Relevance as difference-making: a generalized theory of relevance and its applications

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Abstract
In this paper a generalized account of relevance as difference-making is developed. It is argued that relevance should not be considered as a particular relation, but as a (higher-order) property of instances of arbitrary relations: namely the property that variations of the relata of the relation instance make a difference for its truth. This generalized account of relevance can be fruitfully applied in many domains, such as (i) logical reasoning with applications to explanation, confirmation, verisimilitude, is-ought inference, (ii) probabilistic reasoning with applications to explanation and confirmation, (iii) nomological and causal implication, (iv) communication, (v) grounding and (vi) essentiality. In conclusion, difference-making relevance is a highly unifying and fruitful philosophical concept.

Keywords Difference-making · Replacement criterion · Logical relevance · Probabilistic relevance · Relevant communication · Relevant grounding · Relevant predication

1 Introduction: examples of relevance in different areas

In the last 50 years the study of *relevance* has become an important field in many areas, most prominently in relevance logic, but also in probability theory, explanation, confirmation, metaphysical grounding and communication. However, approaches to relevance have been criticized as being ad-hoc or one-sided, designed for particular domains and purposes and not transferable to other ones.[1] It seems that

[1] For criticisms of relevance logic cf. Copeland (1979), Burgess (1981, 1983), Meyer (1985), Read (1988, 2), Schurz (1999, sec. 2.3); for statistical relevance cf. Hempel (1977, 109); for communicative relevance cf. Woods (1992); for relevance in general cf. Botting (2013).

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in all these accounts of relevance it does not really become clear what kind of entity *relevance in itself* could be. In this paper I will argue that a reason for this problem lies in the fact that in received accounts, relevance has always been considered as a *particular relation* between individuals, propositions or other entities—such as logico-deductive, probabilistic, explanatory, communicative, metaphysical or practical relevance. In this perspective it is hard to see what relevance in general could be—in abstraction from its specific domain and purposes. Recently, grounding has been suggested as a general relevant relation (cf. Correia & Schnieder, 2012). But not all relevant relations are grounding relations. For example, communicative relevance can hardly be conceived as a kind of grounding. Moreover, it is not possible to understand both explanation and confirmation as grounding relations, since grounding is asymmetric but explanation and confirmation are typically inverse relations, i.e., an evidence E confirms a theory T iff T explains E. Thus, grounding is more specific than relevance. Orlowska & Weingartner (1986) have tried to find axioms common to all domain-specific relevance relations, arriving at the meagre result that relevance is a reflexive relation. But even reflexivity fails in certain domains, e.g., in communicative relevance (see Sect. 9).

Is there anything non-trivial that can be said about relevance in general? *Yes* there is. According to the suggestion of this paper, relevance should not be considered as some sort of (domain-specific) relation, but as a certain higher-order property of instances (or instantiations) of arbitrary relations: the property of *difference-making*. If an entity a is said to be relevant to another entity b, this is semantically ambiguous, because relevance is always relative to some 'respect'. In our account, this respect is reflected by a given relation. The relation (and the corresponding 'respect') may vary, but judging a as relevant for b with respect to a relation **R** means always the same, namely that a makes a difference for b's being **R**-related.

To make this idea formally precise, I will assume throughout this paper that **R** is a binary relation between a *domain* **A** and a *co-domain* **B**, $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$. The relation **R** may vary; so "**A**" and "**B**" do not designate a particular (co-)domain but are *variables* for arbitrary (co-)domains of a variable relation **R**. My proposal is this: An entity $a \in \mathbf{A}$ stands in a relevant relation **R** to an entity $b \in \mathbf{B}$ iff a**R**b holds and the fact that a rather than something else is related (or paired) with b *makes a difference* for b's being **R**-related, and for the properties of b that depend on this relationship. Since these properties may vary from domain to domain, the most general formulation of the difference-making criterion is this:

(1) Relevance as difference-making (informal explication 1):
A given (true) relation instance a**R**b is relevant w.r.t. its first relatum (or argument), in short **A**-relevant (or domain-relevant), iff the fact that a rather than something else is related to b makes a difference for the truth value of the relation instance. Similarly for the relevance of a**R**b w.r.t. its second relatum (or argument), called **B**-relevance (or co-domain-relevance).

The difference-making idea of relevance expresses an strongly entrenched intuition of common sense and philosophical tradition. Among others, John St. Mill developed a difference-making account of causal relevance (cf. Mill 1865, book III, ch. viii, "second canon"). In twentieth century philosophy of science the idea of probabilistic relevance as difference making was first developed by Jeffrey (1969) and Salmon (1970). Our formulation in (1) above is closely leaning towards standard formulations of the difference making approach in contemporary philosophy. Strevens, for example, characterizes a factor c as explanatory relevant to an explanandum e iff the factor c "makes a difference to an explanandum e", in comparison with "a nonactual scenario in which c is not present" (2008, 55; see also Woodward, 2016, sec. 3.3). Note that in the formulation "a rather than something else is related to b", the word "related" is understood as being paired as an attempt to create an **R**-instance (e.g., to create an entailment, or an explanation, etc.); this attempt is successful if a is paired with b but not if some other (arbitrary) x is paired with b.

There is an equivalent and more succinct formulation of the difference-making criterion that will be used in his paper:

(2) Relevance as difference-making (equivalent explication 2):
The true relation instance a**R**b is **A**-relevant iff the replacement of a in a**R**b by 'other' elements x∈**A** makes this instance false (and similarly for **B**-relevance)—where these 'other' elements can be specified in several ways differing in strength (see below).

In line with (2), this account has been called the *replacement criterion* of relevance in earlier work on deductive relevance (Schurz, 1991a). This paper develops a generalization of this account that is applicable to all sorts of domains. The account developed in this paper is a basically conservative extension of this earlier work on deductive relevance.[2]

Since the relation **R** may vary, the ontological nature of **R**-instances may vary, too. What can be generally said is that a relation instance a**R**b is regarded as a proposition, or a state of affairs, expressed by the sentence "a**R**b", namely that a stands in relation **R** to b, and if this proposition is true respectively this state of affairs obtains, then the relation instance a**R**b is a fact (cf. Textor, 2021). All this applies likewise to the relevance of a given relation instance, since this relevance is defined in terms of the truth and falsity of (this and other other) relation instances. The ontological nature of the entities in **R**'s domain **A** and co-domain **B** depends on the domain of application. Let us briefly elucidate some such applications.

For example, in the domain of logic, **R** is the relation of deductive inference, **A** is the class of all sets of sentences (or propositions) Γ, Δ … of a suitably specified

[2] Together with a simplification concerning premise relevance: Condition (34.2) in Schurz (1991a) is removed, because its work is taken over by the decomposition of the premise set into its relevant (content) elements. Schurz (1991a) requires the latter step only for the application to verisimilitude and unification (explanation), while the application to hypothetico-deductive confirmation is introduced in Schurz (1994).

language *Lang*, and **B** is the class of all sentences (or propositions) A, B … of *Lang*. An inference or entailment relation is premise-relevant (i.e., **A**-relevant) iff the premises cannot be replaced by *arbitrary* other premises salva validitate, i.e., under preservation of the entailment's validity. This is equivalent with requiring that the premises are necessary for entailing the conclusion. In this latter version the idea of premise-relevance was also the starting idea of Anderson and Belnap's relevance logic (1975, 18). In the area of probability theory, Salmon (1971, 33f.) developed the notion of *statistical relevance*. Here the relation **R** is a conditional statistical probability p(B|A) between event types A and B, and this relation is relevant if $p(B|A) \neq p(B)$, i.e., conditioning on A makes a difference for B's probability. The corresponding replacement operation is A's elimination or A's replacement by a tautology. A more refined version of probabilistic difference making is employed in Woodward's interventionist account of causation (Woodward, 2003, 59).

The notions of deductive and probabilistic relevance play an important role in accounts of hypothetico-deductive and probabilistic confirmation. The idea of difference-making is further involved in Mackie's explication of a cause as an INUS condition (Mackie, 1975), i.e., an insufficient but necessary part of an unnecessary but sufficient condition. Here A_i counts as causal factor for B iff A_i makes a difference for the truth of a nomological implication of the form $(A_1 \wedge \ldots \wedge A_n) \rightarrow B$, i.e. the implication becomes false iff A_i is eliminated from the conjunction, or replaced by something different. A version of difference-making is employed also in Lewis' counterfactual account of causation, according to which an event A is a cause of B iff A relevantly counterfactually implies B, in the sense that $A \rightsquigarrow B$ but $\neg A \rightsquigarrow \neg B$ (Lewis, 1973; for an overview on difference-making accounts to causation cf. Menzies, 2004).

In linguistics, relevance based on difference-making figures as a basic ingredient of the theory of *communication* of Sperber and Wilson (1996), in the tradition of Grice (1975). Here the underlying relation **R** holds between the *utterance* of a proposition P and a *context* C consisting of the background beliefs that are shared by speaker and hearer, where the proposition P causes the hearer to acquire an information I that is implied by P and C. Sperber and Wilson consider the utterance of P as *relevant* in context C iff the information I goes *beyond* what C and P imply separately; thus both P and C must make a difference for the information acquired by the hearer.

Our final application will be the notion of *essential* properties that goes back to Aristotle and plays an important role in modal metaphysics (cf., e.g., Fine, 1994). Here the relation is that of predication between a property F and an individual a, and the predication of F is essential, or property-relevant, iff a's possession of F makes a difference to a's self-identity. The dual notion of the individual-relevance of predication yields the notion of an individual essence.

These are some of the most important applications, but the account is not restricted to them. In principle, the proposed notion of relevance may be applied to any given binary relation, and often enough it will express an important feature of its relation instances. As a final example consider the relation of *support* between people. That *Jim* supports *John* is **B**-relevant if Jim gives the kind of support he conveys to John not to any other person, so that it makes a difference for Jim's support activity whether John or someone else is the support-addressee. Vice versa, the support-relation is **A**-relevant iff no just anyone supports John in the way Jim does, so that it makes a difference for John's being supported whether he requests support from Jim or someone else. In the first case, Jim has a support-preference for John and in the latter case, John is support-dependent on Jim.

Recapitulating the ontological question, in most of the applications mentioned above the entities in **A** and **B** are propositions or sentences (or sets of those). In other applications, these entities are event types (in statistical probability), states of affairs or facts (in grounding), properties and individuals (in relevant predication), uttered propositions and belief contexts (in communication), or simply individuals (in the example of the support relation). More on the characterization of relations and their (co-)domains will be said in Sect. 5.

Summarizing, we regard it as the major advantage of the proposed approach to relevance that it extracts a common property of relevant relation instances, defined via difference-making, that is found again and again in fruitful applications of relevance in different domains. We do not at all deny, however, that alternative approaches that work out relevant relations in a domain-specific manner can be an important complementation of the proposed account. More on the relation of our account to alternative approaches will be said in Sects. 2, 4, 5 and 8.

The rest of this paper is organized as follows. The next three sections present a tour through applications of difference-making relevance to logical inference and compare them with accounts of relevance logic. In these sections the important distinction between minimal, component-wise and essential relevance will be introduced (these sections are lengthy because in the literature studies of logical relevance are dominant). After that, the general definition of relevance will be stated in Sect. 5. In the following sections, its application to the other domains are studied.

2 Relevance in logical reasoning

Throughout the following we assume that A, B,… range over sentences of a language *Lang*; Γ, Δ,… over sets of sentences; P, Q,… over atomic sentences (propositional variables); F, G,…over (n-placed) predicates; a, b… over individual constants; and x, y,… over individual variables.

Relevance logic in the tradition of Anderson and Belnap (1975) starts with the attempt to get rid of two famous cases of irrelevant but classically valid inference (⊢) or corresponding (material) implication (→):

(3) Verum Ex Quodlibet (VEQ): $\{A\} \vdash B\lor\neg B$ (Implication: $A \to (B\lor\neg B)$).
Relevance analysis: The inference or implication is (maximally) *premise-irrelevant*, because $\{X\} \vdash B\lor\neg B$ resp. $X \to (B\lor\neg B)$ is valid for every formula X, since $B\lor\neg B$ is L-true (short for logically true). Thus, it is possible to replace the premise A by arbitrary X *salva validitate*.

(4) Ex Falso Quodlibet (EFQ): $\{A\wedge\neg A\} \vdash B$ (Implication: $(A\wedge\neg A) \rightarrow B$). Relevance analysis: This inference or implication is (maximally) *conclusion-irrelevant*, since $\{A\wedge\neg A\} \vdash X$ resp. $(A\wedge\neg A) \rightarrow X$ is valid for every formula X, as $A\wedge\neg A$ is L-false. Thus, it is possible to replace the conclusion B by arbitrary X salva validitate.

This starting idea of (ir)relevance fits well with the difference-making account, with $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$ being the relation of entailment between sets of sentences and sentences, i.e. $\mathbf{A} = \text{Pow}(Lang)$ and $\mathbf{B} = Lang$. We call the corresponding notion of relevance *minimal* premise- resp. conclusion-relevance, because it is the weakest notion of relevance developed in this paper.

(5) (Definition.) A valid inference $\Gamma \vdash A$ is *minimally* premise-relevant iff Γ is not replaceable by every other set Δ *salva validitate*, i.e. under preservation of the validity. In other words, there exists at least one Δ such that $\Delta \nvdash A$.
(6) (Definition.) A valid inference $\Gamma \vdash A$ is minimally conclusion-relevant iff A is not replaceable by every other formula B salva validitate. In other words, there exists at least one B such that $\Gamma \nvdash B$.

Note that the notion of "minimal premise-relevance" is the negation of "maximal premise-irrelevance", and similarly for conclusion-relevance.

A natural strengthening of minimal premise-relevance is the application of replacements not to the premise set as a whole, but to each particular premise. What one then gets is a criterion for the deductive non-redundancy of each premise. We speak here of premise-relevance (as opposed to 'minimal' premise relevance):

(7) (Definition.) A valid inference $\Gamma \vdash A$ is premise-relevant iff no premise $P \in \Gamma$ is replaceable in Γ by *every* other formula X *salva validitate*.
This condition is equivalent with: iff no premise $P \in \Gamma$ is *eliminable* salva validitate, or in other words, iff every premise P in Γ is needed for entailing A, i.e., $\forall P \in \Gamma: \Gamma - \{P\} \nvdash A$.

Criterion (7) in the 'premise in use' version is the starting point of Anderson and Belnap's relevance logic (1975, 21). Inferences with irrelevant premises are the culprit of the irrelevant inference VEQ, not only in its logical version, but also in its material version, since from A, $\underline{B} \vdash A$ the inference $A \vdash (B \rightarrow A)$ follows by conditional proof, but in the first inference the underlined premise B is superfluous. Also, Strevens' account of explanatory relevance is built upon on this notion of premise relevance. In his "modified classical approach" Strevens models causal explanations as deductive arguments whose premise set Γ consists of causal laws and conditions, and whose conclusion (explanandum) E is premise-relevantly entailed by Γ (2004, 162f.); this account is then strengthened by further criteria going beyond premise-relevance. So for Strevens, the relation Γ-**R**-E is the deduction of a (typically singular) explanandum E from a set Γ of (singular) explanatory factors and causal laws.

The irrelevance of EFQ does not rely on superfluous premises, but is attributed by Anderson and Belnap (1975, 151) to the maximal arbitrariness of the conclusion in the sense of difference making. Until this point Anderson and Belnap's *relevance logic* agrees with a competing approach to logical relevance, the approach of logical *relevance criteria*, which is exemplified by the account of this paper. The two accounts are based on substantially different research programs. The program of relevance logic intends to explicate the notion of relevant inference by means of a *non-classical* logic, weaker than classical logic, whose valid inferences should simultaneously be relevant. In contrast, the program of relevance criteria distinguishes between validity and relevance and considers relevance as a *filter* that separates those (classically) valid inferences that are relevant from those which are not. The idea underlying the latter research program is that while classical validity is well suited for a logically adequate notion of inference, relevance should be used as a criterion for efficient information-processing in *applications* of logical inference. The difference between the two programs is intimately connected with the considerations in the beginning of this section: Should relevance be considered as a specific type of relation, in this case as a non-classical inference relation, or should it be considered as a property of instances of the classical inference relation that distinguishes relevant instances from irrelevant ones?

Prima facie one could ask: why should the two programs be in conflict? Ideally they should agree, in the sense that the 'right' relevance logic produces exactly those inferences as valid that are filtered out as relevant from the set of all classically valid inferences. Unfortunately this is not possible. Relevance logics require certain 'adequacy' properties of inference relations such as substitutive closure or proof-theoretic axiomatizability, but these properties come into conflict with criteria of relevance in the sense of difference making. A detailed analysis of these conflicts is given in Sect. 4; here we mention just one conflict that emerges from Anderson and Belnap's formalization of the premise-in-use idea by the condition that a relevant inference or implication should have a proof in which all premises are *used*. The problem is that one may design *detour* proofs that make superficial use of a superfluous premise. For example, the inference "A, $\underline{\mathrm{B}}$ ⊢ A" contains the superfluous premise B, and its proof "1: A premise, 2: B premise, 3: A from 1 by reiteration" makes no use of premise 2. However, the detour proof "1: A premise, 2: B premise, 3: A∨¬B from 1 by addition, 4: ¬¬B from B by double negation, 5: A from 2, 4 by disjunctive syllogism" makes use of premise B; so it seems that the premise-in-use idea breaks down. To avoid this brake-down, Anderson and Belnap decide to consider the rule of disjunctive syllogism (DS) A∨B, ¬B/A as well as Modus Ponens (MP) A→B, A/B as invalid (ibid., 165f.), although DS and MP are intuitively strongly entrenched and clearly relevant according to the difference-making account. According to the criterion of replacement relevance, the truly irrelevant step in the above detour reasoning is addition: A/A∨¬B, but additions are regarded as valid in relevance logics of the Anderson/Belnap type. A similar detour proof is possible for (VEQ): A ⊢ B∨¬B; Anderson and Belnap block the latter by restricting the rule of conjunction (Con) A, B/A∧B (ibid., 271), while according to the replacement criterion the truly irrelevant step in this detour proof is simplification: A∧B/A (cf. Schurz, 1999, sec. 2.3).

There are many different relevance logics and not all of them are banning DS. But before we can come to a closer comparison (in Sect. 4), we have to develop the account of replacement relevance. A couple of problems need to be solved. First of all, applying premise-relevance merely to the premises but not to their conjunctive components seems arbitrary, because premises can be conjoined without changing the substance of an entailment, but this operation may turn a premise-irrelevant into a premise-relevant inference. For example, $\{p, \underline{r}, p{\rightarrow}q\} \vdash q$ is premise-irrelevant but $\{p{\wedge}r, p{\rightarrow}q\} \vdash q$ is premise-relevant—here and in what follows, irrelevant formulas or components of them are underlined. Therefore we must protect the criterion of premise-relevance against the operation of forming conjunctions. One way of doing this is to decompose the premise set into its conjunctive components. This leads to an important strengthening of the concept of relevance, in which the operation of replacement is applied to conjunctive subformulas of the inference. As an example, consider:

(8) Irrelevant premise components:
(i) $A{\wedge}\underline{B} \vdash A$.
(ii) $\neg(\neg A{\vee}\neg\underline{B}) \vdash A$.

The inference (8i) is conjunctively premise-irrelevant, since the underlined conjunct B is replaceable salva validitate. But conjunctions may be hidden by transforming them into L-equivalent non-conjunctions, as in (8ii). There are two methods to solve this problem:

first, by transforming every premise set into a conjunction of conjunctive elements (in our example by transforming $\neg(\neg A{\vee}\neg B)$ into $A{\wedge}B$), and

second, by applying the replacement criterion not just to conjunctive but to arbitrary subformulas.

The second method is called *component-wise* replacement (and relevance) and is applied in the inference (8ii), in which the underlined subformula B is replaceable by any other formula salva validitate. The replacement criterion chooses the second method, because it leads to the most general criterion, best suited for solving 'paradoxes of irrelevance'. For example, an irrelevant conjunctive subformula may stand in the scope of an existential quantifier, as in

(9) $\exists x(Fx{\wedge}\underline{Gx}) \vdash \exists xFx$.

$\exists x(Fx{\wedge}Gx)$ cannot be logically decomposed into a conjunction, since $\exists xFx{\wedge}\exists xGx$ is weaker than $\exists x(Fx{\wedge}Gx)$, but the subformula Gx is nevertheless irrelevant in this inference.

Moreover, the elimination-criterion applies prima facie only to premise-relevance. The replacement idea is more general than the elimination idea, because it applies straightforwardly to both premise- and conclusion-relevance. Applying the replacement criterion to conclusion-components covers one of the most important culprits of paradoxes of irrelevance:

(10) *Irrelevant disjunctive weakening*: A ⊢ A∨ X̲ (the underlined subformula X is replaceable by any other formula salva validitate).
Examples: (i) The satellite will crash into the Atlantic/Therefore the satellite will crash into the Atlantic or into London.
(ii) The letter should be posted/Therefore the letter should be posted or it should be burned (Ross' paradox, Ross, 1941).

Communicating true but irrelevant disjunctions may have disastrous effects in communications. In accordance with Grice's maxims (1975, 51), the information (10)(i) causes in the hearer the wrong expectation that X is a possibility against which she should protect herself, and the information in (10)(ii) that X is a legitimate option for action. Moreover, irrelevant disjuncts play a crucial role in Hesse's paradox of confirmation (Hesse, 1970, 50): If E expresses an empirical evidence and H an entirely unrelated hypothesis H (e.g., creationism), then provably E∨H confirms H (hypothetically-deductively as well as probabilistically; see below); but since E∨H is logically contained in E, it seems that we must also say that E confirms H, which is absurd. For these reasons, several philosophers, including Gemes (1993, 1994a), Yablo (2014), Fine (2017, 641) and myself have argued that an irrelevant disjunctive weakening A∨X does not express a *content part* of the premise A, as opposed to a conjunctive simplification X that does expresses a content part of the premise A∧X. Besides irrelevant disjunctions, many other examples of irrelevant conclusions are covered by the component-wise application of the replacement criterion as developed by Schurz and Weingartner,[3] such as A ⊢ ¬A̲→B, ∀x(Fx→Gx) ⊢ ∀x((Fx∧H̲x̲)→Gx), etc.

3 The replacement criterion of deductive relevance and its applications

Several other proposals have been developed within the program of relevance criteria (see also Sect. 4). What they have in common is that they are explicated within classical logic and can be considered as varieties of the difference-making account to relevance. An important predecessor of the replacement criterion is the criterion of Körner (1947) as elaborated by Cleave (1973/74). This criterion considers an inference or implication as relevant iff no subformula of it is replaceable by its *negation* salva validitate.[4] Replacements by negations are an important variety of difference-making relevance that will be called *essential* (as opposed to simple) relevance in Sect. 5. In the test for essential relevance, a component c of the respective

[3] Cf., e.g., Schurz and Weingartner (1987, 2010), Schurz (1991a, 1999), Weingartner (2000), Schurz and Schippers (2020).

[4] Cleave's criterion applies to replacements in arbitrary formulas salva logical content; in application to inferences this amounts to the stated version.

relatum a (or b) of the relation instance a**R**b is either replaced by an *opposite* entity c*—which in application to entailments is the negation of c—or by an empty element 0—which in application to entailments corresponds to the elimination of c, or to its replacement by a suitably defined 'empty' subformula, which depending on the case is either a tautology (⊤) or a contradiction (⊥). Tautological or contradictory subformulas are either logically eliminable or the whole formula is contractible to ⊤ or ⊥, by iteration of the L-equivalences A∧⊤ ↔ A, A∧⊥ ↔ ⊥, A∨⊤ ↔ ⊤, A∨⊥ ↔ A; so the replacement of a subformula occurrence by an empty formula is equivalent with its elimination.

In (classical) propositional logic, the relevance conditions based on replacements of subformulas (i) by arbitrary other formulas, (ii) by their negations, and (iii) by empty formulas are provably equivalent (a precise version of this result is proved in the *appendix*). In predicate logic, however, replacements by negations (ii) are weaker than (i). For example, in the inference

(11) Fa, ¬Fb ⊢ ∃x<u>Fx</u>

the underlined subformula Fx is replaceable salva validitate by ¬Fx; yet the inference is conclusion-relevant and a salva validitate replacement of Fx by an arbitrary other predicate Gx is not possible. The equivalence of (i) and (iii) still holds in predicate logic, but only for single replacements, while it breaks down for multiple replacements that will be needed below. Therefore the relevance criterion based on replacements by arbitrary subformulas is preferable in predicate logic.

There is a further important equivalent version of the replacement criterion. In predicate logic one replaces subformulas $A(t_1,\ldots,t_n)$ by arbitrary other subformulas $B(t_1,\ldots,t_n)$ in the same individual terms t_i. This condition is provably equivalent with the replacement of (atomic) predicates by other predicates with the same place-number. This equivalence (proved in Schurz & Weingartner, 1987, proposition 1) makes it possible to formulate the replacement criterion in a much simpler way. To cover propositional variables, the latter ones are regarded as zero-placed predicates.

Before we can state the final definition of replacement-relevance, a further subtlety has to be mentioned. So far we have restricted the attention to the replacement of *single* subformula or predicate occurrences in the premises or conclusion. There is a symmetry between the two: an irrelevant premise-conjunct corresponds to an irrelevant conclusion-disjunct, since A∧<u>B</u> ⊢ C iff A ⊢ C∨¬<u>B</u>. In spite of this logical nicety, reasons of application force us to break this symmetry. There are important cases of conclusion-irrelevance due to the simultaneous replaceability of several occurrences of a predicate or subformula:

(12) Multiple-replacement irrelevance of conclusions:
A∧B ⊢ (A∧<u>C</u>)∨(B∧¬<u>C</u>).
A ⊢ (A→<u>C</u>) → <u>C.</u>

In (12) C is replaceable salva validitate on both occurrences, but not on only one occurrence. To cover these cases of conclusion-irrelevance we must allow for multiple replacements, i.e. replacements of several (one-or-more) occurrences of a predicate (or subformula) by arbitrary other predicates (or subformulas). In contrast, for premise-relevance this generalization would be *inadequate*. This follows from the different roles of premises and conclusion in inferential information processing: One extracts *one* piece of information, the conclusion, from *different* pieces of information, the premises. For this reason, the premises must be *allowed* to contain concepts which are not contained in the conclusion, as in the following example:

(13) *Modus Pones* is premise-relevant, in spite of $\underline{A} \wedge (\underline{A} \rightarrow B) \vdash B$.

In contrast, multiple conclusion-relevance implies that a relevant conclusion must not contain concepts (predicates) that are not contained in the premises.[5] The latter condition has been called the Aristoteles-Parry-Weingartner criterion of conclusion-relevance (Schurz, 1991a, 416).

Summarizing, we obtain the following compact definitions of component-wise premise- and conclusion-relevance:

(14) (Definition.) Assume $\Gamma \vdash C$. Then:

(14.1) $\Gamma \vdash_{pr} C$ ($\Gamma \vdash C$ is *component-wise* premise-relevant, in short p-relevant) iff no single predicate occurrence F in Γ is replaceable by any other predicate F* (with the same place number) salva validitate.

(14.2) $\Gamma \vdash_{cr} C$ ($\Gamma \vdash C$ is component-wise conclusion-relevant, in short c-relevant) iff no predicate F is replaceable on some of its occurrences in C by any other predicate F* (with the same place number) salva validitate.

(14.3) $\Gamma \vdash_{pcr} C$ ($\Gamma \vdash C$ is p- and c-relevant) iff $\Gamma \vdash_{pr} C$ and $\Gamma \vdash_{cr} C$.

Examples: ("$_{\text{-ir}}$" for "irrelevant"): $p \wedge q \vdash_{cr} p$; $p \wedge \underline{q} \vdash_{p\text{-}ir} p$; $p \wedge \neg p \vdash_{pr} q$; $p \wedge \neg p \vdash_{c\text{-}ir} \underline{p}$; $p, (p \rightarrow q) \vdash_{pcr} q$; $p, \neg p \vee q \vdash_{pcr} q$; $p \vdash_{pr} p \vee q$; $p \vdash_{p\text{-}ir} p \vee \underline{q}$; $p \vdash_{p\text{-}ir} p \vee \underline{p}$; $p \vdash_{c\text{-}ir} \neg p \rightarrow \underline{q}$, $(p \vee q) \rightarrow r, p \vdash_{cr} r$; $(p \vee \underline{q}) \rightarrow r, p \vdash_{p\text{-}ir} r$; $\forall xFx \vdash_{pcr} Fa$; $\forall xFx \vdash_{pcr} \exists xFx$; $\forall x(Fx \rightarrow Gx), Fa \vdash_{pcr} Ga$, $\forall x(Fx \rightarrow Gx) \vdash_{c\text{-}ir} \forall y((Fx \wedge \underline{H}x) \rightarrow (Gx \wedge \underline{H}x))$.

Without going into detail we mention that irrelevant occurrences of the identity sign (≡) in an inference are handled by replacing these ≡-occurrences by an arbitrary new binary relation E and adding the equality axioms Ax_E for E to the premises (equivalence axioms for Exy and substitution of E-equivalent terms restricted to predicates in the inference; cf. Schurz, 1997, ch. 10). For example, $Fa \vdash a{\equiv}b \rightarrow Fb$ is conclusion-irrelevant w.r.t. identity, since $Ax_E, Fa \vdash \underline{E}ab \rightarrow Fb$ holds for every equality symbol E.

[5] *Proof:* Assume $\Gamma \vdash C$ and the predicate F occurs in C but not in Γ. Let X* result from X by a uniform replacement of F in X by an arbitrary predicate G. By the rule of uniform substitution, $\Gamma \vdash C$ implies $\Gamma^* \vdash C^*$, and since F doesn't occur in Γ, this implies $\Gamma \vdash C^*$, i.e., $\Gamma \vdash C$ is conclusion-irrelevant.

The relevance criterion as defined in (14) is *hyperintensional*, i.e., not invariant w.r.t. L-equivalent transformations of the premise set or the conclusion, respectively. For example, $(p\rightarrow q)\wedge p \vdash q$ is p-relevant but $p\wedge \underline{q} \vdash$ is p-irrelevant, although $(p\rightarrow q)\wedge p$ and $p\wedge q$ are L-equivalent. Likewise, $(p\vee q)\wedge r \vdash p\vee(q\wedge r)$ is c-relevant but $(p\vee q)\wedge r \vdash (p\vee q)\wedge(\underline{p}\vee r)$ is c-irrelevant, although $p\vee(q\wedge r)$ and $(p\vee q)\wedge(p\vee r)$ are L-equivalent. These results are okay, because when we assess the (ir)relevance of an inference, we take the premises and the conclusion *as stated*. However, in applications of deductive relevance to the verisimilitude of a theory, or to its confirmation or explanatory power w.r.t. given pieces of evidence, one usually wants a criterion that holds independently of how the theory (premise set) or the evidence (conclusion) is formulated. In other words, one looks for an intensional notion of relevant inference. This is achieved by transforming the premises and conclusion of an inference into a certain normal form in which are all irrelevancies due to redundant formulations are eliminated, so that only the 'essential' irrelevancies due to their content remain. As such normal form we use the representation of formulas (or formula sets) by non-redundant conjunctions of their smallest conjunctive parts, the so-called *content elements* (earlier they were called 'relevant elements'). If the relevance criterion is always applied to the same normal form of an inference, its results will be invariant under arbitrary L-equivalent transformations of the inference. The definition of content elements is based on the notion of c-relevance combined with a criterion of conjunctive elementariness:

(15) (Definition.)

(15.1) C is a *content element* of a formula or formula set Γ iff C is a c-relevant consequence of Γ that is *elementary* in the sense of not being L-equivalent with a conjunction $C_1\wedge\ldots\wedge C_n$ of conjuncts all of which are *shorter* than C.

(15.2) A *relevant representation* Γ_r of a sentence or sentence set Γ is a non-redundant conjunction of content elements of Γ that is L-equivalent to Γ (where non-redundancy means that no conjunctive part of Γ_r follows logically from the rest of Γ_r).

The shortness criterion in def. 15.1 is relativized to a language with ¬, ∧, ∨,∃ and ∀ as logical primitives; defined symbols are eliminated. The (ir)relevance of inferences does not change if a subset of {¬, ∧, ∨,→,∃, ∀} is chosen as set of primitives. Equivalence ↔, however, must not be included because it would change the (ir)relevance diagnosis, since $p\leftrightarrow q \vdash p\rightarrow q$ is p-relevant but $(p\rightarrow q) \wedge (\underline{q}\rightarrow \underline{p}) \vdash p\rightarrow q$ is p-irrelevant.

Importantly, every Γ has a relevant representation Γ_r (Schurz, 2014, prop. 3.12–1). Thus, the set of Γ's content elements preserves Γ's classical content.

Examples of content elements, where CE(A) = the set of content elements of A: $CE(p) = CE((p\vee q) \wedge (p\vee\neg q)) = \{p\}$, $CE(p\wedge q) = CE(p\wedge(p\rightarrow q)) = \{p,q\}$, $CE((p\rightarrow q)\wedge(q\rightarrow r)) = \{\neg p\vee q, \neg q\vee r, \neg p\vee r\}$, $CE(Fa) = \{Fa, \exists xFx\}$, $CE(\forall xFx) = \{\forall xFx\} \cup \{Fa_i: a_i \text{ is an individual constant}\} \cup \{\exists xFx\}$.

In combination with the notion of an relevant representation, the account of replacement relevance has been shown to be highly fruitful. By using the tools of this account, problems of irrelevance that emerged in various contexts of applied logical philosophy can be solved in a uniform way. As a first example, consider the concept of *deductive-nomological* (dn) *explanation*. According to its standard definition (Hempel & Oppenheim, 1948), a dn-explanation is a deductive argument L, A ⊢ E whose premise set consists of general laws L and particular antecedens conditions A that together entail the explanandum E. As Hempel and Oppenheim (1948, 273ff.) and later authors discovered, this definition allows various instances of trivial or counterintuitive 'explanations'. The four basic cases of irrelevant dn-arguments are: (1) *Complete self explanations* "L,A/E" where A ⊢ E, i.e., E follows already from A alone, (2) *partial self explanations*, e.g. "∀x(Fx→Gx),Fa∧Ha/Ga∧Ha", where one conjunctive component of E follows from A alone, (3) *complete (or partial) theoretical explanations*, where E (or a conjunctive component of E) follows from T alone, and (4) *irrelevant explanandum-weakenings*, as in "∀x(Fx→Gx), Fa/Ga∨Ha". (1)–(3) have been mentioned the first time in Hempel (1948, 275) and (4) in Gärdenfors (1976, 425). All these and similar counterexamples (cf. Schurz, 1996, sec. 4.2; 2014, sec. 6.4.2) can be avoided by applying the relevance condition as follows:

(16) A *dn-explanation* L, A ⊢ E in the sense of Hempel and Oppenheim (1948) has to satisfy the following condition: there is a relevant representation L_r, A_r and E_r of L, A and E, respectively, such that $L_r, A_r \vdash_{pcr} C$ holds for every conjunct C of E_r.[6]

Next we turn to *hypothetico-deductive* (hd) *confirmation*. According to the standard account, a piece of evidence E hd-confirms a hypothesis H iff (a) H ⊢ E, where (b) H is consistent and (c) ⊬ E (cf. Glymour, 1981, 35).[7] This leads to various counter-intuitive results due to irrelevant components of the hypothesis or the evidence. We already mentioned the Hesse-paradox of confirmation that relies on irrelevant disjuncts in the evidence; Grimes (1990) and Gemes (1993) called this problem "tacking by disjunction". The dual confirmation paradox is based on irrelevant premise conjuncts and has been called "tacking by conjunction". Here the evidence E confirms E∧X, i.e. the conjunction of E with an arbitrary hypothesis X consistent with E (e.g. X = creationism), because E∧X entails E (Lakatos, 1970, 46; Glymour, 1981, 67). This is particularly fatal in connection with the so-called consequence condition (Hempel, 1945, 31), according to which logical consequences of confirmed hypotheses are confirmed, too – which would yield the absurd result that E confirms X. These and similar problems of hd-confirmation can be avoided

[6] Since not every formula (set) possesses a *unique* relevant representation, the definition refers to *some* relevant representations. The application of pc-relevance to every conjunct of E_r is needed to eliminate partial self-explanations or partial theoretical explanations.

[7] Often this definition is relativized to a background knowledge B (Glymour (1981, 35). Then it says: (a) H, B ⊢ E, (b) H is consistent with B, and (c) B⊬ E.

by applying replacement relevance to hd-confirmation as follows (cf. Schurz, 1994, 184f., (HD-2*)):

(17) (Definition.) E *hd-confirms* H iff $H_r \vdash_{pcr} E_r$ holds for some relevant representation H_r and E_r of H and E, respectively.[8]

A further application is Popper's notion of verisimilitude, or truthlikeness. With this notion, Popper intended to work out the following idea. Although most scientific theories are strictly speaking false (because they contain errors of idealization or approximation), some of them are *closer to the truth* than others. According to Popper's original explication (1963, 233f.), a theory T is at least as close to the truth as a theory T*, in short $T \geq T^*$, iff (a) T*'s true consequences are a subset of T's true consequences and (b) T's false consequences are a subset of T*'s false consequences (the notion of $T > T^*$ is defined as $T \geq T^* \wedge \neg(T^* \geq T)$). Tichý (1974) and Miller (1974) diagnosed a fatal flaw in Popper's definition of truthlikeness: it leads to the result that a false theory T can impossibly be closer to the truth than any other theory T*, in contradiction to Popper's idea of truthlikeness. To see this, let T be a false theory that implies the false consequence f. Now, if T would be closer to the truth than some other theory T*, then conditions (a) and (b) above must hold and one of two cases has to obtain:

Case 1: T has a true consequence t which is not implied by T*. But then, T must also have the false consequence $t \wedge f$ which is *not* a consequence of T*. So condition (b) above is violated and $T > T^*$ cannot hold.

Case 2: T* has a false consequence f* which is not implied by T. But then T* would also have the true consequence $\neg f \vee f^*$, which cannot be implied by T (since T implies f but not f*). So condition (a) above is violated and $T > T^*$ cannot hold, too.

According to the diagnosis of Schurz and Weingartner (1987, 2010), this counterintuitive result rests on the fact that the classical consequences of a theory include irrelevant disjunctions and redundant conjunctions. In case 1 $t \wedge f$ is obviously not a "new" false content part of T beyond T's content parts t and f, but simply a *redundant* repetition of the two. By decomposing consequences into content elements this pitfall is avoided. In case 2, the disjunctive weakening $\neg f \vee f^*$ is an *irrelevant* consequence of T*, because ¬f is replaceable salva validitate; thus the pitfall of case 2 is avoided by requiring theory consequences to be relevant. Schurz and Weingartner (1987, 2010) show that a well-functioning notion of truthlikeness is obtained by replacing the logical consequences of a theory by its content elements as follows:

(18) (Definition.) Let T_{te} and T_{fe} be the set of all true respectively false content elements of a theory T. Then: A theory T is *at least as close to the truth* as a theory

[8] If the definition is relativized to a background knowledge B (cf. fn. 7), then the inference $B, H_r \vdash E_r$ must be c-relevant, and p-relevant in regard to H but not in regard to B, since B may contain superfluous information.

T* iff (a) T_{te} is (logically) as least as strong as T^*_{te} and (b) T^*_{fe} is at least as strong as T_{fe}.

Note that for a functioning notion of verisimilitude only the notion of content elements (and, thus, of conclusion-relevance) is needed, but not that of premise-relevance. Content elements are also needed to explicate the notion of *unification* afforded by a theory T w.r.t. to a belief system (cf. Schurz, 1991a, sec. 5.1.3).

As a final application of replacement relevance we mention the *is-ought* problem. According to Hume's famous thesis, no consistent set of descriptive (non-normative) premises can logically imply a non-tautologous normative conclusion. Thereby, a purely descriptive sentences is one that does not contain the obligation operator "O". Moreover, a purely normative sentence is a Boolean combination of elementary normative sentences of the form OA. It is not difficult to prove Hume's thesis for purely descriptive premises and purely normative conclusions. But how should one deal with *mixed* sentences such as $p \lor Oq$? Prior (1960) even construed a 'paradox' out of mixed sentences: He argued that if $p \lor Oq$ is counted as normative then the inference $p \vdash p \lor Oq$ would violate Hume's thesis, and if $p \lor Op$ is counted as non-normative, then the inference $\neg p, p \lor Oq \vdash Oq$ would violate Hume's thesis; so Hume's thesis would be violated in any case. Schurz (1997) proposes to solve this problem by replacing Prior's dichotomic classification by the trichotomic classification of purely descriptive, purely normative and mixed sentences and applying the replacement criterion of relevance to inferences from purely descriptive premises to mixed conclusions. A mixed conclusion is called *completely Ought-irrelevant* iff all its predicate occurrences that lie in the scope of an obligation operator are uniformly replaceable by arbitrary other predicates, salva validitate of the inference – as in the examples $p \vdash p \lor O\underline{q}$ and $p \vdash p \land (O(\underline{q} \land \underline{r}) \rightarrow O\underline{q})$. Schurz proves the following result for all standard systems of alethic-deontic logics (1997, 92, theorem 1):

(19) Inferences from purely descriptive premises to mixed conclusions do not violate the spirit of Hume's thesis, because their conclusion is *completely Ought-irrelevant*.

4 Comparison with other accounts

In the previous section it was demonstrated that the replacement criterion of logical relevance has fruitful applications in a multitude of domains in which failures of irrelevance undermined successful explications of philosophically important concepts. The price of this strength in detecting failures of irrelevance is that several traditional adequacy properties of relevant logical inference are lost. Relevance logics require these properties for reasons of semantic and proof-theoretic efficiency – whence we call them efficiency properties. They subsume at least the following three:

(i) Closure under substitution (i.e., schematicity),
(ii) Decidability or at least recursive axiomatizability, and
(iii) Transitivity of logical implication, or cut rule (for multiple premises inferences).

Criteria of relevance are conflict with these logical efficiency properties. In order to save these properties, relevance logics are forced to rehabilitate intuitively irrelevant inferences or to reject intuitively relevant ones. The conflict with closure under substitution emerges already at very early steps of Anderson and Belnap's relevance logic.[9] For example, the irrelevant inference schemata VEQ and EFQ have the instances.

(20) VEQ-instance: $\underline{\mathrm{A}} \vdash \mathrm{A}\vee\neg\mathrm{A}$, valid in relevance logic.
(21) EFQ-instance: $\mathrm{A}\wedge\neg\mathrm{A} \vdash \underline{\mathrm{A}}$, valid in relevance logic.

(20) is simultaneously an instance of addition (Add): $\mathrm{A} \vdash \mathrm{A}\vee\mathrm{B}$ and (21) an instance of simplification (Simp): $\mathrm{A}\wedge\mathrm{B} \vdash \mathrm{A}$, and Add and Simp are valid in Anderson and Belnap's relevance logics **E** (entailment) and **R** (relevant implication). Since **E** and **R** require closure under substitution, these systems must accept the inferences (20) and (21) as valid, although they are irrelevant, as the underlined formula A may be replaced by any other formula X salva validitate (cf. Anderson & Belnap, 1975, 152, 154; Schurz, 1991a, 413). In contrast, according to the replacement criterion of logical relevance, not the relevant but the irrelevant inferences are closed under substitution.

Is the failure of substitutive closure a problem for the relevance criterion? We think not. The failure arises only for properly *homomorphic* substitutions, in which atomic sentences (or predicates) are replaced by sentences (or open formulas) that are mutually logically dependent and, thus, restrict the freedom of interpretation. Deductive (replacement) relevance is still closed under *isomorphic* substitutions, in which atomic sentences (or predicates) are bijectively replaced by other atomic sentences (or predicates), or alternatively by sentences (or open formulas) that are mutually independent and do not restrict the freedom of interpretation (for details cf. Schurz, 2001, sec. 3 and Schurz, 2023). Thus, deductive relevance is still a *matter of form*, in the sense that it merely depends on logical structure but not on the interpretation of primitive non-logical terms. As argued in Schurz (ibid.), it is a frequently heard misunderstanding that logical formality presupposes closure under homomorphic substitution; all what is required for formality is closure under isomorphic substitution. The reason why deductive relevance is not closed under homomorphic substitution is that its definition involves a non-derivability or equivalently a consistency condition, and the latter one is not closed under homomorphic substitution. For example, $p\wedge q$ is consistent but its homomorphic substitution instance $p\wedge\neg p$ is inconsistent. Yet consistency is clearly a formal notion, and so is deductive relevance. An

[9] These observations apply likewise to Leitgeb's (2019) system HYPE, whose logical consequences without $\{\bot, \top\}$ coincide with Anderson and Belnap's first degree entailments.

example of a formal logic that is not closed under homomorphic substitution is Carnap's modal logic **C** (Hendry & Pokriefka, 1985; Schurz, 2001).

The second traditional adequacy condition is recursive axiomatizability. It is connected with closure under substitution. The standard notion of logical deducibility refers to the *existence* of a recursively defined proof. For this reason, the theorems and inferences of a standard logic are closed under substitution, given that all axioms and rules of the logic are so closed. Following this idea, Anderson and Belnap (1975, 21) defined a proof of $\Gamma \vdash A$ in a relevance logic $\mathbf{L_R}$ as one in which all premises are used. But this definition is undermined by the existence of roundabout proofs of EFQ and VEQ, as explained in Sect. 2. In contrast, according to the replacement criterion the premise-*irrelevance* of an inference $\Gamma \vdash A$ refers to the existence of a proof of $\Gamma \vdash A$ in which *not* all premises are used; therefore not the premise-relevant but the premise-irrelevant inferences are closed under substitution, and the same observation applies to conclusion-irrelevance (cf. Schurz, 1999, sec. 2.2). Moreover, the fact that the relevance of a valid inference depends on the non-validity of other inferences implies a restriction for the recursive axiomatizability or enumerability of relevantly valid inferences: The set of conclusion-relevant inferences of an axiomatizable classical logic **L** with inference relation $\vdash_{\mathbf{L}}$ can only be recursively enumerable (r.e.) if $\vdash_{\mathbf{L}}$ is decidable (and the same fact holds for premise-relevance). For illustration, consider the inference

(22) $P \wedge A \vdash_{\mathbf{L}} P$ where P is atomic and P's nonlogical terms are not in A.

The inference in (22) is conclusion-relevant exactly if A is consistent. Thus, if conclusion-relevant inferences are r.e., then the set of non-theorems of **L** are r.e. and thus **L** must be decidable (for a detailed proof see Schurz, 1991a, theorem (27)).

Does this fact create a strong disadvantage for the criterion replacement relevance? We don't think so. Ideally a logic **L** should be decidable, and in this case **L**'s relevant inferences are decidable and, thus, r.e. The problem arises only for full first order logic $\mathbf{L}_1$ that is not decidable but merely r.e.; so $\mathbf{L}_1$'s relevant inferences are not r.e. But in spite of $\mathbf{L}_1$'s undecidablity an answer to the validity-question has been found for *most* know first order inferences. Moreover, there are many other important logical systems that depend on non-derivability conditions and, thus, are not r.e. in full first order logic – for example, the maximally general first order theorems (Schurz 1995) or the theorems of Carnap's modal logic **C** (Schurz, 2001).

The third traditional adequacy condition is the transitivity of inference. It is easy to see that conclusion-relevant inference is *not transitive*. Here is an example:

(23) Non-transitivity of c(onclusion)-relevant inference:
- (i) $(A \vee B \wedge C \vdash A \vee (B \wedge C)$ is c-relevant,
- (ii) $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$ is c-relevant, but
- (iii) $(A \vee B) \wedge C \vdash (A \vee B) \wedge (\underline{A} \vee C)$ is c-irrelevant.

Similar counter-transitive examples can be given for p-relevance (Schurz 1999a, 33).

The conflict between deductive relevance and logical efficiency properties does not mean that it is not reasonable to search for a logical inference relation that is as relevant as possible, in the sense of an optimal *compromise* between logical efficiency and relevance. Concerning closure under substitution, Smiley (1959) proposed to define a (minimally relevant) entailment as a *substitution instance* of a classically valid inference schema that has consistent premises and a non-provable conclusion. The notion of entailment obtained by this step is closed under substitution, on the cost that irrelevant inferences have to be accepted. From the viewpoint of applications this is a drawback, because what matters in applications are logical arguments and not the maximally general schemata which these arguments instantiate. For example, $p \wedge \neg p \vdash p \vee \neg p$ can hardly be conceived as minimally premise- or conclusion-relevant, although it is a valid Smiley-entailment.

Tennant (1983) developed a recursive axiomatization of Smiley's account of entailment based on the notion of a perfect sequent (sequents $\Gamma \vdash \Delta$ have sets of conclusions that are semantically interpreted as disjunctions). A valid sequent $\Gamma \vdash \Delta$ is *perfect* iff it does not have a valid proper subsequent $\Gamma' \vdash \Delta'$ (i.e. $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ and $\Gamma' \cup \Delta' \subset \Gamma \cup \Delta$). Tennant defines an entailment as a substitution instance of a perfect sequent. His account has three advantages. *First,* compared to Anderson and Belnap's relevance logic, Tennant accepts DS and MP as valid. *Second*, althought perfect sequents are not r.e. (Brauer, 2020, 447), Tennant-entailments are r.e. (Tennnant 1984, 193, "completeness theorem"). Transitivity or cut rule is violated for Tennant-entailments in cases where the united premises/conclusions produce superfluous elements or inconsistent premise-sets or valid conclusion-sets (ibid.,191, corollary 4). *Third*, Tennant entailment avoids irrelevant premises and irrelevant sequent-conclusions. But this stronger relevance-property is not robust under conjoning premises or disjoining sequent-conclusions. Thus, $A,B \vdash A$ is Tennant irrelevant but $A \wedge B \vdash A$ is Tenannt-relevant, and similarly, $A \vdash A,B$ is Tenannt-irrelevant but $A \vdash A \vee B$ is Tennant-relevant, which seems unplausible (cf. Brauer, 2020, 443).[10]

In conclusion, Tennant's relevance logic is much closer to replacement relevance than Anderson and Belnap's relevance logic, but it is not more suited to handle the problems of irrelevance in the applications presented in Sect. 3, because all of these problems are caused by irrelevant disjuncts or conjuncts. These problems can only be handled by systems that avoid the inferences of addition and simplification, but the relevance logics of Anderson/Belnap and of Tennant accept these inferences.

However, some weaker systems of relevant logics have been developed that avoid addition and/or simplification. Already Nelson (1930) objected to the law of simplification $p \wedge q \vdash p$ on the ground that a part of the premise, namely q, is entirely irrelevant to the conclusion. Parry (1993) developed an axiomatic system of "analytic implication" that avoids the axiom of addition $p \vdash p \vee q$. Parry's logic is based on Kant's idea of analyticity, according to which all "contents" (or predicates) of

[10] Brauer (2020) adds to Tennant (1984) a weaker notion of relevance, according to which a formula A is (weakly) relevant in $\Gamma \vdash A$ if it occurs in a perfect valid subsequent of $\Gamma \vdash A$; in application to premises this corresponds to the notion of "conditional premise-relevance" in Schurz (1999, sec. 3.5).

the conclusion must be contained in the contents (or predicates) of the premises. Parry's logic, also called *containment* logic, is much closer to replacement relevance than relevance logics. But because it satisfies the mentioned logical closure properties, it has to accept irrelevant inferences. For example, all substitution instances of irrelevant inferences whose conclusion-predicates are contained in the premises are Parry-valid, even if they are instances of EFQ or VEQ or of irrelevant additions. In applied contexts this is inacceptable, since one would hardly say that a conjunction $P_1 \wedge \ldots \wedge P_n$ of independent facts P_i (relevantly) contains the conclusion $(P_1 \wedge \ldots \wedge P_n) \vee X$ for every X in the propositional variables $P_1, \ldots, P_n$. For hd-confirmation this would imply the counterintuitive result that the universal statement "All animals have lungs and breath air" is confirmed by the set of observations "All animals observed so far have either lungs and breathe air, or don't have lungs and don't breath air".

Further disadvantageous features of Parry (1933) are discussed by Angell (1989) who devised an improved version of Parry's containment logic. More recently, Fine (2017) developed a ground-theoretic logic of exact entailment in the spirit of Nelson (1930) and a logic of containment that is a modification of the system of Angell (1989). Fine's notion of exact entailment (he simply says "entailment") is a logical approximation of premise-relevance. His notion of containment is an approximation of conclusion-relevance. Thus $p \wedge q \vdash p$ is a containment but not an exact entailment, and $p \vdash p \vee q$ is an exact entailment but not a containment. Among all systems of relevance logic that satisfy the mentioned efficiency properties, Fine's systems are the closest approximations of logical replacement relevance known to me. Yet, because of the explained conflict Fine's system has to include irrelevant inferences and to exclude relevant inferences; a closer discussion of Fine's system is given in Sect. 8 on grounding.

In conclusion, all logics of relevant inference can be considered as compromises between the idea of relevance as difference making and logical efficiency properties.[11] These compromises can not only be optimized by improving the logic of relevance, but also by modifying the relevance criterion in a direction that allows a smoother logical handling of it. For example, in Sect. 3 we have met two versions of replacement relevance, one based on single and the other based on multiple replacements; while the former has nicer logical properties (e.g., it satisfies deduction theorem), the latter is more powerful in eliminating cases of irrelevance. Another important alternative criterion of replacement relevance has been developed by Gemes (1993, 1994a, 1997). For Gemes (1993, 481), the conclusion C of an inference $\Gamma \vdash C$ is relevant – or in his words, is a content part of Γ – iff Γ and A are contingent (neither inconsistent nor L-true) and Γ entails no formula C* that is logically stronger than C but constructible from C's atomic formulas. So, in the terminology of difference-making relevance, Gemes' variations are replacements of C by

[11] This diagnosis is further confirmed by Rott's (2022) logic of difference-making conditionals. Rott reports in his abstract that "the connective thus defined violates almost all traditional principles of conditional logic".

logically stronger formulas constructed from C's atomic formulas.[12] Gemes' notion of content part is a notion of conclusion-relevance. To apply his account to premise-relevance in the context of hd-confirmation, Gemes decomposes the premise set (or 'theory') into an L-equivalent non-redundant set of content parts Γ* and requires that no premise in Γ* is eliminable salva validitate (1993, 486). Gemes' concept of content part satisfies one of the three logical efficiency properties: it is transitive; but it is neither recursively enumerable (by the argument for (22)) nor closed under homomorphic substitution (e.g., p is a content part of p∧q but not of p∧¬p). Gemes (1994b, 1997, sec. 7) highlights several differences between his criterion and the Schurz-Weingartner criterion presented in def. (14). But for the majority of examples, Gemes' and Schurz-Weingartner's criterion agree. Concerning the differences, Gemes points out ∃xFx is a relevant consequence (according to def. 14) but not a Gemesian content part of ∀xFx and of Fa. Gemes sees this as a disadvantage for hd-confirmation, but Schurz (1994b) objects that Popperian basic statements of the form ∃xF(x,s) ("some object at location s has property F") do hd-confirm the generalization ∀x∀sF(x,s) and the singular statement F(a,s); for this reason they should be considered as relevant consequences. Moreover, Gemes' account does not exclude irrelevant parts in ∃-scopes; so ∃xFx Gemes-confirms ∃x(Fx∧<u>Gx</u>) (Schurz 2005b).

5 General relevance as a property of relation-instances

After this extensive tour through versions of criteria for difference making relevance in comparison to relevance logics, we turn to the generalization of the idea as explained in the introduction. Thus, we consider difference making relevance as a property of instances of an arbitrary binary relation **R** between a domain **A** and a co-domain **B** (the account is generalizable to n-placed relations, but we restrict ourselves to binary relations). As explained, in application to deductive inferences, **A** is the set of possible premise sets, **B** the set of possible conclusions, and **R** the relation of valid inference. But now, **R** ⊆ **A**×**B** is an arbitrary relation between arbitrary sets of entities **A** and **B**, with a, a_i,… ∈ **A** and b, b_i,… ∈ **B**. We first turn to the notion of minimal relevance, meaning that the given relation instance is not preserved under arbitrary variations of its relata (or arguments). We use x resp. y as variables for elements of **A** resp. **B**.

[12] In an equivalent version, C is a Gemesian content part of Γ iff Γ ⊢ C and every disjunct of the relevant disjunctive normal form (RDN) of C is a subconjunction of the RDN of Γ (Gemes 1997, (D1)). An RDN of a formula A is a disjunctive normal form of A in A's relevant atomic formulas; these are those atomic formulas in A whose truth value can make a difference to the truth value of A. A further variation of Gemes' relevance criterion is that of Stelzner (1992) and Yablo (2014); their definitions of content part (or relevant consequence) use the notion of *minimal* disjunctive normal form instead of Gemes' RDN. As a consequence, p∨q is a content part of p∧q according to Stelzner and Yablo, but not according to Gemes.

(24) (Definition.) *Minimal relevance.* Let a**R**b be a true instance of **R**. Then:

(24.1) a**R**b is a minimally **A**-*relevant* instance of **R** iff ¬∀x∈**A**: x**R**a.
In other words: for some x∈**A**, replacing a by x in a**R**b makes a difference to the truth value of the relation instance.
Correspondingly, a**R**b is a maximally **A**-*irrelevant* instance of **R** iff ∀x: x**R**b.
In other words, a can be replaced in a**R**b by every x∈**A** *salva veritate,* i.e., under preservation of the truth of the relation instance.

(24.2) a**R**b is a minimally **B**-*relevant* instance of **R** iff ¬∀y∈**B**: a**R**y.
(The equivalent reformulations, in analogy to 24.1, are obvious.)

Next we turn to the generalized notion of *component-wise* **A**- and **B**-relevance. This definition refers to the "components" of the left relata a_i and the right relata b_i of the relation. Thereby it is assumed that these components are specified in a domain-dependent way that preserves their 'type' in the following sense: Let C(a) denote the set of components of a∈**A**, C(**A**) the set of components of elements of **A**, and a[c:x] the result of replacing some c∈C(a) in a by x∈C(**A**). Then a[c:x] must in turn be an element of **A**. Likewise for C(b) and b[c:y]. For example, in the domain of logical reasoning, the premises are formulas and their components are *single* predicate occurrences. If they are replaced by arbitrary other predicates with the same place number, the result are again formulas.

(25) (Definition.) *Component-wise relevance.* Let a**R**b be a true instance of **R**.

(25.1) a**R**b is a component-wise **A**-*relevant* instance of **R** iff ¬∃c∈C(a)∀x∈C(**A**): a[c:x]**R**b.
Condition (25.1) says in other words that no component of a can be replaced by arbitrary type-equivalent **A**-elements salva veritate of the relation instance.
Correspondingly, a**R**b is a component-wise **A**-*irrelevant* instance of **R** iff some component of a can be replaced by arbitrary type-equivalent **A**-elements salva veritate.

(25.2) a**R**b is a component-wise **B**-*relevant* instance of **R** iff ¬∃c∈C(b)∀y∈C(**B**): a**R**b[c:y].

We finally turn to the generalized notion of *essential* **A**- or **B**-relevance, in which the relata of the relation instance are replaced by opposite or by empty elements. The notion can be explicated in a minimal and in a component-wise version.

(26) (Definition.) *Essential relevance*. Let a**R**b be a true instance of **R**.
Assume the domain **A** contains an empty element 0 and for each of its elements z an opposite element z*. Then:

(26.1) (i) a**R**b is an essential minimal **A**-*relevant* instance of **R** iff ¬(0**R**b ∨ a*Rb).
In other words, it is not possible to replace a by an empty or an opposite element salva veritate of the relation instance.
(ii) a**R**b is an essential component-wise **A**-*relevant* instance of **R** iff ¬∃c∈C(a): (a[c:0]**R**b ∨ a[c:c*]**R**b).
In other words, it is not possible to replace a component of a by an empty or opposite element salva veritate of the relation instance.

(26.2) (i) a**R**b is an essential minimal **B**-*relevant* instance of **R** iff ¬(a**R**0 ∨ a**R**b*).
(ii) aRb is an essential component-wise *B-relevant* instance of R iff ¬∃c∈C(b): (a**R**b[c:0] ∨ a**R**b[c:c*])

The characterization of the ontological nature of a relation **R**⊆**A**×**B** is a domain-specific task; nevertheless some remarks on this task are appropriate. The nature of **R** is determined by its *intension* or meaning. This intension determines not only the *extension* of **R** in a given world, but also its natural domain **A** and co-domain **B**. Roughly speaking, the natural (co-)domain of **R** consists of entities belonging to the right semantic category: those pairs of objects <x,y> whose being **R**-related is semantically possible. For example, because the relation of logical inference is understood in terms of truth value preservation, the entities in its natural (co-) domain are statements or propositions (see below), but neither meaningless strings of symbols, nor chairs or stones. Likewise, since "support" is a relation between goal-directed agents, the (co-)domain of this relation consists of goal-directed agents, naturally persons.

Often one considers minor variations of the extension of **R** that do not change **R**'s intension. For example, although the (co-)domain of the relation of support consists of persons; for certain purposes one may also consider this relation over the domain of animals; or one may restrict it to the subdomain of one's relatives. When we judge the *relevance* of a relation instance a**R**b, the (co-)domain of **R** is assumed to be the natural one. In all of our applications the natural (co-)domain of **R** is well-defined, but this need not always be so. In any case, the (co-)domain has to be *fixed*, because the class of admissible variations depend on it. For example, that *my brother supports me* is **A**-relevant over the domain of all persons (since not every person would support me), but it is irrelevant over the domain of my relatives (as all of them would support me). In conclusion, the proposed account of relevance understands a relation formally as a triple <**R**,**A**,**B**> with **R**⊆**A**×**B**, where **A** and **B** are the natural domain and co-domain of **R** in accordance with **R**'s intension.

As remarked above, the relation of entailment may either be defined over sentences (which was assumed in Sects. 2 and 3), or alternatively over propositions. The two settings are related by the correspondence $\Gamma \vdash A$ iff $[\Gamma] \vdash [A]$, where [A] is the proposition expressed by the sentence A, and $[\Gamma]=\{[A]: A \in \Gamma\}$. The two settings change the relevance of classical entailments not significantly, but "a little bit", because L-equivalent sentences express the same proposition. Since the same classical proposition can be formulated in redundant ways, and redundancies are a form of irrelevance, the only way to obtain a non-ambiguous notion of relevant propositional entailment is to assume that premises and conclusions are formulated non-redundantly, i.e., are replaced by their "relevant representations". We did this in Sect. 3 when we applied relevant entailment to explanations or confirmations. On the other hand, with a more fine-grained *hyperintensional* notion of proposition – such as the notion of an "articulated proposition" developed in Schurz (1991b) – one can obtain a unique correspondence between sentences and propositions that mirrors all sorts of sentential irrelevance at the level of propositions.

We conclude this section with a brief comparison of our approach to relevance with other possible approaches. Generally speaking, "relevance" is an unsharp natural language concept that can be explicated in different ways. So we don't think that there is only one 'true' concept of relevance. The important question is, rather, which *explication* of relevance is most adequate. We think that if one aims at general explication of relevance that works out what relevance is in itself, then the explication of relevance as difference making is most appropriate. This diagnosis is supported by four facts:

(1) The idea of relevance as difference making is well entrenched both in common sense and in philosophical tradition.
(2) The idea has fruitful applications is many domains. As worked out in Sect. 3, difference-making accounts of relevance have been successfully employed to solve problems in the logical reconstruction of philosophically important concepts such as explanation, confirmation, verisimilitude, or is-ought inference. The following sections will present similarly fruitful applications in the areas of probability and confirmation, nomological and causal implication, communication, grounding and essentiality. In all these domains, the explication of core concepts utilizes the replacement criterion of relevance.
(3) The general explication of relevance proposed by our account *unifies* these domain-specific applications of difference-making relevance. This unification is more than a mere analogy or a pattern. It leads to a generalized theory of relevance that explicates precisely what is common to all domain-specific accounts of relevance: there is a domain-specific relation, and important applications of this relation require the relation instances to be relevant in the sense of difference making.

(4) The adequacy of the proposed explication of relevance is further corroborated by two facts. (4a) Many of the domain-specific accounts of difference-making relevance have been developed and successfully employed *independently* and often in ignorance from each other. So the general notion is not ad hoc; it rather offers an explanation *why* these domain-specific accounts were fruitful – because they instantiated the idea of relevance as difference making. Moreover (4b), the proposed approach predicts successfully that this idea can be fruitfully applied to further domains. In fact, some of the applications mentioned in the remaining sections are new, e.g., the application to modal aspects of the predication relation that yields metaphysical core concepts such as essential properties and individual essences.

For these reasons we think that the proposed explication of relevance as difference-making is highly successful, according to the Carnapian success criteria of explications (Carnap, 1950, 7), namely (i) similarity with the pre-scientific intuitions, (ii) logical clarity and systematicity, (iii) fruitfulness and (iv) simplicity, which includes unification in the sense that one simple principle unifies manifold applications in different domains.

All this does not imply the impossibility of alternative approaches to relevance that explicate relevance as a domain-related but still sufficiently general relation, such as relevant entailment or relevant grounding, rather than a property of arbitrary relation instances as in our approach. However, as mentioned in Sect. 4, there is a *conflict* between the logical principles required for a domain-specific relation **R**, typically certain efficiency or closure principles, and the application of the replacement criterion to this relation. This application *filters* out the relevant instances of **R**, leading to a proper subrelation $\mathbf{R}_{\mathrm{rel}} \subset \mathbf{R}$ that typically does *no longer* fulfill the required logical principles. For example, in the area of relevant entailment we have seen that the logical properties of substitutive and transitive closure of inferences are violated by the subset of relevant entailments. Similar conflicts arise also in other domains, for example in the application to the relation of grounding (Sect. 8).

The conflict between logical principles required for a relation **R** and their violation by the relevance-restricted subrelation $\mathbf{R}_{\mathrm{rel}}$ is a further reason that speaks for the proposed approach that separates relevance as a higher-order property of **R**-instances from the relation itself. This does not mean that it is not reasonable to search for a subrelation **R*** of **R** that *does* satisfy the required logical principles and is *as relevant as possible*. Different measures of this sort have been presented in the discussion of relevance logics in Sect. 4, for example, Smiley's (1959) proposal to obtain substitutive closure by accepting all substitution instances of relevant inferences. By these measures one has to accept irrelevant substitution instances as an unavoidable cost. Nevertheless, the enterprise of searching for a "logic of relevant **R**s" (where **R** may be entailment, grounding, or whatever) constitutes a highly sensible *complementary* approach to the approach of this paper, that strives for an optimal *compromise* between logico-structural principles and relevance requirements. In this sense, Fine (2017, 648) calls his notion of a regular proposition a "compromise between … exact relevance … and … monotonicity". On the other hand, the supplementation of these domain-related accounts by an account of relevance in general, as developed in this paper, is of obvious importance – not only because it extracts relevance a unifying property, but also because it can explain why in these domain-related accounts irrelevant relation instances have to be accepted, in order to satisfy the logical principles required for **R**.

6 Probabilistic relevance

To keep things simple, we will first develop the notion of relevance for the concept of *objective-statistical* probability, denoted by lower-case p(C|A), that applies to open formulas A, C of a predicate language expressing types of events. Precisely the same notion of relevance can be applied to the notion of subjective-epistemic probability, denoted by upper-case P(C|A), that applies to propositions C, A and will be considered in the last part of this section.

In the case of probabilistic relevance, the underlying relation **R** between event types A∈**A** and C∈**B** is an assertion about the conditional probability p(C|A), for example that it has a certain value (p(C|A) = v). In what follows A denotes the *antecedent* event and C the *consequent* event of the conditional probability. We assume that both formulas are monadic and the individual variable x is suppressed, i.e. "A" stands short for "Ax" (a generalization to relational properties is possible but omitted). To grant that conditional probabilities are definable in the standard way (p(C|A) = p(C∧A)/p(A)), we assume that the prior probability of all consistent formulas is positive. We first turn to antecedent-relevance, abbreviated as A-*relevance*; component-wise A-relevance and C-relevance (consequent-relevance) are introduced below. (We use the intuitive terms A- and C-relevance; in our general terminology the former is **A**-relevance and the latter **B**-relevance).

The difference-making criterion defines a conditional probability as A-relevant iff conditionalization on A *changes* the probability value of C, i.e., iff p(C|A) ≠ p(C) holds. The conditional probability relation is *positively* resp. *negatively* A-relevant if this change is positive (p(C|A) > p(C)) respectively negative (p(C|A) < p(C)). If A is probabilistically relevant for C, one also says that C and A are probabilistically dependent or correlated; otherwise C and A are independent or uncorrelated.

In this formulation, the underlying replacement operation is that of *elimination*, or replacement by a tautology (⊤). An equivalent replacement operation is the replacement by *negation*, because of the following fact that is implied by the law of mixed probabilities (p(C) = p(C|A)·p(A) + p(C|¬A)·p(¬A)), given that p(A) ∉ {0,1}:

(27) p(C|A) = p(C) iff p(C|A) = p(C|¬A).

It is no longer adequate, however, to replace A by arbitrary other formulas X, because different from deductive inferences, conditional probabilities are *non-monotonic*, i.e., a high p(C|A) does not entail anything about the value of p(C|A∧X) (while A ⊢ C implies A∧X ⊢ C). Therefore p(C|A) = p(C) = p(C|¬A) is compatible with the existence of other properties X that increase or decrease C's probability, p(C|X) ≠ p(C). In conclusion, the adequate notion of relevance for conditional probabilities is that of *essential* relevance, defined by replacements by empty or opposite elements. Minimal essential antecedent-relevance is defined as follows:

(28) (Definition.)

(28.1) The *conditional probability* $p(C|A)$ is minimally *essentially* antecedent-relevant, in short A-relevant, iff $p(C|A) \neq p(C) = p(C|\top)$, iff $p(C|A) \neq p(C|\neg A)$ (so $\top$ is the empty element and $\neg A$ the opposite element).
In other words, it is not possible to replace A in $p(C|A)$ by $\top$ or by $\neg A$ *salva probabilitate*, i.e. under preservation of the value of $p(C|A)$.

(28.2) $p(C|A)$ is *positively/negatively* A-relevant iff $p(C|A) > / < p(C)$, iff $p(C|A) > / < p(C|\neg A)$.

The condition of probabilistic A-relevance was proposed by Salmon (1971) as a necessary correction of Hempel's inductive-statistical (is) model of explanation. An is-explanation is a probabilistic argument "$p(E|A) = r$, Aa $//_r$ Ea" where (i) "$p(E|A) = r$" is a true or accepted probabilistic law, (ii) the value r is sufficiently high (where r is transferred to the single case as an "inductive" (epistemic) probability, as indicated by "$//_r$"), and (iii) the antecedens Aa is maximally specific for the explanandum Ea. In Hempel's is-model (1965, 381ff) the statistical probability $p(E|A)$ was required to be high. Salmon objected against Hempel's is-model that a high conditional probability is not enough: A must be probabilistically relevant to E. One of Salmon's examples was the discovery that the probability of recovering from a cold within seven days (=C), given that one takes high doses of vitamin C during the cold (=A), is very high. But later it was discovered that A is probabilistically irrelevant to C, $p(C|A) = p(C)$ (Salmon, 1971, 83, fn. 20). Since probabilistic relevance is a necessary condition for *causal* relevance, the researchers' original recommendation that people should regularly take high doses of vitamin C was withdrawn.

Many authors on probabilistic explanations agreed with Salmon's relevance condition, but they required that probabilistic explanations have to be *positively* relevant, i.e. $p(C|A) > p(C)$ (e.g., Gärdenfors, 1980). In contrast, Salmon argued that even factors that are *negatively* relevant must be mentioned in a probabilistic explanation (cf. Schurz, 1996, sec. 6). More precisely, Salmon required that the antecedent A should be a conjunction of all and only those causal factors that are probabilistically relevant to E (Salmon, 1971, 63; 1984, 37). Thus, Salmon's model assumes the antecedent A to consist of a conjunction of factors $A_1 \wedge \ldots \wedge A_n$, in which each factor is relevant. To explicate this stronger condition we need to generalize the notion of essential A-relevance to *component-wise* (essential) A-relevance, as follows[13]:

[13] The causal aspect of Salmon's condition—that each relevant factor A_i must be a cause, as opposed to an effect or co-effect—goes beyond statistical relevance.

(29) (Definition.)

Let A be a conjunction $A_1 \wedge \ldots \wedge A_n$, and let $A[A_i/\neg A_i]$ resp. $A[A_i/\top]$ denote the result of replacing the conjunct A_i in A by ¬A resp. by ⊤. Then:

(29.1) The conditional probability p(C|A) is *component-wise essentially* antecedent-relevant, in short A_{comp}-relevant, iff for every A_i: $p(C|A) \neq p(C|A[A_i|\top])$, iff for every A_i: $p(C|A) \neq p(C|A[A_i/\neg A_i])$.

Thus, it is not possible to replace in p(C|A) an elementary conjunct A_i of A by ⊤ or by $\neg A_i$ *salva probabilitate*.

(29.2) p(C|A) is positively/negatively A-relevant w.r.t. the conjunct A_i iff $p(C|A) > / < p(C|A[A_i|\top])$, iff $p(C|A) > / < p(C|A[A_i/\neg A_i])$.

In statistics, the A-relevance of p(C|A) w.r.t. an antecedent-factor A_i is also called a conditional or partial correlation between A_i and C.

Component-wise probabilistic A-relevance is restricted to conjunctive components. The generalization of this condition to arbitrary subformulas of A would be too strong, because due to probabilistic non-monotonicity, $p(C|A \vee B) = p(C|A \vee \neg B)$ does not exclude that p(C|A) is different from $p(C|A \vee B)$ and $p(C|A \vee \neg B)$.

We next turn to probabilistic consequent-relevance, or C-relevance. In probabilistic C-relevance the consequent is replaced, which leads to a condition very different from A-relevance. Replacing C by its negation leads to the natural explication of C-relevance as $p(C|A) \neq p(\neg C|A)$, which is equivalent with $p(C|A) \neq 1/2$, i.e. C's conditional probability is different from 1/2. In contrast, replacing C by ⊤ or ⊥, resp., does not make sense because it would lead to the conditions $p(C|A) \neq 1 = p(\top|A)$ or $p(C|A) \neq 0 = p(\bot|A)$, resp., which is implausible. Thus, we define:

(30) (Definition.)

(30.1) The conditional probability p(C|A) is minimally essentially consequent-relevant, in short C-relevant, iff $p(C|A) \neq p(\neg C|A)$, iff $p(C|A) \neq 1/2$.

(30.2) p(C|A) is positively/negatively C-relevant iff $p(C|A) > / < p(\neg C|A)$, iff $p(C|A) > / < 1/2$.

In Hempel's is-model, $p(C|A) > 1/2$ is granted by the requirement of high probability. Later accounts (e.g., Tuomela, 1981, 276) weakened Hempel's high-probability requirement to the condition $p(C|A) > 1/2$, i.e. positive probabilistic C-relevance of A.

The application of C-relevance to conjunctive components of $C = C_1 \wedge \ldots \wedge C_n$ is problematic, because in this case the equivalence with possessing a probability value different from 1/2 breaks down. What makes good sense, however, is to apply the notion of probabilistic A-relevance to every conjunct of a conjunctive consequent, as follows:

(31) (Definition.) The conditional probability $p(C_1 \wedge \ldots \wedge C_n | A)$ is A-relevant to all C-conjuncts, in short A-C_i-relevant, iff for all C_i the conditional probability $p(C_i|A)$ is A-relevant, i.e. $p(C_i|A) \neq p(C_i)$ resp. $p(C_i|A) \neq p(C_i|\neg A)$ holds.

Probabilistic relevance is of vital importance for another central concept in philosophy of science: *probabilistic* confirmation. The standard Bayesian account of confirmation can be expressed in terms of difference-making positive relevance as follows:

(32) A piece of evidence E *confirms* a hypothesis H iff $P(H|E) > P(H)$, i.e., the conditional probability $P(H|E)$ is positively E-relevant (in the sense of def. 28.2, where the evidence E is identified with the antecedent A).

The two major paradoxes of confirmation mentioned for hd-confirmation, tacking by conjunction and irrelevant disjunctive weakening, obtain for probabilistic confirmation as well. Different strategies of avoiding these problems have been discussed in the literature (Fitelson, 2002). Schippers and Schurz (2020) propose to solve the problem of tacking by conjunction by decomposing the hypothesis H into a conjunction of content elements and requiring Bayesian confirmation for each content element of H. This amounts to a version of the probabilistic relevance condition (31). Schippers and Schurz (2020) call this type of confirmation *genuine* confirmation and demonstrate its advantages.

Disjunctive weakenings of the evidence can be avoided by requiring E to be represented by a conjunction of content elements, since if E is a basic evidence, then the disjunction $E \vee X$ is not a content element of E. Note that it would be too strong to require positive relevance for all content elements of E, since E is supposed to comprise all relevant evidence, and it may be that some parts of the total evidence are negatively relevant to H, although their totality is positively relevant. This leads to the following definition of relevant probabilistic confirmation:

(33) (Definition.) An evidence E *relevantly confirms* a hypothesis H iff there is a relevant representation H_r of H and E_r of E (in the sense of def. 15.2) satisfying the condition that the conditional probability $P(H_r|E_r)$ is H_i-E_r-relevant in the sense of def. 31, i.e., for every conjunct H_i of H_r it holds that $P(H_i|E_r) > P(H_i)$.

7 Relevance of strictly universal laws

In this application, the domain **A** consists of conjunctions of basic (i.e. unnegated or negated) monadic predicate formulas, abbreviated as $A_i x$, and the co-domain **B** consist of basic formulas Cx. The basic formulas express types of events, and the relation $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$ expresses a *nomological implication*, i.e. a universal and lawlike implication of the form $\forall x(Ax \rightarrow Cx)$, where $Ax = A_1x \wedge \ldots \wedge A_nx$. Again we refer to Ax as the antecedent and Cx as the consequent predicate. As pointed out by John

Stuart Mill (1865, book III, ch. viii), in order to count as a *law* a universal implications must not only be true, but it must also be component-wise antecedent-relevant, in the sense that no one of its conjuncts is eliminable or arbitrarily replaceable:

(34) (Definition.) A *true nomological implication* $\forall x(A_1x\wedge\ldots\wedge A_nx \rightarrow Cx)$ is component-wise antecedent-relevant (A-relevant) iff no conjunctive component A_ix of Ax can be replaced by any other formula Xx *salva veritate*.
Equivalently: … iff no conjunctive component A_ix of Ax can be eliminated or replaced by a tautology, *salva veritate*.

Since strict implications are governed by deductive relations, eliminative replacements and replacements by arbitrary subformulas are equivalent. Salmon (1971, 34) illustrated true but irrelevant nomological implications by the following example:

(35) Males <u>who take birth control pills</u> will never get pregnant.

The underlined antecedent factor is A-irrelevant and, thus, cannot play a causal role for the consequent.

Since Ax is a sufficient but unnecessary condition for the effect Cx (assuming that Cx may be caused in different ways), definition (34) may be regarded as an explication of Mackie's account of a cause as an INUS condition (Mackie, 1975): each causal factor A_ix of an A-relevant causal implication $\forall x(A_1x\wedge\ldots\wedge A_nx \rightarrow Cx)$ is an insufficient but necessary part of an unnecessary but sufficient condition for Cx.

A generalization of the relevance condition for nomological implications to consequent-relevance is possible if one considers consequents consisting of disjunctions of events, $Cx = C_1x\vee \ldots \vee C_nx$; we omit this consideration.

8 Grounding and relevance

In recent years the concept of logical grounding has been developed, based on the idea that ground-theoretic entailments between statements (or propositions) should be based on containment relations between the facts that verify these statements, so-called verifiers or truthmakers. Different theories of grounding have been put forward. In what follows we focus on the account of Fine (2012, 2017) because it is most closely related to matters of p(remise)- and c(onclusion)-relevance.[14] The verifying 'facts' are understood as *possible* facts, so-called *states* (Fine, 2017, 627f.). The basic notion is the relation of verification between sets of states (denoted by small letters) and propositions (denoted by capital letters). Small letters p, q, … denote *primitive* states, a, b … denote arbitrary states and α, β, … denote *sets* of states. Recall that P, Q,… stand for atomic sentences; A, B, … for arbitrary

[14] Correira's (2014) account captures p-relevance (exact entailment) but not c-relevance (containment); Schnieder's (2021) account captures a version of containment, but not exact entailment.

sentences and Γ, Δ, … for sets of sentences. In what follows we assume that each primitive sentence P has a unique verifier p; this assumption is not made in Fine's general account but it simplifies the picture and doesn't affect matters of relevance. [A] denotes the *verification set* of A, i.e. the set of all verifiers of A; it is interpreted as A's content or relevant proposition. By our assumption [P] = p, [Q] = q, etc. The *fusion* of two states a, b is here denoted as a·b. More generally, Π(α) is the fusion of a set of states $\alpha = \{a_1, a_2, \ldots\}$ (ibid., 646). The fusion operation mirrors conjunction at the ontological level, thus [P∧Q] = {p·q}. Every state a has the form Π(α) for some α. In contrast, a disjunction P∨Q has no ontological correlate; rather, [P∨Q] is defined as the set of states verifying either P or Q or both, i.e., [P∨Q] = {p, q, p·q}. Thus, verification sets are understood as disjunctions of verification possibilities. A state a is called a *part* of b iff a is a subfusion of b, and it is *proper* part if a ≠ b; thus p and q are proper parts of p·q. More generally, for every β ⊆ α, Π(α − β) is a part of Π(α). "$\leq_p$" stands for the relation of part and "$<_p$" for "proper part"; if $a \leq_p b$ we also say that b *contains* a. For matters of logical smoothness, Fine (2017, 628) assumes that relevant propositions [A] are *regular*, i.e. closed under fusions and intermediate parts of their elements (if a, b ∈ [A], then: a·b ∈ [A], and if $a <_p c <_p b$ then c ∈ [A]). Thus, with Clos(α) denoting the regular closure of a set of verifiers α, the recursive clauses for the verification sets of conjunctions and disjunctions are as follows:

(36) (i) [A∧B] = Clos([A]·[B]), where [A]·[B] = {a·b: a∈[A], b∈[B]}.
(ii) [A∨B] = Clos([A] ∪ [B]).

Verifiers are designed to be "exact", i.e. "relevant as a whole"; thus the verification relation ⊨ is non-monotonic (ibid., 626): p·q ⊭ P, and if a ⊨ A and $a <_p b$, then b ⊨ A does not generally hold, though it may hold for some substitution instances. *Negation* is more difficult because in Fine's system there are no straightforward negative facts (which seems reasonable).[15] Yet, we make here the idealizing assumption that each negated primitive fact ¬P has a unique 'negative' verifier $\overline{p}$; this simplifies the picture but doesn't affect matters of relevance.

Based on this verification relation, Fine introduces the two relevant entailment relations mentioned in sec. 4 as follows:

(37) (i) A *exactly entails* B, abbreviated as $A \leq_e B$, iff [A] ⊆ [B], i.e., iff every verifier of A is a verifier of B.
(ii) A *contains* B, abbreviated as $A \leq_c B$, iff ∀a∈[A] ∃b∈ [B]: $a \leq_p b$ and ∀b∈[B] ∃a∈[A]: $a \leq_p b$, i.e., iff every A-verifier contains some B-verifier and every B-verifier is contained in some A-verifier.

Definitions (i) and (ii) are extended to sets of premises Γ (replacing A) by treating them as conjunctions, i.e. if $\Gamma = \{A_1, A_2, \ldots,\}$, then $[\Gamma] = \{a_1 \cdot a_2 \cdot \ldots : a_1 \in [A_1],$

[15] Fine (2017, 632–5) offers two non-equivalent ways of handling negations, bilateral propositions consisting of verifiers and falsifiers, and exclusion relations between states.

$a_2 \in [A_2], \ldots\}$. As remarked in Sect. 4, exact entailments reflect the idea of p(remise)-relevance; thus $P \leq_e P \vee Q$ but not $P \wedge Q \leq_e P$. Containments express the idea of c(onclusion)-relevance, i.e., $P \wedge Q \leq_c P$ but not $P \leq_c P \vee Q$. Fine's system satisfies all standard requirement for a logic (discussed in Sect. 4) in a beautiful way, but on the cost of some remarkable deviations from difference making intuitions of relevance. Concerning the omission of relevant inferences, Fine's system excludes DS (and likewise, MP) as exact entailment as well as containment, since $[\neg P \wedge (P \vee Q)] = \mathrm{Clos}(\{\overline{p}\} \cdot \{p,q,p \cdot q\}) = \{\overline{p} \cdot p,\ \overline{p} \cdot q,\ \overline{p} \cdot p \cdot q\}$ and $[Q] = \{q\}$; so not only condition (37)(i), but also condition (37)(ii) is violated.[16] Concerning irrelevant inferences that have to be accepted, it follows from substitutive closure that Fine's system includes irrelevant substitution instances of EFQ (e.g., $P \wedge \neg P \leq_c \underline{P}$) and of VEQ (e.g., $\underline{P} \leq_e P \vee \neg P$). Moreover, the inference $P \wedge Q \vdash P \vee Q$ counts both as an exact entailment and as a containment, since $[P \wedge Q] = \{p \cdot q\}$ and $\{P \vee Q] = \{p,q,p \cdot q\}$, although this inference is both p-irrelevant and c-irrelevant. If exact entailment were applied to confirmation contexts, this would have the counterintuitive consequence that the fact that all observed animals are carnivores *or* herbivores hd-confirms the hypothesis that all animals are carnivores *and* herbivores; and in explanation contexts it would have the consequence that the hypothesis that all animals are carnivores and herbivores explains why all observed animals are carnivores or herbivores. Another example of a p-irrelevant exact entailment is $\underline{P} \wedge (P \vee Q) <_e P \vee Q$, which holds since $[P \wedge (P \vee Q)] = \{p,\ p \cdot q\} \subset \{p,\ q,\ p \cdot q\} = [P\ Q]$.

Since not all exact entailments are p-relevant, it is interesting to filter out the subset of p-relevant exact entailments, or grounds (a similar project could be pursued for c-relevant containments). This question has been investigated by Krämer and Roski (2017), in regard to the notion of *partial* ground. For simplicity we explain Krämer and Roski's notion (ibid., 1194f.) for *basic* statements, abbreviated as B_i, which are atomic statements P_i or their negations $\neg P_i$ (the notion generalizes to conjunctions of basic statements). We write $[B_i] = b_i$, so b_i is the unique verifier of B_i (p_i or $\overline{p}_i$). A *conjunctive* statement has the form $C = B_1 \wedge \ldots \wedge B_n$ with $[C] =_{def} \{c\}$ and $c = b_1 \cdot \ldots \cdot b_n$. Following Fine (2012) and Krämer and Roski (2017), we call a conjunctive statement C a (weak) *full* ground of a statement A iff C''s verifier c is an A-verifier, $c \in [A]$. For any state $c = \Pi(\alpha)$, $c - b_i$ denotes $\Pi(\alpha - \{b_i\})$, i.e. the elimination of b_i from the fusion-components of c. For example, $p \cdot q - p = q$. $\Pi(\varnothing)$ is the empty state, abbreviated as 0. With these notions, Krämer and Roski's definition (ibid., 1195) is this:

[16] Fine's system excludes DS for the following reasons. The premise conjunction $\neg P \wedge (P \vee Q)$ is analyzed via its verification set by the rule of $\wedge\vee$-distribution as $(\neg P \wedge P) \vee (\neg P \wedge Q)$. Definition (37) implies that the relevance of inferences with disjunctive premises obeys the rule of *case distinction*: $X \vee Y \vdash C$ is relevant (in the sense of $\leq_e$ or $\leq_c$) iff both $X \vdash C$ and $Y \vdash C$ are relevant. But $\neg P \wedge P \vdash Q$ is maximally irrelevant, so DS is neither an instance of $\leq_e$ nor of $\leq_c$. A similar argument against DS can already be found in Anderson and Belnap (1975, 165f.). However, that *some* proof of an inference (here DS) involves irrelevant steps (here EFQ) does not mean that the inference itself is irrelevant, if these irrelevant steps can be avoided. Indeed, the natural reasoning from $(\neg P \wedge P) \vee (\neg P \wedge Q)$ to Q does not proceed by case distinction, but by the law of *contraction*: "$\neg P \wedge P$ is impossible, so the only remaining possibility is $\neg P \wedge Q$, which entails Q". The latter proof does no longer involve EFQ.

(38) A basic statement B is a *partial ground* of A, abbreviated as $B \leq_{part} A$, iff there is a full ground C of A such that b is part of c (where c = [C] and b = [B]); if it holds in addition that c − b is *not* a verifier of A, then B is a *relevant* (or difference-making) partial ground of A, abbreviated as $B \leq_{rel\text{-}part} A$.

For example, P is a relevant partial ground of P∧Q, since [P] = p, [P∧Q] = {p·q}, and p is a part of p·q but p·q − p = q ∉ [P∧Q]. P is a relevant full (and thus partial) ground of P∨Q, since [P] = {p} and there is a verifier of [P∨Q] = {p,q,p·q}, namely p, such that p is a part of p and p − p = 0 ∉ [P∨Q].

As Krämer and Roski show, while the partial ground relation is transitive, that of a relevant partial ground is no longer transitive. They demonstrate this fact by the following counterexample (where "$B \leq_{irrel\text{-}part} A$" stands short for B being an irrelevant partial ground of A):

(39) (i) $P \leq_{rel\text{-}part} P \wedge Q$, since [P] = {p}, [P∧Q] = {p·q}, p is a part of p·q, and (p·q − p) = q ∉ [P∧Q].
(ii) $P \wedge Q \leq_{rel\text{-}part} Q \vee (P \wedge Q)$, since [P∧Q] = {p·q}, [Q∨(P∧Q)] = {q, p·q} and p·q is an (improper) part of p·q while p·q − p·q = 0 ∉ [Q∨(P∧Q)].
(iii) However, $P \leq_{irel\text{-}part} Q \vee (P \wedge Q)$, since [P] = {p} and the only verifier in [Q∨(P∧Q)] = {q, p·q} that contains p is p·q, but p·q − p = q ∈ [Q∨(P∧Q)], so p is irrelevant.

Krämer and Roski's counterexample conveys further support to the general observation of Sect. 4, that the application of a difference-making criterion of relevance to a particular relation, here the relation of partial ground, will often fail to satisfy the logico-structural laws governing the relation. In this line, Krämer and Roski discuss extensively whether or not the notion of relevant partial ground should still be regarded as a form of (partial) grounding, in spite of its intransitivity. They conclude that relevant grounding should better be considered as a kind of "good grounding" rather than a relation of grounding (which should be transitive). This conclusion fits with the strategy of this paper that distinguishes matters concerning the truth of a relation instance from matters concerning its relevance.

9 Relevant communication

In a simplified reconstruction of the communication theory of Sperber and Wilson (1996), the underlying relation $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$ is the relation of *communication* that holds between the utterance of a proposition P by a speaker that is received by a hearer, and a context C consisting in the background beliefs shared by speaker and hearer. Thus **A** is the domain of speaker-hearer-directed utterances and **B** the co-domain of possible shared background beliefs. Sperber and Wilson discuss a variety of conditions for the relevance of an uttered proposition in a context. One of their major conditions is reconstructed in this section, namely that the utterance of P must have relevant contextual implications in the following sense: the information I that is acquired by the hearer because of the speaker's utterance of P is entailed by P and C, but neither by P nor by C alone (cf. Sperber & Wilson, 1996, 107f.). Thus, both P and C make a difference for the information acquired by the hearer. For example, if the context C contains Jim's previous remark to his spouse Mary "I feel sick" as well as the shared knowledge "if a person is sick and the doctor is called, then the doctor will take steps to cure the person", then Mary's reply "I'll call the doctor" (P) is relevant in this context, because it causes Jim to acquire the new information "the doctor will take steps to cure me" (which is inferable from P and C but not from one of the two alone). On the other hand, if Mary would reply "The phone rings", then this would be an obvious failure of communicative relevance.

With this background, Sperber and Wilson's notion of communicative relevance can be explicated as follows:

(40) (Definition.) Let P be a proposition uttered by a speaker towards a hearer and C be the conjunction of all beliefs (propositions) shared by speaker and hearer in a given communication context. Then:
The utterance of P in C is a *relevant communication* iff it causes the hearer to acquire an elementary propositional information I such that $P, C \vdash I$ is a premise-relevant entailment, i.e. {P,C} but neither P nor C alone entail I.

The requirement of the *elementariness* of the acquired information I (in the sense of def. 15.1) is a proposed improvement of Sperber and Wilson's account. Without this requirement the definition would get trivialized by the fact that for every P and C there is at least one 'information' that is neither entailed by P nor by C, namely the *conjunction* $P \wedge C$ (cf. also the critique of Woods, 1992). Sperber and Wilson (1996, 97) are aware of this danger of trivialization and try to avoid it by restricting logical inferences to *elimination* rules (that eliminate but don't introduce logical connectives), which excludes the rule of conjunction. But this restriction seems too strong, because it excludes too many relevant inferences, e.g. contraposition $A \rightarrow B / \neg B \rightarrow \neg A$ and others more. A better way to solve the problem is to formulate the relevance criterion for an elementary piece of information I, as in definition (40).

10 Relevant predication and essentiality

Our final application is the notion of *essential properties*. Here the relation $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{B}$ is that of predication between a domain **A** of (assumedly monadic) properties and a co-domain **B** of individuals. A property F of an individual a is considered as (metaphysically) essential to a iff possessing the property F makes a difference to a's self-identity (cf., e.g., Fine, 1994). This is equivalent with saying that the individual a can impossibly loose the property F, because this would turn it into a different individual. In this way, the notion of an essential property can be regarded as a kind of difference-making relevance condition. In the domain of metaphysics one is not so much interested in factual truths, but in the modal status of these truths. Therefore metaphysical relevance considers replacements *salva possibilitate*. The precise definition of an essential property in terms of metaphysical property-relevance is given in definition (41.1) below.

The corresponding notion of metaphysical individual-relevance asserts that the property F can impossibly apply to an individual different from a. This notion, too, gives us an important metaphysical notion, namely that of F as containing an *individual essence* of a (Plantinga, 1974, 70), i.e. a conjunction of properties of a that uniquely differentiates a from all other individuals. Here are the formal definitions:

(41) (Definition.) Let Fa (for "a is F") be a true predication. Then:

(41.1) F is *essential property* of a iff the predication Fa is essentially *property-relevant* in the following sense: F cannot be replaced in Fa by ¬F *salva possibilitate*, i.e. ¬Fa is metaphysically impossible.

(41.2) F contains an *individual essence* of a iff the predication Fa is *individual-relevant* in the following sense: a cannot be replaced in Fa by a different individual b salva possibilitate, i.e. Fb is metaphysically impossible.

11 Conclusion

In this paper we studied the account of relevance as difference-making from a generalized perspective. We argued that relevance should not be considered as a particular relation between certain types of entities, but as a (higher-order) property of instances of arbitrary first order relations, namely the property that variations of the relata of the relation instance make a difference for its truth. We showed that this general account of relevance can be fruitfully applied in a variety of domains, such as (i) logico-deductive reasoning with applications to deductive-nomological explanation, hypothetico-deductive confirmation, verisimilitude and is-ought inference, (ii) probabilistic reasoning with applications to probabilistic explanation and confirmation, (iii) nomological and causal implication, (iv) grounding, (v) communication and (vi) metaphysical notions of essentiality. In all these domains we found applications of difference-making relevance that are precise instantiations of the general account of relevance explicated in Sect. 5. Many more applications are possible that could not be included in this paper. We conclude that the notion of difference-making relevance is a highly unifying and fruitful philosophical concept with applications in virtually all fields.

12 Appendix

If material implications (X→Y) are replaced by equivalent disjunctions (¬X∨Y), then a subformula occurrence X in a formula is defined as *positive/negative* iff it occurs in the nested scopes of an even/odd number of negation symbols (for a recursive definition see Kleene, 1967, 124). Replacing a positive/negative subformula occurrence X in A by a logically stronger formula B strengthens/weakens the formula A, *respectively* (i.e., if positive it strengthens it and if negative it weakens it). A positive/negative occurrence of a subformula of a *premise* is defined as *empty* if it is ⊤/⊥, respectively. Dually, a positive/negative occurrence of a subformula of the *conclusion* is empty if it is ⊥/⊤, respectively. The reason for this definition is that the replacement of a subformula in a premise by an empty formula should make the premise *weaker*, and its replacement in the conclusion should make the conclusion *stronger*.

Theorem Assume Γ ⊢ A is propositionally valid. Let X be a *single* occurrence of a subformula either in Γ or in A. Let (Γ ⊢ A)[X|Y] be the result of replacing this X-occurrence in Γ ⊢ A by the formula Y, and let (Γ ⊢ A)[X|0] be the result of replacing this X-occurrence by an empty formula 0. Then the following three conditions are equivalent:
(i) ∀Y: (Γ ⊢ A)[X|Y], (ii) (Γ ⊢ A)[X|¬X] and (iii) (Γ ⊢ A)[X|0].

Proof We write Γ[X|Y] ⊢ A[X|Y] for the inference (Γ ⊢ A)[X|Y]; note that because "X" denotes a *single* X-occurrence, either in A or in Γ nothing gets replaced by "[X|Y]". The proof proceeds by a circular chain of implications (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (ii): This implication is obtained by letting Y = ¬X.

For (ii) ⇒ (iii): Let v be an arbitrary truth valuation. We write v(Γ) = true if v(B) = true for all premises B in Γ; otherwise v(Γ) = false. Two cases are possible, each dividing into two subcases:

(Case 1:) v(X) = true and 0 = ⊤, or v(X) = false and 0 = ⊥. In both subcases v(0) agrees with v(X). Therefore, v(Γ[X|0]) agrees with v(Γ) and v(A[X|0]) with v(A). Since Γ ⊢ A is valid, the condition "if v(Γ) = true, then v(A) = true" holds. Thus, the condition "if v(Γ[X|0]) = true, then v(A[X|0]) = true" holds, too (for arbitrary v); so condition (iii) holds.

(Case 2:) v(X) = true and 0 = ⊥, or v(X) = false and 0 = ⊤. In both subcases v(0) agrees with v(¬X). Therefore, v(Γ[X|0]) agrees with v(Γ[X|¬X]) and v(A[X|0]) with v(A)[X|¬X]). By condition (ii), Γ[X|¬X] ⊢ A[X|¬X] is valid; so the condition "if v(Γ[X|¬X]) = true, then v(A[X|¬X]) = true" holds. Thus, the condition "if v(Γ[X|0]) = true, then v(A[X|0]) = true" holds, too, whence condition (iii) holds.

For (iii) ⇒ (i): We have two cases. (Case 1:) X occurs in Γ and not in A. Then by the definition of an empty formula, Γ[X|0] is logically weaker than Γ[X|Y]. So Γ[X|Y] ⊢ Γ [X|0] ⊢ A (by (iii)), which proves condition (i). (Case 2:) X occurs in A and not in Γ. Then by the definition of an empty formula, A[X|0] is logically stronger than A[X|Y]; so Γ ⊢ A[X|0] (by (iii)) ⊢ A[X|Y], which proves condition (i). Q.E.D.

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