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Mathematische Zeitschrift (2025) 310:57
https://doi.org/10.1007/s00209-025-03690-1

Mathematische Zeitschrift

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Received: 2 November 2022 / Accepted: 17 January 2025

Abstract

We adapt methods from quiver representation theory and Hall algebra techniques to the counting of representations of virtually free groups over finite fields. This gives rise to the computation of the E-polynomials of $\mathbf{GL}_d(\mathbb{C})$-character varieties of virtually free groups. As examples we discuss counting of representations of the groups $\mathbb{D}_\infty$, $\mathbf{PSL}_2(\mathbb{Z})$, $\mathbf{SL}_2(\mathbb{Z})$, $\mathbf{GL}_2(\mathbb{Z})$ and $\mathbf{PGL}_2(\mathbb{Z})$.

Mathematics Subject Classification 20C07 · 16G99 · 14L30 · 14D20 · 32S35

1 Introduction

Arithmetic representation growth deals with counting the number of representations of algebras over finite fields. More precisely it is the study of the following *counting functions:* For $\mathcal{A}$ a finitely generated $\mathbb{F}_q$-algebra and $d \in \mathbb{N}_0, \alpha \in \mathbb{N}_{\geq 1}$ define

$$\begin{aligned} r_d^{\mathrm{ss},\mathcal{A}}(q^\alpha) &:= \#\mathrm{ssim}_d\left(\mathcal{A} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^\alpha}\right) \\ r_d^{\mathrm{sim},\mathcal{A}}(q^\alpha) &:= \#\mathrm{sim}_d\left(\mathcal{A} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^\alpha}\right) \\ r_d^{\mathrm{absim},\mathcal{A}}(q^\alpha) &:= \#\mathrm{absim}_d\left(\mathcal{A} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^\alpha}\right) \end{aligned} \tag{1}$$

Here we denote by $\mathrm{iso}_d(\mathcal{B}) \supseteq \mathrm{ssim}_d(\mathcal{B}) \supseteq \mathrm{sim}_d(\mathcal{B}) \supseteq \mathrm{absim}_d(\mathcal{B})$ for each d the sets of isomorphism classes of all, of all semisimple, of all simple and of all absolutely simple left modules $\mathcal{M}$ over a K-algebra $\mathcal{B}$ of dimension $\dim_K(\mathcal{M}) = d$. Recall that a left $\mathcal{B}$-module $\mathcal{M}$ is called *absolutely simple* if it is simple and $\mathrm{End}_{\mathcal{B}}(\mathcal{M}) = K$ or equivalently if $\mathcal{M} \otimes_K \overline{K}$ is simple for the algebraic closure $\overline{K} \supseteq K$.

(1) defines functions $r_d^{\mathrm{ss},\mathcal{A}}, r_d^{\mathrm{sim},\mathcal{A}}, r_d^{\mathrm{absim},\mathcal{A}}$ on all q-powers. We call these functions counting functions, as they count the semisimple, simple and absolutely simple modules/representations of $\mathcal{A}$ over $\mathbb{F}_q$ up to isomorphism. If the algebra $\mathcal{A}$ is understood, we will usually drop it from the notation.

The counting functions (1) have been studied by S. Mozgovoy and M. Reineke in the cases $\mathcal{A} = \mathbb{F}_q\vec{Q}$ the path algebra of a finite quiver and $\mathcal{A} = \mathbb{F}_q[F_a]$ the group algebra of

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a finitely generated free group. One of their main results is the following theorem (see [21, Thm. 6.2] and [17, Thm. 1.1]).[1]

Theorem 1.1 *If $\mathcal{A}$ is the path algebra of a finite quiver or the group algebra of a finitely generated free group (over $\mathbb{F}_q$ respectively), then there are polynomials $R_d^{\mathrm{ss}}, R_d^{\mathrm{absim}} \in \mathbb{Z}[s]$ and $R_d^{\mathrm{sim}} \in \mathbb{Q}[s]$ fulfilling*

$$\forall\, \alpha \geq 1 : R_d^{\mathrm{absim}}(q^\alpha) = r_d^{\mathrm{absim}}(q^\alpha)\ ,\ R_d^{\mathrm{sim}}(q^\alpha) = r_d^{\mathrm{sim}}(q^\alpha)\ ,\ R_d^{\mathrm{ss}}(q^\alpha) = r_d^{\mathrm{ss}}(q^\alpha) \quad (2)$$

We call such polynomials realizing the counting functions (1) *counting polynomials.* More important than the mere existence of counting polynomials is the fact that Mozgovoy–Reineke obtained certain generating formulas (see [17, Thm. 1.2]) which enable the practical computation of the counting polynomials (in low dimensions).

The main goal of this paper is to generalize Theorem 1.1 as well as the above mentioned generating formulas to the case where $\mathcal{A} = \mathbb{F}_q[\mathcal{G}]$ is the group algebra of a finitely generated virtually free group (see Theorem 5.4 below). Furthermore we will investigate a few structural properties of the counting polynomials and relate these to the geometry of GIT moduli spaces of representations. SageMath code designed by the author for the practical computation of the counting polynomials is provided at [10].

The main reason why we are studying virtually free groups in this paper is W. Dicks's characterization of hereditary group algebras (see [4, Thm. 1]): If $\mathcal{H}$ is a finitely generated group and K a field, then the group algebra $K[\mathcal{H}]$ is (left) hereditary[2] if and only if $\mathcal{H}$ is virtually free and contains no elements of order char (K). For a given finitely generated virtually free group $\mathcal{H}$ there are only finitely many prime numbers for which $K[\mathcal{H}]$ is not hereditary, which we will call *non-suitable prime numbers*. The hereditariness of the group algebra will be needed to make certain Hall algebra techniques available (see Lemma 4.2 below), which have also been used by Mozgovoy and Reineke when proving Theorem 1.1.

This paper is organized as follows: we start by recalling most of the relevant preknowledge on virtually free groups and algebraic geometry within Sect. 2. Afterwards we discuss some invariants like dimension vectors and the homological Euler form in the context of representations of virtually free groups in Sect. 3. In Sect. 4 we review some Hall algebra methods, before we discuss the main result 5.4 in Sect. 5. Section 6 is devoted to hands-on examples and we conclude in Sect. 7 with discussing a few structural properties of the counting polynomials.

2 Preliminaries

In this section we summarize the preknowledge from group theory, representation theory and algebraic geometry needed within this paper. All the results discussed in the Sects. 2.2–2.4 are non-original and probably well-known to the experts.

2.1 Virtually free groups

In this subsection we recall some group theoretic notions, define the class of groups that we will work with in this paper (see Definition 2.1) and prove that this class of groups coincides

[1] In the quiver case Mozgovoy–Reineke's result was in the more general context of counting absolutely stable representation of a fixed dimension vector. We state it in a weaker form here for expository purposes.

[2] Since every group is isomorphic to its opposite, left and right hereditaryness are equivalent.

with the class of finitely generated virtually free groups (see Lemma 2.4). To understand (most of) the content of the rest of this paper it is not necessary to comprehend all technicalities in this subsection—Definition 2.1 and Remark 2.5 should be sufficient to be able to read through all of the following sections.

Given three groups $\mathcal{E}, \mathcal{F}, \mathcal{H}$ as well as injective group homomorphisms $\iota : \mathcal{F} \hookrightarrow \mathcal{H}$, $\kappa : \mathcal{F} \hookrightarrow \mathcal{E}$ we denote their pushout in the category of groups by $\mathcal{H} *_{\mathcal{F}} \mathcal{E}$ and call it the *amalgamated free product* of $\mathcal{H}$ and $\mathcal{E}$ over $\mathcal{F}$. Recall from e.g. [12, §IV.2] that if $\mathcal{H} = \langle X \mid S \rangle$ and $\mathcal{E} = \langle Y \mid T \rangle$ are presentations of the groups $\mathcal{H}$ and $\mathcal{E}$ in terms of generators and relations, then a presentation of $\mathcal{H} *_{\mathcal{F}} \mathcal{E}$ is given by

$$\left\langle X \cup Y \mid S \cup T \cup \{\kappa(f) \cdot \iota(f)^{-1} \mid f \in \mathcal{F}\} \right\rangle \tag{3}$$

If we are given two embeddings $\iota, \kappa : \mathcal{F} \hookrightarrow \mathcal{H}$ with the same codomain instead, we may consider the induced embeddings

$$\iota', \kappa' : \mathcal{F} \hookrightarrow \mathcal{H} * C_\infty, \quad \iota'(f) := \iota(f), \quad \kappa'(f) := t^{-1}\kappa(f)t$$

where t denotes the generator of the infinite cyclic group C_∞. We denote the coequalizer of

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook, shift left, "\iota'"] \arrow[r, hook, shift right, "\kappa'"'] & \mathcal{H} * C_\infty \end{tikzcd}$$

by $\mathcal{H}*_{\mathcal{F}}^{\iota,\kappa}$ (see e.g. [24, §I.1.1, Prop. 1] for the existence of colimits in the category of groups). This construction is known as the *Higman–Neumann–Neumann extension* (or *HNN extension*) of $\mathcal{H}$ by $\mathcal{F}$ (see e.g. [12, §IV.2]). If $\mathcal{H} = \langle X \mid S \rangle$ is a presentation, then a presentation of $\mathcal{H}*_{\mathcal{F}}^{\iota,\kappa}$ is given by

$$\left\langle X \cup \{t\} \mid S \cup \{t^{-1} \cdot \kappa(f) \cdot t \cdot \iota(f)^{-1} \mid f \in \mathcal{F}\} \right\rangle \tag{4}$$

(see e.g. [12, §IV.2]).

Even though our description of amalgamated free products and HNN extensions make sense if the homomorphisms ι and κ are not injective, we will follow the usual convention in group theory and only consider the case where they are. However, the general machinery of this paper works for non-injective ι, κ too (see the Remarks 2.5 and 2.7 below).

A group $\mathcal{G}$ is called *virtually free* if it contains a finite index subgroup $\mathcal{H}$ which itself is a free group. However, the theory of Bass–Serre provides an equivalent characterization of virtually free groups (see [24, §II.2.6], [8, Thm. 1]): A finitely generated group $\mathcal{G}$ is virtually free if and only if it is isomorphic to the *fundmental group* $\pi_1(\mathcal{G}_Q)$ *of a finite graph of finite groups* $\mathcal{G}_Q$.

Since we will only use the notions of graphs of groups and their fundamental groups explicitly in the proof of Lemma 2.4 below, we will only sketch their definitions and give references for more details. Instead we derive a rigorous notion of *decomposition* from it that will be more suitable for our purposes. A *graph of groups* $\mathcal{G}_Q$ consists of a connected, non-empty undirected graph Q, a group $\mathcal{G}_i$ for each vertex i in Q and a group $\mathcal{G}'_j$ together with two injective group homomorphisms

$$\mathcal{G}_{s(j)} \overset{\iota_j}{\hookleftarrow} \mathcal{G}'_j \overset{\kappa_j}{\hookrightarrow} \mathcal{G}_{t(j)}$$

for each edge j in Q, where $s(j)$ and $t(j)$ are the vertices adjacent to the edge j (see [24, §I.4.4, Def. 8]). $\mathcal{G}_Q$ is called a *finite graph of finite groups* if the graph Q has finitely many vertices and edges and if all of the groups $\mathcal{G}_i$ and $\mathcal{G}'_j$ are finite.

For any graph of groups $\mathcal{G}_Q$ one can associate its *fundamental group* $\pi_1(\mathcal{G}_Q)$ which is a group that contains all the groups $\mathcal{G}_i$ and $\mathcal{G}'_j$ as subgroups (see [24, §I.5.1]). Since we will not work with this construction explicitly, we will not discuss it here, but give two basic examples instead:

(1) If Q has a single edge labeled $j = 1$ connecting two distinct vertices labeled $s(1) = 0$ and $t(1) = 1$, then a graph of groups $\mathcal{G}_Q$ with underlying graph Q consists of three groups $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_1'$ and two injective group homomorphisms

$$\iota_1 : \mathcal{G}_1' \hookrightarrow \mathcal{G}_0, \quad \kappa_1 : \mathcal{G}_1' \hookrightarrow \mathcal{G}_1$$

The fundamental group of this graph of groups is given by the amalgamated free product $\mathcal{G}_0 *_{\mathcal{G}_1'} \mathcal{G}_1$.

(2) If Q has a single edge labeled $j = 1$ which is a loop at a single vertex $i = 0$, then a graph of groups $\mathcal{G}_Q$ with underlying graph Q consists of two groups $\mathcal{G}_0$ and $\mathcal{G}_1'$ and two injective group homomorphisms

$$\mathcal{G}_1' \underset{\kappa_1}{\overset{\iota_1}{\rightrightarrows}} \mathcal{G}_0$$

The fundamental group of this graph of groups is given by the HNN extension $\mathcal{G}_0 *_{\mathcal{G}_1'}^{\iota_1,\kappa_1}$.

For all other (finite) graphs Q the fundamental groups of graphs of groups with underlying graph Q can be obtained from combinations of the two basic examples above. We will see this concept in the proof of Lemma 2.4 below.

We now want to define the notion of groups which are *decomposable into finitely many finite groups*. However, we will afterwards prove that this property is equivalent to the group being finitely generated virtually free, which is why we will not use this name after the current subsection anymore.

Definition 2.1 We say that a group $\mathcal{G}$ arises from a group $\mathcal{H}$ by *loop attachment* if $\mathcal{G}$ is isomorphic to an HNN extension $\mathcal{H}*_{\mathcal{F}}^{\iota,\kappa}$, where $\mathcal{F}$ is some finite group. We say that $\mathcal{G}$ arises from $\mathcal{H}$ by *non-loop attachment* if it is isomorphic to an amalgamated free product $\mathcal{H} *_{\mathcal{F}} \mathcal{E}$, where $\mathcal{F}$ and $\mathcal{E}$ are some finite groups.

We say that a group $\mathcal{G}$ is *decomposable into finitely many finite groups* if it arises from a finite group $\mathcal{G}_0$ by a finite sequence of loop and non-loop attachments, i.e. if there are $I, J \in \mathbb{N}_0$, finite groups $\mathcal{G}_0, \ldots, \mathcal{G}_I$ and $\mathcal{G}_1', \ldots, \mathcal{G}_{I+J}'$ and an injective map $\phi : \{1, \ldots, I\} \to \{1, \ldots, I + J\}$ such that $\mathcal{G} = \mathcal{H}_{I+J}$ where $\mathcal{H}_0 := \mathcal{G}_0$ and for $1 \le j \le I + J$ we set

$$\mathcal{H}_j := \begin{cases} \mathcal{H}_{j-1} *_{\mathcal{G}_j'} \mathcal{G}_i, & \text{if } j = \phi(i) \text{ for some } i \\ \mathcal{H}_{j-1} *_{\mathcal{G}_j'}^{\iota_j,\kappa_j}, & \text{if } j \notin \phi(\{1, \ldots, I\}) \end{cases}$$

for some given injective group homomorphisms $\iota_j : \mathcal{G}_j' \hookrightarrow \mathcal{H}_{j-1}$ and

$$\begin{cases} \kappa_j : \mathcal{G}_j' \hookrightarrow \mathcal{G}_i, & \text{if } j = \phi(i) \text{ for some } i \\ \kappa_j : \mathcal{G}_j' \hookrightarrow \mathcal{H}_{j-1}, & \text{if } j \notin \phi(\{1, \ldots, I\}) \end{cases}$$

Note that I denotes the number of non-loop attachments and J the number of loop attachments. Each of the finite groups $\mathcal{G}_i$ and $\mathcal{G}_j'$ is embedded into $\mathcal{G}$ as a subgroup in a canonical way.

See Sect. 2.2 below for examples. We will now prove two lemmas that will simplify our notation and which we will use in Lemma 2.4 to show that a group is decomposable into finitely many finite groups if and only if it is finitely generated virtually free.

Lemma 2.2 *Let $\mathcal{G}$ be a group that is decomposable into finitely many finite groups. If we fix integers $I, J \in \mathbb{N}_0$ as well as finite subgroups $\mathcal{G}_i, \mathcal{G}'_j \subseteq \mathcal{G}$, injective group homomorphisms ι_j, κ_j and a map ϕ as in Definition 2.1, then for each $1 \leq j \leq I + J$ one can find alternative embeddings $\overline{\iota_j}, \overline{\kappa_j}$ as well as integers $0 \leq s(j), t(j) \leq I$ such that $\overline{\iota_j}(\mathcal{G}'_j) \subseteq \mathcal{G}_{s(j)}$ and $\overline{\kappa_j}(\mathcal{G}'_j) \subseteq \mathcal{G}_{t(j)}$ as subgroups of $\mathcal{G}$ and such that $s(j) = t(j)$ if and only if $j \notin \phi(\{1, \ldots, I\})$.*

Proof First recall that every finite subgroup of an amalgamated free product $\overline{\mathcal{G}} = \mathcal{H} *_{\mathcal{F}} \mathcal{E}$ is conjugated to a subgroup of $\mathcal{H}$ or $\mathcal{E}$ (see [24, §I.4.3, Cor.]) and every finite subgroup of an HNN extension $\overline{\mathcal{G}} = \mathcal{H}*_{\mathcal{F}}^{\iota,\kappa}$ is conjugated to a subgroup of $\mathcal{H}$. (The latter is a consequence of [7, Thm. 4].) Note that a subgroup $\mathcal{F}'$ being conjugated to a subgroup of $\mathcal{H} \subseteq \overline{\mathcal{G}}$ means nothing but that there is an inner automorphism $\Phi \in \mathrm{Aut}\left(\overline{\mathcal{G}}\right)$ such that $\Phi(\mathcal{F}') \subseteq \mathcal{H}$.

Now observe that if $\mathcal{G} = \mathcal{H} *_{\mathcal{F}} \mathcal{E}$ is an amalgamated free product defined by injective group homomorphisms $\iota : \mathcal{F} \hookrightarrow \mathcal{H}$, $\kappa : \mathcal{F} \hookrightarrow \mathcal{E}$ and $\Phi : \mathcal{H} \to \overline{\mathcal{H}}$ is a group isomorphism, then the amalgamated free product $\overline{\mathcal{G}} = \overline{\mathcal{H}} *_{\mathcal{F}} \mathcal{E}$ defined by $\bar{\iota} := \Phi \circ \iota$ and $\overline{\kappa} := \kappa$ admits an isomorphism $\overline{\Phi} : \mathcal{G} \to \overline{\mathcal{G}}$ which extends Φ. Analogously if $\mathcal{G} = \mathcal{H}*_{\mathcal{F}}^{\iota,\kappa}$ is an HNN extension and $\Phi : \mathcal{H} \to \overline{\mathcal{H}}$ is an isomorphism, then the HNN extension $\overline{\mathcal{G}} = \overline{\mathcal{H}}*_{\mathcal{F}}^{\bar{\iota},\bar{\kappa}}$ with $\bar{\iota} := \Phi \circ \iota, \overline{\kappa} := \Phi \circ \kappa$ is the coequalizer of

$$\mathcal{F} \underset{(\Phi*\mathrm{id})\circ\kappa'}{\overset{(\Phi*\mathrm{id})\circ\iota'}{\rightrightarrows}} \overline{\mathcal{H}} * C_\infty$$

which admits an isomorphism $\overline{\Phi} : \mathcal{G} \to \overline{\mathcal{G}}$ extending Φ.

We will now apply the above facts iteratively for each $1 \leq j \leq I + J$ to prove the claim. For $j = 1$ we already have $\iota_1(\mathcal{G}'_1) \subseteq \mathcal{G}_0$ and $\kappa_1(\mathcal{G}'_1) \subseteq \mathcal{G}_i$ for some $1 \leq i \leq I$ or $\kappa_1(\mathcal{G}'_1) \subseteq \mathcal{G}_0$, so there is nothing to do and we may set $\overline{\iota_1} := \iota_1, \overline{\kappa_1} := \kappa_1$, $s(1) := 0$ and $t(1) := i$ or $t(1) := 0$ respectively. So assume $j \geq 2$ and that for all $1 \leq k < j$ we already have integers $s(k), t(k)$ such that $\iota_k(\mathcal{G}'_k) \subseteq \mathcal{G}_{s(k)}$ and $\kappa_k(\mathcal{G}'_k) \subseteq \mathcal{G}_{t(k)}$.

First consider the case that $j = \phi(i)$ for some i. Then $\kappa_j(\mathcal{G}'_j) \subseteq \mathcal{G}_i$ and we may set $t(j) := i, \overline{\kappa_j} := \kappa_j$. Moreover by applying the first paragraph of the proof multiple times we get an inner automorphism Φ of $\mathcal{H}_{j-1}$ such that $\Phi(\iota_j(\mathcal{G}'_j))$ is contained in $\mathcal{G}_{s(j)}$ for some $0 \leq s(j) \leq I$. By the second paragraph we may replace ι_j by $\overline{\iota_j} := \Phi \circ \iota_j$ and obtain an amalgamated free product $\overline{\mathcal{H}_j} = \mathcal{H}_{j-1} *_{\mathcal{G}'_j} \mathcal{G}_i$ defined by $\overline{\iota_j}, \overline{\kappa_j}$ which is isomorphic to $\mathcal{H}_j = \mathcal{H}_{j-1} *_{\mathcal{G}'_j} \mathcal{G}_i$ defined by ι_j, κ_j. By applying the second paragraph iteratively, we obtain an isomorphism $\mathcal{G} \cong \overline{\mathcal{G}}$ where $\overline{\mathcal{G}}$ is defined by the same finite groups $\mathcal{G}_0, \ldots, \mathcal{G}_I, \mathcal{G}'_1, \ldots, \mathcal{G}'_{I+J}$ but different embeddings ι_k, κ_k into $\overline{\mathcal{G}}$ for $j \leq k \leq I + J$. Moreover $\overline{\mathcal{G}}$ satisfies $\iota_k(\mathcal{G}'_k) \subseteq \mathcal{G}_{s(k)}$ and $\kappa_k(\mathcal{G}'_k) \subseteq \mathcal{G}_{t(k)}$ for $1 \leq k \leq j$.

Conversely consider the case that $j \notin \phi(\{1, \ldots, I\})$. Analogously to the first case we obtain an inner automorphism Φ of $\mathcal{H}_{j-1}$ such that $\Phi(\iota_j(\mathcal{G}'_j))$ is contained in $\mathcal{G}_{s(j)}$ for some $0 \leq s(j) \leq I$ and using the HNN part of the second paragraph we can replace $\mathcal{H}_j = \mathcal{H}_{j-1}*_{\mathcal{G}'_j}^{\iota_j,\kappa_j}$ by $\overline{\mathcal{H}_j} := \mathcal{H}_{j-1}*_{\mathcal{G}'_j}^{\overline{\iota_j},\overline{\kappa_j}}$ for $\overline{\iota_j} := \Phi \circ \iota_j, \overline{\kappa_j} := \Phi \circ \kappa_j$. In this case we set $t(j) := s(j)$, because the images of $\mathcal{G}'_j$ in $\overline{\mathcal{H}_j}$ are the same due to the coequalizer construction. Again we may apply the second paragraph of the proof iteratively to obtain a $\overline{\mathcal{G}}$ defined by the same $\mathcal{G}_0, \ldots, \mathcal{G}_I, \mathcal{G}'_1, \ldots, \mathcal{G}'_{I+J}$ and different embeddings ι_k, κ_k into $\overline{\mathcal{G}}$ for $j \leq k \leq I + J$, which is isomorphic to $\mathcal{G}$ and satisfies $\iota_k(\mathcal{G}'_k) \subseteq \mathcal{G}_{s(k)}$ and $\kappa_k(\mathcal{G}'_k) \subseteq \mathcal{G}_{t(k)}$ for $1 \leq k \leq j$.

By repeating the procedure of the last two paragraphs for $j + 1, j + 2, \ldots, I + J$ we produce a group $\overline{\mathcal{G}}$ isomorphic to $\mathcal{G}$ with the claimed properties. □

Lemma 2.3 *Let $\mathcal{G}$ be a group that is decomposable into finitely many finite groups. Fix $I, J \in \mathbb{N}_0$ as well as finite groups $\mathcal{G}_i$, $\mathcal{G}'_j$, a map ϕ and injective group homomorphisms ι_j, κ_j as in Definition 2.1. By Lemma 2.2 we may assume that there are maps $s, t : \{1, \dots, I+J\} \to \{0, \dots, I\}$ such that $\iota_j(\mathcal{G}'_j) \subseteq \mathcal{G}_{s(j)}$ and $\kappa_j(\mathcal{G}'_j) \subseteq \mathcal{G}_{t(j)}$ for all j and such that $s(j) = t(j)$ if and only if $j \notin \phi(\{1, \dots, I\})$. Choose a presentation $\mathcal{G}_i = \langle X_i \mid S_i \rangle$ of the finite group $\mathcal{G}_i$ for each $0 \le i \le I$. Then a presentation of $\mathcal{G}$ is given by*

$$\mathcal{G} = \left\langle \bigcup_{i=0}^{I} X_i \cup \{t_j \mid 1 \le j \le I+J, s(j) = t(j)\} \mid \bigcup_{i=0}^{I} S_i \cup \bigcup_{j=1}^{I+J} T_j \right\rangle \tag{5}$$

with

$$T_j := \begin{cases} \left\{\kappa_j(f) \cdot \iota_j(f)^{-1} \mid f \in \mathcal{G}'_j\right\}, & \text{if } s(j) \neq t(j) \\ \left\{t_j^{-1} \cdot \kappa_j(f) \cdot t_j \cdot \iota_j(f)^{-1} \mid f \in \mathcal{G}'_j\right\}, & \text{if } s(j) = t(j) \end{cases}$$

Proof The proof follows directly from using the presentations of amalgamated free products (see (3)) and HNN extensions (see (4)). □

From the presentation (5) one can see that the order of loop attachments and non-loop attachments can be changed a lot without changing the group $\mathcal{G}$ up to isomorphism. For example it is possible to first perform all the non-loop attachments and then all the loop attachments, because we obtain the same generators and relations via Lemma 2.3 up to reordering. So after possibly changing the numbering of the groups $\mathcal{G}_i$ and $\mathcal{G}'_j$ we can always write $\mathcal{G}$ as

$$\mathcal{G} \cong \left(\dots \left(\left(\dots \left(\left(\mathcal{G}_0 *_{\mathcal{G}'_1} \mathcal{G}_1 \right) *_{\mathcal{G}'_2} \mathcal{G}_2 \right) \cdots \right) *_{\mathcal{G}'_{I+1}}^{\iota_{I+1}, \kappa_{I+1}} \right) \cdots \right) *_{\mathcal{G}'_{I+J}}^{\iota_{I+J}, \kappa_{I+J}} \tag{6}$$

where I is the number of amalgamated free products and J the number of HNN extensions in (6) and we have maps $s, t : \{1, \dots, I+J\} \to \{0, \dots, I\}$ which fulfill

$$t(j) = \begin{cases} j, & \text{if } j \le I \\ s(j), & \text{if } j > I \end{cases} \quad \text{and} \quad s(j) \in \{0, \dots, j-1\}, \text{if } j \le I \tag{7}$$

such that the inclusions among the finite groups $\mathcal{G}_i$, $\mathcal{G}'_j$ (canonically considered as subgroups of $\mathcal{G}$) satisfy

$$\mathcal{G}_{s(j)} \overset{\iota_j}{\hookleftarrow} \mathcal{G}'_j \overset{\kappa_j}{\hookrightarrow} \mathcal{G}_{t(j)} \tag{8}$$

We will conclude this subsection with the proof that a group is decomposable into finitely many finite groups if and only if it is finitely generated virtually free. Afterwards we will throughout the paper choose a decomposition into finite groups of the form (6) for every finitely generated virtually free group occurring and work with this specific fixed decomposition (see Remark 2.5 below).

Lemma 2.4 *A group $\mathcal{G}$ is decomposable into finitely many finite groups if and only if it is finitely generated and virtually free.*

Proof First assume that the group $\mathcal{G}$ is finitely generated virtually free. Hence, $\mathcal{G}$ is isomorphic to the fundamental group $\pi_1(\mathcal{G}_Q)$ of a finite graph of finite groups $\mathcal{G}_Q$. To show that $\mathcal{G}$ is decomposable into finitely many finite groups, we will recall an iterative construction of $\pi_1(\mathcal{G}_Q)$ which the author learned from [11, Def. 3]:

Choose a maximal subtree T in the graph Q, i.e. a maximal connected subgraph without loops. Since Q is connected, T contains all vertices of Q. Let $I \in \mathbb{N}_0$ be the number of edges in T and $J \in \mathbb{N}_0$ be the number of edges in Q not contained in T. Since T is a tree, the number of vertices is $I + 1$.

Now label the edges in T from 1 to I and the vertices by the numbers $0, 1, \ldots, I$ such that the edge 1 connects the vertices 0 and 1, the edge 2 connects the vertex 2 to vertex 0 or 1 and so forth. Afterwards label the edges not contained in T from $I + 1$ to $I + J$. This gives rise to maps $s, t : \{1, \ldots, I + J\} \to \{0, \ldots, I\}$ fulfilling (7) where the edge labeled by $1 \leq j \leq I + J$ connects the vertices $s(j)$ and $t(j)$. With this notation the graph of finite groups $\mathcal{G}_Q$ is given by finite groups $\mathcal{G}_i$ for $0 \leq i \leq I$ and $\mathcal{G}'_j$ for $1 \leq j \leq I + J$ together with injective group homomorphisms

$$\mathcal{G}_{s(j)} \overset{\iota_j}{\hookleftarrow} \mathcal{G}'_j \overset{\kappa_j}{\hookrightarrow} \mathcal{G}_{t(j)}$$

as in (8).

Now define a group $\mathcal{G}''$ by the following two iterative processes: For $j = 1$ define a new graph $\overline{Q}$ by replacing the edge j and its two adjacent vertices $s(j)$ and $t(j)$ by a new vertex such that all other edges adjacent to the deleted vertices are instead adjacent to the new vertex. Define a new graph of groups $\overline{\mathcal{G}}_{\overline{Q}}$ with underlying graph $\overline{Q}$ and the group $\mathcal{G}_{s(j)} *_{\mathcal{G}'_j} \mathcal{G}_{t(j)}$ attached to the new vertex while the groups attached to the other vertices and edges stay the same. Now repeat this process for $j = 2, \ldots, I$.

After this first iterative process we are given a graph of groups $\mathcal{G}_Q$ where Q has J loops labeled from $I + 1$ to $I + J$ and a single vertex with a group $\mathcal{G}'$ attached to it. Now apply the following second iterative process to it: For $j = I + 1$ define a new graph $\overline{Q}$ by deleting the edge j and a new graph of groups $\overline{\mathcal{G}}_{\overline{Q}}$ with underlying graph $\overline{Q}$ and the group $\mathcal{G}' *_{\mathcal{G}'_j}^{\iota_j, \kappa_j}$ attached to its vertex while the groups attached to the other edges stay the same. Now repeat this process for $j = I + 2, \ldots, I + J$.

In the end we will have a graph with no edges and a single vertex and a group $\mathcal{G}''$ attached to it. This group is given by

$$\mathcal{G}'' = \left(\cdots \left(\left(\cdots \left(\left(\mathcal{G}_0 *_{\mathcal{G}'_1} \mathcal{G}_1 \right) *_{\mathcal{G}'_2} \mathcal{G}_2 \right) \cdots \right) *_{\mathcal{G}'_{I+1}}^{\iota_{I+1}, \kappa_{I+1}} \right) \cdots \right) *_{\mathcal{G}'_{I+J}}^{\iota_{I+J}, \kappa_{I+J}}$$

so it is by construction decomposable into finitely many finite groups. We will show that this group $\mathcal{G}''$ is isomorphic to $\mathcal{G} \cong \pi_1(\mathcal{G}_Q)$.

For each $0 \leq i \leq I$ let $\mathcal{G}_i = \langle X_i \mid S_i \rangle$ be a presentation of the finite group $\mathcal{G}_i$. Using Lemma 2.3 we obtain the presentation

$$\mathcal{G} = \left\langle \bigcup_{i=0}^{I} X_i \cup \{t_{I+1}, \ldots, t_{I+J}\} \mid \bigcup_{i=0}^{I} S_i \cup \bigcup_{j=1}^{I+J} T_j \right\rangle$$

for $\mathcal{G}''$.

If we now add the additional generators $t_1, \ldots, t_I$ and $\overline{t_1}, \ldots, \overline{t_{I+J}}$ to this presentation, but also add a suitable set of relations U, we will not change the group $\mathcal{G}''$ and end up with the following alternative presentation

$$\mathcal{G}'' = \left\langle \bigcup_{i=0}^{I} X_i \cup \{t_1, \overline{t_1}, \ldots, t_{I+J}, \overline{t_{I+J}}\} \mid \bigcup_{i=0}^{I} S_i \cup \bigcup_{j=1}^{I+J} T_j \cup U \right\rangle$$

where $U := \{t_i \cdot \overline{t_i} \mid 1 \leq i \leq I + J\} \cup \{t_i \mid 1 \leq i \leq I\}$. Note that this is precisely the presentation in terms of generators and relations given in the definition of the fundamental group $\pi_1(\mathcal{G}_Q)$ in [24, §I.5.1]. So $\mathcal{G}''$ is isomorphic to the finitely generated virtually free group $\mathcal{G}$.

Conversely assume that $\mathcal{G}$ is decomposable into finitely many finite groups. From the discussion after Lemma 2.3 we know that $\mathcal{G}$ is of the form (6). Arguing as for $\mathcal{G}''$ in the first part of the proof we see that $\mathcal{G}$ is isomorphic to the fundamental group of a finite graph of finite groups and therefore finitely generated virtually free.

□

We summarize this subsection in the following remark.

Remark 2.5 In Definition 2.1 we have defined the notion of groups which are *decomposable into finitely many finite groups*. Roughly speaking these are those groups which can be patched together from finitely many finite groups only using amalgamated free products and HNN extensions. More concretely every group which is decomposable into finitely many finite groups can be written as

$$\mathcal{G} \cong \left(\cdots\left(\left(\cdots\left(\left(\mathcal{G}_0 *_{\mathcal{G}'_1} \mathcal{G}_1\right) *_{\mathcal{G}'_2} \mathcal{G}_2\right)\cdots\right) *_{\mathcal{G}'_{I+1}}^{\iota_{I+1},\kappa_{I+1}}\right)\cdots\right) *_{\mathcal{G}'_{I+J}}^{\iota_{I+J},\kappa_{I+J}} \tag{9}$$

where I is the number of amalgamated free products and J the number of HNN extensions in (9) and we have maps $s, t : \{1, \ldots, I + J\} \to \{0, \ldots, I\}$ which fulfill

$$t(j) = \begin{cases} j, & \text{if } j \leq I \\ s(j), & \text{if } j > I \end{cases} \quad \text{and} \quad s(j) \in \{0, \ldots, j-1\}, \quad \text{if } j \leq I \tag{10}$$

such that the inclusions among the finite groups $\mathcal{G}_i, \mathcal{G}'_j$ (canonically considered as subgroups of $\mathcal{G}$) satisfy

$$\mathcal{G}_{s(j)} \overset{\iota_j}{\hookleftarrow} \mathcal{G}'_j \overset{\kappa_j}{\hookrightarrow} \mathcal{G}_{t(j)} \tag{11}$$

In Lemma 2.4 we have seen that a group is decomposable into finitely many finite groups if and only if it is finitely generated virtually free. Throughout this paper we fix a finitely generated virtually free group $\mathcal{G}$ and a decomposition (9) as well as maps s, t satisfying (10) and (11). Since we will almost always work with the decomposition (9) and almost never with the existence of a finite index free subgroup, the readers may feel free to replace *finitely generated virtually free* with *decomposable into finitely many finite groups* everywhere in the rest of the paper.

Note that in the definition of groups which are decomposable into finitely many finite groups one could also allow non-injective group homomorphisms ι_j, κ_j in the decomposition (9) as well. However, this definition would be equivalent to Definition 2.1: We will see below that the group algebra $\mathbb{C}[\mathcal{G}]$ of $\mathcal{G}$ is hereditary even for non-injective ι_j, κ_j (see Sect. 2.3, in particular Remark 2.7). Hence, by Dicks's characterization of hereditary group algebras $\mathcal{G}$ must be virtually free (see Sect. 1).

2.2 Examples of virtually free groups

Although the general description (9) of our fixed virtually free group $\mathcal{G}$ might look intimidating, the examples we want to keep in mind are quite down-to-earth. Trivially every finite group is virtually free. Some of the easiest non-trivial examples are the groups

$\Gamma_{a,b} := C_a * C_b$ which by definition are free products of finite cyclic groups C_a and C_b. Prominent examples of this class are the infinite dihedral group $\mathbb{D}_\infty = \Gamma_{2,2}$ and $\Gamma_{2,3}$ which is isomorphic to $\mathbf{PSL}_2(\mathbb{Z})$ (see [24, §I.1.5]). We may enlarge this class of examples by picking a common divisor c of a and b as well as embeddings $C_c \hookrightarrow C_a, C_b$. This gives rise to the virtually free group $C_a *_{C_c} C_b$. A prominent example here is $C_4 *_{C_2} C_6$ which is isomorphic to $\mathbf{SL}_2(\mathbb{Z})$ (again see [24, §I.1.5]).

The arithmetic groups $\mathbf{GL}_2(\mathbb{Z})$ and $\mathbf{PGL}_2(\mathbb{Z})$ are virtually free as well—they arise as $\mathbb{D}_4 *_{C_2 \times C_2} \mathbb{D}_6$ and $\mathbb{D}_2 *_{C_2} \mathbb{D}_3$ (see [29, Thm. 23.1] for the isomorphism to $\mathbf{GL}_2(\mathbb{Z})$, the isomorphism to $\mathbf{PGL}_2(\mathbb{Z})$ follows by dividing out the center $Z(\mathbf{GL}_2(\mathbb{Z})) \cong C_2$). To define the inclusions of $C_2 \times C_2$ and C_2 we consider the presentation

$$\mathbb{D}_c = \langle s, t \mid s^2 = t^2 = 1 = (st)^c \rangle$$

of the dihedral group. If $c = 2a$ is even, we embed $C_2 \times C_2$ into $\mathbb{D}_{2a}$ by sending the 2 generators to s and $(st)^a$. For arbitrary c we embed C_2 into $\mathbb{D}_c$ by sending the generator to s.

Since the intersection of two finite index subgroups is again of finite index and subgroups of free groups are free, every finite index subgroup of a virtually free group is again virtually free. Hence, all congruence subgroups of the four above mentioned arithmetic groups are virtually free as well. Another class of examples are of course the free groups: The free group F_a on a generators arises by taking $J = a$ trivial HNN extensions of the trivial group, i.e. in terms of our description (9) set $I = 0$ and all $\mathcal{G}_i, \mathcal{G}'_j$ to be the trivial group.

2.3 Group algebras of virtually free groups

In [11, §2] L. Le Bruyn discusses an analogue of graphs of groups for algebras: For $\mathcal{A}, \mathcal{C}$ two K-algebras and $\iota, \kappa : \mathcal{C} \hookrightarrow \mathcal{A}$ injective K-algebra homomorphisms we consider the induced embeddings

$$\iota', \kappa' : \mathcal{C} \hookrightarrow \mathcal{A} *_K K[t, t^{-1}], \quad \iota'(f) := \iota(f), \quad \kappa'(f) := t^{-1}\kappa(f)t$$

where $*_K$ denotes the coproduct of K-algebras. We define the *HNN extension* $\mathcal{A}*_{\mathcal{C}}^{\iota,\kappa}$ of $\mathcal{A}$ by $\mathcal{C}$ as the coequalizer of

$$\mathcal{C} \overset{\iota'}{\underset{\kappa'}{\rightrightarrows}} \mathcal{A} *_K K[t, t^{-1}]$$

We are mostly interested in HNN extensions of algebras, because they arise as group algebras of HNN extensions of groups: The group algebra functor $K[-]$ is a left adjoint, hence, it preserves colimits. So for every HNN extension of groups we obtain an isomorphism $K[\mathcal{H}*_{\mathcal{F}}^{\iota,\kappa}] \cong K[\mathcal{H}]*_{K[\mathcal{F}]}^{\iota,\kappa}$. (We denote the induced algebra homomorphisms $K[\iota], K[\kappa] : K[\mathcal{F}] \to K[\mathcal{H}]$ simply by ι and κ.) Moreover applying the functor $K[-]$ to our decomposition (9) we get a K-algebra isomorphism between $K[\mathcal{G}]$ and

$$\left(\ldots\left(\left(\ldots\left(\left(K[\mathcal{G}_0] *_{K[\mathcal{G}'_1]} K[\mathcal{G}_1]\right) *_{K[\mathcal{G}'_2]} K[\mathcal{G}_2]\right)\ldots\right) *_{K[\mathcal{G}'_{I+1}]}^{\iota_{I+1},\kappa_{I+1}}\right)\ldots\right) *_{K[\mathcal{G}'_{I+J}]}^{\iota_{I+J},\kappa_{I+J}} \quad (12)$$

Analogous to [11, §2] we say that a K-algebra $\mathcal{A}$ is the *fundamental algebra of a finite graph of finite dimensional semisimple K-algebras* if there are $I, J \in \mathbb{N}_0$, maps $s, t : \{1, \ldots, I + J\} \to \{0, \ldots, I\}$ fulfilling (10) as well as finite dimensional semisimple K-algebras $\mathcal{A}_0, \ldots, \mathcal{A}_I$ and $\mathcal{A}'_1, \ldots, \mathcal{A}'_{I+J}$ and K-algebra embeddings $\iota_j : \mathcal{A}'_j \hookrightarrow \mathcal{A}_{s(j)}$, $\kappa_j : \mathcal{A}'_j \hookrightarrow \mathcal{A}_{t(j)}$ such that $\mathcal{A}$ is isomorphic to

$$\left(\ldots\left(\left(\ldots\left(\left(\mathcal{A}_0 *_{\mathcal{A}'_1} \mathcal{A}_1\right) *_{\mathcal{A}'_2} \mathcal{A}_2\right)\ldots\right) *_{\mathcal{A}'_{I+1}}^{\iota_{I+1},\kappa_{I+1}}\right)\ldots\right) *_{\mathcal{A}'_{I+J}}^{\iota_{I+J},\kappa_{I+J}} \quad (13)$$

The group algebra $K[\mathcal{G}]$ of the finitely generated virtually free group $\mathcal{G}$ is the fundamental algebra of a finite graph of finite dimensional semisimple K-algebras whenever char (K) is not a prime number dividing the order of one of the finite groups $\mathcal{G}_i$, $0 \leq i \leq I$, since $K[\mathcal{G}]$ is isomorphic to (12).

Recall from Sect. 1 that the group algebra $K[\mathcal{H}]$ of a finitely generated group $\mathcal{H}$ over a field K is hereditary if and only if $\mathcal{H}$ is virtually free and contains no elements of order char (K). Using the decomposition (9) one can show that $K[\mathcal{G}]$ is hereditary if and only if char (K) does not divide the orders of the groups $\mathcal{G}_0, \ldots, \mathcal{G}_I$, because every finite subgroup $\mathcal{F} \subseteq \mathcal{G}$ (i.e. in particular for $\mathcal{F}$ cyclic of prime order) is conjugated to a subgroup of one of the groups $(\mathcal{G}_i)_i$.

Recall that a K-algebra $\mathcal{A}$ is called *formally smooth* if its Hochschild cohomology $HH^a(\mathcal{A}, -)$ vanishes in degree $a \geq 2$. This is equivalent to $\mathcal{A}$ satisfying a lifting property along square-zero extensions of K-algebras (see [28, Prop. 9.3.3]). Every formally smooth K-algebra is left and right hereditary (use e.g. [28, Lemma 9.1.9]) and the fundamental algebra of a finite graph of finite dimensional semisimple K-algebras is formally smooth (see [11, Thm. 1]). So a group algebra $K[\mathcal{H}]$ of a finitely generated group $\mathcal{H}$ is formally smooth if and only if it is hereditary, i.e. if and only if $\mathcal{H}$ is virtually free and contains no elements of order char (K).

For (parts of) the machinery of this paper to work it is crucial that $K[\mathcal{G}]$ is formally smooth, i.e. char (K) has to be zero or a suitable prime. To make things more convenient we will moreover assume that K is large enough which brings us to the notion of suitable fields:

Let $\mathcal{C}$ be a finite dimensional semisimple K-algebra, e.g. $\mathcal{C} = K[\mathcal{F}]$ for $\mathcal{F}$ a finite group of order coprime to char (K). By Artin-Wedderburn theory we know that $\mathcal{C}$ is (isomorphic to) a product of matrix algebras

$$\mathbf{M}_{\delta_0}(\mathcal{D}_0) \times \cdots \times \mathbf{M}_{\delta_{c-1}}(\mathcal{D}_{c-1}) \tag{14}$$

with $\mathcal{D}_0, \ldots, \mathcal{D}_{c-1}$ finite dimensional division K-algebras. We say that $\mathcal{C}$ is *completely split* if all simple left $\mathcal{C}$-modules are absolutely simple or equivalently if $\mathcal{D}_\gamma \cong K$ for all $0 \leq \gamma < c$, i.e. $\mathcal{C}$ is completely split if and only if it is of the form

$$\mathbf{M}_1(K)^{c_1} \times \mathbf{M}_2(K)^{c_2} \times \cdots \times \mathbf{M}_e(K)^{c_e} \tag{15}$$

for non-negative integers $e, c_1, \ldots, c_e$. Note that all left $\mathcal{C} \otimes_K F$-modules for every field extension $F \supseteq K$ are defined over $\mathcal{C}$.

Remark 2.6 Our notion of completely split semisimple algebras is closely related to the notion of *separable* algebras. Recall from e.g. [11, §1] that a K-algebra $\mathcal{C}$ is called separable if it is a finite dimensional semisimple K-algebra, say of the form (14), such that the center $Z(\mathcal{D}_\gamma)$ is a separable field extension of K for all $0 \leq \gamma < c$. It is immediate from the definitions that a completely split semisimple K-algebra is separable. Moreover one can show that for every separable K-algebra $\mathcal{C}$ there is a finite field extension $F \supseteq K$ such that $\mathcal{C} \otimes_K F$ is completely split (take a finite normal extension $F \supseteq K$ containing the centers $Z(\mathcal{D}_\gamma)$ and use that $Z(\mathcal{D}_\gamma) \otimes_K F \cong F^{[Z(\mathcal{D}_\gamma):K]}$). For the purposes of this paper it is sufficient to restrict to those separable algebras which are completely split.

We say that a field K is of *suitable characteristic* for the virtually free group $\mathcal{G}$ if $\mathcal{G}$ contains no elements of order char (K). We call a field K *suitable* for $\mathcal{G}$ if it is perfect, of suitable characteristic and $K[\mathcal{F}]$ is completely split for every finite subgroup $\mathcal{F} \subseteq \mathcal{G}$.

Note that being suitable is a relative notion—it depends on which virtually free group it refers to. The readers may convince themselves that every algebraic field extension of a suitable field is again suitable and (using that every finite subgroup of $\mathcal{G}$ is contained in a $\mathcal{G}_i$ up to conjugation) that every perfect field of suitable characteristic admits a finite extension that is suitable.

Remark 2.7 Within this paper we will study representations of finitely generated virtually free groups over suitable finite fields. However, all of our methods apply to the more general case of algebras of the form (13) for $\mathcal{A}_i$ and $\mathcal{A}'_j$ completely split finite dimensional semisimple $\mathbb{F}_q$-algebras with ι_j, κ_j (not necessarily injective) $\mathbb{F}_q$-algebra homomorphisms. Note that such algebras are formally smooth (and in particular left and right hereditary), because the proof of [11, Thm. 1] also applies if ι_j, κ_j in (13) are not necessarily injective. Moreover we do not assume injectivity in all results involved in the proof of the main Theorem 5.4.

2.4 Geometric methods

The algebro-geometric methods in this paper are written in the language of schemes. Since we will work almost entirely with (affine) schemes of finite type over a perfect field K, those readers who are less comfortable with schemes may instead think of the associated $\overline{K}$-varieties and see K-valued points as fixed points of the natural Galois action of $\mathrm{Aut}_K\left(\overline{K}\right)$, connected/irreducible components as orbits of the natural Galois action on the connected/irreducible components, etc. For this whole subsection fix a field K.

2.4.1 Counting rational points

Let C be a commutative ring and X be a C-scheme. For each commutative C-algebra B, we will denote by $X(B)$ the set of B*-valued points* of X, i.e. the set of C-scheme morphisms $\mathrm{Spec}\,(B) \to X$. If $C = K$ is a field, we also use the term rational points for the K-valued points $X(K)$.

Now assume C is of finite type over $\mathbb{Z}$ and X is separated and of finite type over C. A polynomial $P \in \mathbb{Z}[s]$ is called *counting polynomial* of X if for every ring homomorphism $C \to \mathbb{F}_q$ to a finite field, we have $\#X(\mathbb{F}_q) = P(q)$. We say that X is *polynomial count* if X admits a counting polynomial.

Example 2.8 The general linear group (scheme) $\mathbf{GL}_d$ is polynomial count, its counting polynomial is given by $P_{\mathbf{GL}_d} := \prod_{\delta=0}^{d-1}(s^d - s^\delta)$.

Note that counting polynomials are unique and that the reduction X_{red} of a polynomial count scheme X is again polynomial count with the same counting polynomial. We will need the following two facts on polynomial count schemes:

Lemma 2.9 *Let C be a commutative ring and X be a separated finite type C-scheme.*

a) *If $P \in \mathbb{Q}(s)$ is a rational function and $\#X(\mathbb{F}_q) = P(q)$ for each homomorphism $C \to \mathbb{F}_q$, then P lies in the subring $\mathbb{Z}[s]$ (and is a counting polynomial of X).*
b) *If C is a subring of $\mathbb{C}$ and X is a polynomial count C-scheme with counting polynomial P, then $P(xy) \in \mathbb{Z}[x, y]$ is the E-polynomial of the analytification $X(\mathbb{C}) = (X \times_C \mathrm{Spec}\,(\mathbb{C}))^{an}$ and $P(1)$ is the Euler characteristic of $X(\mathbb{C})$. (See* [6, Appendix] *for the definition of E-polynomials.)*

See [21, Prop. 6.1] for a proof of (a) and [6, Appendix, Thm. 6.1.2] and [21, Prop. 6.1] for a proof of (b). The notion of E-polynomials only occurs as an application/motivation, readers only interested in the counting of representations over finite fields may ignore it.

2.4.2 Geometry of representation spaces

The schemes we discuss in this paper arise from the representation spaces of algebras, whose construction we will now recall. Recall that the functor $\mathbf{M}_d : \underline{\mathrm{CAlg}}_K \to \underline{\mathrm{Alg}}_K$ from the category $\underline{\mathrm{CAlg}}_K$ of commutative K-algebras to the category $\underline{\mathrm{Alg}}_K$ of K-algebras which sends C to the matrix algebra $\mathbf{M}_d(C)$ is a right adjoint (see e.g. [20, Ch.IV, Thm. 1.1] for a proof). We denote its left adjoint by $\mathcal{R}_d$. For $\mathcal{A}$ a finitely generated K-algebra we call $\mathrm{Rep}_d(\mathcal{A}) := \mathrm{Spec}\,(\mathcal{R}_d(\mathcal{A}))$ the *d-th representation space* of $\mathcal{A}$. $\mathrm{Rep}_d(\mathcal{A})$ is an affine finite type K-scheme admitting a natural bijection

$$\mathrm{Rep}_d(\mathcal{A})(C) \cong \underline{\mathrm{Alg}}_K\,(\mathcal{A}, \mathbf{M}_d(C)) \tag{16}$$

for all commutative K-algebras C, i.e. $\mathrm{Rep}_d(\mathcal{A})$ represents the functor $\underline{\mathrm{CAlg}}_K \to \underline{\mathrm{Set}}$, $C \mapsto \underline{\mathrm{Alg}}_K\,(\mathcal{A}, \mathbf{M}_d(C))$. Denote the image of $x \in \mathrm{Rep}_d(\mathcal{A})(C)$ under (16) by ρ_x. The right hand side of (16) admits a natural $\mathbf{GL}_d(C)$-action via conjugation for each C, hence, the general linear group (scheme) $\mathbf{GL}_{d,K}$ acts on $\mathrm{Rep}_d(\mathcal{A})$ in terms of a K-scheme morphism $\sigma : \mathbf{GL}_{d,K} \times_K \mathrm{Rep}_d(\mathcal{A}) \to \mathrm{Rep}_d(\mathcal{A})$.

For a group scheme action $\sigma : G \times_K X \to X$ of a linear algebraic group G on a separated finite type scheme X we have two notions of the orbit of a point: If x is a C-valued point of X, then we have the orbit $G(C).x \subseteq X(C)$—which we call the *set-theoretic orbit* of x—as well as the *algebro-geometric orbit* $\mathbb{O}_x$. The latter is defined as the image of the orbit map

$$\vartheta_x := (\sigma \circ (\mathrm{id}_G \times x), \mathrm{pr}_2) : G \times_K \mathrm{Spec}\,(C) \to X \times_K \mathrm{Spec}\,(C)$$

If $x \in X(F)$ is an F-valued point for $F \supseteq K$ a field extension, then the orbit $\mathbb{O}_x \subseteq X \times_K \mathrm{Spec}\,(F)$ is locally closed (see [13, Prop. 1.65(b)]) and we may consider it as a reduced locally closed subscheme.

The two notions of orbits are closely related to each other: If $x \in \mathrm{Rep}_d(\mathcal{A})(K)$ is a K-valued point of a representation space, $F \supseteq K$ a field extension and x' the F-valued point associated to x via pulling it back along $\mathrm{Spec}\,(F) \to \mathrm{Spec}\,(K)$, then $\mathbb{O}_x(F) = \mathbf{GL}_d(F).x'$.[3] Moreover we have the following related lemma.

Lemma 2.10 *Let G be a linear algebraic group over K acting on a separated finite type K-scheme X. Denote the algebraic closure of K by $\overline{K}$. If $x \in X(K)$ is a K-valued point and $x' \in X(\overline{K})$ is the associated $\overline{K}$-valued point, then $\mathbb{O}_{x'}$ and $\mathbb{O}_x \times_K \mathrm{Spec}\,(\overline{K})$ coincide as locally closed subsets of $X \times_K \mathrm{Spec}\,(\overline{K})$. If K is perfect, they even coincide as locally closed subschemes.*

Proof Since the orbit map $\vartheta_{x'}$ is the base change of ϑ_x, we obtain a factorization

$$\begin{tikzcd}
G \times_K \mathrm{Spec}\,(\overline{K}) \arrow[rr, "\vartheta_{x'}"] \arrow[dr, "\varphi"'] & & X \times_K \mathrm{Spec}\,(\overline{K}) \\
& \mathbb{O}_x \times_K \mathrm{Spec}\,(\overline{K}) \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Here ι is a locally closed embedding (see [27, Tag 01JY]) and φ is surjective by [27, Tag 01S1]. Hence, the subset $\mathbb{O}_x \times_K \mathrm{Spec}\,(\overline{K})$ coincides with $\mathbb{O}_{x'}$ as the latter is the image of $\vartheta_{x'}$.

[3] In general the inclusion $\subseteq$ is wrong for group scheme actions. For representation spaces it holds, because representations of an algebra have no *twisted forms* i.e. if $\mathcal{M}, \mathcal{N}$ are left $\mathcal{A}$-modules and $\mathcal{M} \otimes_K F \cong \mathcal{N} \otimes_K F$ for some field extension $F \supseteq K$, then $\mathcal{M}$ and $\mathcal{N}$ are already isomorphic by the Noether-Deuring theorem (see e.g. [3, Thm. 29.11] for a proof).

If K is perfect, then $\mathbb{O}_x \times_K \operatorname{Spec}\left(\overline{K}\right)$ is reduced (see [27, Tag 020I]). Since the reduced subscheme structure on a locally closed subset is unique, $\mathbb{O}_{x'}$ and $\mathbb{O}_x \times_K \operatorname{Spec}\left(\overline{K}\right)$ must coincide. $\square$

The purpose of the action on $\operatorname{Rep}_d(\mathcal{A})$ is that K-valued points $x, y \in \operatorname{Rep}_d(\mathcal{A})(K)$ are in the same (set-theoretic) orbit if and only if the representations ρ_x and ρ_y are isomorphic, i.e. there is a natural bijection $\operatorname{iso}_d(\mathcal{A}) \cong {}^{\operatorname{Rep}_d(\mathcal{A})(K)}/_{\mathbf{GL}_d(K)}$.

Within this paper we will usually not distinguish strictly between the set-theoretic orbit $\mathbf{GL}_d(K).x$ of a K-valued point x, the isomorphism class of the corresponding representation $\rho_x : \mathcal{A} \to \mathbf{M}_d(K)$ and the isomorphism class of its associated left module, which we denote by $\mathcal{M}_x$. Given a left $\mathcal{A}$-module $\mathcal{M}$ we denote the K-valued point corresponding to $\mathcal{M}$ under a given choice of basis by $x_{\mathcal{M}}$. If $\mathcal{M}$ is the left module associated to the point x, we will also sometimes denote the algebro-geometric orbit $\mathbb{O}_x$ by $\mathbb{O}_{\mathcal{M}}$.

Similar to the algebro-geometric orbits one defines the *stabilizer* $S(x)$ of a C-valued point $x \in \operatorname{Rep}_d(\mathcal{A})(C)$ geometrically as the fibre product defined by the pullback square

$$\begin{tikzcd}
S(x) \arrow[r] \arrow[d] & \operatorname{Spec}(C) \arrow[d, "{(x,\mathrm{id})}"] \\
\mathbf{GL}_{d,K} \times_K \operatorname{Spec}(C) \arrow[r, "\vartheta_x"'] & \operatorname{Rep}_d(\mathcal{A}) \times_K \operatorname{Spec}(C)
\end{tikzcd}$$

$S(x)$ is a closed C-subgroup scheme of $\mathbf{GL}_{d,K} \times_K \operatorname{Spec}(C) \cong \mathbf{GL}_{d,C}$. Its B-valued points for any commutative C-algebra B are given by

$$S(x)(B) \cong \{g \in \mathbf{GL}_d(B) \mid g.x' = x'\} = \operatorname{Aut}_{\mathcal{A}\otimes_K B}(\mathcal{M}_x \otimes_C B) \tag{17}$$

where $x' \in \operatorname{Rep}_d(\mathcal{A})(B)$ is the B-valued point associated to x. So $S(x)(B)$ is nothing but the (set-theoretic) stabilizer subgroup in $\mathbf{GL}_d(B)$ of the point x'.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of finitely generated K-algebras, then functoriality gives us an induced K-scheme morphism $\operatorname{Rep}_d(\mathcal{B}) \to \operatorname{Rep}_d(\mathcal{A})$ for each d which realizes the restriction of scalars functor geometrically. We denote this morphism by φ^*. Since $\operatorname{Rep}_d(-)$ is the composition of the contravariant equivalence $\operatorname{Spec}(-)$ and the left adjoint functor $\mathcal{R}_d(-)$, it is a left adjoint functor from the category of (finitely generated) K-algebras to the opposite category of affine (finite type) K-schemes, hence, it maps colimits of (finitely generated) K-algebras to limits of affine (finite type) K-schemes. For example $\operatorname{Rep}_d(\mathcal{A} *_K \mathcal{B}) \cong \operatorname{Rep}_d(\mathcal{A}) \times_K \operatorname{Rep}_d(\mathcal{B})$ for $\mathcal{A}, \mathcal{B}$ two finitely generated K-algebras. Moreover (16) shows that

$$\operatorname{Rep}_d(\mathcal{A} \otimes_K F) \cong \operatorname{Rep}_d(\mathcal{A}) \times_K \operatorname{Spec}(F)$$

are naturally isomorphic F-schemes for all field extensions $F \supseteq K$.

The geometry of representation spaces and their orbits plays a substantial role within this paper. We will frequently make use of the following fundamental facts (see e.g. [9, §2.3] for (c) and (d) in the case $\mathcal{A} = \mathbb{C}\vec{Q}$ the path algebra of a quiver). Due to a lack of reference that does not assume the ground field to be algebraically closed, we will provide some proofs.

Proposition 2.11 *Let K be a field, $d \in \mathbb{N}_0$ and $\mathcal{A}$ a finitely generated K-algebra.*

(a) *If $\mathcal{A}$ is formally smooth, then* $\operatorname{Rep}_d(\mathcal{A})$ *is a regular scheme.*
(b) *If $x \in \operatorname{Rep}_d(\mathcal{A})(F)$ is an F-valued point for a field extension $F \supseteq K$, then $\mathbb{O}_x$ is geometrically irreducible and in particular connected.*

(c) *If K is perfect and $0 \to \mathcal{N} \to \mathcal{W} \to \mathcal{M} \to 0$ a short exact sequence of finite dimensional left $\mathcal{A}$-modules, then $\mathbb{O}_{\mathcal{N}\oplus\mathcal{M}} \subseteq \overline{\mathbb{O}_{\mathcal{W}}}$.*
(d) *If K is perfect, then an orbit $\mathbb{O}_{\mathcal{M}} \subseteq \mathrm{Rep}_d(\mathcal{A})$ is closed if and only if the corresponding left $\mathcal{A}$-module $\mathcal{M}$ is semisimple.*

Proof About (a): We will first show that the K-scheme $\mathrm{Rep}_d(\mathcal{A})$ is formally smooth in the sense of [27, Tag 02H0]. So assume we are given a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}_d(\mathcal{A}) \arrow[d] & \mathrm{Spec}\,({}^{B}/_{\mathfrak{b}}) \arrow[l] \arrow[d, "\phi"] \\ \mathrm{Spec}\,(K) & \mathrm{Spec}\,(B) \arrow[l] \end{tikzcd} \tag{18}$$

of schemes where B is a commutative ring, $\mathfrak{b} \subseteq B$ is a square-zero ideal and ϕ is given by the canonical projection $B \to {}^{B}/_{\mathfrak{b}}$. We have to show that there is a scheme morphism $\mathrm{Spec}\,(B) \to \mathrm{Rep}_d(\mathcal{A})$ fitting into the diagram (18) such that it still commutes. Using the natural bijection (16) this is equivalent to finding a K-algebra homomorphism $\mathcal{A} \to \mathbf{M}_d(B)$ letting the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr] & \mathbf{M}_d({}^{B}/_{\mathfrak{b}}) \\ & \mathbf{M}_d(B) \arrow[u] \end{tikzcd}$$

commute. The latter exists by the lifting property of formally smooth K-algebras.

So the K-scheme $\mathrm{Rep}_d(\mathcal{A})$ is formally smooth. However, by [27, Tags 02H6 & 01TX] this is equivalent to it being a smooth K-scheme and smooth K-schemes are always regular (see [27, Tag 056S]).

About (b): Let $\overline{F}$ be the algebraic closure of F and x' the $\overline{F}$-valued point associated to x. By Lemma 2.10, it suffices to show that $\mathbb{O}_{x'} \subseteq \mathrm{Rep}_d(\mathcal{A}) \times_K \mathrm{Spec}\,(\overline{F})$ is irreducible. The latter follows from $\mathbb{O}_{x'}$ being the image of the general linear group $\mathbf{GL}_{d,\overline{F}}$ under the continuous map $\vartheta_{x'}$.

About (c) and (d): If K is algebraically closed, then one may argue analogously to [9, Thm. 2.7 & 2.10]. In general one could reformulate the proof in the language of schemes and it would still work. However, we will instead reduce the general case to the special case of an algebraically closed ground field.

Using Lemma 2.12 below we see that to prove (c) it suffices to show that

$$\mathbb{O}_{\mathcal{N}\oplus\mathcal{M}} \times_K \mathrm{Spec}\,(\overline{K}) \subseteq \overline{\mathbb{O}_{\mathcal{W}} \times_K \mathrm{Spec}\,(\overline{K})}$$

and to prove (d) it suffices to show that

$$\mathbb{O}_{\mathcal{M}} \times_K \mathrm{Spec}\,(\overline{K}) \subseteq \mathrm{Rep}_d(\mathcal{A}) \times_K \mathrm{Spec}\,(\overline{K}) = \mathrm{Rep}_d(\mathcal{A} \otimes_K \overline{K})$$

is closed if and only if $\mathcal{M}$ is semisimple.

By Lemma 2.10, we have $\mathbb{O}_{\mathcal{N}\oplus\mathcal{M}} \times_K \mathrm{Spec}\,(\overline{K}) = \mathbb{O}_{(\mathcal{N}\oplus\mathcal{M})\otimes_K\overline{K}}$, which is contained in $\overline{\mathbb{O}_{\mathcal{W}\otimes_K\overline{K}}} = \overline{\mathbb{O}_{\mathcal{W}} \times_K \mathrm{Spec}\,(\overline{K})}$ by applying (c) over $\overline{K}$. So we have proven (c) in general.

Again using Lemma 2.10 we have $\mathbb{O}_{\mathcal{M}} \times_K \mathrm{Spec}\,(\overline{K}) = \mathbb{O}_{\mathcal{M}\otimes_K\overline{K}}$, which is closed if and only if $\mathcal{M} \otimes_K \overline{K}$ is a semisimple left module over $\mathcal{A} \otimes_K \overline{K}$. Here we have used that we already know that (d) holds over the algebraically closed field $\overline{K}$. Furthermore semisimplicity of $\mathcal{M} \otimes_K \overline{K}$ is equivalent to $\mathcal{M}$ being semisimple (analogous to [21, Lemma 4.2(2)]). So we have proven (d).

□

Lemma 2.12 *Let K be a perfect field with algebraic closure $\overline{K}$, X a finite type K-scheme and $U, V \subseteq X$ geometrically irreducible locally closed subschemes. Denote the base change of X to $\overline{K}$ by $X' := X \times_K \operatorname{Spec}(\overline{K})$. Note that $U' := U \times_K \operatorname{Spec}(\overline{K})$ and $V' := V \times_K \operatorname{Spec}(\overline{K})$ are irreducible locally closed subschemes of X' (see [27, Tag 01JY]).*

a) *$U \subseteq X$ is closed if and only if $U' \subseteq X'$ is closed.*
b) *$U \subseteq \overline{V}$ if and only if $U' \subseteq \overline{V'}$.*

Proof About (a): $U \subseteq X$ is closed if and only if the open embedding $U \to \overline{U}$ is an isomorphism. Analogously $U' \subseteq X'$ is closed if and only if $U' \to \overline{U'}$ is an isomorphism. Note that $\overline{U} \times_K \operatorname{Spec}(\overline{K})$ coincides with $\overline{U'} = \overline{U \times_K \operatorname{Spec}(\overline{K})}$, i.e. taking closures commutes with base change:

Consider the projection $\pi : X' \to X$ for which we have $U' = \pi^{-1}(U)$. Note that π is a base change of $\operatorname{Spec}(\overline{K}) \to \operatorname{Spec}(K)$, hence, it is surjective and flat, i.e. in particular open (see [27, Tags 01S1, 01U9 & 01UA]). Now use that a map $\varphi : Z \to Y$ between topological spaces is continuous (resp. open) if and only if $\varphi^{-1}(\overline{W}) \supseteq \overline{\varphi^{-1}(W)}$ (resp. $\varphi^{-1}(\overline{W}) \subseteq \overline{\varphi^{-1}(W)}$) holds for all subsets $W \subseteq Y$.

So $U' \to \overline{U'}$ is the base change of $U \to \overline{U}$ along $\operatorname{Spec}(\overline{K}) \to \operatorname{Spec}(K)$. Since $\operatorname{Spec}(\overline{K}) \to \operatorname{Spec}(K)$ is an fpqc covering, $\overline{U'} = \overline{U} \times_K \operatorname{Spec}(\overline{K}) \to \overline{U}$ is an fpqc covering as well and the claim follows from the fact that being an isomorphism is fpqc-local on the base (see [27, Tag 02L4]; see [27, Tags 022B, 00VH & 02KO] for the relevant definitions).

About (b): By the definition of closures it suffices to show that

$$\overline{U} \subseteq \overline{V} \Leftrightarrow \overline{\pi^{-1}(U)} \subseteq \overline{\pi^{-1}(V)} \tag{19}$$

Since π is surjective, the left hand side of (19) is equivalent to $\pi^{-1}(\overline{U}) \subseteq \pi^{-1}(\overline{V})$. The latter is equivalent to the right hand side of (19) again by π being continuous and open.

□

2.4.3 E-polynomials of moduli spaces of representations

Since the isomorphism classes of representations of $\mathcal{A}$ are parametrized by orbits of representation spaces, it is natural to define moduli spaces of representations of $\mathcal{A}$ in terms of quotients of representation spaces. We denote the *GIT quotient* of $\operatorname{Rep}_d(\mathcal{A})$ by

$$M(\mathcal{A}, d) := \operatorname{Rep}_d(\mathcal{A}) /\!\!/ \mathbf{GL}_{d,K} = \operatorname{Spec}\left(\mathcal{R}_d(\mathcal{A})^{\mathbf{GL}_{d,K}}\right)$$

(see e.g. [18, §1.2, Thm. 1.1] for the definition).

If the field K is finite or algebraically closed,[4] then there is a natural bijection $M(\mathcal{A}, d)(F) \cong \operatorname{ssim}_d(\mathcal{A} \otimes_K F)$ for every algebraic field extension $F \supseteq K$. So for such K we call $M(\mathcal{A}, d)$ the *(GIT) moduli space of d-dimensional semisimple representations* of $\mathcal{A}$. It contains a (possibly empty) open subscheme $M^{\text{absim}}(\mathcal{A}, d)$ which for K as above admits a natural bijection $M^{\text{absim}}(\mathcal{A}, d)(F) \cong \operatorname{absim}_d(\mathcal{A} \otimes_K F)$ for every algebraic field extension $F \supseteq K$. Accordingly we call $M^{\text{absim}}(\mathcal{A}, d)$ the *(GIT) moduli space of d-dimensional absolutely simple representations* of $\mathcal{A}$.

[4] More generally it would be sufficient to require that K is perfect and has trivial Brauer group.

Note that for $K = \mathbb{F}_q$ a finite field the counting functions $r_d^{\mathrm{ss},\mathcal{A}}$ and $r_d^{\mathrm{absim},\mathcal{A}}$ count the rational points of these GIT moduli spaces. So whenever counting polynomials R_d^{ss} and R_d^{absim} as in Theorem 1.1 exist, they are in fact counting polynomials of these GIT moduli spaces.

Now assume that $\mathcal{A} = \mathcal{A}' \otimes \mathbb{F}_q := \mathcal{A}' \otimes_{\mathbb{Z}} \mathbb{F}_q$ is defined over $\mathbb{Z}$ by a finitely generated $\mathbb{Z}$-algebra $\mathcal{A}'$ (e.g. $\mathcal{A}' = \mathbb{Z}[\mathcal{G}]$ for $\mathcal{G}$ a finitely generated group or $\mathcal{A}' = \mathbb{Z}\vec{Q}$ for $\vec{Q}$ a finite quiver). One can define representation spaces and GIT moduli spaces of $\mathcal{A}'$ as $\mathbb{Z}$-schemes using Seshadri's generalization of geometric invariant theory (see [25]). In this way we obtain $\mathbb{Z}$-schemes $M(\mathcal{A}', d)$ and $M^{\mathrm{absim}}(\mathcal{A}', d)$ such that for $F = \mathbb{C}$ and $F = \overline{\mathbb{F}_p}$ for all primes p in an open subset of Spec $(\mathbb{Z})$ we have

$$\begin{aligned} M(\mathcal{A}', d) \times \mathrm{Spec}\,(F) &\cong M(\mathcal{A}' \otimes F, d), \\ M^{\mathrm{absim}}(\mathcal{A}', d) \times \mathrm{Spec}\,(F) &\cong M^{\mathrm{absim}}(\mathcal{A}' \otimes F, d) \end{aligned}$$

(see [1, Appendix B, Thm. B.3]).

Hence, using Lemma 2.9(a) we see that if there are rational functions $R_d^{\mathrm{ss}}, R_d^{\mathrm{absim}} \in \mathbb{Q}(s)$ which satisfy (2), they must already belong to $\mathbb{Z}[s]$. Furthermore using Lemma 2.9(b) we see that whenever the counting polynomials exist, the E-polynomials of $M(\mathcal{A}' \otimes \mathbb{C}, d)^{\mathrm{an}}$ and $M^{\mathrm{absim}}(\mathcal{A}' \otimes \mathbb{C}, d)^{\mathrm{an}}$ are given by $R_d^{\mathrm{ss}}(xy)$ and $R_d^{\mathrm{absim}}(xy)$.

When $\mathcal{A}' = \mathbb{Z}[\mathcal{G}]$ is the group algebra of a finitely generated group $\mathcal{G}$, (the analytification of) the moduli space $M(\mathbb{C}[\mathcal{G}], d)$ is also called the $\mathbf{GL}_d(\mathbb{C})$-*character variety* of $\mathcal{G}$ and denoted by $X_{\mathcal{G}}(\mathbf{GL}_d(\mathbb{C}))$. Since our methods enable us to compute the counting polynomials explicitly (e.g. using the SageMath code [10]), we obtain a new approach to determine the E-polynomials of $\mathbf{GL}_d(\mathbb{C})$-character varieties of virtually free groups. (In fact we can more generally compute the E-polynomials of the connected components of the character varieties individually.)

2.4.4 Associated fibre spaces and special groups

We now want to recall the construction of associated fibre spaces. Most of the facts we are collecting here can e.g. be found in [22]. Let G be a linear algebraic group over K, $H \subseteq G$ a closed subgroup and X an affine K-scheme endowed with an H-action. We define an induced H-action on $G \times_K X$ via the natural transformation

$$H(C) \times G(C) \times X(C) \to G(C) \times X(C), \quad h.(g, x) := (gh^{-1}, h.x)$$

for any commutative K-algebra C. This is a free action and its respective quotient[5] $G \times^H X := {}^{G \times_K X}/_{H}$ is called the *associated G-fibre space.*

If $(g, x) \in G(C) \times X(C)$ is a C-valued point, we denote its image in $(G \times^H X)(C)$ by $g * x$. We have a natural morphism $X \to G \times^H X$ given by $x \mapsto 1 * x$. If Y is a K-scheme with G-action and $\varphi : X \to Y$ is an H-equivariant morphism, then we obtain a unique G-equivariant morphism φ' such that φ factorizes as

$$X \to G \times^H X \xrightarrow{\varphi'} Y \tag{20}$$

For G and H (geometrically) irreducible we have that $G \times^H X$ is irreducible/connected if and only if X is irreducible/connected. Moreover we have the following useful lemma (see [26, §II.3.7, Lemma 4]).

[5] In the case of free actions we will denote the GIT quotient with a single / instead of $/\!\!/$.

Lemma 2.13 *Let $\varphi : Y \to {}^{G}/_{H}$ be a G-equivariant morphism. If $e \in {}^{G}/_{H}(K)$ is the K-point which is the image of the unit of G and $X := \varphi^{-1}(e)$ with inclusion map $\iota : X \to Y$, then the induced map $\iota' : G \times^H X \to Y$ from (20) is a G-equivariant isomorphism.*

We also want to recall the notion of special algebraic groups. A linear algebraic group G over K is *special* if the quotient map $\pi : X \to {}^{X}/_{G}$ for any affine finite type K-scheme X with a free G-action is Zariski-locally a trivial bundle, i.e. there is an open covering ${}^{X}/_{G} = \bigcup_\alpha U_\alpha$ such that for each α there is a G-equivariant isomorphism Φ_α admitting a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\Phi_\alpha"] \arrow[dr, "\pi"'] & & G \times_K U_\alpha \arrow[dl, "\mathrm{pr}_2"] \\ & U_\alpha & \end{tikzcd} \tag{21}$$

(This is equivalent to the ordinary definition by [22, §4.3, Théorème 2] and by the fact that all such quotient maps are principal fibre bundles (see [18, §0.4, Prop. 0.9]).)

Example 2.14 Let $\mathcal{C}$ be a finite dimensional semisimple K-algebra which is completely split, i.e. $\mathcal{C}$ is of the form (15). Denote its (absolutely) simple left modules by $\mathcal{L}_0, \ldots, \mathcal{L}_{c-1}$. Each finite dimensional left $\mathcal{C}$-module $\mathcal{M}$ is semisimple, i.e.

$$\mathcal{M} \cong \mathcal{L}_0^{\oplus m(0)} \oplus \cdots \oplus \mathcal{L}_{c-1}^{\oplus m(c-1)}$$

for some $m \in \mathbb{N}_0^c$. Using the bijection (17) and $\mathrm{End}_{\mathcal{A}}\left(\mathcal{L}_\gamma\right) = K$ we obtain an isomorphism $S(x_{\mathcal{M}})(B) \cong \mathrm{Aut}_{\mathcal{C}\otimes_K B}(\mathcal{M} \otimes_K B) \cong \mathbf{GL}_{m(0)}(B) \times \cdots \times \mathbf{GL}_{m(c-1)}(B)$ for every commutative K-algebra B. Hence, $S(x_{\mathcal{M}}) \cong \mathbf{GL}_{m(0),K} \times \cdots \times \mathbf{GL}_{m(c-1),K}$ which is a special linear algebraic group. (This can be seen e.g. from the classification of special algebraic groups in [5].)

We will later need the following well-known lemma.

Lemma 2.15 *If H is a special linear algebraic group acting freely on an affine finite type K-scheme X with quotient ${}^{X}/_{H}$, then the canonical injection*

$$ {}^{X(F)}/_{H(F)} \to \left({}^{X}/_{H}\right)(F) \tag{22}$$

is bijective for every field extension $F \supseteq K$.

Proof The quotient map $\pi : X \to {}^{X}/_{H}$ is a prototypical example of what is called a *torsor* or *principal fibre bundle* for the relative group scheme $H \times_K {}^{X}/_{H} \to {}^{X}/_{H}$ (see e.g. [27, Tag 0497] or [22, §2.2] for the definition). Now let $y \in \left({}^{X}/_{H}\right)(F)$ be an F-valued point for some field extension $F \supseteq K$. Then $\pi^{-1}(y) \to \mathrm{Spec}\,(F)$ is a torsor for $H \times_K \mathrm{Spec}\,(F)$ as it fits into a Cartesian square

$$\begin{tikzcd} \pi^{-1}(y) \arrow[r] \arrow[d] & \mathrm{Spec}\,(F) \arrow[d, "y"] \\ X \arrow[r, "\pi"'] & {}^{X}/_{H} \end{tikzcd}$$

and because being a torsor is stable under base change.

So the set $\pi^{-1}(y)(F)$, which is canonically identified with the set of sections of $\pi^{-1}(y) \to \mathrm{Spec}\,(F)$, is either empty or carries a simply transitive action of the group $H(F)$. Hence, the map (22) is injective for H any linear algebraic group.

Now assume H is special. This means that the torsor $\pi : X \to X/H$ is locally trivial with respect to the Zariski topology. Pick an open covering $X/H = \bigcup_\alpha U_\alpha$ admitting equivariant isomorphisms Φ_α as in (21). Since Φ_α is an equivariant isomorphism, it induces an $H(F)$-equivariant bijection $\pi^{-1}(U_\alpha)(F) \to H(F) \times U_\alpha(F)$. Taking quotients by $H(F)$ we obtain a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha)(F)/H(F) \arrow[rr, "\overline{\Phi_\alpha}"] \arrow[dr, "\overline{\pi}"'] & & H(F) \times U_\alpha(F)/H(F) \arrow[dl, "\overline{\mathrm{pr}_2}"] \\ & U_\alpha(F) & \end{tikzcd}$$

Since the induced maps $\overline{\mathrm{pr}_2}$ and $\overline{\Phi_\alpha}$ are bijective, the induced map $\overline{\pi}$ must be bijective too. As the map (22) is given by glueing together the bijective maps $\overline{\pi} : \pi^{-1}(U_\alpha)(F)/H(F) \to U_\alpha(F)$, it has to be bijective as well, which concludes the proof.

□

Note that Lemma 2.15 in particular applies to the case where X/H is an associated fibre space $G \times^H X$.

3 Some invariants of virtually free groups

For the whole section fix a perfect field K.

3.1 Dimension vectors

We will now associate to every finitely generated K-algebra $\mathcal{A}$ a commutative monoid[6] $\mathcal{T}(\mathcal{A})$ together with a monoid homomorphism $|\,.\,| : \mathcal{T}(\mathcal{A}) \to \mathbb{N}_0$ which generalizes the dimension vector monoid from quiver representation theory (see Example 3.1(b) below). For $d \in \mathbb{N}_0$ we denote by $\mathcal{T}_d(\mathcal{A})$ the set of connected components $Z \subseteq \mathrm{Rep}_d(\mathcal{A})$ containing a rational point, i.e. $Z(K) \neq \emptyset$. As a set we define $\mathcal{T}(\mathcal{A})$ as the disjoint union

$$\mathcal{T}(\mathcal{A}) := \bigsqcup_{d \geq 0} \mathcal{T}_d(\mathcal{A})$$

and we define the map $|\,.\,|$ via $|\mathcal{T}_d(\mathcal{A})| = d$. To define the monoid structure on $\mathcal{T}(\mathcal{A})$ we consider the direct sum map

$$\oplus_{c,d} : \mathrm{Rep}_c(\mathcal{A}) \times_K \mathrm{Rep}_d(\mathcal{A}) \to \mathrm{Rep}_{c+d}(\mathcal{A})$$

For $(Z, Z') \in \mathcal{T}_c(\mathcal{A}) \times \mathcal{T}_d(\mathcal{A})$ both Z and Z' are connected and contain a rational point, hence, are both geometrically connected by [27, Tag 04KV]. Therefore the product $Z \times_K Z'$ is connected by [27, Tag 0385]. So there is a unique connected component $Z + Z' \in \mathcal{T}_{c+d}(\mathcal{A})$ containing $\oplus_{c,d}(Z \times_K Z')$.

[6] i.e. a set M with a binary operation $+ : M \times M \to M$ which is associative, commutative and admits a neutral element 0.

The monoid $\mathcal{T}(\mathcal{A})$ has been studied in the past under other names like component semigroup (see e.g. [11, §4]). Since we want to emphasize the analogy to dimension vectors, we will refer to it as the *dimension vector monoid* of $\mathcal{A}$ and call the elements $m \in \mathcal{T}_d(\mathcal{A})$ *dimension vectors of total dimension* d. Usually we will think of dimension vectors as abstract monoid elements. Whenever we want to refer to the connected component (associated to) m as a geometric object, we will denote it by $\mathrm{Rep}_m(\mathcal{A})$.

Since the orbit $\mathbb{O}_{\mathcal{M}}$ associated to a left $\mathcal{A}$-module $\mathcal{M}$ is connected, it belongs to a unique connected component. Denote the corresponding dimension vector by $\underline{\dim}(\mathcal{M})$.

The dimension vector monoid $\mathcal{T}(\mathcal{A})$ is contravariant functorial in $\mathcal{A}$: If $\varphi : \mathcal{A} \to \mathcal{B}$ is a K-algebra homomorphism and $m \in \mathcal{T}(\mathcal{B})$ a dimension vector, then denote by $\mathcal{T}(\varphi)(m)$ the dimension vector associated to the connected component which contains the image of $\mathrm{Rep}_m(\mathcal{B})$ under φ^*. This defines a monoid homomorphism $\mathcal{T}(\varphi) : \mathcal{T}(\mathcal{B}) \to \mathcal{T}(\mathcal{A})$.

The homomorphism $\mathcal{T}(\varphi)$ induces maps $\mathcal{T}_d(\varphi) : \mathcal{T}_d(\mathcal{B}) \to \mathcal{T}_d(\mathcal{A})$ for all $d \in \mathbb{N}_0$, because restriction of scalars preserves the vector space dimension of modules. Since every bijective monoid homomorphism is an isomorphism, $\mathcal{T}(\varphi)$ is an isomorphism if and only if the map $\mathcal{T}_d(\varphi)$ is bijective for every d.

Example 3.1

a) Let $\mathcal{C}$ be a finite dimensional semisimple K-algebra. We assume that $\mathcal{C}$ is completely split, i.e. of the form (15). All left $\mathcal{C} \otimes_K F$-modules for every field extension $F \supseteq K$ are defined over $\mathcal{C}$. Therefore the (finitely many) algebro-geometric orbits of the K-valued points cover $\mathrm{Rep}_d(\mathcal{C})$ and all of them are connected and closed by Proposition 2.11. We deduce that the orbits of the K-valued points are nothing but the connected components and that the map $\mathrm{iso}_d(\mathcal{C}) \to \mathcal{T}_d(\mathcal{C})$, $[\mathcal{M}] \mapsto \underline{\dim}(\mathcal{M})$ is bijective for every $d \in \mathbb{N}_0$. Since there is precisely one simple left $\mathcal{C}$-module for every matrix algebra factor in (15), we have established a monoid isomorphism

$$\mathcal{T}(\mathcal{C}) \cong \mathbb{N}_0^c$$

where we define $c := c_1 + \cdots + c_e$. In fact, $\mathcal{T}(\mathcal{C})$ is nothing but the submonoid of the Grothendieck group $K_0(\mathcal{C})$ generated by the equivalence classes of the (absolutely) simples in this situation. If $m_\gamma \in \mathbb{N}_0^c$ is the γ-th standard basis vector for $0 \leq \gamma < c$, then its image $|m_\gamma|$ under the homomorphism $|\,.\,| : \mathcal{T}(\mathcal{C}) \to \mathbb{N}_0$ is given by $|m_\gamma| = \epsilon$ for the unique $1 \leq \epsilon \leq e$ with $c_1 + \cdots + c_{\epsilon-1} \leq \gamma < c_1 + \cdots + c_\epsilon$. If $\mathcal{C} = K[\mathcal{F}]$ is the group algebra of a finite group $\mathcal{F}$ over a suitable field K, then the monoid homomorphism $|\,.\,|$ (or more specifically its extension to the Grothendieck group) is often called *degree map*.

b) Let $K\vec{Q}$ be the path algebra of a finite quiver $\vec{Q}$ with vertex set $\mathsf{v}(\vec{Q})$, $\mathcal{C} \subseteq K\vec{Q}$ the subalgebra spanned by the paths of length zero and $\iota : \mathcal{C} \hookrightarrow K\vec{Q}$ the homomorphism given by inclusion. We will briefly outline how our notion of dimension vector generalizes the well-known notion from quiver representation theory (see e.g. [11, §4]):

$\mathcal{C} \cong K^{\mathsf{v}(\vec{Q})}$ is a completely split finite dimensional semisimple K-algebra, so by (a) we have

$$\mathcal{T}(\mathcal{C}) \cong \mathbb{N}_0^{\mathsf{v}(\vec{Q})}$$

which is usually called the dimension vector monoid of the quiver $\vec{Q}$. We want to show that $\mathcal{T}(\iota) : \mathcal{T}(K\vec{Q}) \to \mathcal{T}(\mathcal{C})$ is a monoid isomorphism. For $m \in \mathbb{N}_0^{\mathsf{v}(\vec{Q})}$ we have a $\mathbf{GL}_{d,K}$-equivariant K-scheme isomorphism

$$\mathrm{Rep}_m(\mathcal{C}) \cong \mathbf{GL}_{d,K}/H$$

where $H \cong \prod_{v \in \mathsf{v}(\vec{Q})} \mathbf{GL}_{m(v),K}$, because $\operatorname{Rep}_m(\mathcal{C})$ is an algebro-geometric orbit as discussed in (a). The fibre $Y := (\iota^*)^{-1}(x) \subseteq \operatorname{Rep}_{|m|}\left(K\vec{Q}\right)$ of any K-rational point $x \in \operatorname{Rep}_m(\mathcal{C})(K)$ represents the functor

$$C \mapsto \prod_{(v \xrightarrow{a} w) \in \mathsf{a}(\vec{Q})} \mathbf{M}_{m(w) \times m(v)}(C)$$

i.e. Y is an affine space, in particular geometrically connected. So by Lemma 2.13 we see that

$$(\iota^*)^{-1}(\operatorname{Rep}_m(\mathcal{C})) \cong \mathbf{GL}_{d,K} \times^H Y$$

is connected. So for each $d \in \mathbb{N}_0$ the preimages of the connected components of $\operatorname{Rep}_d(\mathcal{C})$ form a finite partition of $\operatorname{Rep}_d\left(K\vec{Q}\right)$ into closed connected subsets, i.e. the preimages have to be the connected components of $\operatorname{Rep}_d\left(K\vec{Q}\right)$.

The following proposition and corollary give a complete description of the dimension vector monoid of the group algebra $K[\mathcal{G}]$ of a finitely generated virtually free group $\mathcal{G}$ over a suitable field K.

Proposition 3.2 *Let $\mathcal{A}$ be a finitely generated K-algebra, $\mathcal{B}$ and $\mathcal{C}$ completely split finite dimensional semisimple K-algebras and $\varphi_1 : \mathcal{C} \to \mathcal{B}$, $\varphi_2, \varphi_3 : \mathcal{C} \to \mathcal{A}$ K-algebra homomorphisms.*

a) *Consider the K-algebra pushout $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ given by φ_1, φ_2. The commutative square*

$$\begin{array}{ccc} \mathcal{T}(\mathcal{A} *_{\mathcal{C}} \mathcal{B}) & \longrightarrow & \mathcal{T}(\mathcal{B}) \\ \downarrow & & \downarrow{\scriptstyle \mathcal{T}(\varphi_1)} \\ \mathcal{T}(\mathcal{A}) & \xrightarrow[\mathcal{T}(\varphi_2)]{} & \mathcal{T}(\mathcal{C}) \end{array} \tag{23}$$

is a pullback square of commutative monoids.

b) *If $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} *_K K[t, t^{-1}]$ is the canonical K-algebra embedding and $\Phi : \mathcal{A} *_K K[t, t^{-1}] \to \mathcal{A} *_K K[t, t^{-1}]$ the K-algebra automorphism $\Phi(f) := t^{-1} f t$, then $\mathcal{T}(\iota_{\mathcal{A}})$ is an isomorphism and $\mathcal{T}(\Phi) = \mathrm{id}$.*

c) *Consider the HNN extension $\mathcal{A} *_{\mathcal{C}}^{\varphi_2, \varphi_3}$ given by φ_2, φ_3. The diagram*

$$\mathcal{T}(\mathcal{A} *_{\mathcal{C}}^{\varphi_2, \varphi_3}) \to \mathcal{T}(\mathcal{A}) \underset{\mathcal{T}(\varphi_3)}{\overset{\mathcal{T}(\varphi_2)}{\rightrightarrows}} \mathcal{T}(\mathcal{C}) \tag{24}$$

is an equalizer diagram of commutative monoids.

Proof About (a): (23) induces a homomorphism $\theta : \mathcal{T}(\mathcal{A} *_{\mathcal{C}} \mathcal{B}) \to \mathcal{T}(\mathcal{A}) \times_{\mathcal{T}(\mathcal{C})} \mathcal{T}(\mathcal{B})$ with

$$\mathcal{T}(\mathcal{A}) \times_{\mathcal{T}(\mathcal{C})} \mathcal{T}(\mathcal{B}) = \{(m, n) \in \mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{B}) \mid \mathcal{T}(\varphi_2)(m) = \mathcal{T}(\varphi_1)(n)\}$$

θ is an isomorphism if and only if its restriction $\theta_d : \mathcal{T}_d(\mathcal{A} *_{\mathcal{C}} \mathcal{B}) \to \mathcal{T}_d(\mathcal{A}) \times_{\mathcal{T}_d(\mathcal{C})} \mathcal{T}_d(\mathcal{B})$ is bijective for every $d \in \mathbb{N}_0$. Denote the natural homomorphisms $\mathcal{A} \to \mathcal{A} *_{\mathcal{C}} \mathcal{B}$ and $\mathcal{B} \to \mathcal{A} *_{\mathcal{C}} \mathcal{B}$ by $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ and the connected components of $\operatorname{Rep}_d(\mathcal{A})$ and $\operatorname{Rep}_d(\mathcal{B})$ by $X_0, \ldots, X_a$ and $Y_0, \ldots Y_b$ respectively.

Since the contravariant functor $\mathrm{Rep}_d(-)$ maps colimits to limits, we have a natural isomorphism $\mathrm{Rep}_d(\mathcal{A} *_{\mathcal{C}} \mathcal{B}) \cong \mathrm{Rep}_d(\mathcal{A}) \times_{\mathrm{Rep}_d(\mathcal{C})} \mathrm{Rep}_d(\mathcal{B})$. This yields a decomposition

$$\mathrm{Rep}_d(\mathcal{A} *_{\mathcal{C}} \mathcal{B}) = \bigsqcup_{\substack{0\le\alpha\le a,\\ 0\le\beta\le b}} X_\alpha \times_{\mathrm{Rep}_d(\mathcal{C})} Y_\beta$$

into open and closed subsets. Hence, each connected component $\mathrm{Rep}_v(\mathcal{A} *_{\mathcal{C}} \mathcal{B})$ associated to a dimension vector $v \in \mathcal{T}(\mathcal{A} *_{\mathcal{C}} \mathcal{B})$ lies in a unique $X_\alpha \times_{\mathrm{Rep}_d(\mathcal{C})} Y_\beta$ and one checks that the homomorphism θ is given by $\theta_d(v) = (m_\alpha, n_\beta)$ where $m_\alpha \in \mathcal{T}(\mathcal{A})$ and $n_\beta \in \mathcal{T}(\mathcal{B})$ are the dimension vectors (associated to) X_α and Y_β. We claim that $\mathrm{Rep}_{m_\alpha}(\mathcal{A}) \times_{\mathrm{Rep}_d(\mathcal{C})} \mathrm{Rep}_{n_\beta}(\mathcal{B}) = X_\alpha \times_{\mathrm{Rep}_d(\mathcal{C})} Y_\beta$ is connected and contains a rational point for each $(m_\alpha, n_\beta) \in \mathcal{T}(\mathcal{A}) \times_{\mathcal{T}(\mathcal{C})} \mathcal{T}(\mathcal{B})$ which proves that θ_d is bijective.

Denote by $u := \mathcal{T}(\varphi_2)(m_\alpha) = \mathcal{T}(\varphi_1)(n_\beta) \in \mathcal{T}(\mathcal{C})$ the dimension vector lying below (m_α, n_β). By construction both $\mathrm{Rep}_{m_\alpha}(\mathcal{A})$ and $\mathrm{Rep}_{n_\beta}(\mathcal{B})$ map into the connected component $Z := \mathrm{Rep}_u(\mathcal{C})$ and we obtain an isomorphism $X_\alpha \times_{\mathrm{Rep}_d(\mathcal{C})} Y_\beta \cong X_\alpha \times_Z Y_\beta$. Since m_α and n_β are dimension vectors, there are K-valued points $x \in \mathrm{Rep}_{m_\alpha}(\mathcal{A})(K)$ and $y \in \mathrm{Rep}_{n_\beta}(\mathcal{B})(K)$. We set $z := \varphi_2^*(x)$.

By Example 3.1(a) we have $Z = \mathbb{O}_z \cong \mathbf{GL}_{d,K}/S(z)$ and $Y_\beta = \mathbb{O}_y \cong \mathbf{GL}_{d,K}/S(y)$. Since $\mathbb{O}_z(K) = \mathbf{GL}_d(K).z$ and φ_1^* restricts to a $\mathbf{GL}_{d,K}$-equivariant map $Y_\beta \to Z$, we may assume without loss of generality that $\varphi_1^*(y) = z = \varphi_2^*(x)$ (for $\varphi_1^*(y) = g.z$ we may replace y by $g^{-1}.y$). So $X_\alpha \times_Z Y_\beta(K)$ is non-empty.

Furthermore taking fibres we obtain a commutative diagram

$$\begin{tikzcd}
X_\alpha \times_Z Y_\beta \arrow[rr, "\overline{\iota_{\mathcal{B}}}"] \arrow[dd] & & \mathbf{GL}_{d,K}/S(y) \arrow[dd, dashed] & \\
& \overline{\iota_{\mathcal{B}}}^{-1}(y) \arrow[ul] \arrow[rr] \arrow[dd] & & \mathrm{Spec}\,(K) \arrow[ul] \arrow[dd, "\mathrm{id}"] \\
X_\alpha \arrow[rr, dashed, "\overline{\varphi}"] & & \mathbf{GL}_{d,K}/S(z) & \\
& \overline{\varphi}^{-1}(z) \arrow[ul] \arrow[rr] & & \mathrm{Spec}\,(K) \arrow[ul, dashed]
\end{tikzcd} \tag{25}$$

where $\overline{\varphi}$ and $\overline{\iota_{\mathcal{B}}}$ are given as the restrictions of φ_2^* and $\iota_{\mathcal{B}}^*$. The bottom, top and back squares of (25) are pullback squares, hence, the front square is too and we obtain an isomorphism

$$\overline{\iota_{\mathcal{B}}}^{-1}(y) \cong \overline{\varphi}^{-1}(z)$$

Applying Lemma 2.13 we obtain isomorphisms

$$X_\alpha \cong \mathbf{GL}_{d,K} \times^{S(z)} \overline{\varphi}^{-1}(z), \quad X_\alpha \times_Z Y_\beta \cong \mathbf{GL}_{d,K} \times^{S(y)} \overline{\iota_{\mathcal{B}}}^{-1}(y)$$

So since X_α is connected by assumption, $\overline{\iota_{\mathcal{B}}}^{-1}(y) \cong \overline{\varphi}^{-1}(z)$ and $X_\alpha \times_Z Y_\beta$ are connected too.

About (b): Since $\mathrm{Rep}_d(K[t, t^{-1}]) \cong \mathbf{GL}_{d,K}$ and $\mathrm{Rep}_d(K) \cong \mathrm{Spec}\,(K)$ are connected, we have isomorphisms $\mathcal{T}(K[t, t^{-1}]) \cong \mathbb{N}_0 \cong \mathcal{T}(K)$ given by $|\,.\,|$ respectively. So $\mathcal{T}(\iota_{\mathcal{A}})$ is an isomorphism by part a).

Now if $(x, g) \in \mathrm{Rep}_d(\mathcal{A})(K) \times \mathbf{GL}_d(K) = \mathrm{Rep}_d(\mathcal{A} *_K K[t, t^{-1}])(K)$ is a K-valued point, then $\Phi^*(x, g) = g^{-1}.(x, g) \in \mathbb{O}_{(x,g)}(K)$. So (x, g) and $\Phi^*(x, g)$ lie in the same

connected component, because $\mathbb{O}_{(x,g)}$ is connected. This proves $\mathcal{T}(\Phi) = \mathrm{id}$.
About (c): Denote the natural projection $\mathcal{A} *_K K[t, t^{-1}] \to \mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}$ by π. By construction of the HNN extension we have $\pi \circ \iota_{\mathcal{A}} \circ \varphi_2 = \pi \circ \Phi \circ \iota_{\mathcal{A}} \circ \varphi_3$. So using part b) we obtain that $\theta := \mathcal{T}(\iota_{\mathcal{A}}) \circ \mathcal{T}(\pi) : \mathcal{T}(\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}) \to \mathcal{T}(\mathcal{A})$ factorizes over

$$\mathrm{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3)) = \{m \in \mathcal{T}(\mathcal{A}) \mid \mathcal{T}(\varphi_2)(m) = \mathcal{T}(\varphi_3)(m)\} \subseteq \mathcal{T}(\mathcal{A})$$

We will show that this induces an isomorphism $\mathcal{T}(\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}) \to \mathrm{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3))$.

Denote the connected components of $\mathrm{Rep}_d(\mathcal{A})$ by $X_0, \ldots, X_a$. As for a) we obtain a natural isomorphism $\mathrm{Rep}_d(\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}) \cong \mathrm{Eq}((\iota_{\mathcal{A}} \circ \varphi_2)^*, (\Phi \circ \iota_{\mathcal{A}} \circ \varphi_3)^*)$ and a decomposition

$$\mathrm{Rep}_d(\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}) = \bigsqcup_{0 \le \alpha \le a} (\iota_{\mathcal{A}}^* \circ \pi^*)^{-1}(X_\alpha)$$

into open and closed subsets $(\iota_{\mathcal{A}}^* \circ \pi^*)^{-1}(X_\alpha) = (\pi^*)^{-1}(X_\alpha \times_K \mathbf{GL}_{d,K})$ and it remains to show that $(\pi^*)^{-1}(X_\alpha \times_K \mathbf{GL}_{d,K})$ is connected and contains a rational point if $X_\alpha = \mathrm{Rep}_{m_\alpha}(\mathcal{A})$ corresponds to a dimension vector $m_\alpha \in \mathrm{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3))$.

We first check using the universal property of the equalizer $\mathrm{Rep}_d(\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3})$ that a K-valued point $(x, g) \in X_\alpha(K) \times \mathbf{GL}_d(K) = (X_\alpha \times_K \mathbf{GL}_{d,K})(K)$ lies in the image of the closed embedding π^* if and only if

$$\rho_x \circ \varphi_2 = (\iota_{\mathcal{A}} \circ \varphi_2)^*(x, g) = (\Phi \circ \iota_{\mathcal{A}} \circ \varphi_3)^*(x, g) = g^{-1}.(\rho_x \circ \varphi_3) \tag{26}$$

The rational points associated to $\rho_x \circ \varphi_2$ and $\rho_x \circ \varphi_3$ belong to the same connected component $Z \subseteq \mathrm{Rep}_d(\mathcal{C})$, because we assumed $m_\alpha \in \mathrm{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3))$. Again using Example 3.1(a) we know that $Z = \mathbb{O}_z \cong \mathbf{GL}_{d,K}/S(z)$ for some $z \in \mathrm{Rep}_d(\mathcal{C})(K)$. Hence, there is a $g \in \mathbf{GL}_d(K)$ satisfying $\rho_x \circ \varphi_2 = g^{-1}.(\rho_x \circ \varphi_3)$ which yields a rational point in $(\pi^*)^{-1}(X_\alpha \times_K \mathbf{GL}_{d,K})$.

Now denote the restriction of $(\pi \circ \iota_{\mathcal{A}} \circ \varphi_2)^*$ to $(\pi^*)^{-1}(X_\alpha \times_K \mathbf{GL}_{d,K})$ by ψ. The criterion (26) yields that $\psi^{-1}(z) \cong (\varphi_2^*)^{-1}(z) \times_K S(z)$. So as $S(z)$ is geometrically irreducible by Example 2.14, $\psi^{-1}(z)$ is connected if and only if $(\varphi_2^*)^{-1}(z)$ is (see [27, Tag 0385]). We now again use Lemma 2.13 to obtain isomorphisms

$$X_\alpha \cong \mathbf{GL}_{d,K} \times^{S(z)} (\varphi_2^*)^{-1}(z), \quad (\pi^*)^{-1}(X_\alpha \times_K \mathbf{GL}_{d,K}) \cong \mathbf{GL}_{d,K} \times^{S(z)} \psi^{-1}(z)$$

So $(\pi^*)^{-1}(X_\alpha \times_K \mathbf{GL}_{d,K})$ is connected, because the connected component X_α is. □

Corollary 3.3 *If $\mathcal{G}$ is the finitely generated virtually free group given by (9) and K is a suitable field for $\mathcal{G}$, then $\mathcal{T}(K[\mathcal{G}])$ is given by*

$$\left\{(m_i)_i \in \prod_{i=0}^{I} \mathcal{T}(K[\mathcal{G}_i]) \mid \forall\, 1 \le j \le I + J : \mathcal{T}(\iota_j)(m_{s(j)}) = \mathcal{T}(\kappa_j)(m_{t(j)})\right\} \tag{27}$$

Proof The claim follows from the decomposition (12) of $K[\mathcal{G}]$ and Example 3.1(a) by repeatedly applying part (a) and (c) of Proposition 3.2. □

Remark 3.4 The inclusion maps $K[\mathcal{G}_{s(j)}] \overset{\iota_j}{\hookleftarrow} K[\mathcal{G}'_j] \overset{\kappa_j}{\hookrightarrow} K[\mathcal{G}_{t(j)}]$, $1 \le j \le I + J$, from (11) form a diagram of K-algebras and applying the contravariant functor $\mathcal{T}$ to it gives a diagram of (free) commutative monoids. Equation (27) is by construction a limit of this diagram of commutative monoids. Of course every other limit of it is naturally isomorphic to (27), e.g. one could also embed $\mathcal{T}(K[\mathcal{G}])$ into $\prod_{i=0}^{I} \mathcal{T}(K[\mathcal{G}_i]) \times \prod_{j=1}^{I+J} \mathcal{T}(K[\mathcal{G}'_j])$ mapping $(m_i)_i$ to $((m_i)_i, (\mathcal{T}(\iota_j)(m_{s(j)}))_j)$.

An important consequence of Corollary 3.3 is that the dimension vector monoid $\mathcal{T}(K[\mathcal{G}])$ does not depend on the choice of the suitable field K: First let $\mathcal{H}$ be a finite group and $\mathcal{F} \subseteq \mathcal{H}$ be a subgroup. Using well-known arguments from the representation theory of finite groups (see [23, §14.6, §15.1 & Prop. 43 in §15.5]) and Example 3.1(a) one first shows that the monoids $\mathcal{T}(K[\mathcal{F}])$ and $\mathcal{T}(K[\mathcal{H}])$ as well as the homomorphism $\mathcal{T}(K[\mathcal{H}]) \to \mathcal{T}(K[\mathcal{F}])$ do not depend on K. So since $\mathcal{T}(K[\mathcal{G}])$ is the limit of a diagram of monoids which itself does not depend on K, $\mathcal{T}(K[\mathcal{G}])$ does not depend on K as well. We will therefore drop K from the notation and simply write $\mathcal{T}(\mathcal{G})$.

We conclude our current discussion of dimension vectors with a few general remarks. However, the readers may feel free to skip forward to Sect. 6 for some hands-on examples at this point. We first note another immediate consequence of the isomorphism (27): $\mathcal{T}(\mathcal{G})$ is equipped with an embedding into the free commutative monoid $\prod_i \mathcal{T}(\mathcal{G}_i)$. While this does not imply that $\mathcal{T}(\mathcal{G})$ has to be free itself, it at least shows that the monoid $\mathcal{T}(\mathcal{G})$ has the cancellation property, i.e. it embeds as a submonoid into its associated group which is free Abelian of finite rank.

Moreover $\mathcal{T}(\mathcal{G})$ comes with a canonical homomorphism $\mathcal{T}(\mathcal{G}) \to \mathcal{T}(\mathcal{G}_i)$ for each $0 \le i \le I$ and a canonical homomorphism $\mathcal{T}(\mathcal{G}) \to \mathcal{T}(\mathcal{G}'_j)$ for each $1 \le j \le I+J$. We say that $m \in \mathcal{T}(\mathcal{G})$ *lies over* $m_i \in \mathcal{T}(\mathcal{G}_i)$ for $0 \le i \le I$ and $u_j \in \mathcal{T}(\mathcal{G}'_j)$ for $1 \le j \le I+J$ if these are the images of m under the canonical homomorphisms. Note that these images uniquely determine m due to the isomorphism (27).

For $c \in \mathbb{N}_{\ge 1}$ and $m \in \mathcal{T}(\mathcal{G})$ we write $c|m$ if there is an $n \in \mathcal{T}(\mathcal{G})$ fulfilling $m = c.n = n + \cdots + n$. Such an n is necessarily unique and we denote it by $m/c := n$. Moreover $\{c \in \mathbb{N}_{\ge 1} \mid c|m\}$ is a finite set—this as well as the uniqueness of m/c are immediate consequences of the embedding $\mathcal{T}(\mathcal{G}) \hookrightarrow \prod_i \mathcal{T}(\mathcal{G}_i)$. We denote

$$\gcd(m) := \max\{c \in \mathbb{N}_{\ge 1} \mid c|m\} = \operatorname{lcm}\{c \in \mathbb{N}_{\ge 1} \mid c|m\} \tag{28}$$

Another important property of dimension vectors is that they are additive on short exact sequences which is the content of the next lemma.

Lemma 3.5 *Let $\mathcal{A}$ be a finitely generated K-algebra. For every short sequence*

$$0 \to \mathcal{N} \to \mathcal{W} \to \mathcal{M} \to 0$$

of left $\mathcal{A}$-modules we have $\underline{\dim}(\mathcal{W}) = \underline{\dim}(\mathcal{N}) + \underline{\dim}(\mathcal{M})$.

Proof Recall that $\underline{\dim}(\mathcal{W})$ is defined as (the element of $\mathcal{T}(\mathcal{A})$ associated to) the connected component containing the algebro-geometric orbit $\mathbb{O}_{\mathcal{W}}$ and that $\underline{\dim}(\mathcal{N})+\underline{\dim}(\mathcal{M})$ is by definition the connected component containing $\mathbb{O}_{\mathcal{N}\oplus\mathcal{M}}$. These two connected components have to coincide, because connected components are closed and we have the inclusion $\mathbb{O}_{\mathcal{N}\oplus\mathcal{M}} \subseteq \overline{\mathbb{O}_{\mathcal{W}}}$ by Proposition 2.11(c). □

We end this subsection with a comparison between the dimension vector monoid $\mathcal{T}(\mathcal{A})$ of a finitely generated K-algebra $\mathcal{A}$ and the Grothendieck group $K_0(\mathcal{A})$ associated to $\mathcal{A}$. Even if $\mathcal{T}(\mathcal{A})$ does not have the cancellation property, we may consider its associated Abelian group which we denote $\mathcal{T}(\mathcal{A})^{\mathrm{gp}}$. From Lemma 3.5 we see that the map

$$\operatorname{iso}(\mathcal{A}) \overset{\underline{\dim}}{\to} \mathcal{T}(\mathcal{A}) \to \mathcal{T}(\mathcal{A})^{\mathrm{gp}}$$

extends to a unique group homomorphism $K_0(\mathcal{A}) \to \mathcal{T}(\mathcal{A})^{\mathrm{gp}}$ which is surjective: If $m - n \in \mathcal{T}(\mathcal{A})^{\mathrm{gp}}$ is any element, where $m, n \in \mathcal{T}(\mathcal{A})$ are dimension vectors, then the corresponding connected components have to contain rational points x, y. If $\mathcal{M}_x, \mathcal{M}_y$ are the respective left $\mathcal{A}$-modules, then the element $[\mathcal{M}_x] - [\mathcal{M}_y] \in K_0(\mathcal{A})$ is mapped to $m - n$.

In the special case where $\mathcal{A}$ is a completely split finite dimensional semisimple K-algebra, this group homomorphism is in fact an isomorphism (see Example 3.1(a)). However, usually the group $\mathcal{T}(\mathcal{A})^{\mathrm{gp}}$ of dimension vectors is much smaller than the Grothendieck group—the dimension vector of a module carries slightly more information than its vector space dimension, but much less information than its isomorphism class.

3.2 Homological Euler form

We now want to discuss another object which again has a well-known analogue in quiver representation theory: the *(homological) Euler form.* For $\mathcal{A}$ a (left hereditary) K-algebra and finite dimensional left $\mathcal{A}$-modules $\mathcal{M}, \mathcal{N}$ we define

$$\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}} := \dim_K \left(\mathrm{Hom}_{\mathcal{A}}\left(\mathcal{M}, \mathcal{N}\right)\right) - \dim_K \left(\mathrm{Ext}^1_{\mathcal{A}}\left(\mathcal{M}, \mathcal{N}\right)\right)$$

Example 3.6 Let $\mathcal{C}$ be a completely split finite dimensional semisimple K-algebra. Recall that $\mathcal{C}$ may be written as (15). $\mathcal{C}$ admits precisely $c := c_1 + \cdots + c_e$ pairwise non-isomorphic (absolutely) simple modules—choose a representative $\mathcal{L}_\gamma$ for each isomorphism class. For two arbitrary finite dimensional left $\mathcal{C}$-modules

$$\mathcal{M} = \bigoplus_{\gamma=0}^{c-1} \mathcal{L}_\gamma{}^{\oplus m(\gamma)}, \quad \mathcal{N} = \bigoplus_{\gamma=0}^{c-1} \mathcal{L}_\gamma{}^{\oplus n(\gamma)}$$

we compute the homological Euler form

$$\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}} = \dim_K \left(\mathrm{Hom}_{\mathcal{C}}\left(\mathcal{M}, \mathcal{N}\right)\right) = \sum_{\gamma=0}^{c-1} m(\gamma) n(\gamma)$$

by using Schur's Lemma for the absolutely simple modules $\mathcal{L}_\gamma$. For $m := \underline{\dim}(\mathcal{M})$ and $n := \underline{\dim}(\mathcal{N})$ we also introduce the notation $\langle m, n \rangle_{\mathcal{C}} := \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}}$ which is well-defined, since $\underline{\dim} : \mathrm{iso}_d(\mathcal{C}) \to \mathcal{T}_d(\mathcal{C})$ is bijective by Example 3.1(a).

As for the dimension vector monoid we now want to compute the homological Euler form of the group algebra of a finitely generated virtually free group $\mathcal{G}$ over a suitable field.

Proposition 3.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be left hereditary finitely generated K-algebras, $\mathcal{C}$ a finite dimensional semisimple K-algebra and $\varphi_1 : \mathcal{C} \to \mathcal{B}$, $\varphi_2, \varphi_3 : \mathcal{C} \to \mathcal{A}$ K-algebra homomorphisms.*

a) *Consider the pushout given by φ_1, φ_2 and let $\mathcal{M}, \mathcal{N}$ be finite dimensional left $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$-modules. The homological Euler form of $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ is given by*

$$\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A} *_{\mathcal{C}} \mathcal{B}} = \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}} + \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{B}} - \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}}$$

b) *Consider the HNN extension $\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}$ and let $\mathcal{M}, \mathcal{N}$ be finite dimensional left $\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}$-modules. The homological Euler form of $\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}$ is given by*

$$\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}*_{\mathcal{C}}^{\varphi_2,\varphi_3}} = \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}} - \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}}$$

Proof Let $\mathcal{D}$ be a finitely generated K-algebra and $\mathcal{W}$ a K-linear $(\mathcal{D}, \mathcal{D})$-bimodule, i.e. a left $\mathcal{D} \otimes_K \mathcal{D}^{\mathrm{op}}$-module. We consider the K-linear map $\eta : \mathcal{W} \to \mathrm{Der}_K(\mathcal{D}, \mathcal{W})$ which sends $w \in \mathcal{W}$ to its inner derivation $\eta(w) = (f \mapsto f \underset{\rightarrow}{.} w - w \underset{\leftarrow}{.} f)$ and obtain an exact sequence

$$0 \to \mathrm{Ker}(\eta) \to \mathcal{W} \xrightarrow{\eta} \mathrm{Der}_K(\mathcal{D}, \mathcal{W}) \to \mathrm{Coker}(\eta) \to 0 \tag{29}$$

For the bimodule $\mathcal{W} = \mathrm{Hom}_K(\mathcal{M}, \mathcal{N})$ we obtain $\mathrm{Ker}(\eta) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$ and $\mathrm{Coker}(\eta) \cong \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$ (see [28, Lemma 9.1.9 & Lemma 9.2.1]). Hence, (29) yields

$$\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{D}} = \dim_K(\mathcal{M}) \cdot \dim_K(\mathcal{N}) - \dim_K \mathrm{Der}_K(\mathcal{D}, \mathrm{Hom}_K(\mathcal{M}, \mathcal{N}))$$

So we can reformulate the claimed identity a) as

$$\dim_K \mathrm{Der}_K(\mathcal{D}, \mathcal{W}) = \dim_K \mathrm{Der}_K(\mathcal{A}, \mathcal{W}) + \dim_K \mathrm{Der}_K(\mathcal{B}, \mathcal{W}) - \dim_K \mathrm{Der}_K(\mathcal{C}, \mathcal{W})$$

for $\mathcal{D} = \mathcal{A} *_{\mathcal{C}} \mathcal{B}$ and $\mathcal{W} = \mathrm{Hom}_K(\mathcal{M}, \mathcal{N})$ and the claimed identity b) takes the form

$$\dim_K \mathrm{Der}_K(\mathcal{D}, \mathcal{W}) = \dim_K \mathrm{Der}_K(\mathcal{A}, \mathcal{W}) + \dim_K(\mathcal{W}) - \dim_K \mathrm{Der}_K(\mathcal{C}, \mathcal{W})$$

for $\mathcal{D} = \mathcal{A} *_{\mathcal{C}}^{\varphi_2, \varphi_3}$ and $\mathcal{W} = \mathrm{Hom}_K(\mathcal{M}, \mathcal{N})$.

For the identity (a) we use that $\mathrm{Der}_K(\mathcal{D}, \mathcal{W})$ is the K-vector space pullback induced by φ_1^* and φ_2^*, hence, $\mathrm{Der}_K(\mathcal{D}, \mathcal{W})$ is the kernel of the map

$$({\varphi_2}^*, -{\varphi_1}^*) : \mathrm{Der}_K(\mathcal{A}, \mathcal{W}) \oplus \mathrm{Der}_K(\mathcal{B}, \mathcal{W}) \to \mathrm{Der}_K(\mathcal{C}, \mathcal{W})$$

which is surjective, because $\mathcal{C}$ is separable, i.e. every derivation of $\mathcal{C}$ is inner. (See Remark 2.6 for the definition of separable algebras and [2, Prop. 4.2] for the fact that an algebra over a field is separable if and only if all of its derivations are inner.)

The identity (b) is proven similarly:

$$\mathrm{Der}_K(\mathcal{D}, \mathcal{W}) \xrightarrow{\pi^*} \mathrm{Der}_K\left(\mathcal{A} *_K K[t, t^{-1}], \mathcal{W}\right) \underset{(\varphi_2')^*}{\overset{(\varphi_3')^*}{\rightrightarrows}} \mathrm{Der}_K(\mathcal{C}, \mathcal{W})$$

is an equalizer diagram of vector spaces, i.e. $\mathrm{Der}_K(\mathcal{D}, \mathcal{W})$ is the kernel of the map

$$\mathrm{Der}_K\left(\mathcal{A} *_K K[t, t^{-1}], \mathcal{W}\right) \xrightarrow{(\varphi_2')^* - (\varphi_3')^*} \mathrm{Der}_K(\mathcal{C}, \mathcal{W})$$

which is surjective as well, because $\mathcal{C}$ is a separable K-algebra. This proves b), since we have an isomorphism

$$\mathrm{Der}_K\left(\mathcal{A} *_K K[t, t^{-1}], \mathcal{W}\right) \cong \mathrm{Der}_K(\mathcal{A}, \mathcal{W}) \oplus \mathcal{W}, \quad \delta \mapsto (\delta \circ \iota_{\mathcal{A}}, \delta(t))$$

(Recall that we assume the field K to be perfect within Sect. 3. Note that this proof only needs K to be perfect so that $\mathcal{C}$ is separable. This could be avoided by instead assuming that $\mathcal{C}$ is separable from the start.) $\square$

Corollary 3.8 *If $\mathcal{G}$ is the finitely generated virtually free group given by (9) and K is suitable for $\mathcal{G}$, then $\langle -, - \rangle_{K[\mathcal{G}]}$ is given by*

$$\langle \mathcal{M}, \mathcal{N} \rangle_{K[\mathcal{G}]} = \sum_{i=0}^{I} \langle m_i, n_i \rangle_{K[\mathcal{G}_i]} - \sum_{j=1}^{I+J} \langle u_j, v_j \rangle_{K[\mathcal{G}'_j]} \tag{30}$$

where $\underline{\dim}(\mathcal{M})$ is the dimension vector lying over $m_i \in \mathcal{T}(\mathcal{G}_i)$ for $0 \le i \le I$ and $u_j \in \mathcal{T}(\mathcal{G}'_j)$ for $1 \le j \le I + J$ and $\underline{\dim}(\mathcal{N})$ is lying over $n_i \in \mathcal{T}(\mathcal{G}_i)$ and $v_j \in \mathcal{T}(\mathcal{G}'_j)$ respectively.

Proof Repeatedly apply Proposition 3.7 to the decomposition (12). □

Since the righthand side of the formula (30) only depends on the dimension vectors $\underline{\dim}(\mathcal{M})$ and $\underline{\dim}(\mathcal{N})$ and is $\mathbb{N}_0$-linear in both arguments, Corollary 3.8 yields that the homological Euler form induces a well-defined $\mathbb{N}_0$-bilinear map

$$\langle -, - \rangle_{K[\mathcal{G}]} : \mathcal{T}(\mathcal{G}) \times \mathcal{T}(\mathcal{G}) \to \mathbb{Z}$$

Furthermore we see from formula (30) that $\langle -, - \rangle_{K[\mathcal{G}]}$ does not depend on K. So we will simply denote it by $\langle -, - \rangle_{\mathcal{G}}$. Moreover (30) combined with Example 3.6 shows that $\langle -, - \rangle_{\mathcal{G}}$ is symmetric.

As before we postpone explicit examples to Sect. 6, but the readers may feel free to skip forward to it now.

3.3 Counting representation spaces

Now assume $K = \mathbb{F}_q$ is finite. We want to show in this subsection that the connected components $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$, $m \in \mathcal{T}(\mathcal{G})$ are polynomial count if $\mathbb{F}_q$ is suitable for $\mathcal{G}$. As before we start with the case of semisimple algebras.

Example 3.9 Let $\mathcal{C}$ be a completely split finite dimensional semisimple $\mathbb{F}_q$-algebra with (absolutely) simple left modules $\mathcal{L}_0, \ldots, \mathcal{L}_{c-1}$. If $\mathcal{M}$ is a finite dimensional left $\mathcal{C}$-module of dimension vector $m = \underline{\dim}(\mathcal{M}) = \sum_\gamma m(\gamma).\underline{\dim}(\mathcal{L}_\gamma)$ we know from Example 3.1(a) that $\mathrm{Rep}_m(\mathcal{C}) = \mathbb{O}_{\mathcal{M}} \cong \mathbf{GL}_{|m|,K}/S(x_{\mathcal{M}})$. Since $S(x_{\mathcal{M}}) \cong \prod_\gamma \mathbf{GL}_{m(\gamma),\mathbb{F}_q}$ is special (see Example 2.14), we obtain

$$\#\,\mathrm{Rep}_m(\mathcal{C})(\mathbb{F}_{q^\alpha}) = \#\mathbf{GL}_{|m|}(\mathbb{F}_{q^\alpha})/S(x_{\mathcal{M}})(\mathbb{F}_{q^\alpha}) = \frac{P_{\mathbf{GL}_{|m|}}}{\prod_{\gamma=0}^{c-1} P_{\mathbf{GL}_{m(\gamma)}}}(q^\alpha)$$

Using Lemma 2.9(a) we see that the rational function

$$P_m^{\mathcal{C}} := \frac{P_{\mathbf{GL}_{|m|}}}{\prod_{\gamma=0}^{c-1} P_{\mathbf{GL}_{m(\gamma)}}}$$

is in fact a counting polynomial for $\mathrm{Rep}_m(\mathcal{C})$. Note that the vector space dimension $|m| = \dim_{\mathbb{F}_q}(\mathcal{M})$ is given by $\sum_\gamma m(\gamma)\dim_{\mathbb{F}_q}(\mathcal{L}_\gamma)$ as $|\,.\,|$ is a monoid homomorphism.

Similar to $\mathcal{T}(\mathcal{G})$ and $\langle -, - \rangle_{\mathcal{G}}$ we give a full description of the counting polynomials of $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$.

Proposition 3.10 *Let $\mathcal{A}$ be a finitely generated $\mathbb{F}_q$-algebra, $\mathcal{B}$ and $\mathcal{C}$ completely split finite dimensional semisimple $\mathbb{F}_q$-algebras and $\varphi_1 : \mathcal{C} \to \mathcal{B}$, $\varphi_2, \varphi_3 : \mathcal{C} \to \mathcal{A}$ homomorphisms of $\mathbb{F}_q$-algebras. For $d \in \mathbb{N}_0$ fix dimension vectors $m \in \mathcal{T}_d(\mathcal{A})$, $n \in \mathcal{T}_d(\mathcal{B})$ and $u \in \mathcal{T}_d(\mathcal{C})$.*

a) *Consider the pushout $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ given by φ_1, φ_2 and assume that (m, n) is a dimension vector in $\mathcal{T}(\mathcal{A}) \times_{\mathcal{T}(\mathcal{C})} \mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{A} *_{\mathcal{C}} \mathcal{B})$ lying over u. If* $\mathrm{Rep}_m(\mathcal{A})$ *admits a counting polynomial $P_m^{\mathcal{A}}$, then the rational function $P_m^{\mathcal{A} *_{\mathcal{C}}^{\varphi_2,\varphi_3}} := P_m^{\mathcal{A}} P_{\mathbf{GL}_d} / P_u^{\mathcal{C}}$ is a counting polynomial for* $\mathrm{Rep}_{(m,n)}(\mathcal{A} *_{\mathcal{C}} \mathcal{B})$.
b) *Consider the HNN extension $\mathcal{A} *_{\mathcal{C}}^{\varphi_2,\varphi_3}$ and assume that m is an element of the equalizer* $\mathrm{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3)) = \mathcal{T}(\mathcal{A} *_{\mathcal{C}}^{\varphi_2,\varphi_3})$ *lying over u. If* $\mathrm{Rep}_m(\mathcal{A})$ *admits a counting polynomial $P_m^{\mathcal{A}}$, then the rational function $P_m^{\mathcal{A} *_{\mathcal{C}}^{\varphi_2,\varphi_3}} := P_m^{\mathcal{A}} P_{\mathbf{GL}_d} / P_u^{\mathcal{C}}$ is a counting polynomial for* $\mathrm{Rep}_m(\mathcal{A} *_{\mathcal{C}}^{\varphi_2,\varphi_3})$.

Proof About (a): As in the proof of Proposition 3.2(a) we may express $\mathrm{Rep}_m(\mathcal{A})$ and $\mathrm{Rep}_{(m,n)}(\mathcal{A} *_{\mathcal{C}} \mathcal{B})$ as associated fibre spaces

$$\mathrm{Rep}_m(\mathcal{A}) \cong \mathbf{GL}_{d,\mathbb{F}_q} \times^{S(z)} Y, \quad \mathrm{Rep}_{(m,n)}(\mathcal{A} *_{\mathcal{C}} \mathcal{B}) \cong \mathbf{GL}_{d,\mathbb{F}_q} \times^{S(y)} Y$$

for $S(y) \subseteq S(z) \subseteq \mathbf{GL}_{d,\mathbb{F}_q}$ the stabilizers of points $y \in \mathrm{Rep}_n(\mathcal{B})(\mathbb{F}_q)$, $z \in \mathrm{Rep}_u(\mathcal{C})(\mathbb{F}_q)$ and Y an affine finite type $\mathbb{F}_q$-scheme with $S(z)$-action. Since $S(y)$ and $S(z)$ are special, we may use (22) to obtain

$$\#\,\mathrm{Rep}_m(\mathcal{A} *_{\mathcal{C}} \mathcal{B})(\mathbb{F}_{q^\alpha}) = \frac{P_{\mathbf{GL}_d}(q^\alpha)}{\#S(y)(\mathbb{F}_{q^\alpha})}\#Y(\mathbb{F}_{q^\alpha}) = \frac{P_{\mathbf{GL}_d}(q^\alpha)}{\#S(y)(\mathbb{F}_{q^\alpha})}\frac{\#S(z)(\mathbb{F}_{q^\alpha})}{P_{\mathbf{GL}_d}(q^\alpha)}P_m^{\mathcal{A}}(q^\alpha)$$

Using Example 3.9 this proves part (a).

About (b): As above we use the proof of Proposition 3.2(c) to obtain

$$\mathrm{Rep}_m(\mathcal{A}) \cong \mathbf{GL}_{d,\mathbb{F}_q} \times^{S(z)} Y, \quad \mathrm{Rep}_m(\mathcal{A} *_{\mathcal{C}}^{\varphi_2,\varphi_3}) \cong \mathbf{GL}_{d,\mathbb{F}_q} \times^{S(z)} \left(Y \times_{\mathbb{F}_q} S(z)\right)$$

for $S(z) \subseteq \mathbf{GL}_{d,\mathbb{F}_q}$ the stabilizer of a point $z \in \mathrm{Rep}_u(\mathcal{C})(\mathbb{F}_q)$ and we calculate

$$\#\,\mathrm{Rep}_m(\mathcal{A} *_{\mathcal{C}}^{\varphi_2,\varphi_3})(\mathbb{F}_{q^\alpha}) = P_{\mathbf{GL}_d}(q^\alpha)\#Y(\mathbb{F}_{q^\alpha}) = P_{\mathbf{GL}_d}(q^\alpha)\frac{\#S(z)(\mathbb{F}_{q^\alpha})}{P_{\mathbf{GL}_d}(q^\alpha)}P_m^{\mathcal{A}}(q^\alpha)$$

□

Corollary 3.11 *If $\mathcal{G}$ is the finitely generated virtually free group given by (9) and $\mathbb{F}_q$ is suitable for $\mathcal{G}$, then*

$$P_m^{\mathcal{G}} := P_{\mathbf{GL}_d}{}^J \frac{\prod_{i=0}^{I} P_{m_i}^{\mathcal{G}_i}}{\prod_{j=1}^{I+J} P_{u_j}^{\mathcal{G}'_j}} = P_{\mathbf{GL}_d} \frac{\prod_{j=1}^{I+J}\prod_{\gamma=0}^{c_j-1} P_{\mathbf{GL}_{u_j(\gamma)}}}{\prod_{i=0}^{I}\prod_{\beta=0}^{b_i-1} P_{\mathbf{GL}_{m_i(\beta)}}} \tag{31}$$

is a counting polynomial for $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$, where $m \in \mathcal{T}(\mathcal{G})$ is the dimension vector lying over $m_i \in \mathcal{T}(\mathcal{G}_i) \cong \mathbb{N}_0^{b_i}$ for $0 \leq i \leq I$ and over $u_j \in \mathcal{T}(\mathcal{G}'_j) \cong \mathbb{N}_0^{c_j}$ for $1 \leq j \leq I+J$ and $P_{m_i}^{\mathcal{G}_i} := P_{m_i}^{\mathbb{F}_q[\mathcal{G}_i]}$ as well as $P_{u_j}^{\mathcal{G}'_j} := P_{u_j}^{\mathbb{F}_q[\mathcal{G}'_j]}$ are given by Example 3.9.

Proof We obtain $P_m^{\mathcal{G}}$ by repeatedly applying Proposition 3.10 to our decomposition (12) of $K[\mathcal{G}]$ (note that J is the number of HNN-extensions involved in (12)). The second expression comes from Example 3.9 by cancelling out the $P_{\mathbf{GL}_d}$ occurring in the numerator and denominator of the fraction. □

The formula (31) in particular shows that the polynomials $P_m^{\mathcal{G}}$ are independent of the choice of a finite suitable field for $\mathcal{G}$.

4 Hall algebra methods

Consider the field $\mathbb{Q}(s)$ of rational functions in the variable s as well as its subring

$$\mathbb{Q}[s]_{(s-q)} = \{P/Q \in \mathbb{Q}(s) \mid Q(q) \neq 0\}$$

where q is some fixed prime power. The $\mathbb{Q}$-algebra homomorphisms

$$\mathbb{Q}(s) \hookleftarrow \mathbb{Q}[s]_{(s-q)} \overset{\mathrm{ev}_q}{\twoheadrightarrow} \mathbb{Q}$$

induce homomorphisms of $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-algebras

$$\mathbb{Q}(s)[\mathcal{T}(\mathcal{G})] \hookleftarrow \mathbb{Q}[s]_{(s-q)}[\mathcal{T}(\mathcal{G})] \overset{\mathrm{ev}_q}{\twoheadrightarrow} \mathbb{Q}[\mathcal{T}(\mathcal{G})] \tag{32}$$

The homomorphism $|\,.\,| : \mathcal{T}(\mathcal{G}) \to \mathbb{N}_0$ endows every $\mathcal{T}(\mathcal{G})$-graded algebra $\mathcal{C}$ with an $\mathbb{N}_0$-grading $\mathcal{C} = \bigoplus_{d\geq 0} \mathcal{C}_d$ where $\mathcal{C}_d$ is spanned by all homogeneous elements with degree in $\mathcal{T}_d(\mathcal{G})$. In particular $\mathcal{C}$ carries a canonical topology where the ideals $\mathfrak{i}_d := \bigoplus_{\delta\geq d} \mathcal{C}_\delta$ form a neighbourhood basis of 0. The completion $\widehat{\mathcal{C}}$ with respect to this is a topological algebra whose underlying topological module is given by $\prod_{\delta\geq 0} \mathcal{C}_\delta$. We have the following facts on completions of (graded) algebras:

Lemma 4.1 *Let A be a commutative ring and $\mathcal{C}, \mathcal{C}'$ be $\mathcal{T}(\mathcal{G})$-graded A-algebras. Denote their completions by $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}'}$.*

a) *An element $(f_\delta)_{\delta\geq 0} \in \widehat{\mathcal{C}}$ is invertible if and only if $f_0 \in \mathcal{C}_0$ is invertible.*
b) *Every graded homomorphism $\mathcal{C} \to \mathcal{C}'$ extends uniquely to a continuous algebra homomorphism $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$.*

So by taking completions of (32) we obtain continuous $\mathbb{Q}$-algebra homomorphisms

$$\mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!] \hookleftarrow \mathbb{Q}[s]_{(s-q)}[\![\mathcal{T}(\mathcal{G})]\!] \overset{\mathrm{ev}_q}{\twoheadrightarrow} \mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!] \tag{33}$$

We now define a second multiplication on the monoid algebras considered above: The so called *twisted multiplication* on $\mathbb{Q}[\mathcal{T}(\mathcal{G})]$ is given by bilinear extension of

$$t^m * t^n := q^{-\langle m,n\rangle_{\mathcal{G}}}.t^{m+n}$$

Analogously we define $t^m * t^n := s^{-\langle m,n\rangle_{\mathcal{G}}}.t^{m+n}$ on $\mathbb{Q}(s)[\mathcal{T}(\mathcal{G})]$ and $\mathbb{Q}[s]_{(s-q)}[\mathcal{T}(\mathcal{G})]$. Note that the powers of q and s are well-defined for negative exponents as q is non-zero and $s \notin (s-q)$. We denote the resulting $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-algebras by $\mathbb{Q}^{q\text{-tw}}[\mathcal{T}(\mathcal{G})]$, $\mathbb{Q}(s)^{\mathrm{tw}}[\mathcal{T}(\mathcal{G})]$ and $\mathbb{Q}[s]_{(s-q)}{}^{\mathrm{tw}}[\mathcal{T}(\mathcal{G})]$. As for the monoid algebras we have $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-algebra homomorphisms analogous to (32) and continuous $\mathbb{Q}$-algebra homomorphisms like (33) between their twisted versions.

The twisted monoid algebras are in fact isomorphic to their untwisted counterparts. To construct explicit isomorphisms between them we need a monoid homomorphism $\mathbb{Y} : \mathcal{T}(\mathcal{G}) \to \mathbb{Z}$ which satisfies

$$\langle m, m\rangle_{\mathcal{G}} \equiv \mathbb{Y}(m) \pmod 2 \quad \forall\, m \in \mathcal{T}(\mathcal{G})$$

We construct a distinguished $\mathbb{Y}$ to show existence, but everything that follows does not depend on this choice. For a finite group $\mathcal{F}$ we have an identification $\mathcal{T}(\mathcal{F}) \cong \mathbb{N}_0^c$ and may take $\mathbb{Y}(m) := \sum_{\gamma=0}^{c-1} m(\gamma)$. For the general case of $\mathcal{G}$ we can mimic our computation of the Euler form and define

$$\mathbb{Y}(m) := \sum_{i=0}^{I}\sum_{\gamma=0}^{c_i-1} m_i(\gamma) - \sum_{j=1}^{I+J}\sum_{\gamma=0}^{c'_j-1} u_j(\gamma)$$

where $m \in \mathcal{T}(\mathcal{G})$ is the dimension vector lying over $m_i \in \mathbb{N}_0^{c_i} \cong \mathcal{T}(\mathcal{G}_i)$ and over $u_j \in \mathbb{N}_0^{c'_j} \cong \mathcal{T}(\mathcal{G}'_j)$. We may now define a $\mathbb{Q}$-vector space isomorphism

$$\mathcal{S} : \mathbb{Q}^{q\text{-tw}}[\mathcal{T}(\mathcal{G})] \to \mathbb{Q}[\mathcal{T}(\mathcal{G})], \quad \mathcal{S}(t^m) := q^{\frac{1}{2}(\langle m,m\rangle_{\mathcal{G}} - \mathbb{Y}(m))}t^m$$

and isomorphisms $\mathbb{Q}(s)^{\mathrm{tw}}[\mathcal{T}(\mathcal{G})] \cong \mathbb{Q}(s)[\mathcal{T}(\mathcal{G})]$, $\mathbb{Q}[s]_{(s-q)}{}^{\mathrm{tw}}[\mathcal{T}(\mathcal{G})] \cong \mathbb{Q}[s]_{(s-q)}[\mathcal{T}(\mathcal{G})]$ via $\mathcal{S}(t^m) := s^{\frac{1}{2}(\langle m,m\rangle_{\mathcal{G}} - \mathbb{Y}(m))} t^m$. We call each of the maps $\mathcal{S}$ *shift operator.* By construction the shift operators preserve the $\mathcal{T}(\mathcal{G})$-grading and using that $\langle -, -\rangle_{\mathcal{G}}$ is symmetric one can deduce that they are isomorphisms of graded algebras. Hence, they extend uniquely to continuous algebra isomorphisms between the completed monoid algebras.

Shift operators like $\mathcal{S}$ have already appeared in Mozgovoy-Reineke's treatment of the free group case in [17]. To get rid of the *correction form* $\mathbb{Y}$ we could also define a shift operator by $\mathcal{S}'(t^m) := q^{\frac{1}{2}\langle m,m\rangle_{\mathcal{G}}} t^m$. This would mean however that we have to work with $\mathbb{Q}[\sqrt{q}]$ instead of $\mathbb{Q}$, $\mathbb{Q}[\sqrt{q}](\sqrt{s})$ instead of $\mathbb{Q}(s)$ etc.

Now fix a finite field $K = \mathbb{F}_q$ which is suitable for $\mathcal{G}$. We briefly recall the construction of the *finitary Hall algebra* $\mathbf{H}(\mathcal{A})$ of a finitely generated $\mathbb{F}_q$-algebra $\mathcal{A}$:

Denote by $\mathrm{iso}(\mathcal{A}) := \bigsqcup_{d\geq 0} \mathrm{iso}_d(\mathcal{A})$ the set of all isomorphism classes of finite dimensional left $\mathcal{A}$-modules. (Analogously we denote by $\mathrm{ssim}(\mathcal{A})$, $\mathrm{sim}(\mathcal{A})$ and $\mathrm{absim}(\mathcal{A})$ the sets of all isomorphism classes of semisimple, simple and absolutely simple modules respectively.) $\mathbf{H}(\mathcal{A})$ is defined as the free $\mathbb{Q}$-vector space on the basis $\mathrm{iso}(\mathcal{A})$. The multiplication of two basis elements $[\mathcal{M}], [\mathcal{N}] \in \mathrm{iso}(\mathcal{A})$ is defined as $[\mathcal{M}] \cdot [\mathcal{N}] = \sum_{[\mathcal{W}]} F^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}[\mathcal{W}]$ (where the sum is running over all $[\mathcal{W}] \in \mathrm{iso}(\mathcal{A})$) with structure coefficients

$$F^{\mathcal{W}}_{\mathcal{M},\mathcal{N}} := \#\{\mathcal{L} \subseteq \mathcal{W} \text{ left } \mathcal{A}\text{-submodule} \mid \mathcal{L} \cong \mathcal{N}, \mathcal{W}/\mathcal{L} \cong \mathcal{M}\}$$

Note that the multiplication is well-defined, because the set $\mathrm{iso}_d(\mathcal{A})$ is finite for all $d \in \mathbb{N}_0$. An upper bound for its number of elements would be $q^{d^2 a}$ where $a \in \mathbb{N}_0$ is the cardinality of some finite set of $\mathbb{F}_q$-algebra generators of $\mathcal{A}$, because a representation $\mathcal{A} \to \mathbf{M}_d(\mathbb{F}_q)$ is uniquely determined by the images of the generators.

Since dimension vectors are additive on short exact sequences, $\mathbf{H}(\mathcal{A})$ is $\mathcal{T}(\mathcal{A})$-graded—$\mathbf{H}_m(\mathcal{A})$ is the $\mathbb{Q}$-linear span of $\{[\mathcal{M}] \in \mathrm{iso}(\mathcal{A}) \mid \underline{\dim}(\mathcal{M}) = m\}$. In particular the homomorphism $|.| : \mathcal{T}(\mathcal{A}) \to \mathbb{N}_0$ induces an $\mathbb{N}_0$-grading $\mathbf{H}(\mathcal{A}) = \bigoplus_{\delta\geq 0} \mathbf{H}_\delta(\mathcal{A})$ where $\mathbf{H}_\delta(\mathcal{A}) = \bigoplus_{m\in\mathcal{T}_\delta(\mathcal{A})} \mathbf{H}_m(\mathcal{A})$. As for the monoid algebras (32) we may complete $\mathbf{H}(\mathcal{A})$ with respect to this $\mathbb{N}_0$-grading. Denote the completed finitary Hall algebra by $\mathbf{H}(\!(\mathcal{A})\!)$.

We consider the element $\varepsilon := \sum_{[\mathcal{M}]\in \mathrm{iso}(\mathcal{A})} [\mathcal{M}] \in \mathbf{H}(\!(\mathcal{A})\!)$ which is a multiplicative unit by Lemma 4.1(a). It was shown by M. Reineke in [21, Lemma 3.4] that the coefficients $e_{\mathcal{M}}$ of the inverse $\varepsilon^{-1} = \sum_{[\mathcal{M}]} e_{\mathcal{M}}[\mathcal{M}]$ are given by

$$\begin{cases} \prod\limits_{[\mathcal{L}]\in \mathrm{sim}(\mathcal{A})} (-1)^{a_{\mathcal{L}}} \# \mathrm{End}_{\mathcal{A}}(\mathcal{L})^{a_{\mathcal{L}}(a_{\mathcal{L}}-1)/2}, & \text{if } \mathcal{M} = \bigoplus\limits_{[\mathcal{L}]\in \mathrm{sim}(\mathcal{A})} \mathcal{L}^{\oplus a_{\mathcal{L}}} \text{ semisimple} \\ 0, & \text{if } \mathcal{M} \text{ not semisimple} \end{cases} \tag{34}$$

The following lemma is essentially due to M. Reineke (see [21, Lemma 3.3]).

Lemma 4.2 *Let $\mathcal{A} = \mathbb{F}_q[\mathcal{G}]$ be the group algebra of the finitely generated virtually free group $\mathcal{G}$ over the suitable field $\mathbb{F}_q$. The $\mathbb{Q}$-linear map*

$$\int : \mathbf{H}(\mathbb{F}_q[\mathcal{G}]) \to \mathbb{Q}^{\,q\text{-}tw}[\mathcal{T}(\mathcal{G})], \quad \int([\mathcal{M}]) := \frac{1}{\#\,\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M})} t^{\underline{\dim}(\mathcal{M})} \tag{35}$$

is a homomorphism of $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-algebras.

Proof Since the proof is completely analogous to [21, Lemma 3.3], we will not give all details. Since (35) is a homomorphism of $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-vector spaces, it suffices to show that $\int([\mathcal{M}]) \cdot \int([\mathcal{N}]) = \int([\mathcal{M}] \cdot [\mathcal{N}])$ for all $[\mathcal{M}], [\mathcal{N}] \in \mathrm{iso}(\mathbb{F}_q[\mathcal{G}])$.

Denote $m := \underline{\dim}(\mathcal{M}), n := \underline{\dim}(\mathcal{N})$. By Lemma 3.5, the coefficients $F^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}$ are zero unless $\underline{\dim}(\mathcal{W}) = m + n$. So both $\int([\mathcal{M}]) \cdot \int([\mathcal{N}])$ and $\int([\mathcal{M}] \cdot [\mathcal{N}])$ are in the $\mathbb{Q}$-linear span of t^{m+n} and it remains to prove the formula

$$\sum_{\substack{[\mathcal{W}]\in \mathrm{iso}(\mathbb{F}_q[\mathcal{G}]),\\ \underline{\dim}(\mathcal{W})=m+n}} \frac{F^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}}{\#\,\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{W})} = \frac{q^{-\langle m,n\rangle_{\mathcal{G}}}}{\#\left(\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M}) \times \mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{N})\right)}$$

Recall from Sect. 1 that $\mathbb{F}_q[\mathcal{G}]$ is hereditary, because $\mathbb{F}_q$ is suitable. So by the definition of the homological Euler form, we have

$$q^{-\langle m,n\rangle_{\mathcal{G}}} = \frac{\#\,\mathrm{Ext}^1_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})}{\#\,\mathrm{Hom}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})}$$

and it suffices to prove the so called *Riedtmann formula*

$$\#\left(\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M}) \times \mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{N})\right) F^{\mathcal{W}}_{\mathcal{M},\mathcal{N}} = \frac{\#\,\mathrm{Ext}^1_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})_{\mathcal{W}}\,\#\,\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{W})}{\#\,\mathrm{Hom}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})} \tag{36}$$

for all $[\mathcal{W}]$ with $\underline{\dim}(\mathcal{W}) = m + n$, where $\mathrm{Ext}^1_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})_{\mathcal{W}} \subseteq \mathrm{Ext}^1_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})$ denotes the set of equivalence classes of extensions with middle term $\mathcal{W}$.

Let $\mathcal{P}^{\mathcal{W}}_{\mathcal{M},\mathcal{N}} \subseteq \mathrm{Hom}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{N},\mathcal{W}) \times \mathrm{Hom}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{W},\mathcal{M})$ be the set of short exact sequences $0 \to \mathcal{N} \to \mathcal{W} \to \mathcal{M} \to 0$. The group $\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M}) \times \mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{N})$ acts freely on $\mathcal{P}^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}$ via $(\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}).(\varphi,\theta) := (\varphi \circ \Phi_{\mathcal{N}}^{-1}, \Phi_{\mathcal{M}} \circ \theta)$, where the number of orbits is given by $F^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}$. Hence, the left hand side of (36) can be identified with $\#\mathcal{P}^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}$.

On the other hand $\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{W})$ acts on $\mathcal{P}^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}$ via

$$\Phi_{\mathcal{W}}.(\varphi,\theta) := (\Phi_{\mathcal{W}} \circ \varphi, \theta \circ \Phi_{\mathcal{W}}^{-1})$$

and the set of orbits can be identified with $\mathrm{Ext}^1_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})_{\mathcal{W}}$. Moreover the orbit of (φ,θ) has $\#\,\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{W})/\#\,\mathrm{Hom}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N})$ elements, because

$$\mathrm{Hom}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M},\mathcal{N}) \to S((\varphi,\theta)), \quad \omega \mapsto \mathrm{id}_{\mathcal{W}} + \varphi \circ \omega \circ \theta$$

defines a group isomorphism to the stabilizer subgroup $S((\varphi,\theta)) \subseteq \mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{W})$. This proves that the right hand side of (36) coincides with $\#\mathcal{P}^{\mathcal{W}}_{\mathcal{M},\mathcal{N}}$ as well.

□

The map $\int$ is called a *Hall algebra integral.* By Lemma 4.1(b) it extends uniquely to a continuous $\mathbb{Q}$-algebra homomorphism between the completions.

We summarize the situation with the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Q}(s)^{\mathrm{tw}}[\![\mathcal{T}(\mathcal{G})]\!] \arrow[d, "\mathcal{S}", "\cong"'] & \mathbb{Q}[s]_{(s-q)}{}^{\mathrm{tw}}[\![\mathcal{T}(\mathcal{G})]\!] \arrow[l, hook'] \arrow[r, two heads, "\mathrm{ev}_q"] \arrow[d, "\mathcal{S}", "\cong"'] & \mathbb{Q}^{\text{q-tw}}[\![\mathcal{T}(\mathcal{G})]\!] \arrow[d, "\mathcal{S}", "\cong"'] & \mathbf{H}(\!(\mathbb{F}_q[\mathcal{G}])\!) \arrow[l, two heads, "\int"'] \\
\mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!] & \mathbb{Q}[s]_{(s-q)}[\![\mathcal{T}(\mathcal{G})]\!] \arrow[l, hook'] \arrow[r, two heads, "\mathrm{ev}_q"] & \mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!] &
\end{tikzcd}$$

Most of the actual computations we are interested in happen in the ring $\mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!]$ while our knowledge of the representation theory of $\mathbb{F}_q[\mathcal{G}]$ comes from the completed Hall algebra $\mathbf{H}(\!(\mathbb{F}_q[\mathcal{G}])\!)$. So the rough idea for proving results like the main theorem is the following: First we observe an interesting identity in $\mathbf{H}((\mathbb{F}_q[\mathcal{G}]))$, then we map it to $\mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!]$ and search for a certain kind of lift of it along ev_q. (Mostly we want the lift to be independent of the choice of the prime power q.) Afterwards we can manipulate the obtained identity within $\mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!]$.

5 Counting polynomials

After introducing a lot of machinery we now come back to our original objective of counting functions and relate them to our machinery. Let $\mathbb{F}_q$ be a suitable finite field for $\mathcal{G}$. For each dimension vector $m \in \mathcal{T}(\mathcal{G})$ define the *refined counting functions*

$$\begin{aligned} r_m^{\mathrm{absim}}(q^\alpha) &:= \#\{[\mathcal{M}] \in \mathrm{absim}(\mathbb{F}_{q^\alpha}[\mathcal{G}]) \mid \underline{\dim}(\mathcal{M}) = m\} \\ r_m^{\mathrm{sim}}(q^\alpha) &:= \#\{[\mathcal{M}] \in \mathrm{sim}(\mathbb{F}_{q^\alpha}[\mathcal{G}]) \mid \underline{\dim}(\mathcal{M}) = m\} \\ r_m^{\mathrm{ss}}(q^\alpha) &:= \#\{[\mathcal{M}] \in \mathrm{ssim}(\mathbb{F}_{q^\alpha}[\mathcal{G}]) \mid \underline{\dim}(\mathcal{M}) = m\} \end{aligned} \tag{37}$$

The refined counting functions r_m^{ss} and r_m^{absim} again count the rational points of GIT moduli spaces. We describe these using the following lemma.

Lemma 5.1 *All connected components* $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}]) \subseteq \mathrm{Rep}_{|m|}(\mathbb{F}_q[\mathcal{G}])$ *are* $\mathbf{GL}_{|m|,\mathbb{F}_q}$*-invariant closed subschemes. Their GIT quotients*

$$M(\mathbb{F}_q[\mathcal{G}], m) := \mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}]) /\!\!/ \mathbf{GL}_{|m|,\mathbb{F}_q}$$

are the connected components of $M(\mathbb{F}_q[\mathcal{G}], |m|)$. *Moreover there is a* $\mathbf{GL}_{|m|,\mathbb{F}_q}$*-invariant open subscheme* $\mathrm{Rep}_m^{absim}(\mathbb{F}_q[\mathcal{G}]) \subseteq \mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ *for each* $m \in \mathcal{T}(\mathcal{G})$ *such that*

$$M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m) := \mathrm{Rep}_m^{absim}(\mathbb{F}_q[\mathcal{G}]) /\!\!/ \mathbf{GL}_{|m|,\mathbb{F}_q} = M(\mathbb{F}_q[\mathcal{G}], m) \cap M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], |m|)$$

The connected components of $M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], d)$ *are given by those* $M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m)$, $m \in \mathcal{T}_d(\mathcal{G})$, *which are non-empty. Moreover all of the spaces* $\mathrm{Rep}_m^{absim}(\mathbb{F}_q[\mathcal{G}])$, $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$, $M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m)$ *and* $M(\mathbb{F}_q[\mathcal{G}], m)$ *are irreducible if non-empty.*

Proof By Proposition 2.11(a) the representation space $\mathrm{Rep}_{|m|}(\mathbb{F}_q[\mathcal{G}])$ is regular. Hence, its connected components $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ are in fact even irreducible (see e.g. [27, Tags 033M & 0569]). Since furthermore $\mathbf{GL}_{|m|,\mathbb{F}_q}$ is geometrically irreducible, $\mathbf{GL}_{|m|,\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ must be irreducible (see [27, Tag 038F]). So the image of the restricted action

$$\mathbf{GL}_{|m|,\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}]) \subseteq \mathbf{GL}_{|m|,\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{Rep}_{|m|}(\mathbb{F}_q[\mathcal{G}]) \to \mathrm{Rep}_{|m|}(\mathbb{F}_q[\mathcal{G}])$$

is irreducible and in particular connected and contained in the connected component $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$, which proves the $\mathbf{GL}_{|m|,\mathbb{F}_q}$-invariance of the connected components.

Since the quotient map $\mathrm{Rep}_{|m|}(\mathbb{F}_q[\mathcal{G}]) \to M(\mathbb{F}_q[\mathcal{G}], |m|)$ is surjective and maps pairwise disjoint invariant closed subsets to pairwise disjoint closed subsets (see e.g. [18, Proof of Thm. 1.1]), the sets $M(\mathbb{F}_q[\mathcal{G}], m)$ for $m \in \mathcal{T}_d(\mathcal{G})$ form a partition of $M(\mathbb{F}_q[\mathcal{G}], d)$ into finitely many pairwise disjoint closed connected subsets, i.e. they must be the connected components.

Now note that the subgroup of scalar matrices $\mathbb{G}_{\mathrm{m}} \subseteq \mathbf{GL}_{|m|,\mathbb{F}_q}$ acts trivially. Hence, there is a natural $\mathbf{PGL}_{|m|,\mathbb{F}_q}$-action on $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ for each $m \in \mathcal{T}(\mathcal{G})$. The subsets $\mathrm{Rep}_m^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}])$ are given by the respective locus of stable points with respect to this $\mathbf{PGL}_{|m|,\mathbb{F}_q}$-action, which is always invariant and open (see e.g. [18, §1.4], note that what nowadays is called *stable point* is called *properly stable point* within [18]).

To prove the claim about the connected components of $M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], d)$ we now observe that for each $m \in \mathcal{T}(\mathcal{G})$ the open subset $\mathrm{Rep}_m^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}]) \subseteq \mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ is either empty or irreducible and in particular connected. The rest of the proof is now analogous to the second paragraph above.

□

The refined counting functions r_m^{ss} and r_m^{absim} count the rational points of the connected components of our moduli spaces discussed in Lemma 5.1, i.e. for all $\alpha \geq 1$ we have

$$r_m^{\mathrm{ss}}\left(q^\alpha\right) = \#M(\mathbb{F}_q[\mathcal{G}], m)(\mathbb{F}_{q^\alpha}), \quad r_m^{\mathrm{absim}}\left(q^\alpha\right) = \#M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m)(\mathbb{F}_{q^\alpha}) \tag{38}$$

We can recover the original counting functions (1) from the refined ones via the formula $r_d^{\mathrm{xyz}} = \sum_{|m|=d} r_m^{\mathrm{xyz}}$. Moreover we define

$$r_{m,c}^{\mathrm{sim}}(q^\alpha) := \#\{[\mathcal{M}] \in \mathrm{sim}(\mathbb{F}_{q^\alpha}[\mathcal{G}]) \mid \underline{\dim}(\mathcal{M}) = m, \dim_{\mathbb{F}_{q^\alpha}}\left(\mathrm{End}_{\mathbb{F}_{q^\alpha}[\mathcal{G}]}(\mathcal{M})\right) = c\}$$

Analogously to [21, §4] we obtain the identities

$$r_m^{\mathrm{absim}}(q^\alpha) = r_{m,1}^{\mathrm{sim}}(q^\alpha), \quad r_{m,c}^{\mathrm{sim}}(q^\alpha) = \begin{cases} \frac{1}{c}\sum_{\gamma|c}\mu(\gamma) r_{m/c}^{\mathrm{absim}}(q^{\alpha c/\gamma}), & \text{if } c|m \\ 0, & \text{else} \end{cases} \tag{39}$$

(The first identity holds just by the definition of absolutely simple modules, the second identity can be obtained from Galois descent and Möbius inversion.) Here $\mu : \mathbb{N}_{\geq 1} \to \{-1, 0, 1\}$ denotes the (classical) Möbius function. Since $\mathcal{T}(\mathcal{G})$ embeds into a free commutative monoid, $\mathcal{C}[\![\mathcal{T}(\mathcal{G})]\!]$ can be embedded into a formal power series ring $\mathcal{C}[\![t_1, \ldots, t_a]\!]$, i.e. we may interpret the elements of $\mathcal{C}[\![\mathcal{T}(\mathcal{G})]\!]$ as formal power series. Important examples are

$$r^{\mathrm{xyz}}(q^\alpha) := \sum_{m \in \mathcal{T}(\mathcal{G})} r_m^{\mathrm{xyz}}(q^\alpha) t^m \in \mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!]$$

where $\mathrm{xyz} \in \{\mathrm{absim}, \mathrm{sim}, \mathrm{ss}\}$. Our goal is to lift the power series $r^{\mathrm{xyz}}(q^\alpha)$ reasonably along the homomorphism ev_{q^α} from equation (33), the coefficients R_m^{xyz} of such a lift R^{xyz} will be the counting polynomials we are aiming for.

We now briefly recall the construction of plethystic exponentials and logarithms. First note that $\mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!]$ is a local ring with maximal ideal

$$\mathfrak{m} := \left\{ \sum_{m \in \mathcal{T}(\mathcal{G})} f_m t^m \in \mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!] \mid f_0 = 0 \right\}$$

which is open. The subset $1 + \mathfrak{m}$ is open as well and is a topological group with respect to multiplication. $(\mathfrak{m}, +)$ and $(1 + \mathfrak{m}, \cdot)$ are isomorphic as topological groups, mutually inverse continuous isomorphisms are given by

$$\mathfrak{m} \underset{\exp}{\overset{\log}{\rightleftarrows}} 1 + \mathfrak{m}, \quad \exp(f) := \sum_{\alpha \geq 0} \frac{f^\alpha}{\alpha!}, \quad \log(1 + f) := \sum_{\beta \geq 1} \frac{(-1)^{\beta+1}}{\beta} f^\beta$$

Note that exp and log are equally well-defined for $\mathbb{Q}[s]_{(s-q)}[\![\mathcal{T}(\mathcal{G})]\!]$ and $\mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!]$ and that they commute with the homomorphisms (33), e.g. $\exp \circ \mathrm{ev}_q(f) = \mathrm{ev}_q \circ \exp(f)$ for each $f \in \mathfrak{m} \cap \mathbb{Q}[s]_{(s-q)}[\![\mathcal{T}(\mathcal{G})]\!]$.

For each $a \in \mathbb{N}_{\geq 1}$ we consider the *Adams operation*

$$\psi_a : \mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!] \to \mathbb{Q}(s)[\![\mathcal{T}(\mathcal{G})]\!], \quad \psi_a\left(\sum_m f_m t^m\right) := \sum_m f_m(s^a) t^{a.m}$$

which is a continuous $\mathbb{Q}$-algebra homomorphism. They give rise to the mutually inverse continuous group automorphisms

$$\mathfrak{m} \underset{\Psi}{\overset{\Psi^{-1}}{\rightleftarrows}} \mathfrak{m}, \quad \Psi(f) := \sum_{\alpha \geq 1} \frac{\psi_\alpha(f)}{\alpha}, \quad \Psi^{-1}(f) = \sum_{\beta \geq 1} \mu(\beta) \frac{\psi_\beta(f)}{\beta}$$

(see e.g. [15, Lemma 20]).

The *plethystic exponential* and *plethystic logarithm* are defined by $\mathrm{Exp} := \exp \circ \Psi$ and $\mathrm{Log} := \Psi^{-1} \circ \log$. They are by definition mutually inverse continuous group isomorphisms, i.e. they in particular fulfill the usual functional equations

$$\mathrm{Exp}(f + g) = \mathrm{Exp}(f)\,\mathrm{Exp}(g), \quad \mathrm{Log}(fg) = \mathrm{Log}(f) + \mathrm{Log}(g)$$

Moreover the same identities hold for convergent infinite sums and products. Exp and Log can alternatively be defined on $\mathbb{Q}((s))[\![\mathcal{T}(\mathcal{G})]\!]$ where $\mathbb{Q}((s))$ denotes the field of formal Laurent series. By some calculations in $\mathbb{Q}((s))[\![\mathcal{T}(\mathcal{G})]\!]$ one can prove

$$\mathrm{Exp}\left(\frac{1}{1 - s^c} t^m\right) = \sum_{b \geq 0} \left(\prod_{\beta=1}^{b} (1 - s^{c\beta})\right)^{-1} .t^{b.m} \tag{40}$$

(See e.g. [14, Lemma 2.2] for the case $c = 1$, then use $\mathbb{Q}(s) \cong \mathbb{Q}(s^c)$.) Using the theorem of Krull–Remak–Schmidt for a product factorization of the power series $r^{\mathrm{ss}}(q)$ and the second formula in (39) one can prove the following lemma.

Lemma 5.2 *If q is the number of elements in the finite suitable field $\mathbb{F}_q$, then the power series*

$$E(q) := \sum_{\substack{m \in \mathcal{T}(\mathcal{G}), \\ \beta \geq 1}} \frac{1}{\beta} r_m^{\mathrm{absim}}(q^\beta) t^{\beta.m}$$

is convergent in $\mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!]$ and satisfies $\exp(E(q)) = r^{\mathrm{ss}}(q)$.

See [15, Lemma 5] for the completely analogous proof in the case of absolutely indecomposables instead of absolutely simples. In Theorem 5.4 we will reformulate this lemma in terms of the plethystic exponential Exp.

We are now ready to prove the existence of counting polynomials for the refined counting functions (37). We begin our proof with a lemma about the element $\varepsilon^{-1} = \sum_{[\mathcal{M}]} e_{\mathcal{M}}[\mathcal{M}]$ discussed at (34).

Lemma 5.3 *Let $\mathbb{F}_q$ be suitable for $\mathcal{G}$. Denote by $\int : \mathbf{H}((\mathbb{F}_q[\mathcal{G}])) \to \mathbb{Q}^{q\text{-}tw}[\![\mathcal{T}(\mathcal{G})]\!]$ the Hall algebra integral defined in (35). We consider $\int \left(\varepsilon^{-1}\right) \in \mathbb{Q}^{q\text{-}tw}[\![\mathcal{T}(\mathcal{G})]\!]$ as an element of $\mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!]$ within this lemma. This element satisfies*

$$\log\left(\int \left(\varepsilon^{-1}\right)\right) = \sum_{m \in \mathcal{T}(\mathcal{G})} \sum_{\delta | m} \frac{1}{\delta(1 - q^\delta)} r_{m/\delta}^{\mathrm{absim}}\left(q^\delta\right) t^m$$

Proof Using that the coefficients $e_{\mathcal{M}}$ of ε^{-1} are given by (34), a computation completely analogous to the proof of [16, Thm. 4.2] shows

$$\sum_{[\mathcal{M}] \in \mathrm{iso}(\mathbb{F}_q[\mathcal{G}])} \frac{e_{\mathcal{M}}}{\#\,\mathrm{Aut}_{\mathbb{F}_q[\mathcal{G}]}(\mathcal{M})} t^{\underline{\dim}(\mathcal{M})} = \prod_{\substack{m \in \mathcal{T}(\mathcal{G}), \\ c | m}} \left(\sum_{b \geq 0} \left(\prod_{\beta=1}^{b} (1 - q^{c\beta})\right)^{-1} .t^{b.m}\right)^{r_{m,c}^{\mathrm{sim}}(q)}$$

in $\mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!]$. So we may apply (40) to obtain

$$\int \left(\varepsilon^{-1}\right) = \prod_{\substack{m \in \mathcal{T}(\mathcal{G}),\\ c|m}} \left(\mathrm{ev}_q \circ \mathrm{Exp}\left(\frac{1}{1-s^c} t^m\right)\right)^{r^{\mathrm{sim}}_{m,c}(q)}$$

By applying log and using $\mathrm{Exp} = \exp \circ \Psi$ we deduce

$$\log\left(\int \left(\varepsilon^{-1}\right)\right) = \sum_{\substack{m \in \mathcal{T}(\mathcal{G}),\\ c|m}} r^{\mathrm{sim}}_{m,c}(q) \left(\log \circ \mathrm{ev}_q \circ \exp \circ \Psi\left(\frac{1}{1-s^c} t^m\right)\right)$$

If we now use the identity $\exp \circ \mathrm{ev}_q = \mathrm{ev}_q \circ \exp$ and the definitions of ev_q and Ψ, this formula simplifies to

$$\log\left(\int \left(\varepsilon^{-1}\right)\right) = \sum_{\substack{m \in \mathcal{T}(\mathcal{G}),\\ c|m}} r^{\mathrm{sim}}_{m,c}(q) \sum_{\beta \geq 1} \frac{1}{\beta(1-q^{c\beta})} t^{\beta.m}$$

Applying the second formula in (39) yields

$$\log\left(\int \left(\varepsilon^{-1}\right)\right) = \sum_{m \in \mathcal{T}(\mathcal{G})} \sum_{\beta \geq 1} \sum_{c|m} \sum_{\gamma|c} \frac{1}{c\beta(1-q^{c\beta})} \mu(\gamma) r^{\mathrm{absim}}_{m/c}\left(q^{c/\gamma}\right) t^{\beta.m} \tag{41}$$

The rest of the proof is done by a substitution. Note that the index set

$$\left\{(m, \beta, c, \gamma) \in \mathcal{T}(\mathcal{G}) \times \mathbb{N}^3_{\geq 1} \mid c|m, \gamma|c\right\}$$

of the sum in (41) is in bijection with the set

$$\left\{(n, a, \delta, \gamma) \in \mathcal{T}(\mathcal{G}) \times \mathbb{N}^3_{\geq 1} \mid \gamma|a\right\}$$

via the mutually inverse bijections $(m, \beta, c, \gamma) \mapsto (m/c, \beta\gamma, c/\gamma, \gamma)$ and $(n, a, \delta, \gamma) \mapsto ((\delta\gamma).n, a/\gamma, \delta\gamma, \gamma)$. Substitution with respect to it shows that

$$\log\left(\int \left(\varepsilon^{-1}\right)\right) = \sum_{n \in \mathcal{T}(\mathcal{G})} \sum_{a \geq 1} \sum_{\delta \geq 1} \sum_{\gamma|a} \frac{1}{a\delta(1-q^{a\delta})} \mu(\gamma) r^{\mathrm{absim}}_{n}\left(q^{\delta}\right) t^{(a\delta).n}$$

If we now use, that the Möbius function satisfies

$$\sum_{\gamma|a} \mu(\gamma) = \begin{cases} 1, & a = 1 \\ 0, & a > 1 \end{cases}$$

we obtain the simplified formula

$$\log\left(\int \left(\varepsilon^{-1}\right)\right) = \sum_{n \in \mathcal{T}(\mathcal{G})} \sum_{\delta \geq 1} \frac{1}{\delta(1-q^{\delta})} r^{\mathrm{absim}}_{n}\left(q^{\delta}\right) t^{\delta.n}$$

The proof ends by another substitution with respect to the bijection

$$\mathcal{T}(\mathcal{G}) \times \mathbb{N}_{\geq 1} \to \{(m, \delta) \in \mathcal{T}(\mathcal{G}) \times \mathbb{N}_{\geq 1} \mid \delta|m\}, \quad (n, \delta) \mapsto (\delta.n, \delta)$$

□

To formulate our main result below we define the power series

$$F := \mathcal{S}\left(\sum_{m\in\mathcal{T}(\mathcal{G})} \frac{P_m^{\mathcal{G}}}{P_{\mathbf{GL}_{|m|}}} t^m\right) \in \mathbb{Q}[s]_{(s-q^\alpha)}[\![\mathcal{T}(\mathcal{G})]\!] \tag{42}$$

where $\alpha \in \mathbb{N}_{\geq 1}$ is arbitrary and $P_m^{\mathcal{G}}$ are the polynomials defined in (31). Note that F does not depend on the given integer α.

Theorem 5.4 *Let $\mathbb{F}_q$ be suitable for the finitely generated virtually free group $\mathcal{G}$. Define the power series*

$$R^{\mathrm{absim}} := (1-s)\,\mathrm{Log}\left(\mathcal{S}^{-1}\left(F^{-1}\right)\right), \quad R^{\mathrm{ss}} := \mathrm{Exp}\left(R^{\mathrm{absim}}\right) \tag{43}$$

for F as defined in (42) and denote their coefficients by R_m^{absim} and R_m^{ss} respectively. For each dimension vector $m \in \mathcal{T}(\mathcal{G})$ these coefficients satisfy $R_m^{\mathrm{absim}}, R_m^{\mathrm{ss}} \in \mathbb{Z}[s]$ and

$$\forall\, \alpha \geq 1 : R_m^{\mathrm{absim}}\left(q^\alpha\right) = r_m^{\mathrm{absim}}\left(q^\alpha\right), \quad R_m^{\mathrm{ss}}\left(q^\alpha\right) = r_m^{\mathrm{ss}}\left(q^\alpha\right) \tag{44}$$

In fact, $R_m^{\mathrm{absim}}, R_m^{\mathrm{ss}}$ are the unique polynomials satisfying (44).

Proof The uniqueness statement follows immediately from the fact that two polynomials in a single variable have to coincide if they take the same values on infinitely many arguments. For the rest of the claim it suffices to show that $R_m^{\mathrm{absim}}, R_m^{\mathrm{ss}} \in \mathbb{Q}[s]_{(s-q^\alpha)}$ for each α and that $R_m^{\mathrm{absim}}, R_m^{\mathrm{ss}}$ fulfill (44), because by (38) this would show that $R_m^{\mathrm{absim}}, R_m^{\mathrm{ss}} \in \mathbb{Q}[s]_{(s-q^\alpha)}$ are rational functions counting the rational points of separated finite type $\mathbb{F}_q$-schemes. So by Lemma 2.9(a) they would automatically be in the polynomial ring $\mathbb{Z}[s]$. For each $\alpha \geq 1$ we consider the continuous $\mathbb{Q}$-algebra homomorphism

$$\mathcal{S} \circ \int : \mathbf{H}(\!(\mathbb{F}_{q^\alpha}[\mathcal{G}])\!) \to \mathbb{Q}[\![\mathcal{T}(\mathcal{G})]\!]$$

Using that $\mathrm{Aut}\left(\mathcal{M}_x\right) \cong S(x)(\mathbb{F}_{q^\alpha})$ by (17) and $\#(\mathbf{GL}_{|m|}(\mathbb{F}_{q^\alpha}).x) = P_{\mathbf{GL}_{|m|}}(q^\alpha)/\# S(x)(\mathbb{F}_{q^\alpha})$ for $x \in \mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])(\mathbb{F}_{q^\alpha})$ and $m \in \mathcal{T}(\mathcal{G})$, we compute

$$\int(\varepsilon) = \sum_{\substack{m\in\mathcal{T}(\mathcal{G}),\\ \underline{\dim}(\mathcal{M})=m}} \frac{1}{\#\,\mathrm{Aut}\,(\mathcal{M})} t^m = \sum_{m\in\mathcal{T}(\mathcal{G})} \frac{\#\,\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])(\mathbb{F}_{q^\alpha})}{P_{\mathbf{GL}_{|m|}}(q^\alpha)} t^m = \mathrm{ev}_{q^\alpha}\left(\mathcal{S}^{-1}(F)\right)$$

Hence, $\int\left(\varepsilon^{-1}\right) = \mathrm{ev}_{q^\alpha}\left(\mathcal{S}^{-1}\left(F^{-1}\right)\right)$ for $\alpha \geq 1$. Since we have a power series $\int\left(\varepsilon^{-1}\right) \in \mathbb{Q}^{q\text{-tw}}[\![\mathcal{T}(\mathcal{G})]\!]$ for each power q^α, we consider the expression $\int\left(\varepsilon^{-1}\right)$ as a function in q-powers and denote its value in q^α by $\int\left(\varepsilon^{-1}\right)_{|q^\alpha}$.

Now define for $m \in \mathcal{T}(\mathcal{G})$ and $\alpha \geq 1$

$$\Lambda_m := \sum_{\delta | m} \frac{1}{\delta(1-s^\delta)} R_{m/\delta}^{\mathrm{absim}}\left(s^\delta\right) \in \mathbb{Q}(s)\,,\ \ \lambda_m(q^\alpha) := \sum_{\delta | m} \frac{1}{\delta(1-q^{\alpha\delta})} r_{m/\delta}^{\mathrm{absim}}\left(q^{\alpha\delta}\right) \in \mathbb{Q}$$

By definition of Ψ we have $\sum_m \Lambda_m t^m = \Psi\left((1-s)^{-1} R^{\mathrm{absim}}\right) = \log \circ \mathcal{S}^{-1}\left(F^{-1}\right)$. On the other hand we have

$$\sum_{m\in\mathcal{T}(\mathcal{G})} \lambda_m\left(q^\alpha\right) t^m = \log\left(\int \left(\varepsilon^{-1}\right)_{|q^\alpha}\right) = \mathrm{ev}_{q^\alpha} \circ \log \circ \mathcal{S}^{-1}\left(F^{-1}\right)$$

by Lemma 5.3, where we use that log commutes with the evaluation homomorphism ev_{q^α}. Hence, $\Lambda_m(q^\alpha) = \lambda_m(q^\alpha)$ holds for all m, α. Via induction on $\gcd(m)$ it can now be seen that $R_m^{\mathrm{absim}} \in \mathbb{Q}[s]_{(s-q^\alpha)}$ and $R_m^{\mathrm{absim}}(q^\alpha) = r_m^{\mathrm{absim}}(q^\alpha)$ for all $\alpha \geq 1$. (For the definition of $\gcd(m)$ see (28) above.)

We deduce the claim for R^{ss} from Lemma 5.2. Since we have already proven with the last paragraph that $R_m^{\mathrm{absim}} \in \mathbb{Z}[s]$ for all m, we have that $\Psi\left(R^{\mathrm{absim}}\right) \in \mathbb{Q}[s][\![\mathcal{T}(\mathcal{G})]\!]$. Hence, $R^{\mathrm{ss}} = \exp \circ \Psi\left(R^{\mathrm{absim}}\right) \in \mathbb{Q}[s][\![\mathcal{T}(\mathcal{G})]\!]$ too. Moreover one checks immediately that $E(q^\alpha) = \mathrm{ev}_{q^\alpha} \circ \Psi\left(R^{\mathrm{absim}}\right)$ for all α. So Lemma 5.2 shows that $\mathrm{ev}_{q^\alpha}\left(R^{\mathrm{ss}}\right) = \exp(E(q^\alpha)) = r^{\mathrm{ss}}(q^\alpha)$ for all $\alpha \geq 1$. □

Note that the counting polynomials are independent of the choice of the suitable field $\mathbb{F}_q$, because all objects involved in (42) and (43) are. As already stated in Remark 2.7 all statements in Theorem 5.4 hold in the more general setting of representations of algebras of the form (13) for $\mathcal{A}_i$ and $\mathcal{A}'_j$ completely split finite dimensional semisimple $\mathbb{F}_q$-algebras with ι_j, κ_j (not necessarily injective) K-algebra homomorphisms.

The proof of the following corollary is immediate from Theorem 5.4 and (39).

Corollary 5.5 *Let $\mathbb{F}_q$ be suitable for $\mathcal{G}$. For $m \in \mathcal{T}(\mathcal{G})$, $c \geq 1$ define*

$$R_{m,c}^{\mathrm{sim}} := \begin{cases} \frac{1}{c}\sum_{\gamma|c} \mu(\gamma)\, R_{m/c}^{\mathrm{absim}}\left(s^{c/\gamma}\right), & \text{if } c|m \\ 0, & \text{else} \end{cases}$$

and $R_m^{\mathrm{sim}} := \sum_{c|m} R_{m,c}^{\mathrm{sim}}$. The polynomials $R_{m,c}^{\mathrm{sim}}, R_m^{\mathrm{sim}} \in \mathbb{Q}[s]$ satisfy

$$\forall \alpha \geq 1 : R_{m,c}^{\mathrm{sim}}\left(q^\alpha\right) = r_{m,c}^{\mathrm{sim}}\left(q^\alpha\right), \quad R_m^{\mathrm{sim}}\left(q^\alpha\right) = r_m^{\mathrm{sim}}\left(q^\alpha\right)$$

6 Examples

6.1 Examples for Sect. 3

In this section we want to provide explicit examples of the objects discussed within this paper. As all invariants we associated to a virtually free group are derived from the invariants associated to its finite subgroups, we will start with applying the Examples 3.1(a), 3.6 and 3.9 to explicit finite groups. Since our invariants are independent of the choice of a suitable field K, we may without loss of generality work over $K = \mathbb{C}$.

6.1.1 Example: finite Abelian groups

Assume $\mathcal{F}$ is a finite Abelian group of order $\#\mathcal{F} = a$. Every (absolutely) simple representation of $\mathcal{F}$ is of dimension 1. Hence, $\mathbb{C}[\mathcal{F}] \cong \mathbb{C}^a$, $\mathcal{T}(\mathcal{F}) \cong \mathbb{N}_0^a$ and $|\,.\,| : \mathbb{N}_0^a \to \mathbb{N}_0$ is given by $|m| = \sum_\alpha m(\alpha)$. $\langle -, - \rangle_\mathcal{F}$ and $P_m^\mathcal{F}$ are given by

$$\langle m, n \rangle_\mathcal{F} = \sum_{\alpha=0}^{a-1} m(\alpha) n(\alpha), \quad P_m^\mathcal{F} := p_{\mathbf{GL}_{|m|}} / \textstyle\prod_{\alpha=0}^{a-1} p_{\mathbf{GL}_{m(\alpha)}} \tag{45}$$

More generally: If $\mathcal{F}$ is any finite group, then $\langle -, - \rangle_\mathcal{F}$ and $P_m^\mathcal{F}$ are given by (45) where a is the number of generators of the free commutative monoid $\mathcal{T}(\mathcal{F})$.

6.1.2 Example: dihedral groups

Now consider the dihedral group $\mathbb{D}_c$ of order $2c$. First consider the case $c = 2a$ even: There are 4 (absolutely) simple representations of dimension 1 and $a-1$ (absolutely) simple representations of dimension 2. Hence, $\mathbb{C}[\mathbb{D}_{2a}] \cong \mathbb{C}^4 \times \mathbf{M}_2(\mathbb{C})^{a-1}$, $\mathcal{T}(\mathbb{D}_{2a}) = \mathbb{N}_0^{a+3}$ and $|m| = \sum_{\gamma=0}^{3} m(\gamma) + 2\sum_{\gamma=4}^{a+2} m(\gamma)$.

If $c = 2a+3$ is odd, we have 2 (absolutely) simple representations of dimension 1 and $a+1$ of dimension 2. So we have $\mathbb{C}[\mathbb{D}_{2a+3}] \cong \mathbb{C}^2 \times \mathbf{M}_2(\mathbb{C})^{a+1}$, $\mathcal{T}(\mathbb{D}_{2a+3}) = \mathbb{N}_0^{a+3}$ and $|m| = \sum_{\gamma=0}^{1} m(\gamma) + 2\sum_{\gamma=2}^{a+2} m(\gamma)$.

6.1.3 Example: amalgamated free products of cyclic groups

We now consider the amalgamated free product $C_a *_{C_c} C_b$. Denote the embeddings of C_c by $\iota : C_c \hookrightarrow C_a$ and $\kappa : C_c \hookrightarrow C_b$. For each (absolutely) simple representation of C_c there are a/c ones of C_a and b/c ones of C_b which are restricted to it:

We only show the case for ι as the other one is analogous. By basic arithmetic we may always pick the generators of $C_c = \langle s \mid s^c = 1\rangle$ and $C_a = \langle t \mid t^a = 1\rangle$ such that $\iota(s) = t^{a/c}$, because ι maps each generator of C_c to an element of order c. Let $\xi_a \in \mathbb{C}$ be a primitive a-th root of unity and set $\xi_c := \xi_a^{a/c}$ which is a primitive c-th root of unity.

The a pairwise non-isomorphic (absolutely) simple representation of C_a are given by $\rho_\alpha : \mathbb{C}[C_a] \to \mathbf{M}_1(\mathbb{C})$, $\rho_\alpha(t) := \xi_a^\alpha$, for $0 \le \alpha < a$. Restricting ρ_α along ι to C_c gives $\rho_\alpha \circ \iota : \mathbb{C}[C_c] \to \mathbf{M}_1(\mathbb{C})$, $\rho_\alpha \circ \iota(s) := (\xi_a^\alpha)^{a/c} = \xi_c^\alpha$. We now conclude by observing that for all $\gamma \in \{0, 1, \ldots, c-1\}$ we have $\#\{0 \le \alpha < a \mid \alpha \equiv \gamma \pmod c\} = a/c$.

After reordering the basis elements of $\mathcal{T}(C_a) = \mathbb{N}_0^a$ we may assume that $\mathcal{T}(\iota)$ is given by $m \mapsto (\sum_{\delta=0}^{a/c-1} m(\gamma + \delta c))_\gamma$ as well as the analogous formula for $\mathcal{T}(\kappa)$. Hence, by Corollary 3.3 $\mathcal{T}(C_a *_{C_c} C_b)$ is given by

$$\mathbb{N}_0^a \times_{\mathbb{N}_0^c} \mathbb{N}_0^b = \left\{(m, n) \in \mathbb{N}_0^a \times \mathbb{N}_0^b \mid \forall\, 0 \le \gamma < c : \sum_{\delta=0}^{a/c-1} m(\gamma + \delta c) = \sum_{\epsilon=0}^{b/c-1} n(\gamma + \epsilon c)\right\}$$

with $|(m, n)| = \sum_\alpha m(\alpha) = \sum_\beta n(\beta)$. By Corollary 3.8 and (45) the Euler form is given by

$$\begin{aligned}\langle (m, n), (u, v)\rangle_{C_a *_{C_c} C_b} &= \sum_{\alpha=0}^{a-1} m(\alpha)u(\alpha) + \sum_{\beta=0}^{b-1} n(\beta)v(\beta)\\ &\quad - \sum_{\gamma=0}^{c-1}\sum_{\delta=0}^{a/c-1}\sum_{\epsilon=0}^{b/c-1} m(\gamma + \delta c)v(\gamma + \epsilon c)\end{aligned}$$

Note that by permuting the entries of $\mathbb{N}_0^a \times \mathbb{N}_0^b$ we obtain a monoid isomorphism $\mathbb{N}_0^a \times_{\mathbb{N}_0^c} \mathbb{N}_0^b \cong \left(\mathbb{N}_0^{a/c} \times_{\mathbb{N}_0} \mathbb{N}_0^{b/c}\right)^c \cong \mathcal{T}(C_{a/c} * C_{b/c})^c$.

6.1.4 Example: $\mathbf{PGL}_2(\mathbb{Z})$ and $\mathbf{GL}_2(\mathbb{Z})$

Our last two examples in this subsection are the groups $\mathbf{PGL}_2(\mathbb{Z}) \cong \mathbb{D}_2 *_{C_2} \mathbb{D}_3$ and $\mathbf{GL}_2(\mathbb{Z}) \cong \mathbb{D}_4 *_{C_2 \times C_2} \mathbb{D}_6$. Using Corollary 3.3 and the computation of $\mathcal{T}(\mathbb{D}_c)$ above, one can compute that $\mathcal{T}(\mathbf{PGL}_2(\mathbb{Z}))$ is isomorphic to

$$\left\{(m, n) \in \mathbb{N}_0^4 \times \mathbb{N}_0^3 \mid m(0) + m(1) = n(0) + n(2), m(2) + m(3) = n(1) + n(2)\right\}$$

with $|(m,n)| = \sum_\gamma m(\gamma) = n(0) + n(1) + 2n(2)$ and that $\mathcal{T}(\mathbf{GL}_2(\mathbb{Z}))$ is isomorphic to $\{(m,n) \in \mathbb{N}_0^5 \times \mathbb{N}_0^6 \mid (*)\}$ where $(*)$ are the four relations

$$m(0)+m(1) = n(0)+n(4), m(4) = n(1)+n(5) = n(2)+n(5), m(2)+m(3) = n(3)+n(4)$$

and with $|(m,n)| = 2m(4) + \sum_{\gamma=0}^3 m(\gamma) = 2n(4) + 2n(5) + \sum_{\delta=0}^3 n(\delta)$.

6.2 Examples of counting polynomials

We now want to present some examples for the counting polynomials. A first trivial example are the counting polynomials of a finite group $\mathcal{F}$: Here $\underline{\dim} : \mathrm{iso}(\mathbb{C}[\mathcal{F}]) \to \mathcal{T}(\mathcal{F})$ is bijective by Example 3.1. Hence, $R_m^{\mathrm{ss},\mathcal{F}} = 1$ for all $m \in \mathcal{T}(\mathcal{F})$ and $R_m^{\mathrm{absim},\mathcal{F}} = 1$ if the unique $[\mathcal{M}] \in \mathrm{iso}(\mathbb{C}[\mathcal{F}])$ of $\underline{\dim}(\mathcal{M}) = m$ is (absolutely) simple and zero otherwise.

For $\mathcal{G}$ an arbitrary finitely generated virtually free group given by (9) one first needs to compute the free commutative monoids $\mathcal{T}(\mathcal{G}_i)$ and $\mathcal{T}(\mathcal{G}'_j)$ as well as the homomorphisms $\mathcal{T}(\iota_j)$ and $\mathcal{T}(\kappa_j)$ between them as we have done above for some examples, i.e. one has to classify the representation theory of these finite groups e.g. over $\mathbb{C}$. The rest of the computation of the counting polynomials can be done by a computer, e.g. using the SageMath code [10]. All of the examples below (and in fact many more) have been computed in this way.

6.2.1 Example: (generalized) infinite dihedral group

First consider the group $\mathcal{G}_c := C_{2c} *_{C_c} C_{2c}$. $\mathcal{G}_c$ is a finite central extension of the infinite dihedral group $\mathbb{D}_\infty = C_2 * C_2$. We will therefore call the groups $\mathcal{G}_c$, $c \geq 1$, *generalized infinite dihedral groups*. As discussed above their dimension vector monoids can be written as $\mathcal{T}(\mathcal{G}_c) \cong \left(\mathbb{N}_0^2 \times_{\mathbb{N}_0} \mathbb{N}_0^2\right)^c$. For the dimension vector $m = (m_0, \ldots, m_{c-1})$ we have

$$R_m^{\mathrm{absim}} = \begin{cases} 1, & \text{if } |m| = 1 \\ s-2, & \text{if } \exists\, \gamma \text{ s.t. } m_\gamma = (1,1,1,1) \;\&\; m_\delta = (0,0,0,0)\ \forall\, \delta \neq \gamma \\ 0, & \text{else} \end{cases} \tag{46}$$

In particular all absolutely simple representations of $\mathcal{G}_c$ over a suitable field occur in dimension 1 or 2. The group $\mathcal{G}_c$ is among the few groups for which it is possible to determine all the polynomials R_m^{absim} explicitly. In fact, we not only count but classify all absolutely simple representations of $\mathcal{G}_c$ in Sect. 6.4 below.

6.2.2 Example: $\mathbf{PSL}_2(\mathbb{Z})$

We now consider $\mathbf{PSL}_2(\mathbb{Z}) \cong C_2 * C_3$ with $\mathcal{T}(\mathbf{PSL}_2(\mathbb{Z})) \cong \mathbb{N}_0^2 \times_{\mathbb{N}_0} \mathbb{N}_0^3$. For $|m| \leq 4$ those $R_m^{\mathrm{absim},\mathbf{PSL}_2(\mathbb{Z})}$ which are non-zero are listed below.

m	$R_m^{\mathrm{absim},\mathbf{PSL}_2(\mathbb{Z})}$	m	$R_m^{\mathrm{absim},\mathbf{PSL}_2(\mathbb{Z})}$
$((1,0),(1,0,0))$	1	$((1,1),(1,0,1))$	$s-2$
$((1,0),(0,1,0))$	1	$((1,1),(0,1,1))$	$s-2$
$((1,0),(0,0,1))$	1	$((2,1),(1,1,1))$	s^2-3s+3
$((0,1),(1,0,0))$	1	$((1,2),(1,1,1))$	s^2-3s+3
$((0,1),(0,1,0))$	1	$((2,2),(2,1,1))$	s^3-3s^2+5s-4
$((0,1),(0,0,1))$	1	$((2,2),(1,2,1))$	s^3-3s^2+5s-4
$((1,1),(1,1,0))$	$s-2$	$((2,2),(1,1,2))$	s^3-3s^2+5s-4

For $|m| \leq 5$ all non-zero $R_m^{\mathrm{absim},\mathbf{PSL}_2(\mathbb{Z})}$ in a given total dimension $|m|$ coincide. However, from total dimension $|m| = 6$ on this fails as the following polynomials show.

m	$R_m^{\mathrm{absim},\mathbf{PSL}_2(\mathbb{Z})}$	m	$R_m^{\mathrm{absim},\mathbf{PSL}_2(\mathbb{Z})}$
$((4,2),(2,2,2))$	$s^5 - 4s^4 + 6s^3 - 7s^2 + 9s - 6$	$((3,3),(1,3,2))$	$s^5 - 3s^4 + 5s^3 - 7s^2 + 9s - 6$
$((3,3),(3,2,1))$	$s^5 - 3s^4 + 5s^3 - 7s^2 + 9s - 6$	$((3,3),(2,1,3))$	$s^5 - 3s^4 + 5s^3 - 7s^2 + 9s - 6$
$((3,3),(2,3,1))$	$s^5 - 3s^4 + 5s^3 - 7s^2 + 9s - 6$	$((3,3),(1,2,3))$	$s^5 - 3s^4 + 5s^3 - 7s^2 + 9s - 6$
$((3,3),(3,1,2))$	$s^5 - 3s^4 + 5s^3 - 7s^2 + 9s - 6$	$((2,4),(2,2,2))$	$s^5 - 4s^4 + 6s^3 - 7s^2 + 9s - 6$
$((3,3),(2,2,2))$	$s^7 + 3s^6 - 10s^5 + 3s^4 + 14s^3 - 27s^2 + 35s - 23$		

The examples above suggest that there are symmetries on the sets $\mathcal{T}_d(\mathbf{PSL}_2(\mathbb{Z}))$ along which the counting polynomials stay the same. This is indeed the case for all of the groups $C_a *_{C_c} C_b$ and we will discuss these symmetries in Sect. 7 below.

6.2.3 Example: $\mathbf{SL}_2(\mathbb{Z})$

Recall that $\mathbf{SL}_2(\mathbb{Z})$ is isomorphic to the amalgamated free product $C_4 *_{C_2} C_6$. The polynomials $R_m^{\mathrm{absim},\mathbf{SL}_2(\mathbb{Z})}$ are basically the same as those for $\mathbf{PSL}_2(\mathbb{Z})$. More generally we have the following result.

Proposition 6.1 *Let $a, b \in \mathbb{N}_{\geq 1}$ be natural numbers, $c \in \mathbb{N}_{\geq 1}$ be a common divisor of a and b and $C_a *_{C_c} C_b$ be the amalgamated free product of the respective cyclic groups defined by injective group homomorphisms $C_c \hookrightarrow C_a, C_b$. If $m = (m_0, \dots, m_{c-1}) \in \mathcal{T}(C_{a/c} * C_{b/c})^c \cong \mathcal{T}(C_a *_{C_c} C_b)$ is any dimension vector, then we have*

$$R_m^{\mathrm{absim}, C_a *_{C_c} C_b} = \begin{cases} R_{m_\gamma}^{\mathrm{absim}, C_{a/c} * C_{b/c}}, & \text{if } \exists\, \gamma \text{ s.t. } m_\delta = 0\ \forall\, \delta \neq \gamma \\ 0, & \text{else} \end{cases}$$

Proof Let $\mathbb{F}_q$ be a suitable field for $C_a *_{C_c} C_b$ and $\rho : \mathbb{F}_q[C_a *_{C_c} C_b] \to \mathbf{M}_d(\mathbb{F}_q)$ be an absolutely simple representation. Since $\mathbb{F}_q$ is suitable, there are primitive a-th, b-th and c-th roots of unity. We consider the presentation

$$C_a *_{C_c} C_b = \langle f, g \mid f^{a/c} = g^{b/c}, f^a \rangle$$

and denote $h := f^{a/c} = g^{b/c}$. Since $h \in C_c = Z(C_a *_{C_c} C_b)$ is in the center and the only endomorphisms of ρ are scalar multiples of the identity, $\rho(h) = z.\mathbb{1}_d$ is a scalar matrix, where z is a c-th root of unity. Note that z only depends on the isomorphism class of ρ, because $z.\mathbb{1}_d \in \mathbf{GL}_d(\mathbb{F}_q)$ is a fix point of the conjugation action. Hence, we have a well-defined map $\phi : \mathrm{absim}_d(\mathbb{F}_q[C_a *_{C_c} C_b]) \to \mu_c(\mathbb{F}_q), [\mathcal{M}] \to z_{\mathcal{M}}$, where $\mu_c(\mathbb{F}_q)$ denotes the group of c-th roots of unity in $\mathbb{F}_q$.

Let $m = (m_0, \dots, m_{c-1}) \in \mathcal{T}(C_{a/c} * C_{b/c})^c$ be the dimension vector of ρ. Recall that every representation of C_c is given by a diagonalizable matrix with eigen values from $\mu_c(\mathbb{F}_q)$ associated to its generator h, that the dimension vector of a representation of C_c counts the multiplicities of the eigen values and that the dimension vector uniquely determines the isomorphism class of the representation (see Example 3.1(a)). The fact that $\rho(h)$ is a scalar matrix shows that ρ restricted to C_c is a direct sum of d copies of the same one-dimensional representation. This shows that m_γ is zero for all but one $0 \leq \gamma < c$.

Furthermore note that we may consider $\mathrm{absim}_d(\mathbb{F}_q[C_{a/c} * C_{b/c}])$ as a subset of $\mathrm{absim}_d(\mathbb{F}_q[C_a *_{C_c} C_b])$ via restriction along the surjective homomorphism

$$\pi : C_a *_{C_c} C_b \to C_{a/c} * C_{b/c}$$

i.e. we map a representation $\overline{\rho}$ of $C_{a/c} * C_{b/c}$ to $\overline{\rho} \circ \pi$. This identifies the set $\mathrm{absim}_d(\mathbb{F}_q[C_{a/c} * C_{b/c}])$ with the fibre $\phi^{-1}(1)$, because h generates $C_c = \mathrm{Ker}(\pi)$.

We now construct a bijection between an arbitrary fibre $\phi^{-1}(z)$, $z \in \mu_c(\mathbb{F}_q)$, and the fibre $\phi^{-1}(1)$. First choose $z_a \in \mu_a(\mathbb{F}_q)$ and $z_b \in \mu_b(\mathbb{F}_q)$ satisfying $z_a^{a/c} = z$ and $z_b^{b/c} = z$. If $\rho : \mathbb{F}_q[C_a *_{C_c} C_b] \to \mathbf{M}_d(\mathbb{F}_q)$ is an absolutely simple representation, whose isomorphism class is in $\phi^{-1}(z)$, then define a representation $\overline{\rho}$ via

$$\overline{\rho}(f) := z_a^{-1}.\rho(f), \quad \overline{\rho}(g) := z_b^{-1}.\rho(g)$$

$\overline{\rho}$ is again absolutely simple and the map $\rho \mapsto \overline{\rho}$ induces the required bijection $\phi^{-1}(z) \to \phi^{-1}(1) = \mathrm{absim}_d(\mathbb{F}_q[C_{a/c} * C_{b/c}])$. Hence,

$$R_m^{\mathrm{absim}, C_a *_{C_c} C_b}(q) = R_{m_\gamma}^{\mathrm{absim}, C_{a/c} * C_{b/c}}(q)$$

for all suitable fields $\mathbb{F}_q$. In particular the polynomials must coincide. $\square$

However, the analogous statement for the counting polynomials $R_m^{\mathrm{ss}, C_a *_{C_c} C_b}$ is false.

6.3 Counting polynomials of character varieties

We now want to give examples for the counting polynomials $R_d^{\mathrm{ss}, \mathcal{G}}$. Recall that these give the E-polynomials of the character varieties $X_{\mathcal{G}}(\mathbf{GL}_d(\mathbb{C})) = M(\mathbb{C}[\mathcal{G}], d)$ as discussed in Sect. 2.4.3. All the polynomials listed in this subsection can be computed by evaluating the formulas (42) and (43) for the dimension vector monoids computed in Sect. 6.1. The computations have been carried out using the SageMath code [10].

d	$R_d^{\mathrm{ss}, \mathbf{PSL}_2(\mathbb{Z})}$
1	6
2	$3s + 15$
3	$2s^2 + 12s + 26$
4	$3s^3 + 9s^2 + 24s + 39$
5	$6s^4 + 6s^3 + 24s^2 + 36s + 54$
6	$s^7 + 3s^6 - 2s^5 + 25s^4 + 56s^2 + 41s + 71$
7	$6s^8 + 12s^7 - 30s^6 + 54s^5 + 36s^3 + 54s^2 + 66s + 90$
8	$3s^{11} + 9s^{10} + 9s^9 - 33s^8 + 66s^7 - 60s^6 + 81s^5 + 24s^4 + 33s^3 + 93s^2 + 66s + 111$

The highest d for which the author has computed $R_d^{\mathrm{ss}, \mathbf{PSL}_2(\mathbb{Z})}$ so far is $d = 12$. $R_{12}^{\mathrm{ss}, \mathbf{PSL}_2(\mathbb{Z})}$ is given by $s^{25} + 3s^{24} + 18s^{23} + 38s^{22} + 67s^{21} + 48s^{20} - 49s^{19} - 210s^{18} - 186s^{17} + 329s^{16} + 738s^{15} - 1131s^{14} + 141s^{13} + 264s^{12} + 657s^{11} - 1067s^{10} + 542s^9 - 216s^8 + 753s^7 - 786s^6 + 508s^5 + 313s^4 - 224s^3 + 476s^2 - 143s + 215$.

d	$R_d^{\mathrm{ss}, \mathbf{PSL}_2(\mathbb{Z})}$
1	12
2	$6s + 66$
3	$4s^2 + 60s + 232$
4	$6s^3 + 51s^2 + 282s + 615$
5	$12s^4 + 60s^3 + 288s^2 + 876s + 1356$
6	$2s^7 + 6s^6 - 4s^5 + 144s^4 + 264s^3 + 1062s^2 + 2092s + 2636$
7	$12s^8 + 36s^7 - 24s^6 + 132s^5 + 624s^4 + 864s^3 + 2916s^2 + 4212s + 4680$
8	$6s^{11} + 18s^{10} + 18s^9 + 12s^8 + 324s^7 - 369s^6 + 1122s^5 + 1575s^4 + 2532s^3 + 6366s^2 + 7620s + 7761$

d	$R_d^{\mathrm{ss},\mathbf{GL}_2(\mathbb{Z})}$
1	4
2	$s+14$
3	$8s+28$
4	$3s^2+26s+56$
5	$20s^2+56s+88$
6	$s^4+8s^3+59s^2+101s+147$
7	$8s^4+36s^3+128s^2+156s+212$
8	$2s^6+6s^5+34s^4+96s^3+223s^2+242s+323$
9	$4s^7+16s^6-8s^5+148s^4+140s^3+400s^2+320s+440$
10	$s^9+8s^8+20s^7+23s^6+35s^5+306s^4+206s^3+647s^2+435s+628$

d	$R_d^{\mathrm{ss},\mathbf{PGL}_2(\mathbb{Z})}$
1	4
2	14
3	$4s+28$
4	$s^2+13s+55$
5	$8s^2+32s+84$
6	$6s^3+18s^2+60s+132$
7	$4s^4+16s^3+44s^2+96s+180$
8	$s^6+5s^5+11s^4+40s^3+64s^2+152s+253$
9	$4s^7+12s^6-20s^5+80s^4+16s^3+156s^2+188s+324$
10	$6s^8+22s^7-16s^5+154s^4-6s^3+256s^2+242s+426$
11	$4s^{10}+20s^9+36s^8-72s^7+72s^6+56s^5+100s^4+148s^3+228s^2+372s+524$
12	$s^{13}+4s^{12}+19s^{11}+27s^{10}-25s^9-15s^8+209s^7-268s^6+303s^5+178s^4+60s^3+438s^2+420s+659$

6.4 Classification for generalized infinite dihedral groups

We will now classify all absolutely simple representations of $\mathcal{G}_c = C_{2c} *_{C_c} C_{2c}$ over a suitable ground field. This will in particular prove that the counting polynomials $R_m^{\mathrm{absim},\mathcal{G}_c}$ are given by (46). Recall that the dimension vector monoid of $\mathcal{G}_c$ is given by $\mathcal{T}(\mathcal{G}_c) \cong \left(\mathbb{N}_0^2 \times_{\mathbb{N}_0} \mathbb{N}_0^2\right)^c$.

Proposition 6.2 *Let K be a suitable field for $\mathcal{G}_c$, i.e.* char (K) *does not divide $2c$ and K is perfect and contains a primitive $2c$-th root of unity. Denote its group of $2c$-th roots of unity by $\mu_{2c}(K)$. Consider the presentation $\mathcal{G}_c = \langle f, g \mid f^2 = g^2, f^{2c} = 1\rangle$. In dimension 1 all representations $\rho : \mathcal{G}_c \to \mathbf{GL}_1(K)$ are absolutely simple and pairwise non-isomorphic. They are given by the set $\{(x, y) \in \mu_{2c}(K) \mid x^2 = y^2\}$ via the bijection $\rho \mapsto (\rho(f), \rho(g))$.*

All other absolutely simple representations ρ of $\mathcal{G}_c$ have dimension 2 and their isomorphism classes are in bijection with the set

$$\{(\overline{x}, y) \mid \overline{x} \in \mu_{2c}(K)/\{\pm 1\},\ y \in K \setminus \{\pm x\}\}$$

where an explicit representative is given by

$$(\rho(f), \rho(g)) = \left(\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \begin{pmatrix} y & 1 \\ x^2 - y^2 & -y \end{pmatrix}\right) \tag{47}$$

Proof The case of dimension 1 is elementary. For dimension $d \geq 2$ we first note that $\rho(f)$ and $\rho(g)$ are diagonalizable with eigen values in $\mu_{2c}(K)$, because char (K) is suitable.

Now take $h := f^2 = g^2$. As $h \in Z(\mathcal{G}_c)$ is in the center, $\rho(h) = z.\mathbb{1}_d$ is a scalar matrix with $z \in \mu_c(K)$ if ρ is absolutely simple. Denote the two square roots of z by $\pm x$. By construction $\rho(f)$ and $\rho(g)$ have no eigen values except for $\pm x$. Since a common eigen vector of $\rho(f)$ and $\rho(g)$ would contradict the simplicity of ρ, each eigen value of $\rho(f)$ and $\rho(g)$ has to have multiplicity ≥ 1. So we may assume without loss of generality that

$$\rho(f) = \begin{pmatrix} x.\mathbb{1}_{d_1} & 0 \\ 0 & -x.\mathbb{1}_{d_2} \end{pmatrix} \tag{48}$$

with $d_1 = d - d_2 \neq 0, d$. Note that the stabilizer subgroup $S := S(\rho(f)) \subseteq \mathbf{GL}_d(K)$ of the matrix $\rho(f)$ with respect to the conjugation action on $\mathbf{GL}_d(K)$ is given by block diagonal matrices and can be canonically identified with $\mathbf{GL}_{d_1}(K) \times \mathbf{GL}_{d_2}(K)$. Consider the set

$$Y := \left\{ V \in \mathbf{GL}_d(K) \mid \left(\begin{pmatrix} x.\mathbb{1}_{d_1} & 0 \\ 0 & -x.\mathbb{1}_{d_2} \end{pmatrix}, V \right) \in \mathrm{Rep}_d^{\mathrm{absim}}(K[\mathcal{G}_c])(K) \right\}$$

of matrices which together with the matrix (48) define an absolutely simple representation. The orbits of the conjugation action of S on Y are in canonical bijection with the $\mathbf{GL}_d(K)$-orbits of the subset $X \subseteq \mathrm{Rep}_d^{\mathrm{absim}}(K[\mathcal{G}_c])(K)$ given by

$$X := \left\{ \rho' \in \mathrm{Rep}_d^{\mathrm{absim}}(K[\mathcal{G}_c])(K) \mid \mathbf{GL}_d(K).\rho'(f) = \mathbf{GL}_d(K). \begin{pmatrix} x.\mathbb{1}_{d_1} & 0 \\ 0 & -x.\mathbb{1}_{d_2} \end{pmatrix} \right\}$$

because X is canonically in $\mathbf{GL}_d(K)$-equivariant bijection with the associated fibre space $\mathbf{GL}_d(K) \times^S Y$. (Note that X is the subset of all absolutely simple representations for which the matrix $\rho'(f)$ is conjugated to (48).) So it suffices to show that the set of S-orbits of Y is empty for $d \geq 3$ and for $d = 2$ given by

$$\left\{ \begin{pmatrix} y & 1 \\ x^2 - y^2 & -y \end{pmatrix} \in \mathbf{GL}_2(K) \mid y \in K \setminus \{\pm x\} \right\}$$

Let $\rho(g) = \left(\begin{smallmatrix} L & M \\ N & W \end{smallmatrix}\right)$ be any element of Y where $L \in \mathbf{M}_{d_1 \times d_1}(K)$, $M \in \mathbf{M}_{d_1 \times d_2}(K)$ etc. The action of $\mathbf{GL}_{d_1}(K) \times \mathbf{GL}_{d_2}(K) \cong S$ on Y is given by

$$(t_1, t_2). \begin{pmatrix} L & M \\ N & W \end{pmatrix} = \begin{pmatrix} t_1 L t_1^{-1} & t_1 M t_2^{-1} \\ t_2 N t_1^{-1} & t_2 W t_2^{-1} \end{pmatrix}$$

Since ρ is simple, we know that $t_1 M t_2^{-1}, t_2 N t_1^{-1} \neq 0$ for all (t_1, t_2) as otherwise the linear subspace $0 \times K^{d_2}$ or $K^{d_1} \times 0$ of K^d would define a non-trivial subrepresentation contradicting the simplicity of ρ.

For $d = 2$ we have $d_1 = d_2 = 1$ and may take $(t_1, t_2) = (1, M)$ to get $(t_1, t_2).\rho(g) = \left(\begin{smallmatrix} L' & 1 \\ N' & W' \end{smallmatrix}\right)$. As the multiplicity of both eigen values $\pm x$ of $\rho(g)$ must be 1 as well, we have $\mathrm{Tr}(\rho(g)) = 0$. Using this and $\rho(g)^2 = z.\mathbb{1}_2$ we obtain that $y := L' = -W'$ and $N' = x^2 - y^2$. This proves the claim for dimension 2.

Now assume ρ were an absolutely simple representation of dimension $d \geq 3$. First we note that $d_1 = d_2$: Denote by c_1 and c_2 the multiplicities of the eigen values $\pm x$ for the matrix $\rho(g)$. We have seen above that $0 < c_1, c_2 < d$. Since every simultaneous eigen vector of $\rho(f)$ and $\rho(g)$ would span a subrepresentation of ρ, the multiplicities have to fulfill

$$c_\gamma + d_\delta \leq d \quad \forall\, 1 \leq \gamma, \delta \leq 2 \tag{49}$$

The inequalities (49) yield that $d = 2r$ is even and $r = c_1 = c_2 = d_1 = d_2$. Furthermore we may assume for $\rho(g) = \left(\begin{smallmatrix} L & M \\ N & W \end{smallmatrix}\right)$ with $L, M, N, W \in \mathbf{M}_{r \times r}(K)$ that $M = \mathbb{1}_r$:

By standard linear algebra arguments we may find $(t_1, t_2) \in \mathbf{GL}_r(K)^2$ s.t. $t_1 M t_2^{-1} = \left(\begin{smallmatrix} \mathbb{1}_{\mathrm{rk}(M)} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and to obtain $M = \mathbb{1}_r$ it remains to show that $\mathrm{rk}(M) = r$. We write $L = \left(\begin{smallmatrix} L_1 & L_2 \\ L_3 & L_4 \end{smallmatrix}\right)$ and $W = \left(\begin{smallmatrix} W_1 & W_2 \\ W_3 & W_4 \end{smallmatrix}\right)$ as block matrices with $L_1, W_1 \in \mathbf{M}_{\mathrm{rk}(M) \times \mathrm{rk}(M)}(K)$. With the straightforward computation

$$z.\mathbb{1}_d = \rho(g)^2 = \begin{pmatrix} * & * & L_1 + W_1 & W_2 \\ * & * & L_3 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

we see that $W_2 = 0$. This means that the last $r - \mathrm{rk}(M)$ basis elements span a subrepresentation, so by simpleness of ρ we have $r = \mathrm{rk}(M)$.

By again using $\rho(g)^2 = z.\mathbb{1}_d$ we may deduce $\rho(g) = \begin{pmatrix} L & \mathbb{1}_r \\ z.\mathbb{1}_r - L^2 & -L \end{pmatrix}$. Now let $v \in \overline{K}^r$ be an eigen vector of L considered as a matrix over $\overline{K}$. One checks easily that $\binom{v}{0}$ and $\binom{0}{v}$ span a two dimensional subrepresentation of the base extension $\rho \otimes_K \overline{K}$ which contradicts our assumption that ρ is absolutely simple. □

Proof of (46) Recall from Sect. 6.1.3 that the dimension vector monoid $\mathcal{T}(\mathcal{G}_c) \cong (\mathbb{N}_0^2 \times_{\mathbb{N}_0} \mathbb{N}_0^2)^c$ is given by

$$\left\{(m_0, \dots, m_{c-1}) \in \left(\mathbb{N}_0^2 \times \mathbb{N}_0^2\right)^c \mid \forall\, 0 \leq \gamma < c : m_\gamma(0) + m_\gamma(1) = m_\gamma(2) + m_\gamma(3)\right\}$$

Since all one-dimensional representations are absolutely simple and pairwise non-isomorphic, $R_m^{\mathrm{absim}} = 1$ for all $m \in \mathcal{T}_1(\mathcal{G}_c)$. Moreover from Proposition 6.2 we know that $R_m^{\mathrm{absim}} = 0$ for $|m| \neq 1, 2$ as there are only absolutely simple representations of $\mathcal{G}_c$ in dimension 1 and 2.

Now assume $\rho := \rho_{x,y}$ is the absolutely simple representation of $\mathcal{G}_c$ of dimension 2 given by (47). Then $\rho(h) = z.\mathbb{1}_2$ for $h := f^2 = g^2$ where $z := x^2$ is a c-th root of unity. Hence, ρ restricted to C_c has a single simple subrepresentation up to isomorphism occurring with multiplicity 2, i.e. there is a $\gamma \in \{0, 1, \dots, c-1\}$ such that

$$m_\delta(0) + m_\delta(1) = m_\delta(2) + m_\delta(3) = \begin{cases} 2, & \text{if } \delta = \gamma \\ 0, & \text{else} \end{cases}$$

However, as in the proof of Proposition 6.2 both eigen values $\pm x$ must have multiplicity 1 for both $\rho(f)$ and $\rho(g)$, i.e. $m_\gamma = (1, 1, 1, 1)$.

If we choose a primitive $2c$-th root of unity ξ_{2c}, then

$$\underline{\dim}(\rho_{\xi_{2c}^\gamma, y}) = (m_0, \dots, m_{c_1}) \quad \text{with} \quad m_\delta = \begin{cases} (1, 1, 1, 1), & \text{if } \delta = \gamma \\ 0, & \text{else} \end{cases}$$

for all $0 \leq \gamma < c$. (Note that the representations $\rho_{\xi_{2c}^\gamma, y}$ and $\rho_{\xi_{2c}^{c+\gamma}, y}$ are isomorphic as $\xi_{2c}^c = -1$.) Hence, $\#(\mathrm{Rep}^{\mathrm{absim}}_{\underline{\dim}(\rho_{x,y})}(\mathbb{F}_q[\mathcal{G}_c])(\mathbb{F}_q)) = \#(\mathbb{F}_q \backslash \{\pm x\}) = q - 2$. □

7 Structural properties

We now discuss some of the main structural properties of the counting polynomials: their degree and the symmetries occuring among them. As before denote by $\mathcal{G}$ the finitely generated virtually free group fixed throughout this paper and denote the counting polynomials $R_m^{\mathrm{absim},\mathcal{G}}$ and $R_m^{\mathrm{ss},\mathcal{G}}$ simply by R_m^{absim} and R_m^{ss}.

7.1 Degrees of counting polynomials

Proposition 7.1 *Let $m \in \mathcal{T}(\mathcal{G})$ be an arbitrary dimension vector and $\mathbb{F}_q$ be suitable for $\mathcal{G}$. The polynomial R_m^{ss} is monic of degree* $\dim M(\mathbb{F}_q[\mathcal{G}], m)$. *If $R_m^{\mathrm{absim}} \neq 0$, then R_m^{absim} is monic too and of the same degree*

$$\dim M(\mathbb{F}_q[\mathcal{G}], m) = \dim M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m) = 1 - \langle m, m \rangle_{\mathcal{G}} \tag{50}$$

Proof We first recall a well-known theorem about counting polynomials which is due to S. Lang and A. Weil: Let X be a polynomial count $\mathbb{F}_q$-scheme. If X is geometrically irreducible, then its counting polynomial is monic of degree $\dim(X)$ (see [19, Thm. 7.7.1]). This proves the claim on R_m^{ss}, since it is a counting polynomial of $M(\mathbb{F}_q[\mathcal{G}], m)$ which is geometrically irreducible, because the connected component $\mathrm{Rep}_m(\overline{\mathbb{F}_q}[\mathcal{G}])$ surjects onto $M(\overline{\mathbb{F}_q}[\mathcal{G}], m) \cong M(\mathbb{F}_q[\mathcal{G}], m) \times_{\mathbb{F}_q} \mathrm{Spec}\left(\overline{\mathbb{F}_q}\right)$ and is irreducible by Lemma 5.1. (In particular we have $R_m^{\mathrm{ss}} \neq 0$ as $M(\mathbb{F}_q[\mathcal{G}], m)(\mathbb{F}_q)$ is non-empty, because $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])(\mathbb{F}_q)$ is.)

The claim on R_m^{absim} is proven analogously by replacing $\mathrm{Rep}_m(\overline{\mathbb{F}_q}[\mathcal{G}])$ with its open subscheme $\mathrm{Rep}_m^{\mathrm{absim}}(\overline{\mathbb{F}_q}[\mathcal{G}])$ and it remains to prove the two equations in (50): The first equation follows from $M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m) \subseteq M(\mathbb{F}_q[\mathcal{G}], m)$ being open and non-empty if $R_m^{\mathrm{absim}} \neq 0$. For the second equation we note that there is an induced $\mathbf{PGL}_{|m|,\mathbb{F}_q}$-action on representation spaces that operates freely on $\mathrm{Rep}_m^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}])$ and that its quotient $\mathrm{Rep}_m^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}])/\mathbf{PGL}_{|m|,\mathbb{F}_q}$ is isomorphic to $M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m)$. Hence,

$$\dim M^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}], m) = \dim \mathrm{Rep}_m^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}]) - \dim \mathbf{PGL}_{|m|,\mathbb{F}_q}$$

Moreover we have $\dim \mathrm{Rep}_m^{\mathrm{absim}}(\mathbb{F}_q[\mathcal{G}]) = \dim \mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$, because $\mathrm{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ is geometrically irreducible. So the second equation in (50) is equivalent to the identity $\deg P_m^{\mathcal{G}} = |m|^2 - \langle m, m \rangle_{\mathcal{G}}$ which can be verified using our general formula (31). □

7.2 Symmetries of counting polynomials

In Sect. 6.2 we have seen that the counting polynomials are invariant with respect to certain symmetries on the dimension vectors of some virtually free groups $\mathcal{G}$. More specifically there is a finite group $S_{\mathcal{G}}$ acting on $K[\mathcal{G}]$ for K suitable and by functoriality on each $\mathcal{T}_d(\mathcal{G})$, $d \in \mathbb{N}_0$ such that $R_m^{\mathrm{absim}} = R_n^{\mathrm{absim}}$ and $R_m^{\mathrm{ss}} = R_n^{\mathrm{ss}}$ if $m, n \in \mathcal{T}_d(\mathcal{G})$ belong to the same $S_{\mathcal{G}}$-orbit. We will now sketch the construction of this group and its action on $K[\mathcal{G}]$. While the general procedure works for arbitrary finitely generated virtually free groups we will only make it explicit in the special case that the finite groups $\mathcal{G}_i$ occurring in (9) are Abelian.

Let K be a field which is suitable for $\mathcal{G}$. Hence, all of the finite dimensional group algebras in (12) are of the form

$$\mathcal{C} \cong \mathbf{M}_1(K)^{c_1} \times \mathbf{M}_2(K)^{c_2} \times \cdots \times \mathbf{M}_e(K)^{c_e}$$

We construct the group $S_{\mathcal{G}}$ and its action iteratively and we start with the case of (group algebras of) finite groups: The symmetric group S_{c_ϵ} acts naturally on $\mathbf{M}_\epsilon(K)^{c_\epsilon}$ via $\tau.(M_1, \ldots, M_{c_\epsilon}) = (M_{\tau(1)}, \ldots, M_{\tau(c_\epsilon)})$ for each $1 \leq \epsilon \leq e$, hence, $S_{\mathcal{C}} := S_{c_1} \times \cdots \times S_{c_e}$ acts on $\mathcal{C}$ via K-algebra automorphisms.

Now assume A, B, C are finite groups acting via K-algebra automorphisms on K-algebras $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and assume we are given group homomorphisms $A, B \to C$ and K-algebra homomorphisms $\mathcal{C} \to \mathcal{A}, \mathcal{B}$ which are A- and B-equivariant. Then $A \times_C B$ acts naturally on $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ via K-algebra automorphisms.

Finally assume we have group homomorphisms $\varphi, \theta : A \to C$ and K-algebra homomorphisms $\iota, \kappa : \mathcal{C} \to \mathcal{A}$ such that ι is A-equivariant with respect to φ and κ via θ. Then $\mathrm{Eq}(\varphi, \theta) \subseteq A$ acts naturally on $\mathcal{A}*_{\mathcal{C}}^{\iota,\kappa}$ via K-algebra automorphisms.

All of the discussion so far works without any assumptions on the involved algebras. However, to iteratively get an induced action on $K[\mathcal{G}]$ from the actions on the group algebras $K[\mathcal{G}_i]$ and $K[\mathcal{G}'_j]$ we need group homomorphisms $S_{\mathcal{G}_{s(j)}} \to S_{\mathcal{G}'_j} \leftarrow S_{\mathcal{G}_{t(j)}}$ for each j such that the embeddings $K[\mathcal{G}_{s(j)}] \hookleftarrow K[\mathcal{G}'_j] \hookrightarrow K[\mathcal{G}_{t(j)}]$ become $S_{\mathcal{G}_{s(j)}}$- and $S_{\mathcal{G}_{t(j)}}$-equivariant. If $\mathcal{G}'_j$ is the trivial group for each j, this obstruction is trivial and we obtain an action of $\prod_{i=0}^{I} S_{\mathcal{G}_i}$ on $K[\mathcal{G}]$.

However, in general this is a non-trivial combinatorial task which is why we assume from now on that $\mathcal{G}_i$ is Abelian for each $0 \le i \le I$.[7] Hence, each of the $\mathcal{C}$ above is of the form K^c, $c = \dim_K(\mathcal{C})$ with an action of the symmetric group S_c. We consider an injective K-algebra homomorphism $\iota : K^c \hookrightarrow K^b$ and denote by e'_γ, $0 \le \gamma < c$ the γ-th standard basis vector of K^c and by e_β, $0 \le \beta < b$ the β-th standard basis vector of K^b. Both $(e'_\gamma)_\gamma$ and $(e_\beta)_\beta$ are systems of pairwise orthogonal central primitive idempotents. Moreover note that all idempotent elements of K^b are of the form

$$\sum_{\beta \in \mathbb{J}} e_\beta$$

for some subset $\mathbb{J} \subseteq \{0, 1, \dots, b-1\}$. Hence, there is a partition

$$\mathbb{I} := \{0, 1, \dots, b-1\} = \bigsqcup_{\gamma=0}^{c-1} \mathbb{I}_\gamma \tag{51}$$

such that $\iota(e'_\gamma) = \sum_{\beta \in \mathbb{I}_\gamma} e_\beta$, because the elements $(\iota(e'_\gamma))_\gamma$ are pairwise orthogonal idempotents summing up to $\iota(1) = 1$.

Now recall that each (absolutely) simple K^b-module is isomorphic to precisely one of the principal ideals $K^b \cdot e_\beta = e_\beta \cdot K^b$ and that the (absolutely) simple K^c-modules analogously are given by $K^c \cdot e'_\gamma = e'_\gamma \cdot K^c$. By construction of the partition (51) we have $\iota^*(K^b \cdot e_\beta) \cong K^c \cdot e'_\gamma$ for each $\beta \in \mathbb{I}_\gamma$. Now consider the subgroup

$$\overline{S_b} := \{\tau \in S_b \mid \forall\, 0 \le \gamma < c : \exists!\, 0 \le \overline{\tau}(\gamma) < c : \tau(\mathbb{I}_\gamma) = \mathbb{I}_{\overline{\tau}(\gamma)}\} \tag{52}$$

of permutations τ on the set $\mathbb{I}$ that preserve the partition (51). We obtain a group homomorphism $\overline{S_b} \to S_c$, $\tau \mapsto \overline{\tau}$ with respect to which the K-algebra embedding ι is $\overline{S_b}$-equivariant.

To obtain the group $S_{\mathcal{G}}$ and its action we now apply the above procedure for all $1 \le j \le I + J$ to the algebra homomorphisms $\iota_j : K[\mathcal{G}'_j] \hookrightarrow K[\mathcal{G}_{s(j)}]$, $\kappa_j : K[\mathcal{G}'_j] \hookrightarrow K[\mathcal{G}_{t(j)}]$ and the groups $S_{\mathcal{G}_{s(j)}}$, $S_{\mathcal{G}_{t(j)}}$ and $S_{\mathcal{G}'_j}$. This replaces the finite groups $S_{\mathcal{G}_i}$ by subgroups $\overline{S_{\mathcal{G}_i}} \subseteq S_{\mathcal{G}_i}$ analogously to (52) admitting group homomorphisms $\overline{S_{\mathcal{G}_{s(j)}}} \to S_{\mathcal{G}'_j} \leftarrow \overline{S_{\mathcal{G}_{t(j)}}}$ for each j with respect to which ι_j and κ_j become equivariant. (Note that some of the groups $\overline{S_{\mathcal{G}_i}}$ might end up being trivial depending on the occurring combinatorics.) Now deploying the iterative process described above, where for each $1 \le j \le I$ we form a fibre product over $S_{\mathcal{G}'_j}$ and for each $I + 1 \le j \le I + J$ we form an equalizer over $S_{\mathcal{G}'_j}$, we obtain the group $S_{\mathcal{G}}$ as a limit of the groups $\overline{S_{\mathcal{G}_i}}$ and the groups $S_{\mathcal{G}'_j}$ with an induced action on $K[\mathcal{G}]$ via K-algebra automorphisms.

[7] Of course one could also consider a hybrid situation where for each j we have $\mathcal{G}'_j$ being trivial or $\mathcal{G}_{s(j)}$ and $\mathcal{G}_{t(j)}$ being Abelian.

By functoriality every group action on $K[\mathcal{G}]$ via K-algebra automorphisms yields an induced action on $M(\mathbb{F}_q[\mathcal{G}], d)$, $M^{\text{absim}}(\mathbb{F}_q[\mathcal{G}], d)$ and $\mathcal{T}_d(\mathcal{G})$ for each $d \in \mathbb{N}_0$. If $m, n \in \mathcal{T}_d(\mathcal{G})$ lie in the same orbit, then we obtain isomorphisms

$$M^{\text{absim}}(\mathbb{F}_q[\mathcal{G}], m) \cong M^{\text{absim}}(\mathbb{F}_q[\mathcal{G}], n), \quad M(\mathbb{F}_q[\mathcal{G}], m) \cong M(\mathbb{F}_q[\mathcal{G}], n)$$

So in particular the counting polynomials for m and n coincide.

Example 7.2 We consider the case of $\mathcal{G} = C_a *_{C_c} C_b$ for $a, b \geq 2$, c a common divisor of a, b. As discussed in Sect. 6.1.3 we may reorder the basis elements of $\mathbb{N}_0^a$, $\mathbb{N}_0^b$ and $\mathbb{N}_0^c$ such that $\mathcal{T}(\iota) : \mathbb{N}_0^a \to \mathbb{N}_0^c$ is given by $m \mapsto (\sum_{\delta=0}^{a/c-1} m(\gamma + \delta c))_\gamma$ and analogously for $\mathcal{T}(\kappa)$. Hence, the partitions (51) of $\mathbb{I} := \{0, \dots, a-1\}$ and $\mathbb{J} := \{0, 1, \dots, b-1\}$ are given by $\mathbb{I}_\gamma = \{\alpha \mid \alpha \equiv \gamma \pmod c\}$, $\mathbb{J}_\gamma = \{\beta \mid \beta \equiv \gamma \pmod c\}$ for all $0 \leq \gamma < c$. This determines the subgroups $\overline{S_a} \subseteq S_a$ and $\overline{S_b} \subseteq S_b$.

The action of $S_{\mathcal{G}} = \overline{S_a} \times_{S_c} \overline{S_b}$ on $\mathcal{T}(C_a *_{C_c} C_b) \cong \mathbb{N}_0^a \times_{\mathbb{N}_0^c} \mathbb{N}_0^b$ coincides with the restriction of the natural $S_a \times S_b$-action on $\mathbb{N}_0^a \times \mathbb{N}_0^b$.

Remark 7.3 Note that our construction of the finite group $S_{\mathcal{G}}$ and its action depends on the choice of decomposition (9) which we fixed for our finitely generated virtually free group $\mathcal{G}$. However, since the sets of dimension vectors $\mathcal{T}_d(\mathcal{G})$ and the counting polynomials R_m^{absim} and R_m^{ss} up to reordering only depend on the isomorphism class of $\mathcal{G}$, the symmetries among the counting polynomials also do not depend on the choice of decomposition. An interesting question to consider would be how the groups and their actions obtained from the above procedure for different decompositions are related.

Acknowledgements Almost all results of this paper have been obtained as part of the author's PhD under the supervision of M. Reineke and B. Klopsch and the author is deeply indebted to his supervisors for all their help, support, kindness and patience. Special thanks also go to L. Hennecart, S. Mozgovoy and S. Schröer for useful discussions on parts of the content of this paper. Moreover the author thanks the Ruhr-University Bochum and most of its members for being a welcoming scientific home during his PhD. During his PhD the author's research was financed by the research training group *GRK 2240: Algebro-Geometric Methods in Algebra, Arithmetic and Topology*, which is funded by the Deutsche Forschungsgemeinschaft. Moreover the author thanks the anonymous referee for a very thorough review of the original manuscript and many useful questions and suggestions.

Funding Open Access funding enabled and organized by Projekt DEAL.

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