On the Representations of Quasi-semisimple Profinite Groups, Compact p-adic Analytic Groups, and Baumslag-Solitar Groups

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Margherita Piccolo
aus Padova, Italien

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1. Prof. Dr. Benjamin Klopsch

2. Prof. Dr. Britta Späth

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Abstract

This dissertation is cumulative; it consists of three parts that deal with representation growth of infinite groups. We study the asymptotic behaviour of representations in three different contexts. Namely, for quasi-semisimple profinite groups, compact p-adic analytic groups, and Baumslag-Solitar groups.

In the first part of the thesis, we give some preliminaries that will help to understand the subject of the following chapters.

In the second part of the thesis, we examine quasi-semisimple profinite groups. Notably, we prove that every positive real number occurs as the abscissa of convergence for a suitable group within this class.

The third part of the thesis focuses on the representation zeta functions of principal congruence subgroups and certain extensions of the group $\mathrm{SL}_2^m(\mathcal{O})$, the m-th principal congruence subgroup of the special linear group of degree 2 over a compact discrete valuation ring $\mathcal{O}$ of characteristic 0 and residue characteristic p, for permissible m. We show that the considered zeta functions have the zeta function of $\mathrm{SL}_2^m(\mathcal{O})$ as a factor. This research has been conducted in collaboration with Moritz Petschick.

In the fourth and final part, we address Baumslag-Solitar groups, investigating their absolutely irreducible representations over finite fields. The primary outcome is the enumeration and description of absolutely irreducible representations of dimension n for a Baumslag-Solitar group $\mathrm{BS}(x, y)$ with coprime parameters. This work has been conducted in collaboration with Iker de las Heras.

Abstract

Questa dissertazione è cumulativa; consiste di tre parti che trattano della crescita delle rappresentazioni dei gruppi infiniti. Studiamo il comportamento asintotico delle rappresentazioni in tre contesti diversi. In particolare, per i gruppi profiniti quasi-semisemplici, i gruppi analitici p-adici compatti e i gruppi di Baumslag-Solitar.

Nella prima parte della tesi, forniamo alcune nozioni preliminari che aiuteranno a comprendere l'argomento dei capitoli successivi.

Nella seconda parte della tesi, esaminiamo i gruppi profiniti quasi-semisemplici. In particolare, dimostriamo che ogni numero reale positivo si verifica come ascissa di convergenza per un gruppo adeguato all'interno di questa classe.

La terza parte della tesi si concentra sulle funzioni zeta di rappresentazione dei sottogruppi di congruenza principali e di alcune estensioni del gruppo $\mathrm{SL}_2^m(\mathcal{O})$, il m-esimo sottogruppo di congruenza principale del gruppo speciale lineare di grado 2 su un anello compatto $\mathcal{O}$ di valutazione discreta di caratteristica 0 e caratteristica residua p, per valori ammissibili di m. Dimostriamo che le funzioni zeta considerate hanno la funzione zeta di $\mathrm{SL}_2^m(\mathcal{O})$ come fattore. Questa lavoro è stato condotto in collaborazione con Moritz Petschick.

Nella quarta e ultima parte, affrontiamo i gruppi di Baumslag-Solitar, investigando le loro rappresentazioni assolutamente irriducibili su campi finiti. Il risultato principale è l'enumerazione e la descrizione delle rappresentazioni assolutamente irriducibili di dimensione n per un gruppo di Baumslag-Solitar $\mathrm{BS}(x, y)$ con parametri coprimi. Questo lavoro è stato condotto in collaborazione con Iker de las Heras.

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Isaia, per essere la mia Thelma, per condividere gioie e dolori di questo dottorato, per opporti all'uso di LaTeX ma aiutandomi, quando non funzionava, a trovare una soluzione, per essere il mio compagno di viaggio preferito (anche se non vuoi leggere le guide) e per esserci sempre.

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Grazie.

Contents

. . . la mathématique, de ce point de vue, n'est pas autre chose qu'un art; une espèce de sculpture dans une matière extrêmement dure et résistance (comme certains porphyres employés parfois, je crois, par les sculpteurs).

André Weil

Tu parles d'art et de matière dure; mais je ne puis concevoir en quoi consiste cette matière. Les arts proprement dits ont une matière qui existe au sens physique du mot. La poésie même a pour matière le langage regardé comme ensemble de sons. La matière de l'art mathématique est une métaphore; et à quoi répond cette métaphore?

. . . la matière de ton travail, ne serait-ce pas l'ensemble des travaux mathématiques antérieurs, avec le langage et le système de signes qui en résulte?

Simone Weil

Introduction

This dissertation explores the representation growth of infinite groups, focusing on quasi-semisimple profinite groups, compact p-adic analytic groups, and Baumslag-Solitar groups. It consists of three distinct parts, each examining the asymptotic behaviour of representations within these varied contexts. We begin the introduction with an ample overview over different concepts and themes that enter and inform the more specific research problems which we are about to discuss. In this way we establish in a straight and practical way a basic framework for our work. Following this, we detail our specific contributions within this wider context.

General concepts and themes

> Roughly speaking, asymptotic group theory can be thought of as groups viewed from some distance. The finer details disappear, and the rough lines become the main focus. If in group theory one often aims at full classification (say, classifying finite simple groups or doubly transitive permutation groups), then in asymptotic group theory we would be happy with a classification up to finitely many (unspecified) exceptions. If in group theory one often likes to compute certain invariants, in asymptotic group theory we would be happy with finding out the asymptotic behavior of these invariants. If in group theory one often studies a single finite group, in asymptotic group theory we often study an infinite family of finite groups or sometimes the set of finite quotients of some given infinite group.

Aner Shalev, April 2001 [115]

Profinite groups

A profinite group is formed by considering an inverse limit of a suitably coherent collection of finite groups. From one perspective, investigating a profinite group is equivalent to studying an infinite family of finite groups. However, a profinite group is also a compact topological group, and this compactness simplifies the handling of infinite number of problems by bringing them, with its usual magic, to a more tractable scale.

The concept of profinite groups was introduced into number theory in the early 20^{th} century. Initially, the group of p-adic integers $\mathbb{Z}_p$ was utilized as a tool to study congruences, simplifying the analysis by consolidating infinitely many congruences modulo p^n, where

$n \in \mathbb{N}_*$, with a single equation over $\mathbb{Z}_p$. In this way, a profinite group replaces infinitely many hypotheses concerning various small objects with a single hypothesis about one large object. The large object - the group of p-adic integers in this case - can then be examined using algebraic or arithmetic techniques. This process of "mathematical reification" has a long tradition - as in the construction of the complex numbers - and continues to be a fundamental strategy in modern mathematical research - such as in category theory, where mathematical structures are treated as objects in a category, and relationships between them are treated as morphisms or in algebraic geometry with Grothendieck's introduction of schemes.

Profinite groups gained complexity through Krull's work, revealing that the Galois group of an infinite algebraic Galois extension is naturally a profinite group. This group is a compact topological group, whose structure is fully determined by the finite Galois groups of all its finite Galois subextensions. This realization led to the sophisticated modern formulations of class field theory by Chevalley, Artin, and Tate.

Nowadays, most questions concerning profinite groups posed by group theorists arise in contexts that either connect information about a suitable collection of finite groups forming an inverse system, lead to new results about abstract groups, or treat them as topological groups in their own right. Some relevant resources on this topic are [132], [105], and [111].

Finite simple groups and finite p-groups

The classification of finite simple groups is one of the most significant mathematical achievements of the past century. These groups exhibit a high degree of structure and are relatively infrequent. A rather weak way to express this rarity is to state that, for any given order, there are at most two non-isomorphic simple groups. Since every finite group can be constructed by gluing together simple groups, it is often possible to reduce a group theoretic problem to a question on simple groups. The classification of finite simple groups has revolutionized the study of finite groups, which in turn has led to numerous results concerning infinite groups with various finiteness conditions. One notable class is residually finite groups. Often, questions about these groups can be reduced to the asymptotic properties of their finite images, with profinite groups serving as a natural tool for this analysis.

The classification states that every non-abelian finite simple group falls in one of the following families

(1) Alternating groups of rank at least 5;
(2) Finite simple groups of Lie type;
(3) Sporadic groups.

For an introductory overview of finite simple groups we refer to Wilson [133] and the references therein. We will also give some more information in Section 1.3.

When the classification had a realistic hope to be completed, the main experts on the subject reunited in a conference in Santa Cruz in 1979. Here is an extract from the preface of the proceeding of the conference, cf. [24].

> In the last year or so there have been widespread rumors that group theory is finished, that there is nothing more to be done. It is not so. While it is true that we are tantalizingly close to that pinnacle representing the classification of finite simple groups, one should remember that only by reaching the top can one properly look back and survey the neighbouring territory. It was the task of the Santa Cruz conference not only to describe the tortuous route which brings us so close to the summit of classification, but also to chart out more accessible paths, ones which might someday be open to the general mathematical public.

Santa Cruz conference Geoffrey Mason
Chicago, June 1980

At the other side of the extensive spectrum of all finite groups are the groups known as finite p-groups, where p is a prime number. In general, finite p-groups have numerous normal subgroups. Since all the composition factors of a finite p-group are cyclic and of order p, understanding these factors provides no additional information beyond the group's order. The Polynomial On Residue Classes (PORC) conjecture pertains to the enumeration of these groups. Let $f(n)$ denote the number of groups of order n. In 1969, Higman proved that for any positive integer n, the number of groups of order p^n is bounded by a polynomial in p. Additionally, he formulated his renowned PORC conjecture regarding the form of the function $f(p^n)$, which specifies the number of groups of order p^n. Higman conjectured that for each n, there exists an integer N (depending on n) such that for p in a fixed residue class modulo N, the function $f(p^n)$ is a polynomial in p. The conjecture has been proved correct for $n \leqslant 7$. See Vaughan-Lee's seminal paper [125] for an overview on the subject.

Quasi-semisimple profinite groups and compact p-adic analytic groups

The profinite groups that in some sense can be called "low and wide" and "very non-abelian" are the so called quasi-semisimple profinite groups. Following [86], we say that a profinite group Q is *quasi-semisimple* if it is perfect, that is, Q equals the closure of its derived group, and if Q modulo its center that is $Q/Z(Q)$ is isomorphic to a Cartesian product of finite non-abelian simple groups. A *semisimple* profinite group is a quasi-semisimple profinite group with trivial center.

On the contrary, compact p-adic analytic groups are "tall and thin" and in a certain sense "close to nilpotent". These groups were examined at length by Lazard [78] as the non-archimedean counterparts to Lie groups, i.e. the compact topological groups which are analytic over $\mathbb{Z}_p$. Lazard's significant accomplishment was demonstrating that the class of compact p-adic analytic Lie groups can be characterized in a relatively simple group-theoretic manner, thus resolving the p-adic counterpart of Hilbert's fifth problem. The group-theoretic elements of his research were revisited and reinterpreted in the 1980s by Lubotzky and Mann. By introducing the concept of unifomly powerful pro-p groups, they were able to derive most of the group-theoretic outcomes of Lazard's theory without relying on complex "analytic" methods. This approach, along with further advancements, is detailed in the book [31]. However, the class of uniformly powerful pro-p groups turned out to differ from what Lazard originally intended. Klopsch [73] clarified this by recovering saturable groups as envisioned by Lazard, thereby generalizing the concept of uniformly powerful pro-p groups. Lazard's method focuses on creating a correspondence between saturated pro-p groups and specific $\mathbb{Z}_p$-Lie lattices through the use of exponential and logarithm maps. This correspondence was clarified by Klopsch [73] and further studied by González-Sánchez [40]. We refer to [75, Chapter 1] for a gentle introduction to compact p-adic Lie groups.

S-arithmetic groups and congruence subgroup property

Some compact p-adic analytic groups can be seen as "local" parts of suitable S-arithmetic groups.

Let k be a number field and S be a finite set of places of k including all archimedean ones. Let $\mathcal{O}$ be the ring of integers of k and consider the ring of S-integers $\mathcal{O}_S$ consisting of all the elements of k which have absolute value smaller than or equal to 1 at the places outside S. For instance, if S consists only of archimedean places, then $\mathcal{O}_S$ coincides with $\mathcal{O}$. Let $\mathbf{G}$ be a connected simply connected simple linear algebraic group. In Chapter 1, we will discuss in more details linear algebraic groups. Although Chapter 1 focuses on the linear algebraic groups defined over an algebraically closed field of prime characteristic p, and here we consider fields of characteristic zero, many definitions and a large part of the basic theory are the same or rather similar, irrespective of the underlying characteristic. For our purpose it is convenient to fix an embedding of $\mathbf{G}$ into $\mathbf{GL}_n$. The *principal S-congruence subgroups* of $\mathbf{G}(\mathcal{O}_S)$ are of the form $\ker(\mathbf{G}(\mathcal{O}_S) \to \mathbf{G}(\mathcal{O}_S/\mathfrak{i}))$ for non zero ideals $\mathfrak{i} \trianglelefteq \mathcal{O}_S$, where the *$S$-arithmetic group* $\mathbf{G}(\mathcal{O}_S)$ is the group of $\mathcal{O}_S$-rational points with respect to the chosen embedding of $\mathbf{G}$ into $\mathbf{GL}_n$. It is interesting to compare the profinite completion $\widehat{\mathbf{G}(\mathcal{O}_S)}$ and the so called *congruence completion* $\overline{\mathbf{G}(\mathcal{O}_S)}$ of $\mathbf{G}(\mathcal{O}_S)$, given by the S-congruence subgroups. There exists a short exact sequence

$$1 \to C(\mathbf{G}, S) \to \widehat{\mathbf{G}(\mathcal{O}_S)} \to \overline{\mathbf{G}(\mathcal{O}_S)} \to 1$$

where $C(\mathbf{G}, S)$ is called the congruence kernel. We say that the group $\mathbf{G}$ has the *congruence subgroup property* (CSP) with respect to S is the congruence kernel $C(\mathbf{G}, S)$ is finite. Let $\widehat{\mathcal{O}}_S$ denote the profinite completion of the ring $\mathcal{O}_S$. Then $\widehat{\mathcal{O}}_S = \prod_{\nu \notin S} \mathcal{O}_\nu$, where $\mathcal{O}_\nu$ is the completion of $\mathcal{O}_S$ with respect to the place ν. The Strong Approximation Theorem yields

$$\overline{\mathbf{G}(\mathcal{O}_S)} \cong \prod_{\nu \notin S} \mathbf{G}(\mathcal{O}_\nu).$$

The groups $\mathbf{G}(\mathcal{O}_\nu)$ are prototypes of compact p-adic analytic groups, where p coincides with the residue characteristic of $\mathcal{O}_\nu$. Considering this picture, they are considered as the inverse limit of the "local" part of the global S-algebraic groups $\mathbf{G}(\mathcal{O}_S)$ which we picture as $\mathbf{G}(\mathcal{O}_S / \mathfrak{p}^m)$, where $\mathfrak{p}$ is the prime ideal corresponding to the place ν.

Although the representation theory of S-arithmetic groups yields beautiful results connecting the "global" picture with the "local" one, we will not delve further into the structure of S-arithmetic groups, as our research focuses more on the compact p-adic analytic side. For more references on the congruence subgroup property consult [124] and [103].

Representation and character theory

At its core, representation theory seeks to understand groups by studying their actions on vector spaces. A *representation* of a group G is a homomorphism from G to the group of invertible matrices, $\mathrm{GL}(V)$, where V is a vector space over a field. This allows the abstract elements of G to be represented concretely as matrices, facilitating the application of linear algebraic techniques to group theory.

A fundamental tool in representation theory is the *character* of a (finite dimensional) representation, which is a function that assigns to each group element the trace of the corresponding matrix. The theory of group characters, initiated and developed by Frobenius at the end of the 19^{th} century, has played a crucial role in understanding the structure of finite groups. The collection of characters of all irreducible representations of a finite group forms the *character table*, which encodes significant information about the group's structure. For instance, the number of irreducible characters of degree 1 is equal to the index of the commutator subgroup of the group. Consequently, a group is perfect if and only if the only irreducible character of degree 1 is the trivial character.

The insights provided by advanced methods from character theory have profound implications. They allow mathematicians to classify groups, understand their properties, and explore the symmetries in various mathematical and physical contexts. Representation theory not only provides a powerful framework for addressing problems in pure mathematics, but also finds applications in physics, chemistry, and beyond, where the symmetry and structure of systems play a crucial role.

Character tables of finite groups of Lie type

We give a brief overview of the evolution of the study of charcater tables of finite groups of Lie type by following the detailed account given by Bonnafé [15].

The hystory of charcater tables of groups of Lie type dates back to the beginning of the 20^{th} century when, as early as 1907, Schur [109] and Jordan [67] determined in a uniform way the character tables of the infinite families of general linear groups $\mathrm{GL}_2(q)$ and special linear groups $\mathrm{SL}_2(q)$. In 1951, Steinberg [121] expanded on this by computing the character tables of $\mathrm{GL}_3(q)$ and $\mathrm{GL}_4(q)$ though general constructions of representations of $\mathrm{GL}_n(q)$, now known as unipotents, see [120]. Then, in 1955, Green [49] achieved a significant combinatorial accomplishment by algorithmically determining the character table of $\mathrm{GL}_n(q)$ for all degrees $n \in \mathbb{N}_*$ and all field sizes q. Progress on understanding the special linear group was slower. The character table of $\mathrm{PSL}_3(q)$ was obtained in 1921 by Brinkmann. Lehrer's 1971 thesis provides part of the character table for $\mathrm{SL}_4(q)$. The character table of $\mathrm{SL}_3(q)$ was completed in 1973, and Lehrer's subsequent work addressed the group $\mathrm{SL}_n(q)$ all degrees $n \in \mathbb{N}_*$ and all field sizes q. Although not explicitly stated, his work provided all the necessary information to complete the character table of $\mathrm{SL}_n(q)$ all prime degrees $n \in \mathbb{N}_*$ and all field sizes q.

A seminal paper by Deligne and Lusztig [29] in 1976 marked significant progress in the character theory of finite groups of Lie type. Here, finite groups of Lie type means the groups of fixed points of a connected simple algebraic group $\mathbf{G}$ under a Steinberg endomorphism F. Deligne and Lusztig's approach was inspired by a calculation of Drinfeld showing that the discrete series of $\mathrm{SL}_2(q)$ appears in the l-adic cohomology of the variety defined by the equation $xy^q - yx^q = 1$. It involved the idea and means for using the geometric structure of $\mathbf{G}$ to produce varieties on which the finite group $\mathbf{G}^F$ acts in a way that provides access to all its irreducible characters. Since 1975, building on this approach, Lusztig has produced several thousand pages of densely written research on the subject. He achieved a major breakthrough in 1984 [89] where he successfully parametrized the irreducible characters of $\mathbf{G}^F$ in the spirit of Langlands' program, [18].

Subgroup growth

Before delving into the world of representation growth, let us take a little diversion to talk about subgroup growth, which is somewhat antecedent and an elder sibling to the growth of representations.

For every positive integer $n \in \mathbb{N}_*$, let $a_n(G)$ denote the number of subgroups of index n of a group G. If G is a topological group we count only open subgroups. The subgroup growth of G is determined by the arithmetic function $n \mapsto a_n(G)$. Over the past four decades, efforts to understand and describe the relationship between a group's algebraic structure and its subgroup growth have evolved into a full fledged branch of infinite group theory. A key result in this field is the characterization of finitely generated residually finite groups of polynomial subgroup growth as those that are virtually soluble of finite rank, cf. [79, Theorem 5.1].

A profinite group G is said to have *polynomial subgroup growth* if it has at most n^a open subgroup of finite index n for some $a \in \mathbb{N}$ across all $n \in \mathbb{N}_*$. Profinite groups with polynomial subgroup growth were characterised by Segal and Shalev [112] as groups having closed normal subgroups $S, G_1 \trianglelefteq_c G$ with $S \leqslant G_1$ such that S is prosoluble of finite rank, G/G_1 is finite, and G_1/S is a quasi-semisimple group of bounded type such that the sequence of the orders of the finite simple groups satisfy some combinatorial condition called "gcd", cf. [79, Theorem 10.3]. A profinite quasi-semisimple group Q is said to be of *bounded type* if the sequence of finite non-abelian simple groups that appears in the Cartesian product isomorphic to $Q/Z(Q)$ is a sequence of finite simple groups of Lie type of bounded rank, each occurring with bounded multiplicity. For example the Cartesian product $\prod_{i=1}^{\infty} \mathrm{PSL}_2(p_i)$ for a strictly increasing sequence of primes p_i.

On the other hand, the finitely generated pro-p groups with polynomial subgroup growth are precisely the pro-p groups of finite rank, i.e. the p-adic analytic pro-p groups, cf. [82, Theorem B].

As a reference for an exposition of subgroup growth see the monograph [86].

Dirichlet generating functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display", cf. [131]. With this idea in mind, Dirichlet generating functions are the perfect tools to study the growth of arithmetic sequences. Some of the pioneers of the application of Dirichlet generating function on the study of groups are Grunewald, Segal, and Smith [55] who introduced a zeta function for the purpose of studying the subgroup growth in some specific classes of infinite groups. With the successful outcome of their investigation, the subject of subgroup growth became a flourishing area of research.

Igusa zeta functions

Let k and n be two positive integers and consider k polynomials with integer coefficients in n variables, i.e. $f_1, \dots, f_k \in \mathbb{Z}[x_1, \dots, x_n]$. Let p be a prime and consider $N_{i,p}$ to be the number of common solutions of the congruences $f_j \equiv 0 \bmod p^i$ for every $j \in [k]$. We consider the Poincaré series

$$P_{f_1,\dots,f_k,p}(T) = \sum_{i=0}^{\infty} N_{i,p} T^i.$$

Igusa [61, 62] proved for $k = 1$ that the Poincaré series is a rational function, i.e. a quotient of two polynomials in T. Furthermore, Igusa illustrated how to express Poincaré series associated with p-adic varieties as integrals with respect to the additive Haar measure on $\mathbb{Z}_p$. In many instances, the problem of computing local Dirichlet generating functions can be transformed into the problem of computing certain Poincaré series, which are, in turn, described through suitable p-adic integrals generalising Igusa local zeta functions.

Representation growth

The primary focus of this thesis is on representation growth, which refers to the increase in the number of irreducible representations of a group in relation to the dimension of the underlying spaces.

Let $r_n(G)$ denote the number of isomorphism classes of n-dimensional complex irreducible representations of a group G. The Dirichlet zeta function that encodes this sequence is called the *representation zeta function* of G and we denote it by $\zeta_G(s)$. If the sequence grows polynomially, the zeta function converges absolutely on some complex half-plane. When $r_n(G)$ is non-zero for infinitely many n, the abscissa of convergence provides the polynomial *degree* of the growth of the summative sequence $(\sum_{n=1}^{N} r_n(G))_{N\in\mathbb{N}_*}$. This is one of the most studied invariants in the subject and the correlation with values and groups is still somehow mysterious. Many other analytic aspects of zeta functions could give some introspective information on groups. Namely the existence of a meromorphic continuation in the entire complex plane, or the existence and location of zeros and poles. The whole subject is relatively new, as very few papers on representation zeta functions of infinite groups are older than twenty years, so much is left to discover. For an introductory survey on this subject, see [73].

Representation growth of S-arithmetic groups. Local and global

We consider the arithmetic subgroups of semisimple algebraic groups defined over number fields, as we did before. More precisely, we consider groups Γ which are commensurable to $\mathbf{G}(\mathcal{O}_S)$, where $\mathbf{G}$ is a connected, simply connected semisimple algebraic group defined over a number field K and $\mathcal{O}_S$ is the ring of S-integers in k for a finite set S of places of k including all the archimedean ones. Let Γ be of this form. Lubotzky and Martin showed that Γ has PRG if and only if Γ has the congruence subgroup property CSP, modulo a standard conjecture, see [84]. Suppose that Γ possesses these properties. Then according to a result of Larsen and Lubotzky [77, Proposition 1.3], the representation zeta function of Γ admits an Euler product decomposition. Indeed, if $\Gamma = \mathbf{G}(\mathcal{O}_S)$ and if the congruence kernel of Γ is trivial, this decomposition takes the form

$$\zeta_\Gamma(s) = \zeta_{\mathbf{G}(\mathbb{C})}(s)^{|k:\mathbb{Q}|} \cdot \prod_{\nu\notin S} \zeta_{\mathbf{G}(\mathcal{O}_\nu)}(s), \tag{0.0.1}$$

where the product extends over all places ν of k which are not in S. Here each archimedean factor $\zeta_{\mathbf{G}(\mathbb{C})}(s)$ enumerates the finite-dimensional, irreducible rational representations of the algebraic group $\mathbf{G}(\mathbb{C})$ and their contribution to the Euler product reflects Margulis super-rigidity. By $\mathcal{O}_\nu$ we denote the ring of integers in the completion k_ν of k at the non-archimedean place ν. The Euler product over the factors captures the representations of Γ with finite image. The groups are FAb compact p-adic analytic groups. An important family of examples of arithmetic groups with the CSP are the special linear groups of degree at least 3. Several of the results of [77] concern the abscissae of convergence of the local representation zeta functions occurring as Euler factors on the right-hand side of (0.0.1) for suitable arithmetic groups Γ. Of particular interest is the dependence of these abscissae on natural invariants, such as the Lie rank of the ambient group of Γ or the place ν at which Γ is completed. With regard to abscissae of convergence of the global representation zeta functions for arithmetic groups, Avni proved in [9] that, for an arithmetic group Γ with the CSP, the abscissa of convergence of is always a rational number. Moreover, Avni, Klopsch, Onn, and Voll [12] showed that the abscissae of convergence of representation zeta functions of S-arithmetic groups are invariant under base extensions of the underlying number field and so the invariant $\alpha(\mathbf{G}(\mathcal{O}_S))$ is determined by the absolute root system of $\mathbf{G}$.

Summary of main results and outline of the thesis

We reproduce here our contributions divided into their respective chapters. More refined introductions are given in the corresponding introductions to the individual chapters.

Chapter 1: Representation growth of quasi-semisimple profinite groups

In the first chapter, we study the representation growth of quasi-semisimple profinite groups. Recall that a profinite group G is called quasi-semisimple if G is perfect and $G/Z(G) \cong \prod_i S_i$, where each S_i is a finite non-abelian simple group. A semisimple profinite group is a quasi-semisimple profinite group with trivial center.

Kassabov and Nikolov [68] studied Cartesian products of alternating groups and they proved that for every positive real number a, there exists a Cartesian product of alternating groups which has polynomial representation growth of the chosen degree a. This is achieved by constructing a finitely generated Cartesian product of alternating groups using a function $f\colon \mathbb{N}_{\geqslant 5} \to \mathbb{N}$ that controls the number copies $f(n)$ of the alternating groups $\mathrm{Alt}(n)$ of degree n appearing in the Cartesian product. By choosing carefully the function so that the group is finitely generated, one can "play" with it and achieve every degree of representation growth.

Cartesian products of finite (simple) groups of Lie type, offer a richer variety of choices. One can vary the root systems, Lie ranks, and/or the defining fields. Avni, Klopsch, Onn, and Voll [12] considered some quasi-semisimple profinite groups in their study of the representation zeta functions of S-arithmetic groups. To explain it, consider the example of the ring of S-integers $\mathbb{Z}_S$ of the field $\mathbb{Q}$ for a finite set of places S including the archimedean place and at least the primes 2 and 3. The S-arithmetic group $\mathrm{SL}_2(\mathbb{Z}_S)$ has the congruence subgroup property, leading to the profinite completion

$$\widehat{\mathrm{SL}_2(\mathbb{Z}_S)} \cong \mathrm{SL}_2(\widehat{\mathbb{Z}_S}) \cong \prod_{p\notin S} \mathrm{SL}_2(\mathbb{Z}_p).$$

Thus, we naturally find a quasi-semismple profinite group by considering the short exact sequence

$$1 \to K \to \mathrm{SL}_2(\widehat{\mathbb{Z}_S}) \to \prod_{p\notin S} \mathrm{SL}_2(p) \to 1,$$

where $K = \prod_{p\notin S} K_p$ and K_p is the principal congruence subgroup $\ker(\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(p))$.

Inspired by the work of Avni, Klopsch, Onn, and Voll, Klopsch and García Rodríguez constructed Cartesian products of simple groups of the form $\mathrm{SL}_{p^\beta}(p^\gamma)$ or $\mathrm{SU}_{p^\beta}(p^\gamma)$ for fixed p and fixed β, and across infinitely many values of γ, or products of $\mathrm{Sp}_{2\eta}(2^\gamma)$, $\mathrm{Spin}^+_{2\eta}(2^\gamma)$, or $\mathrm{Spin}^-_{2\eta}(2^\gamma)$ for fixed η and growing γ, achieving PRG of any chosen degree $a \in \mathbb{R}_{>0}$.

Our work aims to generalise this result by allowing more flexibility in the choice of quasi-semisimple profinite groups. Additionally, Kassabov and Nikolov provided a criterion for semisimple profinite groups to be profinite completion. A finitely generated profinite group G is a *profinite completion* if there exists an abstract finitely generated residually finite group H such that its profinite completion is isomorphic to G. The finitely generated Cartesian products of alternating groups, based on Kassabov and Nikolov's result, are naturally profinite completions. Our result shows that semisimple profinite groups can have a specified degree of PRG. Specifically, we prove the following result.

Theorem A. *For every real number $a > 0$, there exist quasi-semisimple profinite groups G such that $\alpha(G) = a$. Additionally, G can be chosen with flexibility concerning the root system, the Lie ranks, and the defining field of the non-abelian composition factors. Furthermore, there are semisimple profinite groups with $\alpha(G) = a$ that are profinite completions.*

This is Theorem 1.1.1 in Chapter 1.

Chapter 2: Representation growth of compact p-adic analytic groups

Representation zeta functions of p-adic analytic groups arise as local factors of representation zeta functions of arithmetic groups. One of the landmark results of the theory is given by Jaikin-Zapirain [66] who proved that the representation zeta function of a FAb compact p-adic analytic group can always be expressed in terms of finitely many rational functions in p^{-s} over $\mathbb{Q}$. Unfortunately, the explicit calculation of the integrals arising from concrete examples is a very hard task. Although the representation zeta functions of $\mathrm{SL}_2(\mathbb{Z}_p)$ and $\mathrm{SL}_3(\mathbb{Z}_p)$ are known [10, 11, 66], the higher-rank examples remain mysterious; even the abscissae of convergence of the groups $\mathrm{SL}_n(\mathbb{Z}_p)$ for $n \geqslant 5$ have not been computed, though there exist partial results for $\mathrm{SL}_4(\mathbb{Z}_p)$ and some general bounds, cf. [2, 10, 17, 77, 134].

In a joint project with Jan Moritz Petschick, we aimed to provide new examples of representation zeta functions of compact p-adic analytic groups. This is our attempt to break the logjam of explicit computations of representation zeta functions by considering new kinds of compact p-adic analytic groups. Before stating our theorem, let p denote a fixed prime number, and use the symbol $\mathbf{p}$ to indicate the prime p if it is odd, or 4 in the case where the prime is 2. Our main result is the following.

Theorem B. *Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0 and residue characteristic p. Let H be a potent subgroup of $\mathrm{SL}_2^1(\mathcal{O})$, let V be an $\mathcal{O}\,H$-module of finite $\mathcal{O}$-rank n, and let $\sigma\colon H \to \mathrm{GL}_n^m(\mathcal{O})$ be a faithful finite-dimensional $\mathcal{O}$-representation of H on V such that $H^\sigma \cap \mathrm{GL}_n^{m+1}(\mathcal{O}) \leq (H^\sigma)^{\mathbf{p}}$, for a permissible m. Assume furthermore that the semi-direct product $G = H \ltimes_\sigma V$ is FAb. Then*

$$\zeta_G(s) = \zeta_H(s) \cdot \zeta_H^G(s-1),$$

where $\zeta_H^G(s)$ is the zeta function associated to the representation $\mathrm{Ind}_H^G(\mathbb{1})$.

This is Theorem 2.1.1 in Chapter 2.

The representation $\mathrm{Ind}_H^G(\mathbb{1})$ is an example of an admissible representation. Zeta functions associated with admissible representations of p-adic analytic groups were recently defined by Kionke and Klopsch [70] as generalizations of representations zeta functions of groups. Some of the tools of our investigation are the Kirillov orbit method, p-adic integration, and the study of representations of semidirect products.

Moreover, using the product given by Theorem B, we computed the following representation zeta functions of non-semisimple compact p-adic analytic groups.

Theorem C. *Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0, residue characteristic an odd prime p, and residue field cardinality q. Then in the different settings described below the following hold.*

(a) Let $H_n^m = \mathrm{SL}_n^m(\mathcal{O})$ with permissible $m \in \mathbb{N}_*$ and $n \in \mathbb{N}_{\geqslant 2}$, and consider $G_n^m = H_n^m \ltimes \mathcal{O}^2$, where the semidirect product is formed with respect to the natural action. Then

$$\zeta_{H_n^m}^{G_n^m}(s) = q^{mn}\frac{(1-q^{-n(1+s)})}{(1-q^{-ns})}.$$

In particular, for $n=2$ we obtain

$$\zeta_{G_2^m}(s) = \zeta_{H_2^m}(s)\zeta_{H_2^m}^{G_2^m}(s-1) = q^{5m}\frac{(1-q^{-2s})(1-q^{-2-s})}{(1-q^{1-s})^2(1+q^{1-s})}.$$

(b) For simplicity, only in this case, let $\mathcal{O}$ be an unramified extension of $\mathbb{Z}_p$ and let $p \geqslant 7$. For $k \in \mathbb{N}_{\geqslant 1}$, consider G_k of the form $H_k \ltimes \mathcal{O}^2$, where

$$H_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2^1(\mathcal{O}) \mid a \equiv d \equiv 1 \bmod p^k, c \equiv 0 \bmod p^k \right\}.$$

Then
$$\zeta_{H_k}^{G_k}(s) = q^{2+ks}\frac{(1-q^{-1-s})(1-q^{-s}-q^{-1-2s}+q^{-1-(k+1)s})}{(1-q^{-s})^2(1+q^{-s})},$$
and we have
$$\zeta_{G_k}(s) = q^{5+ks}\frac{(1-q^{-2-s})(1-q^{-s})(1-q^{1-s}-q^{1-2s}+q^{k-(k+1)s})}{(1-q^{1-s})^3(1+q^{1-s})}.$$

(c) *For* $G_n^m = \mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^{2n}$ *with* $n \in \mathbb{N}_*$ *and permissible* $m \in \mathbb{N}_*$, *where we regard* $\mathcal{O}^2$ *as the natural module and consider the diagonal action of* $\mathrm{SL}_2^m(\mathcal{O})$ *on* $\mathcal{O}^{2n} \cong \bigoplus_{i=1}^n \mathcal{O}^2$, *we have*
$$\zeta_{G_n^m}(s) = q^{2nm-1}\frac{(1-q^{-s})(1+q-(q^{n-1}-1)q^{2-2s}-q^{n+2-3s}-q^{n+2-4s})}{(1-q^{(n+1)-2s})(1-q^{2n-3s})}\zeta_{\mathrm{SL}_2^m(\mathcal{O})}.$$
In particular, depending on the value of n, *the abscissa of convergence may be determined by either of the two uniformly varying factors in the denominator, i.e. for* $n \leqslant 3$ *the abscissa of convergence is* $\alpha(G_n^m) = \frac{n+1}{2}$, *and for* $n \geqslant 3$ *the abscissa of convergence is* $\alpha(G_n^m) = \frac{2n}{3}$.

(d) *For* $G_n^m = \mathrm{SL}_2^m(\mathcal{O}) \ltimes (\mathrm{Sym}^2(\mathcal{O}^2))^n$ *and* $H^m = \mathrm{SL}_2^m(\mathcal{O})$ *with permissible* $m \in \mathbb{N}_*$ *and* $n \in \mathbb{N}_{\geqslant 2}$, *we have*
$$\zeta_{H^m}^{G_n^m}(s) = q^{3nm-1}(1-q^{-s})(1-q^{-1-s})\frac{(q^{-s}+q^{n-2s}+(1+q^{-s})q+(1+q^{-s})q^{n-1-2s})}{(1-q^{n-2s})(1-q^{3(n-1)-3s})}.$$

(e) *For* $G^m = \mathrm{SL}_2^m(\mathbb{Z}_2) \ltimes \mathrm{Sym}^2(\mathbb{Z}_2^2)$ *and* $H^m = \mathrm{SL}_2^m(\mathbb{Z}_2)$ *with permissible* $m \in \mathbb{N}_*$, *we have*
$$\zeta_{H^m}^{G^m}(s-1) = 2^{3m+1}\frac{(1-2^{-s})(2^{3-s}+(1-2^{-s}))}{1-2^{3-s}}.$$

This is Theorem 2.1.2 in Chapter 2, where some of the notation is explained in more detail.

Chapter 3: Representation growth of Baumslag-Solitar groups

In the last chapter, we write on a work conducted in collaboration with Iker de las Heras, which has a different flavour with respect to the previous ones. We focus on absolute irreducible representations over finite fields of Baumslag-Solitar groups. A Baumslag-Solitar group is a two-generator one-relator group given by the presentation
$$\mathrm{BS}(x,y) = \langle a, b \mid a^x = b^{-1}a^y b\rangle,$$
where $x, y \in \mathbb{Z}\setminus\{0\}$.

These groups were introduced in 1962 by Baumslag and Solitar in [14] to provide the first examples of finitely presented non-Hopfian groups, and have since been a rich source of examples and counterexamples in group theory (recall that a group G is called *Hopfian* if it does not have any proper quotient that is isomorphic to G).

Let $r_n^{\mathrm{abs}}(G,\mathbb{F}_q)$ be the number of non-isomorphic absolutely irreducible n-dimensional representations of a group G over $\mathbb{F}_q$, where q is a power of a prime. We prove the following result.

Theorem D. *Let $x, y \in \mathbb{Z}$ be such that* $\gcd(x,y) = 1$. *Then*

$$r_n^{\mathrm{abs}}(\mathrm{BS}(x,y),\mathbb{F}_q) = \frac{q-1}{n} \sum_{l} \varphi(l),$$

where l runs through all the positive integers satisfying the following conditions

$$\begin{cases} \gcd(l,xy) = 1, & \\ q^n \equiv 1 & \pmod{l}, \\ x^n \equiv y^n & \pmod{l}, \\ x^m \not\equiv y^m & \pmod{l}, \quad \textit{for every } m \in [n-1], \\ y^r \equiv x^r q & \pmod{l}, \quad \textit{for some } r \in [n]; \end{cases}$$

and φ is Euler's totient function.

This is Theorem 3.1.1 in Chapter 3.

This work was inspired by the work of Mozgovoy and Reineke [97], who studied the absolute representation growth over finite fields of free groups. They found some polynomials that describe this growth and, moreover, they related those polynomials to the formula describing the subgroup growth. Baumslag-Solitar groups are not free, but they are somehow close to being so, as they satisfy just one relation. However, the methods Mozgovoy and Reineke are using, do not seem to apply to other kind of groups such as Baumslag-Solitar groups, so we use different methods.

We conclude the chapter with an application to a different type of representation zeta function. More specifically, we look at zeta functions akin to the classic Weil zeta functions, called the *Weil representation zeta functions* defined in 2024, by Corob Cook, Kionke, and Vannacci [26] as

$$\zeta_G^W(s) = \exp\left(\sum_{p\in\mathbb{P}} \sum_{n\geq 1} \sum_{j\geq 1} \frac{r_n^{\mathrm{abs}}(G,\mathbb{F}_{p^j})}{j} \cdot p^{-snj} \cdot \frac{p^{nj}-1}{p^j-1} \right).$$

Statement on the author's contribution to shared research

I declare that the research and creation of ideas for Chapters 2 and 3 were equally shared between myself and the respective coauthors at all levels. Initially we worked together "at the blackboard" and then we carried our the "solidification process" in close exchange with one another. The writing of Chapters 2 and 3 in this thesis is based on manuscripts that were written jointly by the coauthors and subsequently improved by me during the preparation of this thesis. A previous account of the joint work presented in Chapter 2 appears in Jan Moritz Petschick's thesis [102].

Preliminaries

In this chapter, we introduce some notation and present standard definitions and useful results that will be used throughout the thesis. We divide the chapter as follows. In the first section, we present general notations on rings and groups. In the second section, we fix the notations for local fields of characteristic 0 and we discuss some classical results regarding $\mathcal{O}$-lattices. The main sources are [100] and [76]. In the third section, we provide a brief overview of profinite groups with a few of the main results on the subject, and at the end we introduce the notion of Haar measure for topological groups. As references we use the monographs [132] and [105] for profinite groups, and [34] for the Haar measure. In the fourth and fifth sections, we introduce the universe of representations and characters for both, finite and infinite (topological) groups. We rely on the monographs [63] and [60] for finite groups, and on [123] and [35] for locally compact topological groups. In the subsequent section, we outline the theory of central extensions of finite groups based on the monographs [63], [57], and [60]. In the seventh and last section, we introduce Dirichlet generating functions in different contexts. We give an overview of some of the main results in the area and we focus on the Dirichlet generating functions related to representations of groups, known as representation zeta functions. We refer to the monographs [64], [100], [96], and [107], the seminal papers [74], [126] and references therein.

0.1 General notation

We write $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ for the set of natural numbers (non-negative integers) and $\mathbb{N}_* = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}$ for the set of the non-zero natural numbers (positive integers). For every $n \in \mathbb{N}_*$, we write $[n]$ for the set $\{1, \dots, n\}$. As is customary, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$ are respectively the rings of integers and p-adic integers, the fields of rational, real, complex, and p-adic numbers. For q a power of a prime p, let $\mathbb{F}_q$ be the finite field of q elements. An algebraic closure of $\mathbb{F}_q$ is denoted by $\mathbb{F}$; its characteristic and the inclusion $\mathbb{F}_q \subseteq \mathbb{F}$ will be clear from the context. We will mainly use the letter k for a generic field, $\mathcal{O}$ for a compact discrete valuation ring of characteristic 0 and residue characteristic p, a prime, and K for the field of fractions of $\mathcal{O}$, which constitutes a finite extension of $\mathbb{Q}_p$. If l is an extension of a field k, then $|l : k|$ is the degree of the extension. If $\mathfrak{i}$ is an ideal of $\mathcal{O}$ then we write $\mathfrak{i} \trianglelefteq \mathcal{O}$.

Let G be a group. As usual $H \leqslant G$ indicates that H is a subgroup of G and, if H is normal in G, we write $H \trianglelefteq G$. The index of a subgroup H in G is denoted by $|G : H|$. For two elements g, h of G, we write $h^g = g^{-1}hg$ and ${}^g h = h^{g^{-1}} = ghg^{-1}$ for the conjugation action, the commutator of two elements is $[h, g] = h^{-1}h^g$, the center of G is $Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$, and the centralizer of an element $g \in G$ is $C_G(g) = \{h \in G \mid gh = hg\}$. The *Frattini subgroup* $\Phi(G)$ is the intersection of all maximal subgroups of G, and equals G if G has no maximal subgroups. It is well-known fact that $\Phi(G)$ is equal to the set of all "non-generating" elements of G. The *lower central series* of a group G is defined by $\gamma_1(G) = G$ and $\gamma_k(G) = [\gamma_{k-1}(G), G]$ for $k \geqslant 2$.

0.2 Topological groups and profinite completions

A *topological group* is a set G which is equipped with the structure of both, a group and a topological space, and for which the map

$$G \times G \to G, \quad (x, y) \to xy^{-1}$$

is continuous. A tolopological group G satisfies the following elementary properties, cf. [132, Lemma 0.3.1].

- If H is a subgroup containing a non-empty open subset U of G then H is open in G.
- G is Hausdorff if and only if $\{1\}$ is a closed subset of G; and if K is a normal subgroup of G then G/K is Hausdorff if and only if K is closed in G. If G is totally disconnected, then G is Hausdorff.

If a subgroup H is normal and open in G, we write $H \trianglelefteq_o G$ and when it is closed we write $H \leqslant_c G$.

A *profinite group* is a compact totally disconnected topological group; its open subgroups form a base of neighbourhoods of the identity. Let $\mathcal{C}$ be a non-empty class of finite groups. We call a group G a *pro-$\mathcal{C}$ group* if it is an inverse limit of $\mathcal{C}$-groups. The topology of G is inherited from the product topology on the Cartesian product of the $\mathcal{C}$-groups into which G embeds as a closed subgroup. For example, $\mathcal{C}$ could be the class of all finite groups. In this case, a pro-$\mathcal{C}$ group is simply a profinite group. Other examples are when $\mathcal{C}$ is the class of finite p-groups, where p is a fixed prime, when $\mathcal{C}$ is the class of finite cyclic groups, or when $\mathcal{C}$ is the class of finite solvable groups. In these cases, a pro-$\mathcal{C}$ group is commonly called a *pro-p*, *procyclic*, or *prosolvable* group, respectively.

We have the following result, cf. [132, Theorem 1.2.3], [31, Proposition 1.3], and [105, Theorem 2.1.3]

Lemma 0.2.1. *Let G be a topological group and let us assume that $\mathcal{C}$ is a non-empty class of finite groups that is closed for subgroups, direct products, and quotients. Put $\mathcal{N} = \{N \mid N \trianglelefteq_o G, G/N \in \mathcal{C}\}$. Then the following are equivalent:*

(i) *G is pro-$\mathcal{C}$ group;*

(ii) *G is isomorphic (as a topological group) to a closed subgroup of a Cartesian product of $\mathcal{C}$-groups;*

(iii) *G is compact and $\bigcap_{N\in\mathcal{N}} N = 1$;*

(iv) *G is compact and totally disconnected, and $G/L \in \mathcal{C}$ for every $L \trianglelefteq_o G$;*

(v) *G is isomorphic (as a topological group) to $\varprojlim_{N\in\mathcal{N}} G/N$.*

Let G be an abstract group and let $\mathcal{C}$ be a non-empty class of finite groups. Put

$$\mathcal{N} = \{N \trianglelefteq G \mid |G:N| < \infty,\, G/N \in \mathcal{C}\}.$$

We call a subset of G *open* if and only if it is a union of cosets Kg of subgroups $K \in \mathcal{N}$ with $g \in G$. Then G becomes a topological group, for the pro-$\mathcal{C}$ topology.

The *pro-$\mathcal{C}$ completion* of G is the inverse limit

$$\widehat{G}^{\mathcal{C}} = \varprojlim_{N\in\mathcal{N}} G/N.$$

The *profinite completion* $\widehat{G}$ of a group G is the completion of G with respect to the family of all normal subgroups of finite index. A group G is *residually finite* if the intersection of all its subgroups of finite index is trivial. The natural map from G to its profinite completion is injective if and only if G is residually finite.

Let G be a profinite group. For a subset X of G, we denote by $\overline{X}$ the topological closure of X, i.e. $\overline{X} = \bigcap_{N\trianglelefteq_o G} XN$. A subset X of G is said to *generate G topologically* if G is the closure of the abstract subgroup generated by X, i.e. if the abstract subgroup $\langle X\rangle$ of G generated by X is dense in G. We say that G is *finitely generated* if there exists a finite set X that generates G topologically. A profinite group G is called *procyclic* if it contains an element x which generates the group G. We have the following result, see [132, Proposition 4.1.1 and Lemma 4.1.5] or [105, Lemma 2.4.1].

Lemma 0.2.2. *(a) Let G be a profinite group and let X be a subset of G. Consider the projection maps $\pi_N : G \to G/N$, where N runs over all open normal subgroups of G.*

(i) *If X generates G topologically, then $\pi_N(X)$ generates G/N, for each open normal subgroup N of G.*

(ii) *If $\pi_N(X)$ generates G/N for each open normal subgroup N of G, then X generates G topologically.*

(b) Let $\{G_i \mid i \in I\}$ be a family of finite groups and let H be a closed subgroup of $G = \prod_{i\in I} G_i$. For each $i \in I$ write $\pi_i : G \to G_i$ for the projection map. Then

(i) *$H = G$ if and only if for each finite subset $\{i_1,\dots,i_r\}$ of I of size r, the map*

$$H \to G_{i_1} \times \cdots \times G_{i_r}, \quad h \mapsto (\pi_{i_1}(h),\dots,\pi_{i_r}(h))$$

is surjective.

(ii) *Suppose in addition that for all pairs of distinct elements $i, j \in I$, the groups G_i, G_j have no isomorphic composition factors. Then $H = G$ if and only if for each $i \in I$ the restriction of π_i to H, i.e. the map*

$$H \to G_i, \quad h \mapsto \pi_i(h),$$

is surjective.

Example 0.2.3. The groups $\prod_{n\geqslant 5} \mathrm{Alt}(n)$ and $\prod_{p\in\mathcal{P}} \mathrm{PSL}_2(p)$ are 2-generator profinite groups, where $\mathcal{P}$ is the set of primes greater than or equal to 5. This follows essentially from Lemma 0.2.1.

Let G be a profinite group. If G is finitely generated, we define the *minimal number of generators* of G as

$$d(G) = \min\{|X| \mid X \subseteq G, X \text{ generates } G\}.$$

If a profinite group G is not finitely generated, then we write $d(G) = \infty$.

If G is a finitely generated profinite group, the *(Prüfer) rank* of G is defined as

$$\mathrm{rk}(G) = \sup\{d(H) \mid H \leqslant_c G\}.$$

Usually, for a closed subgroup H of G one can say little about $d(H)$. Nevertheless, there are groups for which the rank is finite. It is a theorem that a profinite group G is *p-adic analytic* if it contains an open pro-p subgroup U such that $\mathrm{rk}(U) < \infty$, cf. [31, Theorem 8.1]. A representative example of such groups is $\mathrm{GL}_n(\mathbb{Z}_p)$, see [31] for more details. We will work with compact p-adic analytic groups in Chapter 2.

0.2.1 Haar measure for locally compact groups

A non-empty collection of sets, which is closed under countable unions and complements, is called a *σ-algebra*. For a topological space X, the σ-algebra generated by its open sets is called the *Borel σ-algebra* associated to X. The sets in the Borel σ-algebra are called *measurable sets*. A *measure* on the Borel σ-algebra $\mathcal{B}$, is a function $\mu : \mathcal{B} \to [0, \infty]$ with $\mu(\emptyset) = 0$ which is countably additive, i.e. if $\{B_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{B}$, then $\mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j)$. For convenience we assume without further mentioning that all measures are neither trivial (that is all sets have measure 0), nor assign the measure ∞ to all non-empty sets. A subset B is called *inner regular* if $\mu(B) = \sup\{\mu(K) \mid K \subseteq B, K \text{ compact}\}$, and *outer regular* if $\mu(B) = \inf\{\mu(U) \mid B \subseteq U, U \text{ open}\}$. The measure μ is called *regular* if every open subset is inner regular and every measurable subset is outer regular. For a topological group G, a *left Haar measure* μ on G is a regular measure that is left invariant, i.e. for every $g \in G$ and $B \subseteq G$ measurable, we have $\mu(gB) = \mu(B)$. Haar [56] and von Neumann [101] proved that for locally compact second countable groups, a left Haar measure exists and if μ and ν are left Haar measures, then there exists $c > 0$ such that $\mu = c\nu$. The general case for locally compact groups was settled by Weil. See [34, Theorem 10.5, Theorem 10.14, and Section 10.4] and the references therein. A group is called *unimodular* if the left Haar measure is also a right Haar measure. Compact groups are unimodular, see [34, Proposition 10.16]. For this reason we do not need to distinguish between left and right Haar measure on profinite groups.

0.3 Local fields and $\mathcal{O}$-lattices

Let K be a finite extension field of $\mathbb{Q}_p$ and $\mathcal{O}$ its ring of integers. By [100, Proposition 5.2], the finite extensions of $\mathbb{Q}_p$ produce all (non-archimedean) local fields of characteristic zero up to isomorphism. Let $\mathfrak{p}$ be the unique maximal ideal of $\mathcal{O}$, $\kappa = \mathcal{O}/\mathfrak{p}$ the residue class field of cardinality q, $\mathcal{O}^*$ the group of units, and π an uniformizer so that $\mathfrak{p} = \pi\mathcal{O}$. The normalized (exponential) valuation is denoted by $v_{\mathfrak{p}} \colon K \to \mathbb{Z} \cup \{\infty\}$, and the normalized absolute value by

$$|\cdot|_{\mathfrak{p}} = q^{-v_{\mathfrak{p}}(\cdot)}.$$

For $n \in \mathbb{N}$ we consider the subgroups $U^{(n)}$ of $\mathcal{O}^*$, called *groups of principal units*, which are defined as $U^{(0)} = \mathcal{O}^*$ and for $n \geqslant 1$ as

$$U^{(n)} = 1 + \mathfrak{p}^n = \left\{ x \in K^* \mid |1 - x|_{\mathfrak{p}} < \frac{1}{q^{n-1}} \right\}.$$

The descending chain

$$\mathcal{O}^* = U^{(0)} \supseteq U^{(1)} \supseteq U^{(2)} \supseteq \cdots$$

is a base of neighbourhoods of the element 1 of $K^* := K \setminus \{0\}$ consisting of compact open subgroups. Then there is a uniquely determined (non-injective) continuous homomorphism $\log \colon K^* \to K$ such that $\log(p) = 0$ and

$$\log(1 + x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{x^i}{i}, \quad \text{for} \quad (1 + x) \in U^{(1)},$$

see [100, Proposition 5.4]. Let $e = e(\mathcal{O}, \mathbb{Z}_p)$ be the *ramification index* of $\mathcal{O}$ so that $p\mathcal{O} = \mathfrak{p}^e$. We have the following result, cf. [100, Proposition 5.5].

Proposition 0.3.1. *Let $\mathcal{O}$ be the ring of integers of a local field K of characteristic zero, ramification index e, and with maximal ideal $\mathfrak{p} \trianglelefteq \mathcal{O}$. For every $n \in \mathbb{N}_*$, let $U^{(n)} = 1 + \mathfrak{p}^n$ be the group of principal units. Then the power series*

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \quad \textit{and} \quad \log(1 + x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{x^i}{i},$$

yield, for $n > e/(p-1)$, mutually inverse isomorphisms of groups

$$\mathfrak{p}^n \underset{\log}{\overset{\exp}{\rightleftarrows}} U^{(n)}.$$

Let $d \in \mathbb{N}_*$. An $\mathcal{O}$-lattice Λ in K^d is a free $\mathcal{O}$-submodule of K^d of rank d. We present the Cartan decomposition, cf. [76, Theorem 7.39].

Theorem 0.3.2. *Let K be a non-archimedean local field with a ring of integers $\mathcal{O}$. Let π be an uniformizer of $\mathcal{O}$ and $n \in \mathbb{N}_*$. Then every element A of the group $\mathrm{GL}_n(K)$ has a decomposition $A = PDQ$ with $P, Q \in \mathrm{GL}_n(\mathcal{O})$ and D a diagonal matrix of the form*

$$D = \mathrm{diag}(\pi^{k_1}, \dots, \pi^{k_n}),$$

for some integers $k_i \in \mathbb{Z}$. Up to permutation of diagonal entries, the matrix D is unique.

Let $\mathfrak{g}$ be an $\mathcal{O}$-lattice and $\mathfrak{h}$ an $\mathcal{O}$-sublattice of $\mathfrak{g}$. Let $\xi : \mathfrak{g} \to \mathfrak{h}$ be an $\mathcal{O}$-linear isomorphism between $\mathfrak{g}$ and $\mathfrak{h}$. We can represent ξ by a matrix A of $\mathrm{GL}_n(K)$ with respect to an $\mathcal{O}$-basis for $\mathfrak{g}$. By Theorem 0.3.2, there are two $\mathcal{O}$-bases for $\mathfrak{g}$ such that with respect to these bases ξ is represented by a diagonal matrix $D = \mathrm{diag}(\pi^{k_1}, \dots, \pi^{k_n})$. From this we obbtain the following corollary.

Corollary 0.3.3. *Let $\mathfrak{g}$ be an $\mathcal{O}$-lattice and $\mathfrak{h}$ an $\mathcal{O}$-sublattice. Let $\xi : \mathfrak{g} \to \mathfrak{h}$ be an $\mathcal{O}$-linear isomorphism between $\mathfrak{g}$ and $\mathfrak{h}$. Then ξ extends to an endomorphism of the K-vector space $K \otimes_{\mathcal{O}} \mathfrak{g}$ such that*

$$|\det \xi|_{\mathfrak{p}}^{-1} = |\mathfrak{g} : \mathfrak{h}|.$$

0.4 Representation theory

Let G be a group, and let k be a field. A (finite dimensional) *k-representation* of G is a group homomorphism

$$\rho \colon G \to \mathrm{GL}(V)$$

where V is a (finite dimensional) k-vector space.

The dimension of the space V is called the *dimension* of the representation or its *degree*. Two representations ρ_1 and ρ_2 of G, supported on the same space V, are said to be *isomorphic* if there exists an invertible linear transformation P of V such that $\rho_2(g)(v) = (P \circ \rho_1(g) \circ P^{-1})(v)$ for all $g \in G$ and $v \in V$. Representations on different spaces V_1 and V_2 are isomorphic if there is an equivalent linear isomorphism between V_1 and V_2. A representation is called *irreducible* if the supporting space V is non-trivial and has no proper non-zero subspaces that are stable under the action of the representation. For a finite group G, Maschke's Theorem states that if the characteristic of the scalar field k of V does not divide the order of G, then every representation can be described as a direct sum of irreducible representations. The set of finite dimensional irreducible k-representations of

G, up to equivalence, is denoted by $\mathrm{Irr}_k(G)$. When k is the field of complex numbers, we simply write $\mathrm{Irr}(G)$.

The kernel of a representation ρ of a group G on V is $\ker \rho = \{g \in G \mid \rho(g) = \mathrm{id}_V\}$ as expected. If N is a normal subgroup of a group G and ρ is a k-representation of G with $N \subseteq \ker \rho$, then there is a unique k-representation $\bar{\rho}$ of G/N defined by $\bar{\rho}(gN) = \rho(g)$. Note that ρ is irreducible if and only if $\bar{\rho}$ is. Sometimes we will make no distinction notationally between ρ and $\bar{\rho}$.

When the scalar field k is not algebraically closed, an irreducible k-representation can become reducible as an l-representation, where l is a field extension of k. If an irreducible representation stays irreducible over all algebraic field extensions, it is called *absolutely irreducible*.

From now on, let us assume that the field k is the field of complex numbers $\mathbb{C}$, unless specified otherwise.

If G is a finite group, the study of representations (and characters) of G is a well established research area, see for example [63] and [60]. If the group G is infinite and we consider a topology on it, then we require ρ to be continuous. We say that ρ is *finite* if it has finite image. In the case of profinite groups, the following lemma shows that every representation has finite image.

Lemma 0.4.1. *Let G be a profinite group. Then every continuous finite dimensional complex representation of G factors over an open normal subgroup. Hence, every representation of G has finite image.*

Proof. Let $\rho\colon G \to \mathrm{GL}(V)$ be a continuous finite dimensional complex representation of G. Consider a neighbourhood B of the identity in $\mathrm{GL}(V)$ such that the only subgroup contained in B is the trivial group containing the identity element. Such a neighbourhood B exists because $\mathrm{GL}(V)$ has "no small subgroups", as defined in [123, Definition 6.13 and Section 21]. Since G is a profinite group, $\rho^{-1}(B)$ is an open neighbourhood of the identity in G, which contains an open normal subgroup N. Then $\rho(N) \subseteq B$ and so it is trivial. In conclusion, ρ factors through G/N, which is finite. $\square$

From now on, if G is a topological group it is tacitly understood that representations are continuous.

Moreover, for compact groups, Weyl proved that all finite dimensional complex representations are unitary, i.e. given a finite dimensional complex representation ρ of a compact group G, the underlying complex vector space admits an inner product such that $\rho(g)$ is a unitary operator for every $g \in G$. Compare with [128, Section 22] or [123, Section 14]. This leads to an analogous result of Maschke's Theorem for compact groups as every unitary representation of a compact group G is a direct sum of irreducible representations, see [35, Theorem 5.2].

0.4.1 Representations of semi-direct products

Let A and H be two subgroups of the finite group G, with A normal in G. Let us assume the following extra conditions:

(i) A is abelian.

(ii) G is the semidirect product of H by A, i.e. $G = H \ltimes A$.

Since A is abelian, its irreducible representations are of degree one and form a group $X = \mathrm{Hom}(A, \mathbb{C}^*)$. The group G acts on X as follows

$$(\chi^g)(a) = \chi(gag^{-1}) \text{ for } g \in G,\ \chi \in X,\ a \in A.$$

Let $(\chi_i)_{i\in X/H}$ be a system of representatives for the orbits of H in X. For each $i \in X/H$, let H_i be the stabilizer of χ_i, i.e. the subgroup of H consisting of those elements h such that $\chi_i^h = \chi_i$; and let $G_i = H_i \cdot A$ be the corresponding subgroup of G. Extend the function χ_i to G_i by setting

$$\chi_i(ha) = \chi_i(a) \text{ for } a \in A,\ h \in H_i.$$

Since $\chi_i^h = \chi_i$ for all $h \in H_i$, χ_i becomes a representation of degree one of G_i. Now, let ρ be an irreducible representation of H_i; by composing ρ with the canonical projection $G_i \to H_i$, we obtain an irreducible representation $\tilde{\rho}$ of G_i. Finally, by taking the tensor product of χ_i and $\tilde{\rho}$, we obtain an irreducible representation $\tilde{\rho} \otimes \chi_i$ of G_i; let $\theta_{\rho,i}$ be the corresponding induced representation of G as defined in Section 0.5.2. We have the following result, see [113, Proposition 25].

Proposition 0.4.2. *(a) $\theta_{\rho,i}$ is irreducible.*

(b) If $\theta_{\rho,i}$ and $\theta_{\rho',i'}$ are isomorphic, then $i = i'$ and ρ is isomorphic to ρ'.

(c) Every irreducible representation of G is isomorphic to one of the $\theta_{\rho,i}$.

0.5 Character theory

Let ρ be a (finite dimensional) $\mathbb{C}$-representation of a group G. The ($\mathbb{C}$-)*character* χ of G, *afforded* by ρ, is the function derived from the trace of $\chi(g)$, i.e. defined as $\chi(g) = \mathrm{tr}(\rho(g))$. Equivalent $\mathbb{C}$-representations of G afford equal characters, since the trace operator is invariant under conjugation by an invertible transformation. The *degree* of a a character χ is $\chi(1)$ and equals the dimension of ρ, where χ is afforded by ρ. Additionally, the character of an irreducible representation is called an *irreducible* character. It is convenient to use $\mathrm{Irr}(G)$ also to refer to the set of irreducible characters of G. For profinite groups this is justified by the fact that equivalent classes of irreducible representations over $\mathbb{C}$ are in one-to-one correspondence with irreducible characters.

For a finite group G, the number of irreducible characters equals the number of conjugacy classes of G, see [63, Corollary 2.5] or [60, Theorem 3.12]. Moreover, the relation

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$$

holds, see [63, Corollary 2.7] or [60, Theorem 3.7]. The *inner product* of two characters χ and ϑ of G is defined as

$$\langle \chi, \vartheta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\vartheta(g)}. \tag{0.5.1}$$

For a finite group G with a character χ, the kernel of χ is defined as

$$\ker \chi = \{g \in G \mid \chi(g) = \chi(1)\}.$$

If χ is afforded by the representation ρ, then $\ker \chi = \ker \rho$.

Let N be a normal subgroup of G, and χ a character of G with $N \subseteq \ker \chi$. Then χ is constant on cosets of N in G. In this case, the induced function $\bar{\chi}$ on G/N defined by $\bar{\chi}(gN) = \chi(g)$ is a character of G/N. Moreover, $\chi \in \mathrm{Irr}(G)$ if and only if $\bar{\chi} \in \mathrm{Irr}(G/N)$, see [63, Lemma 2.22]. It follows that a non-trivial group G is simple if and only if $\ker \chi = 1$ for all non-trivial irreducible characters χ of G.

The first orthogonality relation for finite groups ("row orthogonality") is the following, see [63, Corollary 2.14] or [60, Thereom 3.4].

Proposition 0.5.1. *Let* $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ *be the set of irreducible characters of a finite group* G*. Then the following equality holds*

$$\langle \chi_i, \chi_j \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij},$$

where δ_{ij} *denotes the Kronecker delta.*

The second orthogonal relation ("column orthogonality") is the following, see [63, Theorem 2.18] or [60, Theorem 3.10].

Proposition 0.5.2. *Let* g, h *be elements of a finite group* G*. If* g *is not conjugate to* h *in* G*, then*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = 0.$$

If g *is conjugate to* h *in* G*, the sum is equal to* $|C_G(g)|$*, the cardinality of the centralizer of* g*.*

Recall from Section 0.2.1 that for a compact group G, there exists a Haar measure μ. We normalize μ so that $\mu(G) = 1$. Let $C_c(G)$ be the space of compactly supported continuous functions on G. A linear functional I is called *positive* on $C_c(G)$ if $I(f) \geqslant 0$ whenever $f \geqslant 0$.

❀

The Riesz representation Theorem [34, Theorem 7.2] says that if I is a positive linear functional on $C_c(G)$, there is a unique measure μ on G such that $I(f) = \int f d\mu$. In the case of compact groups this is our normalized Haar measure μ. Thus "averaging over G", which for a function f on a finite group G is simply $\frac{1}{|G|}\sum_{t\in G} f(t)$, can be achieved for a compact group G by means of the integral $\int_G f(t)d\mu$ with respect to the normalized Haar measure μ. Most of the properties of representations and characters of finite groups carry over to representations and characters of compact groups with the integral formalism replacing sums. For example, the inner product of two characters χ and ϑ of G is

$$\langle \chi, \vartheta\rangle_G = \int_G \chi(t)\overline{\vartheta(t)}dt.$$

Compare with [35, Proposition 5.3, Theorem 5.8, Peter-Weyl Theorem 5.12, and Proposition 5.23].

0.5.1 Characters of direct products

Let H and K be two finite groups, and let $G = H \times K$ be their direct product. Let φ and ϑ be characters of H and K, respectively, and let V and W be the $\mathbb{C}$-vector spaces supporting the corresponding representations. Define $\chi = \varphi \boxtimes \vartheta$ by $\chi(hk) = \varphi(h)\vartheta(k)$ for $h \in H$ and $k \in K$. The character χ is a character of G and corresponds to a natural "tensor product" representation of $G = H \times K$ supported by the $\mathbb{C}$-vector space $V \otimes_{\mathbb{C}} W$. This leads to the following theorem, see [63, Theorem 4.21] or [60, Theorem 8.1].

Theorem 0.5.3. *Let $G = H \times K$ be a finite group. Then the characters $\varphi\boxtimes\vartheta$ for $\varphi \in \mathrm{Irr}(H)$ and $\vartheta \in \mathrm{Irr}(K)$ are precisely the irreducible characters of G.*

0.5.2 Induced characters and Clifford theory

A frequently used method for constructing characters of a finite group G is the process of induction introduced by Frobenius. Let H be a subgroup of G and let ϑ be a character of H. The *induced character* $\mathrm{Ind}_H^G(\vartheta)$ on G, is given by

$$\mathrm{Ind}_H^G(\vartheta)(g) = \frac{1}{|H|}\sum_{x\in G} \vartheta^\circ(xgx^{-1}) \quad \text{for } g \in G,$$

with ϑ° being defined as $\vartheta^\circ(h) = \vartheta(h)$ if $h \in H$ and $\vartheta^\circ(y) = 0$ if $y \notin H$. Given a character χ of G, we obtain the character $\mathrm{Res}_H^G(\chi)$ of H by restriction.

The following result is also known to be the Frobenious reciprocity theorem, see [63, Lemma 5.2] or [60, Theorem 17.3].

Theorem 0.5.4. *Let $H \leqslant G$ be finite groups and suppose that ϑ is a character of H and χ is a character of G. Then*

$$\langle \vartheta, \mathrm{Res}_H^G(\chi)\rangle_H = \langle \mathrm{Ind}_H^G(\vartheta), \chi\rangle_G.$$

Clifford theory aids in understanding the representations of a group G in terms of the representations of its normal subgroups. If $N \trianglelefteq G$ and $\vartheta \in \mathrm{Irr}(N)$, we define for each $g \in G$ a character ϑ^g of N by setting $\vartheta^g(x) = \vartheta(x^{g^{-1}})$ where $x \in N$ and we call ϑ^g a G-conjugate of ϑ. The following theorem is due to Clifford, see [63, Theorem 6.2] or [60, Theorem 19.3].

Theorem 0.5.5. *Let $N \trianglelefteq G$ be finite groups and let $\chi \in \mathrm{Irr}(G)$. Let ϑ be an irreducible constituent of $\mathrm{Res}_N^G(\chi)$ and suppose $\vartheta = \vartheta_1, \dots, \vartheta_t$ are the distinct G-conjugates of ϑ. Then*

$$\mathrm{Res}_N^G(\chi) = e \sum_{i=1}^{t} \vartheta_i,$$

where $e = \langle \mathrm{Res}_N^G(\chi), \vartheta \rangle_N$.

A consequence of this theorem is given by the following result, see [63, Corollary 6.7].

Corollary 0.5.6. *Let $H \trianglelefteq G$ be finite groups and suppose that $\chi \in \mathrm{Irr}(G)$ with $\langle \mathrm{Res}_H^G(\chi), 1_H \rangle_H \neq 0$. Then $H \subseteq \ker \chi$.*

In the same setting of Theorem 0.5.5, let

$$I_G(\vartheta) = \{g \in G \mid \vartheta^g = \vartheta\},$$

called the *inertia subgroup* of ϑ in G.

Let $N \trianglelefteq G$ and suppose that $\vartheta \in \mathrm{Irr}(N)$ is G-invariant, i.e. $I_G(\vartheta) = G$. For each irreducible constituent χ of $\mathrm{Ind}_N^G(\vartheta)$, it holds that $\mathrm{Res}_N^G \chi = e(\chi)\vartheta$, where $e(\chi)$ is a positive integer called the *ramification* of χ. In general, the values $e(\chi)$ are the degrees of irreducible projective representations of G/N that we discuss in Section 0.6.1.

0.6 Central extensions of groups

A *central extension* (Γ, π) of a group G is a (possibly infinite) group Γ together with a homomorphism π from Γ onto G such that $\ker \pi \subseteq Z(\Gamma)$. For a finite group G, we say that $\tilde{G}$ is a *covering group* of G if $\tilde{G}$ is a central extension of G with the property that $Z(\tilde{G}) \subseteq [\tilde{G}, \tilde{G}]$ and $\tilde{G}/Z(\tilde{G})$ is isomorphic to G. If the centre has order $2, 3$, etc., then the covering group is often referred to as a *double, triple,* etc., *cover* as appropriate.

0.6.1 Projective representations and Schur multipliers

For a group G, let V be a $\mathbb{C}$-vector space of dimension n, where $n \in \mathbb{N}_*$. Let $\rho : G \to \mathrm{GL}(V)$ be such that for all $g, h \in G$, there exists a scalar $\alpha(g, h) \in \mathbb{C}$ such that

$$\rho(g)\rho(h) = \rho(gh)\alpha(g, h).$$

Then ρ is a *projective* ($\mathbb{C}$-)*representation* of G. We call a projective representation ρ *irreducible* if it is non-trivial and has no proper non-zero subspaces that are stable under the action of the representation ρ. Two projective representations ρ_1 and ρ_2 of G, on the same vector space V are *isomorphic* if there exists an invertible linear transformation P of V and $b(g) \in \mathbb{C}^*$, for $g \in G$, such that for all $g \in G$ and all $v \in V$ we have $\rho_2(g)(v) = b(g)(P \circ \rho_1(g) \circ P^{-1})(v)$. The *degree* of a projective representation is the dimension of the vector space V, and the function $\alpha : G \times G \to \mathbb{C}$ is the associated *factor set* of ρ. The latter is a special instance of a more general notion which we briefly recall. Consider an abelian group A and a group G. We regard A as a G-module with respect to the trivial action. An A-*factor set* of G is a function $\alpha : G \times G \to A$ such that

$$\alpha(gh, x)\alpha(g, h) = \alpha(g, hx)\alpha(h, x),$$

for all $g, h, x \in G$; it is easily seen that this implies

$$\alpha(g, 1) = \alpha(1, g) = \alpha(1, 1),$$

for all $g \in G$.

The set of A-factor sets of G forms a group under pointwise multiplication. In the context of group cohomology, this group is denoted by $Z^2(G, A)$, the group of 2-*cocycles*. If $\mu : G \to A$ is an arbitrary function, we can define $\delta(\mu) : G \times G \to A$ by

$$\delta(\mu)(g, h) = \mu(g)\mu(h)\mu(gh)^{-1}.$$

It is easily checked that δ is a homomorphism from the group of A-valued functions on G into $Z^2(G, A)$. The image of δ is the subgroup $B^2(G, A) \subseteq Z^2(G, A)$ which is called the group of 2-*coboundaries*. The factor group

$$Z^2(G, A)/B^2(G, A)$$

is isomorphic to the *second cohomology group* $H^2(G, A)$. We say that two A-factor sets of G are *equivalent* if they are congruent modulo $B^2(G, A)$, making $H^2(G, A)$ the set of equivalence classes of A-factor sets on G. If we take A to be the group $\mathbb{C}^*$, we define the *Schur multiplier* of G as the group $H^2(G, \mathbb{C}^*)$, denoted by $M(G)$.

We have the following result which connects the Schur multiplier $M(G)$ of a group G to a central extension, cf. [60, Remark 20.9] or [57, Theorem 1.2].

Lemma 0.6.1. *For every finite perfect group G there exists a covering group $\tilde{G}$ such that*

$$H/K \cong G,$$

where $K \leqslant Z(\tilde{G})$ and $K \cong M(G)$.

Example 0.6.2. Consider the alternating groups $\mathrm{Alt}(n)$, which are simple, i.e. for $n \geqslant 5$. The order of the Schur multiplier of $\mathrm{Alt}(n)$ is two for all $n \geqslant 5$, except for $n = 6, 7$, where the Schur multipliers have order 6, see [63, Problem 11.17] or [48, Theorem 5.2.3] or [57, Theorem 2.11]. When the order of the Schur multiplier of $\mathrm{Alt}(n)$ is two, we denote its double cover by $2 \cdot \mathrm{Alt}(n)$.

Example 0.6.3. Let $n \in \mathbb{N}_*$ and consider the symmetric group $\mathrm{Sym}(n)$. The Schur multiplier is trivial if $n \leqslant 3$ and it is cyclic of order 2 otherwise. For $n \geqslant 4$ there exist two non isomorphic (except when n is equal to 6) covering groups $\widetilde{\mathrm{Sym}(n)}$ and $\overline{\mathrm{Sym}(n)}$ of $\mathrm{Sym}(n)$. See [57, Theorem 2.7, Theorem 2.8, Theorem 2.9, and Theorem 2.11].

Remark 0.6.4. *A complete list of all Schur multipliers of all finite simple groups can be found in* [44, Table 4.1].

Schur multipliers of direct products of perfect groups are direct products of Schur multipliers, cf. [59, Theorem 25.10].

Theorem 0.6.5. *Let H and G be two perfect groups. Then*

$$M(G \times H) \cong M(G) \times M(H)$$

The following result is based on [63, Lemma 11.9].

Lemma 0.6.6. *Let (Γ, π) be a central extension of G, with $A = \ker(\pi)$. Let X be a set of representatives for G modulo A in Γ, and write $X = \{x_g \mid g \in G\}$, where $\pi(x_g) = g$. Define $\alpha : G \times G \to A$ by $x_g x_h = \alpha(g,h) x_{gh}$. Then $\alpha \in Z^2(G, A)$, and its equivalence class is independent of the choice of X.*

Let (Γ, π) be a central extension such that $A = \ker(\pi)$ is finite. Let X and $\alpha \in Z^2(G, A)$ be as in Lemma 0.6.6. We construct the homomorphism

$$\eta : \mathrm{Irr}(A) \to M(G)$$

by defining $\eta(\lambda) = \overline{\lambda(\alpha)}$, where $\lambda(\alpha) \in Z^2(G, \mathbb{C}^*)$ is defined by $\lambda(\alpha)(g,h) = \lambda(\alpha(g,h))$ and the bar denotes the canonical map $Z^2(G, \mathbb{C}^*) \to H^2(G, \mathbb{C}^*) = M(G)$. The map η is independent of the choice of X and it is called the *standard map*.

Let (Γ, π) be a finite central extension of G. Let ρ be a projective $\mathbb{C}$-representation of G. We say that ρ can be *lifted* to Γ if there exists an ordinary representation θ of Γ and a function $\mu : \Gamma \to \mathbb{C}^*$ such that

$$\theta(x) = \rho(\pi(x))\mu(x)$$

for all $x \in \Gamma$. Furthermore, (Γ, π) has the *projective lifting property* for G if every projective $\mathbb{C}$-representation of G can be lifted to Γ.

We end this section with a result of Schur, see [63, Theorem 11.17].

Theorem 0.6.7. *Given G, there exists a finite central extension (Γ, π) which has the projective lifting property for G. Furthermore, (Γ, π) can be chosen such that*

$$\ker(\pi) = A \cong M(G)$$

and the standard map $\eta : \mathrm{Irr}(A) \to M(G)$ is an isomorphism.

0.7 Dirichlet generating functions

The Riemann zeta function is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$, using the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Originally it was introduced by Euler with real values of s, who observed that for real $s > 1$ the following holds

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}. \tag{0.7.1}$$

The present notation and the notion of $\zeta(s)$ as a function of the complex variable s are due to Riemann, who wrote only one paper [106] on number theory, but that one was truly epoch making and justifies $\zeta(s)$ being called the "Riemann zeta function".

The Riemann zeta function has been a crucial tool in exploring number-theoretic properties, particularly those related to prime numbers. It can be analytically continued to the entire complex plane, cf. [64, Theorem 1.2] or [100, Corollary 1.7], and it plays a key role in the proof of the Prime Number Theorem, cf. [64, Theorem 12.2], that was given independently by Hadamard and de la Vallée Poussin.

Dirichlet generalized the Riemann zeta function by attaching a coefficient a_n to each term n^{-s}. Using as coefficients Dirichlet characters, he proved that for any two coprime positive integers a and d, there are infinitely many primes of the form $a + nd$, where n is also a positive integer, cf. [100, Dirichlet's Prime Number Theorem 5.14].

An example of a Dirichlet generating function is the Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a} \lhd \mathcal{O}} N(\mathfrak{a})^{-s} = \sum_{n=1}^{\infty} a_n^{\lhd}(\mathcal{O}) n^{-s},$$

where $\mathcal{O}$ is the ring of integers of a number field K, $N(\mathfrak{a}) = |\mathcal{O} : \mathfrak{a}|$ is the norm of the ideal $\mathfrak{a}$, and $a_n^{\lhd}(\mathcal{O})$ is the number of ideals of $\mathcal{O}$ of norm n. The Dedekind zeta function of $\mathbb{Q}$ is in fact the Riemann zeta function. A Dedekind ring $\mathcal{O}$ has the property that every non-zero ideal factorizes uniquely as a product of non-zero prime ideals. We denote by $\mathcal{P}_{\mathcal{O}}$ the set of non-zero prime ideals of $\mathcal{O}$. As for the Riemann zeta function, we have the Euler

product decomposition
$$\zeta_K(s) = \prod_{\mathfrak{p}\in\mathcal{P}_{\mathcal{O}}} \zeta_{K,\mathfrak{p}}(s),$$
where for a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, the Euler factor is defined as
$$\zeta_{K,\mathfrak{p}}(s) = \sum_{i=0}^{\infty} N(\mathfrak{p}^i)^{-s} = (1 - N(\mathfrak{p})^{-s})^{-1}.$$

One result that illustrates the power of a generating function is the class number formula which relates some invariants of the number field K to the residue of its Dedekind zeta function $\zeta_K(s)$ at $s = 1$, cf. [100, Corollary 5.11].

0.7.1 Analytic number theoretic background

We commence this section with a basic summation result, which is often used in analytic number theory, see [64, Appendix A.21].

Proposition 0.7.1. *Let M, N be real numbers with $M < N$. Let $x_1, \dots, x_r$ be real numbers with $M \leqslant x_1 < \cdots < x_r \leqslant N$, let $a(x_1), \dots, a(x_r)$ be complex numbers, and put $A(t) = \sum_{x_k \leqslant t} a(x_k)$ for $t \in [M, N]$. Further, let $g : [M, N] \to \mathbb{C}$ be a differentiable function. Then*
$$\sum_{k=1}^{r} a(x_k)g(x_k) = A(N)g(N) - \int_M^N A(t)g'(t)dt.$$

An *arithmetic function* is a function $f\colon \mathbb{N}_* \to \mathbb{C}$. It is called *multiplicative* if it is not the zero function and $f(mn) = f(m)f(n)$ for all coprime positive integers m and n; it is called *strongly multiplicative* if it is not the zero function and $f(mn) = f(m)f(n)$ for all positive integers m and n. Consider the Dirichlet generating function associated with f,
$$L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}.$$

In order to see that the series $L_f(s)$ converges uniformly in a certain half plane of $\mathbb{C}$, possibly empty, we use Proposition 0.7.1 on partial summation. First, let us exclude the case of the empty set of convergence for $L_f(s)$. It is in fact happening when there is no $s \in \mathbb{C}$ for which $L_f(s)$ converges. Assume then that $L_f(s)$ converges for some $s \in \mathbb{C}$ and define
$$\alpha(f) = \inf\{\sigma \mid \text{there exists } s \in \mathbb{C} \text{ with } \Re(s) = \sigma \text{ and such that } L_f(s) \text{ coverges}\}.$$

Clearly by definition $L_f(s)$ diverges if $\Re(s) < \alpha(f)$. To prove that $L_f(s)$ converges for $\Re(s) > \alpha(f)$, take such a complex number s and choose s_0 such that $\Re(s) > \Re(s_0) > \alpha(f)$ and $L_f(s_0)$ converges. Write $s = s' + s_0$. Moreover, there is a constant $C > 0$ such that

$|\sum_{n=1}^{N} f(n)n^{-s_0}| \leqslant C$ for all $N \geqslant 1$. Let us now take a positive integer N and put

$$A(x) = \sum_{n \leqslant x} f(n)n^{-s_0} \quad \text{for } x \in \mathbb{R}.$$

Then, by Proposition 0.7.1, we obtain

$$\sum_{n=1}^{N} f(n)n^{-s} = \sum_{n=1}^{N} f(n)n^{-s_0}n^{-s'} = A(N)N^{-s'} - \int_1^N A(t)(-s')t^{-s'-1}dt.$$

Consider a compact subset of $\{s \in \mathbb{C} \mid \Re(s) > \Re(s_0)\}$. Then there are $\sigma > 0, B > 0$ such that $\Re(s') \geqslant \sigma$, $|s'| \leqslant B$ for s' such that $s = s' + s_0$ and s is in the chosen compact set. Thus we have

$$\begin{aligned}
\left|\sum_{n=1}^{N} f(n)n^{-s}\right| &\leqslant |A(N)N^{-s'}| + |s'| \int_1^N |A(t)t^{-s'-1}|dt \\
&\leqslant C \cdot N^{-\sigma} + B \int_1^N C \cdot t^{-\sigma-1}dt = C \cdot N^{-\sigma} + B \cdot C \cdot \sigma^{-1}(1 - N^{-\sigma}) \\
&\leqslant C + B \cdot C \cdot \sigma^{-1},
\end{aligned}$$

which is an upper bound independent of s and N. Hence, $L_f(s)$ converges for all $s \in \mathbb{C}$ with $\Re(s) > \alpha(f)$.

We showed that there exists a number $\alpha(f) \in \mathbb{R} \cup \{\pm\infty\}$ such that $L_f(s)$ converges for all $s \in \mathbb{C}$ with $\Re(s) > \alpha(f)$ and diverges for all $s \in \mathbb{C}$ with $\Re(s) < \alpha(f)$. The number $\alpha(f)$ is called the *abscissa of convergence* for $L_f(s)$.

The Dirichlet functions associated with multiplicative or strongly multiplicative functions can be written as Euler products, see [96, Theorem 1.9].

Theorem 0.7.2. *Let f be a multiplicative function. Let $s \in \mathbb{C}$ be such that $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely. Then*

$$L_f(s) = \prod_{p \text{ prime}} \left(\sum_{j=0}^{\infty} f(p^j)p^{-js} \right).$$

Further, $L_f(s) \neq 0$ as soon as $\sum_{j=0}^{\infty} f(p^j)p^{-js} \neq 0$ for every prime p.

Moreover, if the function f is strongly multiplicative, then we have

$$L_f(s) = \prod_{p \text{ prime}} \left(1 - f(p)p^{-s}\right)^{-1}.$$

The Dirichlet functions that we are interested in have non-negative coefficients. Landau proved that such Dirichlet function cannot be continued to an analytic function beyond the boundary of their half-plane of convergence. We have indeed the following result, cf. [96, Theorem 1.7]

Lemma 0.7.3. *Let $f\colon \mathbb{N}_* \to \mathbb{R}$ be an arithmetic function with $f(n) \geqslant 0$ for all n. Suppose that $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ has abscissa of convergence α. Then $L_f(s)$ cannot be continued to an analytic function on any open set containing $\{s \in \mathbb{C} \mid \Re(s) > \alpha\} \cup \{\alpha\}$.*

0.7.2 Zeta functions of groups

Dirichlet zeta functions are used in the context of group theory to study the asymptotic behaviour of algebraic information related to a group. As we have seen in the introduction, the pioneers of this field include Grunewald, Segal, and Smith [55], who introduced a zeta function for the purpose of studying the subgroup growth in certain classes of infinite groups. The study of subgroup growth was further explored in various classes of groups, see the monograph [86].

As we pointed out in the introduction, one of the most significant achievements in the study of subgroup growth was made in 1993 by Lubotzky, Mann, and Segal [83], who characterized among finitely generated residually finite groups those with polynomial subgroup growth as the virtually solvable groups of finite rank.

With the successful outcome of these early investigations, the subject of subgroup growth matured quickly, leading to studies of other algebraic information about groups through "zeta function methods".

A couple of valuable resources offering an overview of the subject are [126] and [127].

0.7.3 Representation zeta functions

A representation over $\mathbb{C}$ of a profinite group G is a continuous homomorphism from our group G to the group of automorphisms of a $\mathbb{C}$-vector space.

Let $r_n(G)$ be the number of irreducible n-dimensional complex representations of G up to equivalence, and let $R_n(G) = \sum_{j=1}^{n} r_j(G)$, be the summatory function. A group is *representation rigid* if $r_n(G)$ is finite for every $n \in \mathbb{N}_*$. It is easy to see that if a profinite group G is rigid, then it is FAb, meaning: every open subgroup H of G has finite abelianization $H/[H,H]$. However, the converse is generally not true, except in the case of finitely generated profinite groups. Indeed, we have the following result, cf. [13, Proposition 2].

Proposition 0.7.4. *A finitely generated profinite group is representation rigid if and only if it is FAb.*

In the realm of compact p-adic analytic groups, being FAb corresponds to the associated $\mathbb{Q}_p$-Lie algebra $\log(G)$ being perfect, cf. Section 2.1. On the other hand, every finitely generated quasi-semisimple profinite group is rigid, as it is FAb, cf. Section 1.1.

If the growth rate of $R_n(G)$ is polynomial, i.e., there exists a constant a such that $R_n(G) = O(n^a)$, then G is said to have *polynomial representation growth* (PRG). For a rigid groups G, we encode the sequence $(r_n(G))_{n\geqslant 1}$ in a Dirichlet generating function, called the *representation zeta function* of G, which is of the form

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s},$$

where s is a complex variable. Whenever the sequence $(r_n(G))_{n\geqslant 1}$ grows polynomially, the zeta function $\zeta_G(s)$ converges in a right half-plane of the complex plane $\mathbb{C}$ defined as $\{s \in \mathbb{C} \mid \Re(s) > \alpha\}$, as we have seen in Section 0.7.1. The abscissa of convergence $\alpha(G)$ of $\zeta_G(s)$ is the infimum $\alpha \in \mathbb{R}$ for which convergence occurs. If G admits only finitely many isomorphism classes of irreducible complex representations, then $\alpha(G) = -\infty$ and the representation zeta function $\zeta_G(s)$ is holomorphic on the entire complex plane. Otherwise, the abscissa of convergence satisfies

$$\alpha(G) = \limsup_{n\to\infty} \frac{\log R_n(G)}{\log n}.$$

Hence, a group G has PRG if and only if $\alpha(G)$ is finite, and $\alpha(G)$ is indeed the minimal polynomial degree of growth, i.e. $R_n(G) = O(n^{\alpha(G)+\varepsilon})$ for $\varepsilon > 0$.

This value is one of the most studied invariants in the subject, yet its correlation with group properties remains somehow mysterious. Various analytic aspects of zeta functions could provide some insights into groups. Namely the existence of a meromorphic continuation to the entire complex plane or the existence and location of zeros and poles. The field is relatively new, as very few papers on representation zeta functions of infinite groups are dating back more than twenty years, indicating that much remains to be explored.

Rationality

In the context that is considered in [42], a Dirichlet function $L_f(s)$ is said to be *finitely rational with respect to a prime* p if it has non-empty domain of convergence and admits a meromorphic continuation of the form

$$L_f(s) = \sum_{i=1}^{r} m_i^{-s} F_i(p^{-s}),$$

where $m_1, \ldots, m_r$ are finitely many suitable positive integers and $F_1, \ldots, F_r \in \mathbb{Q}(X)$ are rational functions. In [66], Jaikin-Zapirain established rationality results for the representation zeta functions of FAb compact p-adic analytic groups using tools from model theory. Specifically, the representation zeta function of a FAb compact p-adic analytic pro-p group is rational in p^{-s} in the above sense. This result was extended by Stasinski and Zordan [119] for the prime 2 case.

In [70, Section 5] Kionke and Klopsch proved that the zeta functions of induced representations of potent pro-p groups are finitely rational with respect to p. Additional examples for rationality are found in [70, Proposition 6.5.].

0.7.4 Weil representation zeta functions

In 1949, André Weil formulated a series of significant conjectures about counting points on varieties over finite fields, cf. [129].

Let q be a prime power. For each integer $n \geq 1$, let $\mathbb{F}_{q^n}$ be the extension of $\mathbb{F}_q$ of degree n. Let $V/\mathbb{F}_q$ be a projective variety, defined as the set of solutions to

$$f_1(x_0, \ldots, x_n) = \cdots = f_m(x_0, \ldots, x_n) = 0,$$

where $f_1, \ldots, f_m$ are homogeneous polynomials with coefficients in $\mathbb{F}_q$. Then $V(\mathbb{F}_{q^n})$ is the set of $\mathbb{F}_{q^n}$-rational points of V, i.e. points which admit coordinates in $\mathbb{F}_{q^n}$. We encode the number of points of $V(\mathbb{F}_{q^n})$, across all $n \geq 1$, into a generating function. The *Weil zeta function* of $V/\mathbb{F}_q$ is the power series

$$Z(V/\mathbb{F}_q; T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#V(\mathbb{F}_{q^n})T^n}{n}\right).$$

We have the following result, cf. [116, Theorem 2.2].

Theorem 0.7.5. *Let $V/\mathbb{F}_q$ be a smooth projective variety of dimension n.*

(a) Rationality:
$$Z(V/\mathbb{F}_q; T) \in \mathbb{Q}(T).$$

(b) Functional Equation: There exists an integer $\varepsilon = \varepsilon(V)$, called the Euler characteristic of V, such that
$$Z(V/\mathbb{F}_q; 1/q^nT) = \pm q^{n\varepsilon/2}T^{\varepsilon}Z(V/\mathbb{F}_q; T).$$

(c) "Riemann Hypothesis": The zeta function factors as

$$Z(V/\mathbb{F}_q; T) = \frac{P_1(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)},$$

with each $P_i(T) \in \mathbb{Z}[T]$, where

$$P_0(T) = 1 - T \quad \textit{and} \quad P_{2n}(T) = 1 - q^nT,$$

and such that for every $0 \leq i \leq 2n$, the polynomial $P_i(T)$ factors over $\mathbb{C}$ as

$$P_i(T) = \prod_{j=1}^{b_i}(1 - \alpha_{ij}T) \quad \textit{with} \quad |\alpha_{ij}| = q^{i/2}.$$

The quantity b_i, i.e., the degree of $P_i(T)$, coincides with the i-th Betti number of V.

This theorem was posed as a conjecture by Weil in 1949 in the previously mentioned article [129] and proven by him for curves and abelian varieties. In 1960 Dwork [32] established the rationality of the Weil zeta function. Shortly thereafter, the ℓ-adic cohomology theory developed by M. Artin, Grothendieck and others, see [54], [3], [4], [5], [53], [52], and [28], was used to provide another proof of rationality and to establish the functional equation. In 1973, Deligne [27] proved the last of the Weil Conjectures, the Riemann hypothesis for the Weil zeta function.

We will use zeta functions akin to the classic Weil zeta functions to encode the representation growth of the metabelian Baumslag-Solitar group $\mathrm{BS}(1,-1)$ in Section 3.3.

Chapter 1

Representation growth of quasi-semisimple profinite groups

1.1 Introduction

A profinite group G is said to be *quasi-semisimple* if G is perfect, i.e. G is equal to the closure of its derived group, and $G/Z(G) \cong \prod_i S_i$ where S_i is a family of finite non-abelian simple groups. A *semisimple* profinite group is a quasi-semisimple profinite group with trivial center. We denote the class of semisimple profinite groups by $\mathcal{C}_{\mathcal{S}}$. According to the classification of finite simple groups, every finite non-abelian simple group is either an alternating group $\mathrm{Alt}(n)$, for $n \geqslant 5$, or a simple group of Lie type, or else one of 26 sporadic groups, cf. [133]. In the context of this discussion, we further narrow our focus to two distinct subclasses within $\mathcal{C}_{\mathcal{S}}$. These subclasses are $\mathcal{A}_{\mathcal{S}}$ and $\mathcal{L}_{\mathcal{S}}$, comprising Cartesian products of alternating groups $\mathrm{Alt}(n)$ with $n \geqslant 5$ and finite simple groups of Lie type, respectively. Quasi-simple groups are central extensions of finite simple groups, cf. Section 0.6. Moreover, for finite perfect groups, the Schur multiplier of a direct product of groups is the direct product of their Schur multipliers, cf. Theorem 0.6.5. It follows that quasi-semisimple profinite groups are quotients of Cartesian products of quasi-simple groups. We denote the class of Cartesian products of quasi-simple groups of Lie type by $\mathcal{L}_{\mathcal{Q}}$.

For a (profinite) group G and $n \in \mathbb{N}_*$, let $r_n(G)$ be the number of non-isomorphic irreducible n-dimensional complex representations of G. We have seen in Section 0.7.3, that a finitely generated profinite group is representation rigid if and only if it is FAb, i.e. every open subgroup H of G has finite abelianization $H/[H, H]$, see Proposition 0.7.4. In particular, finitely generated semisimple profinite groups G are always representation rigid so that $r_n(G)$ is finite for all $n \in \mathbb{N}_*$. More generally, since quasi-semisimple profinite groups are quotients of Cartesian products of quasi-simple groups, we can deduce that also finitely generated quasi-semisimple groups are always representation rigid.

If G is rigid we encode the sequence $(r_n(G))_{n\geqslant 1}$ in the representation zeta function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s},$$

where s is a complex variable. The abscissa of convergence $\alpha(G)$ is the infimum positive real number such that $\zeta_G(s)$ converges in a right half-plane of $\mathbb{C}$, possibly empty, delimited by $\Re(s) > \alpha(G)$.

Kassabov and Nikolov [68] proved that for every positive real number a, there exists a group $G \in \mathcal{A}_{\mathcal{S}}$ such that G has PRG and $\alpha(G) = a$. Moreover, they proved that such groups are profinite completions in the following sense. A finitely generated profinite group G is a *profinite completion* if there exists a finitely generated abstract group H such that G is isomorphic to the profinite completion $\widehat{H}$. Even if it is generally difficult to understand if a profinite group is a profinite completion, Kassabov and Nikolov found a criterion for that to happen in the class of semisimple profinite groups $\mathcal{C}_{\mathcal{S}}$. We will report their result in more detail in Section 1.8.

Let $\mathbb{Z}_S$ be the ring of S-integers of the field $\mathbb{Q}$, with respect to a finite set of places S. If S contains the archimedean place and at least one non-archimedean place (i.e. a prime), the S-arithmetic group $\mathrm{SL}_2(\mathbb{Z}_S)$ has the congruence subgroup property so its profinite completion $\widehat{\mathrm{SL}_2(\mathbb{Z}_S)}$ is isomorphic to

$$\mathrm{SL}_2(\widehat{\mathbb{Z}_S}) \cong \prod_{p\notin S}^{\infty} \mathrm{SL}_2(\mathbb{Z}_p).$$

We consider the short exact sequence

$$1 \to K \to \mathrm{SL}_2(\widehat{\mathbb{Z}_S}) \to \prod_{p\notin S} \mathrm{SL}_2(p) \to 1.$$

The kernel is the group $K = \prod_{p\notin S} K_p$, where K_p is the principal congruence subgroup $\ker(\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(p))$. Avni, Klopsch, Onn, and Voll [12] use this connection, more generally for other Lie types with different rank and finite extensions of $\mathbb{Q}$, in order to study the representation growth of S-arithmetic groups. Their analysis of the representation growth of groups of the form $\prod_{p\notin S} \mathrm{SL}_2(p)$, for a suitable set S, has inspired and led our work on representation of more general quasi-semisimple profinite groups.

Our main result is the following generalization of a result by Klopsch and García Rodríguez [37].

Theorem 1.1.1. *For every real number $a > 0$, there exist quasi-semisimple profinite groups G such that $\alpha(G) = a$. Additionally, G can be chosen with flexibility concerning the Chevalley type, the defining field, and the Lie ranks of the composition factors that are simple groups of Lie type. Furthermore, there are even semisimple profinite groups with $\alpha(G) = a$ that are profinite completions.*

This generalizes the work of Klopsch and García Rodríguez, who proved that for every real number $a > 0$, there exists a group in $\mathcal{L}_{\mathbb{Q}}$ having PRG of degree a. Their construction involves Cartesian products of some specific groups of Lie type with fixed Lie rank. More precisely, they considered products of $\mathrm{SL}_{p^\beta}(p^\gamma)$ or $\mathrm{SU}_{p^\beta}(p^\gamma)$ for a fixed prime p and positive integer β, and across infinitely many values of γ, or products of $\mathrm{Sp}_{2\eta}(2^\gamma)$, $\mathrm{Spin}^+_{2\eta}(2^\gamma)$, or $\mathrm{Spin}^-_{2\eta}(2^\gamma)$ for a fixed η and increasing γ.

Our Theorem 1.1.1 improves this result by constructing groups with PRG such that $\alpha(G) = a$ which could involve Cartesian products of quasi-simple and properly simple groups of Lie type of any preferred Chevalley type, with corresponding field of cardinality of any power of a prime, or with any preferred Lie rank. Furthermore, with the freedom of the choice of ranks of the finite simple groups involved, we can construct finitely generated semisimple profinite groups which are profinite completions.

Additionally, we prove in Section 1.8 that the representation theory of semisimple profinite groups that are profinite completions is equivalent to the representation theory of corresponding abstract groups.

Blueprint of the chapter

The chapter is structured as follows. In Section 1.2, we provide an overview of algebraic groups. Then, in Section 1.3, we present finite groups of Lie type and finite simple groups of Lie type. We also prove some standard results regarding the number of generators for finite products of groups of Lie type, see Theorem 1.3.19. Next, in Section 1.4, we give an overview of the representation theory of finite groups of Lie type, highlighting the connection with representations of finite simple groups of Lie type, as shown in Proposition 1.4.10.

The technical core of the chapter is presented in Section 1.5, where we approximate the Dirichlet polynomials associated with the representations of finite (quasi-)simple groups of Lie type, in Theorem 1.5.6. We then provide an illustrative example involving $\mathrm{SL}_2(q)$ and $\mathrm{PSL}_2(q)$ in Section 1.6. In Section 1.7, we explore the connection between polynomial representation growth and finite generation. In particular, we prove Proposition 1.7.2 and Theorem 1.7.3. Then, in Section 1.8, we discuss Kassabov and Nikolov's results on profinite completions of semisimple profinite groups, Theorem 1.8.1, and we prove that the representation theory of the abstract groups and their profinite completions given by (quasi)-semisimple profinite groups is the same, as explained in Proposition 1.8.5. Finally, in the last section, we prove our main result: Theorem 1.1.1.

1.2 Algebraic groups

Algebraic groups can be defined over a general field k. However, here we focus on the case when the underlying field k is algebraically closed of characteristic p, a prime. Affine algebraic groups are defined as affine varieties which are equipped with a group structure in such a way that the binary group operation and inversion are continuous maps. The topology is induced by the Zariski topology which is defined on k^n. It can be shown that affine algebraic groups are exactly the closed subgroups of the general linear group $\mathbf{GL}_n$ over $\mathbb{F}$ with $n \in \mathbb{N}_*$, hence in this setting the concepts of linear and affine coincide. The structure theory of semisimple linear algebraic groups was developed in the mid-20th century and culminated in the classification of semisimple linear algebraic groups over an algebraically closed field, a result attributed to Chevalley. This work was first made available via the "Séminaire sur la classification des groupes de Lie algébriques" at the École Normale Supérieure in Paris between 1956 and 1958 [22]. Similar to the classification of complex semisimple Lie algebras by Cartan and Killing, Chevalley demonstrated that semisimple linear algebraic groups are determined up to isomorphism by a set of combinatorial data, mainly based on a root system (as with semisimple Lie algebras) and a dual root system. Importantly, this set of combinatorial data is independent of the characteristic of the underlying field.

In this section, we present an overview of the theory of linear algebraic groups defined over an algebraically closed field of positive charactersitic following mainly the monographs [92], [38], [30], and [21].

Let $\mathbb{F} = \overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, where p is a prime. We consider linear algebraic groups $\mathbf{G}$ over $\mathbb{F}$ and we drop the word linear for the rest of the text. As first examples, we mention the additive group $\mathbb{G}_a$ isomorphic to $\mathbb{F}$ and the multiplicative group $\mathbb{G}_m$ isomorphic to $\mathbb{F}^\times$. Since algebraic groups are exactly the closed subgroups of the general linear group $\mathbf{GL}_n$ over $\mathbb{F}$ with $n \in \mathbb{N}_*$, we see that every element g of an algebraic group $\mathbf{G}$ over $\mathbb{F}$ has finite order. We define g to be *semisimple* if the order of g is prime to p and *unipotent* if the order of g is a power of p. Then for every $g \in \mathbf{G}$, there exists a unique decomposition $g = g_s g_u = g_u g_s$ where g_s is semisimple and g_u is unipotent, called the *Jordan decomposition of* g. If an algebraic group $\mathbf{G}$ consists entirely of unipotent elements we say that $\mathbf{G}$ is a *unipotent group*. For instance, the additive group $\mathbb{G}_a$ is a unipotent group and the multiplicative group $\mathbb{G}_m$ is a group that consists of semisimple elements. We define a *torus* $\mathbf{T}$ to be an algebraic group isomorphic to a direct product of copies of $\mathbb{G}_m$. The connected component of $\mathbf{G}$ containing the identity, denoted by $\mathbf{G}^\circ$, is a closed normal subgroup of finite index in $\mathbf{G}$. Moreover, $\mathbf{G}$ is the union of finitely many cosets $g\mathbf{G}^\circ$ of the irreducible subgroup $\mathbf{G}^\circ$, cf. [92, Proposition 1.13]. The *dimension* of an algebraic group $\mathbf{G}$ is $\dim(\mathbf{G}) = \dim(\mathbf{G}^\circ)$. A *Borel subgroup* of an algebraic group $\mathbf{G}$, is a closed, connected, solvable subgroup $\mathbf{B}$ of $\mathbf{G}$ which is maximal with respect to all these properties. All Borel subgroups of $\mathbf{G}$ are conjugate [92, Theorem 6.4]. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$, i.e. a subtorus of $\mathbf{G}$ which is maximal with respect to inclusion. Then all maximal tori of $\mathbf{G}$ are conjugate as they are contained in Borel subgroups. We define the *rank* of an algebraic group $\mathbf{G}$ as the dimension of a maximal torus $\mathbf{T}$ of $\mathbf{G}$ and we denote it by $\mathrm{rk}(\mathbf{G})$.

An algebraic group $\mathbf{G}$ is called *reductive* if the maximal closed connected normal unipotent subgroup $R_u(\mathbf{G})$ of $\mathbf{G}$, called the *unipotent radical*, is trivial. The maximal closed connected solvable normal subgroup $R(\mathbf{G})$ of an algebraic group is called the *radical* of $\mathbf{G}$ and $R_u(\mathbf{G}) \leqslant R(\mathbf{G}) \leqslant \mathbf{G}^\circ$. We say that a connected reductive algebraic group $\mathbf{G}$ is *semisimple* if $R(\mathbf{G}) = 1$. Finally, we say that an algebraic group is *simple* if it is connected, non-trivial, and if it has no closed connected normal subgroups other than $\{1\}$ and $\mathbf{G}$. We remark that simple groups $\mathbf{G}$ are connected and semisimple, and that semisimple groups $\mathbf{G}$ are connected and reductive. We have the following properties for connected reductive and semisimple groups, cf. [92, Proposition 6.20, Theorem 8.21, and Corollary 8.22].

Theorem 1.2.1. *Let $\mathbf{G}$ be a connected reductive algebraic group. Then*

(a) $R(\mathbf{G}) = Z(\mathbf{G})^\circ$ *is a torus;*
(b) $R(\mathbf{G}) \cap [\mathbf{G}, \mathbf{G}]$ *is finite;*
(c) $[\mathbf{G}, \mathbf{G}]$ *is semisimple;*
(d) $\mathbf{G} = R(\mathbf{G})[\mathbf{G}, \mathbf{G}] = Z(\mathbf{G})^\circ[\mathbf{G}, \mathbf{G}]$.

Moreover, if $\mathbf{G}$ is also semisimple, then

(e) $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$*;*
(f) there exist (connected) simple algebraic groups $\mathbf{G}_1, \ldots, \mathbf{G}_r$ such that $\mathbf{G} = \mathbf{G}_1 \cdot \ldots \cdot \mathbf{G}_r$.

Let us consider a connected reductive group $\mathbf{G}$ and let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$. Let $X = X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ be the *character group* of $\mathbf{T}$ and $Y = Y(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ the *cocharacter group* of $\mathbf{T}$. Associated with X, we have a finite set of roots $\Phi = \Phi(\mathbf{T})$, called *root system*, and for Y we have a finite set of coroots $\Phi^\vee = \Phi^\vee(\mathbf{T})$, called *coroot system*. We consider the finite dimensional real vector spaces $X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R}$ and $\langle \Phi \rangle_\mathbb{R} = \mathbb{Z}\Phi \otimes_\mathbb{Z} \mathbb{R}$. When $\mathbf{G}$ is connected and semisimple, then $\langle \Phi \rangle_\mathbb{R} = X_\mathbb{R}$, see [92, Proposition 9.2]. The root system Φ has a subset Δ which is a *base* of the real vector space $\langle \Phi \rangle_\mathbb{R}$ and such that any element of Φ can be written as linear combination of elements of the base with coefficients which are all non-negative or all non-positive integers. This involves a choice, and based on this choice we introduce the subset Φ^+ of Φ given by roots which are non-negative linear combinations of Δ is called the *system of positive roots* of Φ with respect to the base Δ. The *Weyl group* of $\mathbf{G}$ with respect to $\mathbf{T}$ is $W = N_\mathbf{G}(\mathbf{T})/C_\mathbf{G}(\mathbf{T})$. The quadruple $(X, \Phi, Y, \Phi^\vee)$ forms a *root datum* with *perfect pairing* $\langle \cdot, \cdot \rangle \colon X \times Y \to \mathbb{Z}$. One of the characterizations of a root datum is that there exists a bijection $\Phi \to \Phi^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. For every element w of the Weyl group W, let $\dot{w}$ be a representative in $N_\mathbf{G}(\mathbf{T})$. The Weyl group W acts naturally on X by $(w.\chi)(t) \coloneqq \chi(t^{\dot{w}})$ for all $w \in W$, $\chi \in X$, $t \in \mathbf{T}$, and on Y by $(w.\gamma)(c) \coloneqq \gamma(c)^{\dot{w}^{-1}}$ for all $w \in W$, $\gamma \in Y$, $c \in \mathbb{G}_m$. Using this action we can identify W as a subset of $\mathrm{Aut}(X)$ and of $\mathrm{Aut}(Y)$. The set of roots $\Phi = \Phi(\mathbf{T})$, is W-stable, by [92, Proposition 8.4]. Let s_α be a reflection on $X_\mathbb{R}$ along the root α stabilizing Φ. Then we can write $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$, cf. [92, Proposition 8.20 and Proposition 9.2].

Let S be the set of reflections $\{s_\alpha \mid \alpha \in \Phi\}$, let S^* be the free monoid on S; that is the set of words on S. We define the *length* function $l \colon W \to \mathbb{N}$ with respect to S in the following way. We set $l(1) = 0$ and we consider a word $s_1 \cdots s_k \in S^*$ which express a non-trivial element $w \in W$. The word $s_1 \cdots s_k \in S^*$ is called a reduced expression for w if it has minimal length among the words representing w. We then write $l(w) = k$. Note that we do not have to take into account inverses since $s^2 = 1$ for all $s \in S$. Moreover, by [92, Proposition A.21], we have an equivalent description of the length function that is

$$l(w) = |\{\alpha \in \Phi^+ \mid w.\alpha \in \Phi^-\}|.$$

We have the following structure theorem for connected reductive groups, cf. [92, Theorem 8.17].

Theorem 1.2.2. *Let $\mathbf{G}$ be a connected reductive algebraic group and $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ with associated root system Φ. Then*

(a) $\dim \mathbf{G} = |\Phi| + \mathrm{rk}(\mathbf{G})$*;*
(b) For each $\alpha \in \Phi$ there exists a morphism of algebraic groups $u_\alpha \colon \mathbb{G}_a \to \mathbf{G}$, which induces an isomorphim onto $u_\alpha(\mathbb{G}_a)$ such that $tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c)$, for all $t \in \mathbf{T}$, $c \in \mathbb{F}$. If u' is a morphism with the same properties, then there is a unique $a \in \mathbb{F}^\times$ with $u'(c) = u_\alpha(ac)$ for all $c \in \mathbb{F}$;
(c) $\mathbf{G} = \langle \mathbf{T} \cup \bigcup_{\alpha\in\Phi} \mathbf{U}_\alpha \rangle$, where $\mathbf{U}_\alpha := \mathrm{im}(u_\alpha)$ for $\alpha \in \Phi$;
(d) For $w \in W$ with preimage $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$, we have $\dot{w}U_\alpha\dot{w}^{-1} = U_{w.\alpha}$;
(e) $Z(\mathbf{G}) = \bigcap_{\alpha\in\Phi} \ker \alpha$.

The subgroup $\mathbf{U}_\alpha = \mathrm{im}(u_\alpha)$ for $\alpha \in \Phi$ of $\mathbf{G}$ is called the *root subgroup* associated with α.

Remark 1.2.3. *If the set of roots is the empty set, that is in the case when our connected reductive group is a torus $\mathbf{T}$, then the intersection of the kernels of roots described in Theorem 1.2.2* (e) *is the empty intersection which we interpret as $\mathbf{T}$. Hence, the center is equal to the whole torus $\mathbf{T}$, as expected since tori are abelian.*

Connected semisimple algebraic groups are classified by isomorphism classes of root data, by a fundamental result of Chevalley. We report this result in the formulation given in [92, Theorem 9.13].

Theorem 1.2.4. *Two connected semisimple algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum there exists a connected semisimple algebraic group which realizes it. This group is simple if and only if its root system is indecomposable.*

Indecomposable root systems are classified up to isomorphism. The types are called A_n with $n \geqslant 1$, B_n with $n \geqslant 2$, C_n with $n \geqslant 3$, D_n with $n \geqslant 4$, E_6, E_7, E_8, F_4, and G_2, see [92, Theorem 9.6]. The types of root systems correspond to the Dynkin diagrams listed in Table 1.1.

To describe all the possible root data for connected semisimple algebraic groups, we introduce $\Omega = \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$, and we view X as a subgroup of Ω, since $X \cong \mathrm{Hom}(Y, \mathbb{Z})$

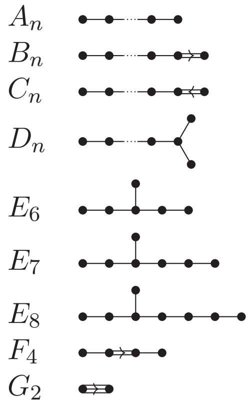

Table 1.1: Dynkin diagrams of indecomposable root systems

by [92, Proposition 3.6] and the homomorphism from $\mathrm{Hom}(Y, \mathbb{Z})$ to Ω is injective. In this way, we have that $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$. The root data with fixed root system Φ are classified by subgroups $X/\mathbb{Z}\Phi$ of $\Omega/\mathbb{Z}\Phi$. The two extremes of the spectrum of possibilities for a fixed root system Φ, are when $X = \Omega$ (which is equivalent to $\Phi^\vee$ spanning Y over $\mathbb{Z}$), in which case $\mathbf{G}$ is said to be *simply connected*, and when $X = \mathbb{Z}\Phi$, in which case $\mathbf{G}$ is said to be of *adjoint type*. An *isogeny* of connected algebraic groups is a surjective homomorphism with finite central kernel. If such a morphism exists between two algebraic groups $\mathbf{G}$ and $\mathbf{H}$, then the groups $\mathbf{G}$ and $\mathbf{H}$ are called *isogenous*. We have the following result stated as [92, Proposition 9.15], cf. [38, Proposition 1.5.8].

Proposition 1.2.5. *Let $\mathbf{G}$ be a semisimple algebraic group with maximal torus $\mathbf{T}$ and associated root system Φ. Let $\mathbf{G}_{\mathrm{sc}}$ be a simply connected semisimple algebraic group with maximal torus $\mathbf{T}_1$ and let $\mathbf{G}_{\mathrm{ad}}$ be an adjoint semisimple algebraic group with maximal torus $\mathbf{T}_2$ both with associated root system Φ. Then there exist natural isogenies*

$$\mathbf{G}_{\mathrm{sc}} \xrightarrow{\pi_1} \mathbf{G} \xrightarrow{\pi_2} \mathbf{G}_{\mathrm{ad}}$$

such that $\pi_1(\mathbf{T}_1) = \mathbf{T}$ and $\pi_2(\mathbf{T}) = \mathbf{T}_2$.

The semisimple algebraic groups corresponding to the same root system Φ are called the *isogeny types* corresponding to Φ. If $\mathbf{G}$ is connected reductive, then we say that $\mathbf{G}$ is simply connected (resp. adjoint) if the semisimple group $[\mathbf{G}, \mathbf{G}]$ is simply connected (resp. adjoint).

1.2.1 Duality of algebraic groups

We introduce now the dual of a compact reductive algebraic group that we will use for the description of the decomposition of characters of finite groups of Lie type.

Let $\mathbf{G}$ be a connected reductive algebraic group, $\mathbf{T}$ a maximal torus of $\mathbf{G}$ with associated root datum $(X, \Phi, Y, \Phi^\vee)$. Then $(Y, \Phi^\vee, X, \Phi)$ is also a root datum, see for example [21, Proposition 4.2.1] or [38, Lemma 1.2.3]. A connected reductive algebraic group $\mathbf{G}^*$ is said to be in *duality* with $\mathbf{G}$ if there exists a maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ such that the associated root datum $(X^*, \Phi^*, Y^*, (\Phi^*)^\vee)$ is isomorphic to $(Y, \Phi^\vee, X, \Phi)$. More precisely, if there exists an isomorphism $\delta\colon X \to Y^*$ such that $\delta(\Phi) = (\Phi^*)^\vee$ and $\langle \chi, \alpha^\vee \rangle = \langle \delta(\alpha)^\vee, \delta(\chi) \rangle$ for all $\chi \in X$ and $\alpha \in \Phi$. The dual of a connected semisimple algebraic group is connected and semisimple, and the dual of a connected semisimple algebraic group of adjoint type is of simply connected type and conversely. If $\mathbf{G}$ with maximal torus $\mathbf{T}$ is in duality with $\mathbf{G}^*$ with maximal torus $\mathbf{T}^*$, and if $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ is the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$ and $W^\vee = \langle s_\alpha^\vee \mid \alpha \in \Phi \rangle$ is the Weyl group of $\mathbf{G}^*$ with respect to $\mathbf{T}^*$, then there exists an isomorphism $\delta_W\colon W \to W^\vee$ such that $\delta_W(s_\alpha) = s_\alpha^\vee$ for all $\alpha \in \Phi$ and $\langle w^{-1}\chi, \gamma \rangle = \langle \chi, \delta_W(w)(\gamma) \rangle$ for all $w \in W, \chi \in X$, and $\gamma \in Y$, see [38, Lemma 1.2.3 and Remark 1.5.19].

1.2.2 Centralizers of semisimple elements

The centralizer of a semisimple element g_s is a closed subgroup of an algebraic group $\mathbf{G}$ by [92, Proposition 5.2], and it is in general not connected. By [92, Corollary 6.11], every semisimple element g_s of a connected algebraic group $\mathbf{G}$ lies in a maximal torus $\mathbf{T}$. Let w be an element of the Weyl group W associated with $\mathbf{T}$ and write $\dot{w}$ for a choice of preimage in $N_{\mathbf{G}}(\mathbf{T})$ mapping to $w \in W = N_{\mathbf{G}}(\mathbf{T})/C_{\mathbf{G}}(\mathbf{T})$. We have the following structure theorem for centralizers of semisimple elements, see [92, Proposition 14.1 and Theorem 14.2], [21, Proposition 3.5.2, Theorem 3.5.3, and Theorem 3.5.4], [38, Subsection 2.2.13], or [30, Proposition 3.5.1 and Proposition 3.5.3].

Theorem 1.2.6. *Let $\mathbf{G}$ be a connected reductive algebraic group and let $g_s \in \mathbf{G}$ be a semisimple element. Then g_s lies in $C^\circ_{\mathbf{G}}(g_s)$. Moreover, let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ containing g_s, Φ the root system of $\mathbf{G}$ associated with $\mathbf{T}$, and W the Weyl group associated with $\mathbf{T}$. Let $\Psi = \{\alpha \in \Phi \mid \alpha(g_s) = 1\}$ and $\mathbf{U}_\alpha$ be the root subgroup associated with $\alpha \in \Phi$. Then*

(i) $C^\circ_{\mathbf{G}}(g_s) = \langle \mathbf{T} \cup \bigcup_{\alpha \in \Psi} \mathbf{U}_\alpha \rangle$.
(ii) $C_{\mathbf{G}}(g_s) = \langle \mathbf{T} \cup \bigcup_{\alpha \in \Psi} \mathbf{U}_\alpha \cup \bigcup_{w \in W, g_s^w = g_s} \langle \dot{w} \rangle \rangle$.

Furthermore, $C^\circ_{\mathbf{G}}(g_s)$ is reductive of maximal rank with root system Ψ.

Consider $\mathbf{G}$, g_s, and $\mathbf{T}$ as in the setting of Theorem 1.2.6. We then have that the centralizer $C_{\mathbf{G}}(g_s)$ is determined by $\Psi \subseteq \Phi$ and by the subgroup $\{w \in W \mid g_s^w = g_s\}$. For both of these, there are just finitely many possibilities and they are all just depending on the root system Φ. Thus we have the following result, cf. [92, Corollary 14.3].

Corollary 1.2.7. *Let $\mathbf{G}$ be a connected reductive algebraic group. Then up to conjugation, there exist only finitely many different centralizers of semisimple elements in $\mathbf{G}$. Moreover, the number of centralizers of semisimple elements is bounded by a constant that depends only on the root system Φ of $\mathbf{G}$.*

We have a sufficient criterion for when $C_{\mathbf{G}}(g_s)$ is connected due to Steinberg. We refer to the formulation given in [21, Theorem 3.5.6], cf. [92, Theorem 14.16] and [38, Theorem 2.2.14].

Theorem 1.2.8. *Let $\mathbf{G}$ be a connected reductive algebraic group whose derived subgroup is simply connected. Let g_s be a semisimple element of $\mathbf{G}$. Then $C_{\mathbf{G}}(g_s)$ is connected.*

As an application of this theorem, we report the following result. It uses the duality to connect the condition of having connected centralizers of semisimple elements in $\mathbf{G}^*$ with the condition for the center of $\mathbf{G}$ being connected, cf. [21, Theorem 4.5.9] or [38, Subsection 2.5.10].

Theorem 1.2.9. *Let $\mathbf{G}$ be a connected reductive algebraic group in which $Z(\mathbf{G})$ is connected. Let $\mathbf{G}^*$ be the dual group of $\mathbf{G}$ and g_s be a semisimple element of $\mathbf{G}^*$. Then $C_{\mathbf{G}^*}(g_s)$ is connected.*

The following result takes into consideration centralizers of two algebraic groups $\mathbf{G}$ and $\mathbf{H}$ that are isogenous, see [38, Subsection 1.3.10 (e)].

Lemma 1.2.10. *Let $\varphi\colon \mathbf{G} \to \mathbf{H}$ be an isogeny between algebraic groups. If $g \in \mathbf{G}$ then φ maps $C^\circ_{\mathbf{G}}(g)$ onto $C^\circ_{\mathbf{H}}(\varphi(g))$.*

Lastly, we have the following result on the number of connected components of a centralizer of a semisimple element that follows from Theorem 1.2.8, see [92, Proposition 14.20], [38, Theorem 2.2.14], or [30, Remark 3.5.2].

Proposition 1.2.11. *Let $\mathbf{G}$ be a semisimple algebraic group, $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ the natural isogeny from a simply connected group of the same type as $\mathbf{G}$. Then for every semisimple element $g_s \in \mathbf{G}$, the group of connected components $C_{\mathbf{G}}(g_s)/C_{\mathbf{G}}(g_s)^\circ$ is isomophic to a subgroup of $\ker(\pi) \leqslant Z(\mathbf{G}_{\mathrm{sc}})$, which is finite. Moreover, the exponent of $C_{\mathbf{G}}(g_s)/C_{\mathbf{G}}(g_s)^\circ$ divides the order of g_s.*

1.2.3 Parabolic and Levi subgroups

Let $\mathbf{G}$ be a connected reductive algebraic group, $\mathbf{T} \leqslant \mathbf{G}$ a maximal torus contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. We have the following structure theorem, cf. [92, Theorem 11.1].

Theorem 1.2.12. *Let $\mathbf{G}$ be a connected reductive algebraic group with maximal torus $\mathbf{T}$ contained in a Borel subgroup $\mathbf{B}$. Let X be the associated character group, Φ the root system, and W the Weyl group. For every $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the root subgroup associated with α. Then the following hold.*

(a) There exists a base Δ of Φ with positive root system $\Phi^+ \subseteq \Phi$ such that

$$\mathbf{B} = \mathbf{T} \cdot \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$$

for every fixed order of the factors $\mathbf{U}_\alpha$.

(b) $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$.
(c) $\mathbf{G} = \langle \mathbf{T} \cup \bigcup_{\alpha \in \pm\Delta} \mathbf{U}_\alpha \rangle$.

Let $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of generating reflections of the Weyl group W. For every subset $I \subseteq S$, the *standard parabolic subgroup* W_I of W is

$$W_I = \langle s \mid s \in I \rangle.$$

Any conjugate of a standard parabolic subgroup is a *parabolic subgroup* of W. The corresponding *parabolic subsystem* of roots is

$$\Phi_I = \Phi \cup \sum_{\alpha \in \Delta_I} \mathbb{Z}\,\alpha,$$

where $\Delta_I = \{\alpha \in \Delta \mid s_\alpha \in I\}$. A *standard parabolic subgroup* $\mathbf{P}_I$ of $\mathbf{G}$ is given by

$$\mathbf{P}_I = \langle \mathbf{T} \cup \bigcup_{\alpha \in \Phi^+ \cup \Phi_I} \mathbf{U}_\alpha \rangle.$$

A *parabolic subgroup* of $\mathbf{G}$ is any subgroup conjugate of a standard parabolic subgroup. For $I \subseteq S$ define

$$\mathbf{U}_I = \prod_{\alpha \in \Phi^+ \setminus \Phi_I} \mathbf{U}_\alpha \qquad \text{and} \qquad \mathbf{L}_I = \langle \mathbf{T} \cup \bigcup_{\alpha \in \Phi_I} \mathbf{U}_\alpha \rangle.$$

In this situation, $\mathbf{L}_I$ is a *complement* to $\mathbf{U}_I$. We have the following structure theorem for parabolic subgroups, cf. [92, Proposition 12.6].

Proposition 1.2.13. *Let $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of generating reflections of the Weyl group W. Let $I \subseteq S$ and $\mathbf{P}_I$ be a standard parabolic subgroup of $\mathbf{G}$ with respect to I. Then $R_u(\mathbf{P}_I) = \mathbf{U}_I$, and $\mathbf{L}_I$ is a complement to $\mathbf{U}_I$, so $\mathbf{P}_I = \mathbf{U}_I \rtimes \mathbf{L}_I$. In particular, $\mathbf{L}_I$ is reductive with root system Φ_I. Moreover, all closed complements to $\mathbf{U}_I$ are conjugate to $\mathbf{L}_I$ in $\mathbf{P}_I$ and $\mathbf{L}_I = C_{\mathbf{G}}(Z(\mathbf{L}_I)^\circ)$.*

The decomposition $\mathbf{P}_I = \mathbf{U}_I \rtimes \mathbf{L}_I$ is called the *Levi decomposition* of the parabolic subgroup $\mathbf{P}_I$, and $\mathbf{L}_I$ is called the *standard Levi complement* of $\mathbf{P}_I$. The conjugates of standard Levi complements are called *Levi subgroups* of $\mathbf{G}$.

Remark 1.2.14. *The Proposition 1.2.13 is telling us in particular that a Levi subgroup $\mathbf{L}$ is the centralizer of its central torus $Z(\mathbf{L})^\circ$. Moreover,* [92, Proposition 12.10] *states that also the converse is true: if $\mathbf{S}$ is a torus of a connected reductive algebraic group $\mathbf{G}$, then $C_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup of $\mathbf{G}$.*

1.3 Finite (quasi-)simple groups of Lie type

Finite simple groups can be traced back to Galois who introduced the concept of normal subgroups and knew that the alternating groups $\mathrm{Alt}(n)$ are simple for $n \geqslant 5$. With the

Jordan-Hölder theorem, finite simple groups became central in the study of finite groups as they constitute a set of invariants of a finite group. The classification of the finite simple groups was announced somewhat prematurely around 1980 as a result of several decades of extraordinary work by many mathematicians. Part of the original proof is covered by a two-volume exposition by Gorenstein [45, 46] which provides an outline of what is entailed rather than detailed proofs. Unfortunately, Gorenstein passed away before completing the third volume of the series. The proof of the classification of finite simple groups is not an ordinary proof due to its complexity and its length, which is around 10,000 pages. Therefore, many mathematicians at the time were quite distrustful of the truthfulness of the classification. Indeed, in the next twenty years, some gaps were found. The majority of them were quickly fixed, but the so-called "quasithin groups" were not adequately dealt with in the original proof. At last, in 2004, Aschbacher and Smith [7, 8] classified quasithin groups closing all the remaining gaps in the classification of finite simple groups, see [6]. The history of the classification is rich and multifaceted, with numerous monographs offering comprehensive insights and analyses. One of the articles in this area is "A Brief History of the Classification of the Finite Simple Groups" by Ronald Solomon [117], where he provides a concise yet comprehensive summary of the classification theorem and its historical context. Lastly, we mention the comprehensive multi-volume series by Daniel Gorenstein, Richard Lyons, and Ronald Solomon that provides a detailed account of the proof of the classification theorem for finite simple groups, see for example [47] and [48].

In this section, we give an introduction to finite (simple) groups of Lie type, mainly following [38], [30], and [92].

Finite simple groups of Lie type can be described as quotients of some finite groups of Lie type, which are quasi-simple, by their center. As we have seen in Section 1.2, algebraic groups are affine varieties over an algebraically closed field $\mathbb{F}$ of characteristic p, a prime; and they are closed subgroups of $\mathbf{GL}_n$ for some $n \geqslant 1$. Let q be a power of the prime p and $\mathbb{F}_q \subseteq \mathbb{F}$ be a finite subfield of q elements. Let $\mathbf{G}$ be an algebraic group and let $\iota\colon \mathbf{G} \to \mathbf{GL}_n$ be an injective homomorphism for some $n \geqslant 1$. We say that $\mathbf{G}$ has a $\mathbb{F}_q$-*structure* if $\iota(\mathbf{G})$ is stable under the *standard Frobenius map*.

$$F_q\colon \mathbf{GL}_n \to \mathbf{GL}_n, \quad (a_{ij}) \mapsto (a_{ij}^q).$$

In this case, there is a group homomorphism $F\colon \mathbf{G} \to \mathbf{G}$ such that $\iota \circ F = F_q \circ \iota$. The homomorphism F is called *Frobenius endomorphism* corresponding the the $\mathbb{F}_q$-structure, see [38, Section 1.4] or cf. [30, Section 4.1]. The group of fixed points $\mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$ is a finite group. If $\mathbf{G}$ is a connected reductive algebraic group, then $\mathbf{G}^F$ is called a *finite group of Lie type*. Note that we are not considering in this exposition the finite groups of Lie type called Suzuki groups and Ree groups since we are restricting to Frobenius endomorphisms instead of considering more general Steinberg maps. We define the *Lie* *rank* of a finite group of Lie type $\mathbf{G}^F$ to be equal to the rank of $\mathbf{G}$. We want to consider groups of Lie type of growing Lie rank so we exclude Suzuki and Ree groups. Moreover it was noted by Avni, Klopsch, Onn, and Voll that Ree groups may behave differently, cf. [12, Remark 3.6].

A subgroup $\mathbf{H}$ of $\mathbf{G}$ is said to be F-stable if $F(h) \in \mathbf{H}$ for all $h \in \mathbf{H}$. Let $\mathbf{T}$ be an F-stable maximal torus, X the character group of $\mathbf{T}$, Y the group of cocharacters of $\mathbf{T}$, and Φ, $\Phi^\vee$ the roots and coroots with respect to $\mathbf{T}$. The action of F on the character group $X = X(\mathbf{T})$ and the cocarachter group $Y = Y(\mathbf{T})$ is given by $F(\chi)(t) := \chi(F(t))$ for $\chi \in X$ and $t \in \mathbf{T}$ and by $F(\gamma)(c) := F(\gamma(c))$ for $\gamma \in Y$ and $c \in \mathbb{F}$. Consider the real vector space $X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R}$. By [92, Proposition 22.2], [30, Proposition 4.2.3 and Lemma 4.2.5], or [38, Lemma 1.4.17 and Proposition 1.4.19], there exists a positive integer β and $\phi \in \mathrm{Aut}(X_\mathbb{R})$ of order β such that $F = q\phi$ on $X_\mathbb{R}$ and $F^\beta|_X = q^\beta\,\mathrm{id}_X$. In particular, F induces a graph automorphism ϕ on the Dynkin diagram of Φ. Let Γ_Φ be the Dynkin diagram associated to the root system Φ. The non-trivial groups of diagram automorphisms of connected Dynkin diagrams given by the action of a Frobenius endomorphism are collected in Table 1.2, cf. [92, Table 11.1].

Φ	$A_n (n \geqslant 2)$	$D_n (n \geqslant 5)$	D_4	E_6
$\mathrm{Aut}(\Gamma_\Phi)$	C_2	C_2	$\mathrm{Sym}(3)$	C_2

Table 1.2: Graph automorphisms of connected Dynkin diagrams

Remark 1.3.1. *If $\mathbf{G}$ is simple, we can classify the finite groups of Lie type $\mathbf{G}^F$ according to their isogeny type, root system Φ, and the parameters q and β, see [92, Section 22.2]. In the literature, the parameter β of the classification that we report here, corresponds to to the automorphism τ of the root system Φ stabilising Φ^+; compare [12, Section 3.1] and references therein. If $\mathbf{G}$ is additionally of simply connected type, then the finite groups of Lie type $\mathbf{G}^F$ are parametrized by q, Φ, and β. Moreover, the groups $\mathbf{G}^F$ are quasi-simple groups (i.e. it is perfect and the quotient by its centre is simple), except in a few cases described in the following remark, cf. [92, Theorem 24.17] or [30, Remark 4.3.4 and Classification 4.3.6].*

Remark 1.3.2. *Let $\mathbf{G}$ be a simply connected simple algebraic group with Frobenius endomorphism F. Then, unless $\mathbf{G}^F$ is one of*

$$\mathrm{SL}_2(2),\ \mathrm{SL}_2(3),\ \mathrm{SU}_3(2),\ \mathrm{Sp}_4(2),\ \mathrm{G}_2(2),$$

the group $\mathbf{G}^F$ is perfect and $\mathbf{G}^F/Z(\mathbf{G}^F)$ is a finite simple group.

Remark 1.3.3. *In the setting of the previous remark, $\mathbf{G}^F$ is a covering group of the simple group $\mathbf{G}^F/Z(\mathbf{G}^F)$ and $Z(\mathbf{G}^F)$ is the Schur multiplier $M(\mathbf{G}^F/Z(\mathbf{G}^F))$ of $\mathbf{G}^F/Z(\mathbf{G}^F)$, cf.* [92, Remark 24.19] *and* [48, Tables 6.1.2 and 6.1.3].

For a simple algebraic group $\mathbf{G}$, we have

$$Z(\mathbf{G}^F) = Z(\mathbf{G})^F, \tag{1.3.1}$$

see [92, Corollary 24.13] or [21, Proposition 3.6.8]. Moreover for $\mathbf{G}$ of simply connected type, the center $Z(\mathbf{G}^F)$ is a cyclic group whose order we denote by z in the table below, except for the case of root systems D_n with $n \geqslant 4$ even and β equal to 1, where we have a product of two cyclic groups both of order $(2, q-1)$. According to Remark 1.3.1, we denote by $L(q)$ the quasi-simple groups, where by L we mean a choice of the root system Φ and the order β of the graph automorphism of the Dynkin diagram, which are associated to the corresponding quasi-simple groups $\mathbf{G}^F$. Moreover, we denote by $S(q)$ the simple group given by the quotient of $L(q)$ by their center. Table 1.3 describes the correspondence with the classical names of groups of Lie type, cf. [133, Section 1.2] or [92, Table 22.1 and Table 24.2].

Φ	β		$L(q)$	z	$S(q)$	Exceptions
A_{n-1}	1	$n \geqslant 2$	$\mathrm{SL}_n(q)$	$(n, q-1)$	$\mathrm{PSL}_n(q)$	$\mathrm{PSL}_2(2), \mathrm{PSL}_2(3)$
B_n	1	$n \geqslant 3$, q odd	$\mathrm{Spin}_{2n+1}(q)$	$(2, q-1)$	$\mathrm{P\Omega}_{2n+1}(q)$	
C_n	1	$n \geqslant 2$	$\mathrm{Sp}_{2n}(q)$	$(2, q-1)$	$\mathrm{PSp}_{2n}(q)$	$\mathrm{PSp}_4(2)$
D_n	1	$n \geqslant 4$ even	$\mathrm{Spin}^+_{2n}(q)$	$(2, q-1)^2$	$\mathrm{P\Omega}^+_{2n}(q)$	
D_n	1	$n \geqslant 5$ odd	$\mathrm{Spin}^+_{2n}(q)$	$(4, q-1)$	$\mathrm{P\Omega}^+_{2n}(q)$	
E_6	1		$(E_6)_{\mathrm{sc}}(q)$	$(3, q-1)$	$E_6(q)$	
E_7	1		$(E_7)_{\mathrm{sc}}(q)$	$(2, q-1)$	$E_7(q)$	
E_8	1		$E_8(q)$	1	$E_8(q)$	
F_4	1		$F_4(q)$	1	$F_4(q)$	
G_2	1		$G_2(q)$	1	$G_2(q)$	$G_2(2)$
A_{n-1}	2	$n \geqslant 3$	$\mathrm{SU}_n(q)$	$(n, q+1)$	$\mathrm{PSU}_n(q)$	$\mathrm{PSU}_3(2)$
D_n	2	$n \geqslant 4$ even	$\mathrm{Spin}^-_{2n}(q)$	$(2, q-1)$	$\mathrm{P\Omega}^-_{2n}(q)$	
D_n	2	$n \geqslant 5$ odd	$\mathrm{Spin}^-_{2n}(q)$	$(4, q-1)$	$\mathrm{P\Omega}^-_{2n}(q)$	
D_4	3		${}^3D_4(q)$	1	${}^3D_4(q)$	
E_6	2		$({}^2E_6)_{\mathrm{sc}}(q)$	$(3, q+1)$	${}^2E_6(q)$	

Table 1.3: Simple and quasi-simple groups of Lie type

The following theorem of Lang and Steinberg, cf. [92, Theorem 21.7] or [30, Theorem 4.2.9], serves as the essential means for extending results from algebraic groups $\mathbf{G}$ to the finite groups $\mathbf{G}^F$ consisting of fixed points under a Frobenius endomorphism F.

Theorem 1.3.4. *Let $\mathbf{G}$ be a connected algebraic group over $\mathbb{F}$ with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Then the morphism*

$$\mathcal{L} : \ : \mathbf{G} \to \mathbf{G}, \qquad g \mapsto F(g)g^{-1},$$

is surjective.

Let $\mathbf{G}$ be a connected reductive algebraic group, $\mathbf{T}$ a maximal torus, and $F\colon \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism. We recall the action of F on the character group $X = X(\mathbf{T})$ and the cocarachter group $Y = Y(\mathbf{T})$ is given by $F(\chi)(t) = \chi(F(t))$ for $\chi \in X$ and $t \in \mathbf{T}$, and by $F(\gamma)(c) = F(\gamma(c))$ for $\gamma \in Y$ and $c \in \mathbb{F}$. We have the following result, cf. [92, Corollary 21.8] or [38, Lemma 1.4.14 and Lemma 1.4.15].

Corollary 1.3.5. *Let $\mathbf{G}$ be a connected reductive algebraic group with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ and consider the cyclic group $\langle F \rangle$ generated by F. Then, in the semidirect product $\mathbf{G} \rtimes \langle F \rangle$, the coset $\mathbf{G}.F$ of F consists of a single conjugacy class, that is, $\mathbf{G}.F = F^{\mathbf{G}}$. In particular, the groups of fixed points $\mathbf{G}^{gF}$ and $\mathbf{G}^F$ are $\mathbf{G}$-conjugate for every $g \in \mathbf{G}$.*

We have seen in Proposition 1.2.5, that for a fixed root system Φ, we have different semisimple groups corresponding to the different isogeny types. Moreover, by [92, Proposition 22.7], all the Frobenius endomorphism of semisimple algebraic groups are induced by Frobenius endomorphisms of simply connected groups, cf. [38, Proposition 1.5.9]. By abuse of notation, we will call all Frobenius endomorphisms of different isogeny types corresponding to the same root system Φ by the same F. We have the following result, see [92, Proposition 24.21].

Proposition 1.3.6. *Let $\mathbf{G}$ be a simple algebraic group and $\pi : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ the natural isogeny from a group of simply connected type with central kernel. Let F be a Frobenius endomorphism on $\mathbf{G}_{\mathrm{sc}}$ normalizing $\ker(\pi)$. Then*

$$\pi(\mathbf{G}_{\mathrm{sc}}^F) \cong \mathbf{G}_{\mathrm{sc}}^F / \ker(\pi)^F.$$

In particular, if $\mathbf{G}_{\mathrm{sc}}^F$ is perfect then $[\mathbf{G}^F, \mathbf{G}^F] = \pi(\mathbf{G}_{\mathrm{sc}}^F) \cong \mathbf{G}_{\mathrm{sc}}^F / \ker(\pi)^F$, and $[\mathbf{G}_{\mathrm{ad}}^F, \mathbf{G}_{\mathrm{ad}}^F]$ is simple.

Corollary 1.3.7. *If $\mathbf{G}$ is simple of adjoint type, then $\ker(\pi) = Z(\mathbf{G}_{\mathrm{sc}})$, where π is the map of Proposition 1.3.6. Using (1.3.1), we have that*

$$[\mathbf{G}_{\mathrm{ad}}^F, \mathbf{G}_{\mathrm{ad}}^F] \cong \mathbf{G}_{\mathrm{sc}}^F / Z(\mathbf{G}_{\mathrm{sc}}^F).$$

1.3.1 Duality of finite groups of Lie type

We have seen in Section 1.2.1 the concept of duality for connected reductive algebraic groups. Now we extend this notion to finite groups of Lie type. Let $\mathbf{G}$ and $\mathbf{G}^*$ be connected reductive algebraic groups with Frobenius maps F and F^* respectively. An F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ is called *maximally split* if it is contained in an F-stable Borel subgroup of $\mathbf{G}$. The pairs $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ are in duality if there exist a maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$ and a maximally split torus $\mathbf{T}_0^*$ of $\mathbf{G}^*$, with associated root data

$(X, \Phi, Y, \Phi^\vee)$ and $(X^*, \Phi^*, Y^*, (\Phi^*)^\vee)$, such that $\mathbf{G}$ and $\mathbf{G}^*$ are in duality with respect to $\mathbf{T}_0$ and $\mathbf{T}_0^*$ and the isomorphism $\delta : X \to Y^*$ satisfies $\delta(\chi \circ F|_{\mathbf{T}_0}) = F^*|_{\mathbf{T}_0^*} \circ \delta(\chi)$ for all $\chi \in X$, see [38, Definition 1.5.17] and [21, Section 4.3].

1.3.2 Cardinalities

Recall that we are considering algebraic groups defined over an algebraically closed field $\mathbb{F}$ of characteristic p, a prime. A finite group of Lie type is defined as the group of fixed points of $\mathbf{G}$ by a Frobenius endomorphism F. The order of a finite group of Lie type is given by a polynomial in q, where q is a power of the prime p, defined by the Frobenius endomorphism. First, we present a result that describes the order of F-fixed points of maximal tori, see [92, Proposition 25.2], [21, Proposition 3.3.5], or [38, Proposition 1.6.6].

Proposition 1.3.8. *Let $\mathbf{G}$ be a connected reductive algebraic group with Frobenius endomorphism F defining an $\mathbb{F}_q$-structure. Let $\mathbf{T}$ be an F-stable maximally split torus and let $X = X(\mathbf{T})$ be the group of characters of $\mathbf{T}$. Then*

$$|\mathbf{T}^F| = |\det_{X_\mathbb{R}}(F - 1)| = |\det_{X_\mathbb{R}}(q\phi - 1)|,$$

where $X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R}$, and $F|_X = q\phi$. In particular, $|\mathbf{T}^F|$ is a monic polynomial in the variable q with non-zero constant term and degree $\dim(\mathbf{T})$.

We now report a classical bound for the cardinality of a torus in a finite group of Lie type. This is a well know result that we record here with its proof.

Corollary 1.3.9. *Let $\mathbf{T}$ be an F-stable maximal torus of a connected reductive algebraic group $\mathbf{G}$. Then*

$$(q-1)^{\dim \mathbf{T}} \leqslant |\mathbf{T}^F| \leqslant (q+1)^{\dim \mathbf{T}}.$$

Proof. Let $X = X(\mathbf{T})$ be the group of characters of T and $X_\mathbb{R} = X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R}$ the real vector space associated to X. The action of F on X is given by $F(\chi)(t) = \chi(F(t))$ for $\chi \in X$ and $t \in \mathbf{T}$. We have seen that we can describe the action of F on $X_\mathbb{R}$ as the product of q and $\phi \in \mathrm{Aut}(X_\mathbb{R})$. Then we write $F|_X = q\phi$. The possible orders of ϕ are collected in Table 1.2. Since ϕ has finite order, all its eigenvalues are roots of unity. Let r be the dimension of $\mathbf{T}$. By Proposition 1.3.8, the order of the group of F-fixed points in $\mathbf{T}$ is given by a monic polynomial in q with non-zero constant term. Moreover, since the morphism ϕ has finite order, then there exist roots of unity $\varepsilon_1, \dots, \varepsilon_r$ such that

$$|\mathbf{T}^F| = (q - \varepsilon_1) \cdots (q - \varepsilon_r).$$

Hence the result follows immediately by taking absolute values of the factors of the right-hand suce and using the triangle inequality. □

The number of F-stable maximal tori of $\mathbf{G}$ is given by the following theorem due to Steinberg, cf. [92, Theorem 25.5] or [21, Theorem 3.4.1].

Theorem 1.3.10. *Let $\mathbf{G}$ be a connected reductive algebraic group with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Then the number of F-stable maximal tori of $\mathbf{G}$ is $q^{2|\Phi^+|}$.*

Let H be a group and σ an (abstract group) automorphism of H. We say that h_1, h_2 are σ-conjugate if there exists an element $x \in H$ with $h_2 = \sigma(x)h_1x^{-1}$. The equivalence classes for this relation are called *σ-conjugacy classes.*

Let $\mathbf{G}$ be a connected reductive algebraic group with Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$, $\mathbf{T} \leqslant \mathbf{G}$ an F-stable maximally split torus with Weyl group W. We have seen in Corollary 1.3.5 that the coset $\mathbf{G}.F$ of F in $\mathbf{G} \rtimes F$ consists of a single conjugacy class. Now, let $w \in W$ and fix a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ of w. We define

$$\begin{aligned} wF\colon \mathbf{T} &\to \mathbf{T} \\ g &\mapsto \dot{w}^{-1}F(g)\dot{w}. \end{aligned}$$

Since $\mathbf{T}$ is F-stable, so is $N_{\mathbf{G}}(\mathbf{T})$. Hence F induces naturally an automorphism $\sigma_F\colon W \to W$, which for very $g \in N_{\mathbf{G}}(\mathbf{T})$ sends $g\mathbf{T}$ to $F(g)\mathbf{T}$. Moreover, σ_F satisfies the condition $\sigma_F(w) = \phi^{-1} \circ w \circ \phi$, for $w \in W$, see [38, Section 1.6.1]. Thus, we can regard ϕ as an automorphism of W and consider the semidirect product $\tilde{W} = W \rtimes \langle\tilde{\phi}\rangle$. We interpret W as a subgroup of $\tilde{W}$ and $\tilde{\phi} \in \tilde{W}$ is essentially ϕ. Hence, in $\tilde{W}$, we have the identity

$$\tilde{\phi}w\tilde{\phi}^{-1} = \phi(w) \quad \text{for all } w \in W.$$

Note that $\tilde{\phi}(w\tilde{\phi})\tilde{\phi}^{-1} = w^{-1}(w\tilde{\phi})w$ for $w \in W$. The ϕ-conjugacy classes of W corresponds to usual conjugacy classes of $\tilde{W}$ via the map $w \mapsto w\tilde{\phi}$, cf. [38, Remark 2.1.9].

There is a connection between certain conjugacy classes of maximal tori and ϕ-conjugacy classes in the Weyl group. More precisely, we have the following result, see [92, Proposition 25.1] or [21, Proposition 3.3.3].

Proposition 1.3.11. *Let $\mathbf{G}$ be connected reductive algebraic group with Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$, $\mathbf{T} \leqslant \mathbf{G}$ an F-stable maximal torus with Weyl group W and X group of characters of $\mathbf{T}$. There is a natural bijection*

$$\left\{\begin{matrix} G^F\text{-conjugacy classes of} \\ F\text{-stable maximal tori of } \mathbf{G} \end{matrix}\right\} \longleftrightarrow \{\phi\text{-conjugacy classes in } W\},$$

where $F|_X = q\phi$.

Remark 1.3.12. *Let $\mathbf{T}$ and $\mathbf{T}_1$ be two F-stable maximal tori of $\mathbf{G}$. All maximal tori are conjugate, hence there exists $g \in \mathbf{G}$ such that $\mathbf{T}_1 = g^{-1}\mathbf{T}g$. Thus we have $F(g^{-1})\mathbf{T}F(g) = F(\mathbf{T}_1) = \mathbf{T}_1 = g^{-1}\mathbf{T}g$. Using Theorem 1.3.4, $F(g)g^{-1}$ is an element of $\mathbf{G}$ and in particular $F(g)g^{-1} \in N_{\mathbf{G}}(\mathbf{T})$. Let $w_g = F(g)g^{-1}\mathbf{T}$ be the corresponding element of the Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/C_{\mathbf{G}}(\mathbf{T})$. The torus $\mathbf{T}_1$ corresponds to the ϕ-conjugacy class of $w_g \in W$. We write $\mathbf{T}_{w_g} = \mathbf{T}_1 = g^{-1}\mathbf{T}g$.*

If $\mathbf{T}$ is also maximally split, the ϕ-conjugacy class of w_g is called the type of the F-stable maximal torus $\mathbf{T}_{w_g}$. If $\mathbf{T}_{w_g}$ is of type w_g, the pair $(\mathbf{T}_{w_g}, F)$ is sent by g^{-1}-conjugation to the pair $(\mathbf{T}, wF)$. Compare with [30, Section 4.2], [38, Section 1.6.4], *or* [92, Section 25.1].

If we want to compute the cardinality of $\mathbf{T}_w^F$, one uses Proposition 1.3.8 to get the following result, cf. [92, Proposition 25.3], [30, Proposition 4.4.9], or [38, Proposition 1.6.6].

Proposition 1.3.13. *Let $\mathbf{G}$ be a connected reductive algebraic group with Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$, $\mathbf{T} \leqslant \mathbf{G}$ an F-stable maximal torus with Weyl group W and character group X. Then for $w \in W$ we have*

$$|\mathbf{T}_w^F| = |\det_{X_{\mathbb{R}}}(wF - 1)| = |\det_{X_{\mathbb{R}}}(q - (w\phi)^{-1})|,$$

where $X_{\mathbb{R}} = X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $F|_X = q\phi$.

As for the order of a finite group of Lie type, we have the following result, see [38, Theorem 1.6.7 and Remark 1.6.15] or [92, Corollary 24.6].

Proposition 1.3.14. *Let $\mathbf{G}$ be a connected reductive algebraic group with Frobenius endomorphism F defining an $\mathbb{F}_q$-structure. Let $\mathbf{T}$ be an F-stable maximally split torus and Φ be the associated root system. Then there exists a monic polynomial $r(x) \in \mathbb{Q}[x]$ with non-zero constant term and of degree $|\Phi|/2$ which is independent of q, but depends on the Weyl group W and the graph automorphism ϕ associated to F, such that*

$$|\mathbf{G}^F| = q^{|\Phi|/2}\, |\mathbf{T}^F|\, r(q).$$

In particular, $|\mathbf{G}^F|$ is a monic polynomial in the variable q and of degree $\dim(\mathbf{G})$.

We have the following result on the dual, see [21, Proposition 4.4.4] or [38, Example 1.6.19].

Proposition 1.3.15. *Let $\mathbf{G}$ be a connected reductive algebraic group with Frobenius endomorphism F defining an $\mathbb{F}_q$-structure. If $(\mathbf{G}, F)$ is in duality with $(\mathbf{G}^*, F^*)$, then*

$$|\mathbf{G}^F| = |\mathbf{G}^{*F^*}|.$$

In conclusion of this section, we introduce the *complete root datum* $\mathbb{G} = ((X, \Phi, Y, \Phi^\vee), \phi W)$ for a finite group of Lie type $\mathbf{G}^F$ given by the root datum of $\mathbf{G}$ with respect to a maximal torus $\mathbf{T}$ with group of characters $X = X(\mathbf{T})$, and the coset $\phi W = \{\phi \circ w \mid w \in W\}$, where W is the Weyl group of $\mathbf{G}$ and ϕ is the automorphism of $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$ induced by F, cf. [38, Definition 1.6.10] or [92, Definition 22.10]. If we fix the root complete root datum, it make sense to talk about a "series of finite groups of Lie type" $\{\mathbb{G}(q)\}$, indexed by a parameter q which is a power of a prime p, cf. [38, Remark 1.6.12].

1.3.3 Regular elements

Let $\mathbf{G}$ be a connected reductive algebraic group. An element $g \in \mathbf{G}$ is called *regular* if the dimension of its centralizer is minimal among all the elements of $\mathbf{G}$. The connected centralizer of a semisimple regular element g is a maximal torus $\mathbf{T}$ that contains g and a semi-simple element is regular if and only if it is contained in only one maximal torus, see [30, Proposition 12.1.6] and [92, Corollary 14.10]. Moreover, regular semisimple elements are dense in $\mathbf{G}$, see [30, Corollary 12.1.9] and [92, Corollary 14.10]. This implies that, considering a Frobenius endomorphism F which defines an $\mathbb{F}_q$-structure on $\mathbf{G}$, the set of regular elements in $\mathbf{G}^F$ is non-empty for q large enough. More precisely we have the following result, cf. [38, Lemma 2.3.11].

Lemma 1.3.16. *Let $\mathbf{T}$ be an F-stable maximal torus of a connected reductive algebraic group $\mathbf{G}$ and let $\mathbf{T}_{\mathrm{reg}}$ be the subset of regular elements contained in $\mathbf{T}$. Then there is a constant $C > 0$, that depends only on the root datum of $\mathbf{G}$ such that*

$$|\mathbf{T}_{\mathrm{reg}}^F|/|\mathbf{T}^F| \geqslant 1 - C/q.$$

1.3.4 Generation

Let G be a group. We denote by $d(G)$ the minimal number of generators of G.

Proposition 1.3.17. *Let G be a finite perfect group. Then the center $Z(G)$ is contained in the Frattini subgroup $\Phi(G)$.*

Proof. Suppose that the center $Z(G)$ is not contained in the Frattini subgroup $\Phi(G)$. The Frattini subgroup of G is the intersection of all maximal proper subgroups of G. Then there exists a maximal subgroup M of G that does not contain $Z(G)$. By maximality of M, it follows that $G = MZ(G)$. This leads to a contradiction with G being perfect, since $[G, G] = [MZ(G), MZ(G)] \subseteq M$ which is strictly contained in G. $\square$

As a consequence of the classification of finite simple groups, every finite non-abelian simple group is 2-generated, cf. [86, Window 2, Theorem 3]. Using the previous proposition we have the following result.

Corollary 1.3.18. *Every covering group of a finite non-abelian simple group is 2-generated.*

We will consider Cartesian products of finite quasi-simple and non-abelian simple groups. Let G be a finite group, and $S_1, S_2, \ldots, S_r$ the different non-abelian simple groups (if any) that are images of G. Let λ_i, with $i \in \{1, \ldots, r\}$ be the highest power of S_i such that $S_i^{\lambda_i}$ is an image of G. Then by [130], we have

$$d(G^b) = \max\big(d(G), b\, d(G/[G,G]), d(S_1^{\lambda_1 b}), \ldots, d(S_r^{\lambda_r b})\big) \quad \text{for } b \in \mathbb{N}_*. \tag{1.3.2}$$

If G is a finite non-abelian simple group, then the number of generators of $G^{|G|^b}$ for every $b \in \mathbb{N}$, is given by the theorem in [130]. If G is not simple, then the number of generators of G^b for every $b \in \mathbb{N}$, can be obtained by a combination of Corollary 1.3.18, (1.3.2), and the theorem in [130]. We combine them in the following result.

Theorem 1.3.19. *Let G be a non-abelian finite simple group or more generally any central cover of a such group. Then $d(G^{|G|^b}) = b + 2$ for all $b \in \mathbb{N}$.*

1.4 Characters of finite (simple) groups of Lie type

The characters of finite groups of Lie type are described by the theory of Deligne and Lusztig which decomposes the set of irreducible characters into series. Such a partition is indexed by representatives of conjugacy classes of semisimple elements in the dual group. We give an overview of Deligne-Lusztig theory and of the most relevant results. We refer the reader to [21], [30], and [38] for more details.

Let $\mathbf{G}$ be a connected reductive algebraic group, F a Frobenius endomorphism, $\mathbf{T}$ an F-stable maximal torus, and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. A *virtual character* of a group G is an integral linear combination of irreducible characters of G. Deligne and Lusztig, in [29], [87], and [88], constructed a virtual character denoted by $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, that is called a *Deligne-Lusztig character*. Two Deligne–Lusztig characters are either equal or orthogonal to each other, see [38, Corollary 2.2.10], [30, Corollary 9.3.1 (iii)], or [21, Corollary 7.3.7 and Theorem 7.3.8]. If $\mathbf{T}_1$ and $\mathbf{T}_2$ are two F-stable maximal tori of $\mathbf{G}$ and $\theta_1 \in \mathrm{Irr}(\mathbf{T}_1^F)$ and $\theta_2 \in \mathrm{Irr}(\mathbf{T}_2^F)$, we have $R_{\mathbf{T}_1}^{\mathbf{G}}(\theta_1) = R_{\mathbf{T}_2}^{\mathbf{G}}(\theta_2)$ if and only if there exists some $g \in \mathbf{G}^F$ such that $\mathbf{T}_1^g = \mathbf{T}_2$ and $\theta_1^g = \theta_2$. Moreover, there are a scalar product and a degree formula, which we present in the following proposition, cf. [30, Corollary 9.3.1, Lemma 7.1.6, and Proposition 10.2.2], [38, Theorem 2.2.8, Corollary 2.2.9, Theorem 2.2.12, and Proposition 2.5.18], or [21, Theorem 7.3.4 and Theorem 7.5.1].

Proposition 1.4.1. *Let $\mathbf{G}$ be a connected reductive algebraic group and let F be a Frobenius endomorphism corresponding to an $\mathbb{F}_q$-structure, where q is a power of a prime p. For an F-stable maximal torus $\mathbf{T}$, let W be the Weyl group associated to it.*

- ***Scalar product formula.*** *Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be F-stable maximal tori. Let $\theta_1 \in \mathrm{Irr}(\mathbf{T}_1^F)$ and $\theta_2 \in \mathrm{Irr}(\mathbf{T}_2^F)$. Then*

$$\langle R_{\mathbf{T}_1}^{\mathbf{G}}(\theta_1), R_{\mathbf{T}_2}^{\mathbf{G}}(\theta_2)\rangle_{\mathbf{G}^F} = \left|\{g \in \mathbf{G}^F \mid g\mathbf{T}_1 g^{-1} = \mathbf{T}_2 \text{ and } {}^g\theta_1 = \theta_2\}\right| / |\mathbf{T}_1^F|.$$

- ***Degree formula.*** *Let $\mathbf{T}$ be an F-stable maximal torus of type $w \in W$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. Then*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1) = (-1)^{l(w)} |\mathbf{G}^F : \mathbf{T}^F|_{p'},$$

  where $l(w)$ is the length of w.

- ***Bounds for Deligne-Lusztig characters.*** *For every* $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ *and* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$*, we have*

$$-|W|^{1/2} \leqslant \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi \rangle \leqslant |W|^{1/2}.$$

The number of $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ *that occur in* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *is at most* $|W|$.

A notable insight in Deligne-Lusztig theory was to point out the relevance of the dual group $\mathbf{G}^*$. It is made clear in the following result, cf. [21, Proposition 4.4.1], [38, Lemma 2.5.7], or [30, 11.1.7 and 11.1.14].

Proposition 1.4.2. *Let* $\mathbf{G}$ *be a connected reductive algebraic group,* F *a Frobenius endomorphism, and* $\mathbf{T}$ *an* F*-stable maximal torus of* $\mathbf{G}$*. Let* $\mathbf{G}^*$ *be a connected reductive group with maximal torus* $\mathbf{T}^*$ *and let* F^* *be a Frobenius endomorphism of* $\mathbf{G}^*$ *such that* $(\mathbf{G}, F)$ *and* $(\mathbf{G}^*, F^*)$ *are in duality. Then the dual map* $\delta : X \to Y^*$ *gives rise to an isomorphism between* $\mathbf{T}^{*F^*}$ *and* $\mathrm{Irr}(\mathbf{T}^F)$.

Using this result, one can relate a pair of the form $(\mathbf{T}, \theta)$ with a corresponding semisimple element g_s in $\mathbf{T}^{*F^*}$, leading to a corresponding pair of the type $(\mathbf{T}^*, g_s)$. More precisely, we have the following result, see [30, Proposition 11.1.16] and cf. [38, Corollary 2.5.14].

Proposition 1.4.3. *There is a canonical bijection between the* $\mathbf{G}^F$*-orbits of pairs* $(\mathbf{T}, \theta)$*, where* $\mathbf{T}$ *is an* F*-stable maximal torus of* $\mathbf{G}$ *and* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$*, and* $\mathbf{G}^{*F^*}$*-orbits of pairs* $(\mathbf{T}^*, g_s)$*, where* g_s *is a semisimple element of* $\mathbf{G}^{*F^*}$ *and* $\mathbf{T}^*$ *is an* F^**-stable maximal torus containing* g_s.

Using Proposition 1.4.3, given a pair $(\mathbf{T}, \theta)$ and a corresponding pair $(\mathbf{T}^*, g_s)$ in the dual group, we denote the Deligne-Lusztig character by

$$R_{\mathbf{T}^*}^{\mathbf{G}}(g_s) = R_{\mathbf{T}}^{\mathbf{G}}(\theta).$$

The *rational series* $\mathcal{E}(\mathbf{G}^F, g_s)$ of irreducible characters of $\mathbf{G}^F$ associated with g_s is the set of irreducible characters of $\mathbf{G}^F$ which occur in some Deligne-Lusztig character $R_{\mathbf{T}^*}^{\mathbf{G}}(g_s)$, for some F^*-stable maximal torus $\mathbf{T}^*$, where $g_s \in \mathbf{T}^{*F^*}$ is semisimple. The series $\mathcal{E}(\mathbf{G}^F, 1)$ corresponding to the trivial element is the set of *unipotent characters* of $\mathbf{G}^F$. We have the following decomposition of characters, cf. [30, Proposition 11.3.2], [38, Theorem 2.6.2], or [30, Proposition 12.4.4].

Theorem 1.4.4. *If* $g_1, g_2 \in \mathbf{G}^{*F^*}$ *are semisimple and conjugate in* $\mathbf{G}^{*F^*}$*, then* $\mathcal{E}(\mathbf{G}^F, g_1) = \mathcal{E}(\mathbf{G}^F, g_2)$*. There is a partition*

$$\mathrm{Irr}(\mathbf{G}^F) = \bigsqcup_{g_s} \mathcal{E}(\mathbf{G}^F, g_s),$$

where g_s *runs over a set of representatives of the conjugacy classes of semisimple elements in* $\mathbf{G}^{*F^*}$.

The non-unipotent characters of $\mathbf{G}^F$ are related to unipotent characters of a (usually) smaller reductive group, which is the centralizer $C_{\mathbf{G}^*}(g_s)^{F^*}$ of the semisimple element g_s of the corresponding rational series. If the centre $Z(\mathbf{G})$ is not connected, then the centralizer $C_{\mathbf{G}^*}(g_s)$ may not be connected, see Theorem 1.2.9. We denote by $\mathcal{E}(C_{\mathbf{G}^*}(g_s)^{F^*}, 1)$ the set of irreducible constituents of the induced unipotent characters of the connected part of the centralizer, see the discussion before [30, Theorem 11.5.1] and in [38, Remark 2.6.26].

The relation between non-unipotent characters and unipotent characters of centralizers of semisimple elements is called *Jordan decomposition of characters* of the finite group $\mathbf{G}^F$. Recall, for $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ in duality, that the cardinality of the group $\mathbf{G}^F$ equals to $\mathbf{G}^{*F^*}$, cf. Proposition 1.3.15. We state the Jordan decompostion in the following theorem, cf. [30, Theorem 11.5.1 and Proposition 11.5.6] or [38, Theorem 2.6.22 and Remark 2.6.26].

Theorem 1.4.5. *Let $\mathbf{G}$ be a connected reductive algebraic group with a Frobenius endomorphism F. For every semisimple element $g_s \in \mathbf{G}^{*F^*}$, there is a bijection*

$$\psi_{g_s} : \mathcal{E}(\mathbf{G}^F, g_s) \to \mathcal{E}(C_{\mathbf{G}^*}(g_s)^{F^*}, 1).$$

Moreover, for every $\chi \in \mathcal{E}(\mathbf{G}^F, g_s)$ we have

$$\chi(1) = |\mathbf{G}^{*F^*} : C_{\mathbf{G}^*}(g_s)^{F^*}|_{p'}\ \psi_{g_s}(\chi)(1), \tag{1.4.1}$$

where $|\cdot|_{p'}$ denotes the p-prime part of a natural number.

It is natural to further investigate the unipotent characters, as we have just seen that many questions about arbitrary irreducible characters can be reduced to problems concerning unipotent characters. The study of unipotent characters of $\mathbf{G}^F$ can be reduced to the case when $\mathbf{G}$ is simple of adjoint type, cf. [38, Proposition 2.3.15 and Remark 4.2.1]. A detailed description of unipotent characters can be found in [21, Sections 13.8 and 13.9] or in [38, Chapter 4]. In particular, the degrees of all irreducible unipotent characters have been determined. We summarize the results on unipotent characters that are relevant to us in the following remarks.

Remark 1.4.6. *For each finite group of Lie type $\mathbf{G}^F$, the number of irreducible unipotent characters $k_{\mathbf{G}}$ depends only on the associated root system Φ. Furthermore, there exist $k_{\mathbf{G}}$ polynomials $f_{\mathbf{G}^F,1}(x), \ldots, f_{\mathbf{G}^F,k_{\mathbf{G}}}(x) \in \mathbb{Q}[x]$, depending only on Φ such that the degrees of the unipotent characters are precisely $f_{\mathbf{G}^F,1}(q), \ldots, f_{\mathbf{G}^F,k_{\mathbf{G}}}(q)$. Additionally, among these polynomials, only one has degree zero. This is the constant polynomial 1, indicating the degree of the trivial character, see* [21, Sections 13.8 and 13.9] *or* [38, Sections 4.3, 4.4, and 4.5].

Example 1.4.7. For the type A_n, the polynomials that describe the degrees of the unipotent characters are parametrized by partitions of $n+1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a partition of $n+1$, with $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$, and let

$$\lambda_1 = \alpha_1,\ \lambda_2 = \alpha_2 + 1,\ \lambda_3 = \alpha_3 + 2,\ \ldots\ , \lambda_m = \alpha_m + m - 1.$$

Then the degree of the unipotent character χ^α corresponding to α is given by

$$\chi^\alpha(1) = \frac{\prod_{i=1}^{n+1}(q^i - 1)\prod_{i<j}(q^{\lambda_j} - q^{\lambda_i})}{q^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots+1}\prod_{i=1}^{m}\prod_{k=1}^{\lambda_i}(q^k - 1)},$$

where m is the number of parts of α, see [21, Section 13.8] or [38, Proposition 4.3.2].

Remark 1.4.8. *The trivial character and the Steinberg character of a finite group of Lie type $\mathbf{G}^F$ are unipotent, see* [38, Example 2.3.9].

We are interested in the irreducible representations of finite simple groups of Lie type. In view of Remark 1.3.2, we are investigating

$$\operatorname{Irr}(\mathbf{G}^F/Z(\mathbf{G}^F)) = \{\chi \in \operatorname{Irr}(\mathbf{G}^F) \mid Z(\mathbf{G}^F) \subseteq \ker \chi\}. \tag{1.4.2}$$

For every irreducible character $\chi \in \operatorname{Irr}(\mathbf{G}^F)$, there exists a pair $(\mathbf{T}, \theta)$ such that $\langle R^{\mathbf{G}}_{\mathbf{T}}(\theta), \chi\rangle \neq 0$, see [38, Corollary 2.2.19]. Moreover, the character values of an irreducible character χ of $\mathbf{G}^F$ on the elements of the centre $Z(\mathbf{G})^F$ are completely determined by the character values of the corresponding irreducible representation θ of a torus $\mathbf{T}$ such that χ is a non-trivial irreducible component of $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$. More precisely we have the following result, see [38, Proposition 2.2.20].

Proposition 1.4.9. *Let $\mathbf{G}$ be a connected reductive algebraic group, $\mathbf{T} \subseteq \mathbf{G}$ an F-stable maximal torus, and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$. Let $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ be such that $\langle R^{\mathbf{G}}_{\mathbf{T}}(\theta), \chi\rangle \neq 0$. Then $\chi(z) = \theta(z)\chi(1)$ for all $z \in Z(\mathbf{G})^F$.*

The following proposition describes the connection between representations whose kernel contains the center and rational series.

Proposition 1.4.10. *Let $\mathbf{G}$ be a simply connected simple algebraic group and let F be a Frobenius endomorphism. Let $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ and let $g_s \in \mathbf{G}^{*F^*}$ be a semisimple element such that χ is contained in the rational series $\mathcal{E}(\mathbf{G}^F, g_s)$. Then*

$$Z(\mathbf{G}^F) \subseteq \ker \chi \iff g_s \in [\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}].$$

Proof. Let $\mathbf{T}^*$ be a F^*-stable maximal torus of $\mathbf{G}^*$ such that $g_s \in \mathbf{T}^{*F^*}$ and $\chi \in R^{\mathbf{G}}_{\mathbf{T}^*}(g_s)$. By the duality described in Proposition 1.4.3, the pair $(\mathbf{T}^*, g_s)$ corresponds to a pair $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$. Recall that $R^{\mathbf{G}}_{\mathbf{T}^*}(g_s) = R^{\mathbf{G}}_{\mathbf{T}}(\theta)$. By Proposition 1.4.9, we have

$$Z(\mathbf{G}^F) \subseteq \ker \chi \iff Z(\mathbf{G}^F) \subseteq \ker \theta.$$

Our claim follows by using [99, Lemma 4.4 (ii)], which states that for a semisimple element g_s of $[\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}]$ and if $|Z(\mathbf{G}_{\mathrm{sc}}^F)| = |\mathbf{G}_{\mathrm{sc}}^F/[\mathbf{G}_{\mathrm{ad}}^F, \mathbf{G}_{\mathrm{ad}}^F]|$ then all $\chi \in \mathcal{E}(\mathbf{G}^F, g_s)$ restrict trivially on $Z(\mathbf{G}_{\mathrm{sc}}^F)$. The equality between the cardinality of the center and of the abelianisation of $\mathbf{G}_{\mathrm{sc}}^F$ follows from Corollary 1.3.7, which gives $[\mathbf{G}_{\mathrm{ad}}^F, \mathbf{G}_{\mathrm{ad}}^F] \cong \mathbf{G}_{\mathrm{sc}}^F/Z(\mathbf{G}_{\mathrm{sc}}^F)$. □

Let us restrict to the setting of Remark 1.3.2, i.e. we consider the groups of fixed points of a simply connected simple algebraic group under a Frobenius endomorphism. As before, we denote such groups by $L(q)$. A lower bound for the dimension of any non-trivial irreducible representation of such groups of Lie type is given by the following proposition, cf. [80, Proposition 3.1 (ii)].

Proposition 1.4.11. *There is an absolute constant $d > 0$ such that for any finite quasi-simple group $L(q)$ as defined above of rank $r = \operatorname{rk}\Phi$, where Φ is the root system associated with $L(q)$, every non-trivial irreducible character χ of $L(q)$ satisfies*

$$\chi(1) > dq^r.$$

Remark 1.4.12. *Observe that, simply by (1.4.2), this lower bound for non-trivial representations of groups of Lie type $L(q)$, is valid also for $S(q)$.*

1.5 Approximations

A Dirichlet generating function associated to an arithmetic sequence $\{a_n\}_{n=1}^{\infty}$ is given by

$$\sum_{n=1}^{\infty} a_n n^{-s},$$

where s is a complex variable, cf. Section 0.7.

We introduce a notion of approximating Dirichlet generating functions with non-negative integer coefficients, as defined in [12, Definition 2.4], which we will use to study representation zeta functions.

Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be Dirichlet generating functions with non-negative integer coefficients and with abscissae of convergences α_f and α_g. Let $C \in \mathbb{R}$ and let $\sigma_0 \in \mathbb{R}$ with $\sigma_0 \geq \max\{\alpha_f, \alpha_g, 0\}$. We write

$$f \lesssim_C g \quad \text{for} \quad \sigma > \sigma_0$$

if $f(\sigma) \leq C^{1+\sigma} g(\sigma)$ for every $\sigma \in \mathbb{R}$ with $\sigma > \sigma_0$. If $\alpha_f = \alpha_g$ and if $f \lesssim_C g$ for $\sigma > \max\{0, \alpha_f\}$, then we do not specify the domain and we write $f \lesssim_C g$. If $f \lesssim_C g$ and $g \lesssim_C f$ then we write

$$f \sim_C g.$$

Representation zeta functions of quasi-semisismple profinite groups can be "approximated by" products of Dirichlet polynomials. We denote by A the collection of all finite subsets $a \subset \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{>0} \cup \{(0,0)\}$ and $A^+ = \{a \in A \mid (0,0) \notin a\}$. Define for $a \in A$ and for $q \in \mathbb{N}_{\geqslant 2}$, the Dirichlet polynomial

$$\xi_{a,q}(s) = \sum_{(m,n)\in a} q^{m-ns}. \tag{1.5.1}$$

For $a, b \in A$, we can compare the associated Dirichlet polynomials by studying the inclusions of the "north-west"-Newton polytopes $\mathcal{N}(a)$ and $\mathcal{N}(b)$ associated with a and b respectively, i.e. the convex hulls of $\bigcup\{u + (\mathbb{R}_{\leqslant 0} \times \mathbb{R}_{\geqslant 0}) \mid u \in a\}$ and of $\bigcup\{u + (\mathbb{R}_{\leqslant 0} \times \mathbb{R}_{\geqslant 0}) \mid u \in b\}$.

Remark 1.5.1. *As noted in* [12, Remark 2.7], *we have $\mathcal{N}(a) \subset \mathcal{N}(b)$ if and only if there exits $C \in \mathbb{R}$ such that $\xi_{a,q}(s) \lesssim_C \xi_{b,q}(s)$ for all $q \in \mathbb{N}_{\geqslant 2}$. Moreover, if $\mathcal{N}(a) \subset \mathcal{N}(b)$ we can take $C = |a|$.*

We will use these approximations to compute the abscissa of convergence of the representation zeta functions of Cartesian products of groups. In particular, we will make use of the following result, see [12, Lemma 2.5].

Lemma 1.5.2. *Let f, g be Dirichlet generating functions with abscissae of convergence α_f, α_g. Suppose we can find two sequences $\{f_m\}_{m=1}^{\infty}$ and $\{g_m\}_{m=1}^{\infty}$ of Dirichlet generating functions with vanishing constant terms, such that $f = \prod_{m=1}^{\infty}(1+f_m)$ and $g = \prod_{m=1}^{\infty}(1+g_m)$. Let β_m denote the abscissa of convergence of g_m for every $m \in \mathbb{N}_*$. Furthermore, assume that for each $\varepsilon > 0$, there is $C(\varepsilon) \in \mathbb{R}_{>0}$ such that, for all m, $f_m \lesssim_{C(\varepsilon)} g_m$ for $\sigma > \beta_m + \varepsilon$. Then $\alpha_f \leqslant \alpha_g$.*

Remark 1.5.3. *For any two sequences $(x_j)_{j\in\mathbb{N}_*}$ and $(y_j)_{j\in\mathbb{N}_*}$ of positive real numbers, the product*

$$\prod_{j=1}^{\infty}(1 + x_j + y_j)$$

converges if and only if $\prod_{j=1}^{\infty}(1 + x_j)$ and $\prod_{j=1}^{\infty}(1 + y_j)$ converge individually.

The following result concerns the Dirichlet polynomial of a finite group of Lie type $\mathbf{G}^F$, given in [12, Theorem 3.1]. Our notation of $\mathbf{G}^F$ replaces the notation used in [12] of $\mathbf{G}(\mathbb{F}_q)$. Moreover, as in [12], a Lie type refers to a pair (Φ, τ), where Φ is a root system and τ is an automorphism preserving a choice of positive roots Φ^+. A finite group of Lie type $\mathbf{G}^F$ has Lie type (Φ, τ) if the algebraic group $\mathbf{G}$ has root system Φ and the action of the Frobenius endomorphism induces the same action of τ on the root system Φ; compare [30, Chapter 4] and [118, Chapter 15]. We report on [12, Theorem 3.1] in the special case of interest to us.

Theorem 1.5.4. *Let Φ be a non-trivial irreducible root system, and let $\mathcal{L}_\Phi$ denote the collection of Lie types with underlying root system Φ. Let $\mathcal{Q}$ be the set of all prime powers. Then there exist a constant $C \in \mathbb{R}$, a finite set $a(\Phi) \in A^+$, and $a((\Phi, \tau), q) \in A^+$ for $((\Phi, \tau), q) \in \mathcal{L}_\Phi \times \mathcal{Q}$ such that the following hold:*

(i) $a((\Phi,\tau),q) \subseteq a(\Phi)$ *for all* $((\Phi,\tau),q) \in \mathcal{L}_\Phi \times \mathcal{Q}$,

(ii) for every finite group of Lie type $\mathbf{G}^F$ *which has Lie type* $(\Phi,\tau) \in \mathcal{L}_\Phi$ *and where* $\mathbf{G}$ *is a connected simply connected simple algebraic group defined over an algebraic closure* $\mathbb{F}$ *of* $\mathbb{F}_q$*, and* F *is a Frobenius endomorphism defining an* $\mathbb{F}_q$*-structure (with* $q > 3$*),*

$$\zeta_{\mathbf{G}^F}(s) - 1 \sim_C \xi_{a((\Phi,\tau),q),q}(s).$$

Moreover, $(\mathrm{rk}(\Phi), |\Phi^+|) \in a(\Phi)$*, and we have*

$$\zeta_{\mathbf{G}^F}(s) \sim_C 1 + q^{\mathrm{rk}(\Phi)-|\Phi^+|s}.$$

The proof of this theorem uses the Deligne-Lusztig decomposition of characters of finite groups of Lie type discussed in Section 1.4. Indeed, using Theorem 1.4.4, we can write the representation zeta function of a finite group of Lie type $\mathbf{G}^F$ as follows

$$\zeta_{\mathbf{G}^F}(s) = \sum_{g_s \in \mathcal{G}_s} \sum_{\chi \in \mathcal{E}(\mathbf{G}^F, g_s)} \chi(1)^{-s},$$

where $\mathcal{G}_s$ is a set of representatives of $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements g_s of $\mathbf{G}^{*F^*}$. We write

$$\zeta^{\mathrm{unip}}_{\mathbf{G}^F}(s) = \sum_{\chi \in \mathcal{E}(\mathbf{G}^F, 1)} \chi(1)^{-s}$$

for the sum of degrees of unipotent characters. This sum can be approximated by 1 with an appropriate constant by the following result, cf. [12, Proposition 3.5].

Proposition 1.5.5. *Let* Φ *be a non-trivial root system. Then there exist a constant* $C \in \mathbb{R}$*, and a finite set* $b(\Phi) \in A^+$ *such that for every* $\mathbf{G}$ *a connected reductive algebraic group with underlying root system* Φ*, and* F *a Frobenius endomorphism defining an* $\mathbb{F}_q$*-structure, there exists a finite set* $b(\Phi,\tau) \in A^+$ *such that* $b(\Phi,\tau) \subseteq b(\Phi)$ *and*

$$\zeta^{\mathrm{unip}}_{\mathbf{G}^F}(s) - 1 \sim_C \xi_{b(\Phi,\tau),q}(s) \quad \textit{and} \quad \zeta^{\mathrm{unip}}_{\mathbf{G}^F}(s) \sim_C 1.$$

The proof of this result is based on the study of unipotent characters. By Remark 1.4.6, we know that for each $\mathbf{G}^F$, there exists a number $k_{\mathbf{G}} \in \mathbb{N}$, that depends only on Φ and F, such that $\mathbf{G}^F$ has $k_{\mathbf{G}}$ irreducible unipotent characters. Then we can choose

$$b(\Phi,\tau) = \{(0, \deg f_{\mathbf{G}^F,i}) \mid 2 \leqslant i \leqslant k_{\mathbf{G}}\} \in A^+.$$

Now, we use Proposition 1.4.10 and the ideas from the proof of Theorem 1.5.4 to demonstrate that we achieve the same approximation with a different constant for finite simple groups of Lie type. We follow the framework developped in [12, Section 3]. Unlike in [12], we employ rational series instead of geometric ones. This choice was influenced by a mistake in a previous version of [30], which was subsequently corrected. Additionally, we provide more details in some critical passages concerning certain counting bounds. Finally, our setting necessitates different conditions, which are then resolved and bounded similarly to the approach in [12].

Theorem 1.5.6. *Let Φ be a non-trivial irreducible root system, and let $\mathcal{L}_\Phi$ denote the collection of Lie types with underlying root system Φ. Let $\mathcal{Q}$ be the set of all prime powers. Then there exist a constant $C \in \mathbb{R}$, a finite set $d(\Phi) \in A^+$, and $d((\Phi, \tau), q) \in A^+$ for $((\Phi, \tau), q) \in \mathcal{L}_\Phi \times \mathcal{Q}$ such that the following hold:*

(i) $d((\Phi, \tau), q) \subseteq d(\Phi)$ for all $((\Phi, \tau), q) \in \mathcal{L}_\Phi \times \mathcal{Q}$,

(ii) for every finite simple group of Lie type $\mathbf{G}^F/Z(\mathbf{G}^F)$, where $\mathbf{G}^F$ has Lie type $(\Phi, \tau) \in \mathcal{L}$ and where $\mathbf{G}$ is a connected simply connected simple algebraic group defined over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, F is a Frobenius endomorphism defining an $\mathbb{F}_q$-structure (with $q > 3$),

$$\zeta_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) - 1 \sim_D \xi_{d((\Phi,\tau),q),q}(s).$$

Furthermore, $(\mathrm{rk}(\Phi), |\Phi^+|) \in d(\Phi)$, *and we have*

$$\zeta_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) \sim_D 1 + q^{\mathrm{rk}(\Phi) - |\Phi^+| s}.$$

Proof. Let $\mathcal{G}_s$ be a set of representatives for the $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements g_s of $\mathbf{G}^{*F^*}$. Then we can write the representation zeta function of $\mathbf{G}^F$ as

$$\zeta_{\mathbf{G}^F}(s) = \sum_{g_s \in \mathcal{G}_s} \sum_{\chi \in \mathcal{E}(\mathbf{G}^F, g_s)} \chi(1)^{-s}.$$

Note that $\mathrm{Irr}(\mathbf{G}^F/Z(\mathbf{G}^F)) = \{\chi \in \mathrm{Irr}(\mathbf{G}^F) \mid Z(\mathbf{G}^F) \subseteq \ker \chi\}$. By Proposition 1.4.10, the representation zeta function of the group $\mathbf{G}^F/Z(\mathbf{G}^F)$ is

$$\zeta_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) = \sum_{g_s \in \bar{\mathcal{G}}_s} \sum_{\chi \in \mathcal{E}(\mathbf{G}^F, g_s)} \chi(1)^{-s}, \tag{1.5.2}$$

where $\bar{\mathcal{G}}_s$ is a set of representatives of $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements $g_s \in \mathbf{G}^{*F^*}$ such that $g_s \in [\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}]$. Without loss of generality we assume that $1 \in \bar{\mathcal{G}}_s$. Unipotent characters always restrict trivially to $Z(\mathbf{G}^F)$, and so the representation zeta function of $\mathbf{G}^F/Z(\mathbf{G}^F)$ restricted to unipotent characters is

$$\zeta^{\mathrm{unip}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) = \sum_{\chi \in \mathcal{E}(\mathbf{G}^F, 1)} \chi(1)^{-s} = \zeta^{\mathrm{unip}}_{\mathbf{G}^F}(s). \tag{1.5.3}$$

Moreover, by Proposition 1.5.5, there exists a constant $D_0 \in \mathbb{R}$ that depends only on Φ, and a finite set $b(\Phi, \tau) \in A^+$ such that

$$\zeta^{\mathrm{unip}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) - 1 = \zeta^{\mathrm{unip}}_{\mathbf{G}^F}(s) - 1 \sim_{D_0} \xi_{b(\Phi,\tau),q}(s) \quad \text{and} \quad \zeta^{\mathrm{unip}}_{\mathbf{G}^F}(s) \sim_{D_0} 1.$$

We are then left to deal with the non-unipotent characters

$$\zeta^{\mathrm{nu}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) = \sum_{g_s \in \bar{\mathcal{G}}_s \setminus \{1\}} \sum_{\chi \in \mathcal{E}(\mathbf{G}^F, g_s)} \chi(1)^{-s}.$$

By (1.4.1), we can decompose the characters in the following way

$$\zeta^{\mathrm{nu}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) = \sum_{g_s \in \bar{\mathcal{G}}_s \setminus \{1\}} |\mathbf{G}^{*F^*} : C_{\mathbf{G}^*}(g_s)^{F^*}|_{p'}^{-s} \cdot \zeta^{\mathrm{unip}}_{C_{\mathbf{G}^*}(g_s)^{F^*}}(s).$$

When $Z(\mathbf{G})$ is not connected, the unipotent characters of $C_{\mathbf{G}^*}(g_s)^{F^*}$ are the irreducible constituents of the induced unipotent characters of the connected part of the centralizer $C^\circ_{\mathbf{G}^*}(g_s)^{F^*}$, as we discussed in Section 1.4. Moreover, note that the index $|C_{\mathbf{G}^*}(g_s) : C^\circ_{\mathbf{G}^*}(g_s)|$ is bounded by a constant $D_1 \in \mathbb{R}$ that depends only on Φ by Proposition 1.2.11. Hence, employing again Proposition 1.5.5 for the unipotent part with a constant $D_2 \in \mathbb{R}$, we have

$$\zeta^{\mathrm{nu}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) \sim_{D_1 D_2} \sum_{g_s \in \bar{\mathcal{G}}_s \setminus \{1\}} |\mathbf{G}^{*F^*} : C^\circ_{\mathbf{G}^*}(g_s)^{F^*}|_{p'}^{-s}.$$

Let $\mathbf{K}_1, \ldots, \mathbf{K}_N$ be algebraic subgroups of $\mathbf{G}^*$ with associated root systems $\Psi_1, \ldots, \Psi_N$ respectively, which form a set of representatives for the $\mathbf{G}^{*F^*}$-conjugacy classes of connected parts $C^\circ_{\mathbf{G}^*}(g_s)$ of the centralizers of non-trivial semisimple elements $g_s \in \mathbf{G}^{*F^*}$. By Corollary 1.2.7, the number of centralizers $C_{\mathbf{G}}(g_s)$ of semisimple elements g_s is bounded by a constant that depends only on the root system Φ of $\mathbf{G}$. Every semisimple element g_s lies in $C^\circ_{\mathbf{G}}(g_s)$ by Theorem 1.2.6, and the torus $C^\circ_{\mathbf{G}}(g_s)$ is an F-stable maximal torus of $C_{\mathbf{G}}(g_s)$ and also of $\mathbf{G}$. Using Proposition 1.3.11, the number of $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori of $\mathbf{G}$ is bounded by the size of the Weyl group W, which depends only on Φ. Hence, the number N is bounded by a constant depending only on Φ. By Proposition 1.3.14 and Theorem 1.2.2 part (a), there is a constant D_3 such that

$$|\mathbf{G}^{*F^*} : C^\circ_{\mathbf{G}^*}(g_s)^{F^*}|_{p'} \sim_{D_3} q^{\dim \mathbf{G} - |\Phi^+| - \dim \mathbf{K}_i + |\Psi_i^+|} = q^{|\Phi^+| - |\Psi_i^+|}.$$

Here the notation $\sim$ means that the quantities on the different sides differ by a constant. With $D_4 := D_1 D_2 D_3$, we have

$$\zeta^{\mathrm{nu}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) \sim_{D_4} \sum_{i=1}^{N} |\mathcal{G}_i| \; q^{-(|\Phi^+| - |\Psi_i^+|)s}, \tag{1.5.4}$$

where $\mathcal{G}_i$ is a set of representatives g_s of a $\mathbf{G}^{*F^*}$-conjugacy class such that $g_s \in [\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}]$ and $C^\circ_{\mathbf{G}^*}(g_s)$ is $\mathbf{G}^{*F^*}$-conjugate to $\mathbf{K}_i$. Fix $i \in \{1, \ldots, N\}$. If $g_s \in \mathcal{G}_i$, then there exists an element g'_s which is a $\mathbf{G}^{*F^*}$-conjugate of g_s and such that $C^\circ_{\mathbf{G}^*}(g'_s) = \mathbf{K}_i$. Any other different element in the $\mathbf{G}^{*F^*}$-conjugacy class of g_s, such that their connected centralizer is equal to $\mathbf{K}_i$, is a conjugate by an element of $N_{\mathbf{G}^{*F^*}}(C^{\circ}_{\mathbf{G}^*}(g'_s))\backslash C^{\circ}_{\mathbf{G}^*}(g'_s)^{F^*}$. Employing [80, Lemma 2.2(ii)], there exists a constant D_5 which depends only on Φ, that bounds the number of elements of the form of g'_s. Thus

$$|\mathcal{G}_i| \sim_{D_5} |\{g_s \in \mathbf{G}^{*F^*} \,|\, g_s \text{ semisimple}, \, g_s \in [\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}], \text{and } C^{\circ}_{\mathbf{G}^*}(g_s) = \mathbf{K}_i\}|. \tag{1.5.5}$$

We denote the set on the right by $\mathcal{H}_i$. Let $\mathbf{G}^*_{\mathrm{sc}}$ be the simply connected cover of $\mathbf{G}^*$. Then by Proposition 1.2.5, we have an isogeny $\pi : \mathbf{G}^*_{\mathrm{sc}} \to \mathbf{G}^*$ with central kernel and by Proposition 1.3.6, whenever ${\mathbf{G}^*_{\mathrm{sc}}}^{F^*}$ is perfect, then $\pi({\mathbf{G}^*_{\mathrm{sc}}}^{F^*}) = [\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}]$, so we can find an element $h_i \in {\mathbf{G}^*_{\mathrm{sc}}}^{F^*}$ such that $\pi(h_i) = g_s$ for $g_s \in \mathcal{H}_i$. By Lemma 1.2.10, the isogeny π maps $C^{\circ}_{\mathbf{G}^*_{\mathrm{sc}}}(h_i)$ onto $C^{\circ}_{\mathbf{G}^*}(g_s)$. Write $\mathbf{H}_i = C^{\circ}_{\mathbf{G}^*_{\mathrm{sc}}}(h_i)$ and consider the set

$$\mathcal{H}_i^{\mathrm{sc}} = \{h_s \in {\mathbf{G}^*_{\mathrm{sc}}}^{F^*} \,|\, h_s \text{ semisimple and } C^{\circ}_{\mathbf{G}^*_{\mathrm{sc}}}(h_s) = \mathbf{H}_i\}.$$

The set $\mathcal{H}_i^{\mathrm{sc}}$ is non-empty because $h_i \in \mathcal{H}_i^{\mathrm{sc}}$.

Let $\mathbf{T}_i$ be a maximal torus of $\mathbf{H}_i$ and so also maximal in $\mathbf{G}^*_{\mathrm{sc}}$. Let Δ_i be the set of roots of $\mathbf{H}_i$ with respect to the torus $\mathbf{T}_i$ and Λ_i be the set of roots of $\mathbf{G}^*_{\mathrm{sc}}$ with respect to $\mathbf{T}_i$. Note that Λ_i is isomorphic to Φ. For every $\alpha \in \Delta_i$, the identity $\alpha(h_i) = 1$ holds, and by Theorem 1.2.6, we have

$$\mathbf{H}_i = \langle \mathbf{T}_i \cup \bigcup_{\alpha \in \Delta_i} \mathbf{U}_\alpha \rangle.$$

Consider h_s in $\mathcal{H}_i^{\mathrm{sc}}$, and observe that $h_s \in Z(\mathbf{H}_i)^{F^*}$. The algebraic group $\mathbf{H}_i$ is a connected reductive group by Theorem 1.2.6. Hence, $R(\mathbf{H}_i) = Z(\mathbf{H}_i)^{\circ}$ is a torus, $[\mathbf{H}_i, \mathbf{H}_i]$ is semisimple, $Z(\mathbf{H}_i)^{\circ} \cap [\mathbf{H}_i, \mathbf{H}_i]$ is finite, and $\mathbf{H}_i = Z(\mathbf{H}_i)^{\circ}[\mathbf{H}_i, \mathbf{H}_i]$, by Theorem 1.2.1. Thus, we can describe the center of $\mathbf{H}_i$ as $Z(\mathbf{H}_i) = Z(\mathbf{H}_i)^{\circ}A$ where $A \leqslant Z([\mathbf{H}_i, \mathbf{H}_i])$. Since $[\mathbf{H}_i, \mathbf{H}_i]$ is semisimple, the group $Z([\mathbf{H}_i, \mathbf{H}_i])$ is finite and so is A. The order and the exponent e of A are bounded by a constant that depends on Δ_i. Recall that the exponent of a group is the least common multiple of the orders of all elements of the group.

We distinguish between three cases. If the dimension of $Z(\mathbf{H}_i)$ is zero, we have just finitely many elements in $\mathcal{H}_i^{\mathrm{sc}}$, and this number is bounded by the cardinality of $Z(\mathbf{H}_i)$, which is finite and bounded by a constant that just depends on Δ_i. Thus there exists a constant C_1 such that

$$|\mathcal{H}_i^{\mathrm{sc}}| \sim_{C_1} 1.$$

In the second case, we assume that $\dim(Z(\mathbf{H}_i)) > 0$, and that $\mathbf{H}_i$ is a Levi subgroup of $\mathbf{G}^*_{\mathrm{sc}}$. Then by Proposition 1.2.13, $\mathbf{H}_i = C_{\mathbf{G}^*_{\mathrm{sc}}}(Z(\mathbf{H}_i)^{\circ})$ holds. Since $|Z(\mathbf{H}_i)/Z(\mathbf{H}_i)^{\circ}| \leqslant |A|$ is bounded by some constant depending on Λ_i, by [51, Corollary 9.7.9], we count first the elements in

$$\mathcal{T}_H = Z(\mathbf{H}_i)^{\circ} \setminus \{g \in Z(\mathbf{H}_i)^{\circ} \mid C^{\circ}_{\mathbf{G}^*_{\mathrm{sc}}}(g) \neq \mathbf{H}_i\}. \tag{1.5.6}$$

If $C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(g) \neq \mathbf{H}_i$, this means that $C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(g) \supsetneq \mathbf{H}_i$, and we have that $\beta(g) = 1$ for all $\beta \in \Delta_i$ and for some $\beta \in \Lambda_i \setminus \Delta_i$. By Theorem 1.2.2, we have that $Z(\mathbf{H}_i) = \bigcap_{\alpha\in\Delta_i} \ker\alpha$, so we can rewrite (1.5.6) as

$$(\bigcap_{\alpha\in\Delta_i} \ker\alpha)^\circ \setminus \bigcup_{\beta\in\Lambda_i\setminus\Delta_i} \ker\beta.$$

If $\dim(\ker\beta \cap Z(\mathbf{H}_i)) = \dim(Z(\mathbf{H}_i))$ for some $\beta \in \Lambda_i$, then $Z(\mathbf{H}_i)^\circ \subseteq \ker\beta$, and hence

$$U_\beta \subseteq C_{\mathbf{G}^*_{sc}}(Z(\mathbf{H}_i)^\circ) = \mathbf{H}_i,$$

which leads to a contradiction to our assumption. Thus for every $\beta \in \Lambda_i \setminus \Delta_i$ we have that $\dim(\ker\beta \cap Z(\mathbf{H}_i)) \lneqq \dim(Z(\mathbf{H}_i))$. Using the bound given by Corollary 1.3.9 for the torus $Z(\mathbf{H}_i)^{\circ F^*}$, there exists a constant C_2 that depends only on Δ_i such that

$$|\mathcal{H}_i^{\mathrm{sc}}| \sim_{C_2} q^{\dim Z(\mathbf{H}_i)}.$$

In the third case, we have $\dim(Z(\mathbf{H}_i)) > 0$ and we have that the subgroup $\mathbf{H}_i$ is not a Levi subgroup of $\mathbf{G}^*_{\mathrm{sc}}$. Then, again by Proposition 1.2.13 and Remark 1.2.14, the subgroup $C_{\mathbf{G}^*_{\mathrm{sc}}}(Z(\mathbf{H}_i)^\circ)$ is a Levi subgroup of $\mathbf{G}^*_{\mathrm{sc}}$ that we denote by $\mathbf{L}_i$. In particular, $\mathbf{H}_i \lneqq C_{\mathbf{G}^*_{\mathrm{sc}}}(Z(\mathbf{H}_i)^\circ) = \mathbf{L}_i$. Moreover, note that for every $t \in Z(\mathbf{H}_i)^\circ$ we have $C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(t) \geqslant C_{\mathbf{G}^*_{\mathrm{sc}}}(Z(\mathbf{H}_i)^\circ) = \mathbf{L}_i \gneqq \mathbf{H}_i$. Then the elements in $\mathcal{H}_i^{\mathrm{sc}}$ corresponds to elements in $Z(\mathbf{H}_i)$ of the form $x = ta$ where $t \in Z(\mathbf{H}_i)^\circ$ and a is a non-trivial element of A and such that $C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(x) = \mathbf{H}_i$. The latter condition is satisfied if $C^\circ_{\mathbf{L}_i}(a) = \mathbf{H}_i$ and $C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(t^e) = \mathbf{L}_i$, where e is the exponent of A since

$$\begin{aligned}\mathbf{H}_i \leqslant C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(at) \leqslant C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}((at)^e) \cap C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(at) &= C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(t^e) \cap C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(at) \\ &= \mathbf{L}_i \cap C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(at) = C^\circ_{\mathbf{L}_i}(a).\end{aligned}$$

This means that we need to count the elements l_s in $Z(\mathbf{H}_i)^\circ$ such that $C_{\mathbf{G}^*_{\mathrm{sc}}}(l_s) = \mathbf{L}_i$. Equivalently, we need to count the elements in

$$\mathcal{T}_L = Z(\mathbf{H}_i)^\circ \setminus \{g \in Z(\mathbf{H}_i)^\circ \mid C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(g) \neq \mathbf{L}_i\}. \tag{1.5.7}$$

As for $\mathcal{T}_H$, and using again Corollary 1.3.9 for the torus $Z(\mathbf{H}_i)^{\circ F^*}$, there is a constant C_L which depends only on Δ_i which is such that

$$|\mathcal{T}_L^{F^*}| \sim_{C_L} q^{\dim Z(\mathbf{H}_i)}.$$

The subset of A whose elements have centralizers in $\mathbf{L}_i$ which are equal to $\mathbf{H}_i$, is still a finite set. Lastly the condition $t^e \in \mathcal{T}_L$ does not influence the count of elements in $Z(\mathbf{H}_i)$ of the form $x = ta$ with $C^\circ_{\mathbf{G}^*_{\mathrm{sc}}}(x) = \mathbf{H}_i$, since $\dim(\mathcal{T}_L) = \dim(Z(\mathbf{H}_i)^\circ)$. Hence there exists a constant C_3 such that

$$|\mathcal{H}_i^{\mathrm{sc}}| \sim_{C_3} q^{\dim Z(\mathbf{H}_i)}.$$

Put $C = C_1C_2C_3$ to have all the cases covered, and we see that for every fixed $\mathbf{H}_i$ the approximation

$$|\mathcal{H}_i^{\mathrm{sc}}| \sim_C q^{\dim Z(\mathbf{H}_i)}$$

holds. By Proposition 1.3.6, in the case when ${\mathbf{G}_{\mathrm{sc}}^*}^{F^*}$ is perfect, we have $\pi({\mathbf{G}_{\mathrm{sc}}^*}^{F^*}) = [\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}]$. Thus we have a surjection $\mathcal{H}_i^{\mathrm{sc}} \twoheadrightarrow \mathcal{H}_i$ induced by π, whose fibres have order at most $|Z(\mathbf{G}_{\mathrm{sc}}^*)|$. Note that $\dim Z(\mathbf{L}_i) = \dim Z(\mathbf{K}_i)$, and so it follows that

$$|\mathcal{H}_i| \sim_{C|Z(\mathbf{G}_{\mathrm{sc}}^*)|} q^{\dim Z(\mathbf{K}_i)}.$$

Using (1.5.4) and (1.5.5), and defining D_6 to be the product $D_5C|Z(\mathbf{G}_{\mathrm{sc}}^*)|$, we obtain

$$\zeta^{\mathrm{nu}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) \sim_{D_4D_6} \sum_{i=1}^{N} q^{\dim Z(\mathbf{K}_i)-(|\Phi^+|-|\Psi_i^+|)s}.$$

Since $\dim Z(\mathbf{K}_i) \leqslant \operatorname{rk}\Phi$ for $1 \leqslant i \leqslant N$ and moreover, using [80, Lemma 2.5], we have

$$\frac{\dim Z(\mathbf{K}_i)}{|\Phi^+| - |\Psi^+|} \leqslant \frac{\operatorname{rk}\Phi}{|\Phi^+|}. \tag{1.5.8}$$

By Lemma 1.3.16, we see that for q large enough (depending on Φ), there exist regular semisimple elements in ${\mathbf{G}_{\mathrm{sc}}^*}^{F^*}$. Employing again Proposition 1.3.6, the regular elements in ${\mathbf{G}_{\mathrm{sc}}^*}^{F^*}$ are sent by π to regular elements in $[\mathbf{G}^{*F^*}, \mathbf{G}^{*F^*}]$. Then there exists an $i \in \{1, \ldots, N\}$ such that

$$(\dim Z(\mathbf{K}_i), |\Phi^+| - |\Psi_i^+|) = (\operatorname{rk}\Phi, |\Phi^+|). \tag{1.5.9}$$

Let $a^*((\Phi,\tau),q)$ be the set $\{(\dim Z(\mathbf{K}_i), |\Phi^+| - |\Psi_i^+|) \mid 1 \leqslant i \leqslant N\}$. Then we have

$$\zeta^{\mathrm{nu}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) \sim_{D_4D_6N} \sum_{(m,n)\in a^*((\Phi,\tau),q)} q^{m-ns}, \tag{1.5.10}$$

where N is added because we could have different $\mathbf{K}_i$ and $\mathbf{K}_j$ for $i \neq j$ which lead to the same data. The decomposition of characters of Theorem 1.4.4 and (1.5.2) imply that

$$\zeta_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) = \zeta^{\mathrm{unip}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) + \zeta^{\mathrm{nu}}_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s).$$

Using (1.5.3) in combination with (1.5.10), Proposition 1.5.5 yields $b(\Phi,\tau) \subset b(\Phi) \in A^+$, and a constant $D \in \mathbb{R}$ such that

$$\zeta_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) - 1 \sim_D \xi_{b(\Phi,\tau)\cup a^*((\Phi,\tau),q),q}(s).$$

We set $d((\Phi,\tau),q) = b(\Phi,\tau) \cup a^*((\Phi,\tau),q)$ and this concludes part (ii) of the theorem.

Assertion (i) follows from the observation that $d((\Phi,\tau),q) \subset a((\Phi,\tau),q)$ of Theorem 1.5.4 for all $((\Phi,\tau),q) \in \mathcal{L}_\Phi \times \mathcal{Q}$ and by setting $d(\Phi) = a(\Phi)$.

As for the last part, we define the set $d_0((\Phi,\tau),q) = d((\Phi,\tau),q) \cup \{(0,0)\} \in A$ and $f(\Phi) = \{(0,0), (\mathrm{rk}(\Phi), |\Phi^+|)\}$. Then by (1.5.9), we have $\xi_{d_0((\Phi,\tau),q),q}(s) \gtrsim_{N+1} \xi_{f(\Phi),q}(s)$ and by Remark 1.5.1 and (1.5.8) we have $\xi_{d_0((\Phi,\tau),q),q}(s) \lesssim_{N+1} \xi_{f(\Phi),q}(s)$ noting that $|d_0((\Phi,\tau),q)| \leqslant N+1$. Thus, for the same constant D as before, we have

$$\zeta_{\mathbf{G}^F/Z(\mathbf{G}^F)}(s) \sim_D 1 + q^{\mathrm{rk}(\Phi)-|\Phi^+|s}.$$

□

The work of Avni, Klopsch, Onn, and Voll was a refiniment of the results of Liebeck and Shalev in [79] and [80], who proved some asymptotic results on the Dirichlet polynomials of finite (quasi-)simple groups. We highlight the following, cf. [80, Theorem 1.1].

Theorem 1.5.7. *Let Φ be a non-trivial irreducible root system and τ an automorphism of Φ stabilising a choice of Φ^+. For every $\mathbf{G}$ a connected simply connected simple algebraic group over with underlying root system Φ, and F a Frobenius endomorphism inducing τ, write $L(q)$ for $\mathbf{G}^F$, as in the notation of Table 1.3, to underline the dependency on q. Then for every real number $t > 2\,\mathrm{rk}(\Phi)/|\Phi|$, it holds that*

$$\zeta_{L(q)}(t) \to 1 \quad \text{as } q \to \infty.$$

Moreover, for $t < 2\,\mathrm{rk}(\Phi)/|\Phi|$, we have $\zeta_{L(q)}(t) \to \infty$ as $q \to \infty$.

Furthermore, there is a result for groups of Lie type of the form $L(q)$ with unbounded Lie rank, cf. [80, Theorem 1.2].

Theorem 1.5.8. *Fix a real number $t > 0$. Then there is an integer $r(t)$ such that for quasi-simple groups $L(q)$ of Lie rank $r \geqslant r(t)$, we have*

$$\zeta_{L(q)}(t) \to 1 \quad \text{as } |L(q)| \to \infty.$$

For alternating groups, the following holds, cf. [79, Corollary 2.7]; the latter follows from a result on symmetric groups, cf. [79, Theorem 1.1].

Theorem 1.5.9. *Fix a real number $t > 0$. Then*

$$\zeta_{\mathrm{Alt}(n)}(t) \to 1 \quad \text{as } n \to \infty.$$

Moreover, $\zeta_{\mathrm{Alt}(n)}(t) = 1 + O(n^{-t})$.

Regarding the central covers of alternating groups $\mathrm{Alt}(n)$, the Schur multiplier has order 2 for $n \geqslant 5$, except for $n = 6$ and $n = 7$, where the Schur multipliers have order 6, as noted in Example 0.6.2. We consider alternating groups $\mathrm{Alt}(n)$ with Schur multiplier of order 2 and we refer to such double covers by $2.\mathrm{Alt}(n)$.

The irreducible representations of $2.\mathrm{Alt}(n)$ correspond to projective representations of $\mathrm{Alt}(n)$, cf. Section 0.6.1. The projective representations of the symmetric and alternating groups were first examined by Schur [108]. He proved that we can distingush between two types of irreducible representations of $2.\mathrm{Alt}(n)$: the irreducible representations $\chi \in \mathrm{Irr}(2.\mathrm{Alt}(n))$ with $Z(2.\mathrm{Alt}(n)) \subseteq \ker\chi$, which correspond to the irreducible representations of $\mathrm{Alt}(n)$, cf. Section 0.5; and remaining irreducible representations that we call *irreducible spin representations*, following [122], and we denoted the set of them by $\mathrm{Irr}_s(\mathrm{Alt}(n))$. By [122, Corollary 7.5], we see that the irreducible spin representations of $2.\mathrm{Alt}(n)$ are parametrized by the partitions of n with distinct parts, i.e. the partitions of n for which no number occurs more than once and a possible sign choice. More precisely, if we denote by $p(n)$ the number of partitions of n then the number of spin representations is bounded by 2 times $p(n)$. Then we use a weak form of a theorem proved by Hardy and Ramanujan in 1918, which gives an estimation of $p(n)$, cf. [98, Theorem 6.10] and we get

$$|\mathrm{Irr}_s(\mathrm{Alt}(n))| \leqslant 2 \cdot p(n) \leqslant 2 \cdot \exp(\pi\sqrt{2n/3}) \leqslant 3^{\pi\sqrt{2/3}n^{1/2}},$$

for all sufficiently large n. The degree of an irreducible spin representation is at least $2^{(n-3)/2}$, by [72, Corollary 3.2]. Hence

$$\sum_{\chi \in \mathrm{Irr}_s(2.\mathrm{Alt}(n))} \chi(1)^{-s} \leqslant 3^{\pi\sqrt{2/3}n^{1/2}} 2^{-(n-3)s/2} \leqslant 3^{-ns/3}$$

for all sufficiently large n. Hence we proved the following result.

Lemma 1.5.10. *Fix a real number $t > 0$. Then*

$$\zeta_{2.\mathrm{Alt}(n)}(t) \to 1 \quad \textit{as } n \to \infty.$$

Moreover, $\zeta_{2.\mathrm{Alt}(n)}(t) = 1 + O(n^{-t})$.

By considering Theorem 1.5.7, Theorem 1.5.8, Theorem 1.5.9, and Lemma 1.5.10, we obtain the following corollary, cf. [80, Corollary 1.4 (ii)].

Corollary 1.5.11. *Let $\mathcal{G}$ be the set of alternating groups $Alt(n)$, double covers of alternating groups $2.\mathrm{Alt}(n)$, finite simple groups of Lie type of unbounded Lie rank $S(q)$, and quasi-simple groups of Lie type of unbounded Lie rank $L(q)$. Given any $\varepsilon > 0$, there exists $m = m(\varepsilon)$, such that if $G \in \mathcal{G}$ of rank at least m, then*

$$r_n(G) < n^{\varepsilon} \quad \textit{for all } n \in \mathbb{N}_*.$$

1.6 Illustrated example of $\mathrm{SL}_2(q)$ and $\mathrm{PSL}_2(q)$

The character table of $\mathrm{SL}_2(q)$ was first explicitely described independently by Jordan [67] and Shur [109] in 1907 with ad hoc arguments. After the development of Deligne-Lusztig theory, Bonnafé published a book on the representations of $\mathrm{SL}_2(q)$ over the field k [16] with the aim of giving an illustrative example of the general theory. He deals with representations of $\mathrm{SL}_2(q)$ both in the non-defining and defining characteristic, i.e. when the characteristic of k is not the characteristic of the group, including the case when k has characteristic 0, and otherwise. For more references in the defining characteristic case, we point to [38, Example 2.1.17, Example 2.2.30, Remark 2.6.19, Example 4.3.3, and Example 4.8.25], and [30, Section 12.5].

The character table of $\mathrm{PSL}_2(p)$, where p is a prime, was firstly described by Frobenius in 1896, cf. [38, Table 2.4]. Since $\mathrm{PSL}_2(q)$ is the quotient of $\mathrm{SL}_2(q)$ by the center $Z(\mathrm{SL}_2(q))$, the character table of $\mathrm{PSL}_2(q)$ can be deduced from the one of $\mathrm{SL}_2(q)$. We explain how to do so with the degrees of the irreducible characters using Proposition 1.4.10.

Let p be an odd prime, q a power of p, $\mathbb{F}$ an algebraically closed field of characteristic p that contains $\mathbb{F}_q$, the field of q elements, and consider the group $\mathrm{SL}_2(q)$. This group can be described as the group of fixed points of the simply connected simple algebraic group $\mathbf{SL}_2$ over $\mathbb{F}$, with respect to the standard Frobenius endomorphism F, which raises the matrix entries to their q^{th} power. The subgroup $\mathbf{T}$ of $\mathbf{SL}_2$ of diagonal matrices is an F-stable maximal torus. Let $X = X(\mathbf{T}) \cong \mathbb{Z}$ be the character group and $Y = Y(\mathbf{T}) \cong \mathbb{Z}$ be the cocharacter group of $\mathbf{T}$. Since they are infinite cyclic groups we denote by χ the generator of X and by $\chi^\vee$ the generator of Y. The character χ is defined as

$$\chi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t,$$

for $t \in \mathbb{F}$. We have just two roots $\Phi = \{\pm\alpha\}$, where $\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^2$, and $\mathbb{Z}\Phi = \langle 2\chi \rangle$.

The Weyl group $W(\mathbf{T})$ has two elements $\{1, w\}$ and acts via signed permutation matrices on $\mathbf{T}$. The element w can be represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As we have seen in Section 1.3, the action of F on the character group $X = X(\mathbf{T})$ and the cocarachter group $Y = Y(\mathbf{T})$ is given by $F(\chi)(t) = \chi(F(t))$ for $\chi \in X$ and $t \in \mathbf{T}$ and by $F(\gamma)(c) = F(\gamma(c))$ for $\gamma \in Y$ and $c \in \mathbb{F}$. We observe that $F|_X = q\,\mathrm{id}_X$.

By Theorem 1.3.10, the group $\mathbf{SL}_2$ has q^2 F-stable maximal tori, and by Proposition 1.3.11, we have two $\mathrm{SL}_2(q)$-conjugacy classes of F-stable maximal tori.

We denote by T the set of F-fixed points of $\mathbf{T}$. For the other $\mathrm{SL}_2(q)$-conjugacy class of maximal tori, we have T_w which is the set isomorphic to the set of fixed points $\mathbf{T}^{wF}$. That is to say, the pair $(\mathbf{T}, wF)$ corresponds to the pair $(\mathbf{T}_w, F)$ where $\mathbf{T}_w$ is of type w, cf. Remark 1.3.12. The group T is formed by diagonal matrices $\mathrm{diag}(\mu, \mu^{-1})$ where $\mu \in \mathbb{F}_q^*$ and the group T_w is isomorphic to the group formed by diagonal matrices $\mathrm{diag}(\eta, \eta^q)$ with $\eta^{q+1} = 1$ and $\eta \in \mathbb{F}_{q^2}^*$.

Duality

Following Section 1.2.1, the dual group of $\mathbf{SL}_2$ with respect to $\mathbf{T}$ is the algebraic group of adjoint type $\mathbf{PGL}_2$ over $\mathbb{F}$ with maximal torus $\mathbf{T}^*$. When we consider the Frobenius endomorphism F, note that $\mathbf{T}$ is maximally split as it is contained in the Borel subgroup B of $\mathbf{SL}_2$ given by the upper triangular matrices, which is F-stable. Hence, following Section 1.3.1, the pair of $(\mathbf{SL}_2, F)$ and $(\mathbf{PGL}_2, F^*)$ are in duality with respect to $\mathbf{T}$ and $\mathbf{T}^*$, where F^* is also the standard Frobenius endomorphism which the matrix entries to their q^{th} powers. We denote by $W^* = \{1, w^*\}$ the Weyl group with respect to $\mathbf{T}^*$. Moreover, W^* is isomorphic to the Weyl group W. Compare with [92, Remark 8.10] and Section 1.2.1.

Cardinalities

In order to compute the cardinality of $\mathrm{SL}_2(q)$, let us compute the cardinalities of its maximal tori. As we have seen before, $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{SL}_2$ and $F|_X = q\,\mathrm{id}_X$. Thus, by Proposition 1.3.8, we have that $|T| = |\mathbf{T}^F| = |\det_{X_\mathbb{R}}(F-1)| = |\det_{X_\mathbb{R}}(q\,\mathrm{id}_X - 1)| = q-1$.

As for the torus T_w, we use Proposition 1.3.13 to compute the cardinality. We have that $|T_w| = |\mathbf{T}^{wF}| = |\det_{X_\mathbb{R}}(wF - 1)| = |\det_{X_\mathbb{R}}(q - (w\,\mathrm{id}_X)^{-1})| = q+1$. Altogether, using Proposition 1.3.14, we have that

$$|\mathrm{SL}_2(q)| = q(q+1)(q-1).$$

Lastly, by Proposition 1.3.15 we also have that

$$|\mathrm{PGL}_2(q)| = |\mathrm{SL}_2(q)| = q(q+1)(q-1).$$

In the group $\mathrm{PGL}_2(q)$, we also have two maximal tori T^* and $T^*_{w^*}$. By duality we have that $T^* \cong T$ and $T^*_{w^*} \cong T_w$.

$\mathrm{PGL}_2(q)$-conjugacy classes of semisimple elements

We describe the $\mathrm{PGL}_2(q)$-conjugacy classes of semisimple elements in $\mathrm{PGL}_2(q)$ and we choose a set of representatives.

- First, the identity element forms a $\mathrm{PGL}_2(q)$-conjugacy class just by itself.

- Now, for $a \in \mathbb{F}_q^*$, consider the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

If $a^2 \neq 1$, then the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}$$

are different, but they lie in the same $\mathrm{PGL}_2(q)$-conjugacy class. Since $|\mathbb{F}_q^* \setminus \{\pm 1\}| = q-3$, we have $\frac{q-3}{2}$ classes with a representative of that form.
- If $a^2 = 1$, then we either have the identity matrix, which we have already considered, or the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Let $N \colon \mathbb{F}_{q^2}^* \to \mathbb{F}_q^*$ be the norm map and $\mathrm{Tr} : \mathbb{F}_{q^2} \to \mathbb{F}_q$ be the trace map. The map N is surjective and has kernel of size $q+1$. Let $\delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and such that $\delta^q = -\delta$, i.e. $\mathrm{Tr}(\delta) = 0$. Then the characteristic polynomial of a representative of the form

$$\begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix},$$

is $x^2 - \delta^2 = x^2 + N(\delta)$ and has distinct roots δ and $-\delta$ in $\mathbb{F}_{q^2}$. The number of such polynomials is $q-1$, since the norm map is surjective. Moreover, since in $\mathrm{PGL}_2(q)$ the elements

$$\begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & c^2\delta^2 \\ 1 & 0 \end{pmatrix},$$

with $c \in \mathbb{F}_q^*$, are conjugate, we have just one conjugacy class with such a representative.
- As for the last type of semisimple elements, we have $q^2 - q$ elements in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Moreover, there are $q-1$ non-zero elements with $\mathrm{Tr}(\lambda) = 0$. Thus we have $(q^2 - q) - (q-1) = (q-1)^2$ elements $\lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\mathrm{Tr}(\lambda) \neq 0$. Since λ and λ^q give rise to the same element of $\mathrm{PGL}_2(q)$, and the elements

$$\begin{pmatrix} \mathrm{Tr}(\lambda) & -N(\lambda) \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{Tr}(\lambda) & -c^2 N(\lambda) \\ 1 & 0 \end{pmatrix},$$

with $c \in \mathbb{F}_q^*$, are conjugate, we have $\frac{q-1}{2}$ conjugacy classes with such representatives.

We define Ω to be a set of representatives of equivalence classes of elements $a \in \mathbb{F}_q^*$ such that $a^2 \neq 1$, where two elements a_1 and a_2 of $\mathbb{F}_q^*$ are equivalent if and only if either a_1 equals a_2 or their product $a_1 a_2$ equals one. As we have just seen $|\Omega| = \frac{q-3}{2}$. Moreover, we define Λ to be a set of representatives of equivalence classes of elements $\lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\lambda^q \neq -\lambda$ and where λ_1 is equivalent to λ_2 if and only if $\mathrm{Tr}(\lambda_1) = \mathrm{Tr}(\lambda_2)$ and there exists a $c \in \mathbb{F}_q^*$ such that $N(\lambda_1) = c^2 N(\lambda_2)$. As we have just seen $|\Lambda| = \frac{q-1}{2}$. Finally, we choose an element $\delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, such that $\delta^q = -\delta$. With these choices, the set

$$\mathcal{G}_s = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \;\middle|\; a \in \Omega \right\} \cup \left\{ \begin{pmatrix} \lambda + \lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix} \;\middle|\; \lambda \in \Lambda \right\}$$

is a set of representatives of $\mathrm{PGL}_2(q)$-conjugacy classes of semisimple elements in $\mathrm{PGL}_2(q)$.

Centralizers of semisimple elements in $\mathbf{PGL}_2$

We use Theorem 1.2.6 to describe the centralizers of elements in $\mathcal{G}_s$. Among them, the non-trivial elements which lie in $\mathbf{T}^*$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

The first element has a non-connected centraliser as the conjugation by $w^* \in W^*$ leaves the elements invariant, and the latter one has connected centralizer. We have

$$C_{\mathbf{PGL}_2}\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\right) = C^\circ_{\mathbf{PGL}_2}\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\right) = C^\circ_{\mathbf{PGL}_2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \mathbf{T}^*$$

As for the remaining non-trivial elements of $\mathcal{G}_s$, recall that by Corollary 1.3.5, the group $\mathrm{PGL}_2(q)(= \mathbf{PGL}_2^{F^*})$ is $\mathbf{PGL}_2$-conjugate to the group $\mathbf{PGL}_2^{gF^*}$ for every $g \in \mathbf{PGL}_2$. Let $g_0 \in \mathbf{PGL}_2$ be the element such that

$$\begin{pmatrix} \lambda + \lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}^{g_0} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^q \end{pmatrix},$$

in $\mathbf{PGL}_2$. By Theorem 1.3.4, we have a surjective morphism

$$\mathcal{L} \colon \mathbf{PGL}_2 \to \mathbf{PGL}_2, \quad g \mapsto F(g)g^{-1},$$

that yields an element $\dot{w}^* \in \mathbf{PGL}_2$ such that $\dot{w}^* = F(g_0){g_0}^{-1}$. In particular,

$$\dot{w}^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

corresponds to the element w^* of the Weyl group W^*, justifying our notation. According to Remark 1.3.12, the torus $g_0^{-1}\mathbf{T}^* g_0$ is a torus of type w^*, hence $g_0^{-1}\mathbf{T}^* g_0 = \mathbf{T}^*_{w^*}$. The pair

$(\mathbf{T}^*_{w^*}, F^*)$ is sent by g_0^{-1}-conjugation to the pair $(\mathbf{T}^*, w^*F)$. The elements

$$\begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda+\lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}$$

belong to $\mathbf{T}^*_{w^*}$. We can now use Theorem 1.2.6 to describe the centralizers. We have that the centralizer of the first element is not connected as the conjugation by w^* leaves it invariant, whereas the centralizer of the latter element is connected. Hence we have

$$C_{\mathbf{PGL}_2}\left(\begin{pmatrix} \lambda+\lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}\right) = C^\circ_{\mathbf{PGL}_2}\left(\begin{pmatrix} \lambda+\lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}\right) =$$
$$C^\circ_{\mathbf{PGL}_2}\left(\begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix}\right) = \mathbf{T}^*_{w^*} \cong \mathbf{T}^*.$$

In particular, all non-trivial semisimple elements are regular, cf. Section 1.3.3.

Finally, we compute

$$\left|\mathrm{PGL}_2(q) : C_{\mathbf{PGL}_2}\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\right)^{F^*}\right|_{p'} = \left|\mathrm{PGL}_2(q) : \mathbf{T}^{*F^*}\right|_{p'} = (q+1), \tag{1.6.1}$$

$$\left|\mathrm{PGL}_2(q) : C_{\mathbf{PGL}_2}\left(\begin{pmatrix} \lambda+\lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}\right)^{F^*}\right|_{p'} = \left|\mathrm{PGL}_2(q) : \mathbf{T}^{*\,F^*}_{w^*}\right|_{p'} = (q-1), \tag{1.6.2}$$

$$\left|\mathrm{PGL}_2(q) : C_{\mathbf{PGL}_2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)^{F^*}\right|_{p'} = \left|\mathrm{PGL}_2(q) : \langle \mathbf{T}^* \cup \langle \dot{w}^* \rangle\rangle^{F^*}\right|_{p'} = \frac{q+1}{2}, \tag{1.6.3}$$

$$\left|\mathrm{PGL}_2(q) : C_{\mathbf{PGL}_2}\left(\begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix}\right)^{F^*}\right|_{p'} = \left|\mathrm{PGL}_2(q) : \langle \mathbf{T}^*_{w^*} \cup \langle \dot{w}^* \rangle\rangle^{F^*}\right|_{p'} = \frac{q-1}{2}. \tag{1.6.4}$$

as $w^* \in W^*$ has order 2.

Characters of $\mathrm{SL}_2(q)$

The torus T is a finite abelian group with $q-1$ elements. Hence the number of irreducible characters of T is equal to the cardinality of T, as the group of irreducible characters of T isomorphic to T itself. More precisely, the set $\mathrm{Irr}(T)$ is given by the trivial character 1_T (corresponding to the trivial element in $\mathbb{F}_q^*$), the character α_0 that is the unique linear character of order 2 of T (corresponding to the element -1 in $\mathbb{F}_q^*$), and characters α with $\alpha^2 \neq 1$ (for all the other elements of $\mathbb{F}_q^*$). Proposition 1.4.2 tells us that we have an isomorphism between $\mathrm{Irr}(T)$ and T^*, and Proposition 1.4.3 gives us the following identifications

$$(T, 1_T) \leftrightarrow \left(T^*, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad (T, \alpha_0) \leftrightarrow \left(T^*, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

$$(T,\alpha) \leftrightarrow \left(T^*, \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\right)$$

where $a \in \mathbb{F}_q^*$ such that $a^2 \neq 1$.

The set $\mathrm{Irr}(T_w)$ instead is given by the trivial character 1_{T_w}, the character θ_0 that is the unique linear character of order 2 of T_w, and characters θ with $\theta^2 \neq 1$. By the same argument as before, we have the following identifications

$$(T_w, 1_{T_w}) \leftrightarrow \left(T^*_{w^*}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad (T_w, \theta_0) \leftrightarrow \left(T^*_{w^*}, \begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix}\right),$$

$$(T_w, \theta) \leftrightarrow \left(T^*_{w^*}, \begin{pmatrix} \lambda + \lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}\right),$$

where $\delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ is the one that we fixed before and $\lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, and such that $\mathrm{Tr}(\lambda) \neq 0$.

Let g_s be an element of $\mathcal{G}_s$. The corresponding rational series $\mathcal{E}(\mathrm{SL}_2(q), g_s)$ of irreducible characters of $\mathrm{SL}_2(q)$ is the set of irreducible characters of $\mathrm{SL}_2(q)$ which occur in some Deligne-Lusztig character $R^{\mathbf{G}}_{\mathbf{T}^*}(g_s)$, for some F^*-stable maximal torus $\mathbf{T}^*$ where g_s is semisimple and $g_s \in \mathbf{T}^{*F^*}$. By Proposition 1.4.1, the number of characters in each Deligne-Lusztig character $R^{\mathbf{G}}_{\mathbf{T}^*}(g_s)$ is at most 2. Up to $\mathrm{PGL}_2(q)$-conjugacy, we have two F^*-stable maximal tori, i.e. $\mathbf{T}^*$ and $\mathbf{T}^{F^*}_{w^*}$.

- The trivial element lies both in T^* and T^*_w. By Proposition 1.4.1 we have

$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)(1) = R^{\mathbf{SL}_2}_{\mathbf{T}}(1_T)(1) = (-1)^{l(1)}|\,\mathrm{SL}_2(q) : T|_{p'} = q+1$$

$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*_{w^*}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)(1) = R^{\mathbf{SL}_2}_{\mathbf{T}_w}(1_{T_w})(1) = (-1)^{l(w)}|\,\mathrm{SL}_2(q) : T_w|_{p'} = q-1$$

  The characters involved are the unipotent characters of $\mathrm{SL}_2(q)$. Example 1.4.7 tells us that we have two unipotent characters, since the number of unipotent characters is in bijection with the partitions of 2, namely $\alpha = (1,1)$ and $\beta = (2)$. Using the formula of the degree of χ^α, having $\lambda_1 = 1$ and $\lambda_2 = 2$, we get

$$\chi^\alpha(1) = \frac{\left(\prod_{i=1}^2 (q^i - 1)\right)(q^2 - q)}{\prod_{i=1}^2 \prod_{k=1}^{\lambda_i} (q^k - 1)} = q.$$

  For the other partition, we have

$$\chi^\beta(1) = \frac{\prod_{i=1}^2 (q^i - 1)}{\prod_{k=1}^2 (q^k - 1)} = 1.$$

  These are exactly the trivial and the Steinberg character that are irreducible unipotent characters, cf. Remark 1.4.8. Hence we have

$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = R^{\mathbf{SL}_2}_{\mathbf{T}}(1_T) = 1_{\mathrm{SL}_2(q)} + \mathrm{St}$$
$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*_{w^*}}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = R^{\mathbf{SL}_2}_{\mathbf{T}_w}(1_{T_w}) = \mathrm{St} - 1_{\mathrm{SL}_2(q)}.$$

- The semisimple element $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ lies in T^*. By Proposition 1.4.1 we have

$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)(1) = R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha_0)(1) = (-1)^{l(1)}|\,\mathrm{SL}_2(q) : T|_{p'} = q+1$$

$$\langle R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha_0), R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha_0)\rangle_{\mathrm{SL}_2(q)} = \left|\{g \in \mathrm{SL}_2(q) \mid g\mathbf{T}g^{-1} = \mathbf{T} \text{ and } {}^g\alpha_0 = \alpha_0\}\right| / |T| = 2.$$

Thus, $R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha_0)$ is not an irreducible character. As afore mentioned, since by Proposition 1.4.1 we have at most two irreducible characters for each Deligne-Lusztig character, we have 2 irreducible characters that we call $R_+(\alpha_0)$ and $R_-(\alpha_0)$. Moreover, by Theorem 1.4.5 and (1.6.3), we have

$$R_+(\alpha_0)(1) = R_-(\alpha_0)(1) = \frac{q+1}{2}.$$

- The semisimple elements $\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}$ with $a \in \Omega$ lie in T^*. By Proposition 1.4.1 we have

$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*}\left(\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}\right)(1) = R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha)(1) = (-1)^{l(1)}|\,\mathrm{SL}_2(q) : T|_{p'} = q+1$$

and

$$\langle R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha), R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha)\rangle_{\mathrm{SL}_2(q)} = \left|\{g \in \mathrm{SL}_2(q) \mid g\mathbf{T}g^{-1} = \mathbf{T} \text{ and } {}^g\alpha = \alpha\}\right| / |T| = 1.$$

Hence $R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha)$ is an irreducible character of $\mathrm{SL}_2(q)$. We denote it by $R(\alpha) = R^{\mathbf{SL}_2}_{\mathbf{T}}(\alpha)$. Using (1.6.1), we have

$$R(\alpha)(1) = q+1.$$

- The semisimple element $\begin{pmatrix}0 & \delta^2\\ 1 & 0\end{pmatrix}$ lies in T^*_w. By Proposition 1.4.1 we have

$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*_{w^*}}\left(\begin{pmatrix}0 & \delta^2\\ 1 & 0\end{pmatrix}\right)(1) = R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta_0)(1) = (-1)^{l(w)}|\,\mathrm{SL}_2(q) : T_w|_{p'} = q-1$$

and

$$\langle R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta_0), R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta_0)\rangle_{\mathrm{SL}_2(q)} = \left|\{g \in \mathrm{SL}_2(q) \mid g\mathbf{T}_w g^{-1} = \mathbf{T}_w \text{ and } {}^g\theta_0 = \theta_0\}\right| / |T_w| = 2.$$

Thus, $R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta_0)$ is not an irreducible character. We have 2 irreducible characters that we call $R^w_+(\theta_0)$ and $R^w_-(\theta_0)$. Moreover, by Theorem 1.4.5 and (1.6.4), we have

$$R^w_+(\theta_0)(1) = R^w_-(\theta_0)(1) = \frac{q-1}{2}.$$

- The semisimple elements $\begin{pmatrix} \lambda+\lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}$ with $\lambda \in \Lambda$. By Proposition 1.4.1 we have

$$R^{\mathbf{PGL}_2}_{\mathbf{T}^*_{w^*}}\left(\begin{pmatrix} \lambda+\lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}\right)(1) = R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta)(1) = (-1)^{l(w)}|\,\mathrm{SL}_2(q) : T_w|_{p'} = q-1,$$

and

$$\langle R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta), R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta)\rangle_{\mathrm{SL}_2(q)} = \left|\{g \in \mathrm{SL}_2(q) \mid g\mathbf{T}_w g^{-1} = \mathbf{T}_w \text{ and } {}^g\theta = \theta\}\right| / |T_w| = 1.$$

Hence $R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta)$ is an irreducible character of $\mathrm{SL}_2(q)$. We denote it by $R^w(\theta) = R^{\mathbf{SL}_2}_{\mathbf{T}_w}(\theta)$. Using (1.6.2), we have that

$$R^w(\theta)(1) = q-1.$$

Since

$$|\,\mathrm{SL}_2(q)| = 1^2 + q^2 + 2\left(\frac{q+1}{2}\right)^2 + \frac{q-3}{2}(q+1)^2 + 2\left(\frac{q-1}{2}\right)^2 + \frac{q-1}{2}(q-1)^2, \quad (1.6.5)$$

we conclude that

$$\begin{aligned} \mathrm{Irr}(\mathrm{SL}_2(q)) = \{1_{\mathrm{SL}_2(q)}, \mathrm{St}\} \,\dot\cup\, \{R_+(\alpha_0), R_-(\alpha_0)\} \,\dot\cup\, \{R(\alpha) \mid \alpha \in \mathrm{Irr}(T), \alpha^2 \neq 1\} \\ \dot\cup \{R^w_+(\theta_0), R^w_-(\theta_0)\} \,\dot\cup\, \{R^w(\theta) \mid \theta \in \mathrm{Irr}(T_w), \theta^2 \neq 1\}. \end{aligned}$$

Now, we divide the characters with respect to the rational series as in Theorem 1.4.4. With an analysis on the degrees that we got so far, we deduce that the series of unipotent characters is

$$\mathcal{E}\left(\mathrm{SL}_2(q), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \{1_{\mathrm{SL}_2(q)}, \mathrm{St}\},$$

and the other rational series are

$$\mathcal{E}\left(\mathrm{SL}_2(q), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \{R_+(\alpha_0), R_-(\alpha_0)\}, \text{ for } \alpha_0 \in \mathrm{Irr}(T) \setminus \{1_T\},\ \alpha_0^2 = 1;$$
$$\mathcal{E}\left(\mathrm{SL}_2(q), \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\right) = \{R(\alpha)\}, \text{ for } a \in \mathbb{F}_q^*,\ a^2 \neq 1, \text{ and } \alpha \in \mathrm{Irr}(T),\ \alpha^2 \neq 1;$$
$$\mathcal{E}\left(\mathrm{SL}_2(q), \begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix}\right) = \{R_+^w(\theta_0), R_-^w(\theta_0)\}, \quad \begin{array}{l} \text{for a } \delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q,\ \delta^q = -\delta, \\ \text{and } \theta_0 \in \mathrm{Irr}(T_w) \setminus \{1_{T_w}\},\ \theta_0^2 = 1; \end{array}$$
$$\mathcal{E}\left(\mathrm{SL}_2(q), \begin{pmatrix} \lambda + \lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}\right) = \{R^w(\theta)\}, \quad \begin{array}{l} \text{for } \lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q,\ \lambda^q \neq -\lambda, \\ \text{and } \theta \in \mathrm{Irr}(T_w),\ \theta^2 \neq 1. \end{array}$$

This yields the Lusztig decomposition

$$\mathrm{Irr}(\mathrm{SL}_2(q)) = \bigsqcup_{g_s \in \mathcal{G}_s} \mathcal{E}(\mathrm{SL}_2(q), g_s),$$

as we stated in Theorem 1.4.4.

For q odd, we write the table of orders and multiplicities in Table 1.4, cf. [16, Table 5.4] and [30, Table 12.1].

$\mathrm{SL}_2(q),\ q$ odd		
Character χ	Degree n	Multiplicity $r_n(\mathrm{SL}_2(q))$
$1_{\mathrm{SL}_2(q)}$	1	1
St	q	1
$R(\alpha),\ \alpha^2 \neq 1$	$q+1$	$\frac{q-3}{2}$
$R^w(\theta),\ \theta^2 \neq 1$	$q-1$	$\frac{q-1}{2}$
$R_\sigma(\alpha_0),\ \sigma \in \{\pm 1\}$	$\frac{q+1}{2}$	2
$R_\sigma^w(\theta_0),\ \sigma \in \{\pm 1\}$	$\frac{q-1}{2}$	2

Table 1.4: Representations of $\mathrm{SL}_2(q)$ for q odd

Characters of $\mathrm{PSL}_2(q)$

Using Proposition 1.4.10, we look at the representatives of the $\mathrm{PGL}_2(q)$-conjugacy classes of semisimple elements associated with the characters and we check whether they satisfy the conditions of being in the commutator subgroup of $\mathrm{PGL}_2(q)$. By Corollary 1.3.7, there is a natural isomorphism between $[\mathrm{PGL}_2(q), \mathrm{PGL}_2(q)]$ and the group $\mathrm{PSL}_2(q)$. Indeed,

$$\begin{aligned}
[\mathrm{PGL}_2(q), \mathrm{PGL}_2(q)] &= [\mathrm{GL}_2(q)/Z(\mathrm{GL}_2(q)), \mathrm{GL}_2(q)/Z(\mathrm{GL}_2(q))] \\
&= [\mathrm{GL}_2(q), \mathrm{GL}_2(q)] \cdot Z(\mathrm{GL}_2(q))/Z(\mathrm{GL}_2(q)) \\
&\cong [\mathrm{GL}_2(q), \mathrm{GL}_2(q)]/([\mathrm{GL}_2(q), \mathrm{GL}_2(q)] \cap Z(\mathrm{GL}_2(q))) \\
&= \mathrm{SL}_2(q)/Z(\mathrm{SL}_2(q))
\end{aligned}$$

Hence, if we consider a lift g in $\mathrm{GL}_2(q)$ of a representative of a $\mathrm{PGL}_2(q)$-conjugacy class of a semisimple element $gZ(\mathrm{GL}_2(q)) \in \mathrm{PGL}_2(q)$, we need to check that g belongs to $\mathrm{SL}_2(q)$.

Let us discuss each case by studying the representatives in $\mathcal{G}_s$.

- First, let us consider the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ that corresponds to the representations $R_+(\alpha_0)$ and $R_-(\alpha_0)$, where $\alpha_0 \in \mathrm{Irr}(T)$ and such that $\alpha_0^2 = 1$. Hence

$$\begin{aligned}
Z(\mathrm{SL}_2(q)) \subseteq \ker R_\pm(\alpha_0) &\iff \text{there exists } b \in \mathbb{F}_q^* \text{ such that } b^2 = -1 \\
&\iff q \equiv 1 \bmod 4.
\end{aligned}$$

- Let us consider the element $\begin{pmatrix} 0 & \delta^2 \\ 1 & 0 \end{pmatrix}$, with $\delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $\mathrm{Tr}(\delta) = 0$. The associated characters are $R_+^w(\theta_0)$ and $R_-^w(\theta_0)$, where $\theta_0 \in \mathrm{Irr}(T_w)$ and such that $\theta_0^2 = 1$. Note that δ^2 has order $q-1$ in $\mathbb{F}_q^*$, so it is a generator of the cyclic group $\mathbb{F}_q^*$. Then, we have

$$\begin{aligned}
Z(\mathrm{SL}_2(q)) \subseteq \ker R_\pm^w(\theta_0) &\iff \text{there exists } b \in \mathbb{F}_q^* \text{ such that } b^2\delta^2 = -1 \\
&\iff \text{there exists } j \in \mathbb{N} \text{ such that } (\delta^2)^{2j}\delta^2 = (\delta^2)^{\frac{q-1}{2}} \\
&\iff q \equiv 3 \bmod 4.
\end{aligned}$$

- For the semisimple elements $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, where $a \in \mathbb{F}_q^*$ and $a^2 \neq 1$, and the corresponding $\alpha \in \mathrm{Irr}(T)$ with $\alpha^2 \neq 1$, we have

$$\begin{aligned}
Z(\mathrm{SL}_2(q)) \subseteq \ker R(\alpha) &\iff \text{there exists } b \in \mathbb{F}_q^* \text{ such that } ab^2 = 1 \\
&\iff a \in (\mathbb{F}_q^*)^2.
\end{aligned}$$

  - If $q \equiv 1 \bmod 4$, then $-1 \in (\mathbb{F}_q^*)^2$. Thus we have $|(\mathbb{F}_q^*)^2 \setminus \{\pm 1\}| = \frac{q-5}{2}$ possibilities for a. Moreover, since in $\mathrm{PGL}_2(q)$ the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}$$

are in the same $\mathrm{PGL}_2(q)$-conjugacy class, we are left with $\frac{q-5}{4}$ characters of degree $q+1$.

- On the other hand, if $q \equiv 3 \bmod 4$ then -1 is not a square in $\mathbb{F}_q$ and so we have $\frac{q-3}{4}$ characters of degree $q+1$.

- Finally, for $\begin{pmatrix} \lambda+\lambda^q & -\lambda\lambda^q \\ 1 & 0 \end{pmatrix}$, with $\lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $\mathrm{Tr}(\lambda) \neq 0$, and the corresponding $\theta \in \mathrm{Irr}(T_w)$ with $\theta^2 \neq 1$, we have

$$\begin{aligned} Z(\mathrm{SL}_2(q)) \subseteq \ker R^w(\theta) &\iff \text{there exists } b \in \mathbb{F}_q^* \text{ such that } \lambda^{q+1}b^2 = 1 \\ &\iff N(\lambda) \in (\mathbb{F}_q^*)^2. \end{aligned}$$

The number of elements $\lambda \in \mathbb{F}_{q^2}^*$ such that $N(\lambda) \in (\mathbb{F}_q^*)^2$, is

$$|\ker(N)| \cdot |(\mathbb{F}_q^*)^2| = (q+1) \cdot \frac{q-1}{2}.$$

Since all the elements of $\mathbb{F}_q^*$ have the property that their norm is a square, we need to subtract $q-1$ elements from the ones that we have counted so far. Then we have

$$(q+1) \cdot \frac{q-1}{2} - (q-1) = \frac{(q-1)^2}{2}$$

elements $\lambda \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$ such that $N(\lambda) \in (\mathbb{F}_q^*)^2$. The last condition that we need to take into account is for the trace of λ to be non-zero. If that is not true, i.e. $\mathrm{Tr}(\lambda) = 0$, then the characteristic polynomial of our representative

$$\begin{pmatrix} 0 & -N(\lambda) \\ 1 & 0 \end{pmatrix}$$

is of the form $x^2 + N(\lambda)$. This polynomial is irreducible in $\mathbb{F}_q[x]$ if and only if $-N(\lambda)$ is not a square in $\mathbb{F}_q^*$.

- If $q \equiv 1 \bmod 4$, then $(-1) \in (\mathbb{F}_q^*)^2$, so then $-N(\lambda) \in (\mathbb{F}_q^*)^2$ and so this cannot happen since we are restricting to elements in $\mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$. Moreover, since λ and λ^q give rise to the same representative, and the elements

$$\begin{pmatrix} \mathrm{Tr}(\lambda) & -N(\lambda) \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{Tr}(\lambda) & -c^2N(\lambda) \\ 1 & 0 \end{pmatrix},$$

with $c \in \mathbb{F}_q^*$, are conjugate, we have $\frac{q-1}{4}$ conjugacy classes with such representatives.

- Instead, if $q \equiv 3 \bmod 4$, then the characteristic polynomial $x^2 + N(\lambda)$ is irreducible and so we need to subtract $q-1$, which corresponds to the number of irreducible polynomials of the form $x^2 + N(\lambda)$. Thus we get

$$\frac{(q-1)^2}{2} - (q-1) = (q-1) \cdot \frac{q-3}{2}$$

elements $\lambda \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$ such that $N(\lambda) \in (\mathbb{F}_q^*)^2$ and $\mathrm{Tr}(\lambda) \neq 0$. By the same argument as above, we divide this number by $2(q-1)$ and we get $\frac{q-3}{4}$ conjugacy class with such representatives.

We then have Table 1.5.

$\mathrm{PSL}_2(q),\ q \equiv 1 \bmod 4$			$\mathrm{PSL}_2(q),\ q \equiv 3 \bmod 4$		
Character χ	Degree n	Mult. r_n	Character χ	Degree n	Mult. r_n
1	1	1	1	1	1
St	q	1	St	q	1
$R(\alpha),\ \alpha^2 \neq 1$	$q+1$	$\frac{q-5}{4}$	$R(\alpha),\ \alpha^2 \neq 1$	$q+1$	$\frac{q-3}{4}$
$R^w(\theta),\ \theta^2 \neq 1$	$q-1$	$\frac{q-1}{4}$	$R^w(\theta),\ \theta^2 \neq 1$	$q-1$	$\frac{q-3}{4}$
$R_\sigma(\alpha_0),\ \sigma \in \{\pm 1\}$	$\frac{q+1}{2}$	2	$R_\sigma^w(\theta_0),\ \sigma \in \{\pm 1\}$	$\frac{q-1}{2}$	2

Table 1.5: Representations of $\mathrm{PSL}_2(q)$ for q odd

To verify that everything is correct, for $q \equiv 1 \bmod 4$, we compute

$$|\,\mathrm{PSL}_2(q)| = 1^2 + q^2 + \frac{q-5}{4}(q+1)^2 + \frac{q-1}{4}(q-1)^2 + 2\left(\frac{q+1}{2}\right)^2, \tag{1.6.6}$$

and, for $q \equiv 3 \bmod 4$, we compute

$$|\,\mathrm{PSL}_2(q)| = 1^2 + q^2 + \frac{q-3}{4}(q+1)^2 + \frac{q-3}{4}(q-1)^2 + 2\left(\frac{q-1}{2}\right)^2. \tag{1.6.7}$$

Representation growth

Let q be a prime power congruent to 1 modulo 4. Then

$$\zeta_{\mathrm{PSL}_2(q)}(s) = 1 + q^{-s} + \frac{q-5}{4}(q+1)^{-s} + \frac{q-1}{4}(q-1)^{-s} + 2\left(\frac{q+1}{2}\right)^{-s}.$$

and

$$\zeta_{\mathrm{SL}_2(q)}(s) = 1 + q^{-s} + \frac{q-3}{2}(q+1)^{-s} + \frac{q-1}{2}(q-1)^{-s} + 2\left(\frac{q+1}{2}\right)^{-s} + 2\left(\frac{q-1}{2}\right)^{-s}$$

We consider the quasi-semisimple profinite groups

$$G_1 = \prod_{j\geq 1} \mathrm{PSL}_2(q^j) \quad \text{and} \quad \widetilde{G_1} = \prod_{j \geq 1} \mathrm{SL}_2(q^j).$$

Note that the center of $\widetilde{G_1}$ is an infinite group of exponent 2. Their representation zeta functions are

$$\zeta_{G_1}(s) = \prod_{j\geq 1}\left(1 + q^{-js} + \frac{q^j-5}{4}(q^j+1)^{-s} + \frac{q^j-1}{4}(q^j-1)^{-s} + 2\left(\frac{q^j+1}{2}\right)^{-s}\right)$$

and

$$\zeta_{\widetilde{G_1}}(s) = \prod_{j\geq 1}\left(1 + q^{-js} + \frac{q^j-3}{2}(q^j+1)^{-s} + \frac{q^j-1}{2}(q^j-1)^{-s}\right.$$
$$\left. + 2\left(\frac{q^j+1}{2}\right)^{-s} + 2\left(\frac{q^j-1}{2}\right)^{-s}\right).$$

By Remark 1.5.3, for any two sequences $(x_j)_{j\in\mathbb{N}_*}$ and $(y_j)_{j\in\mathbb{N}_*}$ of positive real numbers, the product

$$\prod_{j\geq 1}(1 + x_j + y_j)$$

converges if and only if $\prod_{j\geq 1}(1+x_j)$ and $\prod_{j\geq 1}(1+y_j)$ converge individually. Thus, the abscissae of convergence of $\zeta_{G_1}(s)$ and $\zeta_{\widetilde{G_1}}(s)$ are the same as the one of the product $\prod_{j\geq 1} q^{j(1-s)}$, which converges if and only if $\sum_{j\geq 1} q^{j(1-s)}$ does. Therefore we have

$$\alpha(G_1) = \alpha(\widetilde{G_1}) = 1.$$

Let now $\mathcal{P}_1$ be the set of primes congruent to 1 modulo 4 and consider

$$G_2 = \prod_{p\in\mathcal{P}_1}\mathrm{PSL}_2(p) \quad \text{and} \quad \widetilde{G_2} = \prod_{p\in\mathcal{P}_1}\mathrm{SL}_2(p).$$

Then the representation zeta functions are

$$\zeta_{G_2}(s) = \prod_{p\in\mathcal{P}_1}\left(1 + p^{-s} + \frac{p-5}{4}(p+1)^{-s} + \frac{p-1}{4}(p-1)^{-s} + 2\left(\frac{p+1}{2}\right)^{-s}\right)$$

and

$$\zeta_{\widetilde{G_2}}(s) = \prod_{p\in\mathcal{P}_1}\left(1 + p^{-s} + \frac{p-3}{2}(p+1)^{-s} + \frac{p-1}{2}(p-1)^{-s}\right.$$
$$\left. + 2\left(\frac{p+1}{2}\right)^{-s} + 2\left(\frac{p-1}{2}\right)^{-s}\right)$$

that converge if and only if $\prod_{p\in\mathcal{P}_1}(1+p^{1-s})$ does. Since

$$\prod_{p\in\mathcal{P}_1}(1+p^{1-s}) = \frac{\prod_{p\in\mathcal{P}_1}(1-p^{-(1-s)})^{-1}}{\prod_{p\in\mathcal{P}_1}(1-p^{-2(1-s)})^{-1}},$$

and by the Chebotarev Density Theorem the set $\mathcal{P}_1$ has positive analytic density, we see that

$$\alpha\left(\prod_{p\in\mathcal{P}_1}(1+p^{1-s})\right) = \alpha\left(\frac{\zeta(s-1)}{\zeta(2s-2)}\right).$$

From this we conclude that

$$\alpha(G_2) = \alpha(\widetilde{G_2}) = 2.$$

The groups G_1 and $\widetilde{G_1}$ are the kind of groups that we will consider in the next section, while G_2 and $\widetilde{G_2}$ are more of the flavour of groups considered by [12].

1.7 Polynomial representation growth

Recall that a group G has polynomial representation growth if the function $R_n(G)$, which counts the number of irreducible complex representations of G of dimension at most n, up to isomorphism, grows at most polynomially in n. For quasi-semisimple profinite groups, we establish that this condition is equivalent to the polynomial growth of the number of simple factors which occur in G and admit at least one non-trivial representation of dimension at most n.

Definition 1.7.1. Let G be an infinite countable based quasi-semisimple profinite group, i.e. a perfect profinite group such that $G/Z(G) \cong \prod_{j\in\mathbb{N}_*} G_j$ where the groups G_j are finite, non-abelian, and simple. Define

$$M_n(G) = \{j \in \mathbb{N}_* \mid R_n(G_j) > 1\} \quad \text{and} \quad m_n(G) = |M_n(G)|.$$

We build upon the idea presented in [37, Theorem 6.1] and improve it by proving the following result.

Proposition 1.7.2. *Let G be an infinite countable based quasi-semisimple profinite group, i.e. a perfect profinite group such that $G/Z(G) \cong \prod_{j\in\mathbb{N}_*} G_j$ where the groups G_j are finite simple and non-abelian. Then G has polynomial representation growth if and only if $m_n(G)$ is polynomially bounded.*

Proof. The function $R_n(G)$ counts the number of irreducible representations of dimension at most n, up to isomorphism, and $m_n(G)$ counts the number of indices in the Cartesian product isomorphic to $G/Z(G)$ for which the corresponding factor has at least one non-trivial representation of dimension at most n. Every n-dimensional irreducible representation of

$G/Z(G)$ is a tensor product of irreducible representations of a finite number of factors G_j such that the product of the dimensions is n, by Lemma 0.4.1 and Theorem 0.5.3. Thus,

$$R_n(G) \geqslant R_n(G/Z(G)) \geqslant m_n(G),$$

and so if $R_n(G)$ grows at most polynomially, so does $m_n(G)$.

Conversely, let $m_n(G)$ be polynomially bounded, i.e. there exists a constant $b \in \mathbb{N}_*$ such that $m_n(G) \leqslant n^b$ for all $n \in \mathbb{N}_*$. As we noticed in the introduction, we can see G as quotient of the Cartesian product $\widetilde{G} = \prod_{j\in\mathbb{N}_*} \widetilde{G_j}$ of all covers $\widetilde{G_j}$ of the finite non-abelian simple groups G_j and in particular we have $m_n(G) = m_n(\widetilde{G})$. Let ρ be an irreducible n-dimensional representation of $\widetilde{G}$. Then there is a unique decomposition of ρ such that $\rho \cong \rho_{i_1} \boxtimes \cdots \boxtimes \rho_{i_t}$ for certain distinct $i_1, \ldots, i_t \in \mathbb{N}_*$, where each ρ_{i_j} is a non-trivial n_{i_j}-dimensional irreducible representation of $\widetilde{G_{i_j}}$ and $n = n_{i_1} \cdots n_{i_t}$ with $n_{i_j} > 1$. By [84, Lemma 4.7], there exists a constant $\mu \in \mathbb{N}_*$ such that the number of possibilities for n to be written as the product just described is at most n^μ. Moreover, given a configuration $(n_{i_1}, \ldots, n_{i_t})$, the number of factors $\widetilde{G_l}$ of $\widetilde{G}$ such that $\widetilde{G_l}$ has a representation of dimension n_{i_j} is at most $m_{n_{i_j}}(G) \leqslant n_{i_j}^b$ by assumption. Hence, the number of choices for the indices $(i_1, \ldots, i_t) \in (\mathbb{N}_*)^t$ is bounded by $\prod_{j=1}^t n_{i_j}^b = n^b$. It remains to prove that $r_n(\widetilde{G_l})$ is polynomially bounded for each finite quasi-simple group $\widetilde{G_l}$. For Suzuki and Ree groups, the result follows directly from [80, Table 1]. For quasi-simple of unbounded ranks, we use Corollary 1.5.11. For quasi-simple groups of Lie type of bounded rank, the result is given by [80, Corollary 1.4 (i)]. The remaining groups are just finitely many so we can bound the number of their representations by a constant, yielding the result. □

Now, we proceed to prove the next theorem, following the idea of [37, Theorem 6.4].

Theorem 1.7.3. *Let G be a quasi-semisimple profinite group such that $G/Z(G)$ is a Cartesian product of finite simple groups. If G has polynomial representation growth then G is finitely generated as a profinite group.*

Proof. By Proposition 1.3.17, we know that the elements in the center of G do not contribute to the number of generators of G. Hence, we consider $G/Z(G)$ and denote it H. According to Proposition 1.7.2, $m_n(H) \leqslant n^b$ for some constant $b \in \mathbb{N}_*$. Let $f(S)$ denote the number of copies of S appearing as a composition factor of H. Note that we can write $H = \prod_S S^{f(S)}$, where S runs along the non-abelian finite simple groups. Since for every finite simple group S, there exists a non-trivial irreducible representation of dimension less than $|S|$, it follows that $f(S) \leqslant m_{|S|}(H) \leqslant |S|^b$ for every S. Using Theorem 1.3.19, we conclude that $S^{f(S)}$ is generated by $b+2$ elements. Utilizing Lemma 0.2.2, we then deduce that H is topologically finitely generated and, by our initial consideration, it follows that also G is also finitely generated. □

The result is reminiscent of a similar phenomenon concerning subgroup growth, which states that every profinite group with polynomial subgroup growth is finitely generated, cf. [86, Theorem 10.6]. While we do not have an analogous result for all profinite groups, our founding establishes it within the realm of quasi-semisimple profinite groups.

1.8 Profinite completions

A finitely generated profinite group G is termed a *profinite completion* if there exists a finitely generated residually finite group H such that its profinite completion $\widehat{H}$ is isomorphic to G. It is generally a difficult question whether a finitely generated profinite group is a profinite completion. One might wonder whether every finitely generated profinite group can be realized as the completion of a finitely generated residually finite abstract group. However, a brief consideration reveals that this is not the case: for example, the p-adic integers $\mathbb{Z}_p$ are not the profinite completion of any finitely generated group.

An *upper composition factor* of a group G is a composition factor of some finite image of G. Segal [110] has shown that any collection of non-abelian finite simple groups can serve as the upper composition factors of a profinite completion. He was inspired by the work of Grigorchuk [50] and constructed a branch group H such that its profinite completion has the desired properties. There are only a few more specific classes of profinite groups known to be profinite completions. Among them are the exmples given by Pyber [104], who studied profinite completions similar to

$$\widehat{\mathbb{Z}} \times \prod_{\substack{j \text{ odd}, \\ j \geqslant 7}} \mathrm{Alt}(j).$$

Analogously, Lubotzky, Pyber, and Shalev constructed profinite completions of the form

$$\widehat{\mathbb{Z}} \times \prod_{n=1}^{\infty} \mathrm{PSL}_{d(n)}(q),$$

with q a fixed prime power and $d(n)$ an arbitrary strictly increasing sequence of integers greater than or equal to 2, cf. [85, Proposition 4.2].

Pyber posed the question of whether the factor $\widehat{\mathbb{Z}}$ is inevitable when talking about profinite completions involving a Cartesian product of finite simple groups. This was disproved by Kassabov, who, together with Nikolov, provided the following characterization in the realm of semisimple profinite groups, establishing the conditions under which a profinite group is a profinite completion, cf. [68, Theorem 1.4].

Theorem 1.8.1. *Let $G = \prod_{j \in \mathbb{N}_*} S_j$ where $\{S_j\}_{j=1}^{\infty}$ is an infinite family of finite non-abelian simple groups such that $\mathrm{rk}(S_j) \to \infty$. If G is topologically finitely generated, then it is a profinite completion.*

In the theorem, the rank $\mathrm{rk}(G)$ of a finite non-abelian simple group G is the integer n for $\mathrm{Alt}(n)$ and it is the Lie rank for finite simple groups of Lie type, i.e. if $G = \mathbf{G}^F/Z(\mathbf{G}^F)$, then the Lie rank is the rank of $\mathbf{G}$.

Question 1.8.2. *Is Theorem 1.8.1 true also for quasi-simsimple profinite groups of unbounded rank?*

With our Theorem 1.1.1, which we will prove in the next section, and with Theorem 1.8 of Kassabov and Nikolov [68], we have a construction which for every $a > 0$, gives a semisimple profinite group G such that it has PRG of degree a and it is a profinite completion of a finitely generated group H, cf. Theorem 1.1.1. Clearly, H can be taken to be residually finite.

The next natural question is to understand the representations of the abstract group H. All representations of H of finite image factor through representations of G. We prove in Proposition 1.8.5 that all the representation of H are of finite image, providing a complete description of the representations of H in terms of representations of G. First, recall that G is a linear group over a field k if G is a subgroup of $\mathrm{GL}_n(k)$ for some positive integer n. If G is finitely generated, then $G \leqslant \mathrm{GL}_n(R)$ where R is some finitely generated subring of k. First, we report a result by Mal'cev, cf. [86, Window 7, Proposition 8].

Proposition 1.8.3. *If G is a finitely generated linear group, then G is residually finite.*

Then we report also the following result of Platonov, cf. [86, Window 7, Proposition 9].

Proposition 1.8.4. *Let G be a finitely generated linear group over a field k. If $\operatorname{char} k = p$, then G is virtually residually p-finite. If $\operatorname{char} k = 0$, then G is virtually residually p-finite for all but finitely many primes.*

We are now ready to prove the following result.

Proposition 1.8.5. *Let H be a finitely generated residually finite group such that the profinite completion of H is a semisimple profinite group. Then all the complex representations of H are finite.*

Proof. By contradiction, let $\rho : H \to \mathrm{GL}_n(\mathbb{C})$ be an infinite representation of H and let Γ be the image of H. The group H is finitely generated, hence so is Γ. Using Proposition 1.8.3, since Γ is a linear group, we have that Γ is residually finite. Applying Proposition 1.8.4 to the field of the complex numbers, there exists a prime p and a normal subgroup of finite index N of Γ that is residually p-finite. Let K be a normal subgroup of N such that the quotient N/K is a non-trivial finite p-group and let M be the core of K in Γ. Hence we can describe M in the following way

$$M = K^{g_1} \cap \cdots \cap K^{g_r},$$

where $g_1, \ldots, g_r$ are cosets representatives for N in Γ. The quotient of N modulo M injects into the direct product of finite p-groups of the form

$$N/K^{g_1} \times \cdots \times N/K^{g_r}.$$

Thus, the finite quotient Γ/M of Γ has composition factors isomorphic to a cyclic group of order p. By the third isomorphism theorem, also H has such factors, thus violating the hypothesis that the profinite completion of H is semisimple. $\square$

Remark 1.8.6. *If Theorem 1.8.1 were true more generally for quasi-semisimple profinite groups with non-trivial centers, then Proposition 1.8.5 could be easily generalized to quasi-semisimple profinite groups that have composition factors with centers of bounded order. One could choose a prime p in the previous proof such that it does not divide the order of any center of the quasi-simple groups involved in G. Comparing with Table 1.3, the only type which has centers of unbounded order is the type A_n.*

1.9 Constructing quasi-semisimple profinite groups with specified representation growth rates

The first result that we prove is an intermediate step that we will use to prove our main Theorem 1.1.1. We follow the idea of the proof of [37, Theorem 6.3] and we adapt it for our more general setting.

Theorem 1.9.1. *For every $a \in \mathbb{R}_{>0}$, there exists a quasi-semisimple profinite group G that has polynomial representation growth and such that $\alpha(G) = a$. Moreover, we can choose G to be a semisimple profinite group involving just one Lie type with fixed rank and increasing defining field, i.e. of the form $G = \prod_{j\in\mathbb{N}_*} S(q^j)^{f(j)}$. If $\widetilde{G}$ is a quasi-semisimple profinite group such that there exists a surjection of $\widetilde{G}$ onto G, then $\alpha(G) = \alpha(\widetilde{G})$.*

Proof. Let $a \in \mathbb{R}_{>0}$ and let Φ be a non-trivial irreducible root system and τ an automorphisms of Φ stabilising a choice of Φ^+. Let $\mathfrak{Q}$ denote the set of all powers of a prime p. Consider a connected simply connected simple algebraic groups defined over an algebraic closure of characteristic p with associated root system Φ. We consider a family of Frobenious endomorphisms F defining $\mathbb{F}_q$-structures of $\mathbf{G}$ with $q \in \mathfrak{Q}$ and inducing the automorphism τ on the root system Φ. According to Remark 1.3.1, we are defining a family $\{L(q) \mid q \in \mathfrak{Q}, q \neq 2, 3\}$ of quasi-simple groups of Lie type. Denote the simple groups of Lie type given by the quotient $L(q)/Z(L(q))$ by $S(q)$, cf. Remark 1.3.2. Let r be the Lie rank of $S(q)$ which equals to the Lie rank of $L(q)$, i.e. $r = \mathrm{rk}(\mathbf{G}) = \mathrm{rk}(\Phi)$. Using Theorem 1.5.7, we choose Φ such that $a > 2r/|\Phi|$.

Let $(a_j)_{j\in\mathbb{N}_*}$ be a sequence of natural numbers such that $\left|\frac{a_j}{j} - a\right| \leq \frac{1}{j}$ for every $j \in \mathbb{N}_*$. Then a_j/j converges to a as j tends to infinity. Consider the following function

$$\begin{aligned} f\colon \mathbb{N}_* &\to \mathbb{N}_* \\ j &\mapsto \begin{cases} q^{\frac{r(a_j|\Phi|/r-2j)}{2}} & \text{if } j \geq \frac{|\Phi|}{2(r-1)} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Define a constant D such that

$$f(j) = q^{\frac{r(a_j|\Phi|/r-2j)}{2}} \leq q^{\frac{j(|\Phi|a-2)}{2}} = q^{jD},$$

and consider the groups

$$G = \prod_{j=1}^{\infty} S(q^j)^{f(j)} \quad \text{and} \quad \widetilde{G} = \prod_{j=1}^{\infty} L(q^j)^{f(j)}.$$

It suffices to show that G and $\widetilde{G}$ have polynomial representation growth and that $\alpha(G)$ equals $\alpha(\widetilde{G})$. By Proposition 1.4.11 and Remark 1.4.12, there exists an absolute constant d greater than 0 such that, if $S(q^j)$ and $L(q^j)$ have an irreducible representation of dimension n, then $n > dq^{jr}$. Hence, there exists a constant B such that

$$j < \frac{\log(n/d)}{r\,\log q} \leq B\,\log n.$$

Recall that $m_n(G)$ counts the number of non-abelian composition factors of a quasi-semisimple profinite group G for which we have at least one non-trivial representation of dimension at most n. Thus

$$m_n(G) = m_n(\widetilde{G}) < B\,(\log n)\,f(B(\log n)) \leq B\,(\log n)\,q^{DB\log n} = B\,(\log n)\,n^{DB\log q},$$

which means that $m_n(G)$ and $m_n(\widetilde{G})$ are polynomially bounded. By Proposition 1.7.2, it follows that G and $\widetilde{G}$ have PRG.

Recall the definition of a Dirichlet polynomial (1.5.1). By Theorem 1.5.6, there exist a constant $D \in \mathbb{R}$ that depends only on Φ and a finite set $d((\Phi,\tau),q^j) \in A^+$ such that

$$\zeta_{S(q^j)}(s) - 1 \sim_D \xi_{d((\Phi,\tau),q^j),q^j}(s).$$

Similarly, by Theorem 1.5.4 there exist a constant $C \in \mathbb{R}$ and a finite set $a(\Phi) \in A^+$ such that

$$\zeta_{L(q^j)}(s) - 1 \sim_C \xi_{a((\Phi,\tau),q^j),q^j}(s).$$

Let

$$\xi_G = \prod_{j=1}^{\infty} \left(1 + \xi_{d((\Phi,\tau),q^j),q^j}\right) \quad \text{and} \quad \xi_{\widetilde{G}} = \prod_{j=1}^{\infty} \left(1 + \xi_{a((\Phi,\tau),q^j),q^j}\right).$$

By Remark 1.5.3, the asbcissa of convergence of ξ_G and $\xi_{\widetilde{G}}$ respectively is the maximum of the abscissae of convergence where instead of the Dirichlet polynomials $\xi_{a((\Phi,\tau),q^j),q^j}$ and $\xi_{d((\Phi,\tau),q^j),q^j}$, we consider only one monomial. Hence, the functions ξ_G and $\xi_{\widetilde{G}}$ converge if and only if

$$\prod_{j=1}^{\infty} \left(1 + q^{j(m-ns)}\right) \quad \text{and} \quad \prod_{j=1}^{\infty} \left(1 + q^{j(\tilde{m}-\tilde{n}s)}\right)$$

converge for all $(m,n) \in a((\Phi,\tau),q^j)$ and $(\tilde{m},\tilde{n}) \in d((\Phi,\tau),q^j)$. Hence

$$\alpha(\xi_G) = \max\{m/n \mid (m,n) \in \bigcup_{j=1}^{\infty} a((\Phi,\tau),q^j)\}$$

and

$$\alpha(\xi_{\widetilde{G}}) = \max\{\tilde{m}/\tilde{n} \mid (\tilde{m},\tilde{n}) \in \bigcup_{j=1}^{\infty} d((\Phi,\tau),q^j)\}.$$

Note that the maximum is well defined by Theorem 1.5.4 (i) and Theorem 1.5.6 (i). Choosing q large enough, we can use (1.5.8), (1.5.9), [12, (3.16)], and [12, (3.17)], to conclude that

$$\alpha(\xi_G) = \alpha(\xi_{\widetilde{G}}) = \frac{r}{|\Phi^+|}.$$

Thus, an application of Lemma 1.5.2 gives

$$\alpha(G) = \alpha(\widetilde{G}) = \alpha\left(\prod_{j=1}^{\infty}(1+q^{jr(1-\frac{|\Phi|}{2r}s)})^{f(j)}\right).$$

The latter product converges if and only if the series

$$\sum_{j=1}^{\infty} f(j)\, q^{jr(1-\frac{|\Phi|}{2r}s)} = \sum_{j>N} q^{\frac{r(a_j|\Phi|/r-2j)}{2}} q^{jr(1-\frac{|\Phi|}{2r}s)} = \sum_{j>N} q^{\frac{j|\Phi|}{2}(\frac{a_j}{j}-s)}$$

converges, where $N = \frac{|\Phi|}{2(r-1)}$ comes from the definition of f. We claim that the latter series converges if and only if $\sigma > a$ where σ is the real part of s. Suppose first that $\sigma > a$ and recall that a_j/j tends to a as j goes to infinity. Then there exist $\varepsilon > 0$ such that $\frac{a_j}{j} - \sigma < -\varepsilon$ and so

$$\sum_{j>N} q^{\frac{j|\Phi|}{2}(\frac{a_j}{j}-\sigma)} \leq \sum_{j>N} q^{\frac{j|\Phi|}{2}(-\varepsilon)}.$$

Then we have an upper bound with a geometric series of ratio smaller than 1, which converges.

On the other hand, if $\sigma < a$, then we can find $\varepsilon > 0$ such that $\frac{a_j}{j} - \sigma > \varepsilon$ and hence

$$\sum_{j>N} q^{\frac{j|\Phi|}{2}(\frac{a_j}{j}-\sigma)} \geq \sum_{j>N} q^{\frac{j|\Phi|}{2}\varepsilon}$$

where the latter series diverges. □

With Theorem 1.9.1, we constructed groups whose non-abelian composition factors have fixed rank. This condition for semisimple profinite groups is in contrast with the hypothesis of Theorem 1.8.1. Therefore, it remains to address groups whose non-abelian composition factors have unbounded ranks.

Proof of Theorem 1.1.1. Let $(a_m)_{m\in\mathbb{N}_*}$ be a sequence positive real numbers that converges to a from below and let $(H_m)_{m\in\mathbb{N}_*}$ be a sequence of groups as in Theorem 1.9.1 such that the simple factors are all of the same Lie type with Lie rank r_m and $\alpha(H_m) = a_m$. Moreover, let the sequence of ranks $(r_m)_{m\in\mathbb{N}_*}$ be a strictly increasing sequence. We construct a semisimple profinite group H whose non-abelian composition factors have unbounded rank and with $\alpha(H) = a$, by taking the product of some quotients of the groups H_m. We write

$$H_m = \prod_{j=1}^{\infty} S_{m,j}.$$

First, consider the group H_1 with factors $S_{1,j}$ ordered by size, and define the sequence $x_n = \frac{\log R_n(H_1)}{\log n}$. By construction, we know that

$$a_1 = \limsup_{n\to\infty} \frac{\log R_n(H_1)}{\log n} = \limsup_{n\to\infty} x_n.$$

By the definition of limit superior in combination with $a_1 < a$, there are at most finitely many x_n greater than a. Deleting those factors in H_1, we create a group, that we will call again H_1, such that for all $n \in \mathbb{N}_*$ we have

$$\frac{\log R_n(H_1)}{\log n} \leqslant a.$$

Again by the properties of the limit superior, we know that there exists a subsequence x_{k_n} of x_n for which

$$x_{k_n} > \limsup_{n\to\infty} x_n - \varepsilon = a_1 - \varepsilon,$$

for any $\varepsilon > 0$ and for all $n \in \mathbb{N}_*$. Thus there exists $n(1) \in \mathbb{N}_*$ so that

$$\frac{\log R_{n(1)}(H_1)}{\log n(1)} \geqslant a_1 - 1.$$

We define $\widetilde{H_1}$ to be the product of finitely many groups $S_{1,j}$ such that

$$R_{n(1)}(H_1) = R_{n(1)}(\widetilde{H_1}).$$

The next step is to understand how we modify H_2, taking into account that we will have to make the product with H_1. As before, we write

$$H_2 = \prod_{j=1}^{\infty} S_{2,j},$$

such that $\alpha(H_2) = a_2$ and $\operatorname{rk} S_{2,j} = r_2$ for every $j \in \mathbb{N}_*$ with $r_2 > r_1$. Note that $\alpha(\widetilde{H_1} \times H_2) = \alpha(H_2)$. We change again H_2, and afterwards still call it H_2, deleting the first finitely many factors with representations of "small" dimension in such way that

$$R_{n(1)}(\widetilde{H_1} \times H_2) = R_{n(1)}(\tilde{H}_1).$$

As before, we delete again finitely many factors of H_2 so we have that for all $n \in \mathbb{N}_*$

$$\frac{\log R_n(\widetilde{H_1} \times H_2)}{\log n} \leqslant a,$$

using the fact that $a_2 < a$. Recalling the definition of the limit superior, we find $n(2) \in \mathbb{N}_*$ and $n(2) > n(1)$ such that

$$\frac{\log R_{n(2)}(\widetilde{H_1} \times H_2)}{\log n(2)} \geqslant a_2 - \frac{1}{2}.$$

Finally, we pick a finite subproduct of H_2, say $\tilde{H}_2$, such that

$$R_{n(2)}(\widetilde{H_1} \times H_2) = R_{n(2)}(\widetilde{H_1} \times \widetilde{H_2}).$$

We repeat this process inductively constructing the group

$$H = \prod_{m=1}^{\infty} \widetilde{H}_m$$

where every factor $\widetilde{H}_m$ is a finite product of finite non abelian simple groups $\prod_{j \in I} S_{m,j}$ for some finite set $I \subset \mathbb{N}_*$ with $\operatorname{rk} S_{m,j} = r_m$, and where we choose the sequence $(r_m)_{m \in \mathbb{N}_*}$ to be strictly increasing. Moreover, there exists a strictly increasing sequence of natural numbers $(n(j))_{j \in \mathbb{N}_*}$ such that

$$\frac{\log R_{n(j)}(\widetilde{H}_1 \times \cdots \times \widetilde{H}_j)}{\log n(j)} \geqslant a_j - \frac{1}{j},$$

and such that for $n(j-1) < n \leqslant n(j)$ with $j \in \mathbb{N}_{>1}$, we have

$$R_n(H) = R_n(\widetilde{H}_1 \times \cdots \times \widetilde{H}_j).$$

We want to prove that $\alpha(H) = a$. Observe that

$$\frac{\log R_{n(j)}(H)}{\log n} = \frac{\log R_{n(j)}(\widetilde{H}_1 \times \cdots \times \widetilde{H}_j)}{\log n} \geqslant a_j - \frac{1}{j},$$

thus

$$\limsup_{j \to \infty} \frac{\log R_{n(j)}(H)}{\log n} = \limsup_{j \to \infty} \frac{\log R_{n(j)}(\widetilde{H}_1 \times \cdots \times \widetilde{H}_j)}{\log n} \geqslant \lim_{j \to \infty} \left(a_j - \frac{1}{j} \right) = a.$$

On the other hand, for $n(j-1) < n \leqslant n(j)$ with $j \in \mathbb{N}_{>1}$, we have

$$R_n(H) = R_n(\widetilde{H}_1 \times \cdots \times \widetilde{H}_j)$$

and by construction, we deduce that $\frac{\log R_n(H)}{\log n} \leqslant a$. This gives $\alpha(H) \leqslant a$.

Hence, we have constructed an infinite countable based semisimple profinite group H that has polynomial representation growth and hence it is finitely generated by Theorem 1.7.3. Moreover, its simple factors have unbounded rank and so by Theorem 1.8.1, it is a profinite completion. □

Chapter 2

Representation growth of compact p-adic analytic groups

Based on joint work with Jan Moritz Petschick

2.1 Introduction

A topological group G is called *representation rigid*, or just rigid, if the number of isomorphism classes of n-dimensional continuous irreducible complex representations of G is finite for every positive integer $n \in \mathbb{N}_*$. Given a rigid group G, we write $r_n(G)$ for the number of irreducible representations of dimension n and $R_N(G)$ for the sum $\sum_{n=1}^{N} r_n(G)$. It is a fundamental goal of representation theory to understand the growth and general behaviour of the resulting arithmetic sequences $(r_n(G))_{n \in \mathbb{N}_*}$ and $(R_N(G))_{N \in \mathbb{N}_*}$. If G is such that the latter sequence grows polynomially, we say that G has *polynomial representation growth* (PRG). The study of representation growth is inspired by the study of subgroup growth, which analyses the growth of the arithmetic sequence $a_n(G)$, where $a_n(G)$ denotes the number of subgroups of index n. Grunewald, Segal, and Smith [55] initiated a systematic study of groups with polynomial subgroup growth, examining the arithmetic of the sequence using number theoretic methods. In a similar spirit, for a group G having PRG, we encode the sequence $(r_n(G))_{n \in \mathbb{N}_*}$ using a Dirichlet generating function which is defined on a right half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha(G)\}$ delimited by the *abscissa of convergence* $\alpha(G)$. This function is called the *representation zeta function of* G and it is given by

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}.$$

By the theory of Dirichlet generating functions, if G has polynomial representation growth, the abscissa of convergence $\alpha(G)$ coincides with the degree of growth of the sequence $(R_N(G))_{N \in \mathbb{N}_*}$, cf. Section 0.7 for more details.

We consider the representation zeta functions of certain compact p-adic analytic groups and, in particular, we consider profinite groups. Zeta functions of this kind appear, for instance, as the local factors in the Euler product decomposition of the representation zeta functions associated to arithmetic groups G with the congruence subgroup property [77], where the congruence subgroup property ensures polynomial growth of representations for arithmetic groups, cf. [84]. In the realm of finitely generated profinite groups, rigid groups are algebraically characterised by the FAb property: a topological group is FAb if every open subgroup H has finite abelianisation $H/[H,H]$, see [13, Proposition 2]. For a p-adic analytic group G, this characterisation can be carried over to its Lie algebra, employing Lazard's correspondence between saturable pro-p groups and saturable $\mathbb{Z}_p$-Lie lattices, cf. [43, 78]. In particular, a p-adic analytic group G is FAb and hence rigid if and only if it has an open FAb saturable pro-p subgroup U; and a saturable pro-p subgroup U is FAb if and only if the $\mathbb{Z}_p$-Lie lattice $\log(U)$ is FAb. Furthermore, the latter happens if and only if the $\mathbb{Q}_p$-Lie algebra $\log(U) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is perfect, cf. [11, Proposition 2.1].

According to Jaikin-Zapirain [66] for odd primes and Stasinski and Zordan [119] for the prime 2 case, the representation zeta function of a FAb compact p-adic analytic group is a finite sum

$$\sum_{i=1}^{r} n_i^{-s} f_i(p^{-s}),$$

where $n_1, \ldots, n_r$ are positive integers and $f_1, \ldots, f_r \in \mathbb{Q}(t)$ are rational functions. There are only a few examples where this computation has been carried out explicitly, see [10,11,66,134]. Specifically, the representation zeta functions of the groups $\mathrm{SL}_2(\mathbb{Z}_p)$ and $\mathrm{SL}_3(\mathbb{Z}_p)$, as well as their principal congruence subgroups and those of $\mathrm{SL}_4(\mathbb{Z}_p)$, are known. Nevertheless, already the groups $\mathrm{SL}_n(\mathbb{Z}_p)$ for $n > 4$ remain mysterious; even the abscissae of convergence of their respective zeta functions are not determined, although some bounds exist, cf. [2, 10, 17, 77].

The main technique for the computation of representation zeta functions of compact p-adic analytic groups can be separated into two steps, using two very different techniques. The first step is to compute the zeta function of a uniformly potent normal pro-p subgroup of finite index, using the Kirillov orbit method, see e.g. [41,58], by evaluating a p-adic integral related to the structure of the associated Lie lattice. For any odd prime p, a pro-p group G is called *potent* if $\gamma_{p-1}(G) \subseteq G^p$, where $\gamma_k(G)$ denotes the k^{th} term of the lower central series of G, cf. Section 0.1, and G^k the subgroup generated by k^{th} powers, for $k \in \mathbb{N}_*$. For $p = 2$, we instead demand that $[G, G] \subseteq G^4$. If G is potent, finitely generated, and torsion-free, we call it *uniformly potent* for short; such groups are a straightforward generalisation of uniformly powerful groups as consider e.g. in [31].

After the p-adic integral associated to a uniformly potent normal pro-p subgroup has been computed, the second step is to perform (approximative) Clifford theory to extend the analysis of representations from uniformly potent normal pro-p groups to the encompassing compact p-adic analytic group. Both steps are described in detail in [11].

Even though the situation for the first step is backed-up by strong theoretical results, the actual computation of the involved p-adic integrals is highly complicated. Here we restrict ourselves to the computation of representation zeta functions of uniformly potent pro-p groups and ignore the second step that one can take additionally.

As we mention above, a key ingredient in this context is the Lie correspondence between the groups involved and certain $\mathcal{O}$-Lie lattices. Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0 and residue characteristic p, with uniformiser π. Put $\mathfrak{p} = \pi\,\mathcal{O}$, denote by q the cardinality of the residue field $\kappa = \mathcal{O}/\mathfrak{p}$, and let K be the field of fractions of $\mathcal{O}$, which constitutes a finite extension of $\mathbb{Q}_p$. Let $\mathfrak{g}$ be a $\mathbb{Z}_p$-Lie lattice, i.e. a Lie ring over $\mathbb{Z}_p$ that is also a free $\mathbb{Z}_p$-module of finite rank. A $\mathbb{Z}_p$-Lie lattice $\mathfrak{g}$ is called *potent* if $\gamma_{p-1}(\mathfrak{g}) \subseteq p\,\mathfrak{g}$ for $p > 2$ and $\gamma_2(\mathfrak{g}) \subseteq 4\,\mathfrak{g}$ for $p = 2$, as analogously defined for pro-p groups. For every finite extension K of $\mathbb{Q}_p$ and every maximal ideal $\mathfrak{p} \trianglelefteq \mathcal{O}$, we consider the $\mathcal{O}$-Lie lattice $\mathfrak{g}_\mathfrak{p} = \mathfrak{g} \otimes_{\mathbb{Z}_p} \mathcal{O}$. For $m \in \mathbb{N}$, its principal congruence sublattices are $\mathfrak{g}_{\mathfrak{p},m} = \pi^m \cdot \mathfrak{g}_\mathfrak{p}$. The $\mathcal{O}$-Lie lattice $\mathfrak{g}_{\mathfrak{p},m}$ is called *potent* if it is potent as a $\mathbb{Z}_p$-Lie lattice. The Hausdorff series defines a group multiplication on $\mathfrak{g}_{\mathfrak{p},m}$ so then we can define the group $\exp(\mathfrak{g}_{\mathfrak{p},m})$. The Lie lattice $\mathfrak{g}_{\mathfrak{p},m}$ is potent for all sufficiently large integers m, so that $G_{\mathfrak{p},m} = \exp(\mathfrak{g}_{\mathfrak{p},m})$ is a uniformly potent pro-p group. We call such positive integers m *permissible* for $\mathfrak{g}_\mathfrak{p}$, cf. [11, Proposition 2.3]. If $\mathcal{O}$ is unramified over $\mathbb{Z}_p$ and $p > 2$, every $m \in \mathbb{N}_*$ is permissible for every $\mathcal{O}$-Lie lattice $\mathfrak{g}_\mathfrak{p}$, and in the same way, if $p = 2$ every $m \geqslant 2$ is permissible, cf. [11, Section 2.1].

Our main result concerns the representation zeta function of the semi-direct product of a suitable subgroup H of the first principal congruence subgroup $\mathrm{SL}_2^1(\mathcal{O})$ of $\mathrm{SL}_2(\mathcal{O})$ acting continuously on an $\mathcal{O}H$-module $V \cong \mathcal{O}^n$ of finite $\mathcal{O}$-rank. Due to the condition for potency, we introduce the symbol $\mathbf{p}$, signifying p in the odd and 4 in the even case. More precisely, we prove the following.

Theorem 2.1.1. *Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0 and residue characteristic p. Let H be a potent subgroup of $\mathrm{SL}_2^1(\mathcal{O})$, let V be an $\mathcal{O}\,H$-module of finite $\mathcal{O}$-rank n, and let $\sigma\colon H \to \mathrm{GL}_n^m(\mathcal{O})$ be a faithful finite-dimensional $\mathcal{O}$-representation of H on V such that $H^\sigma \cap \mathrm{GL}_n^{m+1}(\mathcal{O}) \leq (H^\sigma)^\mathbf{p}$, for a permissible m. Assume furthermore that the semi-direct product $G = H \ltimes_\sigma V$ is FAb. Then*

$$\zeta_G(s) = \zeta_H(s) \cdot \zeta_H^G(s-1),$$

where $\zeta_H^G(s)$ is the zeta function associated to the representation $\mathrm{Ind}_H^G(\mathbb{1})$.

In the product, the factor $\zeta_H^G(s)$, which is the zeta function associated to the representation of G induced from the trivial representation of H, is a kind of zeta function which was recently introduced and studied in detail by Kionke and Klopsch in [70]. This kind of zeta function is a generalisation of the representation zeta function related to a group, see Section 2.2.2 for more details. In the special case of $H = \mathrm{SL}_2^m(\mathcal{O})$, for a permissible m, the specific function appearing as a factor above may be seen as the zeta function associated to the action of $\mathrm{SL}_2^m(\mathcal{O})$ on the cosets of $\mathrm{SL}_2^m(\mathcal{O})$ in $\mathrm{SL}_2^m(\mathcal{O}) \ltimes V$, in natural correspondence to V. However, the decomposition of the zeta function as such a product cannot be expected in a more general setting: we give an example of a potent pro-p group where the corresponding equation fails to hold, showing that this behaviour is not universal, cf. Example 2.4.2.

To obtain Theorem 2.1.1, we use the following description of the representation zeta function of all potent subgroups of $\mathrm{SL}_2^1(\mathcal{O})$. Consider a potent pro-$p$-subgroup $H$of $\mathrm{SL}_2^1(\mathcal{O})$, and let K be a open potent pro-p subgroup of H. Then

$$\zeta_K(s) = |H : K| \cdot \zeta_H(s).$$

In fact, we investigate a more general situation in Theorem 2.3.1, using a 'weak' analogue of Lie algebra isomorphisms. It is unclear how to characterise subgroups fulfilling similar equations – i.e. with 'asymptotically the same representation theory' meaning that the zeta functions differ only by a constant factor – in general. We present a class of subgroups of $\mathrm{SL}_3(\mathcal{O})$ of the desired kind in Proposition 2.3.3.

Building on the aforementioned results, we explicitly compute the representation zeta functions of various semi-direct products, which we present in the following.

Theorem 2.1.2. *Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0, residue characteristic an odd prime p, and residue field cardinality q. Then in the different settings described below the following hold.*

(a) Let $H_n^m = \mathrm{SL}_n^m(\mathcal{O})$ with permissible $m \in \mathbb{N}_*$ and $n \in \mathbb{N}_{\geqslant 2}$, and consider $G_n^m = H_n^m \ltimes \mathcal{O}^2$, where the semidirect product is formed with respect to the natural action. Then

$$\zeta_{H_n^m}^{G_n^m}(s) = q^{mn} \frac{(1 - q^{-n(1+s)})}{(1 - q^{-ns})}.$$

In particular, for $n = 2$ we obtain

$$\zeta_{G_2^m}(s) = \zeta_{H_2^m}(s)\zeta_{H_2^m}^{G_2^m}(s-1) = q^{5m} \frac{(1 - q^{-2s})(1 - q^{-2-s})}{(1 - q^{1-s})^2(1 + q^{1-s})}.$$

(b) For simplicity, only in this case, let $\mathcal{O}$ be an unramified extension of $\mathbb{Z}_p$ and let $p \geqslant 7$. For $k \in \mathbb{N}_{\geqslant 1}$, consider G_k of the form $H_k \ltimes \mathcal{O}^2$, where

$$H_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2^1(\mathcal{O}) \mid a \equiv d \equiv 1 \bmod p^k, c \equiv 0 \bmod p^k \right\}.$$

Then

$$\zeta_{H_k}^{G_k}(s) = q^{2+ks} \frac{(1 - q^{-1-s})(1 - q^{-s} - q^{-1-2s} + q^{-1-(k+1)s})}{(1 - q^{-s})^2(1 + q^{-s})},$$

and we have

$$\zeta_{G_k}(s) = q^{5+ks} \frac{(1 - q^{-2-s})(1 - q^{-s})(1 - q^{1-s} - q^{1-2s} + q^{k-(k+1)s})}{(1 - q^{1-s})^3(1 + q^{1-s})}.$$

(c) For $G_n^m = \mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^{2n}$ with $n \in \mathbb{N}_*$ and permissible $m \in \mathbb{N}_*$, where we regard $\mathcal{O}^2$

as the natural module and consider the diagonal action of $\mathrm{SL}_2^m(\mathcal{O})$ *on* $\mathcal{O}^{2n} \cong \bigoplus_{i=1}^n \mathcal{O}^2$, *we have*

$$\zeta_{G_n^m}(s) = q^{2nm-1}\frac{(1-q^{-s})(1+q-(q^{n-1}-1)q^{2-2s}-q^{n+2-3s}-q^{n+2-4s})}{(1-q^{(n+1)-2s})(1-q^{2n-3s})}\zeta_{\mathrm{SL}_2^m(\mathcal{O})}.$$

In particular, depending on the value of n*, the abscissa of convergence may be determined by either of the two uniformly varying factors in the denominator, i.e. for* $n \leqslant 3$ *the abscissa of convergence is* $\alpha(G_n^m) = \frac{n+1}{2}$*, and for* $n \geqslant 3$ *the abscissa of convergence is* $\alpha(G_n^m) = \frac{2n}{3}$.

(d) For $G_n^m = \mathrm{SL}_2^m(\mathcal{O}) \ltimes (\mathrm{Sym}^2(\mathcal{O}^2))^n$ *and* $H^m = \mathrm{SL}_2^m(\mathcal{O})$ *with permissible* $m \in \mathbb{N}_*$ *and* $n \in \mathbb{N}_{\geqslant 2}$*, we have*

$$\zeta_{H^m}^{G_n^m}(s) = q^{3nm-1}(1-q^{-s})(1-q^{-1-s})\frac{(q^{-s}+q^{n-2s}+(1+q^{-s})q+(1+q^{-s})q^{n-1-2s})}{(1-q^{n-2s})(1-q^{3(n-1)-3s})}.$$

(e) For $G^m = \mathrm{SL}_2^m(\mathbb{Z}_2) \ltimes \mathrm{Sym}^2(\mathbb{Z}_2^2)$ *and* $H^m = \mathrm{SL}_2^m(\mathbb{Z}_2)$ *with permissible* $m \in \mathbb{N}_*$*, we have*

$$\zeta_{H^m}^{G^m}(s-1) = 2^{3m+1}\frac{(1-2^{-s})(2^{3-s}+(1-2^{-s}))}{1-2^{3-s}}.$$

We prove this result in Section 2.4. These are the first examples of representation zeta functions of non-semisimple compact p-adic analytic groups.

2.2 Preliminaries

2.2.1 Tensor products, induced representations, inflations, and extensions

Denote by $\mathrm{Irr}(G)$ the set of (isomorphism classes of) finite dimensional irreducible complex representations $\sigma\colon G \to \mathrm{GL}(W)$ of a group G, for short (σ, W). In the following all groups considered are profinite, and all representations are continuous. In this situation every representation decomposes as a direct sum of irreducible representations, and all irreducible representations factor through a finite quotient of G, cf. Section 0.4.

A representation (σ, W) is *smooth* if the map $G \times W \to W$ is continuous when W is equipped with the discrete topology and G is equipped with its natural topology. It is a well known fact that for a profinite group G every smooth representation decomposed as a direct sum of smooth irreducible constituents and that the smooth irreducible representations of G are precisely the finite dimensional irreducible continuous representations of G, see for example [70, Lemma 2.1] and the references therein.

Given two representations (σ, W) and (φ, U) of G, their *tensor product* $(\sigma \otimes \varphi, W \otimes_{\mathbb{C}} U)$ is the unique representation of G on $W \otimes_{\mathbb{C}} U$ satisfying

$$(\sigma \otimes \varphi)(g).(w \otimes u) = \sigma(g).w \otimes \varphi(g).u \quad \text{for } g \in G, w \in W, u \in U.$$

For a profinite group G, let $\mathcal{C}^\infty(G, W)$ be the space of all continuous functions from G to a $\mathbb{C}$-vector space W equipped with the discrete topology. Given a closed subgroup H of a profinite group G and a representation (σ, W) of H, we define a representation $\varrho = \mathrm{Ind}_H^G(\sigma)$ of G on

$$U_\sigma = \{f \in \mathcal{C}^\infty(G, W) \mid \forall\, h \in H\, \forall\, x \in G : f(hx) = \sigma(h).f(x)\}$$

via right translation

$$(\varrho(g).f)(x) = ({}^g f)(x) = f(xg) \quad \text{for } g, x \in G, f \in U_\sigma.$$

The representation ϱ is called the representation *induced by* σ. Following Kionke and Klopsch [70, Section 2.3] we call the representation 'induced' rather than 'co-induced', which is the terminology of Serre in [114, Section I 2.5].

There is a one-to-one correspondence between the isomorphism classes of smooth irreducible representations of a profinite group G and its irreducible complex characters. If H is open in G, and χ_σ is the character associated to a finite dimensional representation (σ, W) of H, the character corresponding to the induced representation can be expressed as the function

$$\mathrm{Ind}_H^G(\chi_\sigma) \colon \gamma \mapsto \sum_{\substack{x \in G/H \\ \gamma^x \in H}} \chi_\sigma(\gamma^x), \quad \text{for } \gamma \in G.$$

Given a group G, an (arbitrary) homomorphism $\varphi : G \to H$ and a representation σ of H we define the *inflation of* σ *along* φ by $\mathrm{Inf}_H^{G,\varphi}(\sigma) = \varphi\sigma$. If φ is implicit or its choice clear from the context, we will drop it from the superscript.

Lastly, given a closed subgroup H of G and a representation σ of H, we say that σ is *extendable* if there exists a representation $\tilde{\sigma}$ of G such that $\tilde{\sigma}|_H$ equals σ.

2.2.2 Integral formalism for potent pro-p groups

A smooth representation σ of a profinite group G is called *strongly admissible*, if its decomposition into irreducible subrepresentations,

$$\sigma = \bigoplus_{\varphi \in \mathrm{Irr}(G)} m(\sigma, \varphi)\varphi,$$

is such that there are only finitely many constituents (counted with multiplicity) of every dimension, i.e. that the multipliers $m(\sigma, \varphi) \in \mathbb{N}$ in the formula above fulfil

$$\sum_{\substack{\varphi \in \mathrm{Irr}(G) \\ \dim(\varphi) = d}} m(\sigma, \varphi) \in \mathbb{N}$$

for all $d \in \mathbb{N}_*$. Given a strongly admissible representation σ of G, one forms the formal Dirichlet series

$$\zeta_\sigma(s) = \sum_{\varphi \in \mathrm{Irr}(G)} m(\sigma, \varphi) \dim(\varphi)^{-s},$$

called the *zeta function associated to* σ. If $\rho = \mathrm{Ind}_1^G(\mathbb{1})$ is the *regular representation* of a group G, i.e. the (right) translation action of G on the space of continuous functions $G \to \mathbb{C}$, then every irreducible representation φ of G appears with multiplicity $m(\rho, \varphi) = \dim(\varphi)$. If G is rigid then the regular representation is strongly admissible and we have the formal identity

$$\zeta_\rho(s) = \sum_{\varphi \in \mathrm{Irr}(G)} \dim(\varphi)^{1-s} = \zeta_G(s-1), \tag{2.2.1}$$

as noted in [70, Example 2.5]. In this way, the zeta functions of strongly admissible representations generalise the representation zeta functions of groups.

Assume that G is finitely generated and let H be a closed subgroup of G. Then, according to [70, Theorem A], the representation $\mathrm{Ind}_H^G(\mathbb{1}_H)$ is strongly admissible if and only if the group G is *FAb relative to* H, i.e. if the quotient $K/(H \cap K)[K, K]$ is finite for every open subgroup K of G.

Since we are only concerned with representations of the form $\mathrm{Ind}_H^G(\mathbb{1}_H)$ for $H \leq_{\mathrm{c}} G$, we simplify our notation. We call this function the *relative zeta function of G with respect to H* and denote it by $\zeta_H^G(s)$.

Recall that for any odd prime p, a pro-p group G is called potent if $\gamma_{p-1}(G) \subseteq G^p$, where $\gamma_k(G)$ denotes the k^{th} term of the lower central series of G and G^k the subgroup generated by k^{th} powers, for $k \in \mathbb{N}$, and for $p = 2$, we instead demand that $[G, G] \leq G^4$. If the pro-p group G is potent, finitely generated, and torsion-free, we call it uniformly potent. Every uniformly potent group is a *saturable* group in the sense of Lazard, see [40, 73, 78]. González-Sánchez [40] gave a characterization of saturable groups in terms of potent filtrations. For a saturable pro-p group G, one associates a saturable $\mathbb{Z}_p$-Lie lattice $\mathfrak{g} = \log(G)$, which coincides with G as a topological space. There exists a precise Lie correspondence between G and $\mathfrak{g}$; see [43]. Recall that a $\mathbb{Z}_p$-Lie lattice is a Lie ring over $\mathbb{Z}_p$ that is also a free module of finite rank over $\mathbb{Z}_p$. As a consequence of the characterization of González-Sánchez, uniformly powerful pro-p groups and, more generally, torsion free finitely generated pro-p groups G with $\gamma_p(G) \subseteq \Phi(G)^p$ are saturable, where $\Phi(G)$ denotes the Frattini subgroup of G, cf. [40, Corollary 5.4].

Furthermore, recall that a $\mathbb{Z}_p$-Lie lattice $\mathfrak{g}$ is potent if $\gamma_{p-1}(\mathfrak{g}) \subseteq p\,\mathfrak{g}$ for $p > 2$ and $\gamma_2(\mathfrak{g}) \subseteq 4\,\mathfrak{g}$ for $p = 2$. If G is a saturable pro-p group and $\mathfrak{g}$ the associated $\mathbb{Z}_p$-Lie lattice, then G is potent if and only if $\mathfrak{g}$ is potent. We remak that every saturable pro-p group of dimension less than p is potent, cf. [43]. Moreover, Klopsch proved that every insoluble maximal p-adic analytic just-infinite pro-p group of dimension less than $p-1$ is saturable, see [73].

Recall also that $\mathcal{O}$ is a compact discrete valuation ring of characteristic 0 and residue characteristic p, with uniformiser π. The valuation ideal of $\mathcal{O}$ is $\mathfrak{p} = \pi\,\mathcal{O}$, q denotes the cardinality of the residue field $\kappa = \mathcal{O}\,/\mathfrak{p}$, and K the field of fractions of $\mathcal{O}$, which constitutes a finite extension of $\mathbb{Q}_p$.

For a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, recall the definition of the $\mathcal{O}$-Lie lattice $\mathfrak{g}_\mathfrak{p}$ as the tensor product $\mathfrak{g} \otimes_{\mathbb{Z}_p} \mathcal{O}$. This Lie lattice $\mathfrak{g}_\mathfrak{p}$ is potent as a $\mathbb{Z}_p$-Lie lattice if the original $\mathbb{Z}_p$-Lie lattice $\mathfrak{g}$ is potent.

Using the Lie lattices associated to a uniformly potent group G and a closed potent subgroup H, one can apply the Kirillov orbit method to describe the irreducible representations of G. This is done by considering the orbits under the co-adjoint action of G on the Pontryagin dual of the $\mathbb{Z}_p$-Lie lattice $\log(G)$, with some additional measures necessary to address the case of the even prime. For a detailed explanation, see [70, Section 4].

To state the p-adic integral that results from this description, we need some further notation. Assume that the potent $\mathcal{O}$-Lie lattice $\mathfrak{h}$ is a direct summand of an $\mathcal{O}$-Lie lattice $\mathfrak{g}$. We choose an $\mathcal{O}$-basis $\mathfrak{Y}$ of $\mathfrak{h}$ and extend this basis to $\mathfrak{X} \cup \mathfrak{Y}$, an $\mathcal{O}$-basis of $\mathfrak{g}$. The *commutator matrix of* $\mathfrak{g}$ *with respect to* $\mathfrak{X} \cup \mathfrak{Y}$ is the skew-symmetric $\dim(\mathfrak{g})$-by-$\dim(\mathfrak{g})$ matrix with entries in $\mathfrak{g}$ given by

$$\mathrm{Com}(\mathfrak{g}, \mathfrak{X} \cup \mathfrak{Y}) = \big([b, b']\big)_{(b,b') \in (\mathfrak{X} \cup \mathfrak{Y})^2}.$$

Let $w \colon \mathfrak{g} \to K$ be an $\mathcal{O}$-linear functional. Write $\mathrm{Com}(\mathfrak{g}, \mathfrak{X} \cup \mathfrak{Y})w$ for the entry-wise application of w. This results in a skew-symmetric matrix with entries in K.

For a commutative ring R, recall that the determinant of every skew-symmetric matrix $T = (t_{i,j})_{1 \le i,j \le n} \in \mathrm{Mat}_{n \times n}(R)$ is the square (as a polynomial) of the *Pfaffian determinant*, which must necessarily be trivial in case n is odd, and which, for even numbers $n = 2k$, is defined by

$$\mathrm{pf}(T) = \frac{1}{2^k k!} \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{l=1}^{k} t_{(2l-1)\sigma,(2l)\sigma}.$$

A *Pfaffian minor* of a skew-symmetric matrix T is the Pfaffian determinant of a principal submatrix given by the rows and columns with index in some fixed subset of $[n]$ of even cardinality.

We write $\mathrm{Pfaff}(T)$ for the set of all Pfaffian minors of a skew-symmetric matrix T. Finally, let $|\cdot|_\mathfrak{p}$ be the norm with respect to the $\mathfrak{p}$-adic valuation and for a subset $S \subseteq K$, define

$$\|S\|_\mathfrak{p} = \max\{|s|_\mathfrak{p} \mid s \in S\}.$$

Theorem 2.2.1. [70, Propsition C and Proposition 4.6] *Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic* 0, *residue characteristic p, and residue field cardinality q. Let $\mathfrak{p}$ be the valuation ideal, π an uniformiser so that $\mathfrak{p} = \pi\,\mathcal{O}$, and K the field of fractions of $\mathcal{O}$. Let $\mathfrak{g}$ be an $\mathcal{O}$-Lie lattice and let $m \in \mathbb{N}$ be such that $G = \exp(\pi^m \mathfrak{g})$ is a uniformly potent pro-p group. Let $\mathfrak{h}$ be direct summand of the $\mathcal{O}$-Lie lattice which corresponds to the potent subgroup $H = \exp(\pi^m \mathfrak{h}) \leqslant_c G$ such that G is FAb relative to H. Write $d = \dim_\mathcal{O} \mathfrak{g} - \dim_\mathcal{O} \mathfrak{h}$.*

Let $\mathfrak{Y}$ be a basis of $\mathfrak{h}$ and $\mathfrak{X} \cup \mathfrak{Y}$ an $\mathcal{O}$-basis of $\mathfrak{g}$. Let $K\mathfrak{X}^*$ be the subspace spanned by the part $\mathfrak{X}^*$ of the dual basis of $\mathfrak{X} \cup \mathfrak{Y}$ belonging to $\mathfrak{X}$ in the dual space $\mathfrak{g}^* = \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, K)$. Then

$$\zeta_H^G(s) = q^{md} \int\limits_{K\mathfrak{X}^*} \|\mathrm{Pfaff}(\mathrm{Com}(\mathfrak{g}, \mathfrak{X} \cup \mathfrak{Y})w)\|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(w),$$

where μ denotes the Haar measure of $K\mathfrak{X}^*$ satisfying $\mu(K\mathfrak{X}^* \cap \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})) = 1$.

As noted in (2.2.1), we have the identity $\zeta_G(s) = \zeta_1^G(s+1)$, hence one obtains a corresponding integral formula for the representation zeta function of G, made explicit in the following corollary.

Corollary 2.2.2. *Let G be a uniformly potent FAb pro-p group of dimension d and let $\mathfrak{X}$ be a basis of the associated $\mathbb{Z}_p$-Lie lattice $\mathfrak{g}$. Then*

$$\zeta_G(s) = \int_{\mathfrak{g}^*} \|\mathrm{Pfaff}(\mathrm{Com}(\mathfrak{g}, \mathfrak{X})w)\|_{\mathfrak{p}}^{-2-s} \mathrm{d}\mu(w),$$

where $\mathfrak{g}^* = \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, K)$ is the dual space of $\mathfrak{g}$ and μ denotes the Haar measure of $\mathfrak{g}^*$, satisfying $\mu(\mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})) = 1$.

2.2.3 Representation theory of semi-direct products

We make the following standing assumptions for the remainder of this section. Let G be a compact topological group and H a closed subgroup such that G decomposes (continuously) as the semi-direct product $G = H \ltimes V$, where V is an abelian closed normal subgroup of G. We describe the irreducible (continuous) representations of G in terms of those of H, using the classic description by Mackey [90] for semi-direct products, cf. Section 0.4.1.

Given a subgroup Δ of Γ and a representation σ of Γ, we say that σ is *extendable* if there exists a representation $\tilde{\sigma}$ of Γ such that $\tilde{\sigma}|_\Delta = \sigma$. In general, such *extensions* may not exist; however, for a pair (G, V) as described above, a representation $\sigma \in \mathrm{Irr}(V)$ can always be extended to the group $H_\sigma = \mathrm{Stab}_H(\sigma) \ltimes V$, where H acts on $\mathrm{Irr}(V)$ by $\sigma^h(v) = \sigma(v^{h^{-1}})$ for all $v \in V$, $h \in H$, and $\mathrm{Stab}_H(\sigma)$ is the stabilizer of σ in H. Set

$$\mathrm{Ext}_V^{H_\sigma}(\sigma)(hv) = \sigma(v)$$

for all $h \in \mathrm{Stab}_H(\sigma)$. In fact,

$$\mathrm{Ext}_V^{H_\sigma}(\sigma)(hvh'v') = \sigma(v^{h'}v') = \sigma(v)\sigma(v') = \mathrm{Ext}_V^{H_\sigma}(\sigma)(hv)\,\mathrm{Ext}_V^{H_\sigma}(\sigma)(h'v'),$$

so $\mathrm{Ext}_V^{H_\sigma}(\sigma)(hv)$ defines a representation of $H_\sigma = \mathrm{Stab}_H(\sigma) \ltimes V$.

Using this terminology, we may describe the structural form of irreducible representations of G, cf. [113, Proposition 25].

Proposition 2.2.3. *Let G be the semi-direct product $H \ltimes V$. Let $\mathcal{X}$ be a set of representatives of the orbits of the action of H on $\mathrm{Irr}(V)$. Assume that all stabilizers of characters in $\mathcal{X}$ are of finite index in H, and assume that the set $\mathcal{X}$ is countable. Since V is abelian, the irreducible characters coincide with irreducible representations of V. For each character χ of $\mathcal{X}$, let K_χ be the stabiliser of χ in H. Then every irreducible representation of G is of the form*

$$\mathrm{Ind}_{K_\chi \ltimes V}^{G}\left(\mathrm{Inf}_{K_\chi}^{K_\chi \ltimes V}(\tau) \otimes \mathrm{Ext}_{V}^{K_\chi \ltimes V}(\chi)\right), \tag{$\dagger$}$$

for some $\chi \in \mathcal{X}$, and $\tau \in \mathrm{Irr}(K_\chi)$. Two representations of this form are equivalent only if they are given by the same pair (χ, τ).

For compact groups, one may define a generalisation of the usual inner product on the set of irreducible characters for finite groups, by setting

$$\langle \chi, \theta \rangle_G = \int_G \chi(g)\overline{\theta(g)}\mathrm{d}\mu(g), \quad \text{for } \chi, \theta \in \mathrm{Irr}(G),$$

where μ denotes the normalised (left-)Haar measure of G, compare with Section 0.2.1 and Section 0.5. As in the setting of finite groups, it is still true that, given an irreducible component θ of χ, the value of $\langle \chi, \theta \rangle_G$ equals the multiplicity of θ appearing in the decomposition of χ. For us, the following equality will be of use.

Proposition 2.2.4. *In the set-up of the previous proposition, let $\chi \in \mathcal{X}$, K_χ be the stabilizer of χ in H, and τ be an irreducible representation of K_χ. Denote by θ_τ the character associated to τ and by $\theta_{\tau,\chi}$ the character associated to the representation $\mathrm{Inf}_{K_\chi}^{K_\chi \ltimes V}(\tau) \otimes \mathrm{Ext}_{V}^{K_\chi \ltimes V}(\chi)$ of $K_\chi \ltimes V$. Then we have*

$$\langle \mathrm{Ind}_{K_\chi \ltimes V}^{G}(\theta_{\tau,\chi}), \mathrm{Ind}_{H}^{G}(\mathbb{1}_H) \rangle_G = \langle \mathrm{Ind}_{K_\chi}^{H}(\theta_\tau), \mathbb{1}_H \rangle_H.$$

Proof. Let $\mathcal{R}_{K_\chi}$ be the set of representatives of cosets of K_χ in H. The set $\mathcal{R}_{K_\chi}$ may be viewed also as the set of representatives of $K_\chi \ltimes V$ in G. Then for $x \in \mathcal{R}_{K_\chi}$ and $h \in H$, the element h^x is contained in $K_\chi \ltimes V$ precisely when it is contained in K_χ. Since H is closed, by [35, Theorem 6.10], Frobenius reciprocity yields

$$\begin{aligned}
&\langle \mathrm{Ind}_{K_\chi \ltimes V}^{G}(\theta_{\tau,\chi}), \mathrm{Ind}_{H}^{G}(\mathbb{1}_H) \rangle_G \\
&\qquad = \langle \mathrm{Res}_{H}^{G} \mathrm{Ind}_{K_\chi \ltimes V}^{G}(\theta_{\tau,\chi}), \mathbb{1}_H \rangle_H \\
&\qquad = \int_H \sum_{\substack{x \in \mathcal{R}_{K_\chi} \\ (h)^x \in K_\chi \ltimes V}} \theta_{\tau,\chi}(h^x)\mathrm{d}\mu(h) \\
&\qquad = \int_H \sum_{\substack{x \in \mathcal{R}_{K_\chi} \\ h^x \in K_\chi}} \theta_\tau(h^x) \cdot \chi(1)\mathrm{d}\mu(h) \\
&\qquad = \int_H \mathrm{Ind}_{K_\chi}^{H}(\theta_\tau)(h) \cdot \mathbb{1}_H(h)\, \mathrm{d}\mu(h)
\end{aligned}$$

$$= \langle \mathrm{Ind}_{K_\chi}^H(\theta_\tau), \mathbb{1}_H \rangle_H.$$

□

2.3 Zeta functions of subgroups and semi-direct products

2.3.1 A condition for subgroups to be thetyspectral

Let G be a uniformly potent pro-p group. Assume that we understand its representation zeta function – what can we deduce about the representation zeta function of an open potent subgroup H? Are there circumstances where G essentially dictates the representation zeta function of H in terms of its own zeta function? One such case is well-known. Given a uniformly potent pro-p, the subgroup G^{p^k} generated by the $(p^k)^{\mathrm{th}}$ powers of G (which is indeed equal to the set of these powers) gives rise to the same representation zeta function as the full group G does, up to a constant factor, see [42, Proposition 6]. We call two groups with this property *thetyspectral* and two groups with identical representation zeta functions *isospectral*.

Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0 and residue characteristic p, with uniformiser π. The valuation ideal is $\mathfrak{p} = \pi\,\mathcal{O}$. Denote by q the cardinality of the residue field $\kappa = \mathcal{O}/\mathfrak{p}$, and let K be the field of fractions of $\mathcal{O}$, which constitutes a finite extension of $\mathbb{Q}_p$.

Let $\mathfrak{g}$ be a potent and saturable $\mathcal{O}$-Lie lattice and let $\mathfrak{h}$ be a open potent and saturable $\mathcal{O}$-Lie sublattice of $\mathfrak{g}$. Assume that $G = \exp(\mathfrak{g})$ is a uniformly potent pro-p group (otherwise choose a permissible m and work instead with $\pi^m\mathfrak{g}$) Using the Lazard correspondence, $H = \exp(\mathfrak{h})$ is an open uniformly potent subgroup of G. The Lie lattices $\mathfrak{h}$ and $\mathfrak{g}$ considered as $\mathbb{Z}_p$-Lie lattices are the Lie lattices associated to H and G. Since the sets underlying the Lie lattices are equal to the sets underlying the groups and there is a correspondence between subgroups and Lie sublattices of saturable pro-p groups, see [73, Theorem 1.4], the index of $\mathfrak{h}$ in $\mathfrak{g}$ is the index of H in G. This index is a finite number, hence $\mathfrak{h}$ has the same $\mathcal{O}$-dimension as $\mathfrak{g}$. Thus, since $\mathfrak{g}$ is torsion-free, there exists a homomorphism $\xi\colon \mathfrak{g} \to \mathfrak{h}$ of $\mathcal{O}$-modules, that does (in general) not preserve the Lie bracket. This homomorphism naturally extends to a K-isomorphism between $\mathfrak{g}_K = \mathfrak{g} \otimes_{\mathcal{O}} K$ and $\mathfrak{h}_K = \mathfrak{h} \otimes_{\mathcal{O}} K$. Of course, $\mathfrak{g}_K \cong \mathfrak{h}_K$, and, for the same reason, $\mathfrak{g}^* \cong \mathfrak{h}^*$, where $\mathfrak{g}^* = \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, K)$ and $\mathfrak{h}^* = \mathrm{Hom}_{\mathcal{O}}(\mathfrak{h}, K)$.

Let $\mathfrak{X}$ be a basis for $\mathfrak{g}$. If ξ is indeed an isomorphism of Lie lattices, we have

$$\mathrm{Pfaff}(([x_i, x_j]w)_{(x_i,x_j)\in\mathfrak{X}\times\mathfrak{X}}) = \mathrm{Pfaff}(([(x_i)\xi, (x_j)\xi]\xi^{-1}w)_{(x_i,x_j)\in\mathfrak{X}\times\mathfrak{X}})$$

and thus, as seen directly from the integral formulation of Corollary 2.2.2, G and H are isospectral. However, a weaker condition is sufficient to establish thetyspectrality.

Theorem 2.3.1. *Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0 and with residue characteristic p. Denote the maximal ideal of $\mathcal{O}$ by $\mathfrak{p}$ and let K be the field of* *fractions of $\mathcal{O}$. Let $\mathfrak{g}$ be a potent and saturable $\mathcal{O}$-Lie lattice such that $\mathfrak{g}$ as $\mathbb{Z}_p$-Lie lattice is FAb. Let $\mathfrak{h}$ be a open potent and saturable $\mathcal{O}$-Lie sublattice of $\mathfrak{g}$ and write $\mathfrak{g}_K = \mathfrak{g} \otimes_{\mathcal{O}} K$ and $\mathfrak{h}_K = \mathfrak{h} \otimes_{\mathcal{O}} K$. Assume that $G = \exp(\mathfrak{g})$ is a uniformly potent pro-p group and consider $H = \exp(\mathfrak{h})$, an open uniformely potent subgroup of G. Let $\beta\colon \mathfrak{g}_K \wedge \mathfrak{g}_K \to \mathfrak{g}_K$ be the linear map induced by the Lie bracket of $\mathfrak{g}$, and let $\xi\colon \mathfrak{g} \to \mathfrak{h}$ be an $\mathcal{O}$-linear isomorphism, which extends to $\xi\colon \mathfrak{g}_K \to \mathfrak{h}_K$*

If there exists a K-linear map $\psi\colon \mathfrak{g}_K \to \mathfrak{h}_K$ such that

$$(\xi \wedge \xi)\beta = \beta\psi,$$

then G and H are thetyspectral with factor $|\det(\psi\xi^{-1})|_{\mathfrak{p}}^{-1}$.

Proof. We use the integral formalism introduced earlier. Put $d = \dim_{\mathcal{O}}(\mathfrak{g})$, let $\mathfrak{X}$ be a $\mathcal{O}$-basis for $\mathfrak{g}$ and let φ be the isomorphism from K^d to $\mathfrak{g}$ mapping the standard basis B of K^d to $\mathfrak{X}$. Of course, $(\mathfrak{X})\xi = (B)\varphi\xi$ is an $\mathcal{O}$-basis for $\mathfrak{h}$, and we may write

$$\begin{aligned}
\zeta_H(s) &= \int_{(K^d)^*} \|\mathrm{Pfaff}\left(([(b)\varphi\xi, (b')\varphi\xi](\varphi\xi)^{-1}w)_{(b,b')\in B\times B}\right)\|_{\mathfrak{p}}^{-2-s}\mathrm{d}\mu(w) \\
&= \int_{(K^d)^*} \|\mathrm{Pfaff}\left(((b)\varphi \wedge (b')\varphi)\beta\psi\xi^{-1}\varphi^{-1}w)_{(b,b')\in B\times B}\right)\|_{\mathfrak{p}}^{-2-s}\mathrm{d}\mu(w) \\
&= |\det((\psi\xi^{-1})^*)|_{\mathfrak{p}}^{-1} \int_{((\psi\xi^{-1})^{\varphi})^*(K^d)^*} \|\mathrm{Pfaff}\left([(b)\varphi, (b')\varphi]\varphi^{-1}w\right)_{(b,b')\in B^2})\|_{\mathfrak{p}}^{-2-s}\mathrm{d}\mu(w),
\end{aligned}$$

where the last step is a simple change of variables induced by the K-linear map $(\varphi(\psi\xi^{-1})\varphi^{-1})^*$, and μ is the normalized Haar measure such that $\mu((\mathcal{O}^d)^*) = 1$. One easily verifies that $((\varphi(\psi\xi^{-1})\varphi^{-1})^*(K^d)^* = (K^d)^*$ and

$$\det((\varphi(\psi\xi^{-1})\varphi^{-1})^*) = \det((\psi\xi^{-1})^*) = \det(\psi\xi^{-1}).$$

Using φ, we have an isomorphism between $(K^d)^*$ and $\mathfrak{g}^*$ and so, comparing the last integral with the one describing $\zeta_G(s)$ from Corollary 2.2.2, we have then proven the theorem. □

We mainly apply the above theorem to the m^{th} principal congruence subgroup of $\mathrm{SL}_2(\mathcal{O})$, where m is permissible for $\mathfrak{sl}_2(\mathcal{O})$. Recall that the m^{th} principal congruence subgroup of $\mathrm{SL}_2(\mathcal{O})$ is given by

$$\ker(\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/\mathfrak{p}^m\mathcal{O})), \tag{2.3.1}$$

and, more generally, the m^{th} principal congruence subgroup of $\mathrm{SL}_n(\mathcal{O})$ is given by

$$\ker(\mathrm{SL}_n(\mathcal{O}) \to \mathrm{SL}_n(\mathcal{O}/\mathfrak{p}^m\mathcal{O})). \tag{2.3.2}$$

For $n = 2$, the groups are special since their Lie lattices have dimension three, whence the linear map β induced by the Lie bracket is a linear map between spaces of the same dimension. This is exploited in the following result.

Corollary 2.3.2. *Let $G^m = \mathrm{SL}_2^m(\mathcal{O})$ be the m^{th} principal congruence subgroup where $m \in \mathbb{N}_*$ is permissible for $\mathfrak{sl}_2(\mathcal{O})$. Let H be an open potent subgroup of G^m. Then*

$$\zeta_H(s) = |G^m : H| \cdot \zeta_{G^m}(s).$$

Proof. Using the notation of Theorem 2.3.1, the $\mathcal{O}$-Lie lattice corresponding to G^m is $\mathfrak{g}^m = \pi^m \cdot \mathfrak{sl}_2(\mathcal{O})$. Let $\mathfrak{g}_K^m = \mathfrak{g}^m \otimes_{\mathcal{O}} K$ be the associated K-Lie algebra. Since the linear map $\beta\colon \mathfrak{g}_K^m \wedge \mathfrak{g}_K^m \to \mathfrak{g}_K^m$ induced by the Lie bracket is non degenerate, the image of $\mathfrak{g}^m \wedge \mathfrak{g}^m$ is of finite index in $\mathfrak{g}^m$. Crucially, since $\dim_{\mathcal{O}}(\mathfrak{g}^m) = 3$, the dimension of the exterior square $\mathfrak{g}^m \wedge \mathfrak{g}^m$ is also 3. Thus as a K-linear map, β is invertible. Let $\mathfrak{h}$ be a potent $\mathcal{O}$-Lie sublattice of $\mathfrak{g}^m$, $\mathfrak{h}_K = \mathfrak{h} \otimes_{\mathcal{O}} K$ the K-Lie algebra, $H = \exp(\mathfrak{h})$ an open potent subgroup of G^m, $\xi : \mathfrak{g}^m \to \mathfrak{h}$ an isomorphism of $\mathcal{O}$-modules, and $\xi\colon \mathfrak{g}_K^m \to \mathfrak{h}_K$ its extension. Clearly the K-linear map $\psi\colon \mathfrak{g}_K^m \to \mathfrak{h}_K$ defined by

$$\psi = \beta^{-1}(\xi \wedge \xi)\beta$$

meets the conditions of Theorem 2.3.1. Its determinant satisfies

$$\det(\psi) = \det(\beta^{-1})\det(\xi \wedge \xi)\det(\beta) = \det(\xi \wedge \xi) = \det(\xi)^{\dim_{\mathcal{O}}(\mathfrak{g})-1} = \det(\xi)^2,$$

thus the factor $\det(\psi\xi^{-1})$ is equal to $\det(\xi)$. Using Corollary 0.3.3, we have $|\det(\xi)|_{\mathfrak{p}}^{-1} = |\mathfrak{g}^m : \mathfrak{h}| = |G^m : H|$. □

Theorem 2.3.1 allows us to recover some known identities. Given an $\mathcal{O}$-Lie lattice $\mathfrak{g}$ such that $\mathfrak{g} \otimes_{\mathcal{O}} K$ is perfect, and defining $\mathfrak{g}^m$ to be the $\mathcal{O}$-Lie lattice $\pi^m \cdot \mathfrak{g}$, we choose $m \in \mathbb{N}$ permissible, i.e. big enough so that $\mathfrak{g}^m$ is potent and saturable, cf. [11, Proposition 2.3]. Let d be the $\mathcal{O}$-dimension of the Lie lattice $\mathfrak{g}$ and let $G^m = \exp(\mathfrak{g}^m)$ be the uniformly potent pro-p group associated with $\mathfrak{g}^m$. Consider an integer $k \in \mathbb{N}$ and the uniformly potent subgroup G^{m+k} of G^m, i.e. by construction given by $G^{m+k} = \exp(\pi^{m+k} \cdot \mathfrak{g})$. The representation zeta function of the subgroup of G^{m+k} is equal to $q^{dk} \cdot \zeta_{G^m}(s)$, where q is the cardinality of the residue field $\kappa = \mathcal{O} / \mathfrak{p}$, and so in particular, a power of p. A similar behaviour was first described in [66] and then in [11, Proposition 3.1], and a variation of this result [42, Proposition 6] was used to prove the main result in [42]. We can derive the statement as follows. We consider the map ξ that is given by scalar multiplication by π^k. Thus

$$(v \wedge w)(\xi \wedge \xi)\beta = [\pi^k v, \pi^k w] = \pi^{2k}[v, w] = (v, w)\beta\xi^2, \quad \text{for } v, w \in \mathfrak{g}^m$$

i.e. we may choose ξ^2 as our ψ. Clearly $\det(\psi\xi^{-1}) = \det(\xi) = \pi^{dk}$ and $|\det(\xi)|_{\mathfrak{p}}^{-1} = q^{dk}$ and we recover the result mentioned above.

Furthermore, Theorem 2.3.1 can be used to compute the zeta functions of several subgroups of the group $G^m = \mathrm{SL}_3^m(\mathcal{O})$, the m^{th} principal congruence subgroup of $\mathrm{SL}_3(\mathcal{O})$ with m permissible for $\mathfrak{sl}_3(\mathcal{O})$, cf. [11, Proposition 2.3], where $\mathfrak{g} = \mathfrak{sl}_3(\mathcal{O})$ denotes the $\mathcal{O}$-Lie lattice of 3×3-matrices with trace zero. The ring $\mathcal{O}$ is the ring of integers of a finite extension K of $\mathbb{Q}_p$, with maximal ideal $\mathfrak{p} \trianglelefteq \mathcal{O}$ generated by an uniformiser π. Put $\mathfrak{g}^m = \pi^m \mathfrak{g}$, so then $G^m = \exp(\mathfrak{g}^m)$. For $i, j \in \{1, 2, 3\}$, let $E^m_{i,j}$ be the 3×3-matrix with entry π^m at position (i, j) and zero otherwise. We choose $\mathfrak{X} = \{h_{12}, h_{23}, e_{12}, e_{13}, e_{23}, f_{21}, f_{31}, f_{32}\}$ as an $\mathcal{O}$-basis for $\mathfrak{g}^m$, where

$$\begin{aligned}
h_{12} &= E^m_{1,1} - E^m_{2,2}, & e_{12} &= E^m_{1,2}, & f_{21} &= E^m_{2,1},\\
h_{23} &= E^m_{2,2} - E^m_{3,3}, & e_{13} &= E^m_{1,3}, & f_{31} &= E^m_{3,1},\\
& & e_{23} &= E^m_{2,3}, & f_{32} &= E^m_{3,2}.
\end{aligned}$$

In this case $\mathfrak{g}^m$, regarded as a $\mathbb{Z}_p$-Lie lattice, is the lattice associated to G^m. Let $\mathfrak{h}$ be a potent $\mathcal{O}$-sublattice of $\mathfrak{g}^m$. Put $H = \exp(\mathfrak{h})$ and note that it is an open uniformly potent subgroup of G^m. Consider the $\mathcal{O}$-linear isomorphism $\xi\colon \mathfrak{g}^m \to \mathfrak{h}$. We restrict to sublattices $\mathfrak{h}$ that arise from maps ξ which may be represented by a diagonal matrix of the form $\operatorname{diag}(\pi^{k_1}, \ldots, \pi^{k_8})$ with respect to the $\mathcal{O}$-basis $\mathfrak{X}$ of $\mathfrak{g}^m$, for some $k_1, \ldots, k_8 \in \mathbb{N}$. It is easy to see that $\xi \wedge \xi$ may be represented by $\operatorname{diag}(\pi^{k_1+k_2}, \pi^{k_1+k_3}, \ldots, \pi^{k_7+k_8})$ using the standard ordering for the basis induced by $\mathfrak{X}$ on the exterior square. Comparing the matrix representation of the bracket β and the disturbed $(\xi \wedge \xi)\beta$, we find

$$[(e_{12})\xi, (f_{21})\xi] = \pi^{k_3+k_6+m} h_{12} = (\pi^m h_{12})\psi = ([e_{12}, f_{21}])\psi,$$

and similarly

$$\begin{aligned}
(\pi^m h_{23})\psi &= \pi^{k_5+k_8+m} h_{23}, & (\pi^m e_{12})\psi &= \pi^{k_1+k_3+m} e_{12}, & (\pi^m e_{13})\psi &= \pi^{k_1+k_4+m} e_{13},\\
(\pi^m e_{23})\psi &= \pi^{k_1+k_5+m} e_{23}, & (\pi^m f_{21})\psi &= \pi^{k_1+k_6+m} f_{21}, & (\pi^m f_{31})\psi &= \pi^{k_1+k_7+m} f_{31},\\
& & (\pi^m f_{32})\psi &= \pi^{k_1+k_8+m} f_{32}
\end{aligned}$$

as necessary conditions for ψ, using the first occurrences of each variable in the commutator matrix. But every variable appears twice or thrice, thus we find the conditions

$$\begin{aligned}
k_3 + k_6 &= k_4 + k_7 = k_5 + k_8, & k_1 + k_3 &= k_2 + k_3 = k_4 + k_8,\\
k_1 + k_4 &= k_2 + k_4 = k_3 + k_5, & k_1 + k_5 &= k_2 + k_5 = k_4 + k_6,\\
k_1 + k_6 &= k_2 + k_6 = k_5 + k_7, & k_1 + k_7 &= k_2 + k_7 = k_6 + k_8,\\
k_1 + k_8 &= k_2 + k_8 = k_3 + k_7.
\end{aligned}$$

A minimal equivalent set of equations is given by

$$\begin{aligned}
k_1 = k_2 &= k_6 - k_7 + k_8, & k_3 &= k_6 - 2k_7 + 2k_8,\\
k_4 &= 2k_6 - 3k_7 + 2k_8, & k_5 &= 2k_6 - 2k_7 + k_8.
\end{aligned} \tag{2.3.3}$$

Considering the above, we have proven the following statement.

Proposition 2.3.3. *Let p be an odd prime and let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0 with residue field cardinality q, a power of p, valuation ideal $\mathfrak{p}$, and choose an uniformiser π. Put $G^m = \mathrm{SL}_3^m(\mathcal{O})$ for permissible $m \in \mathbb{N}$, as above, and let H be the subgroup of G^m corresponding to a potent and saturable $\mathcal{O}$-Lie sublattice generated by*

$$\pi^{k_1} h_{12}, \pi^{k_2} h_{23}, \pi^{k_3} e_{12}, \pi^{k_4} e_{13}, \pi^{k_5} e_{23}, \pi^{k_6} f_{21}, \pi^{k_7} f_{31}, \pi^{k_8} f_{32}$$

for some positive integers $k_i \in \mathbb{N}$ for $i \in \{1, \dots, 8\}$, necessarily adhering to the conditions (2.3.3) and m permissible for $\mathfrak{sl}_3(\mathcal{O})$. Then H is thetyspectral in G^m with factor $|\prod_{i=1}^{8} \pi^{k_i}|_{\mathfrak{p}}^{-1} = q^{\sum_{i=1}^{8} k_i} = |G^m : H|$.

Note that all subgroups arising in this way – i.e. that are subject to the conditions above – are of index $q^{\dim_{\mathcal{O}} \mathrm{SL}_3(\mathcal{O}) n}$ in $\mathrm{SL}_3^1(\mathcal{O})$, for some positive integer $n \in \mathbb{N}_*$. Of course, the proposition above only deals with a very special case. It is an interesting question which open uniformly potent subgroups of $\mathrm{SL}_3^1(\mathcal{O})$ are thetyspectral and which factors may appear aside from 1 and the index of the subgroup. Furthermore, it seems likely that there are many non-thetyspectral subgroups of $\mathrm{SL}_3^1(\mathcal{O})$, but no concrete examples are known at a present state.

2.3.2 Semi-direct products and potency

Lemma 2.3.4. *Let H be a uniformly potent pro-p group, and let $\sigma \colon H \to \mathrm{GL}_n^m(\mathcal{O})$ be a finite-dimensional $\mathcal{O}$-representation of H with image in the m^{th} congruence subgroup, where m is permissible for $\mathfrak{gl}_n(\mathcal{O})$, which is the $\mathcal{O}$-Lie lattice of $n \times n$-matrices. Then the semi-direct product $G = H \ltimes_\sigma \pi^m \mathcal{O}^n$ is a uniformly potent pro-p group.*

Proof. The semi-direct product G is clearly torsion-free and finitely generated. We have to prove that $\gamma_{p-1}(G) \subseteq G^p$ for odd primes p and that $[G, G] \subseteq G^4$ in case $p = 2$. In the odd case, the $(p-1)^{\mathrm{st}}$ term of the lower central series is generated by $\gamma_{p-1}(H)$ and $[\pi^m \mathcal{O}^n, H, \dots, H]$, since $[\pi^m \mathcal{O}^n, H] \subseteq \pi^m \mathcal{O}^n$, an abelian group; in case $p = 2$ the same holds for $[H, H]$ and $[\pi^m \mathcal{O}^n, H]$. The inclusion $\gamma_{p-1}(H) \subseteq H^p \subseteq (H \ltimes \pi^m \mathcal{O}^n)^p = G^p$ (respectively $[H, H] \subseteq G^4$) follows since H is potent. Let $h \in H$ and $v \in \pi^m \mathcal{O}^n$. Then

$$[v, h] = v^{-1} v^h = v^{(h)\sigma - \mathrm{Id}_{n \times n}}.$$

But our assumption that $(h)\sigma$ is contained in $\mathrm{GL}_n^m(\mathcal{O})$ implies that $(h)(\sigma - \mathrm{Id}_{n \times n}) = \pi^m g$ for some $g \in \mathrm{Mat}_n(\mathcal{O})$, hence $[v, h] = \pi^m v^g \in \pi^{2m} \mathcal{O}^n$. The subgroup of p^{th} powers is a normal subgroup, hence $[\pi^m \mathcal{O}^n, H, \dots, H] \subseteq \pi^{2m} \mathcal{O}^n \subseteq (H \ltimes \pi^m \mathcal{O}^n)^p \subseteq G^p$. $\square$

Lemma 2.3.5. *Let G be uniformly potent pro-p group. Let H and $K \leq G$ be two closed potent subgroups. Then $H \cap K$ is potent.*

Proof. Write $\mathfrak{g}$ for the $\mathbb{Z}_p$-Lie lattice $\log(G)$. Since H and K are potent, they correspond to potent $\mathbb{Z}_p$-Lie sublattices $\mathfrak{h}$ and $\mathfrak{k}$. It is enough to prove that the intersection $\mathfrak{h} \cap \mathfrak{k}$ is potent;

if it is, it corresponds to a potent subgroup of G, which is necessarily equal to $H \cap K$, since the underlying sets of the Lie lattices and groups are the same. But this follows from

$$\gamma_{p-1}(\mathfrak{h} \cap \mathfrak{k}) \subseteq \gamma_{p-1}(\mathfrak{h}) \cap \gamma_{p-1}(\mathfrak{k}) \subseteq p\mathfrak{h} \cap p\mathfrak{k} = p(\mathfrak{h} \cap \mathfrak{k}),$$

in case of an odd prime, and similarly in the case $p = 2$. $\square$

Lemma 2.3.6. *Let G be a finitely generated torsion-free pro-p group, and let $\sigma\colon G \to \mathrm{GL}_n^m(\mathcal{O})$ be a faithful finite-dimensional $\mathcal{O}$-representation of G such that $G^\sigma \cap \mathrm{GL}_n^{m+1}(\mathcal{O}) \leq (G^\sigma)^{\mathbf{p}}$, for a permissible m. Let χ be a (continuous) irreducible representation of $\pi^m \mathcal{O}^n$. Then the stabiliser $\mathrm{Stab}_G(\chi)$ with respect to the action induced by σ is an open potent subgroup of the uniformly potent group G.*

Proof. Without loss of generality, we identify G and its image under σ. Looking at the inclusion

$$[G, G] = [G \cap \mathrm{GL}_n^m(\mathcal{O}), G \cap \mathrm{GL}_n^m(\mathcal{O})] \leq G \cap \mathrm{GL}_n^{m+1}(\mathcal{O}) \leq G^{\mathbf{p}},$$

we see that G is potent. Since χ is continuous, it factors over some finite-index subgroup of $\pi^m \mathcal{O}^n$; hence its kernel contains a subgroup of the form $\pi^{k+m} \mathcal{O}^n$ for some $k \in \mathbb{N}$. If $g \in G \cap \mathrm{GL}_n^{k+m}(\mathcal{O})$, then g stabilises χ, since χ cannot detect the action of g on its argument. Hence $G_k = G \cap \mathrm{GL}_n^{k+m}(\mathcal{O}) \leq \mathrm{Stab}_G(\chi)$.

If $k = 0$, the representation is trivial and the statement follows immediately. If $k = 1$, since G is contained in $\mathrm{GL}_n^m(\mathcal{O})$, the full (potent) group stabilises χ. Thus, assume $k > 1$. Since χ factors over $\pi^{k+m} \mathcal{O}^n$, it induces a representation of $\mathcal{O}^n / \pi^{k+m} \mathcal{O}^n$; which can be described by an element $x \in (\mathcal{O}^n / \pi^{k+m} \mathcal{O}^n)^*$ of the dual. The stabiliser of χ fits into the exact sequence

$$\{1\} \to G_k \to \mathrm{Stab}_G(\chi) \to \mathrm{Stab}_{G/G_k}(x) \to \{1\}.$$

Every orbit of $\pi^m \mathcal{O}^n$ under the action of the group $\mathrm{GL}_n(\mathcal{O})$ contains an element of the form $(\pi^{j+m}, 0, \ldots, 0)$ for some $j \in \mathbb{N}$. Thus, under conjugation by an appropriate element of $\mathrm{GL}_n(\mathcal{O})$, we may assume that x is of this form. It is easy to see that its stabiliser is the intersection of G/G_k with the general affine group $\mathrm{GA}_n(\mathcal{O} / \pi^{k+m} \mathcal{O}) \cong \mathrm{GL}_{n-1}(\mathcal{O} / \pi^{k+m} \mathcal{O}) \ltimes \pi^m(\mathcal{O} / \pi^{k+m} \mathcal{O})^{n-1}$. Lifting to G, we find that every element of G is (non-uniquely) a product of an element of G_k and an element of $G \cap \mathrm{GA}_n(\mathcal{O})$. Since $\mathrm{GA}_n(\mathcal{O}) \cong \mathrm{GL}_{n-1}(\mathcal{O}) \ltimes \pi^m \mathcal{O}^{n-1}$, both G_k and $G \cap \mathrm{GA}_n(\mathcal{O})$ are potent groups by Lemma 2.3.4 and Lemma 2.3.5.

As a consequence of our assumption, the intersection $G_k = G \cap \mathrm{GL}_n^{k+m}(\mathcal{O})$ is contained in the subgroup of $\mathbf{p}^{\mathrm{th}}$ powers of G. Moreover, using that $G = G_k \cdot G \cap \mathrm{GA}_n(\mathcal{O})$, we have that $\gamma_{p-1}(G) = [G, \ldots, G] \subseteq G_k \cdot \gamma_{p-1}(G \cap \mathrm{GA}_n(\mathcal{O})) \subseteq G^{\mathbf{p}}$. Furthermore,

$$[G, G] \subseteq [G \cap \mathrm{GA}_n(\mathcal{O}), G \cap \mathrm{GA}_n(\mathcal{O})] \cdot [G \cap \mathrm{GA}_n(\mathcal{O}), G_k] \cdot [G_k, G_k].$$

Thus, all together, we find $[G \cap \mathrm{GA}_n(\mathcal{O}), G_k] \leq G^{\mathbf{p}}$. Since $G \cap \mathrm{GA}_n(\mathcal{O})$ is potent, we deduce that $\mathrm{Stab}_G(\chi)$ is potent. $\square$

The assumptions for the last lemma seem rather technical. It would be interesting to describe all actions such that the point stabilisers are potent. However, Lemma 2.3.6 is strong enough to establish that the point stabilisers of irreducible representations under the action induced by

(1) the natural action of $\mathrm{SL}_n^m(\mathcal{O})$ or $\mathrm{GL}_n^m(\mathcal{O})$, for $n, m \in \mathbb{N}$ with permissible m,
(2) the symmetric square of the natural action of $\mathrm{SL}_2^m(\mathcal{O})$, and
(3) the direct sums of representations

fulfil the assumptions.

We are now able to prove our main result.

Theorem 2.3.7. *Let $\mathcal{O}$ be a compact discrete valuation ring of characteristic 0 and residue characteristic p, and let $V = \mathcal{O}^n$ be an $\mathcal{O}$-lattice of rank n. Let H be a potent subgroup of $\mathrm{SL}_2^1(\mathcal{O})$, and let $\sigma\colon H \to \mathrm{GL}_n^m(\mathcal{O})$ be a faithful finite-dimensional $\mathcal{O}$-representation of H on V, such that $H^\sigma \cap \mathrm{GL}_n^{m+1}(\mathcal{O}) \leq (H^\sigma)^{\mathbf{p}}$, for permissible m. Assume furthermore that the semi-direct product $G = H \ltimes_\sigma V$ is FAb. Then*

$$\zeta_G(s) = \zeta_H(s) \cdot \zeta_H^G(s-1),$$

where $\zeta_H^G(s)$ is the zeta function associated to the representation $\mathrm{Ind}_H^G(\mathbb{1})$.

Proof. Let $\mathcal{X}$ be a set of representatives of the H-orbits in $\mathrm{Irr}(V)$. By Proposition 2.2.3, all irreducible representations of G are parametrized by the representatives $\chi \in \mathcal{X}$ and the irreducible representations of each stabilizer K_χ for $\chi \in \mathcal{X}$, and are of the form

$$\mathrm{Ind}_{K_\chi \ltimes V}^G(\mathrm{Inf}_{K_\chi}^{K_\chi \ltimes V}(\tau) \otimes \mathrm{Ext}_V^{K_\chi \ltimes V}(\chi)).$$

The dimension of such a representation is given by the product $|H : K_\chi| \cdot \dim(\tau)$. Thus

$$r_n(G) = \sum_{\substack{a,b\in\mathbb{N}_* \\ ab=n}} \sum_{\substack{\chi\in\mathcal{X} \\ |H:K_\chi|=a}} r_b(K_\chi).$$

By Lemma 2.3.6 the group K_χ is a uniformly potent pro-p group, hence Corollary 2.3.2 implies $r_b(K_\chi) = |H : K_\chi| \cdot r_b(H)$.

Now to every $\chi \in \mathcal{X}$ we may associate the $|H : K_\chi|$-dimensional representation $\mathrm{Ind}_{K_\chi \ltimes V}^G(\mathrm{Ext}_V^{K_\chi \ltimes V}(\chi))$. In view of Proposition 2.2.4, these are precisely the $|H : K_\chi|$-dimensional irreducible constituents of $\mathrm{Ind}_H^G(\mathbb{1}_H)$, hence

$$r_n(G) = \sum_{\substack{a,b\in\mathbb{N}_* \\ ab=n}} \sum_{\substack{\chi\in\mathcal{X} \\ |H:K_\chi|=a}} a \cdot r_b(H) = \sum_{\substack{a,b\in\mathbb{N}_* \\ ab=n}} r_a(G, H) \cdot a \cdot r_b(H).$$

We see that the numbers $r_n(G)$ result from the Dirichlet convolution of the arithmetic sequences $a \cdot r_a(G,H)$, with $a \in \mathbb{N}_*$ and $r_b(H)$, with $b \in \mathbb{N}_*$. The factor a corresponds to a shift in the Dirichlet generating function of the sequence $r_a(G,H)$, i.e. $\sum_{a\in\mathbb{N}_*} r_a(G,H)\cdot a\cdot a^{-s} = \zeta_H^G(s-1)$. Since the generating function of a Dirichlet convolution is the product of the corresponding generating functions, this concludes the proof. □

2.4 Examples

In this section we prove Theorem 2.1.2. Each example proves a bullet point of the theorem.

With Theorem 2.1.1 established, we aim to compute the representation zeta functions of semi-direct products with $\mathrm{SL}_2^m(\mathcal{O})$, for permissible m, contributing to the small but growing library of examples. However, mirroring the difficulties of computing the zeta-functions (or even the abscissae of convergence) of p-adic analytic groups of higher dimension, the relative zeta-functions associated to high-dimensional modules of $\mathrm{SL}_2^m(\mathcal{O})$ correspond to p-adic integrals that are cumbersome to compute. To reduce the complexity somewhat, we restrict ourselves to odd primes, except in the last example.

We begin with some generalities. Using the terminology of Theorem 2.2.1, we find that the Lie lattice $\mathfrak{v}$ spanned by $\mathfrak{X}$, i.e. such that $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{h}$, corresponds to the abelian subgroup V. Thus, the commutator matrix can be written as a block matrix of the form

$$\mathrm{Com}(\mathfrak{g}, \mathfrak{X} \cup \mathfrak{Y}) = \begin{pmatrix} 0 & A \\ -A^\intercal & \mathrm{Com}(\mathfrak{h}, \mathfrak{Y}) \end{pmatrix},$$

where A is the matrix $A = ([x,y])_{(x,y)\in\mathfrak{X}\times\mathfrak{Y}}$. This simplifies the integral described in Corollary 2.2.2. Recall that we integrate over the subspace $W = K\mathfrak{X}^*$ spanned by the part $\mathfrak{X}^*$ of a dual basis of $\mathfrak{g}$, associated to the basis $\mathfrak{X}$ of $\mathfrak{v}$. Moreover, since $\mathfrak{h}$ is a sublattice, the entries of $\mathrm{Com}(\mathfrak{h}, \mathfrak{Y})$ are elements of in $\mathfrak{h}$. Thus, given $w \in W$, after entry-wise application we find a matrix of the form

$$\begin{pmatrix} 0 & Aw \\ -A^\intercal w & 0 \end{pmatrix}.$$

We have to compute the Pfaffian minors of this matrix. Every principal submatrix is still of the form $\left(\begin{smallmatrix} 0 & B \\ -B^\intercal & 0 \end{smallmatrix}\right)$, for some submatrix B of Aw obtained by deleting rows and columns. The determinant (and hence the Pfaffian determinant) is 0 if B is not a square matrix; otherwise the determinant of such a matrix is equal to $\det(B)^2$, and its Pfaffian determinant equals $\det(B)$. Thus, the set $\mathrm{Pfaff}(\mathrm{Com}(\mathfrak{g}, \mathfrak{X} \cup \mathfrak{Y})w)$ is equal to the set $\mathrm{Min}(Aw)$ of all minors of Aw. All in all, we find

$$\zeta_H^G(s) = \int_{\mathfrak{v}^*} \|\mathrm{Min}(Aw)\|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(w).$$

Example 2.4.1. For our first example, we aim to compute the representation zeta function of the group $G_2^m = \mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^2$ for a suitable $m \in \mathbb{N}_*$. The representation zeta function of $\mathrm{SL}_2(\mathcal{O})$ was computed by Jaikin-Zapirain in [66] and the representation zeta functions of its principal congruence subgroups $\mathrm{SL}_2^m(\mathcal{O})$, for permissible $m \in \mathbb{N}$, were computed by Avni, Klopsch, Onn, and Voll in [10]. Hence, using Theorem 2.1.1, it remains to compute the relative zeta function $\zeta_{\mathrm{SL}_2^m(\mathcal{O})}^{G_2^m}(s)$. Taking a step back, we even compute $\zeta_{H_n^m}^{G_n^m}(s)$ for $G_n^m = \mathrm{SL}_n^m(\mathcal{O}) \ltimes \mathcal{O}^n$ and $H_n^m = \mathrm{SL}_n^m(\mathcal{O})$ for all $n \geqslant 2$ and permissible m simultaneously.

Consider the special affine group $\tilde{G}_n = \mathrm{SL}_n(\mathcal{O}) \ltimes \mathcal{O}^n$ and the embedding into $\mathrm{SL}_{n+1}(\mathcal{O})$. We define the m^{th} congruence subgroup of $\tilde{G}_n$ as the intersection of $\mathrm{SL}_n(\mathcal{O}) \ltimes \mathcal{O}^n$ with the m^{th} congruence subgroup $\mathrm{SL}_{n+1}^m(\mathcal{O})$ of $\mathrm{SL}_{n+1}(\mathcal{O})$ defined in (2.3.2), i.e. the m^{th} congruence subgroup of $\tilde{G}_n$ equals $\tilde{G}_n^m = \mathrm{SL}_n^m(\mathcal{O}) \ltimes \pi^m \mathcal{O}^n = H_n^m \ltimes \pi^m \mathcal{O}^n$. If m is permissible for $\mathfrak{sl}_n(\mathcal{O})$, i.e. the group $\mathrm{SL}_n^m(\mathcal{O}) = \exp(\pi^m \mathfrak{sl}_n(\mathcal{O}))$ is uniformly potent, then, by Lemma 2.3.4, also the group $\tilde{G}_n^m$ is uniformly potent. The group $\tilde{G}_n^m$ is isomorphic to G_n^m, thus both groups have the same representation zeta function.

For $\mathfrak{h}_n^m = \mathfrak{sl}_n(\pi^m \mathcal{O})$, let $\mathfrak{g}_n^m = \mathfrak{h}_n^m \oplus \mathfrak{v}$ be the Lie sublattice of $\mathfrak{sl}_{n+1}(\pi^m \mathcal{O})$ consisting of all matrices of the form

$$\begin{pmatrix} \mathfrak{sl}_n(\pi^m \mathcal{O}) & \pi^m \mathcal{O}^n \\ 0 & 0 \end{pmatrix}.$$

Writing $E_{i,j}^m$ for the matrix with entry π^m at position (i,j) and all other entries equal to 0, we define $u_k^m = E_{k,n+1}^m$ for $k \in [n]$, $h_l^m = (E_{l,l}^m - E_{l+1,l+1}^m)$, for $l \in [n-1]$, $e_{i,j}^m = E_{i,j}^m$, for $1 \leq i < j \leq n$, and $f_{s,t}^m = E_{s,t}^m$, for $1 \leq t < s \leq n$. Our choice of basis for $\mathfrak{g}_n^m$ is given by $\mathfrak{X} \cup \mathfrak{Y}$, where

$$\begin{aligned} \mathfrak{X} &= \{u_k^m \mid k \in [n]\} \quad \text{and} \\ \mathfrak{Y} &= \{h_l^m \mid l \in [n-1]\} \cup \{e_{i,j}^m \mid 1 \leq i < j \leq n\} \cup \{f_{s,t}^m \mid 1 \leq t < s \leq n\}. \end{aligned}$$

Note that $\mathfrak{Y}$ forms a basis for $\mathfrak{h}_n^m$. Moreover, by construction $\tilde{G}_n^m = \exp(\mathfrak{g}_n^m)$. The Lie lattice $\mathfrak{g}_n^m$, has $\mathcal{O}$-dimension $n^2 + n - 1$.

In order to simplify the notation we fix m and n, we write $G = G_n^m$ and we drop the upper notation m from the elements of the basis $\mathfrak{X} \cup \mathfrak{Y}$.

By the considerations above, we may restrict our attention to the partial commutator matrix $A = ([x,y])_{(x,y) \in \mathfrak{X} \times \mathfrak{Y}}$. Note the following identities,

$$[h_l, u_k] = \begin{cases} \pi^m u_k & \text{if } k = l, \\ -\pi^m u_k & \text{if } k = l+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[e_{i,j}, u_k] = \begin{cases} \pi^m u_i & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases} \qquad [f_{s,t}, u_k] = \begin{cases} \pi^m u_s & \text{if } k = t, \\ 0 & \text{otherwise,} \end{cases}$$

which yield the following description of A, in which the rows correspond to the elements $u_1, \dots, u_n$ in the given order, the columns of the matrix A_h correspond to $h_1, \dots, h_{n-1}$, the columns of A_j correspond to $e_{1,j}, \dots, e_{j-1,j}, f_{j+1,j}, \dots, f_{n,j}$ for $j \in [n]$,

$$A = -\pi^m \begin{pmatrix} A_h & A_1 & \dots & A_n \end{pmatrix}$$

with

$$A_h = \begin{pmatrix} \mathrm{diag}(u_1, \dots, u_{n-1}) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mathrm{diag}(u_2, \dots, u_n) \end{pmatrix} \in \mathrm{Mat}_{n,n-1}(\mathfrak{o})$$

and A_j the $(n \times (n-1))$-matrix with all zero entries except for the j^{th} row, which corresponds to the $(n-1)$-ordered tuple of elements $u_1, \dots, u_n$, excluding u_j.

As an example, the matrix A for $n = 4$ is of the form

$$-\pi^m \begin{pmatrix} u_1 & 0 & 0 & u_2 & u_3 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_2 & u_2 & 0 & 0 & 0 & 0 & u_1 & u_3 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u_3 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & u_2 & u_4 & 0 & 0 & 0 \\ 0 & 0 & -u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & u_2 & u_3 \end{pmatrix}.$$

We claim that all minors of A have determinant of the form $\pm \pi^{km} \prod_{i=0}^{k-1} u_{j_i}$ for $j_i \in [n]$ and $k \in [n]$ or equal to 0. In particular, we can reach all the products with the same u_i for $i \in [n]$. The statement is easily true for $k \in [n-1]$ by looking at the part of the matrix A corresponding to $\begin{pmatrix} A_1 & \dots & A_n \end{pmatrix}$. When the whole matrix A is considered, one proves the claim by using an inductive argument and noticing that in every row of the matrix A we have all the elements $u_1, \dots, u_n$.

Now we may calculate

$$\zeta_H^G(s) = \int_{K\mathfrak{X}^*} \|\mathrm{Min}(Aw)\|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(w).$$

First notice that all $u_i^* \in \mathfrak{X}^*$ appear with a factor π^m in the integrand. Thus, by change of variables $u_i^* \mapsto \pi^m u_i^*$, we find

$$\begin{aligned} \zeta_H^G(s) &= q^{mn} \int_{K\mathfrak{X}^*} \|\mathrm{Min}(\pi^{-m} Aw)\|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(w) \\ &= q^{mn} \int_{K\mathfrak{X}^*} \left\| \left\{ \prod_{k=0}^{l} (u_{j_k}) w \mid l, j_k \in \{0, \dots, n\}, j_k \neq 0 \text{ for all } k \right\} \right\|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(w). \end{aligned}$$

Notice that, since the constant polynomial 1 is a minor of $\pi^{-m} Aw$, if $w \in \mathcal{O}\,\mathfrak{X}^*$, the maximal valuation is achieved at $|1|_\mathfrak{p} = 1$. Hence we write

$$\begin{aligned}\zeta_H^G(s) &= q^{mn}\int_{\mathcal{O}\,\mathfrak{X}^*} 1\,\mathrm{d}\mu(w) + q^{mn}\int_{K\mathfrak{X}^*\smallsetminus\mathcal{O}\,\mathfrak{X}^*} \|\mathrm{Min}(\pi^{-m}wA)\|_\mathfrak{p}^{-1-s}\mathrm{d}\mu(w)\\ &= q^{mn}\left(1+\int_{K\mathfrak{X}^*\smallsetminus\mathcal{O}\,\mathfrak{X}^*} \|\mathrm{Min}(\pi^{-m}wA)\|_\mathfrak{p}^{-1-s}\mathrm{d}\mu(w)\right).\end{aligned}$$

We consider the following partition

$$K\mathfrak{X}^*\smallsetminus\mathcal{O}\,\mathfrak{X}^* = \bigcup_{j=1}^{\infty}(\pi^{-j}\,\mathcal{O}\,\mathfrak{X}^*)\smallsetminus(\pi^{1-j}\,\mathcal{O}\,\mathfrak{X}^*) \tag{2.4.1}$$

into sets where the minimum of the valuations of the coordinates of the integral is constant and equal to $-j$. Thus on each set the maximal possible norm is achieved at a power u_i^n for some $i \in [n]$, and it is equal to q^{jn}. Setting $t = q^{-s}$, we calculate

$$\begin{aligned}\zeta_H^G(s) &= q^{mn}\left(1+\sum_{j=1}^{\infty}\int_{(\pi^{-j}\,\mathcal{O}\,\mathfrak{X}^*)\smallsetminus(\pi^{1-j}\,\mathcal{O}\,\mathfrak{X}^*)} q^{-jn}t^{jn}\mathrm{d}\mu(w)\right)\\ &= q^{mn}\left(1+\sum_{j=1}^{\infty}(q^{jn}-q^{(j-1)n})q^{-jn}t^{jn}\right)\\ &= q^{mn}\left(1+(1-q^{-n})\sum_{j=1}^{\infty}t^{jn}\right)\\ &= q^{mn}\frac{(1-q^{-n}t^n)}{(1-t^n)}.\end{aligned}$$

Using [10, Theorem 1.2] for p odd, we have

$$\zeta_{\mathrm{SL}_2^m(\mathcal{O})}(s) = q^{3m}\frac{(1-q^{-2}t)}{(1-qt)}.$$

Consequently, the representation zeta function of $G = \mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^2$, when G is potent, is given by

$$\zeta_G(s) = \zeta_H(s)\zeta_H^G(s-1) = q^{5m}\frac{(1-t^2)(1-q^{-2}t)}{(1-qt)^2(1+qt)}.$$

In particular, the abscissa of convergence is $\alpha(G) = \alpha(H) = 1$.

Example 2.4.2. For simplicity, consider an unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$. In the general case, we could do similar computations with an extra m to ensure the permissibility. We continue to use π as uniformiser to maintain consistency with the notation and to emphasize that the general case can also be handled.

Let G be the group $\mathrm{SL}_2^1(\mathcal{O}) \ltimes \pi\,\mathcal{O}^2$ and consider the $\mathcal{O}$-Lie lattice $\mathfrak{g}$ generated by the elements h, e, f, u, and v, where

$$h = \begin{pmatrix} \pi & 0 & 0 \\ 0 & -\pi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & \pi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 0 & \pi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 0 & 0 \\ \pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \pi \\ 0 & 0 & 0 \end{pmatrix}.$$

Then one can check that $G = \exp(\mathfrak{g})$. We want to construct a subgroup of G which is not thetyspectral with G.

Let $\mathfrak{g}_k$ be the $\mathcal{O}$-Lie sublattice of $\mathfrak{g}$ generated by the elements $\pi^k h, e, \pi^k f, u$, and v. We can write $\mathfrak{g}_k = \mathfrak{h}_k \oplus \mathfrak{v}$, where $\mathfrak{h}_k$ is the $\mathcal{O}$-Lie sublattice generated by the elements $\pi^k h, e$, and $\pi^k f$. Since $\mathfrak{g}$ is potent, it easily follows that $\mathfrak{g}_k$ is potent as well, for every $k \geqslant 1$. For the sake of simplicity, we assume that $p \geqslant 7$ to ensure also the saturability, cf. [43, Theorem 4.6].

Let $G_k = \exp(\mathfrak{g}_k)$ and $H_k = \exp(\mathfrak{h}_k)$. The group G_k has the structure of a semi-direct product. Indeed, $G_k = H_k \ltimes \pi\,\mathcal{O}^2$.Now by Theorem 2.1.1 and Corollary 2.3.2, we may write

$$\zeta_{G_k}(s) = \zeta_{H_k}(s) \cdot \zeta_{H_k}^{G_k}(s-1) = |G_k : H_k| \cdot \zeta_{\mathrm{SL}_2^1(\mathcal{O})}(s) \cdot \zeta_{H_k}^{G_k}(s-1).$$

Consider the quotient of G_k by $\mathrm{SL}_2^k(\mathcal{O}) \ltimes \pi^k\,\mathcal{O}^2$. The subgroup $H_k(\mathrm{SL}_2^k(\mathcal{O}) \ltimes \pi^k\,\mathcal{O}^2)$ is the group of upper uni-triangular matrices with one non-trivial entry at position $(1,2)$. Thus, the index of H_k in $\mathrm{SL}_2^1(\mathcal{O})$ is q^{k-1}. It remains to calculate the relative zeta function.

We compute the matrix $A = ([a,b])_{a \in \{u,v\}, b \in \{\pi^k h, e, \pi^k f\}}$,

$$A = \begin{pmatrix} -\pi^{k+1} u & -\pi v & 0 \\ \pi^{k+1} v & 0 & -\pi^{k+1} u, \end{pmatrix}$$

and the set of minors

$$\mathrm{Min}(wA) = \left\{1, w(\pi^{k+1}u), w(\pi v), w(\pi^{k+1}v), w(\pi^{k+1}uv), w(\pi^{2k+2}u^2), w(\pi^{k+2}v^2)\right\}.$$

After a change of basis $u \mapsto \pi^{k+1}u, v \mapsto \pi v$ the integral describing the relative zeta function has the form

$$\zeta_{H_k}^{G_k}(s) = q^{k+2} \int_{K^2} \|1, u, v, uv, u^2, \pi^k v^2\|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v).$$

Since $|u|_{\mathfrak{p}} \leq |u^2|_{\mathfrak{p}}$ whenever $|u^2|_{\mathfrak{p}} \geq |1|_{\mathfrak{p}}$, the polynomial u is irrelevant. Comparing the valuations of the remaining polynomials, we determine that the maximum is reached by 1 in the area $\mathcal{O}^2$, by u^2 within $\bigcup_{m \in \mathbb{N}} \pi^{-m}\,\mathcal{O}^\times \times \pi^{-m}\,\mathcal{O}$, by uv within $\bigcup_{m \in \mathbb{N}} \pi^{-m}\,\mathcal{O}^\times \times \pi^{-m-k}\,\mathcal{O} \setminus \pi^{-m+1}\,\mathcal{O}$, by $\pi^k v^2$ within $\bigcup_{m \in \mathbb{N}} \pi^{-m}\,\mathcal{O} \times \pi^{-m-k}\,\mathcal{O}^\times$, and by v in the area $\mathcal{O} \times (\pi^{-k}\,\mathcal{O} \setminus \pi\,\mathcal{O})$.

These areas overlap. We cut up K^2 into pieces on which a fixed polynomial is maximal, see Fig. 2.1 for comparison, such that

$$\zeta_{H_k}^{G_k}(s) = \int_{A_1} |1|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) + \int_{A_2} |u^2|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) + \int_{A_3} |uv|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) \\ + \int_{A_4} |\pi^k v^2|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) + \int_{A_5} |v|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v).$$

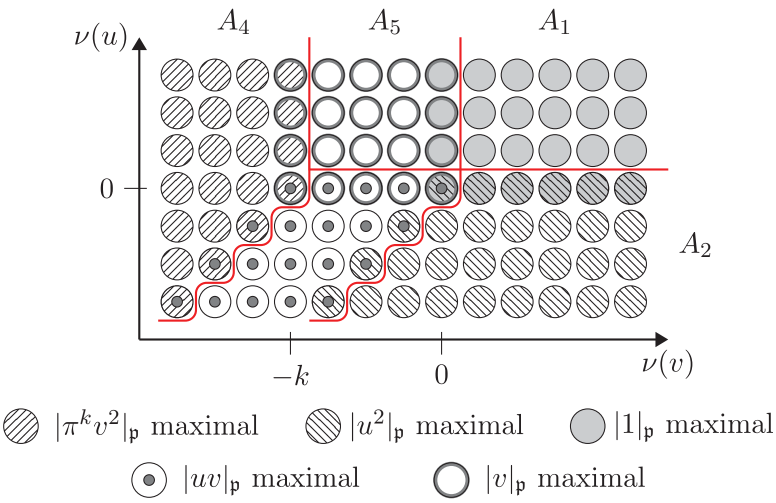

Figure 2.1: A sketch of the partition of K^2 we use for the computation of Example 2.4.2, in case $k = 4$. Every circle represents a subset of fixed valuation, i.e. of the form $\pi^a \mathcal{O}^\times \times \pi^b \mathcal{O}^\times$ of K^2.

Area A_1, where $|1|_{\mathfrak{p}}$ is maximal, is defined as $\pi \mathcal{O}^2$, hence

$$\int_{A_1} |1|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) = q^{-2}.$$

Area A_2 is defined as $\bigcup_{j\in\mathbb{N}} \pi^{-j} \mathcal{O}^\times \times \pi^{-j+1} \mathcal{O}$, such that

$$\begin{aligned} \int_{A_2} |u^2|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) &= \sum_{j=0}^{\infty} \int_{\pi^{-j} \mathcal{O}^\times \times \pi^{-j+1} \mathcal{O}} |u^2|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) \\ &= (1-q^{-1}) q^{-1} \sum_{j=0}^{\infty} t^{2j} \\ &= q^{-1} \frac{(1-q^{-1})}{1-t^2}, \end{aligned}$$

where $t = q^{-s}$ as in the previous example. We put

$$A_3 = \bigcup_{j\in\mathbb{N}} \pi^{-j} \mathcal{O}^\times \times (\pi^{-j-k+1} \mathcal{O} \smallsetminus \pi^{-j+1} \mathcal{O})$$

and calculate

$$\int_{A_3} |uv|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) = \sum_{j=0}^{\infty} \int_{\pi^{-j}\,\mathcal{O}^\times} |u|_{\mathfrak{p}}^{-1-s} \sum_{l=0}^{k-1} \int_{\pi^{-j-l}\,\mathcal{O}^\times} |v|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(v)\mathrm{d}\mu(u).$$

Evaluation of the inner sum yields

$$\sum_{l=0}^{k-1} \int_{\pi^{-j-l}\,\mathcal{O}^\times} |v|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(v) = \sum_{l=0}^{k-1} (1-q^{-1}) t^{j+l} = (1-q^{-1}) t^j \frac{1-t^k}{1-t}.$$

Thus the integral over A_3 is equal to the following expression

$$\begin{aligned}\int_{A_3} |uv|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) &= (1-q^{-1}) \frac{1-t^k}{1-t} \sum_{j=0}^{\infty} t^j \int_{\pi^{-j}\,\mathcal{O}^\times} |u|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u) \\ &= (1-q^{-1})^2 \frac{1-t^k}{(1-t)(1-t^2)}.\end{aligned}$$

The fourth area is defined by $A_4 = \bigcup_{j \in \mathbb{N}} \pi^{-j}\,\mathcal{O} \times \pi^{-j-k}\,\mathcal{O}^\times$, and we compute

$$\begin{aligned}\int_{A_4} |\pi^k v^2|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) &= \sum_{j=0}^{\infty} \int_{\pi^{-j}\,\mathcal{O} \times \pi^{-j-k}\,\mathcal{O}^\times} |\pi^k v^2|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) \\ &= \frac{(1-q^{-1})t^k}{1-t^2}.\end{aligned}$$

Finally, the last area $A_5 = \pi\,\mathcal{O} \times (\pi^{-k+1}\,\mathcal{O} \setminus \pi\,\mathcal{O})$ gives rise to

$$\begin{aligned}\int_{A_4} |v|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(u,v) = \int_{\pi\,\mathcal{O}} \mathrm{d}\mu(u) \sum_{l=0}^{k-1} \int_{\pi^{-l}\,\mathcal{O}^\times} |v|_{\mathfrak{p}}^{-1-s} \mathrm{d}\mu(v) &= \frac{1-q^{-1}}{q} \sum_{l=0}^{k-1} t^l \\ &= \frac{q-1}{q^2} \frac{1-t^k}{1-t}.\end{aligned}$$

The sum of the five integrals calculated above factorises to

$$\zeta_{H_k}^{G_k}(s) = t^{-k} q^2 \frac{(1-q^{-1}t)(1-t-q^{-1}t^2+t^{k+1}q^{-1})}{(1-t)^2(1+t)},$$

and overall, we have

$$\zeta_{G_k}(s) = t^{-k} q^5 \frac{(1-q^{-2}t)(1-t)(1-qt-qt^2+q^k t^{k+1})}{(1-qt)^3(1+qt)}.$$

Thus, the subgroup G_k of G and G are not thetyspectral. Indeed, the quotient of the respective zeta functions is

$$\frac{\zeta_{G_k}(s)}{\zeta_G(s)} = t^{-k} \frac{(1-qt-qt^2+q^k t^{k+1})}{(1-qt)(1+t)}.$$

Example 2.4.3. Let $n \in \mathbb{N}_*$ be a positive integer. We consider the uniformly potent pro-p group $G = \mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^{2n}$, where the semi-direct product is defined with respect to the diagonal action of $\mathrm{SL}_2^m(\mathcal{O})$ on $\mathcal{O}^{2n} \cong \bigoplus_{i=1}^n \mathcal{O}^2$ and m is permissible for $\mathfrak{sl}_2(\mathcal{O})$. We shall calculate $\zeta_G(s)$. In the case $n = 1$, this (partially) recovers Example 2.4.1.

The Lie lattice $\mathfrak{g}$ associated to G has $\mathcal{O}$-dimension $3 + 2n$. As in the previous examples, we embed $\mathfrak{g}$ into $\pi^m \mathfrak{sl}_{2n+1}$ as the sublattice generated by $\{h, e, f, u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$, where the first three basis elements are the block diagonal matrices

$$h = \sum_{i=1}^n E^m_{2i-1,2i-1} - E^m_{2i,2i}, \quad e = \sum_{i=1}^n E^m_{2i-1,2i}, \quad f = \sum_{i=1}^n E^m_{2i,2i-1},$$

and

$$u_i = E^m_{2n+1,2i-1}, \quad v_i = E^m_{2n+1,2i}$$

for $i \in [n]$ are the basis elements corresponding to $\mathcal{O}^{2n}$. For convenience, let

$$\mathfrak{U} = \{u_i \mid i \in [n]\}, \quad \mathfrak{V} = \{v_i \mid i \in [n]\}, \quad \mathfrak{X} = \mathfrak{U} \cup \mathfrak{V}, \quad \text{and} \quad \mathfrak{Y} = \{h, e, f\}.$$

We have the following partial commutator matrix $-A^\intercal = \ ([y, x])_{(y,x) \in \mathfrak{Y} \times \mathfrak{X}}$:

$$-A^\intercal = \pi^m \begin{pmatrix} u_1 & -v_1 & \cdots & u_n & -v_n \\ v_1 & 0 & \cdots & v_n & 0 \\ 0 & u_1 & \cdots & 0 & u_n \end{pmatrix}.$$

The set of Pfaffian polynomials, ordered by the size of the respective submatrices, is

$$\{1\} \cup \{\pi^m a \mid a \in \mathfrak{X}\} \cup \{\pi^{2m} ab \mid a, b \in \mathfrak{X}\} \cup \{\pi^{3m} a \det(b, c) \mid a \in \mathfrak{X}, b, c \in \mathfrak{U}, b \neq c\},$$

where we set $\det(u_r, u_q) = u_r v_q - u_q v_r$. The monomials of degree 1 do not contribute to integral, since we have $|a|_\mathfrak{p} \leq \max\{|1|_\mathfrak{p}, |a^2|_\mathfrak{p}\}$ for all $a \in \mathcal{O}$. Similarly, the only monomials of degree 2 that are relevant for the value of the integral are the unmixed ones, i.e. the monomials of the form $\pi^{2m} a^2$ for $a \in \mathfrak{X}$. Thus we set

$$P_1 = \{x^2 \mid x \in \mathfrak{X}\}, \quad P_2 = \{a \det(b, c) \mid a \in \mathfrak{X}, b, c \in \mathfrak{U}, b \neq c\},$$

and, as it will be useful later,

$$P_2' = \{\det(b, c) \mid b, c \in \mathfrak{U}, b \neq c\}.$$

Distinguishing between $\mathcal{O}^{2n}$ – where the maximum is attained by $|1|_\mathfrak{p}$ – and its complement $K^{2n} \setminus \mathcal{O}^{2n}$ and afterwards transforming the integral over the latter area by substituting $\pi^m x$

for x, with $x \in \mathfrak{X}$, we find

$$\zeta_H^G(s) = q^{2nm} \left(1 + \sum_{j=1}^{\infty} \int_{\pi^{-j}(\mathcal{O}^{2n} \setminus \pi\,\mathcal{O}^{2n})} \|P_1 \cup P_2\|_{\mathfrak{p}}^{-s-1} \mathrm{d}\mu(\mathfrak{X}) \right)$$
$$= q^{2nm} \left(1 + \sum_{j=1}^{\infty} (q^{n-1}t)^{2j} A(t,j,n) \right),$$

using again the convention $t = q^{-s}$ and setting

$$A(t,j,n) = \int_{\mathcal{O}^{2n} \setminus \pi\,\mathcal{O}^{2n}} \|P_1 \cup \pi^{-j} P_2\|_{\mathfrak{p}}^{-s-1} \mathrm{d}\mu(\mathfrak{X})$$
$$= \int_{\mathcal{O}^{2n} \setminus \pi\,\mathcal{O}^{2n}} \|\{1\} \cup \pi^{-j} P_2'\|_{\mathfrak{p}}^{-s-1} \,\mathrm{d}\mu(\mathfrak{X}).$$

The second equality is due to the fact that on the area of integration there is at least one invertible variable x in $\mathfrak{X}$, hence the maximum attained on the set P_1 is always equal to $|x|_{\mathfrak{p}} = |1|_{\mathfrak{p}}$, and since $|a \det(b,c)|_{\mathfrak{p}} \leq |x \det(b,c)|_{\mathfrak{p}} = |\det(b,c)|_{\mathfrak{p}}$ for a, b, c as in P_2. To further evaluate the integral, we partition the area of integration by considering the first index $i \in [n]$ such that either u_i or v_i is invertible, i.e.

$$\mathcal{O}^{2n} \setminus \pi\,\mathcal{O}^{2n} = \bigcup_{k=1}^{n} \pi\,\mathcal{O}^{2(k-1)} \times (\mathcal{O}^2 \setminus \pi\,\mathcal{O}^2) \times \mathcal{O}^{2(n-k)}.$$

Consider one of these subsets, say $I_k = \pi\,\mathcal{O}^{2(k-1)} \times (\mathcal{O}^2 \setminus \pi\,\mathcal{O}^2) \times \mathcal{O}^{2(n-k)}$ for some $k \in [n]$. At least one variable among u_k and v_k is a unit. Consider the integral transformation by substituting v_l with $\tilde{v}_l = \det(u_k, u_l)$ in case u_k is invertible, and u_l with $\tilde{u}_l = \det(u_l, u_k)$ otherwise, for $l \in [n] \setminus \{k\}$. Note that this is a concatenation of translations and multiplications with invertibles, hence the corresponding Jacobian has a determinant with valuation 1. Furthermore, I_k is invariant under this transformation, i.e. if $v_l, u_l \in \pi\,\mathcal{O}$, then also $\tilde{v}_l, \tilde{u}_l \in \pi\,\mathcal{O}$.

We have to rewrite the polynomials $\det(u_l, u_r)$ for $l, r \in [n] \setminus \{k\}$ with $l \neq r$ as polynomials in the substituted variables. An easy computation shows that

$$\det(u_l, u_r) = u_l v_r - u_r v_l = u_k^{-1}(u_l \tilde{v}_r - u_r \tilde{v}_l), \quad \text{resp.} \quad \det(u_l, u_r) = v_k^{-1}(\tilde{u}_l v_r - \tilde{u}_r v_l).$$

In both cases it is plain to see that $\det(u_l, u_r)$ is a multiple of $\tilde{v}_r$ or $\tilde{v}_l$ (and $\tilde{u}_r$ or $\tilde{u}_l$, respectively), hence

$$|\det(u_l, u_r)|_{\mathfrak{p}} \leq \|\{\tilde{v}_l, \tilde{v}_r\}\|_{\mathfrak{p}}, \quad \text{resp.} \quad |\det(u_l, u_r)|_{\mathfrak{p}} \leq \|\{\tilde{u}_l, \tilde{u}_r\}\|_{\mathfrak{p}}.$$

For convenience, we write z_l for the new variables $\tilde{v}_l$ or $\tilde{u}_l$ in case $0 < l < k$, and $z_l = \tilde{v}_{l+1}$, resp. $z_l = \tilde{u}_{l+1}$ for $n > l \geq k$. Set $\mathfrak{Z} = \{z_l \mid l \in [n-1]\}$. Thus the integral over I_k is reduced to

$$\int_{I_k} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}) = (1-q^{-2})q^{1-k}\int_{\pi\,\mathcal{O}^{k-1}\times\mathcal{O}^{n-k}} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}).$$

The integral on the right will appear, in a similar form, also in Example 2.4.4. Write

$$B(t,j,n,k) = \int_{\pi\,\mathcal{O}^{k-1}\times\mathcal{O}^{n-k}} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}).$$

To evaluate such integral, we write it as a difference:

$$\int_{\mathcal{O}^{n-1}} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}) - \int_{(\mathcal{O}^{k-1}\setminus\pi\,\mathcal{O}^{k-1})\times\mathcal{O}^{n-k}} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}).$$

The first (positive) integral may be computed as follows,

$$\begin{aligned}
\int_{\mathcal{O}^{n-1}} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}) &= \int_{\pi^j\,\mathcal{O}^{n-1}} \|1\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}) \\
&\qquad + \sum_{r=0}^{j-1}\int_{\pi^r(\mathcal{O}^{n-1}\setminus\pi\,\mathcal{O}^{n-1})} \|\pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}) \\
&= q^{-j(n-1)} + \left(1-q^{-(n-1)}\right)(q^{-1}t)^j\sum_{r=0}^{j-1}(q^{(2-n)}t^{-1})^r \\
&= q^{-j(n-1)}\left(1+\left(1-q^{-(n-1)}\right)\frac{1-(q^{(n-2)}t)^j}{1-q^{(n-2)}t}q^{n-2}t\right),
\end{aligned}$$

and the second integral easily evaluates to

$$\int_{(\mathcal{O}^{k-1}\setminus\pi\,\mathcal{O}^{k-1})\times\mathcal{O}^{n-k}} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}^{-s-1}\,\mathrm{d}\mu(\mathfrak{z}) = (1-q^{1-k})(q^{-1}t)^j.$$

Thus we have

$$B(t,j,n,k) = q^{-j(n-1)}\left(1+\left(1-q^{-(n-1)}\right)\frac{1-q^{j(n-2)}t^j}{1-q^{(n-2)}t}q^{n-2}t\right) - (1-q^{1-k})(q^{-1}t)^j.$$

Overall we find

$$\begin{aligned}
A(t,j,n) &= \sum_{k=1}^{n}\int_{I_k} \|\{1\} \cup \pi^{-j}\mathfrak{z}\|_{\mathfrak{p}}\,\mathrm{d}\mu(\mathfrak{z}) \\
&= (1-q^{-n})\left(q^{-j(n-1)}(1+q^{-1})\left(1+\left(1-q^{-(n-1)}\right)\frac{1-q^{j(n-2)}t^j}{1-q^{(n-2)}t}q^{n-2}t\right)\right. \\
&\qquad\left. -(1-q^{-(n-1)})q^{-j-1}t^j\right)
\end{aligned}$$

$$= q^{-j(n-1)} \left(\frac{(1-q^{-n})(1-q^{-2})}{1-q^{n-2}t} \right. \\ \left. - (q^{-1}t)^j \left(q^{-1} \frac{(1-q^{-n})(1-q^{1-n})(1+q^{n-1}t)}{1-q^{n-2}t} \right) \right).$$

We now multiply this expression with the remaining coefficients and we take the sum over j, arriving at

$$\sum_{j=1}^{\infty} (q^{(n-1)}t)^{2j} A(t,j,n) = (1-q^{-n}) \frac{q^2+q^3+(q^{n+1}-q^2)t-(1+q^n)q^n t^3}{(1-q^{(n-1)}t^2)(1-q^{(2n-3)}t^3)} q^{n-4}t^2.$$

In consequence, the desired function is equal to

$$\zeta_G(s) = q^{2nm}(1-t) \frac{(1+q^{-1}-qt^2+q^nt^2-q^{n+1}t^3-q^{n+1}t^4)}{(1-q^{(n+1)}t^2)(1-q^{2n}t^3)} \zeta_{\mathrm{SL}_2^m(\mathcal{O})}.$$

By Theorem 2.1.1 we have, for n greater than or equal to 3,

$$\alpha(\mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^{2n}) = \frac{2n}{3},$$

while $\alpha(\mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^2) = 1$ and $\alpha(\mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^4) = \frac{3}{2}$. Thus the family of groups $\mathrm{SL}_2^m(\mathcal{O}) \ltimes \mathcal{O}^{2n}$, varying in n, gives rise to zeta functions with two poles each, whose position behaves regularly with respect to n, and both families of poles possess members that dictate the abscissae, i.e. the degree of growth, for some groups in the family.

Example 2.4.4. Let p be an odd prime and consider the uniformly potent pro-p group

$$G_n^m = \mathrm{SL}_2^m(\mathcal{O}) \ltimes (\mathrm{Sym}^2(\mathcal{O}^2))^n$$

with the diagonal action, where m is permissible. Denote $\mathrm{SL}_2^m(\mathcal{O})$ by H^m. The $\mathcal{O}$-dimension of the associated Lie lattice $\mathfrak{g}_n$ is $3(n+1)$ so we embed $\mathfrak{g}_n$ into $\pi^m\mathfrak{sl}_{3n+1}$ as the sublattice generated by

$$\{h, e, f, u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_n, v_n, w_n\},$$

where the first three basis elements are the block diagonal matrices

$$h = 2\sum_{i=1}^{n} (E^m_{3i-2,3i-2} - E^m_{3i,3i}),$$

$$e = \sum_{i=1}^{n} (2E^m_{3i-2,3i-1} + E^m_{3i-1,3i}), \quad f = \sum_{i=1}^{n} (E^m_{3i-1,3i-2} + 2E^m_{3i,3i-1}),$$

and the other elements, which correspond to the abelian factor, appear in the last column of the matrices

$$u_i = E^m_{3i-2,4}, \quad v_i = E^m_{3i-1,4}, \quad w_i = E^m_{3i,4}.$$

We partition the basis elements as follows

$$\mathfrak{U} = \{u_i \mid i \in [n]\}, \quad \mathfrak{V} = \{v_i \mid i \in [n]\}, \quad \mathfrak{W} = \{w_i \mid i \in [n]\},$$

$$\mathfrak{X} = \mathfrak{U} \cup \mathfrak{V} \cup \mathfrak{W}, \quad \text{and} \quad \mathfrak{Y} = \{h, e, f\}.$$

We have the following partial commutators matrix

$$-A^{\intercal} = \begin{pmatrix} 2\pi^m u_1 & 0 & -2\pi^m w_1 & \cdots & 2\pi^m u_n & 0 & -2\pi^m w_n \\ 2\pi^m v_1 & \pi^m w_1 & 0 & \cdots & 2\pi^m 2v_n & \pi^m w_n & 0 \\ 0 & \pi^m u_1 & 2\pi^m v_1 & \cdots & 0 & \pi^m u_n & 2\pi^m v_n. \end{pmatrix}.$$

The set of Pfaffian polynomials is the following

$$\begin{aligned} \{1\} \ \cup \ &\{\pi^m a \mid a \in \mathfrak{X}\} \cup \{\pi^{2m} ab \mid a, b \in \mathfrak{X}\} \\ &\cup \{\pi^{3m} u \det(u_i, v_j), p^{3m} v \det(v_j, w_l), p^{3m} w \det(w_l, u_i), \\ &\quad \pi^{3m} u \det(u_i, w_l), \pi^{3m} v \det(v_j, u_i), \pi^{3m} w \det(w_l, v_j), \\ &\quad \pi^{3m} u \det(v_i, w_l), \pi^{3m} v \det(w_l, u_i), \pi^{3m} w \det(u_i, v_j), \\ &\quad \pi^{3m}(u_i v_j w_l - u_l v_i w_j) \mid u \in \mathfrak{U}, v \in \mathfrak{V}, w \in \mathfrak{W}, i, j, l \in [n] \text{ pairwise different}\} \end{aligned}$$

where $\det(a_i, b_j) = a_i b_j - a_j b_i$. As in Example 2.4.3, the monomials of degree one do not play any role in the computation of the representation zeta function as for the mixed monomials of degree two. Hence, we are left with the following set of Pfaffians monomials modulo π^{2m} for the first set and π^{3m} for the second one.

$$\begin{aligned} P_1 =& \{a^2 \mid a \in \mathfrak{X}\} \\ P_2 =& \{u \det(u_i, v_j), v \det(v_j, w_l), w \det(w_l, u_i), \\ &\quad u \det(u_i, w_l), v \det(v_j, u_i), w \det(w_l, v_j), \\ &\quad u \det(v_i, w_l), v \det(w_l, u_i), w \det(u_i, v_j), \\ &\quad (u_i v_j w_l - u_l v_i w_j) \mid u \in \mathfrak{U}, v \in \mathfrak{V}, w \in \mathfrak{W}, i, j, l \in [n] \text{ pairwise different}\}. \end{aligned}$$

As for the previous examples, by partitioning K^{3n} into

$$K^{3n} = \mathcal{O}^{3n} \cup \bigcup_{j=1}^{\infty} \pi^{-j} \mathcal{O}^{3n} \setminus \pi^{-j+1} \mathcal{O}^{3n},$$

and applying the change of variables $y = \pi^m x$ where $x \in \mathfrak{X}$, we get the following

$$\zeta_{H^m}^{G_n^m}(s) = q^{3nm} \left(1 + \sum_{j=1}^{\infty} \int_{\pi^{-j} \mathcal{O}^{3n} \setminus \pi^{-j+1} \mathcal{O}^{3n}} \|P_1 \cup P_2\|_{\mathfrak{p}}^{-s-1} \mathrm{d}\mu(\mathfrak{X}) \right)$$

$$= q^{3nm}\left(1+\sum_{j=1}^{\infty} q^{j(3n-2)}t^{2j}\tilde{A}(t,j,n)\right),$$

with $t = q^{-s}$ and

$$\begin{aligned}\tilde{A}(t,j,n) &= \int_{\mathcal{O}^{3n}\setminus\pi\,\mathcal{O}^{3n}} \|P_1\cup\pi^{-j}P_2\|_{\mathfrak{p}}^{-s-1}\mathrm{d}\mu(\mathfrak{X})\\ &= \int_{\mathcal{O}^{3n}\setminus\pi\,\mathcal{O}^{3n}} \|\{1\}\cup\pi^{-j}P_2\|_{\mathfrak{p}}^{-s-1}\mathrm{d}\mu(\mathfrak{X}).\end{aligned}$$

We decompose the domain of integration in to n parts, isolating one triple of variables (u_k, v_k, w_k) for $k\in[n]$ as follows

$$\tilde{A}(t,j,n) = \sum_{k=1}^{n}\int_{\pi\,\mathcal{O}^{3(k-1)}\times(\mathcal{O}^3\setminus\pi\,\mathcal{O}^3)\times\mathcal{O}^{3(n-k)}} \|\{1\}\cup\pi^{-j}P_2\|_{\mathfrak{p}}^{-s-1}\mathrm{d}\mu(\mathfrak{X}).$$

The triple (u_k, v_k, w_k) is in $\mathcal{O}^3\setminus\pi\,\mathcal{O}^3$, hence at least one of them is invertible. Without loss of generality, suppose that u_k is invertible. Thus for all $i, j\in[n]$ with $i\neq j$, and $l\in[n]$, we have

$$\begin{aligned}\|\{u_l\det(u_i,v_j), v_l\det(u_i,v_j), w_l\det(u_i,v_j)\}\|_{\mathfrak{p}} &= |u_k\det(u_i,v_j)|_{\mathfrak{p}},\\ \|\{u_l\det(u_i,w_j), v_l\det(u_i,w_j), w_l\det(u_i,w_j)\}\|_{\mathfrak{p}} &= |u_k\det(u_i,w_j)|_{\mathfrak{p}},\\ \|\{u_l\det(v_i,w_j), v_l\det(v_i,w_j), w_l\det(v_i,w_j)\}\|_{\mathfrak{p}} &= |u_k\det(v_i,w_j)|_{\mathfrak{p}}.\end{aligned}$$

We now perform a linear change of variables whenever u_k is involved in those determinants and in this way we obtain $2(n-1)$ new variables given by $\tilde{v}_i = \det(u_k, v_i)$ and $\tilde{w}_i = \det(u_k, w_i)$ for $i\in[n]\setminus\{k\}$. Rewriting our integrand, we find it to be the minimum of the norms of the functions

$$\begin{aligned}&\tilde{v}_i, \tilde{w}_i, \det(v_k,\tilde{w}_j), \det(u_i,\tilde{v}_j), \det(u_i,\tilde{w}_j), \det(\tilde{v}_i,\tilde{w}_j),\\ &u_i\tilde{v}_j\tilde{w}_l - u_l\tilde{v}_i\tilde{w}_j + u_iu_j\det(v_k,\tilde{w}_l) - u_iu_l\det(v_k,\tilde{w}_j),\end{aligned}$$

where $i, j, l\in[n]\setminus\{k\}$ are pairwise different. A comparison of norms shows that the only relevant polynomials are the monomials $\tilde{v}_i$ and $\tilde{w}_i$ with $i\in[n]\setminus\{k\}$. Consequently, we may consider a new set of variables $\mathfrak{Z} = \{z_1,\ldots,z_{2(n-1)}\}$, yielding

$$\begin{aligned}A_{3n}(j,n) &= (1-\frac{1}{q^3})\sum_{k=1}^{n}\frac{1}{q^{k-1}}\int_{\pi\,\mathcal{O}^{2(k-1)}\times\mathcal{O}^{2(n-k)}} \|\{1\}\cup\pi^{-j}\mathfrak{Z}\|_{\mathfrak{p}}^{-s-1}\mathrm{d}\mu(\mathfrak{Z})\\ &= (1-\frac{1}{q^3})\sum_{k=1}^{n}\frac{1}{q^{k-1}}B(j, 2n-1, 2k-1),\end{aligned}$$

where $B(j,n,k)$ is defined as in Example 2.4.3. Thus, by routine calculations we get

$$\begin{aligned} A_{3n}(j,n) &= q^{-2j(n-1)}\frac{(1-q^{-3})(1-q^{-n})(1-q^{-1}t)}{(1-q^{-1})(1-q^{2n-3}t)} \\ &\quad - q^{-j}t^j(1-q^{-n})(1-q^{1-n})\left(q^{-1}\frac{(1+q^{-1}+q^{-n})+(1+q+q^n)q^{n-2}t}{1-q^{2n-3}t}\right) \end{aligned}$$

This leads to

$$\zeta_{H^m}^{G_n^m}(s) = q^{3nm-1}(1-t)(1-q^{-1}t)\frac{(t+q^nt^2+(1+t)q+(1+t)q^{n-1}t^2)}{(1-q^nt^2)(1-q^{3(n-1)}t^3)}.$$

As in Example 2.4.3, the zeta function $\zeta_{H^m}^{G_n^m}$ associated to the induced representation $\mathrm{Ind}_{H^m}^{G_n^m}(\mathbb{1}_{H^m})$ has two poles whose position depends on n. In conclusion, we have

$$\alpha(\mathrm{SL}_2^m(\mathcal{O}) \ltimes (\mathrm{Sym}^2(\mathcal{O}^2))^n) = n,$$

for n greater than or equal to 2, and $\alpha(\mathrm{SL}_2^m(\mathcal{O}) \ltimes (\mathrm{Sym}^2(\mathcal{O}^2))) = \frac{3}{2}$.

Example 2.4.5. Finally, we consider a special case of the last example – namely the representation zeta function of the semi-direct product $G = \mathrm{SL}_2^m(\mathbb{Z}_p) \ltimes \mathrm{Sym}^2(\mathbb{Z}_p^2)$ for $m \in \mathbb{N}_*$ – but for the case of the prime $p = 2$.

The image of $H = \mathrm{SL}_2^m(\mathbb{Z}_2)$ under the representation corresponding to the symmetric square is given by

$$\left\{ \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \right\} \leqslant \mathrm{GL}_3(\mathbb{Z}_2).$$

Let $\mathfrak{g}$ be the 6-dimensional Lie lattice associated to G over $\mathbb{Z}_2$. We choose as basis $\mathfrak{X} \cup \mathfrak{Y}$, where $\mathfrak{X} = \{u,v,z\}$ and $\mathfrak{Y} = \{h,x,y\}$ are bases for $2^m\,\mathbb{Z}_2^3$ and $2^m{\cdot}\mathfrak{sl}_2(\mathbb{Z}_2)$ as in Example 2.4.1. We embed this Lie lattice into $2^m{\cdot}\mathfrak{sl}_4(\mathbb{Z}_2)$, and we compute the following partial commutator matrix with columns corresponding to the elements of Y and rows corresponding to elements of X

$$A = 2^m \begin{pmatrix} 2u & 2v & 0 \\ 0 & z & u \\ -2z & 0 & 2v \end{pmatrix}.$$

The set of minors of A is given by

$$\begin{aligned} \mathrm{Min}(A) = \{&1, 2^mu, 2^{m+1}u, 2^{m+1}v, 2^mz, 2^{m+1}z, \\ &2^{2m+1}uv, 2^{2m+2}uv, 2^{2m+1}uz, 2^{2m+1}vz, 2^{2m+2}vz, \\ &2^{2m+1}u^2, 2^{2m+2}v^2, 2^{2m+1}z^2\}. \end{aligned}$$

Clearly, the polynomials $2^{m+1}u, 2^{m+1}z, 2^{2m+2}uv, 2^{2m+2}vz$ are irrelevant, as is $2^{m+1}v$. Also,

$\|2^m u\|_2 \le \|1, 2^{2m+1}u^2\|_2$, thus $2^m u$ and, similarly, $2^m z$ may be ignored. Finally, as seen before, the mixed terms never attain the maximum (alone). After scaling u and z by a factor 2^m and v by a factor 2^{m+1}, it remains to consider the integral

$$\begin{aligned}
\zeta_H^G(s) &= 2^{3m+1} \int_{\mathbb{Q}_2^3} \|\{1, 2u^2, v^2, 2z^2\}\|_2^{-1-s} \mathrm{d}\mu(u,v,z) \\
&= 2^{3m+1}(1 + \sum_{j=1}^{\infty} 2^{3j} \int_{\mathbb{Z}_2^3 \setminus 2\,\mathbb{Z}_2^3} \|\{1, 2^{1-2j}u^2, 2^{-2j}v^2, 2^{1-2j}z^2\}\|_2^{-1-s} \mathrm{d}\mu(u,v,z) \\
&= 2^{3m+1} \left(1 + \sum_{j=1}^{\infty} 2^{3j}(1-2^{-2})2^{(2j-1)(-1-s)} + (1-2^{-1})2^{2j(-1-s)-2}\right) \\
&= 2^{3m-1} \frac{(2-t)(2^3 t - 2t + 2 + t)}{1-2t^2} \\
&= 2^{3m-1} \frac{(2-t)(2^3 t + (2-t))}{1-2t^2}.
\end{aligned}$$

Thus the zeta function of G is

$$\zeta_G(s) = \zeta_H^G(s-1)\zeta_H(s) = 2^{3m+1} \frac{(1-t)(2^3 t + (1-t))}{1-2^3 t^2} \cdot \zeta_H(s).$$

It is interesting to compare the relative zeta function with the corresponding case $n = 1$ of the previous example, which gives (for odd primes p)

$$\zeta_G(s) = p^{3m} \frac{1-t^2}{1-p^3 t^2} \cdot \zeta_H(s).$$

The case of the prime 2 behaves differently from the odd prime cases due to the structure constants of the associated Lie lattice.

Chapter 3

Representation growth of Baumslag-Solitar groups

Based on joint work with Iker de las Heras

3.1 Introduction

A Baumslag-Solitar group is a two-generator one-relator group given by the presentation

$$\mathrm{BS}(x, y) = \langle a, b \mid a^x = b^{-1} a^y b \rangle, \tag{3.1.1}$$

where $x, y \in \mathbb{Z} \setminus \{0\}$. These groups were introduced in 1962 by Baumslag and Solitar in [14] to provide the first examples of finitely presented non-Hopfian groups, and have since been a rich source of examples and counterexamples in group theory (recall that a group G is called *Hopfian* if it does not have any proper quotient that is isomorphic to G). In 1940 Mal'cev [91] proved that residual finiteness of a finitely generated group implies Hopficity. More precisely for Baumslag-Solitar groups it was shown in [14] and [94] that:

- $\mathrm{BS}(x, y)$ is residually finite (and hence Hopfian) if and only if $|x| = 1$ or $|y| = 1$ or $|x| = |y|$.
- $\mathrm{BS}(x, y)$ is Hopfian if and only if it is residually finite or $\pi(x) = \pi(y)$, where $\pi(x)$ is the set of prime divisors of x.

Note that $\mathrm{BS}(x, y)$, $\mathrm{BS}(y, x)$, and $\mathrm{BS}(-x, -y)$ are isomorphic so we can assume, when it is convenient, that x and y satisfy the condition $|y| \geqslant x > 0$.

A wealth of information about the residual properties of Baumslag-Solitar groups can be found in [95].

In the last decades, much attention has been devoted to the study of different asymptotic invariants of the Baumslag-Solitar groups. For instance, a great deal of results concerning the word growth of such groups can be found in [33], [23], [36], [19], or [1]; furthermore, explicit computations of several subgroup growth functions were provided in [39], [20], and [69].

We set out to study finite dimensional linear representations of Baumslag-Solitar. A complete characterisation of irreducible representations over $\mathbb{C}$ of the Baumslag-Solitar groups with $\gcd(x, y) = 1$ and not both x and y equal to ± 1, was given by McLaury in [93]. McLaury's strategy is to examine image of a representation $\rho\colon \mathrm{BS}(x, y) \to \mathrm{GL}_n(\mathbb{C})$ and analyse its Zariski closure as a subgroup of $\mathrm{GL}_n(\mathbb{C})$, cf. Remark 3.1.3. This work entered the picture of our research after we completed all the work presented in this chapter. Therefore, while working, we were unaware of McLaury's results, which later turned out to align with our investigation. On the one hand, this is reassuring regarding the validity of our results; on the other hand, it makes our work seem less innovative. The main difference lies in the fact that we focus on the absolute representation growth over the field $\mathbb{F}_q$ of q elements, where q is a prime power, for the Baumslag-Solitar groups $\mathrm{BS}(x, y)$ with $\gcd(x, y) = 1$. Our approach is less geometrical than McLaury's and includes additional information derived from the different base fields of the considered representations.

Recall that an *n-dimensional linear representation over $\mathbb{F}_q$* of the Baumslag-Solitar group $\mathrm{BS}(x, y)$ is a group homomorphism

$$\rho : \mathrm{BS}(x, y) \to \mathrm{GL}_n(\mathbb{F}_q),$$

and that a representation ρ is called *absolutely irreducible* if it is irreducible over the algebraic closure of $\mathbb{F}_q$ which we denote by $\mathbb{F}$. Two absolutely irreducible representations ρ and ρ' of dimension n are equivalent over $\mathbb{F}_q$ if there exists an isomorphism of vector spaces $f : \mathbb{F}_q^n \to \mathbb{F}_q^n$ such that $f(\rho(g)(v)) = \rho'(g)(f(v))$ for every $v \in \mathbb{F}_q^n$ and $g \in \mathrm{BS}(x, y)$. Compare with Section 0.4.

Let $r_n^{\mathrm{abs}}(\mathrm{BS}(x, y), \mathbb{F}_q)$ be the number of non-isomorphic absolutely irreducible n-dimensional representations of $\mathrm{BS}(x, y)$ over $\mathbb{F}_q$. Our main result is the following.

Theorem 3.1.1. *Let $x, y \in \mathbb{Z}$ be such that $\gcd(x, y) = 1$. The number of non-isomorphic absolutely irreducible representations over $\mathbb{F}_q$ of dimension n, where q is a prime power, is*

$$r_n^{\mathrm{abs}}(\mathrm{BS}(x, y), \mathbb{F}_q) = \frac{q-1}{n} \sum_{l} \varphi(l),$$

where l runs through all positive integers satisfying the following conditions

$$\begin{cases} \gcd(l, xy) = 1, & \\ q^n \equiv 1 & \pmod{l}, \\ x^n \equiv y^n & \pmod{l}, \\ x^m \not\equiv y^m & \pmod{l}, \quad \text{for every } m \in [n-1], \\ y^r \equiv x^r q & \pmod{l}, \quad \text{for some } r \in [n]; \end{cases} \tag{3.1.1}$$

and φ is Euler's totient function.

Remark 3.1.2. *If l satisfies the conditions in (3.1.1), then it also satisfies*

$$x^t \equiv y^t q \pmod{l} \text{ for some } t \in [n].$$

Indeed, we have $x^r \equiv y^r q^{n-1} \pmod{l}$ for some $r \in [n]$, which is equivalent to $x^t \equiv y^t q \pmod{l}$ with $t = -r + n$.

Remark 3.1.3. *McLaury's result* [93, Theorem 5.1 and Corollary 5.2] *states that there is an n-dimensional irreducible $\mathbb{C}$-representation ρ such that $\rho(a)$ is a diagonal matrix $\operatorname{diag}(\lambda, \lambda^s, \dots, \lambda^{s^{n-1}})$ with λ a primitive l-root of unity and $x \equiv ys \bmod l$, and $\rho(b)$ is a permutation matrix associated to a cycle of length n multiplied by a complex number c, if and only if l divides $x^n - y^n$ and does not divide $x^k - y^k$ for any $k \in [n-1]$. This analysis aligns with our result in Theorem 3.1.1 and the findings in Section 3.2.*

In the last part of the chapter, we discuss Weil representation zeta functions of Baumslag-Solitar groups. This type of representation zeta function has been investigated quite extensively by Corob Cook, Kionke, and Vannacci in [26]. We compute the Weil zeta function of the metabelian group $\mathrm{BS}(1, -1)$. It is an open challenge to transform the explicit information in Theorem 3.1.1 into a statement about the Weil representation zeta function of $\mathrm{BS}(x, y)$ for more general x, y with $\gcd(x, y) = 1$. See Section 0.7.4 for an introduction to the classic Weil zeta function.

Notation

We denote by $\mathrm{Sym}(n)$ the symmetric group of degree n. We will always write q for a prime power and $\mathbb{F}$ for the algebraic closure of a field $\mathbb{F}_q$ of q elements..

3.2 Counting absolutely irreducible representations

Lemma 3.2.1. *Let q be a power of a prime p, A and $B \in \mathrm{GL}_n(\mathbb{F}_q)$, and $x, y \in \mathbb{Z}$ such that $\gcd(x, y) = 1$. Suppose $A^x = B^{-1} A^y B$ and that the action of $\langle A, B \rangle$ on $\mathbb{F}^n$ is irreducible. Then, A is diagonalizable over $\mathbb{F}$.*

Proof. As A, A^x, and A^y commute, there exists a single matrix P such that $P^{-1}AP$, $P^{-1}A^xP$, and $P^{-1}A^yP$ are upper triangular matrices of Jordan canonical form over $\mathbb{F}$. We may choose a basis $\mathcal{A} = \{a_1, \dots, a_n\}$ of $\mathbb{F}^n$ such that A, A^x and A^y are upper triangular matrices with respect to $\mathcal{A}$. Since $\gcd(x, y) = 1$, either x or y is coprime to p. Without loss of generality, suppose that x is coprime to p. The order of A divides the order of $\mathrm{GL}_n(\mathbb{F}_q)$ which is

$$|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2}(q^n - 1)(q^{n-1} - 1) \cdots (q - 1).$$

Hence, writing $q = p^f$ for some $f \in \mathbb{N}_*$, we can write the order of A as $p^a k$ where $a \leqslant fn(n-1)/2$ and k is coprime to p. If a is zero, then A is directly diagonalizable.

Otherwise, since x is coprime to p, the order of A^x is $p^a k_x$ where k_x divides k. If y is not coprime to p then the order of A^y will be a product of a smaller power of p times some number coprime to p. However this is not possible since A^x and A^y are conjugated by B. Hence, x and y are both coprime to p. From this it follows that the Jordan blocks of A^x and A^y are of the same dimension and stand in the same position within the matrix since A^x and A^y are powers of the same matrix A.

The characteristic polynomials of A^x and A^y may be decomposed into $\mathbb{F}_q$-irreducible polynomials of degree smaller than or equal to n, so all the roots of these polynomials certainly lie in $\mathbb{F}_{q^{n!}}$. Note that $A^{x(q^{n!}-1)}$ and $A^{y(q^{n!}-1)}$ are upper triangular matrices whose diagonal entries are all equal to 1 and set $W = \ker(A^{x(q^{n!}-1)} - I_n)$ and $W' = \ker(A^{y(q^{n!}-1)} - I_n)$. Clearly $W \cap W'$ is not empty since $a_1 \in W \cap W'$. We have

$$(A^{y(q^{n!}-1)} - I_n)BW = B(A^{x(q^{n!}-1)} - I_n)W = 0,$$

so $BW \subseteq W'$. Similarly $B^{-1}W' \subseteq W$ and hence $W' = BW$. Furthermore, since x and y are coprime to p, we have that $W = W'$. It follows that W is a non-zero invariant subspace of $\mathbb{F}^n$ under the action of $\langle A, B\rangle$. Hence, the irreducibility of the action of $\langle A, B\rangle$ implies that $W = W' = \mathbb{F}^n$. This implies $A^{x(q^{n!}-1)} = A^{y(q^{n!}-1)} = I_n$.

Since $x(q^{n!} - 1)$ is coprime to p, it follows that A is diagonalizable over $\mathbb{F}$. □

Lemma 3.2.2. *Let y be a non-zero integer, $B \in \mathrm{GL}_n(\mathbb{F})$, and $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{F}^*$. Then the following assertions are equivalent.*

(i) $A = B^{-1}A^yB$ and $\langle A, B\rangle$ acts irreducibly on $\mathbb{F}^n$;

(ii) The elements λ_i are pairwise distinct and B is a monomial matrix associated to a cycle $\sigma \in \mathrm{Sym}(n)$ of order n such that $\lambda_i = \lambda_{\sigma(i)}^y$.

Moreover, if one of the equivalent assertion holds, then

$$\{\lambda_1, \ldots, \lambda_n\} = \{\lambda, \lambda^y, \lambda^{y^2}, \ldots, \lambda^{y^{n-1}}\} \tag{3.2.1}$$

and $\lambda^{y^n} = \lambda$ for every $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$.

Proof. Suppose that $A = B^{-1}A^yB$ holds and that $\langle A, B\rangle$ acts irreducibly on $\mathbb{F}^n$. By similarity, A and A^y have the same eigenvalues, so we have equality between the elements $\lambda_1, \ldots, \lambda_n$ and $\lambda_1^y, \ldots, \lambda_n^y$, up to reordering. Moreover, we deduce, by the irreducibility of the action, that the conjugation by B induces a permutation $\sigma \in \mathrm{Sym}(m)$ on the distinct eigenvalues. More precisely, suppose that $\lambda_1, \ldots, \lambda_m$ are distinct and that for each $j \in \{m+1, \ldots, n\}$ there exists $i \in \{1, \ldots, m\}$ such that $\lambda_j = \lambda_i$. Then the conjugation by B induces a permutation of the elements $\lambda_1, \ldots, \lambda_m$ which gives $\sigma \in \mathrm{Sym}(m)$. We want to prove that m equals n. Suppose this is not the case, and without loss of generality suppose that the algebraic multiplicity k_1 of the eigenvalue λ_1 is greater than one. Consider the eigenspace W_1 of A associated with the eigenvalue λ_1. Let d_1 be the dimension of W_1.

Using $A = B^{-1}A^yB$, we see that BW_1 is a subspace of the eigenspace W_1' of A^y with eigenvalue λ_1 still of dimension d_1, since B is invertible. Since $\langle A, B\rangle$ acts irreducibly on $\mathbb{F}^n$, we have that $\lambda_1 \neq \lambda_1^y$ and so $\sigma(1) \neq 1$. Let now W_1' be the eigenspace of A^y associated to the eigenvalue $\lambda_1 = \lambda_{\sigma(1)}^y$. Then using again $A = B^{-1}A^yB$, we have that $B^{-1}W_1'$ is a subspace of the eigespace W_1. Hence, we have an equality $W_1' = BW_1$. If we now consider the eigenspace $W_{\sigma(1)}$ of A with eigenvalue $\lambda_{\sigma(1)}$, we have as before that $BW_{\sigma(1)}$ is the eigenspace of A^y with eigenvalue $\lambda_{\sigma(1)}$. We have that $\lambda_{\sigma(1)} \neq \lambda_{\sigma(1)}^y$ and $\lambda_{\sigma(1)} \neq \lambda_1$. Hence, by an inductive argument, σ must be a cycle of order m, where m is the number of different eigenvalues. Moreover, we deduce that all the eigenspaces must have the same dimension. Notice that B^m is a block diagonal matrix, so we can choose v_1, an eigenvector of B^m, that is also an eigenvector of A in an eigenspace W_1. Moreover, B^rv_1 is also an eigenvector of A for every $r \geq 1$, as it is contained in $W_{\sigma^{-r}(1)}$. Therefore, the subspace

$$\langle v_1, Bv_1, B^2v_1, \ldots, B^{m-1}v_1\rangle$$

is invariant under the action of $\langle A, B\rangle$. This is a contradiction with the irreducibility of the action and so it yields $m = n$, as wanted. Furthermore, this also shows that the blocks of B have all size 1, so that B is a monomial matrix associated to the cycle σ of order n.

Conversely, suppose that the elements of the diagonal of A are distinct and that B is a monomial matrix with associated cycle σ of order n. By assumption, we have that $\lambda_i = \lambda_{\sigma(i)}^y$. Hence, we see that $A = B^{-1}A^yB$ holds. For the irreducibility of the action of $\langle A, B\rangle$, let $\{e_1, \ldots, e_n\}$ be the canonical basis for $\mathbb{F}^n$. Then, since all the eigenvalues of A are distinct, a non-trivial A-invariant subspace of $\mathbb{F}^n$ is of the form $V = \langle e_{i_1}, \ldots, e_{i_\ell}\rangle$ for $i_1, \ldots, i_\ell \in [n]$ with $\ell \in [n]$. Since the associated cycle of B is of order n, B is V-invariant if and only if $\dim V = n$.

In order to prove the equality (3.2.1) in the statement of the lemma, let $\lambda = \lambda_1$. For $n = 1$, there is nothing to prove, so assume $n \geq 2$. Since B is a monomial matrix whose associated permutation is an n-cycle, we deduce from $A = B^{-1}A^yB$ that $\lambda \neq \lambda^y$. $\square$

Notice that with respect to the appropriate basis, we can write $A = \operatorname{diag}(\lambda, \lambda^y, \ldots, \lambda^{y^{n-1}})$ and so the associated permutation of B is the cycle $(n\, n-1 \ldots 1)$. The permutation matrix associated to B is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Remark 3.2.3. *Suppose that one of the equivalent condition of the previous lemma holds. Let $\lambda \in \mathbb{F}$ be such that $A = \operatorname{diag}(\lambda, \lambda^y, \ldots, \lambda^{y^{n-1}})$ and $\lambda^{y^n} = \lambda$. Suppose that the characteristic polynomial $f(X)$ of A is in $\mathbb{F}_q[X]$. The $\mathbb{F}[A]$-module $\mathbb{F}^n$ is cyclic, hence there* *is only one invariant factor, namely* $f = \mathrm{charpol}(A) = \mathrm{minpol}(A)$ *and* A *is conjugate to the companion matrix of* f *(which is the Frobenius normal form). If* $f \in \mathbb{F}_q[X]$*, then* A *is conjugate to an element of* $\mathrm{GL}(\mathbb{F}_{q^n})$*.*

Lemma 3.2.4. *Let* $\mathcal{D}$ *be the group of invertible diagonal matrices over* $\mathbb{F}$*, and let* $B \in \mathrm{GL}_n(\mathbb{F})$ *be a monomial matrix with associated permutation matrix* P *corresponding to a cycle* $\sigma \in \mathrm{Sym}(n)$ *of order* n*. Write* $b = \det(B)$*. Then the orbit of* B *in* $\mathrm{GL}_n(\mathbb{F})$ *under the conjugation action of* $\mathcal{D}$ *is*

$$\{\mathrm{diag}(b_1, \ldots, b_n)P \mid b_1, \ldots, b_n \in \mathbb{F}, \, b_1 \cdots b_n = b\}.$$

Proof. Let $C = \mathrm{diag}(c_1, \ldots, c_n) \in \mathcal{D}$ and $B = DP$ with $D = \mathrm{diag}(d_1, \ldots, d_n)$. Then

$$C^{-1}DPC = DC^{-1}PC = D\,\mathrm{diag}(c_1^{-1}c_{\sigma(1)}, \ldots, c_n^{-1}c_{\sigma(n)})P.$$

Now, take any $L = \mathrm{diag}(b_1, \ldots, b_n)$ with $\det(L) = b$, and define $c_1 = 1$ and $c_{\sigma^{i+1}(1)} = d_{\sigma^i(1)}^{-1} b_{\sigma^i(1)} c_{\sigma^i(1)}$ for $0 \le i \le n-2$. Then, we have $C^{-1}DPC = LP$, and since L was arbitrary with determinant b, the result follows. □

Lemma 3.2.5. *Let* $A \in \mathrm{GL}_n(\mathbb{F}_q)$*, and suppose that* A *is similar over* $\mathbb{F}$ *to a diagonal matrix* $\tilde{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathrm{GL}_n(\mathbb{F})$*, where* λ_i, λ_j *are distinct for all* $i, j \in [n]$ *with* $i \neq j$*. Then, for every* $c \in \mathbb{F}_q^*$*, there exists* $C \in C_{\mathrm{GL}_n(\mathbb{F}_q)}(A)$ *such that* $\det(C) = c$*.*

Proof. We will show that for a fixed $c \in \mathbb{F}_q^*$, there exists a polynomial $f \in \mathbb{F}_q[X]$ such that $\det(f(A)) = c$. Let $Q \in \mathrm{GL}_n(\mathbb{F})$ be such that $Q^{-1}AQ = \tilde{A}$. Then, for every polynomial $h \in \mathbb{F}_q[X]$, we have

$$\det(h(A)) = \det(Q^{-1}h(A)Q) = \det(h(\tilde{A})) = \prod_{i=1}^{n} h(\lambda_i).$$

On the other hand, let $p_A \in \mathbb{F}_q[X]$ be the characteristic polynomial of A, and write $p_A = h_1 \cdots h_m$, where $h_1, \ldots, h_m$ are irreducible polynomials over $\mathbb{F}_q$ of degree $d_1, \ldots, d_m$, respectively. For every $j \in [m]$, we fix a root λ_{r_j} of h_j, where $r_j \in [n]$. Let N_{d_j} be the norm map associated to the field extension $\mathbb{F}_{q^{d_j}} \cong \mathbb{F}_q[X]/(h_j)$ over $\mathbb{F}_q$. Then,

$$\prod_{i=1}^{n} h(\lambda_i) = \prod_{j=1}^{m} \mathrm{N}_{d_j}(h(\lambda_{r_j})).$$

Consider the isomorphism $\mathbb{F}_{q^{d_1}} \cong \mathbb{F}_q[X]/(h_1)$ and let $l \in \mathbb{F}_{q^{d_1}}^*$ be such that $\mathrm{N}_{d_1}(l) = c$. Since the elements λ_i are all distinct, the polynomials $h_1, \ldots, h_m$ are pairwise coprime. Hence, by the Chinese reminder theorem and the field isomorphism just described, the system

$$
\begin{aligned}
h &\equiv l \pmod{h_1} \\
h &\equiv 1 \pmod{h_2} \\
&\vdots \\
h &\equiv 1 \pmod{h_m}
\end{aligned}
$$

has a solution f in $\mathbb{F}_q[X]$. Observe that $f(\lambda_{r_1}) = l$ and $f(\lambda_{r_j}) = 1$ for $j = 2, \ldots, m$, which gives $\det(f(A)) = \mathrm{N}_{d_1}(l) = c$, as desired. □

With these preparations we are ready to establish our main result.

Proof of Theorem 3.1.1. Let us start with the case $x = 1$. Let ρ be an absolutely irreducible representation of dimension n of $\mathrm{BS}(1, y)$. Write A and B for $\rho(a)$ and $\rho(b)$ in $\mathrm{GL}_n(\mathbb{F}_q)$ where a and b are the generators of $\mathrm{BS}(x, y)$ with regard to the presentation (3.1.1). Then $A = B^{-1}A^yB$ and the action of $\langle A, B\rangle$ on $\mathbb{F}^n$ is irreducible. Moreover, since the characteristic polynomial of A lies in $\mathbb{F}_q[X]$, Lemma 3.2.1, Lemma 3.2.2, and Remark 3.2.3 show that A is similar in $\mathrm{GL}_n(\mathbb{F}_{q^n})$ to a matrix $\tilde{A} = \mathrm{diag}(\lambda, \lambda^y, \ldots, \lambda^{y^{n-1}})$, where $\lambda \in \mathbb{F}_{q^n}^*$. Let l be the order of λ in $\mathbb{F}_{q^n}^*$ and consequently the order of $\tilde{A}$ in $\mathrm{GL}_n(\mathbb{F}_{q^n})$. First, since $\tilde{A}$ and $\tilde{A}^y$ have the same order, it follows that

$$\gcd(l, y) = 1. \tag{C1}$$

Moreover, we have

$$q^n \equiv 1 \pmod{l}. \tag{C2}$$

We also know from Lemma 3.2.2 that $\lambda^{y^i} \neq \lambda^{y^j}$ for all $i, j \in [n]$ with $i \neq j$ and that $\lambda^{y^n} = \lambda$, which implies

$$y^n \equiv 1 \pmod{l} \tag{C3}$$

and

$$y^m \not\equiv 1 \pmod{l} \quad \text{for } m \in [n-1]. \tag{C4}$$

Since $A \in \mathrm{GL}_n(\mathbb{F}_q)$, we have $(X - \lambda)\cdots(X - \lambda^{y^{n-1}}) \in \mathbb{F}_q[X]$, and therefore

$$y^r \equiv q \pmod{l} \quad \text{for some } r \in [n] \tag{C5}$$

(compare with [65, Section 4.13]). Equations (C1)–(C5) show that l satisfies all the conditions in (3.1.1).

Let us now consider l satisfying (3.1.1) with $x = 1$. Fix an element $\lambda \in \mathbb{F}_{q^n}^*$ of order l and write $\tilde{A} = \mathrm{diag}(\lambda, \lambda^y, \ldots \lambda^{y^{n-1}})$. We will show that there are $q-1$ absolutely irreducible representations ρ of $\mathrm{BS}(1, y)$ over $\mathbb{F}_q$ such that $\rho(a)$ is similar to $\tilde{A}$ in $\mathrm{GL}_n(\mathbb{F}_{q^n})$.

Observe that the third and fourth conditions in (3.1.1) yield $\lambda^{y^i} \neq \lambda^{y^j}$ for all $i, j \in [n]$ with $i \neq j$ and $\lambda^{y^n} = \lambda$. Therefore, by Lemma 3.2.2, the matrices $\tilde{B}$ for which $\langle \tilde{A}, \tilde{B} \rangle$ acts irreducibly on $\mathbb{F}^n$ and $\tilde{A} = \tilde{B}^{-1} \tilde{A}^y \tilde{B}$ are precisely the monomial matrices over $\mathbb{F}$ associated with the permutation $(1 \dots n)$. By Lemma 3.2.4, the isomorphism classes of the associated representations over $\mathbb{F}$ only depend on the determinant of $\tilde{B}$. Moreover, we know that $\det(\tilde{B}) \in \mathbb{F}_q^*$ as the characteristic polynomial of the monomial matrix $\tilde{B}$ is $X^n - \det(\tilde{B})$. Let us show that for each element $d \in \mathbb{F}_q^*$ there exists a representation as desired such that $\det(\tilde{B}) = d$.

The last condition in (3.1.1) yields that the characteristic polynomial of $\tilde{A}$ is in $\mathbb{F}_q[X]$. Therefore, there exists a Vandermonde matrix $Q \in \mathrm{GL}_n(\mathbb{F}_{q^n})$ such that $A = Q^{-1}\tilde{A}Q \in \mathrm{GL}_n(\mathbb{F}_q)$, which is the companion matrix of the characteristic polynomial of $\tilde{A}$. Concretely, one takes

$$Q = \begin{pmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^n \\ 1 & \lambda^y & \lambda^{2y} & \cdots & \lambda^{ny} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda^{y^{n-1}} & \lambda^{2y^{n-1}} & \cdots & \lambda^{ny^{n-1}} \end{pmatrix}.$$

Moreover, because $\tilde{A}$ and $\tilde{A}^y$ are similar matrices, it follows that A and A^y are also similar, which implies that there exists $B \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $A = B^{-1}A^yB$. Write $\tilde{B} = QBQ^{-1}$ so that $\tilde{A} = \tilde{B}^{-1}\tilde{A}^y\tilde{B}$.

Now, from Lemma 3.2.5, for every $b \in \mathbb{F}_q^*$ there exists $C_b \in C_{\mathrm{GL}_n(\mathbb{F}_q)}(A)$ such that $\det(C_b) = b$. Thus, the matrices C_bB satisfy the condition $A = (C_bB)^{-1}A^y(C_bB)$, and in addition we have $\det(C_bB) = b\det(B)$. We have hence constructed $q-1$ non-isomorphic absolutely irreducible representations of $\mathrm{BS}(1, y)$ for a given λ of order l satisfying (3.1.1), as desired.

Finally, since there are $\varphi(l)$ elements λ in $\mathbb{F}_{q^n}$ of order l, and since choosing λ^{y^i} instead of λ gives an isomorphic representation, it follows that there are $(q-1)\varphi(l)/n$ absolutely irreducible representations for each l satisfying (3.1.1). These are all the isomorphism classes of representations of dimension n, since we showed at the beginning of our proof that the relation $A = B^{-1}A^yB$ and the irreducibility of the action of $\langle A, B \rangle$ on $\mathbb{F}^n$ are equivalent to the conditions (C1)–(C5) applied to the order l of A.

Now remove the assumption that $x = 1$, and let $A, B \in \mathrm{GL}_n(\mathbb{F}_q)$ be such that $A^x = B^{-1}A^yB$ and that the action of $\langle A, B \rangle$ on $\mathbb{F}^n$ is irreducible. Since A^x and A^y are similar matrices, they have the same order. Hence, $\gcd(x, l) = \gcd(y, l)$ for the order l of A. However, since by assumption x and y are coprime, it follows that the order of the matrix $A \in \mathrm{GL}_n(\mathbb{F}_q)$ is coprime to both x and y. Hence, we can find $z \in \mathbb{Z}$ such that $(A^x)^z = A$, and so the condition $A^x = B^{-1}A^yB$ is equivalent to $A = B^{-1}A^{yz}B$. Moreover, since xz is congruent to 1 modulo the order of A, which we called l. It is immediate to see that the conditions (3.1.1) for (x, y) are equivalent to the conditions

(3.1.1) for $(1, yz)$, where z is the inverse of x modulo l. We can thus reduce the problem to the case $x = 1$. $\square$

3.3 Weil representation zeta function

For a profinite group G and a finite field k, we write $r_n^{\mathrm{abs}}(G, k)$ to denote the number of absolutely irreducible representations of G of dimension n and defined over k. We say that G has *unifomly bounded exponential representation growth (over finite fields),* in short UBERG, if there exists a constant $c > 0$ such that $r_n^{\mathrm{abs}}(G, k) \leqslant |k|^{cn}$ for every finite field k. Corob Cook, Kionke, and Vannacci [25] investigated the structure of UBERG groups. In fact, already in 2018, Kionke and Vannacci [71] showed that there is a strict correlation between the study of UBERG groups and probabilistic generation properties of profinite groups. This is based on the fact that a profinite group G equipped with its normalized Haar measure provides a probability space. A profinite group is *positively finitely generated*(PFG) if for some positive integer n the probability $P(G, n)$ that n random elements generate G is positive. A finitely generated profinite group is *positively finitely related* (PFR) if every continuous epimorphism $f : H \to G$ from every finitely generated profinite group H has its kernel $\ker(f)$ positively finitely normally generated in H, i.e. if there exists some positive integer n such that n random elements and their H-conjugates generate $\ker(f)$ with positive probability.

Kionke and Vannacci showed that a finitely presented profinite group is PFR exactly if it has UBERG or equivalently, if the completed group algebra $\hat{\mathbb{Z}}[\![G]\!]$ is *positively finitely generated* (PFG) as a $\hat{\mathbb{Z}}[\![G]\!]$-module, cf. [71, Theorem A and Proposition 6.1].

Moreover, in 2024, Corob Cook, Kionke, and Vannacci [26] introduced the following representation zeta function of an UBERG group G,

$$\zeta_G^{\mathrm{W}}(s) = \exp\left(\sum_{p\in\mathbb{P}}\sum_{n\geq 1}\sum_{j\geq 1}\frac{r_n^{\mathrm{abs}}(G, \mathbb{F}_{p^j})}{j}\cdot p^{-snj}\cdot\frac{p^{nj}-1}{p^j-1}\right).$$

Upon expansion of the exponential series, this is a formal Dirichlet series. It admits an obvious Euler product decomposition

$$\zeta_G^{\mathrm{W}}(s) = \prod_{p\in\mathbb{P}}\exp\left(\sum_{n\geq 1}\sum_{j\geq 1}\frac{r_n^{\mathrm{abs}}(G, \mathbb{F}_{p^j})}{j}\cdot p^{-snj}\cdot\frac{p^{nj}-1}{p^j-1}\right),$$

and it converges on some complex half-plane, see [26, Corollary 2.3]. The formula for $\zeta_G^{\mathrm{W}}(s)$ resembles the Weil zeta function of an algebraic variety V, where the absolutely irreducible $\mathbb{F}_q$-representations of G take the place of the $\mathbb{F}_q$-rational points of V. For this reason, $\zeta_G^{\mathrm{W}}(s)$ is called the *Weil representation zeta function* of G. For more details, see Section 0.7.4. The factor $\frac{p^{nj}-1}{p^j-1}$ that appears in the definition of $\zeta_G^{\mathrm{W}}(s)$ is the cardinality of $\mathbb{P}^{n-1}(\mathbb{F}_{p^j})$ and it appears in the Weil zeta function because for every absolutely irreducible $\mathbb{F}_q$-representation over the vector space V of dimension n, there are $\mathbb{P}^{n-1}(\mathbb{F}_q)$ maximal ideals M in $\mathbb{F}_q[\![G]\!]$ such that $\mathbb{F}_q[\![G]\!]/M \cong V$, as noted by Corob Cook, Kionke, and Vannacci [26].

As previously discussed, there is a strong correlation between the study of UBERG groups and probabilistic generation properties of profinite groups. This correlation has been demonstrated with the Weil zeta function by Corob Cook, Kionke, and Vannacci [26]. They showed that for an UBERG profinite group G and $R = \hat{\mathbb{Z}}[\![G]\!]$ its group ring over $\hat{\mathbb{Z}}$, the following equality holds for sufficiently large integers l:

$$P_R(R, l)^{-1} = \zeta_G^{\mathrm{W}}(l),$$

where $P_R(R, l)$ is the probability that l random elements generate the group ring $R = \hat{\mathbb{Z}}[\![G]\!]$, see [26, Theorem A].

If G is an abstract group, every finite dimensional representation of G over a finite field $\mathbb{F}_q$ factors through a finite quotient. Hence, for a group G whose profinite completion is UBERG, we define $\zeta_G^{\mathrm{W}}(s) = \zeta_{\widehat{G}}^{\mathrm{W}}(s)$.

Example 3.3.1. We compute the Weil representation zeta function for the metabelian group $\mathrm{BS}(1, -1) \cong \mathbb{Z} \rtimes \mathbb{Z}$. Let us first suppose that p is odd and consider $q = p^j$ with $j \in \mathbb{N}_*$. It is a basic fact that absolutely irreducible representations of abelian groups are one dimensional. Consider the presentation

$$\mathrm{BS}(1, -1) = \langle a, b \mid a = b^{-1}a^{-1}b \rangle. \tag{3.3.1}$$

The commutator of the generators b and a is $[b, a] = a^2$ and $[a, b] = a^{-2}$, hence one can see that the abelianisation of $\mathrm{BS}(1, -1)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. To shorten our notation let $G = \mathrm{BS}(1, -1)$. All the absolutely irreducible representations of the abelianisation $G/[G, G]$ are one dimensional and correspond to the irreducible representations of dimension 1 of G, cf. Section 0.4. Since we consider irreducible $\mathbb{F}_q$-representations, we have $2(q - 1)$ non-isomorphic irreducible representations of $G/[G, G]$ of dimension 1. We are left with representations which do not contain $[G, G]$ in their kernel. One could refine the argument given by Corob Cook et. all [26, Example A.5] by considering the normal abelian subgroup $\langle a, b^2 \rangle$ of $\mathrm{BS}(1, -1)$. The counting here needs to account for the field of definition of the representations. Since this approach does not significantly simplify our proof, we will present the results directly by applying our Theorem 3.1.1.

- For every $n \geq 3$ we have $r_n^{\mathrm{abs}}(\mathrm{BS}(1, -1), \mathbb{F}_q) = 0$, from the fourth condition of (3.1.1).
- For $n = 1$ the conditions (3.1.1) reduce to

$$\begin{cases} q \equiv 1 \pmod{l}, \\ 2 \equiv 0 \pmod{l}. \end{cases}$$

Hence, the only possibilities are with $l = 1, 2$. Then

$$r_1^{\text{abs}}(\mathrm{BS}(1,-1), \mathbb{F}_q) = (q-1)(\varphi(1) + \varphi(2)) = 2(q-1).$$

- For $n = 2$ the conditions (3.1.1) reduce to

$$\begin{cases} q^2 \equiv 1 & (\text{mod } l), \\ 1 \not\equiv -1 & (\text{mod } l), \\ 1 \equiv (-1)^r q & (\text{mod } l), \quad \text{for some } r \in [2]. \end{cases}$$

  Since $(q-1, q+1) = 2$, then

$$\begin{aligned} r_2^{\text{abs}}(\mathrm{BS}(1,-1), \mathbb{F}_q) &= \frac{q-1}{2}\left(\sum_{\substack{l \mid q-1 \\ l \neq 1,2}} \varphi(l) + \sum_{\substack{l \mid q+1 \\ l \neq 1,2}} \varphi(l)\right) \\ &= \frac{q-1}{2}\left(q - 1 - \varphi(1) - \varphi(2) + q + 1 - \varphi(1) - \varphi(2)\right) \\ &= \frac{q-1}{2} 2(q-2) \\ &= q^2 - 3q + 2. \end{aligned}$$

Let us now suppose that $p = 2$. We have a slightly different counting as before since for example the abelianisation of $\mathrm{BS}(1,-1)$ is the product of a cyclic group of order 2 and the infinite cyclic group $\mathbb{Z}$ and so we have $2^j - 1$ one dimensional representations over the field $\mathbb{F}_{2^j}$. We report the counting based on our Theorem 3.1.1. Then we have the following.

- For every $n \geq 3$ and $j \geq 1$ one has $r_n^{\text{abs}}(\mathrm{BS}(1,-1), 2^j) = 0$.
- For $n = 1$ the conditions (3.1.1) reduce to

$$\begin{cases} 2^j \equiv 1 & (\text{mod } l), \\ 2 \equiv 0 & (\text{mod } l). \end{cases}$$

  So the only possibility is $l = 1$. Then

$$r_1^{\text{abs}}(\mathrm{BS}(1,-1), \mathbb{F}_{2^j}) = (2^j - 1)\varphi(1) = (2^j - 1).$$

- For $n = 2$ the conditions (3.1.1) reduce to

$$\begin{cases} 2^{2j} \equiv 1 & (\text{mod } l), \\ 1 \not\equiv -1 & (\text{mod } l), \\ 1 \equiv (-1)^r 2^j & (\text{mod } l), \quad \text{for some } r \in [2]. \end{cases}$$

Since $(2^j-1, 2^j+1) = 1$, one has

$$\begin{aligned}
r_2^{\mathrm{abs}}(\mathrm{BS}(1,-1), \mathbb{F}_{2^j}) &= \frac{2^j-1}{2}\left(\sum_{\substack{l \mid 2^j-1 \\ l\neq 1}} \varphi(l) + \sum_{\substack{l \mid 2^j+1 \\ l\neq 1}} \varphi(l)\right) \\
&= \frac{2^j-1}{2}\left(2^j - 1 - \varphi(1) + 2^j + 1 - \varphi(1)\right) \\
&= (2^j-1)(2^j-1).
\end{aligned}$$

We can now compute the Weil representation zeta function with all the information that we collected so far.

$$\begin{aligned}
\zeta^{\mathrm{W}}_{\mathrm{BS}(1,-1)}(s) =& \prod_{p\in\mathbb{P}} \exp\left(\sum_{n\geq 1}\sum_{j\geq 1} \frac{r_n^{\mathrm{abs}}(\mathrm{BS}(1,-1), \mathbb{F}_{p^j})}{j} \cdot p^{-snj} \cdot \frac{p^{nj}-1}{p^j-1}\right) \\
=& \exp\left(\sum_{j\geq 1} \frac{(2^j-1)}{j}\cdot 2^{-sj} + \sum_{j\geq 1} \frac{(2^{2j}-1)(2^j-1)}{j} \cdot 2^{-2sj}\right) \cdot \\
& \cdot \prod_{p \text{ odd}} \exp\left(\sum_{j\geq 1} \frac{2(p^j-1)}{j}\cdot p^{-sj} + \sum_{j\geq 1} \frac{(p^{2j}-1)(p^j-2)}{j}\cdot p^{-2sj}\right) \\
=& \exp\left(\sum_{j\geq 1}\left(\left(\frac{2^{(1-s)j}}{j} + 2\cdot \sum_{p \text{ odd}} \frac{p^{(1-s)j}}{j}\right) - \left(\frac{2^{(-s)j}}{j} + 2\cdot\sum_{p \text{ odd}} \frac{p^{(-s)j}}{j}\right)\right)\right) \\
& \cdot \exp\left(\sum_{j\geq 1}\left(\frac{2^{(-2s)j}}{j} + 2\cdot \sum_{p \text{ odd}} \frac{p^{(-2s)j}}{j}\right)\right) \\
& \cdot \exp\left(-\sum_{j\geq 1}\left(\frac{2^{(1-2s)j}}{j} + \sum_{p \text{ odd}} \frac{p^{(1-2s)j}}{j}\right)\right) \\
& \cdot \exp\left(\sum_{j\geq 1}\left(\frac{2^{(3-2s)j}}{j} + \sum_{p \text{ odd}} \frac{p^{(3-2s)j}}{j}\right)\right) \\
& \cdot \exp\left(-\sum_{j\geq 1}\left(\frac{2^{(2-2s)j}}{j} + 2\cdot \sum_{p \text{ odd}} \frac{p^{(2-2s)j}}{j}\right)\right)
\end{aligned}$$

Note that we can use the logarithmic expansion given by

$$\log(1-x) = -\sum_{i\geqslant 1} \frac{x^i}{i},$$

to get rid of the exponential function. Thus

$$\zeta^{\mathrm{W}}_{\mathrm{BS}(1,-1)}(s) = \left(1-2^{(1-s)}\right)^{-1}\left(\prod_{p\text{ odd}}\left(1-p^{(1-s)}\right)^{-2}\right)\left(1-2^{-s}\right)\left(\prod_{p\text{ odd}}\left(1-p^{-s}\right)^{2}\right)$$
$$\cdot\left(1-2^{-2s}\right)^{-1}\left(\prod_{p\text{ odd}}\left(1-p^{-2s}\right)^{-2}\right)\left(\prod_{p\in\mathbb{P}}\left(1-p^{(1-2s)}\right)\right)$$
$$\cdot\left(\prod_{p\in\mathbb{P}}\left(1-p^{(3-2s)}\right)^{-1}\right)\left(1-2^{(2-2s)}\right)\left(\prod_{p\text{ odd}}\left(1-p^{(2-2s)}\right)^{2}\right).$$

If we multiply $\zeta^{w}_{\mathrm{BS}(1,-1)}(s)$ with a rational function in 2^{-s}, we find an equality that involves the Riemann zeta function, using the Euler product decomposition (0.7.1).

$$\zeta^{\mathrm{W}}_{\mathrm{BS}(1,-1)}(s)\cdot\frac{(1-2^{-s})}{(1-2^{(1-s)})}\cdot\frac{(1-2^{(2-2s)})}{(1-2^{-2s})} = \frac{\zeta(s-1)^2\zeta(2s)^2\zeta(2s-3)}{\zeta(s)^2\zeta(2s-2)^2\zeta(2s-1)}.$$

To express this phenomenon, Corob Cook, Kionke, and Vannacci define the following relation. Given two meromorphic functions f, g on $\mathbb{C}$ we write $f \sim_n g$ if there is a rational function h in $\{p^{-s} \mid p \leqslant n, p \text{ prime}\}$ such that $fh = g$. Hence, our results agree with the computations of Corob Cook, Kionke, and Vannacci in [26, Example A.8].

Example 3.3.2. We want to apply our Theorem 3.1.1 for the case $x = 1$ and $y = p$, i.e. $\mathrm{BS}(1, p)$, where p is a prime. In particular, we focus on the case when the finite field has cardinality p^j where $j \in \mathbb{N}_*$, i.e. $r^{\mathrm{abs}}_n(\mathrm{BS}(1,p), \mathbb{F}_{p^j})$. The conditions (3.1.1) in this case are

$$\begin{cases} \gcd(l,p) = 1, & \\ p^{nj} \equiv 1 \pmod{l}, & \\ p^{n} \equiv 1 \pmod{l}, & \\ p^{m} \not\equiv 1 \pmod{l}, & \text{for every } m \in [n-1], \\ p^{r} \equiv p^{j} \pmod{l}, & \text{for some } r \in [n]. \end{cases}$$

We distinguish two cases depending on whether j is greater or smaller than n.

(i) If $j \leq n$, the last condition has solutions when $r = j$.
(ii) If $j > n$, we write $j = an + b$ where $a \geqslant 0$ and $n > b \geqslant 0$. Then the condition $p^r \equiv p^j \pmod{l}$ is satisfied for $r = b$ if $b \neq 0$ or for $r = n$ if $b = 0$.

Then we have

$$r^{\mathrm{abs}}_n(\mathrm{BS}(1,p), \mathbb{F}_{p^j}) = \frac{p^j - 1}{n}\sum_{t\mid n}\sum_{l\mid p^t-1}\mu(n/t)\varphi(l)$$

$$= \frac{p^j - 1}{n} \sum_{t|n} \mu(n/t)(p^t - 1).$$

In order to compute the Weil representation zeta function $\zeta^{\mathrm{W}}_{\mathrm{BS}(1,p)}(s)$, we should compute $r_n^{\mathrm{abs}}(\mathrm{BS}(1,p), \mathbb{F}_q)$ for q a power of a prime different form p. Already this task seems quite ambitions due to the complexity of the set of solutions of the conditions (3.1.1).

Final considerations

One of the most intriguing examples would be computing the Weil representation zeta function for the group $\mathrm{BS}(2,3)$, which is non-Hopfian. However, such computations are currently beyond our reach. We leave this for future research.

3.4 Relation with subgroup growth

Another intriguing area of research is exploring the potential relationship between representation growth and subgroup growth. This question is inspired by the work of Mozgovoy and Reineke [97], who linked the counting polynomials of absolutely irreducible n-dimensional representations over finite fields $\mathbb{F}_q$ with the number of subgroups of index n in the free groups [97, Lemma 6.4]. They, in turn, were motivated by the concepts of $\mathbb{F}_1$-geometry [81], which interprets the symmetric group $\mathrm{Sym}(n)$ as the group GL_n over a hypothetical field with one element. Therefore,it may be possible to establish a connection between these two types of counting by considering the limit process $q \to 1$.

Baumslag-Solitar groups serve as natural examples of one-relator groups. Although they are not free groups, they are closely related because they are defined by only a single relation.

Let G be a finitely generated groups and let $a_n(G)$ denote the number of subgroups of G of index n. Gelman [39] proved that for x and y non-zero coprime integers, the number of n-index subgroups of a Baumslag-Solitar group $\mathrm{BS}(x,y)$ is

$$a_n(\mathrm{BS}(x,y)) = \sum_{\substack{l|n \\ \gcd(l,xy)=1}} l.$$

The explicit count of subgroups of finite index for a Baumslag-Solitar group $\mathrm{BS}(x,y)$ and our formula for the number of n-dimensional absolutely irreducible representations over finite fields with q elements suggest a potential connection between these two quantities. However, establishing this connection is more challenging than it appears, raising the question of whether such a link is even possible.

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Ich versichere an Eides Statt, dass die Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der “Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist.

Margherita Piccolo
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