Operations on Milnor-Witt K-theory

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Summary

The aim of this thesis is to study operations on Milnor-Witt K-theory $\underline{\mathrm{K}}_*^{\mathrm{MW}}$. This invariant of smooth schemes arises naturally in motivic homotopy theory as the motivic 0-stem $\underline{\pi}_0(\mathbb{1}_k)_*$ of the motivic sphere spectrum $\mathbb{1}_k$, see [72], and many other invariants are modules over it. The starting point for studying operations on Milnor-Witt K-theory is a paper of Vial [96], where the $M_*(k)$-module of all uniformly bounded operations $\underline{\mathrm{K}}_n^{\mathrm{M}} \to M_*$ on Milnor K-theory $\underline{\mathrm{K}}_n^{\mathrm{M}}$ is computed. Here M_* is a cycle module with ring structure in the sense of Rost [86]. It turns out that this module of operations is generated by certain divided power operations. By a result of Morel [73], Milnor-Witt K-theory can be seen as a quadratic refinement of Milnor K-theory. Therefore this thesis deals with a generalization of Vial's aforementioned result to Milnor-Witt K-theory.

In Chapter I, which is a preparatory chapter of this thesis, we give an introduction to motivic homotopy theory and to the origin of Milnor-Witt K-theory. In particular, we give a detailed account of Morel's unstable computation of homotopy sheaves of spheres from [75] where Milnor-Witt K-theory shows up. We also give a rough outline of Morel's stable computation, by which we mean his proof of $\underline{\pi}_0(\mathbb{1}_k)_* = \underline{\mathrm{K}}_*^{\mathrm{MW}}$ from [72].

In Chapter II, the main chapter of this thesis, we first compute all additive and all $\mathbb{G}_m$-stable operations on Milnor-Witt K-theory. After this we construct divided power operations $\underline{\mathrm{K}}_n^{\mathrm{MW}} \to M_*$ for any homotopy module M_*. This is our first main result. Next we study operations on canonical generators $[-_1, \ldots, -_n]$ of $\underline{\mathrm{K}}_n^{\mathrm{MW}}$. Our second main result is a full description of the $M_*(k)$-module of all operations $[-_1, \ldots, -_n] \to M_*$, where M_* is a homotopy module with ring structure. Following a strategy of Garrell [42] from the theory of quadratic forms, we study how a general operation $\underline{\mathrm{K}}_n^{\mathrm{MW}} \to M_*$ changes when adding/subtracting an element of $[-_1, \ldots, -_n]$ to the argument. We refer to these changes as shifts. Using those shifts we compute the $M_*(k)$-module of all operations $\underline{\mathrm{K}}_n^{\mathrm{MW}} \to M_*$, which turns out to be "essentially generated" by divided power operations. This is our next main result. Here we had to restrict to a certain class of homotopy modules, but those still contain all cycle modules. Finally, we retrieve and generalize both Vial's and Garrell's computations of operations on Milnor K-theory and on powers of the fundamental ideal of the Witt ring from [96] and [42] respectively. This also leads to our last main result, which is a description of operations between Milnor, Witt and Milnor-Witt K-theory in fixed degrees.

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Zusammenfassung

Das Ziel dieser Arbeit ist, Operationen auf der Milnor-Witt-K-Theorie $\underline{\mathrm{K}}_*^{\mathrm{MW}}$ zu studieren. Diese Invariante glatter Schemata taucht auf natürliche Weise in der motivischen Homotopie-Theorie als 0-te motivische Homotopiegarbe $\underline{\pi}_0(\mathbb{1}_k)_*$ des motivischen Sphärenspektrums $\mathbb{1}_k$ auf, siehe [72], und viele andere Invarianten sind Moduln darüber. Der Ausgangspunkt des Studiums der Operationen auf der Milnor-Witt-K-Theorie ist ein Artikel von Vial [96], in welchem der $M_*(k)$-Modul aller uniform beschränkten Operationen $\underline{\mathrm{K}}_n^{\mathrm{M}} \to M_*$ auf der Milnor-K-Theorie $\underline{\mathrm{K}}_n^{\mathrm{M}}$ berechnet wird. Hierbei ist M_* ein Zykelmodul mit Ringstruktur nach Rost [86]. Es stellt sich heraus, dass dieser Modul der Operationen von gewissen dividierten Potenzoperationen erzeugt wird. Aufgrund eines Resultates von Morel [73] kann die Milnor-Witt-K-Theorie als quadratische Verfeinerung der Milnor-K-Theorie angesehen werden. Daher behandelt diese Arbeit eine Verallgemeinerung des zuvor [Theorem II.6.7] erwähnten Resultates von Vial zur Milnor-Witt-K-Theorie.
In Kapitel I, welches ein Vorbereitungskapitel dieser Arbeit ist, geben wir eine Einführung in die motivische Homotopietheorie und in den Ursprung der Milnor-Witt-K-Theorie. Insbesondere geben wir Morels instabile Berechnung von Homotopiegarben von Sphären aus [75], bei welchen die Milnor-Witt K-Theorie auftaucht, detailliert wieder. Wir geben auch einen groben Überblick über Morels stabile Berechnung, d.h. seinen Beweis von $\underline{\pi}_0(\mathbb{1}_k)_* = \underline{\mathrm{K}}_*^{\mathrm{MW}}$ aus [72].
In Kapitel II, dem Hauptkapitel dieser Arbeit, berechnen wir zunächst alle additiven und alle $\mathbb{G}_m$-stabilen Operationen auf der Milnor-Witt-K-Theorie. Danach konstruieren wir dividierte Potenzoperationen $\underline{\mathrm{K}}_n^{\mathrm{MW}} \to M_*$ für jeden Homotopiemodul M_*. Dies ist unser erstes Hauptresultat. Als nächstes studieren wir Operationen auf den kanonischen Erzeugern $[-_1, \ldots, -_n]$ von $\underline{\mathrm{K}}_n^{\mathrm{MW}}$. Unser zweites Hauptresultat ist eine vollständige Beschreibung des $M_*(k)$-Moduls aller Operationen $[-_1, \ldots, -_n] \to M_*$, wobei M_* ein Homotopiemodul mit Ringstruktur ist. Einer Strategie von Garrell [42] aus der Theorie der quadratischen Formen folgend studieren wir, wie sich eine allgemeine Operation $\underline{\mathrm{K}}_n^{\mathrm{MW}} \to M_*$ unter der Addition/Subtraktion eines Elementes aus $[-_1, \ldots, -_n]$ im Argument verändert. Die Veränderungen bezeichnen wir als Verschiebungen. Unter Verwendung dieser Verschiebungen berechnen wir den $M_*(k)$-Modul aller Operationen $\underline{\mathrm{K}}_n^{\mathrm{MW}} \to M_*$, welcher sich im Wesentlichen als von unseren dividierten Potenzoperationen erzeugt herausstellt. Dies ist unser nächstes Hauptresultat. Hierbei mussten wir uns auf eine bestimmte Klasse von Homotopiemoduln einschränken, welche aber immer noch die Zykelmoduln enthält. Schlussendlich gewinnen wir Vials und Garrells Berechnungen der Operationen auf der Milnor-Witt-K-Theorie [96] bzw. auf den Potenzen des Fundamentalideals des Wittrings [42] zurück und verallgemeinern diese. Dies führt auch zu unserem letzten Hauptresultat, welches eine Beschreibung der Operationen zwischen der Milnor-, der Witt- und der Milnor-Witt-K-Theorie in festen Graden ist.

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Contents

Conventions, notations and a shortcut

Throughout this thesis we let k be a perfect base field of characteristic not 2. Furthermore we assume all schemes to be separated and of finite type over k and rings are not necessarily commutative. For the convenience of the reader, we give the following table of notations for the main part of the thesis, which is Chapter II:

k	Perfect base field of characteristic not 2
Fld_k	Category of field extensions of k
$\mathrm{Fld}_k^{\mathrm{ftr}}$	Category of field extensions of k with finite transcendence degree
K_*^{M}	Milnor K-theory as a functor on Fld/k or $\mathrm{Fld}^{\mathrm{ftr}}/k$
K_*^{W}	Witt K-theory as a functor on Fld/k or $\mathrm{Fld}^{\mathrm{ftr}}/k$
K_*^{MW}	Milnor-Witt K-theory as a functor on Fld/k or $\mathrm{Fld}^{\mathrm{ftr}}/k$
GW	Grothendieck-Witt ring as a functor on Fld/k or $\mathrm{Fld}^{\mathrm{ftr}}/k$
W	Witt ring as a functor on Fld/k or $\mathrm{Fld}^{\mathrm{ftr}}/k$
I	Fundamental Ideal as a functor on Fld/k or $\mathrm{Fld}^{\mathrm{ftr}}/k$
Sm_k	Category of smooth schemes over k with the Nisnevich topology
Set/k	Category of sheaves (of sets) on Sm/k
Set_*/k	Category of sheaves of pointed sets on Sm/k
Ab/k	Category of abelian sheaves on Sm/k
$\mathrm{Ab}_{\mathbb{A}^1}/k$	Category of strictly $\mathbb{A}^1$-invariant sheaves on Sm/k
$\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]$	Free strictly $\mathbb{A}^1$-invariant sheaf on X
$\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[X]$	Reduced free strictly $\mathbb{A}^1$-invariant sheaf on X
$\underline{\mathrm{K}}_*^{\mathrm{M}}$	Milnor K-theory as a homotopy module
$\underline{\mathrm{K}}_*^{\mathrm{W}}$	Witt K-theory as a homotopy module
$\underline{\mathrm{K}}_*^{\mathrm{MW}}$	Milnor-Witt K-theory as a homotopy module
$\underline{\mathrm{GW}}$	Grothendieck-Witt ring as an unramified sheaf on Sm/k
$\underline{\mathrm{W}}$	Witt ring as an unramified sheaf on Sm/k
$\underline{\mathrm{I}}$	Fundamental ideal as an unramified sheaf on Sm/k
${}_xM_*$	x-torsion of some homotopy module M_*
δ_n	1 if n is odd and 0 if n is even
$\mathrm{Op}_{\mathrm{sp}}$	Operations on field extensions commuting with specialization maps
τ_n	action of $[-1]^{n-1}$ on some homotopy module

The first chapter of this thesis is a brief introduction to motivic homotopy, which also explains where the central objects of this thesis naturally arise. Some readers may for various reasons want to skip this introduction and directly read Chapter II.2. We believe this to be possible and our advise is to at least read Section I.6.2 of Chapter I. There will be a couple of statements at the end of said section which are not necessarily comprehensible without knowing a bit of motivic homotopy theory, but these are not needed for the main results and their proofs in Chapter II.

Introduction

Motivic (or $\mathbb{A}^1$-)homotopy theory is the homotopy theory of schemes. Its study was initiated by Voevodsky, and Morel and Voevodsky in [98] and [76] respectively, and has since been developed further by the work of many.
Although this theory is rather young, it has already had various deep applications in algebraic geometry and algebraic topology. Let us list a few:

- The Milnor and Bloch-Kato conjectures, see [101],[102] and [104]
- Computations of new stable stems, see [56]
- Representability results for algebraic vector bundles, see [13], [14] and [15]

One crucial aspect of motivic homotopy theory is the study of the new (co-)homology theories and their (co-)homology operations. Classically, (co-)homology operations have also been studied extensively and have been used for various applications. Famous examples include so-called Adams operations $\psi^l : K \to K$ on topological K-theory, which were used in Adams' study of vector fields on spheres [4] and Adams' and Atiyah's proof of the Hopf invariant one problem [5]. Also stable operations, which are families of operations on a (co-)homology theory respecting the suspension functor, are of central interest. The most famous example being the (mod p) Steenrod algebra, the algebra of stable cohomology operations in mod p singular cohomology $H^*(-;\mathbb{Z}/p\mathbb{Z})$.
In the motivic world, such operations on mod p motivic cohomology $H^{*,*}(-;\mathbb{Z}/p\mathbb{Z})$ are for example constructed and used by Voevodsky in his proof of the aforementioned Bloch-Kato conjecture [101],[102] and [104]. For a field F, the (mod p) motivic cohomology group $H^{n,n}(\mathrm{Spec}(F);\mathbb{Z}/p\mathbb{Z})$ is given by (mod p) Milnor K-theory $K_n^{\mathrm{M}}(F)/p$ [68], which is an invariant defined by Milnor in his seminal paper [70]. For this theory, Vial [96] determines the $M_*(k)/p$-module of all uniformly bounded operations $K_n^{\mathrm{M}}/p \to M_*$ and the $M_*(k)$-module of uniformly bounded operations $K_n^{\mathrm{M}} \to M_*$. Here, M_* is a so-called cycle module with ring structure, such as Milnor K-theory or algebraic K-theory. In turns out that these modules of operations are spanned by so-called divided power operations. This thesis is a generalization of Vial's results. Let us therefore introduce the main objects.
In topology, the n-sphere S^n can be defined purely in terms of the 1-dimensional sphere S^1, by using the smash product "$\wedge$". The latter is a "tensor product" of pointed topological spaces and it is not difficult to verify that $S^n = (S^1)^{\wedge n}$. Following that idea, we now find that there are multiple 1-dimensional spheres in motivic homotopy. As an amalgamation of objects from algebraic topology and objects from algebraic geometry, we have the topological/simplicial 1-spheres S^1 and also the algebraic 1-dimensional sphere $\mathbb{G}_{\mathrm{m}}$. This results in a bigraded family of spheres $S^{n,m} = (S^1)^{\wedge(n-m)} \wedge \mathbb{G}_m^{\wedge m}$ in motivic homotopy theory. These spheres give rise to bigraded stable homotopy sheaves of spheres $\underline{\pi}_{n,m}(\mathbb{1}_k)$, which are of central interest to motivic, but also to classical stable homotopy theory. The latter is due to work of Levine [61], which in particular yields that $\underline{\pi}_{n,0}(\mathbb{1}_{\mathbb{C}})(\mathbb{C})$ are the usual stable homotopy groups of spheres. So this is what happens if we ignore the algebraic spheres. For the other extreme, Morel [72] showed that for all integers m, the sheaf $\underline{\pi}_{-m,-m}(\mathbb{1}_k)$ has a purely algebraic description, called Milnor-Witt K-theory $\underline{K}_m^{\mathrm{MW}}$ in degree m. This makes use of further work of Morel [73], where it is shown that there is a pullback square

$$\begin{array}{ccc} \underline{K}_m^{\mathrm{MW}} & \longrightarrow & \underline{K}_m^{\mathrm{M}} \\ \downarrow & & \downarrow \\ \underline{I}^m & \longrightarrow & \underline{K}_m^{\mathrm{M}}/2, \end{array}$$

where $\underline{I}$ is the fundamental ideal of the Witt ring of quadratic forms $\underline{W}$. Based on the fact that $\underline{I}^m$ also has an algebraic description called Witt K-theory in degree m, see [73], this pullback square explains the name Milnor-Witt K-theory and describes Milnor-Witt K-theory as a quadratic refinement of Milnor K-theory. This leads to the natural question whether Vial's aforementioned results can be generalized to Milnor-Witt K-theory, which is the main content of my thesis.

The main strategy to get a hold of these operations is the following. Garrel [42] computes the modules of all operations $I^n \to W$ and $I^n \to H^*(-;\mu_2)$ defined over field extensions of a fixed base field. These computations also respect the natural filtrations of W and $H^*(-;\mu_2)$ and thus in particular give all operations $I^n \to I^m$. Garrel relies on Theorem 18.1 of Serre [88], which describes all operations $\mathrm{Pf}_n \to W$ and $\mathrm{Pf}_n \to H^*(-;\mu_2)$ as free modules of rank 2. Here Pf_n are isomorphism classes of n-Pfister forms, which are the canonical choice of generators of I^n. He goes on to study how an operation on I^n changes when adding and subtracting generators $x \in \mathrm{Pf}_n$. In other words, the idea is to start with operations on generators, which in Garrel's case are known due to the aforementioned results of Serre, and then to extend these to the entire theory, even though the operations need not be additive. For Milnor-Witt K-theory, operations on generators are not known, which at first prevents us from following Garrel's strategy. Our first main result gets rid of this obstruction. Let us denote by $[-_1, \ldots, -_n]$ the subsheaf of canonical generators of Milnor-Witt K-theory.

Theorem (Theorem II.3.3)**.** For any homotopy module M_* with ring structure and any positive integer n, the $M_*(k)$-module of operations $[-_1, \ldots, -_n] \to M_*$ is free of rank 2 generated by the constant operation 1 and the action of $[-_1, \ldots, -_n]$ on $1 \in M_*(k)$.

Here, a homotopy module is a certain kind of $\mathbb{Z}$-graded module over Milnor-Witt K-theory given by $\underline{\pi}_{-*,-*}(E)$ for a motivic spectrum E. This includes Milnor-Witt K-theory, Milnor K-theory, Witt K-theory, algebraic K-theory and Hermitian K-theory.

While this result generally enables Garrel's strategy for us, the next issue is that many operations on quadratic forms are already known. For example, exterior power operations were already known to Bourbaki [23]. It was folklore that these are λ-operations, which was finally shown by McGarraghy in [69]. As shown by Garrel [42], these turn out to generate all operations in a suitable sense. Based on Vial's and Garrel's computations, the natural guess is that also the operations on Milnor-Witt K-theory are spanned by similar kinds of operations. This leads us to construct divided power operations for Milnor-Witt K-theory:

Theorem (Theorem II.2.1)**.** Let n be a positive integer, let ℓ be a non-negative integer and let δ_n be 1 if n is odd and 0 if n is even. Moreover, let $h \in K_0^{\mathrm{MW}}(k)$ be the standard hyperbolic form, let M_* be a homotopy module and let $y \in {}_{\delta_n h}M_*(k)$. There are operations $\lambda_l^n \cdot y \colon \underline{\mathrm{K}}_n^{\mathrm{MW}} \to M_*$ which map sums of generators $[a_{1,1}, \ldots, a_{1,n}] + \ldots + [a_{r,1}, \ldots, a_{r,n}]$ to

$$\left(\sum_{1 \le i_1 < \ldots < i_l \le r} [a_{i_1,1}, \ldots, a_{i_1,n}] \cdot \ldots \cdot [a_{i_l,1}, \ldots, a_{i_l,n}] \right) \cdot y.$$

Now that we also have some natural candidates for generators of all operations, we continue translating Garrel's strategy to Milnor-Witt K-theory. For this to work we restrict to $\mathbb{N}$-graded homotopy modules with ring structure M_*, which we call $\mathbb{N}$-graded homotopy algebras. Several other obstructions arise from the non-commutativity of $\underline{K}_*^{\mathrm{MW}}$, which we manage to overcome by a careful study of the differences in Garrel's and our setup. To state the main results, let us define

$$\sigma_n^l = \sum_{j=0}^{\lfloor \frac{l-1}{2} \rfloor} \binom{\lfloor \frac{l-1}{2} \rfloor}{j} [-1]^{n(l-j)} \lambda_{l-j}^n$$

for all integers $l \geq 1$ and $\sigma_n^0 = \lambda_0^n$. These are certain linear combinations of our previous operations, which suit Garrel's strategy. Our next main result is the following.

Theorem (Theorem II.5.6)**.** Let n be a positive integer. For all $\mathbb{N}$-graded homotopy algebras M_*, the map

$$f\colon M_*(k)^2 \times {}_{\delta_n h} M_*(k)^{\mathbb{N}\setminus\{0,1\}} \to \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(\underline{\mathrm{K}}_n^{\mathrm{MW}}, M_*),\ (a_l)_{l\geq 0} \mapsto \sum_{l\geq 0} \sigma_n^l \cdot a_l$$

is an isomorphism of $M_*(k)$-modules, where δ_n is 1 if n is odd and 0 if n is even.

Actually, this result can be refined a bit. The right hand side is equipped with the natural filtration induced by the filtration $F_d M_* = M_{\geq d}$ and also the left hand side can be given a suitable filtration. The isomorphism above is then even an isomorphism of filtered modules. It is not completely obvious that infinite sums of the form $\sum_{l\geq 0} \sigma_n^l \cdot a_l$ with suitable coefficients $(a_l)_{l\geq 0}$ make sense. For this we had to show that while these sums are infinite, they become finite when evaluating at any element of $\underline{K}_n^{\mathrm{MW}}$.
Since Milnor K-theory can be identified with a quotient of Milnor-Witt K-theory, the above theorem also allows us to recover Vial's aforementioned results [96] and to generalize these to non-uniformly bounded operations:

Theorem (Theorem II.6.6)**.** For all positive integers n and all cycle modules with ring structure M_*, we have

$$\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(\underline{\mathrm{K}}_n^{\mathrm{M}}, M_*) \cong \Big\{ \sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_{l\geq 0} \in M_*(k)^2 \times {}_{\delta_n 2}\Big({}_{\tau_n} M_*(k)\Big)^{\mathbb{N}\setminus\{0,1\}} \Big\}$$

as a filtered $M_*(k)$-module. In particular we recover Theorem 5.5 of [96].

Here a cycle module can be defined as a homotopy module on which a certain element, namely the Hopf element $\eta \in K_{-1}^{\mathrm{MW}}(k)$, acts trivially. Similarly, our computation of operations on Milnor-Witt K-theory also allows us to generalize Garrel's [42] computation to $\mathbb{N}$-graded homotopy algebras:

Theorem (Theorem II.6.2)**.** For all positive integers n and all $\mathbb{N}$-graded homotopy algebras M_*, we have

$$\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(\underline{\mathrm{I}}^n, M_*) \cong \Big\{ \sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_{l\geq 0} \in M_*(k) \times {}_h M_*(k)^{\mathbb{N}\setminus\{0\}} \Big\}$$

as a filtered $M_*(k)$-module. In particular we recover Theorem 4.9 of [42] if $M_* = \underline{\mathrm{W}}$ or $M_* = \underline{\mathrm{K}}^{\mathrm{M}}_* /2 \cong \underline{H}^*(-, \mu_2)$.

A direct consequence of this is the following corollary, which describes all operations on Milnor-Witt K-theory in negative degree:

Corollary (Corollary II.6.3 and Corollary II.6.4)**.** Let n be a negative integer. For all $\mathbb{N}$-graded homotopy algebras M_*, the filtered $M_*(k)$-modules $\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(\underline{\mathrm{K}}^{\mathrm{MW}}_0, M_*)$ and $\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(\underline{\mathrm{K}}^{\mathrm{MW}}_n, M_*)$ are given by

$$\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(\underline{\mathbb{Z}}, M_*) \times \Big\{ \sum_{l \geq 0} \overline{\sigma}_1^l \cdot a_l \mid (a_l)_{l \geq 0} \in M_*(k) \times {}_h M_*(k)^{\mathbb{N} \setminus \{0\}} \Big\}$$

and

$$\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(\underline{\mathbb{Z}/2\mathbb{Z}}, M_*) \times \Big\{ \sum_{l \geq 0} \overline{\sigma}_1^l \cdot a_l \mid (a_l)_{l \geq 0} \in M_*(k) \times {}_h M_*(k)^{\mathbb{N} \setminus \{0\}} \Big\},$$

respectively.

Finally, due to the above computations as filtered modules, we obtain the following description of operations between K-theory sheaves in fixed degrees.

Theorem (Theorem II.6.7)**.** Let n be a positive integer. The following table gives a complete list of operations between Milnor, Witt and Milnor-Witt K-theory

A	B	$\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(A, B)$
$\underline{\mathrm{K}}^{\mathrm{M}}_n$	$\underline{\mathrm{K}}^{\mathrm{M}}_m$	$\Big\{ \sum_{l \geq 0} \overline{\sigma}_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in \prod_{\min(\frac{m}{n},1) \geq l \geq 0} K^{\mathrm{M}}_{m-nl}(k) \times \prod_{\frac{m}{n} \geq l \geq 2} {}_{\delta_n 2}({}_{\tau_n} K^{\mathrm{M}}_{m-nl}(k)) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{M}}_n$	$\underline{\mathrm{K}}^{\mathrm{W}}_m$	$\Big\{ \sum_{l \geq 0} \overline{\sigma}_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in \prod_{l=0}^{1} K^{\mathrm{W}}_{m-nl}(k) \times \prod_{l \geq 2} {}_{\delta_n 2}({}_{\tau_n} K^{\mathrm{W}}_{m-nl}(k)) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{M}}_n$	$\underline{\mathrm{K}}^{\mathrm{MW}}_m$	$\Big\{ \sum_{l \geq 0} \overline{\sigma}_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in \prod_{l=0}^{1} K^{\mathrm{MW}}_{m-nl}(k) \times \prod_{l \geq 2} {}_{\delta_n 2}({}_{\tau_n} K^{\mathrm{MW}}_{m-nl}(k)) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{W}}_n$	$\underline{\mathrm{K}}^{\mathrm{M}}_m$	$\Big\{ \sum_{l \geq 0} \overline{\sigma}_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in K^{\mathrm{M}}_m(k) \times \prod_{\frac{m}{n} \geq l \geq 1} {}_2 K^{\mathrm{M}}_{m-nl}(k) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{W}}_n$	$\underline{\mathrm{K}}^{\mathrm{W}}_m$	$\Big\{ \sum_{l \geq 0} \overline{\sigma}_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in \prod_{l \geq 0} K^{\mathrm{W}}_{m-nl}(k) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{W}}_n$	$\underline{\mathrm{K}}^{\mathrm{MW}}_m$	$\Big\{ \sum_{l \geq 0} \overline{\sigma}_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in K^{\mathrm{MW}}_m(k) \times \prod_{l \geq 1} {}_h K^{\mathrm{MW}}_{m-nl}(k) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{MW}}_n$	$\underline{\mathrm{K}}^{\mathrm{M}}_m$	$\Big\{ \sum_{l \geq 0} \sigma_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in \prod_{\min(\frac{m}{n},1) \geq l \geq 0} K^{\mathrm{M}}_{m-nl}(k) \times \prod_{\frac{m}{n} \geq l \geq 2} {}_{\delta_n 2} K^{\mathrm{M}}_{m-nl}(k) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{MW}}_n$	$\underline{\mathrm{K}}^{\mathrm{W}}_m$	$\Big\{ \sum_{l \geq 0} \sigma_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in \prod_{l \geq 0} K^{\mathrm{W}}_{m-nl}(k) \Big\}$
$\underline{\mathrm{K}}^{\mathrm{MW}}_n$	$\underline{\mathrm{K}}^{\mathrm{MW}}_m$	$\Big\{ \sum_{l \geq 0} \sigma_n^l \cdot a_l \,\Big\vert\, (a_l)_l \in \prod_{l=0}^{1} K^{\mathrm{MW}}_{m-nl}(k) \times \prod_{l \geq 2} {}_{\delta_n h} K^{\mathrm{MW}}_{m-nl}(k) \Big\}$,

where τ_n is the action of $[-1]^{n-1}$ on the target.

Chapter I

Motivic Homotopy Theory

In this chapter we will introduce the unstable and stable motivic homotopy categories over a perfect base field k of characteristic not 2, which we, as also mentioned before, fix for the rest of the thesis. Let us note that this is a very common assumption in motivic homotopy theory and that certain results that we will present need this assumption or are only known under this assumption. Moreover, we want to highlight that this is a rather short introduction and hence many fundamental results will not be included here. One such example is the so-called homotopy purity, found as Theorem 2.23 of Section 3 of [76].

The main idea of motivic homotopy is that the affine line $\mathbb{A}^1$ replaces the unit interval $I = [0, 1]$ used in the homotopy theory of topological spaces and that this replacement allows us to do homotopy theory with schemes. In particular $\mathbb{A}^1$ becomes contractible in this theory. As there are quite some interesting invariants of schemes which are not $\mathbb{A}^1$-invairant, i.e., not homotopy invariant in this setting, also non-$\mathbb{A}^1$-invariant variants of motivic homotopy theory are being explored more and more; see [8], [7], [6] and [52]. We will exclusively work with the "classical" $\mathbb{A}^1$-invariant theory. Before we get into more details, let us give a quick summary of how to construct the unstable motivic homotopy category $\mathrm{H}(k)$ to motivate the various sections of this chapter:

- *Restriction to nice spaces*

  We want algebraic K-theory to be homotopy invariant, but this is false in general. By Quillen's fundamental theorem for algebraic K-theory, this is true for regular noetherian schemes though [46]. This suggests that we should not work with all schemes, but rather with a suitable subcategory of schemes. As it turns out, the category Sm_k of smooth schemes which are separated and of finite type over k is the correct choice.

- *Choice of a suitable topology* (Section I.1)

  The Zariski topology is not too well-behaved when trying to replicate notions or results from topology. That is when algebraic geometers often switch to the étale topology, which will not work for our purposes, since algebraic K-theory is known not to satisfy étale descent, see for instance page 3 of [65]. But there is a topology sitting in between these two topologies, which shares the respective good properties and at the same time avoids bad ones. This is the so-called Nisnevich topology [77], which is the default topology on Sm_k for doing homotopy theory. Let us nevertheless note that there are other topologies under consideration as well. One such example is the so-called cdh-topology; see, for instance, [103].

- *Extending the category* (Section I.1)

  Various colimits of spaces show up naturally in homotopy theory, but categories of schemes are very poorly equipped when it comes to colimits. Therefore we consider presheaves on Sm_k, which by the Yoneda lemma gives us the free cocompletion. There are also models with sheaves instead of presheaves due to the sheafification functor being a left adjoint. We will use the latter, as these for instance respect previously existing pushouts.

- *Getting a homotopy theory* (Sections I.2 and I.3)

  A well-behaved model for the usual homotopy theory of topological spaces is given by the homotopy category of simplicial sets, see Theorem I.3.19. The latter are, in some sense, a category-theoretic version of simplicial complexes. We now replace sheaves on Sm_k by sheaves of simplicial sets on Sm_k, which we will call spaces. Equivalently, we may consider simplicial sheaves on Sm_k. This allows us to define a simplicial model structure and hence to get a homotopy theory on smooth schemes.

- *Making $\mathbb{A}^1$ contractible* (Section I.4)

  Although we managed to define a homotopy theory on smooth schemes, it has a purely simplicial nature and does not reflect the main idea. In particular, the affine line $\mathbb{A}^1$ is not yet contractible. Therefore we incorporate this into the simplicial model structure, which yields the so-called $\mathbb{A}^1$-model structure on spaces. The associated homotopy category is the unstable motivic homotopy category $\mathrm{H}(k)$.

Before we begin, let us quickly mention what happens if one tries to follow the same recipe for topological spaces. Here the first step should be the restiction to a suitable category of manifolds. Dugger showed in [34] that this results in a model for the usual homotopy theory of topological spaces.

I.1 Topologies on Schemes

In this section we will deal with the second and third bullet points of our summary. In particular, we will introduce the Nisnevich topology on smooth schemes and the associated notion of Nisnevich (pre-)sheaves.

I.1.1 First examples of Grothendieck Topologies

The Zariski topology is rather coarse, which certainly comes with its downsides. One being the lack of a suitable cohomology theory for schemes over finite fields, which Weil suggested would be instrumental at solving the famous conjectures named after him. This led Grothendieck to categorify open coverings and thus to give birth to the notion nowadays refered to as a Grothendieck topology [9]. In particular, the étale topology was born and with it étale and ℓ-adic cohomology as constructed by Grothendieck and Artin [1], [2] and

[3], which indeed turned out to be crucial in the proofs of the Weil conjectures. Since then, various other Grothendieck topologies on schemes were found and studied. Let us note that even if we ignore motivic homotopy theory, Grothendieck topologies are also of interest to homotopy theorists. They were for instance used to define scissors congruence K-theory spectra by Zakharevich [110] and thus gave rise to higher versions of Hilbert's 3rd problem. Also in the recent proof of the redshift conjecture in chromatic homotopy theory by Burklund, Schlank and Yuan [26] it was crucial to find a suitable Grothendieck topology in the setting of spectral algebraic geometry.
For this subsection we follow Chapter 2.1 of [78].

Definition I.1.1. A Grothendieck topology τ on a category $\mathcal{C}$ consists of collections $\mathrm{Cov}(X)$ of families of morphisms $\{U_i \to X\}_{i\in I}$ for each object $X \in \mathcal{C}$, called coverings of X, subject to the following three conditions:

(i) For every isomorphism $U \to X$, we have $\{U \to X\} \in \mathrm{Cov}(X)$.

(ii) For all coverings $\{U_i \to X\}_{i\in I}$ and all morphisms $Y \to X$, the pullback $U_i \times_X Y$ exists and we have $\{U_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$.

(iii) For all coverings $\{U_i \to X\}_{i\in I}$ and for all coverings $\{U_{ij} \to U_i\}_{j\in J_i}$, $i \in I$, we have $\{U_{ij} \to U_i \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}(X)$.

Some authors would refer to the above notion as a Grothendieck pretopology. In their language, we will not need the full notion of Grothendieck topologies and therefore refrain from defining them. Let us just mention that every Grothendieck pretopology gives rise to a Grothendieck topology and that one can think of a Grothendieck pretopology as a convenient basis of a Grothendieck topology.
Let us give a couple of elementary examples. As for topological spaces, there are two trivial examples of Grothendieck topologies:

Example I.1.2. If $\mathcal{C}$ is an arbitrary category, we can define the indiscrete Grothendieck topology on $\mathcal{C}$ by only allowing isomorphisms as coverings. Since isomorphisms are stable under pullback and composition, this does indeed yield a Grothendieck topology.

Example I.1.3. If $\mathcal{C}$ is a category that has all pullbacks, we can let all families of morphisms be coverings. This certainly satifies axioms (i)-(iii) and thus defines a Grothendieck topology on $\mathcal{C}$, called the discrete Grothendieck topology on $\mathcal{C}$.

Furthermore, every topological space gives rise to a Grothendieck topology:

Example I.1.4. Let X be a topological space and consider the poset category $\mathrm{Op}(X)$ of open subsets of X. For every open subset U of X, we define $\mathrm{Cov}(U)$ to consist of those families of morphisms $\{U_i \to U\}_{i\in I}$ with $\bigcup_{i\in I} U_i = U$. This defines a Grothendieck topology on $\mathrm{Op}(X)$. Let us quickly check the three axioms. The isomorphisms in $\mathrm{Op}(X)$ are exactly the identity maps $\mathrm{id}_U\colon U \to U$, which certainly are coverings. Thus (i) holds. For all open subsets U of X and for each pair of morphisms $U_1 \to U$ and $U_2 \to U$, the pullback $U_1 \times_U U_2$ is given by the intersection $U_1 \cap U_2$, which is again an open subset of X and hence exists in $\mathrm{Op}(X)$. If now $\{U_i \to U\}_{i\in I}$ is a covering, i.e. we have $\bigcup_{i\in I} U_i = U$, then we clearly also have the equality $\bigcup_{i\in I}(U_i \cap U') = U'$ for any other open subset U' of X with $U' \subset U$. In other words, $\{U_i \times_U U' \to U'\}_{i\in I}$ is a covering of U' for any morphism $U' \to U$. This shows that (ii) also holds. Finally, for all open subsets U of X, for all coverings $\{U_i \to U\}_{i\in I}$ of U and for all coverings $\{U_{ij} \to U_i\}_{j\in J_i}$ of U_i, $i \in I$, we have that $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering of U, since $U = \bigcup_{i\in I} U_i = \bigcup_{i\in I}\bigcup_{j\in J_i} U_{ij}$. Therefore also (iii) holds.

Before we give more examples, let us quickly introduce the following notion:

Definition I.1.5. A category $\mathcal{C}$ together with a Grothendieck topology on $\mathcal{C}$ is called a site.

Often, particularly for schemes, there are two sites associated with the same Grothendieck topology, namely a small one local to some fixed geometric object X and a big one on the full category of all such objects.

Example I.1.6. The site associated with the Grothendieck topology given by the usual coverings of a topological space X from Example I.1.4 is called the small site of X.

Example I.1.7. We can also consider the "usual coverings" of topological spaces as coverings on the category of all topological spaces. This yields the big site of topological spaces.

Example I.1.8. If X is a scheme, then the small site of the underlying topological space of X is called the small Zariski site of X and is usually denoted by X_{Zar}. Although we will not define it, there is a notion of equivalence of sites. Up to such an equivalence, the small Zariski site can also be defined in terms of morphisms of schemes over X. Here a cover $\{U_i \to X\}_{i\in I}$ is given by open immersions that are jointly surjective.

Example I.1.9. The "usual coverings" for schemes define a Grothendieck topology on Sch which gives rise to the big Zariski site $\mathrm{Sch}_{\mathrm{Zar}}$.

As already mentioned in the introduction, a very important example of a Grothendieck topology is the étale topology. Let us therefore quickly recall one of the many equivalent definitions of étale morphisms, which we will use to verify some examples later on, see also [89, Tag 02GH].

Definition I.1.10. A morphism $f\colon \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ of affine schemes is called (standard) étale if it is isomorphic to the canonical map $\mathrm{Spec}(R[x]_h/(g)) \to \mathrm{Spec}(R)$ for some polynomials $g, h \in R[x]$, where g is monic and its derivative g' is invertible in $R[x]_h/(g)$.

Although a bit more specialized, this definition is essentially one way of saying that the given morphism is smooth (the invertibility of g' is essentially the Jacobian criterion) and that it is of relative dimension 0 (the number of indeterminates and relations agree), which indeed is one of the more commonly used definitions of étale morphisms also found in loc. cit.

Example I.1.11. Let n be a positive integer not dividing the characteristic of k. Then we claim that the map $f\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1$, $x \mapsto x^n$ is étale. We have a commutative diagram

$$\begin{array}{ccc} k[t] & \xrightarrow{t\mapsto t^n} & k[t,t^{-1}] \\ \downarrow\cong & & \downarrow\cong \\ k[t^n] & \xrightarrow{\text{can}} & k[t^n][s]_s/(s^n - t^n) \end{array}$$

of k-algebras, where the vertical isomorphisms are given by $t \mapsto t^n$ and $t \mapsto s$ respectively. Note that $s^n - t^n \in k[t^n][s]$ is a monic polynomial and that its derivative ns^{n-1} is a unit in $k[t^n][s]_s/(s^n - t^n)$. Indeed, due to the localization at s we have that s^{n-1} is a unit and by our assumption on the characteristic of k also n is a unit. Therefore this diagram shows that f is étale.

For more concrete examples we refer the reader to Section I.1.3. This now allows us to define étale morphisms as standard étale morphisms:

Definition I.1.12. A morphism $f\colon X \to Y$ of schemes is called étale if there exist open affine coverings $Y = \bigcup_{i \in I} V_i$ and $f^{-1}(V_i) = \bigcup_{j_i \in J_i} U_j$ for all $i \in I$ such that each of the morphisms $U_{j_i} \to V_i$ are (standard) étale.

Example I.1.13. Open immersions of schemes are étale. Indeed, we can cover an open subscheme by open affines and then these by basic open subschemes. The latter are given by localizing at the multiplicative set generated by one element, which therefore tells us that we have an étale map by choosing $g = x$ in the definition of étale maps between affine schemes.

We are now able to define étale coverings:

Definition I.1.14. A family of morphisms $\{f_i\colon U_i \to X\}_{i \in I}$ of schemes is called an étale covering if the morphisms f_i are étale for all $i \in I$ and $X = \bigcup_{i \in I} f_i(U_i)$.

Example I.1.15. By Example I.1.13, every Zariski covering is also an étale covering.

Now that we have one definition for étale morphisms and their associated coverings, we are also able to consider the étale sites.

Example I.1.16. There is the small étale site $X_{\text{ét}}$ of a scheme X, which is the full subcategory of Sch_X with objects given by étale morphisms.

Example I.1.17. Taking étale coverings on the category of all schemes yields the big étale site $\mathrm{Sch}_{\text{ét}}$.

There are of course various other sites of interest, such as the ones given by the fppf and fpqc topologies on schemes, which we are not mentioning here, see [89, Tag 021L] and [89, Tag 03NV].

I.1.2 Sheaves on Sites

Before we get to the central example for our purposes, we want to recall how to define sheaves on sites, see for instance Chapter 2.2 of [78]. Let us first recall the general definition of presheaves.

Definition I.1.18. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A $\mathcal{D}$-valued presheaf (or presheaf of "objects of $\mathcal{D}$") on $\mathcal{C}$ is a functor $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

If the category $\mathcal{D}$ is not specified, it is always assumed to the category of sets.

Example I.1.19. A very important class of examples of presheaves on a category $\mathcal{C}$ are the representable presheaves, which by definition are the presheaves isomorphic to $\mathrm{Hom}(-, X)$ for some object $X \in \mathcal{C}$.

This notion of presheaves on a category specializes to the notion of presheaves on a given topological space X by choosing $\mathcal{C}$ as the topology of X considered as a poset category. In this situation we are able to talk about sheaves and not only presheaves, which we can now also generalize based on the notions of Grothendieck topologies and sites.

Definition I.1.20. Let $\mathcal{D}$ be a complete category. A $\mathcal{D}$-valued sheaf on a site $\mathcal{C}$ is a $\mathcal{D}$-valued presheaf $\mathcal{F}$ on $\mathcal{C}$ such that for any $X \in \mathcal{C}$ and any covering $\{U_i \to X\}_{i\in I}$, the diagram

$$\mathcal{F}(X) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer diagram, where the two arrows on the right are induced by the two projections.

We denote the categories of presheaves and sheaves on $\mathcal{C}$ by $\mathrm{PreSh}(\mathcal{C})$ and $\mathrm{Shv}(\mathcal{C})$ respectively. By definition, we have an inclusion functor $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{PreSh}(\mathcal{C})$. Let us quickly mention that sheafification also extends to our general setup:

Proposition I.1.21. *For any site $\mathcal{C}$, the inclusion functor* $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{PreSh}(\mathcal{C})$ *has a left adjoint.*

This left adjoint is called the sheafification functor and its construction, as for example done in [78, Theorem 2.2.4], shows that it commutes with finite limits.

Example I.1.22. An example which we will use very often is the following. For every abelian group A, we can consider the constant presheaf with value A on a site $\mathcal{C}$. Its sheafification will be denoted by $\underline{A}$ and is called the constant sheaf with value A.

Let us also adress representables. Central to algebraic geometry is that every scheme X itself defines a Zariski sheaf $\mathrm{Hom}(-, X)$, obtained from gluing affine opens. This is the basis for the functorial point of view of algebraic geometry and leads to the following notion:

Definition I.1.23. A site $(\mathcal{C}, \tau)$ (or just the Grothendieck topology τ) is called subcanonical if all representable presheaves on $\mathcal{C}$ are sheaves on $\mathcal{C}$.

The name comes from the fact that such a topology is coarser than the so-called canonical topology, which by definition is the finest topology with the property that all representables are sheaves, see [89, Tag 00WO].
The simplest way to establish this property is the following easy observation, which follows directly from the definitions:

Proposition I.1.24. *If $\mathcal{C}$ is a category with two Grothendieck topologies τ and τ' such that τ is coarser than τ' and the site $(\mathcal{C}, \tau')$ is subcanonical, then so is $(\mathcal{C}, \tau)$.*

As any étale covering is a so-called fpqc covering, see for instance [89, Tag 03PF], and it is known that the fpqc topology is subcanonical [89, Tag 03NV], also the étale topology is subcanonical. Therefore we get:

Corollary I.1.25. *All topologies on* Sm_k *which are coarser than the étale topology are subcanonical.*

Remark I.1.26. In the next subsection we will introduce the Nisnevich topology. By definition, it will be coarser than the étale topology and thus be a subcanonical topology.

Although not directly relevant for us, we cannot refrain from quickly mentioning the following. Let X be a scheme and let τ be a Grothendieck topology on schemes. In the same way that the stalks of the structure sheaf $\mathcal{O}_X$ with respect to the Zariski topology are local rings, one might wonder what happens for finer topologies. Among these we have only introduced the étale topology so far, but let us nevertheless also mention what happens in case of the Nisnevich topology:

Example I.1.27. For the étale topology, the local rings are strictly henselian local rings [2]. These are henselian local rings, i.e. local rings for which Hensel's lemma holds, whose residue field is separably closed.

Example I.1.28. As already mentioned, the Nisnevich topology lies in between the Zariski topology and the étale topology. Therefore also its local rings must not quite be strictly henselian local rings, but still local rings. It turns out that these are exactly the henselian local rings [77].

Finally, let us quickly discuss sheaf cohomology on a site $(\mathcal{C}, \tau)$. For this it is, as always, crucial to know that we can choose injective resolutions.

Theorem I.1.29. *The category of abelian sheaves on any site $\mathcal{C}$ has enough injectives. In particular, any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ has an injective resolution $\mathcal{F} \to \mathcal{I}^*$.*

Proof. This is Theorem 19.7.4 of [89, Tag 01DL]. □

Now the usual definition of sheaf cohomology extends, see also [89, Tag 01FT]:

Definition I.1.30. Let $\mathcal{C}$ be a site and let n be a non-negative integer. The n-th cohomology of an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ is the functor

$$H^n(-, \mathcal{F}) = H^n(\mathcal{I}^*(-))\colon \mathcal{C} \to \mathrm{Ab}$$

for an injective resolution $\mathcal{F} \to \mathcal{I}^*$.

Note that these groups do not depend on the chosen injective resolution resulting in a well-defined notion.

Remark I.1.31. If $\mathcal{C}$ is some category of schemes, we have various useful choices of topologies. Therefore we may also consider sheaf cohomology with respect to any of these topologies. To ensure that the reader knows with which topology we are currently working, we will usually add the topology τ as an index of the cohomology groups. So for example, $H^*_{\mathrm{Nis}}(X, \mathcal{F})$ will denote the Nisnevich cohomology groups of some scheme X with respect to an abelian (Nisnevich) sheaf $\mathcal{F}$.

I.1.3 The Nisnevich Topology

In the last subsection we already mentioned some properties of the Nisnevich topology. So let us finally introduce it.

Definition I.1.32. A collection of morphisms of schemes $\{f_i\colon U_i \to X\}_{i\in I}$ is called a Nisnevich covering if the following two conditions hold:

(i) The morphisms f_i are étale for all indices $i \in I$.

(ii) For every point $x \in X$ there exist an index $i \in I$, such that f_i is completely decomposed at x, i.e. for every point $x \in X$ there exist an index $i \in I$, a point $u \in U_i$ with $f_i(u) = x$, such that the morphism on the residue fields $\kappa(x) \to \kappa(u)$ induced by f_i is an isomorphism.

We will denote the collection of all Nisnevich coverings of a given scheme X by $\mathrm{Cov}_{\mathrm{Nis}}(X)$. The original definition of these coverings due to Nisnevich [77] was different, but we will soon see that this is an equivalent description.

Remark I.1.33. Some authors demand that the morphisms f_i are not only étale, but étale and of finite type. This is automatically the case in our setup due to our general assumptions on schemes; see page 1. Since étale morphisms are locally of finite presentation and thus in particular locally of finite type, we only need to observe that they are always quasi-compact, when between separated schemes of finite type. This follows from the fact that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of schemes such that g is separated and $g \circ f$ is quasi-compact, then also f is quasi-compact [89, Tag 050Y].

Let us now have a look at some examples:

Example I.1.34. Every Zariski covering is a Nisnevich covering. Indeed, if $\{f_i\colon U_i \to X\}_{i\in I}$ is a Zariski covering, i.e. a family of open immersions whose images cover X in the usual sense, then it is in particular an étale covering, see I.1.15. By the fact that open immersions always induce isomorphisms on residue fields, it is even a Nisnevich covering.

Example I.1.35. We let n be a positive integer and choose $a \in k^\times$. Suppose that the characteristic of k does not divide n. Then we claim that the morphisms

$$\begin{array}{ccc} & & \mathbb{A}^1 \setminus \{a\} \\ & & \big\downarrow{\scriptstyle i} \\ \mathbb{G}_{\mathrm{m}} & \xrightarrow{(-)^n} & \mathbb{A}^1 \end{array}$$

are an étale covering, which form a Nisnevich covering if and only if $a \in k^{\times n}$, i.e if and only if the element a has an n-th root in k. Let us verify this.
The morphism i is certainly étale, since it is an inclusion of an open subscheme and we have already seen in Example I.1.11 that the power map $(-)^n\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1$, which we will also denote by f, is étale.
Let us now show that for each $b \in \mathbb{A}^1$, at least one of the morphisms f or i is completely decomposed at b. As i is the inclusion of the open subscheme $\mathbb{A}^1 \setminus \{a\}$, i is certainly completely decomposed for all $b \in \mathbb{A}^1 \setminus \{a\}$ and cannot be completely decomposed at $a \in \mathbb{A}^1$, as a has no preimage under i. Therefore the two morphisms f and i form a Nisnevich covering if and only if the morphism f is completely decomposed at a.
If they form a Nisnevich covering, f has to be completely decomposed at a, i.e. there exists a preimage c of a under f, such that $\overline{f}\colon \kappa(a) \to \kappa(c)$ is an isomorphism. In particular, we have $a \in k^{\times n}$.
If a has an n-th root in k, let's say c, then we can choose c as a preimage of a under f and we get the commutative diagram

$$\begin{array}{ccccccc}
\kappa(a) & = & \mathrm{Frac}(k[t]/(t-a)) & \xrightarrow{\cong} & k[t]/(t-a) & \xrightarrow{\cong} & k \\
\downarrow{\scriptstyle \overline{f}} & & & & \downarrow{\scriptstyle t\mapsto t^n} & & \| \\
\kappa(c) & = & \mathrm{Frac}(k[t,t^{-1}]/(t-c)) & \xrightarrow{\cong} & k[t]/(t-c) & \xrightarrow{\cong} & k
\end{array}$$

Since the identity morphism on k and the rows are isomorphisms, so is $\overline{f}$.

Example I.1.36. Let $n \geq 2$ be an integer and suppose that the characteristic of k does not divide n. Then the morphism

$$\mathrm{Spec}(k[t,t^{-1},s]/(s^n-t)) \xrightarrow{f} \mathrm{Spec}(k[t,t^{-1}]) = \mathbb{G}_\mathrm{m}$$

induced by the composition $k[t,t^{-1}] \hookrightarrow k[t,t^{-1},s] \twoheadrightarrow k[t,t^{-1},s]/(s^n-t)$ is an example of an étale covering, which is not a Nisnevich covering. Let us first verify that it is étale.
The polynomial $s^n - t \in k[t,t^{-1}][s] = k[t,t^{-1},s]$ is monic and its derivative ns^{n-1} is a unit in the quotient $k[t,t^{-1},s]/(s^n-t)$. Indeed, the inverse is given by $\overline{n^{-1}st^{-1}}$. Therefore, the morphism f is standard étale.
It remains to show that this morphism is not a Nisnevich covering. For this we consider the generic fiber:

$$\begin{aligned}
f^{-1}(\eta) &= \mathrm{Spec}(k[t,t^{-1},s]/(s^n-t)) \times_{\mathbb{G}_\mathrm{m}} \mathrm{Spec}(k(t)) \\
&\cong \mathrm{Spec}(k[t,t^{-1},s]/(s^n-t) \otimes_{k[t,t^{-1}]} k(t)) \\
&\cong \mathrm{Spec}(k(t)[s]/(s^n-t))
\end{aligned}$$

The polynomial $s^n - t$ is irreducible as an element of $k[t,s]$ by Eisenstein's criterion with respect to the prime ideal $(t) \subset k[t,s]$ and it is also primitive. Therefore it is also irreducible as an element in $\mathrm{Frac}(k[t])[s] = k(t)[s]$ by Gauss's lemma, so that $k(t)[s]/(s^n-t)$ is a field extension of $k(t)$ of degree n. This shows that the generic point $\eta \in \mathbb{G}_\mathrm{m}$ only has one preimage under f (which is the generic point of $\mathrm{Spec}(k[t,t^{-1},s]/(s^n-t))$) and that the induced morphism on the residue fields is the degree n field extension $k(t) \subset k(t)[s]/(s^n-t)$. In other words, f is not completely decomposed at η and hence not a Nisnevich covering.

Example I.1.37. Let us again consider the previous example, i.e. the canonical morphism

$$\mathrm{Spec}(k[t,t^{-1},s]/(s^n-t)) \xrightarrow{f} \mathrm{Spec}(k[t,t^{-1}]) = \mathbb{G}_\mathrm{m}$$

We have already seen that this is an étale covering, which fails to define a Nisnevich covering. It is completely decomposed at $1 \in \mathbb{A}^1$ though:
Clearly the point $(1,1) \in \mathrm{Spec}(k[t,t^{-1},s]/(s^n-t))$ is a preimage of $1 \in \mathbb{G}_\mathrm{m}$ and we get the commutative diagram

$$\begin{array}{ccccc}
\kappa(1) & = & \mathrm{Frac}(k[t,t^{-1}]/(t-1)) & \xrightarrow{\cong} & k \\
\downarrow{\scriptstyle \overline{f}} & & & & \| \\
\kappa((1,1)) & = & \mathrm{Frac}(k[t,t^{-1},s]/(s^n-t,t-1,s-1)) & \xrightarrow{\cong} & k
\end{array}$$

which shows that the induced morphism $\overline{f}\colon \kappa(1) \to \kappa((1,1))$ is an isomorphism. Therefore we can turn the étale covering f into a Nisnevich covering by adding the open inclusion $\mathbb{A}^1 \setminus \{0,1\} \hookrightarrow \mathbb{G}_{\mathrm{m}}$ to our covering (open immersions are completely decomposed at all points of their image as explained in Example I.1.34).

There are various ways of checking if a given family of étale morphisms is a Nisnevich covering:

Proposition I.1.38. *Let X be a smooth scheme and let $\{f_i\colon U_i \to X\}_{i\in I}$ be a family of étale morphisms. The following are equivalent:*

(i) The family $\{f_i\colon U_i \to X\}_{i\in I}$ is a Nisnevich covering.

(ii) For all fields field extensions $k \subset F$, the induced morphism

$$\coprod_{i\in I} U_i(F) \to X(F)$$

is surjective.

(iii) There exists a non-negative integer r and a sequence

$$\emptyset = Z_r \subset Z_{r-1} \subset \ldots \subset Z_1 \subset Z_0 = X$$

of finitely presented closed subschemes, such that for all $0 \leq m \leq r$ the induced morphism

$$\coprod_{i\in I} f_i^{-1}(Z_m \setminus Z_{m+1}) \to Z_m \setminus Z_{m+1}$$

admits a section.

(iv) For all $x \in X$, the induced morphism

$$\coprod_{i\in I} U_i \times_X \mathcal{O}^h_{X,x} \to \mathcal{O}^h_{X,x}$$

admits a section.

Here $\mathcal{O}^h_{X,x}$ denotes the henselization of the local ring $\mathcal{O}_{X,x}$. We will not use the last condition and only included it for the sake of completeness.

Proof. Let us outline how to, for instance, obtain a sequence as in (iii) from the definition of Nisnevich coverings and refer to [49] and [51] for the remaining details of the proof (including the other directions). Given a Nisnevich covering $\{f_i\colon U_i \to X\}_{i\in I}$, we can consider the induced morphism $f\colon \coprod_{i\in I} U_i \to X$. Now we set $Z_0 = X$. Assuming we have constructed Z_i as in (iii), let us see how to obtain Z_{i+1}. Since étale morphisms are stable under base change, we have that $\coprod_{i\in I} U_i \times_X Z_i \to Z_i$ is étale. Using that the f_i are completely decomposed, one now finds a dense open subset $V_i \subset Z_i$ on which $\coprod_{i\in I} U_i \times_X Z_i \to Z_i$ has a section. Letting $Z_{i+1} = (Z_i \setminus V_i)_{\mathrm{red}}$, we obtain the desired sequence which must stabilize as X is noetherian. □

Since we are particularly fond of condition (ii), which also happens to be the original definition by [77], let us revisit some of the examples:

Example I.1.39. Let us verify once more that the two étale morphisms

$$\begin{array}{ccc} & & \mathbb{A}^1 \setminus \{a\} \\ & & \big\downarrow i \\ \mathbb{G}_{\mathrm{m}} & \xrightarrow{f = (-)^n} & \mathbb{A}^1 \end{array}$$

from Example I.1.35 form a Nisnevich covering if and only if $a \in k^{\times n}$. Let $k \subset F$ be a field extension. Using condition (ii), we need to check under which conditions the map

$$F^\times \amalg (F \setminus \{a\}) \xrightarrow{f \amalg i} F$$

is surjective. Certainly all elements of $F \setminus \{a\}$ are in the image via the morphism i. Therefore this map is surjective if and only if a is in the image of f for every field extension $k \subset F$, i.e. if and only if a has a n-th root in $k^\times$.

Example I.1.40. Let us also use condition (ii) to see why the étale morphism

$$\mathrm{Spec}(k[t, t^{-1}, s]/(s^n - t)) \xrightarrow{f} \mathrm{Spec}(k[t, t^{-1}]) = \mathbb{G}_{\mathrm{m}}$$

from Example I.1.36 fails to be a Nisnevich covering. If $k \subset F$ is a field extension, the induced map on F-points is given by

$$\{(a, b) \in F^2 \mid a \neq 0, b^n = a\} \xrightarrow{\mathrm{pr}_1} F^\times,$$

which certainly fails to be surjective in general.

It is now quite easy to verify that Nisnevich coverings give rise to a Grothendieck topology, see for instance [49]:

Proposition I.1.41. *The data of Nisnevich coverings form a Grothendieck topology on* Sm_k*.*

This is the Nisnevich topology, which will be our default topology on smooth schemes from now on. It is actually rather simple to check whether a presheaf on Sm_k is a Nisnevich sheaf. To be able to make this more precise, we introduce the following notion from [76]:

Definition I.1.42. A cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

in Sm_k is called an elementary (or distinguished) Nisnevich square if the following three conditions are fullfilled:

(i) The morphism i is an open immersion.

(ii) The morphism p is étale.

(iii) The morphism $p\colon p^{-1}(X \setminus U)_{\mathrm{red}} \to (X \setminus U)_{\mathrm{red}}$ is an isomorphism.

Example I.1.43. Every pullback of the form

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ U & \hookrightarrow & X \end{array}$$

with U and V covering X clearly defines an elementary Nisnevich square. In particular, the usual covering of $\mathbb{P}^1$ by two $\mathbb{A}^1$'s intersecting in $\mathbb{G}_{\mathrm{m}}$ defines an elementary Nisnevich square.

As the name might suggest, elementary Nisnevich squares yield examples of Nisnevich coverings. Let us verify this:

Lemma I.1.44. *The two morphisms i and p of an elementary Nisnevich square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & \overset{i}{\hookrightarrow} & X \end{array}$$

form a Nisnevich covering of X.

Proof. First note that i and p are both étale. Therefore we just need to verify that, given a point $x \in X$, either i or p is completely decomposed at x. This is clear if $x \in U$, so let us assume that $x \in X \setminus U$. Since $p\colon p^{-1}((X \setminus U)_{\mathrm{red}}) \to (X \setminus U)_{\mathrm{red}}$ is an isomorphism, x has a unique preimage $p^{-1}(x) \in p^{-1}((X \setminus U)_{\mathrm{red}}) \subset V$ and the scheme $p^{-1}((X \setminus U)_{\mathrm{red}})$ can be considered as a closed subscheme of X. Thus the induced map $\kappa(p^{-1}(x)) \to \kappa(x)$ is an isomorphism, since the operation $(-)_{\mathrm{red}}$ does not change residue fields. □

Elementary Nisnevich squares are more than just some class of Nisnevich coverings. It turns out that to check if a given presheaf on Sm_k is a (Nisnevich) sheaf, these are the only coverings we need to deal with [76] (see also [49]):

Theorem I.1.45. *Let $\mathcal{C}$ be a complete category and let $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf on Sm_k. The presheaf $\mathcal{F}$ is a $\mathcal{C}$-valued (Nisnevich) sheaf if and only if $\mathcal{F}(\emptyset)$ is terminal and for all schemes X and all elementary Nisnevich squares*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ U & \longrightarrow & X \end{array}$$

the induced diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(V) \\ \big\downarrow & & \big\downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U \times_X V) \end{array}$$

is a cartesian square in $\mathcal{C}$.

Now that we understand the topology of our choice a bit, we will introduce a general setup for homotopy theory next.

I.2 Homotopical Algebra

In this section we will deal with the technical framework needed to install a homotopy theory on a category (non-∞-categorical version). A standard source for most of this material is the book [48] by Hovey, which we will also generally follow.

I.2.1 Model Categories

A first approach for creating a homotopy theory one might come up with is:

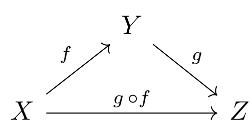
Definition I.2.1. A category with weak equivalences consists of a category $\mathcal{C}$ together with a collection $\mathrm{W}(\mathcal{C})$ of morphisms of $\mathcal{C}$, called weak equivalences, such that

(i) The collection $\mathrm{W}(\mathcal{C})$ contains all isomorphisms.

(ii) For every commutative triangle in $\mathcal{C}$ if two of the three morphisms are contained in $\mathrm{W}(\mathcal{C})$, then so is the third one.

We will usually refer to the property (ii) as two-out-of-three for weak equivalences.

Example I.2.2. For every category $\mathcal{C}$, the collection $\mathrm{Iso}(\mathcal{C})$ of isomorphisms in $\mathcal{C}$ turns $\mathcal{C}$ into a category with weak equivalences. We clearly only need to check (ii). Consider a commutative triangle



in $\mathcal{C}$. If both f and g are isomorphisms, then so is $g \circ f$ with inverse morphism $f^{-1} \circ g^{-1}$. If both f and $g \circ f$ are isomorphisms, then we claim that $f \circ (g \circ f)^{-1}$ is an inverse morphisms of g. Indeed, we clearly have $g \circ (f \circ (g \circ f)^{-1}) = \mathrm{id}_Z$, so that $f \circ (g \circ f)^{-1}$ is a right inverse of g. To see that it is also a left inverse of g, we precompose $(f \circ (g \circ f)^{-1}) \circ g \circ f = f$ with f^{-1}. Analogously we get that f is an isomorphism with inverse $(g \circ f)^{-1} \circ g$ if both g and $g \circ f$ are isomorphisms.

Example I.2.3. Let R be a ring and consider one of the categories $\mathrm{Ch}_{\geq 0}(R)$, $\mathrm{Ch}^+(R)$, $\mathrm{Ch}^-(R)$ or $\mathrm{Ch}^b(R)$, i.e. the category of non-negatively graded, bounded below, bounded above or bounded chain complexes of R-modules. Recall that a morphism $f\colon M_\bullet \to N_\bullet$ between two such chain complexes of R-modules $M_\bullet$ and $N_\bullet$ is called a quasi-isomorphism if the induced morphisms $H_n(f)\colon H_n(M_\bullet) \to H_n(N_\bullet)$ on the homology groups are isomorphisms for all integers n. The collection of all quasi-isomorphisms turns each of these four categories into a category with weak equivalences. Let us quickly explain why:
Since homology is functorial, isomorphisms get mapped to isomorphisms, so that (i) is satisfied. Moreover we have that (ii) holds, since by the previous example isomorphisms satisfy two-out-of-three and quasi-isomorphisms are defined via isomorphisms.
More generally, we can also replace the category of R-modules by an arbitrary abelian category $\mathcal{A}$ and the same arguments apply.

Example I.2.4. As is to be expected, the classical notion of weak (homotopy) equivalences turns the category of topological spaces into a category with weak equivalences. Recall that a continuous map $f\colon X \to Y$ between two topological spaces X and Y is a weak (homotopy) equivalence if the induced map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ on the path components is bijective and if the induced homomorphisms $\pi_n(f,x)\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ on the homotopy groups are isomorphisms of groups for all $x \in X$ and all integers $n \geq 1$. Since both π_0 and π_n are functors, they map isomorphisms to isomorphisms, so that (i) is satisfied. Once again we have that (ii) holds since weak (homotopy) equivalences are defined via isomorphisms.

Example I.2.5. There is of course another suitable notion to turn the category of topological spaces into a category with weak equivalences, namely homotopy equivalences. The collection of homotopy equivalences certainly contains all homeomorphisms and does also satisfy two-out-of-three, since this notion is symmetric with respect to source and target and stable under compositions.

As in the classical situation (see Example I.2.8 below), we would like to invert weak equivalences. For this we introduce:

Definition I.2.6. Let $\mathcal{C}$ be a category and let W be a collection of morphisms of $\mathcal{C}$. A localization of the category $\mathcal{C}$ at the/with respect to the collection W consists of a category $\mathcal{C}[W^{-1}]$ together with a functor $L_W\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$, such that:

(i) For every morphism $w \in W$, the morphism $L_W(w)$ is an isomorphism.

(ii) If $\mathcal{D}$ is a category together with a functor $F\colon \mathcal{C} \to \mathcal{D}$, such that the morphism $F(w)$ is an isomorphism for all $w \in W$, then there exists a unique functor $\overline{F}\colon \mathcal{C}[W^{-1}] \to \mathcal{D}$ making the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & D \\ \downarrow{\scriptstyle L_W} & \nearrow{\scriptstyle \overline{F}} & \\ \mathcal{C}[W^{-1}] & & \end{array}$$

commutative.

The second condition ensures that the category $\mathcal{C}[W^{-1}]$ is unique up to a unique equivalence of categories. Therefore we will speak of the localization with respect to some collection of morphisms instead of a localization with respect to these morphisms.

Definition I.2.7. The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a category with weak equivalences $\mathcal{C}$ is the localization $\mathcal{C}[\mathrm{W}(\mathcal{C})^{-1}]$ of $\mathcal{C}$ at the collection $\mathrm{W}(\mathcal{C})$ of weak equivalences.

It is time for a couple of examples:

Example I.2.8. Localizing the category of topological spaces at either the homotopy equivalences or the weak equivalences, we obtain some category Ho(Top) which one usually refers to as the homotopy category of topological spaces. Here the localization can be constructed explicitly via a so-called calculus of fractions [41] (this is an analog of how one localizes rings in terms of multiplicative subsets). Which category this is now depends on whether we just restrict to some nice subcategory of topological spaces (as usual in homotopy theory) or really all topological spaces. Either way this is a classical object of interest.

Example I.2.9. Let $\mathcal{A}$ be an abelian category. The derived category $D(\mathcal{A})$ arises as a localization of a category with weak equivalences. Here one usually considers a category of chain complexes (see Example I.2.3) or a category $\mathcal{K}(\mathcal{A})$ of chain complexes, where chain homotopy equivalences have been inverted, and then one localizes at the quasi-isomorphisms. Also this makes use of the aforementioned notion of a calculus of fractions. For more details we refer to Chapter 10 of [105].

Looking at our examples, everything seems to work quite well. A problem is that we cannot guarantee the existence of localizations of categories in general, at least without changing our universe. In other words, the typical matter of size issues once again arises. Furthermore, the morphisms given by the general construction are very difficult to control and hence also difficult to work with, see for instance page 147 of [43]. The notion of model categories remedies this issue. Before we can state the definition of a model category, we will need some notions.
Let X and X' be objects of a category $\mathcal{D}$. Recall that X is a retract of X' if there exist morphisms $r\colon X' \to X$ and $s\colon X \to X'$ with $r \circ s = \mathrm{id}_X$. In this case the morphism r is called a retraction of s. The important case for us is the one when $\mathcal{D}$ is a morphism category of some category $\mathcal{C}$. If we spell this out in terms of the category $\mathcal{C}$, this means:

Definition I.2.10. Let $\mathcal{C}$ be a category and let $f\colon X \to Y$ and $g\colon X' \to Y'$ be two morphisms in $\mathcal{C}$. We say that f is a retract of g if there exists a commutative diagram of the form

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_X}{\frown} & & \\ X & \longrightarrow & X' & \longrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & Y' & \longrightarrow & Y \\ & & \underset{\mathrm{id}_Y}{\smile} & & \end{array}$$

Based on this we now introduce:

Definition I.2.11. A model category consists of a category $\mathcal{C}$ together with three subcategories $\mathrm{W}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{Cof}(\mathcal{C})$ of $\mathcal{C}$, called weak equivalences, fibrations and cofibrations of $\mathcal{C}$, such that

(MC1) The category $\mathcal{C}$ is complete and cocomplete, i.e. it has all small limits and all small colimits.

(MC2) The collection $\mathrm{W}(\mathcal{C})$ satisfies two-out-of-three.

(MC3) The three collections $\mathrm{W}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{Cof}(\mathcal{C})$ are stable under retracts.

(MC4) For every commutative square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle i} & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & Y' \end{array}$$

in $\mathcal{C}$, where $i \in \mathrm{Cof}(\mathcal{C})$ and $p \in \mathrm{Fib}(\mathcal{C})$, there exists a morphism $\varphi\colon Y \to X'$ making the two resulting triangles in $\mathcal{C}$ commutative if $i \in \mathrm{W}(\mathcal{C})$ or $p \in \mathrm{W}(\mathcal{C})$.

(MC5) Every morphism f in $\mathcal{C}$ has a functorial factorization $f = p \circ i$ where $p \in \mathrm{Fib}(\mathcal{C})$ and $i \in \mathrm{Cof}(\mathcal{C}) \cap \mathrm{W}(\mathcal{C})$, and a functorial factorization of the form $f = p' \circ i'$ where $p' \in \mathrm{Fib}(\mathcal{C}) \cap \mathrm{W}(\mathcal{C})$ and $i' \in \mathrm{Cof}(\mathcal{C})$.

If $(\mathcal{C}, \mathrm{W}(\mathcal{C}), \mathrm{Fib}(\mathcal{C}), \mathrm{Cof}(\mathcal{C}))$ is the datum a model category, one also says that the three collections $\mathrm{W}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{Cof}(\mathcal{C})$ equip/endow the underlying category $\mathcal{C}$ with a model structure. Furthermore, (co-)fibrations that are also weak equivalences are usually called acyclic or trivial (co-)fibrations. In the situation of (MC4) we say that i has the left lifting property with respect to p and that p has the right lifting property with respect to i.

Remark I.2.12. The above definition is not the one given by Quillen in [80]. It turned out, that one can demand stronger versions of Quillen's axioms without loosing any essential examples. Furthermore, the additional assumptions are simply helpful.

Let us now get to some examples:

Example I.2.13. Every complete and cocomplete category $\mathcal{C}$ has three different model structures given by letting one of the three collections $\mathrm{W}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{Cof}(\mathcal{C})$ be the collection of all isomorphisms and by letting the other two be the collection of all morphims. Then clearly all the axioms hold.

Example I.2.14. There are exactly 9 model structures on the category Set, a fact which can be worked out directly from the definition (we refrain from doing that here, but encourage any interested reader to try to prove this). These are:

W(Set)	Fib(Set)	Cof(Set)
Bij	All maps	All maps
Non-$\emptyset$ maps and $\mathrm{id}_\emptyset$	Bij and $\emptyset$ maps	All maps
Non-$\emptyset$ maps and $\mathrm{id}_\emptyset$	Surj and $\emptyset$ maps	Inj
All maps	Bij	All maps
All maps	Surj	Inj
All maps	Bij and $\emptyset$ maps	Non-$\emptyset$ maps and $\mathrm{id}_\emptyset$
All maps	Surj and $\emptyset$ maps	Non-$\emptyset$ Inj and $\mathrm{id}_\emptyset$
All maps	Inj	Surj
All maps	All maps	Bij

Here Inj, Surj and Bij are the collections of injective, surjective and bijective maps. Furthermore, by "$\emptyset$ maps" we mean the collection of inclusions of the empty set into all other sets, which should also explain what we mean by "Non-$\emptyset$ maps" and "Non-$\emptyset$ Inj". This example stems from a mathoverflow comment of Goodwillie, see [45], and was then worked out by Barthel and Antolín Camarena [19]. For those readers willing/able to read German, we also recommend the bachelor thesis of Dratschuk [32].

Example I.2.15. In their recent paper [18], Balchin, Ormsby, Osorno and Roitzheim show that the totally ordered set $[n] = \{0, \dots, n\}$ considered as a category has $\binom{2n+1}{n}$ model structures. Their strategy is to show that the model structures are in bijection with so-called contractible submodels of $[n]$, which they can count via compositions of the integer $n + 1$.

Let us now mention some of the model categories which one encounters more naturally. There are two classical model structures on topological spaces based on the fact that there are two natural candidates for weak equivalences, namely homotopy equivalences and weak (homotopy) equivalences.

Example I.2.16. The Quillen model structure on Top is given by weak (homotopy) equivalences, Serre fibrations and retracts of relative cell complexes as cofibrations, see [80].

Example I.2.17. The Strøm model structure on Top is given by homotopy equivalences, Hurewicz fibrations and closed Hurewicz cofibrations, see [90].

Example I.2.18. Note that if $\mathcal{C}$ is a model category, then so is $\mathcal{C}^{op}$. Here the weak equivalence do not change, but the fibrations and cofibrations get swapped; see Remark 1.1.7 in [48].

There are also various model structures on categories of chain complexes, whose homotopy theories usually go by the name of homological algebra. Here we refer to Chapter 2.3 of [48] for more details and let R be a ring.

Example I.2.19. The injective model structure on $\mathrm{Ch}_{\geq 0}(R)$ is given by the quasi-isomorphisms as weak equivalences, degreewise epimorphisms with injective kernel as fibrations and degreewise monomorphisms as cofibrations. To extend this model structure to the category of all chain complexes, one needs to replace the fibrations by degreewise split surjections with so-called fibrant kernels.

Example I.2.20. The projective model structure on $\mathrm{Ch}_{\geq 0}(R)$ is given by the quasi-isomorphisms as weak equivalences, degreewise epimorphisms as fibrations and degreewise monomorphisms with projective cokernel as cofibrations. Also here we can extend to the category of all chain complexes by replacing the cofibrations by degreewise split injections with so-called cofibrant cokernels.

Our definition of a model category is in some sense rather minimalistic, although not all authors demand that the two factorizations from (MC5) are functorial or that model categories are closed under retracts. Many authors do demand further properties, especially regarding the lifts from (MC4). These follow from our axioms, see Lemma 1.10 of [48]:

Proposition I.2.21. *Let $\mathcal{C}$ be a model category. Then the following hold:*

(i) The fibrations of $\mathcal{C}$ are exactly the morphisms having the right lifting property with respect to acyclic cofibrations of $\mathcal{C}$.

(ii) The acyclic fibrations of $\mathcal{C}$ are exactly the morphisms having the right lifting property with respect to cofibrations of $\mathcal{C}$.

(iii) The cofibrations of $\mathcal{C}$ are exactly the morphisms having the left lifting property with respect to acyclic fibrations of $\mathcal{C}$.

(iv) *The acyclic cofibrations of $\mathcal{C}$ are exactly the morphisms having the left lifting property with respect to fibrations of $\mathcal{C}$.*

Remark I.2.22. Based on this proposition, two of the three collections $\mathrm{W}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{Cof}(\mathcal{C})$ determine the third one. The only case for which this is maybe not immediately clear is the one where $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{Cof}(\mathcal{C})$ are given. The weak equivalences are then exactly the morphisms of $\mathcal{C}$ that can be factorized as an acyclic cofibration followed by an acyclic fibration. Here we use (ii) and (iv) of the proposition to define acyclic (co-)fibrations since we cannot use weak equivalences. Indeed, each morphism, hence also each weak equivalence, can be factorized as a acyclic cofibration followed by a fibration due to (MC5). Therefore also the fibration must be acyclic by two-out-of-three for weak equivalences. On the other hand, two-out-of three also ensures that each morphism that has such a factorization is a weak equivalence.

I.2.2 Homotopy Categories

In the previous subsection we mentioned that categories with weak equivalences are in general not sufficient for a well-behaved notion of homotopy categories. In fact, this was our main reason to introduce model categories. In this subsection we will construct a suitable model of a homotopy category $\mathrm{Ho}(\mathcal{C})$ associated with a model category $\mathcal{C}$ following Chapter 1.2 of [48]. Let us fix some notation. Since any model category $\mathcal{C}$ is complete and cocomplete, we can consider the limit and colimit over the empty diagram. Therefore our category $\mathcal{C}$ has a terminal object and an initial object, which we denote by 1 and $\emptyset$ respectively.

Definition I.2.23. Let $\mathcal{C}$ be a model category. An object X of $\mathcal{C}$ is

(i) fibrant if the morphism $X \to 1$ is a fibration of $\mathcal{C}$.

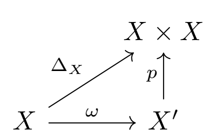
(ii) cofibrant if the morphism $\emptyset \to X$ is a cofibration of $\mathcal{C}$.

(iii) bifibrant if X is both fibrant and cofibrant.

For a model category $\mathcal{C}$, these notions yield three full subcategories $\mathcal{C}_{\mathrm{fib}}$, $\mathcal{C}_{\mathrm{cof}}$ and $\mathcal{C}_{\mathrm{bif}}$ given by the fibrant, cofibrant and bifibrant objects respectively. Note that the axioms of a model category allow us to turn objects into fibrant/cofibrant ones. Indeed, if we consider the map $\emptyset \to X$ for some object $X \in \mathcal{C}$, then we can factorize it as $\emptyset \to QX \to X$, where the first map is a cofibration and the second map is an acyclic fibration. In other words, up to weak equivalence, we can replace any given object X by a cofibrant one. Since our factorizations are assumed to be functorial, this yields a functor $Q\colon \mathcal{C} \to \mathcal{C}_{\mathrm{cof}}$, $X \mapsto QX$, called the cofibrant replacement functor. Analogously we obtain a fibrant replacement functor $R\colon \mathcal{C} \to \mathcal{C}_{\mathrm{fib}}$.

Definition I.2.24. Let $\mathcal{C}$ be a model category and let X be an object of $\mathcal{C}$. An object X' of $\mathcal{C}$ together with

(i) a fibration $p\colon X' \to X \times X$ together with a weak equivalence $\omega\colon X \to X'$ is called a path space object of X if the triangle



commutes, where Δ_X is the diagonal.

(ii) a cofibration $i\colon X \amalg X \to X'$ together with a weak equivalence $\omega\colon X' \to X$ is called a cylinder object of X if the triangle

$$\begin{array}{ccc} X \amalg X & & \\ \downarrow{\scriptstyle i} & \overset{\nabla_X}{\searrow} & \\ X' & \xrightarrow{\omega} & X \end{array}$$

commutes, where ∇_X is the codiagonal.

Note that every object in a model category has both a path object and a cylinder object due to the two factorizations from (MC5). These notions now allow us to define certain notions of homotopies.

Definition I.2.25. Let $f, g\colon X \to Y$ be two parallel morphisms in a model category $\mathcal{C}$.

(i) A left homotopy from f to g relative to a cylinder object (X', i, ω) of X is a morphism $h\colon X' \to Y$ such that the diagram

$$\begin{array}{ccc} X \amalg X & \xrightarrow{(f,g)} & Y \\ \downarrow{\scriptstyle i} & \overset{h}{\nearrow} & \\ X' & & \end{array}$$

commutes. Furthermore, we call f left homotopic to g if there exists a cylinder object (X', i, ω) together with a left homotopy from f to g relative to (X', i, ω).

(ii) A right homotopy from f to g relative to a path space object (Y', p, ω) of Y is a morphism $h\colon X \to Y'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{(f,g)^T} & Y \times Y \\ & \overset{h}{\searrow} & \uparrow{\scriptstyle p} \\ & & Y' \end{array}$$

commutes. Furthermore, we call f right homotopic to g if there exists a path space object (Y', p, ω) together with a right homotopy from f to g relative to (Y', p, ω).

We see that path space objects and right homotopies are dual to cylinder objects and left homotopies. This is both true in an informal and formal sense, where the latter uses that the opposite of a model category is itself a model category; see Example I.2.18. Therefore we can once again focus on one of the two notions by the self-duality of model categories. Since the notion of cylinder objects and left homotopies is closer to the usual definition of homotopies in topology, we prefer to phrase everything in terms of these. Moreover, due to the following lemma, which can be found as part of Proposition 1.2.5 in [48], we do not need to worry about these matters too much:

Lemma I.2.26. *Let $\mathcal{C}$ be a model category and let X and Y be objects of $\mathcal{C}$. If X is cofibrant and Y is fibrant, then for all parallel morphisms $f, g\colon X \to Y$ the following are equivalent:*

(i) The morphism f is left homotopic to the morphism g.

(ii) The morphism f is left homotopic to the morphism g relative to a fixed cylinder object of X.

(iii) The morphism f is right homotopic to the morphism g.

(iv) The morphism f is right homotopic to the morphism g relative to a fixed path space object of Y.

Under the same assumptions we will also get a suitable homotopy relation, which is also part of Proposition 1.2.5 of loc. cit.:

Lemma I.2.27. *Let $\mathcal{C}$ be a model category and let X and Y be objects of $\mathcal{C}$. The left homotopy relation on $\mathrm{Hom}(X, Y)$ is reflexive and symmetric. If X is cofibrant, it is also transitive. In particular, the left homotopy relation defines an equivalence relation on $\mathrm{Hom}(X, Y)$ for cofibrant X.*

So here we are in the situation where we have a suitable notion of homotopies between two parallel morphisms. The set of equivalence classes with respect to the homotopy relation on $\mathrm{Hom}(X, Y)$ will be denoted by $[X, Y]$. To have these between all objects, we will need to restrict to objects of a given closed model category $\mathcal{C}$ that are bifibrant, i.e. both fibrant and cofibrant. We now define $\pi(\mathcal{C}_{\mathrm{bif}})$ to be the category with objects given by the objects of $\mathcal{C}$ that are bifibrant together with morphisms $[X, Y]$ between each two objects X and Y of $\pi(\mathcal{C}_{\mathrm{bif}})$. This is a model for the wanted homotopy category:

Theorem I.2.28. *For every model category $\mathcal{C}$, the category $\pi(\mathcal{C}_{\mathrm{bif}})$ is a homotopy category of $\mathcal{C}$. Furthermore, every isomorphism in $\pi(\mathcal{C}_{\mathrm{bif}})$ is represented by the homotopy class of a weak equivalence under this equivalence of categories.*

Proof. This is Theorem 1.2.10 of [48]. □

Here we have two choices for the functor $\mathcal{C} \to \pi(\mathcal{C}_{\mathrm{bif}})$. We can first use the fibrant replacement functor, then the cofibrant replacement functor and then pass to homotopy classes of maps. We can also first use the cofibrant replacement functor, then the fibrant replacement functor and then pass to homotopy classes. By Theorem 1.2.10 of loc. cit. these two choices are equivalent. Moreover, Theorem I.2.28 does not only tell us that we managed to construct a homotopy category, but also that we did not accidentally invert more morphisms than we wanted to.
Now that we know how to get homotopy categories, we want to see which functors descend to them. For this we first introduce the following:

Definition I.2.29. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors.

(i) The functor F is called a left Quillen functor if it is a left adjoint and preserves cofibrations and acyclic cofibrations.

(ii) The functor G is called a right Quillen functor if it is a right adjoint and preserves fibrations and acyclic fibrations.

(iii) The pair (F, G) is called a Quillen adjunction if $F \dashv G$, and F is a left Quillen functor or G is a right Quillen functor.

Note that by Lemma 1.3.4 of [48], the two conditions within (iii) are equivalent. Based on Quillen functors, we can now define derived functors, where we will make use of our notation from Definition I.2.6.

Definition I.2.30. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

(i) If a functor $F\colon \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, then the composition

$$\mathbb{L}F = \overline{L_W \circ F} \circ \overline{L_W \circ Q}\colon \operatorname{Ho}(\mathcal{C}) \to \operatorname{Ho}(\mathcal{C}_{\mathrm{cof}}) \to \operatorname{Ho}(\mathcal{D})$$

is called the (total) left derived functor of F.

(ii) If a functor $G\colon \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, then the composition

$$\mathbb{R}G = \overline{L_W \circ G} \circ \overline{L_W \circ R}\colon \operatorname{Ho}(\mathcal{D}) \to \operatorname{Ho}(\mathcal{D}_{\mathrm{fib}}) \to \operatorname{Ho}(\mathcal{C})$$

is called the (total) right derived functor of G.

Example I.2.31. By definition, the homotopy categories of our model categories of chain complexes $\mathrm{Ch}_{\geq 0}(R)$ or $\mathrm{Ch}(R)$ are given by the associated derived categories $D_{\geq 0}(R)$ and $D(R)$, see also Chapter 2.3 of [48], and our notion of left/right derived functors retrieves the more classical notion of derived functors between derived categories.

Definition I.2.32. A Quillen adjunction (F, G) is called a Quillen equivalence if $\mathbb{L}F$ or $\mathbb{R}G$ defines an equivalence of categories.

Note that, if one of those two derived functors is an equivalence, then so is the other since it becomes a quasi-inverse of the former. This notion is the right notion of equivalent homotopy theories in this setup.

Example I.2.33. The Quillen and the Strøm model structures on Top are Quillen equivalent, see Chapter 17 of [67].

For another concrete example of a Quillen equivalence we refer the reader to Theorem I.3.19 in the next section.

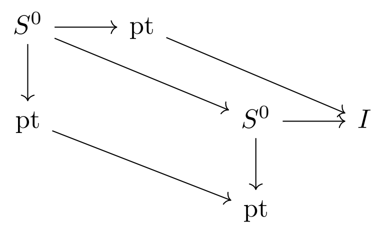
I.2.3 Homotopy Limits and Colimits

Even if we are given a model category and thus also have an associated homotopy category, limits and colimits do not generally work well with our homotopy theory. In this subsection we will rectify this issue, at least for a certain type of index categories. Let us start with an example in the case of topological spaces:

Example I.2.34. Consider the two pushout squares

$$\begin{array}{ccc} S^0 & \longrightarrow & \mathrm{pt} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{pt} \end{array} \qquad\qquad \begin{array}{ccc} S^0 & \hookrightarrow & I \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{pt} & \longrightarrow & S^1 \end{array}$$

We clearly have a morphism of pushout diagrams



where all the three maps are weak equivalences, but there cannot be a weak equivalence pt $\to S^1$ on the level of the pushouts. This shows us that we cannot take (co-)limits of homotopy types in general.

The solution is to consider a version of (co-)limit which works well with respect to the notion of homotopy, which is hence called homotopy (co-)limit. These always exist, see for example the very nice mathoverflow answer [97] of Virili following the construction from [27]:

Theorem I.2.35. *Let $\mathcal{C}$ be a model category and let $\mathcal{I}$ be an index category.*

(i) There exists a model structure on $\mathcal{C}^{\mathcal{I}}$ so that $\mathbb{R}\mathrm{lim}(F)$ exists for any $F \in \mathcal{C}^{\mathcal{I}}$.

(ii) There exists a model structure on $\mathcal{C}^{\mathcal{I}}$ so that $\mathbb{L}\mathrm{colim}(F)$ exists for any $F \in \mathcal{C}^{\mathcal{I}}$.

Definition I.2.36. Let $\mathcal{C}$ be a model category, let $\mathcal{I}$ be an index category and let $D\colon \mathcal{I} \to \mathcal{C}$ be a diagram. The functors from the theorem above are called the homotopy limit $\mathrm{holim}(D)$ of D and the homotopy colimit $\mathrm{hocolim}(D)$ of D respectively.

Setting up the full theory of homotopy (co-)limits is not only not easy, but also not relevant for this document. Moreover, homotopy (co-)limits are generally difficult to compute. We will just focus on a certain class of homotopy (co-)limits, for which the theory is easier. Let us start by treating a concrete kind of diagram of topological spaces. Note that by Example I.2.33, we do not need to worry about the choice of model structure.

Theorem I.2.37. *The homotopy pushout $X \amalg_A^h Y$ of a diagram*

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f} & & \\ X & & \end{array}$$

of topological spaces is modeled by the double mapping cylinder $M(f,g) = X \amalg_{A\times\{0\}} (A \times I) \amalg_{A\times\{1\}} Y$.

Proof. This follows from Example 8.8 of [82] together with the fact that taking the mapping cylinder of a continuous map gives rise to the cofibrant replacement functor, see for instance page 45 of [66]. □

There is an obvious collapse map $X \amalg_A^h Y \to X \amalg_A Y$ to the usual pushout, given by collapsing the cylinder $A \times I$.

Lemma I.2.38. *The collapse map $X \amalg_A^h Y \to X \amalg_A Y$ is a weak equivalence if one of the maps $A \to X$ and $A \to Y$ is a (Hurewicz) cofibration.*

Proof. This is a special case of Example 8.8 of [82]. □

In other words, if $A \to X$ or $A \to Y$ is an inclusion of a nice subspace, then the actual pushout $X \amalg_A Y$ already has the correct homotopy type.
This lemma also indicates how we could try to compute homotopy pushouts. Here we will at first not justify why everything works, but we will resolve this afterwards. Given a diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f} & & \\ X & & \end{array},$$

we can replace one of the two maps $f\colon A \to X$ and $g\colon A \to Y$ by a cofibration whose target is weakly equivalent to the target of f or g respectively. Without loss, let us choose f. As mentioned in the proof of the above theorem, this can be realized by choosing $\widetilde{X}$ as the mapping cylinder of f and the map $\widetilde{f}$ is just the inclusion of $A \hookrightarrow M(f)$. Using the language of model categories, $\widetilde{f}$ is a cofibrant replacement of f. The pushout of the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \widetilde{f}} & & \\ \widetilde{X} & & \end{array}$$

is then the homotopy pushout of the original diagram. If we feel like it, we can of course also do this for both maps, which then directly gives the double mapping cylinder. If A is cofibrant, then the above works. If A is not cofibrant, then we need to assume that the given model category is left-proper, that is, weak equivalences are preserved by pushouts along cofibrations. For a reference see Proposition A.2.4.4 in [62]. While this is a restriction, essentially all model categories that are usually considered turn out to be left-proper.

Example I.2.39. Let us return to the example from the beginning. We consider the diagram

$$\begin{array}{ccc} S^0 & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ \mathrm{pt} & & \end{array}$$

whose maps are certainly not cofibrations. The cofibrant replacement of $S^0 \to \mathrm{pt}$ is the inclusion $S^0 \hookrightarrow I$, so that the pushout of the replaced diagram is given by S^1. Here S^1 is built from a 1-cell I attached to the 0-cell pt. Had we cofibrantly replaced both maps, the resulting S^1 would arise from two 1-cells glued together at their corresponding end points. Either way, we were able to compute the homotopy pushout.

So why does the above recipe work? For this recall that a subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called wide if it contains all objects of $\mathcal{C}$. We now introduce the following notion which nowadays is named after Reedy due to his paper [81], where some first instances of said notion were implicitly studied:

Definition I.2.40. A category $\mathcal{I}$ together with two wide subcategories $\mathcal{I}_+$ and $\mathcal{I}_-$ and a map $\deg\colon \mathrm{Ob}(\mathcal{I}) \to \mathbb{N}$ is called a Reedy category if

(i) For every non-identity morphism $X \to X'$ in $\mathcal{I}_+$ we have $\deg(X) < \deg(X')$.

(ii) For every non-identity morphism $X \to X'$ in $\mathcal{I}_-$ we have $\deg(X) > \deg(X')$.

(iii) Every morphism in $\mathcal{I}$ has a unique factorization by a morphism in $\mathcal{I}_-$ followed by a morphism in $\mathcal{I}_+$.

So, vaguely speaking, we have a degree function deg on the objects of our category and choose positive and negative morphisms. Then we demand that positive morphisms raise the degree, that negative morphisms lower the degree and, that every morphism is uniquely built from positive and negative ones.

Example I.2.41. Every discrete category $\mathcal{I}$ is Reedy by choosing $\deg(X) = 0$ for all $X \in \mathcal{I}$. Here both $\mathcal{I}_+$ and $\mathcal{I}_-$ coincide with the category $\mathcal{I}$.

We can visualize Reedy categories by denoting an object by its degree and by drawing a "+" or a "−" over a non-identity morphism to indicate in which of the two wide subcategories it lives.

Example I.2.42. The index category for pushouts is a Reedy category:

$$\begin{array}{ccc} 0 & \xrightarrow{+} & 1 \\ \big\downarrow{\scriptstyle +} & & \\ 1 & & \end{array}$$

Example I.2.43. Also the following category is Reedy:

$$\begin{array}{ccccc} 2 & \xleftarrow[+]{} & 1 & \xrightarrow{+} & 2 \\ {\scriptstyle +}\big\uparrow & & {\scriptstyle +}\big\uparrow & & {\scriptstyle +}\big\uparrow \\ 1 & \xleftarrow[+]{} & 0 & \xrightarrow{+} & 1 \\ \big\downarrow{\scriptstyle +} & & \big\downarrow{\scriptstyle +} & & \big\downarrow{\scriptstyle +} \\ 2 & \xleftarrow[+]{} & 1 & \xrightarrow{+} & 2 \end{array}$$

This category is the index category for a pushout of pushouts.

The relevance of Reedy categories lies in the simplicity of model structures on functor categories involving the Reedy category, which can be found as Theorem 4.18 in [82]:

Theorem I.2.44. *Let $\mathcal{C}$ be a model category and let $\mathcal{I}$ be a Reedy category. Then the category $\mathcal{C}^{\mathcal{I}}$ has a model structure, where the weak equivalences are exactly the objectwise weak equivalences and both the fibrations and cofibrations are contained in the objectwise fibrations and cofibrations.*

This is the Reedy model structure on $\mathcal{C}^{\mathcal{I}}$. It is quite technical to define the fibrations and cofibrations of the Reedy model structure, which is why we omit this. The relevant aspect of this model structure is the following:

Example I.2.45. The Reedy structure on the index category of a pushout from Example I.2.42 retrieves the statement from Lemma I.2.38 for a general model category $\mathcal{C}$. Moreover, the Reedy structure allows us to compute the homotopy pushout in terms of a cofibrant replacement of one of its maps, as can be found as Example 8.8 in [82]. So the argument why the above recipe works is that the index category for pushouts is Reedy, and that we understand the cofibrations on the level of the associated functor category well enough.

All the homotopy (co-)limits that will show up later can be expressed in terms of homotopy pushouts, so that we are now suited to deal with those.

I.3 Simplicial Stuff

In this section we will deal with those objects which allow us to install a homotopy theory on smooth schemes, namely with simplicial objects in a given category $\mathcal{C}$. Here we will generally follow [44]. Due to Dugger [34], taking simplicial sheaves on a site can even be seen to create a universal homotopy theory in a suitable sense. This is a further justification of Morel's and Voevodsky's approach to the homotopy theory of smooth schemes from [76].

I.3.1 The Simplex Category and Simplicial Objects

We have seen how model categories enable us to get our hands on a homotopy category. To get a meaningful model structure on a category closely related to smooth schemes, we study the so-called simplex category which allows us to define simplicial objects.

Definition I.3.1. The simplex category Δ has as objects the ordered sets

$$[n] = \{0 \leq 1 \leq \ldots \leq n\}$$

for all non-negative integers n together with order-preserving maps as morphisms.

In other words, the simplex category is just a skeleton of the category of non-empty finite ordered sets.
We will now consider two particular kinds of morphisms in Δ. For all integers $n \geq 1$ and all $0 \leq i \leq n$, we define the i-th coface map to $[n]$ to be the morphism

$$\delta_i^n \colon [n-1] \to [n],\ j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

and for all non-negative integers n and all $0 \leq i \leq n$ the i-th codegeneracy map to $[n]$ to be

$$\sigma_i^n \colon [n+1] \to [n],\ j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}.$$

In other words, the morphism δ_i^n is the unique injective order-preserving map $[n-1] \to [n]$ with i not in its image and the morphism σ_i^n is the unique surjective order-preserving map $[n+1] \to [n]$ that hits i twice. The importance of these two types of morphisms comes from the following:

Lemma I.3.2. *Any morphism in the simplex category Δ is a composition of coface and codegeneracy maps.*

Proof. This is the Lemma 1 on page 177 of [64]. □

Any relation between two morphisms in Δ is hence a consequence of relations between coface and codegeneracy maps. Let us therefore list these:

Lemma I.3.3. *The coface and codegeneracy maps satisfy the relations:*

(i) For all $0 \leq i < j \leq n+1$ we have $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$.

(ii) *For all* $0 \leq i < j \leq n-1$ *we have* $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \delta_{j-1}^{n-2}$.

(iii) *For all* $0 \leq i \leq n-1$ *we have* $\sigma_i^{n-1} \circ \delta_i^n = \mathrm{id}_{[n-1]} = \sigma_i^{n-1} \circ \delta_{i+1}^n$.

(iv) *For all* $0 < j+1 < i \leq n$ *we have* $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$.

(v) *For all* $0 \leq i \leq j \leq n-1$ *we have* $\sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n$.

Proof. These follow directly from the definitions. □

Definition I.3.4. Let $\mathcal{C}$ be a category. The category s$\mathcal{C}$ of simplicial objects in $\mathcal{C}$ is the category $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$ of $\mathcal{C}$-valued presheaves on Δ.

Example I.3.5. A simple example of a simplicial object is the following. Fix some object X of a category $\mathcal{C}$. Then the constant functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ with value X is a simplicial object of $\mathcal{C}$. One usually refers to such simplicial objects as discrete simplicial objects.

Considering objects of a category $\mathcal{C}$ as discrete simplicial objects gives us a natural inclusion functor $\mathcal{C} \to \mathrm{s}\mathcal{C}$. This is how we will view smooth schemes as spaces in the next section.

Remark I.3.6. Categories of presheaves are always cocomplete. In fact, given a category $\mathcal{C}$, the category $\mathrm{PreSh}(\mathcal{C})$ is the free cocompletion of $\mathcal{C}$ by the Yoneda lemma. Furthermore, limits and colimits are computed objectwise.

If X is a simplicial object in a category $\mathcal{C}$, we will usually write X_n instead of $X([n])$ and call it the n-th level or the n-simplices of the simplicial object X. By Lemma I.3.2 and Lemma I.3.3, we have:

Proposition I.3.7. *Let $\mathcal{C}$ be a category. The data of a simplicial object in $\mathcal{C}$ is equivalent to a sequence $(X_n)_{n\in\mathbb{N}}$ of objects X_n of $\mathcal{C}$ together with morphisms $d_i^n \colon X_n \to X_{n-1}$ for all integers $n \geq 1$ and all $0 \leq i \leq n$ and morphisms $s_i^n \colon X_n \to X_{n+1}$ for all non-negative integers n and all $0 \leq i \leq n$, such that the following relations hold:*

(i) *For all* $0 \leq i < j \leq n+1$ *we have* $d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}$.

(ii) *For all* $0 \leq i < j \leq n-1$ *we have* $d_i^n \circ s_j^{n-1} = d_{j-1}^{n-2} \circ d_i^{n-1}$.

(iii) *For all* $0 \leq i \leq n-1$ *we have* $d_i^n \circ s_i^{n-1} = \mathrm{id}_{X_n} = d_{i+1}^n \circ s_i^{n-1}$.

(iv) *For all* $0 < j+1 < i \leq n$ *we have* $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$.

(v) *For all* $0 \leq i \leq j \leq n-1$ *we have* $s_i^n \circ s_j^{n-1} = s_{j+1}^n \circ s_i^{n-1}$.

See also Proposition 2 on page 178 of [64]. The morphisms d_i^n are called face maps and the morphisms s_i^n are called degeneracy maps. If an n-simplex lies in the image of any degeneracy map, it is called degenerate. If not, then it is simply called non-degenerate.

Example I.3.8. Any discrete simplicial object X has n-simplices $X_n = X$ together with the identity morphism as face and degeneracy maps.

I.3.2 Simplicial Sets

For us the relevant simplicial objects will be simplicial sets and simplicial (pre-)sheaves on the category Sm_k of smooth schemes. Let us therefore start by giving examples of simplicial sets:

Example I.3.9. For any non-negative integer n, the standard n-simplex is the simplicial set $\Delta^n = \mathrm{Hom}(-, [n])$. In other words, the standard n-simplex is the image of the ordered set $[n]$ under the Yoneda embedding $\Delta \hookrightarrow \mathrm{PreSh}(\Delta) = \mathrm{sSet}$.
There is a unique order-preserving map $[m] \to [0]$ for every non-negative m. Thus the simplicial set Δ^0 consists of a point at every level with the only non-degenerate simplex given by the identity $\mathrm{id}\colon [0] \to [0] \in (\Delta^0)_0$. More generally, Δ^n has no non-degenerate m-simplices for $m > n$ and a unique non-degenerate n-simplex $\mathrm{id}\colon [n] \to [n] \in (\Delta^n)_n$.

Example I.3.10. Given the standard n-simplex Δ^n for some non-negative integer n, we can define its boundary $\partial\Delta^n$. If this was the topological standard n-simplex

$$|\Delta^n| = \{(x_0, \ldots, x_n) \in [0,1]^{n+1} \mid x_0 + \ldots + x_n = 1\} \subset \mathbb{R}^{n+1},$$

we would just throw away the unique face of dimension n to get its boundary. This is essentially also what happens here, except that we also have degenerate versions of that face coming from higher-dimensional simplices. Therefore we set

$$\partial\Delta^n([m]) = \{f \in \Delta^n([m]) \mid f \text{ is not surjective}\}$$

to additionally exclude exactly all of these potentially degenerate n-dimensional faces and consider the boundary $\partial\Delta^n$ as a simplicial subset of Δ^n.

Example I.3.11. For all $0 \leq j \leq n$, the j-th horn Λ^n_j of the standard n-simplex Δ^n is the simplicial subset given by the union of all faces of Δ^n except for the j-th one.

Although we are not going to use that language, we can now also define what an ∞-category (modeled by quasi-categories) is. This notion was originally defined by Boardman and Vogt [21] and has afterwards been developed further by work of Joyal [58] and [59], and also considerably by Lurie [62]. These quasi-categories are defined via the so-called inner horn filling condition:

Definition I.3.12. An ∞-category is a simplicial set X such that for all non-negative integers n and all $0 < j < n$, every morphism $\Lambda^n_j \to X$ can be extended to a morphism $\Delta^n \to X$.

For a justification/explanation of this definition we recommend reading [63, Tag 0001]. Even though we stress once again that we are not going to use the language of ∞-categories here, we certainly recommend the interested reader to at least read these two pages.

Example I.3.13. For any category $\mathcal{C}$, there exists a simplicial set $N(\mathcal{C})$ called the nerve of the category $\mathcal{C}$. The n-th level $N_n(\mathcal{C})$ of $N(\mathcal{C})$ is given by the set $\mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$ of all functors from the ordered set $[n]$, considered as a category, to the category $\mathcal{C}$. In other words, $N_n(\mathcal{C})$ consists of all diagrams of the form

$$X_0 \to X_1 \to \ldots \to X_{n-1} \to X_n$$

in the category $\mathcal{C}$. We now define the i-th face map $d_i^n \colon N_n(\mathcal{C}) \to N_{n-1}(\mathcal{C})$ to be the morphism that "removes" the i-th object of a diagram

$$X_0 \to X_1 \to \ldots \to X_{i-1} \to X_i \to X_{i+1} \to \ldots \to X_{n-1} \to X_n.$$

For all $1 \leq i \leq n-1$ this means, that the i-th face map d_i^n maps

$$X_0 \to X_1 \to \ldots \to X_{i-1} \to X_i \to X_{i+1} \to \ldots \to X_{n-1} \to X_n$$

to the diagram

$$X_0 \to X_1 \to \ldots \to X_{i-1} \to X_{i+1} \to \ldots \to X_{n-1} \to X_n,$$

where the morphism $X_{i-1} \to X_{i+1}$ is the composition $X_{i-1} \to X_i \to X_{i+1}$. The i-th degeneracy map $s_i^n \colon N_n(\mathcal{C}) \to N_{n+1}(\mathcal{C})$ maps a diagram

$$X_0 \to X_1 \to \ldots \to X_{i-1} \to X_i \to X_{i+1} \to \ldots \to X_{n-1} \to X_n$$

to the diagram

$$X_0 \to X_1 \to \ldots \to X_{i-1} \to X_i \to X_i \to X_{i+1} \to \ldots \to X_{n-1} \to X_n,$$

where the morphism $X_i \to X_i$ is given by the identity id_{X_i}. The nerve is actually functorial in $\mathcal{C}$ by mapping a functor $F \colon \mathcal{C} \to \mathcal{D}$ to the sequence of maps $(\mathrm{Hom}([n], F))_{n\in\mathbb{N}}$. The nerve of a category is an ∞-category. Not only is it an ∞-category, but this is exactly how one can consider a (1-)category as an ∞-category since the nerve functor turns out to be fully-faithful and hence is an embedding. Its essential image consists exactly of those ∞-categories that admit unique fillers for inner horns. For more details see [63, Tag 002L] and [63, Tag 003F].

Example I.3.14. For any topological space X, there exists a simplicial set $\mathrm{Sing}(X)$ called the singular simplicial set associated with X. The n-th level $\mathrm{Sing}_n(X)$ of $\mathrm{Sing}(X)$ is given by the set $\mathrm{Hom}(|\Delta^n|, X)$ of continuous maps from the topological n-simplex

$$|\Delta^n| = \{(x_0, \ldots, x_n) \in [0,1]^{n+1} \mid x_0 + \ldots + x_n = 1\} \subset \mathbb{R}^{n+1}$$

to the topological space X. Here the i-th face map is

$$d_i^n \colon \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X),\ f \mapsto f \circ \mathrm{incl}_i,$$

where $\mathrm{incl}_i \colon |\Delta^{n-1}| \to |\Delta^n|$, $(x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$ is the inclusion outside of the i-th position, and the i-th degeneracy map is

$$s_i^n \colon \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n+1}(X),\ f \mapsto f \circ \mathrm{add}_i,$$

where $\mathrm{add}_i \colon |\Delta^{n+1}| \to |\Delta^n|$ is the continuous map defined by

$$(x_0, \ldots, x_{n+1}) \mapsto (x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_{n+1}).$$

This is also functorial in X. If $f \colon X \to Y$ is a continuous map between two topological spaces X and Y, then we have a map $\mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ given by composing with f on each level. For more information we refer to [63, Tag 001Q]

We can also define various operations of simplicial sets. We will collect some important ones in one big definition. For this note that a pointed simplicial set is simplicial set X together with a choice of an element $x \in X_0$.

Definition I.3.15. Let X and Y be simplicial sets.

(i) The product $X \times Y$ is the simplicial set with n-simplices $X_n \times Y_n$ for all non-negative integers n and coordinatewise face and degeneracy maps. If X and Y happen to be pointed with base points x and y respectively, the basepoint of the product is (x, y).

(ii) The wedge sum $(X, y) \vee (Y, y)$ of two pointed simplicial sets (X, x) and (Y, y) is the subsimplicial set of $(X \times Y, (x, y))$ with n-simplices $X_n \times \{y\} \cup \{x\} \times Y_n$ for all non-negative integers n.

(iii) If Y is a simplicial subset of X, then the quotient simplicial set X/Y is the simplicial set with n-simplices X_n/Y_n and face and degeneracy maps induced from the ones of X. If X and Y happen to be pointed with base point x, the basepoint of the quotient is the equivalence class of x.

(iv) The smash product $X \wedge Y$ is the pointed simplicial set $(X \times Y, (x, y))/(X \vee Y)$.

Example I.3.16. The simplicial 1-sphere is the quotient simplicial set $S^1 = \Delta^1/\partial\Delta^1$. Based on this we define the simplicial n-sphere S^n as $(S^1)^{\wedge n}$ for all non-negative integers n. Another model for the n-sphere in positive degree is $\Delta^n/\partial\Delta^n$. These two models turn out to be weakly equivalent, which is a notion that we will introduce now.

Before we introduce a model structure on sSet, there is one last notion that we will use and that implicitly showed up in Example I.3.14, see also [63, Tag 001X].

Definition I.3.17. The geometric realization of a simplicial set $(X_n)_{n\in\mathbb{N}}$ is the quotient

$$\coprod_{n\in\mathbb{N}} (X_n \times |\Delta^n|)/\sim,$$

where each X_n is equipped with the discrete topology, and $\sim$ is the equivalence relation generated by $(x, \mathrm{incl}_i(p)) \sim (d_i(x), p)$ and $(y, \mathrm{add}_i(p)) \sim (s_i(y), p)$ for $x \in X_{n+1}$, $y \in X_{n-1}$ and $p \in |\Delta^n|$, where D_i and S_i are the standard inclusions and collapses of topological simplices.

Now we can introduce weak equivalences, fibrations and cofibrations of simplicial sets:

W(sSet)	=	Morphisms of simplicial sets whose geometric realizations are weak equivalences of topological spaces
Fib(sSet)	=	Morphisms of simplicial sets having the right lifting property with respect to inclusions of horns $\Lambda^n_i \hookrightarrow \Delta^n$ for all positive integers n and all $0 \leq i \leq n$ (also called Kan-fibrations)
Cof(sSet)	=	Monomorphisms of simplicial sets, i.e. levelwise injective maps

Theorem I.3.18. *The three classes* W(sSet), Fib(sSet) *and* Cof(sSet) *endow the category* sSet *of simplicial sets with a model structure. Moreover, a morphisms* $f\colon X \to Y$ *of simplicial sets is an acyclic fibration if and only if it has the right lifting property with respect to inclusions of boundaries* $\partial\Delta^n \hookrightarrow \Delta^n$ *for all non-negative integers* n.

Proof. This is Theorem 3.6.5. of [48]. □

This is usually called the Quillen model structure on simplical sets and from now on we consider sSet as a model category with this model structure. We can now compare this model category with any one of the two Quillen equivalent model structures on the category of topological spaces, see Chapter 1.4 Example 2 of [80]:

Theorem I.3.19. *The functors*

$$\mathrm{sSet} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathrm{Top}\,. \quad (\perp)$$

are a Quillen equivalence.

This adjunction shows that simplicial sets give rise to a combinatorial model for the homotopy theory of topological spaces.
Finally, let us mention that the category of simplicial sets also has a well-behaved notion of internal Hom's. Given two simplicial sets X and Y, we define a presheaf $\underline{\mathrm{Hom}}(X,Y)$ by

$$\begin{array}{ccc} [n] & \longmapsto & \mathrm{Hom}_{\mathrm{sSet}}(X \times \Delta^n, Y) \\ f\big\downarrow & & \big\uparrow \varphi \mapsto \varphi \circ (\mathrm{id}_X \times f_*) \\ [m] & \longmapsto & \mathrm{Hom}_{\mathrm{sSet}}(X \times \Delta^m, Y) \end{array}$$

on the simplex-category Δ. Note that by definition there is a canonical bijection

$$\underline{\mathrm{Hom}}(X,Y)_0 = \mathrm{Hom}_{\mathrm{sSet}}(X,Y).$$

Proposition I.3.20. *For all simplicial sets* Y*, there is an adjunction*

$$\mathrm{sSet} \underset{\underline{\mathrm{Hom}}(Y,-)}{\overset{-\times Y}{\rightleftarrows}} \mathrm{sSet}\,. \quad (\perp)$$

In particular, the category sSet *of simplicial sets is cartesian closed.*

Proof. This can be found in [89, Tag 017H]. □

We will also use the Notation $(-)^Y$ instead of $\underline{\mathrm{Hom}}(Y,-)$. Then the adjunction above yields the usual exponential law

$$(Z^Y)^X \cong Z^{Y\times X}$$

for all simplicial sets X, Y and Z.

I.3.3 Simplicial Homotopy Theory

Now that we have a combinatorial model for the homotopy category of topological spaces, we certainly also want to talk about homotopy groups within that model. This is what we will quickly discuss in this section.
Although we have seen the general setup of homotopies within model categories, we think it is valuable to be a bit more explicit. Recall that the usual homotopy groups $\pi_n(X)$ of a pointed topological space X can be defined as homotopy classes of continuous maps $I^n \to X$ which take the boundary ∂I^n of the n-cube I^n to the base point of X. To translate this definition into the simplicial world, we first have to define homotopies in the style of classical homotopy theory. For this entire subsection we follow Chapters 1.6 and 1.7 of [44].

Definition I.3.21. A homotopy from a simplical map $f\colon X \to Y$ to a simplicial map $g\colon X \to Y$ is a simplicial map $h\colon X \times \Delta^1 \to Y$ making the diagram

$$\begin{array}{ccccc}
X \times \Delta^0 & = & X & & \\
\downarrow{\scriptstyle \mathrm{id} \times d^1} & & & \searrow^{f} & \\
X \times \Delta^1 & & \xrightarrow{\quad h \quad} & & Y \\
\uparrow{\scriptstyle \mathrm{id} \times d^0} & & & \nearrow_{g} & \\
X \times \Delta^0 & = & X & &
\end{array}$$

of simplical sets commutative. If we are additionally given an inclusion $i\colon A \hookrightarrow X$ of simplicial sets such that the compositions $f \circ i$ and $g \circ i$ agree, then we will say that the homotopy h is relative to A if the diagram

$$\begin{array}{ccc}
X \times \Delta^1 & \xrightarrow{h} & Y \\
{\scriptstyle i \times \mathrm{id}}\uparrow & & \uparrow{\scriptstyle f \circ i} \\
A \times \Delta^1 & \xrightarrow{\mathrm{pr}_1} & A
\end{array}$$

commutes as well.

In the language of model categories we would say that Δ^1 gives rise to a cylinder object. As in the homotopy theory of topological spaces, we say that f is homotopic to g if there exists a homotopy from f to g and denote this by $f \simeq g$. We would now like to claim that being homotopic defines an equivalence relation, but this is not true in general as the following example shows.

Example I.3.22. Consider the two simplicial maps $d^1\colon \Delta^0 \to \Delta^1$ and $d^0\colon \Delta^0 \to \Delta^1$. Then the diagram

$$\begin{array}{ccccc}
\Delta^0 \times \Delta^0 & = & \Delta^0 & & \\
\downarrow{\scriptstyle \mathrm{id} \times d^1} & & & \searrow^{d^1} & \\
\Delta^0 \times \Delta^1 & & = & & \Delta^1 \\
\uparrow{\scriptstyle \mathrm{id} \times d^0} & & & \nearrow_{d^0} & \\
\Delta^0 \times \Delta^0 & = & \Delta^0 & &
\end{array}$$

is clearly commutative so that $d^1 \simeq d^0$. When we swap the two simplicial maps on the right there is no simplicial map $\Delta^0 \times \Delta^1 \to \Delta^1$ making the resulting diagram commutative, since such a map would require non-order-preserving maps. Therefore $d^0 \not\simeq d^1$, which means that the homotopy relation is not symmetric.

As we have also seen in the section on homotopy categories, the solution is to add some further assumption:

Proposition I.3.23. *Let Y be a fibrant simplicial set and let X be a simplicial set with a simplicial subset $A \subset X$. Then both the homotopy relation and the homotopy relation relative to A on* $\mathrm{Hom}(X, Y)$ *are equivalence relations.*

Proof. This is Corollary 6.2 of Chapter I of [44]. □

Fibrant simplicial sets are also called Kan complexes. These now allows us to define homotopy groups due to the above proposition.

Definition I.3.24. Let n be positive integer. The n-th (simplicial) homotopy group $\pi_n(X, x)$ of a pointed Kan complex (X, x) is the set of homotopy classes relative to $\partial\Delta^n$ of simplicial maps $\alpha\colon \Delta^n \to X$ which make the diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\alpha} & X \\ \uparrow & & \uparrow{\scriptstyle x} \\ \partial\Delta^n & \longrightarrow & \Delta^0 \end{array}$$

commutative.

Since $S^n = \Delta^n/\partial\Delta^n$ is a model for the simplicial n-sphere, the above definition can equivalently be phrased in terms of homotopy classes of simplical maps $S^n \to X$, which recovers the other usual definition of homotopy groups.
Although we have already called $\pi_n(X, x)$ the homotopy groups of (X, x), we are yet to actually describe the group structure. Since this is rather technical we will refrain from doing that, but let us note that it is of course defined similarly as for topological spaces and results in an analogous theory:

Theorem I.3.25. *Let n be a positive integer and let (X, x) be a pointed Kan complex. There is a natural operation $\pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x)$ turning $\pi_n(X, x)$ into a group with identity element given by the homotopy class of $\Delta^n \to \Delta^0 \xrightarrow{x} X$. If $n \geq 2$, these groups are abelian.*

Proof. This is Theorem 7.2 of Chapter I of [44]. □

To lift this theory to arbitrary simplicial sets, we use that Kan complexes by definition are the fibrant objects for the Quillen model structure on simplicial sets.

Definition I.3.26. Let n be a positive integer. The n-th homotopy group $\pi_n(X, x)$ of a pointed simplicial set (X, x) is the n-th homotopy group of the fibrant replacement (RX, Rx).

This yields a very well-behaved homotopy theory of simplicial sets, see [44]. Moreover, if we were to unwind everything, this definition follows exactly the general setup of Section I.2.2, i.e $\pi_n(X,x)$ has an alternative description as maps in $\mathrm{Ho}(\mathrm{sSet}_*)$ from the simpicial n-sphere to (X,x).

Remark I.3.27. There is, of course, also a homotopy set $\pi_0(X,x)$ of a pointed simplicial set, which is defined exactly as above, but which does not come with a natural group structure. For the sake of simplicity, we will still call it the zeroth homotopy group, so that we can just speak of homotopy groups in all non-negative degrees.

Remark I.3.28. Due to the Quillen equivalence between sSet and Top, these homotopy groups coincide with the usual homotopy groups of the geometric realization of X, see Proposition 3.6.3 of [48]. This gives another way of seeing that the higher homotopy groups are abelian.

We are now interested in the sheaf-theoretic version of this theory, which is what we will study next.

I.4 Unstable Motivic Homotopy Theory

We are finally ready to enter the world of unstable motivic homotopy theory. Here we will start by introducing the basic objects, called spaces, and then we will see a glimpse of how their homotopy theory works. Finally, we study motivic spheres.

I.4.1 Spaces and the $\mathbb{A}^1$-model structure

Let us introduce one of the main objects of study in motivic homotopy theory following [76].

Definition I.4.1. A space is a simplicial (Nisnevich) sheaf on the category Sm_k.

Equivalently, a space can also be defined to be a sheaf of simplicial sets on the category Sm_k of smooth schemes and both points of view are useful. We denote the category of spaces by

$$\mathrm{Spc}(k) = \mathrm{sShv}(\mathrm{Sm}_k) \simeq \mathrm{Shv}(\mathrm{Sm}_k, \mathrm{sSet}).$$

Let us have a look at two main classes of examples.

Example I.4.2. Every simplicial set X gives rise to a space by taking the sheaf associated with the constant presheaf with value X. This space will also be denoted by X.

Example I.4.3. Every sheaf $\mathcal{F}$ on Sm_k gives rise to a space by considering it as a discrete simplicial sheaf on Sm_k. Since every representable presheaf is a sheaf (see Corollary I.1.25), this allows us to consider every smooth scheme X as a space which we will still denote by X.

These two classes also help with understanding what a general space is supposed to be. Spaces are the outcome of merging the categories of simplicial sets and smooth schemes and extending the result to a well-behaved category in which our usual homotopical notions will make sense.
As for topological spaces, we need a notion of pointed spaces for various homotopical constructions and objects. For this we also denote the space $\mathrm{Spec}(k)$ by pt.

Definition I.4.4. A pointed space is a pair (X, x) where X is a space and $x\colon \mathrm{pt} \to X$ is a morphism of spaces.

This yields a category of pointed spaces and morphisms compatible with the basepoints, which we denote by $\mathrm{Spc}_*(k)$. Also here we can give an equivalent definition, namely as a sheaf of pointed simplicial sets.

Remark I.4.5. The following pointed spaces occur so frequently that we do not want to mention their base points all the time:

- $(\mathbb{A}^n, e_1)$ with $e_1 = (1, 0, \ldots, 0)$ and in particular $(\mathbb{A}^1, 1)$
- $(\mathbb{A}^n \setminus \{0\}, e_1)$ and in particular $(\mathbb{G}_{\mathrm{m}}, 1)$
- $(\mathbb{P}^1, \infty)$
- $(S^n, 1)$

Therefore we will drop their base points from our notation unless we happen to choose different ones or unless we want to highlight that we are working in the pointed category.

As in topology, we can rather easily go back and forth between unpointed and pointed spaces in terms of the usual adjunction.

Lemma I.4.6. *The forgetful functor* $\mathrm{Spc}_*(k) \to \mathrm{Spc}(k)$ *has a left adjoint* $\mathrm{Spc}(k) \to \mathrm{Spc}_*(k)$ *given by mapping a space* X *to the space* $X_+ = X \amalg \mathrm{pt}$ *with the additional point as a base point.*

This follows directly from the definitions, but see also page 82 of [76]. Using this notion, we can once again define homotopy groups, just that now they will be sheaves. This should not be too surprising though.

Definition I.4.7. Let n be a non-negative integer. The n-th homotopy sheaf $\underline{\pi}_n(X, x)$ of a pointed space (X, x) is the (Nisnevich) sheafification of the presheaf $U \mapsto \pi_n(X(U), x(U))$.

Here the space under consideration was pointed. To make sure that any space can be pointed, Morel and Voevodsky make use of topos-theoretic points. We do not want to discuss those in more detail, but refer the interested reader to [76].
This gives us a functor $\underline{\pi}_n$, which according to Theorem I.3.25 maps to the category of sheaves of groups Grp/k if $n = 1$, and for $n \geq 2$ maps to the category of abelian sheaves Ab/k. We can now introduce the following weak equivalences, fibrations and cofibrations of simplicial sheaves:

$\mathrm{W}_s(\mathrm{Spc}(k))$ = Morphisms of spaces, which for all choices of compatible base points induces isomorphisms on all simplicial homotopy sheaves.

$\mathrm{Cof}_s(\mathrm{Spc}(k))$ = Monomorphisms of spaces, i.e. objectwise monomorphisms of simplicial sets

$\mathrm{Fib}_s(\mathrm{Spc}(k))$ = Morphisms of spaces having the right lifting property with respect to acyclic cofibrations

Theorem I.4.8. *The three classes* $\mathrm{W}_s(\mathrm{Spc}(k))$*,* $\mathrm{Fib}_s(\mathrm{Spc}(k))$ *and* $\mathrm{Cof}_s(\mathrm{Spc}(k))$ *endow the category* $\mathrm{Spc}(k)$ *of spaces with a model structure.*

Proof. This is Theorem 1.4 together with Remark 1.3 from Chapter 2 of [76]. □

This is the so-called simplicial model structure on spaces and comes together with a simplicial homotopy category $H_s(k) = H_s(\mathrm{Spc}(k))$. As stated at the very beginning of this chapter, this homotopy theory is not quite what we want. We still need to ensure that $\mathbb{A}^1$ becomes contractible. For this we consider the following notion.

Definition I.4.9. A space Y is $\mathbb{A}^1$-local if for smooth schemes X, the map

$$\mathrm{Hom}_{H_s(k)}(X, Y) \to \mathrm{Hom}_{H_s(k)}(X \times \mathbb{A}^1, Y)$$

induced by the projection onto the first component is bijective.

In other words, $\mathbb{A}^1$-local spaces are spaces, for which the affine line already seems contractible. These spaces are also called motivic spaces. Let us denote the full subcategory of $\mathbb{A}^1$-local spaces by $\mathrm{Spc}^{\mathbb{A}^1}(k)$. These now allow us to upgrade our simplicial weak equivalences to $\mathbb{A}^1$-equivalences:

Definition I.4.10. A morphism $f\colon X \to Y$ of spaces is an $\mathbb{A}^1$-weak equivalence if for all $\mathbb{A}^1$-local spaces Z, the map

$$f^*\colon \mathrm{Hom}_{H_s(k)}(Y, Z) \to \mathrm{Hom}_{H_s(k)}(X, Z)$$

given by precomposing with f is a bijection.

We have of course an inclusion functor $\mathrm{Spc}^{\mathbb{A}^1}(k) \hookrightarrow \mathrm{Spc}(k)$. This functor turns out to have a left adjoint with a very useful property, see Theorem 3.2 from Chapter 2 of [76] together with Example 4 directly beneath it:

Theorem I.4.11. *The inclusion functor* $\mathrm{Spc}^{\mathbb{A}^1}(k) \hookrightarrow \mathrm{Spc}(k)$ *has a left adjoint* $L_{\mathbb{A}^1}$*, which is a (left) Bousfield localization. In particular, there is a model structure on spaces given by the following weak equivalences, fibrations and cofibrations:*

$$\begin{aligned} \mathrm{W}_{\mathbb{A}^1}(\mathrm{Spc}(k)) &= \mathbb{A}^1\text{-}\textit{weak equivalences} \\ \mathrm{Fib}_{\mathbb{A}^1}(\mathrm{Spc}(k)) &= \textit{Morphisms of spaces having the right lifting property with respect to acyclic cofibrations} \\ \mathrm{Cof}_{\mathbb{A}^1}(\mathrm{Spc}(k)) &= \textit{Monomorphisms of spaces} \end{aligned}$$

We never explained what a (left) Bousfield localization is. This notion originally due to Bousfield [24] is a localization of a model category, which adds morphisms to the weak equivalences without changing the cofibrations, see for instance Chapter X.3 of [44]. We advise a reader who is unfamiliar with Bousfield localizations, to split this theorem into two parts. Namely, that there is a left adjoint to the inclusion $\mathrm{Spc}_{\mathbb{A}^1}(k) \hookrightarrow \mathrm{Spc}(k)$ and that this yields a model structure as described above. This model structure is the $\mathbb{A}^1$-model structure.

Definition I.4.12. The homotopy category $H(k)$ associated with the $\mathbb{A}^1$-model structure on spaces is the unstable motivic homotopy category.

The forgetful functor $\mathrm{Spc}_*(k) \to \mathrm{Spc}(k)$ also endows the category $\mathrm{Spc}_*(k)$ of pointed spaces with the $\mathbb{A}^1$-model structure, so that we also get a pointed unstable motivic homotopy category $H_*(k)$. As mentioned more generally in Section I.2.2, we use the notation $[(X, x), (Y, y)]$ to denote the set of morphisms $(X, x) \to (Y, y)$ in the category $H_*(k)$ and denote $\mathbb{A}^1$-weak equivalences by $\simeq_{\mathbb{A}^1}$.

Definition I.4.13. Let n be a non-negative integer. The n-th $\mathbb{A}^1$-homotopy sheaf $\underline{\pi}_n^{\mathbb{A}^1}(X, x)$ of a pointed space (X, x) is the sheafification of the presheaf $U \mapsto [S^n \wedge U_+, (X, x)]$.

This is once again functorial in (X, x). These sheaves are sheaves of groups if $n = 1$, and abelian sheaves if $n \geq 2$ by the usual Eckmann-Hilton argument.

Remark I.4.14. Note that the adjunction between unpointed and pointed spaces from Lemma I.4.6 shows that the $\mathbb{A}^1$-connected components $\underline{\pi}_0^{\mathbb{A}^1}(-)$ do not really require base-points. We can also define it as the sheafification of $U \mapsto \mathrm{Hom}_{H(k)}(U, -)$ on the level of unpointed spaces. As in classical homotopy theory, computing these homotopy sheaves is not easy at all. We will nevertheless see a non-trivial computation in Section I.6.3.

Definition I.4.15. A pointed space (X, x) is $\mathbb{A}^1$-n-connected if the sheaf $\underline{\pi}_i^{\mathbb{A}^1}(X, x)$ vanishes for all $0 \leq i \leq n$.

Example I.4.16. In the next section we will see that the smash product "$\wedge$" of simplicial sets can be extended to spaces. In particular, there are spaces of the form $S^n \wedge (X, x)$. By Theorem 1.18 of [75], such spaces are $\mathbb{A}^1$-$(n-1)$-connected.

Proposition I.4.17. *A morphism $f\colon (X, x) \to (Y, y)$ of $\mathbb{A}^1$-connected pointed spaces is an $\mathbb{A}^1$-equivalence iff for all $n > 0$, the induced map*

$$f_*\colon \underline{\pi}_n^{\mathbb{A}^1}(X, x) \to \underline{\pi}_n^{\mathbb{A}^1}(Y, y)$$

is an isomorphism.

Proof. This is Proposition 2.14 from Chapter 3 of [76]. □

I.4.2 From old to new Spaces

Except for the two standard examples of spaces from Examples I.4.2 and I.4.3, we have not seen any examples of spaces. Let us therefore discuss a couple of operations on (pointed) spaces resulting in new examples.

Definition I.4.18. The wedge sum $(X, x) \vee (Y, y)$ of two pointed spaces (X, x) and (Y, y) is the sheafification of the presheaf $U \mapsto (X, x)(U) \vee (Y, y)(U)$.

It is not difficult to observe that the wedge sum is the coproduct of pointed spaces. The desired universal property just lifts from the one on the level of simplicial sets.

Definition I.4.19. The smash product $(X, x) \wedge (Y, y)$ of two pointed spaces (X, x) and (Y, y) is the sheafification of the presheaf $U \mapsto (X, x)(U) \wedge (Y, y)(U)$.

This defines a symmetric monoidal structure on the category $H_*(k)$, i.e. , it is a 'tensor product" of pointed spaces, see Lemma 2.13 from Chapter 3 of [76].

Example I.4.20. By definition we have $S^0 \wedge (X, x) = (X, x)$ for all pointed spaces (X, x). In other words, S^0 is the tensor unit.

Example I.4.21. For any pointed space (X, x), the smash product $S^1 \wedge (X, x)$ is given by the homotopy pushout of the diagram

$$\begin{array}{ccc} (X, x) & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ \mathrm{pt} & & \end{array}$$

i.e. it is the so-called categorical suspension $\Sigma(X,x)$ of (X,x), see for example page 148 of [48]. Since other kinds of suspensions naturally show up in motivic homotopy theory, we will from now on denote this suspension by $\Sigma_{S^1}(X,x)$.

These two operations allow us to expand our collection of examples, which can then once again result in further examples by taking limits or colimits or their derived variants.

Remark I.4.22. The smash product distributes over the wedge sum. This can be seen very directly on the level of pointed simplicial sets. If (X,y), (Y,y) and (Z,z) are pointed simplicial sets, then both $(X,y) \vee (Y,y)) \wedge (Z,y)$ and $((X,x) \wedge (Z,z)) \vee ((Y,y) \wedge (Z,z))$ are quotients of $(X \amalg Y) \times Z \cong (X \times Z) \amalg (Y \times Z)$ by the same equivalence relation. By definition of "$\wedge$" and "$\vee$" for pointed spaces, this distributivity now lifts to the level of pointed spaces.

Let us also mention another example of homotopy pushouts of spaces which we have already introduced under a different name.

Lemma I.4.23. *Every elementary Nisnevich square*

$$\begin{array}{ccc} U \times_X V & \xrightarrow{\mathrm{pr}_2} & V \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle p} \\ U & \xhookrightarrow{i} & X \end{array}$$

is a homotopy pushout square of spaces.

Proof. First note that since i is an open immersion, so is its base change pr_2. The latter is hence a monomorphism of discrete simplicial sheaves and thus a cofibration for the $\mathbb{A}^1$-model structure. Therefore the usual pushout coincides with the homotopy pushout. Moreover, as all these are discrete simplicial sheaves, we just need to verify that X is the pushout in the category of sheaves on Sm_k. Let $\mathcal{F}$ be a test object together with morphisms $t_U \colon U \to \mathcal{F}$ and $t_V \colon V \to \mathcal{F}$ satisfying $\mathrm{pr}_1 \circ t_U = \mathrm{pr}_2 \circ t_V$.

$$\begin{array}{cccc} U \times_X V & \xrightarrow{\mathrm{pr}_2} & V & \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle p} & \searrow{\scriptstyle t_V} \\ U & \xhookrightarrow{i} & X & \\ & & \xrightarrow{t_U} & \mathcal{F} \end{array}$$

By Theorem I.1.45, we know that when we apply F to the elementary Nisnevich square, we obtain a pullback square of sets

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(p)} & \mathcal{F}(V) \\ \downarrow{\scriptstyle \mathcal{F}(i)} & & \downarrow{\scriptstyle \mathcal{F}(\mathrm{pr}_2)} \\ \mathcal{F}(U) & \xrightarrow{\mathcal{F}(\mathrm{pr}_1)} & \mathcal{F}(U \times_X V). \end{array}$$

Via the Yoneda lemma, the morphisms t_U and t_V define elements of $\mathcal{F}(U)$ and $\mathcal{F}(V)$ respectively, and these elements satisfy $\mathcal{F}(\mathrm{pr}_1)(t_U) = \mathcal{F}(\mathrm{pr}_2)(t_V)$ since $\mathrm{pr}_1 \circ t_U = \mathrm{pr}_2 \circ t_V$. Therefore the pair (t_U, t_V) defines an element of the pullback $\mathcal{F}(X)$, which gives us the required morphism $t\colon X \to \mathcal{F}$ compatible with t_U and t_V via the Yoneda lemma. Moreover, since the pair (t_U, t_V) is uniquely determined, this morphism is also unique. □

This can also be found in [76] as Lemma 1.6 of Chapter 3.

I.4.3 Motivic Spheres

In topology there is a family of spheres S^n graded by the natural numbers. Now that we work with spaces, we also have the algebraic sphere $\mathbb{G}_\mathrm{m}$ which is used to build algebraic tori etc. So we have two kinds of spheres. Going back to the topological picture, all the topological spheres are generated by S^1 under the smash product, i.e., we have $S^n = (S^1)^{\wedge n}$. If we are to mimic this, we end up with the following definition of motivic spheres:

Definition I.4.24. Let m and n be non-negative integers with $n \geq m$. The motivic sphere $S^{n,m}$ of bidegree (n, m) is the (pointed) space $(S^1)^{\wedge(n-m)} \wedge \mathbb{G}_m^{\wedge m} = \Sigma_{S^1}^{n-m} \mathbb{G}_m^{\wedge m}$.

Note that there are two conventions for the bidegree. In our convention, which is the more common one since it matches up with the grading in motivic cohomology, we have $S^1 = S^{1,0}$ and $\mathbb{G}_\mathrm{m} = S^{1,1}$ whereas one sometimes finds $\mathbb{G}_\mathrm{m} = S^{0,1}$ as well.

Example I.4.25. By definition $S^{1,1} = \mathbb{G}_\mathrm{m}$ and $S^{n,0} = S^n$ for all non-negative integers n.

By Lemma 2.15 and Example 2.20 from Chapter 3 of [76], there are a couple more spheres which we can understand very concretely:

Lemma I.4.26. *There is an* $\mathbb{A}^1$*-weak equivalence* $S^{2,1} = \Sigma_{S^1} \mathbb{G}_\mathrm{m} \simeq_{\mathbb{A}^1} \mathbb{P}^1$ *in* $\mathrm{Spc}_*(k)$.

Proof. Consider the pointed version of the distinguished Nisnevich square

$$\begin{array}{ccc} \mathbb{G}_\mathrm{m} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & (\mathbb{P}^1, 1) \end{array}$$

from Example I.1.43, which by Lemma I.4.23 defines a homotopy pushout square in $\mathrm{Spc}_*(k)$. Since $\mathbb{A}^1 \simeq_{\mathbb{A}^1} \mathrm{pt}$, we hence get the homotopy pushout square

$$\begin{array}{ccc} \mathbb{G}_\mathrm{m} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & (\mathbb{P}^1, 1) \end{array}$$

so that $\Sigma_{S^1} \mathbb{G}_\mathrm{m} \simeq_{\mathbb{A}^1} (\mathbb{P}^1, 1)$. Since for each point $a \in \mathbb{P}^1$, the automorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix} \in \mathrm{PGL}_2(k) = \mathrm{Aut}(\mathbb{P}^1)$$

maps a to ∞, we have $\Sigma_{S^1} \mathbb{G}_\mathrm{m} \simeq_{\mathbb{A}^1} \mathbb{P}^1$ as claimed since isomorphisms are always weak equivalences. □

Proposition I.4.27. *There is an* $\mathbb{A}^1$*-weak equivalence* $S^{2n-1,n} \simeq_{\mathbb{A}^1} \mathbb{A}^n \setminus \{0\}$ *in* $\mathrm{Spc}_*(k)$ *for all positive integers* n*.*

Proof. We give a proof by induction on $n \geq 1$. As already noted in Example I.4.25, the case $n = 1$ holds by definition. From now on consider $n \geq 2$. The distinguished Nisnevich square

$$\begin{array}{ccc} (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_{\mathrm{m}} & \longrightarrow & \mathbb{A}^n \times \mathbb{G}_{\mathrm{m}} \\ \downarrow & & \downarrow \\ (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^n \setminus \{0\} \end{array}$$

defines a homotopy pushout square in $\mathrm{Spc}_*(k)$ by Lemma I.4.23. Contracting all the affine spaces, we thus get the homotopy pushout square

$$\begin{array}{ccc} (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_{\mathrm{m}} & \longrightarrow & \mathbb{G}_{\mathrm{m}} \\ \downarrow & & \downarrow \\ \mathbb{A}^{n-1} \setminus \{0\} & \longrightarrow & \mathbb{A}^n \setminus \{0\} \end{array}$$

We now compute the homotopy colimit of the diagram

$$\begin{array}{ccccc} \mathrm{pt} & \longleftarrow & \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{A}^{n-1} \setminus \{0\} & \longleftarrow & (\mathbb{A}^{n-1} \setminus \{0\}) \vee \mathbb{G}_{\mathrm{m}} & \longrightarrow & \mathbb{G}_{\mathrm{m}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^{n-1} \setminus \{0\} & \longleftarrow & (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_{\mathrm{m}} & \longrightarrow & \mathbb{G}_{\mathrm{m}} \end{array}$$

in two ways. We can first take the homotopy pushouts of the rows and then the homotopy pushout of the resulting diagram, which is given by the homotopy pushout square

$$\begin{array}{ccc} \mathrm{pt} & \longrightarrow & \mathbb{A}^n \setminus \{0\} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathbb{A}^n \setminus \{0\}. \end{array}$$

We can also take the homotopy pushouts of the columns and then the homotopy pushout of the resulting diagram, which yields the homotopy pushout square

$$\begin{array}{ccc} (\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{G}_{\mathrm{m}} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \Sigma_{S^1}(\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{G}_{\mathrm{m}}\,. \end{array}$$

Thus we have an $\mathbb{A}^1$-weak equivalence $\mathbb{A}^n \setminus \{0\} \simeq_{\mathbb{A}^1} \Sigma_{S^1}(\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{G}_m$ in $\mathrm{Spc}_*(k)$. Finally, using the induction hypothesis, the right hand side is given by

$$S^1 \wedge (\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{G}_{\mathrm{m}} \simeq_{\mathbb{A}^1} S^{1,0} \wedge S^{2n-3,n-1} \wedge S^{1,1} = S^{2n-1,n},$$

which is what we wanted to show. □

We now consider the two families of affine quadrics

$$Q_{2n} = \mathrm{Spec}(k[x_1, \dots, x_n, y_1, \dots, y_n, z]/(x_1y_1 + x_1y_2 + \dots + x_ny_n - z(1+z)))$$

and

$$Q_{2n+1} = \mathrm{Spec}(k[x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}]/(x_1y_1 + x_1y_2 + \dots + x_{n+1}y_{n+1} - 1))$$

for all positive integers n. As also described [11], the projection $Q_{2n+1} \to \mathbb{A}^{n+1} \setminus \{0\}$ onto the first $n+1$ coordinates is a well-known $\mathbb{A}^1$-weak equivalence. By Proposition I.4.27 we thus have that $\mathrm{Q}_{2n+1} \simeq_{\mathbb{A}^1} S^{2n+1,n+1}$ for all positive integers n. In loc. cit. Asok, Doran and Fasel also manage to treat the other case:

Theorem I.4.28. *There is an $\mathbb{A}^1$-weak equivalence $Q_m \simeq_{\mathbb{A}^1} S^{m,\lceil \frac{m}{2} \rceil}$ in $\mathrm{Spc}_*(k)$ for all non-negative integers m.*

In loc. cit. they also find out that for some integers m and n, the motivic sphere $S^{m,n}$ does not have a smooth (affine) model, by which we mean that there exists a smooth (affine) scheme X over k, so that $S^{m,n}$ is $\mathbb{A}^1_k$-weakly equivalent to (X, x) for some $x \in X$. To go into a bit more detail, they show:

Proposition I.4.29. *If $m > 2n$, the motivic sphere $S^{m,n}$ does not have a smooth affine model.*

Let us conclude with a remark explaining why we did not consider general motivic spheres $S^{n,m}$ for the definition of the $\mathbb{A}^1$-homotopy sheaves.

Remark I.4.30. One can, in fact, also define more general $\mathbb{A}^1$-homotopy sheaves by considering the sheafification of $U \mapsto [S^{n,m} \wedge U_+, -]$. As it turns out, these can be expressed fully in terms of the sheaves $\underline{\pi}_n^{\mathbb{A}^1}(-)$ and are hence obsolete, see Theorem I.6.29.

I.5 Stable Motivic Homotopy Theory

The stable world within motivic homotopy theory was first explored and studied by Voevodsky [98] and Jardine [57]. Here the central object is the stable motivic homotopy category SH(k), the homotopy category of motivic spectra.

I.5.1 Motivic spectra and the $\mathbb{A}^1$-Stable Model Structure

As for spaces, let us begin by defining our basic objects of study. Here it is rather common not to include base points in the notation. We will still use our convention from Remark I.4.5 though.

Definition I.5.1. A motivic spectrum is a sequence $E = (E_n)_{n\geq 0}$ of pointed spaces together with morphisms $\sigma_n \colon \mathbb{P}^1 \wedge E_n \to E_{n+1}$ of pointed spaces for all $n \geq 0$.

Example I.5.2. Any pointed space X has an associated motivic spectrum

$$\Sigma^\infty X = (X, \mathbb{P}^1 \wedge X, (\mathbb{P}^1)^{\wedge 2} \wedge X, (\mathbb{P}^1)^{\wedge 3} \wedge X, \dots),$$

called the suspension spectrum of X, where the structure morphisms are just identity morphisms.

Example I.5.3. Although it is a special case of the previous example, we nevertheless wish to highlight the following motivic spectrum due to its importance. The suspension spectrum of $S^{0,0} = S^0$ is called the motivic sphere spectrum, which we will denote by $\mathbb{1}_k$. This notation reflects that it is the unit for the smash product of motivic spectra; see Theorem I.5.26.

Example I.5.4. More generally, we set $\mathbb{1}_k^{n,m} = \Sigma^\infty S^{n,m}$. Here we have $n \geq m \geq 0$ for the motivic sphere $S^{n,m}$ to be defined.

Example I.5.5. Also any unpointed space X has an associated suspension spectrum. It is defined by taking the suspension spectrum of the pointed space $X_+ = X \amalg \mathrm{pt}$ and is denoted by $\Sigma^\infty_+ X$.

Example I.5.6. The zero spectrum 0 is given by $(\mathrm{pt}, \mathrm{pt}, \dots)$ with the evident structure maps. Although we have not introduced morphisms of motivic spectra yet, let us already note that this is indeed the zero object in the category of motivic spectra, which is certainly not difficult to imagine based on its definition.

For further and in particular more interesting examples of motivic specta, we refer to the next subsection.

Definition I.5.7. A morphisms $f \colon E \to E'$ between two motivic spectra is a family of morphisms $f_n \colon E_n \to E'_n$ such that the diagrams

$$\begin{array}{ccc} \mathbb{P}^1 \wedge E_n & \xrightarrow{\sigma_n} & E_{n+1} \\ \big\downarrow{\scriptstyle \mathrm{id}_{\mathbb{P}^1} \wedge f_n} & & \big\downarrow{\scriptstyle f_{n+1}} \\ \mathbb{P}^1 \wedge E'_n & \xrightarrow{\sigma'_n} & E'_{n+1}. \end{array}$$

commute for all non-negative integers n.

This gives us the category $\mathrm{Sp}(k)$ of motivic spectra, which are the central objects of stable motivic homotopy theory. Note that taking suspension spectra of pointed or unpointed spaces yields functors

$$\Sigma^\infty\colon \mathrm{Spc}_*(k) \to \mathrm{Sp}(k) \text{ and } \Sigma^\infty_+\colon \mathrm{Spc}(k) \to \mathrm{Sp}(k).$$

We now quickly introduce a model structure on $\mathrm{Sp}(k)$.

Definition I.5.8. Let E be a motivic spectrum and let n and m be integers. The sheafification $\underline{\pi}_{n,m}(E)$ of the presheaf $\widetilde{\pi}_{n,m}(E)$ which maps a smooth scheme U to the colimit of

$$\ldots \to [(\mathbb{P}^1)^{\wedge r} \wedge S^{n,m} \wedge U_+, E_r] \xrightarrow{(\sigma_r)_* \circ \Sigma_{\mathbb{P}^1}} [(\mathbb{P}^1)^{\wedge(r+1)} \wedge S^{n,m} \wedge U_+, E_{r+1}] \to \ldots$$

for $r > 0$ is called the (n, m)-th motivic stable homotopy sheaf of E.

Note that the stable homotopy sheaves are abelian sheaves due to $\mathbb{P}^1$ being a suspension in $\mathrm{H}_*(k)$, which follows from Theorem 3 of [25] together with the hom-tensor adjunction as in classical stable homotopy theory. Alternatively, using Theorem I.6.29, we see that this colimit is a colimit of higher $\mathbb{A}^1$-homotopy sheaves and hence must be abelian as well. These hence define functors

$$\underline{\pi}_{n,m}\colon \mathrm{Sp}(k) \to \mathrm{Ab}/k$$

by level-wise post-composing with a given morphism of motivic spectra.

Remark I.5.9. There are many different notations for these homotopy sheaves in the literature (mostly on the level of the indices). Therefore we want to provide translations between the ones that we see most frequently:

$$\underline{\pi}_{n,m}(E) = \underline{\pi}_{n-m+(m)}(E) = \underline{\pi}_{n-m+\alpha m}(E) = \underline{\pi}_{n-m}(E)_{-m}$$

While the one from our definition is closest to the topological situation, the other ones have the advantage of separating the two kinds of spheres. Additionally, the one on the very right has a different sign with respect to the algebraic spheres, which gives the right sign for homotopy modules (the kind of objects that we will introduce in the next section). Therefore we will also use this notation as well.

As in the usual category of spectra, the homotopy (pre-)sheaves allow us to define suitable weak equivalences:

Definition I.5.10. A morphism $f\colon E \to E'$ of motivic spectra is called an $\mathbb{A}^1$-stable equivalence if the induced morphisms $\widetilde{\pi}_{n,m}(E) \to \widetilde{\pi}_{n,m}(E')$ are isomorphisms for all $n, m \in \mathbb{Z}$.

Example I.5.11. Any levelwise $\mathbb{A}^1$-weak equivalence, that is, a morphism $f\colon E \to E'$ of motivic spectra such that all the components $f_n\colon E_n \to E'_n$ are $\mathbb{A}^1$-weak equivalences, is an $\mathbb{A}^1$-stable equivalence. In particular, if we have an $\mathbb{A}^1$ weak equivalence $g\colon X \to Y$ between pointed spaces, then the induced morphisms $\Sigma^\infty f\colon \Sigma^\infty X \to \Sigma^\infty Y$ on the level of suspension spectra are $\mathbb{A}^1$-stable equivalences.

We now consider the three classes:

$$\begin{aligned} W(\mathrm{Sp}(k)) &= \mathbb{A}^1\text{-stable equivalences} \\ \mathrm{Fib}(\mathrm{Sp}(k)) &= \text{Morphisms of spectra having the right lifting property with respect to acyclic cofibrations} \\ \mathrm{Cof}(\mathrm{Sp}(k)) &= \text{Levelwise cofibrations of pointed spaces, i.e. levelwise monomorphisms} \end{aligned}$$

Theorem I.5.12. *The three classes* $W(\mathrm{Sp}(k))$, $\mathrm{Fib}(\mathrm{Sp}(k))$ *and* $\mathrm{Cof}(\mathrm{Sp}(k))$ *endow the category* $\mathrm{Sp}(k)$ *of motivic spectra with a model structure.*

Proof. This can be found as Theorem 6.25. in [83]. □

This model structure is called the $\mathbb{A}^1$-stable model structure and its associated homotopy category is the central object of stable motivic homotopy theory:

Definition I.5.13. The homotopy category $\mathrm{SH}(k)$ associated with the $\mathbb{A}^1$-stable model structure on motivic spectra is the stable motivic homotopy category.

I.5.2 Some further examples of Motivic Spectra

In this subsection we want to sketch how to construct two important examples of motivic spectra, the first one being motivic versions of Eilenberg-Mac Lane spectra. For these we generally follow Section 6.1 of Voevodsky's very well written notes [98], and also our notes from talk 4 at Talbot 2023 [107]. Let us quickly recall how to construct Eilenberg-Mac Lane spectra in classical homotopy theory:
Given an abelian group A, there exist CW-complexes $K(A, n)$ for all $n \geq 0$ called Eilenberg-Mac Lane spaces, satisfying

$$\pi_i(K(A,n)) = \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

These spaces come along with equivalences $K(A, n) \simeq \Omega K(A, n+1)$ yielding a spectrum

$$HA = (K(A,0), K(A,1), K(A,2), \dots)$$

via the maps $\Sigma K(A, n) \to K(A, n+1)$ adjoint to the above equivalences. This is the Eilenberg-Mac Lane spectrum associated with the abelian group A and it satisfies

$$\pi_i(HA) = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and represents ordinary cohomology with coefficients in A.
Let us now construct its motivic version. The main question here is how to define motivic Eilenberg-Mac Lane spaces. We could try to copy the topological definition, i.e. we take spaces $K(A, n)$ with

$$\underline{\pi}_i^{\mathbb{A}^1}(K(A,n))(Y) = \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

for every connected smooth scheme Y. Assuming that such spaces exist, we still need structure maps

$$\mathbb{P}^1 \wedge K(A,n) \to K(A,n+1),$$

but there are in general no non-constant such maps in $H_*(k)$. Indeed, we have

$$[\mathbb{P}^1 \wedge X, K(A,n+1)] \cong [X, \Omega_{\mathbb{P}^1} K(A,n+1)]$$

for any space X, where $\Omega_{\mathbb{P}^1}$ is a $\mathbb{P}^1$-loop space, which can be defined as an internal hom. The latter should at least be conceivable to readers familiar with the analogous setting in algebraic topology. We have

$$\underline{\pi}_i^{\mathbb{A}^1}(\Omega_{\mathbb{P}^1} K(A,n+1)) = \underline{\pi}_{i+1}^{\mathbb{A}^1}(K(A,n+1))_{-1} = \begin{Bmatrix} \underline{A}_{-1} & \text{if } i = n \\ \underline{0}_{-1} & \text{else} \end{Bmatrix} = \underline{0}$$

for all i. Here $(-)_{-1}$ is the homotopy contraction, an algebraic incarnation of the derived $\mathbb{G}_m$-loop space which we will introduce in Section I.6.1. If the reader is unfamiliar with it, we advise to view this computation as a black box for now. Thus we have shown that the space $\Omega_{\mathbb{P}^1} K(A,n+1)$ is contractible and hence does not allow any non-constant structure maps. Let us contemplate a bit more about this in the following remark.

Remark I.5.14. The actual cause of the contractibility of the space $\Omega_{\mathbb{P}^1} K(A,n+1)$ was that the sheaf $\underline{A}_{-1}$ is trivial. While this is an issue for defining motivic Eilenberg-Mac Lane spectra representing motivic cohomology as above, this is not an issue in general. There are plenty of sheaves for which the construction $(-)_{-1}$ does not vanish, see Sections I.6.1 and I.6.2, and using these we may find structure maps to define motivic spectra. This does lead to a different kind of Eilenberg-Mac Lane spectra HM_* in motivic homotopy theory [71, page 64], which are indeed defined in analogy to the topological situation. Here the input M_* is a so-called homotopy module, a notion which we will introduce in Section I.6.1.

Let us now focus on motivic Eilenberg-Mac Lane spetra associated to an abelian group again. We need to remedy the above obstruction. Suslin had the idea to use the Dold-Thom theorem [31], a version of which we therefore state:

Theorem I.5.15 (Dold-Thom)**.** *Let (X, x_0) be a pointed CW-complex. Then we have*

$$K_0(\mathrm{SP}^\infty(X,x_0)) \simeq \prod_{n \geq 0} K(\tilde{H}_n(X), n),$$

where $\mathrm{SP}^\infty(X,x_0) = \mathrm{colim}_m (X,x_0)^m / S_m$ is the infinite symmetric product considered as a commutative topological monoid under concatenation and K_0 is its topological group completion.

This version can be found on page 596 of [98]. Taking homotopy groups of the topological spaces above yields the more known version of the Dold-Thom theorem. If we take $X = M(A,n)$ to be a Moore space (essentially an Eilenberg-Mac Lane space for ordinary reduced homology instead of homotopy), we thus have an equivalence

$$K_0(\mathrm{SP}^\infty(M(A,n))) \simeq \prod_{n \geq 0} K(A,n)$$

and can hence get our hands on Eilenberg-Mac Lane spaces in this way. But what is a good motivic candidate for $K_0(\mathrm{SP}^\infty(X, x_0))$? A point in it should be a $\mathbb{Z}$-linear combination of closed points of our given smooth scheme X. From the point of view of algebraic geometry, this sounds like we should use algebraic cycles/correspondences. This idea turns out to work:

Definition I.5.16. The category Cor_k of finite correspondences has smooth schemes as objects and morphisms $\mathrm{Hom}_{\mathrm{Cor}_k}(X, Y) = \mathrm{Cor}_k(X, Y)$ are formal $\mathbb{Z}$-linear combinations of the form $\sum_i m_i[Z_i \hookrightarrow X \times Y]$, where the Z_i are integral closed subschemes and $\mathrm{pr}_1 \colon Z_i \to X$ is finite and surjective onto some irreducible component of X.

The composition of such morphisms is defined as follows:
If $c_1 \in \mathrm{Cor}_k(X, Y)$ and $c_2 \in \mathrm{Cor}_k(Y, Z)$, then $c_1 \times Z$ and $X \times c_2$ intersect properly inside of $X \times Y \times Z$, see Lemma 1.15 of [29] where this is shown over a more general base. This allows us to set

$$c_2 \circ c_1 = (\mathrm{pr}_{1,3})_*((c_1 \times Z) \cdot (X \times c_2)) \in \mathrm{Cor}_k(X, Z),$$

where "$\cdot$" denotes the intersection product. Note that the finiteness is used for the above pushforward.
Note that Cor_k is an additive category with direct sum $X \oplus Y$ given by the disjoint union of X and Y. Therefore we can consider spaces

$$A_{\mathrm{tr}}(X) = \mathrm{Hom}_{\mathrm{Cor}_k}(-, X) \otimes A$$

for any abelian group A and any smooth scheme X, which we will call A-linear representable presheaves with transfer. As we will not introduce presheaves with transfers, any reader unfamiliar with these should just view the above as a name.
Presheaves of the form A_{tr} come together with natural morphisms $j_X \colon X \to A_{\mathrm{tr}}(X)$ defined by mapping $f \in \mathrm{Hom}(Y, X)$ to its graph $[\Gamma_f] \in \mathbb{Z}_{\mathrm{tr}}(X)$ and then tensoring with the abelian group A.
We also define their pointed versions

$$A_{\mathrm{tr}}(X, x_0) = \mathrm{cofib}(A_{\mathrm{tr}}(x_0) \to A_{\mathrm{tr}}(X))$$

and set $A_{\mathrm{tr}}((X_1, x_{0,1}) \wedge \ldots \wedge (X_r, x_{0,r}))$ to be the space

$$\mathrm{cofib}\Big(\coprod_i A_{\mathrm{tr}}\Big(\prod_{j \neq i}(X_j, x_{0,j})\Big) \to A_{\mathrm{tr}}\Big(\prod_j (X_j, x_{0,j})\Big)\Big).$$

Although we have briefly motivated the use of correspondences, let us nevertheless quickly justify these definitions with respect to our goal as done in [92]:

Theorem I.5.17. *We have $\mathbb{Z}_{\mathrm{tr}}(X)(Y)[\frac{1}{p}] \cong K_0(\mathrm{Hom}(Y, \mathrm{SP}^\infty(X))[\frac{1}{p}]$ for all normal connected Y, where p is the exponential characteristic of k.*

So these definitions do indeed provide us with a good candidate for $K_0(\mathrm{SP}^\infty(-))$. To get suitable structure morphisms, we observe that the $\mathbb{Z}$-bilinear map

$$\mathbb{Z}_{\mathrm{tr}}(X) \times \mathbb{Z}_{\mathrm{tr}}(X') \to \mathbb{Z}_{\mathrm{tr}}(X \times X')$$

defined by mapping $([Z],[Z'])$ to its product $[Z\times Z']$, induces a morphism

$$A_{\mathrm{tr}}(X,x_0)\wedge A_{\mathrm{tr}}(X',x_0')\to A_{\mathrm{tr}}((X,x_0)\wedge(X',x_0')).$$

Now we have all the ingredients that we need and can finally define motivic Eilenberg-Mac Lane spectra.

Definition I.5.18. The motivic Eilenberg-Mac Lane space $K(A,2n,n)$ is the A-linear representable presheaf with transfers $A_{\mathrm{tr}}((\mathbb{P}^1)^{\wedge n})$. Furthermore, the motivic Eilenberg-Mac Lane spectrum HA is the sequence of pointed spaces

$$(K(A,0,0),K(A,2,1),K(A,4,2),\dots)$$

together with the structure maps σ_n given by the composite

$$\mathbb{P}^1\wedge A_{\mathrm{tr}}((\mathbb{P}^1,\infty)^{\wedge n})\to A_{\mathrm{tr}}(\mathbb{P}^1,\infty)\wedge A_{\mathrm{tr}}((\mathbb{P}^1,\infty)^{\wedge n})\to A_{\mathrm{tr}}((\mathbb{P}^1,\infty)^{\wedge(n+1)})$$

of $j_{\mathbb{P}^1}\wedge\mathrm{id}$ and the morphism induced by the product map.

So now we have Eilenberg-Mac Lane spectra. Let us now also briefly sketch how to obtain a motivic version of the algebraic K-theory spectrum, following subsection 2.1 of [60], the thesis of Kumar.
We consider Thomason-Trobaugh's algebraic K-theory sheaf $\mathcal{K}$ of [95]. In loc. cit. it is shown that this is a Nisnevich sheaf on Sm_k and hence defines a space. Note that by results of Borelli [22] regular separated Noetherian schemes have an ample family of line bundles, so that Thomason-Trobaugh K-theory agrees with Quillen K-theory [95]. In particular, these two definitions of algebraic K-theory agree in our setting. Since we prefer Quillen's setup, we will hence only use Quillen K-theory from now on.
We can assign a basepoint to $\mathcal{K}$ by choosing the trivial vector bundle of rank 0. A fundamental result of Morel and Voevodsky found as Proposition 3.9 from Chapter 4 of [76] now states:

Theorem I.5.19. *For all pointed smooth schemes (X,x), there are natural isomorphisms*

$$\mathrm{Hom}_{H_*(k)}(S^n\wedge(X,x),\mathcal{K})\cong K_n^{\mathrm{Q}}(X)$$

for all non-negative integers n.

This is the representability of algebraic K-theory in the unstable motivic homotopy category.

Remark I.5.20. It is also possible to give a more concrete model for $\mathcal{K}$. Consider the colimit $\mathrm{GL}=\operatorname{colim}\mathrm{GL}_n$, where the diagram for the colimit is given by the block sum $A\mapsto A\oplus 1$. Then the Nisnevich sheaf $\mathbb{Z}\times\mathrm{BGL}$ is a model for $\mathcal{K}$, see Proposition 3.10 from Chapter 4 of loc. cit.

Using the projective bundle formula in algebraic K-theory from [95], it is now possible to obtain the following:

Theorem I.5.21. *There is a natural map $\sigma\colon\mathbb{P}^1\wedge\mathcal{K}\to\mathcal{K}$.*

Proof. Kumar gives a very detailed construction in his thesis [60, Corollary 2.1.11], from which we get the desired map via the hom-tensor adjunction. □

The result presented in Kumar's thesis can be seen as a motivic Bott periodicity result. As for usual K-theory, this can be used to define the desired K-theory spectrum. As we never talked about the notion of Ω-spectra, let us nevertheless use the adjoint map as in the above theorem.

Definition I.5.22. The (motivic) algebraic K-theory spectrum KGL is the sequence of pointed spaces $(\mathcal{K}, \mathcal{K}, \mathcal{K}, \dots)$ together the map $\sigma\colon \mathbb{P}^1 \wedge \mathcal{K} \to \mathcal{K}$ as structure map on all levels.

Note that while this construction gives a model for the motivic algebraic K-theory spectrum, it does not directly yield an E_∞-structure, i.e. a homotopy coherent multiplication. For our purposes this model is enough though.

I.5.3 More on $\mathrm{SH}(k)$

Here we introduce various operations of motivic spectra and study their properties on the level of the stable motivic homotopy category $\mathrm{SH}(k)$. As before, all of this is completely analogous to the classical stable homotopy theory.

Definition I.5.23. The wedge sum $E \vee E'$ of two motivic spectra E and E' with structure maps $(\sigma_n)_{n\geq 0}$ and $(\sigma'_n)_{n\geq 0}$ respectively is the motivic spectrum $(E_0 \vee E'_0, E_1 \vee E'_1, E_2 \vee E'_2 \dots)$ with structure maps

$$\mathbb{P}^1 \wedge (E_n \vee E'_n) \cong (\mathbb{P}^1 \wedge E_n) \vee (\mathbb{P}^1 \wedge E'_n) \xrightarrow{\sigma_n \vee \sigma'_n} E_{n+1} \vee E'_{n+1}$$

for all non-negative integers n.

So this works exactly as for spaces. In Section I.4.2 we have also introduced smash products of pointed spaces. It is certainly not to difficult to define a mixed smash product:

Definition I.5.24. The smash product $E \wedge (X, x)$ of a motivic spectrum E with a pointed space (X, x) is the motivic spectrum

$$(E_0 \wedge (X, x), E_1 \wedge (X, x), E_2 \wedge (X, x), \dots)$$

with structure maps $\sigma_n \wedge \mathrm{id}_{(X,x)}$, where the σ_n are the structure maps of E, for all non-negative integers n.

These two operations on motivic spectra are clearly functorial.

Theorem I.5.25. *The category* $\mathrm{SH}(k)$ *is a triangulated category with shift functor* $- \wedge S^1$.

Proof. See for instance Theorem 3.10 of [84]. □

As for (non-motivic) spectra, it is not possible to define a well-defined smash product on $\mathrm{Sp}(k)$ for our model of motivic spectra, i.e. for the sequential approach to (motivic) spectra. It does nevertheless work on the level of the stable homotopy category, see Theorem 5.6 of [98] and the discussion beneath it:

Theorem I.5.26. *The category* $\mathrm{SH}(k)$ *has a symmetrical monoidal structure "*$\wedge$*" with unit* $\mathbb{1}_k$ *satisfying:*

(i) *For any motivic spectrum E and any pointed space X, the motivic spectrum $E \wedge \Sigma^\infty X$ is canonically isomorphic to the motivic spectrum $E \wedge X$.*

(ii) *For any motivic spectrum E and any family of motivic spectra E_i, $i \in I$, the motivic spectrum $(\bigoplus_{i\in I} E_i) \wedge E$ is canonically isomorphic to the motivic spectrum $\bigoplus_{i\in I}(E_i \wedge E)$.*

Via this theorem, the above shift functor can also be seen as $- \wedge \mathbb{1}_k^{1,0}$. This means that smashing with $\mathbb{1}_k^{1,0}$ is invertible in $\mathrm{SH}(k)$. This also leads to $\mathbb{1}_k^{n,m}$ being $\wedge$-invertible for all $n \geq m \geq 0$. These inverses can also be defined quite explicitly.

Example I.5.27. We first define a motivic spectrum $\mathbb{1}_k^{-2n,-n}$ for $n \geq 0$ as follows. It is the motivic spectrum

$$(\mathrm{pt}, \dots, \mathrm{pt}, S^0, \mathbb{P}^1, (\mathbb{P}^1)^{\wedge 2}, (\mathbb{P}^1)^{\wedge 3}, \dots)$$

with S^0 being at the n-th entry. The structure maps are the same as for the motivic sphere spectrum together with the canonical map for the first $n-1$ entries. In other words, it is a shifted motivic sphere spectrum.
Now let n and m are arbitrary integers. Since the two tuples $(-2,-1)$ and $(1,1)$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, there exist integers a and b with $a(-2,-1) + b(1,1) = (n,m)$. We now define

$$\mathbb{1}_k^{n,m} = (\mathbb{1}_k^{-2,-1})^{\wedge a} \wedge (\mathbb{1}_k^{1,1})^{\wedge b}.$$

For $n \geq m \geq 0$, these are isomorphic in $\mathrm{SH}(k)$ to the suspension spectra from Example I.5.4. As one might suspect by now, the inverse of $- \wedge \mathbb{1}_k^{n,m}$ is then given by $- \wedge \mathbb{1}_k^{-n,-m}$. For more details we refer to Wickelgren's lecture notes [106] where this approach is used in classical stable homotopy theory.

Remark I.5.28. Given a motivic spectrum E, these generalized motivic sphere spectra can also be used to define the E-homology and E-cohomology. The (n,m)-th E-homology of a spectrum F is $E_{n,m}(F) = [\mathbb{1}_k^{n,m}, E \wedge F]$ and the (n,m)-th E-cohomology of a spectrum F is $E^{n,m}(F) = [E, \mathbb{1}_k^{n,m} \wedge F]$. If $E = HA$ and $F = \Sigma_+^\infty X$ for some scheme X, then $E^{n,m}(F)$ is called motivic cohomology of X with coefficients in A and is denoted by $H^{n,m}(X; A)$. For more details we refer to page 595 of [98].

Finally, let us very briefly talk about ring and module spectra in this setting.

Definition I.5.29. A motivic ring spectrum is a monoid in the category $\mathrm{SH}(k)$.

In other words, it is a motivic spectrum R equipped with a multiplication map $\mu\colon R \wedge R \to R$ and a unit map $\eta\colon \mathbb{1}_k \to R$, such that μ is associative and such that η is unital. The reason for choosing monoid objects is the following. Spectra behave in many ways similarly to abelian groups and rings are monoid objects in the category of abelian groups.

Example I.5.30. The motivic sphere spectrum is a motivic ring spectrum. Here the multiplication is the isomorphism $\mathbb{1}_k \wedge \mathbb{1}_k \to \mathbb{1}_k$ coming from $\mathbb{1}_k$ being the unit for the smash product and the unit map is given by the identity $\mathrm{id}\colon \mathbb{1}_k \to \mathbb{1}_k$.

If we have rings, we can also talk about modules.

Definition I.5.31. A module over a motivic ring spectrum R is a module object $M \in \mathrm{SH}(k)$ over E.

Also here this condition can be spelled out similarly as for ring spectra. Such a module M is a motivic spectrum together with a scalar multiplication/action $\rho\colon R \wedge M \to M$ satisfying associativity and unitality. The prime example for modules over motivic ring spectra and our reason for introducing these notions is the following.

Example I.5.32. Every motivic spectrum E is a module over the motivic sphere spectrum $\mathbb{1}_k$. Here the scalar multiplication is once again the isomorphism $\mathbb{1}_k \wedge E \to E$ coming from the motivic sphere spectrum being the unit for the smash product.

It is not difficult to observe that this example can be upgrades to an equivalence of categories $F\colon \operatorname{Mod}_{\mathbb{1}_k} \to \operatorname{SH}(k)$, where F is the forgetful functor. This can even be upgraded further into a highly coherent setting; see [37] and [38] for the corresponding statements in the topological setting.

I.6 Homotopy Modules

In this section we want to introduce homotopy modules, which arise as a certain abelian subcategory of the motivic stable homotopy category $\mathrm{SH}(k)$. These will be the central objects which we will deal with in Chapter II.

I.6.1 The homotopy t-structure

Let us first talk about t-structures in general. The idea here is the following. Given an abelian category $\mathcal{A}$, we can pass to its derived category $D(\mathcal{A})$, which is a triangulated category that still contains $\mathcal{A}$ as the subcategory of complexes concentrated in degree 0. Introduced by Beĭlinson, Bernstein and Deligne in their study of perverse sheaves [20], a t-structure on a triangulated category now allows to do this in general, i.e. to find an abelian subcategory defined by objects "concentrated in degree 0" within a given triangulated category. We must of course specify which kind of notion of degree we are talking about here.

Definition I.6.1. A (homological) t-structure on a triangulated category $\mathcal{T}$ consists of two full subcategories $\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0} \subset \mathcal{T}$ subject to the following axioms:

(i) For all $X \in \mathcal{T}_{\geq 0}$ and $Y \in \mathcal{T}_{\leq 0}[-1]$ we have $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$;

(ii) There are inclusions $\mathcal{T}_{\geq 0}[1] \subset \mathcal{T}_{\geq 0}$ and $\mathcal{T}_{\leq 0} \subset \mathcal{T}_{\leq 0}[1]$;

(iii) For any $T \in \mathcal{T}$ there is a distinguished triangle

$$X \longrightarrow T \longrightarrow Y \longrightarrow X[1]$$

with $X \in \mathcal{T}_{\geq 0}$ and $Y \in \mathcal{T}_{\leq 0}[-1]$.

As the name suggests, there is also a notion of cohomological t-structures which we will not consider here. One usually denotes $\mathcal{T}_{\leq 0}[n]$ by $\mathcal{T}_{\leq n}$ and $\mathcal{T}_{\geq 0}[n]$ by $\mathcal{T}_{\geq n}$. Given a t-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$ on a triangulated category $\mathcal{T}$, the full subcategory $\mathcal{T}^{\heartsuit} = \mathcal{T}_{\leq 0} \cap \mathcal{T}_{\geq 0}$ of $\mathcal{T}$ is called the heart of the t-structure. Furthermore, the inclusion functors $\mathcal{T}_{\leq n} \to \mathcal{T}$ have right adjoints $\tau_{\leq n}$ and the inclusion functors $\mathcal{T}_{\geq n} \to \mathcal{T}$ have left adjoints $\tau_{\geq n}$, which are called truncation functors, see Proposition 1.3.3 of loc. cit. The proof of this proposition in particular shows the following statement, which we wish to record separately.

Lemma I.6.2. *Let $\mathcal{T}$ be a triangulated category with t-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$. There is a natural bijection $\mathrm{Hom}_{\mathcal{T}}(\tau_{\leq 0} X, A) \to \mathrm{Hom}_{\mathcal{T}}(X, A)$ for objects $X \in \mathcal{T}_{\geq 0}$ and $A \in \mathcal{T}^{\heartsuit}$.*

This innocent looking lemma turns out to be quite useful. Before we finally get to some examples, let us quickly state the following:

Theorem I.6.3. *The heart $\mathcal{T}^{\heartsuit}$ of a t-structure on a triangulated category $\mathcal{T}$ is an abelian category.*

Proof. This is Theorem 1.3.6 of [20]. □

So the category given by the intersection of the objects of non-positive and non-negative degree, i.e. the objects of degree 0, is indeed abelian as mentioned in the introduction.

Remark I.6.4. In search of the conjectural abelian category of mixed motives $\mathrm{MM}(k)$, one hope is to find a suitable t-structure on Voevodsky's derived category of motives $\mathrm{DM}(k)$, which then on the level of its heart would yield a model for $\mathrm{MM}(k)$.

Let us now discuss two standard examples of t-structure, one of which we already mentioned in the introduction:

Example I.6.5. Let $\mathcal{A}$ be an abelian category with associated derived category $D(\mathcal{A})$. Then the pair of full subcategories

$$D(\mathcal{A})_{\geq 0} = \{X \in D(\mathcal{A}) \mid H_i(X) = 0 \text{ for all } i < 0\}$$

and

$$D(\mathcal{A})_{\leq 0} = \{X \in D(\mathcal{A}) \mid H_i(X) = 0 \text{ for all } i > 0\}$$

define a t-structure on $D(\mathcal{A})$, see Example 1.3.2 from [20]. Here our notion of degree is hence the homological degree. In this case we have

$$D(\mathcal{A})_{\geq n} = \{X \in D(\mathcal{A}) \mid H_i(X) = 0 \text{ for all } i < n\}$$

and

$$D(\mathcal{A})_{\leq n} = \{X \in D(\mathcal{A}) \mid H_i(X) = 0 \text{ for all } i > n\},$$

and the heart of the t-structure is isomorphic to the abelian category $\mathcal{A}$, which we hence managed to reconstruct from $D(\mathcal{A})$.
Since this is the guiding example, let us also have a look at the truncation functors. Given a chain complex $C_\bullet \in D(\mathcal{A})$, the truncation $\tau_{\leq n} C_\bullet$ is the chain complex

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{coker}(\partial_{n+1}) \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \ldots$$

with $\mathrm{coker}(\partial_{n+1})$ in degree n. In particular, the chain complex $\tau_{\leq 0} C_\bullet$ is the homology group $H_0(C_\bullet)$ considered as a chain complex in degree 0 for $C \in D(\mathcal{A})_{\geq 0}$. The truncation $\tau_{\geq n} C_\bullet$ is the chain complex

$$\ldots \longrightarrow C_{n+2} \longrightarrow C_{n+1} \longrightarrow \ker(\partial_n) \longrightarrow 0, \longrightarrow 0 \longrightarrow \ldots$$

where once again $\ker(\partial_n)$ is in degree 0.

For those readers familiar with stable homotopy theory we also have the following classical example:

Example I.6.6. The two full subcategories

$$\mathrm{SH}_{\geq 0} = \{E \in \mathrm{SH} \mid \pi_i(E) = 0 \text{ for all } i < 0\}$$

and

$$\mathrm{SH}_{\leq 0} = \{E \in \mathrm{SH} \mid \pi_i(E) = 0 \text{ for all } i > 0\}$$

given by connective and coconnective spectra define a t-structure on the stable homotopy category SH, known as the Postnikov t-structure. The give rise to Postnikov towers in stable homotopy theory. For references see Theorem 5.25 together with Theorem 8.3 of the second chapter of [87]. Here the heart $\mathrm{SH}^\heartsuit$ consists exactly of the Eilenberg-Mac Lane spectra HA for abelian groups A, which yields an equivalence of categories $\pi_0\colon \mathrm{SH}^\heartsuit \to \mathrm{Ab}$ with inverse $H\colon \mathrm{Ab} \to \mathrm{SH}^\heartsuit$.

By Remark 5.3.2 of Morel's seminal paper [72] we now have an analogous t-structure for motivic spectra, where we still use the simplicial degree.

Theorem I.6.7. *The two full subcategories*

$$\mathrm{SH}(k)_{\geq 0} = \{E \in \mathrm{SH}(k) \mid \underline{\pi}_n(E)_* = 0 \text{ for all } n < 0\}$$

and

$$\mathrm{SH}(k)_{\leq 0} = \{E \in \mathrm{SH}(k) \mid \underline{\pi}_n(E)_* = 0 \text{ for all } n > 0\}$$

define a t-structure on the motivic stable homotopy category $\mathrm{SH}(k)$*.*

This is the homotopy t-structure. We now want to have a more algebraic description of its heart

$$\mathrm{SH}(k)^{\heartsuit} = \{E \in \mathrm{SH}(k) \mid \underline{\pi}_n(E)_* = 0 \text{ for all } n \neq 0\}$$

similarly to describing the heart of the Postnikov t-structure in terms of abelian groups. For this we introduce $\mathbb{A}^1$-invariance properties following [76], [99] [100] and [75]:

Definition I.6.8. A sheaf of

(i) sets on Sm_k is called $\mathbb{A}^1$-invariant if for any $X \in \mathrm{Sm}_k$, the morphism

$$H^0_{\mathrm{Nis}}(X, \mathcal{F}) \to H^0_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F})$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is a bijection.

(ii) groups on Sm_k is called strongly $\mathbb{A}^1$-invariant if for any $X \in \mathrm{Sm}_k$, the morphism

$$H^i_{\mathrm{Nis}}(X, \mathcal{F}) \to H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F})$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is a bijection for $i = 0$ and $i = 1$.

(iii) abelian groups on Sm_k is called strictly $\mathbb{A}^1$-invariant if for any $X \in \mathrm{Sm}_k$, the morphism

$$H^i_{\mathrm{Nis}}(X, \mathcal{F}) \to H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F})$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is a bijection for all $i \in \mathbb{N}$.

We are now only interested in abelian sheaves so that all of these notions make sense. In this case we do actually not need to distinguish the two latter ones as the following crucial result of Morel [75, Corollary 5.45] shows.

Theorem I.6.9. *Every strongly $\mathbb{A}^1$-invariant abelian sheaf is strictly $\mathbb{A}^1$-invariant.*

We now denote the full subcategory of Ab/k given by strictly $\mathbb{A}^1$-invariant abelian sheaves by $\mathrm{Ab}_{\mathbb{A}^1}/k$ and give a couple of examples.

Example I.6.10. There are various ways of verifying that $\underline{\mathbb{Z}}$ is strictly $\mathbb{A}^1$-invariant. As a constant sheaf, it is certainly $\mathbb{A}^1$-invariant. Additionally, $\underline{\mathbb{Z}}$ is an example of a so-called sheaf with transfers, which are additive presheaves on finite correspondences (Section I.5.2), whose restriction along the inclusion $\mathrm{Sm}_k \hookrightarrow \mathrm{Cor}_k$ is a sheaf. Indeed, if X is a smooth scheme, we have by definition

$$\mathrm{Cor}_k(X, \mathrm{Spec}(k)) = \bigoplus_{i=1}^{r} \mathbb{Z},$$

where r is the number of connected components of X. Thus the presheaf with transfers $\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}(k))$ is nothing but the constant sheaf with value $\underline{\mathbb{Z}}$ and in particular a sheaf with transfers. In Theorem 5.6 of [99] and Theorem 3.1.12 [100] Voevodsky shows that $\mathbb{A}^1$-invariant sheaves with transfers are strictly $\mathbb{A}^1$-invariant, which hence gives us that $\underline{\mathbb{Z}}$ is strictly $\mathbb{A}^1$-invariant.

Example I.6.11. The abelian sheaf $\mathbb{G}_m$ is strongly and hence also strictly $\mathbb{A}^1$-invariant. Indeed, $H^0_{\mathrm{Nis}}(-, \mathbb{G}_m) = \mathbb{G}_m$ is $\mathbb{A}^1$-invariant. Furthermore we have a canonical isomorphism $H^1_{\mathrm{Nis}}(-, \mathbb{G}_m) \cong \mathrm{Pic}(-)$, see [89, Tag 040D], and the Picard group is known to be $\mathbb{A}^1$-invariant for normal and thus also for smooth schemes, see Chapter II Prop. 6.6 of [47].

Example I.6.12. By Corollary 6.2 of [75], the higher $\mathbb{A}^1$-homotopy sheaves $\underline{\pi}_n^{\mathbb{A}^1}$ are strictly $\mathbb{A}^1$-invariant. Here "higher" means $n \geq 2$ as usual. Moreover, if the $\mathbb{A}^1$-fundamental sheaf $\underline{\pi}_1^{\mathbb{A}^1}$ happens to be abelian, then it is also strictly $\mathbb{A}^1$-invariant by Theorem 6.1 of loc. cit. together with Theorem I.6.9.

Remark I.6.13. One might also wonder which kind of $\mathbb{A}^1$-invariance holds for the sheaf $\underline{\pi}_0^{\mathbb{A}^1}$ of $\mathbb{A}^1$-connected components. Morel's conjecture on $\underline{\pi}_0^{\mathbb{A}^1}$ stated as Conjecture 1.12 in [75] predicted it to be $\mathbb{A}^1$-invariant, but it turns out not to be $\mathbb{A}^1$-invariant in general. This is a recent result of Ayoub [16].

The category $\mathrm{Ab}_{\mathbb{A}^1}/k$ ha a symmetric monoidal structure, see Lemma 6.2.13 of [74] satisfying the usual properties. From now on, whenever we consider tensor products of strictly $\mathbb{A}^1$-invariant sheaves, it will be with respect to this symmetric monoidal structure.
The category $\mathrm{Ab}_{\mathbb{A}^1}/k$ also turns out to be abelian. To understand why this is the case we first consider so-called $\mathbb{A}^1$-local chain complexes following Chapter 6.2 of [75]. Here we make use of the fact that abelian sheaves form an abelian category so that we can consider the associated derived category. In there we can consider any abelian sheaf as a complex concentrated in degree 0, which includes free abelian sheaves $\underline{\mathbb{Z}}[X]$ given by applying to some sheaf X the left adjoint of the inclusion functor $\mathrm{Ab}/k \hookrightarrow \mathrm{Set}/k$.

Definition I.6.14. A chain complex $M_\bullet \in D(\mathrm{Ab}/k)$ is called $\mathbb{A}^1$-local if for all chain complexes $C_\bullet \in \mathrm{Ch}_\bullet(\mathrm{Ab}/k)$, the projection $C_\bullet \otimes \underline{\mathbb{Z}}[\mathbb{A}^1] \to C_\bullet \otimes \underline{\mathbb{Z}} \cong C_\bullet$ induces a bijection $\mathrm{Hom}_{D(\mathrm{Ab}/k)}(C_\bullet, M_\bullet) \to \mathrm{Hom}_{D(\mathrm{Ab}/k)}(C_\bullet \otimes \underline{\mathbb{Z}}[\mathbb{A}^1], M_\bullet)$.

Here the tensor product "$\otimes$" is the sheafification of the tensor product on the level of presheaves. Before we get to an example, let us give a criterion for a chain complex to be $\mathbb{A}^1$-local.

Proposition I.6.15. *A chain complex $M_\bullet \in D(\mathrm{Ab}/k)$ is $\mathbb{A}^1$-local if and only if all its homology sheaves are strictly $\mathbb{A}^1$-invariant.*

Proof. This follows from Theorem I.6.18 as observed in Corollary 6.23 of [75]. □

Example I.6.16. Let $M \in \mathrm{Ab}/k$, considered as a chain complex concentrated in degree 0. Since its homology sheaves are given by

$$H_n(M) = \begin{cases} M & \text{if } n = 0 \\ \underline{0} & \text{else,} \end{cases}$$

we have that the chain complex A is $\mathbb{A}^1$-local if and only if the abelian sheaf M is strictly $\mathbb{A}^1$-invariant.

We denote by $D_{\mathbb{A}^1}(\mathrm{Ab}/k)$ the full subcategory of $D(\mathrm{Ab}/k)$ given by $\mathbb{A}^1$-local chain complexes. As the name suggests, this category is indeed a localization:

Proposition I.6.17. *There is an $\mathbb{A}^1$-localization functor*

$$L_{\mathbb{A}^1} : D(\mathrm{Ab}/k) \to D_{\mathbb{A}^1}(\mathrm{Ab}/k)$$

turning $D_{\mathbb{A}^1}(\mathrm{Ab}/k)$ into a reflective subcategory, i.e $L_{\mathbb{A}^1}$ is left adjoint to the inclusion functor $D_{\mathbb{A}^1}(\mathrm{Ab}/k) \hookrightarrow D(\mathrm{Ab}/k)$.

Proof. This is Corollary 6.19 of [75]. □

Recall that if a fully faithful functor R has a left adjoint L, the counit of the adjunction $\epsilon\colon L \circ R \to \mathrm{id}$ is a natural isomorphism. Therefore, if $M_\bullet$ is an $\mathbb{A}^1$-local chain complex, we have $L_{\mathbb{A}^1} M_\bullet \cong M_\bullet$. Furthermore, Morel [75, Theorem 6.22] shows:

Theorem I.6.18 ($\mathbb{A}^1$-connectivity Theorem in $D(\mathrm{Ab}/k)$)**.** *If a chain complex $M_\bullet \in D(\mathrm{Ab}/k)$ is (-1)-connected, i.e. $M_n = 0$ for all negative integers n, then so is $L_{\mathbb{A}^1} M_\bullet$.*

In loc. cit. Morel immediately concludes:

Corollary I.6.19. *There is a t-structure on $D(\mathrm{Ab}_{\mathbb{A}^1}/k)$ whose heart is exactly the category $\mathrm{Ab}_{\mathbb{A}^1}/k$ of strictly $\mathbb{A}^1$-invariant abelian sheaves. In particular, the category $\mathrm{Ab}_{\mathbb{A}^1}/k$ is abelian.*

Proof. By the previous theorem, the standard t-structure on $D(\mathrm{Ab}/k)$ descends to a t-structure on $D(\mathrm{Ab}_{\mathbb{A}^1}/k)$. Now Proposition I.6.15 shows that its heart is $\mathrm{Ab}_{\mathbb{A}^1}/k$, which by Theorem I.6.3 in particular implies that the category of strictly $\mathbb{A}^1$-invariant abelian sheaves is abelian. □

Let us now also give another central kind of examples of strictly $\mathbb{A}^1$-invariant abelian sheaves.

Example I.6.20. As already used above, there is a free abelian sheaf functor $\underline{\mathbb{Z}}[-]\colon \mathrm{Set}/k \to \mathrm{Ab}/k$ defined as the left adjoint of the inclusion functor $\mathrm{Ab}/k \hookrightarrow \mathrm{Set}/k$. Now Proposition I.6.17 together with Theorem I.6.18 also yields a left adjoint for the inclusion $\mathrm{Ab}_{\mathbb{A}^1}/k \hookrightarrow \mathrm{Ab}/k$. Therefore we can compose these two left adjoints and get a free strictly $\mathbb{A}^1$-invariant sheaf functor $\underline{\mathbb{Z}}_{\mathbb{A}^1}[-]\colon \mathrm{Set}/k \to \mathrm{Ab}_{\mathbb{A}^1}/k$, which of course is once again the left adjoint of the inclusion $\mathrm{Ab}_{\mathbb{A}^1}/k \hookrightarrow \mathrm{Set}/k$.

Example I.6.21. There is also a reduced version $\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[-]$ of the previous example. It is defined by mapping (X, x) to the quotient $\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]/\underline{\mathbb{Z}}_{\mathbb{A}^1}[x] = \underline{\mathbb{Z}}_{\mathbb{A}^1}[X]/\underline{\mathbb{Z}}[x]$ and yields a left adjoint $L_{\mathbb{A}^1}$ to the inclusion $\mathrm{Ab}_{\mathbb{A}^1}/k \hookrightarrow \mathrm{Set}_*/k$. It is compatible with the $\mathbb{A}^1$-localization functor from Proposition I.6.17 via Corollary I.6.19.

Example I.6.22. Since we clearly have an identification

$$\mathrm{Hom}_{\mathrm{Ab}/k}(\underline{\mathbb{Z}}, M) = \mathrm{Hom}_{\mathrm{Ab}/k}(\underline{\mathbb{Z}}[\mathrm{Spec}(k)], M) = \mathrm{Hom}_{\mathrm{Set}/k}(\mathrm{Spec}(k), M)$$

for all strictly $\mathbb{A}^1$-invariant abelian sheaves M, and since we know that $\underline{\mathbb{Z}}$ is strongly $\mathbb{A}^1$-invariant, $\underline{\mathbb{Z}}$ must be the free strictly $\mathbb{A}^1$-invariant sheaf on $\mathrm{Spec}(k)$. In particular, we have $\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[S^0] = \underline{\mathbb{Z}}$.

In the next subsection we will actually see how the sheaf $\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_m^{\wedge n}]$ can be understood more concretely for $n \geq 1$.

Remark I.6.23. Note that by the definition of the tensor product from Lemma 6.2.13 of [74] we have

$$\underline{\mathbb{Z}}_{\mathbb{A}^1}[X \times Y] \cong \underline{\mathbb{Z}}_{\mathbb{A}^1}[X] \otimes \underline{\mathbb{Z}}_{\mathbb{A}^1}[Y]$$

for all sheaves X and Y, that is, the this property of free abelian sheaves lifts to the $\mathbb{A}^1$-invariant setting. Furthermore, by the relations on top of page 19 of [35] together with the definition of $\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[-]$ we have

$$\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[X \wedge Y] \cong \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[(X, x)] \otimes \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[(Y, y)]$$

for all pointed sheaves (X, x) and (Y, y).

A last ingredient we need for our algebraic description is the following construction of Voevodsky [99]:

Definition I.6.24. Let $M \in \mathrm{Ab}/k$. The abelian sheaf $M_{-1} = \ker(M(- \times \mathbb{G}_m) \to M)$, where the morphism is induced by the inclusion of the first factor, is called the (homotopy) contraction of M.

Example I.6.25. If $\underline{A}$ is the constant abelian sheaf with values in an abelian group A, then we have $\underline{A}_{-1} = \underline{0}$ by definition.

Example I.6.26. We have $(\mathbb{G}_m)_{-1} = \underline{\mathbb{Z}}$. While this can be seen more directly, this also follows from the fact that Milnor K-theory is a homotopy module, see Example I.6.43, with $\underline{\mathrm{K}}_1^{\mathrm{M}} = \mathbb{G}_{\mathrm{m}}$ and $\underline{\mathrm{K}}_0^{\mathrm{M}} = \underline{\mathbb{Z}}$.

By its definition, the contraction M_{-1} of an abelian sheaf M fits into the short exact sequence

$$0 \longrightarrow M_{-1} \longrightarrow M(- \times \mathbb{G}_m) \xrightarrow{i_1^*} M \longrightarrow 0,$$

which is a quite beneficial point a view. The morphism i_1^* used to define contractions is often called the evaluation at $1 \in \mathbb{G}_m$. This is based on:

Proposition I.6.27. *For all smooth schemes X and all $M \in \mathrm{Ab}_{\mathbb{A}^1}/k$, the two abelian sheaves $M(- \times X)$ and $M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]}$ are canonically isomorphic.*

Proof. Via the free-forgetful adjunction, the Yoneda lemma gives us a natural identification $M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]} = \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[-], M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]})$. Using the hom-tensor adjunction, the latter sheaf is isomorphic to $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[-] \otimes \underline{\mathbb{Z}}_{\mathbb{A}^1}[X], M) \cong \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[- \times X], M)$, where we also make use of Remark I.6.23. Therefore, using the free-forgetful adjunction and the Yoneda lemma once more, we thus have a canonical isomorphism $M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]} \xrightarrow{\cong} M(- \times X)$ as claimed. □

If one keeps track of all the isomorphisms leading to the isomorphism $M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]} \xrightarrow{\cong} M(- \times X)$ that we have just constructed, it is not difficult to see that the morphism $i_1^* \colon M(-\times \mathbb{G}_m) \to M$ becomes exactly $\mathrm{ev}_1 \colon M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_m]} \to M$, the evaluation at $1 \in \mathbb{G}_m$.

Corollary I.6.28. *For every $M \in \mathrm{Ab}_{\mathbb{A}^1}/k$, the two abelian sheaves $M^{\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_m]}$ and M_{-1} are canonically isomorphic.*

Proof. The short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_{\mathbb{A}^1}[\{1\}] \longrightarrow \underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_m] \longrightarrow \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_m] \longrightarrow 0$$

of abelian sheaves is split. Indeed, the morphism $\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_m] \to \underline{\mathbb{Z}}_{\mathbb{A}^1}[\{1\}] = \underline{\mathbb{Z}}$ induced by the constant map $\mathbb{G}_m \to \{1\}$ is a retraction. Therefore $M^{(-)} = \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(-, M)$ preserves the exactness of the above sequence, which yields the short exact sequence

$$0 \longrightarrow M^{\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_m]} \longrightarrow M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_m]} \longrightarrow M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[\{1\}]} \longrightarrow 0.$$

Under the canonical isomorphism $M^{\underline{\mathbb{Z}}_{\mathbb{A}^1}[\{1\}]} \xrightarrow{\cong} M$, the map on the right is the evaluation at $1 \in \mathbb{G}_m$, which gives us $M^{\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_m]} = \ker(\mathrm{ev}_1) = M_{-1}$. □

From now on, we will use this as the definition of contractions. Furthermore, we denote by M_{-j} the sheaf recursively defined by $(M_{-(j-1)})_{-1}$.
Let us also state where the homotopy contractions show up naturally, which also clarifies in which way they are related to homotopy theory as the name indicates. They are an algebraic incarnation of derived $\mathbb{G}_m$-loop spaces, see Theorem 6.13 of [75]:

Theorem I.6.29. *For all pointed connected $\mathbb{A}^1$-local spaces X and all integers $j \geq 0$ and $n \geq 1$, the sheafification of the presheaf $U \mapsto [S^n \wedge \mathbb{G}_m^{\wedge j} \wedge U_+, X]$ on Sm/k is canonically isomorphic to $\pi_n^{\mathbb{A}^1}(X)_{-j}$.*

In other words, homotopy contractions allow us to express homotopy sheaves with respect to general motivic spheres in terms of homotopy sheaves with respect to simplicial spheres. This is what we mentioned in Remark I.4.30.
Now that we have everything we need, lets us finally define homotopy modules.

Definition I.6.30. A homotopy module (M_*, ϵ_*) is a sheaf $M_* \in (\mathrm{Ab}_{\mathbb{A}^1}/k)^{\mathbb{Z}}$ together with isomorphisms $\epsilon_n \colon M_n \to (M_{n+1})_{-1}$ for all $n \in \mathbb{Z}$. If it additionally has the structure of a graded ring when applied to fields, we call it a homotopy algebra.

So for homotopy modules the contraction is nothing but a shift of degree. There is an obvious notion of morphisms of homotopy modules, which yields the category $\Pi_*(k)$ of homotopy modules. We will however usually drop the isomorphisms ϵ_n from the notation and just say that M_* is a homotopy module.

Theorem I.6.31. *The restriction of the functor*

$$\underline{\pi}_0(-)_* \colon \mathrm{SH}(k) \to \Pi_*(k), E \mapsto \underline{\pi}_0(E)_* = \bigoplus_{m \in \mathbb{Z}} \underline{\pi}_{-m,-m}(E)$$

to the heart of the homotopy t-structure $\mathrm{SH}(k)^{\heartsuit}$ defines an equivalence of categories.

Proof. This is Theorem 5.2.6 of [71]. □

The inverse is an Eilenberg-Mac Lane spectrum construction we hinted at in Remark I.5.14. As a consequence, the category $\Pi_*(k)$ is an abelian category. We say that a homotopy module M_* is associated with a motivic spectrum $E \in \mathrm{SH}(k)$ if $\underline{\pi}_0(E)_* \cong M_*$. Note that we do not demand that E lies in $\mathrm{SH}(k)^{\heartsuit}$.
According to [40] or to Chapter 2.3 of [75], one can define homotopy modules as certain functors on the category of finitely generated field extensions of k, which then extend naturally to all smooth schemes. To make this work, quite some extra data needs to be specified. This includes so-called residue and specialization maps, which we will see a concrete example of in the next subsection. We are namely going to study the homotopy module associated with the motivic sphere spectrum $\mathbb{1}_k$, but from a very hands on perspective.

I.6.2 Milnor-Witt K-theory and further Homotopy Modules

Following Section 3 of our preprint [108], we quickly recall some basics of Milnor-Witt K-theory. This invariant is a prime example of a homotopy module and also gives rise to further examples. For more details we refer to Chapters 3.1 and 3.2 of Morel's book [75].

Definition I.6.32. The Milnor-Witt K-theory ring $K_*^{\mathrm{MW}}(F)$ of a field F is the free unital $\mathbb{Z}$-graded ring generated by symbols $[a]$ of degree 1 for all $a \in F^\times$ and a symbol η of degree -1 subject to the following relations:

(MW1) $[a][1-a] = 0$ for all $a \in F^\times \setminus \{1\}$ (Steinberg relation);

(MW2) $[ab] = [a] + [b] + \eta[a][b]$ for all $a, b \in F^\times$ (Twisted tensor relation);

(MW3) $\eta[a] = [a]\eta$ for all $a \in F^\times$ (Centrality relation);

(MW4) $\eta(2 + \eta[-1]) = 0$ (Witt relation).

In particular, an element of degree n in $K_*^{\mathrm{MW}}(F)$ is given by a $\mathbb{Z}$-linear combination of elements of the form $\eta^d[a_1, \dots, a_r]$ with $r - d = n$, where we denote the product $[a_1] \cdot \dots \cdot [a_r]$ by $[a_1, \dots, a_r]$. Furthermore we set $\langle a \rangle = 1 + \eta[a] \in K_0^{\mathrm{MW}}(F)$ for all $a \in F^\times$, $h = \langle 1 \rangle + \langle -1 \rangle$ and $\epsilon = -\langle -1 \rangle$. A summary of the most essential relations is:

Lemma I.6.33. *We have the following:*

(i) $0 = [1]$ *and* $1 = \langle 1 \rangle$ *as elements of* $K_*^{\mathrm{MW}}(F)$. *In particular,* $h = 2 + \eta[-1]$ *and the relation* $\eta(2 + \eta[-1]) = 0$ *can be rewritten as* $\eta h = 0$.

(ii) $[a, -a] = 0 = [-a, a]$ *for all* $a \in F^\times$.

(iii) $[a, -1] = [a, a] = [-1, a]$ *for all* $a \in F^\times$. *In particular* $\langle a \rangle [a] = \langle -1 \rangle [a]$ *for all* $a \in F^\times$.

(iv) $[ab^{-1}] = [a] - [b]\langle ab^{-1} \rangle$ *for all* $a, b \in F^\times$.

(v) $[a^n] = \sum_{i=0}^{n-1} \langle (-1)^i \rangle [a]$ *for all positive* n *and* $[a^n] = \epsilon \sum_{i=0}^{-(n-1)} \langle (-1)^i \rangle [a]$ *for all negative* n *and all* $a \in F^\times$. *In particular,* $[a^2] = h[a]$ *for all* $a \in F^\times$.

(vi) $\langle a\rangle\langle b\rangle = \langle ab\rangle$ *for all* $a, b \in F^\times$. *Together with (i) this in particular yields that* $\langle a\rangle$ *is a unit with inverse* $\langle a^{-1}\rangle$ *and that* $\epsilon^2 = 1$.

(vii) $\langle a\rangle^2 = \langle a^2\rangle = 1$ *for all* $a \in F^\times$.

(viii) $\langle a\rangle x = x\langle a\rangle$ *for all* $x \in K_*^{\mathrm{MW}}(F)$ *and all* $a \in F^\times$.

(ix) $\langle a\rangle[b] = [ab] - [a]$ *for all* $a, b \in F^\times$.

(x) $xx' = \epsilon^{nm}x'x$ *for all homogeneous elements* $x, x' \in K_*^{\mathrm{MW}}(F)$ *of degrees* n *and* m *respectively, i.e. the ring* $K_*^{\mathrm{MW}}(F)$ *is* ϵ*-graded commutative.*

Proof. These are Lemma 3.5, Lemma 3.7,Corollary 3.8 and Lemma 3.14 of [75]. □

All these relations will be used freely in all of our computations. Therefore we certainly want to encourage the reader to check this list of relations in case that some computation is unclear.

Lemma I.6.34. *For all* $n \geq 1$*, the abelian group* $K_n^{\mathrm{MW}}(F)$ *is generated by elements of the form* $[a_1, \ldots, a_n]$ *with* $a_1, \ldots, a_n \in F^\times$ *and for all* $n \leq 0$*, the abelian group* $K_n^{\mathrm{MW}}(F)$ *is generated by elements of the form* $\eta^n\langle a\rangle$ *with* $a \in F^\times$.

Proof. This is Lemma 3.6 of loc. cit. □

We will mostly make use of this statement in the case $n \geq 1$. Here the proof merely consists of getting rid of powers of η in elements of the form $\eta^d[a_1, \ldots, a_{n+d}]$ by using relation (MW2) often enough, thus resulting in the pure (η-free) symbols as generators. Let us note that a list of relations with respect to these generators was computed by Hutchinson-Tao for $n \geq 2$ in [54] and by Tao/Hutchinson-Tao for $n = 1$ in [94] and [55]. We prefer to use the following standard presentation, which one obtains directly from the definition, see also Lemma 3.4 of [75]:

Lemma I.6.35. *For* $n \geq 1$*, the* n*-th Milnor Witt K-theory group* $K_n^{\mathrm{MW}}(F)$ *of* F *is generated by elements of the form* $\eta^d[a_1, \ldots, a_r]$ *with* $d = r - n \geq 0$ *subject to the relations:*

(i) $\eta^d[a_1, \ldots, a_r] = 0$ *whenever* $a_i + a_{i+1} = 1$ *for some* $1 \leq i \leq r - 1$.

(ii)
$$\begin{aligned}\eta^d[a_1, \ldots, a_{i-1}, bb', a_{i+1}, \ldots, a_r] &= \eta^d[a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_r] \\ &+ \eta^d[a_1, \ldots, a_{i-1}, b', a_{i+1}, \ldots, a_r] \\ &+ \eta^{d+1}[a_1, \ldots, a_{i-1}, b, b', a_{i+1}, \ldots, a_r]\end{aligned}$$

(iii) $2\eta^{d+1}[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{r+2}] + \eta^{d+2}[a_1, \ldots, a_{i-1}, -1, a_{i+1}, \ldots, a_{r+2}] = 0$

Milnor-Witt K-theory is functorial with respect to field extensions. This allows us to view both K_*^{MW} and K_n^{MW} for a fixed integer n as Ab- and Set-valued functors on the categories Fld_k of field extensions and the category $\mathrm{Fld}_k^{\mathrm{ftr}}$ of field extensions of our base field k with finite transcendence degree. The more general definition of Milnor-Witt K-theory of smooth schemes requires certain maps. For this suppose that F is a discretely valued field with valuation ν and fixed uniformizing element π. We denote the associated valuation ring by $\mathcal{O}_\nu$ and the residue field by $\kappa(\nu)$.

Theorem I.6.36. *There is exactly one homomorphism*

$$\partial_\nu^\pi : K_*^{\mathrm{MW}}(F) \to K_{*-1}^{\mathrm{MW}}(\kappa(\nu))$$

of graded abelian groups with the following three properties:

(i) $\partial_\nu^\pi(\eta x) = \eta \partial_\nu^\pi(x)$ *for all* $x \in K_*^{\mathrm{MW}}(F)$.

(ii) $\partial_\nu^\pi([\pi, u_2, \ldots, u_n]) = [\overline{u_2}, \ldots, \overline{u_n}]$ *for all* $u_2, \ldots, u_n \in \mathcal{O}_\nu^\times$.

(iii) $\partial_\nu^\pi([u_1, u_2, \ldots, u_n]) = 0$ *for all* $u_1, \ldots, u_n \in \mathcal{O}_\nu^\times$.

Proof. This is Theorem 3.15 of [75]. □

This homomorphism is called residue map and the composition

$$s_\nu^\pi : K_*^{\mathrm{MW}}(F) \xrightarrow{[-\pi]\cdot} K_{*+1}^{\mathrm{MW}}(F) \xrightarrow{\partial_\nu^\pi} K_*^{\mathrm{MW}}(\kappa(\nu)) \xrightarrow{\langle\overline{-1}\rangle\cdot} K_*^{\mathrm{MW}}(\kappa(\nu))$$

is called specialization map and is a homomorphism of graded rings. Let $a \in F^\times$ and write $a = \pi^n u$ for some unit $u \in \mathcal{O}_\nu^\times$. As it turns out, the specialization map can also be defined as the unique homomorphism $K_*^{\mathrm{MW}}(F) \to K_*^{\mathrm{MW}}(\kappa(\nu))$ of graded rings mapping $[\pi^n u]$ to $[\overline{u}]$ and η to η, see page 57 of loc. cit. Some useful relations of these two kinds of maps are:

Proposition I.6.37. *For all* $u \in \mathcal{O}_\nu^\times$ *and all* $x \in K_*^{\mathrm{MW}}(F)$ *we have:*

(i) $\partial_\nu^\pi([u]x) = \epsilon[\overline{u}]\partial_\nu^\pi(x)$ *and* $s_\nu^\pi([u]x) = [\overline{u}]s_\nu^\pi(x)$.

(ii) $\partial_\nu^\pi(\langle u\rangle x) = \langle\overline{u}\rangle\partial_\nu^\pi(x)$ *and* $s_\nu^\pi(\langle u\rangle x) = \langle\overline{u}\rangle s_\nu^\pi(x)$.

(iii) $\partial_\nu^{u\pi}(x) = \langle\overline{u}\rangle\partial_\nu^\pi(x)$ *and* $s_\nu^{u\pi}(x) = s_\nu^\pi(x) + \epsilon[\overline{u}]\partial_\nu^\pi(x)$. *In particular, both the residue map and the specialization map do generally depend on the choice of the uniformizing element* π.

Proof. The first two relations for residue maps are Proposition 3.17 in [75] and the corresponding ones for the specialization maps follow immediately from the definition and the respective relations for the residue map. This clarifies (i) and (ii). The first formula of (iii) is Remark 1.9 in [39]. We will quickly prove the second one. Let $u \in \mathcal{O}_\nu^\times$ and $x \in K_*^{\mathrm{MW}}(F)$. Then

$$\begin{aligned} s_\nu^{u\pi}(x) = \langle\overline{-1}\rangle\partial_\nu^{u\pi}([-u\pi]x) = \langle\overline{-1}\rangle\langle\overline{u}\rangle\partial_\nu^\pi([-u\pi]x) &= \langle\overline{-1}\rangle\langle\overline{u}\rangle\partial_\nu^\pi((\langle u\rangle[-\pi] + [u])x) \\ &= \langle\overline{-1}\rangle(\partial_\nu^\pi([-\pi]x) + \epsilon\langle\overline{u}\rangle[\overline{u}]\partial_\nu^\pi(x)) \\ &= \langle\overline{-1}\rangle\partial_\nu^\pi([-\pi]x) - \langle\overline{u}\rangle[\overline{u}]\partial_\nu^\pi(x) \\ &= s_\nu^\pi(x) + \epsilon[\overline{u}]\partial_\nu^\pi(x), \end{aligned}$$

where we use that the residue map is a group homomorphism which satisfies the two formulas from (i), (ii) and (iii). □

Recall that every closed point $p \in \mathbb{A}^1$, or equivalently every monic irreducible polynomial $f \in F[t]$, gives rise to a discrete valuation on $F(t)$, which measures the divisibility with respect to f. We will denote this valuation by v_p or v_f and the associated residue map with respect to the uniformizer f by $\partial^p_{v_p}$ or $\partial^f_{v_f}$. These residue maps allow us to express Milnor-Witt K-theory of $F(t)$ in terms of Milnor-Witt K-theory of F and Milnor-Witt K-theory of the residue fields $\kappa(v_p)$:

Theorem I.6.38. *There is a split short exact sequence*

$$0 \longrightarrow K_*^{\mathrm{MW}}(F) \xrightarrow{i_*} K_*^{\mathrm{MW}}(F(t)) \xrightarrow{\oplus_p \partial^p_{\nu_p}} \bigoplus_{p \in \mathbb{A}^1} K_{*-1}^{\mathrm{MW}}(\kappa(v_p)) \longrightarrow 0$$

of graded $K_*^{\mathrm{MW}}(F)$-modules, where i_* is the map induced by the inclusion $F \subset F(t)$.

Proof. This is Theorem 3.24 of [75]. □

It is not difficult to observe that a retraction of i_* is given by the specialization map $s^t_{v_t}$. This kind of sequence is usually refered to as Milnor's short exact sequence due to Milnor's seminal paper [70], where he constructs this type of sequence for both Milnor K-theory and Witt rings of quadratic forms.
For a scheme X we denote by $X^{(c)}$ the set of its points of codimension c. Recall that one can define a discrete valuation ring to be a normal noetherian local domain of dimension 1. Therefore, if we are given a smooth irreducible scheme X, any point $x \in X^{(1)}$ gives rise to a discrete valuation v_x on $\mathrm{Frac}(\mathcal{O}_{X,x}) = k(X)$. Indeed, the local ring is noetherian since X is locally noetherian. It is normal since X is smooth, and its dimension is given by $\dim(\mathcal{O}_{X,x}) = \mathrm{codim}(x) = 1$. If X is reducible, then the same holds for all codimension 1 points $y \in X^{(1)}$ in the closure of a given codimension 0 point $x \in X^{(0)}$. In particular, we get residue maps $\partial^{\pi_y}_{v_y} \colon K_*^{\mathrm{MW}}(\kappa(x)) \to K_{*-1}^{\mathrm{MW}}(\kappa(y))$ for any choice of uniformizing elements π_y.

Definition I.6.39. The n-th Milnor-Witt K-theory group of a smooth scheme X is

$$K_n^{\mathrm{MW}}(X) = \ker\Bigg(\bigoplus_{x \in X^{(0)}} K_n^{\mathrm{MW}}(\kappa(x)) \overset{\oplus \partial^{\pi_y}_{\nu_y}}{\longrightarrow} \bigoplus_{y \in X^{(1)}} K_{n-1}^{\mathrm{MW}}(\kappa(y)) \Bigg)$$

Note that this does not depend on the choices of uniformizers by Proposition I.6.37 adn Lemma I.6.33, and is hence well-defined. If we are given a morphism $f \colon X \to Y$ between smooth schemes, one can define a pullback map $f^* \colon K_n^{\mathrm{MW}}(Y) \to K_n^{\mathrm{MW}}(X)$ as follows:
As a morphism between smooth schemes, f is a local complete intersection morphism and thus factorizes as a regular embedding $j \colon X \to Z$ followed by a smooth morphism $g \colon Z \to Y$, see [89, Tag 068E]. The idea is to proceed by constructing the desired pullback maps for j and g. For g this is relatively straight-forward and is really just given by a pullback of symbols. For the morphism j, one reduces to the codimension 1 case via a choice of a regular sequence and realizes the pullback by using specialization maps. Finally, there is work to be done to verify that everything is independent of all the choices that were made. For more details we refer to Chapter 3.2 of [75]. The resulting n-th Milnor-Witt K-theory sheaf will be denoted by $\underline{\mathrm{K}}_n^{\mathrm{MW}}$, whereas we will still use the notation K_n^{MW} for its "restriction" to the category $\mathrm{Fld}_k^{\mathrm{ftr}}$. An alternative definition of $\underline{\mathrm{K}}_n^{\mathrm{MW}}$ for positive n can be given due to the following result of Morel:

Theorem I.6.40. *Let M be a strictly $\mathbb{A}^1$-invariant abelian sheaf and let n be a positive integer. The map*

$$[-]^*\colon \operatorname{Hom}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_n^{\mathrm{MW}}, M) \to \operatorname{Hom}_{\mathrm{Set}_*/k}(\mathbb{G}_{\mathrm{m}}^{\wedge n}, M)$$

induced by the universal symbol $[-]\colon \mathbb{G}_{\mathrm{m}}^{\wedge n} \to \underline{\mathrm{K}}_n^{\mathrm{MW}}$, $(a_1, \dots, a_n) \mapsto [a_1, \dots, a_n]$ is a natural bijection in M. In other words, $\underline{\mathrm{K}}_n^{\mathrm{MW}}$ is the reduced free strictly $\mathbb{A}^1$-invariant abelian sheaf $\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}^{\wedge n}]$ on the sheaf of pointed sets $\mathbb{G}_{\mathrm{m}}^{\wedge n}$.

Proof. This is Theorem 3.37 of [75]. □

There is also a similar description in degree 0, which is Theorem 3.46 of loc. cit.:

Theorem I.6.41. *Let M be a strictly $\mathbb{A}^1$-invariant abelian sheaf. The map*

$$\langle - \rangle^*\colon \operatorname{Hom}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_0^{\mathrm{MW}}, M) \to \operatorname{Hom}_{\mathrm{Set}/k}(\mathbb{G}_{\mathrm{m}}/2, M),$$

induced by the universal form $\langle - \rangle\colon \mathbb{G}_{\mathrm{m}}/2 \to \underline{\mathrm{K}}_0^{\mathrm{MW}}$, $a \mapsto \langle a \rangle$ is a bijection. In other words, $\underline{\mathrm{K}}_0^{\mathrm{MW}}$ is the free strongly $\mathbb{A}^1$-invariant abelian sheaf $\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}/2]$ on the sheaf $\mathbb{G}_{\mathrm{m}}/2$.

Proof. As sheaves of sets we have $\mathbb{G}_{\mathrm{m}}/2 = \operatorname{coeq}(2, 0)$, where $n\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ is the n-th power map of the group operation for $n = 0, 2$. Therefore a morphism $\mathbb{G}_{\mathrm{m}}/2 \to M$ of sheaves corresponds to a morphism $\mathbb{G}_{\mathrm{m}} \to M$ of sheaves that coequalizes the two morphisms 2 and 0. The free-forgetful adjunction now yields that morphisms $\mathbb{G}_{\mathrm{m}} \to M$ of sheaves that equalize the two morphisms 2 and 0 are in natural bijection to morphisms $\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}] \to M$ of abelian sheaves that coequalize $\underline{\mathbb{Z}}_{\mathbb{A}^1}[2]$ and $\underline{\mathbb{Z}}_{\mathbb{A}^1}[0]$. Under the isomorphism $\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}] \cong \underline{\mathbb{Z}} \oplus \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}]$, these once again correspond to morphisms $\underline{\mathbb{Z}} \oplus \widetilde{\underline{\mathbb{Z}}}[\mathbb{G}_{\mathrm{m}}] \to M$ that coequalize

$$\begin{pmatrix} \mathrm{id}_{\underline{\mathbb{Z}}} & 0 \\ 0 & \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[2] \end{pmatrix} \text{ and } \begin{pmatrix} \mathrm{id}_{\underline{\mathbb{Z}}} & 0 \\ 0 & 0 \end{pmatrix}.$$

By Theorem I.6.40 and Lemma 3.14 of [75], these morphisms in turn correspond to morphisms $\underline{\mathbb{Z}} \oplus \underline{\mathrm{K}}_1^{\mathrm{MW}} \to M$ that coequalize the matrices

$$\begin{pmatrix} \mathrm{id}_{\underline{\mathbb{Z}}} & 0 \\ 0 & h \end{pmatrix} \text{ and } \begin{pmatrix} \mathrm{id}_{\underline{\mathbb{Z}}} & 0 \\ 0 & 0 \end{pmatrix}.$$

In other words, these morphisms are exactly the morphisms

$$\underline{\mathbb{Z}} \oplus \underline{\mathrm{K}}_1^{\mathrm{W}} \cong \underline{\mathbb{Z}} \oplus \underline{\mathrm{K}}_1^{\mathrm{MW}}/h \to M$$

of abelian sheaves, where we already make use of Example I.6.44 from below. Finally, we precompose with the isomorphism

$$\underline{\mathrm{K}}_0^{\mathrm{MW}} = \underline{\mathrm{GW}} \to \underline{\mathbb{Z}} \oplus \underline{\mathrm{I}} = \underline{\mathbb{Z}} \oplus \underline{\mathrm{K}}_1^{\mathrm{W}}$$

given by splitting off the rank, see Example I.6.44 and the discussion after Theorem I.6.46, and thus end up with morphisms $\underline{\mathrm{K}}_0^{\mathrm{MW}} \to M$ of abelian sheaves. We leave it to particularly motivated readers to verify that all these identifications together yield the map induced by the universal form. □

Now that we have seen the basics of Milnor-Witt K-theory, let us finally give some examples of homotopy modules:

Example I.6.42. Milnor-Witt K-theory $\underline{\mathrm{K}}_*^{\mathrm{MW}}$ is the homotopy module associated with the motivic sphere spectrum $\mathbb{1}_k$ as shown by Morel [72], the motivic spectrum $\tilde{H}\mathbb{Z}$ representing Milnor-Witt motivic cohomology, see e.g. Déglise and Fasel [30, Theorem 4.2.3] and the algebraic special linear cobordism spectrum MSL by work of Yakerson [109, Proposition 3.6.3].

In particular, the $\mathbb{1}_k$-module structure on any motivic spectrum E gives rise to a $\underline{\mathrm{K}}_*^{\mathrm{MW}}$-module structure on the homotopy module $\underline{\pi}_0(E)_*$, so that every homotopy module is equipped with such structure. This can also be seen via Chapter 2.3 of [75] or Feld's theory of Milnor-Witt cycle modules, see [40]. In particular, there is a canonical choice of contraction isomorphisms given by the aforementioned action:
Theorem I.6.40 together with Lemma I.6.28 gives us, that for any homotopy module M_* and for any integer n, the contraction $(M_n)_{-1}$ can be identified with $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}, M_n)$. On the level of the latter sheaf we can now consider the morphism

$$M_{n-1} \to \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}, M_n),\ x \mapsto (y \mapsto yx),$$

which by Lemma 2.48 of [75] is an isomorphism for all integers n and thus gives us contraction isomorphisms by the discussion above. On the other hand, if we are given a homotopy module M_*, we can also use the contractions to define the module structure, see for instance appendix A of [17]. As this action in positive degree is the most crucial for this work, let us quickly explain it. As above, the contraction isomorphisms give us

$$M_m \cong \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_n^{\mathrm{MW}}, M_{n+m})$$

which via the hom-tensor adjunction corresponds to an action $\underline{\mathrm{K}}_n^{\mathrm{MW}} \otimes M_m \to M_{n+m}$. This is the desired action.

Example I.6.43. By its very definition, the quotient $\underline{\mathrm{K}}_*^{\mathrm{MW}}/\eta$ is Milnor K-theory $\underline{\mathrm{K}}_*^{\mathrm{M}}$. Since the category of homotopy modules is abelian and thus has such quotients, also $\underline{\mathrm{K}}_*^{\mathrm{M}}$ is a homotopy module. Here the $\underline{\mathrm{K}}_*^{\mathrm{MW}}$-action and the residue and specialization maps are given via the quotient map. Milnor K-theory also arises as the homotopy module associated with the motivic Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ and the algebraic cobordism spectrum MGL, see Theorem 3.4 of [93] and Remark 3.10 of [50]. Additionally, the quotient $\underline{\mathrm{K}}_*^{\mathrm{M}}/2$ is the homotopy module associated with $H\mathbb{Z}/2$.

Example I.6.44. A second quotient we consider is $\underline{\mathrm{K}}_*^{\mathrm{W}} \cong \underline{\mathrm{K}}_*^{\mathrm{MW}}/h$, called Witt K-theory. It was defined by Morel in terms of generators and relations similar to Milnor-Witt K-theory [73] and the isomorphism $\underline{\mathrm{K}}_*^{\mathrm{W}} \xrightarrow{\cong} \underline{\mathrm{K}}_*^{\mathrm{MW}}/h$ is given by mapping η to $\eta + h\underline{\mathrm{K}}_*^{\mathrm{MW}}$ and a symbol $\{a\}$ to the class $-[a]+h\underline{\mathrm{K}}_*^{\mathrm{MW}}$. Furthermore, Morel showed that Witt K-theory is nothing but the graded ring of powers of the fundamental ideal $\bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{I}}^n$, where by convention $\underline{\mathrm{I}}^n = \underline{\mathrm{W}}$ for negative n. Here the isomorphism identifies pure symbols $[a_1, \ldots, a_n]$ of length n with n-Pfister forms $\langle\langle a_1, \ldots, a_n\rangle\rangle$ and the multiplication by η with the inclusions $\underline{\mathrm{I}}^{n+1} \hookrightarrow \underline{\mathrm{I}}^n$.

Example I.6.45. Algebraic K-theory $\underline{\mathrm{K}}_*^{\mathrm{Q}}$ is a homotopy module, which arises from the algebraic K-theory spectrum KGL. This follows from Theorem I.5.19 and the definition of KGL.

Now that we have a couple of examples at hand, we can also give another description of Milnor-Witt K-theory, namely as follows. The resolution of the Milnor conjecture of quadratic forms by Orlov-Vishik-Voevodsky [79] gives a commutative diagram

$$\begin{array}{ccccc}
\underline{\mathrm{K}}^{\mathrm{MW}}_* & & \xrightarrow{\qquad\qquad} & & \underline{\mathrm{K}}^{\mathrm{M}}_* \\
\Big\downarrow & & & & \Big\downarrow \\
 & & & & \underline{\mathrm{K}}^{\mathrm{M}}_*/2 \\
\Big\downarrow & & & \overset{\cong}{\nearrow} & \\
\underline{\mathrm{K}}^{\mathrm{W}}_* & \twoheadrightarrow & \underline{\mathrm{K}}^{\mathrm{W}}_*/\eta\underline{\mathrm{K}}^{\mathrm{W}}_{*+1} & &
\end{array}$$

where the bottom right map is given by $[a_1, \ldots, a_n] + \eta\underline{\mathrm{K}}^{\mathrm{W}}_{*+1} \mapsto [a_1, \ldots, a_n] + 2\underline{\mathrm{K}}^{\mathrm{M}}_*$. Morel [73] shows that this diagram fully describes Milnor-Witt K-theory:

Theorem I.6.46. *The above diagram is a pullback square.*

In loc. cit. he proves that this is a pullback square when applied to fields, which then by the content of Chapter 2 and 3 of [75] extends to the case of sheaves, as all the occuring maps are compatible with the respective residue/specialization maps. In degree 0, this in particular recovers the classical pullback square

$$\begin{array}{ccc}
\underline{\mathrm{GW}} & \longrightarrow & \underline{\mathbb{Z}} \\
\Big\downarrow & & \Big\downarrow \\
\underline{\mathrm{W}} & \longrightarrow & \underline{\mathbb{Z}/2\mathbb{Z}}
\end{array}$$

and shows that $\underline{\mathrm{K}}^{\mathrm{MW}}_n \cong \underline{\mathrm{W}}$ for negative integers n. The description of Milnor-Witt K-theory as a pullback of Milnor- and Witt K-theory over their common base allows us to study operations $\underline{\mathrm{K}}^{\mathrm{MW}}_n \to \underline{\mathrm{K}}^{\mathrm{MW}}_m$ by studying operations $\underline{\mathrm{K}}^{\mathrm{MW}}_n \to \underline{\mathrm{K}}^{\mathrm{R}}_m$, where the later can stand for Milnor K-theory, Milnor K-theory mod 2 or Witt K-theory.
By [40] or by Chapter 2.3 of [75], homotopy modules come with residue and specialization maps, which also satisfy the properties of Proposition I.6.37. Furthermore, it is shown in loc. cit. that the following two properties hold:

Proposition I.6.47. *For any homotopy module M_* and any transcendental element t over k, the map $M_*(k) \to M_{*+1}(k(t))$, $x \mapsto [t]x$ is injective with left-inverse $\partial^t_{\nu_t}$. In particular, if $[a]x = 0$ for all field extensions $k \subset F$ and all $a \in F$, then $x = 0$.*

Proposition I.6.48. *For any homotopy module M_* and any field extension $k \subset F$, there is a split short exact sequence*

$$0 \longrightarrow M_*(F) \xrightarrow{i_*} M_*(F(t)) \xrightarrow{\oplus_p \partial^p_{\nu_p}} \bigoplus_{p \in \mathbb{A}^1} M_{*-1}(\kappa(v_p)) \longrightarrow 0$$

of graded $M_*(F)$-modules, where i_* is the map induced by the inclusion $F \subset F(t)$.

These properties will be used to compute operations on generators of Milnor-Witt K-theory in Section II.3.
The last ingredient we need is the following. Sheaves defined on objects as in Definition I.6.39, and on morphisms as in the discussion after Definition I.6.39, are called unramified and their construction is the main content of Chapter 2.1 of [75].

Example I.6.49. In Lemma 6.4.4 of [74] Morel observed that any strictly $\mathbb{A}^1$-invariant abelian sheaf is unramified. In particular this yields that homotopy modules are unramified.

All the sheaves that we consider from now on will be strictly $\mathbb{A}^1$-invariant and hence unramified. We refrain from giving a precise definition of unramified sheaves, but rather advise the reader to blackbox the proposition below which in particular allows us to ignore the notion of being unramified.
Given two unramified sheaves M and N, we can "restrict" them to the category $\mathrm{Fld}_k^{\mathrm{ftr}}$ of field extensions of k with finite transcendence degree. Morel observes in Theorem 2.11 together with Definition 2.9 of [75] that unramified sheaves always come with specialization maps as we have introduced below Theorem I.6.36 for Milnor-Witt K-theory, but defined on the level of valuation rings. For homotopy modules we even have those on the level of fields, as we have seen. Therefore we can consider those morphisms of unramified sheaves restricted to $\mathrm{Fld}_k^{\mathrm{ftr}}$ which commute with those specialization maps. By this we mean morphisms $\varphi\colon\ M|_{\mathrm{Set}^{\mathrm{Fld}_k^{\mathrm{ftr}}}} \to N|_{\mathrm{Set}^{\mathrm{Fld}_k^{\mathrm{ftr}}}}$ making the diagram

$$\begin{array}{ccc} M|_{\mathrm{Set}^{\mathrm{Fld}_k^{\mathrm{ftr}}}}(\mathcal{O}_F) & \xrightarrow{\varphi} & N|_{\mathrm{Set}^{\mathrm{Fld}_k^{\mathrm{ftr}}}}(\mathcal{O}_F) \\ \big\downarrow s_\nu^\pi & & \big\downarrow s_\nu^\pi \\ M|_{\mathrm{Set}^{\mathrm{Fld}_k^{\mathrm{ftr}}}}(\kappa(\nu)) & \xrightarrow{\varphi} & N|_{\mathrm{Set}^{\mathrm{Fld}_k^{\mathrm{ftr}}}}(\kappa(\nu)). \end{array}$$

commutative for every finitely generated field extension $k \subset F$ equipped with a discrete valuation ν, valuation ring $\mathcal{O}_F$, residue field $\kappa(\nu)$ containing k and every choice of uniformizing element π. In particular, although the morphism is only defined on the level of certain field extensions, it is supposed to respect discrete valuation rings. The set of such morphisms will be denoted by $\mathrm{Op}_{\mathrm{sp}}(M, N)$.

Proposition I.6.50. *For all unramified sheaves M and N we have an identification*

$$\mathrm{Hom}_{\mathrm{Set}/k}(M, N) = \mathrm{Op}_{\mathrm{sp}}(M, N).$$

Moreover, the same identification holds for quotients of $\underline{\mathrm{K}}_n^{\mathrm{MW}}$.

Proof. This is Remark 2.15 in [75]. □

Therefore we will from now on mostly restrict to the category $\mathrm{Fld}_k^{\mathrm{ftr}}$ of field extensions with finite transcendence degree of our base field k and work in this more concrete setting.

I.6.3 Morel's Unstable and Stable Computations

We will give a detailed account of Morel's computation of $\underline{\pi}_n^{\mathbb{A}^1}(S^n \wedge \mathbb{G}_m^{\wedge i})$ from Chapter 6.3 of [75], and then also give a very brief outline of Morel's computation of $\underline{\pi}_0(\mathbb{1}_k)_*$ from [72]. The

latter computation is the original proof of the fact that Milnor-Witt K-theory is a homotopy module.
Let us now start with the aforementioned unstable computation. The general idea is to use a motivic Hurewicz theorem to translate the computation of motivic homotopy groups to certain homology groups. The latter are what we will introduce now following Chapter 6.2 of [75]. We want to define the $\mathbb{A}^1$-singular chain complex $C_\bullet^{\mathbb{A}^1}(X)$ of a space X, which is the motivic analogue of the singular chain complex. For this we make use of the sheaf-theoretic Dold-Kan correspondence or at least of one of the involved functors therein, which we will recall now.

Definition I.6.51. Let $A \in \mathrm{sAb}/k$ be a simplicial abelian sheaf. Its normalized chain complex $NA_\bullet \in \mathrm{Ch}_\bullet(\mathrm{Ab}/k)$ is given by $NA_n = \bigcap_{i=0}^{n-1} \ker(d_i)$ together with the differentials $\partial_n = (-1)^n d_n$, where the $d_i\colon A_n \to A_{n-1}$ are the face maps.

Note that such chain complexes are (-1)-connected, since there are no negative-dimensional simplices. Moreover, this construction is functorial and we denote the corresponding functor by $N\colon\ \mathrm{sAb}/k \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab}/k)$.

Theorem I.6.52 (Sheaf-theoretic Dold-Kan correspondence)**.** *The normalized chain complex functor $N\colon\ \mathrm{sAb}/k \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab}/k)$ yields a Quillen equivalence, where* sAb *is equipped with the Quillen model structure and* $\mathrm{Ch}_{\geq 0}(\mathrm{Ab}/k)$ *is equipped with the projective model structure.*

We are not aware of a reference for the sheaf-theoretic version, but it can be proven in the usual way. For this we recommend [63, Tag 00QQ]. In particular, for all $A \in \mathrm{sAb}/k$ the simplicial homotopy sheaves $\pi_n(A)$ coincide with the homology sheaves $H_n(NA_\bullet)$ for all integers n.

Definition I.6.53. The $\mathbb{A}^1$-singular chain complex of a space X is

$$C_\bullet^{\mathbb{A}^1}(X) = L_{\mathbb{A}^1} N\underline{\mathbb{Z}}[X]_\bullet,$$

i.e the $\mathbb{A}^1$-localization of the normalized chain complex of the free simplical abelian sheaf on X. Its homology sheaves $H_n^{\mathbb{A}^1}(X) = H_n(C_\bullet^{\mathbb{A}^1}(X))$ are the $\mathbb{A}^1$-homology sheaves.

The chain complex $N\underline{\mathbb{Z}}[X]_\bullet$ is often denoted by $C_\bullet(X)$ and we just call it the chain complex associated with X. This gives us $C_\bullet^{\mathbb{A}^1}(X) = L_{\mathbb{A}^1} C_\bullet(X)$.

Remark I.6.54. Since $C_\bullet(X)$ by definition is (-1)-connected, so is the chain complex $L_{\mathbb{A}^1} C_\bullet(X) = C_\bullet^{\mathbb{A}^1}(X)$ by Theorem I.6.18 (the $\mathbb{A}^1$-connectivity Theorem). Therefore $\mathbb{A}^1$-homology sheaves vanish in negative degree.

Example I.6.55. Let us compute the $\mathbb{A}^1$-homology of a point. As a space, $\mathrm{Spec}(k)$ has n-simplices $\mathrm{Spec}(k)$ for all $n \geq 0$ together with the identity morphism as face and degeneracy maps. Therefore also the simplicial abelian sheaf $\underline{\mathbb{Z}}[\mathrm{Spec}(k)]$ has n-simplices $\underline{\mathbb{Z}}[\mathrm{Spec}(k)] \cong \underline{\mathbb{Z}}$ for all $n \geq 0$ together with identity maps. This gives

$$C_\bullet(\mathrm{Spec}(k)) \cong \ \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

with $\underline{\mathbb{Z}}$ in degree 0, the chain complex associated with the strictly $\mathbb{A}^1$-invariant abelian sheaf $\underline{\mathbb{Z}}$ from Example I.6.10. By our observation from Example I.6.16 this means that $C_\bullet(\mathrm{Spec}(k))$ is $\mathbb{A}^1$-local, so that $L_{\mathbb{A}^1} C_\bullet(\mathrm{Spec}(k)) \cong C_\bullet(\mathrm{Spec}(k))$. Hence we have

$$H_n^{\mathbb{A}^1}(\mathrm{Spec}(k)) = \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 0 \\ 0 & \text{else,} \end{cases}$$

which clearly resembles the singular homology of a point.

Note that the same arguments show that $C_\bullet(X)$ is given by $\underline{\mathbb{Z}}[X]$ concentrated in degree 0 for any $X \in \mathrm{Sm}/k$. The actual information therefore really comes from the $\mathbb{A}^1$-localization.

Definition I.6.56. The n-th reduced $\mathbb{A}^1$-homology sheaf of a space X is the abelian sheaf $\tilde{H}_n^{\mathbb{A}^1}(X) = \ker(H_n^{\mathbb{A}^1}(X) \to H_n^{\mathbb{A}^1}(\mathrm{Spec}(k)))$.

If X is a pointed space, we have

$$H_n^{\mathbb{A}^1}(X) = \begin{cases} \tilde{H}_n^{\mathbb{A}^1}(X) \oplus \underline{\mathbb{Z}} & \text{if } n = 0 \\ \tilde{H}_n^{\mathbb{A}^1}(X) & \text{else} \end{cases}$$

by the previous example. Let us list a couple of properties of $\mathbb{A}^1$-homology, which all have familiar analogues in the theory of singular homology.

Proposition I.6.57. *For all $X \in \mathrm{Set}/k$ there is a canonical isomorphism $H_0^{\mathbb{A}^1}(X) \cong \underline{\mathbb{Z}}_{\mathbb{A}^1}[X]$ and in particular an isomorphism $\tilde{H}_0^{\mathbb{A}^1}(X) \cong \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[X]$.*

Proof. Let X be Nisnevich sheaf considered as a space and let A be a strictly $\mathbb{A}^1$-invariant abelian sheaf. There are identifications

$$\mathrm{Hom}_{\mathrm{Set}/k}(X, A) = \mathrm{Hom}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[X], A) = \mathrm{Hom}_{D_{\mathbb{A}^1}(\mathrm{Ab}/k)}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[X], A),$$

where the last one makes use of Example I.6.16. Since $\underline{\mathbb{Z}}_{\mathbb{A}^1}[X]$ is $\mathbb{A}^1$-local, we furthermore have $\underline{\mathbb{Z}}_{\mathbb{A}^1}[X] \cong L_{\mathbb{A}^1}\, \underline{\mathbb{Z}}_{\mathbb{A}^1}[X]$ and thus

$$\mathrm{Hom}_{D_{\mathbb{A}^1}(\mathrm{Ab}/k)}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[X], A) = \mathrm{Hom}_{\mathrm{Ab}/k}(\tau_{\leq 0} L_{\mathbb{A}^1}\, \underline{\mathbb{Z}}_{\mathbb{A}^1}[X], A)$$

by Lemma I.6.2. Now the $\mathbb{A}^1$-connectivity Theorem yields $\tau_{\leq 0} L_{\mathbb{A}^1}\, \underline{\mathbb{Z}}_{\mathbb{A}^1}[X] \cong H_0^{\mathbb{A}^1}(X)$ as seen in Example I.6.5. Thus

$$\mathrm{Hom}_{\mathrm{Set}/k}(X, A) = \mathrm{Hom}_{D_{\mathbb{A}^1}(\mathrm{Ab}/k)}(H_0^{\mathbb{A}^1}(X), A) = \mathrm{Hom}_{\mathrm{Ab}/k}(H_0^{\mathbb{A}^1}(X), A),$$

which finishes the proof. □

Remark I.6.58. In light of the above proposition we have a natural extension of the definition of the functors $\underline{\mathbb{Z}}_{\mathbb{A}^1}$ and $\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}$ to all spaces, namely in terms of the 0th $\mathbb{A}^1$-homology sheaves.

The $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}$ commutes with the S^1-suspension functor Σ_{S^1}, as Morel observes on page 164 of [75]. A consequence of this is the suspension isomorphism:

Proposition I.6.59. *For all integers n and all pointed spaces X, there is a natural isomorphism $\tilde{H}_n^{\mathbb{A}^1}(X) \to \tilde{H}_{n+1}^{\mathbb{A}^1}(\Sigma_{S^1} X)$ of abelian sheaves.*

The last tool of $\mathbb{A}^1$-homology we need is an analogue of the Hurewicz theorem. If X is a space, we consider the evident morphism

$$\underline{\pi}_n^{\mathbb{A}^1}(X) = \underline{\pi}_n(L_{\mathbb{A}^1} X) \to \underline{\pi}_n(\underline{\mathbb{Z}}[L_{\mathbb{A}^1} X]).$$

Via Dold-Kan the latter sheaf can be identified with $H_n(N\underline{\mathbb{Z}}[L_{\mathbb{A}^1} X]_\bullet)$, which we can map further to $H_n(L_{\mathbb{A}^1} N\underline{\mathbb{Z}}[L_{\mathbb{A}^1} X]_\bullet)$. But by page 161 of [75] the natural map

$$H_n^{\mathbb{A}^1}(X) = H_n(L_{\mathbb{A}^1} N\underline{\mathbb{Z}}[X]_\bullet) \to H_n(L_{\mathbb{A}^1} N\underline{\mathbb{Z}}[L_{\mathbb{A}^1} X]_\bullet)$$

is an equivalence. Therefore we can consider its inverse and obtain the desired Hurewicz map $\underline{\pi}_n^{\mathbb{A}^1}(X) \to H_n^{\mathbb{A}^1}(X)$.

Theorem I.6.60 ($\mathbb{A}^1$-Hurewicz Theorem)**.** *Let X be a pointed $\mathbb{A}^1$-connected space.*

(i) The Hurewicz morphism $\underline{\pi}_1^{\mathbb{A}^1}(X) \to H_1^{\mathbb{A}^1}(X)$ is the initial morphism from $\underline{\pi}_1^{\mathbb{A}^1}(X)$ to a strictly $\mathbb{A}^1$-invariant abelian sheaf. Moreover, it is an isomorphism if $\underline{\pi}_1^{\mathbb{A}^1}(X)$ is an abelian sheaf.

(ii) If $n \geq 2$ and X is $\mathbb{A}^1$-$(n-1)$-connected, then $H_i^{\mathbb{A}^1}(X)$ vanishes for all $0 \leq i \leq n-1$ and the Hurewicz morphism $\underline{\pi}_n^{\mathbb{A}^1}(X) \to H_n^{\mathbb{A}^1}(X)$ is an isomorphism. Moreover, the Hurewicz morphism $\underline{\pi}_{n+1}^{\mathbb{A}^1}(X) \to H_{n+1}^{\mathbb{A}^1}(X)$ is an epimorphism.

Proof. This is Theorem 6.35 together with Theorem 6.37 of [75]. □

Note that while recent results of Choudhury and Hogadi [28] show that the Hurewicz morphism is an epimorphism in degree 1, it is still unknown if it is the abelianization morphism in general.

Corollary I.6.61. *For all pointed spaces X and all integers $n \geq 2$, we have a canonical isomorphism $\underline{\pi}_n^{\mathbb{A}^1}(\Sigma_{S^1}^n X) \xrightarrow{\cong} \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[X]$ of abelian sheaves.*

Proof. Let $n \geq 2$ and let X be a pointed space. Then $\Sigma_{S^1}^n X$ is $\mathbb{A}^1$-$(n-1)$-connected as seen in Example I.4.16, and the Hurewicz morphism

$$\underline{\pi}_n^{\mathbb{A}^1}(\Sigma_{S^1}^n X) \xrightarrow{\cong} H_n^{\mathbb{A}^1}(\Sigma_{S^1}^n X)$$

is an isomorphism. Applying the suspension isomorphism n times gives us

$$H_n^{\mathbb{A}^1}(\Sigma_{S^1}^n X) = \widetilde{H}_n^{\mathbb{A}^1}(\Sigma_{S^1}^n X) \cong \widetilde{H}_0^{\mathbb{A}^1}(X)$$

which by Proposition I.6.57 proves the claim. □

Example I.6.62. For all integers $i \geq 0$ and $n \geq 2$, the previous Corollary yields

$$\underline{\pi}_n^{\mathbb{A}^1}(S^n \wedge \mathbb{G}_m^{\wedge i}) = \underline{\pi}_n^{\mathbb{A}^1}(\Sigma_{S^1}^n \mathbb{G}_m^{\wedge i}) \cong \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_m^{\wedge i}] = \begin{cases} \underline{\mathbb{Z}} & \text{if } i = 0 \\ \underline{\mathrm{K}}_i^{\mathrm{MW}} & \text{if } i \geq 1, \end{cases}$$

where the second case makes use of Theorem I.6.40. Since we have the $\mathbb{A}^1$-equivalence $\mathbb{A}^{n+1}\backslash\{0\} \simeq_{\mathbb{A}^1} \Sigma_{S^1}^n \mathbb{G}_m^{\wedge n+1}$ from Proposition I.4.27, this gives us $\underline{\pi}_n^{\mathbb{A}^1}(\mathbb{A}^{n+1}\backslash\{0\}) \cong \underline{\mathrm{K}}_{n+1}^{\mathrm{MW}}$ for all $n \geq 2$.

In light of the recently established $\mathbb{P}^1$-Freudenthal theorem of Asok, Bachmann and Hopkins [10], this unstable computation also gives the stable one for certain fields:

Theorem I.6.63. *Suppose that the characteristic of k is 0. There is a canonical isomorphism $\underline{\pi}_0(\mathbb{1})_m \xrightarrow{\cong} \underline{\mathrm{K}}_m^{\mathrm{MW}}$ of abelian sheaves for all integers m.*

Proof. Let m and r be integers with $r \geq m$. By definition we then have

$$\underline{\pi}_0(\mathbb{1})_m(U) = \mathrm{colim}_{r \geq m}[S^r \wedge \mathbb{G}_m^{\wedge r-m} \wedge U_+, S^r \wedge \mathbb{G}_m^{\wedge r}]$$

for all smooth schemes U on the level of presheaves before Nisnevich sheafification. Hence Theorem I.6.29 together with the $\mathbb{P}^1$-Freudenthal theorem of [10] yields that the sheaf $\underline{\pi}_0(\mathbb{1})_m$ is isomorphic to $\underline{\pi}_r^{\mathbb{A}^1}(S^r \wedge \mathbb{G}_m^{\wedge r})_{-(r-m)}$ for some $r >> m$. As seen in Example I.6.62, this sheaf is given by $(\underline{\mathrm{K}}_r^{\mathrm{MW}})_{-(r-m)}$. Now the latter sheaf is canonically isomorphic to $\underline{\mathrm{K}}_m^{\mathrm{MW}}$ by section 3 of [75] together with Lemma 2.48 of loc. cit. This finishes the proof. □

Let us now very quickly outline the stable computation. Although we have already used this notation for the abstract symbols of Milnor-Witt K-theory, let us nevertheless introduce the following notation for any finitely generated field extension $k \subset F$:

$$[a] = \Sigma^\infty(S^0 \to \mathbb{G}_\mathrm{m}, -1 \mapsto a) \in [\mathbb{1}_k, \Sigma^\infty \mathbb{G}_\mathrm{m}](F) = \underline{\pi}_0(\mathbb{1}_k)_1(\mathrm{Spec}(F)) \text{ for } a \in F^\times$$

$$\eta = \Sigma^\infty(\mathbb{A}^2 \setminus \{0\} \twoheadrightarrow \mathbb{P}^1) \in [\Sigma^\infty \mathbb{G}_\mathrm{m}, \mathbb{1}_k](F) \cong [\mathbb{1}_k, (\Sigma^\infty \mathbb{G}_m)^{\wedge -1}](F) = \underline{\pi}_0(\mathbb{1}_k)_{-1}(\mathrm{Spec}(F))$$

where for the latter map we used both Lemma I.4.26 and Proposition I.4.27, together with the invertibility of the generalized motivic sphere spectra from Example I.5.27. This suggests how we want to define a map $\underline{\mathrm{K}}_*^{\mathrm{MW}} \to \underline{\pi}_0(\mathbb{1}_k)_*$. That this actually works is one of Morel's results from [72] based on previous results of Hu and Kriz [53]:

Theorem I.6.64 (Morel)**.** *The Milnor-Witt relations hold in $\underline{\pi}_0(\mathbb{1}_k)_*$. In particular, there is a well-defined morphism $\underline{\mathrm{K}}_*^{\mathrm{MW}} \to \underline{\pi}_0(\mathbb{1}_k)_*$ given by $[a] \mapsto [a]$ and $\eta \mapsto \eta$.*

Proof. Druzhinin [33] proves this in the more general setup of an arbitrary base scheme. □

We claim that this is an isomorphism. To show that, Morel constructs a morphism in the opposite direction. This morphism is essentially a byproduct of the fact that the pullback of $\underline{\mathrm{K}}_*^{\mathrm{M}}$ and $\underline{\mathrm{K}}_*^{\mathrm{W}}$ along their common quotient $\underline{\mathrm{K}}_*^{\mathrm{M}}/2$ is a homotopy module. In other words, Morel obtains a natural morphism $\underline{\pi}_0(\mathbb{1}_k)_* \to \underline{\mathrm{K}}_*^{\mathrm{M}} \times_{\underline{\mathrm{K}}_*^{\mathrm{M}}/2} \underline{\mathrm{K}}_*^{\mathrm{W}}$, but the latter object is isomorphic to $\underline{\mathrm{K}}_*^{\mathrm{MW}}$ as we have seen in Theorem I.6.46. This turns out to be the inverse to the morphism from the above theorem, see [72].

Chapter II

Operations on Milnor-Witt K-theory

This chapter is the main part of this thesis, which can also be found as sections 4-9 in our preprint [108]. We start with the very formal computations of additive operations on Milnor-Witt K-theory, from which we also obtain the $\mathbb{G}_m$-stable ones.

II.1 Warmup: The Additive and $\mathbb{G}_m$-stable Operations

The computations of this section can be easily deduced from the results in [75], but their proofs are not recorded very well. Therefore we will quickly deal with those. The key ingredient is Theorem I.6.40.

Corollary II.1.1. *Let n be a positive integer and let M_* be a homotopy module. For all integers m, the abelian sheaf $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_n^{\mathrm{MW}}, M_m)$ is isomorphic to M_{m-n}. In particular, we have $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_n^{\mathrm{MW}}, \underline{\mathrm{K}}_m^{\mathrm{MW}}) \cong \underline{\mathrm{K}}_{m-n}^{\mathrm{MW}}$ for all integers m.*

Proof. Theorem I.6.40 together with Remark I.6.23 yields $\underline{\mathrm{K}}_n^{\mathrm{MW}} = \underline{\widetilde{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}^{\wedge n}] \cong \underline{\widetilde{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}]^{\otimes n}$. Therefore we get $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_n^{\mathrm{MW}}, M_m) \cong \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathbb{Z}}, (M_m)_{-n}) \cong (M_m)_{-n}$ via the hom-tensor adjunction. Since M_* is a homotopy module, this is just M_{m-n}. □

As we already observed just after Example I.6.42, the isomorphism

$$K_{m-n}^{\mathrm{MW}}(k) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_n^{\mathrm{MW}}, \underline{\mathrm{K}}_m^{\mathrm{MW}})$$

is given as follows. It maps an element x to the multiplication with x, and for general homotopy modules, it maps an element to the action of $\underline{\mathrm{K}}_n^{\mathrm{MW}}$ on said element.

Corollary II.1.2. *Let M_* be a homotopy module. For all integers m, the abelian sheaf $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_0^{\mathrm{MW}}, M_m)$ is isomorphic to $M_m \oplus {}_h M_{m-1}$. In particular, we have an isomorphism $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_0^{\mathrm{MW}}, \underline{\mathrm{K}}_m^{\mathrm{MW}}) \cong \underline{\mathrm{K}}_m^{\mathrm{MW}} \oplus {}_h\underline{\mathrm{K}}_{m-1}^{\mathrm{MW}}$ for all integers m.*

Proof. We have an isomorphism $\underline{\mathrm{GW}} \xrightarrow{\cong} \underline{\mathbb{Z}} \oplus \underline{\mathrm{I}}$ by splitting off the rank, which we can translate to an isomorphism $\underline{\mathrm{K}}_0^{\mathrm{MW}} \xrightarrow{\cong} \underline{\mathbb{Z}} \oplus \underline{\mathrm{K}}_1^{\mathrm{MW}}/h$ on the level of Milnor-Witt K-theory. This gives

$$\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_0^{\mathrm{MW}}, M_m) \cong \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathbb{Z}}, M_m) \oplus \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}/h, M_m)$$

with the first summand being M_m. The latter one is the kernel of

$$h^*\colon \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}, M_m) \to \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}, M_m),$$

which under the isomorphism $\underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}, M_m) \cong M_{m-1}$ from the previous Corollary is ${}_hM_{m-1}$ as claimed. □

It is also not difficult to keep track of the isomorphisms in this case:

$$\mathrm{Hom}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_0^{\mathrm{MW}}, M_m) = \mathrm{rk} \cdot M_m(k) \oplus (\langle - \rangle \mapsto [-]) \cdot {}_hM_{m-1}(k)$$

Note that the latter map is not well-defined by itself. It really requires an element in the kernel of h. Indeed, whenever we are given an element $a \in F^\times$ for some field extension $k \subset F$, then $\langle a^2 \rangle = \langle 1 \rangle$. If we want a well-defined map we thus have

$$[a]h \cdot y = [a^2] \cdot y = [1] \cdot y = 0 \cdot y = 0$$

for all elements $y \in M_{m-1}(k)$. Since a was arbitrary, Proposition I.6.47 thus yields $h \cdot y = 0$. So if we want to respect the relations of the form $\langle a^2 \rangle = \langle 1 \rangle$, we have no choice but to act on h-torsion elements. To give an actual counterexample, consider M_* to be Milnor K-theory. By the fact that $h \bmod \eta = 2 \bmod \eta$, the well-definedness of the above map for $y = 1$ would imply that $K_1^{\mathrm{M}}(F) \cong F^\times$ is 2-torsion, which is clearly false in general.
For $M_* = \underline{\mathrm{K}}_*^{\mathrm{MW}}$, the multiplication of an element of $\underline{\mathrm{K}}_0^{\mathrm{MW}}$ with an element $x \in K_m^{\mathrm{MW}}(k)$ is given by $\mathrm{rk} \cdot x + (\langle - \rangle \mapsto [-])\eta x$, which allows us to write

$$\mathrm{Hom}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_0^{\mathrm{MW}}, \underline{\mathrm{K}}_m^{\mathrm{MW}}) = \mathrm{id} \cdot K_m^{\mathrm{MW}}(k) \oplus (\langle - \rangle \mapsto [-]) \cdot {}_h\mathrm{K}_{m-1}^{\mathrm{MW}}(k).$$

Multiplication with a fixed element of suitable degree does of course also give us operations $K_{-n}^{MW} \to K_m^{\mathrm{MW}}$. For those readers interested in stable operations, note that in light of Theorem I.6.29, we see that the isomorphism

$$\underline{\mathrm{K}}_{n-1}^{\mathrm{MW}} \cong \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}, \underline{\mathrm{K}}_n^{\mathrm{MW}}) = \underline{\mathrm{Hom}}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}], \underline{\mathrm{K}}_n^{\mathrm{MW}}) = (\underline{\mathrm{K}}_n^{\mathrm{MW}})_{-1}$$

given by multiplication is exactly the $\mathbb{G}_{\mathrm{m}}$-suspension isomorphism. Therefore Corollary II.1.1 yields that a $\mathbb{G}_{\mathrm{m}}$-stable operation of degree m of Milnor-Witt K-theory, i.e. a family of operations respecting the suspension isomorphisms, needs to be a constant sequence of multiplications with a fixed element $x \in K_m^{\mathrm{MW}}(k)$, which certainly are well-defined operations. Let us record this observation:

Corollary II.1.3. *The $\mathbb{G}_{\mathrm{m}}$-stable operations of degree m on Milnor-Witt K-theory are exactly the constant sequences $(x \cdot \mathrm{id})_{n \in \mathbb{Z}}$ with $x \in K_m^{\mathrm{MW}}(k)$.*

We have not dealt with all additive operations on negative degree Milnor-Witt K-theory here. We did not see a direct way to easily compute these and will hence not do that in this section. We will still see all operations on negative degree Milnor-Witt K-theory (and hence in particular the additive ones) though.

II.2 The Divided Power Operations λ_n^l

We will now introduce the operations which turn out to "essentially generate" all operations on Milnor-Witt K-theory. Here "essentially" will mean that we have to allow certain infinite linear combinations, which we will explain later. For the entire section we let n be a positive integer and all natural transformations/operations are considered to be between Set-valued functors. Recall that we can reduce to this setup by Proposition I.6.50. If $k \subset F$ is a field extension and

$$x = [a_{1,1}, \ldots, a_{1,n}] + \ldots + [a_{r,1}, \ldots, a_{r,n}] \in K_n^{\mathrm{MW}}(F)$$

is a sum of pure symbols, we call

$$\lambda_n^l(x) = \sum_{1 \leq i_1 < \ldots < i_l \leq r} [a_{i_1,1}, \ldots, a_{i_1,n}] \cdot \ldots \cdot [a_{i_l,1}, \ldots, a_{i_l,n}] \in K_{ln}^{\mathrm{MW}}(F)$$

the l-th divided power of x, where we allow l to be any non-negative integer. Since Milnor-Witt K-theory is non-commutative, this expression is in general not even well-defined for a fixed element x, although it is when l or n is even. We can remedy this. As also done in the theory of λ-rings, it is sometimes easier to define all λ-operations λ^l at once in terms of one power series $\Lambda = \sum \lambda^l t^l$. To be able to define this on a Milnor-Witt K-theory group, we specify the desired value on generators $[a_1, \ldots, a_n]$ and extend this to arbitrary elements of K_n^{MW} via the formula from Proposition II.2.3. This is a usual identity of divided power operations, which we hence certainly want to have for out yet to be defined operations as well. So this is the approach we take. If we let δ_n be 1 if n is odd and 0 if n is even, the following then turns out to give us well-defined divided power maps:

Theorem II.2.1. *Let $k \subset F$ be a field extension and let M_* be a homotopy module. Furthermore, let $S_n(F)$ be the set of symbols $\eta^d[a_1, \ldots, a_{d+n}]$, where d is a non-negative integer and $a_1, \ldots, a_{d+n} \in F^\times$. For any $y \in {}_{\delta_n h}M_*(k)$, the map*

$$\Lambda_n \cdot y \colon \mathbb{Z}^{\oplus S_n(F)} \to M_*(F)[[t]]$$

given by mapping $\sum_{i=1}^r m_i \eta^{d_i}[a_{i,1}, \ldots, a_{i,d_i+n}]$ to

$$\prod_{i=1}^r \prod_{J \subset \{1,\ldots,d_i+1\}} \left(1 + \Big[\prod_{j \in J} a_{i,j}, a_{i,d_i+2}, \ldots, a_{i,d_i+n}\Big] t\right)^{(-1)^{e_{d_i,J}} \cdot m_i} \cdot y,$$

where $e_{d_i,J} = d_i + 1 - |J|$, is a well-defined map, which factorizes through the quotient map $\mathbb{Z}^{\oplus S_n(F)} \twoheadrightarrow K_n^{\mathrm{MW}}(F)$.

Proof. If n is odd, we have $y = \langle 1 \rangle \cdot y = \epsilon \cdot y$ since $y \in {}_hM_*(k)$. Therefore the products in $\Lambda_n \cdot y$ are independent of their order, which results in the well-definedness of this map. If n is even, $\Lambda_n \cdot y$ is well-defined without restrictions on y by the fact that K_{2*}^{MW} is commutative. Therefore we are either way in a commutative setting and will from now on freely change the order within the occuring products.
To show that $\Lambda_n \cdot y$ factorizes through the quotient map $\mathbb{Z}^{\oplus S_n(F)} \twoheadrightarrow K_n^{\mathrm{MW}}(F)$, we need to verify the three relations from Lemma I.6.35. Let $\eta^d[a_1, \ldots, a_{d+n}] \in \mathbb{Z}^{\oplus S_n(F)}$ be a generator

of the Steinberg relation, i.e. we have that $a_{i+1} = 1 - a_i$ for some $1 \leq i \leq d+n-1$. If $i \leq d$, we can permute the a_j's in the image and may thus assume that $i = 1$. Denoting the tuple $(a_{d+2}, \ldots, a_{d+n})$ by $a_{\underline{d}}$, the product

$$\Lambda_n \cdot y(\eta^d[a_1, \ldots, a_{d+n}]) = \prod_{J \subset \{1, \ldots, d+1\}} \left(1 + \Big[\prod_{j \in J} a_j, a_{\underline{d}}\Big] t\right)^{(-1)^{e_{d,J}}} \cdot y$$

can be rewritten as

$$\prod_{I \subset \{3, \ldots, d+1\}} \frac{\left(1 + \left[\prod_{i \in I} a_i, a_{\underline{d}}\right] t\right)^{(-1)^{e_{d,I}}} \left(1 + \left[a_1 a_2 \prod_{i \in I} a_i, a_{\underline{d}}\right] t\right)^{(-1)^{e_{d,I}}}}{\left(1 + \left[a_1 \prod_{i \in I} a_i, a_{\underline{d}}\right] t\right)^{(-1)^{e_{d,I}}} \left(1 + \left[a_2 \prod_{i \in I} a_i, a_{\underline{d}}\right] t\right)^{(-1)^{e_{d,I}}}} \cdot y.$$

It therefore suffices to show that

$$(1 + [b]t)(1 + [a(1-a)b]t) = (1 + [(1-a)b]t)(1 + [ab]t)$$

for all $a, b \in F^\times$, where we are in a commutative setting. This amounts to showing the equality of the linear and quadratic coefficients of both sides. Using the Steinberg relation, the linear coefficient on the left hand side is

$$[b] + [a(1-a)b] = [b] + [1-a] + [ab] + \eta[1-a, ab] = [b] + [1-a] + [ab] + \eta[1-a, b],$$

which coincides with $[(1-a)b] + [ab]$, the one from the right hand side. The quadratic one on the left hand side is

$$[b, a(1-a)b] = [b, 1-a] + [b, ab] + \eta[b, 1-a, ab] = [b, 1-a] + [b, ab] + \eta[b, 1-a, b],$$

whereas the quadratic coefficient on the right hand side is

$$[(1-a)b, ab] = \eta[1-a, b, ab] + [1-a, ab] + [b, ab] = \eta[1-a, b, b] + [1-a, b] + [b, ab].$$

Since we are in a commutative setting, these two agree. If $i = d+1$, the same style of argument works, although it is simpler in this case. One cannot ignore the contributions of $a_{\underline{d}}$ though. Finally, if $i \geq d+2$, the statement is clear. Therefore the map $\Lambda_n \cdot y$ factorizes through the quotient map $\mathbb{Z}^{\oplus S_n(F)} \twoheadrightarrow \mathbb{Z}^{\oplus S_n(F)}/R_{\mathrm{st}}$, where R_{st} is the subgroup defined by the Steinberg relation. By abuse of notation we still denote the induced map on the quotient $\mathbb{Z}^{\oplus S_n(F)}/R_{\mathrm{st}} \to M_*(F)[[t]]$ by $\Lambda_n \cdot y$. Let us now verify that the twisted tensor relation is respected. For this we consider a generator

$$\begin{aligned}\eta^d[a_1, \ldots, a_{i-1}, bb', a_{i+1}, \ldots, a_{d+n}] &- \eta^d[a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{d+n}] \\ &- \eta^d[a_1, \ldots, a_{i-1}, b', a_{i+1}, \ldots, a_{d+n}] \\ &- \eta^{d+1}[a_1, \ldots, a_{i-1}, b, b', a_{i+1}, \ldots, a_{d+n}]\end{aligned}$$

of the twisted tensor relation in $\mathbb{Z}^{\oplus S(F)}/R_{\mathrm{st}}$ and set $D_i = \{1, \ldots, d+1\} \setminus \{i\}$. Using that we are in a commutative setting, we may once again assume that $i = 1$ since the case $i \geq d+2$

is trivial. Furthermore, we can ignore the contribution of the tuple $(a_{d+2}, \ldots, a_{d+n}) = a_{\underline{d}}$ as seen above. This reduces the task to showing that the product of

$$\prod_{J\subset\{2,\ldots,d+1\}} \frac{\Big(1+\Big[\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d,J}}}\Big(1+\Big[b\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d,J}}}\Big(1+\Big[b'\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d,J}}}}{\Big(1+\Big[bb'\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d,J}}}\Big(1+\Big[\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d,J}}}\Big(1+\Big[\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d,J}}}}$$

and

$$\prod_{J\subset\{2,\ldots,d+1\}} \frac{\Big(1+\Big[b\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d+1,J}}}\ \Big(1+\Big[b'\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d+1,J}}}}{\Big(1+\Big[\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d+1,J}}}\ \Big(1+\Big[bb'\prod_{j\in J} a_j\Big]t\Big)^{(-1)^{e_{d+1,J}}}}$$

is 1, which it clearly is. This gives us an induced map $\mathbb{Z}^{\oplus S_n(F)}/R_{\mathrm{st,tt}} \to M_*(F)[[t]]$, which we will still denote by $\Lambda_n \cdot y$. Here $R_{\mathrm{st,tt}}$ is the subgroup defined by the generators of the Steinberg and twisted tensor relation. Finally, let us check the Witt relation. We pick a generator

$$\eta^{d+2}[a_1, \ldots, a_{i-1}, -1, a_{i+1}, \ldots, a_{d+2+n}] + 2\eta^{d+1}[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{d+2+n}]$$

considered as an element of the group $\mathbb{Z}^{\oplus S_n(F)}/R_{\mathrm{st,tt}}$, and set $D_i = \{1, \ldots, d+3\} \setminus \{i\}$ and $a_{\underline{d}} = (a_{d+4}, \ldots, a_{d+2+n})$. As before, we can reduce to the case that $i = 1$. After cancellation, this generator is now mapped to

$$\prod_{J\subset\{2,\ldots,d+3\}} \Big(1+\Big[-\prod_{j\in J} a_j, a_{\underline{d}}\Big]t\Big)^{(-1)^{e_{d+1,J}}}\Big(1+\Big[\prod_{j\in J} a_j, a_{\underline{d}}\Big]t\Big)^{(-1)^{e_{d+1,J}}} \cdot y,$$

which agrees with

$$\prod_{J\subset\{2,\ldots,d+3\}} \Big(1+\Big(\Big[\prod_{j\in J} a_j, a_{\underline{d}}\Big]+\Big[-\prod_{j\in J} a_j, a_{\underline{d}}\Big]\Big)t\Big)^{(-1)^{e_{d+1,J}}} \cdot y$$

by the fact that $[a, -a] = 0$ for all $a \in F^\times$. We also have

$$[a] + [-a] = [a] + [-1] + [a] + \eta[a, -1] = [-1] + [a](2 + \eta[-1]) = [-1] + [a]h$$

for all $a \in F^\times$, which yields

$$\Big[\prod_{j\in J} a_j, a_{\underline{d}}\Big] + \Big[-\prod_{j\in J} a_j, a_{\underline{d}}\Big] = [-1, a_{\underline{d}}] + \Big[\prod_{j\in J} a_j, a_{\underline{d}}\Big]h = [-1, a_{\underline{d}}] + h\sum_{j\in J}[a_j, a_{\underline{d}}]$$

for all $J \subset \{2, \ldots, d+3\}$. The latter summand does not contribute outside of degree 1 since $[-1] \in \ker(h)$ and $[a, a] = [a, -1]$ for all $a \in F^\times$. Thus we are left with

$$\frac{\sum_{d-|J|\text{ even}} \sum_{j\in J} h[a_j, a_{\underline{d}}]t + \prod_{d-|J|\text{ even}} \big(1 + [-1, a_{\underline{d}}]t\big)}{\sum_{d-|J|\text{ odd}} \sum_{j\in J} h[a_j, a_{\underline{d}}]t + \prod_{d-|J|\text{ odd}} \big(1 + [-1, a_{\underline{d}}]t\big)} \cdot y,$$

which by a simple counting argument is $1 \cdot y = y$. This finishes the proof. □

From the definition of $\Lambda_n \cdot y$ it is clear that this map is functorial in F since y is defined over k. It might not be clear how $\Lambda_n \cdot y$ relates to the divided powers as introduced before.

Definition II.2.2. Let l be a non-negative integer and let $y \in {}_{\delta_n h}M_*(k)$ for some homotopy module M_*. The l-th divided power operation on K_n^{MW} associated with y is the operation $K_n^{\mathrm{MW}} \to M_*$ given by taking the coefficient of $(\Lambda_n \cdot y)(x)$ of degree l for all $x \in K_n^{\mathrm{MW}}(F)$ and all field extensions $k \subset F$.

We denote the l-th divided power operation on K_n^{MW} associated with y by $\lambda_n^l \cdot y$, which as before is not only a notation, but allows us to work with the in general non-defined operation λ_n^l as long as we act on $y \in {}_{\delta_n h}M_*(k)$ in the end. Of course, $\lambda_n^0 = 1$ and $\lambda_n^1 = \mathrm{id}$ are defined for all y. Furthermore we will just refer to an l-th divided power on K_n^{MW} when speaking about $\lambda_n^l \cdot y$ for some homotopy module M_* and $y \in {}_{\delta_n h}M_*(k)$. By its definition, we get:

Proposition II.2.3. *Let $k \subset F$ be a field extension. We have*

$$\lambda_n^l \cdot y(x + x') = \sum_{i=0}^{l} \lambda_n^i(x)\lambda_n^{l-i}(x') \cdot y$$

for all elements $x, x' \in K_n^{\mathrm{MW}}(F)$.

Here the expression on the right hand side is to be read as $\sum_{i=0}^{l} \lambda_n^i(x) \cdot (\lambda_n^{l-i}(x') \cdot y)$ and thus exists by first applying the above theorem for the field extension $k \subset F$ and then for the trivial field extension $F \subset F$.

Corollary II.2.4. *Let $k \subset F$ be a field extension. If*

$$x = [a_{1,1}, \ldots, a_{1,n}] + \ldots + [a_{r,1}, \ldots, a_{r,n}] \in K_n^{\mathrm{MW}}(F)$$

is a sum of pure symbols, then

$$\lambda_n^l \cdot y(x) = \sum_{1 \le i_1 < \ldots < i_l \le r} [a_{i_1,1}, \ldots, a_{i_1,n}] \cdot \ldots \cdot [a_{i_l,1}, \ldots, a_{i_l,n}] \cdot y.$$

This justifies the name and also once again explains why $\Lambda_n \cdot y$ is defined the way it is. As state before, an arbitrary element of $K_n^{\mathrm{MW}}(F)$ is first rewritten in terms of pure symbols and then one extends the desired formula from the previous corollary via Proposition II.2.3 to negative signs. The element y is still needed for it to map to M_* and to be well-defined in the case of odd n, of course.
Let us conclude this section by giving some further concrete examples of operations and explaining how they can be expressed in terms of the operations from Definition II.2.2.

Example II.2.5. We do have the squaring map $\varphi\colon K_1^{\mathrm{MW}} \to K_2^{\mathrm{MW}}, x \mapsto x^2$. By the fact that $[a]^2 = [a, -1]$ and $[a, b] = [b, a]\epsilon$ for all elements a and b of some field extension $k \subset F$, this operation can be seen to coincide with $\lambda_1^2 \cdot (1 + \epsilon) + \lambda_1^1 \cdot [-1]$. Note that here the element $1 + \epsilon$ indeed lies in the kernel of h, since $h = 1 - \epsilon$ and $\epsilon^2 = 1$.

II.3 Operations on Generators of Milnor-Witt K-theory

In this section we will state the basic tools needed for our computations later. For these we will consider certain operations on Milnor K-theory. Let M_* be a homotopy module and let $y \in M_*(k)$. For a positive integer n and a non-empty ordered subset $\{i_1, \ldots, i_l\} \subset \{1, \ldots, n\}$, we define the operation

$$[-_{i_1}, \ldots, -_{i_l}] \cdot y \colon (K_1^{\mathrm{M}})^n \to M_*$$

by mapping tuples $(a_1, \ldots, a_n) \in \mathbb{G}_m^n(F) \cong (K_1^{\mathrm{M}}(F))^n$ to $[a_{i_1}, \ldots, a_{i_l}] \cdot y$ for every field extension $k \subset F$. Furthermore we set this operation to be the constant operation with value $1 \cdot y$ in the case that $l = 0$. These operations clearly commute with specialization maps and turn out to generate all such operations $(K_1^{\mathrm{M}})^n \to M_*$, essentially by Theorem 3.18 of Vial [96] with minor adaptations to generalize to homotopy modules:

Theorem II.3.1 (Vial). *Let M_* be a homotopy algebra and let n be a positive integer. The $M_*(k)$-module $\mathrm{Op}_{\mathrm{sp}}((K_1^{\mathrm{M}})^n, M_*)$ of operations $(K_1^{\mathrm{M}})^n \to M_*$ commuting with specialization maps is given by the free $M_*(k)$-module*

$$\bigoplus_{l=0}^{n} \bigoplus_{1 \le i_1 < \ldots < i_l \le n} [-_{i_1}, \ldots, -_{i_l}] \cdot M_*(k).$$

Before we get to the proof, let us quickly remind the reader what the notation $\mathrm{Op}_{\mathrm{sp}}(-,-)$ means. These are natural transformations which commute with specialization maps, as discussed at the very end of Section I.6.2. As we have seen there, these are exactly the kind of operations on the level of field extensions which assemble into morphisms between sheaves.

Proof. Let us first consider the case $n = 1$. Using Proposition I.6.50, we can prove this statement on the level of sheaves, i.e., it suffices to show that

$$\mathrm{Hom}_{\mathrm{Set}/k}(\underline{\mathrm{K}}_1^{\mathrm{M}}, M_*) = \mathrm{Hom}_{\mathrm{Set}/k}(\mathbb{G}_{\mathrm{m}}, M_*)$$

is given by $[-]M_*(k) \oplus M_*(k)$. By the free-forgetful adjunction, the above coincides with

$$\mathrm{Hom}_{\mathrm{Ab}_{\mathbb{A}^1}/k}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}], M_*) = \mathrm{Hom}_{\mathrm{Ab}/k}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}], M_*),$$

but this is not difficult to compute. The proof of Corollary I.6.28 together with Theorem I.6.40 gives us the splitting $\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}] \cong \widetilde{\underline{\mathbb{Z}}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}] \oplus \underline{\mathbb{Z}} \cong \underline{\mathrm{K}}_1^{\mathrm{MW}} \oplus \underline{\mathbb{Z}}$, which yields

$$\mathrm{Hom}_{\mathrm{Ab}/k}(\underline{\mathbb{Z}}_{\mathbb{A}^1}[\mathbb{G}_{\mathrm{m}}], M_*) \cong \mathrm{Hom}_{\mathrm{Ab}/k}(\underline{\mathrm{K}}_1^{\mathrm{MW}}, M_*) \oplus \mathrm{Hom}_{\mathrm{Ab}/k}(\underline{\mathbb{Z}}, M_*).$$

This is $[-]M_*(k) \oplus M_*(k)$ via Corollay II.1.1, as claimed.
As for the proof of Theorem 3.18 of [96], we now conclude by induction on $n \ge 1$. The case $n = 1$ has already been treated. Let us therefore assume that the claim is true for all positive integers $l \le n$ for some $n \ge 1$. Let $\varphi \in \mathrm{Op}_{\mathrm{sp}}((K_1^{\mathrm{M}})^{n+1}, M_*)$, let $k \subset F$ be a field extension and let $x \in (K_1^{\mathrm{M}}(F))^n$ be a fixed element. Then $\varphi((x,-)) \colon K_1^{\mathrm{M}} \to M_*$ defines an operation over the field F, which we will denote by φ_x. By induction hypothesis/the previous step, there exist $a_x, b_x \in M_*(F)$ such that $\varphi_x = [-]a_x + b_x$. The assignments $x \mapsto a_x$ and $x \mapsto b_x$ define operations in $\mathrm{Op}_{\mathrm{sp}}((K_1^{\mathrm{M}})^n, M_*)$ and are by induction hypothesis hence given

by $M_*(k)$-linear combinations of $[-_{i_1}, \ldots, -_{i_l}] \cdot 1$ for $0 \leq l \leq n$ and $1 \leq i_1 < \ldots < i_l \leq n$. Together with $\varphi_x = [-]a_x + b_x$ this implies that φ is of the claimed form. It remains to show that the operations of the form $[-_{i_1}, \ldots, -_{i_l}] \cdot 1$ are linearly independent. Suppose

$$\varphi = \sum_{l=0}^{n} \sum_{1 \leq i_1 < \ldots < i_l \leq n} [-_{i_1}, \ldots, -_{i_l}] \cdot a_{i_1,\ldots,i_l} = 0$$

where $a_{i_1,\ldots,i_l} \in M_*(k)$ for $0 \leq l \leq n$ and $1 \leq i_1 < \ldots < i_l \leq n$. We fix one ordered subset $\{j_1, \ldots, j_s\} \subset \{1, \ldots, n\}$ and consider the finitely generated field extension $k \subset k(t_{j_1}, \ldots, t_{j_s}) = F$. We set $t_j = 0$ for all $j \in \{1, \ldots, n\} \setminus \{j_1, \ldots, j_s\}$ and denote by $\underline{t}$ the element $(t_1, \ldots, t_n) \in (K_1^{\mathrm{M}}(F))^n$. Then we have

$$a_{j_1,\ldots,j_s} = \partial^{t_{j_s}}_{\nu_{t_{j_s}}} \circ \ldots \circ \partial^{t_{j_1}}_{\nu_{t_{j_1}}} (\varphi(\underline{t})) = \partial^{t_{j_s}}_{\nu_{t_{j_s}}} \circ \ldots \circ \partial^{t_{j_1}}_{\nu_{t_{j_1}}} (0) = 0,$$

which we had to show. □

Remark II.3.2. This proof in particular shows that we do not need to distinguish between operations defined on all field extensions or only on finitely generated ones. Since Theorem II.3.1 is the very first ingredient of our computation, we will thus from now on just speak about operations without specifying the underlying category of field extensions.

We now consider the subfunctor $[-_1, \ldots, -_n] \subset K_n^{\mathrm{MW}}$ which for every field extension $k \subset F$ is given by $[F^\times, \ldots, F^\times]$, the pure symbols with entries from $F^\times$. Note that we use this notation both for this subfunctor and for the operations arising in the previous Theorem. According to Lemma I.6.34, this subfunctor encodes exactly the canonical generators of K_n^{MW} and our goal is to understand the operations on these generators. To express what it means for such operations to commute with specialization maps (the latter do not restrict to $[-_1, \ldots, -_n]$), we do the following. Since $[-_1, \ldots, -_n]$ is the image of the universal symbol $[-]\colon \mathbb{G}_{\mathrm{m}}^{\wedge n} \to K_n^{\mathrm{MW}}$, the operations $[-_1, \ldots, -_n] \to M_*$ correspond to operations $\mathbb{G}_m^n \to M_*$ which factorize through $(K_1^{\mathrm{M}})^n = \mathbb{G}_m^n \twoheadrightarrow \mathbb{G}_{\mathrm{m}}^{\wedge n} \to [-_1, \ldots, -_n]$. The latter map will also be called universal symbol and denoted by u. We know what it means for the latter operations to commute with specialization maps and can restrict to those. This gives us a definition of $\mathrm{Op}_{\mathrm{sp}}([-_1, \ldots, -_n], M_*)$, which we will now determine using the previous theorem.

Theorem II.3.3. *For any homotopy algebra M_* and any positive integer n, the $M_*(k)$-module $\mathrm{Op}_{\mathrm{sp}}([-_1, \ldots, -_n], M_*)$ is free of rank 2 generated by the constant operation 1 and $[-_1, \ldots, -_n] \cdot 1$.*

Proof. The $M_*(k)$-module $\mathrm{Op}_{\mathrm{sp}}([-_1, \ldots, -_n], M_*)$ is the submodule of $\mathrm{Op}_{\mathrm{sp}}((K_1^{\mathrm{M}})^n, M_*)$ given by those operations which factorize through the universal symbol

$$u\colon (K_1^{\mathrm{M}})^n = \mathbb{G}_m^n \twoheadrightarrow \mathbb{G}_{\mathrm{m}}^{\wedge n} \to [-_1, \ldots, -_n].$$

According to Theorem II.3.1, the $M_*(k)$-module $\mathrm{Op}_{\mathrm{sp}}((K_1^{\mathrm{M}})^n, M_*)$ is

$$\bigoplus_{l=0}^{n} \bigoplus_{1 \leq i_1 < \ldots < i_l \leq n} [-_{i_1}, \ldots, -_{i_l}] M_*(k)$$

and it is clear that its submodule $M_*(k) \oplus [-_1, \ldots, -_n]M_*(k)$ consists of operations that factorize through u. It remains to show that these are the only ones, which we will do by induction on n.
For $n = 1$ the statement coincides with the $n = 1$ case of Theorem II.3.1 and is thus already shown. We now assume that the statement is true up to some positive integer n. Let $\varphi \in \mathrm{Op}_{\mathrm{sp}}([-_1, \ldots, -_{n+1}], M_*)$, let $k \subset F$ be a field extension and let $x \in (K_1^{\mathrm{M}}(F))^n$. Then $\varphi_x = \varphi(x, -_{n+1})$ defines an operation $K_1^{\mathrm{M}} \to M_*$ defined over the field F, and is by the previous theorem hence given by $b_x + [-_{n+1}]b'_x$ for some elements $b_x, b'_x \in M_*(F)$. We now let ψ and ψ' denote the operations $(K_1^{\mathrm{M}})^n \to M_*$ over k, given by mapping $x \in (K_1^{\mathrm{M}}(F))^n$ to b_x and b'_x respectively.

Step 1: The operation ψ is constant. In particular, the operation $[-_{n+1}]\psi'$ factorizes through u.
By the fact that $[1] = 0$ in Milnor-Witt K-theory, we have $[x, 1] = [x', 1]$ for all field extensions $k \subset F$ and all elements $x, x' \in (F^\times)^n = (K_1^{\mathrm{M}}(F))^n$. Since the operation $\varphi = \psi + [-_{n+1}]\psi'$ factorizes through u, this gives us

$$\psi(x) = \psi(x) + [1]\psi'(x) = \varphi(x, 1) = \varphi(x', 1) = \psi(x') + [1]\psi'(x') = \psi(x').$$

In other words, the operation ψ is constant. Therefore, if we consider ψ as an operation $(K_1^{\mathrm{M}})^{n+1} \to M_*$, it factorizes through $u\colon (K_1^{\mathrm{M}})^{n+1} \to [-_1, \ldots, -_{n+1}]$. Since the operations which factorize through u are a $M_*(k)$-module and in particular a group, also the element $[-_{n+1}]\psi' = \varphi - \psi$ factorizes through u.

Step 2: The operation ψ' factorizes through $u\colon (K_1^{\mathrm{M}})^n \to [-_1, \ldots, -_n]$.
Let $k \subset F$ be a field extension and consider $(a_1, \ldots, a_n), (a'_1, \ldots, a'_n) \in (K_1^{\mathrm{M}}(F))^n$ such that $[a_1, \ldots, a_n] = [a'_1, \ldots, a'_n]$. Thus, if t is a transcendental element over F, we also have $[a_1, \ldots, a_n, t] = [a'_1, \ldots, a'_n, t] \in K_{n+1}^{\mathrm{MW}}(F(t))$. Since the operation $[-_{n+1}]\psi'$ facorizes through $u\colon (K_1^{\mathrm{M}})^n \to [-_1, \ldots, -_n]$, we get $[t]\psi'(a_1, \ldots, a_n) = [t]\psi'(a'_1, \ldots, a'_n)$, which yields the equality $\psi'(a_1, \ldots, a_n) = \psi'(a'_1, \ldots, a'_n)$ by Proposition I.6.47.

Step 3: The operation φ is of the wanted form.
Using step 1 and 2 and the induction hypothesis, there exist $x, y, z \in M_*(k)$ with

$$\varphi = x + [-_{n+1}](y + [-_1, \ldots, -_n]z) = x + [-_{n+1}]y + [-_1, \ldots, -_{n+1}]\epsilon^n z.$$

Renaming $\epsilon^n z = z' \in M_*(k)$, it remains to show that $y = 0$. Since both the elements φ and $\varphi - [-_{n+1}]y = x + [-_1, \ldots, -_{n+1}]z'$ factorize through u, so does the operation $[-_{n+1}]y$. Let $k \subset F$ be a field extension and let $a_1, \ldots, a_{n-1} \in F^\times$. Furthermore let t be transcendental over F. Then we have

$$[a_1, \ldots, a_{n-1}, t, t] = [a_1, \ldots, a_{n-1}, t, -1] \in K_{n+1}^{\mathrm{MW}}(F(t))$$

and hence $[t]y = [-1]y$. Applying the residue map $\partial_{\nu_t}^t$ now yields $y = 0$ since $[-1]y$ is defined over F. □

II.4 Shifts for Operations on Milnor-Witt K-theory

Now that we computed the operations on the generators of Milnor-Witt K-theory, we follow Garrel's strategy from [42] and measure how an operation changes by adding or subtracting generators in order to understand all operations. For this we will now restrict to $\mathbb{N}$-graded homotopy algebras M_*. For these we consider the separated filtration given by $F_d M_* = M_{\geq d}$, where $d \geq 0$. Note that being $\mathbb{N}$-graded in particular gives us that the filtration pieces $F_d M_*$ define ideals in M_*. Furthermore the filtration endows the $M_*(k)$-module $\operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$ with a separated filtration given by

$$F_d \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*) = \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, F_d M_*)$$

for all non-negative integers d.

Remark II.4.1. Even though it is per se not an example of an $\mathbb{N}$-graded homotopy algebra, all of the following arguments will also work for the Witt ring W together with the separated filtration given by powers of the fundamental ideal I and the usual residue and specialization morphisms, as for example found in [36]. Here the K_*^{MW}-action is the multiplication action after passing to the quotient $K_*^{\mathrm{W}} \cong I^*$.

Recall that n is a positive integer. In the following proposition, which corresponds to Proposition 3.1 of [42], we will make use of the n-th negative shift of a filtration, which is commonly denoted by $[-n]$. We stress this to ensure that the reader does not confuse this shift with a symbol of Milnor-Witt K-theory. Furtermore, let us note that we will define two kinds of shifts for operations, a positive one and a negative one, but we will define them at the same time using the symbol "$\pm$".

Proposition II.4.2. *For all $\mathbb{N}$-graded homotopy algebras M_*, there exist unique morphisms*

$$\partial^{\pm} \colon \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*) \to \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)[-n] = \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_{*-n})$$

of filtered $M_*(k)$-modules, such that

$$\varphi(x \pm [\underline{a}]) = \varphi(x) \pm [\underline{a}]\partial^{\pm}(\varphi)(x)$$

for all $\varphi \in \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$, $x \in K_n^{\mathrm{MW}}(F)$, $\underline{a} \in (F^\times)^n$ and all field extensions $k \subset F$.

Proof. Consider $\varphi \in \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$. Furthermore let $k \subset F$ be a field extension, let $x \in K_n^{\mathrm{MW}}(F)$ and let $\underline{a} \in (L^\times)^n$ for some field extension $F \subset L$.
We set $\psi(\varphi)_x^{\pm}([\underline{a}]) = \varphi(x \pm [\underline{a}])$, which yields an operation

$$\psi(\varphi)_x^{\pm} \in \operatorname{Op}_{\mathrm{sp}}([-_1, \ldots, -_n], M_*)$$

defined over F. Theorem II.3.3 now gives $\psi(\varphi)_x^{\pm} = [-_1, \ldots, -_n]a_x + b_x$ for some elements $a_x, b_x \in M_*(F)$. Since $0 \in [F^\times, \ldots, F^\times]$ we have

$$\varphi(x) = \varphi(x \pm 0) = \psi(\varphi)_x^{\pm}(0) = 0 \cdot a_x + b_x = b_x.$$

Setting $\partial^{\pm}(\varphi)(x) = a_x$ therefore does the job and also clarifies that $\partial^{\pm}$ is unique with the wanted property. Furthermore, $\partial^{\pm}$ is by definition clearly a morphism of $M_*(k)$-modules. It remains to verify that $\partial^{\pm}$ respects the respective filtrations. If $\varphi \in \operatorname{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, F_d M_*)$ for some integer d, then for any element $x \in K_n^{\mathrm{MW}}(F)$ we have

$$\varphi(x \pm [\underline{a}]) = \varphi(x) \pm [\underline{a}]\partial^{\pm}(\varphi)(x) \in F_d M_*(L)$$

for all $\underline{a} \in (L^\times)^n$. Hence $\partial^{\pm}(\varphi)(x)$ lives in $F_{d-n} M_*(F)$, which finishes the proof. ∎

We will usually denote $\partial^{\pm}(\varphi)$ by $\varphi^{(\pm)}$ and refer to them as positive and negative shifts of φ. Let us record some useful examples, which correspond to Propositions 3.3 and 3.4 of [42]:

Proposition II.4.3. *For all $\mathbb{N}$-graded homotopy algebras M_* and all $y \in {}_{\delta_n h}M_*(k)$ we have*

(i) $\lambda_n^0 \cdot 1^{(\pm)} = 0$ *and* $\lambda_n^1 \cdot 1^{(\pm)} = \lambda_n^0 \cdot 1$;

(ii) $\lambda_n^l \cdot y^{(+)} = \lambda_n^{l-1} \cdot y$ *for all integers* $l \geq 2$;

(iii) $\lambda_n^l \cdot y^{(-)} = \sum_{i=0}^{l-1}(-1)^{l-(i+1)}[-1]^{n(l-(i+1))}\lambda_n^i \cdot y$ *for all integers* $l \geq 2$;

Proof. This is a direct consequence of Proposition II.2.3. For the convenience of the reader we will nevertheless do the computations. Let $k \subset F$ be a field extension, let $x \in K_n^{\mathrm{MW}}(F)$ and let $\underline{a} \in (F^\times)^n$. We have

$$\lambda_n^0 \cdot 1(x \pm [\underline{a}]) = 1 = \lambda_n^0 \cdot 1(x) \text{ and } \lambda_n^1 \cdot 1(x \pm [\underline{a}]) = (x \pm [\underline{a}]) \cdot 1 = \lambda_n^1 \cdot 1(x) \pm [\underline{a}] \cdot 1,$$

which shows (i). Now let $l \geq 2$. The computation

$$\lambda_n^l \cdot y(x + [\underline{a}]) = \sum_{i+j=l} \lambda_n^i(x)\lambda_n^j([\underline{a}]) \cdot y = (\lambda_n^l(x) + \lambda_n^{l-1}(x)[\underline{a}]) \cdot y = \lambda_n^l \cdot y(x) + [\underline{a}]\lambda_n^{l-1} \cdot y(x)$$

shows claim (ii) and the computation

$$\lambda_n^l \cdot y(x - [\underline{a}]) = \sum_{i+j=l} \lambda_n^i(x)\lambda_n^j(-[\underline{a}]) \cdot y = \lambda_n^l \cdot y(x) + [\underline{a}]\sum_{i=0}^{l-1}(-1)^{l-i}[-1]^{n(l-i-1)}\lambda_n^i \cdot y(x)$$

using Proposition II.2.3 shows (iii) by rewriting the second summand as

$$[\underline{a}]\sum_{i=0}^{l-1}(-1)^{l-i}[-1]^{n(l-i-1)}\lambda_n^i \cdot y(x) = -[a]\sum_{i=0}^{l-1}(-1)^{l-(i+1)}[-1]^{n(l-(i+1))}\lambda_n^i \cdot y(x).$$

□

We will now relate the two shifts to one another. In analogy to Proposition 3.2 of [42], we obtain the following result.

Lemma II.4.4. *Let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$ for some $\mathbb{N}$-graded homotopy algebra M_*. Then we have*

(i) $(\varphi^{(+)})^{(-)} = \epsilon^n(\varphi^{(-)})^{(+)}$;

(ii) $(\varphi^{(+)})^{(+)} \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, {}_{\delta_n h}M_*)$;

(iii) $\varphi^{(+)} - \varphi^{(-)} = [-1]^n(\varphi^{(+)})^{(-)}$;

Proof. Let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$, let $k \subset F$ be a field extension, let $x \in K_n^{\mathrm{MW}}(F)$ and let $\underline{a}, \underline{b} \in (F^\times)^n$. We can compute $\varphi(x + [\underline{a}] - [\underline{b}])$ in two ways. We get

$$\varphi(x + [\underline{a}] - [\underline{b}]) = \varphi(x) + [\underline{a}]\varphi^{(+)}(x) - [\underline{b}](\varphi^{(-)}(x) + [\underline{a}](\varphi^{(-)})^{(+)}(x))$$

by first applying the defining formula of ∂^- and then the one of ∂^+ and

$$\varphi(x + [\underline{a}] - [\underline{b}]) = \varphi(x) - [\underline{b}]\varphi^{(-)}(x) + [\underline{a}](\varphi^{(+)}(x) - [\underline{b}](\varphi^{(+)})^{(-)}(x))$$

if we use the other order. Hence we have

$$[\underline{a}, \underline{b}](\varphi^{(+)})^{(-)}(x) - [\underline{b}, \underline{a}](\varphi^{(-)})^{(+)}(x) = 0.$$

Using $[\underline{b}, \underline{a}] = \epsilon^{n^2}[\underline{a}, \underline{b}] = \epsilon^n[\underline{a}, \underline{b}]$ and choosing $F = k(\underline{t}, \underline{s})$ and $\underline{a} = \underline{t}$ and $\underline{b} = \underline{s}$ for some transcendental elements $\underline{t}$ and $\underline{s}$ over k, we thus have $(\varphi^{(+)})^{(-)} = \epsilon^n(\varphi^{(-)})^{(+)}$ by Proposition I.6.47, which shows (i). Similarly one gets

$$[\underline{a}, \underline{b}]\delta_n h(\varphi^{(+)})^{(+)}(x) = 0,$$

which then by Proposition I.6.47 yields that $(\varphi^{(+)})^{(+)} \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, {}_{\delta_n h}M_*)$ as claimed in (ii). Finally, let us show (iii). Setting $\underline{a} = \underline{b}$, we get

$$\varphi(x) = \varphi(x) - [\underline{a}]\varphi^{(-)}(x) + [\underline{a}](\varphi^{(+)}(x) - [\underline{a}](\varphi^{(+)})^{(-)}(x))$$

from the second formula above and therefore

$$[\underline{a}]([-1]^n(\varphi^{(+)})^{(-)}(x) - (\varphi^{(+)} - \varphi^{(-)})) = 0$$

by using that $[\underline{a}, \underline{a}] = [\underline{a}][-1]^n$. Choosing $F = k(\underline{t})$ and $a_i = t_i$ for some transcendental elements $t_1, \dots, t_n$ over k therefore once again completes the argument by Proposition I.6.47. □

So the two shifts can in general not be applied independently of their order. This is different from the situation in [42] and is an actual obstruction. As we will see in the next section, we will eventually be able to commute them though.
We will also consider quotients of $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$. For typographical reasons, we will occasionally also denote operations respecting specialization maps from K_n^{MW} to some $\mathbb{N}$-graded homotopy algebra M_* by $\mathrm{Op}^n_{\mathrm{sp}}(M_*)$. The following is our version of Proposition 7.8 of [42]:

Proposition II.4.5. *For all $\mathbb{N}$-graded homotopy algebras M_* and for all non-negative integers d, the morphisms $\partial^\pm$ induce morphisms*

$$\mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_{d+n}M_*) \xrightarrow{\overline{\partial^\pm}} \mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_d M_*)$$

of $M_*(k)/F_dM_*(k)$-modules whose kernels are $M_*(k)/F_{d+n}M_*(k)$. In particular, the kernels of $\partial^\pm$ are the submodule $M_*(k)$ of constant operations.

Proof. If $\varphi, \psi \in \mathrm{Op}^n_{\mathrm{sp}}(M_*)$ are two operations whose images $\overline{\varphi}$ and $\overline{\psi}$ in the quotient by $\mathrm{Op}^n_{\mathrm{sp}}(F_{d+n}M_*)$ coincide, then we also have $\overline{\varphi^{(\pm)}} = \overline{\psi^{(\pm)}}$ in $\mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_dM_*)$ since $\partial^\pm$ maps $\mathrm{Op}^n_{\mathrm{sp}}(F_{d+n}M_*)$ to $\mathrm{Op}^n_{\mathrm{sp}}(F_dM_*)$ by Proposition II.4.2. Therefore $\overline{\partial^\pm}$ is a well-defined.
Let $\varphi \in \mathrm{Op}^n_{\mathrm{sp}}(M_*)$ with $\varphi^{(\pm)} \in \mathrm{Op}^n_{\mathrm{sp}}(F_dM_*)$. Furthermore let $k \subset F$ be a field extension, let $\underline{a} \in (F^\times)^n$ and let $x \in K^{\mathrm{MW}}_n(F)$. We have

$$\varphi(x \pm [\underline{a}]) = \varphi(x) \pm [\underline{a}]\varphi^{(\pm)}(x) = \varphi(x) \bmod F_{d+n}M_*(F)$$

and know that every element $x \in K^{\mathrm{MW}}_n(F)$ can be written as a sum and difference of elements in $[F^\times, \dots, F^\times]$. Therefore we get

$$\varphi(x) = \varphi(0) \bmod F_{d+n}M_*(F)$$

by repeating the previous computation, so that $\overline{\varphi} \in M_*(k)/F_{d+n}M_*(k)$ is a constant operation. Since such elements certainly are in the kernel of $\overline{\partial^\pm}$, this shows the first claim.
Now let φ be in the kernel of $\partial^\pm$. Furthermore let $k \subset F$ be a field extension and let $x \in K^{\mathrm{MW}}_n(F)$. Then we have $\varphi(x) - \varphi(0) \in F_{d+n}M_*(F)$ for all non-negative integers d, so that $\varphi(x) = \varphi(0)$ by the fact that the intersection $\bigcap_{d \geq 0} F_{d+n}M_*(F)$ is trivial. Since elements of $M_*(k)$ clearly are in the kernel of $\partial^\pm$, we are done. □

II.5 Computing the Operations

Using the previously defined shifts we finally start with computing operations. As in Proposition 8.1 of [42], we will first deal with quotients with respect to the filtration and then lift these computations using the separatedness of the filtration. As in the last section, we let n be a positive integer.

Proposition II.5.1. *For all $\mathbb{N}$-graded homotopy algebras M_* and all non-negative integers d, the $M_*(k)/F_dM_*(k)$-module $\mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_dM_*)$ is generated by residue classes of the $\lambda^i_n \cdot a$ for $ni < d$ with $a \in {}_{\delta_n h}M_*(k)$ if $i \geq 2$.*

Proof. We give a proof by induction on $d \geq 0$. For $d = 0$ we have $F_dM_* = M_*$, so that the quotient $\mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_0M_*)$ is the trivial module over the zero ring. This is certainly generated by the empty set.
Suppose that the statement is true for non-negative integers up to some integer d and let $\varphi \in \mathrm{Op}^n_{\mathrm{sp}}(M_*)$. We denote the image of φ under the quotient map

$$\mathrm{Op}^n_{\mathrm{sp}}(M_*) \twoheadrightarrow \mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_{d+1}M_*)$$

by $\overline{\varphi}$. Its positive shift $\overline{\varphi}^{(+)}$ lies in $\mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_{d+1-n}M_*)$ and can hence by the induction hypothesis be written as $\overline{\varphi}^{(+)} = \sum_{0 \leq i \leq \frac{d+1-n}{n}} \overline{\lambda^i_n \cdot a_i}$ for some $a_i \in M_*(k)$, which for $i \geq 2$ lie in ${}_{\delta_n h}M_*(k)$. We now consider the operation

$$\psi = \varphi - \sum_{0 \leq i \leq \frac{d+1-n}{n}} \lambda^{i+1}_n \cdot a_i \in \mathrm{Op}^n_{\mathrm{sp}}(M_*)$$

which is well-defined since $\overline{a_1}$ has a representative from ${}_{\delta_n h}M_*(k)$. Indeed, we can write

$$\overline{a_1} = ((\overline{\varphi})^{(+)})^{(+)} - \Big(\Big(\sum_{2\leq i\leq \frac{d+1-n}{n}} \overline{\lambda_n^{i+1}\cdot a_i}\Big)^{(+)}\Big)^{(+)},$$

so that Lemma II.4.4 yields $\delta_n h\cdot\overline{a_1} = 0$. Thus $\delta_n h a_1 = b$ for some element $b \in F_{d+1-n}M_*(k)$. Since $\delta_n h$ has degree 0, the multiplication $\delta_n h\colon M_*(k) \to M_*(k)$ is a homomorphism of graded rings so that in particular

$$\delta_n h(M_*(k)) \cap F_{d+1-n}M_*(k) = \delta_n h(F_{d+1-n}M_*(k)).$$

Therefore we have $b = \delta_n h b'$ for some $b' \in F_{d+1-n}M_*(k)$ and hence $\delta_n h(a_1 - b') = 0$. The element $a_1 - b' \in {}_{\delta_n h}M_*(k)$ is the wanted representative of $\overline{a_1}$. From now on we denote this representative by a_1.
By the definition of ψ and Proposition II.4.3 we have $\overline{\psi}^{(+)} = 0$, which yields $\overline{\psi} = \overline{a_{-1}}$ for some element $a_{-1} \in M_*(k)$ according to Proposition II.4.5. Here $\overline{a_{-1}}$ denotes the residue class of a_{-1} modulo $F_dM_*(k)$ considered as a constant operation. Thus $\overline{\varphi} = \sum_{-1\leq i\leq \frac{d+1-n}{n}} \overline{\lambda_n^{i+1}\cdot a_i}$ as wanted. ☐

We are finally ready to improve Lemma II.4.4. Moreover, we clarify the relation between the $\delta_n h$-torsion elements and higher shifts.

Corollary II.5.2. *For all $\mathbb{N}$-graded homotopy algebras M_* and all non-negative integers d, we have*

(i) $(\overline{\varphi}^{(+)})^{(-)} = (\overline{\varphi}^{(-)})^{(+)}$ *for all* $\overline{\varphi} \in \mathrm{Op}^n_{\mathrm{sp}}(M_*)/\mathrm{Op}^n_{\mathrm{sp}}(F_dM_*)$ *and all odd n. In particular,* $(\varphi^{(+)})^{(-)} = (\varphi^{(-)})^{(+)}$ *holds for all operations* $\varphi \in \mathrm{Op}^n_{\mathrm{sp}}(M_*)$ *independent of the parity of n;*

(ii) $(\varphi^{(+)})^{(-)}, (\varphi^{(-)})^{(+)} \in \mathrm{Op}_{\mathrm{sp}}(K^{\mathrm{MW}}_n, {}_{\delta_n h}M_*)$ *for all operations* $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K^{\mathrm{MW}}_n, M_*)$.

(iii) $\varphi^{(\pm)} \in \mathrm{Op}_{\mathrm{sp}}(K^{\mathrm{MW}}_n, {}_{\delta_n h}M_*(k))$ *for all* $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K^{\mathrm{MW}}_n, {}_{\delta_n h}M_*(k))$*;*

Proof. In light of Lemma II.4.4, all statements are only of interest to us in the case that n is odd. The first part of (i) follows directly from the previous statement together with Proposition II.4.3. Now for the second part, let $\varphi \in \mathrm{Op}^n_{\mathrm{sp}}(M_*)$. Using the first part, the difference $(\varphi^{(+)})^{(-)} - (\varphi^{(-)})^{(+)}$ defines an element of $\mathrm{Op}^n_{\mathrm{sp}}(F_dM_*)$ for every non-negative integer d and hence lies in the intersection $\bigcap_{d\geq 0}\mathrm{Op}^n_{\mathrm{sp}}(F_dM_*) = 0$.
Let us now prove the second statement. Part (i) together with Lemma II.4.4 yields

$$\epsilon^n(\varphi^{(-)})^{(+)} = (\varphi^{(+)})^{(-)} = (\varphi^{(-)})^{(+)}$$

so that we have

$$\delta_n h(\varphi^{(-)})^{(+)} = (1-\epsilon^n)(\varphi^{(-)})^{(+)} = 0 = (1-\epsilon^n)(\varphi^{(+)})^{(-)} = \delta_n h(\varphi^{(+)})^{(-)},$$

which is what we wanted to show.

For (iii) let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, {}_hM_*)$, let $k \subset F$ be a field extension, let $x \in K_n^{\mathrm{MW}}(F)$ and let $\underline{a} \in (F^\times)^n$. The operation $\varphi^{(\pm)}$ is defined via the equation

$$\varphi(x \pm [\underline{a}]) = \varphi(x) \pm [\underline{a}]\varphi^{(\pm)}(x),$$

which gives us

$$\pm[\underline{a}]\varphi^{(\pm)}(x) = \varphi(x \pm [\underline{a}]) - \varphi(x) \in {}_hM_*(F).$$

Hence we have

$$[\underline{a}](\pm h)\varphi^{(\pm)}(x) = h(\pm[\underline{a}])\varphi^{(\pm)}(x) = 0,$$

which as seen so often yields $h\varphi^{(\pm)}(x) = 0$ by Lemma I.6.47. In other words we have $\varphi^{(\pm)} \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, {}_hM_*)$. □

In particular, we may apply the two shifts independently of their order and can define $\varphi^{(+m,-n)}$ as the operation φ shifted m times with respect to ∂^+ and n times with respect to ∂^-.

As mentioned before, the operations λ_n^l turn out to essentially generate all operations. To be able to make the word "essentially" precise, we introduce the following operations following Proposition 4.6 of [42]:

$$\sigma_n^l = \sum_{j=\lfloor \frac{l}{2}\rfloor+1}^{l} \binom{\lfloor\frac{l-1}{2}\rfloor}{j-\lfloor\frac{l}{2}\rfloor-1}[-1]^{n(l-j)}\lambda_n^j = \sum_{j=0}^{\lfloor\frac{l-1}{2}\rfloor}\binom{\lfloor\frac{l-1}{2}\rfloor}{j}[-1]^{nj}\lambda_n^{l-j}$$

for all integers $l \geq 1$ and we additionally set $\sigma_n^0 = \lambda_n^0$.

Remark II.5.3. Note that the transition matrix from the λ_n^l to the σ_n^l is an upper triangular matrix with 1's on the diagonal. Hence it is invertible and the modules of operations generated by $\lambda_n^0, \ldots, \lambda_n^l$ and $\sigma_n^0, \ldots, \sigma_n^l$ coincide. This can of course also be seen explicitly. Solving the defining equation of σ_n^l for λ_n^l, we obtain

$$\lambda_n^l = \sigma_n^l - \sum_{j=1}^{\lfloor\frac{l-1}{2}\rfloor}\binom{\lfloor\frac{l-1}{2}\rfloor}{j}[-1]^{nj}\lambda_n^{l-j}.$$

On the right hand side only λ_n^d's of lower degree than l show up, for which we can plug in the same formula of lower degree. Although we have not checked in detail, we believe that this yields the formula

$$\lambda_n^l = \sum_{j=2}^{l}(-1)^j\binom{\lfloor\frac{l+j}{2}\rfloor}{j}[-1]^{j-2}\sigma_n^{l+2-j}.$$

Once again we need to know the shifts of these operations:

Proposition II.5.4. *Let M_* be an $\mathbb{N}$-graded homotopy algebra and let $y \in {}_{\delta_n h}M_*(k)$. Then we have*

(i) $(\sigma_n^0 \cdot 1)^{(\pm)} = 0$ *and* $(\sigma_n^1 \cdot 1)^{(\pm)} = \sigma_n^0 \cdot 1$;

(ii) $(\sigma_n^l \cdot y)^{(+)} = \sigma_n^{l-1} \cdot y$ *and* $(\sigma_n^l \cdot y)^{(-)} = (\sigma_n^{l-1} - [-1]^n \sigma_n^{l-2}) \cdot y$ *for* $l \geq 2$ *even;*

(iii) $(\sigma_n^l \cdot y)^{(+)} = (\sigma_n^{l-1} + [-1]^n \sigma_n^{l-2}) \cdot y$ *and* $(\sigma_n^l \cdot y)^{(-)} = \sigma_n^{l-1} \cdot y$ *for* $l \geq 2$ *odd;*

Proof. This is just a computation using Proposition II.4.3, which in particular already contains part (i). Let us therefore focus on (ii) and (iii). Let $l \geq 2$ be even and write $l = 2d$. Then the operation $\sigma_n^l \cdot y$ is given by

$$\sigma_n^l \cdot y = \sum_{j=d+1}^{2d} \binom{d-1}{j-d-1} [-1]^{n(2d-j)} \lambda_n^j \cdot y,$$

which yields

$$(\sigma_n^l \cdot y)^{(+)} = \sum_{j=d+1}^{2d} \binom{d-1}{j-d-1} [-1]^{n(2d-j)} \lambda_n^{j-1} \cdot y.$$

If $l \geq 2$ is odd, we write it as $l = 2d+1$ and get

$$\sigma_n^l \cdot y = \sum_{j=d+1}^{2d+1} \binom{d}{j-d-1} [-1]^{n(2d+1-j)} \lambda_n^j \cdot y,$$

which results in

$$(\sigma_n^l \cdot y)^{(+)} = \sum_{j=d+1}^{2d+1} \binom{d}{j-d-1} [-1]^{n(2d+1-j)} \lambda_n^{j-1} \cdot y.$$

For $l = 2d \geq 2$ we directly get

$$(\sigma_n^l \cdot y)^{(+)} = \sum_{j=d+1}^{2d} \binom{d-1}{j-d-1} [-1]^{n(2d-j)} \lambda_n^{j-1} \cdot y = \sum_{j=d}^{2d-1} \binom{d-1}{j-d} [-1]^{n(2d-j-1)} \lambda_n^j \cdot y,$$

which is exactly $\sigma_n^{l-1} \cdot y$ as written above. If $l = 2d+1 \geq 2$ is odd, we need to compare

$$(\sigma_n^l \cdot y)^{(+)} = \sum_{j=d+1}^{2d+1} \binom{d}{j-d-1} [-1]^{n(2d+1-j)} \lambda_n^{j-1} \cdot y = \sum_{j=d}^{2d} \binom{d}{j-d} [-1]^{n(2d-j)} \lambda_n^j \cdot y$$

with

$$\sum_{j=d+1}^{2d} \binom{d-1}{j-d-1} [-1]^{n(2d-j)} \lambda_n^j \cdot y + [-1] \sum_{j=d}^{2d-1} \binom{d-1}{j-d} [-1]^{n(2d-j-1)} \lambda_n^j \cdot y.$$

Now these two terms agree by the standard recurrence relation for binomial coefficients. The two formulas for the negative shifts can be shown similarly, although we want to remark that this case is more painful. □

These computations of shifts will be used freely from now on. We will form infinite sums of our operations and hence need to know that this is well-defined. As in Proposition 4.7 of [42], the key is that these sums become finite whenever evaluated:

Proposition II.5.5. *Let M_* be an $\mathbb{N}$-graded homotopy algebra. For all elements y of ${}_{\delta_n h}M_*(k)$, all field extensions $k \subset F$ and all elements $x \in K_n^{\mathrm{MW}}(F)$, we have $\sigma_n^l \cdot y(x) = 0$ for all but finitely many $l \geq 0$.*

Proof. Let M_* be an $\mathbb{N}$-graded homotopy algebra, let $y \in {}_{\delta_n h}M_*(k)$, let $k \subset F$ be a field extension and let $x \in K_n^{\mathrm{MW}}(F)$. Then x can be written as

$$x = [\underline{a_1}] + \ldots + [\underline{a_r}] - [\underline{b_1}] - \ldots - [\underline{b_s}]$$

for some elements $\underline{a_1}, \ldots, \underline{a_r}, \underline{b_1}, \ldots, \underline{b_s} \in (F^\times)^n$ and some non-negative integers r and s. We claim that $\sigma_n^l \cdot y(x) = 0$ for all $l \geq 2\max(r,s) + 1$.
First note that we may assume $r = s$ by adding or subtracting $[1]^n = 0$ enough times. We now prove the claim by induction on $r \geq 0$. If $r = 0$, we have $x = 0$ and hence clearly $\sigma_n^l \cdot y(x) = 0$ for all $l \geq 1$.
Let us now assume that the claim is true for all non-negative integers up to some $r - 1$. We consider an element of the form

$$x = [\underline{a_1}] + \ldots + [\underline{a_r}] - [\underline{b_1}] - \ldots - [\underline{b_r}]$$

which we will also write as $x = x' + [\underline{a_r}] - [\underline{b_r}]$. Using Proposition II.5.4 we now get that

$$\sigma_n^l \cdot y(x) = \sigma_n^l \cdot y(x') + [\underline{a_r}](\sigma_n^l \cdot y)^{(+)}(x') - [\underline{b_r}](\sigma_n^l \cdot y)^{(-)}(x') + [\underline{a_r}, \underline{b_r}](\sigma_n^l \cdot y)^{(+,-)}(x')$$

is some combination of elements of the form $\sigma_n^d \cdot y(x')$ with $d \geq l - 2$ and therefore vanishes if $l \geq 2r + 1$ by the induction hypothesis. □

We define a filtration on $M_*(k)^2 \times {}_{\delta_n h}M_*(k)^{\mathbb{N}\setminus\{0,1\}}$ via taking

$$F_d(M_*(k)^2 \times {}_{\delta_n h}M_*(k)^{\mathbb{N}\setminus\{1,2\}}) = \{(a_l)_{l\geq 0} \mid a_l \in F_{\max(d-nl,0)}M_*(k) \text{ for all } l \geq 0\}$$

to be the d-th piece of the filtration. This allows us to present our second main result, which corresponds to Theorem 4.9 of [42].

Theorem II.5.6. *For all $\mathbb{N}$-graded homotopy algebras M_* and all positive integers n, the two maps*

$$f\colon M_*(k)^2 \times {}_{\delta_n h}M_*(k)^{\mathbb{N}\setminus\{0,1\}} \to \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*),\ (a_l)_{l\geq 0} \mapsto \sum_{l\geq 0} \sigma_n^l \cdot a_l$$

and

$$g\colon \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*) \to M_*(k)^2 \times {}_{\delta_n h}M_*(k)^{\mathbb{N}\setminus\{0,1\}},\ \varphi \mapsto (\varphi^{(+\lfloor \frac{l+1}{2} \rfloor, -\lfloor \frac{l}{2} \rfloor)}(0))_{l\geq 0}$$

are mutually inverse isomorphisms of filtered $M_*(k)$-modules.

Proof. First note that the map f is well-defined by the previous Proposition, and that g is well-defined by Corollary II.5.2. Furthermore, these two maps are clearly morphisms of $M_*(k)$-modules which preserve the respective filtrations since σ_n^l takes values in $F_l M_*$ for non-negative integers l and each application of $\partial^\pm$ shifts the filtration by n as seen in Proposition II.4.2.

Next we show that f is a right inverse of g. Let $(a_l)_{l\geq 0} \in M_*(k)^2 \times {}_{\delta_n h}M_*(k)^{\mathbb{N}\setminus\{0,1\}}$. Note that by Proposition II.4.5, we can pretend that $f((a_l)_{l\geq 0})$ is a finite sum to compute its image under the map g. If d is even, we claim that

$$(f((a_l)_{l\geq 0}))^{(+\lfloor\frac{d+1}{2}\rfloor,-\lfloor\frac{d}{2}\rfloor)} = \left(\sum_{l\geq 0}\sigma_n^l \cdot a_l\right)^{(+\lfloor\frac{d+1}{2}\rfloor,-\lfloor\frac{d}{2}\rfloor)} = \sum_{l\geq 0}\sigma_n^l \cdot a_{d+l}$$

according to Proposition II.5.4. Here we apply $\partial^{(+,-)}$ multiple times. To see that the outcome is as claimed, first apply ∂^+ if l is even, and ∂^- first is l is odd, where we make use of Corollary II.5.2 to choose the desired order of ∂^+ and ∂^-. Then Proposition II.5.4 directly yields the result. If d is odd, we simply need to compute the positive shift of this operation, which by the same Proposition is

$$\sigma_n^0 \cdot a_d + \sigma_n^1 \cdot a_{d+1} + (\sigma_n^2 + [-1]\sigma_n^1)\cdot a_{d+2} + \sigma_n^3 \cdot a_{d+3} + (\sigma_n^4 + [-1]\sigma_n^3)\cdot a_{d+4} + \dots$$

Plugging in 0 in both cases hence gives $g(f((a_l)_{l\geq 0})) = (a_l)_{l\geq 0}$ as wanted.
Finally we show that the kernel of g is trivial. Let $\varphi \in \ker(g)$, in other words we have $\varphi^{(+\lfloor\frac{l+1}{2}\rfloor,-\lfloor\frac{l}{2}\rfloor)}(0) = 0$ for all non-negative integers l. By Proposition II.5.1 and the definition of the operations σ_n^l, we have $\overline{\varphi} = \sum_{i=-1}^{d-1}\overline{\sigma_n^{i+1}\cdot a_i}$ for some $\overline{a_0},\dots,\overline{a_{d-1}} \in M_*(k)/F_dM_*(k)$, where we consider φ modulo $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, F_dM_*)$. Therefore we get

$$a_l = \varphi^{(+\lfloor\frac{l+1}{2}\rfloor,-\lfloor\frac{l}{2}\rfloor)}(0) = 0 \text{ modulo } F_dM_*(k)$$

for all $0 \leq l \leq d-1$. Thus all the a_i live in $F_dM_*(k)$. Since this is true for all non-negative integers d and the filtration $(F_dM_*(k))_{d\geq 0}$ is separated, we have $\varphi = 0$. □

Corollary II.5.7. *For every integer m, the $K_*^{\mathrm{M}}(k)$-module $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_{\geq m}^{\mathrm{M}})$ is given by*

$$\prod_{l=0}^{1}\sigma_n^l \cdot K_{\geq m-nl}^{\mathrm{M}}(k) \times \prod_{l\geq 2}\sigma_n^l \cdot {}_{\delta_n 2}K_{\geq m-nl}^{\mathrm{M}}(k).$$

In particular we have that the abelian group $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_m^{\mathrm{M}})$ is given by

$$\prod_{\min(\frac{m}{n},1)\geq l\geq 0}\sigma_n^l \cdot K_{m-nl}^{\mathrm{M}}(k) \times \prod_{\frac{m}{n}\geq l\geq 2}\sigma_n^l \cdot {}_{\delta_n 2}K_{m-nl}^{\mathrm{M}}(k)$$

Corollary II.5.8. *For every integer m, the $K_*^{\mathrm{M}}(k)/2$-module $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_{\geq m}^{\mathrm{M}}/2)$ is given by*

$$\prod_{l\geq 0}\sigma_n^l \cdot K_{\geq m-nl}^{\mathrm{M}}(k)/2.$$

In particular the abelian group $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_m^{\mathrm{M}}/2)$ is given by

$$\prod_{\frac{m}{n}\geq l\geq 0}\sigma_n^l \cdot K_{m-nl}^{\mathrm{M}}(k)/2.$$

Via Remark II.4.1 we also have:

Corollary II.5.9. *For every integer n, the $W(k)$-module $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, I^m)$ is given by*

$$\prod_{l\geq 0} \sigma_n^l \cdot I^{m-nl}(k).$$

Based on these corollaries and the pullback square of Milnor-Witt K-theory, we can circumvent the assumption that the homotopy algebras we work with are $\mathbb{N}$-graded and also get the operations on Milnor-Witt K-theory.

Corollary II.5.10. *For all integers m, the abelian group $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_m^{\mathrm{MW}})$ is given by*

$$\prod_{l=0}^{1} \sigma_n^l \cdot K_{m-nl}^{\mathrm{MW}}(k) \times \prod_{l\geq 2} \sigma_n^l \cdot {}_{\delta_n h}K_{m-nl}^{\mathrm{MW}}(k).$$

Proof. The pullback diagram

$$\begin{array}{ccc} K_m^{\mathrm{MW}} & \longrightarrow & K_m^{\mathrm{W}} \\ \downarrow & & \downarrow \\ K_m^{\mathrm{M}} & \longrightarrow & K_m^{\mathrm{M}}/2 \end{array}$$

gives rise to the pullback diagram

$$\begin{array}{ccc} \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_m^{\mathrm{MW}}) & \longrightarrow & \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_m^{\mathrm{W}}) \\ \downarrow & & \downarrow \\ \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_m^{\mathrm{M}}) & \longrightarrow & \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, K_m^{\mathrm{M}}/2) \end{array}$$

of operations. By the previous Corollaries, it therefore suffices to show that

$$\begin{array}{ccc} \prod_{l=0}^{1} \sigma_n^l \cdot K_{m-nl}^{\mathrm{MW}}(k) \times \prod_{l\geq 2} \sigma_n^l \cdot {}_{\delta_n h}K_{m-nl}^{\mathrm{MW}}(k) & \longrightarrow & \prod_{l\geq 0} \sigma_n^l \cdot \mathrm{K}_{n-ml}^{\mathrm{W}}(k) \\ \downarrow & & \downarrow \\ \prod_{l=0}^{1} \sigma_n^l \cdot K_{m-nl}^{\mathrm{M}}(k) \times \prod_{\frac{m}{n}\geq l\geq 2} \sigma_n^l \cdot {}_{\delta_n 2}K_{m-nl}^{\mathrm{M}}(k) & \longrightarrow & \prod_{\frac{m}{n}\geq l\geq 0} \sigma_n^l \cdot K_{m-nl}^{\mathrm{M}}(k)/2 \end{array}$$

is a pullback square, which is clear by the fact that pullbacks and products commute. □

II.6 Garrel's and Vial's Operations, and Operations in Non-positive Degree

Now that we understand operations on Milnor-Witt K-theory, let us reprove the known results on operations on Milnor K-theory by Vial [96] and Witt K-theory by Garrel [42]. Let r be a positive integer. Given some further integers s_{i_d} indexed by a subset $\{i_1, \ldots, i_j\} \subset \{1, \ldots, r\}$, we denote by $e_{(s_{i_d})}$ the number of even and by $o_{(s_{i_d})}$ the number of odd integers among $(s_{i_d}) = (s_{i_1}, \ldots, s_{i_j})$.

Lemma II.6.1. *Let n be a positive integer, let M_* be an $\mathbb{N}$-graded homotopy algebra and let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$. Then we have*

$$\varphi\Big(x + h\sum_{i=1}^{r}(-1)^{s_i}[\underline{a_i}]\Big) = \varphi(x) + \sum_{j=1}^{r} h^j \sum_{1\leq i_1<\ldots<i_j\leq r}(-1)^{\sum_{d=1}^{j} s_{i_d}}\prod_{d=1}^{j}[\underline{a_{i_d}}]\varphi^{(+e_{(s_{i_d})},-o_{(s_{i_d})})}(x)$$

for all $x \in K_n^{\mathrm{MW}}(F)$, $\underline{a_1}, \ldots, \underline{a_r} \in (F^\times)^n$, all field extensions $k \subset F$ and all positive integers r and $s_1, \ldots, s_r$.

Proof. We give a proof by induction on $r \geq 1$. Let $k \subset F$ be a field extension and let $x \in K_n^{\mathrm{MW}}(F)$. If $a_1, \ldots, a_n \in F^\times$, then

$$\begin{aligned}\varphi(x \pm h[a_1, \ldots, a_n]) = \varphi(x \pm [a_1^2, a_2, \ldots, a_n]) &= \varphi(x) \pm [a_1^2, a_2, \ldots, a_n]\varphi^{(\pm)}(x) \\ &= \varphi(x) \pm h[a_1, \ldots, a_n]\varphi^{(\pm)}(x),\end{aligned}$$

which clarifies the $r = 1$ case. Now suppose the statement is true for some positive integer r and let $\underline{a_1}, \ldots, \underline{a_{r+1}} \in (F^\times)^n$. Then we have

$$\varphi\Big(x + h\sum_{i=1}^{r+1}(-1)^{s_i}[\underline{a_i}]\Big) = \varphi\Big(x + h\sum_{i=1}^{r}(-1)^{s_i}[\underline{a_i}]\Big) \pm h[\underline{a_{r+1}}]\varphi^{(\pm)}\Big(x + h\sum_{i=1}^{r}(-1)^{s_i}[\underline{a_i}]\Big).$$

Using the induction hypothesis for both summands and regrouping everything clearly yields the claimed formula. □

We denote by $\overline{\sigma}_n^l \cdot y$ the operations on the quotient $K_n^{\mathrm{MW}}/hK_n^{\mathrm{MW}}$ induced by $\sigma_n^l \cdot y$ if they are well-defined. Since the isomorphism $K_n^{\mathrm{MW}}/hK_n^{\mathrm{MW}} \to K_n^W (\cong I^n)$ maps $[\underline{a}] + hK_n^{\mathrm{MW}}$ to $-\{\underline{a}\}$ (or further to $-\langle\langle \underline{a}\rangle\rangle$), the operation $\overline{\sigma}_n^l \cdot y$ corresponds to the operation $g_n^l \cdot y$ of Garrel, but does not coincide with it under the above isomorphism due to the change of sign. The operations $g_n^l \cdot y$ are defined via certain operations $f_n^d \cdot y$, which are the ones corresponding to our operations of the form $\lambda_n^d \cdot y$. We can also define the operations $f_n^d \cdot y$ on the level of Milnor-Witt K-theory, by mapping $[\underline{a}]$ to $(1 + [\underline{a}]t)^{-1} \cdot y$ instead of $(1 + [\underline{a}]t) \cdot y$ and then repeating the proof of Proposition II.2.1. Then one obtains the relation

$$f_n^l \cdot y = (-1)^l \sum_{i=0}^{l-1}\binom{l-1}{i}[-1]^{ni}\lambda_n^{l-i} \cdot y$$

for all positive integers l and n by a simple induction. Alternatively we could also use Remark 7.3 of [42]. The same formula also holds if one starts with $\lambda_n^l \cdot y$ and wishes to express it via operations of the form $f_n^d \cdot y$. This allows us to go back and forth between our operations and the ones of Garrel.

Theorem II.6.2. *For all positive integers n and all $\mathbb{N}$-graded homotopy algebras M_*, we have*

$$\mathrm{Op}_{\mathrm{sp}}(I^n, M_*) \cong \Big\{\sum_{l\geq 0}\overline{\sigma}_n^l \cdot a_l \mid (a_l)_{l\geq 0} \in M_*(k) \times {}_hM_*(k)^{\mathbb{N}\setminus\{0\}}\Big\}$$

as a filtered $M_*(k)$-module. In particular we recover Theorem 4.9 of [42] if $M_* = W$ or $M_* = K_*^{\mathrm{M}}/2 \cong H^*(-, \mu_2)$.

Proof. We need to determine those operations $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$ satisfying

$$\varphi(x + hK_n^{\mathrm{MW}}(F)) = \varphi(x)$$

for all $x \in K_n^{\mathrm{MW}}(F)$ and all field extensions $k \subset F$. By the previous Lemma and Proposition II.5.4, operations of the form $\sum_{l\geq 0} \sigma_n^l \cdot a_l$ with $(a_l)_{l\geq 0} \in M_*(k) \times {}_hM_*(k)^{\mathbb{N}\setminus\{0\}}$ do exactly that. Therefore it remains to show that these are the only such operations. For this we let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$ with $\varphi(x + hK_n^{\mathrm{MW}}(F)) = \varphi(x)$ for all $x \in K_n^{\mathrm{MW}}(F)$ and all field extensions $k \subset F$. Picking $\pm h[b_1, \ldots, b_n] \in hK_n^{\mathrm{MW}}(F)$, Lemma II.6.1 tells us that $\pm h[b_1, \ldots, b_n]\varphi^{(\pm)}(x) = 0$. Thus we have $h\varphi^{(\pm)}(x) = 0$ due to Lemma I.6.47, which in light of Theorem II.5.6 and Proposition II.5.4 means that $\varphi = \sum_{l\geq 0} \sigma_n^l \cdot a_l$ with sequence of coefficients from $M_*(k) \times {}_hM_*(k)^{\mathbb{N}\setminus\{0\}}$ as claimed. □

Corollary II.6.3. *For all* $\mathbb{N}$*-graded homotopy algebras* M_* *we have*

$$\mathrm{Op}_{\mathrm{sp}}(K_0^{\mathrm{MW}}, M_*) = \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z}, M_*) \times \Big\{ \sum_{l\geq 0} \overline{\sigma}_1^l \cdot a_l \mid (a_l)_{l\geq 0} \in M_*(k) \times {}_hM_*(k)^{\mathbb{N}\setminus\{0\}} \Big\}$$

as a filtered $M_*(k)$*-module.*

Proof. Since $K_0^{\mathrm{MW}} \cong \mathrm{GW} \cong \mathbb{Z} \times I$ by splitting off the rank, we get

$$\mathrm{Op}_{\mathrm{sp}}(K_0^{\mathrm{MW}}, M_*) = \mathrm{Op}_{\mathrm{sp}}(\mathbb{Z}, M_*) \times \mathrm{Op}_{\mathrm{sp}}(I, M_*) = \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z}, M_*) \times \mathrm{Op}_{\mathrm{sp}}(I, M_*).$$

The previous theorem now yields the desired formula. □

Corollary II.6.4. *For all negative integers* n *and all* $\mathbb{N}$*-graded homotopy algebras* M_**, we have*

$$\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*) = \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z}/2\mathbb{Z}, M_*) \times \Big\{ \sum_{l\geq 0} \overline{\sigma}_1^l \cdot a_l \mid (a_l)_{l\geq 0} \in M_*(k) \times {}_hM_*(k)^{\mathbb{N}\setminus\{0\}} \Big\}$$

as a filtered $M_*(k)$*-module.*

Proof. For negative n we have $K_n^{\mathrm{MW}} \cong \mathrm{W} = \mathrm{GW}/h$. Since we know the operations on $\mathrm{GW} \cong K_0^{\mathrm{MW}}$ by the previous corollary and since h gets mapped to $(2, [-1])$ under the isomorphism $\mathrm{GW} \cong \mathbb{Z} \times I \cong \mathbb{Z} \times K_1^{\mathrm{MW}}/h$, we get that $\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$ is the product of

$$\{\varphi \in \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z}, M_*) \mid \varphi(x) = \varphi(x + 2\mathbb{Z}) \text{ for all } x \in \mathbb{Z}\} = \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z}/2, M_*\}$$

and the group of those operations $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_1^{\mathrm{MW}}/h, M_*)$ satisfying

$$\varphi(x) = \varphi(x + [-1]K_0^{\mathrm{MW}}(F)/h)$$

for all $x \in K_1^{\mathrm{MW}}(F)/h$ and all field extensions $k \subset F$. Now we determine the latter group. Let φ be an element of it and let $k \subset F$ be a field extension. Furthermore, consider a generator $\langle a\rangle \in K_0^{\mathrm{MW}}(F)/h$. Then we have $\pm[-1]\langle a\rangle = \pm([-a] - [a])$, which gives us

$$\varphi(x) = \varphi(x \pm ([-a] - [a])) = \varphi(x) \pm [-a]\varphi^{(\pm)}(x) \mp [a]\varphi^{(\mp)}(x)$$

for all via Proposition II.4.2 and thus

$$\pm[-a]\varphi^{(\pm)}(x) = \pm[a]\varphi^{(\mp)}(x)$$

for all $x \in K_1^{\mathrm{MW}}(F)/h$. Setting $a = 1$, this yields $\varphi^{(\pm)}(x) \in {}_{\tau_2}({}_hM_*(F))$, where we denote by τ_n the action of $[-1]^{n-1}$ and make use of the previous corollary. As in Theorem II.6.2 we thus have $\varphi = \sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l$ where the sequence of coefficients $(a_l)_{l\geq 0}$ lives in $M_*(k) \times {}_{\tau_2}({}_hM_*(k))^{\mathbb{N}\setminus\{0\}}$. On the other hand, if $\varphi = \sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l$ with sequence of coefficients $(a_l)_{l\geq 0} \in M_*(k) \times {}_{\tau_2}({}_hM_*(k))^{\mathbb{N}\setminus\{0\}}$, then Proposition II.5.4 together with $[-a] = [a] + [-1] + \eta[a, -1]$ for all $a \in F^\times$ and all field extensions $k \subset F$ yields

$$\pm[-a]\varphi^{(\pm)}(x) = \pm[a]\varphi^{(\mp)}(x)$$

for all $x \in K_1^{\mathrm{MW}}(F)/h$. In other words, such φ is in the abelian group which we are interested in. This finishes the proof.

□

Let us now deal with Vial's operations. For this we will first derive a formula which explains what happens if we add elements of the form $\pm\eta[a, b, \underline{c}]$ before applying an operation, at least when η acts trivially on M_*. Note that such M_* are equivalent to Rost's notion of cycle modules [86] with ring structure, see Remark 2.50 of [75] or Section 12 of [40] together with Theorem 4.0.1 of [40].

Lemma II.6.5. *Let n be a positive integer, let M_* be a cycle module with ring structure and let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$. Then we have*

$$\varphi(x \pm \eta[a, b, c_1, \dots, c_{n-1}]) = \varphi(x) - [a, b, c_1, \dots, c_{n-1}][-1]^{n-1}\varphi^{(\mp 2)}(x)$$

for all $x \in K_n^{\mathrm{MW}}(F)$, $a, b, c_1, \dots, c_{n-1} \in F^\times$ and all field extensions $k \subset F$.

Proof. Let $k \subset F$ be a field extension, let $a, b, c_1, \dots, c_{n-1} \in F^\times$, $x \in K_n^{\mathrm{MW}}(F)$ and let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$. We set $\underline{c} = (c_1, \dots, c_{n-1})$. Then we have

$$\eta[a, b, \underline{c}] = [ab, \underline{c}] - [a, \underline{c}] - [b, \underline{c}]$$

which gives us

$$\begin{aligned}\varphi(x \pm \eta[a, b, \underline{c}]) = {} & \varphi(x) \pm [ab, \underline{c}]\varphi^{(\pm)}(x) \mp [a, \underline{c}]\varphi^{(\mp)}(x) \mp [b, \underline{c}]\varphi^{(\mp)}(x) \\ & - [ab, \underline{c}, a, \underline{c}]\varphi^{(+,-)}(x) - [ab, \underline{c}, b, \underline{c}]\varphi^{(+,-)}(x) + [a, \underline{c}, b, \underline{c}]\varphi^{(\mp 2)}(x) \\ & \pm [ab, \underline{c}, a, \underline{c}, b, \underline{c}]\varphi^{(\pm 1, \mp 2)}(x)\end{aligned} \qquad (\star)$$

by applying Proposition II.4.2 various times. Note that since η acts trivially on M_* and that hence the action of $h = 2 + \eta[-1]$ and 2 agree, Corollory II.5.2 tells us that we are working with 2-torsion modules after shifting twice. Thus the second line of $(\star)$ is given by

$$-(2[a, b, \underline{c}][-1]^{n-1} + [-1]^n([a, \underline{c}] + [b, \underline{c}]))\varphi^{(+,-)}(x) + [a, b, \underline{c}][-1]^{n-1}\varphi^{(\mp 2)}(x),$$

and using Proposition II.4.4 (iii), we can replace the first line of the right hand side of $(h\star)$ by

$$\varphi(x) + [-1]^n([a, \underline{c}] + [b, \underline{c}])\varphi^{(+,-)}(x).$$

Therefore we have

$$\begin{aligned}\varphi(x \pm \eta[a,b,\underline{c}]) = \varphi(x) - 2[a,b,\underline{c}][-1]^{n-1}\varphi^{(+,-)}(x) + [a,b,\underline{c}][-1]^{n-1}\varphi^{(\mp 2)}(x) \\ \pm 2[a,b,\underline{c}][-1]^{n-1}[-1]^n\varphi^{(\pm 1,\mp 2)}(x).\end{aligned}$$

Proposition II.4.4 (iii) now gives us

$$\begin{aligned}\pm 2[a,b,\underline{c}][-1]^{n-1}[-1]^n\varphi^{(\pm 1,\mp 2)}(x) = 2[a,b,\underline{c}][-1]^{n-1}\varphi^{(+,-)}(x) \\ - 2[a,b,\underline{c}][-1]^{n-1}\varphi^{(\mp 2)}(x))\end{aligned}$$

which yields $\varphi(x \pm \eta[a,b,c]) = \varphi(x) - [a,b,\underline{c}][-1]^{n-1}\varphi^{(\mp 2)}(x)$ as claimed. □

As also observed by Garrel in [42] with respect to the mod 2 case, Vial forgot to explicitly mention that his operations $K_n^{\mathrm{M}} \to M_*$ are uniformly bounded. Here M_* is a cycle module. Such operations by definition map to $M_{\leq m}$ for some integer m, but as we have already observed with respect to operations on Milnor-Witt K-theory, there are operations which are not bounded in this sense. Therefore we will be able to find more operations than are listed in [96]. From now on we denote the action of $[-1]^{n-1}$ on some homotopy module M_* by τ_n. As for the operations on Witt K-theory, we denote by $\overline{\sigma}_n^l \cdot y$ the operations on the quotient $K_n^{\mathrm{M}} = K_n^{\mathrm{MW}}/\eta K_{n+1}^{\mathrm{MW}}$ induced by $\sigma_n^l \cdot y$ if they are well-defined.

Theorem II.6.6. *For all positive integers n and all cycle modules with ring structure M_*, we have*

$$\mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{M}}, M_*) = \Big\{\sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_{l\geq 0} \in M_*(k)^2 \times {}_{\delta_n 2}\Big({}_{\tau_n}M_*(k)\Big)^{\mathbb{N}\setminus\{0,1\}}\Big\}$$

as a filtered $M_*(k)$-module. In particular we recover Theorem 5.5 of [96].

Proof. We need to find the operations $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$ satisfying

$$\varphi(x + \eta K_{n+1}^{\mathrm{MW}}(F)) = \varphi(x)$$

for all $x \in K_n^{\mathrm{MW}}(F)$ and all field extensions $k \subset F$. Since every element of $\eta K_{n+1}^{\mathrm{MW}}(F)$ for a field extension $k \subset F$ can be written as a sum of elements of the form $\pm\eta[a_1,\ldots,a_{n+1}]$, operations of the form $\sum_{l\geq 0}\sigma_n^l \cdot a_l$ with coefficients $(a_l)_{l\geq 0} \in M_*(k)^2 \times {}_{\delta_n 2}({}_{\tau_{n-1}}M_*(k))^{\mathbb{N}\setminus\{0,1\}}$ do that by the previous Lemma and Proposition II.5.4. Here the $\delta_n 2$-torsion comes from the fact that $h = 2 + \eta[-1]$ becomes 2 in the quotient $K_*^{\mathrm{M}} = K_*^{\mathrm{MW}}/\eta K_{*+1}^{\mathrm{MW}}$.
Now we show that these are the only such operations. Let $\varphi \in \mathrm{Op}_{\mathrm{sp}}(K_n^{\mathrm{MW}}, M_*)$ with $\varphi(x + \eta K_{n+1}^{\mathrm{MW}}(F)) = \varphi(x)$ for all $x \in K_n^{\mathrm{MW}}(F)$ and all field extensions $k \subset F$. Picking $\pm\eta[a_1,\ldots,a_{n+1}]$, Lemma II.6.5 gives us that

$$[a_1,\ldots,a_{n+1}][-1]^{n-1}\varphi^{(\mp 2)}(x) = 0.$$

Therefore we have $[-1]^{n-1}\varphi^{(\mp 2)}(x) = 0$ due to Lemma I.6.47, which by Theorem II.5.6 and Proposition II.5.4 means that $\varphi = \sum_{l\geq 0}\sigma_n^l \cdot a_l$ with coefficients $(a_l)_{l\geq 0}$ from the product $M_*(k)^2 \times {}_{\delta_n 2}({}_{\tau_n}M_*(k))^{\mathbb{N}\setminus\{0,1\}}$.
For the "in particular part", note that it does not matter whether we consider linear combinations of the operations λ_n^l or σ_n^l when working with uniformly bounded operations. This follows from Remark II.5.3. □

We can of course also compute operations $K_n^{\mathrm{W}} \cong I^n \to K_m^{\mathrm{MW}}$ and $K_n^{\mathrm{M}} \to K_m^{\mathrm{MW}}$ analogously as we did for Corollary II.5.10. Together with Proposition I.6.50 and the Corollaries of Theorem II.5.6 this yields the following table on the level of sheaves:

Theorem II.6.7. *For all positive integers n, the following table gives a complete list of operations of degree (n, m) between Milnor, Witt and Milnor-Witt K-theory*

A	B	$\mathrm{Hom}_{\mathrm{Shv}(\mathrm{Sm}_k^{\mathrm{Nis}})}(A, B)$
$\underline{\mathrm{K}}_n^{\mathrm{M}}$	$\underline{\mathrm{K}}_m^{\mathrm{M}}$	$\left\{\sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_l \in \prod_{\min(\frac{m}{n},1)\geq l\geq 0} K^{\mathrm{M}}_{m-nl}(k) \times \prod_{\frac{m}{n}\geq l\geq 2} {}_{\delta_n 2}({}_{\tau_n}K^{\mathrm{M}}_{m-nl}(k))\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{M}}$	$\underline{\mathrm{K}}_m^{\mathrm{W}}$	$\left\{\sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_l \in \prod_{l=0}^{1} K^{\mathrm{W}}_{m-nl}(k) \times \prod_{l\geq 2} {}_{\delta_n 2}({}_{\tau_n}K^{\mathrm{W}}_{m-nl}(k))\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{M}}$	$\underline{\mathrm{K}}_m^{\mathrm{MW}}$	$\left\{\sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_l \in \prod_{l=0}^{1} K^{\mathrm{MW}}_{m-nl}(k) \times \prod_{l\geq 2} {}_{\delta_n 2}({}_{\tau_n}K^{\mathrm{MW}}_{m-nl}(k))\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{W}}$	$\underline{\mathrm{K}}_m^{\mathrm{M}}$	$\left\{\sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_l \in K^{\mathrm{M}}_{m}(k) \times \prod_{\frac{m}{n}\geq l\geq 1} {}_{2}K^{\mathrm{M}}_{m-nl}(k)\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{W}}$	$\underline{\mathrm{K}}_m^{\mathrm{W}}$	$\left\{\sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_l \in \prod_{l\geq 0} K^{\mathrm{W}}_{m-nl}(k)\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{W}}$	$\underline{\mathrm{K}}_m^{\mathrm{MW}}$	$\left\{\sum_{l\geq 0} \overline{\sigma}_n^l \cdot a_l \mid (a_l)_l \in K^{\mathrm{MW}}_{m}(k) \times \prod_{l\geq 1} {}_{h}K^{\mathrm{MW}}_{m-nl}(k)\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{MW}}$	$\underline{\mathrm{K}}_m^{\mathrm{M}}$	$\left\{\sum_{l\geq 0} \sigma_n^l \cdot a_l \mid (a_l)_l \in \prod_{\min(\frac{m}{n},1)\geq l\geq 0} K^{\mathrm{M}}_{m-nl}(k) \times \prod_{\frac{m}{n}\geq l\geq 2} {}_{\delta_n 2}K^{\mathrm{M}}_{m-nl}(k)\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{MW}}$	$\underline{\mathrm{K}}_m^{\mathrm{W}}$	$\left\{\sum_{l\geq 0} \sigma_n^l \cdot a_l \mid (a_l)_l \in \prod_{l\geq 0} K^{\mathrm{W}}_{m-nl}(k)\right\}$
$\underline{\mathrm{K}}_n^{\mathrm{MW}}$	$\underline{\mathrm{K}}_m^{\mathrm{MW}}$	$\left\{\sum_{l\geq 0} \sigma_n^l \cdot a_l \mid (a_l)_l \in \prod_{l=0}^{1} K^{\mathrm{MW}}_{m-nl}(k) \times \prod_{l\geq 2} {}_{\delta_n h}K^{\mathrm{MW}}_{m-nl}(k)\right\}$,

where τ_n is the action of $[-1]^{n-1}$ on the target.

Since algebraic K-theory agrees with Milnor K-theory in degree 1, Theorem II.6.6 also gives us all operations $\underline{\mathrm{K}}_1^{\mathrm{Q}} \to \underline{\mathrm{K}}_*^{\mathrm{Q}}$ and $\underline{\mathrm{K}}_1^{\mathrm{Q}} \to \underline{\mathrm{K}}_m^{\mathrm{Q}}$ for arbitrary m. Let us record what our results yield for higher degrees.

Remark II.6.8. If $n \geq 2$, we still obtain a large set of operations $\underline{\mathrm{K}}_n^{\mathrm{Q}} \to \underline{\mathrm{K}}_*^{\mathrm{Q}}$ and $\underline{\mathrm{K}}_n^{\mathrm{Q}} \to \underline{\mathrm{K}}_m^{\mathrm{Q}}$. There is the so-called Suslin-Hurewicz map $\underline{\mathrm{K}}_n^{\mathrm{Q}} \to \underline{\mathrm{K}}_n^{\mathrm{M}}$, which can be defined using the $\mathbb{A}^1$-fiber sequence

$$\mathbb{A}^{n+1} \setminus \{0\} \to \mathrm{BGL}_n \to \mathrm{BGL}_{n+1},$$

coming from the canonical inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$, see [12], which directly yields a map on the level of sheaves. On the level of fields, this map had already been defined by Suslin before [91]. Let us quickly sketch his construction for a field F:
We take the definition $K_n^{\mathrm{Q}}(F) = \pi_n(\mathrm{BGL}(F)^+)$ and apply the Hurewicz map. The target is then the homology of $\mathrm{BGL}(F)^+$, but the plus construction does not change the homology, so

that we actually get a map to $H_n(\mathrm{BGL}(F))$. Here we are in the stable range, so that this coincides with $H_n(\mathrm{BGL}_n(F))$. Suslin shows that the quotient $H_n(\mathrm{BGL}_n(F))/H_n(\mathrm{BGL}_{n-1}(F))$ is given by Milnor K-theory, which then yields the desired map $K_n^{\mathrm{Q}}(F) \to K_n^{\mathrm{M}}(F)$ by composing with the quotient map

$$H_n(\mathrm{BGL}_n(F)) \twoheadrightarrow H_n(\mathrm{BGL}_n(F))/H_n(\mathrm{BGL}_{n-1}(F)) \cong K_n^{\mathrm{M}}(F).$$

It should be possible to verify that this map on fields extends to a map on sheaves, but this has not been done as far as we know. Therefore we will take the first definition and consider the map $\underline{\mathrm{K}}_n^{\mathrm{Q}} \to \underline{\mathrm{K}}_n^{\mathrm{M}}$.
Theorem II.6.6 now also yields all operations $\underline{\mathrm{K}}_n^{\mathrm{Q}} \to \underline{\mathrm{K}}_*^{\mathrm{Q}}$ factorizing over the Suslin-Hurewicz map. Note that if we were only interested in uniformly bounded ones, then Vial's aforementioned results suffice here. Either way this does of course raise the question what the image of the Suslin-Hurewicz map is. On page 370 of [91] Suslin conjectured that said image is given by $(n-1)!K_n^{\mathrm{M}}(F)$ for any infinite field F. He also showed that the $n = 3$ case of this conjecture is equivalent to the Milnor conjecture on quadratic forms in degree 3, thus justifying the interest in his conjecture. For more on the current state of this still widely open conjecture, we refer the reader to [12], where the authors also prove the case $n = 5$ for fields of characterstic not 2 or 3. Under the same assumptions the $n = 4$ case was more recently proven by Röndigs [85].

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