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Statistics and Probability Letters 226 (2025) 110458

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

# Semistable distributions as marginals of operator stable laws

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ARTICLE INFO

Dedicated to the memory of Hans-Peter Scheffler who untimely passed away during the revision process of this article.

*MSC:*
primary 60E07
secondary 60E10

*Keywords:*
Operator stable distribution
One-dimensional marginal distribution
Semistable distribution
Fourier transform
Spectral representation
Log-periodic function

ABSTRACT

An example of Meerschaert (1990) shows that univariate marginals of an operator stable distribution are not necessarily stable distributions, but turned out to be semistable as shown by Meerschaert and Scheffler (1999). We characterize all semistable distributions that can appear as an univariate marginal of an operator stable law in terms of the spectral measure.

## 1. Introduction

Many natural phenomena exhibit an intrinsic scaling structure. To model the scaling by linear operators provides a flexible tool for multivariate random systems with dependent components and different scaling in each coordinate. In particular, operator self-similar stochastic processes and random fields have been proven to be useful models in many applications such as anomalous diffusion in porous media, stock market prices, or dynamics of microbes, and in a variety of diverse further fields such as electrical engineering, image processing, computer network traffic, or astrophysics; see Cohen et al. (2010) and the literature cited therein. Operator stable laws naturally appear as the distribution of an operator self-similar Lévy process at a fixed time and they are often a building block of further operator self-similar stochastic processes and random fields with heavy tails. For practical information on stochastic modeling with operator scaling we refer to Cohen et al. (2010) for stochastic processes and Biermé et al. (2007) for random fields. The one-dimensional marginal distributions provide useful information of the random phenomena, e.g. the distribution of a portfolio in a multivariate stock market model, or the concentration in a layer of a porous media flow. For multivariate stable laws, i.e. for diagonal operators, it is well known and easy to see that all one-dimensional marginals have stable distributions. The surprising fact that for general operator stable laws also semistable distributions that are not stable can appear as marginals is not completely understood. Our study aims to give a full answer to this phenomenon in and thus contributes to the fine structure of operator stable laws and to the relevance of semistable distributions.

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https://doi.org/10.1016/j.spl.2025.110458
Received 23 March 2025; Received in revised form 5 May 2025; Accepted 13 May 2025
Available online 30 May 2025

A probability measure $\nu$ in $\mathbb{R}^d$ is called *operator stable* with *exponent* $E \in \mathbb{R}^{d\times d}$ if it is infinitely divisible and for all $t > 0$ there exists $a(t) \in \mathbb{R}^d$ such that

$$\nu^{*t} = t^E \nu * \delta_{a(t)}, \tag{1.1}$$

where $\nu^{*t}$ denotes the $t$-fold convolution power, $t^E\nu$ is the pushforward measure of $\nu$ under the linear operator $t^E = \sum_{k=0}^{\infty} \frac{(\log t)^k}{k!} E^k$, and $\delta_x$ denotes the Dirac measure in $x \in \mathbb{R}^d$. In case (1.1) is only fulfilled for some $t = c > 1$, thus inductively for all $t \in c^{\mathbb{Z}} = \{c^k : k \in \mathbb{Z}\}$, the probability measure $\nu$ is called $(c^E, c)$-*operator semistable*. For details on operator stable and semistable laws we refer to the monograph (Meerschaert and Scheffler, 2001). Throughout this paper we will assume that $\nu$ is not supported on any lower dimensional hyperplane, called *fullness* of $\nu$, and we will exclude Gaussian components of $\nu$ in which case the real part of any eigenvalue of an exponent belongs to $(\frac{1}{2}, \infty)$. In case $d = 1$ we thus have $E = 1/\alpha$ for some $\alpha \in (0, 2)$ and $\nu$ is simply called $\alpha$-stable, respectively $(c^{1/\alpha}, c)$-semistable. In general, the exponent $E$ of an operator stable or operator semistable distribution $\nu$ is not unique due to symmetries $\{A \in \mathrm{GL}(\mathbb{R}^d) : A\nu = \nu\}$ of the distribution. If $E$ is an exponent and $A$ is a symmetry for $\nu$, also $AEA^{-1}$ is an exponent for $\nu$. Although exponents are not unique, the collection of real parts of the eigenvalues of any exponent $E$ for fixed $\nu$ is unique including their multiplicity. In case $d \geq 2$ we may choose a *commuting exponent* $E$ that commutes with every symmetry. The existence of a commuting exponent is proven in Hudson et al. (1986) for operator stable laws and in Hazod et al. (1998) for operator semistable laws, where in the latter case we may have to restrict considerations to $(c^{kE}, c^k)$-operator semistability for some $k \in \mathbb{N}$; see also Theorem 7.2.1 in Meerschaert and Scheffler (2001) and Theorem 1.11.6 in Hazod and Siebert (2001). In general, $k \geq 2$ may be necessary in the operator semistable situation as an example in Hazod et al. (1998) shows.

It is known by an example in Meerschaert (1990) that univariate marginals of an operator stable law are not necessarily stable distributions. In fact it is shown that there exists an operator stable distribution whose marginals are not necessarily in the domain of attraction of an univariate stable distribution. Later it turned out that such univariate marginals of an operator stable law are always semistable distributions, including the stable distributions, as shown in Theorem 1 of Meerschaert and Scheffler (1999). Our aim is to investigate to what extent the converse relation holds, i.e. to characterize all semistable distributions that can appear as an univariate marginal of an operator stable law.

Due to Theorem 3.2 in Meerschaert and Veeh (1993), in an appropriate basis $\mathbb{R}^d$ decomposes into $E$-invariant subspaces such that $E$ is a block diagonal matrix $E = E_1 \oplus \cdots \oplus E_p$, where each block $E_i = S_i + N_i$ is the sum of a semisimple matrix $S_i$, i.e. diagonalizable over the complex numbers, and a nilpotent matrix $N_i$ such that $N_i S_i = S_i N_i$. Moreover, the semisimple part $S_i$ is either diagonal or block diagonal with blocks of the form

$$B_i = \begin{pmatrix} 1/\alpha_i & -b_i \\ b_i & 1/\alpha_i \end{pmatrix} \quad \text{for some } \alpha_i \in (0,2) \text{ and } b_i > 0 \tag{1.2}$$

corresponding to complex conjugate eigenvalues of $E$. In the latter case the nilpotent part $N_i$ is given by

$$N_i = \begin{pmatrix} 0 & & & 0 \\ I & \ddots & & \\ & \ddots & \ddots & \\ 0 & & I & 0 \end{pmatrix} \text{, where } I \in \mathbb{R}^{2\times 2} \text{ is the identity matrix.} \tag{1.3}$$

By Theorem 1 in Meerschaert and Scheffler (1999), a semistable marginal can only appear in this latter case when projecting onto a one-dimensional subspace generated by a vector $e_0 \neq 0$ belonging to the kernel of the transpose $N_i^\top$. Thus (1.3) shows that only the first two coordinates of $e_0$ with respect to the basis representation corresponding to $E_i$ can be nonzero. By first projecting the operator stable law onto this two-dimensional subspace, we may restrict our considerations to dimension $d = 2$ without loss of generality.

In dimension $d = 2$, we can conclude that the commuting exponent $E$ is either $E = D + N$ for a diagonal matrix $D$ and a nilpotent matrix $N$, or $E$ is given by the semisimple part in (1.2) and, according to $N = 0$ in this situation, takes the form

$$E = \begin{pmatrix} 1/\alpha & -b \\ b & 1/\alpha \end{pmatrix} \quad \text{for some } \alpha \in (0,2) \text{ and } b > 0. \tag{1.4}$$

In the latter case we have the freedom to choose any $e_0 \in \mathbb{R}^2 \setminus \{0\}$ for the projection and by elementary calculations we get

$$t^E = t^{1/\alpha} \begin{pmatrix} \cos(b \log t) & -\sin(b \log t) \\ \sin(b \log t) & \cos(b \log t) \end{pmatrix} =: t^{1/\alpha} R(b \log t), \tag{1.5}$$

where $R(\varphi)$ corresponds to a counterclockwise rotation by the angle $\varphi$; see Lemma 2.2.3 in Meerschaert and Scheffler (2001) for details. Taking into account the above considerations, Theorem 1 in Meerschaert and Scheffler (1999) can be restated in dimension $d = 2$ as follows.

**Theorem 1.1.** *Let $\nu$ be a full operator stable law in $\mathbb{R}^2$ with commuting exponent $E$ and without Gaussian component. For $e_0 \in \mathbb{R}^2 \setminus \{0\}$ let $T_0(x) = \langle x, e_0\rangle$ and denote by $\nu_0 = T_0(\nu)$ the corresponding marginal distribution.*

(a) *If $E = D + N$ for a diagonal matrix $D = 1/\alpha \cdot I$ and a nilpotent matrix $N$, then $\nu_0$ is an $\alpha$-stable distribution for every $e_0$ belonging to the kernel of $N^\top$.*
(b) *If $E$ is semisimple of the form (1.4), then $\nu_0$ is a $(c^{1/\alpha}, c)$-semistable distribution for every $e_0 \in \mathbb{R}^2 \setminus \{0\}$, where $c = e^{2\pi/b}$.*

According to part (b) of Theorem 1.1 we have to choose $b = 2\pi/\log c$ in (1.4) so that $c = e^{2\pi/b}$ to be able to find an operator stable law with a genuine $(c^{1/\alpha}, c)$-semistable marginal distribution. It is also possible to generalize Theorem 1.1 from the operator stable to the operator semistable case with essentially the same proof as in Meerschaert and Scheffler (1999).

**Lemma 1.2.** *Let $\nu$ be a full $(c^E, c)$-operator semistable law in $\mathbb{R}^2$ for some $c > 1$ with commuting exponent $E$ and without Gaussian component. Let $\nu_0 = T_0(\nu)$ with $T_0(x) = \langle x, e_0\rangle$ for some $e_0 \in \mathbb{R}^2 \setminus \{0\}$.*

(a) *If $E = D + N$ for a diagonal matrix $D = 1/\alpha \cdot I$ and a nilpotent matrix $N$, then $\nu_0$ is a $(c^{1/\alpha}, c)$-semistable distribution for every $e_0$ belonging to the kernel of $N^\top$.*
(b) *If $E$ is semisimple of the form (1.4) with $b = 2\pi/\log c$, then $\nu_0$ is a $(c^{1/\alpha}, c)$-semistable distribution for every $e_0 \in \mathbb{R}^2 \setminus \{0\}$, where $c = e^{2\pi/b}$.*

**Proof.** (a) We have $N^\top e_0 = 0$ and hence $t^{N^\top} e_0 = e_0$ for all $t > 0$. Then $t^{E^\top} e_0 = t^D t^{N^\top} e_0 = t^{1/\alpha} e_0$ for all $t > 0$. Furthermore, (1.1) for $t = c$ implies

$$\nu_0^{*c} = T_0(\nu^{*c}) = T_0(c^E \nu * \delta_{a(c)}) = T_0(c^E \nu) * \delta_{T_0(a(c))}. \tag{1.6}$$

For the Fourier transform we get

$$\begin{aligned}\widehat{T_0(c^E\nu)}(s) &= \int_{\mathbb{R}^2} e^{isT_0(x)}\, d(c^E\nu)(x) = \int_{\mathbb{R}^2} e^{is\langle c^E x, e_0\rangle}\, d\nu(x) = \int_{\mathbb{R}^2} e^{is\langle x, c^{E^\top} e_0\rangle}\, d\nu(x)\\ &= \int_{\mathbb{R}^2} e^{isc^{1/\alpha}T_0(x)}\, d\nu(x) = \widehat{T_0(\nu)}(c^{1/\alpha}s) = \widehat{\nu}_0(c^{1/\alpha}s) = \widehat{(c^{1/\alpha}\nu_0)}(s)\end{aligned}$$

for all $s \in \mathbb{R}$, showing that $T_0(c^E\nu) = (c^{1/\alpha}\nu_0)$. By (1.6) we get $\nu_0^{*c} = (c^{1/\alpha}\nu_0) * \delta_{T_0(a(c))}$ showing that $\nu_0$ is $(c^{1/\alpha}, c)$-semistable.

(b) Since (1.6) holds, by (1.5) and $R(b \log c) = R(2\pi) = I$ for $c = e^{2\pi/b}$ we get for the Fourier transform

$$\widehat{T_0(c^E\nu)}(s) = \int_{\mathbb{R}^2} e^{is\langle c^E x, e_0\rangle}\, d\nu(x) = \int_{\mathbb{R}^2} e^{isc^{1/\alpha}\langle R(b\log c)x, e_0\rangle}\, d\nu(x) = \int_{\mathbb{R}^2} e^{isc^{1/\alpha}T_0(x)}\, d\nu(x) = \widehat{(c^{1/\alpha}\nu_0)}(s)$$

for all $s \in \mathbb{R}$, again showing that $T_0(c^E\nu) = (c^{1/\alpha}\nu_0)$. As in part (a) we conclude that $\nu_0$ is $(c^{1/\alpha}, c)$-semistable. □

It is easy to see that also the following converse of Lemma 1.2 is true. Given a $(c^{1/\alpha}, c)$-semistable distribution $\nu_0$ for some $c > 1$ and $\alpha \in (0, 2)$ there is a $(c^E, c)$-operator semistable law $\nu$ in $\mathbb{R}^2$ such that $\nu_0$ appears as a marginal distribution of $\nu$. Simply choose $\nu = \nu_0 \otimes \nu_0$ as the product measure which is $(c^E, c)$-operator semistable with diagonal exponent $E = 1/\alpha \cdot I$ and take $e_0 = (1, 0)^\top$ or $e_0 = (0, 1)^\top$ to get $\nu_0 = T_0(\nu)$ by projection. Note that for diagonal exponents the question whether operator semistability of a probability measure is characterized by semistability of its marginal distributions has been fully answered in Maejima and Samorodnitsky (1999) on $\mathbb{R}^d$ and the results have been extended to $p$-adic vector spaces in Maejima and Shah (2006) with even more general scaling automorphisms.

The remaining question is whether a given semistable distribution can appear as a marginal of an operator stable law. We will show in Section 2 that this is not true in general and characterize all semistable distributions that can appear as a marginal of an operator stable law $\nu$ in $\mathbb{R}^2$ in terms of the spectral representation of $\nu$.

## 2. Main results

We first give a general projection result for infinitely divisible laws in $\mathbb{R}^d$ without Gaussian component. Due to the Lévy-Khintchine representation, the Fourier transform $\widehat{\nu}$ of an infinitely divisible probability measure $\nu$ in $\mathbb{R}^d$ without Gaussian component can be written as $\widehat{\nu}(x) = \exp(\psi(x))$ for all $x \in \mathbb{R}^d$, where the *Lévy exponent* $\psi$ is given by

$$\psi(x) = i\langle x, a\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle x,y\rangle} - 1 - \frac{i\langle x, y\rangle}{1 + \|y\|^2}\right) d\phi(y) \tag{2.1}$$

for some unique $a \in \mathbb{R}^d$ and a *Lévy measure* $\phi$, i.e. a $\sigma$-finite measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d\setminus\{0\}} \min\{1, \|x\|^2\}\, d\phi(x) < \infty$; see Meerschaert and Scheffler (2001) or Sato (1999) for details. Since $a \in \mathbb{R}^d$ and the Lévy measure $\phi$ are unique, we may identify $\nu \sim [a, 0, \phi]$, where 0 corresponds to the absent Gaussian part.

Let $\nu \sim [a, 0, \phi]$ and $\nu_0 = T_0(\nu)$ with $T_0(x) = \langle x, e_0\rangle$ for some $e_0 \in \mathbb{R}^d \setminus \{0\}$. Since $\nu_0 \sim [a_0, 0, \phi_0]$ is infinitely divisible, the Lévy exponent $\psi_0$ of $\nu_0$ fulfills

$$\exp(\psi_0(t)) = \widehat{\nu}_0(t) = \widehat{T_0(\nu)}(t) = \int_{\mathbb{R}} \exp(its)\, d(T_0(\nu))(s) = \int_{\mathbb{R}^d} \exp(i\langle x, t\cdot e_0\rangle)\, d\nu(x) = \widehat{\nu}(t\cdot e_0) = \exp(\psi(t\cdot e_0)).$$

Hence by the Lévy-Khintchine representation (2.1) we have

$$\begin{aligned}&ita_0 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{its} - 1 - \frac{its}{1+s^2}\right) d\phi_0(s) = \psi_0(t) = \psi(t\cdot e_0) = i\langle t\cdot e_0, a\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle t\cdot e_0, x\rangle} - 1 - \frac{i\langle t\cdot e_0, x\rangle}{1+\|x\|^2}\right) d\phi(x)\\ &= it\left(T_0(a) + \int_{\mathbb{R}^d\setminus\{0\}} \left(\frac{T_0(x)}{1+T_0(x)^2} - \frac{T_0(x)}{1+\|x\|^2}\right) d\phi(x)\right) + \int_{\mathbb{R}\setminus\{0\}} \left(e^{its} - 1 - \frac{its}{1+s^2}\right) d(T_0(\phi))(s).\end{aligned}$$

Note that the first integral on the right-hand side exists, since elementary calculations yield

$$\left| \frac{T_0(x)}{1+T_0(x)^2} - \frac{T_0(x)}{1+\|x\|^2} \right| \le C \cdot \min\{1, \|x\|^2\}$$

for some constant $C > 0$. Uniqueness of the representation directly implies:

**Lemma 2.1.** *Let $\nu \sim [a, 0, \phi]$ be an infinitely divisible law in $\mathbb{R}^d$ and $T_0(x) = \langle x, e_0 \rangle$ for some $e_0 \in \mathbb{R}^d \setminus \{0\}$. Then an infinitely divisible distribution $\nu_0 = [a_0, 0, \phi_0]$ in $\mathbb{R}$ fulfills $T_0(\nu) = \nu_0$ if and only if*

$$\phi_0 = T_0(\phi) \quad \text{and} \tag{2.2}$$

$$a_0 = T_0(a) + \int_{\mathbb{R}^d \setminus \{0\}} \left( \frac{T_0(x)}{1+T_0(x)^2} - \frac{T_0(x)}{1+\|x\|^2} \right) d\phi(x). \tag{2.3}$$

Thus, to construct an infinitely divisible law $\nu \sim [a, 0, \phi]$ with marginal $\nu_0$, it suffices to construct a Lévy measure $\phi$ such that $T_0(\phi) = \phi_0$ and to set

$$a = \|e_0\|^{-2} \left( a_0 - \int_{\mathbb{R}^d \setminus \{0\}} \left( \frac{T_0(x)}{1+T_0(x)^2} - \frac{T_0(x)}{1+\|x\|^2} \right) \phi(dx) \right) \cdot e_0. \tag{2.4}$$

Now, let $\nu \sim [a, 0, \phi]$ be an operator stable law with exponent $E$ in which case the Lévy measure fulfills $t^E \phi = t \cdot \phi$ for all $t > 0$. To analyze (2.2) in this situation, we will use the following spectral representation of the Lévy measure from Theorem 7.2.5 in Meerschaert and Scheffler (2001). For all Borel sets $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ we have

$$\phi(B) = \int_{\{\|\theta\|_0 = 1\}} \int_0^\infty 1_B(r^E \theta) \frac{dr}{r^2} \, d\sigma(\theta), \tag{2.5}$$

where the *spectral measure* $\sigma$, which uniquely determines $\phi$, is a bounded Borel measure on the unit sphere with respect to $\|\cdot\|_0$. The norm $\|\cdot\|_0$ fulfills that $t \mapsto \|t^E x\|_0$ is strictly increasing for all $x \neq 0$ and can be chosen by Lemma 6.1.5 in Meerschaert and Scheffler (2001) (in combination with Theorem 6.1.7 of Meerschaert and Scheffler, 2001) as

$$\|x\|_0 = \int_0^1 \|t^E x\| \frac{dt}{t}.$$

As argued in the Introduction, in dimension $d = 2$ we can restrict our considerations to commuting exponents $E$ of the form (1.4) with $b = 2\pi / \log c$. Then (1.5) implies $\|t^E x\| = t^{1/\alpha} \|x\|$ and we easily calculate $\|x\|_0 = \alpha \|x\|$ for all $x \in \mathbb{R}^2$. Hence, for simplicity we will choose $\|\cdot\|_0$ as the Euclidean norm $\|\cdot\|$ in the sequel. In this case the spectral measure is given by $\sigma(A) = \phi\{t^E \theta : \theta \in A, t > 1\}$ for Borel sets $A \subseteq \mathbb{T} = \{\|\theta\| = 1\}$ and captures the dependence structure, whereas the exponent $E$ is responsible for the scaling.

Now, let $\nu_0 \sim [a_0, 0, \phi_0]$ be a $(c^{1/\alpha}, c)$-semistable distribution in $\mathbb{R}$ for some $c > 1$ and $\alpha \in (0, 2)$. Then by Corollary 7.4.4 in Meerschaert and Scheffler (2001) the Lévy measure $\phi_0$ is given by

$$\phi_0(t, \infty) = t^{-\alpha} h_1(\log t) \quad \text{and} \quad \phi_0(-\infty, -t) = t^{-\alpha} h_2(\log t) \tag{2.6}$$

for all $t > 0$ and some non-negative $\log(c^{1/\alpha})$-periodic and bounded functions $h_1, h_2$ such that $h_1 + h_2 > 0$. This result goes back to Kruglov (1972, Theorem 1) or Mejzler (1973, Theorem 4.1). Since $T_0(\phi) = \phi_0$ by Lemma 2.1, using (2.6), (2.5), the representation of $R$ in (1.5) and a change of variables $s = t^{-\alpha} r$ we get

$$\begin{aligned} h_1(\log t) &= t^\alpha \cdot \phi_0\{s : s > t\} = t^\alpha \cdot T_0(\phi)\{s : s > t\} = t^\alpha \cdot \phi\{x : T_0(x) > t\} = t^\alpha \int_{\{\|\theta\|=1\}} \int_0^\infty \mathbf{1}_{\{T_0(r^E\theta) > t\}} \frac{dr}{r^2} \, d\sigma(\theta) \\ &= t^\alpha \int_{\{\|\theta\|=1\}} \int_0^\infty \mathbf{1}_{\{T_0(R(b \log r)\theta) > r^{-1/\alpha} t\}} \frac{dr}{r^2} \, d\sigma(\theta) = \int_{\{\|\theta\|=1\}} \int_0^\infty \mathbf{1}_{\{T_0(R(b \log s + b\alpha \log t)\theta) > s^{-1/\alpha}\}} \frac{ds}{s^2} \, d\sigma(\theta). \end{aligned} \tag{2.7}$$

Write $\theta = (\cos\varphi, \sin\varphi)^\top =: T(\varphi)$, where $T : [0, 2\pi) \to \{\|\theta\| = 1\}$ is the transformation to polar coordinates, and define the bounded Borel measure $\mu := T^{-1}(\sigma)$ in $[0, 2\pi)$. Recall the periodic functions $h_1, h_2$ from (2.6) which determine the positive and negative tail of the Lévy measure $\phi_0$. For $s > 0$ we further introduce the $2\pi$-periodic non-negative functions

$$h_1^*(s) := h_1\left(-\frac{s}{b\alpha}\right) \quad \text{and} \quad f_1(s) := \int_0^\infty \mathbf{1}_{\{T_0(T((b \log r - s)(\mathrm{mod}\, 2\pi))) > r^{-1/\alpha}\}} \frac{dr}{r^2}$$

as well as

$$h_2^*(s) := h_2\left(-\frac{s}{b\alpha}\right) \quad \text{and} \quad f_2(s) := \int_0^\infty \mathbf{1}_{\{T_0(T((b \log r - s)(\mathrm{mod}\, 2\pi))) < -r^{-1/\alpha}\}} \frac{dr}{r^2}.$$

**Lemma 2.2.** *With the above notations we have $h_1^* = f_1 * \mu$ and $h_2^* = f_2 * \mu$. Hence for the Fourier coefficients we get*

$$\widehat{h_1^*}(k) = \widehat{f_1}(k) \cdot \widehat{\mu}(k) \quad \text{and} \quad \widehat{h_2^*}(k) = \widehat{f_2}(k) \cdot \widehat{\mu}(k) \quad \text{for all } k \in \mathbb{Z}.$$

**Proof.** First note that by (1.5) we have

$$R(b \log r - s) T(\varphi) = \begin{pmatrix} \cos(b \log r - s) & -\sin(b \log r - s) \\ \sin(b \log r - s) & \cos(b \log r - s) \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \cos(\varphi + b \log r - s) \\ \sin(\varphi + b \log r - s) \end{pmatrix} = T((\varphi + b \log r - s)(\mathrm{mod}\, 2\pi)).$$

From (2.7) and $b\frac{\log c}{2\pi}=1$ it follows that

$$h_1^*(s)=h_1\left(-\frac{s}{b\alpha}\right)=\int_{\{\|\theta\|=1\}}\int_0^\infty \mathbf{1}_{\{T_0(R(b\log r-s)\theta)>r^{-1/\alpha}\}}\frac{dr}{r^2}\,d\sigma(\theta)=\int_0^{2\pi}\int_0^\infty \mathbf{1}_{\{T_0(R(b\log r-s)T(\varphi))>r^{-1/\alpha}\}}\frac{dr}{r^2}\,d\mu(\varphi)$$
$$=\int_0^{2\pi}\int_0^\infty \mathbf{1}_{\{T_0(T((\varphi+b\log r-s)(\mathrm{mod}\,2\pi)))>r^{-1/\alpha}\}}\frac{dr}{r^2}\,d\mu(\varphi)=\int_0^{2\pi}f_1(s-\varphi)\,d\mu(\varphi)=(f_1*\mu)(s).$$

A similar calculation also shows that $h_2^*=f_2*\mu$ holds, concluding the proof. $\square$

We now analyze the function $f_1$ and its Fourier coefficients in more detail. Write $e_0=\|e_0\|\cdot(\cos\theta_0,\sin\theta_0)^\top$ for some $\theta_0\in[0,2\pi)$, i.e. $\theta_0=T^{-1}(e_0/\|e_0\|)$. Then we get

$$T_0(T((b\log r-s)(\mathrm{mod}\,2\pi)))=\left\langle\begin{pmatrix}\cos(b\log r-s)\\ \sin(b\log r-s)\end{pmatrix},e_0\right\rangle=\|e_0\|\cdot\cos(b\log r-s-\theta_0). \tag{2.8}$$

**Lemma 2.3.** *Let $g_1$ be the $2\pi$-periodic non-negative function given by*

$$g_1(t):=\int_1^\infty \mathbf{1}_{\{\cos(b\log r-t)>r^{-1/\alpha}\}}\frac{dr}{r^2},\quad t\in\mathbb{R}.$$

*Then $g_1$ has Fourier coefficients*

$$\widehat{g_1}(k)=\frac{2}{1-ikb}\int_0^{\pi/2}\cos(kt)\cdot(\cos t)^{\alpha(1-ikb)}\,dt,\quad k\in\mathbb{Z},$$

*and for the function $f_1$ we have*

$$f_1=\|e_0\|^\alpha\left(g_1*\delta_{(-\theta_0-\alpha b\log\|e_0\|)(\mathrm{mod}\,2\pi)}\right).$$

**Proof.** For the Fourier coefficients of $g_1$ we get by Fubini's theorem and a change of variables $s=t-b\log r$

$$\widehat{g_1}(k)=\int_0^{2\pi}e^{ikt}\int_1^\infty \mathbf{1}_{\{\cos(b\log r-t)>r^{-1/\alpha}\}}\frac{dr}{r^2}\,dt=\int_1^\infty\int_0^{2\pi}e^{ikt}\,\mathbf{1}_{\{\cos(b\log r-t)>r^{-1/\alpha}\}}\,dt\,\frac{dr}{r^2}$$
$$=\int_1^\infty e^{ikb\log r}\int_{-\pi}^{\pi}e^{iks}\,\mathbf{1}_{\{\cos(s)>r^{-1/\alpha}\}}\,ds\,\frac{dr}{r^2}=2\int_1^\infty e^{ikb\log r}\int_0^{\pi}\cos(ks)\,\mathbf{1}_{\{\cos(s)>r^{-1/\alpha}\}}\,ds\,\frac{dr}{r^2}$$
$$=2\int_1^\infty r^{ikb}\int_0^{\pi/2}\cos(ks)\,\mathbf{1}_{\{\cos(s)>r^{-1/\alpha}\}}\,ds\,\frac{dr}{r^2},$$

where we used that $s\mapsto\mathbf{1}_{\{\cos(s)>r^{-1/\alpha}\}}$ is an even function that vanishes on $[\frac{\pi}{2},\pi]$. Another application of Fubini's theorem gives us

$$\widehat{g_1}(k)=2\int_0^{\pi/2}\cos(ks)\int_{(\cos s)^{-\alpha}}^\infty r^{ikb-2}\,dr\,ds=\frac{2}{1-ikb}\int_0^{\pi/2}\cos(ks)\cdot(\cos s)^{\alpha(1-ikb)}\,ds.$$

For $f_1$ from Lemma 2.2 we get using (2.8) and a change of variables $w=\|e_0\|^\alpha r$

$$f_1(s)=\int_0^\infty \mathbf{1}_{\{\|e_0\|\cos(b\log r-s-\theta_0)>r^{-1/\alpha}\}}\frac{dr}{r^2}=\|e_0\|^\alpha\int_0^\infty \mathbf{1}_{\{\cos(b\log(\|e_0\|^{-\alpha}w)-s-\theta_0)>w^{-1/\alpha}\}}\frac{dw}{w^2}$$
$$=\|e_0\|^\alpha\int_1^\infty \mathbf{1}_{\{\cos(b\log w-s-\alpha b\log\|e_0\|-\theta_0)>w^{-1/\alpha}\}}\frac{dw}{w^2},$$

which shows that $f_1(s-\theta_0-\alpha b\log\|e_0\|)=\|e_0\|^\alpha g(s)$. An application of the Fourier transform easily gives us $f_1=\|e_0\|^\alpha\cdot\left(g_1*\delta_{(-\theta_0-\alpha b\log\|e_0\|)(\mathrm{mod}\,2\pi)}\right)$. $\square$

**Remark 2.4.** A similar calculation shows that the function

$$g_2(t):=\int_1^\infty \mathbf{1}_{\{\cos(b\log r-t)<-r^{-1/\alpha}\}}\frac{dr}{r^2},\quad t\in\mathbb{R},$$

fulfills $f_2=\|e_0\|^\alpha\left(g_2*\delta_{(-\theta_0-\alpha b\log\|e_0\|)(\mathrm{mod}\,2\pi)}\right)$ and for every $k\in\mathbb{Z}$ has Fourier coefficient $\widehat{g_2}(k)=(-1)^k\widehat{g_1}(k)$. Since $(-1)^k=e^{ik\pi}$, we easily get $f_2(t)=f_1(t-\pi)$ and by Lemma 2.2 we conclude $h_2^*(t)=h_1^*(t-\pi)$, thus $h_2(t)=h_1(t-\frac{1}{2}\log(c^{1/\alpha}))$ for all $t\in\mathbb{R}$. Hence $h_2$ is a phase shift of $h_1$ by half the period. This will also follow from our main theorem below, therefore we will not provide the detailed calculations.

Our main result is a combination of the Lemmas 2.1–2.3 that we rather formulate on $\mathbb{T}=\{\|\theta\|=1\}\subseteq\mathbb{R}^2$ than on $[0,2\pi)$. Therefore, we introduce the functions

$$h_{\mathbb{T}}=h_1^*\circ T^{-1}\quad\text{and}\quad g_{\mathbb{T}}=g_1\circ T^{-1}$$

with the above transformation $T:[0,2\pi)\to\mathbb{T}$ to polar coordinates.

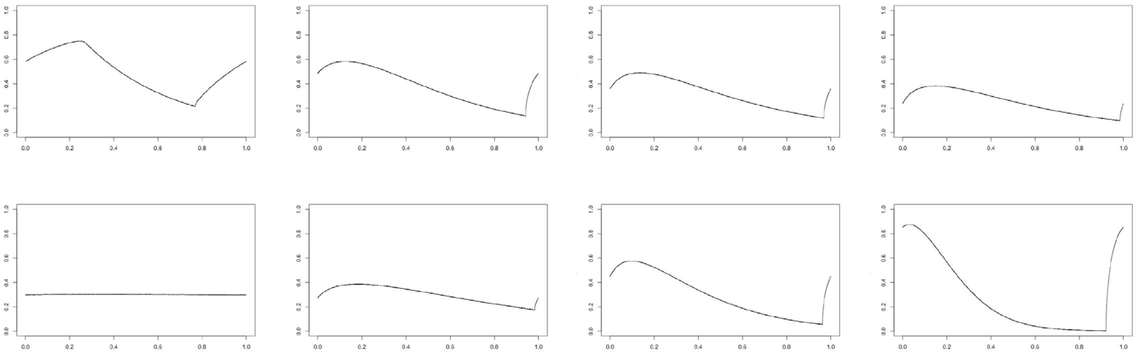

**Fig. 1.** Numerical approximation of the graph of $g_1(2\pi t)$ for $t \in [0,1]$ and various values of $\alpha \in (0,2)$ and $c > 1$. Top row: $c = 12$ and (from left to right) $\alpha = 0.01$, $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.99$. Bottom row: $\alpha = 1.2$ and (from left to right) $c = 1.1$, $c = 5$, $c = 42$, $c = 6456$.

**Theorem 2.5.** *Let $\nu = [a, 0, \phi]$ be an operator stable law in $\mathbb{R}^2$ with commuting exponent $E$ of the form (1.4) for $b = 2\pi/\log c > 0$ and with corresponding spectral measure $\sigma$. Let $\nu_0 = [a_0, 0, \phi_0]$ be a $(c^{1/\alpha}, c)$-semistable distribution in $\mathbb{R}$ with Lévy measure $\phi_0$ given by (2.6) and corresponding $2\pi$-periodic functions $h_1^*$ and $h_2^*$. Then for every $e_0 = \|e_0\| \cdot (\cos\theta_0, \sin\theta_0)^\top \in \mathbb{R}^2 \setminus \{0\}$ and $T_0(x) = \langle x, e_0\rangle$ we have $T_0(\nu) = \nu_0$ if and only if the following three conditions hold:*

(i) *For the positive tail of $\phi_0$ we have*

$$h_{\mathbb{T}} = \|e_0\|^\alpha \left(\delta_{\|e_0\|^{-\alpha E} e_0} * g_{\mathbb{T}} * \sigma\right),$$

*where the function $g_{\mathbb{T}} : \mathbb{T} \to (0,\infty)$ only depends on $E$ and is uniquely given by the Fourier coefficients $\widehat{g_{\mathbb{T}}}(k) = \widehat{g}_1(k)$ from Lemma 2.3 with $b = \frac{2\pi}{\log c}$.*

(ii) *For the negative tail of $\phi_0$ we have $h_2(t) = h_1(t - \frac{1}{2}\log(c^{1/\alpha}))$ for all $t \in \mathbb{R}$.*

(iii) *The drift coefficients $a_0 \in \mathbb{R}$ and $a \in \mathbb{R}^2$ fulfill (2.3).*

**Proof.** By Lemma 2.1 we have to show that $\phi_0 = T_0(\phi)$ is equivalent to the fulfillment of (i) and (ii). Since $\phi_0 = T_0(\phi)$ is equivalent to (2.7) together with a corresponding result for $h_2$, it is equivalent to the statement of Lemma 2.2. Since $\mu = T^{-1}(\sigma)$, the combination of Lemmas 2.2 and 2.3 shows equivalence to the fulfillment of (i) and a corresponding statement for $h_2$. Note that the Dirac measure in Lemma 2.3 has to be interpreted on $\mathbb{T}$ as a rotation by

$$\begin{aligned}\begin{pmatrix}\cos(-\theta_0 - \alpha b \log\|e_0\|)\\ \sin(-\theta_0 - \alpha b\log\|e_0\|)\end{pmatrix} &= R\big(b\log(\|e_0\|^{-\alpha})\big)\cdot\begin{pmatrix}\cos(\theta_0)\\ \sin(\theta_0)\end{pmatrix}\\ &= \big(\|e_0\|^{-\alpha}\big)^{1/\alpha} R\big(b\log(\|e_0\|^{-\alpha})\big)\cdot\|e_0\|\begin{pmatrix}\cos(\theta_0)\\ \sin(\theta_0)\end{pmatrix} = \|e_0\|^{-\alpha E}e_0 \in \mathbb{T},\end{aligned}$$

where we used (1.5). Finally, the above mentioned corresponding statement of (i) for $h_2$ is (ii), since the roles of $h_1$ and $h_2$ interchange when switching from $e_0$ to $-e_0$. This only has an effect on the Dirac measure in (i), which changes to $\delta_{-\|e_0\|^{-\alpha E}e_0}$ and gives the phase shift by half the period. □

The phase translation in Theorem 2.5(ii) shows that not every semistable distribution in $\mathbb{R}$ can appear as a marginal of an operator stable law in $\mathbb{R}^2$. The following remark shows that even $h_1$ cannot be an arbitrary $\log(c^{1/\alpha})$-periodic function such that $t \mapsto t^{-\alpha}h_1(\log t)$ is non-increasing.

**Remark 2.6.** Note that a change of variables $s = r^{-1}$ for $g_1$ in Lemma 2.3 gives

$$g_1(t) = \int_1^\infty \mathbf{1}_{\{\cos(b\log r - t) > r^{-1/\alpha}\}}\frac{dr}{r^2} = \int_0^1 \mathbf{1}_{\{\cos(-b\log s - t) > s^{1/\alpha}\}}\,ds = \lambda\left(\left\{s \in (0,1) : \big(\cos(-b\log(s\,e^{t/b}))\big)^\alpha > s\right\}\right), \tag{2.9}$$

which for fixed $t \in \mathbb{R}$ is the Lebesgue measure of the set of points in the unit interval where the continuous function $s \mapsto \big(\cos(-b\log(s\,e^{t/b}))\big)^\alpha$ lies above the diagonal. Varying the scaling factor $e^{t/b}$ of this function, continuously changes the value of the Lebesgue measure for this set in the variable $t$, which shows that $g_1$ is continuous. As a consequence of Theorem 2.5(i) also $h_{\mathbb{T}}$ and thus $h_1$ are continuous. This rules out semistable distributions with a discontinuous tail function of the Lévy measure $\phi_0$. E.g., for $c = 2$ and $\alpha = 1$ the semistable limit distribution of successive gains in the St. Petersburg game appearing in Martin-Löf (1985) has a discrete Lévy measure $\phi_0$ on $2^{\mathbb{Z}}$ with $\phi_0(\{2^k\}) = 2^{-k}$ for all $k \in \mathbb{Z}$ such that for $t \in [2^n, 2^{n+1})$ we get by (2.6)

$$h_1(\log t) = t\cdot\phi_0(t,\infty) = t\sum_{k=n+1}^\infty 2^{-k} = t\cdot 2^{-n}$$

for all $n \in \mathbb{Z}$, which is a sawtooth function with discontinuities in $2^{\mathbb{Z}}$.

**Example 2.7.** To get an impression of how the periodic function $g_1$ looks like, we plotted the graph of $g_1(2\pi t)$ for $t \in [0,1]$ numerically using (2.9) for various values of $\alpha \in (0,2)$ and $c > 1$ in Fig. 1.

In case $\sigma = \delta_{-\|e_0\|^{-\alpha E} e_0}$ all these functions are valid examples for the function $h_1^* = g_1$ by Theorem 2.5(i). Note that the Lévy measure $\phi$ of the operator stable law $\nu$ is then concentrated on the orbit $\{-s^E e_0 : s > 0\}$, whereas by (2.6) and Theorem 2.5(ii) the Lévy measure $\phi_0$ of the semistable distribution $\nu_0 = T_0(\nu)$ is fully determined by the periodic function $h_1(t) = h_1^*(-b\alpha t) = g_1(-b\alpha t)$.

## Data availability

No data was used for the research described in the article.

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