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Absolutely continuous edge spectrum of topological insulators with an odd time-reversal symmetry

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Abstract
We show that non-trivial two-dimensional topological insulators protected by an odd time-reversal symmetry have absolutely continuous edge spectrum. To accomplish this, we establish a time-reversal symmetric version of the Wold decomposition that singles out extended edge modes of the topological insulator.

Keywords Time-reversal symmetry · Topological insulator · Absolutely continuous edge spectrum · Wold-von-Neumann decomposition

Mathematics Subject Classification 47B93 · 47A53 · 81Q10 · 81S99

1 Introduction

Hall insulators support extended chiral edge modes [17]. In a free electron description, these edge modes are associated with absolutely continuous spectrum filling the bulk gap of the single particle Hamiltonian. The presence of absolutely continuous spectrum has been proven for the Landau Hamiltonian with weak disorder and a steep edge potential or appropriate half-plane boundary conditions using Mourre estimates [9, 12, 15, 18, 24], and recently for Hall insulators on the lattice using index theory [8].

The question naturally arises whether such extended modes are also present in topological insulators that are protected by an odd time-reversal symmetry, and for which the Hall conductance vanishes. Because of the time-reversal symmetry any left moving edge mode has a companion right moving edge mode so Mourre estimates, which apply only when the edge modes are strictly chiral, cannot be applied in a straighforward way to answer this question. In this note, we use index theory to show

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that absolutely continuous edge spectrum filling the bulk gap follows from a non-trivial $\mathbb{Z}_2$-valued bulk index.

We appeal to the bulk-edge correspondence for time-reversal invariant topological insulators [2, 7, 14, 16] which links the bulk index to an edge index associated with a time-reversal symmetric unitary acting on the edge modes, the "edge unitary," which is a smooth function of the edge Hamiltonian. Inspired by [4], we prove a symmetric Wold decomposition for such unitaries which, when applied to the edge unitary, identifies counter propagating edge channels which are localized near the edge. This implies in particular that the absolutely continuous spectrum of this unitary covers the whole unit circle if the edge index is non-trivial. The Hamiltonian describing the system with edge is then shown to inherit this absolutely continuous spectrum from the edge unitary.

The symmetric Wold decomposition is the crucial technical novelty presented in this paper. This decomposition applies to any unitary U and projection P such that $U^*PU - P$ is compact, and $\tau U \tau^* = U^*$ while $\tau P \tau^* = P$ for an anti-unitary τ with $\tau^2 = -1$. To any such pair, we can assign a $\mathbb{Z}_2$-valued index which counts the parity of Kramers pairs moved by U into/out of the range of P. If this $\mathbb{Z}_2$-valued index is non-trivial, then we can find a unitary W which is a compact perturbation of U, and the symmetric Wold decomposition identifies a shift and counter shift in W, i.e. one can decompose $W \simeq S \oplus S^* \oplus W_{\mathrm{triv}}$ where S is the bilateral shift and W_{triv} is a unitary that leaves the range of P invariant. If $U^*PU - P$ is actually trace-class then we can take W to be a trace-class perturbation of U. Since W contains a shift its absolutely continuous spectrum covers the whole unit circle, and since absolutely continuous spectrum is stable under trace-class perturbations [20, 25], the same is true for U.

2 Setup and results

2.1 Edge spectrum of time-reversal symmetric topological insulators

We consider free electrons moving on the lattice $\mathbb{Z}^2$ modeled by a bulk Hamiltonian H on $\ell^2(\mathbb{Z}^2, \mathbb{C}^n)$ that is exponentially local in the sense that there are constants $C < \infty$, $\xi > 0$ such that

$$\|P_{\vec{x}} H P_{\vec{y}}\| \le C\mathrm{e}^{-\|\vec{x}-\vec{y}\|/\xi} \tag{2.1}$$

for all $\vec{x}, \vec{y} \in \mathbb{Z}^2$, where $P_{\vec{x}}$ denotes the projection onto the site at $\vec{x}$.

Moreover, we take H to be invariant under an **odd time-reversal symmetry**, i.e. there is an anti-unitary operator τ with $\tau^2 = -\mathbb{1}$ such that $\tau H \tau^* = H$. The time-reversal symmetry is further assumed to act "on-site" meaning that $\tau X_i \tau^* = X_i$ for $i = 1, 2$ where X_i denotes the position operator in the i-direction. We further assume that H has a **bulk gap**, i.e. that for some open interval $\Delta \subset \mathbb{R}$

$$\Delta \cap \sigma(H) = \emptyset.$$

For any $\mu \in \Delta$ we denote the Fermi projection by $P_F = \chi_{\le\mu}(H)$.

We define a unitary that models the insertion of a unit of magnetic flux at the origin in the bulk of the system:

$$U_B := \mathrm{e}^{\mathrm{i} \arg \vec{X}} \tag{2.2}$$

with $\vec{X} = (X_1, X_2)$ the vector of position operators and $\arg(x_1, x_2) = \arctan(x_2/x_1)$. It follows from the discussion around Lemma 1 in [1] that the difference $A_B := U_B P_F U_B^* - P_F$ is compact, so we can define a $\mathbb{Z}_2$-valued **bulk index**

$$\operatorname{ind}_2^B(H, \Delta) := \dim \ker(A_B - \mathbb{1}) \mod 2. \tag{2.3}$$

This bulk index for odd time-reversal invariant topological insulators was first defined in an equivalent form in [26], the form given here first appeared in [21]. It is a non-commutative extension of the Kane-Mele invariant [19] and related to the index of a pair of projections introduced in [6]. Note that this index only depends on the gap Δ but not on the choice of $\mu \in \Delta$.

With the bulk Hamiltonian H, we associate a half-space Hamiltonian $\hat{H}$ on the half-space $\ell^2(\mathbb{Z} \times \mathbb{N}, \mathbb{C}^n)$ that is also exponentially local and agrees with the bulk Hamiltonian in the bulk in the sense that

$$\| P_{\vec{x}} (\iota^\dagger H \iota - \hat{H}) P_{\vec{y}} \| \leq C \mathrm{e}^{-\|\vec{x}-\vec{y}\|/\xi - y_2/\xi'} \tag{2.4}$$

for some $C < \infty, \xi, \xi' > 0$ and all $\vec{x}, \vec{y} \in \mathbb{Z} \times \mathbb{N}$. Here $\iota : \ell^2(\mathbb{Z} \times \mathbb{N}, \mathbb{C}^n) \to \ell^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^n)$ denotes the injection that is induced by the natural inclusion of the half-space lattice $\mathbb{Z} \times \mathbb{N}$ in the bulk lattice $\mathbb{Z} \times \mathbb{Z}$. Since the time-reversal symmetry τ acts on-site, it naturally restricts to the half-space and we denote this restriction also by τ. We assume that the half-space Hamiltonian is also time-reversal invariant, i.e. $\tau \hat{H} \tau^* = \hat{H}$.

The main result of the paper is the following:

Theorem 2.1 *If* $\operatorname{ind}_2^B(H, \Delta) = 1$ *then*

$$\Delta \subset \sigma_{ac}(\hat{H}), \tag{2.5}$$

i.e. the half-space Hamiltonian $\hat{H}$ *has absolutely continuous spectrum everywhere in the bulk gap.*

Remark 2.2 We recall that the classification of gapped free-fermion systems that are constrained by on-site unitary, time-reversal, particle-hole and/or chiral symmetries reduces to the classification of gapped free-fermion systems belonging to one of the symmetry classes of the tenfold way [3]. The classification is summarized in the periodic table of topological insulators and superconductors of [23] of which we reproduce the entries for two-dimensional systems in Table 1.

Note that all symmetry classes with a potentially non-trivial $\mathbb{Z}_2$-valued index in two dimensions have an odd time-reversal symmetry, and have their $\mathbb{Z}_2$-index given by the same bulk index (2.3), see Theorem 2 of [22]. It follows that our Theorem 2.1 applies to all these classes.

Table 1 The symmetry classes S of the tenfold way for two-dimensional topological insulators and superconductors

S	A	D	C	AI	AII	AIII	BDI	CI	CII	DIII
τ^2				$\mathbb{1}$	$-\mathbb{1}$		$\mathbb{1}$	$\mathbb{1}$	$-\mathbb{1}$	$-\mathbb{1}$
$\mathbf{I}(\mathsf{S})$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}_2$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}_2$

The first row indicates the presence of a time-reversal symmetry, and whether this time-reversal symmetry squares to $\mathbb{1}$ or to $-\mathbb{1}$. Our main theorem applies to those classes for which $\tau^2 = -\mathbb{1}$, i.e. classes with an odd time-reversal. The second row gives the index group $\mathbf{I}(\mathsf{S})$ that classifies the gapped phases within each symmetry class. Red: the symmetry classes with an odd time-reversal symmetry for which we show absolutely continuous edge spectrum for non-trivial $\mathbb{Z}_2$-index. Blue: symmetry classes with a potentially non-trivial Chern invariant for which absolutely continuous edge spectrum was proved in [8]. Note that while class CII has a well-defined $\mathbb{Z}_2$-index due to the presence of odd time-reversal symmetry, it always vanishes due to the other symmetries in the class

Similarly, for all symmetry classes with a $\mathbb{Z}$-valued index in two dimensions the index is given by the bulk Chern number, see [22, Theorem 1]. For all those symmetry classes absolute continuity of the edge spectrum follows from [8]. Theorem 2.1 above together with [8] therefore covers all non-trivial cases of two-dimensional free-fermion topological insulators and superconductors, see Table 1.

2.2 Time-reversal symmetric Wold decomposition

In order to prove Theorem 2.1 we will construct in Sect. 3.1 an edge unitary U_E that acts on the edge modes of the half-space Hamiltonian $\hat{H}$. The main theorem will then follow from an application of the following time-reversal symmetric extension of Theorem 2.1 of [4] to the edge unitary:

Theorem 2.3 *Let U be a unitary and P a projection acting on a separable Hilbert space $\mathcal{H}$ such that $A = UPU^* - P$ is compact. Moreover, assume that $\tau U \tau^* = U^*$ and $\tau P \tau^* = P$ for an odd time-reversal symmetry τ. Then there exists a unitary W with $U - W$ compact and $\tau W \tau^* = W^*$ such that:*

- *If* $\dim\ker(A - \mathbb{1}) \mod 2 = 0$, *then* $[W, P] = 0$.
- *If* $\dim\ker(A - \mathbb{1}) \mod 2 = 1$, *then there is a decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ such that the projection P decomposes as $P = P' \oplus P''$ and the unitary W decomposes as $W = S \oplus W_{\mathrm{triv}}$ where $[W_{\mathrm{triv}}, P''] = 0$ and the unitary S which we define in Eq.* (2.7) *consists of two opposite shift operators and therefore has absolutely continuous spectrum covering the whole unit circle.*

Moreover, if A is Schatten-p, then so is $U - W$.

Since the absolutely continuous spectrum of an operator is stable under trace-class perturbations, Theorem 2.3 immediately implies

Corollary 2.4 *Let U be a unitary and P a projection such that $A = UPU^* - P$ is trace-class. Moreover, assume that $\tau U \tau^* = U^*$ and $\tau P \tau^* = P$ for an odd time-*

reversal symmetry τ. Then, if $\dim\ker(A - \mathbb{1}) \mod 2 = 1$ *the absolutely continuous spectrum of U covers the whole unit circle.*

The unitary S appearing in the statement of Theorem 2.3 is defined as follows: Consider the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ with orthonormal basis $\{|x, \pm\rangle\}_{x\in\mathbb{Z}}$ labeled by position and σ_z-eigenstates. On this Hilbert space, consider the odd time-reversal symmetry

$$\tau = \bigoplus_{x\in\mathbb{Z}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} K, \tag{2.6}$$

where K is complex conjugation with respect to the basis $\{|x, \pm\rangle\}_{x\in\mathbb{Z}}$. Then S acts as the right shift on the spin-up sector and the left shift on the spin-down sector, i.e.

$$S|x, \pm\rangle = |x \pm 1, \pm\rangle \tag{2.7}$$

for all $x \in \mathbb{Z}$. Since S contains two copies of the shift operator, it has absolutely continuous spectrum covering the whole unit circle. Moreover, it is straightforward to verify that the τ above is an odd time-reversal symmetry for S, i.e., that $\tau S\tau^* = S^*$.

3 Proof of absolute continuity of edge spectrum

3.1 Edge index and bulk-edge correspondence

In the proof of Theorem 2.1, we will also appeal to the bulk-boundary correspondence, which we briefly recall here.

Let $g : \mathbb{R} \to [0, 1]$ be a smooth non-increasing function interpolating from 1 to 0 such that its derivative is supported in the bulk gap Δ. We have

$$P_F = g(H). \tag{3.1}$$

Consider now the edge unitary $U_E := W_g(\hat{H})$ where W_g is the function

$$W_g : \mathbb{R} \to \mathbb{C} : x \mapsto \mathrm{e}^{2\pi \mathrm{i} g(x)}. \tag{3.2}$$

This unitary is local and supported near the edge of the half-space. We denote by $\hat{\Pi}_1$ the projection on the upper right quadrant $\{\vec{x} \in \mathbb{Z} \times \mathbb{N} \mid x_1 \geq 0\}$, then

Lemma 3.1 *The commutator $[U_E, \hat{\Pi}_1]$ is trace class.*

This lemma follows immediately from Lemmas A.2. and A.3. in [13]. It follows that $A_E := U_E\hat{\Pi}_1 U_E^* - \hat{\Pi}_1 = [U_E, \hat{\Pi}_1]U_E^*$ is also trace class and in particular compact so the index

$$\operatorname{ind}_2^E(H, \Delta) := \dim\ker(A_E - \mathbb{1}) \mod 2 \tag{3.3}$$

is well defined. This **edge index** was first defined in [14].

By bulk-boundary correspondence, the bulk and edge indices are equal:

Theorem 3.2 (Theorem 2.11 of [14]) *Under the above assumptions on H we have*

$$\operatorname{ind}_2^B(H,\Delta)=\operatorname{ind}_2^E(H,\Delta). \tag{3.4}$$

Remark 3.3
- The edge index may a priori depend on the boundary conditions defining the half-space Hamiltonian $\hat{H}$. The bulk-edge correspondence, Theorem 3.2, implies that this is not the case, justifying our notation $\operatorname{ind}_2^E(H,\Delta)$.
- In [14], the bulk and edge indices are given as (odd) Fredholm indices [5, 26]. If $[U,P]$ is compact then $F = PUP + P^\perp$ is a Fredholm operator and its odd Fredholm index is $\dim\ker F \mod 2$. The equivalence to our $\dim\ker(A-\mathbb{1}) \mod 2$ with $A=UPU^*-P$ is easily established by noting that $\psi\in\ker F$ if and only if $U\psi\in\ker(A-\mathbb{1})$.

3.2 Proof of theorem 2.1

The following is a verbatim copy of the corresponding proof in [8], except for the remark that now $\sigma_{ac}(U_E)=\mathbb{S}^1$ follows, through Corollary 2.4, from an odd time-reversal symmetry and a non-trivial $\mathbb{Z}_2$-valued edge index.

By Theorem 3.2, $\operatorname{ind}_2^B(H,\Delta)=1$ implies $\operatorname{ind}_2^E(H,\Delta)=\dim\ker(U_E\hat{\Pi}_1U_E^*-\hat{\Pi}_1-\mathbb{1}) \mod 2=1$. Since τ acts locally, $\tau\hat{\Pi}_1\tau^*=\hat{\Pi}_1$, and from $\tau\hat{H}\tau^*=\hat{H}$ we get $\tau U_E\tau^*=\tau W_g(\hat{H})\tau^*=\overline{W}_g(\hat{H})=U_E^*$. Moreover, $U_E\hat{\Pi}_1U_E^*-\hat{\Pi}_1$ is trace-class by Lemma 3.1. It therefore follows from Corollary 2.4 that the absolutely continuous spectrum of the edge unitary U_E is the whole unit circle.

We can choose g in such a way that $x\mapsto W_g(x):=e^{2\pi i g(x)}$ is a smooth function that satisfies $\Delta=W_g^{-1}\big(\mathbb{S}^1\backslash\{1\}\big)$ and such that W_g is invertible on Δ.

Now, let $P_\Delta=\chi_\Delta(\hat{H})$ be the spectral projection of $\hat{H}$ on the interval Δ. Since the absolutely continuous spectrum of $W_g(\hat{H})$ covers the whole unit circle and W_g differs from 1 only on Δ, there is an absolutely continuous spectral measure μ_{ac} of $W_g(P_\Delta\hat{H}P_\Delta)$ that is supported on the whole unit circle. By spectral mapping (Proposition 8.12 of [11]), we have that $\mu_{\mathrm{ac}}\circ W_g$ is a spectral measure for $(W_g)|_\Delta^{-1}(W_g(P_\Delta\hat{H}P_\Delta))=P_\Delta\hat{H}P_\Delta$. The function W_g is smooth and maps the interval Δ into the unit circle, so we see that $\mu_{\mathrm{ac}}\circ W_g$ is an absolutely continuous measure supported on the entire closed interval $\overline{\Delta}$. (Supports of measures are closed sets.) This means that $\sigma_{\mathrm{ac}}(P_\Delta\hat{H}P_\Delta)=\overline{\Delta}$, and hence, $\Delta\subset\sigma_{\mathrm{ac}}(\hat{H})$ as required. □

4 Proof of the symmetric Wold decomposition

In this section, we prove Theorem 2.3. We fix a unitary U and a projection P such that $A=UPU^*-P$ is compact. Moreover, we require $\tau U\tau^*=U^*$ and $\tau P\tau^*=P$ for some odd time-reversal symmetry τ. We further write $Q=UPU^*$, and for $I\subset\mathbb{R}$ we denote by E_I the range of the spectral projection $\chi_I(A)$ of A. We also write $E_\lambda=E_{\{\lambda\}}$ for the λ-eigenspace of A.

4.1 Spectral symmetry and Kramers degeneracy

Following [6], we introduce the operator $B = \mathbb{1} - P - Q$. One easily checks that

$$A^2 + B^2 = \mathbb{1}, \quad \text{and} \quad AB + BA = 0. \tag{4.1}$$

Lemma 4.1 *For $\lambda \notin \{-1, 0, +1\}$ the operator B maps E_λ isomorphically to $E_{-\lambda}$. In particular,* $\dim E_\lambda = \dim E_{-\lambda}$.

Proof Let $A\phi = \lambda\phi$ for $\lambda \notin \{-1, 0, +1\}$. Since $AB = -BA$ we have

$$AB\phi = -BA\phi = -\lambda\phi \tag{4.2}$$

and since $B^2 = \mathbb{1} - A^2$ we have that $B^2\phi = (1 - \lambda^2)\phi \neq 0$, because $\lambda \notin \{-1, +1\}$. Thus $B\phi \neq 0$, and B maps E_λ injectively into $E_{-\lambda}$ and vice versa. Since A is compact, E_λ is finite dimensional and it follows that B maps E_λ isomorphically to $E_{-\lambda}$ □

Define $\tilde{\tau} = U\tau$ which satisfies $\tilde{\tau}^2 = U\tau U\tau = UU^*\tau^2 = -\mathbb{1}$. From straightforward calculations we get

Lemma 4.2 *We have*

$$\tilde{\tau} P \tilde{\tau}^* = Q, \quad \tilde{\tau} Q \tilde{\tau}^* = P. \tag{4.3}$$

It follows that $\tilde{\tau} B \tilde{\tau}^* = B$ and $\tilde{\tau} A \tilde{\tau}^* = -A$, hence $\tilde{\tau} E_\lambda = E_{-\lambda}$ for all eigenvalues λ of A.

Lemma 4.3 *For $\lambda \notin \{-1, 0, 1\}$, the spaces E_λ have even dimension.*

Proof From Lemma 4.1 we have that B is a linear isomorphism from E_λ to $E_{-\lambda}$ so $(B^*B)^{-1/2}B$ is a unitary from E_λ to $E_{-\lambda}$. By Lemma 4.2, $\tilde{\tau}$ maps $E_{-\lambda}$ to E_λ so the anti-unitary $\theta = (B^*B)^{-1/2}B\tilde{\tau}$ maps E_λ to itself and is odd:

$$\theta^2 = \left((B^*B)^{-1/2}B\tilde{\tau}\right)^2 = (B^*B)^{-1}B^2\tilde{\tau}^2 = -(B^*B)^{-1}B^*B = -\mathbb{1} \tag{4.4}$$

where we used $\tilde{\tau}^2 = -\mathbb{1}$ and $B\tilde{\tau} = \tilde{\tau}B$ (cf. Lemma 4.2). By Lemma A.1 this leads to Kramers degeneracy, i.e. $\dim E_\lambda$ is even. □

4.2 Decoupling

The vanishing of $A = UPU^* - P$ means that the unitary U leaves the subspaces $\operatorname{Ran} P$ and $\operatorname{Ran} P^\perp$ invariant. In this case, we call U "decoupled" (with respect to P). Vice versa, if A does not vanish, U "couples" $\operatorname{Ran} P$ and $\operatorname{Ran} P^\perp$. The following proposition states that we can always decouple U if the $+1$-eigenspace of A is even-dimensional and almost decouple U if it is odd-dimensional:

Proposition 4.4 *Let U and P be as above. Then there exists a unitary W with $U - W$ compact, $\tau W \tau^* = W^*$ and:*

- *If* $\dim\ker(A-\mathbb{1}) \mod 2 = 0$, *then* $WPW^* - P = 0$.
- *If* $\dim\ker(A-\mathbb{1}) \mod 2 = 1$, *then*

$$WPW^* - P = \Pi_+ - \Pi_- \tag{4.5}$$

where Π_+ *and* Π_- *are one-dimensional projections.*

Moreover, if $[U, P]$ *is Schatten-*p*, then* $U - W$ *is also Schatten-*p*.*

The remainder of this subsection is devoted to construct the W in this proposition. Taking W to be of the form $W = VU$ we first note that the decoupling condition $[W, P] = 0$ translates to

$$PV = VQ \tag{4.6}$$

where $Q = UPU^*$. We call such a V a "decoupler". The symmetry constraint on W implies that V has to satisfy $\tilde{\tau}V\tilde{\tau}^* = V^*$, where $\tilde{\tau} = U\tau$ as above. Moreover, since we want $U - W$ compact, we must have $V - \mathbb{1}$ compact.

If $\dim\ker(A-\mathbb{1}) \mod 2 = 1$ we will not be able to find a V satisfying these requirements. Although we do not prove it here, this is actually impossible.

We construct V in two steps. First, following [10] we construct a decoupler on the orthogonal complement of $E = E_{+1} \oplus E_{-1}$. In the second step we try to construct a decoupler on E. This turns out to be possible only if $\dim\ker(A-\mathbb{1}) \mod 2 = 0$. In the other case we can only decouple W up to a two-dimensional subspace.

4.2.1 Decoupling on the orthogonal complement of $E_{+1} \oplus E_{-1}$

Define the operator

$$X = B(\mathbb{1} - 2Q) = (\mathbb{1} - 2P)B = \mathbb{1} - P - Q + 2PQ. \tag{4.7}$$

This operator satisfies the decoupling condition, i.e.,

$$PX = XQ = PQ, \tag{4.8}$$

Moreover, $X - \mathbb{1} = PA - AQ$ is compact because A is, and if $[U, P]$ is Schatten-p, then so is $A = [U, P]U^*$ and therefore also X. Yet, X is not unitary:

$$XX^* = X^*X = \frac{X + X^*}{2} = B^2. \tag{4.9}$$

It follows however from this equation that X is normal and that its spectrum is contained in the circle $\{z \in \mathbb{C} : (\operatorname{Im} z)^2 + (\operatorname{Re} z - 1/2)^2 = 1/4\}$.

The kernel of X is precisely $E = E_{+1} \oplus E_{-1}$ since it coincides with the kernel of $X^*X = B^2 = A^2 - \mathbb{1}$. On its orthogonal complement $E^\perp = E_{(-1,1)}$ we define a unitary $\widetilde{V}$ by

$$\widetilde{V} := (X^*X)^{-1/2}|_{E^\perp} X|_{E^\perp}, \tag{4.10}$$

which is well-defined because X is normal, and $(X^*X)^{-1/2}$ is strictly positive on $E^\perp$. The unitary $\widetilde{V}$ has the same eigenvectors as $X|_{E^\perp}$ but with the eigenvalues rescaled to have modulus one.

Since $X - \mathbb{1}$ is compact, the spectrum of X consists of eigenvalues of finite multiplicity, possibly accumulating at 1. One easily sees that $\widetilde{V} - \mathbb{1}$ is also compact. In fact, if $X - \mathbb{1}$ is Schatten-p, then so is $\widetilde{V} - \mathbb{1}$, see Lemma B.1. Moreover, since by (4.8) X maps the range of Q into the range of P, so does its partial isometry. Since $\widetilde{V}$ is the partial isometry of $X|_{E^\perp}$ it follows that

$$P|_{E^\perp}\widetilde{V} = \widetilde{V}Q|_{E^\perp} \tag{4.11}$$

where we use that both P and Q leave $E^\perp$ invariant.

Note now that since $\tau U \tau^* = U^*$, the subspace $E^\perp$ is invariant under $\tilde{\tau} = \tau U$. Indeed, by Lemma 4.2, we have $\tilde{\tau}E_\lambda = E_{-\lambda}$ for any $\lambda \in \mathbb{R}$ so in particular, $E^\perp = E_{(-1,1)}$ is invariant under $\tilde{\tau}$. Moreover, by Lemma 4.2 we have $\tilde{\tau}X\tilde{\tau}^* = X^*$ which implies

$$\tilde{\tau}\widetilde{V}\tilde{\tau}^* = \widetilde{V}^* \tag{4.12}$$

on $E^\perp$. Thus, $\widetilde{V}$ is a decoupler on $E^\perp$ that is compatible with the symmetry condition of W.

4.2.2 Decoupling on $E_{+1} \oplus E_{-1}$

It remains to find an as-good-as-possible decoupler on the remaining subspace $E = E_{+1} \oplus E_{-1}$. Since the restriction of Q to E is the projection onto E_{+1} and the restriction of P to E is the projection onto E_{-1}, such a decoupler has to swap the spaces E_{+1} and E_{-1}. We construct a unitary v on E that achieves this as best as possible. Moreover, we require that $\tilde{\tau}v\tilde{\tau}^* = v^*$. This symmetry induced constrant is crucial: without it a v swapping E_{+1} and E_{-1} can always be found because these spaces have the same dimension by Lemma 4.2.

By Lemma 4.2, $\tilde{\tau}$ maps E_{+1} to E_{-1} and vica versa. Let $\{\phi_1, \ldots, \phi_{2m+k}\}$ be an orthonormal basis of E_{+1} and take accordingly $\{\tilde{\tau}\phi_1, \ldots, \tilde{\tau}\phi_{2m+k}\}$ as orthonormal basis of E_{-1} with $k = 0, 1$ depending on whether $\dim E_{+1}$ is even or odd. If $\dim\ker(A - \mathbb{1}) \bmod 2 = \dim E_{+1} \bmod 2 = 0$ then E_{+1} and E_{-1} are both even dimensional and we take $F_{\mathrm{rest}} = \{0\}$, $F_{+1} = E_{+1}$ and $F_{-1} = E_{-1}$. If $\dim\ker(A - \mathbb{1}) \bmod 2 = \dim E_{+1} \bmod 2 = 1$ then $\dim E_{+1} = \dim E_{-1} = 2m + 1$ are odd and we take $F_{+1} = \mathrm{span}\{\phi_1, \ldots, \phi_{2m}\} \subset E_{+1}$, we take $F_{-1} = \tilde{\tau}F_{+1} = \mathrm{span}\{\tilde{\tau}\phi_1, \ldots \tilde{\tau}\phi_{2m}\}$, and $F_{\mathrm{rest}} = \mathrm{span}\{\phi_{2m+1}, \tilde{\tau}\phi_{2m+1}\}$. In either case, $\tilde{\tau}$ leaves $F_{+1} \oplus F_{-1}$ invariant and in the chosen basis takes the form

$$\tilde{\tau}|_{F_{+1}\oplus F_{-1}} = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} K \tag{4.13}$$

where K is complex conjugation. According to the decomposition $E = F_{\mathrm{rest}} \oplus (F_{+1} \oplus F_{-1})$ we then take v to be

$$v = \mathbb{1} \oplus \begin{bmatrix} 0 & \begin{matrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{matrix} \\ \begin{matrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{matrix} & 0 \end{bmatrix}. \tag{4.14}$$

This leaves F_{rest} invariant, swaps F_{+1} and F_{-1}, and satisfies $\tilde{\tau} v \tilde{\tau}^* = v^*$.

4.2.3 Proof of Proposition 4.4

Let

$$V = v \oplus \widetilde{V} \tag{4.15}$$

with v the (almost) decoupler on $E = E_{+1} \oplus E_{-1}$ from (4.14) and $\widetilde{V}$ the decoupler on $E^{\perp}$ constructed in Sect. 4.2.1. By construction, $\tilde{\tau} V \tilde{\tau}^* = V^*$ so that $W = VU$ satisfies $\tau W \tau^* = W^*$. Moreover, since $\widetilde{V} - \mathbb{1}$ is compact and E is finite dimensional, $V - \mathbb{1}$ is compact and so is $W - U = (V - \mathbb{1})U$. By (4.11) we have that

$$(WPW^* - P)|_{E^{\perp}} = \widetilde{V} Q|_{E^{\perp}} \widetilde{V}^* - P|_{E^{\perp}} = 0 \tag{4.16}$$

where $Q = UPU^*$.

It remains to see how $W^*PW - P$ acts on the subspace E. We have

$$(WPW^* - P)|_E = vQ|_E v^* - P_E = vqv^* - p \tag{4.17}$$

where $q = Q|_E$ and $p = P|_E$ are the projections on E_{+1} and E_{-1}, respectively. If $\dim\ker(A - \mathbb{1}) \mod 2 = 0$, the unitary v swaps E_{+1} and E_{-1} so $vqv^* - p = 0$ and therefore $WPW^* - P = 0$. If $\dim\ker(A - \mathbb{1}) \mod 2 = 1$, we decompose $E = F_{\text{rest}} \oplus (F_{+1} \oplus F_{-1})$ as above. The unitary v in (4.14) leaves F_{rest} invariant and swaps F_{+1} and F_{-1}. Thus $(vqv^* - p)|_{F_{+1} \oplus F_{-1}} = 0$ and $(vqv^* - p)|_{F_{\text{rest}}} = \Pi_+ - \Pi_-$ where Π_+ and Π_- are one-dimensional projections. We conclude that

$$WPW^* - P = \Pi_+ - \Pi_-. \tag{4.18}$$

Finally, we saw in Sect. 4.2.1 that if $[U, P]$ is Schatten-p then $\widetilde{V} - \mathbb{1}$ is Schatten-p. Since E is finite dimensional, $V - \mathbb{1}$ is also Schatten-p. Therefore $W - U = (V - \mathbb{1})U$ is Schatten-p and so is

$$[W, P] = [(W - U), P] + [U, P]. \tag{4.19}$$

□

4.3 The symmetric Wold construction

For two projections P and P' we write $P \preceq P'$ if $\operatorname{Ran} P \subset \operatorname{Ran} P'$. In the proof of the symmetric Wold decomposition we will need the following abstract result:

Proposition 4.5 *Let W be a unitary and P a projection with $\tau W\tau^* = W^*$ and $\tau P\tau^* = P$ for an odd time-reversal symmetry τ and such that*

$$WPW^* - P = \Pi_+ - \Pi_- \tag{4.20}$$

where Π_+ and Π_- are one-dimensional projections. For $k \in \mathbb{Z}$, let $\Pi_+^{(k)} := \mathrm{Ad}_W^{k-1}(\Pi_+)$ and $\Pi_-^{(k)} := \mathrm{Ad}_{W^*}^k(\Pi_-)$. Then:

(1) These projections are mutually orthogonal, i.e.

$$\Pi_\sigma^{(k)}\Pi_{\sigma'}^{(l)} = \delta_{k,l}\delta_{\sigma,\sigma'}\Pi_\sigma^{(k)} \tag{4.21}$$

for all $k, l \in \mathbb{Z}$ and $\sigma, \sigma' \in \{+, -\}$.

(2) For $k \leq 0$ we have $\Pi_+^{(k)}, \Pi_-^{(k)} \preceq P$ while for $k \geq 1$ we have $\Pi_+^{(k)}, \Pi_-^{(k)} \preceq P^\perp$.

(3) For all $k \in \mathbb{Z}$ the projections $\Pi_+^{(k)}$ and $\Pi_-^{(k)}$ form a Kramers pair, i.e.

$$\tau\Pi_\pm^{(k)}\tau^* = \Pi_\mp^{(k)}. \tag{4.22}$$

Proof We first prove the last statement of the proposition. Write $A = \mathrm{Ad}_W(P) - P = \Pi_+^{(1)} - \Pi_-^{(0)}$. From the assumptions we have

$$\begin{aligned}\tau\Pi_+^{(1)}\tau^* - \tau\Pi_-^{(0)}\tau^* &= \tau A\tau^* = W^*PW - P\\ &= -\mathrm{Ad}_{W^*}(A) = \mathrm{Ad}_{W^*}(\Pi_-^{(0)}) - \mathrm{Ad}_{W^*}(\Pi_+^{(1)})\\ &= \Pi_-^{(1)} - \Pi_+^{(0)}\end{aligned} \tag{4.23}$$

so $\tau\Pi_+^{(1)}\tau^* = \Pi_-^{(1)}$ and $\tau\Pi_-^{(0)}\tau^* = \Pi_+^{(0)}$ and the claim follows for $k = 0, 1$. For any other $k \in \mathbb{Z}$, Eq. (4.22) follows from

$$\tau\Pi_+^{(k)}\tau^* = \tau\mathrm{Ad}_W^k(\Pi_+^{(0)})\tau^* = \mathrm{Ad}_{W^*}^k(\Pi_-^{(0)}) = \Pi_-^{(k)}. \tag{4.24}$$

Since all these projections are one-dimensional, it further follows by Kramers degeneracy that

$$\Pi_+^{(k)}\Pi_-^{(k)} = 0 \tag{4.25}$$

for all $k \in \mathbb{Z}$, which proves part of the orthogonality claims. We prove the remaining orthogonality claims in (4.21) and the second statement by induction. The N^{th} induction hypothesis is that the family of one-dimensional projections $\{\Pi_+^{(k)}, \Pi_-^{(k)}\}_{k=-N+1}^N$ is mutually orthogonal, and $\Pi_\pm^{(k)} \preceq P$ for $k = -N-1, \ldots, 0$ while $\Pi_\pm^{(k)} \preceq P^\perp$ for $k = 1, \ldots, N$.

Base case $N = 1$**:** We must show that $\{\Pi_\pm^{(0)}, \Pi_\pm^{(1)}\}$ form an orthogonal set, and $\Pi_\pm^{(0)} \subset P$ while $\Pi_\pm^{(1)} \subset P^\perp$. Since $A = \Pi_+^{(1)} - \Pi_-^{(0)}$ is self adjoint, we get $\Pi_+^{(1)} \perp \Pi_-^{(0)}$. Since conjugation by τ preserves orthogonality we get from (4.22) that also $\Pi_-^{(1)} \perp \Pi_+^{(0)}$. Moreover, $\Pi_+^{(0)} \perp \Pi_-^{(0)}$ and $\Pi_+^{(1)} \perp \Pi_-^{(1)}$ by (4.25). Finally, since

$$A = WPW^* - P = \Pi_+^{(1)} - \Pi_-^{(0)} \tag{4.26}$$

we have $\Pi_-^{(0)} \preceq P$ and $\Pi_+^{(1)} \preceq P^\perp$. Moreover, since P is τ-invariant also $\Pi_+^{(0)} \preceq P$ and $\Pi_-^{(1)} \preceq P^\perp$, which proves the claims about inclusions in P and $P^\perp$, and also the remaining orthogonality claims $\Pi_+^{(0)} \perp \Pi_+^{(1)}$ and $\Pi_-^{(0)} \perp \Pi_-^{(1)}$.

Induction step: We assume the N^{th} induction hypothesis to hold and derive the $(N+1)^{st}$. Let

$$P^{(N)} = P + \sum_{n=1}^{N} \left(\Pi_+^{(n)} + \Pi_-^{(n)}\right). \tag{4.27}$$

By the N^{th} induction hypothesis and (4.22) this is a τ-invariant projection and $P \preceq P^{(N)}$. Consider

$$A^{(N)} := WP^{(N)}W^* - P^{(N)} = \Pi_+^{(N+1)} - \Pi_-^{(N)}. \tag{4.28}$$

Since this is a difference of projections it follows that $\Pi_+^{(N+1)} \preceq (P^{(N)})^\perp \preceq P^\perp$, and by τ-invariance also $\Pi_-^{(N+1)} \preceq P^\perp$. The claims $\Pi_\pm^{(-N)} \preceq P$ follow similarly by setting $P^{(-N)} = P - \sum_{n=0}^{N-1} \left(\Pi_+^{(-n)} + \Pi_-^{(-n)}\right)$ and considering

$$A^{(-N)} := WP^{(-N)}W^* - P^{(-N)} = \Pi_+^{(-N+1)} - \Pi_-^{(-N)}. \tag{4.29}$$

This proves the claims about inclusions in P and $P^\perp$.

To prove the orthogonality claims, note that $(\tau W^n)^2 = -\mathbb{1}$ for any n, and since

$$(\tau W^n)\Pi_+^{(m-n)}(\tau W^n)^* = \tau^* \Pi_+^{(m)} \tau = \Pi_-^{(m)} \tag{4.30}$$

for any m, we see that $\Pi_+^n \perp \Pi_-^{(m)}$ for all $n, m \in \mathbb{Z}$.

It remains to show that $\{\Pi_+^{(n)}\}_{n=-N}^{N+1}$ and $\{\Pi_-^{(n)}\}_{n=-N}^{N+1}$ are othogonal families. We already know that $\Pi_\pm^{N+1}$ is orthogonal to $\Pi_\pm^n$ for $n = -N, \ldots, 0$ because the former is a subprojection of $P^\perp$ while the latter are subprojections of P. Similarly, we know that $\Pi_\pm^{(-N)}$ is orthogonal to $\Pi_\pm^{(n)}$ for $n = 1, \ldots, N+1$.

To see that $\Pi_\pm^{(N+1)}$ is orthogonal to $\Pi_\pm^{(n)}$ for $n = 1, \ldots, N$ we simply note that by induction hypothesis for any such n we have $\Pi_\pm^{(0)} \perp \Pi_\pm^{(N+1-n)}$. Since conjugation by W preserves orthogonality, we conclude that $\Pi_\pm^{(n)} \perp \Pi_\pm^{(N+1)}$ as required. The orthogonality of $\Pi_\pm^{(-N)}$ with $\Pi_\pm^{(n)}$ for $n = -N+1, \ldots, 0$ is proved in the same way. This proves all the required orthogonality relations and thereby concludes the induction step. $\square$

4.4 Proof of Theorem 2.3

If $\dim\ker(A-\mathbb{1}) \mod 2 = 0$, all claims follow directly from Proposition 4.4.

If $\dim\ker(A-\mathbb{1}) \mod 2 = 1$, then Proposition 4.4 provides a unitary W with $U-W$ compact, $\tau W\tau^* = W^*$ and such that

$$\mathrm{Ad}_W(P) - P = \Pi_+ - \Pi_- \tag{4.31}$$

with one-dimensional projections Π_+ and Π_-. Moreover, if $[U, P]$ is Schatten-p, then so is $U-W$.

For $k\in\mathbb{Z}$, let $\Pi_+^{(k)} := \mathrm{Ad}_W^{k-1}(\Pi_+)$ and $\Pi_-^{(k)} := \mathrm{Ad}_{W^*}^{k}(\Pi_-)$. Decompose the Hilbert space as $\mathcal{H} = \mathcal{H}'\oplus\mathcal{H}''$ where $\mathcal{H}' = \bigoplus_{k\in\mathbb{Z},\sigma\in\{+,-\}}\mathrm{Ran}\,\Pi_\sigma^{(k)}$. By Proposition 4.5, both W and P leave $\mathcal{H}'$ and $\mathcal{H}''$ invariant so $W = W'\oplus W''$ and $P = P'\oplus P''$. Since $\mathrm{Ran}\,\Pi_+, \mathrm{Ran}\,\Pi_-\subset\mathcal{H}'$, we have $[W'', P''] = 0$, and it only remains to identify W' with the unitary S described at the end of Sect. 2.2.

Let ϕ_1 be a unit vector spanning $\mathrm{Ran}\,\Pi_+^{(1)}$ and define $\phi_k = W^{k-1}\phi_1$ for all $k\in\mathbb{Z}$. Then ϕ_k spans $\mathrm{Ran}\,\Pi_+^{(k)}$. For each $k\in\mathbb{Z}$, let $\overline{\phi}_k = \tau\phi_k$. Then $\overline{\phi}_k$ spans $\mathrm{Ran}\,\Pi_-^{(k)}$ and $\mathcal{H}' = \mathrm{span}\{\phi_k, \overline{\phi}_k\ :\ k\in\mathbb{Z}\}$. On this space, the unitary W' acts as

$$W'\phi_k = \phi_{k+1},\quad W'\overline{\phi}_k = \overline{\phi}_{k-1} \tag{4.32}$$

for all $k\in\mathbb{Z}$. Indeed, $\phi_{k+1}=W^k\phi_1=W\phi_k$ while $\overline{\phi}_k=\tau\phi_k=\tau W^{k-1}\phi_1=W^{1-k}\tau\phi_1=W^{1-k}\overline{\phi}_1$, so $\overline{\phi}_{k-1}=W^{2-k}\overline{\phi}_1 = W\overline{\phi}_k$. The unitary W' is equivalent to S in (2.7) by the isomorphism $\phi_x\mapsto|x,+\rangle$ and $\overline{\phi}_x\mapsto|x,-\rangle$. This concludes the proof. $\square$

Appendix A Kramers degeneracy

Lemma A.1 *Let θ be an anti-unitary operator on a finite dimensional Hilbert space V with $\theta^2 = -\mathbb{1}$. Then V is even-dimensional and has an orthonormal basis consisting of Kramers pairs, i.e. there is an orthonormal basis $\{\phi_1, \phi_1', \dots\phi_n, \phi_n'\}$ such that $\theta\phi_i = \phi_i'$ and $\theta\phi_i' = -\phi_i$ for all $i = 1,\dots,n$.*

Proof Let ϕ_1 be any vector in V of unit length, and put $\phi_1' = \theta\phi_1$. Then

$$\langle\phi_1, \phi_1'\rangle = \langle\phi_1, \theta\phi_1\rangle = \overline{\langle\theta\phi_1, \theta^2\phi_1\rangle} = -\langle\phi_1, \theta\phi_1\rangle = -\langle\phi_1, \phi_1'\rangle, \tag{A.1}$$

i.e. $\phi_1\perp\phi_1'$ and $\theta\phi_1' = \theta^2\phi_1 = -\phi_1$. Now pick any vector ϕ_2 in the orthongonal complement of $\mathrm{span}\{\phi_1, \phi_1'\}$ and put $\phi_2' = \theta\phi_2$. By the same reasoning as before, $\phi_2\perp\phi_2'$ and $\theta\phi_2' = -\phi_2$. Repeating this construction eventually yields the required basis $\{\phi_1, \phi_1', \dots, \phi_1, \phi_n'\}$. $\square$

Appendix B Schatten-*p* Lemma

Lemma B.1 *Let X be a normal operator with $X - \mathbb{1}$ compact, spectrum contained in the circle $\{z \in \mathbb{C} \,:\, (\operatorname{Im} z)^2 + (\operatorname{Re} z - 1/2)^2 = 1/2\}$ and empty kernel. Let*

$$V := (X^*X)^{-1/2}X. \tag{B.1}$$

Then if $X - \mathbb{1}$ is Schatten-p, so is $V - \mathbb{1}$.

Proof Let $\{\lambda_i\}_{i\in\mathbb{N}}$ be the nonzero eigenvalues of the compact operator $X - \mathbb{1}$ ordered such that $|\lambda_i| \geq |\lambda_{i+1}|$ for all i. This operator is Schatten-p if and only if

$$\sum_{i\in\mathbb{N}} |\lambda_i|^p < \infty. \tag{B.2}$$

The nonzero eigenvalues of $V - \mathbb{1}$ are

$$\mu_i = \frac{\lambda_i + 1}{|\lambda_i + 1|} - 1. \tag{B.3}$$

Note that since $X - \mathbb{1}$ has empty kernel, none of the λ_i equal -1 so the μ_i are always well defined.

For μ_i close to 0 we have $|\mu_i| = |\lambda_i| + \mathcal{O}(|\lambda_i|^2)$ so

$$|\mu_i|^s = |\lambda_i|^s + s\mathcal{O}(|\lambda_i|^2|\lambda_i|^{s-1}) = |\lambda_i|^s + \mathcal{O}(|\lambda_i|^{s+1}). \tag{B.4}$$

Both terms on the right-hand side are summable by assumption, so

$$\sum_{i} |\mu_i|^s < \infty, \tag{B.5}$$

i.e. $V - \mathbb{1}$ is Schatten-p. □

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Declarations

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