover of J. We postulate:

(I) the images of $P \in E(k)$ and $Q \in J(k)$ in $\mathrm{Br}(k)[2]$ under boundary maps ∂_C and ∂_D (see (5.1.1) below or (6.3.1) below), respectively, are equal to α;

(II.1) the images of $P \in E(k)$ and $Q \in J(k)$ in k^*/k^{*2} under the boundary maps δ_E and δ_J (see (5.2.2) below or Section 6.2 below), respectively, are equal and denoted β;

(II.2) the $\widetilde{J}$-torsor $D/(\mathbb{Z}/2\mathbb{Z})$ has a rational point, where $\mathbb{Z}/2\mathbb{Z} \subset J$ acts by translation;

(III) the elliptic curve $\widetilde{E}$ has full two-torsion, i.e. $\widetilde{E}[2](k) = \widetilde{E}[2](k^{\mathrm{alg}})$;

Each assumption is needed critically in one of the outlined steps. The above enumeration corresponds directly to the associated step. For example, assumption (II.2) will be necessary in Step 2.2. The optional additional assumption that E and J are non-isogenous yields a stronger non-existence result, in view of Corollary 4.3.13.

5.1 Constructing the μ_2-cover $Y \to C \times D$

Self-duality of elliptic curves yields the natural identification $\mathrm{Pic}^0_{C/k} = \mathrm{Pic}^0_{E/k} = E$. The rational point $P \in E[2](k)$ corresponds to the class of an invertible sheaf $L_C \in \mathrm{Pic}^0_{C/k}[2](k)$. Similarly, let $L_D \in \mathrm{Pic}^0_{D/k}[2](k)$ be the class of the invertible sheaf corresponding to $Q \in J[2](k)$. The pair $(P, Q) \in \mathrm{Pic}^0_{C/k}(k) \times \mathrm{Pic}^0_{D/k}(k) = \mathrm{Pic}^0_{C\times D/k}(k)$ considered as datum of an invertible sheaf is denoted $\mathrm{pr}_C^* L_C \otimes \mathrm{pr}_D^* L_D$. Recall the commutative diagram (2.3.4) with exact rows, in which we set $H = \mu_2$.

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix} k^*/k^{*2} \\ \oplus \\ k^*/k^{*2} \end{matrix} \arrow[r] \arrow[d, "\mathrm{mult}"] & \begin{matrix} \mathrm{H}^1(C,\mu_2) \\ \oplus \\ \mathrm{H}^1(D,\mu_2) \end{matrix} \arrow[r] \arrow[d, "\mathrm{pr}_C^*+\mathrm{pr}_D^*"] & \begin{matrix} E[2](k) \\ \oplus \\ J[2](k) \end{matrix} \arrow[r, "\partial_E \oplus \partial_J"] \arrow[d, "\cong"] & \begin{matrix} \mathrm{Br}(k)[2] \\ \oplus \\ \mathrm{Br}(k)[2] \end{matrix} \arrow[r] \arrow[d, "+"] & \begin{matrix} \mathrm{H}^2(C,\mu_2) \\ \oplus \\ \mathrm{H}^2(D,\mu_2) \end{matrix} \arrow[d] \\
0 \arrow[r] & k^*/k^{*2} \arrow[r] & \mathrm{H}^1(C\times D,\mu_2) \arrow[r] & (E\times J)[2](k) \arrow[r] & \mathrm{Br}(k)[2] \arrow[r] & \mathrm{H}^2(C\times D,\mu_2)
\end{tikzcd} \tag{5.1.1}$$

The cokernel of $k^*/k^{*2} \to \mathrm{H}^1(X, \mu_2)$ is canonically isomorphic to the Picard group $\mathrm{Pic}(X)$, cf. Kummer theory of Section 1.4, which we apply to $X = C$, D and $C \times D$. The obstruction in $\mathrm{Br}(k)[2]$ then measures whether the point on the Picard scheme comes from an actual invertible sheaf. If so, the choice of global nowhere vanishing section of its second tensor power determines the μ_2-torsor, corresponding to a pre-image in $\mathrm{H}^1(X, \mu_2)$. Since the obstructions for L_C and L_D are equal by virtue of assumption (I), the contribution from the two factors cancel and the obstruction vanishes on the product.

Lemma 5.1.1. *$\mathscr{L} = \mathrm{pr}_C^* L_C \otimes \mathrm{pr}_D^* L_D$ is an invertible sheaf of order 2 on $C \times D$.*

Notation 5.1.2. Pick a global section $\mathcal{O}_Y \xrightarrow{\sim} \mathscr{L}^{\otimes 2}$. Let $Y \to C \times D$ be a μ_2-torsor corresponding to the pair $(\mathscr{L}, s)$.

If α is trivial, then we can argue similarly on the factors that there are μ_2-torsors $\widetilde{C} \to C$ and $\widetilde{D} \to D$, by choice of sections of $L_C^{\otimes 2}$ and $L_D^{\otimes 2}$. Choosing the sections correctly, we can exhibit Y as the quotient $Y \cong (\widetilde{C} \times \widetilde{D})/\mu_2$, cf. Theorem 2.3.18. Conversely, non-vanishing of α prohibits Y being a quotient of this form. The stronger results Theorem 2.3.20 and Corollary 2.3.23 eliminate further possibilities for Y to be isomorphic to a quotient of a product of two para-elliptic curves, the latter relying on the assumption that E and J are non-isogenous.

Let A be the associated abelian surface of Y. Then the sequence

$$1 \longrightarrow \mu_2 \longrightarrow A \longrightarrow E \times J \longrightarrow 0,$$

is short exact. The obstruction plays no role in the setting of abelian varieties, due to the existence of rational points, cf. Proposition 2.3.12. We thus define covers $\widetilde{E} \to E$ and $\widetilde{J} \to J$ such that A is the quotient $A = (\widetilde{E} \times \widetilde{J})/\mu_2$. In some sense, although the para-elliptic curves $\widetilde{C}$ and $\widetilde{D}$ may not exist, their associated elliptic curves do exist. For later reference we recall from Proposition 2.3.11 that the following diagram is commutative with exact rows and exact first column.

$$\begin{tikzcd}
 & 1 \arrow[d] & & & \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, "\Delta"] & \widetilde{E} \times \widetilde{J} \arrow[r] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mu_2 \times \mu_2 \arrow[r] \arrow[d, "\mathrm{mult}"] & \widetilde{E} \times \widetilde{J} \arrow[r] \arrow[d] & E \times J \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & A \arrow[r] \arrow[uur, "\mathrm{id}"', crossing over] & E \times J \arrow[r] & 0 \\
 & 1 & & &
\end{tikzcd} \tag{5.1.2}$$

5.2 Constructing the involution on Y

In this step we construct two involutions on Y, the first being a *translation involution* σ_{transl}, and the second being a *family of sign involutions* σ_{sgn} over C; see the corresponding subsections below. Let σ be the composition $\sigma = \sigma_{\text{transl}} \circ \sigma_{\text{sgn}}$ on Y. Taking for granted that the description over an algebraically closed field is as in the introduction, we can already formulate the following result.

Proposition 5.2.1. *The automorphism σ is an involution on Y. After base-change to an algebraic closure it lifts to an involution*

$$(x, y) \longmapsto (x + \widetilde{P}, \psi(y) + \widetilde{Q}), \tag{5.2.1}$$

on $\widetilde{C} \times \widetilde{D}$, where $\widetilde{P} \in \widetilde{E}[2](k^{\text{alg}})$ and $\widetilde{Q} \in \widetilde{J}[2](k^{\text{alg}})$, and where ψ is a sign involution on $\widetilde{D}^{\text{alg}}$.

Proof. Suppose without loss of generality that $k = k^{\text{alg}}$. Since then the obstruction α vanishes, we regard Y as a quotient of $\widetilde{C} \times \widetilde{D}$. The involutions σ_{transl} and σ_{sgn} lift to the cover $\widetilde{C} \times \widetilde{D}$ by construction, and σ is of the form as described above; see Proposition 5.2.4 and Theorem 5.2.12 below. ☐

From this it follows that σ indeed defines a $\mathbb{Z}/2\mathbb{Z}$-action on Y, whose properties we study in the next step. First we study two involutions σ_{trans} and σ_{sgn} in more detail.

5.2.1 A translation automorphism

The automorphisms which 'act by translation' on a para-abelian variety correspond directly to the rational points on the associated abelian variety A of Y; therefore this step is essentially concerned with the existence of a suitable rational two-torsion point on A. Note that diagram (5.1.2) contains a multitude of short exact sequences, with maps between them. By naturality, there are morphisms between the long exact sequences, as follows.

$$\begin{tikzcd}
1 \arrow[r] & \mu_2(k) \arrow[r] \arrow[d, "\Delta"] & \widetilde{E}(k) \times \widetilde{J}(k) \arrow[r] \arrow[d, "\text{id}"] & A(k) \arrow[r] \arrow[d] & k^*/k^{*2} \arrow[r] \arrow[d, "\Delta"] & \cdots \\
1 \arrow[r] & \mu_2(k) \times \mu_2(k) \arrow[r] \arrow[d, "\text{mult}"] & \widetilde{E}(k) \times \widetilde{J}(k) \arrow[r] \arrow[d] & E(k) \times J(k) \arrow[r, "\delta_E \times \delta_J"] \arrow[d, "\text{id}"] & k^*/k^{*2} \times k^*/k^{*2} \arrow[r] \arrow[d, "\text{mult}"] & \cdots \\
1 \arrow[r] & \mu_2(k) \arrow[r] & A(k) \arrow[r] \arrow[uur, "\text{id}"', crossing over] & E(k) \times J(k) \arrow[r] & k^*/k^{*2} \arrow[r] & \cdots
\end{tikzcd} \tag{5.2.2}$$

Our interest is directed to the boundary maps δ_E and δ_J. Recall that assumption (II.1) states that $\delta_E(P) = \delta_J(Q) = \beta$ in k^*/k^{*2}. Multiplied together the obstruction equals β^2, which equals the trivial element of k^*/k^{*2}. From (5.2.2) we can directly conclude that there is a pre-image in $A(k)$.

Lemma 5.2.2. *There is an element $R \in A(k)$ mapping to $(P, Q) \in E(k) \times J(k)$.*

The abelian variety A acts freely and transitively on Y. The element $R \in A(k)$ hence defines a automorphism of Y, which after base-change to an algebraic closure and choice of a rational point determining an isomorphism $Y \cong A$ corresponds to the translation map $x \mapsto x + R$. The following result is important to have a well-defined $\mathbb{Z}/2\mathbb{Z}$-action.

Lemma 5.2.3. *The rational point R is two-torsion.*

Proof. This can be verified after base-change to an algebraic closure. Consider the diagram (5.2.2) with k^{alg}-points instead. Then there is a pre-image $(\widetilde{P}, \widetilde{Q}) \in \widetilde{E}(k^{\text{alg}}) \times \widetilde{J}(k^{\text{alg}})$ of (P, Q) that maps to R. It suffices to verify that $\widetilde{P}$ and $\widetilde{Q}$ are two-torsion. This follows from considering a pre-image of $\widetilde{P}$ under the dual isogeny $E \to \widetilde{E}$, which hence doubles to P. Since P lies in the kernel of this dual isogeny, it indeed follows that $2\widetilde{P} = 0$. The argument for $\widetilde{Q}$ is analogous. ☐

Translation by R hence defines an involution on Y, which we denote by σ_{transl}. The following result is clear.

Proposition 5.2.4. *If $\alpha = 0$ and $\beta = 1$, then σ_{transl} lifts to an involution on $\widetilde{C} \times \widetilde{D}$ of the form $(x, y) \mapsto (x + \widetilde{P}, y + \widetilde{Q})$, for $(\widetilde{P}, \widetilde{Q}) \in \widetilde{E}(k) \times \widetilde{J}(k)$.*

We note the similarities between this step and Step 1: the bottom two rows of (5.2.2) in some sense resemble (5.1.1). In both cases, there are cohomological obstructions in $\mathrm{H}^i(k, \mu_2)$ for $i = 1, 2$ on the two factors. Because the sum is trivial, the obstruction vanishes on the product $C \times D$, which yields the existence of some object, i.e. either a rational point in $A(k)$ or an invertible sheaf on $C \times D$.

Although for our purposes it would be sufficient to treat the case where $\beta = 1$ in k^*/k^{*2} is trivial, the added generality of a non-vanishing β can be potentially helpful, since there can be non-trivial relationships between the cohomology classes of α and β, cf. Remark 6.3.29. The explicit example we construct in Section 5.4 does admittedly have $\beta = 1$.

5.2.2 A family of sign involutions

We now construct a *family of sign involutions* on Y, considered as *family of para-elliptic curves* over C. Throughout we treat C as base scheme. Although we developed the theory of para-abelian varieties and the associated abelian variety in Section 2.1 over a field, it is treated in §4 of [LS23] over general base schemes. Note that op. cit. is set in the context of algebraic spaces, which is convenient for the representability of Picard schemes and automorphism group schemes. In our context, we stay in the realm of schemes, since separated *group algebraic spaces* (i.e. group objects in the category of algebraic spaces) over a noetherian base of dimension 1 are schematic by Thm. 4.B. of [Ana73].

The para-abelian surface Y is elliptic in the sense of Section 2.3 because of the morphism $Y \to C$. On the level of associated abelian varieties, there is hence a short exact sequence

$$0 \longrightarrow \widetilde{J} \longrightarrow A \longrightarrow E \longrightarrow 0,$$

see (2.3.1). Therefore $\widetilde{J}$ acts on Y through its inclusion in A. The action is by translation and preserves the fibres of the morphism $Y \to C$. Let $\widetilde{J}_C$ be the base-change $\widetilde{J} \times C$, considered as scheme over C. Then $\widetilde{J}_C$ acts on $Y \to C$, also considered as scheme over C. This action gives $Y \to C$ the structure of a $\widetilde{J}_C$-torsor, hence defines a cohomology class $[Y/C] \in \mathrm{H}^1(C, \widetilde{J}_C)$. We generalise the notion of a sign involution to the setting where C is a base scheme.

Definition 5.2.5. An involution of Y over C is a *family of sign involutions* if it restricts to a sign involution in every fibre.

We first show that we may consider only the generic fibres without loss of generalisation. Let $K = \kappa(C)$ be the function field and let $\eta = \mathrm{Spec}(K)$ denote the generic point of C. Let Y_η denote the generic fibre, which is a para-elliptic curve over K. It is a torsor under $\widetilde{J}_K = \widetilde{J} \times \mathrm{Spec}(K)$.

Proposition 5.2.6. *A sign involution on the generic fibre Y_η extends to a family of sign involutions on Y over C.*

Proof. Let k'/k be a finite separable field extension such that Y and hence C have k'-valued points. They admit the structure of abelian varieties over k'. From Prop. 8 on p. 15 of [BLR90] it follows that the base change $Y' \to C'$ is the Néron model of its generic fibre. Furthermore Thm. 1 on p. 176 of op. cit. asserts that the Néron model of $Y_\eta \to \mathrm{Spec}(K)$ exists. Let $Z \to C$ denote the Néron model, then the natural map $Y \to Z$ is an isomorphism after base change to k' by loc. cit. □

Since the generic fibre Y_η is a para-elliptic curve over the function field K, the existence of sign involutions is governed by whether the cohomology class of Y_η in the Weil–Châtlet group $\mathrm{H}^1(K, \widetilde{J}_K)$ is two-torsion; see Corollary 2.2.13.

Remark 5.2.7. The weaker fact that the existence of a sign involution on the generic fibre implies the existence of a family of sign involutions may be shown using a cohomological argument. Suppose that the period of Y_η is 2. Note that the restriction map $\mathrm{H}^1(C, \widetilde{J}_C) \to \mathrm{H}^1(K, \widetilde{J}_K)$ is injective, because a rational point on the generic fibre of a $\widetilde{J}_C$-torsor Z over C induces a rational map $C \dashrightarrow Z$ that extends to a morphism by the valuative criterion of properness. Then the cohomology class $[Y/C] \in \mathrm{H}^1(C, \widetilde{J}_C)$ is two-torsion. The equality $[Y/C] = -[Y/C]$ determines a scheme theoretic isomorphism $\psi\colon Y \xrightarrow{\sim} Y$ such $\psi(x+y) = \psi(x) - y$, for $x \in Y(S)$ and $y \in \widetilde{J}(S)$, and S a scheme over C. One can then verify fppf-locally that ψ indeed determines a family of sign involutions.

We control the order of $[Y_\eta]$ in $\mathrm{H}^1(K, \widetilde{J}_K)$ using the μ_2-torsor $Y \to C \times D$, or its generic fibre $Y_\eta \to D_K$. It turns out that, although the cover $\widetilde{D} \to D$ may be obstructed to exist by the cohomology class α, the para-elliptic curve Y_η plays the role of $\widetilde{D}_K$ over the function field K, in the following sense.

Lemma 5.2.8. *Under the natural map $\widetilde{J}_K \to J_K$, the cohomology class $[Y_\eta]$ in $\mathrm{H}^1(K, \widetilde{J}_K)$ maps to the cohomology class $[D_K]$ in $\mathrm{H}^1(K, J_K)$.*

Proof. Note that the μ_2-action of the torsor $Y \to C \times D$ coincides with the translation action of $\mu_2 \subset \widetilde{J} \subset A$. Since $Y/\mu_2 \cong C \times D$, it follows on the level of generic fibres of the canonical morphisms to C that $D_K \cong Y_\eta/\mu_{2,K} = Y_\eta \wedge^{\mu_{2,K}} \widetilde{J}_K$ as J_K-torsors. □

The existence of such a cohomology class is explained by the following fact, which can also be seen as a consequence, though we provide a separate proof.

Proposition 5.2.9. *The Brauer class α vanishes in* $\mathrm{Br}(K)$.

Proof. The Brauer class α as element of $\mathrm{Br}(k)$ is assumed to obstruct the existence of an invertible sheaf on C; see assumption (I). Consider the five term exact sequence (1.3.5) associated to the Leray–Serre spectral sequence for C with coefficients in $\mathbb{G}_m$:

$$0 \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}_{C/k}(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(C).$$

Exactness implies that α lies in the kernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$. Since $\mathrm{Br}(C) \subset \mathrm{Br}(K)$, it follows indeed that α vanishes in the Brauer group of the function field of C. □

The vanishing of α in $\mathrm{Br}(K)$ means that the classes $L_C \otimes K$ and $L_D \otimes K$ define invertible sheaves of period 2 on C_K and D_K respectively. Therefore the their corresponding μ_2-covers can also be defined, uniquely up to an element of K^*/K^{*2}. We rephrase Lemma 5.2.8 in a different way, which is equivalent in light of the lifting property: for any choice of μ_2-torsor $\widetilde{D} \to D$, the curve $\widetilde{D}$ is para-elliptic and can be equipped with the structure of a $\widetilde{J}$-torsor, whose cohomology class maps to $[D_K] \in \mathrm{H}^1(K, J_K)$; see Theorems 1.3.4 and 1.3.5. The following formulation allows us to give a more concrete proof.

Lemma 5.2.10. *The $\mu_{2,K}$-torsor $Y_\eta \to D_K$ corresponds to the invertible sheaf $L_D \otimes K$.*

Proof. Consider the following Cartesian square:

$$\begin{tikzcd}
Y_\eta \arrow[r, "\pi_\eta"] \arrow[d, "i_Y"] & D_K \arrow[d, "i_D"] \\
Y \arrow[r, "\pi"] & C \times D
\end{tikzcd}$$

Note that $\pi_* \mathcal{O}_Y = \mathcal{O}_{C\times D} \oplus \mathscr{L}$, since $\pi \colon Y \to C \times D$ is a μ_2-torsor attached to $\mathscr{L}$. Restricting to the generic fibres, we also find that

$$\pi_{\eta,*}\mathcal{O}_{Y_\eta} = \pi_{\eta,*} i_Y^* \mathcal{O}_Y \cong i_D^* \pi_* \mathcal{O}_Y = i_D^*(\mathcal{O}_{C\times D} \oplus \mathscr{L}) = \mathcal{O}_{D_K} \oplus i_D^* \mathscr{L}.$$

Here the isomorphism follows from Prop. III.9.3 of [Har13], since π is proper and the open immersion i_D is flat. It hence suffices to show that there is an isomorphism $i_D^*\mathscr{L} \cong L_D \otimes K$. To this end, we may without loss of generality base-change the ground field and suppose that k is algebraically closed. Then the obstruction α vanishes, so $\mathscr{L}$ is isomorphic to a tensor product $\mathscr{L} = \mathrm{pr}_C^* L_C \otimes \mathrm{pr}_D^* L_D$, where L_C and L_D are invertible sheaves on C and D, respectively. Since

$$\begin{tikzcd}
D_K \arrow[r, "p_{D_K}"] \arrow[d, "i_D"] & \mathrm{Spec}(K) \arrow[d, "i_\eta"] \\
C \times D \arrow[r, "\mathrm{pr}_C"] & C
\end{tikzcd}$$

is commutative, it follows that

$$i_D^*\mathscr{L} = i_D^* \mathrm{pr}_C^* L_C \oplus i_D^* \mathrm{pr}_D^* L_D = p_{D_K}^* i_\eta^* L_C \oplus (L_D \otimes K) = L_D \otimes K. \qquad \square$$

Instead of choosing a single sign involution on $Y_\eta = \widetilde{D}_K$ over K, for technical reasons we need to pick a sign involution on $\widetilde{D}_{k'}$ for each field extension k'/k where α vanishes in $\mathrm{Br}(k')$, but *in a consistent manner.* For each such k'/k, pick a μ_2-torsor $\widetilde{D}_{k'} \to D'$ corresponding to $Q \in J[2](k')$. In some sense, $\widetilde{D}_{k'}$ should be thought of as the base change of the scheme '$\widetilde{D}$', even though the latter is obstructed from existing as a torsor. The lifting property then equips it with the structure of a $\widetilde{J}'$-torsor, extending the action of μ_2; see Theorem 1.3.5.

Proposition 5.2.11. *Let k'/k be a field extension such that α vanishes in $\mathrm{Br}(k')$. The cohomology class $[\widetilde{D}_{k'}] \in \mathrm{H}^1(k', \widetilde{J}')$ maps to $[D] \in \mathrm{H}^1(k', J')$ under the natural map induced by $\widetilde{J}' \to J$. Therefore, assumption (II.2) implies that $2[\widetilde{D}_{k'}] = 0$ in $\mathrm{H}^1(k, \widetilde{J}')$, so that it admits a sign-involution.*

Proof. The first statement follows from the lifting property Theorem 1.3.4. The composition of the dual isogenies $\widetilde{J} \to J \to \widetilde{J}$ is multiplication by 2. Therefore $2[\widetilde{D}_{k'}]$ is the image of $[D']$ under the natural map $\mathrm{H}^1(k', J') \to \mathrm{H}^1(k', \widetilde{J}')$. This image is the cohomology class of the quotient $D'/(\mathbb{Z}/2\mathbb{Z})$, which has a rational point by (II.2). The existence of a sign involution then follows from Corollary 2.2.13. ☐

To construct the sign involutions in a consistent manner, we *fix* an isomorphism $D/(\mathbb{Z}/2\mathbb{Z}) \cong \widetilde{J}$ over k by choice of rational point. Then, for any k'/k such that α vanishes in $\mathrm{Br}(k')$, the composition $\widetilde{D}_{k'} \to \widetilde{D}' \to \widetilde{J}$ is a two-covering, which induces a sign involution on $\widetilde{D}_{k'}$ through (2.2.5) of Section 2.2. If k''/k' is a further field extension, then the sign involution on $\widetilde{D}_{k''}$ constructed in this manner is the base-change of the sign involution on $\widetilde{D}_{k'}$.

Theorem 5.2.12. *There exists a family of sign involutions σ_{sgn} on Y over C may be chosen satisfying the following property. Let k'/k be a field extension such that α vanishes in $\mathrm{Br}(k')$ and pick curves $\widetilde{C}'$ and $\widetilde{D}'$ such that $Y' = (\widetilde{C}' \times \widetilde{D}')/\mu_2$. Then there is a sign involution ψ on $\widetilde{D}'$ such that the following diagram is commutative.*

$$\begin{tikzcd}
\widetilde{C}' \times \widetilde{D}' \arrow[r, "\mathrm{id}\times\psi"] \arrow[d] & \widetilde{C}' \times \widetilde{D}' \arrow[d] \\
Y' \arrow[r, "\sigma'_{\mathrm{sign}}"] & Y'
\end{tikzcd}$$

Proof. Consider the sign involution constructed above on the generic fibre $Y_\eta = \widetilde{D}_K$. It extends to a family of sign involutions σ_{sgn} on Y by Proposition 5.2.6.

Let k'/k be a field extension such that α vanishes in $\mathrm{Br}(k')$, so that the factors $\widetilde{C}_{k'}$ and $\widetilde{D}_{k'}$ exist over k'.

We consider Y' as a scheme over C', as before. The base-change along $\widetilde{C}_{k'} \to C'$ is the cover $\widetilde{C}_{k'} \times_{C'} Y' = \widetilde{C}_{k'} \times \widetilde{D}_{k'}$, hence the base-change of σ'_{sgn} determines a morphism $\Psi \colon \widetilde{C}_{k'} \times \widetilde{D}_{k'} \to \widetilde{C}_{k'} \times \widetilde{D}_{k'}$ over $\widetilde{C}_{k'}$ such that the required diagram is commutative. Restricting Ψ to the generic fibre determines a sign involution on $\widetilde{D}_{\widetilde{K}'}$ over the function field $\widetilde{K}' = \kappa(\widetilde{C}_{k'})$. By construction, it is the base change of a sign involution ψ on $\widetilde{D}_{k'}$ over k'. ☐

Remark 5.2.13. The choice of isomorphism $D/(\mathbb{Z}/2\mathbb{Z}) \cong \widetilde{J}$ in sense determines a sign involution on '$\widetilde{D}$', except that this scheme is obstructed from existing by the non-vanishing of α. Instead, fixing the isomorphism determines consistent sign involutions on each $\widetilde{D}_{k'}$.

The choice of *consistent* sign involutions is crucial in above proof to assert that Ψ is of the form $\mathrm{id} \times \psi$. It is in general not true that a family of sign involutions on a product $\widetilde{C} \times \widetilde{D}$ over $\widetilde{C}$ is of the form $\mathrm{id} \times \psi$ for a sign involution on $\widetilde{D}$: consider for example the case where $\widetilde{C} = \widetilde{D} = E$ are elliptic curves, then the involution $(x, y) \mapsto (x, x - y)$ is not of the required form. Such an involution would cause the proof of Proposition 5.2.1 to break, as then $(x, y) \mapsto (x + P, x - y + Q)$ would no longer be an involution.

This problem disappears if we assume that C and D are not geometrically isogenous. This can be seen using the *scheme of sign involutions* of [BDS24], cf. the discussion surrounding Theorem 2.2.14. Let E be an elliptic curve and let $K = \kappa(E)$ be its function field. Restricting above problematic involution to the generic fibre E_K, it corresponds to a K-point on the scheme of sign involutions $\mathrm{Inv}^{\mathrm{sgn}}_{E_K/K} = \mathrm{Inv}^{\mathrm{sgn}}_{E/k} \otimes k$, namely the base-change to K of the inclusion of the generic point of E. If C and D are geometrically not isogenous, then a $\kappa(\widetilde{C})'$-point on $\mathrm{Inv}^{\mathrm{sgn}}_{\widetilde{D}_{k'}/k'} \otimes \kappa(\widetilde{C})'$ comes from a closed point on the scheme of sign involutions.

5.3 Constructing the bielliptic surface

We equip Y with the $\mathbb{Z}/2\mathbb{Z}$-action determined by the involution σ.

Proposition 5.3.1. *The quotient $X = Y/(\mathbb{Z}/2\mathbb{Z})$ is a bielliptic surface of type (a2), and $Y \to X$ is a canonical cover.*

Proof. We base-change to an algebraic closure. The μ_2-action on $\widetilde{C} \times \widetilde{D}$ determined by the covering $\widetilde{C} \times \widetilde{D} \to Y$ commutes with the action of $\mathbb{Z}/2\mathbb{Z}$ determined by σ. It follows that $X = (\widetilde{C} \times \widetilde{D})/(\mu_2 \times \mathbb{Z}/2\mathbb{Z})$. The action of $\mathbb{Z}/2\mathbb{Z}$ is given by (5.2.1), and the action of μ_2 is by translation. We see that the action of $\mu_2 \times \mathbb{Z}/2\mathbb{Z}$ coincides with type (a2) of the BdF-classification Theorem 3.4.4, so indeed X is a bielliptic surface, and $Y \to X$ is a choice of canonical cover. □

There are compatible actions on the factors C and D. Equip C with the $\mathbb{Z}/2\mathbb{Z}$-action given by translation by $P \in E[2](k)$, and equip D with the $\mathbb{Z}/2\mathbb{Z}$-action given by translation by $Q \in J[2](k)$, composed with the sign involution. Note that the maps $Y \to C$ and $Y \to D$ are $\mathbb{Z}/2\mathbb{Z}$-equivariant. Let $P = C/(\mathbb{Z}/2\mathbb{Z})$ and $B = D/(\mathbb{Z}/2\mathbb{Z})$. Note that P is a torsor under $E/(\mathbb{Z}/2\mathbb{Z}) = \widetilde{E}$.

Proposition 5.3.2. *The induced maps $X \to P$ and $X \to B$ are the Albanese fibration and the other fibration, respectively.*

Proof. This is clear after base-change to an algebraic closure. Indeed, the involutions $x \mapsto x + \widetilde{P}$ and $y \mapsto \psi + \widetilde{Q}$ on the factors $\widetilde{C}$ and $\widetilde{D}$ induce above involutions on the quotients P and B, respectively. And from the structure theorem Theorem 3.4.1 it follows that the two fibrations are given by the projections

$$\begin{tikzcd}
 & \widetilde{C}/(\mu_2 \times \mathbb{Z}/2\mathbb{Z}) = C/(\mathbb{Z}/2\mathbb{Z}) = P \\
\dfrac{\widetilde{C} \times \widetilde{D}}{\mu_2 \times \mathbb{Z}/2\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
 & \widetilde{D}/(\mu_2 \times \mathbb{Z}/2\mathbb{Z}) = D/(\mathbb{Z}/2\mathbb{Z}) = B
\end{tikzcd}$$

This finishes the proof. □

Given our construction of the bielliptic surface X, it should not be surprising that it admits no BdF-cover over the canonical cover Y. In order to show this rigorously, we return to the fundamental diagram of the lifting property of Section 4.2. In this context, we identify $\widetilde{E} = F = \mathrm{Pic}^{\alpha}_{X/k}$, so the diagram (4.2.4) becomes

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & & \\
\cdots \arrow[r] & k^*/k^{*2} \arrow[r] \arrow[d] & \mathrm{H}^1(k, \widetilde{E}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^1(k, E) \arrow[r] \arrow[d] & \mathrm{Br}(k)[2] \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(k, \widetilde{E}[2]) \arrow[r] \arrow[d] & \mathrm{H}^1(k, \widetilde{E}) \arrow[r, "2"] \arrow[d] \arrow[ur] & \mathrm{H}^1(k, \widetilde{E}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^2(k, \widetilde{E}[2]) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[uurrr, rounded corners, "\delta"] & \mathrm{H}^1(k, E) \arrow[r] \arrow[uur, "\mathrm{id}"'] & \mathrm{H}^1(k, \widetilde{E}) \arrow[r] & \mathrm{H}^2(k, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d] & \cdots \\
 & & & & \vdots &
\end{tikzcd} \tag{5.3.1}$$

Theorem 5.3.3. *The bielliptic surface X is obstructed to have a BdF-cover over Y by α.*

Proof. The choice of canonical cover $Y \to X$ is equivalent to the choice of $[C] \in \mathrm{H}^1(k, E)$ mapping to $[P] \in \mathrm{H}^1(k, \widetilde{E})$. By Lemma 1.3.3, its image in $\mathrm{Br}(k)[2]$ equals $\partial_C(P) \in \mathrm{Br}(k)[2]$, which is α. $\square$

Of course, it may still be possible that X admits a BdF-cover over a different choice of canonical cover. The obstruction for a different choice of canonical cover differs by the image of an element of $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$ under the boundary map δ, so that X nevertheless admits a BdF-cover if and only if α is contained in the image of δ. As such, we invoke the last unused assumption.

Corollary 5.3.4. *If $\alpha \neq 0$ then the bielliptic surface X does not admit a Bagnera–de Franchis cover.*

Proof. Assumption (III) states that E has full two-torsion. This implies that the short exact sequence $0 \to \mu_2 \to \widetilde{E}[2] \to \mathbb{Z}/2\mathbb{Z} \to 0$ splits, so that the boundary map δ is 0. By injectivity of $\mathrm{Br}(k)[2] \to \mathrm{H}^2(k, \widetilde{E}[2])$ the image of $[P] \in \mathrm{H}^2(k, E[2])$ is also nonzero. $\square$

5.4 Constructing the building blocks

The construction of the bielliptic surface X depends on elliptic curves E and J, with torsors C and D and rational points P and Q, respectively, satisfying suitable properties. As a fourth or zeroth step, we construct suitable examples of these building blocks, under the assumption that the characteristic is not 2. This construction relies on a detailed study of para-elliptic curves and certain boundary maps if $p \neq 2$, which we postpone to Chapter 6. We briefly describe the main results that we invoke from the upcoming chapter.

- Corollary 6.1.22: an explicit description of a para-elliptic curve of period two;
- Proposition 6.2.1: an explicit formula of the boundary maps δ_E and δ_J;
- Corollary 6.3.19: the calculation of the boundary maps ∂_C and ∂_D on two-torsion points.

Proof of Theorem 5.0.1. Suppose that $\mathrm{Br}(k)[2] \neq 0$. By the Merkurjev–Suslin theorem [Mer81; Wad86] the two-torsion of the Brauer group is generated by cohomology classes of quaternion algebra's. Pick a non-trivial quaternion algebra $\alpha = (\lambda, \mu)_k$. Without loss of generality suppose that $\lambda \neq -1$, by multiplying λ by a nonzero square, if necessary.

Let $a = \lambda + \lambda^{-1}$ and define the elliptic curve E by the affine equation

$$\begin{aligned} E\colon y^2 &= x^3 + ax^2 + x \\ &= x(x+\lambda)(x+\lambda^{-1}). \end{aligned} \tag{5.4.1}$$

The requirement that $\lambda \neq -1$ is necessary for E to be an elliptic curve. We enumerate the roots of the right hand side cubic polynomial as $\alpha_1 = 1$, $\alpha_2 = -\lambda$, $\alpha_3 = -\lambda^{-1}$.

Consider the rational two-torsion point $P = (0, 0)$. We define the E-torsor $C = E \wedge^{\mu_2} k(\sqrt{\mu})$, where μ_2 acts on E by translation by $(0, \lambda)$. In other words, the cohomology class $[C]$ in $\mathrm{H}^1(k, E)[2]$ is the image of $\mu \in k^*/k^{*2}$ under the natural map $\mathrm{H}^1(k, \mu_2) \to \mathrm{H}^1(k, E[2]) \to \mathrm{H}^1(k, E)[2]$, where $\mu_2 \cong \mathbb{Z}/2\mathbb{Z}$ is considered as a subgroup scheme of $E[2]$ through the 2-torsion point $(0, \lambda)$. In notation of Section 6.1, the cohomology class of $[C]$ equals $v(\mu, 1, \mu)$. By Corollary 6.1.22 below, we can describe C explicitly through the system of equations

$$\begin{cases} -\lambda t_0^2 - \mu t_1^2 + \ \ t_2^2 = 0; \\ -\lambda^{-1} t_0^2 - \mu t_1^2 + \mu t_3^2 = 0, \end{cases}$$

in $\mathbb{P}^4$. Alternatively, since C is the twist of E by a quadratic Galois extension along translation by a two-torsion point, we are in the situation of Example 6.1.24, at least after translating P to the origin $(0, 0)$ by applying the substitution $x \mapsto x - \lambda$. Then the elliptic curve E has affine Weierstraß equation given by $y^2 = x^3 - (2\lambda + \lambda^{-1})x^2 + (\lambda^2 + 1)x$, so that an explicit affine equation of C is given by

$$\mu W^2 = \lambda^2 Z^4 - 2(2\lambda + \lambda^{-1})\mu Z^2 + \mu^2.$$

There is a considerable amount of choice for the elliptic curve J. Let J be an elliptic curve of the form

$$\begin{aligned} J\colon y^2 &= x^3 + \mu(1+b^2)x^2 + \mu^2 b^2 x \\ &= x(x+\mu)(x+\mu b^2), \end{aligned}$$

where $b \neq 0$, 1 is an otherwise arbitrary element of k. By choosing b sufficiently general, it should be possible for the elliptic curves J and E to not be geometrically isogenous. Consider the rational two-torsion point $Q = (0, 0)$. Now define the J-torsor D as $D = J \wedge^{\mathbb{Z}/2\mathbb{Z}} k(\sqrt{\lambda})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on J by translation by the two-torsion point $(0, -\mu)$. Similarly to C, it can be described by the system of equations

$$\begin{cases} -\lambda t_0^2 - \mu t_1^2 + \ \ t_2^2 = 0; \\ -\lambda^{-1} t_0^2 - \mu t_1^2 + \mu t_3^2 = 0, \end{cases}$$

in $\mathbb{P}^4$, or by the affine equation

$$\lambda W^2 = \mu^2 b^4 Z^4 - 2\lambda\mu(b^2 - 2)Z^2 + \lambda^2.$$

We verify that this datum satisfies the assumptions (I), (II.1), (II.2), and (III), in this order. For the computation of the Brauer classes associated to P and Q we invoke Corollary 6.3.19 below: it follows directly that

$$\begin{aligned} \partial_C((0,0)) &= (\lambda, \mu)_k + (\lambda^{-1}, 1)_k = (\lambda, \mu)_k; \\ \partial_D((0,0)) &= (\mu, \lambda)_k + (b^2\mu, 1)_k = (\mu, \lambda)_k = (\lambda, \mu)_k. \end{aligned}$$

For the computations of the images of P and Q in k^*/k^{*2} under the boundary maps δ_E and δ_J, respectively, we invoke Proposition 6.2.1 to see that

$$\delta_E((0,0)) = 1 \quad \text{and} \quad \delta_J((0,0)) = b^2\mu^2,$$

which are both squares and hence equal $\beta = 1$ in k^*/k^{*2}.

We now verify assumption (II.2). By construction, the cohomology class $[D]$ in $\mathrm{H}^1(k, J)$ is the image of the cohomology class $[\operatorname{Spec} k(\sqrt{\mu})/\operatorname{Spec} k]$ in $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$ under the natural map determined by the inclusion of the subgroup scheme $\mathbb{Z}/2\mathbb{Z} \subset J$ through $Q \in J[2](k)$. In fact, the short exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to J \to \widetilde{J} \to 0$ induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{H}^1(k, J) \longrightarrow \mathrm{H}^1(k, \widetilde{J}) \longrightarrow \cdots,$$

so that $[D]$ is contained in the kernel. This shows that $D \wedge^J \widetilde{J} = D/(\mathbb{Z}/2\mathbb{Z}) \cong \widetilde{J}$ has a rational point.

Finally consider assumption (III). Following Section 6.2, equations for $\widetilde{E}$ are well-known, see (6.2.1). In our case $\widetilde{E}$ is the elliptic curve given by

$$\begin{aligned} \widetilde{E} \colon y^2 &= x^3 - 2ax^2 + (a^2 - 4)x \\ &= x(x - a - 2)(x - a + 2), \end{aligned}$$

which has full two-torsion. □

Remark 5.4.1. We verify concretely that in the above construction, the boundary map δ of (5.3.1) vanishes, as is key in the proof of Corollary 5.3.4. Enumerating the roots of $f = x^3 + ax^2 + x$ by $\alpha_1 = 0$, $\alpha_2 = -\lambda$, $\alpha_3 = -\lambda^{-1}$, then in the notation of Corollary 6.1.22 the cohomology class $[C] \in \mathrm{H}^1(k, E)$ equals $v(\mu, 1, \mu)$ as in above proof. Let $[C_\nu]$ denote its translate by the image of $[\nu] \in k^*/k^{*2} = \mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$ in $\mathrm{H}^1(k, E)$, where $\mathbb{Z}/2\mathbb{Z} \to E$ is induced by the two-torsion point $(0, 0) \in E(k)$. Then $[C_\nu] = v(\mu, \nu, \nu\mu)$, so we calculate its obstruction for $(0, 0) \in E[2](k)$ through Corollary 6.3.19: it equals

$$(\lambda, \nu\mu)_k + (\lambda^{-1}, \nu)_k = (\lambda, \nu\mu)_k + (\lambda, \nu)_k = (\lambda, \mu)_k = \alpha.$$

Note that this does not depend on the choice of λ, corresponding to the fact that $\delta = 0$; see Proposition 4.2.11.

Remark 5.4.2. The theory of two-descent of Chapter 6 breaks down in characteristic 2. A technical obstacle is that the two-torsion group scheme of an elliptic curve E is no longer étale, so that $E[2]$ cannot be seen as a restriction of scalars of μ_2. Nevertheless, it could very well be possible to produce examples of para-elliptic curves with a given quaternion algebra as obstruction. (For a definition and discussion of quaternion algebras in characteristic two, see §6 of [Voi21].) If so, then it seems likely that Theorem 5.0.1 extends to characteristic 2 as well.

Chapter 6

The arithmetic of para-elliptic curves

In the previous chapter we constructed a bielliptic surface of type (a2) over a field with $p \neq 2$ that does not admit a Bagnera–de Franchis cover. This construction relies on para-elliptic curves C and D as 'building blocks', with certain assumptions regarding their cohomological obstructions. We postponed the justification of the construction. In this Chapter we develop the theory of the arithmetic of para-elliptic curves from an algebro-geometric perspective sufficiently to obtain the required results, as alluded to at the start of Section 5.4. All results in this chapter are known to the expert in number theory.

In Section 6.1 we develop the theory of *two-descent* if $p \neq 2$ and use it to give explicit descriptions of para-elliptic curves of period 2. We discuss the relation to the theory of *descent by two-isogeny* and describe the boundary map δ_E in Section 6.2. Finally Section 6.3 is concerned with the boundary map ∂_C.

6.1 Two-descent on elliptic curves

In this section we summarise the arithmetic theory of *descent* on elliptic curves from the perspective of algebraic geometry. We warn the reader that the only similarity with Grothendiecks theory of descent (including Galois descent, étale descent and fppf-descent) is the similar name. Historically, it arose from attempts to compute the group of rational points of elliptic curves over number fields. Geometrically, it concerns the study of torsors under elliptic curves of a given period $n > 1$. We first motivate the theory from the perspective of number theory, then treat the theory of two-descent in more detail from an algebro-geometric perspective.

Let E be an elliptic curve over a *number field* k. The celebrated Mordell–Weil theorem states that the group of rational points $E(k)$ is finitely generated. It is therefore isomorphic as an abstract group to $\mathbb{Z}^r \oplus E(k)_{\text{tors}}$; see Thm. VIII.6.7 of [Sil09]. The integer $r \geq 0$ is called the *rank* of the elliptic curve E over k. It is an important invariant in the arithmetic of elliptic curves, arising, for example, in the famous Birch–Swinnterton-Dyer conjecture. Determining the rank is in general a difficult computational problem, which is in some sense equivalent to solving the Diophantine equation set out by the Weierstraß equation of E. Indeed, the torsion part can be computed effectively using the Nagell–Lutz theorem; see e.g. Cor. VIII.7.2 of op. cit. The theory of *n-descent* provides a strategy for computing the rank.

The proof of the Mordell–Weil theorem depends crucially on the following fact: for any $n > 1$, the quotient $E(k)/nE(k)$ is finitely generated; see §VIII.3.2 of op. cit. This fact is often referred to as the *weak Mordell–Weil theorem.* The proof of the Mordell–Weil theorem is constructive given generators of $E(k)/nE(k)$: knowledge of the latter is sufficient to compute generators of $E(k)$ algorithmically. We study the quotient with cohomological methods. In analogy to Kummer theory, consider the short exact sequence

$$0 \longrightarrow E[n] \longrightarrow E \xrightarrow{n} E \longrightarrow 0. \tag{6.1.1}$$

Its long exact sequence yields the short exact sequence

$$0 \longrightarrow E(k)/nE(k) \longrightarrow \mathrm{H}^1(k, E[n]) \longrightarrow \mathrm{H}^1(k, E)[n] \longrightarrow 0. \tag{6.1.2}$$

We can therefore identify the quotient $E(k)/nE(k)$ with the kernel of the natural map $\mathrm{H}^1(k, E[n]) \to \mathrm{H}^1(k, E)$. This is the starting point of the theory of *n-descent*, but the description is too abstract for many computational purposes: we are instead interested in a more concrete description of this abstract group. The theory of n-descent attempts to find an explicit isomorphism between $\mathrm{H}^1(k, E[n])$ and a sufficiently concrete abstract group, and to describe the image of any element in the Weil–Châtelet group $\mathrm{H}^1(k, E)$.

We warm the reader that theory of n-descent, though related, is ultimately different from the theory called called *descent by n-isogeny*, despite the similar sounding names. We briefly describe the latter: let $f\colon E_1 \to E_2$ be an isogeny of elliptic curves of degree n with kernel K. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow E_1 \longrightarrow E_2 \longrightarrow 0,$$

whose long exact sequence yields

$$0 \longrightarrow E_2/f(E_1(k)) \longrightarrow \mathrm{H}^1(k, K) \longrightarrow \mathrm{H}^1(k, E_1)[f_*] \longrightarrow 0.$$

This short exact sequence refines the short exact sequence (6.1.2). We briefly summarise the theory of descent by two-isogeny in Section 6.2.

So far, the short exact sequences can be constructed over any ground field. In a number theoretic context, the ground field k is often assumed to be a number field. Then one usually refines the short exact sequence (6.1.2) by taking the *Hasse principle* into account. It wrongly states that a para-elliptic curve has a rational point if and only if it has a point over all completions k_v. Many counterexamples are known, the most famous of which is due to Selmer and is described by the cubic curve $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ over $\mathbb{Q}$; see [Sel51]. For other examples, see e.g. §18 of [Cas91] or pp. 331–334 of [SC21]. The short exact sequence (6.1.2) can be refined by considering only those cohomology classes that violate the Hasse principle. To that end, we define the *Tate–Shafarevich group* and the *n-Selmer group* as

$$\begin{aligned} Ш(E/k) &\coloneqq \operatorname{Ker}\Big(\mathrm{H}^1(k, E) \longrightarrow \prod_v \mathrm{H}^1(k_v, E_v)\Big); \\ \mathrm{Sel}^{(n)}(E/k) &\coloneqq \operatorname{Ker}\Big(\mathrm{H}^1(k, E[n]) \longrightarrow \prod_v \mathrm{H}^1(k_v, E_v)\Big), \end{aligned}$$

respectively, where v ranges over all places of k and where $E_v = E \otimes k_v$. Note that an element of the Shafarevich group determines a para-elliptic curve that has a rational point over every completion, hence produces a counterexample to the Hasse principle. Then the short exact sequence (6.1.2) is often replaced by the short exact sequence

$$0 \longrightarrow E(k)/nE(k) \longrightarrow \mathrm{Sel}^{(n)}(E/k) \longrightarrow Ш(E/k)[n] \longrightarrow 0.$$

In the following sections we work in a more general setting, where the ground field may not be global, so we do not consider the Shafarevich and Selmer groups further.

From now on we treat the case of *two-descent* only, corresponding to setting $n = 2$. Descent by n-isogeny for $n > 2$ has been studied in the literature: for a non-exhaustive list, we refer to [GT22] for $n = 3$, [Fis01] for $n = 5$ and 7, and [DSS00; SS04] for arbitrary prime numbers. The theory is considerably more technical if $n > 2$.

We cover the theory of two-descent from an algebro-geometric perspective, where we work over a general base scheme on which 2 is invertible for as long as possible. From a certain point onwards, we work over fields of characteristic not 2 only, we restrict ourselves to a ground field. The theory of two-descent is well-known topic in number theory. Most number theoretic books with an interest in computing the group of rational points contain elementary expositions: see for example §X of [Sil09] or §8 of [Hus04]. We note that the theory of two-descent seems to generalise quite well to Jacobians of hyperelliptic curves, as for example in [Sch95; CV15]. The remainder of this section essentially follows and expand upon App. A of [Cre16].

The assumption that 2 is invertible seems to be necessary for our approach. It would be interesting to additionally generalise the theory to characteristic 2 as well, since it could potentially be used in constructing construct an ordinary bielliptic surface of type (a2) with a nonzero obstruction to admit a Bagnera–de Franchis cover, cf. Chapter 5.

Let S be a locally noetherian base scheme on which 2 is invertible. Let E be an elliptic curve over S. Let X be the relatively affine open subscheme of $E[2]$ by removing the identity section and let $i\colon X \to E[2]$ be the inclusion. Recall the Weil pairing, which on two-torsion points is a perfect alternating pairing

$$e_2\colon E[2] \times E[2] \longrightarrow \mu_2,$$

with $e_2(P,P) = 1$ for all $P \in E[2](T)$ and all schemes T; see e.g. §2.8 of [KM85] for the definition over a base scheme. For each scheme T, we define a map

$$\begin{aligned} E[2](T) \times \operatorname{Hom}(T,X) &\longrightarrow \mu_2(T); \\ (P,\varphi) &\longmapsto e_2(P, i \circ \varphi). \end{aligned}$$

This defines a morphism $E[2] \times X \to \mu_2$, so by the universal property of the restriction of scalars also a map $E[2] \to \operatorname{Res}_{X/S} \mu_{2,X}$.

Example 6.1.1 (Universal two-torsion point). Suppose $S = \operatorname{Spec}(k)$ and $E\colon y^2 = f(x)$ is given in a short Weierstraß form. Then the two-torsion subgroup scheme is $\operatorname{Spec}(k) \sqcup \operatorname{Spec}(A)$, where $A = k[x]/(f)$. Let θ denote the image of x in A. Then the A-valued two-torsion point $i \circ \varphi$ is given in coordinates by $(\theta, 0)$.

Example 6.1.2. Suppose that $P \in E[2](S)$ is nonzero, defining a subgroup scheme $\mathbb{Z}/2\mathbb{Z} \subset E[2]$. Let $Y = E[2] \setminus (\mathbb{Z}/2\mathbb{Z})$, so that $X = \operatorname{Spec}(S) \sqcup Y$ as a scheme. Then the natural map $E[2] \to \operatorname{Res}_{X/S} \mu_{2,X} = \mu_2 \times \operatorname{Res}_{Y/S} \mu_{2,Y}$ is induced by the pair of maps

$$\begin{aligned} E[2] &\longrightarrow \mu_2, & Q &\longmapsto e_2(Q,P); \quad \text{and} \\ E[2] \times Y &\longrightarrow \mu_2, & (Q,\phi) &\longmapsto e_2(Q, j \circ \phi), \end{aligned}$$

where $j\colon Y \to E[2]$ denotes the inclusion. In particular, the composition $\mathbb{Z}/2\mathbb{Z} \to E[2] \to \mu_2 \times \operatorname{Res}_{Y/S} \mu_{2,Y}$ is given by $P \mapsto (1, -1)$.

Example 6.1.3. If $E[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$, then X consists of the disjoint union of three copies of $\operatorname{Spec}(S)$, corresponding to the three non-identity points P_1, P_2, P_3 of $E[2]$. It follows that in this case the map $E[2] \to \operatorname{Res}_{X/S} \mu_{2,X} = \mu_2^3$ is given by $P_i \mapsto (e_2(P_i, P_j))_{j=1}^3$, where $e_2(P_i,P_j) = 1$ if and only if $i = j$ and that $e_2(P_i, P_j) = -1$ otherwise.

Recall that there is a *norm map* $\operatorname{Res}_{X/S} \mu_{2,X} \to \mu_2$, sometimes called a *trace map*; see [Stacks, Tag 03SH]. If $U \to S$ is fppf such that the base-change $X_U \to S$ is a disjoint union of copies of U, then the restrictions of scalars $\operatorname{Res}_{X_U/U} \mu_{2,X_U}$ is isomorphic to a product of copies of $\mu_{2,U}$ and the norm map is given by multiplication $\prod \mu_{2,U} \to \mu_{2,U}$. Taking this as defining property, the norm map may then be constructed by fppf-descent. Alternatively, the norm map is a consequence of p_* being left adjoing to p^* since $p\colon X \to S$ is finite étale; see loc. cit.

Proposition 6.1.4. *The sequence*

$$0 \longrightarrow E[2] \longrightarrow \operatorname{Res}_{X/S} \mu_{2,X} \longrightarrow \mu_2 \longrightarrow 1 \tag{6.1.3}$$

is short exact.

Proof. Exactness can be verified on the geometric points of S. Thus suppose without loss of generality that $S = \operatorname{Spec}(k)$ for an algebraically closed field k. We choose an enumeration P_1, P_2, P_3 of non-trivial elements of $E[2](k)$, which induces isomorphisms $X \cong \operatorname{Spec}(k^3)$ and $\operatorname{Res}_{X/S} \mu_{2,X} \cong \mu_2^3$. Since all group schemes in question are finite étale, we identify them with finite abstract groups.

Under these isomorphisms, the map $E[2] \to \operatorname{Res}_{X/S} \mu_{2,X} \cong \{\pm 1\}^3$ maps the point P_i to $(e_2(P_i,P_j))_{j=1}^3$. Since $e_2(P_i, P_j) = 1$ if and only if $i = j$, this map is injective and the image is the subgroup of elements having an odd number of positive entries. The norm map $\operatorname{Res}_{X/S} \mu_{2,X} \to \mu_2$ is the multiplication map $\{\pm 1\}^3 \to \{\pm 1\}$, which indeed has the desired kernel. □

Remark 6.1.5. Let $n > 2$ and let k be a field whose characteristic exponent p is coprime to n. Let $X = E[n] \setminus \{0\}$ and let $A = \Gamma(X, \mathcal{O}_X)$. Although one can define an injective map $E[n] \to \operatorname{Res}_{A/k} \mu_{n,A}$ through the Weil-paring in an entirely analogous way, it is not true that the cokernel is isomorphic to μ_n, as one can see by comparing orders. This already indicates that the theory of n-descent for $n > 2$ is of a more technical nature than the theory of two-descent.

Lemma 6.1.6. *The short exact sequence* (6.1.3) *induces a short exact sequence*

$$0 \longrightarrow \mathrm{H}^1(S, E[2]) \longrightarrow \mathrm{H}^1(X, \mu_{2,X}) \longrightarrow \mathrm{H}^1(S, \mu_2) \longrightarrow 0. \tag{6.1.4}$$

Proof. Since all group schemes in question are smooth, we may compute their cohomology in the étale topology. The long exact sequence associated to the short exact sequence (6.1.3) is

$$\cdots \longrightarrow \mu_2(X) \xrightarrow{\mathrm{Norm}_{X/S}} \mu_2(S) \longrightarrow \mathrm{H}^1(S, E[2]) \longrightarrow \mathrm{H}^1(S, \mathrm{Res}_{X/S}\, \mu_{2,X}) \xrightarrow{\mathrm{Norm}_{X/S,*}} \mathrm{H}^1(S, \mu_2) \longrightarrow \cdots$$

Since the compositions $\mathrm{H}^i(S, \mu_2) \to \mathrm{H}^i(S, \mathrm{Res}_{X/S}\, \mu_{2,X}) \to \mathrm{H}^i(S, \mu_2)$ is multiplication by 3, which is an isomorphism, it follows that the norm maps are surjective. This is refered to as the 'méthode de la trace'; see [Stacks, Tag 03SH]. We obtain the short exact sequence.

$$0 \longrightarrow \mathrm{H}^1(S, E[2]) \longrightarrow \mathrm{H}^1(S, \mathrm{Res}_{X/S}\, \mu_{2,X}) \xrightarrow{\mathrm{Norm}_{X/S,*}} \mathrm{H}^1(S, \mu_2) \longrightarrow 0.$$

We conclude by the natural isomorphism $\mathrm{H}^1(S, \mathrm{Res}_{X/S}\, \mu_{2,X}) = \mathrm{H}^1(X, \mu_{2,X})$. Indeed, if $p\colon X \to S$ is the structure morphism, then the Weil restriction $\mathrm{Res}_{X/S}$ equals the push-forward p_* on abelian sheaves in the étale topology. Note that p_* is an exact functor in the étale topology; see [Stacks, Tag 03QP]. It follows directly that $\mathrm{H}^1(S, p_*\mathscr{F}) = \mathrm{H}^1(X, \mathscr{F})$ for all sheaves $\mathscr{F}$ on X in the étale topology. □

The cohomology groups $\mathrm{H}^1(S, \mu_2)$ and $\mathrm{H}^1(X, \mu_{2,X})$ are studied through Kummer theory; see Section 1.4. Let $R = \Gamma(S, \mathcal{O}_S)$ and let $A = \Gamma(X, \mathcal{O}_X)$. The Kummer sequences of S and X are related by the norm map, in the sense that the following diagram is a morphism of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Res}_{X/S}\, \mu_{2,X} \arrow[r] \arrow[d, "\mathrm{Norm}_{X/S}"] & \mathrm{Res}_{X/S}\, \mathbb{G}_{m,X} \arrow[r, "2"] \arrow[d, "\mathrm{Norm}_{X/S}"] & \mathrm{Res}_{X/S}\, \mathbb{G}_{m,X} \arrow[r] \arrow[d, "\mathrm{Norm}_{X/S}"] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathbb{G}_m \arrow[r, "2"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

The long exact sequences are hence compatible through the norm maps. It follows that the following diagram is a morphism of short exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & A^*/A^{*2} \arrow[r] \arrow[d, "\mathrm{Norm}_{X/S}"] & \mathrm{H}^1(X, \mu_{2,X}) \arrow[r] \arrow[d, "\mathrm{Norm}_{X/S,*}"] & \mathrm{Pic}(X)[2] \arrow[r] \arrow[d, "\mathrm{Norm}_{X/S,*}"] & 0 \\
1 \arrow[r] & R^*/R^{*2} \arrow[r] & \mathrm{H}^1(S, \mu_2) \arrow[r] & \mathrm{Pic}(S)[2] \arrow[r] & 0
\end{tikzcd} \tag{6.1.5}$$

Proposition 6.1.7. *If* $\mathrm{Pic}(X)[2] = 0$ *then* $\mathrm{Pic}(S)[2] = 0$*. Furthermore the short exact sequence* (6.1.4) *is isomorphic to the short exact sequence*

$$0 \longrightarrow \mathrm{H}^1(S, E[2]) \longrightarrow A^*/A^{*2} \xrightarrow{\mathrm{Norm}_{A/R}} R^*/R^{*2} \longrightarrow 1. \tag{6.1.6}$$

Proof. Recall from the proof of Lemma 6.1.6 that the norm map $\mathrm{Norm}_{X/S,*}\colon \mathrm{H}^1(X, \mu_{2,X}) \to \mathrm{H}^1(S, \mu_2)$ is surjective by the méthode de la trace. The snake lemma applied to (6.1.5) hence implies that $\mathrm{Pic}(S)[2] = 0$. Therefore the natural maps $A^*/A^{*2} \to \mathrm{H}^1(X, \mu_{2,X})$ and $R^*/R^{*2} \to \mathrm{H}^1(S, \mu_2)$ are isomorphisms. □

Example 6.1.8. We continue in the setting of Example 6.1.2. Let $B = \Gamma(Y, \mathcal{O}_Y)$, so that $A \cong R \times B$. Since the composition $\mathbb{Z}/2\mathbb{Z} \to E[2] \to \mu_2 \times \mathrm{Res}_{Y/S}\, \mu_{2,Y}$ is given by $P \mapsto (1, -1)$, it follows that the composition in cohomology $R^*/R^{*2} = \mathrm{H}^1(S, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1(S, E[2]) \to A^*/A^{*2} \cong R^*/R^{*2} \times B^*/B^{*2}$ is given by $d \mapsto (1, d)$. This provides an explicit description of the image of all elements in the image of $\mathrm{H}^1(S, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1(S, E[2])$.

Remark 6.1.9. If $E[2](S) = (\mathbb{Z}/2\mathbb{Z})^2$ then there is an isomorphism $A \cong R^3$. The norm map $A^*/A^{*2} \to R^*/R^{*2}$ corresponds to the multiplication map $(R^*/R^{*2})^3 \to R^*/R^{*2}$.

Under the assumption that $\mathrm{Pic}(X)[2] = 0$, the short exact sequence (6.1.6) hence concretely describes the cohomology group $\mathrm{H}^1(S, E[2])$ in terms of the units of A and R. In order to take this assumption into account, from now on we suppose that $S = \mathrm{Spec}(k)$ is the spectrum of a field of characteristic not 2. Then $\mathrm{Pic}(A) = \mathrm{H}^1(k, \mathrm{Res}_{A/k}\, \mathbb{G}_{m,A})$ equals the Galois cohomology $\mathrm{H}^1(k, A^*)$, which vanishes by a version of Hilbert theorem 90; see Exc. 2, p. 152 of [Ser79].

Working over a base field, E now admits an affine Weierstraß equation of the form $E\colon y^2 = f(x)$ for some separable monic degree 3 polynomial $f \in k[x]$, so that $A = k[x]/(f)$ and $X = \mathrm{Spec}(A)$. We let θ

denote the image of x in A. We substitute the middle term in the fundamental short exact sequence (6.1.2). An explicit description of the inclusion $E(k)/2E(k) \to A^*/A^{*2}$ is computed in e.g. Thm. 1.1 of [Sch95], where a similar statement is shown in the context of hyperelliptic curves. We omit the computation of the composition $E(k)/2E(k) \to \mathrm{H}^1(k, E[2]) \to A^*/A^{*2}$; we instead refer to loc. cit. for a proof of the following result.

Theorem 6.1.10 (Two-descent)**.** *Above isomorphism of* $\mathrm{H}^1(k, E[2]) \cong \mathrm{Ker}(\mathrm{Norm}_{A/k})$ *induces an isomorphism of short exact sequences between* (6.1.2) *and*

$$0 \longrightarrow E(k)/2E(k) \xrightarrow{x-\theta} \mathrm{Ker}\left(\mathrm{Norm}_{A/k}\colon A^*/A^{*2} \to k^*/k^{*2}\right) \xrightarrow{v} \mathrm{H}^1(k, E)[2] \longrightarrow 0. \tag{6.1.7}$$

The map $x - \theta$ is given as follows. Outside of the rational two-torsion points it is induced by

$$E(k) \setminus E[2](k) \longrightarrow A^*; \quad (x, y) \longmapsto x - \theta.$$

For any root α of f, write $f(x) = (x - \alpha) \cdot g(x)$ for some separable quadratic polynomial $g \in k[x]$. Let $B = k[x]/(g)$, so that $A = k \times A$ by the Chinese remainder theorem. Denote the image of x in B by ϑ. The image of $(\alpha, 0)$ under the composition $E(k)/2E(k) \xrightarrow{x-\theta} A^*/A^{*2} \cong k^*/k^{*2} \times B^*/B^{*2}$ is represented by $(\mathrm{Norm}_{B/k}(\alpha - \vartheta), \alpha - \vartheta)$.

Remark 6.1.11. Let α_1, α_2, α_3 be the distinct roots of f in some algebraic closure of k. The norm of the element $x - \theta$ is the product

$$\mathrm{Norm}_{A/k}(x - \theta) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) = f(x) = y^2,$$

which is indeed a nonzero square if $(x, y) \notin E[2](k)$. Similarly, if $(x, y) \in E[2](k)$, then the norm of $x - \theta$ is 0. Another way to see this is under the isomorphism $A^*/A^{*2} \cong k^*/k^{*2} \times B^*/B^{*2}$, whence the element $x - \theta$ maps to $(0, x - \vartheta)$. The description of the map $x - \theta$ is different on the rational two-torsion of E for this reason. Instead replace the first component of $(0, x - \theta)$ by any element of k^* such that the norm is a square. Indeed, $\mathrm{Norm}_{B/k}(x - \vartheta)$ is a canonical choice satisfying this property.

Example 6.1.12. Suppose that f factors as a product of linear polynomials $f = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$, for certain α_1, α_2, α_3 in k. By evaluating θ at the three dinstinct roots of f, we obtain an isomorphism $\mathrm{Ker}(\mathrm{Norm}\colon A^*/A^{*2} \to k^*/k^{*2}) \cong \mathrm{Ker}(\mathrm{mult}\colon (k^*/k^{*2})^3 \to k^*/k^{*2})$. Under this isomorphism, the image of any non two-torsion $P = (x, y)$ in $E(k)$ under the $x - \theta$ map is the tuple $(x - \alpha_1, x - \alpha_2, x - \alpha_3)$. Suppose $P = (\alpha_i, 0)$ is a rational two-torsion point. Without loss of generality set $i = 1$. We determine its image in $(k^*/k^{*2})^3$ through Theorem 6.1.10 without computing the norm $\mathrm{Norm}_{B/k}(x - \vartheta)$. Indeed, we only use that the entries corresponding to α_2 and α_3 are equal to $x - \alpha_2$ and α_3, respectively. The first entry then needs to be well-chosen such that their product is a square: for example, we may choose

$$(x - \theta)(P) = \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}, \alpha_1 - \alpha_2, \alpha_1 - \alpha_3\right)$$

in $(k^*/k^{*2})^3$; cf. Prop. X.1.4 of [Sil09].

Example 6.1.13. Suppose α is a root of f. Factor $f(x) = (x - \alpha) \cdot g(x)$ and define B as in Theorem 6.1.10, so that $A = k \times B$. Let $d \in k^*/k^{*2}$ and note that elements of the form $(1, d) \in k^*/k^{*2} \times B^*/B^{*2}$ have square norm. We describe the map v explicitly on elements of the above form: by Example 6.1.8, the para-elliptic curve is the contracted product $C = E \wedge^{\mathbb{Z}/2\mathbb{Z}} k(\sqrt{d})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on E by translation by $(\alpha, 0)$.

More generally, any element $a \in A^*$ of square norm determines a para-elliptic curve C_a of period 2, defined abstractly through the cohomology class $v(a)$. It is valuable to have a more concrete description of the para-elliptic curve: we describe explicit equations for C_a as an intersection of two quadrics in $\mathbb{P}^4$. This description is very classical, tracing back to at least Mordell in §16 of [Mor69]. We mentioned these descriptions in the proof of Theorem 5.0.1. We expand upon the proof of Thm. A.1 in [Cre16].

Theorem 6.1.14. *Let $a \in A^*$ be an element of square norm. For $i = 0$, 1, 2 define the quadratic forms Q_i by $\sum_{i=0}^{2} Q_i(z_0, z_1, z_2)\theta^i = a \cdot (z_0 + z_1\theta + z_2\theta^2)^2$. Then the underlying scheme of the para-elliptic curve $v(a)$ is described by the intersection of two quadrics C_a*

$$\begin{cases} Q_1(z_0, z_1, z_2) + z_3^2 = 0; \\ Q_2(z_0, z_1, z_2) = 0, \end{cases} \tag{6.1.8}$$

in $\mathbb{P}^3$.

Heuristically, this system of equations for C_a is directly related to the short exact sequence (6.1.7). Note that C_a has a rational point if and only if the element a lies in the image of the map $x - \theta$. If we ignore the subtleties surrounding the rational two-torsion points and assume that there is a point $(x, y) \in E(k)$ with $y \neq 0$ such that $az^2 = x - \theta$ has a solution in z, then writing $z = z_0 + z_1\theta + z_2\theta^2$ yields a solution to $Q_1(z_0, z_1, z_2) = -1$ and $Q_2(z_0, z_1, z_2) = 0$, hence a solution to the system (6.1.8) with $z_3 = 1$. The converse is heuristically true as well: any rational point on (6.1.8) with $z_3 = 1$ defines an element $z = z_0 + z_1\theta + z_2\theta^2$ such that $az^2 = x - \theta$ for some $x \in k$. Let $y = c\,\mathrm{Norm}_{A/k}(z)$, then from Remark 6.1.11 it follows out that (x, y) is a rational point on E. If z is invertible then $y \neq 0$, so it maps to the element a under the map $x - \theta$.

Of course this heuristic argument does not suffice. There are two technical problems, the most obvious one being issues concerning the closed subschemes of E and C with $y = 0$ and $z_3 = 0$, respectively. A more substantial issue is that two non-isomorphic E-torsors may have the same set of *splitting fields*, meaning that they obtain rational points over the same set of extension fields. This is explored in Section 2.2 under the assumption that $\mathrm{End}(E) = \mathbb{Z}$; see Proposition 2.2.9.

Lemma 6.1.15. *For each $\ell \in A^*$ there is an isomorphism $\phi_\ell \colon C_a \xrightarrow{\sim} C_{a\ell^2}$, satisfying $\phi_{\ell_1 \cdot \ell_2} = \phi_{\ell_1} \circ \phi_{\ell_2}$.*

Proof. Multiplication by ℓ yields a linear bijection $A \to A$. Identifying $A = k^3$ through the basis 1, θ, θ^2, the element ℓ defines an invertible 3×3 matrix, which induces an automorphism of $\mathbb{P}^2$ equipped with homogeneous coordinates z_0, z_1, z_2. Extend this automorphism to $\mathbb{P}^3$ by $z_3 \mapsto z_3$. By construction this automorphism maps the curve $C_{a\ell^2}$ to C_a. Let ϕ_ℓ be its inverse. □

Remark 6.1.16. If $\ell^2 = 1$ then ϕ_ℓ defines an involution on C_a. In fact, since the above Lemma is functorial, this defines an action of $\mathrm{Res}_{A/k}\,\mu_{2,A}$ on C_a. We return to this action during the course of the proof of Lemma 6.1.21.

Remark 6.1.17. Consider the natural map from C_a to the conic $Q_2(z_0, z_1, z_2) = 0$ in $\mathbb{P}^2$. It is the quotient by the involution $z_3 \mapsto -z_3$. Note that there are geometric fixpoints, either by noting that ramification occurs due to Riemann–Hurwitz formula, or by setting $z_3 = 0$ and noting that the intersection of the two quadrics $Q_1(z_0, z_1, z_2) = Q_2(z_0, z_1, z_2) = 0$ is nonempty. This again shows that any para-elliptic curve of period 2 admits a sign involution; cf. Corollary 2.2.13.

If k is algebraically closed then there is an isomorphism $A \cong k^3$ of rings, so every element is a square. This allows us to identify C_a with C_1. In this case we may also drop the subscript from the notation and simply denote the scheme by C.

Lemma 6.1.18. *The scheme C_a described by the system of equations* (6.1.8) *is a smooth genus-one curve.*

Proof. Without loss of generality suppose that k is an algebraically closed field and that $a = 1$. We use the Jacobi criterion to verify that C is smooth. The partial derivatives of the quadratic forms Q_i can be calculated through

$$\sum_{i=0}^{2} \frac{\partial Q_i}{\partial z_j}\theta^i = \frac{\partial (z_0 + z_1\theta + z_2\theta^2)^2}{\partial z_j} = 2(z_0 + z_1\theta + z_2\theta^2)\theta^j, \quad (j = 0, 1, 2). \tag{6.1.9}$$

Let $(z_0 : z_1 : z_2 : z_3)$ be a rational point of C. It obeys the quadratic equations $Q_1(z_0, z_1, z_2) = -z_3^2$ and $Q_2(z_0, z_1, z_2) = 0$. We define $x = Q_0(z_0, z_1, z_2)$. The Jacobi matrix is

$$\begin{pmatrix} \frac{\partial Q_1}{\partial z_0}(z_0, z_1, z_2) & \frac{\partial Q_1}{\partial z_1}(z_0, z_1, z_2) & \frac{\partial Q_1}{\partial z_2}(z_0, z_1, z_2) & 2z_3 \\ \frac{\partial Q_2}{\partial z_0}(z_0, z_1, z_2) & \frac{\partial Q_2}{\partial z_1}(z_0, z_1, z_2) & \frac{\partial Q_2}{\partial z_2}(z_0, z_1, z_2) & 0 \end{pmatrix}. \tag{6.1.10}$$

We show that it has rank 2.

First suppose that z is invertible. Since $\{1, \theta, \theta^2\}$ forms a basis of A as a k-vector space, so do the elements of (6.1.9) for $j = 0$, 1, 2. Therefore the 3×3 matrix

$$\left(\frac{\partial Q_i}{\partial z_j}(z_0, z_1, z_2)\right)_{i,j=0}^{2}$$

is invertible. Disregarding the last column of (6.1.10), it follows that the remaining 2×3 submatrix is a submatrix of an invertible matrix and hence has full rank.

Suppose now that $z_3 = 1$. If (6.1.10) does not have full rank, then the bottom row vanishes and $\partial Q_2/\partial z_j(z_0, z_1, z_2) = 0$ for $j = 0$, 1, 2. From (6.1.9) it follows that the θ^2-coefficient of $2z\theta^j$ is 0 for $j = 0$, 1, and 2. This successively implies that $0 = z_2 = z_1 = z_0$, which violates (6.1.8).

The two cases treated above are sufficient. Indeed, if $z_3 = 0$ then $z^2 = x$ is an element of k, which follows from $Q_1(z_0, z_1, z_2) = Q_2(z_0, z_1, z_2) = 0$. Since z is nonzero and A is reduced, it follows that z^2 is also nonzero, hence is an invertible element of k. Note that z is invertible if and only if z^2 is.

Now we show that C indeed is one-dimensional, for which we assume without loss of generality that k is algebraically closed. By dimension theory it is clear that $\dim C \geq 1$. The intersection with the hyperplane $z_3 = 0$ yields the intersection of two quadrics $Q_1(z_0, z_1, z_2) = Q_2(z_0, z_1, z_2) = 0$ in $\mathbb{P}^2$. Then we have seen that $z = z_0 + z_1\theta + z_2\theta^2$ satisfies $z^2 \in k^*$. By the ring isomorphism $A \cong k^3$, those elements are determined up to k^*-action by the four elements $(1, \pm 1, \pm 1) \in k^3$. Therefore $C \cap \{z_3 = 0\}$ is zero-dimensional, meaning that C is at most 1-dimensional.

In summary, we have shown that C is a smooth complete intersection. By the adjunction formula it follows that the dualizing sheaf is $\omega_C \cong \mathcal{O}_C$ so indeed the genus of C equals one. ☐

Remark 6.1.19. During the course of the proof, we have seen that the locus where $z_3 = 0$ consists of four geometric points. The locus where z is not invertible consists of the points where $z_3 = 1$ and $Q_0(z_0, z_1, z_2)$ is a root of f. Outside of these finitely many points, z is invertible and $z_3 \neq 0$ happen simultaneously.

Lemma 6.1.20. *Let $a \in A^*$ be an element of square norm. Let $c \in k^*$ such that* $\mathrm{Norm}_{A/k}(a) = c^2$. *The map $C_a \to E$ defined by*

$$(z_0 : z_1 : z_2 : 1) \longmapsto (Q_0(z_0, z_1, z_2), c\,\mathrm{Norm}_{A/k}(z_0 + z_1\theta + z_2\theta^2)) = (x, y),$$

is a twisted form of the multiplication by 2 *map $E \to E$.*

Proof. To see that this map is well-defined on the open neighbourhood of C_a where $z_3 = 1$, we note that $a(z_0 + z_1\theta + z_2\theta^2)^2 = Q_0(z_0, z_1, z_2) - \theta$ and taking norms on both sides yields

$$\begin{aligned} y^2 &= (c\,\mathrm{Norm}_{A/k}(z_0 + z_1\theta + z_2\theta^2))^2 \\ &= \mathrm{Norm}_{A/k}(a(z_0 + z_1\theta + z_2\theta^2)^2) \\ &= \mathrm{Norm}_{A/k}(Q_0(z_0, z_1, z_2) - \theta) \\ &= \prod_{i=1}^{3}(Q_0(z_0, z_1, z_2) - \alpha_i) \\ &= f(Q_0(z_0, z_1, z_2)) = f(x), \end{aligned}$$

where α_i denote the three roots of f, cf. Remark 6.1.11.

Since C_a is a smooth curve, the above rational map extends to a surjective morphism $C_a \to E$ between genus-one curves, which is an isogeny of para-elliptic curves. On the level of function fields, the element $x \in \kappa(E)$ maps to $Q_0 \in k[z_0, z_1, z_2]$ considered as rational function on $C_a \subset \mathbb{P}^3$. The unique pole of x is the point at infinity ∞, whereas the poles of Q_0 arise whenever $z_3 = 0$. Therefore the fibre of $C_a \to E$ over the point at infinity is the hyperplane section $C_a \cap \{z_3 = 0\}$. In the proof of Lemma 6.1.18 we have seen that the intersection $C_a \cap \{z_3 = 0\}$ consists of four geometric points. Moreover, the (scheme theoretic) intersection has length four, since it is the intersection of the two quadrics $Q_1(z_0, z_1, z_2) = Q_2(z_0, z_1, z_2) = 0$, by Bézout's theorem. It follows that $C_a \cap \{z_3 = 0\}$ is étale of length four, from which we deduce that $C_a \to E$ is étale of degree 4.

Base-change to an algebraic closure and fix a rational point of C with $z_3 = 0$, e.g. the point $(1 : 0 : 0 : 0)$, making $C \to E$ into an isogeny of elliptic curves. Its kernel consists of the hyperplane $C \cap \{z_3 = 0\}$, but by the definition of the group law, the points in a hyperplane section add up to 0. This eliminates the case that the kernel is cyclic of order 4, thus it must be isomorphic to the Klein four-group and therefore $C \to E$ is isomorphic to $E \xrightarrow{2} E$ after base change to k^{alg}. ☐

In the terminology of Section 2.2, we have shown that $C_a \to E$ is a *two-covering* of E. The 2-coverings are classified by the cohomology group $\mathrm{H}^1(k, E[2])$, so Lemma 6.1.20 associates to any $a \in \mathrm{Ker}(\mathrm{Norm}_{A/k} \colon A^*/A^{*2} \to k^*/k^{*2})$ a cohomology class in $\mathrm{H}^1(k, E[2])$. In order to prove Theorem 6.1.14, we show that this is inverse to the isomorphism determined by Proposition 6.1.7.

Lemma 6.1.21. *Under the above isomorphism* $\mathrm{H}^1(k, E[2]) \cong \operatorname{Ker}(\mathrm{Norm}_{A/k} \colon A^*/A^{*2} \to k^*/k^{*2})$*, the cohomology class* $[C_a \to E]$ *maps to* a*. Therefore* $[C_a] = v(a)$ *in* $\mathrm{H}^1(k, E)[2]$*.*

From this, Theorem 6.1.14 follows directly. Indeed, the map $\mathrm{H}^1(k, E[2]) \to \mathrm{H}^1(k, E)[2]$ is simply induced by the forgetful map, that maps a cohomology class $[C_a \to E]$ to the cohomology class of a para-elliptic curve $[C_a]$.

Proof. We chase the following isomorphisms:

$$\begin{tikzcd}
{[C_a \to E]} \arrow[d, mapsto] & \in & \mathrm{H}^1(k, E[2]) \arrow[d, "\cong"] \\
{[P] = \{Q_1 = Q_2 = 0\}} \arrow[d, mapsto] & \in & \mathrm{H}^1(k, \operatorname{Ker}(\mathrm{Norm}_{A/k} \colon \operatorname{Res}_{A/k} \mu_{2,A} \to \mu_2)) \arrow[d, "\cong"] \\
{[R] = \operatorname{Spec} \dfrac{k[z_0, z_1, z_2]}{(Q_0 - 1, Q_1, Q_2)}} & \in & \operatorname{Ker}\left(\mathrm{Norm}_{A/k,*} \colon \mathrm{H}^1(k, \operatorname{Res}_{A/k} \mu_{2,A}) \to \mathrm{H}^1(k, \mu_2)\right) \\
{\operatorname{Spec} \dfrac{A[z]}{(az^2 - 1)}} \arrow[u, mapsto] \arrow[d, mapsto] & \in & \operatorname{Ker}\left(\mathrm{Norm}_{A/k,*} \colon \mathrm{H}^1(A, \mu_{2,A}) \to \mathrm{H}^1(k, \mu_2)\right) \arrow[u, "\cong"] \arrow[d, "\cong"] \\
{a^{-1} \equiv a} & \in & \operatorname{Ker}\left(\mathrm{Norm}_{A/k} \colon A^*/A^{*2} \to k^*/k^{*2}\right)
\end{tikzcd}$$

We start at the top and work downwards. Although we can indeed consider a 2-covering $[C_a \to E]$ as a cohomology class in $\mathrm{H}^1(k, E[2])$, the corresponding $E[2]$-torsor is actually the fibre over the identity element of E. In this case it is described by the intersection of two quadrics $Q_1(z_0, z_1, z_2) = Q_2(z_0, z_1, z_2) = 0$, denoted P. Note that $E[2]$ and $\operatorname{Ker}(\mathrm{Norm}_{A/k} \colon \operatorname{Res}_{A/k} \mu_{2,A} \to \mu_2)$ both act freely and transitively on P with their respective actions, as can be verified after base-change to an algebraic closure. There one sees also that these two actions agree under the isomorphism $E[2] \xrightarrow{\sim} \operatorname{Ker}(\mathrm{Norm}_{A/k} \colon \operatorname{Res}_{A/k} \mu_{2,A} \to \mu_2)$ of (6.1.3).

Consider the affine scheme

$$R = \operatorname{Spec} \frac{k[z_0, z_1, z_2]}{(Q_0(z_0, z_1, z_2) - 1, Q_1(z_0, z_1, z_2), Q_2(z_0, z_1, z_2))},$$

equipped with a transitive action by $\operatorname{Res}_{A/k} \mu_{2,A}$. It parametrises elements $z \in A$ satisfying $az^2 = 1$. Note the difference with the scheme P, whose set of k'-points for an arbitrary field extension k'/k consists of the nonzero elements $z \in A \otimes k'$ such that $az^2 \in k'$, up to rescaling by $(k')^*$. In this latter case however $x = az^2$ is square in k', since taking norms results in $c^2 \mathrm{Norm}_{A \otimes k'/k'}(z)^2 = x^3$, so that $x = (c \mathrm{Norm}(z)/x)^2$. One can thus always find a representative of z, unique up to multiplication by ± 1, such that $az^2 = 1$. This defines a morphism $P \to R$, which is equivariant with respect to the action of $\operatorname{Ker}(\mathrm{Norm}_{A/k} \colon \operatorname{Res}_{A/k} \mu_{2,A} \to \mu_2)$. It follows that also the cohomology class $[P]$ maps to $[R]$.

Note that R is isomorphic to $\operatorname{Res}_{A/k} \operatorname{Spec} A[z]/(az^2 - 1)$, by the explicit construction of the restriction of scalars of affine schemes. This isomorphism clearly respects the action of $\operatorname{Res}_{A/k} \mu_{2,A}$. Last of all, by Kummer theory the torsor $A[z]/(az^2 - 1)$ over A corresponds to the element $a^{-1} \in A^*/A^{*2}$, which equals a modulo squares. ☐

We also investigate the special case where f is completely reducible, meaning that the elliptic curve E has full two-torsion. The equations for this case are also very classical, see e.g. Eqtn. 24.10 of [Cas66] or p. 70 of the more modern [Cas91].

Corollary 6.1.22. *Suppose* f *factors as a product* $f = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ *for certain* $\alpha_i \in k$*. Suppose the ring isomorphism* $A \xrightarrow{\sim} k^3$ *by evaluating* θ *at* α_i *maps* a *to the triple* (a_1, a_2, a_3)*, so the product* $a_1 a_2 a_3$ *is a square in* k*. Then the para-elliptic curve* $v(a)$ *is described by the system of equations*

$$\begin{cases} (\alpha_2 - \alpha_1) t_0^2 - a_1 t_1^2 + a_2 t_2^2 = 0; \\ (\alpha_3 - \alpha_1) t_0^2 - a_1 t_1^2 + a_3 t_3^2 = 0. \end{cases} \tag{6.1.11}$$

Proof. The system (6.1.8) can be summarized as $a(z_0 + z_1\theta + z_2\theta^2)^2 = Q_0(z_0, z_1, z_2) - z_3^2\theta + 0 \cdot \theta^2$ in A. We apply the ring isomorphism $A \xrightarrow{\sim} k^3$ by evaluating at $\theta \mapsto \alpha_i$ for $i = 1, 2, 3$ to find the system of equations

$$a_i(z_0 + z_1\alpha_i + z_2\alpha_i^2)^2 = Q_0(z_0, z_1, z_2) - z_3^2\alpha_i. \quad (i = 1, 2, 3)$$

Taking differences of equations, we find the system

$$\begin{cases} a_2(z_0 + z_1\alpha_2 + z_2\alpha_2^2)^2 - a_1(z_0 + z_1\alpha_1 + z_2\alpha_1^2)^2 = -z_3^2(\alpha_2 - \alpha_1); \\ a_3(z_0 + z_1\alpha_3 + z_2\alpha_3^2)^2 - a_1(z_0 + z_1\alpha_1 + z_2\alpha_1^2)^2 = -z_3^2(\alpha_3 - \alpha_1), \end{cases} \tag{6.1.12}$$

We show that conversely any solution to (6.1.12) also satisfies (6.1.8). Indeed applying the identity $\sum_{j=0}^{2} Q_j(z_0, z_1, z_2) \cdot \alpha_i^j = a_i(z_0 + z_1\alpha_i + z_2\alpha_i^2)^2$ twice per equation, the above system is written equivalently as

$$Q_1(z_0, z_1, z_2) \cdot (\alpha_i - \alpha_1) + Q_2(z_0, z_1, z_2) \cdot (\alpha_i^2 - \alpha_1^2) = -z_3^2(\alpha_i - \alpha_1). \quad (i = 2, 3)$$

Dividing by $\alpha_i - \alpha_1$, it follows that $Q_2(z_0, z_1, z_2) \cdot (\alpha_i + \alpha_1) = -z_3^2 - Q_1(z_0, z_1, z_2)$ for $i = 2$, 3. Note that the right hand side is independent of i. Since the roots of f are distinct, this is only possible if $Q_2(z_0, z_1, z_2) = 0$. Then $Q_1(z_0, z_1, z_2) = -z_3^2$ follows directly.

Since the Vandermonde matrix $(\alpha_i^j)_{i,j=1,0}^{3,2}$ is invertible, we can perform a change of basis in (6.1.12) by setting $t_i = z_0 + z_1\alpha_i + z_2\alpha_i^2$ for $i = 1, 2, 3$. Also letting $t_0 = z_3$ we arrive at the equations (6.1.11). ☐

Remark 6.1.23. In Section 6.3 below we use visually more appealing notation. Let E be an ellliptic curve with affine Weierstraß equation of the form $E\colon y^2 = x(x - \alpha)(x - \beta)$ and let $(u, v, w) \in (k^*)^3$ be a triple such that the product uvw is square. In this notation, the system (6.1.11) becomes

$$\begin{cases} \alpha t_0^2 - u t_1^2 + v t_2^2 = 0; \\ \beta t_0^2 - u t_1^2 + w t_3^2 = 0. \end{cases} \tag{6.1.13}$$

Example 6.1.24. We continue in the setting of Example 6.1.13, where we set $\alpha = 0$. Then $f(x) = x \cdot g(x)$, where g is a quadratic polynomial of the form $g(x) = x^2 + ax + b$ for certain $a, b \in k$. Recall that the element $a \in A$ corresponds to the pair $(1, d)$ under the isomorphism $A = k \times B$. The curve C_a is described by the equation (6.1.8), which parametrises solutions to $a(z_0 + z_1\theta + z_2\theta^2)^2 = Q_0(z_0, z_1, z_2) - z_3^2\theta + 0 \cdot \theta^2$ in A. Since $A = k \times B$ by substituting 0 and ϑ for θ, this is equivalent to the system given by $z_0^2 = Q_0(z_0, z_1, z_2)$ and

$$d(z_0 + z_1\vartheta + z_2\vartheta^2)^2 = Q_0(z_0, z_1, z_2) - z_3^2\vartheta.$$

The first equation is vacuous: $Q_0(z_0, z_1, z_2) = z_0^2$ is clear from the definition because $f(x) = x^3 + ax^2 + bx$ is divisible by x. Since $\vartheta^2 = -a\vartheta - b$, the equation becomes $d(z_0 - bz_2 + (z_1 - az_2)\vartheta)^2 = z_0^2 - z_3^2\vartheta$. Consider the coordinates $w_0 = z_0/d$ and $w_1 = z_1 - az_2$ and $w_2 = z_0 - bz_2$ and $w_3 = z_2/d$. Comparing coefficients results in the system of equations

$$\begin{cases} dw_0^2 = w_2^2 - bw_1^2; \\ dw_3^2 = 2w_1w_2 + aw_1^2. \end{cases}$$

Multiplying the top equation by $4w_1^2$, we substitute the bottom equation into it to find the equation

$$\begin{aligned} 4dw_0^2w_1^2 &= (dw_3^2 - aw_1^2)^2 - 4bw_1^4 \\ &= (a^2 - 4b)w_1^4 - 2adw_1^2w_3^2 + d^2w_3^4. \end{aligned}$$

Setting $w_3 = 1$ and $W = w_0w_1$ and $Z = w_1$, this coincides with the quartic affine equation

$$dW^2 = (a^2 - 4b)Z^4 - 2adZ^2 + d^2, \tag{6.1.14}$$

which is very classical; see e.g. Ex. III.4.5 of [Sil09]. It can be obtained much more directly using the description of $C = E \wedge^{\mathbb{Z}/2\mathbb{Z}} k(\sqrt{d})$ by computing the quotient through invariant theory. We illustrate on the affine patch of E obtained by removing the point at infinity and the two-torsion point $(0, 0)$. The translation involution is given by

$$x \longmapsto \frac{b}{x} \quad \text{and} \quad y \longmapsto \frac{-by}{x^2}.$$

Thus C is the quotient of $E \otimes k(\sqrt{d})$ by the involution $x \mapsto \sqrt{d}b/x$ and $y \mapsto -b\sqrt{d}y/x^2$. Note that the following elements are invariant, and in fact generate the invariant ring

$$Z = \sqrt{d} \cdot \frac{x}{y} \quad \text{and} \quad W = \sqrt{d}\Big(x - \frac{b}{x}\Big)\Big(\frac{x}{y}\Big)^2, \tag{6.1.15}$$

and they indeed satisfy the relation (6.1.14). After base-change to $k(\sqrt{d})$, the equations (6.1.15) determine an isomorphism $E \otimes k(\sqrt{d}) \xrightarrow{\sim} C \otimes k(\sqrt{d})$.

6.2 Descent by two-isogeny

Let k be a ground field in which 2 is invertible. Suppose that E has a rational two-torsion point, in which case it is described by an affine Weierstraß equation of the form

$$E \colon y^2 = x^3 + ax^2 + bx,$$

having the rational point $(0, 0)$ of order 2. This point defines a subgroup scheme $\mathbb{Z}/2\mathbb{Z} \subset E[2]$. The quotient $\widetilde{E} = E/(\mathbb{Z}/2\mathbb{Z})$ is an elliptic curve isogenous to E. An affine Weierstraß equation for $\widetilde{E}$ is well-known: it is given by

$$\widetilde{E} \colon y^2 = x^3 + a'x^2 + b'x, \tag{6.2.1}$$

where $a' = -2a$ and $b' = a^2 - 4b$; see for example Ex. III.4.5 of [Sil09]. Let $\phi \colon E \to \widetilde{E}$ be the isogeny, and let $\phi^\vee : \widetilde{E} \to E$ be the dual isogeny.

We refine the theory of two-descent by replacing the short exact sequence (6.1.1) by the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow E \xrightarrow{\phi} \widetilde{E} \longrightarrow 0.$$

There is a similar short exact sequence for the dual isogeny $\phi^\vee$. These two short exact sequences are explicitly related to the short exact sequence (6.1.1) for $n = 2$, as follows: they sit inside a 3×3 commutative diagram with exact rows and exact first column.

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & E \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"] & \widetilde{E} \arrow[r] \arrow[d, "\phi^\vee"] & 0 \\
0 \arrow[r] & E[2] \arrow[r] \arrow[d] & E \arrow[r, "2"] \arrow[d, "\phi"'] & E \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \widetilde{E} \arrow[r, "\phi^\vee"] \arrow[uur, "\mathrm{id}"] & E \arrow[r] & 0 \\
 & 1 & & &
\end{tikzcd}$$

Taking long exact sequences, the following diagram is commutative with exact rows and exact middle column.

$$\begin{tikzcd}
 & & & \vdots \arrow[d] & & & \\
\cdots \arrow[r] & E(k) \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"] & \widetilde{E}(k) \arrow[r, "\delta_{\widetilde{E}}"] \arrow[d] & k^*/k^{*2} \arrow[r] \arrow[d] & \mathrm{H}^1(k,E) \arrow[r, "\phi_*"] \arrow[d, "\mathrm{id}"] & \mathrm{H}^1(k,\widetilde{E}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & E(k) \arrow[r, "2"] \arrow[d] & E(k) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^1(k,E[2]) \arrow[r] \arrow[d] & \mathrm{H}^1(k,E) \arrow[r, "2"] \arrow[d] & \mathrm{H}^1(k,E) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & \widetilde{E}(k) \arrow[r, "\phi^\vee"] \arrow[uur, "\mathrm{id}"'] & E(k) \arrow[r, "\delta_E"] & k^*/k^{*2} \arrow[r] \arrow[d] & \mathrm{H}^1(k,\widetilde{E}) \arrow[r, "\phi^\vee_*"] \arrow[uur, "\mathrm{id}"'] & \mathrm{H}^1(k,E) \arrow[r] & \cdots \\
 & & & \vdots & & &
\end{tikzcd} \tag{6.2.2}$$

The middle exact row has been studied in Section 6.1, where we established an explicit isomorphism $\mathrm{H}^1(k, E[2]) \cong \mathrm{Ker}(A^*/A^{*2} \to k^*/k^{*2})$ as described in Theorem 6.1.10. An equally concrete description of the boundary map δ_E follows.

Proposition 6.2.1 (Descent by two-isogeny)**.** *The boundary map δ_E is given by*

$$\delta_E(P) = \begin{cases} x & \textit{if } P = (x, y) \textit{ with } x \neq 0; \\ b & \textit{if } P = (0, 0); \\ 1 & \textit{if } P \textit{ is the identity element.} \end{cases}$$

Proof. We use the commutativity of (6.2.2) and the explicit description of the $x-\theta$ map of Theorem 6.1.10 to compute the boundary map δ_E. Let $A^*/A^{*2} \to k^*/k^{*2}$ be induced by the point $(0, 0)$. The following diagram is commutative:

$$\begin{tikzcd}
E(k) \arrow[r, "x-\theta"] \arrow[d, "\mathrm{id}"] & \mathrm{Ker}(\mathrm{Norm}_{A/k}\colon A^*/A^{*2} \to k^*/k^{*2}) \arrow[d] \\
E(k) \arrow[r, "\delta_E"] & k^*/k^{*2}
\end{tikzcd}$$

We use exactness of the middle column of (6.2.2) to compute the vertical map $\mathrm{H}^1(k, E[2]) \to k^*/k^{*2}$: it suffices to describe the image of $k^*/k^{*2} \to \mathrm{H}^1(k, E[2])$ under above isomorphisms. This is done in Example 6.1.8: with its notation, we have seen that the composition $k^*/k^{*2} \to \mathrm{H}^1(k, E[2]) \to A^*/A^{*2} \to k^*/k^{*2} \times B^*/B^{*2}$ is given by $d \mapsto (1, d)$, so that by exactness the map to k^*/k^{*2} is projection onto the first factor. Recall here that $B = k[x]/(x^2 + ax + b)$; let ϑ be the image of x in B.

If $P = (x, y)$ is a non two-torsion point, meaning $y \neq 0$, then $(x - \theta)(P) = x - \theta$, and the projection to k^*/k^{*2} simply evaluates θ at 0. If $P = (0, 0)$ then $(x-\theta)(P)$ equals $(\mathrm{Norm}_{B/k}(\vartheta), -\vartheta)$ in $k^*/k^{*2} \times B^*/B^{*2}$, which maps to $\mathrm{Norm}_{B/k}(-\vartheta) = b$. The last case is $P = (\alpha, 0)$, where $\alpha = \alpha_1$ is a nonzero root of x^2+ax+b. The other roots of f are given by $\alpha_2 = -a - \alpha$ and $\alpha_3 = 0$. Under the isomorphism $A^*/A^{*2} \cong (k^*/k^{*2})^3$, we have computed in Example 6.1.12 that $x - \theta$ maps the rational point $(\alpha, 0)$ to a tuple

$$\left(\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}, \alpha_1 - \alpha_2, \alpha_1 - \alpha_3\right) = \left(\frac{\alpha - b/\alpha}{\alpha}, \alpha - b/\alpha, \alpha\right)$$

in $(k^*/k^{*2})^3$. Its image in k^*/k^{*2} is the third entry α, which is indeed the x-coordinate of P. ☐

Note that the setup is symmetrical: by interchanging the elliptic curves E and $\widetilde{E}$, and the isogenies ϕ and $\phi^\vee$, we conclude with an analogous description of the boundary map $\delta_{\widetilde{E}}$, cf. Ex. X.4.8 and Prop. X.4.9 of [Sil09].

Corollary 6.2.2. *The boundary map $\delta_{\widetilde{E}}$ is given by*

$$\delta_{\widetilde{E}}(P) = \begin{cases} x & \textit{if } P = (x, y) \textit{ with } x \neq 0; \\ b' = b^2 - 4a & \textit{if } P = (0, 0); \\ 1 & \textit{if } P \textit{ is the point at infinity.} \end{cases}$$

6.3 The relative Brauer group

Let k be a ground field. Let X be a scheme such that the Picard functor $\mathrm{Pic}_{X/k}$ is representable. It does not quite represent the functor $T \mapsto \mathrm{Pic}(X \times T)$, since it does not satisfy the sheaf property; instead, the Picard scheme represents its sheafification in, say, the fppf-topology. As a consequence, there is a discrepancy between the k-points of the Picard scheme and the Picard group of X. This discrepancy is measured through the seven-term exact sequence of the Leray–Serre spectral sequence:

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/k}(k) \xrightarrow{\partial_X} \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}_{X/k}) \longrightarrow \mathrm{H}^3(k, \mathbb{G}_m), \tag{6.3.1}$$

cf. Section 1.3. Here $\mathrm{Br}_1(X) \coloneqq \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{alg}}))$ is called the *algebraic Brauer group.*

In this section we study the boundary map ∂_C, where C is a para-elliptic curve. As motivation, we note that the non-vanishing of ∂_X is an obstruction to the existence of a rational point on X, since the choice of point provides a splitting $\mathrm{Br}(X) \to \mathrm{Br}(k)$. There is also a close connection to the period-index problem. We follow the article [ÇK12] closely. The approach is based on a number of closely related bilinear pairings, some of which are mostly applicable in a theoretical setting, while others allow for concrete computations. Much of the theory seems to generalise naturally to the case where X is a torsor under the Jacobian of a hyperelliptic curve; see [CV15; Cre16]. Although we do not work in this generality, we occasionally reference the statements contained in ibid.

Definition 6.3.1. Let $Y \to X$ be a morphism of schemes. The *relative Brauer group* is defined as

$$\mathrm{Br}(X/Y) = \mathrm{Ker}(\mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X)).$$

By exactness of (6.3.1), the relative Brauer group equals the image of ∂_X.

Let $D \in \mathrm{Pic}_{X/k}(k)$ be a k-point on the Picard scheme such that $\partial_X(D)$ is a nonzero element of $\mathrm{Br}(k)$. Then D is obstructed to arise from an invertible sheaf on X. It is possible to make sense of the linear system $|D|$ as a Brauer–Severi variety, whose Brauer class equals $\partial_X(D)$; see Lem. 2.2 of [Cre16]. Note that if D arises from an invertible sheaf, then the linear system is isomorphic to projective space, and the rational points parametrise effective Cartier divisors linearly equivalent to D. The lack of rational points indicates that D does not arise from an invertible sheaf.

Example 6.3.2. Let E be an elliptic curve. Since E has a rational point, the boundary map ∂_E vanishes and all k-points on $\mathrm{Pic}_{E/k}$ arise from invertible sheaves. On the other hand, if C is an E-torsor, then alghough there is a natural isomorphism of Picard schemes $\mathrm{Pic}_{C/k} = \mathrm{Pic}_{E/k}$, the boundary map ∂_C may be nonzero, so the relative Brauer group $\mathrm{Br}(C/k)$ is non-trivial.

We now specialise to the above case where $X = C$ is a para-elliptic curve, with associated elliptic curve E. Then the above seven-term exact sequence simplifies.

Proposition 6.3.3. *The sequence*

$$0 \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}_{E/k}(k) \xrightarrow{\partial_C} \mathrm{Br}(k) \longrightarrow \mathrm{Br}(C) \xrightarrow{r_C} \mathrm{H}^1(k, E) \longrightarrow \mathrm{H}^3(k, \mathbb{G}_m). \tag{6.3.2}$$

is exact.

Proof. This is the seven-term exact sequence (6.3.1), but with two adjustments. First of all, since $\dim(C) = 1$ it follows from Tsen's theorem (see [Gro66b, Cor. 1.2] or [CS21, Thm. 1.2.14]) that $\mathrm{Br}(C^{\mathrm{alg}}) = 0$, so that $\mathrm{Br}_1(C) = \mathrm{Br}(C)$. Furthermore in this case, the long exact sequence of the *split* short exact sequence

$$0 \longrightarrow E \longrightarrow \mathrm{Pic}_{E/k} \longrightarrow \mathbb{Z} \longrightarrow 0$$

implies that $\mathrm{H}^1(k, \mathrm{Pic}_{E/k})$ is naturally isomorphic to the Weil–Châtelet group $\mathrm{H}^1(k, E)$. □

Example 6.3.4. Let C_a be the para-elliptic curve given by the system of equations (6.1.8). As was shown in Remark 6.1.17, there is a morphism from the para-elliptic curve C_a maps to the Brauer–Severi curve determined by the quadric $Q_2(z_0, z_1, z_2) = 0$ in $\mathbb{P}^2$. By Thm. 3.4 of [Lie17], the Brauer class of this quadric is an element of the relative Brauer group of C_a.

The image of $E(k) = \mathrm{Pic}^0_{E/k}(k)$ under ∂_C is a subgroup of the relative Brauer group $\mathrm{Br}(C/k)$. Although in some cases there is an equality $\partial_C(E(k)) = \mathrm{Br}(C/k)$, in general it defines a strict subgroup, whose index is an interesting arithmetic invariant of C pertaining to the so-called *period-index problem.* Recall that the *period* $\mathrm{per}(C)$ of C is defined as the order of the cohomology class $[C]$ in the Weil–Châtelet group $\mathrm{H}^1(k, E)$.

Notation 6.3.5. The *index* $\mathrm{ind}(C)$ of a para-elliptic curve C is

$$\mathrm{ind}(C) = \gcd\{[\kappa(P) : k] \mid P \text{ is a closed point of } C\}.$$

Remark 6.3.6. One can similarly consider the so-called *separable index*, which is defined analogously with the additional restraint that the residue field of P be separable over k. In our context the two notions coincide: Thm. 4 of [Lic68] states that the separable index equals the index for para-elliptic curves.

The discrepancy between the period and the index measures the surjectivity of ∂_C when restricted to $E(k) = \mathrm{Pic}^0_{C/k}(k)$, in the following sense; see Rk. 2.2 of [ÇK12] or Prop. 2.4 of [Cla04].

Proposition 6.3.7. *The period* $\mathrm{per}(C)$ *divides the index* $\mathrm{ind}(C)$*. Furthermore, the sequence*

$$0 \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^0_{C/k}(k) \xrightarrow{\partial_C} \mathrm{Br}(C/k) \longrightarrow \frac{\mathrm{per}(C)\mathbb{Z}}{\mathrm{ind}(C)\mathbb{Z}} \longrightarrow 0$$

is exact.

In fact, although the period and the index certainly do not have to be equal, there are a number of results in the direction that they cannot be 'too different'. This can be thought of as a 'near surjectivity' of $\partial_C\colon \mathrm{Pic}^0_{C/k}(k) \to \mathrm{Br}(C/k)$.

Proposition 6.3.8. *The period and the index have the same set of prime divisors. In fact,* $\mathrm{ind}(C)$ *divides* $\mathrm{per}(C)^2$*.*

Remark 6.3.9. For higher dimensional para-abelian varieties it is still true that $\mathrm{per}(X)$ divides $\mathrm{ind}(X)$, at least if the period is coprime to the characteristic exponent. We note that Cor. 11 of [Cla04] states that $\mathrm{ind}(X)$ divides $\mathrm{per}(X)^{2g}$, where $g = \dim(X)$.

Above bound is sharp for para-elliptic curves, although it is not particularly easy to construct examples where equality $\mathrm{ind}(C) = \mathrm{per}(C)^2$ holds. The first examples were constructed by Lang and Tate in [LT58], but see also the short paper [Cas63] for an example of a para-elliptic curve of period 2 and index 4 over a number field.

It turns out that the most fruitful method for studying the map ∂_C is through a number of bilinear pairings, as explained in the exposition §3 of [ÇK12], which contains the relevant pairings and sketches their connections. We start with a seemingly unrelated bilinear pairing, called the *evaluation pairing.*

Definition 6.3.10. The *evaluation pairing* is the bilinear pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{eval}}\colon \mathrm{Br}(E) \times E(k) \longrightarrow \mathrm{Br}(k),$$

defined through the pullback $\langle \alpha, P\rangle = P^*(\alpha)$, where we regard P as a morphism $P\colon \mathrm{Spec}(k) \to E$.

The evaluation pairing is usually quite concrete. If the Brauer group coincides with the cohomological Brauer group, then any Brauer class α is represented by an Azumaya algebra $\mathcal{A}$, up to Morita equivalence. Then the pullback $P^*(\alpha)$ corresponds to the central simple algebra $\mathcal{A} \otimes \kappa(P)$ over $\kappa(P) = k$. In the context of regular schemes of dimension at most 2, the two Brauer groups are naturally isomorphic, as follows from Cor. 2.2 of [Gro66a]. We may of course also apply the well-known result by Gabber, that the Brauer group equals the cohomological Brauer group if there exists an ample invertible sheaf; for a proof we refer to [Jon]. For more details on the coincidence of the Brauer group and the cohomological Brauer group we mention the doctoral thesis [Fis21].

If $C = E$, then the map $r = r_E$ in (6.3.2) relates the Brauer group $\mathrm{Br}(E)$ with the Weil–Châtelet group $\mathrm{H}^1(k, E)$. In fact, it is 'nearly' an isomorphism, in the following sense.

Definition 6.3.11. Given $P \in E(k)$, define

$$\mathrm{Br}(E, P) = \mathrm{Ker}(\langle\cdot, P\rangle_{\mathrm{eval}}\colon \mathrm{Br}(E) \longrightarrow \mathrm{Br}(k)).$$

Of course E has a distinguished rational point, namely the point at infinity $\infty \in E(k)$ that serves as the identity element for the group operation.

Proposition 6.3.12. *The map r induces an isomorphism* $\mathrm{Br}(E,\infty) \to \mathrm{H}^1(k,E)$.

Proof. The evaluation map $\langle\cdot,\infty\rangle_{\mathrm{eval}}\colon \mathrm{Br}(E) \to \mathrm{Br}(k)$ is a retraction of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(E)$, which is hence injective. Therefore r restricts to an injective map $\mathrm{Br}(E,\infty) \to \mathrm{H}^1(k,E)$ with the same image.

The surjectivity of r follows from a computation using the Leray–Serre spectral sequence $E_2^{r,s} = \mathrm{H}^r(k, R^s p_* \mathbb{G}_m) \Rightarrow \mathrm{H}^{r+s}(E,\mathbb{G}_m)$, where $p\colon E \to \mathrm{Spec}(k)$ is the structure morphism. The existence of the rational point ∞ again yields retractions to the maps $\mathrm{H}^n(k,\mathbb{G}_m) \to \mathrm{H}^n(E,\mathbb{G}_m)$, which are injective. It follows that $E_2^{n,0} = E_\infty^{n,0}$ and therefore that $d_2^{n,1} = 0$ for all n. This implies that $E_\infty^{2,0} = E_2^{2,0}$ and $E_\infty^{1,1} = E_2^{1,1}$. We saw before that $E_2^{0,2} = 0$ by Tsen's Theorem, so also $E_\infty^{0,2} = 0$. The filtration of $\mathrm{H}^2(E,\mathbb{G}_m)$ now directly gives the short exact sequence

$$0 \longrightarrow \mathrm{H}^2(k,\mathbb{G}_m) \longrightarrow \mathrm{H}^2(E,\mathbb{G}_m) \longrightarrow \mathrm{H}^1(k,R^1p_*\mathbb{G}_m) \longrightarrow 0,$$

which is the desired surjectivity. ☐

This induces a pairing between the Weil–Châtelet group and the Brauer group of k.

Definition 6.3.13. The *Tate pairing* (sometimes called the *Tate–Lichtenbaum* or *Lichtenbaum–Tate pairing*) is the bilinear pairing $\mathrm{H}^1(k,E) \times E(k) \to \mathrm{Br}(k)$ defined by

$$\langle [C], P\rangle_{\mathrm{Tate}} = \langle \alpha, P\rangle_{\mathrm{eval}},$$

where $\alpha \in \mathrm{Br}(E,\infty)$ is a Brauer class such that $r(\alpha) = [C]$.

A precise description of Tate's original definition [Tat58] and its relation with the above definition can be found in §3.1 and §3.2 of [ÇK12]. The above bilinear pairings surprisingly capture the boundary map δ_C. This is originally due to Lichtenbaum in §of [Lic68] by a cocycle computation. The following is Thm. 3.1 of [ÇK12].

Theorem 6.3.14. *Let E be an elliptic curve with rational point $P \in E(k)$ and let C be an E-torsor. Then*

$$\langle [C], P\rangle_{\mathrm{Tate}} = \partial_C(P). \tag{6.3.3}$$

That is: the Tate pairing coincides with the boundary map ∂_C.

Remark 6.3.15. Bilinearity of the Tate pairing implies that

$$\partial_{C_1 \wedge^E C_2}(P) = \partial_{C_1}(P) + \partial_{C_2}(P),$$

for $[C_1],[C_2] \in \mathrm{H}^1(k,E)$.

In contrast to the boundary map ∂_C, the evaluation pairing is reasonably simple to compute if one is given an explicit description of an Azumaya algebra $\mathcal{A}$ corresponding to the Brauer class $\alpha \in \mathrm{Br}(E,\infty)$. This motivates a further study of the Brauer class $\mathcal{A}$ attached to an explicit C by the inverse of r, especially in the case where C is a para-elliptic curve of period 2, as described in Section 6.1. Fix $a \in A^*/A^{*2}$ having square norm and suppose the cohomology class $v(a) \in \mathrm{H}^1(k,E)[2]$ of (6.1.7) equals $[C]$. In [CV15] the authors give the following explicit description for $\mathcal{A}$ as a certain corestriction, relying on a cocycle computation. For an elementary background on corestrictions of central simple algebras, see [Tig87].

Proposition 6.3.16. *The corestriction*

$$\mathrm{Cor}_{\kappa(E)\otimes A/\kappa(E)}(x-\theta, a)_{\kappa(E)\otimes A}. \tag{6.3.4}$$

defines a class in $\mathrm{Br}(E,\infty)$*, which maps to the cohomology class* $[C]$ *under* r.

Proof. By Thm. 1.1 of loc. cit., (6.3.4) defines a class in $\mathrm{Br}(C)[2]$ and hence defines a map $\gamma\colon A^*/A^{*2} \to \mathrm{Br}(C)[2]$. As was remarked in the proof of Prop. 5.1 of op. cit., the composition $A^*/A^{*2} \xrightarrow{\gamma} \mathrm{Br}(C)[2] \xrightarrow{r} \mathrm{H}^1(k,E)$ equals $A^*/A^{*2} \xrightarrow{v} \mathrm{H}^1(k,E)[2] \subset \mathrm{H}^1(k,E)$, by comparing their Prop. 3.2 and Lem. 4.6. (Note that the Υ in loc. cit. is 0 in our context.) ☐

Applying the evaluation pairing yields the boundary map ∂_C. Under the condition that $P = (x, y) \in E(k)$ is not two-torsion, the element $x - \theta$ is invertible in A and hence $(x - \theta, a)_A$ defines a quaternion algebra over A. Thus in this case, the evaluation pairing is simply given by substituting the desired x-coordinate.

Corollary 6.3.17. *Let $a \in A^*/A^{*2}$ such that $v(a) = [C]$, and let $P = (x, y) \in E(k) \setminus E[2](k)$. Then*

$$\partial_C(P) = \mathrm{Cor}_{A/k}(x - \theta, a)_A$$

It can be helpful to have an explicit description of this corestriction as a tensor product of quaternion algebras. This is possible using Rosset–Tate reciprocity; see Prop. 2.4 of [CV15]. For sake of notation, let $K = \kappa(E)$ denote the function field of E.

Theorem 6.3.18. *Suppose $a \in A^* \setminus k^*$. Let $g \in k[x]$ be the polynomial of minimal degree such that $g(\theta) = a$. Set $r_0 = f$ and $r_1 = g$. Then inductively define r_{i+2} as the remainder of r_i upon division by r_{i+1}, i.e. the polynomial such that $r_{i+2} \equiv r_i \bmod r_{i+1}$ with $\deg(r_{i+2}) < \deg(r_{i+1})$. Let c_i be the leading coefficient of r_i and let n be the smallest positive integer such that $r_{n+2} = 0$. Then*

$$\mathrm{Cor}_{K\otimes A/K}\left((x - \theta, a)_{K\otimes A}\right) = \left(\sum_{i=0}^{n} (r_{i+1}, r_i)_K\right) + \left(\sum_{i=0}^{n} (c_{i+1}, c_i)_K\right).$$

The above allows us to algorithmically calculate the obstructions ∂_C of k-points of E, as follows. Let $(f_1, f_2)_K$ be a quaternion algebra in $\mathrm{Br}(E)$, where $f_1, f_2 \in K$, and let $P \in E$ be a rational point such that f_1 and f_2 are regular at P, i.e. $f_1, f_2 \in \mathcal{O}_{E,P}$. The evaluation pairing is computed by $\langle (f_1, f_2)_K, P\rangle_{\mathrm{eval}} = (f_1(P), f_2(P))_k$ in $\mathrm{Br}(k)$.

We return to the context of Remark 6.1.23, where we do not necessarily need the full strength of Theorem 6.3.18. The following Brauer classes are also computed in §6.2 of [Cre16].

Corollary 6.3.19 (Obstructions of two-torsion points of para-elliptic curves of period two)**.** *Suppose that E has an affine Weierstraß equation of the form $y^2 = x(x - \alpha)(x - \beta)$ for certain $\alpha, \beta \in k$. Let $(u, v, w) \in (k^*)^3$ such that the product uvw is a square. Let C be the para-elliptic curve corresponding to the cohomology class $v(a)$, cf. Corollary 6.1.22. Then the obstructions of the 2-torsion points of E equal*

$$\begin{aligned}\partial_C((0,0)) &= (-\alpha, w)_k + (-\beta, v)_k\\ \partial_C((\alpha,0)) &= (\alpha, w)_k + (\alpha - \beta, u)_k\\ \partial_C((\beta,0)) &= (\beta, v)_k + (\beta - \alpha, u)_k.\end{aligned}$$

Proof. In this case evaluation of θ at 0, α and β determines an isomorphism $A \xrightarrow{\sim} k^3$. As such, the corestriction of Proposition 6.3.16 equals the tensor product $\mathcal{A} = (x, u)_K \otimes (x - \alpha, v)_K \otimes (x - \beta, w)_K$, see Lem. 2.2 of [Kra10]. From the equation $y^2 = x(x-\alpha)(x-\beta)$ it follows that $(x, u)_K \cong ((x-\alpha)(x-\beta), u)_K$. This yields Brauer equivalences

$$\mathcal{A} \cong ((x - \alpha)(x - \beta), u)_K \otimes_K (x - \alpha, v)_K \otimes_K (x - \beta, uv)_K \sim (x - \alpha, uv)_K \otimes_K (x - \beta, v)_K.$$

In a similar way, we find Brauer equivalences

$$\begin{aligned}\mathcal{A} &\sim (x, uv)_K \otimes_K (x - \beta, u)_K;\\ \mathcal{A} &\sim (x, v)_K \otimes_K (x - \alpha, u)_K.\end{aligned}$$

By Theorem 6.3.14, the value of $\partial_C(P)$ evaluated a the two-torsion point P is computed by substituting the x-coordinate of P into one of the above expressions. ☐

In a similar way we treat the case where f decomposes as a linear factor times an irreducible quadratic, as in Example 6.1.24.

Corollary 6.3.20. *Suppose that E has an affine Weierstraß equation of the form $y^2 = x^3 + Ax^2 + Bx$ for certain $A, B \in k$. Let $a \in A^*/A^{*2}$ be an element of square norm mapping to the pair (u, d). Assume that $d \in k^*$, so that u is a square. Let C be the para-elliptic curve corresponding to the cohomology class $v(a)$ of Corollary 6.1.22. The k-rational point $(0, 0) \in E[2](k)$ is 2-torsion, and its obstruction is*

$$\partial_C((0,0)) = (B, d)_k.$$

Proof. The assumption that a has square norm implies that ud^2 is square, hence u is a square. Thus without loss of generality we may assume that $u = 1$. The corestriction of Proposition 6.3.16 equals the tensor product

$$\mathcal{A} = (x, u)_K \otimes_K \operatorname{Cor}_{\frac{K[\theta]}{(\theta^2+A\theta+B)}/K}(x - \theta, d) \sim (x^2 + Ax + B, d)_K,$$

in which the first tensor factor vanishes, and the equivalence of the second tensor factor follows from the so-called *projection formula*; see Thm. 3.2 of [Tig87]. By Theorem 6.3.14 we may calculate $\partial_C((0, 0))$ by substituting $x = 0$. ☐

Example 6.3.21. We also compute the corestriction using the description of Theorem 6.3.18 if E has an affine Weierstraß equation of the form $y^2 = x^3 + B$. Under the isomorphism $A = k[x]/(x^3 + Bx) = k \times k[x]/(x^2 + B)$ the element $a = \frac{1-d}{B}\theta^2 + 1$ maps to the pair $(1, d)$. Let $[C] = v(a)$. We thus iteratively calculate

$$\begin{aligned} r_0 &= x^3 + B, & c_0 &= 1; \\ r_1 &= \frac{1-d}{B}x^2 + 1, & c_1 &= \frac{1-d}{B}; \\ r_2 &= \frac{-Bd}{1-d}x, & c_2 &= \frac{-Bd}{1-d}; \\ r_3 &= 1, & c_3 &= 1. \end{aligned}$$

Then the corestriction is the sum of the quaternion algebra's $(r_0, r_1) + (r_1, r_2) + (c_1, c_2)$. We calculate $\partial_C((0, 0))$ by substituting $x = 0$. Note that $r_1(0) = 1$, so the first two terms vanish. Since $(\alpha, (1 - \alpha))_k \cong (\alpha, -\alpha)_k \cong \operatorname{Mat}_{2\times 2}(k)$, it follows that

$$\partial_C((0, 0)) = ((1 - d)B, -(1 - d)B \cdot d)_k = ((1 - d)B, d)_k = (B, d)_k.$$

This particular special case can also be treated in a completely elementary fashion, i.e. without the use of cohomology. We continue in the setting of Example 6.1.24, i.e. for a curve $C = E \wedge^{\mathbb{Z}/2\mathbb{Z}} k(\sqrt{d})$, for a non-square $d \in k^*/k^{*2}$ and a subgroup scheme $\mathbb{Z}/2\mathbb{Z} \subset E$, without loss of generality generated by $(0, 0) \in E(k)$.

Lemma 6.3.22. *In terms of the equation* (6.1.14)*, the curve C admits the sign involutions $\sigma_1 \colon (Z, W) \mapsto (-Z, W)$ and $\sigma_2 \colon (Z, W) \mapsto (Z, -W)$. They differ by translation by the 2-torsion point $(0, 0) \in E[2](k)$.*

Proof. Both involutions are indeed sign involutions, since fixed points occur $Z = 0$ and $W = 0$, respectively. We need to show that the composition of the sign involutions $(Z, W) \mapsto (-Z, -W)$ equals translation by $(0, 0) \in E(k)$. Conceptually this follows from the description $C = E \wedge^{\mathbb{Z}/2\mathbb{Z}} k(\sqrt{d})$, since translation by $(0, 0)$ equals the Galois involution in the second factor, so the equations (6.1.15) show that Z and W both get mapped to their negatives. Alternatively it follows by a short computation, since after base-change to $k(\sqrt{d})$ we may apply the isomorphism $E \otimes k(\sqrt{d}) \xrightarrow{\sim} C \otimes k(\sqrt{d})$ of (6.1.15). Translation by $(0, 0)$ on E is given by

$$x \longmapsto \frac{b}{x}, \qquad y \longmapsto \frac{-by}{x^2},$$

thus $Z = x/y$ and $W = x - b/x$ get mapped to

$$\frac{x}{y} \longmapsto \frac{b/x}{-by/x^2} = -\frac{x}{y} \qquad \text{and} \qquad x - \frac{b}{x} \longmapsto \frac{b}{x} - x = -\Big(x - \frac{b}{x}\Big).$$

Therefore translation by $(0, 0)$ corresponds to mapping Z and W to their negatives. ☐

Proposition 6.3.23. *The rational 2-torsion point $(0, 0)$ on the genus-one curve* (6.1.14) *is obstructed by the Brauer class $\alpha = (d, B)_k$ to come from an element of* $\operatorname{Pic}(C)$.

Remark 6.3.24. In the special case that E is of the form $y^2 = x^3 - 4abx$ with torsor $y^2 = ax^4 + b$ the associated quaternion algebra is $(a, b)_k$, as also calculated using a different method in Ex. 5.2.1 of [ÇK12] and [Han03].

Proof. Each sign involution on C defines a morphism to a Brauer–Severi curve, given by the quotient map. The quotient by $(Z, W) \mapsto (Z, -W)$ defines a map from C to the curve with equation $w = \frac{A^2-4B}{d}z^4 - 2Az^2 + d$, which is isomorphic to $\mathbb{P}^1$. The other sign involution may produce a non-trivial Brauer–Severi curve: it induces a map from C to the conic $B\colon w^2 = \frac{A^2-4B}{d}z^2 - 2Az + d$. We perform a change of variables: let $X = z - \frac{Ad}{A^2-4B}$ and $Y = w$, so that B is also described by the equation

$$\begin{aligned} Y^2 &= \frac{A^2-4B}{d}\Big(X + \frac{Ad}{A^2-4B}\Big)^2 - 2A\Big(X + \frac{Ad}{A^2-4B}\Big) + d \\ &= \frac{A^2-4B}{d}X^2 + 2AX + \frac{A^2d}{A^2-4B} - 2aX - \frac{2A^2d}{A^2-4B} + d \\ &= \frac{A^2-4B}{d}X^2 + d - \frac{A^2d}{A^2-4B}. \end{aligned}$$

Recall that any Brauer–Severi variety has a corresponding element in the Brauer group, called its Brauer class, cf. [Lie17]. Since the equation for B is in a standard form, we read off that the Brauer class of B is given by the quaternion algebra $\big(\frac{A^2-4B}{d}, d - \frac{A^2d}{A^2-4B}\big)_k$. Although performing variable changes for the Brauer–Severi curve is equivalent to constructing isomorphisms between quaternion algebra's, the latter are computationally more convenient, hence we restrict our calculations to those. We recall that for $\alpha, \beta, \beta' \in k^*$ there is an isomorphism of quaternion algebra's $(\alpha, \beta)_k \cong (\alpha, \beta')_k$ if and only if β/β' is of the form $x^2 - \alpha y^2$ for certain $x, y \in k$; see Exc. 5.19 of [Voi21] or Thm. 5.1 of [Con]. As a special case, there is an isomorphism $(\alpha, \beta)_k \cong (\alpha, -\alpha\beta)_k$. Furthermore for $\alpha, \beta, \gamma \in k^*$ there is an isomorphism $(\alpha, \beta)_k \cong (\alpha, \gamma^2\beta)_k$. It therefore follows that

$$\begin{aligned} \Big(\frac{A^2-4B}{d}, d - \frac{A^2d}{A^2-4B}\Big)_k &= (d(A^2-4B), d((A^2-4B)^2 - A^2(A^2-4B)))_k \\ &= (d(A^2-4B), -4Bd(A^2-4B))_k \\ &= (d(A^2-4B), B)_k \quad \text{(by } (\alpha, -4\alpha\beta)_k \cong (\alpha, -\alpha\beta)_k \cong (\alpha, \beta)_k\text{);} \\ &= (d, B)_k \qquad\qquad\quad \text{(since } A^2 - 4B \text{ is of the form } x^2 - By^2\text{).} \end{aligned}$$

Let L be the pullback of $\mathcal{O}_B(1)$ along the quotient map $C \to B$ and let $\mathscr{N}$ be the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ along $C \to \mathbb{P}^1$, both considered as element of $\mathrm{Pic}_{C/k}(k)$. According to Thm. 3.4 of [Lie17], the obstruction of L to come from $\mathrm{Pic}(C)$ is the Brauer class $(d, B)_k \in \mathrm{Br}(k)[2]$, whereas $\mathscr{N}$ is an actual invertible sheaf. Let $\mathscr{L} = L \otimes \mathscr{N}^\vee \in \mathrm{Pic}^0_{C/k}(k)$, which maps to

$$\delta(L \otimes \mathscr{N}^\vee) = \delta(L) - \delta(\mathscr{N}) = (d, B)_k \in \mathrm{Br}(k)[2].$$

It thus suffices to show that $\mathscr{L}$ is 2-torsion and maps to $(0, 0) \in E(k)$ under the isomorphism $\mathrm{Pic}^0_{C/k}(k) = E(k)$. Without loss of generality suppose that k is algebraically closed. Let P and Q be closed fixed points for the sign involutions σ_1 and σ_2 respectively, so that $L = \mathcal{O}_C(2P)$ and $N = \mathcal{O}_C(2Q)$. Let $r \in E(k)$ be any point such that $2r = (0, 0)$, then

$$\sigma_2(P + r) = \sigma_1(P + r) + (0, 0) = \sigma_1(P) + r + (0, 0) = P + r,$$

so $\mathscr{L} = L \otimes \mathscr{N}^\vee = \mathcal{O}_C(2P - 2Q)$ corresponds to $2r = (0, 0) \in E(k)$. □

Remark 6.3.25. Alternatively, to compute that $\mathcal{O}_C(2P - 2Q)$ corresponds to $(0, 0) \in E(k)$, one may use the isomorphism of (6.1.15). We pick the $k(\sqrt{d})$-valued points $P = (0, -\sqrt{d})$ and $Q = \big(\sqrt{d/(A + 2\sqrt{B})}, 0\big)$ on C, that map to the $k(\sqrt{d})$-valued points $p = (0, 0)$ and $q = \big(\sqrt{B}, \sqrt{B\sqrt{B} + A\sqrt{B} + B}\big)$ of E respectively. Diligent elliptic curve arithmetic one verifies that $2q = p = (0, 0)$.

The study of the boundary map ∂_C through the Tate pairing and the evaluation pairing has been quite fruitful: the above results certainly suffice in the context of the computation of Section 5.4. For completeness, we briefly mention a third pairing related to the Tate and evaluation pairing. Let n be a positive integer coprime to the characteristic exponent p, which should be thought of as the period of the para-elliptic curve C.

Definition 6.3.26. The cup product $\smile$ and the Weil pairing $e_n\colon E[n]\otimes E[n]\to\mu_n$ induce a bilinear pairing

$$(\cdot,\cdot)_n\colon\ \mathrm{H}^1(k,E[n])\times\mathrm{H}^1(k,E[n])\xrightarrow{\smile}\mathrm{H}^2(k,E[n]\otimes E[n])\xrightarrow{e_{n,*}}\mathrm{H}^2(k,\mu_n)=\mathrm{Br}(k)[n].$$

Remark 6.3.27. The Weil pairing depends substantially on n: Prop. 8.1.e of [Sil09] states that the Weil pairings satisfy the compatibility law for the Weil pairing $e_{nm}(P,Q)=e_n(P,mQ)$, where $P\in E[n](k)$ and $Q\in E[nm](k)$, where n and m are positive integers coprime to the characteristic exponent. Letting $i\colon E[n]\to E[nm]$ be the inclusion, then the above pairing inherits the compatibility $(i_*\gamma,\delta)_{nm}=(\gamma_1,m\gamma_2)_n$, where $\gamma\in\mathrm{H}^1(k,E[n])$ and $\delta\in\mathrm{H}^1(k,E[nm])$.

The above pairing is related to the Tate pairing. This was shown in Prop. 9 of [Bas72]. Given an elliptic curve E, recall that we have the natural map in cohomology $\mathrm{H}^1(k,E[n])\to\mathrm{H}^1(k,E)[n]$, as well as the boundary map $\delta_n\colon E(k)\to\mathrm{H}^1(k,E[n])$.

Theorem 6.3.28. *Let $[C]\in\mathrm{H}^1(k,E)[n]$ be the cohomology class of a para-elliptic curve with a lift $\gamma\in\mathrm{H}^1(k,E[n])$. Furthermore, let $P\in E(k)$ be a rational point. Then*

$$\langle[C],P\rangle_{\mathrm{Tate}}=(\gamma,\delta_n(P))_n.$$

Remark 6.3.29 (Refinement to cyclic isogenies). Suppose $\mathbb{Z}/n\mathbb{Z}\subset E[n]$ is a subgroup scheme, whose quotient $E[n]/(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to μ_n by self-duality of elliptic curves. Suppose that the cohomology class γ lies in the image of $\mathrm{H}^1(k,\mathbb{Z}/n\mathbb{Z})\to\mathrm{H}^1(k,E[n])$. It is not difficult to see that in this case the pairing $(\gamma,\delta_n(P))_n$ only depends on the image of $\delta_n(P)$ under the natural map $\mathrm{H}^1(k,E[n])\to\mathrm{H}^1(k,\mu_n)$, which is $\delta_E(P)$ by the obvious generalisation of (6.2.2). The Weil pairing restricts to $e_n\colon\mathbb{Z}/n\mathbb{Z}\otimes\mu_n\xrightarrow{\sim}\mu_n$, in which case we have

$$\langle[C],P\rangle_{\mathrm{Tate}}=e_n(\gamma\cup\delta_E(P));$$

see Prop. 4.1 of [ÇK12].

Let $n=2$, so also assume $p\neq 2$. This tool allows us to give another proof of Corollary 6.3.17. The main idea is the standard fact that the cup product $\mathrm{H}^1(k,\mu_2)\times\mathrm{H}^1(k,\mu_2)\to\mathrm{H}^2(k,\mu_2)$ maps a pair of elements $\alpha,\beta\in k^*/k^{*2}$ to the quaternion algebra $(\alpha,\beta)_k$. We fix an isomorphism $\mu_2\cong\mathbb{Z}/2\mathbb{Z}$. Multiplication provides an isomorphism $\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z}\xrightarrow{\sim}\mathbb{Z}/2\mathbb{Z}$. We let $\star$ denote the induced isomorphism $\mu_2\otimes\mu_2\xrightarrow{\sim}\mu_2$, which is given on k^{sep}-points by

$$1\star 1=1,\qquad 1\star -1=1,\qquad -1\star 1=1,\qquad -1\star -1=-1.$$

Lemma 6.3.30. *Recall the map $E[2]\to\mathrm{Res}_{A/k}\,\mu_{2,A}$ of Section 6.2. The following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{H}^1(k,E[2])\times\mathrm{H}^1(k,E[2]) \arrow[r] \arrow[d, "\smile"] & \mathrm{H}^1(k,\mathrm{Res}_{A/k}\,\mu_{2,A})\times\mathrm{H}^1(k,\mathrm{Res}_{A/k}\,\mu_{2,A}) \arrow[d, "\smile"] \\
\mathrm{H}^2(k,E[2]\otimes E[2]) \arrow[d, "e_{2,*}"] & \mathrm{H}^2(k,\mathrm{Res}_{A/k}\,\mu_{2,A}\otimes\mathrm{Res}_{A/k}\,\mu_{2,A}) \arrow[d, "\star_{A,*}"] \\
\mathrm{H}^2(k,\mu_2) & \mathrm{H}^2(k,\mathrm{Res}_{A/k}\,\mu_{2,A}) \arrow[l, "\mathrm{Norm}_{A/k,*}"']
\end{tikzcd}\tag{6.3.5}$$

Proof. Since all group schemes in question are smooth, we may calculate all cohomology groups in the étale cohomology, which is simply the Galois cohomology of the field k. We verify the statement by a cocycle computation in Galois cohomology. Fix a separable closure k^{sep}. We let σ and τ denote arbitrary elements of the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. Represent an arbitrary element of $\mathrm{H}^1(k,E[2])\times\mathrm{H}^1(k,E[2])$ by the pair of 1-cocycles (ψ,φ) with values in $E[n]$.

We first calculate the downward composition, not involving A. The cup-product is represented by the 2-cocycle $(\sigma,\tau)\mapsto\psi(\sigma)\otimes\sigma(\varphi(\tau))$, which under the Weil pairing maps to the 2-cocycle with values in μ_2 given by

$$(\sigma,\tau)\longmapsto e_2(\psi(\sigma),\sigma(\varphi(\tau)))=\begin{cases}+1 & \text{if }\sigma^{-1}(\psi(\sigma))=\varphi(\tau),\\ -1 & \text{otherwise.}\end{cases}\tag{6.3.6}$$

Recall that the map $E[2] \to \operatorname{Res}_{A/k} \mu_{2,A}$ is given on k^{sep}-points by $P \mapsto e_2((\theta, 0), P)$, where $(\theta, 0) \in E(A)$; see Example 6.1.1. As such, the cup-product is represented by the 2-cocycle with values in $\mu_2(A \otimes k^{\mathrm{sep}}) \otimes \mu_2(A \otimes k^{\mathrm{sep}})$ given by

$$(\sigma, \tau) \longmapsto e_2((\theta, 0), \psi(\sigma)) \otimes e_2((\theta, 0), \sigma(\varphi(\tau))). \tag{6.3.7}$$

Let P_1, P_2, $P_3 \in E[2](k^{\mathrm{sep}})$ be the three non-identity two-torsion points. Recall that, under the canonical isomorphism $\mu_2(A \otimes k^{\mathrm{sep}}) \cong \mu_2(k^{\mathrm{sep}})^3$, the Weil pairing $e_2((\theta, 0), P_i)$ is given by a permutation of the tuple $(+1, -1, -1)$, where the positive entry is in the ith index, q.v. Example 6.1.3. Applying the pushforward of $\star_A$ corresponds to applying $\star$ componentwise. We calculate that

$$\begin{aligned} &(+1, -1, -1) \star (+1, -1, -1) = (+1, -1, -1) \text{ has norm } +1\text{; and,} \\ &(+1, -1, -1) \star (-1, +1, -1) = (+1, +1, -1) \text{ has norm } -1. \end{aligned}$$

The other possibilities for $\star$ follow by permutations. We conclude that the norm of $e_2((\theta, 0), P_i) \otimes e_2((\theta, 0), P_j)$ equals 1 if and only if $i = j$. In other words, (6.3.7) maps to the 2-cocycle (6.3.6), as desired. ☐

Proof of Corollary 6.3.17. Let $\gamma \in \mathrm{H}^1(k, E[2])$ mapping to $[C] = v(a)$. Then $\partial_C(P) = (\gamma, \delta_n(P))_n$. By definition, the composition $\mathrm{H}^1(k, E[2]) \times \mathrm{H}^1(k, E[2]) \to \mathrm{H}^2(k, \mu_2)$ of Lemma 6.3.30 is the pairing $(\cdot, \cdot)_n$. We therefore chase the diagram (6.3.5) along the rightward composition.

Under the isomorphisms $\mathrm{H}^1(k, \operatorname{Res}_{A/k} \mu_{2,A}) \cong \mathrm{H}^1(A, \mu_{2,A}) \cong A^*/A^{*2}$, the elements γ and $\delta_n(P)$ map to a and $(x - \theta)(P)$, respectively, by Theorem 6.1.10. The cup-product map $A^*/A^{*2} \times A^*/A^{*2} \to \mathrm{H}^2(A, \mu_{2,A})$ maps a pair (u, v) to the quaternion algebra $(u, v)_A$. Hence in our case, the image in $\mathrm{H}^2(A, \mu_{2,A})$ is the quaternion algebra $((x - \theta)(P), a)_A$. The norm map corresponds directly to the corestriction. Finally, since $P = (x, y)$ is not a two-torsion point, we have $(x - \theta)(P) = x - \theta$. ☐

Chapter 7

Cohomological invariants through the Néron–Severi group scheme

7.1 The torsion of the Néron–Severi group scheme

Let X be a proper geometrically integral and geometrically normal scheme X, so that the Picard scheme $\mathrm{Pic}_{X/k}$ is representable by a proper scheme. The *Néron–Severi group* is defined as the quotient group $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. It is a theorem of Severi, called the theorem of the base, that $\mathrm{NS}(X)$ is finitely generated as abstract group. We study a group scheme theoretic version by replacing the Picard group by the Picard scheme.

Notation 7.1.1. The *étale Néron–Severi group scheme* $\mathrm{NS}^{\text{ét}}_{X/k}$ is the quotient $\mathrm{NS}^{\text{ét}}_{X/k} = \mathrm{Pic}_{X/k} / \mathrm{Pic}^0_{X/k}$.

In the literature this quotient is sometimes instead called the *Néron–Severi group scheme.* Our choice of terminology is motivated by the following property.

Proposition 7.1.2. *The group scheme* $\mathrm{NS}^{\text{ét}}_{X/k}$ *is étale.*

Proof. Note that $\mathrm{NS}^{\text{ét}}_{X/k} = \mathrm{Pic}_{X/k} / \mathrm{Pic}^0_{X/k} = \pi_0(\mathrm{Pic}_{X/k})$ is naturally the group scheme of connected components of the Picard scheme, which is étale by Thm. 2.4.1 of [Bri17]. ☐

Although this may seem to be a desirable property for a Néron–Severi group scheme, we are on the contrary unable to observe certain essential infinitesimal parts. The Picard scheme $\mathrm{Pic}_{X/k}$ may be non-reduced in positive characteristic. In the quotient, the non-reducedness is offset by the non-reducedness of the connected component $\mathrm{Pic}^0_{X/k}$, which explains why $\mathrm{NS}^{\text{ét}}_{X/k}$ is reduced. We are therefore unable to observe the infinitesimal remnants in the étale Néron–Severi group scheme. We instead consider the quotient $\mathrm{Pic}_{X/k} / \mathrm{Pic}^{\alpha}_{X/k}$, where $\mathrm{Pic}^{\alpha}_{X/k}$ denotes the maximal abelian subvariety. Recall that we may identify $\mathrm{Pic}^{\alpha}_{X/k}$ with the reduced subscheme $(\mathrm{Pic}^0_{X/k})_{\mathrm{red}}$ by Lem. 3.3.7 of [Bri17], using the fact that the Picard scheme is proper; cf. Section 3.2. The reduced subscheme of a group scheme in general does not inherit the structure of a group scheme over imperfect ground fields. Since the maximal abelian subvariety does not have these flaws, it is for this reason that it is more natural to work with it instead.

Definition 7.1.3. The *Néron–Severi group scheme* $\mathrm{NS}_{X/k}$ is the quotient $\mathrm{NS}_{X/k} = \mathrm{Pic}_{X/k} / \mathrm{Pic}^{\alpha}_{X/k}$.

Remark 7.1.4. Since the formation of the Picard scheme and its maximal abelian subvariety commute with base change, so does the Néron–Severi group scheme.

Remark 7.1.5. The third isomorphism theorem implies that $\mathrm{NS}^{\text{ét}}_{X/k} = \mathrm{NS}_{X/k} / \mathrm{NS}^0_{X/k}$. Therefore the étale Néron–Severi group scheme is naturally isomorphic to the group scheme of connected components $\pi_0(\mathrm{NS}_{X/k})$ of the Néron–Severi group scheme, justifying our earlier statement regarding the unobserved infinitesimal parts.

A related definition is that of the *numerical group scheme.* Following §3 of [LS23] we define it as the quotient $\mathrm{Num}_{X/k} = \mathrm{Pic}_{X/k} / \mathrm{Pic}^{\tau}_{X/k}$. The most essential difference between the numerical group scheme and the Néron–Severi group scheme is that $\mathrm{Num}_{X/k}(k')$ is free group for every field extension

k'/k, whereas $\mathrm{NS}_{X/k}(k')$ may contain torsion elements. Otherwise they are closely related: the groups of k'-points are both finitely generated abstract groups, whose ranks of the free parts are both equal to the Picard number ρ of X, and furthermore the cardinality of the torsion subgroup of $\mathrm{NS}_{X/k}(k')$ is bounded in k'; see Exp. XIII, Thm. 5.1 of [SGA 6]. It follows that the torsion part of $\mathrm{NS}^{\tau}_{X/k}$ is finite. An explicit bound for the order of $\mathrm{NS}^{\tau}_{X/k}$ is given in Thm. 5.10 of [Kwe22]. Note that $\mathrm{NS}^{\tau}_{X/k}$ is the kernel of the quotient map $\mathrm{NS}_{X/k} \to \mathrm{Num}_{X/k}$, and by the third isomorphism theorem is hence isomorphic to the quotient $\mathrm{Pic}^{\tau}_{X/k} / \mathrm{Pic}^{\alpha}_{X/k}$. Throughout this chapter we consider the following shorthand notation.

Notation 7.1.6. Let $\Gamma = \mathrm{NS}^{\tau}_{X/k} = \mathrm{Pic}^{\tau}_{X/k} / \mathrm{Pic}^{\alpha}_{X/k}$ be the torsion subgroup scheme of $\mathrm{NS}_{X/k}$.

The group scheme Γ is also particularly interesting in the study of the group scheme $\mathrm{Pic}^{\tau}_{X/k}$ from a structural perspective. In the theory of algebraic groups, one often studies a group scheme (resp. an algebraic group) G through its *affinisation* $G^{\mathrm{aff}} = \operatorname{Spec} \Gamma(G, \mathcal{O}_G)$, which inherits the structure of a group scheme (resp. an algebraic group); see for example the theory as developed in [Bri17; Mil17], but see also §7 of [LS23]. Note that there is a natural morphism $G \to G^{\mathrm{aff}}$, which is a surjective group scheme homomorphism. Its kernel, often denoted G_{ant}, is often called the *anti-affinisation*, which is *anti-affine*, i.e. $(G_{\mathrm{ant}})^{\mathrm{aff}} = 0$. In this way, any group scheme is an extension of an affine group scheme by an anti-affine group scheme. Applying this to the group scheme $G = \mathrm{Pic}^{\tau}_{X/k}$, the following result further motivates the study of Γ.

Proposition 7.1.7. *The anti-affinisation* $(\mathrm{Pic}^{\tau}_{X/k})_{\mathrm{ant}}$ *coincides with the maximal abelian subvariety* $\mathrm{Pic}^{\alpha}_{X/k}$. *Therefore the quotients* $(\mathrm{Pic}^{\tau}_{X/k})^{\mathrm{aff}}$ *and* Γ *are isomorphic.*

Proof. We crucially use the standing assumption that $\mathrm{Pic}^{\tau}_{X/k}$ is proper. It follows that the affinisation is proper and anti-affine, so also smooth and connected by Lem. 3.3.2 of [Bri17]. We conclude that there is an inclusion $(\mathrm{Pic}^{\tau}_{X/k})_{\mathrm{ant}} \subset \mathrm{Pic}^{\alpha}_{X/k}$. The other inclusion holds generally, since the canonical morphism $\mathrm{Pic}^{\alpha}_{X/k} \to (\mathrm{Pic}^{\tau}_{X/k})^{\mathrm{aff}}$ from a proper scheme to an affine scheme is constant; cf. p. 24 of [LS23]. ☐

We now specialise to bielliptic surfaces. Remarkably, they are the only class within the tetrachotomy of minimal smooth surfaces of $\mathrm{kod}(X) = 0$ where torsion in the Néron–Severi group scheme may occur. We motivate the importance of the torsion of the Néron–Severi group scheme for bielliptic surfaces especially.

For exposition, suppose first that $k = \mathbb{C}$. Then there is a chain of isomorphisms

$$\mathrm{NS}(X)_{\mathrm{tors}} = \mathrm{Pic}^{\tau}(X)/\mathrm{Pic}^{0}(X) \cong \mathrm{H}^1(X, \mathbb{Z})_{\mathrm{tors}} \cong \mathrm{H}^2(X, \mathbb{Z})_{\mathrm{tors}} \cong \mathrm{Br}(X)_{\mathrm{tors}}, \tag{7.1.1}$$

where the first and last isomorphisms rely on the long exact sequence associated to the exponential short exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$ and vanishing of $h^2(\mathcal{O}_X)$, whereas the middle isomorphism follows from the Universal Coefficient Theorem, cf. §1 of [Fer+22] and §2.1 of [Boa21]. Note that $\mathrm{NS}(X)_{\mathrm{tors}}$ is hence also a topological invariant of the analytification of X.

The torsion subgroup $\mathrm{H}^1(X, \mathbb{Z})_{\mathrm{tors}}$ was determined over the field of complex numbers in [Ser90] according to the type of the bielliptic surface X. Explicit generators of $\mathrm{H}^2(X^{\mathrm{an}}, \mathbb{Z})_{\mathrm{tors}}$ can be found in [Fer+22]: for later reference we note that the generators are differences of multiple fibres of the elliptic fibration $g\colon X \to \mathbb{P}^1_{\mathbb{C}}$. Over arbitrary ground fields, a chain of isomorphisms comparable to (7.1.1) hold in étale and crystalline cohomology; q.v. Ch. 1, §2 of [Boa21].

Let k again be an arbitrary ground field. We compute the torsion subgroup scheme of $\mathrm{NS}_{X/k}$ in a large number of cases of the Bagnera–de Franchis classification of bielliptic surfaces. Namely, the bielliptic surfaces with a smooth Albanese, and the quasi-bielliptic surfaces of Jacobian type.

This is of importance for the algebraic de Rham cohomology of those bielliptic surfaces in view of [Suw83], in which the author determines the Hodge and de Rham numbers for surfaces for which the Albanese dimension equals $h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X)$ in terms of $\mathrm{NS}^{\tau}_{X/k}$. The Hodge and de Rham numbers were previously computed for bielliptic surfaces with smooth Albanese, quasi-bielliptic surfaces in characteristic 3 (both §4 of [Lan79]), and quasi-bielliptic surfaces in characteristic 2 of type (a1) (see §9 of [Sch21b]). To this list we add the four other Jacobian quasi-bielliptic surfaces in characteristic 2, which are classical or supersingular of type (c1) or (d).

As discussed in Chapter 4, it is a general principle that a bielliptic surface is studied best through its canonical covers and its BdF-covers. We apply this strategy to the torsion of its Néron–Severi group scheme. We state the critical lemma in a general form, which applies to both situations at once.

Lemma 7.1.8. *Let X be a proper geometrically integral surface with Albanese $f\colon X \to P$. Let $\widetilde{P} \to P$ be an isogeny of para-abelian varieties with kernel K and let $\widetilde{X} = X \times_P \widetilde{P} \to X$ be the pullback torsor. Consider the induced K-action on the Picard scheme $\mathrm{Pic}^\tau_{\widetilde{X}/k}$ by pullback of invertible sheaves. Then the pullback map $\mathrm{Pic}^\tau_{X/k} \to \mathrm{Pic}^\tau_{\widetilde{X}/k}$ induces an inclusion $\Gamma \to \mathrm{Coker}(\mathrm{Pic}^\tau_{\widetilde{P}/k} \to (\mathrm{Pic}^\tau_{\widetilde{X}/k})^K)$. Furthermore, it is an isomorphism if K satisfies one of the additional assumptions of Theorem 1.3.14.*

Proof. The pullback square for $\widetilde{X}$ induces a commutative square on Picard schemes.

$$\begin{tikzcd}
\mathrm{Pic}^\tau_{P/k} \arrow[r] \arrow[d, "f^*"] & \mathrm{Pic}^\tau_{\widetilde{P}/k} \arrow[d] \\
\mathrm{Pic}^\tau_{X/k} \arrow[r] & \mathrm{Pic}^\tau_{\widetilde{X}/k}
\end{tikzcd}$$

Recall that f^* induces an isomorphism $\mathrm{Pic}^\tau_{P/k} \xrightarrow{\sim} \mathrm{Pic}^\alpha_{X/k}$. The kernels of the horizontal arrows are isomorphic to $K^\vee$ due to Theorem 1.3.14. The induced map between kernels $K^\vee \to K^\vee$ is injective by injectivity of f^*. Since the group schemes are finite of the same order, it is therefore an isomorphism. The images of the horizontal maps are contained inside $(\mathrm{Pic}^\tau_{\widetilde{P}/k})^K$ and $(\mathrm{Pic}^\tau_{\widetilde{X}/k})^K$, respectively. The group scheme K acts by translations on $\widetilde{P}$, so its induced action on the Picard scheme is trivial. The map on K-invariants is hence $\mathrm{Pic}^\tau_{P/k} \to \mathrm{Pic}^\tau_{\widetilde{P}/k}$, which is an isogeny of abelian varieties, hence surjective. We conclude that the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & K^\vee \arrow[r] \arrow[d, "\cong"] & \mathrm{Pic}^\tau_{P/k} \arrow[r] \arrow[d, "f^*"] & \mathrm{Pic}^\tau_{\widetilde{P}/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^\vee \arrow[r] & \mathrm{Pic}^\tau_{X/k} \arrow[r] & (\mathrm{Pic}^\tau_{\widetilde{X}/k})^K &
\end{tikzcd}$$

Functoriality of cokernels of the horizontal maps yields $\Gamma = \mathrm{Coker}(f^*) \to \mathrm{Coker}(\mathrm{Pic}^\tau_{\widetilde{P}/k} \to (\mathrm{Pic}^\tau_{\widetilde{X}/k})^K)$, which is injective by the snake lemma. Surjectivity holds under one of the additional assumptions of Theorem 1.3.14, since then the map $\mathrm{Pic}^\tau_{X/k} \to (\mathrm{Pic}^\tau_{\widetilde{X}/k})^K$ is surjective. ☐

We apply the above Lemma successively to a canonical cover and a BdF-cover of a bielliptic surface, for the latter assuming it exists. First fix a canonical cover $\pi\colon Y \to X$, with Stein factors C and D. The Picard schemes $A^\vee = \mathrm{Pic}^\tau_{Y/k}$, $E^\vee = \mathrm{Pic}^\tau_{C/k}$ and $J^\vee = \mathrm{Pic}^\tau_{D/k}$ inherit N-actions through the pullback of invertible sheaves on Y, C and D, respectively. Note that, since N acts by translations on C, the resulting action on $E^\vee$ is trivial.

Proposition 7.1.9. *Suppose X is not a supersingular quasi-bielliptic surface of type (d). Then the pullback map $\pi^*\colon \mathrm{Pic}^\tau_{X/k} \to A^\vee$ induces an isomorphism*

$$\Gamma \xrightarrow{\sim} \mathrm{Coker}(E^\vee \to (A^\vee)^N).$$

If X is supersingular quasi-bielliptic of type (d), then the induced map $\Gamma \to \mathrm{Coker}(E^\vee \to (A^\vee)^N)$ is injective.

Remark 7.1.10. We show below that for supersingular quasi-bielliptic surfaces of type (d) both the domain and codomain are trivial, hence the map is is also an isomorphism in this case.

Proof. Apply Lemma 7.1.8 to a canonical cover $\pi\colon Y \to X$. The natural map $\Gamma \to \mathrm{Coker}(E^\vee \to (A^\vee)^N)$ is always injective. If X is not a supersingular quasi-bielliptic surface of type (d), then N is isomorphic to either $\mathbb{Z}/n\mathbb{Z}$, μ_n or α_p, each of which satisfies the additional assumptions of Theorem 1.3.14. Therefore surjectivity also holds. ☐

Even though this cokernel can be quite mysterious, we relate it naturally to $\widetilde{J}^\vee$ through the following short exact sequence, that we proved in the coarse of Proposition 3.5.21 in the setting of an algebraically closed ground field, but whose proof used only the existence of a BdF-cover; cf. (3.5.3). We also note the similarity to the dual short exact sequence (2.3.1) if X has a smooth Albanese.

Lemma 7.1.11. *Suppose that X admits a BdF-cover. Then the sequence*

$$0 \longrightarrow E^\vee \longrightarrow A^\vee \longrightarrow \widetilde{J}^\vee \longrightarrow 0 \tag{7.1.2}$$

induced by the natural maps $A \to E$ and $\widetilde{J} \to A$, is short exact.

This results in the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & E^\vee \arrow[r] \arrow[d, "\mathrm{id}"] & (A^\vee)^N \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^\vee \arrow[r] & A^\vee \arrow[r] & \widetilde{J}^\vee \arrow[r] & 0
\end{tikzcd}$$

The natural map $\Gamma \to \widetilde{J}$ is injective as a consequence of the snake lemma. Applying Lemma 7.1.8 to a Bagnera–de Franchis cover of X actually yields a stronger result.

Proposition 7.1.12. *Suppose that X admits a Bagnera–de Franchis cover $\widetilde{C} \times \widetilde{D} \to X$. The pullback map $\mathrm{Pic}^\tau_{X/k} \to \widetilde{E}^\vee \times \widetilde{J}^\vee$ induces an injective morphism $\Gamma \to (\widetilde{J}^\vee)^G$.*

Proof. A BdF-cover $Z \to X$ is the pullback of a G-cover $\widetilde{C} \to P$. Then Lemma 7.1.8 yields an injective map $\Gamma \to \mathrm{Coker}(\mathrm{Pic}^\tau_{\widetilde{C}/k} \to (\mathrm{Pic}^\tau_{\widetilde{C}\times\widetilde{D}/k})^G) = (\widetilde{J}^\vee)^G$. □

The above statements, in particular Proposition 7.1.9, allow us to Γ concretely in most cases of the Bagnera–de Franchis classification. We cover two special cases, the first being bielliptic surfaces with $d = 1$, in which case the notions of canonical cover and BdF-cover coincide. The following resulted is stated in Prop. 4.1 of [Lan79] for bielliptic surfaces with a smooth Albanese; see also Lem. 8.1.2 of [Sko01] for the specific case of tame characteristic (a1). We emphasise that our statement uniformly holds for all bielliptic surfaces of Jacobian type, including quasi-bielliptic surfaces.

Theorem 7.1.13. *Let X be bielliptic surface with $d = 1$. Then $\Gamma = (J^\vee)^N$.*

Proof. The condition that $d = 1$ implies that $G = N$, and that a canonical cover $Y \to X$ is a BdF-cover, because the natural map $Y \to C \times D$ is an isomorphism. On the level of Picard schemes, this means that $A^\vee = E^\vee \times J^\vee$, so $\Gamma = \mathrm{Coker}(E^\vee \to (A^\vee)^N) = (J^\vee)^N$. □

We now treat the second special case, namely where the bielliptic surface X is assumed to have a smooth Albanese. This special case is manageable due to the absence of certain critical behaviour, as discussed in Section 4.3. For example, the map $Y \to C \times D$ is an isogeny of para-abelian varieties, the associated map of abelian surfaces $A \to E \times J$ has kernel H, hence $Y \to C \times D$ is an H-torsor, and the quotient map $\widetilde{D} \to D$ is an H-torsor.

Recall that we have equipped $A^\vee$, $E^\vee$ and $J^\vee$ with the N-action arising form the pullback of invertible sheaves on Y, C and D, respectively. Since X may not have a BdF-cover, we cannot equip $\widetilde{E}^\vee$ and $\widetilde{J}^\vee$ with a G-action in a similar way. However, $\widetilde{E}^\vee$ and $\widetilde{J}^\vee$ are quotients of $E^\vee$ and $J^\vee$, respectively, by the N-stable subgroup scheme $H^\vee$. In this way, we still equip them with a canonical N-action, which can then be extended trivially to a G-action. Note that if X does admit a BdF-cover then the two actions agree: this is clear for $\widetilde{E}$ since the G-action is trivial; for $\widetilde{J}$ this relies on the fact that $\widetilde{D} \to D$ is an H-torsor, so the induced map $\mathrm{Pic}^0_{D/k} \to \mathrm{Pic}^0_{\widetilde{D}/k}$ agrees with the quotient map by Theorem 1.3.14.

Theorem 7.1.14. *Let X be a bielliptic surface with smooth Albanese. Suppose that it admits a Bagnera–de Franchis cover. Then there is a natural isomorphism*

$$\Gamma = (J^\vee)^N / H^\vee.$$

Proof. Consider the short exact sequence (7.1.2). The latter term $\widetilde{J}^\vee$ is defined to be the quotient $J^\vee/H^\vee$. Since the Albanese map is smooth, it follows from Proposition 4.3.4 that $A^\vee = (E^\vee \times J^\vee)/H^\vee$, where $H^\vee \subset E^\vee \times J^\vee$ is the anti-diagonal embedding of the subgroup schemes $H^\vee \subset E^\vee$ and $H^\vee \subset J^\vee$. In particular, taking the quotient commutes with the projection to $J^\vee$, so that the following diagram is commutative.

$$\begin{tikzcd}
E^\vee \times J^\vee \arrow[r] \arrow[d] & J^\vee \arrow[d] \\
A^\vee \arrow[r] & \widetilde{J}^\vee
\end{tikzcd}$$

Note that all maps are surjective. Consider hence the kernels: the above diagram extends to the following commutative diagram, with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H^\vee \arrow[r, "\mathrm{id}"] \arrow[d] & H^\vee \arrow[d] & \\
0 \arrow[r] & E^\vee \arrow[r] \arrow[d, "\mathrm{id}"] & E^\vee \times J^\vee \arrow[r] \arrow[d] & J^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^\vee \arrow[r] & A^\vee \arrow[r] \arrow[d] & \widetilde{J}^\vee \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We note that the maps in the above diagram are N-equivariant; consider the induced diagram of N-invariants. Since the N-action on the kernels $E^\vee$ and $H^\vee$ is trivial, Lemma 7.3.3 applies in the top row and the left column, which results in the surjectivity of the maps $E^\vee \times (J^\vee)^N \to (J^\vee)^N$ and $E^\vee \times (J^\vee)^N \to (A^\vee)^N$. (For the former map, this is quite clear.) The other maps may not be surjective on N-invariants, so consider the cokernels of $(A^\vee)^N \to (\widetilde{J}^\vee)^N$ and $(J^\vee)^N \to (\widetilde{J}^\vee)^N$, denoted ∂ and δ, respectively. Thus the following diagram is commutative with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H^\vee \arrow[r, "\mathrm{id}"] \arrow[d] & H^\vee \arrow[d] & \\
0 \arrow[r] & E^\vee \arrow[r] \arrow[d, "\mathrm{id}"] & E^\vee \times (J^\vee)^N \arrow[r] \arrow[d] & (J^\vee)^N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^\vee \arrow[r] & (A^\vee)^N \arrow[r] \arrow[d] & (\widetilde{J}^\vee)^N \arrow[r, "\partial"] \arrow[d, "\delta"] & \mathrm{Coker} \\
 & & 0 & \mathrm{Coker} &
\end{tikzcd} \tag{7.1.3}$$

Note that the cokernel of $E^\vee \to (A^\vee)^N$ is isomorphic to $\mathrm{Ker}(\partial)$. Now a simple diagram chase shows that $\mathrm{Ker}(\partial) = \mathrm{Ker}(\delta)$, which is canonically isomorphic to $(J^\vee)^N/H^\vee = (J^\vee)^N/H^\vee$. ☐

Remark 7.1.15. Note that Theorems 7.1.13 and 7.1.14 agree whenever both are applicable: if X is a bielliptic surface with smooth Albanese and $d = 1$, then $\widetilde{J}^\vee = J^\vee$ and $H = 0$.

In the two cases described above, a concrete application of Theorem 7.1.13 or 7.1.14 requires an explicit description of the invariants of $J^\vee$. If the ground field k is algebraically closed, this fixed locus is computable using the explicit description of the action of G on $\widetilde{D}$ of the Bagnera–de Franchis classification of Section 3.4. We postpone the computations to Section 7.3, but already state the results here. Although our method is unable to compute Γ for a supersingular bielliptic surface of type (b2), this case was treated by Lang in [Lan79] through a detailed study of quasi-elliptic fibrations in characteristic 3; see Thm. 3.2 and the comments on its p. 489.

Theorem 7.1.16. *Let k be an algebraically closed ground field. The different isomorphism classes of the torsion of the Néron–Severi group scheme Γ are tabulated in Table 7.1. The variable λ arising for supersingular quasi-bielliptic surfaces of type (a1) refers to the corresponding variable in the Bagnera–de Franchis classification Theorem 3.4.12. The values with a question mark are conjectural; see Conjecture 7.1.23.*

Remark 7.1.17. Note that the order of the group scheme Γ seems to be constant among bielliptic surfaces of the same type, at least in all cases that we can confirm. This would not have been the case if we had considered the torsion part of the étale Neron–Severi group scheme $\mathrm{NS}^{\text{ét},\tau}_{X/k}$, because non-smooth group schemes occur in Table 7.1.

Type	Tame Char.	Ordinary Biell.	Classical Q.-Biell.	Supersing. Q.-Biell.
(a1)	$(\mathbb{Z}/2\mathbb{Z})^2$	$J^\vee[2] = \begin{cases} \mu_2 \times \mathbb{Z}/2\mathbb{Z} \\ M_2 \end{cases}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\begin{cases} \alpha_2 \times \mathbb{Z}/2\mathbb{Z} & \text{if } \lambda \neq 0; \\ \alpha_4 & \text{if } \lambda = 0; \end{cases}$
(a2)	$\mathbb{Z}/2\mathbb{Z}$	μ_2	$\mathbb{Z}/2\mathbb{Z}$?	α_2?
(b1)	$\mathbb{Z}/3\mathbb{Z}$	α_3	$\mathbb{Z}/3\mathbb{Z}$	α_3
(b2)	0	$\nexists$	0	$\nexists$
(c1)	$\mathbb{Z}/2\mathbb{Z}$	α_2	$\mathbb{Z}/2\mathbb{Z}$	$\nexists$
(c2)	0	$\nexists$	0?	$\nexists$
(d)	0	0	0	0

Table 7.1: The group scheme $\Gamma = \mathrm{NS}^\tau_{X/k}$ in all cases of the Bagnera–de Franchis classification

Example 7.1.18. The failure of the fixed locus functor to be right-exact can be observed explicitly for bielliptic surfaces of non-Jacobian type with a smooth Albanese. For example: let X be a bielliptic surface with smooth Albanese of type (a2). Then the action of $N \cong \mathbb{Z}/2\mathbb{Z}$ on $J^\vee$ and $\widetilde{J}^\vee$ is by the sign involution, hence the N-invariants comprises the two-torsion. Then $J^\vee[2]/H^\vee \neq (J^\vee/H^\vee)[2]$, due to the discrepancy in the orders. In general, we note that if ∂ vanishes if and only if δ vanishes in (7.1.3), as can be shown by a small diagram chase.

Example 7.1.19. We return to the case where $k = \mathbb{C}$ the field of complex numbers, where there is an isomorphism $\mathrm{NS}(X)_{\mathrm{tors}} \cong \mathrm{H}^1(X, \mathbb{Z})_{\mathrm{tors}}$. Since this is a topological invariant, the above table separates a number of types up to homeomorphism. It in fact turns out that bielliptic surfaces of different types are not homeomorphic: they can be further distinguished by the abelianisation of the inner automorphism group of the fundamental group; see Table II of [Iit69].

Example 7.1.20. Suppose again $k = \mathbb{C}$. The torsion subgroup of $\mathrm{H}^1(X, \mathbb{Z})$ is computed by Serrano in [Ser90]. Explicit generators correspond to differences of multiple fibres of the other elliptic fibration $g\colon X \to \mathbb{P}^1_{\mathbb{C}}$, as was shown in [Fer+22]. It is noteworthy that these multiple fibres correspond exactly to the non-singular points of D (which in this context can be identified with $J^\vee$) that are invariant under the action of N.

In the above case distinction we have treated most cases in the Bagnera–de Franchis classification of bielliptic surfaces over an algebraically closed field, but it notably does not apply to the quasi-bielliptic surfaces of types (a2), (b2) and (c2), i.e. the quasi-bielliptic surfaces of non-Jacobian type. The argument crucially breaks down due to the failure of the quotient map $\widetilde{D} \to \widetilde{D}/H = D$ to be an H-torsor, since the action is not free around the cusp. We thus cannot apply Theorem 1.3.14 to conclude that there is a short exact sequence

$$0 \longrightarrow H^\vee \longrightarrow J^\vee \longrightarrow \widetilde{J}^\vee \longrightarrow 0,$$

where the map $J^\vee \to \widetilde{J}^\vee$ is induced by the quotient map $\widetilde{D} \to D$. Indeed, this conclusion is in general simply false.

Proposition 7.1.21. *Let X be a quasi-bielliptic surface with $d > 1$. The induced map $J^\vee \to \widetilde{J}^\vee$ is zero.*

Proof. Without loss of generality suppose that the ground field is algebraically closed. By assumption $\widetilde{D}$ is a rational cuspidal curve. From the proof of Proposition 3.5.18, we observe that the quotient map $\widetilde{D} \to D$ factors over the normalisation $\mathbb{P}^1 \to D$. The induced map on the Picard schemes factors over $\mathrm{Pic}^0_{\mathbb{P}^1/k} = 0$. □

Although the isomorphism class of Γ remains open in the remaining cases of non-Jacobian quasi-bielliptic surfaces in characteristic 2, we do limit the possible options, at least up to twisted forms.

Proposition 7.1.22. *Suppose the ground field k is algebraically closed.*

- *If X is classical quasi-bielliptic of type (a2), then $\Gamma \cong 0$ or $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$.*
- *If X is supersingular quasi-bielliptic of type (a2), then $\Gamma \cong \alpha_2$ or $\alpha_2 \times \mathbb{Z}/2\mathbb{Z}$;*
- *If X is quasi-bielliptic of type (c2), necessarily classical, then $\Gamma = 0$ or $\mathbb{Z}/2\mathbb{Z}$.*

Proof. In each case this follows from Proposition 7.1.12, which states that Γ is a subgroup scheme of the finite group scheme $(J^\vee)^N$. We again postpone the computation of the fixed locus to Section 7.3. We see in Section 7.2 that if X is a supersingular quasi-bielliptic surface of type (a2), then Γ has a non-trivial Frobenius kernel, since in this case $\mathrm{Pic}_{X/k}$ is non-reduced; see Corollary 7.2.5. This limits the possible subgroup schemes of $\alpha_2 \times \mathbb{Z}/2\mathbb{Z}$ to the two cases listed. □

Under the assumption that the order of Γ is constant among bielliptic surfaces of the same type, there is only one isomorphism class for Γ. We thus conjecture the following, indicated in Table 7.1 by a question mark.

Conjecture 7.1.23.

- *If X is classical quasi-bielliptic of type (a2), then $\Gamma = \mathbb{Z}/2\mathbb{Z}$;*
- *If X is supersingular quasi-bielliptic of type (a2), then $\Gamma = \alpha_2$;*
- *If X is quasi-bielliptic of type (c2), necessarily classical, then $\Gamma = 0$.*

7.2 Hodge numbers and de Rham numbers

Let k be an algebraically closed ground field. In this section we investigate the dimension of the Hodge cohomology groups and algebraic de Rham cohomology groups of a bielliptic surface X, originally considered by Grothendieck in [Gro66c]. We also refer to the Stacks project [Stacks, Tag 0FK4].

Both cohomology groups pertain to the sheaf of differentials $\Omega^i_{X/k}$, defined as the ith exterior power $\wedge^i\Omega^1_{X/k}$ of the sheaf of Kähler differentials. They sit in the *de Rham complex*

$$\Omega^\bullet_{X/k} = [\Omega^0_{X/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^2_{X/k} \longrightarrow \cdots],$$

where the boundary map is defined on local sections by the usual formula $d(f_0 \wedge df_1 \wedge \cdots \wedge df_i) = df_0 \wedge df_1 \wedge \cdots \wedge df_i$. The de Rham cohomology is defined as the *hypercohomology* of this complex, i.e. through the cohomology groups

$$\mathrm{H}^i_{\mathrm{dR}}(X) = \mathrm{H}^i(R\Gamma(X, \Omega^\bullet_{X/k})).$$

Let $h^i_{\mathrm{dR}} = \dim \mathrm{H}^i_{\mathrm{dR}}(X)$ be the *de Rham numbers* of X. The *Hodge cohomology groups* $\mathrm{H}^j(X, \Omega^i_{X/k})$ are obtained by considering the sheaf cohomology of the individual constituents of the de Rham complex. The dimensions $h^{ij} = \dim H^j(X, \Omega^i_{X/k})$ are called the *Hodge numbers* of X. These two cohomology groups are related by the *Hodge-to-de Rham spectral sequence* (sometimes called the *Hodge spectral sequence*) $E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(X, \Omega^\bullet_{X/k})$, as in [Stacks, Tag 0FM6].

The Hodge cohomology and de Rham cohomology originally arise from the theory of real and complex manifolds. If X is a regular scheme over $\mathbb{C}$, then the above is a natural generalisation: there are natural isomorphisms $\mathrm{H}^i_{\mathrm{dR}}(X) \cong \mathrm{H}^i_{\mathrm{dR}}(X^{\mathrm{an}})$ and $\mathrm{H}^j(X, \Omega^i_{X/k}) \cong \mathrm{H}^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}})$; see [Gro66c]. Thus in this context *Hodge symmetry* $h^{ij} = h^{ji}$ holds. Furthermore the Hodge-to-de Rham spectral sequence degenerates at the E_1-page, which consequently implies the identity $h^i_{\mathrm{dR}} = \sum_{j=0}^i h^{j,i-j}$. In positive characteristic neither Hodge symmetry nor the degeneration of the Hodge-to-de Rham spectral sequence holds generally. Only the relation $h^{ij} = h^{n-i,n-j}$ obtained form Serre duality generalises to positive characteristic; see [Dob21].

In this section we deduce the Hodge numbers and de Rham numbers of most types of bielliptic surfaces X from the group scheme Γ by invoking the results of [Suw83]. Before we state the main result of op. cit., we introduce the following notation. We denote by $\Gamma[F]$ and $\Gamma[V]$ the kernels of Frobenius and Verschiebung respectively; in characteristic 0 our convention is that both are equal to 0. We also introduce the following measure of size for a p-group scheme.

Definition 7.2.1. The *rank* of a finite p-group scheme G is defined as

$$\mathrm{rk}\, G = \log_p \dim \mathrm{H}^0(G, \mathcal{O}_G).$$

Note that this differs from the *order* $|G| = \dim \mathrm{H}^0(G, \mathcal{O}_G)$ by a base p logarithm. For example, $\mathrm{rk}\, \alpha_p = \mathrm{rk}\, \mathbb{Z}/p\mathbb{Z} = \mathrm{rk}\, \mu_p = 1$. The rank is a natural number exactly by the assumption that G is a p-group scheme. We now state Thm. 1 and Thm. 2 from op. cit.

Theorem 7.2.2. *Let X be a smooth proper surface whose Albanese dimension equals $1-\chi(\mathcal{O}_X)$. Then the Hodge numbers of X are given by*

$$\begin{pmatrix} h^{02} & h^{12} & h^{22} \\ h^{01} & h^{11} & h^{21} \\ h^{00} & h^{10} & h^{20} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} \operatorname{rk}\Gamma[F] & \operatorname{rk}\Gamma[V] & 0 \\ \operatorname{rk}\Gamma[F] & 2\operatorname{rk}\Gamma[V] & \operatorname{rk}\Gamma[F] \\ 0 & \operatorname{rk}\Gamma[V] & \operatorname{rk}\Gamma[F] \end{pmatrix}. \tag{7.2.1}$$

Furthermore, the de Rham numbers of X are given by

$$h^n_{\mathrm{dR}} = \begin{cases} 1 & \text{if } n = 0 \text{ or } 4; \\ 2+\ \ \operatorname{rk}\Gamma[p] & \text{if } n = 1 \text{ or } 3; \\ 2+2\operatorname{rk}\Gamma[p] & \text{if } n = 2. \end{cases}$$

Furthermore, the Hodge-to-de Rham spectral sequence $E_1^{rs} = H^s(X, \Omega^r_X) \Rightarrow \mathrm{H}^{r+s}(X, \Omega^\bullet_X)$ degenerates at the E_1-page if and only if $\operatorname{rk}\Gamma[F] + \operatorname{rk}\Gamma[V] = \operatorname{rk}\Gamma[p]$.

The Hodge numbers are traditionally displayed in the shape of a diamond, the so-called *Hodge diamond*. Below we present the Hodge diamond together with the de Rham numbers of bielliptic surfaces in the following configuration:

$$\begin{array}{ccccc|c} & & h^{00} & & & h^0_{\mathrm{dR}} \\ & h^{10} & & h^{01} & & h^1_{\mathrm{dR}} \\ h^{20} & & h^{11} & & h^{02} & h^2_{\mathrm{dR}} \\ & h^{21} & & h^{12} & & h^3_{\mathrm{dR}} \\ & & h^{22} & & & h^4_{\mathrm{dR}} \end{array}$$

Conventions differ in the precise positioning of the Hodge numbers in the diamond: for example, some authors apply the symmetry $h^{ij} \leftrightarrow h^{ji}$. In a context where Hodge symmetry holds (e.g. over the field of complex numbers) this causes no ambiguity. For us, Hodge symmetry holds if and only if $\operatorname{rk}\Gamma[F] = \operatorname{rk}\Gamma[V]$, as follows from (7.2.1). We note that for a surface satisfying the conditions of Theorem 7.2.2, the Hodge-to de Rham spectral sequence degenerates if and only if each Hodge number h^i_{dR} equals $\sum_{j=0}^{i} h^{j,i-j}$, which is the sum of the corresponding row in the Hodge diamond.

From now on let X be a bielliptic surface. To see that the Theorem of Suwa applies, we simply verify the following, which is clear from Table 3.1.

Lemma 7.2.3. *Let X be a bielliptic surface. Its Albanese dimension equals $h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X)$.*

We compute the remaining Hodge numbers and de Rham numbers for all cases where the isomorphism class of Γ is known, by applying the formulæ of Theorem 7.2.2 to Table 7.1 describing the isomorphism classses of Γ. The result is displayed in Table 7.2. As usual, conjectural values are displayed with a question mark.

The tabulation extends the results of [Lan79], in which the author determines the Hodge and de Rham numbers of bielliptic surfaces with a smooth Albanese, and quasi-bielliptic surfaces in characteristic 3. Furthermore the Hodge and de Rham numbers of quasi-bielliptic surfaces of type (a1), necessarily in characteristic 2, are determined in §9 of [Sch23a] through a careful analysis of the multiple fibres of the other fibration $g\colon X \to \mathbb{P}^1$. It is remarkable that Example 7.1.20 also outlines a connection between the multiple fibres and the group scheme Γ, indicating that there is perhaps a deeper connection. As far as the author is aware, it is still an open question to compute the Hodge and de Rham numbers of non-Jacobian quasi-bielliptic surfaces in characteristic 2.

Remark 7.2.4. Suppose X is an ordinary bielliptic surface of type (a1), (b1) or (c1) over an algebraically closed field. The invertible sheaf $\Omega^1_{X/k}$ may be computed with a similar approach to Proposition 3.4.19. Since the action of G is trivial on both $\Omega^1_{\widetilde{C}/k} = \mathcal{O}_{\widetilde{C}}$ and $\Omega^1_{\widetilde{D}/k} = \mathcal{O}_{\widetilde{D}}$, it follows that $\Omega^1_{X/k}$ is a free sheaf of rank 2. This explains the Hodge diamond.

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

<table>
<tr><th>Type</th><th>Tame Char.</th><th>Ordinary Biell.</th><th>Classical Q.-Biell.</th><th>Supersing. Q.-Biell.</th></tr>
<tr><td>(a1)</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2 & & 2 & & 4\\ 1 & & 4 & & 1 & 6\\ & 2 & & 2 & & 4\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 3 & & 1 & & 4\\ 0 & & 6 & & 0 & 6\\ & 1 & & 3 & & 4\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 3 & & 2 & & 4\\ 1 & & 6 & & 1 & 6\\ & 2 & & 3 & & 4\\ & & 1 & & & 1\end{array}$</td></tr>
<tr><td>(a2)</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 2 & & 3\\ 1 & & 2 & & 1 & 4\\ & 2 & & 1 & & 3\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2? & & 1 & & 3?\\ 0 & & 4? & & 0 & 4?\\ & 1 & & 2? & & 3?\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2? & & 2 & & 3?\\ 1 & & 4? & & 1 & 4?\\ & 2 & & 2? & & 3?\\ & & 1 & & & 1\end{array}$</td></tr>
<tr><td>(b1)</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2 & & 2 & & 3\\ 1 & & 4 & & 1 & 4\\ & 2 & & 2 & & 3\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2 & & 1 & & 3\\ 0 & & 4 & & 0 & 4\\ & 1 & & 2 & & 3\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2 & & 2 & & 3\\ 1 & & 4 & & 1 & 4\\ & 2 & & 2 & & 3\\ & & 1 & & & 1\end{array}$</td></tr>
<tr><td>(b2)</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\nexists$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\nexists$</td></tr>
<tr><td>(c1)</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2 & & 2 & & 3\\ 1 & & 4 & & 1 & 4\\ & 2 & & 2 & & 3\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 2 & & 1 & & 3\\ 0 & & 4 & & 0 & 4\\ & 1 & & 2 & & 3\\ & & 1 & & & 1\end{array}$</td>
<td>$\nexists$</td></tr>
<tr><td>(c2)</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\nexists$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1? & & 1 & & 2?\\ 0 & & 2? & & 0 & 2?\\ & 1 & & 1? & & 2?\\ & & 1 & & & 1\end{array}$</td>
<td>$\nexists$</td></tr>
<tr><td>(d)</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td>
<td>$\begin{array}{ccccc|c} & & 1 & & & 1\\ & 1 & & 1 & & 2\\ 0 & & 2 & & 0 & 2\\ & 1 & & 1 & & 2\\ & & 1 & & & 1\end{array}$</td></tr>
</table>

Table 7.2: The Hodge diamonds and de Rham numbers of bielliptic surfaces

The author does not know whether supersingular quasi-bielliptic surfaces of types (b2) and (c1) also have free $\Omega^1_{X/k}$.

A direct consequence of Theorem 7.2.2 regarding the de Rham number h^{02} pertains to the reducedness of $\mathrm{Pic}^\tau_{X/k}$. Since Γ is a quotient of $\mathrm{Pic}^\tau_{X/k}$, the following result extending Proposition 3.1.27 should not be too surprising.

Corollary 7.2.5. *Let X be a bielliptic surface. The Picard scheme $\mathrm{Pic}^\tau_{X/k}$ is reduced if and only if $\Gamma[F] = 0$.*

Proof. Recall that m denotes the order of ω_X in $\mathrm{Pic}(X)$. By Serre duality we see that

$$\operatorname{rk}\Gamma[F] = h^{02} = \dim \mathrm{H}^2(X, \mathcal{O}_X) = \dim \mathrm{H}^0(X, \omega_X) = \begin{cases} 0 & \text{if } m > 1; \\ 1 & \text{if } m = 1. \end{cases} \tag{7.2.2}$$

We recall from Proposition 3.1.27 that $m = 1$ if and only if $\mathrm{Pic}^\tau_{X/k}$ is non-reduced. □

Remark 7.2.6. It follows that a tabulation of $\operatorname{rk}\Gamma[F]$ is implicit in Table 3.5 describing m. Alternatively, we can read off whether $m = 1$ from the leftmost entry, or, equivalently by Serre duality, the rightmost entry in the Hodge diamonds of Table 7.2.

Corollary 7.2.7. *Let X be a bielliptic surface. The Hodge-to-de Rham spectral sequence does not degenerate at the E_1-page if and only if*

- *X is supersingular of type (a1) or (a2); or*
- *X is ordinary of type (b1) or (c1).*

Proof. Recall that the Hodge-to-de Rham spectral sequence of a bielliptic surface degenerates if and only if the sum of the ith row of the Hodge diamond equals the ith de Rham number h^i_{dR}. For the types of bielliptic surfaces where Γ is known, the result follows from a direct observation of Table 7.2. If the isomorphism class of Γ is not known, we may use Proposition 7.1.22 to limit the possible isomorphism classes, each giving a potential Hodge diamond. For example, if X is supersingular of type (a2) then it follows that the Hodge diamond equals either

$$\begin{array}{ccccc|c} & & 1 & & & 1 \\ & 2 & & 2 & & 3 \\ 1 & & 4 & & 1 & 4 \\ & 2 & & 2 & & 3 \\ & & 1 & & & 1 \end{array} \quad \text{or} \quad \begin{array}{ccccc|c} & & 1 & & & 1 \\ & 3 & & 2 & & 4 \\ 1 & & 6 & & 1 & 6 \\ & 2 & & 3 & & 4 \\ & & 1 & & & 1. \end{array}$$

In either case, the Hodge-to-de Rham spectral sequence does not degenerate. The other cases are similar. ☐

The non-degeneration of the Hodge-to-de Rham spectral sequence has consequences for the liftability of these types of bielliptic surfaces to the second truncation $W_2 = W/p^2W$ of the ring of Witt vectors $W = W(k)$, by invoking the following famous result of Deligne and Illusie; see [DI87].

Theorem 7.2.8. *Let X be a smooth, proper scheme over a perfect field k with $p > 0$ and $\dim(X) \leq p$. If X lifts to $W_2(k)$, then the Hodge-to-de Rham spectral sequence $E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(X, \Omega^\bullet_{X/k})$ degenerates at the E_1-page.*

Alternatively, to prove the weaker result that these classes of bielliptic surfaces are non-liftable to the ring of Witt vectors W, we may apply the criterion Thm. 5.3 of [Sch21b], for which we only need to verify that the connected component Γ^0 contains a copy of α_{p^N} as a direct summand, for some $N \geq 1$, and that $h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X) = 1$. A brief inspection of Table 7.1 shows that the first condition holds, and the second condition holds because the given expression equals $1 - \chi(\mathcal{O}_X)$; see Table 3.1.

Nevertheless, the above discussion does not inhibit the existence of liftings to characteristic 0 in any way: in fact any bielliptic surface with a smooth Albanese admits a projective lifting to characteristic 0 due to Cor. 1.8.9 of [Par10]. On p. xv of op. cit. it is remarked that the same is true for quasi-bielliptic surfaces.

7.3 The fixed locus: a lemma and computations

In course of the proof of Theorem 7.1.14 we postponed proof of surjectivity of certain arrows in (7.1.3).

Definition 7.3.1. A *G-module* is a commutative group scheme M equipped with a G-action by group scheme homomorphisms.

Categorically, this means that the diagram

$$\begin{tikzcd} G \times M \times M \arrow[r, "\Delta_G \times \mathrm{id}_{M \times M}"] \arrow[d, "\mathrm{id}_G \times +"'] & G \times G \times M \times M \cong G \times M \times G \times M \arrow[r, "\alpha \times \alpha"] & M \times M \arrow[d, "+"] \\ G \times M \arrow[rr, "\alpha"] & & M \end{tikzcd}$$

is commutative. Equivalently, the $G(S)$-action on $M(S)$ is required to be by group homomorphisms, for every scheme S. A *homomorphism of G-modules* is defined to be an equivariant homomorphism of group schemes. We note that the category of G-modules is abelian.

Example 7.3.2. Let G be an abstract abelian group and let M be an abstract G-module, i.e. an abstract abelian group equipped with a G-action. Then the constant group scheme M_k is naturally a G_k-module for the constant group scheme G_k.

Lemma 7.3.3. *Let*

$$0 \longrightarrow M' \xrightarrow{i} M_1 \times M_2 \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of G-modules. Suppose that $\mathrm{pr}_{M_1} \circ i$ is injective, and that the G-action on M_1 is trivial. Then the induced map on G-invariants $M_1 \times (M_2)^G \to (M'')^G$ is surjective.

Proof. Consider an fppf-morphism $S \to \mathrm{Spec}(k)$, and note that S is locally noetherian. Let $m'' \in (M'')^G(S) \subset M''(S)$ be an arbitrary element. By surjectivity, there is an fppf-cover $S' \to S$ such that m'' lifts to some $(m_1, m_2) \in (M_1 \times M_2)(S')$, which by exactness is unique up to translation by a unique element of $M'(S')$. We show that m_2 is fixed by the action of $G_{S'}$. Let $S'' \to S'$ be an arbitrary fppf-cover, and let $g \in G(S'')$ be any element. Since m'' is fixed under the G-action, it follows that $g \cdot (m_1, m_2) = (m_1, g \cdot m_2)$ maps to m'', so that there is a unique element $m' \in M'(S'')$ such that$(m_1, g \cdot m_2) = i(m') + (m_1, m_2)$. It follows that the image of $i(m')$ in M_1 vanishes, from which we conclude that $m' = 0$. Whence $g \cdot (x, y) = (x, y)$, as required. □

Remark 7.3.4. Consider the special case where G is a constant group scheme, and where M', M_1, M_2 and M'' are constant G-modules. We may treat G as an abstract group, and the G-modules as abstract G-modules. Then the above result can also be proven using group cohomology, as follows. There is a long exact sequence

$$0 \longrightarrow M' \longrightarrow M_1 \times (M_2)^G \longrightarrow (M'')^G \longrightarrow \mathrm{H}^1(G, M') \longrightarrow \mathrm{H}^1(G, M_1 \times M_2) \longrightarrow \cdots .$$

Since the G-action on M' is trivial, we can identify $\mathrm{H}^1(G, M') = \mathrm{Hom}(G, M')$. Similarly, we find $\mathrm{H}^1(G, M_1 \times M_2) = \mathrm{Hom}(G, M_1) \times \mathrm{H}^1(G, M_2)$. Since the natural map $\mathrm{Hom}(G, M') \to \mathrm{Hom}(G, M_1) \times \mathrm{H}^1(G, M_2)$ is injective, it follows by exactness that $M_1 \times (M_2)^G \to (M'')^G$ is surjective.

From now on let k be an algebraically closed ground field and let X be a bielliptic surface. For the remainder of this section we compute the fixed locus of the action of N on the Picard scheme $J^\vee = \mathrm{Pic}^0_{D/k}$. This proves Theorem 7.1.16, by invoking Theorems 7.1.13 and 7.1.14. It also completes the proof of Proposition 7.1.22.

Proposition 7.3.5. *The fixed locus $(J^\vee)^N$ is tabulated in Table 7.3.*

Type	Tame Char.	Ordinary Biell.	Classical Q.-Biell.	Supersingular Q.-Biell.
(a)	$J^\vee[2]$	$J^\vee[2]$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\begin{cases} \alpha_2 \times \mathbb{Z}/2\mathbb{Z} & \text{if } \lambda \neq 0; \\ \alpha_4 & \text{if } \lambda = 0; \end{cases}$
(b)	$\mathbb{Z}/3\mathbb{Z}$	α_3	$\mathbb{Z}/3\mathbb{Z}$	α_3
(c)	$\mathbb{Z}/2\mathbb{Z}$	α_2	$\mathbb{Z}/2\mathbb{Z}$	$\nexists$
(d)	0	0	0	0

Table 7.3: The fixed locus $(J^\vee)^N$ in all cases of the Bagnera–de Franchis classification

The proof uses the Bagnera–de Franchis classification of bielliptic surfaces Theorems 3.4.4 and 3.4.12. The possible N-actions on D may be read off from the G-actions on the curve $\widetilde{D}$. In effect, we may assume without loss of generality that X is of Jacobian type, i.e. of type (a1), (b1), (c1) or (d).

Let us first treat the case where X has a smooth Albanese; in this case Theorem 7.1.14 is applicable to deduce the isomorphism class of the group scheme Γ. The following result is clear from Theorem 3.4.4.

Lemma 7.3.6. *Suppose X has a smooth Albanese. Then there is an isomorphism $N \cong \mathbb{Z}/n\mathbb{Z}$, and the action of N on $J^\vee$ is an elliptic curve automorphism ω of order n. Then $(J^\vee)^N = J^\vee[1-\omega]$ so that $\Gamma = J^\vee[1-\omega]/H^\vee$.*

To compute the isomorphism class of $J^\vee[1-\omega]$ in all different cases in an elementary fashion, apply §III.10 and App. §A of [Sil09], as done in §5.1 of [Mar22]. Note that the order of $J^\vee[1-\omega]$ equals

$$\left|J^\vee[1-\omega]\right| = \deg(1-\omega) = (1-\zeta_n)(1-\zeta_n^{-1}) = \begin{cases} 4 & \text{if } n=2; \\ 3 & \text{if } n=3; \\ 2 & \text{if } n=4; \\ 1 & \text{if } n=6, \end{cases}$$

where $\zeta_n \in \mathbb{C}$ denotes a primitive nth root of unity. We describe the group scheme Γ in all cases in more detail.

Example 7.3.7 (Type (a1)). In this case $N \cong \mathbb{Z}/2\mathbb{Z}$ and $H = 0$. Since ω is the sign involution on $\widetilde{J}^\vee = J^\vee$, it follows that $\Gamma \cong J^\vee[2]$. In tame characteristics $p \neq 2$ it is indeed true that $J^\vee[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Example 7.3.8 (Type (a2)). In this case $N \cong \mathbb{Z}/2\mathbb{Z}$ and $H \cong \mu_2$, so $\Gamma \cong \widetilde{J}^\vee[2]/(\mathbb{Z}/2\mathbb{Z}) \cong \mu_2$. In the tame characteristics $p \neq 2$ this is indeed coincides with $\mathbb{Z}/2\mathbb{Z}$; see Table 7.1.

Example 7.3.9 (Type (b1)). Here $N \cong \mathbb{Z}/3\mathbb{Z}$ and $H \cong 0$, so $\Gamma \cong \widetilde{J}^\vee[1-\omega]$. According to §5.1 of [Mar22] this is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ if $p \neq 3$ and isomorphic to α_3 if $p = 3$.

Example 7.3.10 (Type (b2)). Here $N \cong \mathbb{Z}/3\mathbb{Z}$ and $H \cong \mu_3$, so $\Gamma \cong \widetilde{J}^\vee[1-\omega]/(\mathbb{Z}/3\mathbb{Z}) = 0$. Note that $\mathbb{Z}/3\mathbb{Z}$ is only a subgroup scheme of $\widetilde{J}^\vee[1-\omega]$ if $p \neq 3$, which is consistent with the fact that there exists no ordinary bielliptic surface of type (b2).

Example 7.3.11 (Type (c1)). Here $N \cong \mathbb{Z}/4\mathbb{Z}$ and $H = 0$, so $\Gamma \cong \widetilde{J}^\vee[1-\omega]$, which by §5.1 of [Mar22] is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $p \neq 2$ or α_2 if $p = 2$.

Example 7.3.12 (Type (c2)). Here $N \cong \mathbb{Z}/4\mathbb{Z}$ and $H \cong \mu_2$, so $\Gamma \cong \widetilde{J}^\vee[1-\omega]/(\mathbb{Z}/2\mathbb{Z}) = 0$. Note that $\mathbb{Z}/2\mathbb{Z}$ is only a subgroup scheme of $J^\vee[1-\omega]$ if $p \neq 2$, but similarly to type (b2), this example only occurs in tame characteristic.

Example 7.3.13 (Type (d)). Here $N \cong \mathbb{Z}/6\mathbb{Z}$ and $H = 0$. We have seen above that the order of $\widetilde{J}^\vee[1-\omega]$ is 1, hence it is the trivial group scheme. Therefore $\Gamma = 0$.

We now treat quasi-bielliptic surfaces. Since the group scheme G may be infinitesimal, we will calculate certain invariant loci of group scheme actions by an infinitesimal group scheme. For *height one* infinitesimal group schemes, there is an equivalence of categories with p-Lie algebra's and derivations; see for example §2 of [Sch21a] or §1 of [Sch07]. One can characterise the fixed point locus using this perspective, as is done for two cases quasi-elliptic cases of type (a1) in characteristic 2 in [Sch21b]. We instead perform a more direct calculation, also since the group scheme G may not be of height 1 in characteristic 2: we observe from Table 3.8 that a μ_4 appears.

Note that $G = \operatorname{Spec}(A)$ is affine. On the side of Hopf algebras, its action on $\mathbb{G}_a$ hence corresponds to a co-action $\alpha^\sharp\colon k[t] \to A \otimes k[t]$. Let $I \subset k[t]$ be an ideal. The closed subscheme $Z = \operatorname{Spec} k[t]/I$ is stable if $\alpha^\sharp$ induces a map $k[t]/I \to A \otimes k[t]/I$. Furthermore Z is invariant if this map equals the inclusion $z \mapsto 1 \otimes z$. Recall that the fixed locus $(J^\vee)^G$ is the closure of the union of invariant subschemes. We note also that, since G acts on $J^\vee \cong \mathbb{G}_a$ by group scheme homomorphisms, the fixed locus naturally inherits the structure of a subgroup scheme.

Lemma 7.3.14. *Let k be algebraically closed and suppose X is a classical quasi-bielliptic surface of type (a1) or (a2). Then $(J^\vee)^N = (\mathbb{Z}/2\mathbb{Z})^2$. If X is of type (a1), then it follows that $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. If X is of type (a2), then the induced N-action on the rational cuspidal curve $D = \widetilde{D}/H$ is as for a classical quasi-bielliptic surface of type (a1), as follows from the actions listed in the Bagnera–de Franchis classification Theorem 3.4.12. Thus assume without loss of generality that X is of type (a1). The action of μ_2 on $J^\vee \cong \mathbb{G}_a$, whose underlying scheme is a copy of $\operatorname{Spec} k[t]$, is induced by the co-action $\alpha^\sharp\colon k[t] \to k[a]/(a^2-1) \otimes k[t]$ given by

$$\begin{aligned} t \longmapsto & a \otimes t + \lambda(a+1) \otimes t^2 + (a+1) \otimes t^4 \\ = & 1 \otimes (\lambda t^2 + t^4) + a \otimes (t + \lambda t^2 + t^4). \end{aligned}$$

for some $\lambda \in k$. It is not difficult to see that any closed subscheme $Z = \operatorname{Spec} k[t]/I$ is stable. Write $I = (f)$ for some polynomial f. The image of t lies in $1 \otimes k[t]/(f)$ if and only if $t + \lambda t^2 + t^4 \equiv 0 \bmod f$; in other words if and only if f divides $t + \lambda t^2 + t^4$. It follows that $(J^\vee)^G \cong \operatorname{Spec} k[t]/(t + \lambda t^2 + t^4) \subset \mathbb{G}_a$. Since $t + \lambda t^2 + t^4$ is separable, this subgroup scheme is isomorphic to a copy of $(\mathbb{Z}/2\mathbb{Z})^2$. ☐

Lemma 7.3.15. *Let k be algebraically closed and suppose X is classical quasi-bielliptic of type (b1) or (b2). Then $(J^\vee)^N = \mathbb{Z}/3\mathbb{Z}$. If X is of type (b1), then $\Gamma \cong \mathbb{Z}/3\mathbb{Z}$.*

Proof. Similarly to the proof of Lemma 7.3.14, assume without loss of generality that X is of type (b1). The action of μ_3 on $J^\vee \cong \mathbb{G}_a$ is induced by the co-action $\alpha^\sharp \colon k[t] \to k[a]/(a^3 - 1) \otimes k[t]$ given by

$$t \longmapsto a \otimes t + (1 - a) \otimes t^3 = 1 \otimes t^3 + a \otimes (t - t^3).$$

Again, any closed subscheme $Z = \operatorname{Spec} k[t]/(f)$ is stable, and invariant if and only if f is a divisor of $t - t^3$. Thus $(J^\vee)^G$ is $\operatorname{Spec} k[t]/(t - t^3) \subset \mathbb{G}_a$, which is the copy of $\mathbb{Z}/3\mathbb{Z}$ generated by 1. ☐

Lemma 7.3.16. *Let X be a classical quasi-bielliptic surface of type (c1) or (c2). Then $(J^\vee)^N = \mathbb{Z}/2\mathbb{Z}$. If X is of type (c1), then $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. Assume without loss of generality that X is of type (c1). The action of μ_4 on $J^\vee \cong \mathbb{G}_a$ is induced by the co-action $\alpha^\sharp \colon k[t] \to k[a]/(a^4 - 1) \otimes k[t]$ given by

$$t \longmapsto 1 \otimes t^4 + a \otimes (t + t^2) + a^2 \otimes (t^2 + t^4).$$

The fixed locus is hence $\operatorname{Spec} k[t]/I \subset \mathbb{G}_a$, where $I = (t + t^2, t + t^4) = (t + t^2)$ since k has characteristic 2. This is the copy of $\mathbb{Z}/2\mathbb{Z}$ generated by 1. ☐

Lemma 7.3.17. *Let X be a classical quasi-bielliptic surface of type (d). Then $\Gamma = (J^\vee)^N = 0$.*

Proof. Without loss of generality let k be algebraically closed. Suppose first that $p = 2$. The action of μ_6 of is given by the action of μ_2 of (a1), combined with the action of μ_3 by $t \mapsto \zeta_3 t$. The invariant locus $(J^\vee)^N$ hence equals the maximal subgroup scheme of $\operatorname{Spec} k[t]/(t + \lambda t^2 + t^4)$ invariant under $t \mapsto \zeta_3 t$, which is 0.

The case that $p = 3$ is similar: in this case the action of μ_6 decomposes as the action of μ_3 of (b1), combined with the action of μ_2 by $t \mapsto -t$. The maximal subgroup scheme of $\operatorname{Spec} k[t]/(t - t^3)$ invariant under $t \mapsto -t$ is 0. ☐

Lemma 7.3.18. *Let k be algebraically closed and suppose X is supersingular quasi-bielliptic of type (a1) or (a2). Let $\lambda \in k$ be as in the Bagnera–de Franchis classification Theorem 3.4.12. Then $\Gamma \cong \alpha_2 \times \mathbb{Z}/2\mathbb{Z}$ if $\lambda \neq 0$, and $\Gamma \cong \alpha_4$ if $\lambda = 0$.*

Proof. Suppose without loss of generality that X is of type (a1). The action of α_2 of $J^\vee \cong \mathbb{G}_a$ is induced by the co-action $\alpha^\sharp \colon k[t] \to k[a]/(a^2) \otimes k[t]$ given by

$$t \longmapsto 1 \otimes t + a \otimes (\lambda t^2 + t^4).$$

Again, any closed subscheme $Z = k[t]/(f)$ is stable. It is invariant if and only if f is a divisor of $\lambda t^2 + t^4 = t^2(\lambda + t^2)$. If $\lambda = 0$ this defines the subgroup scheme $\alpha_4 \subset \mathbb{G}_a$, and if $\lambda \neq 0$ this defines a subgroup scheme isomorphic to $\alpha_2 \times \mathbb{Z}/2\mathbb{Z}$. ☐

Lemma 7.3.19. *Let k be algebraically closed and suppose X is supersingular quasi-bielliptic of type (b1). Then $\Gamma \cong \alpha_3$.*

Proof. The action of α_3 is induced by the co-action $\alpha^\sharp \colon k[t] \to k[a]/(a^3) \otimes k[t]$ given by

$$t \longmapsto 1 \otimes t + a \otimes t^3.$$

The maximal invariant locus is $k[t]/(t^3)$, which is the subgroup scheme $\alpha_3 \subset \mathbb{G}_a$. ☐

Lemma 7.3.20. *Let X be a supersingular quasi-bielliptic surface of type (d). Then $\Gamma = 0$.*

Proof. Without loss of generality suppose k is algebraically closed. Suppose first that $p = 2$. The action of $\alpha_2 \times \mu_3$ is given by the action of α_2 in the supersingular (a1) case, combined with the action of μ_3 given by $t \mapsto \zeta t$, where ζ is a primitive third root of unity. The maximal invariant locus of the action of μ_3 on $\operatorname{Spec} k[t]/(t^4)$ is 0. The case $p = 3$ is similar. In this case the action of μ_2 on $\operatorname{Spec} k[t]/(t^3)$ by $t \mapsto -t$, whose fixed locus is 0. ☐

Bibliography

[AH32] Abraham Adrian Albert and Helmut Hasse. ‘A determination of all normal division algebras over an algebraic number field’. In: *Transactions of the American Mathematical Society* 34.3 (1932), pp. 722–726. DOI: `10.2307/1989375`.

[Ana73] Sivaramakrishna Anantharaman. ‘Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1’. In: *Sur les groupes algébriques.* Mémoires de la Société Mathématique de France 33. Société mathématique de France, 1973, pp. 5–79. DOI: `10.24033/msmf.109`.

[Art77] Michael Artin. ‘Coverings of the Rational Double Points in Characteristic p’. In: *Complex Analysis and Algebraic Geometry: A Collection of Papers Dedicated to K. Kodaira.* Ed. by W. L. Jr. Baily and T. Shioda. Cambridge University Press, 1977, pp. 11–22. ISBN: 9780521217774. DOI: `10.1017/CBO9780511569197.003`.

[Băd01] Lucian Bădescu. *Algebraic surfaces.* Universitext. Springer New York, NY, 2001. ISBN: 978-0-387-98668-5.

[BF07] Giuseppe Bagnera and Michele de Franchis. ‘Sur les surfaces hyperelliptiques’. In: *Comptes rendus hebdomadairs des séances de l’Académie des sciences* 145 (1907), pp. 747–749. URL: `https://gallica.bnf.fr/ark:/12148/bpt6k3099v/f747.item`.

[BF08] Giuseppe Bagnera and Michele de Franchis. ‘Le superficie algebriche le quali ammettono una rappresentazione parametrica mediante funzioni iperellittiche di due argomenti’. In: *Memorie di Mathematica e di Fisica della Società Italiana della Scienze* 3.15 (1908), pp. 251–343. URL: `https://media.accademiaxl.it/memorie/S3-VXV-1908/Bagnera-DeFranchis251-343.pdf`.

[BF10] Giuseppe Bagnera and Michele de Franchis. ‘Le nombre ρ de M. Picard pour les surfaces hyperelliptiques et pour les surfaces irrégulières de genre zéro’. In: *Rendiconti del Circolo Mathematico di Palermo* 30 (1910), pp. 185–238. DOI: `10.1007/BF03014874`.

[Bas72] Mark Ivanovich Bashmakov. ‘The cohomology of Abelian varieties over a number field’. In: *Russian Mathematical Surveys* 27.6 (1972). English translation, pp. 25–70. DOI: `10.1070/rm1972v027n06abeh001392`.

[BS04] Carmen Laura Basile and Alexei Skorobogatov. ‘On the Hasse Principle for bielliptic surfaces’. In: *Number Theory and Algebraic Geometry.* Ed. by Miles Reid and Alexei Skorobogatov. London Mathematical Society Lecture Note Series 303. Cambridge University Press, 2004, pp. 31–40. ISBN: 9780521545181. DOI: `10.1017/CBO9780511734946.003`.

[Bea96] Arnaud Beauville. *Complex algebraic surfaces.* Vol. 34. London Mathematical Society Student Texts. Cambridge University Press, 1996. ISBN: 0-521-49510-5.

[BM90] Curtis Bennett and Rick Miranda. ‘The automorphism groups of the hyperelliptic surface’. In: *The Rocky Mountain Journal of Mathematics* 20.1 (1990), pp. 31–37. DOI: `10.1216/rmjm/1181073156`.

[Ber23] Jakob Bergqvist. ‘The Kummer Constructions in Families’. PhD thesis. Heinrich-Heine-Universität Düsseldorf, July 2023. URN: `nbn:de:hbz:061-20230728-081344-3`.

[BDS24] Jakob Bergqvist, Thuong Dang and Stefan Schröer. ‘Sign involutions on para-abelian varieties’. In: *Indagationes Mathematicae* 35 (3 2024), pp. 570–583. DOI: `10.1016/j.indag.2024.04.006`.

[SGA 6] Pierre Berthelot, Alexander Grothendieck and Luc Illusie, eds. *Séminaire de Géométrie Algébrique du Bois Marie 1966/67. SGA 6.* Théorie des Intersections et Théorème de Riemann–Roch. Ed. by A. Dold and B. Eckmann. Vol. 225. Lecture Notes in Mathematics. Springer Verlag, 1971. ISBN: 3-540-05647-5.

[Boa21] Daniel Boada De Narvaez. 'Moduli of Bielliptic Surfaces'. PhD thesis. Technische Universität München, 2021. URN: `nbn:de:bvb:91-diss-20210429-1618614-0-4`.

[BM76] Enrico Bombieri and David Mumford. 'Enriques' Classification of Surfaces in Char. p, III.' In: *Inventiones mathematicae* 35 (1976), pp. 197–232. URL: `http://eudml.org/doc/142405`.

[BM77] Enrico Bombieri and David Mumford. 'Enriques' Classification of Surfaces in Char. p, II'. In: *Complex Analysis and Algebraic Geometry: A Collection of Papers Dedicated to K. Kodaira.* Ed. by W. L. Jr. Baily and T. Shioda. Cambridge University Press, 1977, pp. 23–42. ISBN: 9780521217774. DOI: `10.1017/CBO9780511569197.004`.

[BLR90] Siegfried Bosch, Lütkebohmert and Michel Raynaud. *Néron Models.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 1990. ISBN: 978-3-540-50587-7.

[Bri17] Michel Brion. 'Some structure theorems for algebraic groups'. In: *Algebraic Groups: Structure and Actions.* Ed. by Mahir Bilen Can. Vol. 94. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 2017, pp. 53–126. ISBN: 978-1-4704-2601-9.

[Bro19] Sylvain Brochard. 'Duality for Commutative Group Stacks'. In: *International Mathematics Research Notices* 2021.3 (2019), pp. 2321–2388. DOI: `10.1093/imrn/rnz161`.

[CF03] Fabrizio Cantanese and Paola Frediani. 'Real hyperelliptic surfaces and the orbifold fundamental group'. In: *Journal of the Institute of Mathematics of Jussieu* 2.2 (2003), pp. 169–233. DOI: `10.1017/S1474748003000070`.

[Cas63] John William Scott Cassels. 'Arithmetic on Curves of Genus 1 (V). Two Counter-Examples'. In: *Journal of the London Mathematical Society* s1-38.1 (1963), pp. 244–248. DOI: `10.1112/jlms/s1-38.1.244`.

[Cas66] John William Scott Cassels. 'Diophantine Equations with Special Reference to Elliptic Curves'. In: *Journal of the London Mathematical Society* 41.1 (1966), pp. 193–291. DOI: `10.1112/jlms/s1-41.1.193`.

[Cas91] John William Scott Cassels. *Lectures on Elliptic Curves.* Vol. 24. London Mathematical Society Student Text. Cambridge University Press, 1991. ISBN: 978-0-521-41517-0.

[Cat03] Fabrizio Catanese. *From Abel's heritage: transcendental objects in algebraic geometry and their algebrization.* 2003. arXiv: `math/0307068 [math.AG]`.

[Cil98] Ciro Ciliberto. 'M. de Franchis and the theory of hyperelliptic surfaces'. In: *Rendiconti del Circolo Matematico di Palermo* 55.Supplementi II (1998): *Studies in the History of Mathematics III.* Ed. by U. Bottazzini, pp. 45–74.

[Cil20] Ciro Ciliberto. *Classification of Complex Algebraic Surfaces.* EMS Series of Lectures in Mathematics. EMS Press, 2020. ISBN: 978-3-03719-210-8.

[ÇK12] Mirela Çiperiani and Daniel Krashen. 'Relative Brauer groups of genus 1 curves'. In: *Israel Journal of Mathematics* 192 (2012), pp. 921–949. DOI: `10.1007/s11856-012-0057-5`.

[Cla06] Pete Louis Clark. 'On elementary equivalence, isomorphism and isogeny'. In: *Journal de théorie des nombres de Bordeaux* 18.1 (2006), pp. 29–58. DOI: `10.5802/jtnb.532`.

[Cla04] Peter Louis Clark. *The period-index problem in WC-groups II: abelian varieties.* 2004. arXiv: `math/0406135 [math.NT]`.

[CS21] Jean-Louis Colliot-Thélène and Alexei Skorobogatov. *The Brauer–Grothendieck Group.* Vol. 71. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Cham, 2021. ISBN: 978-3-030-74247-8.

[CSS97] Jean-Louis Colliot-Thélène, Alexei Skorobogatov and Peter Swinnerton-Dyer. 'Double fibres and double covers: paucity of rational points'. In: *Acta Arithmetica* 79.2 (1997), pp. 113–135. DOI: `10.4064/aa-79-2-113-135`.

[Con] Keith Conrad. *Quaternion Algebras.* URL: `https://kconrad.math.uconn.edu/blurbs/ringtheory/quaternionalg.pdf` (visited on 16/05/2023).

[CD89] François R. Cossec and Igor V. Dolgachev. *Enriques Surfaces I.* Vol. 76. Progress in Mathematics. Birkhäuser Boston, MA, 1989. ISBN: 978-1-4612-8216-7.

[CDL24] François R. Cossec, Igor V. Dolgachev and Christian Liedtke. *Enriques Surfaces I.* Updated version of [CD89] (online only). Apr. 2024. URL: `https://dept.math.lsa.umich.edu/~idolga/EnriquesOne.pdf` (visited on 30/07/2024).

[CLS11] David A. Cox, John B. Little and Henry K. Schenck. *Toric Varieties.* Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, 2011. ISBN: 978-1-4704-7820-9.

[Cre16] Brendan Creutz. 'Relative Brauer groups of torsors of period two'. In: *Journal of Algebra* 459 (2016), pp. 109–132. DOI: `10.1016/j.jalgebra.2016.03.034`.

[CV15] Brendan Creutz and Bianca Viray. 'Two torsion in the Brauer group of a hyperelliptic curve'. In: *Manuscripta Mathematica* 147 (2015), pp. 139–167. DOI: `10.1007/s00229-014-0721-7`.

[CV18] Brendan Creutz and Bianca Viray. 'Degree and the Brauer—Manin obstruction'. In: *Algebra & Number Theory* 12.10 (2018), pp. 2445–2470. DOI: `10.2140/ant.2018.12.2445`.

[DI87] Pierre Deligne and Luc Illusie. 'Relèvements modulo p^2 et décomposition du complexe de de Rham.' In: *Inventiones mathematicae* 89 (1987), pp. 247–270. URL: `https://eudml.org/doc/143480`.

[SGA 3] Michel Demazure and Alexander Grothendieck, eds. *Séminaire de Géométrie Algébrique du Bois Marie 1962/64. SGA 3.* I: Schémas en Groupes. Ed. by A. Dold and B. Eckmann. Vol. 151. Lecture Notes in Mathematics. Springer Verlag, 1970. ISBN: 978-3-540-36336-1.

[DSS00] Z. Djabri, Edward F. Schaefer and N. P. Smart. 'Computing the p-Selmer group of an elliptic curve'. In: *Transactions of the American Mathematical Society* 352 (2000), pp. 5583–5597. DOI: `10.1090/S0002-9947-00-02535-6`.

[Dob21] Remy van Dobben de Bruyn. 'The Hodge ring of varieties in positive characteristic'. In: *Algebra & Number Theory* 15.3 (2021), pp. 729–745. DOI: `10.2140/ant.2021.15.729`.

[EGM] Bas Edixhoven, Gerard van der Geer and Ben Moonen. *Abelian varieties.* In Preparation.

[ES07] Federigo Enriques and Francesco Severi. 'Intorno alle superficie iperellittiche'. In: *Rend. Accad. Lincei.* 5th ser. XVI.1 (1907), pp. 443–453. URL: `http://operedigitali.lincei.it/rendicontiFMN/rol/visabs.php?lang=it&type=mat&fileId=2565`.

[ES08] Federigo Enriques and Francesco Severi. 'Intorno alle superficie iperellittiehe irregolari'. In: *Rend. Accad. Lincei.* 5th ser. XVII.1 (1908), pp. 4–9. URL: `http://operedigitali.lincei.it/rendicontiFMN/rol/visabs.php?lang=it&type=mat&fileId=2762`.

[ES09] Federigo Enriques and Francesco Severi. 'Mémoire sur les surfaces hyperelliptiques'. In: *Acta Mathematica* 32 (1909), pp. 238–392. DOI: `10.1007/BF02403219`. And in: *Acta Mathematica* 33 (1910), pp. 321–403. DOI: `10.1007/BF02393217`.

[Fer+22] Eugenia Ferrari, Sofia Tirabassi, Magnus Vodrup and Jonas Bergström. 'On the Brauer group of bielliptic surfaces'. In: *Documenta Mathematica* 27 (2022), pp. 383–425. DOI: `10.4171/DM/873`.

[Fis21] Johannes Fischer. 'Pinching Azumaya Algebras'. PhD thesis. Heinrich-Heine-Universität Düsseldorf, Nov. 2021. URN: `nbn:de:hbz:061-20220301-110713-0`.

[Fis01] Tom Fischer. 'Some examples of 5 and 7 descent for elliptic curves over $\mathbb{Q}$'. In: *Journal of the European Mathematical Society* 3 (2001), pp. 169–201. DOI: `10.1007/s100970100030`.

[Fog73] John Fogarty. 'Fixed Point Schemes'. In: *American Journal of Mathematics* 95.1 (1973), pp. 35–51. DOI: `10.2307/2373642`.

[Fra36a] Michele de Franchis. 'Dimostrazione del teorema fondamentale sulle superficie iperellittiche'. In: *Rend. Accad. Lincei* 24 (1936), pp. 3–6.

[Fra36b] Michele de Franchis. 'Sulla classificazione delle superficie iperellittiche'. In: *Scritti in onore di Berzolari.* 1936, pp. 613–615.

[Gee88] Gerard van der Geer. *Hilbert modular surfaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Berlin Heidelberg: Springer, 1988. ISBN: 3540176012.

[Gir71] Jean Giraud. *Cohomologie non abélienne.* Vol. 179. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer-Verlag Berlin, Heidelberg, New York, 1971. ISBN: 978-3-540-05307-1.

[GT22] Steven R. Groen and Jaap Top. 'A parametrized set of explicit elements of $Ш(E/\mathbb{Q})[3]$'. In: *Integers* 22 (2022). DOI: `10.5281/zenodo.10999247`.

[EGA II] Alexander Grothendieck. 'Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes'. In: *Publications Mathématiques de l'IHÉS* 8 (1961), pp. 5–222. URL: `http://www.numdam.org/item/PMIHES_1961__8__5_0/`.

[Gro62] Alexander Grothendieck. 'Technique de descente et théorèmes d'existance en géométrie algébrique. VI. Les schémas de Picard : propriétés générales'. In: *Séminaire Bourbaki* 7.Exp. 236 (1962), pp. 221–243. URL: `http://www.numdam.org/item/SB_1961-1962__7__221_0/`.

[EGA III_2] Alexander Grothendieck. 'Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, Seconde partie'. In: *Publications Mathématiques de l'IHÉS* 17 (1963), pp. 5–91. URL: `http://www.numdam.org/item/?id=PMIHES_1963__17__5_0`.

[EGA IV_2] Alexander Grothendieck. 'Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie'. In: *Publications Mathématiques de l'IHÉS* 24 (1965), pp. 5–231. URL: `http://www.numdam.org/item/?id=PMIHES_1965__24__5_0`.

[Gro66a] Alexander Grothendieck. 'Le groupe de Brauer II : Théorie cohomologique'. In: *Dix Exposes Sur La Cohomologie des Schémas.* Ed. by Alexander Grothendieck and N. H. Kuiper. Advanced Studies in Pure Mathematics volume 3. Paris and Amsterdam: Masson & Cie, Editeur and North-Holland Publishing Company, 1966, pp. 67–87.

[Gro66b] Alexander Grothendieck. 'Le groupe de Brauer III : Exemples et compléments'. In: *Dix Exposes Sur La Cohomologie des Schémas.* Ed. by Alexander Grothendieck and N. H. Kuiper. Advanced Studies in Pure Mathematics volume 3. Paris and Amsterdam: Masson & Cie, Editeur and North-Holland Publishing Company, 1966, pp. 88–188.

[Gro66c] Alexander Grothendieck. 'On the de Rham cohomology of algebraic varieties'. In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 29.1 (1966), pp. 95–103. DOI: `10.1007/BF02684807`.

[Han03] Ilseop Han. 'Relative Brauer groups of function fields of curves of genus one'. In: *Communications in Algebra* 31.9 (2003), pp. 4301–4328. DOI: `https://doi.org/10.1081/AGB-120022794`.

[Har13] Robin Hartshorne. *Algebraic geometry.* Vol. 52. Graduate Texts in Mathematics. Springer New York, NY, 2013. ISBN: 978-0-387-90244-9.

[HBN32] Helmut Hasse, Richard Dagobert Brauer and Emmy Noether. 'Beweis eines Hauptsatzes in der Theorie der Algebren'. In: *Journal für die reine und angewandte Mathematik* 167 (1932), pp. 399–404. URL: `https://resolver.sub.uni-goettingen.de/purl?PPN243919689_0167`.

[Has09] Brendan Hassett. 'Rational surfaces over nonclosed fields'. In: *Arithmetic Geometry.* Ed. by Henri Darmon, David Alexandre Ellwood, Brendan Hasssett and Yuri Tschinkel. Vol. 8. Clay Mathematics Proceedings. American Mathematical Society and Clay Mathematics Institute, 2009, pp. 155–210. ISBN: 978-0-8218-4476-2.

[Hau10] Herwig Hauser. 'On the problem of resolution of singularities in positive characteristic (Or: A proof we are still waiting for)'. In: *Bulletin of the American Mathematical Society* 47.1 (2010), pp. 1–30. DOI: `10.1090/S0273-0979-09-01274-9`.

[HS24] Cesar Hilario and Stefan Schröer. 'Generalizations of Quasielliptic Curves'. In: *Épijournal de Géometrie Algébrique* 7 (2024). DOI: `10.46298/epiga.2024.11181`.

[Hir64] Heisuke Hironaka. 'Resolution of singularities of an algebraic variety over a field of characteristic zero. I'. In: *Annals of Mathematics* 79.1 (1964), pp. 109–203. DOI: `10.2307/1970486`. And 'II'. In: *Annals of Mathematics* 79.2 (1964), pp. 205–326. DOI: `10.2307/1970547`.

[HLT20] Katrina Honigs, Luigi Lombardi and Sofia Tirabassi. 'Derived equivalences of canonical covers of hyperelliptic and Enriques surfaces in positive characteristic'. In: *Mathematische Zeitschrift* 295 (2020), pp. 727–749. DOI: `10.1007/s00209-019-02362-1`.

[Hür92] Werner Hürlimann. 'A short proof of the Albert–Brauer–Hasse–Noether theorem'. In: *Astérique* 209 (Journées arithmétiques de Genève 1992), pp. 215–220. URL: `https://smf.emath.fr/publications/short-proof-albert-brauer-hasse-noether-theorem`.

[Hus04] Dale Husemöller. *Elliptic Curves*. 2nd ed. Springer New York, NY, 2004. ISBN: 978-0-387-95490-5.

[Igu55] Jun-Ichi Igusa. 'On some problems in abstract algebraic geometry'. In: *Proceedings of the National Academy of Sciences* 41.11 (1955), pp. 964–967. DOI: `10.1073/pnas.41.11.964`.

[Igu60] Jun-Ichi Igusa. 'Betti and Picard numbers of abstract algebraic surfaces'. In: *Proceedings of the National Academy of Sciences* 46.5 (1960), pp. 724–726. DOI: `10.1073/pnas.46.5.724`.

[Iit69] Shigeru Iitaka. 'Deformations of compact complex surfaces'. In: *Global Analysis. Papers in Honor of K. Kodaira.* Ed. by Donald Clayton Spencer and Shokichi Iyanaga. Vol. 29. Princeton Mathematical Series. University of Tokyo Press and Princeton University Press, 1969, pp. 469–476. ISBN: 9781400871230.

[Jen87] Sven Toft Jensen. 'Picard Schemes of Quotients by Finite Commutative Group Schemes'. In: *Mathematica Scandinavica* 42.2 (1987), pp. 197–210. URL: `https://www.jstor.org/stable/24491293`.

[Jon] Aise Johan de Jong. *A result of Gabber.* URL: `https://www.math.columbia.edu/~dejong/papers/2-gabber.pdf` (visited on 11/09/2024).

[KM85] Nicholas Michael Katz and Barry Mazur. *Arithmetic moduli of elliptic curves.* Annals of Mathematical Studies 108. Princeton University Press, 1985. ISBN: 9780691083520.

[KS21] Shigeyuki Kondō and Stefan Schröer. 'Kummer surfaces associated with group schemes'. In: *Manuscripta Mathematica* 166.3-4 (2021), pp. 323–342. DOI: `10.1007/s00229-020-01257-4`.

[Kra10] Daniel Krashen. 'Corestrictions of Algebras and Splitting Fields'. In: *Transactions of the American Mathematical Society* 362.9 (2010), pp. 4781–4792. DOI: `10.1090/S0002-9947-10-04967-6`. URL: `http://www.jstor.org/stable/25733391`.

[Kwe22] Hyuk Jun Kweon. 'Bounds on the torsion subgroup schemes of Néron–Severi group schemes'. In: *Advances in Mathematics* 409.B (2022). DOI: `10.1016/j.aim.2022.108687`.

[Lan56] Serge Lang. 'Algebraic groups over finite fields'. In: *American Journal of Mathematics* 78.3 (1956), pp. 555–563. DOI: `10.2307/2372673`.

[LS57] Serge Lang and Jean-Pierre Serre. 'Sur Les Revetements Non Ramifies des Varietes Algebriques'. In: *American Journal of Mathematics* 79.2 (1957), pp. 319–330. URL: `http://www.jstor.org/stable/2372685`.

[LT58] Serge Lang and John Tate. 'Principal Homogeneous Spaces Over Abelian Varieties'. In: *American Journal of Mathematics* 80.3 (1958), pp. 659–684. DOI: `10.2307/2372778`.

[Lan79] William Edward Lang. 'Quasi-elliptic surfaces in characteristic three'. In: *Annales scientifiques de l'École Normale Supérieure.* 4th ser. 12.4 (1979), pp. 473–500. DOI: `10.24033/asens.1373`.

[LS23] Bruno Laurent and Stefan Schröer. 'Para-abelian varieties and Albanese maps'. In: *Bulletin of the Brazilian Mathematical Society, New Series* 55.4 (2023). DOI: `10.1007/s00574-023-00378-0`.

[Laz04a] Robert Lazarsfeld. *Positivity in Algebraic Geometry I. Classical setting: Line Bundles and Linear Series.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 2004. ISBN: 978-3-540-22533-1.

[Laz04b] Robert Lazarsfeld. *Positivity in Algebraic Geometry II. Positivity for Vector Bundles and Multiplier Ideals.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 2004. ISBN: 978-3-540-22534-8.

[Lic68] Stephen Lichtenbaum. 'The Period-Index Problem for Elliptic Curves'. In: *American Journal of Mathematics* 90.4 (1968), pp. 1209–1223. DOI: `10.2307/2373297`.

[Lie09] Christian Liedtke. 'A note on non-reduced Picard schemes'. In: *Journal of Pure and Applied Algebra* 213.5 (2009), pp. 737–741. DOI: `https://doi.org/10.1016/j.jpaa.2008.09.001`.

[Lie17] Christian Liedtke. 'Morphisms to Brauer–Severi varieties, with applications to del Pezzo surfaces'. In: *Geometry over nonclosed fields.* Ed. by Fedor Bogomolov, Brendan Hassett and Yuri Tschinkel. Simons Symposia. Springer Cham, 2017, pp. 157–196. ISBN: 978-3-319-49762-4.

[LS10] Christian Liedtke and Stefan Schröer. 'The Néron model over the Igusa curves'. In: *Journal of Number Theory* 130.10 (2010), pp. 2157–2197. DOI: `10.1016/j.jnt.2010.03.016`.

[Mar22] Gebhard Martin. 'Automorphism group schemes of bielliptic and quasi-bielliptic surfaces'. In: *Épijournal de Géométrie Algébrique* Volume 6 (2022). DOI: `10.46298/epiga.2022.7317`. arXiv: `2008.11679v3 [math.AG]`.

[MO67] Hideyuki Matsumura and Frans Oort. 'Representability of Group Functors, and Automorphisms of Algebraic Schemes'. In: *Inventiones mathematicae* 4 (1967), pp. 1–25. DOI: `10.1007/BF01404578`.

[Mat52] Teruhisa Matsusaka. 'On the Algebraic Construction of the Picard Variety (II)'. In: *Japanese journal of mathematics* 22 (1952), pp. 51–62. DOI: `10.4099/jjm1924.22.0_51`.

[Maz92] Barry Mazur. 'The Topology of Rational Points'. In: *Experimental Mathematics* 1.1 (1992), pp. 35–45. DOI: `10.1080/10586458.1992.10504244`.

[Maz95] Barry Mazur. 'Speculations about the topology of rational points: an up-date'. In: *Astérique* 288 (1995), pp. 161–181.

[MS87] Vikram Bhagvandas Mehta and Vasudevan Srinivas. 'Varieties in positive characteristic with trivial tangent bundle'. In: *Compositio Mathematica* 64.2 (1987), pp. 191–212.

[Mer81] Alexander Sergeevich Merkurjev. 'On the norm residue symbol of degree 2'. In: *Doklady Akademii Nauk SSSR* 261.3 (1981), pp. 542–547. URL: `https://www.mathnet.ru/eng/dan44871`.

[Mil80] James Stuart Milne. *Étale Cohomology.* Princeton University Press, 1980. ISBN: 978-0-691-08238-7.

[Mil17] James Stuart Milne. *Algebraic Groups. The Theory of Group Schemes of Finite Type over a Field.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.

[Mor69] Louis Joel Mordell. *Diophantine Equations.* Vol. 30. Pure and Applied Mathematics. Elsevier, 1969. ISBN: 978-0-12-506250-3.

[Mum69] David Mumford. 'Enriques' classification of surfaces in char p. I'. In: *Global Analysis: Papers in Honor of K. Kodaira (PMS-29).* Ed. by Donald Clayton Spencer and Shokichi Iyanaga. Vol. 56. Princeton Mathematical Series. Princeton University Press, 1969, pp. 325–339. ISBN: 9780691648026.

[Mum08] David Mumford. *Abelian Varieties.* Vol. 5. Tata Institute of Fundamental Research Studies in Mathematics, 2008. ISBN: 978-81-85931-86-9.

[MFK94] David Mumford, John Fogarty and Frances Kirwan. *Geometric Invariant Theory.* Springer–Verlag Berlin Heidelberg, 1994. ISBN: 978-3-540-56963-3.

[Mur64] Jacob Pieter "Jaap" Murre. 'On contravariant functors from the category of preschemes over a field into the category of abelian groups (with an application to the picard functor)'. In: *Publications Mathématiques de l'IHÉS* 23 (1964), pp. 5–43. DOI: `10.1007/BF02684309`.

[Nor83] Madhav Vithal Nori. 'The Fundamental Group-Scheme of an Abelian Variety'. In: *Mathematische Annalen* 263.3 (1983), pp. 263–266. DOI: `10.1007/BF01457128`.

[Nov14] Saša Novaković. 'Geometric Tilting Theory and the Amitsur conjecture'. PhD thesis. Heinrich-Heine-Universität Düsseldorf, Sept. 2014. URN: `nbn:de:hbz:061-20150601-103326-4`.

[Nov24] Saša Novaković. 'Absolutely split locally free sheaves on proper k-schemes and Brauer–Severi varieties'. In: *Bulletin des sciences mathématiques* 197 (2024), pp. 1–25. DOI: `10.1016/j.bulsci.2024.103494`.

[Nue25] Howard Nuer. 'Stable sheaves on bielliptic surfaces: from the classical to the modern'. In: *Mathematische Zeitschrift* 309.1 (2025). DOI: `10.1007/s00209-024-03659-6`.

[OR] John O'Connor and Edmund Robertson. MacTutor, University of St Andrews. URL: `https://mathshistory.st-andrews.ac.uk/Biographies/De_Franchis/` (visited on 13/06/2022).

[Oor66] Frans Oort. *Commutative group schemes*. Vol. 15. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1966. ISBN: 978-3-540-03598-5.

[Par10] Holger Partsch. 'The Liftability of Elliptic Surfaces'. PhD thesis. Heinrich-Heine-Universität Düsseldorf, Dec. 2010. URN: `urn:nbn:de:hbz:061-20110131-104522-2`.

[Pot17] Rory Potter. *Derived Autoequivalences of Bielliptic Surfaces*. 2017. arXiv: `1701.01015 [math.AG]`.

[Que71] Clifford Steven Queen. 'Non-conservative function fields of genus one I'. In: *Archiv der Mathematik* 22 (1971), pp. 612–623. DOI: `10.1007/BF01222625`.

[Que72] Clifford Steven Queen. 'Non-conservative function fields of genus one II'. In: *Archiv der Mathematik* 23 (1972), pp. 30–37. DOI: `10.1007/BF01304838`.

[Ray70] Michel Raynaud. 'Spécialisation du foncteur de Picard'. In: *Publications Mathématiques de l'IHÉS* 38 (1970), pp. 27–76. URL: `http://www.numdam.org/item/PMIHES_1970__38__27_0/`.

[Ree23] Fabian Reede. 'Picard schemes of noncommutative bielliptic surfaces'. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 93 (2023), pp. 61–70. DOI: `10.1007/s12188-023-00265-x`.

[Ryb08] Sergey Yur'evich Rybakov. 'Zeta functions of bielliptic surfaces over finite fields'. In: *Matematicheskie Zametki* 83.2 (2008), pp. 246–256. DOI: `10.1134/S0001434608010264`.

[Ryb16] Sergey Yur'evich Rybakov. 'Classification of zeta functions of bielliptic surfaces over finite fields'. In: *Mathematical Notes* 99.3 (2016), pp. 397–405. DOI: `10.1134/S0001434616030081`.

[Sat09] Takakazu Satoh. 'Generating Genus Two Hyperelliptic Curves over Large Characteristic Finite Fields'. In: *Advances in Cryptology – EUROCRYPT 2009. 28th Annual International Conference on the Theory and Applications of Cryptographic Techniques*. Ed. by Antoine Joux. Vol. 5479. Lecture Notes in Computer Science. Cologne, Germany: Springer Berlin, Heidelberg, 2009, pp. 536–553. ISBN: 978-3-642-01000-2. DOI: `10.1007/978-3-642-01001-9_31`.

[Sch95] Edward F. Schaefer. '2-Descent on the Jacobians of Hyperelliptic Curves'. In: *Journal of Number Theory* 51.2 (1995), pp. 219–232. DOI: `10.1006/jnth.1995.1044`.

[SS04] Edward F. Schaefer and Michael Stoll. 'How to do p-descent on an elliptic curve'. In: *Transactions of the American Mathematical Society* 356.3 (2004), pp. 1209–1231. DOI: `10.1090/S0002-9947-03-03366-X`.

[Sch00] Stefan Schröer. 'On contractible curves on normal surfaces'. In: *Journal für die reine und angewandte Mathematik* 524 (2000), pp. 1–15. DOI: `10.1515/crll.2000.057`.

[Sch07] Stefan Schröer. 'Kummer surfaces for the self-product of the cuspidal rational curve'. In: *Journal of Algebraic Geometry* 16.2 (2007), pp. 305–346. DOI: `10.1090/S1056-3911-06-00438-3`.

[Sch08] Stefan Schröer. 'Singularities Appearing on Generic Fibers of Morphisms between Smooth Schemes'. In: *Michigan Mathematical Journal* 56.1 (2008), pp. 55–76. DOI: `10.1307/mmj/1213972397`.

[Sch21a] Stefan Schröer. 'Enriques surfaces with normal K3-like coverings'. In: *Journal of the Mathematical Society of Japan* 73.2 (2021), pp. 433–496. DOI: `10.2969/jmsj/83728372`.

[Sch21b] Stefan Schröer. 'The Deligne—Illusie Theorem and exceptional Enriques surfaces'. In: *European Journal of Mathematics* 7.2 (2021), pp. 489–525. DOI: `10.1007/s40879-021-00451-2`.

[Sch22] Stefan Schröer. *The structure of regular genus-one curves over imperfect fields.* 2022. arXiv: `2211.04073 [math.AG]`.

[Sch23a] Stefan Schröer. 'Albanese maps for open algebraic spaces'. In: *International Mathematics Research Notices* 2024.6 (2023), pp. 4963–5004. DOI: `10.1093/imrn/rnad187`.

[Sch23b] Stefan Schröer. 'There is no Enriques surface over the integers'. In: *Annals of Mathematics* 197.1 (2023), pp. 1–63. DOI: `10.4007/annals.2023.197.1.1`.

[ST23] Stefan Schröer and Nikolaos Tziolas. 'The structure of Frobenius kernels for automorphism group schemes'. In: *Algebra & Number Theory* 17.9 (2023), pp. 1637–1680. DOI: `10.2140/ant.2023.17.1637`.

[Sei87] Wolfgang K. Seiler. 'Global moduli for polarized elliptic surfaces'. In: *Compositio Mathematica* 62.2 (1987), pp. 187–213. URL: `https://eudml.org/doc/89839`.

[Sel51] Ernst Sejersted Selmer. 'The diophantine equation $ax^3+by^3+cz^3 = 0$'. In: *Acta Mathematica* 85 (1951), pp. 203–362. DOI: `10.1007/BF02395746`.

[Sel56] Ernst Sejersted Selmer. 'On the irreducibility of certain trinomials'. In: *Mathematica Scandinavica* 4 (1956), pp. 287–302. DOI: `10.7146/math.scand.a-10478`.

[Ser90] Fernando Serrano. 'Divisors of bielliptic surfaces and embeddings in $\mathbb{P}^4$'. In: *Mathematische Zeitschrift* 203 (1990), pp. 527–533. DOI: `10.1007/BF02570754`.

[Ser73] Jean-Pierre Serre. *A Course in Arithmetic.* Graduate Texts in Mathematics. Springer New York, NY, 1973. ISBN: 978-0-387-90040-7.

[Ser79] Jean-Pierre Serre. *Local Fields.* Trans. by Marvin Jay Greenberg. Vol. 67. Graduate Texts in Mathematics. Springer New York, NY, 1979. ISBN: 978-0-387-90424-5.

[Ser97] Jean-Pierre Serre. *Abelian l-adic representations and elliptic curves.* Advanced Book Classics. CRC Press, 1997. ISBN: 0-201-09384-7.

[Sha66] Igor Rostislavovich Shafarevich. *Lectures on minimal models and birational transformations of two dimensional schemes.* Vol. 37. Tata Institute of Fundamental Research, Bombay, 1966.

[Sha96] Igor Rostislavovich Shafarevich. *Algebraic Geometry II.* Vol. 35. Encyclopaedia of Mathematical Sciences. Springer Berlin, Heidelberg, 1996. ISBN: 978-3-642-64607-2.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves.* Vol. 106. Graduate Texts in Mathematics. Springer New York, NY, 2009. ISBN: 978-0-387-09493-9.

[Sko99] Alexei Skorobogatov. 'Beyond the Manin obstruction'. In: *Inventiones Mathematicae* 135.2 (1999), pp. 399–424. DOI: `10.1007/s002220050291`.

[Sko01] Alexei Skorobogatov. *Torsors and rational points.* Vol. 144. Cambridge University Press, 2001. ISBN: 9780521802376.

[Sko09] Alexei Skorobogatov. 'Descent obstruction is equivalent to étale Brauer-–Manin obstruction'. In: *Mathematische Annalen* 344.501–510 (2009). DOI: `10.1007/s00208-008-0314-4`.

[SC21] Alexei Skorobogatov and Jean-Louis Colliot-Thélène. *The Brauer–Grothendieck group.* Springer Cham, 2021. ISBN: 978-3-030-74247-8.

[Stacks] The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`. 2018.

[Suw83] Noriyuki Suwa. 'De Rham cohomology of algebraic surfaces with $q = -p_a$ in char. p'. In: *Algebraic Geometry. Proceedings of the Japan-France Conference held at Tokyo and Kyoto, October 5-14, 1982.* Ed. by Michel Raynaud and Tetsuji Shioda. Vol. 1016. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1983, pp. 73–85. ISBN: 978-3-540-12685-0.

[Tak20] Teppei Takamatsu. *Reduction of Bielliptic Surfaces.* 2020. arXiv: `2001.06855 [math.AG]`.

[Tat52] John Tate. 'Genus change in inseparable extensions of function fields'. In: *Proceedings of the American Mathematical Society* 3.3 (1952), pp. 400–406. DOI: `10.2307/2031892`.

[Tat58] John Tate. 'WC-groups over $\mathfrak{p}$-adic fields'. In: *Séminaire Bourbaki* 4.Exp. 156 (1958), pp. 265–277. URL: `http://www.numdam.org/item/SB_1956-1958__4__265_0/`.

[Tig87] Jean-Pierre Tignol. 'On the corestriction of central simple algebras'. In: *Mathematische Zeitschrift* 194.2 (1987), pp. 267–274.

[Voi21] John Voight. *Quaternion Algebras*. Vol. 288. Graduate Texts in Mathematics. Springer Cham, 2021. ISBN: 978-3-030-56692-0.

[Wad86] Adrian R. Wadsworth. 'Merkurjev's elementary proof of Merkurjev's theorem'. In: *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory: Part II. Proceedings of the AMS–IMS–SIAM Joint Summer Research Conference held June 12–18, 1983, with support from the National Science Foundation.* Ed. by Spencer J. Bloch, R. Keith Dennis, Eric M. Friedlander and Michael R. Stein. Vol. 55. Contemporary Mathematics. American Mathematical Society, 1986, pp. 741–776. ISBN: 978-0-8218-7642-8. DOI: `10.1090/conm/055.2/1862663`.

[Wit08] Olivier Wittenberg. 'On Albanese torsors and the elementary obstruction'. In: *Mathematische Annalen* 340.4 (2008), pp. 805–838. DOI: `10.1007/s00208-007-0170-7`.

[Zar45] Oscar Zariski. 'Analytical Irreducibility of Normal Varieties'. In: *Annals of Mathematics* 49.2 (1945), pp. 352–361. DOI: `10.2307/1969284`.

[Zim19] Leif Zimmermann. 'The torsion in the cohomology of wild elliptic fibers'. PhD thesis. Heinrich-Heine-Universität Düsseldorf, Apr. 2019. URN: `nbn:de:hbz:061-20190723-110013-0`.