Difference Tropical Geometry

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Abstract

This thesis creates a link between *Tropical Geometry* and *Difference Algebra*. The main result is a difference version of *Kapranov's Theorem*. In this theorem, we extend Kapranov's Theorem to the case of a Laurent difference polynomial with coefficients from a multiplicative valued difference field, where the residue field is an algebraically closed field with a generic automorphism (ACFA). A result of this thesis that plays a critical role in the proof of the Difference Kapranov Theorem, is a difference version of *Newton's Lemma*.

In other results, we provide a combinatorial intuition for some difference tropical objects, namely a difference tropical plane curve and a difference tropical hypersurface.

Kurzfassung

Diese Dissertation schafft eine Verbindung zwischen der *Tropischen Geometrie* und der *Differenzenalgebra*. Das Hauptresultat ist eine Differenzversion von *Kapranovs Theorems*. In diesem Theorem erweitern wir das Kapranov-Theorem auf den Fall eines Laurent-Differenzpolynoms mit Koeffizienten aus einem multiplikativ bewerteten Differenzkörper, dessen Restklassenkörper ein algebraisch geschlossener Körper mit einem generischen Automorphismus (ACFA) ist. Ein Ergebnis dieser Dissertation, das eine wichtige Rolle beim Beweis des Differenz-Kapranov-Theorems spielt, ist eine Differenzversion des *Newton-Lemmas*.

Außerdem geben wir eine kombinatorische Intuition für einige differenz-tropische Objekte, nämlich eine differenz-tropische Ebene Kurve und eine differenz-tropische Hyperfläche.

To those who bring peace and kindness to this world,
to the kindest among them.

Statement of authorship:

I declare under oath that I have produced my thesis independently and without any undue assistance by third parties under consideration of the Principles for the Safeguarding of Good Scientific Practice at Heinrich Heine University Düsseldorf.

Düsseldorf, August 2024
Saba Aliyari

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"The more one's knowledge increases, the greater the worth of one's very being."

— Imam Ali

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Contents

Chapter 1

Introduction

In this thesis, we establish a *difference* version of *tropical geometry* that connects *tropical geometry* to *difference algebra*. In this difference version, instead of studying polynomials, we study difference polynomials. In a difference polynomial, the variables are $x_1, \ldots, x_n$ for $n \in \mathbb{N}$, along with various iterations of an automorphism of the field, σ, on these variables.

Certain objects of difference algebra, such as difference polynomials are complex objects to study. Determining the roots of a difference polynomial, if possible, is no easy task.

Tropical arithmetic and a procedure called *tropicalization* enable us to find tropical analogues of classical mathematical objects, for instance polynomials. Polynomials may have complicated graphs, but tropicalization turns them into piecewise linear graphs, which are much easier objects to study.

This aspect served as a motivation for our project, in which we utilize tropical tools to gain a deeper understanding of difference polynomials and their roots.

This thesis consists of two main parts. In the first part, we define *difference tropical polynomials*. Then we apply graph theory to describe the combinatorial intuition of a *difference tropical curve*.

Subsequently, we define further difference tropical objects. In particular, we define the *tropicalization* of a Laurent difference polynomial in n variables, and *difference tropical hypersurfaces*. Then by using polyhedral geometry, we describe the combinatorics of a difference tropical hypersurface.

In the second part, we prove a difference version of *Kapranov's Theorem*. For a Laurent polynomial f in $K[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$, Kapranov's Theorem builds a bridge between its associated classical hypersurface $V(f)$ and the associated tropical hypersurface $\mathrm{trop}(V(f))$. The main goal of this part is to extend Kapranov's Theorem to the case where f is a Laurent difference polynomial with coefficients from a multiplicative valued difference field.

Tropical geometry is a field that studies polynomials and their geometric properties in the *tropical semiring* $(\mathbb{R}\cup\{\infty\},\oplus,\odot)$, with $\oplus$ being the minimum and $\odot$ being classical addition. This structure is also called the *min-plus algebra*. Although, in this thesis, we work in this setting, but some people work with another isomorphic structure called *max-plus algebra*, in which the addition is defined to be the maximum.

The adjective "tropical" has no deep meaning and it was chosen in honor of Imre Simon who first introduced this structure on $\mathbb{N}$. Since he was a professor at the university of São Paulo near the Tropic of Capricorn, his French colleagues coined this adjective to honor him. Efforts to consolidate the definitions of the theory began in the late 1990's. Imre Simon, Grigory Mikhalkin, and Bernd Sturmfels, along with many other mathematicians, made significant contributions in this area.

Tropical geometry redefines rules of arithmetic and this results in useful mathematics. Using tropicalization, algebro-geometric problems can be converted to combinatorial problems. Then the obtained data in the combinatorial world can be lifted back to the classical case. Therefore, tropical geometry has been a useful tool. For an overview of this field, see, for example, [14].

One of the important theorems that builds this connection is *Kapranov's Theorem*. It forges a connection between algebraic hypersurfaces and *tropical hypersurfaces* in $\mathbb{R}^n$. The statement of this theorem is as follows:
Theorem *(Kapranov's Theorem) If (K,v) is an algebraically closed valued field, with nontrivial valuation, and if $f\in K[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$, then the following sets coincide:*

1. $\mathrm{trop}(V(f))$ *which is a subset of* $\mathbb{R}^n$*;*

2. *the set of all points in* $\mathbb{R}^n$ *for which the initial form of* f *is not a monomial;*

3. *the topological closure of* $\{(v(y_1),\ldots v(y_n)) \mid (y_1,\ldots,y_n) \in V(f)\}$ *in* $\mathbb{R}^n$.

The proof of this theorem can be found, for instance, in [18] and in [17]. The connection made by this theorem is generalized to arbitrary varieties in the *Fundamental Theorem of Tropical Algebraic Geometry* which is a central theorem in this field. In fact, Kapranov's Theorem is considered as a critical step in the proof of the Fundamental Theorem. For further information on this topic, helpful references include [17] and [19].

Difference algebra is an area in mathematics which studies *difference field*s and *difference polynomial*s. A difference field is a field K together with a field automorphism σ. A difference polynomial with coefficients from K in n variables $x_1,\ldots,x_n$, is a polynomial in infinitely many formal variables $\sigma^j(x_i)$, for $i \in \{1,\ldots,n\}$ and $j \in \mathbb{N}$ with σ^j being the j-th iteration of σ. In this case, we say that f is an element of the *ring of difference polynomials* and use the notation $f \in K_\sigma[x_1,\ldots,x_n]$. Even a very simple difference polynomial, such as $x - x^\sigma$, defines a variety that is an infinite field containing $\mathbb{Q}$. So it is challenging to study these objects.

Difference algebra is considered as an analogue to *Differential Algebra*, but in this area difference equations are studied rather than differential equations. difference algebra was first introduced as an independent field of study by Joseph Ritt and Richard Cohn. Later on, Hrushovski applied it in proving *The Manin-Mumford Conjecture* in 2001 [10]. Another interesting application appears in the connection between *algebraic dynamical systems* and difference fields. In fact the knowledge about difference fields leads to a better understanding of algebraic dynamical systems, [20] provides valuable insights on this topic.

A *valued difference field* K, is a valued field together with an automorphism which takes the valuation ring to the valuation ring. The valuation and the automorphism of K can interact in different ways which are explained in Remark 2.0.11. In this thesis, we work with the case of *multiplicative valued difference fields*. In this case the value group is a $\mathbb{Z}[\rho]$-module for ρ being a positive real number and we have

$$\forall x \in K^\times : \ v(\sigma(x)) = \rho \cdot v(x).$$

In this project, we assume that the value group Γ is a divisible subgroup of $\mathbb{R}$. Therefore, it is not difficult to show that any automorphism σ_Γ on Γ is of the following form:

$$\forall x \in K^\times : \quad \sigma_\Gamma(v(x)) = \rho \cdot v(x),$$

with ρ being a positive real number. Moreover, as we will see in Remark 2.0.8, the automorphism σ of K induces an automorphism σ_Γ on Γ such that $\forall x \in K^\times : \quad \sigma_\Gamma(v(x)) = v(\sigma(x))$. This gives the above property of the multiplicative case. We call ρ the *scaling exponent* of σ.
In addition, we assume that Γ is a $\mathbb{Q}(\rho)$-module for ρ being transcendental over $\mathbb{Q}$. Thus it is a $\mathbb{Z}[\rho]$- module. Hence, we work with the case of multiplicative valued difference fields.

For further exploration of difference algebra, we recommend referring to [23], [5], and [16].

Main Results

Here, we present a list of the main results of this thesis. We start with the main results of the first part. There are analogues of these results in the classical nondifference case.

In this theorem, we describe the combinatorics of a difference tropical plane curve which is a subset of $\mathbb{R}^2$ defined by a difference tropical polynomial in two variables.
Theorem: *Any difference tropical plane curve which is not a straight line is a DBWR graph, and vice versa.*

"DBWR" is an abbreviation for "difference balanced weighted rectilinear" graph. It is an object consisting of finitely many vertices and edges (segments and halfrays), such that each edge has a weight. Moreover, there are some conditions on the edges and vertices of a DBWR graph. For the detailed definition see Definition 3.2.14.

See Theorem 3.2.15 for the precise statement and the proof of this result.

Another result of this part is the following proposition:
Proposition: *For a Laurent difference polynomial f, its associated difference*

tropical hypersurface $\operatorname{trop}(V(f))$ *is the support of a pure* $(\Gamma, \mathbb{Q}(\rho))$*-polyhedral complex of dimension* $(n-1)$.

This proposition describes the combinatorics of a difference tropical hypersurface. Its exact statement which also presents a more precise description of $\operatorname{trop}(V(f))$ is Proposition 4.3.1.

In the second part of this thesis, we aim to prove a difference version of Kapranov's Theorem. To achieve this goal, we assume some conditions on the valued difference field K. See Assumption 2.0.45. The Difference Kapranov Theorem (Theorem 6.2.1) states

Theorem: *Suppose* $f \in K_\sigma\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ *is a Laurent difference polynomial.The following sets coincide:*

1. $\operatorname{trop}(V(f)) \subseteq \mathbb{R}^n$ *which is the difference tropical hypersurface associated to* f*;*

2. *the set of all the points* $w \in \mathbb{R}^n$ *for which the initial form* $\operatorname{in}_w(f)$ *is not a monomial;*

3. *the closure of the set* $A = \{(v(y_1), \ldots, v(y_n)) : (y_1, \ldots, y_n) \in V(f)\}$ *in* $\mathbb{R}^n$.

The most challenging part in the proof of this theorem is to prove that the set in (1) is included in the set in (3). This is done with the help of Proposition 6.1.9, which states

Proposition: *Let* $f \in K_\sigma[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ *be a Laurent difference polynomial, and* $\underline{w} = (w_1, \ldots, w_n) \in \Gamma^n$ *such that* $\operatorname{in}_{\underline{w}}(f)$ *is not a monomial. Suppose* $\bar{\alpha}$ *is a root of* $\operatorname{in}_{\underline{w}}(f)$ *in* $(\mathbf{k}^*)^n$*. Then there exists an element* y *in* $(K^*)^n$ *which is a root of* f*, and satisfies the following conditions:*

- $v(y) = \underline{w}$,

- $\forall i,\ 1 \le i \le n : \overline{t^{-w_i} \cdot y_i} = \bar{\alpha}_i$.

In the classical case, in [17], induction is used to prove the analogue of this proposition . In the base case, the assumption of K being an algebraically closed field plays an important role. By using this assumption, it is possible to decompose f into linear factors which finally results in the existence of a root with the desired conditions. In contrast, in our context, concerning Laurent *difference* polynomials, we do not have such a decomposition. As a solution, we proved the *Difference Newton Lemma*(5.0.1) to guarantee the existence of a root of a difference polynomial in one variable.

Theorem: *(Difference Newton Lemma) Assume the same setting as in Assumption 2.0.45. Given $f \in K_\sigma[x]$ is not constant and suppose $b \in K$ such that $f(b) \neq 0$.*
We define $\varepsilon := \max_{\substack{J \\ |J|\geq 1}} \varepsilon_J$, where

$$\varepsilon_J := \frac{1}{|J|_\rho}\big(v(f(b)) - v(f_{(J)}(b))\big).$$

There exists a root $a \in K$ of f such that $v(a-b) = \varepsilon$.

The definitions of $|J|_\rho$ and $f_{(J)}(b)$ are given later in the first chapter.

An essential ingredient to prove this Theorem as well as the Difference Kapranov Theorem, is Theorem 2.0.42. Its statement is given below.

Theorem: *Let k be an ACFA. Suppose f is in $k_\sigma[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and is not a monomial. Then f has a root in $(k^*)^n$.*

One of the main assumptions in this thesis is that the difference residue field is an *ACFA*. This concept comes from *model theory* and model theorists say that a field is a model of ACFA, which is an abbreviation for *Algebraically Closed Field with a generic Automorphism*. A field k is a model of ACFA if σ is an automorphism of k, and it satisfies the conditions of Theorem 2.0.30 which are the axioms presented in 3.1 of [4]. As in this thesis we do not use the model theoretic approach, we simply say that a field is an ACFA. This concept can be considered as a difference version of algebraically closed fields. For the precise definition see Definition 2.0.26.

Outline

- **Chapter 1**

  This chapter is devoted to some preliminaries in difference algebra.

We start with definitions of a difference ring and a difference field. Further, we state the definition of the difference polynomial ring and a Laurent difference polynomial. An interesting result of this chapter is Theorem 2.0.42. It guarantees the existence of a root for a nonmonomial Laurent difference polynomial in n variables with coefficients from an ACFA. The significance of this theorem becomes evident in the proof of the Difference Kapranov Theorem. The key step in the proof of Theorem 2.0.42 is the one variable case which is Lemma 2.0.28. We finish this chapter by fixing our general setting in Assumption 2.0.45.

- **Chapter 2**

  The main result of this chapter is Theorem 3.2.15. It describes the combinatorics of a difference tropical plane curve. To obtain this result, we use graph theory. This chapter consists of two sections. In the first one, we present all graph theory needed. We discuss some basics about planar graphs and duality. In the second section, we start establishing a connection between tropical geometry and difference algebra by introducing difference tropical polynomials and difference tropical plane curves. In particular, we define a difference version of the concept of weight for the edges of a difference tropical plane curve.
  Finally, the definition of a DBWR graph is given. In Theorem 3.2.15, we use this object to show what a difference tropical plane curve looks like.

- **Chapter 3**

  This chapter consists of three sections. In the first one, by introducing more difference tropical objects, we expand the connection we made between tropical geometry and difference algebra. Specifically, we define the tropicalization of a Laurent difference polynomial, a difference tropical hypersurface, and the initial form of a Laurent difference polynomial.

  The second section is devoted to the prerequisites in polyhedral geometry. Particularly, the definitions of a polyhedral complex, and the regular subdivision are given. We also define a difference polyhedral object which is a $(\Gamma, \mathbb{Q}(\rho))$- polyhedral complex. In the last section, we prove the main result of this chapter which is Proposition 4.3.1.

- **Chapter 4**

  Our main objective is to prove the Difference Newton Lemma (Theorem 5.0.1). This theorem plays a critical role in our proof of the Difference Kapranov Theorem. To prove the Difference Newton Lemma, we follow two main steps. Firstly, in Lemma 5.0.4, we prove that we can have a better estimation of a possible root of $f \in K_\sigma[x]$ around a nonroot $b \in K$. Finally, in Proposition 5.0.5, we use the assumption of spherical completeness to find a root for f. In a spherically complete field, any pseudocauchy sequence has a pseudolimit, see [13]. In Proposition 5.0.5, we construct a pseudocauchy sequence whose pseudolimite is a root of f.

- **Chapter 5**

  The main result of this thesis is proved in the second section of **Chapter 5**, namely the Difference Kapranov Theorem (Theorem 6.2.1). The main tool to prove this theorem is Proposition 6.1.9, which is the goal we pursue in the first section of this chapter. To prove this proposition, we start by proving the same statement for a Laurent difference polynomial in one variable. This is done in Lemma 6.1.1 and Lemma 6.1.3. In this case Difference Newton Lemma plays a highlighted role. Afterwards, we assume f is a Laurent difference polynomial in n variables and we impose a condition on it, and in Proposition 6.1.4, we prove a similar statement for f. In Lemma 6.1.6, we prove that this imposed condition does not cause a serious restriction. In other words, if f is in $K_\sigma[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we can associate a Laurent difference polynomial g to f which has that imposed condition and for which the statement above holds. In the proof of the Difference Kapranov Theorem, the tricky part is to prove that the set appearing in (a) is included in the set appearing in (c) and this is done by Proposition 6.1.9.

Chapter 2

Preliminaries

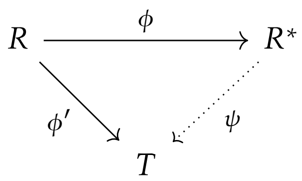
This chapter provides the difference algebra needed for this thesis. The main references here are [23], [21],[4], and [9]. [23] is a well-written lecture note on this topic and is used as a reference in this chapter. Any material presented from this source can also be found in [5] or [9]. Additionally, we establish our general assumptions and present all necessary definitions in this setting.

Definition 2.0.1 ([23], Definition 1.1.1 and Definition 1.1.14)**.** Let R be a commutative ring, and $\sigma : R \longrightarrow R$ be an endomorphism. Then R together with σ is a *difference ring*. It is denoted by (R,σ).

If σ is an automorphism, then (R,σ) is called an *inversive* difference ring.

Definition 2.0.2 ([23], Definition 1.1.2)**.** Let (R,σ) and $(S,\tilde{\sigma})$ be two difference rings. Then $\phi : R \longrightarrow S$ is a *morphism of difference rings*, if it is a morphism of rings for which we have $\phi\sigma = \tilde{\sigma}\phi$.

Proposition 2.0.3 ([23], Proposition 1.1.18)**.** *Let (R,σ) be a difference ring. Then there exists an inversive difference ring (R^*,σ^*), and a morphism of difference rings $\phi : R \longrightarrow R^*$ such that the following universal property is satisfied: If T is an inversive difference ring , and $\phi' : R \longrightarrow T$ is a morphism of difference rings, then there exists a unique morphism $\psi : R^* \longrightarrow T$ such that the following diagram commutes:*



The difference ring R^* is unique up to isomorphism, and is called the inversive closure of R.

Proof. Define

$$S := \{(n, f) \mid n \in \mathbb{N} \text{ and } f \in R\}[1].$$

Then (n, f) is said to be equivalent to (m, g) if and only if there exist $i, j \in \mathbb{N}$, such that

$$(n+i, \sigma^i(f)) = (m+j, \sigma^j(g)).$$

This is an equivalence relation on S. The set of all equivalence classes is denoted by R^*, and satisfies the intended properties of the statement. For the detailed proof, see [23] Proposition 1.1.18. □

Remark 2.0.4. If R is a field then R^* is a field. This is clear from the construction of R^* in the proof of Proposition 1.1.18 in [23].

The following definition is extracted from [4], Definition 2.1.

Definition 2.0.5. A *difference field* is a field K together with an automorphism $\sigma : K \longrightarrow K$. It is denoted by (K, σ).
Sometimes, for simplicity, we write K is a difference field instead of (K, σ) is a difference field.

Remark 2.0.6. In this thesis, we primarily focus on difference fields, where σ is assumed to be an automorphism. We will state explicitly whenever we work with a difference ring, where σ is not necessarily an automorphism.

Definition 2.0.7 ([21], §1)**.** A *valued difference field* is a valued field K together with an automorphism σ satisfying $\sigma(\mathcal{O}_K) = \mathcal{O}_K$, where $\mathcal{O}_K$ is the valuation ring.

Remark 2.0.8. The automorphism of a valued difference field induces two important automorphisms as follows:

[1] In this thesis, we assume that the set of natural numbers $\mathbb{N}$ contains 0.

- Let Γ be the value group of K. Then σ induces an automorphism σ_Γ on Γ as below:

$$\begin{aligned}\sigma_\Gamma : &\Gamma \longrightarrow \Gamma,\\ &\gamma \longmapsto v(\sigma(a)),\end{aligned}$$

  where a is an element of K such that $v(a) = \gamma$.
  In this case, Γ is called the *difference value group* of K.

  The property $\sigma(\mathcal{O}_K) = \mathcal{O}_K$ guarantees that σ_Γ is well-defined. To see this, assume a and a' are two elements of K such that $v(a) = \gamma = v(a')$. We want to prove that $v(\sigma(a)) = v(\sigma(a'))$. $v(a) = v(a')$ means $v\left(\frac{a}{a'}\right) = 0$, or equivalently $\frac{a}{a'} \in \mathcal{O}_K$. From $\sigma(\mathcal{O}_K) = \mathcal{O}_K$, we have $\sigma\left(\frac{a}{a'}\right) \in \mathcal{O}_K$. This means, $v\left(\sigma\left(\frac{a}{a'}\right)\right) \geq 0$ which means $v(\sigma(a)) \geq v(\sigma(a'))$. Similarly, we can show $v(\sigma(a')) \geq v(\sigma(a))$. Hence, $v(\sigma(a')) = v(\sigma(a))$ and σ_Γ is well-defined.
  σ_Γ is also order preserving. Assume $\gamma > \gamma'$ with $\gamma = v(a)$ and $\gamma' = v(a')$ for $a, a' \in K$. This means $v\left(\frac{a}{a'}\right) > 0$ or equivalently $\frac{a}{a'} \in \mathcal{M}$, where by $\mathcal{M}$ we mean the maximal ideal of $\mathcal{O}_K$. As $\mathcal{M}$ contains all non-invertible elements, $\frac{a}{a'}$ is not invertible. Therefore, $\sigma\left(\frac{a}{a'}\right)$ is not invertible, which means $\sigma\left(\frac{a}{a'}\right) \in \mathcal{M}$ or equivalently $v\left(\sigma\left(\frac{a}{a'}\right)\right) > 0$. This gives $v(\sigma(a)) > v(\sigma(a'))$, in fact $\sigma_\Gamma(\gamma) > \sigma_\Gamma(\gamma')$.

- σ also induces an automorphism $\bar{\sigma}$ on $\mathbf{k}$, the residue field of K, as follows:

$$\begin{aligned}\bar{\sigma} : &\mathbf{k} \longrightarrow \mathbf{k},\\ &\bar{a} \longmapsto \overline{\sigma(a)}.\end{aligned}$$

  Then $(\mathbf{k}, \bar{\sigma})$ is called the *difference residue field* of K. Again, from the property $\sigma(\mathcal{O}_K) = \mathcal{O}_K$, $\bar{\sigma}$ is well defined. To see this, assume for two elements $\bar{a}$ and $\bar{b}$ of $\mathbf{k}$, we have $\bar{a} = \bar{b}$. This means $a - b \in \mathcal{M}$. In other words $v(a-b) > 0$. As we discussed for σ_Γ, we have $v(\sigma(a-b)) > 0$. Thus, $\sigma(a) - \sigma(b) \in \mathcal{M}$. Hence, $\overline{\sigma(a)} = \overline{\sigma(b)}$.

Notation 2.0.9. We fix the notations above for the rest of this work. Explicitly, from now on, for a valued field K, we denote the value group of K by Γ, its residue field by $\mathbf{k}$, and the valuation ring by $\mathcal{O}_k$.

As a clarification, we consider an example of a Hahn-field. In general, if a field $\mathbf{k}$ and an ordered abelian group Γ are given, then one can define a valued field whose residue field is $\mathbf{k}$ and whose value group is Γ. This field is called a *Hahn-field* and it is defined as follows:

$$K = \mathbf{k}\left(\left(t^{\Gamma}\right)\right) := \{f(t) = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \mid a_\gamma \in \mathbf{k}, \text{ and } \operatorname{supp}(f) \text{ is well-ordered}\},$$

where $\operatorname{supp}(f) = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$. K is a field with a natural addition and multiplication, and it is a valued field with the following valuation:

$$\begin{aligned} v : K &\longrightarrow \Gamma, \\ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma &\longmapsto \min\left\{\gamma \mid a_\gamma \neq 0\right\}. \end{aligned}$$

Suppose $\mathbf{k}$ is a field and $\bar{\sigma}$ is an automorphism on $\mathbf{k}$. Also assume that an ordered abelian group Γ with an order preserving automorphism σ_Γ on it are given. Using these automorphisms, an automorphism on the corresponding Hahn-field is defined. This automorphism σ of K is defined as follows:

$$\begin{aligned} \sigma : K &\longrightarrow K, \\ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma &\longmapsto \sum_{\gamma \in \Gamma} \bar{\sigma}(a_\gamma) t^{\sigma_\Gamma(\gamma)}. \end{aligned}$$

The following is a concrete example of a Hahn-field.

Example 2.0.10. Let $\mathbf{k}$ be the field of complex numbers, $\mathbb{C}$, and Γ be $\mathbb{R}$, regarded as an ordered abelian group. As it is defined above, the corresponding Hahn-field is

$$K = \mathbb{C}\left(\left(t^{\mathbb{R}}\right)\right) = \{f(t) = \sum_{\gamma \in \mathbb{R}} a_\gamma t^\gamma \mid a_\gamma \in \mathbb{C}, \text{ and } \operatorname{supp}(f) \text{ is well-ordered}\}.$$

Assume $\mathbb{C}$ is considered with the identity automorphism; in this case, $(\mathbb{C}, id)$ is called a constant difference field. If we consider $\mathbb{R}$ as an ordered abelian group, then any automorphism σ_Γ on $\mathbb{R}$ is of the following form:

$$x \mapsto \sigma_\Gamma(x) = \rho \cdot x \text{ for some fixed } \rho > 0.$$

To see this, refer to Remark 2.0.12. Using these two automorphisms, we can define an automorphism σ on the Hahn-field K as follows:

$$\sigma : K \longrightarrow K,$$
$$\sum_{\gamma\in\mathbb{R}} a_\gamma t^\gamma \longmapsto \sum_{\gamma\in\mathbb{R}} a_\gamma t^{\rho\cdot\gamma}.$$

Then (K, v, σ) is a valued difference field.

Remark 2.0.11. The valuation and the automorphism of a valued difference field can interact in different ways. This interaction results in different cases of valued difference fields. Below, we consider three interesting cases:

- The *isometric* case in which we have

$$\forall x \in K^\times : \ v(\sigma(x)) = v(x).$$

- The *contractive* case in which $\forall x \in K^\times$ with $v(x) > 0$, we have

$$\forall n \in \mathbb{N} : \ v(\sigma(x)) > nv(x). \tag{2.0.1}$$

  The eager reader can see [3] to learn more about the isometric case, and [1] to know more about contractive valued difference fields.

- The *multiplicative* case in which the difference value group is a $\mathbb{Z}[\rho]$-module with ρ being a positive real number, and we have

$$\forall x \in K^\times : \ v(\sigma(x)) = \rho \cdot v(x).$$

  This case is well studied in [21].
  Note that the isometric case is a special case of the multiplicative case.

The following remark is well-known. We present a proof for clarity.

Remark 2.0.12. Let Γ be a divisible subgroup of $\mathbb{R}$. Let σ be an automorphism of the ordered abelian group Γ. Then for a fixed positive real number ρ, we have

$$\forall x \in \Gamma, \ \sigma(x) = \rho \cdot x. \tag{2.0.2}$$

Proof. Without loss of generality, we assume that Γ contains 1. Otherwise, we rescale ρ and a similar proof works. We make a case distinction, and we show that 2.0.2 holds for $\rho = \sigma(1)$.

1. $x \in \mathbb{Q}$:

   A rational number x can be written as $x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. Based on the properties of an automorphism, we have

   $$\sigma(1) = \sigma(n \cdot \frac{1}{n}) = n \cdot \sigma(\frac{1}{n}),$$

   which means

   $$\sigma(\frac{1}{n}) = \frac{\sigma(1)}{n}. \tag{2.0.3}$$

   From (2.0.3), for $x = \frac{m}{n}$, we can write

   $$\sigma(\frac{m}{n}) = \sigma(m \cdot \frac{1}{n}) = m \cdot \sigma(\frac{1}{n}) = m \cdot \frac{\sigma(1)}{n} = \frac{m}{n} \cdot \sigma(1).$$

   Since σ is an automorphism of ordered abelian groups, it preserves order. This means that in this case 2.0.2 holds for $\rho = \sigma(1) > 0$.

2. $x \in \mathbb{Q}^c$:

   To discuss this case, firstly, we claim that σ is a continuous map.

   Claim 2.0.13. *If σ is an automorphism of an ordered abelian group Γ, then it is a continuous map.*

   Proof. Let x_0 be an arbitrary element of Γ. Assume $\varepsilon > 0$ is given. There exists $n > 0$ such that $\frac{1}{n} < \varepsilon$. Define $\delta := \sigma^{-1}(\frac{1}{n})$. If $|x - x_0| < \delta$, then we have

   $$\begin{aligned} |x - x_0| < \delta &\iff -\sigma^{-1}(\frac{1}{n}) < x - x_0 < \sigma^{-1}(\frac{1}{n}) \\ &\iff -\frac{1}{n} < \sigma(x - x_0) < \frac{1}{n} \\ &\iff -\frac{1}{n} < \sigma(x) - \sigma(x_0) < \frac{1}{n} \\ &\iff |\sigma(x) - \sigma(x_0)| < \frac{1}{n} < \varepsilon, \end{aligned}$$

   where the second equivalence results from the fact that σ is an order preserving map. ■

For the irrational number x, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q} \subset \Gamma$ converging to x. Thus, from continuity of σ, we can write

$$\sigma(x) = \sigma(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \sigma(x_n) = \lim_{n\to\infty} x_n \cdot \sigma(1) = \sigma(1) \cdot \lim_{n\to\infty} x_n = \sigma(1) \cdot x.$$

Therefore, in this case, 2.0.2 holds for $\rho = \sigma(1) > 0$.

□

Throughout this thesis, we will assume that the difference value group is a divisible subgroup of $\mathbb{R}$. If we consider the induced automorphism on Γ, for a fixed ρ, we have

$$\forall x \in K^{\times}: \ v(\sigma(x)) = \sigma_{\Gamma}(v(x)) = \rho \cdot v(x).$$

This means that, in this thesis, we work with multiplicative valued difference fields. See Assumption 2.0.45. We call ρ the *scaling exponent* of the automorphism σ.

Remark 2.0.14. In Assumption 2.0.45, we will make further assumptions about Γ. Specifically, we assume that Γ is a $\mathbb{Q}(\rho)$-module.

Definition 2.0.15 ([23], Subsection 1.1.4)**.** Let (K,σ) be a difference field. The *difference polynomial ring* over K, in difference variables $x = (x_1,\ldots,x_n)$, is denoted by $K_\sigma[x]$. It is the polynomial ring over K in formal variables $\sigma^i(x_j)$ for $i \in \mathbb{N}$ and $j \in \{1,\ldots,n\}$ where $\sigma^0(x_j) := x_j$. In other words, we have

$$K_\sigma[x] = K\left[\sigma^i(x_j) \mid i \in \mathbb{N},\ j \in \{1,\ldots,n\}\right].$$

Any element of a difference polynomial ring is called a *difference polynomial*.

Note that here by x we mean $(x_1,\ldots,x_n)$, even though in some parts of this work, x may refer to a single variable, which will be clear from the context.

Similarly, we can define the *ring of Laurent difference polynomials* in n difference variables over K, which is denoted by $K_\sigma[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$. Any element of this ring is called a *Laurent difference polynomial*. From the notation, it is clear how to evaluate a Laurent difference polynomial f at an element a of the field K.

There is a natural way to extend the automorphism σ of K, to an endomorphism σ of the difference polynomial ring. It is the unique endomorphism satisfying the following condition:

$$\sigma(\sigma^i(x_j)) := \sigma^{i+1}(x_j).$$

Considering this extension, $(K_\sigma[x], \sigma)$ is a difference ring.

Notation 2.0.16. We commonly use the notation $\sigma(x) = x^\sigma$. By using this notation, the difference monomial $x^{a_0}\sigma(x)^{a_1}\ldots\sigma^m(x)^{a_m}$ in one variable x can be written as $x^{a_0+a_1\sigma+\cdots+a_m\sigma^m}$.

If $\mathbb{Z}[\sigma]$ denotes the set

$$\{\sum_{i=0}^{m} a_i\sigma^i | \, \forall i,\ a_i \in \mathbb{Z} \text{ and } \sigma^i \text{ is the i-th iteration of } \sigma\},$$

by using this notation, all the exponents appearing in a Laurent difference polynomial in n variables are elements of $(\mathbb{Z}[\sigma])^n$, which are called *σ-powers*. Note that in this case, we use the notation $x^{u(\sigma)} := x_1^{u_1(\sigma)}\cdots x_n^{u_n(\sigma)}$. This means, if f is a Laurent difference polynomial in variables $x_1,\ldots,x_n$, it can be written as

$$f(x) = \sum_{u(\sigma)\in\Lambda} c_{u(\sigma)}x^{u(\sigma)},$$

where Λ is a finite subset of $(\mathbb{Z}[\sigma])^n$.

If we consider the subset $\mathbb{N}[\sigma]$ of $\mathbb{Z}[\sigma]$ in which for each i, a_i is an element of $\mathbb{N}$, then the σ-powers appearing in a difference polynomial in n variables are elements of $(\mathbb{N}[\sigma])^n$. This means that any difference polynomial f, in variables $x_1,\ldots,x_n$ can be written as follows:

$$f(x) = \sum_{u(\sigma)\in\Lambda} c_{u(\sigma)}x^{u(\sigma)},$$

with Λ being a finite subset of $(\mathbb{N}[\sigma])^n$.

The following example clarifies Notation 2.0.16.

Example 2.0.17. Consider

$$f(x_1,x_2) = x_1^2x_2 + \sigma(x_1)^4 + x_1\sigma^3(x_2)x_2.$$

If we use Notation 2.0.16, it can be rewritten as:

$$f(x_1, x_2) = x_1^2 x_2 + x_1^{4\sigma} + x_1 x_2^{\sigma^3+1}.$$

This is a difference polynomial in $\mathbb{C}_\sigma[x]$, with $x = (x_1, x_2)$.

Remark 2.0.18. In the case of a difference polynomial in one variable, sometimes it is more convenient to use the following notations (namely Remark 2.0.21) rather than the one defined in Notation 2.0.16.

Notation 2.0.19. J is called a *multi-index*, if it is an element of $\mathbb{N}^{n+1}$. For $J = (j_0, j_1, \ldots, j_n)$, its *length* which is denoted by $|J|$ is defined as follows:

$$|J| = j_0 + j_1 + \cdots + j_n.$$

For a positive real number ρ, the ρ-*length* of J which is denoted by $|J|_\rho$, is defined as follows:

$$|J|_\rho = \rho^0 \cdot j_0 + \rho^1 \cdot j_1 + \cdots + \rho^n \cdot j_n.$$

Throughout the rest of this thesis, when we write $|J|_\rho$, ρ refers to the scaling exponent of σ.

Notation 2.0.20. For an automorphism σ, and an n which is clear from the context, by $\boldsymbol{\sigma}(x)$ we mean the following tuple:

$$\boldsymbol{\sigma}(x) = \big(\sigma^0(x), \sigma(x), \ldots, \sigma^n(x)\big).$$

For a multi-index $J = (j_0, j_1, \ldots j_n)$, by $\boldsymbol{\sigma}^J(x)$, we mean

$$\begin{aligned}\boldsymbol{\sigma}^J(x) &= x^{j_0} \cdot \sigma(x)^{j_1} \cdots (\sigma^n(x))^{j_n} \\ &= x^{j_0 + j_1\sigma + \cdots + j_n\sigma^n}.\end{aligned}$$

Remark 2.0.21. Let f be a difference polynomial in one variable. Using Notation 2.0.20, f is of this form

$$f(x) = \sum_{J \in \Lambda} c_J \boldsymbol{\sigma}^J(x),$$

where Λ is a finite subset of $\mathbb{N}^{n+1}$.

Remark 2.0.22. As an example, the difference monomial $x\sigma(x)$ can be written as $\boldsymbol{\sigma}^{(1,1)}(x)$ using Notation 2.0.20, and it can also be expressed as $x^{1+\sigma}$ using Notation 2.0.16. Throughout this work, for difference polynomials in one variable, we will switch between these two notations.

Remark 2.0.23. Suppose $f \in K_\sigma[x^{\pm 1}]$ is a Laurent difference polynomial. Then it is of the form $f(x) = \sum_{J\in\Lambda} c_J \boldsymbol{\sigma}^J(x)$ where Λ is a finite subset of $\mathbb{Z}^{n+1}$. Given Λ as above, define $J_{\max}$ to be the multi-index such that

$$\forall i,\ \ 0 \le i \le n,\ \ (J_{\max})_i = \begin{cases} 0 & \text{if } \ \forall J \in \Lambda\ j_i \ge 0, \\ \max\{|j_i| \mid j_i < 0\} & \text{if } \ j_i < 0 \text{ for some } \ J \in \Lambda. \end{cases}$$

Multiplying $f(x)$ by $\boldsymbol{\sigma}^{J_{\max}}(x)$ gives a difference polynomial $g(x)$. In other words, we have

$$f(x) \cdot \boldsymbol{\sigma}^{J_{\max}}(x) = g(x) \in K_\sigma[x].$$

As explained in Remark 2.0.18, in the case of difference polynomials in one variable, we may use the notation defined in Remark 2.0.21 for convenience. Therefore, we present a similar remark to Remark 2.0.23 using this notation.

Remark 2.0.24. If f is a Laurent difference polynomial in difference variables $x = (x_1, \dots, x_n)$, using Notation 2.0.16, it is of the following form:

$$f(x) = \sum_{u(\sigma)\in\Lambda} c_{u(\sigma)} x^{u(\sigma)},$$

where Λ is a finite subset of $(\mathbb{Z}[\sigma])^n$. Suppose $u(\sigma) = (u_1(\sigma), \dots, u_n(\sigma))$ is one of the exponents appearing in f. For each i, $1 \le i \le n$, we have

$$u_i(\sigma) = \sum_{j_i=0}^{m_i} a_{j_i} \sigma^{j_i},$$

where for each j_i, $a_{j_i} \in \mathbb{Z}$. Define $|u_i|_\circ(\sigma) = \sum_{j_i=0}^{m_i} \tilde{a}_{j_i} \sigma^{j_i}$ such that $\tilde{a}_{j_i} = |a_{j_i}|$ if a_{j_i} is negative, and $\tilde{a}_{j_i} = 0$ otherwise. Set $|u|_\circ(\sigma) = (|u_1|_\circ(\sigma), \dots, |u_n|_\circ(\sigma))$. Multiplying $f(x)$ by $\prod_{u(\sigma)\in\Lambda} x^{|u|_\circ(\sigma)}$ gives a difference polynomial $g(x)$. In other words, we have

$$f(x) \cdot \prod_{u(\sigma)\in\Lambda} x^{|u|_\circ(\sigma)} = g(x) \in K_\sigma[x_1, \dots, x_n].$$

Definition 2.0.25 ([21], §3)**.** To any difference polynomial in a single variable, $f(x) = \sum_{J\in\Lambda} c_J \boldsymbol{\sigma}^J(x)$, a polynomial $P(\boldsymbol{x}) = \sum_{J\in\Lambda} c_J \boldsymbol{x}^J$ is associated, where $\boldsymbol{x} = (x_0, \dots, x_n)$, so that $f(x) = P(\boldsymbol{\sigma}(x))$.
We use the notation $I! := i_0! i_1! \dots i_n!$ with $I = (i_0, \dots, i_n)$. Then for the multi-index I and a point a, $f_{(I)}(a)$ is defined as follows:

$$f_{(I)}(a) = P_{(I)}(\boldsymbol{\sigma}(a)) = \frac{\partial^{|I|} P(a, \sigma(a), \dots, \sigma^n(a))}{\partial x_0^{i_0} \partial x_1^{i_1} \dots \partial x_n^{i_n}} \cdot \frac{1}{I!}.$$

Note that for any multi-index I, we have $f_{(I)}(0) = c_I$.

Similarly, to any Laurent difference polynomial f, a Laurent polynomial P is associated.

Definition 2.0.26. A difference field (k, σ) is called an *ACFA*, if for any finite system of difference polynomial equations over k with a solution in an extension k' of k, this system has a solution in k. This concept can be considered as a difference version of algebraically closed fields.

To know more on this topic, see [4].

Remark 2.0.27. In the same way as one constructs the algebraic closure of a field, one constructs a difference algebraic closure of a difference field; this is an ACFA.

Unlike the algebraic closure of a field, a difference algebraic closure of a difference field is not unique.

It is clear from the definition of an ACFA that any ACFA is an algebraically closed field.

In the next few pages, we present an important lemma concerning ACFA, which directly follows. We prove it using two distinct methods. Both proofs have their own interest, as each considers ACFA from a different perspective. Prior to each proof, we will provide the necessary materials required for that proof.

Lemma 2.0.28. Let k be an ACFA. Suppose f is a Laurent difference polynomial in one variable with coefficients from k that is not a monomial. Then f has a nonzero root in k.

- **The materials for the first proof:**

  The main reference for this part is [4].

  Definition 2.0.29. Let (K,σ) be a difference field. Then σ extends canonically to an automorphism of $K[x_1,\ldots,x_n]$ (For all i, $1 \leq i \leq n$; $\sigma(x_i) = x_i$). It is denoted by the same notation as σ.

  Let K be an algebraically closed field. By a variety, we mean an irreducible Zariski closed subset of K^n.

  Suppose U is a variety over K. For $I(U) = \{f \in K[x_1,\ldots,x_n] \mid f(U) = 0\}$, $\sigma(I(U))$ is defined as follows:

  $$\sigma(I(U)) = \{\sigma(f) \mid f \in I(U)\}.$$

  In this case, the variety $V(\sigma(I(U)))$ is denoted by U^σ.

  Suppose V is another variety over K, such that $V \subseteq U \times U^\sigma$, and two projection maps are given as follows:

  $$\pi_1 : U \times U^\sigma \to U$$
  $$\pi_2 : U \times U^\sigma \to U^\sigma.$$

  Then the projection of V to U (to U^σ) is called *generically onto*, if $\pi_1(V)$ (if $\pi_2(V)$) is Zariski dense in U (in U^σ).

  The following theorem is not a result of this thesis. It is derived from the axioms in 3.1 and Theorem 3.2 of [4].

  Theorem 2.0.30. *Let (k,σ) be a difference field. Then it is an ACFA if and only if for any two varieties U and V over k, with $V \subseteq U \times U^\sigma$, k satisfies the following conditions:*

  1. *k is algebraically closed;*
  2. *If V projects generically onto U and U^σ, then there exists a point $a = (a_1,\ldots,a_n) \in k^n$, such that $(a,\sigma(a)) \in V$.*

Proof. For the proof, see Theorem 3.2 in [4]. □

Below, in the first proof, we use Theorem 2.0.30 as the main ingredient to prove Lemma 2.0.28.

- **The first proof of Lemma 2.0.28:**

Proof. Let f be a Laurent difference polynomial, such that $\sigma^n(x)$ is the greatest iteration of σ appearing in f. If $n = 0$, f is a Laurent polynomial in one variable with coefficients from an ACFA. Since any ACFA is an algebraically closed field and f is not a monomial, it has a nonzero root in k. Therefore, we assume that $n \neq 0$. We say that the order of f is n. From Definition 2.0.25, there exists a Laurent polynomial P in $k[y_0^{\pm 1}, \ldots, y_n^{\pm 1}]$, for which we have

$$f(x) = P(x, \sigma(x), \ldots, \sigma^n(x)).$$

By Remark 2.0.23, we can multiply f by $\boldsymbol{\sigma}^{J_{\max}}(x)$, and convert it to a difference polynomial in $k_\sigma[x]$. Therefore, from now on, we assume that f is in $k_\sigma[x]$, and is irreducible.
We define the three following sets:

$$\begin{aligned} U &= \{(\underline{r}, s_0) \mid P(\underline{r}) = 0, r_0 \cdots r_n s_0 - 1 = 0\}; \\ \sigma(U) &= \{(\underline{t}, w_0) \mid \sigma(P)(\underline{t}) = 0, t_0 \cdots t_n w_0 - 1 = 0\}; \\ V &= \{(\underline{r}, s_0, \underline{t}, w_0) \in U \times \sigma(U) \mid t_i = r_{i+1} \text{ for } 0 \leq i \leq n-1\}, \end{aligned}$$

where $\underline{r} = (r_0, \ldots, r_n)$, and $\underline{t} = (t_0, \ldots, t_n)$.

To find a nonzero root of f, we apply Theorem 2.0.30. If we prove that V projects generically onto U and $\sigma(U)$, then from this theorem, there exists a point $a = (a_0, \ldots, a_n, b_0)$ such that $(a, \sigma(a)) \in V$. Finally, we prove that a_0 is a root of f.

Now, we prove that V projects generically onto U. Since f, and consequently P, is irreducible, and as the order of f is $n \neq 0$, there are at least two monomials in $\sigma(P)$ with different powers of y_n. Suppose l and m are two different powers of y_n appearing in $\sigma(P)$. Regarding $\sigma(P)$ as a polynomial in y_n with coefficients in $k[y_0, \ldots, y_{n-1}]$, it can be written as

$$\sigma(P) = \sum_{i \in I} g_i(y_0, \ldots, y_{n-1}) y_n^i,$$

where I is a finite subset of $\mathbb{N}$. Define

$$h(y_0,\dots,y_{n-1}) = g_l(y_0,\dots,y_{n-1})\cdot g_m(y_0,\dots,y_{n-1}). \tag{2.0.4}$$

Using the following claim, we prove that V projects generically onto U. Similarly it can be proven that V projects generically onto $\sigma(U)$.

Claim 2.0.31. *Let $(\underline{r},s_0)$ be in U, such that $h(r_1,\dots,r_n)\neq 0$. Then $(\underline{r},s_0)$ is in $\pi_1(V)$.*

Proof. In order to show that $(\underline{r},s_0)\in\pi_1(V)$, we look for a point $(\underline{t},w_0)$, such that $(\underline{r},s_0,\underline{t},w_0)$ is in V. For any i, $0\leq i\leq n-1$, set $t_i = r_{i+1}$. Therefore, we have

$$h(t_0,\dots,t_{n-1}) = h(r_1,\dots,r_n)\neq 0.$$

From (2.0.4), we obtain $g_l(t_0,\dots,t_{n-1})\neq 0$, and $g_m(t_0,\dots,t_{n-1})\neq 0$. This means that $\sigma(P)(t_0,\dots,t_{n-1},y_n)$ is a polynomial in y_n with coefficients in k, with at least two monomials. Since k is an ACFA, it is an algebraically closed field. Hence, this polynomial has a nonzero root t_n in k. This gives the nonzero root $(t_0,\dots,t_n)$ of $\sigma(P)$. Also set $w_0 = \frac{r_0 s_0}{t_n}$. Then, we have

$$\begin{aligned} t_0\cdots t_{n-1}t_n w_0 - 1 &= r_1\cdots r_n t_n\frac{r_0 s_0}{t_n} - 1 \\ &= r_0\cdots r_n s_0 - 1 = 0. \end{aligned}$$

Thus, $(\underline{t},w_0)\in\sigma(U)$, and consequently $(\underline{r},s_0,\underline{t},w_0)\in V$. ∎

Remark 2.0.32. Let $D(h) = k^{n+2}\smallsetminus V(h)$ be the Zariski open set defined by h. Then $D(h)\cap U$ is nonempty.

Proof. As we discussed on the previous page, there are at least two monomials in $\sigma(P)$ with different powers of y_n. Similarly, P has at least two monomilas with different powers of y_n. Suppose l and m are two different powers of y_n appearing in P. Regard P as a polynomial in one variable y_n with coefficients in $k[y_0,\dots,y_{n-1}]$. Then P is of the following form:

$$P = \sum_{i\in I} q_i(y_0,\dots,y_{n-1})y_n^i,$$

where I is a finite subset of $\mathbb{N}$. Define

$$h'(y_0,\dots y_{n-1}) = q_l(y_0,\dots y_{n-1}) \cdot q_m(y_0,\dots y_{n-1}) \cdot y_0 \cdots y_{n-1}.$$

Since h and h' are nonzero polynomials, $h \cdot h'$ has a nonroot $(r_0,\dots r_{n-1})$. Therefore, we have $h(r_0,\dots,r_{n-1}) \neq 0$, which means that this is a point in $D(h)$. On the other hand, we have $h'(r_0,\dots,r_{n-1}) \neq 0$. This means that $P(r_0,\dots r_{n-1},y_n)$ is a nonzero polynomial with coefficients in k. Since k is an algebraically closed field, and P (regarded as a polynomial in one variable y_n) has at least two monomials, it has a nonzero root r_n in k. Hence, $(r_0,\dots,r_{n-1},r_n)$ is a root of P. From the definition of h', it is clear that for all i, $1 \leq i \leq n-1$, we have $r_i \neq 0$. Set $s_0 = \dfrac{1}{r_0 \cdots r_n}$, then $(r_0,\dots,r_n,s_0)$ is a point in U, which is also a point of $D(h)$. Thus, $D(h) \cap U$ is nonempty.

■

From the previous claim, we have

$$D(h) \cap U \subseteq \pi_1(V).$$

By taking the Zariski closure, we obtain the following relation:

$$\overline{D(h)} \cap \overline{U} \subseteq \overline{\pi_1(V)}.$$

Since Zariski open sets are dense, we have $U \subseteq \overline{\pi_1(V)}$. Moreover, from $\overline{\pi_1(V)} \subseteq U$, we obtain $\overline{\pi_1(V)} = U$. This means that $\pi_1(V)$ is Zariski dense in U, and V projects generically onto U. With a similar argument, V projects generically onto $\sigma(U)$. Hence, from Theorem 2.0.30, there exists a point $a = (a_0,\dots,a_n,b_0)$, such that $(a,\sigma(a)) \in V$, where $\sigma(a) = (\sigma(a_0),\dots \sigma(a_n),\sigma(b_0))$, and $a_{i+1} = \sigma(a_i)$. This means that

$$f(a_0) = P(a_0,\sigma(a_0),\dots \sigma^n(a_0)) = P(a_0,a_1,\dots,a_n) = 0.$$

Since $a \in U$, we have $a_0 \neq 0$. Hence, f has a nonzero root a_0 in k.

□

- **The materials for the second proof:**

  The reader can consult [23] as the reference of this part.

  Definition 2.0.33. Let (R,σ) be a difference ring. Then I is called a *difference ideal*, if it is an ideal of R such that $\sigma(I) \subseteq I$.

  The intersection of difference ideals is a difference ideal of R. If F is a subset of R, the intersection of all difference ideals containing F is the smallest difference ideal containing F. It is called the *difference ideal generated* by F, and is denoted by $[F]$.

  Definition 2.0.34. Let I be a difference ideal of a difference ring (R,σ). Then I is called *perfect* if for any element a of R, $a\sigma(a) \in I$ implies $a \in I$.

  The intersection of perfect difference ideals is perfect. Let F be a subset of R. Then the *perfect difference ideal generated* by F, or the *perfect closure* of F is defined as the intersection of all perfect difference ideals containing F, and is denoted by $\{F\}$.

  The perfect closure of a subset F of R is described in a recursive way which is explained in the following remark:

  Remark 2.0.35. Let I be a difference ideal of a difference ring (R,σ). Then the set I' is defined as follows:

  $$I' := \{a \in R \mid \sigma^{i_1}(a)\dots\sigma^{i_n}(a) \in I \text{ for some } i_1 \dots i_n \in \mathbb{N}\}.$$

  Let F be a subset of R. Then $F^{\{1\}}$ is defined as $[F]'$, and $F^{\{i\}}$ is defined as follows:

  $$F^{\{i\}} := [F^{\{i-1\}}]' \text{ for } i \geq 2.$$

  We have $\{F\} = \bigcup_{i\geq 1} F^{\{i\}}$. The truth of this equality is discussed in [23].

  Definition 2.0.36. A difference ideal I of a difference ring (R,σ) is called *transformally prime*, if it is a prime ideal of R for which we have $\sigma^{-1}(I) = I$.

Definition 2.0.37. The *difference spectrum of R*, denoted by $\mathrm{Spec}^\sigma(R)$, is the set of all transformally prime difference ideals of R. This set is usually considered with a topology called *Cohn topology* whose closed sets are as follows:

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}^\sigma(R) \mid \mathfrak{p} \text{ contains } I\},$$

where I is a difference ideal of R.

Remark 2.0.38. Let (R,σ) be a difference ring, and $\mathfrak{p}$ be a transformally prime ideal of R. Denote the complement of $\mathfrak{p}$ by S. This is a multiplicatively closed subset of R, and we have $\sigma(S) \subset S$. Then $R_\mathfrak{p} := S^{-1}R$ is a difference ring, and $S^{-1}\mathfrak{p}$ is its unique maximal ideal.

Consider the quotient of $R_\mathfrak{p}$ by $S^{-1}\mathfrak{p}$. This is denoted by $k(\mathfrak{p})$. It is a field together with an endomorphism which is defined naturally. By Proposition 2.0.3, the inversive closure of $k(\mathfrak{p})$ is a difference field. We denote it by $k(\mathfrak{p})^*$, and call it the *residue difference field at* $\mathfrak{p}$.

Lemma 2.0.39 ([23], Lemma 1.2.21). Let (R,σ) be a difference ring. Suppose $\mathcal{A}$ is the set of all proper perfect difference ideals of R. Let I be a maximal element of $\mathcal{A}$. Then I is transformally prime.

Proof. For the proof, see Lemma 1.2.21 of [23] or page 88 of [5]. □

Lemma 2.0.40 ([23], Proposition 1.2.35). Let (R,σ) be a difference ring. Suppose $\mathcal{A}$ is the set of all perfect difference ideals of R, and $\mathcal{B}$ is the set of all closed subsets of $\mathrm{Spec}^\sigma(R)$. Consider the following map:

$$\begin{aligned} \phi : \mathcal{A} &\longrightarrow \mathcal{B} \\ I &\mapsto V(I). \end{aligned}$$

Then ϕ is an inclusion-reversing bijection.

Proof. The proof can be found in [23], Proposition 1.2.35(ii) or [9], subsection 3.1. □

The following theorem is obtained from Proposition 3.3.4 and Theorem 3.3.8 of [23] and is not a result of this thesis.
One can also see the same result by considering Lemma 2.0.40, Theorem II and Theorem V in Chapter 3 of [5].

Theorem 2.0.41. *Let* (R,σ) *be a difference ring, and* $R_\sigma[x]$ *for* $x = (x_1,\ldots,x_n)$ *be the difference polynomial ring as defined in Definition 2.0.15. If* $\mathrm{Spec}^\sigma(R)$ *is a Noetherian topological space, then* $\mathrm{Spec}^\sigma(R_\sigma[x])$ *is a Noetherian topological space.*

Proof. Proposition 3.3.4 and Theorem 3.3.8 of [23] together provide the proof of this theorem. □

Here, we prove Lemma 2.0.28 with another method. The main idea of this proof is to use the definition of an ACFA field (Definition 2.0.26).

- **The second proof of Lemma 2.0.28:**

Proof. Similar to what we did at the beginning of the first proof, we use again Remark 2.0.23, and we assume that f is an irreducible difference polynomial in $k_\sigma[x]$.

To find a nonzero root of f, we apply the definition of ACFA. This means that it suffices to find an extension L of the field k such that f has a nonzero root in L. Then by Definition 2.0.26 f has a root in k, and we show that this root is nonzero.

Consider the following system of difference equations in $k_\sigma[x,y]$:

$$\begin{cases} f(x) = 0, \\ g(x,y) = xy - 1 = 0, \end{cases} \tag{2.0.5}$$

Set $F = \{f,g\}$ which is a subset of the difference polynomial ring $k_\sigma[x,y]$. By Remark 2.0.35, we consider the perfect difference ideal $\{F\}$ generated by F which is a proper difference ideal. We make a case distinction:

1. $\{F\}$ is maximal: By Lemma 2.0.39, $\{F\}$ is transformally prime. Consider $k(\{F\})^*$, the residue difference field at $\{F\}$. This is a difference field. As $(x+\{F\}, y+\{F\})$ is a solution of 2.0.5 in $k(\{F\})$, this system has a solution in $k(\{F\})^*$. Since k is an ACFA, 2.0.5 has a solution (x_0, y_0) in k. This means that $x_0y_0 - 1 = 0$, and consequently x_0 is a nonzero root of $f(x)$.
2. $\{F\}$ is not maximal: In this case, there exists a proper perfect difference ideal I_1 such that $\{F\} \subset I_1$. If I_1 is maximal, then by repeating the procedure in the first case, 2.0.5 has a solution in $k(I_1)^*$ which gives a nonzero root of $f(x)$. Otherwise, there exists another proper perfect difference ideal I_2 such that $\{F\} \subset I_1 \subset I_2$. Continuing the same method, as long as we do not obtain a maximal perfect difference ideal, we obtain an ascending chain $\{F\} \subset I_1 \subset I_2 \subset \ldots$ of proper perfect ideals containing $\{F\}$. Since k is a difference field, we have $\mathrm{Spec}^{\sigma}(k) = \emptyset$. Therefore, $\mathrm{Spec}^{\sigma}(k)$ is a Noetherian topological space. By Theorem 2.0.41, $\mathrm{Spec}^{\sigma}(k_{\sigma}[x,y])$ is a Noetherian topological space. Using the inclusion-reversing bijection of Lemma 2.0.40, any ascending chain of proper perfect difference ideals of $k_{\sigma}[x,y]$ is finite. This means that this chain stops, and we obtain a maximal perfect difference ideal $\mathcal{M}$ such that $\{F\} \subset \mathcal{M}$. Thus, 2.0.5 has a solution in $k(\mathcal{M})^*$, this gives a nonzero root of $f(x)$ in k.

□

In the following theorem, we extend Lemma 2.0.28 to the case where f is a Laurent difference polynomial in n variables.

Theorem 2.0.42. *Let k be an ACFA. Suppose f is in $k_{\sigma}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and is not a monomial. Then f has a root in $(k^*)^n$.*

Proof. Remark 2.0.23 enables us to convert a Laurent difference polynomial to a difference polynomial. Therefore, we assume that f is an element of $k_{\sigma}[x_1, \ldots, x_n]$.
Since f is not a monomial, we assume that $c_{u(\sigma)}x_1^{u_1(\sigma)} \cdots x_n^{u_n(\sigma)}$ and

$c_{u'(\sigma)} x_1^{u'_1(\sigma)} \cdots x_n^{u'_n(\sigma)}$ are two distinct monomials of f. Therefore, for some i, $1 \le i \le n$, we have $u_i(\sigma) \ne u'_i(\sigma)$. Without loss of generality, we assume that $i = n$, and we have at least two monomials with different σ-powers of x_n.
Regard f as a difference polynomial in one variable x_n, with coefficients in $k_\sigma[x_1, \ldots, x_{n-1}]$. We write f as follows:

$$f = \sum_{k=1}^{N} g_k (x_1, \ldots, x_{n-1}) x_n^{u_{(n,k)}(\sigma)},$$

where $u_{(n,k)}(\sigma)$ denotes distinct σ-powers of x_n appearing in f.
Define $h(x_1, \ldots, x_{n-1}) = g_1 (x_1, \ldots, x_{n-1}) \cdots g_N (x_1, \ldots, x_{n-1})$. The difference polynomial h is nonzero. In the following claim, we want to find $(a_1, \ldots, a_{n-1}) \in (k^*)^{n-1}$ such that $h(a_1, \ldots, a_{n-1}) \ne 0$. In this case, $f(a_1, \ldots, a_{n-1}, x_n)$ is a difference polynomial in one variable with coefficients from an ACFA which is not a monomial. Therefore, from Lemma2.0.28 it has a nonzero root a_n. Hence, $(a_1, \ldots, a_{n-1}, a_n)$ is a root of f in $(k^*)^n$, and the proof is complete.

Claim 2.0.43. *Let k be an ACFA. Suppose h is a nonconstant difference polynomial in $k_\sigma[x_1, \ldots, x_{n-1}]$. Then h has a nonroot in $(k^*)^{n-1}$.*

Proof. We prove this claim by induction over $n-1$. Suppose $n-1 = 1$, and consider the difference polynomial $h(x_1) + b$, where $0 \ne b \ne -c_0$ for c_0 being the constant term of h. This is an element of $k_\sigma[x_1]$, which is not a monomial. Since k is an ACFA, Lemma 2.0.28 implies that this polynomial has a nonzero root α_1. This means that $h(\alpha_1) + b = 0$, and consequently $h(\alpha_1) \ne 0$.
We assume that the statement holds, if the number of variables is less than $n-1$. We prove the statement for h in $k_\sigma[x_1, \ldots, x_{n-1}]$. We write h as a polynomial in one variable x_{n-1}, with coefficients from $k_\sigma[x_1, \ldots, x_{n-2}]$. More precisely, h is of the following form:

$$h = \sum_{k=1}^{M} h_k (x_1, \ldots, x_{n-2}) x_{n-1}^{u_{(n-1,k)}(\sigma)}.$$

For each k, $1 \le k \le M$, we have $h_k \in k_\sigma[x_1, \ldots, x_{n-2}]$. By induction assumption, for each k, there exists $\left(\alpha_{(k,1)}, \ldots, \alpha_{(k,n-2)}\right)$ in $(k^*)^{n-2}$ that is a nonroot of h_k. Therefore, there exists a point $(a_1,\ldots,a_{n-2})$ such that $h(a_1,\ldots,a_{n-2},x_{n-1})$ is a nonzero difference polynomial in one variable x_{n-1}. Thus, from the first step of the induction, it has a nonroot a_{n-1}. This means that $(a_1,\ldots,a_{n-2},a_{n-1})$ is a nonroot of h. ■

As explained before this claim, using the nonroot $(a_1,\ldots,a_{n-1})$ of h from the previous claim, we find a root $(a_1,\ldots,a_{n-1},a_n)\in(k^*)^n$ of f. □

In this thesis, we assume that the field K we work with is spherically complete. Below, we present the definition of spherical completeness. This concept is well known, and one can consult [22] as a reference.

Definition 2.0.44. Let (K,v) be a valued field. We consider a totally ordered collection $\{B_i\}_{i\in I}$ of balls in K. Then K is called *spherically complete*, if for every such collection we have $\bigcap_{i\in I} B_i \neq \emptyset$.
Hahn fields are spherically complete, see [15] and [13].

Assumption 2.0.45. (*General assumptions*) Here are some of the general assumptions we make throughout the present work:

- (K,σ) is a multiplicative valued difference field which is spherically complete and of characteristic zero. We also assume that ρ, the scaling exponent of σ is transcendental.(The importance of this assumption is explained in Definition 4.1.1.)
- The valuation has a splitting. This splitting $\psi:\Gamma\to K$ interacts with σ_Γ as follows:
$$\forall a\in\Gamma:\psi(\sigma_\Gamma(a))=\sigma(\psi(a)).$$
We also use the notation $\psi(a)=t^a$.
- The difference residue field is an ACFA and is of characteristic zero.
- The difference value group Γ is a subgroup of $\mathbb{R}$ that is a $\mathbb{Q}(\rho)$-module.

Example 2.0.46. Let k be an ACFA of characteristic zero. Assume $\mathbb{R}$ is considered as an ordered abelian group with an automorphism σ_Γ such that
$$\forall x\in\mathbb{R}:\ \sigma_\Gamma(x)=\rho\cdot x$$

where ρ is a fixed positive real number which is transcendental. Then $K = k((t^{\mathbb{R}}))$ satisfies the assumptions in Assumption 2.0.45.

Chapter 3

Difference Tropical Polynomials

In this thesis, we assume that the reader is familiar with tropical geometry, and we establish its *difference* version.

In this chapter, we define *difference tropical polynomials* and *difference tropical plane curves*. These are the initial objects based on which *difference tropical geometry* is founded. Then, we provide the definition of a *difference balanced weighted rectilinear graph*, which we abbreviate as *DBWR*. Finally, in Theorem 3.2.15, we prove that there is a one to one correspondence between difference tropical plane curves and difference balanced weighted rectilinear graphs. To prove this theorem, we apply graph theoretical tools. Hence, we start this chapter with some preliminaries in graph theory.

3.1 Graph-Theoretical Prerequisites

In this section, we present graph-theoretical background necessary for the proof of Theorem 3.2.15. The main reference for this section is [2].

3.1.1 Planar Graphs

Planar graphs are broadly discussed in section 10.1 of [2] which is the reference of this subsection.

Definition 3.1.1. A graph G is called *planar* if it can be drawn in the plane,

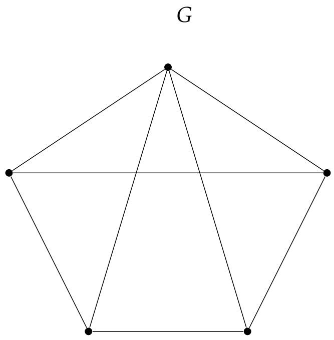


Figure 3.1: The planar graph G obtained from deleting two edges of K_5.

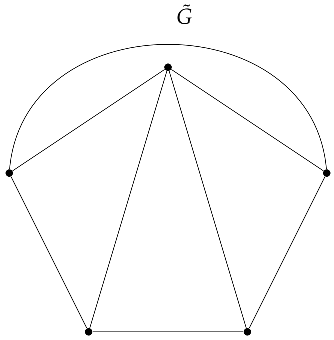


Figure 3.2: $\tilde{G}$ is a planar embedding of G.

such that the edges do not cross each other. This drawing is called a *planar embedding* of G, and is denoted by $\tilde{G}$.

Notation 3.1.2. Let G be a graph. We denote the set of all its vertices by $V(G)$, and the set of all its edges by $E(G)$.

Remark 3.1.3. Let G be a planar graph. Consider $V(\tilde{G})$ to be the set of all points representing the vertices of G, and $E(\tilde{G})$ to be the set of all segments (possibly curved) representing the edges of G. Each vertex of $\tilde{G}$ is only incident with the edges that contain it. In this way, $\tilde{G}$ can be regarded as a graph that is isomorphic to G. It is commonly refereed to as a *plane graph*.

Example 3.1.4. In Figure 3.1, a planar graph G is illustrated. It is obtained from deleting two edges of K_5. The plane graph $\tilde{G}$, in Figure 3.2, is its planar embedding.

In order to review more about planar graphs, we need to recall some definitions from topology.

Definition 3.1.5.
- Let $f : I \to X$ be a continuous map, where I is a closed unit interval and X is a topological space. Then $\text{im}(f)$ is a *curve*.
- If f is defined on a circle, then $\text{im}(f)$ is a *closed curve*.
- If f is one to one, then the curve is called *simple*. Intuitively, this means that the curve does not intersect itself.

Definition 3.1.6. Let A be a subset of the plane. It is said to be *arcwise connected* if and only if any two points of A can be connected via a curve such that the curve lies completely within A.

Theorem 3.1.7 (**The Jordan Curve Theorem**). *Let C be a simple closed curve in the plane. Then, it partitions the rest of the plane into two disjoint open sets, each of which is arc-wise connected.*

Remark 3.1.8. The two open subset of the plane, mentioned in the previous theorem, are called the *interior* and the *exterior* of C. They are denoted by $\text{int}(C)$ and $\text{ext}(C)$ respectively.
This theorem implies that if $a \in \text{int}(C)$ and $b \in \text{ext}(C)$, then any arc connecting a and b intersect C at least once.

3.1.2 Duality

This subsection provides the necessary background on duality in graph theory. All definitions can be found in Section 10.2 of [2]. Finally, by proving Lemma 3.1.22, we obtain Corollary 3.1.25, which plays an important role in the proof of Lemma 3.2.19.

Definition 3.1.9. Suppose G is a plane graph. The Jordan Curve Theorem guarantees that G partitions the rest of the plane into finitely many arcwise connected open sets. Each of these open sets is called a *face*. We denote the set of all faces of G by $F(G)$.

Example 3.1.10. The plane graph $\tilde{G}$ from Example 3.1.4 partitions the rest of the plane into 5 arcwise connected open sets. See Figure 3.2. Therefore, we have $F(\tilde{G}) = \{f_1, f_2, \dots, f_5\}$.

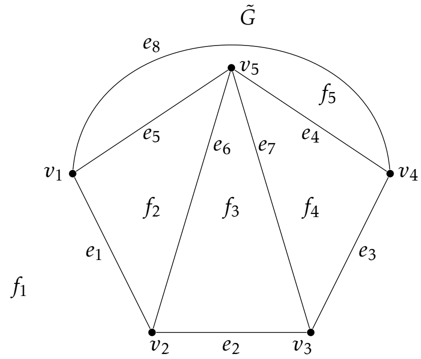


Figure 3.3: The faces of $\tilde{G}$ from Figure 3.2.

The unbounded face f_1, shown in Figure 3.3, is called an *outer face*. Any plane graph has exactly one outer face.

Definition 3.1.11. Let G be a plane graph, and f be one of its faces. Since f is an open set, it makes sense to talk about its boundary from the topological point of view. So the *boundary* of a face is its boundary in the topological meaning. It is denoted by $\partial(f)$. It is commonly said that a face is *incident* with the edges and vertices in its boundary. Two faces f_i and f_j of G are said to be *adjacent*, if e is a common edge in their boundaries. In this case, we use the notation $e \subset \partial(f_i) \cap \partial(f_j)$.

Example 3.1.12. The outer face f_1 of Figure 3.3 is incident with e_1, e_2, e_3, e_8 and v_1, v_2, v_3, v_4. Moreover, f_2, f_3, f_4 and f_5 are adjacent to f_1.

Definition 3.1.13. Let G be a plane graph. We associate a graph G^* to G, which is defined as follows:

- For any face f of G, there exists a corresponding vertex f^* of G^*;
- for any edge e of G, there exists a corresponding edge e^* of G^*;
- the vertices f^* and g^* of G^* are linked by an edge e^* if and only if f and g, the corresponding faces in G, are separated by the edge e of G.

The graph G^* defined as above is called the *dual graph of G*.

Remark 3.1.14. If G is a plane graph, its dual graph G^* can be embedded in the plane naturally. It suffices to draw each vertex f^* of G^* inside the corresponding face f of G. If e^* connects two vertices f^* and g^*, we draw it in a way that it crosses the corresponding edge e only once. Considering this drawing it is intuitively clear that G^* is planar. This specific drawing of G^* is plane and is called the *plane dual* of G. See [2], page 252. For ease of use throughout this thesis, whenever we refer to the dual of a plane graph, we consider the plane dual.

Example 3.1.15. In Figure 3.4, the plane dual $\tilde{G}^*$ of the plane graph $\tilde{G}$ from Example 3.1.4 is illustrated in bold.

Proposition 3.1.16 ([2], Proposition 10.9)**.** *Let G be a plane graph, and G^* be its dual. Then G^* is connected.*

Proof. For the proof, see [2], Proposition 10.9. □

Lemma 3.1.17 ([8], Corollary 6.2.1)**.** *Let G be a connected plane graph, and G^* be its dual. Then we have $G^{**} = G$.*

Assumption 3.1.18. Throughout this thesis, by a subgraph H of a graph G, we mean a subgraph obtained by deleting some edges of G and leaving the vertices intact. In fact, we have $V(H) = V(G)$. So all the subgraphs are assumed to be spanning. Otherwise, it will be stated.

Definition 3.1.19 ([2], section 4.3)**.** Let G be a connected graph, and T be a spanning tree of G. A *cotree* is denoted by $E \setminus T$, and is defined to be the set of all edges in G which are not an edge of T.

Similarly, we can define $E \setminus H$, for any spanning subgraph H of G.

The following notation is borrowed from [2].

Notation 3.1.20. Let G be a plane graph, and G^* be its dual. Suppose $A \subseteq E(G)$ by A^*, we mean the following set:

$$A^* = \{e^* \in E(G^*) \mid e \in A\}.$$

Below, we state a theorem which is well-known in graph theory, but we did not find a good reference for its proof. So, for the convenience of the reader we present a proof.

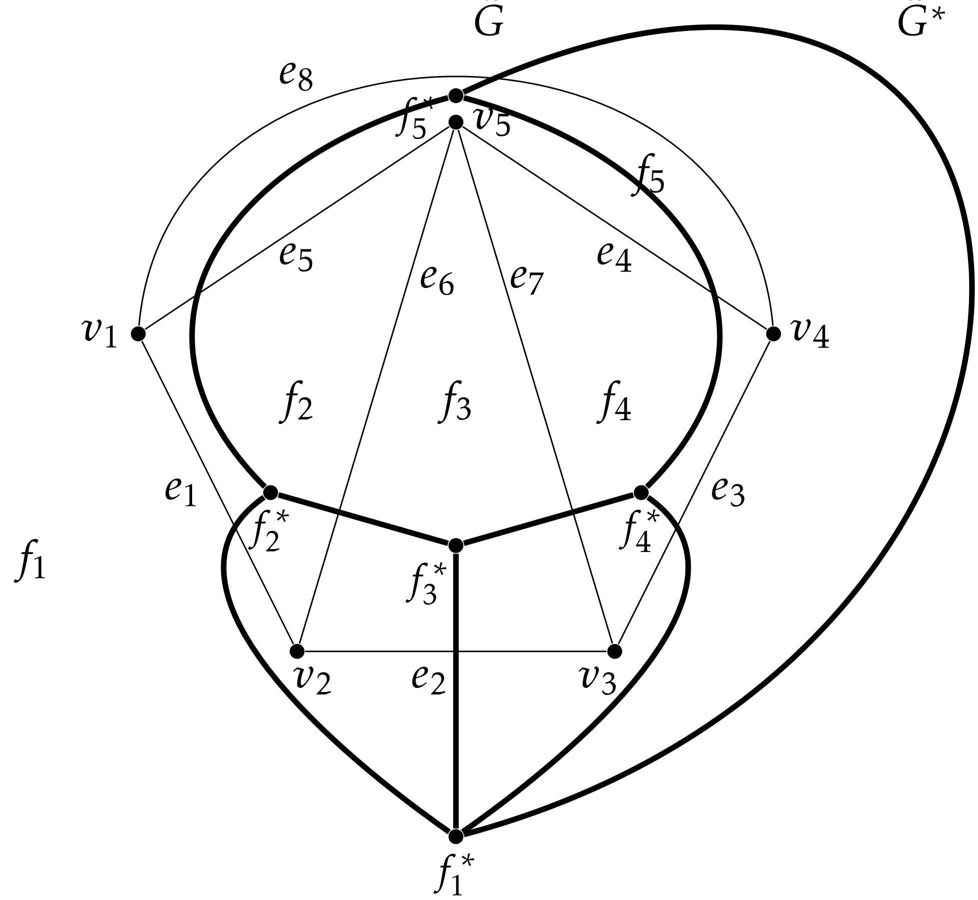


Figure 3.4: The dual graph of $\tilde{G}$ is shown in bold.

Theorem 3.1.21. *Let G be a connected plane graph, and T be a subgraph of G. Consider the subgraph T' of G^*, which is induced by $(E \setminus T)^*$. T is a spanning tree of G if and only if T' is a spanning tree of G^*.*

Proof. $\Longrightarrow$)

Suppose T is a spanning tree of G. We want to prove that T' is a spanning tree of G^*. Equivalently, we prove that it is connected and has no cycle (is acyclic).

- Connectedness:

  Consider the subgraph T of G. As we have assumed, it is a tree. Therefore, it has no cycle. This means that it has one face, namely $\mathbb{R}^2 \setminus T$(the set of all points in $\mathbb{R}^2$, which are neither on an edge nor on a vertex of T). By definition, a face is an arc-wise connected open set. Thus any two points of $\mathbb{R}^2 \setminus T$ can be joined via a curve without crossing T.

  As we fixed in Remark 3.1.14, whenever we refer to G^*, we consider the plane dual. If f^* and g^* are two vertices of G^*, they are drawn in their corresponding faces f and g in $F(G)$. Since T is a subgraph of G, f^* and g^* are neither on an edge nor on a vertex of T. This means that they are in $\mathbb{R}^2 \setminus T$. Therefore, they can be connected via a curve without crossing T. Without loss of generality, we assume that this curve crosses each edge finitely many times. Considering the faces of G, this curve starts from the face $f_0 = f$, passes through a finite sequence of adjacent faces $f_1, \ldots, f_{n-1}$ and ends in $f_n = g$. If we denote the edge separating the faces f_i and f_{i+1} by $e_{i,i+1}$, then this curve crosses the edges $e_{i,i+1}$, with $0 \le i \le n-1$. Therefore, these edges do not belong to $E(T)$. That is to say

  $$\forall i \;\; 0 \le i \le n-1 \; : \; e_{i,i+1} \in E \setminus T.$$

  Thus, this curve gives a walk from f^* to g^* on the graph induced by $(E \setminus T)^*$, which consists of the edges $e^*_{i,i+1}$.

  This means that for any two arbitrary vertices f^* and g^* of G^*, there is a walk, and consequently a path on G^* connecting them. This path consists of the edges in $(E \setminus T)^*$. Hence, T' is a connected subgraph of G^*.

- T' is acyclic:

  Assume the opposite. Suppose C is a cycle in G^* consisting of the edges in $(E \setminus T)^*$.

  Let $f_0^*, f_1^*, \ldots, f_m^*, f_0^*$ be the sequence of vertices on C. So this sequence corresponds to the sequence $f_0, f_1, \ldots, f_m, f_{m+1} = f_0$ of faces in G. Denote the edge connecting the vertices f_i^* and f_{i+1}^* by $e_{i,i+1}^*$. As C consists of the edges in $(E \setminus T)^*$, for each i, with $0 \le i \le m$, $e_{i,i+1}^*$ corresponds to the edge $e_{i,i+1} \in E \setminus T$.

  Since C is a cycle, it can be regarded as a closed curve. Moreover, it is a subgraph of G^*, which is a plane graph, and therefore it is non-intersecting. Thus C can be regarded as a simple curve. Applying Jordan Curve Theorem, we know that C partitions the rest of the plane into two arc-wise connected open sets.

  Let $e_{i,i+1}^*$ be an edge of this cycle whose corresponding edge in G is $e_{i,i+1}$. Based on the explanation in Remark 3.1.14, we know that $e_{i,i+1}^*$, and so C crosses $e_{i,i+1}$ only once. Suppose $e_{i,i+1}$ connects the vertices u and v of G. Without loss of generality, we assume that $u \in \mathrm{int}(C)$, and $v \in \mathrm{ext}(C)$.

  Since u and v are vertices of G, they are also vertices of T. As we assumed, T is a spanning tree. So it is connected. Therefore, there is a path from u to v on T. In other words, there is a path on T from inside of C to outside of C. So this path crosses C. This means that C crosses an edge of T. Hence, it contains an edge of T^*. This contradicts the assumption that C consists of the edges in $(E \setminus T)^*$. Thus, T' is acyclic.

$\Longleftarrow$)

From Remark 3.1.14 and Proposition 3.1.16, G^* is a connected plane graph. To prove this direction, we apply the first direction for the connected plane graph G^*. That is to say, if T' is a spanning tree of G^*, then the graph induced by $(E^* \setminus T')^*$ is a spanning tree of G^{**}. From Lemma

3.1.17, this means that $(E^* \setminus T')^*$ induces a spanning tree of G. To complete the proof, we show that $(E^* \setminus T')^*$ induces T, and therefore T is a spanning tree of G.

Suppose $e^{**} = e \in (E^* \setminus T')^*$. From Notation 3.1.20, this means that $e^* \in (E^* \setminus T')$. In other words, we have $e^* \in E^*$ and $e^* \notin E(T')$. This gives

$$\begin{aligned} e^* \notin E(T') &\iff e^* \notin (E \setminus T)^* \\ &\iff e \notin E \setminus T \\ &\iff e \in E(T). \end{aligned}$$

So $(E^* \setminus T')^*$ gives the set of all edges in T, and induces the graph T. □

The following lemma will be needed in the proof of Corollary 3.1.25, which is one of the main ingredients of the proof of Lemma 3.2.19.

Lemma 3.1.22. Suppose G is a plane graph, and H is a subgraph of G contained in a spanning tree T of G. Assume that H satisfies the following condition:

$$\forall v \in V(H) : \#\{e \in E(H) \mid e \text{ is adjacent to } v\} \neq 1. \tag{3.1.1}$$

Then H has no edges.

Proof. Assume the opposite. Suppose $E(H) \neq \varnothing$. Let e_1 be an edge of H connecting v_0 and v_1. Note that degree of v_0 and v_1 is not one. Therefore, there is another edge e_2 adjacent to v_1 and a vertex v_2. We have finitely many vertices, and the degree of each vertex is not equal to one. So if we continue like this, we will find a vertex v which appears twice on this path. Therefore, H contains a cycle, and this contradicts the fact that it is contained in a tree T. Hence, H does not have any edge.

□

Remark 3.1.23. Let G be a plane graph, and H be a subgarph of G contained in a spanning tree T. Then H satisfies the condition (3.1.1), if and only if for each vertex v of G, and $\{e_1, \dots e_k\}$ the set of all edges adjacent to v, we have the following condition:

$$\text{If } e_1, \dots, e_{k-1} \notin E(H), \text{ then } e_k \notin E(H). \tag{3.1.2}$$

Proof. Suppose H satisfies the condition (3.1.1), and v is a vertex of G, such that the set of all the edges adjacent to v is $\{e_1, \dots, e_k\}$, but the condition (3.1.2) does not hold. Thus, v as a vertex of H has degree 1, and this contradicts the condition (3.1.1).

Conversely, assume that the condition (3.1.2) holds, but there exists a vertex v_0 of H, such that $\#\{e \in E(H) \mid e \text{ is adjacent to } v_0\} = 1$. Suppose e_0 is the single edge adjacent to v_0. This means that all but one edge e_0 are not edges of H, so condition (3.1.2) implies that e_0 is not an edge of H, which is a contradiction. □

Remark 3.1.24. Consider the assumptions of the previous remark, and assume G is connected. From Lemma 3.1.17, we have $G^{**} = G$. This means that, to each vertex v in $V(G)$, a face f of $F(G^*)$ is associated. Besides, to each edge e_i in $E(G)$, an edge e_i^* of $E(G^*)$ is associated. Suppose, H' is the graph induced by $(E \setminus H)^*$. If e_i is an edge of G adjacent to v, such that $e_i \notin E(H)$, we have $e_i \in E \setminus H$. This means that $e_i^* \in (E \setminus H)^*$. Thus, for any face f of $F(G^*)$, if $e_1^*, \dots, e_k^*$ are all the edges in $\partial(f)$, then the dual version of the condition (3.1.2) is as follows:

$$\text{If } e_1^*, \dots, e_{k-1}^* \in E(H^{'}), \text{ then } e_k^* \in E(H^{'}). \tag{3.1.3}$$

Corollary 3.1.25. *Let G be a connected plane graph, and H be a subgraph of G. Suppose G^* is the plane dual of G. Assume $H^{'}$ is the subgraph of G^* induced by $(E \setminus H)^*$. If $H^{'}$ contains a spanning tree T' of G^*, and the condition* (3.1.3) *holds, then $H^{'} = G^*$.*

Proof. Since T' is a spanning tree of G^*, Theorem 3.1.21 implies that $(E^* \setminus T^{'})^*$ induces a spanning tree T of G. Moreover, H is contained in T. To see this, we have:

$$\begin{aligned} &e \in E(H) \Rightarrow e \notin E \setminus H \Rightarrow e^* \notin (E \setminus H)^* \\ \Rightarrow &e^* \notin E(H^{'}) \Rightarrow e^* \notin E(T^{'}) \Rightarrow e^* \in E^* \setminus T^{'} \Rightarrow e = e^{**} \in (E^* \setminus T^{'})^*. \end{aligned}$$

Since $(E^* \setminus T^{'})^*$ induces T, e is an edge of T. Therefore, $H \subset T$.

That is to say that H is a subgraph of G contained in the spanning tree T. As we assumed, condition (3.1.3) holds. From Remark 3.1.24, this condition is dual to the condition (3.1.2). Therefore, Remark 3.1.23 together with Lemma 3.1.22 imply that $E(H) = \emptyset$. In other words $(E \smallsetminus H)^* = E^*$, which means that $(E \smallsetminus H)^*$ induces G^*. Hence, $H' = G^*$. □

3.2 Difference Tropical Polynomials and Curves

In this section, we introduce difference tropical objects based on the framework presented in [11], Section 3. In [11], similar objects in the classical case are discussed. We extend these concepts to the difference case. This is a fundamental step in establishing a connection between tropical geometry and difference algebra. We will define more difference tropical objects in Section 4.1.

Definition 3.2.1. Let $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ be the tropical semiring. Assume σ is an automorphism of ordered abelian groups on $\mathbb{R}$. A *difference tropical monomial* in $x_1, \ldots, x_n$ is a product of variables $x_1, \ldots, x_n$ and some iterations of σ on these variables with a real coefficient.

To clarify this definition, we consider the following example, where we observe how a difference tropical monomial in one variable looks like.

Example 3.2.2. A **difference tropical monomial in the variable x** is of the following form:

$$a \odot x^{\odot r_0} \odot \sigma(x)^{\odot r_1} \odot \sigma^2(x)^{\odot r_2} \odot \cdots \odot \sigma^n(x)^{\odot r_n}, \tag{3.2.1}$$

where $a \in \mathbb{R}$ and by $\sigma^i(x)$ with $1 \leq i \leq n$, we mean the i-th iteration of σ. If we evaluate (3.2.1) in classical arithmetic, then we have

$$a + r_0 x + r_1 \sigma(x) + r_2 \sigma^2(x) + \cdots + r_n \sigma^n(x). \tag{3.2.2}$$

Since σ is an automorphism of ordered abelian groups on $\mathbb{R}$, by using Remark 2.0.12, (3.2.2) can be written as

$$a + r_0 x + r_1 \rho \cdot x + r_2 \rho^2 \cdot x + \cdots + r_n \rho^n \cdot x = a + (r_0 + r_1 \rho + r_2 \rho^2 + \cdots + r_n \rho^n) x = a + g(\rho) x.$$

Here by $g(\rho)$, we mean an element of $\mathbb{Z}[\rho]$, which is the following set:

$$\mathbb{Z}[\rho] = \{\sum_{i=0}^{m} a_i \rho^i \mid \forall i,\ a_i \in \mathbb{Z} \text{ and } m \in \mathbb{N}\},$$

and ρ is assumed to be transcendental over $\mathbb{Q}$. See Assumption 2.0.45. From now on, for simplicity, we avoid using $\odot$ in the exponent, and it will be clear from the context whether the operation is tropical or not.

Example 3.2.3. The following is an explicit example of a difference tropical monomial in variables x_1, x_2 and x_3:

$$\begin{aligned} 3 \odot x_1^2 \odot x_2 \odot \sigma(x_1)^3 \odot \sigma^2(x_3) &= 3 + 2x_1 + x_2 + 3\rho \cdot x_1 + \rho^2 \cdot x_3 \\ &= 3 + (2 + 3\rho)x_1 + x_2 + \rho^2 \cdot x_3. \end{aligned}$$

In fact, a difference tropical monomial in variables $x_1, \dots x_n$ can be evaluated in classical arithmetic as follows:

$$a + g_1(\rho)x_1 + \cdots + g_n(\rho)x_n,$$

where $a \in \mathbb{R}$, and for each $i \in \{1, \dots, n\}$, we have $g_i(\rho) \in \mathbb{Z}[\rho]$. A difference tropical monomial is a function from $\mathbb{R}^n$ to $\mathbb{R}$, and if we evaluate it in classical arithmetic, it is linear.

Definition 3.2.4. A *difference tropical polynomial* is a tropical sum of finitely many difference tropical monomials. We denote it by P.

Note that, as in the classical tropical geometry, distinct difference tropical polynomials can define the same function.

Example 3.2.5. A **difference tropical polynomial in one variable x** is of the following form:

$$\begin{aligned} P(x) &= \bigoplus_{i=1}^{k} a_i \odot x^{r_{i,0}} \odot \sigma(x)^{r_{i,1}} \odot \sigma^2(x)^{r_{i,2}} \odot \cdots \odot \sigma^n(x)^{r_{i,n}} \\ &= \min\{a_1 + g_1(\rho)x, a_2 + g_2(\rho)x, \dots, a_k + g_k(\rho)x\}. \end{aligned}$$

Here n is the greatest iteration of σ among these k difference tropical monomials. If for some j, $0 \le j \le n$, and for some i, $1 \le i \le k$, $\sigma^j(x)$ does not appear in i-th monomial, this means that $r_{i,j}$ is zero. For each $i \in \{1, \dots, k\}$, we have $a_i \in \mathbb{R}$ and $g_i(\rho) \in \mathbb{Z}[\rho]$.

A difference tropical polynomial in n variables is a function $P : \mathbb{R}^n \to \mathbb{R}$. As P is the minimum of linear functions, it is concave and piecewise linear.

Definition 3.2.6. Let P be a difference tropical polynomial. A point a is a *root* of P, if the minimum in definition of P at the point a is attained at least twice.

Definition 3.2.7. The set of the roots of a difference tropical polynomial in two variables is called a *difference tropical plane curve*.

To have a better intuition about difference tropical plane curves, we first see how a difference tropical polynomial in two variables looks like.

A **difference tropical monomial in variables x and y** is of the following form:

$$\begin{aligned}&a \odot x^{r_0} \odot \sigma(x)^{r_1} \odot \cdots \odot \sigma^n(x)^{r_n} \odot y^{s_0} \odot \sigma(y)^{s_1} \odot \cdots \odot \sigma^m(y)^{s_m}\\&= a + g(\rho)x + h(\rho)y.\end{aligned}$$

where $a \in \mathbb{R}$, $m,n \in \mathbb{N}$ and $g(\rho), h(\rho) \in \mathbb{Z}[\rho]$. Similarly, by evaluating a **difference tropical polynomial in x and y** in classical arithmetic, we can see that it is the minimum of finitely many such terms. In other words, it is of the following form:

$$P(x,y) = \min_{i \in \{1,\dots,k\}} \{a_i + g_i(\rho)x + h_i(\rho)y\}.$$

For each i with $1 \le i \le k$, we have $z = a_i + g_i(\rho)x + h_i(\rho)y$ is a plane in $\mathbb{R}^3$. If at a point (x_0, y_0) the minimum is attained more than once, this means that $(x_0, y_0, P(x_0, y_0))$ is on a line (or a vertex) where two planes (or more) meet. Otherwise, it is on one of these k planes. This means that $P(x,y)$ represents a polyhedral surface in $\mathbb{R}^3$. If we project this polyhedral surface on $\mathbb{R}^2$, we obtain a collection of vertices and edges. These edges can have two end points (segments) or just one end point (half rays). For the points on these edges and vertices, P is not linear. In fact, we obtain the set of all points where at least two planes attain the minimum, and this is by definition the difference tropical plane curve associated to P. It is denoted by T_P.

Remark 3.2.8. As we explained above, $(x_0, y_0) \in T_P$ if and only if it is a root of P. Thus, any point $(x,y) \in \mathbb{R}^2 \setminus T_P$ is not a root, and the minimum at this point is attained only once. In other words, for $(x,y) \in \mathbb{R}^2 \setminus T_P$, there exists $i \in \{1, \dots, k\}$ such that $P(x,y) = a_i + g_i(\rho)x + h_i(\rho)y$. More precisely,

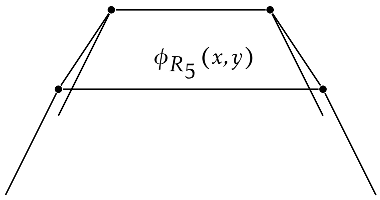


Figure 3.5: The graph of $P(x,y)$ looks like a tent.

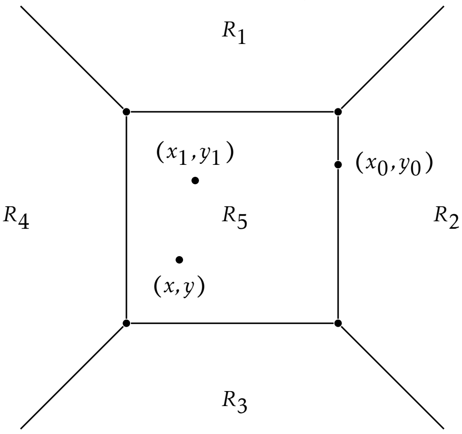


Figure 3.6: T_P is obtained from projecting the graph in Figure 3.5 on $\mathbb{R}^2$.

T_P partitions $\mathbb{R}^2$ into k regions $R_1,\dots R_k$. Let (x_1,y_1) be a point in $\mathbb{R}^2 \setminus T_P$ such that (x,y) and (x_1,y_1) are in the same region R_i. Then $P(x_1,y_1)$ and $P(x,y)$ are on the same plane. We denote this plane by ϕ_{R_i}. This means that $P(x_1,y_1) = a_i + g_i(\rho)x_1 + h_i(\rho)y_1$.

We illustrate this, in an example in Figure 3.5 and Figure 3.6 .

Definition 3.2.9. To each difference tropical polynomial $P(x,y)$, we associate a polygon, which is called a *difference Newton polygon*. Let

$$P(x,y) = \min_{i\in\{1,\dots,k\}} \{a_i + g_i(\rho)x + h_i(\rho)y\}$$

be a difference tropical polynomial, then the difference Newton polygon associated to P is the convex hull of the points $(g_i(\rho),h_i(\rho)) \in \mathbb{R}^2$ with $i \in \{1,\dots,k\}$.

Definition 3.2.10. Let $P(x,y)$ be a difference tropical polynomial, and T_P be its difference tropical plane curve. We associate a *weight* to each edge of T_P. To do so, we assume e is an edge of T_p, which separates two regions R_i and R_j. See Figure 3.7. In this case, R_i and R_j are called *adjacent* or *neighbour* regions. As we denoted in Remark 3.2.8, ϕ_{R_i} and ϕ_{R_j} are the planes which attain the minimum on R_i and R_j respectively. Here, we have

$$\phi_{R_i}(x,y) = g_i(\rho)x + h_i(\rho)y + a_i$$
$$\phi_{R_j}(x,y) = g_j(\rho)x + h_j(\rho)y + a_j.$$

The weight of e is the length of the segment connecting $(g_i(\rho), h_i(\rho))$ and $\big(g_j(\rho), h_j(\rho)\big)$. We denote it by $w_\sigma(e)$.

Remark 3.2.11. In the classical tropical geometry, the definition of weight for an edge e of a tropical plane curve T_P is slightly different. In this case, let e be adjacent to the regions R_i and R_j such that

$$\phi_{R_i} = r_i x + s_i y + a_i$$
$$\phi_{R_j} = r_j x + s_j y + a_j,$$

where r_i, r_j and also s_i, s_j are integers and a_i, a_j are in $\mathbb{R}$. Then the weight of e is defined to be the integer length of the segment connecting (r_i, s_i) and (r_j, s_j) (see [11], page 5).
In the difference case, if σ appears in the presentation of P, then the slope of the segment connecting the points $(g_i(\rho), h_i(\rho))$ and $(g_j(\rho), h_j(\rho))$ is an element of $\mathbb{Q}(\rho)$. Therefore, this segment does not necessarily pass through an integer point, and it is not meaningful to talk about the integer length.

Lemma 3.2.12. *Let $P(x,y)$ be a difference tropical polynomial, and T_P be its difference tropical plane curve. Suppose e is an edge of T_P, which is separating the regions R_i and R_j. Assume ϕ_{R_i} and ϕ_{R_j} are defined as in Definition 3.2.10. If $\vec{v}$ is the vector from $(g_i(\rho), h_i(\rho))$ to $\big(g_j(\rho), h_j(\rho)\big)$, then e and $\vec{v}$ are orthogonal.*

Proof. From the definition of $\vec{v}$, we have $\vec{v} = \big(g_j(\rho) - g_i(\rho), h_j(\rho) - h_i(\rho)\big)$. Since e is an edge of T_p, which is between the regions R_i and R_j, it is the

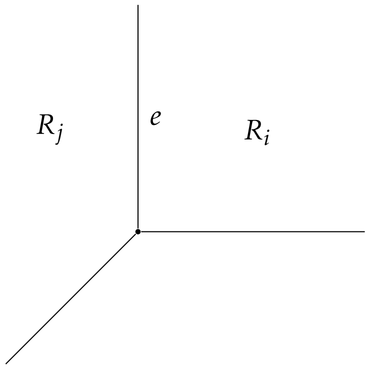


Figure 3.7: If ϕ_{R_i} and ϕ_{R_j} are the ones defined in Definition 3.2.10, we have $w_\sigma(e) = |\big(g_j(\rho) - g_i(\rho), h_j(\rho) - h_i(\rho)\big)|$.

set of all points where ϕ_{R_i} and ϕ_{R_j} are equal and they attain the minimum in $P(x,y)$. Therefore, we have

$$\forall (x,y) \in e, \quad \phi_{R_i}(x,y) = g_i(\rho)x + h_i(\rho)y + a_i = g_j(\rho)x + h_j(\rho)y + a_j = \phi_{R_j}(x,y).$$

This gives the following equation:

$$y = \left(\frac{g_j(\rho) - g_i(\rho)}{h_i(\rho) - h_j(\rho)}\right)x + \frac{a_j - a_i}{h_i(\rho) - h_j(\rho)}.$$

From this equation, the slope of e is $\dfrac{g_j(\rho) - g_i(\rho)}{h_i(\rho) - h_j(\rho)}$. Thus, it is parallel to the vector $\vec{u} = \big(h_i(\rho) - h_j(\rho), g_j(\rho) - g_i(\rho)\big)$. It suffices to show that $\vec{v}$ and $\vec{u}$ are orthogonal, or in other words $\vec{v} \cdot \vec{u} = 0$. We have

$$\begin{aligned}\vec{v} \cdot \vec{u} &= \big(g_j(\rho) - g_i(\rho), h_j(\rho) - h_i(\rho)\big) \cdot \big(h_i(\rho) - h_j(\rho), g_j(\rho) - g_i(\rho)\big) \\ &= g_j(\rho)h_i(\rho) - g_j(\rho)h_j(\rho) - g_i(\rho)h_i(\rho) + g_i(\rho)h_j(\rho) \\ &\quad + h_j(\rho)g_j(\rho) - h_j(\rho)g_i(\rho) - h_i(\rho)g_j(\rho) + h_i(\rho)g_i(\rho) = 0.\end{aligned}$$

This means $\vec{v} \perp \vec{u}$. Therefore, $\vec{v}$ is orthogonal to e. □

Notation 3.2.13. We denote the unit vector along the edge e by $\vec{e}$. So the previous lemma implies that $\vec{v} \perp \vec{e}$.

Definition 3.2.14. Assume

- $\mathcal{V}$ is a finite collection of points in $\mathbb{R}^2$;
- ε_b is a finite collection of segments with their end points in $\mathcal{V}$;
- ε_n is a finite collection of half rays whose end points are in $\mathcal{V}$;

and for any two elements of $\varepsilon_b \cup \varepsilon_n$, their intersection is either an element of $\mathcal{V}$ or is empty. We define a function $w_\sigma : \varepsilon_b \cup \varepsilon_n \to \mathbb{R} \smallsetminus \{0\}$, and for each $e \in \varepsilon_b \cup \varepsilon_n$, we call $w_\sigma(e)$ the weight of e. Then the quadruplet $(\mathcal{V}, \varepsilon_b, \varepsilon_n, w_\sigma)$ is called a *difference weighted rectilinear graph*. The elements of $\mathcal{V}$ are called vertices, and the elements of $\varepsilon_b \cup \varepsilon_n$ are called edges of $G = (\mathcal{V}, \varepsilon_b, \varepsilon_n, w_\sigma)$. A difference weighted rectilinear graph is called *balanced* if it satisfies the following properties:

1. For a fixed real number ρ, which is transcendental over $\mathbb{Q}$, the slope of each edge $e \in \varepsilon_b \cup \varepsilon_n$ is an element of $\mathbb{Q}(\rho)$.

2. There is no vertex $v \in \mathcal{V}$, which is adjacent to exactly two edges of $\varepsilon_b \cup \varepsilon_n$.

3. (**The Balancing Condition**): For each vertex $v \in \mathcal{V}$, we have

$$\sum_{e_i \in \varepsilon(v)} w_\sigma(e_i) \cdot \vec{e}_i = 0,$$

   where $\varepsilon(v) \subset \varepsilon_b \cup \varepsilon_n$ is the set consisting of all the edges, which are adjacent to v, and $\vec{e}_i$ is the unit vector along the edge e_i pointing in the direction away from v.

We abbreviate difference balanced weighted rectilinear graph as *DBWR* graph. By a curve associated to a DBWR graph, we mean the union of its vertices, segments and half rays.

Theorem 3.2.15. *Let T_P be a difference tropical plane curve, which is not a straight line, then there exists a DBWR graph whose associated curve is T_P. Inversely, the associated curve to any DBWR graph is a difference tropical plane curve.*

Proof. The proof of this theorem consists of two main parts. The first one is easier. In this part, we assume T_P is a difference tropical plane curve, and we prove that it is a DBWR graph. The difficult part is to consider a DBWR graph G and find a tropical polynomial P such that $T_P = G$.

Now, we start with the easier part. Let T_P be a difference tropical plane curve defined by the difference tropical polynomial $P(x,y)$. We prove that there exists a DBWR graph such that its associated curve is T_P.

We define a difference weighted rectilinear graph $G = (\mathcal{V}, \varepsilon_b, \varepsilon_n, w_\sigma)$ such that $\mathcal{V}$ is the set of vertices in T_P, ε_b is the set of its segments and ε_n is the set of its half rays. We also assume that for any $e \in \varepsilon_b \cup \varepsilon_n$, $w_\sigma(e)$ coincides with its weight as an edge of T_P.

We want to prove that G is balanced. In order to prove this, it suffices to check whether it satisfies the properties 1, 2 and 3 of Definition 3.2.14.

1. Let e be an edge of T_P, which is between two regions R_i and R_j. Assume $\phi_{R_i} = g_i(\rho)x + h_i(\rho)y + a_i$ and $\phi_{R_j} = g_j(\rho)x + h_j(\rho)y + a_j$ are their corresponding planes. Similar to the calculations in the proof of Lemma 3.2.12, we can use the equation below to calculate the slope of e.

$$\forall (x,y) \in e: \; P(x,y) = \phi_{R_i}(x,y) = \phi_{R_j}(x,y). \tag{3.2.3}$$

   This equation gives the slope of e, which is $\left(\dfrac{g_j(\rho) - g_i(\rho)}{h_i(\rho) - h_j(\rho)}\right)$. So the slope of e is an element of $\mathbb{Q}(\rho)$, and the first property is satisfied.

2. As we discussed earlier, when we project the polyhedral surface defined by $P(x,y)$ on $\mathbb{R}^2$, we obtain T_P. This means that any vertex v of T_P corresponds to the shadow of a vertex u from this polyhedral surface. Since u is a vertex, it is the point where at least three planes and consequently three edges intersect. Therefore, v is also adjacent to the projections of these edges, satisfying the second property.

3. The last property to check is the balancing condition. Let v be a vertex in T_P. In Figure 3.8, T_P is illustrated locally at the vertex v. Suppose v is adjacent to the edges $e_1, e_2, \ldots, e_{n-1}, e_n$ such that they

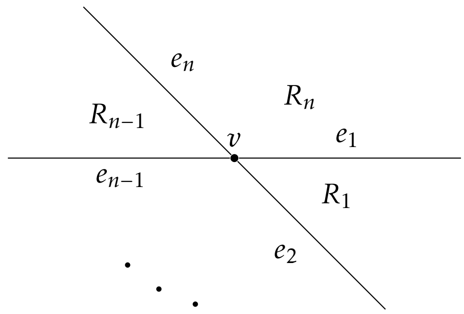


Figure 3.8: The vertex v and all its adjacent edges.

are adjacent to the regions $R_1, R_2, \ldots, R_{n-1}, R_n$ respectively. Assume that for any $i \in \{1, \ldots, n\}$, $\phi_{R_i}(x,y) = g_i(\rho)x + h_i(\rho)y + a_i$ is the plane corresponding to the monomial that obtains the minimum in $P(x,y)$ for any $(x,y) \in R_i$. From Definition 3.2.10 and Lemma 3.2.12, for any edge e_i, there exists a vector $\vec{v}_i = (g_i(\rho) - g_{i-1}(\rho), h_i(\rho) - h_{i-1}(\rho))$ such that $e_i \perp \vec{v}_i$. That is to say,

$$\forall i \in \{2, \ldots, n\},\ w_\sigma(e_i) = |\vec{v}_i| = |(g_i(\rho) - g_{i-1}(\rho), h_i(\rho) - h_{i-1}(\rho))|,$$

and

$$w_\sigma(e_1) = |\vec{v}_1| = |(g_1(\rho) - g_n(\rho), h_1(\rho) - h_n(\rho))|.$$

If we consider the segment connecting $(g_1(\rho), h_1(\rho))$ to $(g_2(\rho), h_2(\rho))$, and then the segment connecting $(g_2(\rho), h_2(\rho))$ to $(g_3(\rho), h_3(\rho))$, etc, then we see that the end of each one is the initial of the next one. This gives

$$\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n = 0.$$

From Notation 3.2.13, $\vec{e}_i$ is a unit vector, which is orthogonal to $\vec{v}_i$. Therefore, for each $i \in \{1, \ldots, n\}$, if we rotate all $\vec{v}_i$s by 90° all in the same direction, we obtain

$$w_\sigma(e_1)\vec{e}_1 + w_\sigma(e_2)\vec{e}_2 + \cdots + w_\sigma(e_n)\vec{e}_n = 0.$$

Hence, the balancing condition holds at vertex v.

So $G = (\mathcal{V}, \varepsilon_b, \varepsilon_n, w_\sigma)$ is the DBWR graph whose associated curve is T_P.

The challenging part of the proof lies ahead. Now, suppose that $G = (\mathcal{V}, \varepsilon_b, \varepsilon_n, w_\sigma)$ is a DBWR graph. Then, we define a difference tropical polynomial P, such that $T_P = G$.

Since G is a finite collection of vertices, segments and half rays in $\mathbb{R}^2$, it partitions $\mathbb{R}^2$ into finitely many sections. We call each of them a *region*. Assume G partitions $\mathbb{R}^2$ into n regions.

The general idea to prove this part is to fix a pattern to assign an affine function ϕ_R to each region R. To do this, we use graph theory. For the prerequisites in graph theory, see Section 3.1. Finally, we show that the tropical sum of these affine functions is the tropical polynomial P we are looking for.

- **Step 1) Choosing a pattern and assigning an affine function to each region:**

  Given that $G = (\mathcal{V}, \varepsilon_b, \varepsilon_n, w_\sigma)$ is a DBWR graph. For each half ray $e \in \varepsilon_n$, we regard it as an edge with an end point in infinity. Then G can be regarded as a plane connected graph $\tilde{G}$, such that $\mathcal{V}$ is the set of its vertices together with the point at infinity, and $\varepsilon_b \cup \varepsilon_n$ is the set of its edges. We denote its dual graph by $\tilde{G}^*$.
  Note that we treat a point at infinity just like a vertex that is far from all other vertices. Therefore, everything that we presented in the graph theory section naturally generalizes to such a graph.

  From this point of view, associating affine functions to the regions of $\mathbb{R}^2 \setminus G$ means associating affine functions to the faces of $\tilde{G}$, and equivalently to the vertices of $\tilde{G}^*$. See Figure 3.9.

  Let T be a spanning tree of $\tilde{G}^*$. See Figure 3.10. Choose a vertex of T, and label it as 1, and also label the corresponding face of $\tilde{G}$ as R_1. Associate an arbitrary affine function ϕ_{R_1} to 1, and equivalently to R_1, as follows:

$$\begin{aligned} \phi_{R_1} : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\mapsto k_1 x + l_1 y + a_1, \end{aligned}$$

  such that $k_1, l_1 \in \mathbb{Q}(\rho)$. Choose one of the adjacent vertices to 1, and label it as 2, and label the corresponding face in $\tilde{G}$ as R_2. Suppose e is the common edge in $\partial(R_1) \cap \partial(R_2)$. We associate the affine function

$$\begin{aligned} \phi_{R_2} : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\mapsto k_2 x + l_2 y + a_2 \end{aligned}$$

to 2, and also to R_2, such that $\left(\frac{k_1 - k_2}{w_\sigma(e)}, \frac{l_1 - l_2}{w_\sigma(e)}\right)$ is a unit vector normal to e pointing to R_2, and $\phi_{R_1}\big|_e = \phi_{R_2}\big|_e$.

Since k_1, l_1 and e are given, it is easy to find k_2, l_2 such that $\left(\frac{k_1 - k_2}{w_\sigma(e)}, \frac{l_1 - l_2}{w_\sigma(e)}\right)$ satisfies the intended conditions. Suppose (x_0, y_0) is a point on e. If we define $a_2 = (k_1 - k_2)x_0 + (l_1 - l_2)y_0 + a_1$, then we have $\phi_{R_1}(x_0, y_0) = \phi_{R_2}(x_0, y_0)$. As it is shown below in Remark 3.2.18,this implies that $\phi_{R_1}\big|_e = \phi_{R_2}\big|_e$. This means that we can define ϕ_{R_2} with the above conditions.

Similarly, we label the other vertices adjacent to 1, and with the same method, we associate an affine function to each of them. We repeat the same procedure for these vertices to assign affine functions to their adjacent vertices. If we continue similarly, we can finally assign an affine function to each vertex of $\tilde{G}^*$, and therefore to each region of $\mathbb{R}^2 \setminus G$.

Definition 3.2.16. Let R_i and R_j be two regions of $\mathbb{R}^2 \setminus G$. We say these two regions are *neighbour* if the boundaries of their corresponding faces in $\tilde{G}$ have an edge in common.

Definition 3.2.17. Suppose we fix a spanning tree of $\tilde{G}^*$, and similar to the above procedure, use it as a pattern to assign an affine function to each region of $\mathbb{R}^2 \setminus G$. Let R_i and R_j be two neighbor regions. Assume ϕ_{R_i} and ϕ_{R_j} be their associated affine functions. Suppose that $e \subset \partial(R_i) \cap \partial(R_j)$, where e is an edge of $\tilde{G}$. We say that the *normal vector condition* holds for (R_i, R_j), and we abbreviate this condition as *NVC*, if we have $\left(\frac{k_i - k_j}{w_\sigma(e)}, \frac{l_i - l_j}{w_\sigma(e)}\right)$ is a unit vector normal to e pointing to R_j , and $\phi_{R_i}\big|_e = \phi_{R_j}\big|_e$. We also say that (i, j) is a *good pair* in $\tilde{G}^*$, if the corresponding regions R_i and R_j satisfy NVC.

Remark 3.2.18. Let the assumptions be as above. Assume R_i and R_j are two neighbour regions, and $e \subset \partial(R_i) \cap \partial(R_j)$. Suppose $\phi_{R_i} = k_i x + l_i y + a_i$ and $\phi_{R_j} = k_j x + l_j y + a_j$ are the corresponding affine functions, and $\vec{v} = \left(\frac{k_i - k_j}{w_\sigma(e)}, \frac{l_i - l_j}{w_\sigma(e)} \right)$ is a unit vector normal to e. If $\phi_{R_i}\big|_e \neq \phi_{R_j}\big|_e$ then there is no point p on e such that $\phi_{R_i}(p) = \phi_{R_j}(p)$.

Proof. Assume the opposite. Suppose $p = (x_0, y_0)$ is a point on e such that $\phi_{R_i}(p) = \phi_{R_j}(p)$. This means

$$\left(\phi_{R_i} - \phi_{R_j}\right)(p) = 0 \tag{3.2.4}$$

$$\Leftrightarrow (k_i - k_j)x_0 + (l_i - l_j)y_0 = a_j - a_i. \tag{3.2.5}$$

Let $q = (x, y)$ be an arbitrarily chosen point on e. Since $\vec{v} = \left(\frac{k_i - k_j}{w_\sigma(e)}, \frac{l_i - l_j}{w_\sigma(e)} \right)$ is a unit vector normal to e, we have

$$\begin{aligned} &\left\langle \left(\frac{k_i - k_j}{w_\sigma(e)}, \frac{l_i - l_j}{w_\sigma(e)} \right), (x - x_0, y - y_0) \right\rangle = 0 \\ \Leftrightarrow &(k_i - k_j)(x - x_0) + (l_i - l_j)(y - y_0) = 0 \\ \Leftrightarrow &(k_i - k_j)x + (l_i - l_j)y = (k_i - k_j)x_0 + (l_i - l_j)y_0. \end{aligned}$$

Together with the equation (3.2.5), we have

$$(k_i - k_j)x + (l_i - l_j)y = a_j - a_i. \tag{3.2.6}$$

That is to say $\phi_{R_i}(x, y) = \phi_{R_j}(x, y)$. Since q is arbitrarily chosen, (3.2.6) means that $\phi_{R_i}\big|_e = \phi_{R_j}\big|_e$, which is a contradiction. Therefore, for any point p on e, we have $\phi_{R_i}(p) \neq \phi_{R_j}(p)$. ∎

- **Step 2) Defining a difference tropical polynomial whose difference tropical plane curve is G:**

  In this step, we want to prove that the tropical sum of the affine functions ϕ_R gives a difference tropical polynomial P, such that $T_P = G$. This is the content of Claim 3.2.20. As an intermediate step to prove this claim, we need Lemma 3.2.19.

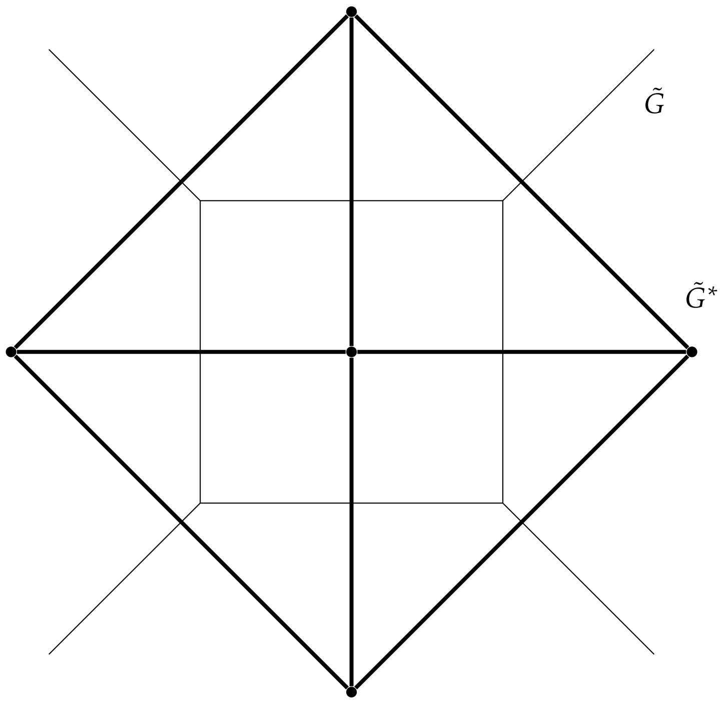


Figure 3.9: G is regarded as a plane connected graph $\tilde{G}$, and its dual graph $\tilde{G}^*$ is shown in bold

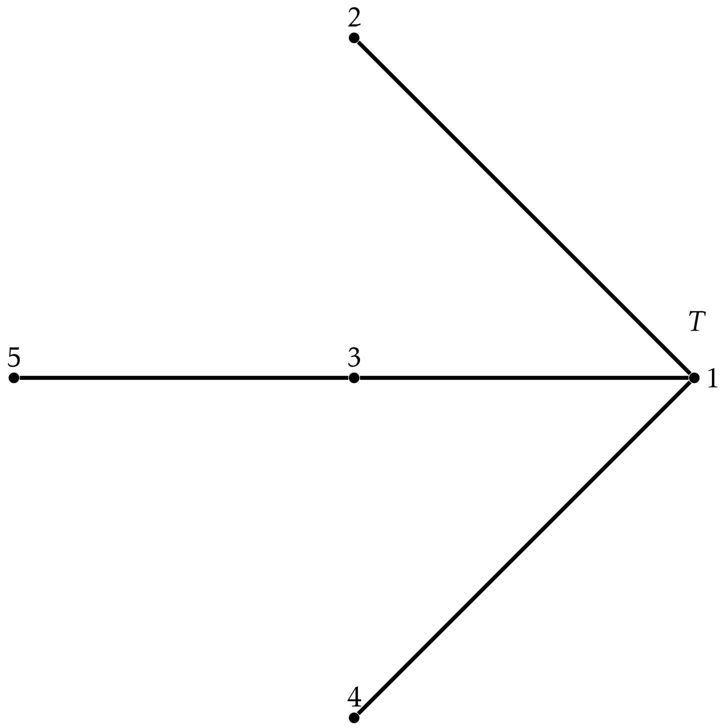


Figure 3.10: We choose and fix a spanning tree T of $\tilde{G}^*$ to determine a pattern.

In this Lemma, we prove that in any DBWR graph G, if we use a fixed spanning tree of $\tilde{G}^*$ as the pattern to assign an affine function to each region of $\mathbb{R}^2 \setminus G$, then for any two neighbour regions NVC holds.

Lemma 3.2.19. *Let G be a DBWR graph. Assume a numbering on the regions of $\mathbb{R}^2 \setminus G$ is chosen. Applying the same method as in Step 1, suppose an affine function is assigned to each region. Then any two neighbour regions of $\mathbb{R}^2 \setminus G$, satisfy NVC.*

Proof. The general idea of the proof is to apply Corollary 3.1.25. Recall that it states

Let G be a connected plane graph, and H be a subgraph of G. Assume H' is the subgraph of G^* induced by$(E \setminus H)^*$, and contains a spanning tree of G^*. If for any face f of G^*, and all edges $e_1^*, \ldots, e_m^*$ in $\partial(f)$, we have the following condition:

$$\text{If } e_1^*, \ldots, e_{m-1}^* \in E(H'), \text{ then } e_m^* \in E(H'). \tag{3.2.7}$$

Then, $H' = G^*$.

Note that the above corollary is a general statement in graph theory, and in this context G refers to a plane graph. This should not be confused with G mentioned in the statement of this lemma, which refers to a DBWR graph.

Regard the DBWR graph G as a plane connected graph $\tilde{G}$. Assume we number the faces of $\tilde{G}$ and vertices of $\tilde{G}^*$ based on the chosen numbering of the regions of $\mathbb{R}^2 \setminus G$.

Suppose H' is a subgraph of $\tilde{G}^*$, which is defined as follows:

$$(i,j) \in E(H') \iff (i,j) \text{ is a good pair.}$$

As we have seen before, in order to assign affine functions to the regions, we fix a spanning tree T of $\tilde{G}^*$, and we use it as our pattern. Clearly, any two adjacent vertices of T make a good pair. Therefore, T is contained in H'.

Define the subgraph H of $\tilde{G}$ as follows:

$$e \in E(H) \iff e \subset \partial(R_s) \cap \partial(R_t), \text{ where } (s,t) \text{ is not a good pair.}$$

This means that if e is an edge of H, the corresponding edge e^* of $\tilde{G}^*$ is not an edge of H'. In other words, we have $(E \setminus H)^* = E(H')$, which means that $(E \setminus H)^*$ induces the subgraph H' of $\tilde{G}^*$.

Define $\tilde{T}$ to be the subgraph of $\tilde{G}$ as follows:

$$e \in \tilde{T} \iff e^* \notin T.$$

Apply Theorem 3.1.21, for $\tilde{G}^*$ and its spanning tree T. Then, we have $\tilde{T}$ is a spanning tree of $(\tilde{G}^*)^* = \tilde{G}$. Moreover, for any edge e of H, we have $e^* \notin H'$, and consequently $e^* \notin T$. This means that $e \in \tilde{T}$, or equivalently H is contained in $\tilde{T}$.

Let v be a vertex in $\tilde{G}$. Without loss of generality, we can assume that the chosen numbering is such that v is the common vertex of $\partial(R_1),\ldots,\partial(R_m)$. Suppose $\{e_1,\ldots e_m\}$ is the set of all edges adjacent to v, and we have

$$\begin{aligned} &e_1 \subset \partial(R_1) \cap \partial(R_2) \\ &\qquad\vdots \\ &e_{m-1} \subset \partial(R_{m-1}) \cap \partial(R_m) \\ &e_m \subset \partial(R_m) \cap \partial(R_1). \end{aligned}$$

Assume the assigned function to the region R_i, with $1 \le i \le m$, is $\phi_{R_i} = k_i x + l_i y + a_i$. Since G is a DBWR graph, the balancing condition holds for v. More precisely, we have

$$\sum_{i=1}^{m} w_\sigma(e_i)\vec{e}_i = 0. \tag{3.2.8}$$

For an edge e_i of $\tilde{G}$, we use the notation e_i^* to denote the corresponding edge in $\tilde{G}^*$. Since $\tilde{G}^{**} = \tilde{G}$, there exists a face f of $\tilde{G}^*$, which corresponds to v and f is incident with $e_1^*,\ldots,e_m^*$. Suppose $e_1^*,\ldots,e_{m-1}^*$ are the edges in H'. This means that $(1,2),\ldots,(m-1,m)$ are good pairs. We want to see if $(m,1)$ is a good pair, or in other words e_m^* is an edge of H'. From these good pairs, we know that

$$\forall 1 \leq i < m-1: \ \vec{v}_i = \left(\frac{k_i - k_{i+1}}{w_\sigma(e_i)}, \frac{l_i - l_{i+1}}{w_\sigma(e_i)} \right),$$

is a unit vector normal to e_i pointing to R_{i+1}. If we rotate each $\vec{v}_i$ by 90°, then we obtain $\vec{e}_i$. Assume $\mathcal{R}$ is the rotation matrix by 90°. This means $\vec{e}_i = \mathcal{R}\vec{v}_i$. Hence, (3.2.8) can be written as follows:

$$\begin{aligned}
0 = \sum_{i=1}^{m} w_\sigma(e_i)\vec{e}_i = \sum_{i=1}^{m-1} w_\sigma(e_i)\vec{e}_i + w_\sigma(e_m)\vec{e}_m &= \sum_{i=1}^{m-1} w_\sigma(e_i)\mathcal{R}\vec{v}_i + w_\sigma(e_m)\vec{e}_m \\
&= \sum_{i=1}^{m-1} \mathcal{R} w_\sigma(e_i) \left(\frac{k_i - k_{i+1}}{w_\sigma(e_i)}, \frac{l_i - l_{i+1}}{w_\sigma(e_i)} \right) + w_\sigma(e_m)\vec{e}_m \\
&= \sum_{i=1}^{m-1} \mathcal{R}(k_i - k_{i+1}, l_i - l_{i+1}) + w_\sigma(e_m)\vec{e}_m \\
&= \mathcal{R}\left(\sum_{i=1}^{m-1} k_i - k_{i+1}, \sum_{i=1}^{m-1} l_i - l_{i+1} \right) + w_\sigma(e_m)\vec{e}_m \\
&= \mathcal{R}(k_1 - k_m, l_1 - l_m) + w_\sigma(e_m)\vec{e}_m.
\end{aligned}$$

This means that

$$(l_m - l_1, k_1 - k_m) + w_\sigma(e_m)\vec{e}_m = 0.$$

From this $\vec{e}_m$ is obtained as

$$\vec{e}_m = \left(\frac{l_1 - l_m}{w_\sigma(e_m)}, \frac{k_m - k_1}{w_\sigma(e_m)} \right),$$

which is a unit vector along the edge e_m pointing in the direction away from v. Rotating $\vec{e}_m$ by 90°, we obtain

$$\vec{v}_m = \left(\frac{k_m - k_1}{w_\sigma(e_m)}, \frac{l_m - l_1}{w_\sigma(e_m)} \right),$$

which is a unit vector normal to e_m pointing to R_1. If we show that $\phi_{R_1}\big|_{e_m} = \phi_{R_m}\big|_{e_m}$, we deduce that $(m,1)$ is a good pair, and e_m^* is an edge of H'. This is clear from Remark 3.2.18. Since $(1,2),\ldots,(m-1,m)$ are good pairs, for any i with $1 \leq i < m-1$, we have

$$\phi_{R_i}\big|_{e_i} = \phi_{R_{i+1}}\big|_{e_i}.$$

We know that v as a vertex of G is a common point on the edges $e_1 \ldots e_m$. Therefore, we have

$$\phi_{R_1}(v) = \phi_{R_2}(v) = \ldots \phi_{R_{m-1}}(v) = \phi_{R_m}(v),$$

or simply we have $\phi_{R_m}(v) = \phi_{R_1}(v)$. From Remark 3.2.18, we conclude that $\phi_{R_1}\big|_{e_m} = \phi_{R_m}\big|_{e_m}$. This means $(m,1)$ is a good pair, and e_m^* is an edge of H'.

This shows that the condition (3.2.7) holds. Thus, from Corollary 3.1.25, $H' = \tilde{G}^*$. This means that for any two adjacent vertices i and j of $\tilde{G}^*$, (i,j) is a good pair. That is to say, for any two neighbour regions of $\mathbb{R}^2 \setminus G$, NVC holds. □

Claim 3.2.20. *The assumed DBWR, G is the difference tropical plane curve associated to the following difference tropical polynomial:*

$$P(x,y) = \bigoplus_{R \text{ is a region of } \mathbb{R}^2 \setminus G} a_R \odot x^{\odot k_R} y^{\odot l_R}.$$

Proof. Suppose $\mathbb{R}^2 \setminus G$ has n regions, and we choose a numbering on them. In order to prove that $T_P = G$, it suffices to show that on each region R_i, we have $\phi_{R_i} < \phi_{R_j}$, where $1 \leq j \leq n$ and $j \neq i$. In other words, we need to show that these affine functions define a polyhedral surface Σ, such that on each region R_i the affine function ϕ_{R_i} attains the minimum. If we prove that Σ is concave, the proof is complete.

Throughout this proof, in order to improve readability, we will be somewhat imprecise. For instance, by ϕ_{R_i}, we sometimes mean the function and sometimes its graph. Additionally, whenever we intersect a two-dimensional object with a three-dimensional one, we mean considering the two-dimensional object in $\mathbb{R}^3$ and then taking the intersection.

Let $\mathcal{Q} : x = x_0$ be a plane parallel to yz-plane. From the definition of Σ, we know that $\mathcal{Q} \cap \Sigma$ is a piecewise linear curve γ. See Figure 3.11 and Figure 3.12. To prove that Σ is concave, equivalently we prove that γ is concave. By concavity of γ, we mean that if we move

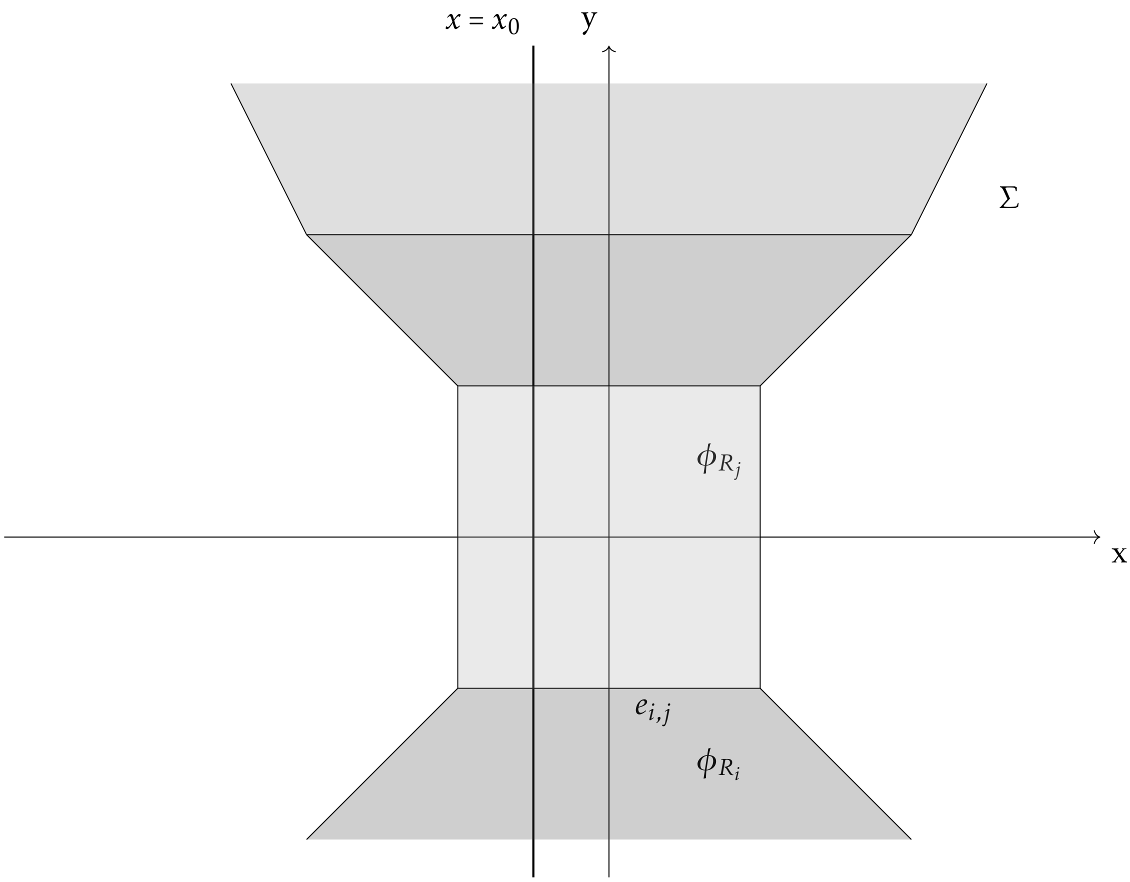


Figure 3.11: A top view of the polyhedral surface Σ and its intersection with the plane $\mathcal{Q} : x = x_0$.

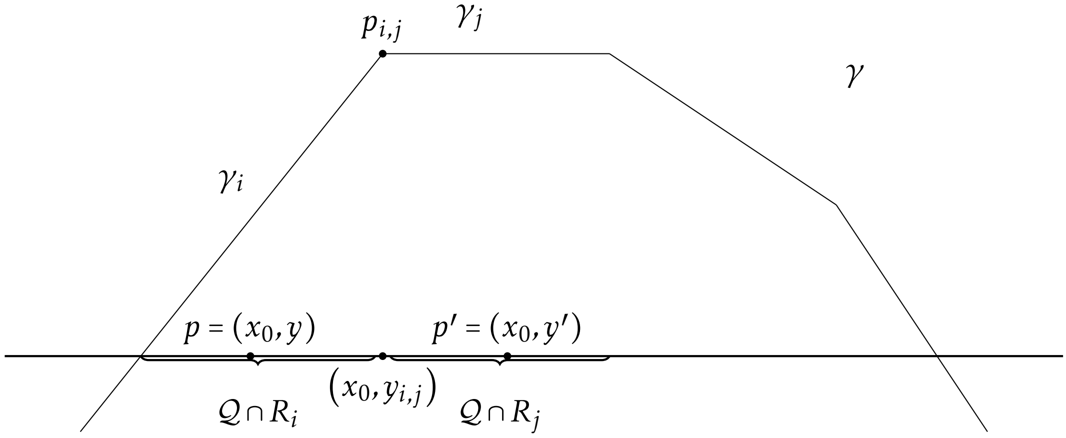


Figure 3.12: The intersection of the polyhedral surface Σ with the plane $\mathcal{Q} : x = x_0$ is a piecewise linear curve γ.

along γ in the direction of the positive y-axis, the slopes of the pieces decrease.

Each linear piece of γ is obtained from the intersection of $\mathcal{Q}$ with one ϕ_{R_i}, where $1 \le i \le n$. If for some i, we have $\mathcal{Q} \cap \phi_{R_i} \neq \varnothing$ then we use the notation $\mathcal{Q} \cap \phi_{R_i} = \gamma_i$, and we denote the slope of γ_i by m_i. Let R_i and R_j be two neighbor regions, such that $\mathcal{Q} \cap \phi_{R_i} \neq \varnothing$ and $\mathcal{Q} \cap \phi_{R_j} \neq \varnothing$. Without loss of generality, we suppose that if $(x_0, y) \in \mathcal{Q} \cap R_i$, and $(x_0, y') \in \mathcal{Q} \cap R_j$, then $y < y'$. Assume $e_{i,j}$ is the edge between these two neighbor regions. Since $\phi_{R_i}\big|_{e_{i,j}} = \phi_{R_j}\big|_{e_{i,j}}$, γ_i and γ_j intersect at a point $p_{i,j}$. More precisely, we have $p_{i,j} = \left(x_0, y_{i,j}, z_{i,j}\right)$, where $z_{i,j} = \phi_{R_i}(x_0, y_{i,j}) = \phi_{R_j}(x_0, y_{i,j})$.

To show that Σ is concave, it suffices to show that $m_i > m_j$. Suppose $p = (x_0, y) \in \mathcal{Q} \cap R_i$ and $p' = (x_0, y') \in \mathcal{Q} \cap R_j$. From $p_{i,j}$ and p, the slope of γ_i is as follows:

$$m_i = \frac{\phi_{R_i}(x_0, y_{i,j}) - \phi_{R_i}(x_0, y)}{y_{i,j} - y} = \frac{k_i x_0 + l_i y_{i,j} + a_i - k_i x_0 - l_i y - a_i}{y_{i,j} - y} = l_i.$$

Similarly, we obtain $m_j = l_j$ from p' and $p_{i,j}$.

Since R_i and R_j are two neighbor regions, NVC holds for $\left(R_i, R_j\right)$. This means that $\left(\dfrac{k_i - k_j}{w_\sigma(e_{i,j})}, \dfrac{l_i - l_j}{w_\sigma(e_{i,j})}\right)$ is a unit vector normal to $e_{i,j}$ pointing to R_j.

As we supposed, moving from $R_i \cap \mathcal{Q}$ to $R_j \cap \mathcal{Q}$ the second coordinates of points increase. Therefore, $l_i > l_j$ that means $m_i > m_j$.

Hence γ is a concave piecewise linear curve. We can make a similar argument to show that if we intersect Σ with $Q' : y = y_0$, we obtain a concave piecewise linear curve γ'. Therefore Σ is a concave polyhedral surface. □

Thus, from these two main steps, we conclude that G represents the difference tropical plane curve defined by the difference tropical polynomial $P(x,y) = \bigoplus_{R \text{ is a region of } \mathbb{R}^2 \setminus G} a_R \odot x^{\odot k_R} y^{\odot l_R}$, which means $T_P = G$. □

In the second part of Theorem 3.2.15, we assumed that G is the curve associated with a DBWR graph. We then defined P such that $T_P = G$. Furthermore, in the following proposition, we show that if we fix a numbering on the regions of $\mathbb{R}^2 \setminus G$, and similar to Step 1 of the previous theorem assign an affine function ϕ_{R_1} to R_1, then the difference tropical polynomial P from the previous proof is determined uniquely.

Proposition 3.2.21. *Let G be a DBWR graph. Assume a numbering on the regions of $\mathbb{R}^2 \setminus G$ is arbitrarily chosen. If we apply the same method as in Step 1 of Theorem 3.2.15, and use a spanning tree T of $\tilde{G}^*$ to assign an affine function ϕ_R to each region R. Then the following statements are equivalent:*

1. *Using any spanning tree to assign the affine functions to the regions, any two neighbour regions of $\mathbb{R}^2 \setminus G$ satisfy NVC;*
2. *If ϕ_R and ϕ'_R are assigned to each region R of $\mathbb{R}^2 \setminus G$, using two distinct spanning trees T and T' of $\tilde{G}^*$, if $\phi_{R_1} = \phi'_{R_1}$, then $\phi_R = \phi'_R$.*

Proof. $(1) \Rightarrow (2)$:

Regard G as a plane connected graph $\tilde{G}$. Assume we number the faces of $\tilde{G}$, and vertices of its dual, $\tilde{G}^*$ based on the chosen numbering of the regions of $\mathbb{R}^2 \setminus G$. Suppose T and T' are two distinct spanning trees of $\tilde{G}^*$. Let R_i be a region of $\mathbb{R}^2 \setminus G$, where its corresponding vertex on $\tilde{G}^*$, and also on T is i. Since both T and T' are connected graphs, there exists a path P on T, and a path P' on T' to reach i from 1.

Suppose by using T and T' as the patterns, we assign affine function $\phi_{R_i} = k_i x + l_i y + a_i$ and $\phi'_{R_i} = k'_i x + l'_i y + a'_i$ respectively, to each region R_i. We want to prove that $\phi_{R_i} = \phi'_{R_i}$. We prove this by using induction on the length of P. Without loss of generality, assume $1, 2, \ldots, i$ are the sequence of vertices on path P, through which we pass, in order to define ϕ_{R_i} for the vertex i.

We assumed, P' is a path on T' from 1 to i, and also $\phi_{R_1} = \phi'_{R_1}$.

Considering the path P, we move from 1 to 2 to define ϕ_{R_2} based on ϕ_{R_1} and NVC. This means that NVC holds for (R_1, R_2) using the coefficients appearing in ϕ_{R_1} and ϕ_{R_2}. More precisely, $\vec{v}_1 = \left(\frac{k_1 - k_2}{w_\sigma(e_1)}, \frac{l_1 - l_2}{w_\sigma(e_1)}\right)$ is a unit vector normal to e_1 pointing to R_2, where $e_1 \subset \partial(R_1) \cap \partial(R_2)$.

On the other hand, from the path P, it is clear that 1 and 2 are adjacent vertices. This means that R_1 and R_2 are neighbour regions. From the assumption in (1), if we consider T' as a pattern, and the corresponding defined affine functions ϕ'_{R_1} and ϕ'_{R_2}, NVC holds for (R_1, R_2) using the coefficients appearing in these affine functions. This means that, $\vec{v'}_1 = \left(\frac{k'_1 - k'_2}{w_\sigma(e_1)}, \frac{l'_1 - l'_2}{w_\sigma(e_1)}\right)$ is a unit vector normal to e_1 pointing to R_2.

Since $\phi_{R_1} = \phi'_{R_1}$, we have $k_1 = k'_1$ and $l_1 = l'_1$. We can write $w_\sigma(e_1)\vec{v}_1 = (k_1 - k_2, l_1 - l_2)$ and $w_\sigma(e_1)\vec{v'}_1 = \left(k_1 - k'_2, l_1 - l'_2\right)$ are vectors of the same length and normal to e_1 pointing to R_2. This implies that $k_2 = k'_2$ and $l_2 = l'_2$. From NVC, we know that $\phi_{R_1}\big|_{e_1} = \phi_{R_2}\big|_{e_1}$, and also $\phi'_{R_1}\big|_{e_1} = \phi'_{R_2}\big|_{e_1}$. Let (x_0, y_0) be a point on e_1. Thus, we have

$$k_1 x_0 + l_1 y_0 + a_1 = k_2 x_0 + l_2 y_0 + a_2, \tag{3.2.9}$$

and we also have

$$k_1 x_0 + l_1 y_0 + a_1 = k'_1 x_0 + l'_1 y_0 + a'_1 = k'_2 x_0 + l'_2 y_0 + a'_2. \tag{3.2.10}$$

Therefore, (3.2.9) and (3.2.10) together give

$$k_2 x_0 + l_2 y_0 + a_2 = k'_2 x_0 + l'_2 y_0 + a'_2.$$

This means that $a_2 = a'_2$, and consequently $\phi_{R_2} = \phi'_{R_2}$. Hence, statement (2) is valid for a path of length one.

Let R be a region on $\mathbb{R}^2 \setminus G$ with the following condition:

The length of the path from 1 to the corresponding vertex of R is $i - 1$.

Then we assume that R satisfies statement (2). We want to prove that even if the length of this path is i instead, R still satisfies statement (2).

From path P, we know that there is a path of length $i-1$ from 1 to $i-1$. Therefore, by the induction assumption, we have $\phi_{R_{i-1}} = \phi'_{R_{i-1}}$.

As P is a path on T, ϕ_{R_i} is assigned to i based on $\phi_{R_{i-1}}$ and NVC. This means that R_{i-1} and R_i satisfy NVC using the coefficients appearing in $\phi_{R_{i-1}}$ and ϕ_{R_i}.

Since i and $i-1$ are adjacent vertices, their corresponding regions are neighbours. Thus from the assumed statement in (1), if we consider any spanning tree as a pattern, for instance T', NVC holds for R_i and R_{i-1}. That is to say, NVC holds for these two regions using the coefficients appearing in ϕ'_{R_i} and $\phi'_{R_{i-1}}$.

With the same argument as in the first step of the induction, since $\phi_{R_{i-1}} = \phi'_{R_{i-1}}$, we can deduce that $\phi_{R_i} = \phi'_{R_i}$.

$(2) \Rightarrow (1)$:

Assume that statement (2) is satisfied. We want to prove that if R_i and R_j are two neighbour regions, such that $e \subset \partial(R_i) \cap \partial(R_j)$, then NVC holds. Suppose $\phi_{R_i} = k_i x + l_i y + a_i$ and $\phi_{R_j} = k_j x + l_j y + a_j$ are the assigned functions to R_i and R_j respectively. We aim to show that $\left(\dfrac{k_i - k_j}{w_\sigma(e)}, \dfrac{l_i - l_j}{w_\sigma(e)}\right)$ is a unit vector normal to e pointing to R_j, and $\phi_{R_i}\big|_e = \phi_{R_j}\big|_e$.

We used the introduced method of Step (1) of Theorem 3.2.15 to assign affine functions to the regions of $\mathbb{R}^2 \setminus G$. We assume that T is the spanning tree that we used as our pattern. Since T is connected, similar to the previous direction of the proof, assume that $1, 2, \ldots, i$ is a path P of T from 1 to i. Also, assume that P' is another path on T from 1 to j. Suppose in the sequence of vertices on this path, k is the vertex adjacent to j, or in other words we have $1, \ldots, k, j$ as the sequence of vertices through which we can reach j from 1. Assume that i and j are not on the same path; otherwise NVC holds and there is nothing left to prove. From T, we define another spanning tree T'. To do so, define $E(T')$ by adding e_{ij} (the edge connecting two vertices i and j) to $E(T)$, and deleting e_{kj} from $E(T)$. The obtained graph T' is obviously a spanning tree, because these changes in the set of edges do not change connectivity and do not create any cycle.

Suppose, we use T' as another pattern, and assign an affine function to each region, with the assumption that $\phi_{R_1} = \phi'_{R_1}$.

Let ϕ'_{R_j} be the assigned function to R_j, using the pattern T'. Note that, the path from 1 to i on T' is the same as the path from 1 to i on T. This means that $\phi_{R_i} = \phi'_{R_i}$. Since on T', i and j are adjacent vertices, we have $\left(\dfrac{k_i - k'_j}{w_\sigma(e)}, \dfrac{l_i - l'_j}{w_\sigma(e)}\right)$ is a unit vector normal to e pointing to R_j, and $\phi'_{R_j}\big|_e = \phi_{R_i}\big|_e$. From the assumption, statement (2) implies that $\phi_{R_j} = \phi'_{R_j}$. That is to say, $k_j = k'_j$ and $l_j = l'_j$. Hence, $\left(\dfrac{k_i - k_j}{w_\sigma(e)}, \dfrac{l_i - l_j}{w_\sigma(e)}\right)$ is a unit vector normal to e pointing to R_j, and $\phi_{R_j}\big|_e = \phi_{R_i}\big|_e$. So R_i and R_j satisfy NVC. □

Chapter 4

Difference Tropical Hypersurfaces

In the previous chapter, we introduced some of the key concepts of difference tropical geometry. In this chapter, we delve deeper into the subject, focusing on the combinatorics of a *difference tropical hypersurface*. This is the content of Proposition 4.3.1. We initiate this chapter by introducing additional difference tropical objects in Section 4.1. Specifically, in this section we see what a difference tropical hypersurface is. To describe its combinatorics, a review of certain concepts in polyhedral geometry becomes necessary. This is presented in Section 4.2.

4.1 Difference Tropical Objects

In this section, we introduce additional difference tropical objects. For clarity, we also present some examples. The reader can find classical analogues of this material in [17].

Definition 4.1.1. Consider a Laurent difference polynomial $f \in K_\sigma[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. So f can be written as $f(x) = \sum_{u(\sigma)\in(\mathbb{Z}[\sigma])^n} c_{u(\sigma)} x^{u(\sigma)}$. To obtain the *tropicalization of* f, we replace classical addition and multiplication operations with tropical ones, $\oplus$ and $\odot$, replace the coefficients with their respective valuations, and replace σ with the induced automorphism on the value group, denoted by σ_Γ. In other words, the tropicalization of

f is defined as follows:

$$\begin{aligned}\operatorname{trop}(f)(w) &= \bigoplus_{u(\sigma_\Gamma)\in(\mathbb{Z}[\sigma_\Gamma])^n} \left(v(c_{u(\sigma)}) \odot w^{u(\sigma_\Gamma)}\right) \\ &= \min_{u(\sigma_\Gamma)\in(\mathbb{Z}[\sigma_\Gamma])^n} \left(v(c_{u(\sigma)}) + u(\rho)\cdot w\right).\end{aligned}$$

Here by $w^{u(\sigma_\Gamma)}$ we mean $\bigodot_{i\in\{1,\dots,n\}} w_i^{u_i(\sigma_\Gamma)}$, for $u(\sigma_\Gamma) = (u_1(\sigma_\Gamma),\dots,u_n(\sigma_\Gamma))$. By using Notation 2.0.16, we have $w_i^{u_i(\sigma_\Gamma)} = u_i(\sigma_\Gamma)(w_i) = u_i(\rho)\cdot w_i$.

By tropicalizing a Laurent difference polynomial f, we obtain a tropical polynomial $\operatorname{trop}(f)$. Each tropical monomial appearing in $\operatorname{trop}(f)$ corresponds to a difference monomial in f. In order to keep this correspondence and avoid possible annihilation of some tropical monomials, we suppose the automorphism of K is such that its scaling exponent, ρ is transcendental.

Remark 4.1.2. Let f be a difference polynomial in one variable. Using the notation defined in Remark 2.0.21, the tropicalization of $f(x) = \sum_{J\in\Lambda} c_J \boldsymbol{\sigma}^J(x)$ is of the following form:

$$\operatorname{trop}(f)(w) = \min_{J\in\Lambda}\{v(c_J) + J\boldsymbol{\sigma}_\Gamma(w)\},$$

where by $\boldsymbol{\sigma}_\Gamma(w)$ we mean $\left(w, \sigma_\Gamma(w), \dots, \sigma_\Gamma^n(w)\right)$, and Λ is a finite subset of $\mathbb{N}^{n+1}$.

Example 4.1.3. Let $f \in \mathbb{C}((t^{\mathbb{R}}))_\sigma[x_1^{\pm1}, x_2^{\pm1}]$ be

$$f(x_1,x_2) = (1+t)x_1x_2^{\sigma^3} + t^2x_2^{\sigma} + 1,$$

and $\rho = \pi$. The tropicalization of f is

$$\operatorname{trop}(f)(w_1,w_2) = \min\{w_1+\rho^3\cdot w_2, 2+\rho\cdot w_2, 0\} = \min\{w_1+\pi^3w_2, 2+\pi w_2, 0\}.$$

Definition 4.1.4. Let f be a Laurent difference polynomial in n variables with coefficients from K. We say w is a *tropical root of* f, if in $\operatorname{trop}(f)(w)$ the minimum is attained at least twice.

A *difference tropical hypersurface*, which is denoted by $\operatorname{trop}(V(f))$, is the set of all tropical roots of a Laurent difference polynomial f. In other words, we have

$$\operatorname{trop}(V(f)) = \{w\in\mathbb{R}^n \mid \text{in } \operatorname{trop}(f)(w) \text{ the minimum is attained at least twice}\}.$$

Definition 4.1.5. Suppose $f(x) = \sum_{u(\sigma)\in(\mathbb{Z}[\sigma])^n} c_{u(\sigma)} x^{u(\sigma)}$ is a Laurent difference polynomial in $K_\sigma[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$.
The *initial form* of f with respect to the point $w \in \mathbb{R}^n$ is a difference polynomial with coefficients in $\mathbf{k}$ that is defined as follows:

$$\mathrm{in}_w(f) = \sum_{\substack{u(\sigma): v(c_{u(\sigma)})+u(\rho)\cdot w \\ =\mathrm{trop}(f)(w)}} \overline{c_{u(\sigma)} t^{-v(c_{u(\sigma)})}} \cdot x^{u(\bar{\sigma})}.$$

In other words, to obtain $\mathrm{in}_w(f)$, we consider those monomials of $\mathrm{trop}(f)$ which achieve the minimum at w. Corresponding to each of these monomials, a monomial appears in $\mathrm{in}_w(f)$.

Example 4.1.6. Consider $f(x_1,x_2) = (1+t)x_1 x_2^{\sigma^3} + t^2 x_2^{\sigma} + 1$ from Example 4.1.3. The initial form of f with respect to $w = (3\pi^2, \frac{-2}{\pi})$ is as follows:

$$\mathrm{in}_w(f) = \bar{1} x_2^{\bar{\sigma}} + \bar{1}.$$

From the definition of the initial form the following lemma can be deduced.

Lemma 4.1.7. Suppose f is a Laurent difference polynomial in $K_\sigma[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$. Then w is a tropical root of f if and only if $\mathrm{in}_w(f)$ is not a monomial.

Proof. $\Longrightarrow$) Since f is an element of $K_\sigma\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$, it is of the form $f = \sum_{u(\sigma)\in(\mathbb{Z}[\sigma])^n} c_{u(\sigma)} x^{u(\sigma)}$. Suppose $w = (w_1,\ldots,w_n)$ is a tropical root of f. By definition, we have

$$\mathrm{trop}(f)(w) = \min_{u(\sigma)\in(\mathbb{Z}[\sigma])^n} (v(c_{u(\sigma)}) + u(\rho)\cdot w),$$

and w attains the minimum in at least two different tropical monomials each of which corresponds to a monomial in $\mathrm{in}_w(f)$. Therefore $\mathrm{in}_w(f)$ has at least two monomials, meaning that $\mathrm{in}_w(f)$ is not a monomial.
$\Longleftarrow$) If $\mathrm{in}_w(f)$ is not a monomial, from the definition of the initial form, each of its monomials corresponds to a tropical monomial in $\mathrm{trop}(f)(w)$ each of which attains the minimum. Thus, $\mathrm{trop}(f)(w)$ achieves the minimum for at least two different tropical monomials. This means w is a tropical root of f. ◻

The following two lemmas are well-known in the context of tropical geometry. See [12] Proposition 2.8, and [17] Lemma 2.6.2. The generalization that f and g are Laurent difference polynomials does not affect their validity. Therefore, we can state them in terms of Laurent difference polynomials.

Lemma 4.1.8. Let f and g be two Laurent difference polynomials in $K_\sigma[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Suppose w is an element of Γ^n, then we have

$$\operatorname{trop}(fg)(w) = \operatorname{trop}(f)(w) + \operatorname{trop}(g)(w).$$

Lemma 4.1.9. Let f and g be two Laurent difference polynomials in $K_\sigma[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Suppose w is an element of Γ^n, then we have

$$\operatorname{in}_w(fg) = \operatorname{in}_w(f)\operatorname{in}_w(g).$$

4.2 Polyhedral Geometry

In this section, we provide some preliminaries on polyhedral geometry. Readers who are familiar with the subject can skip ahead, while those who want to know more can consult the main references for this section, which are [7, 17, 24].

Definition 4.2.1 ([17], Section 2.3)**.** A *polyhedron* is a subset of $\mathbb{R}^n$ that is the intersection of finitely many closed half spaces. It is usually denoted by P. More precisely, it can be described as:

$$P = \{x \in \mathbb{R}^n \mid \mathcal{A} \cdot x \leq \mathcal{B}\},$$

where $\mathcal{A} \in M_{d\times n}(\mathbb{R})$ and $\mathcal{B} \in \mathbb{R}^d$. Here, $\mathcal{A} \cdot x \leq \mathcal{B}$ means that for each i, $1 \leq i \leq d$, we have $(\mathcal{A} \cdot x)_i \leq (\mathcal{B})_i$, where $(\mathcal{A} \cdot x)_i$ and $(\mathcal{B})_i$ denote the i-th coordinates of $\mathcal{A} \cdot x$ and $\mathcal{B}$ respectively.

A specific class of polyhedra are polytopes. A *polytope* is a bounded polyhedron. See Figure 4.2. Below, in Definition 4.2.4, we see an equivalent definition of a polytope. To see this equivalence, refer to Lecture 1 of [24].

Definition 4.2.2 ([17], Definition 2.3.1)**.** Let A be a subset of $\mathbb{R}^n$. A is called *convex* if for any two elements $a, b \in A$, the straight line segment connecting a and b is also in A. More precisely, if for $a, b \in A$, we have $\lambda a + (1-\lambda)b \in A$ where $0 \leq \lambda \leq 1$.

For $U \subseteq \mathbb{R}^n$ the *convex hull* of U, denoted by $conv(U)$, is defined to be the intersection of all convex sets containing U, or equivalently is the smallest convex set containing U.

Remark 4.2.3. The following definition is in fact the definition of a convex polytope. In the literature, a polytope can be convex or nonconvex, but everywhere in this thesis, by a polytope, we mean a convex polytope. Therefore, in this definition, we omit the word "convex".

Definition 4.2.4 ([17], Definition 2.3.1)**.** If U is a finite subset of $\mathbb{R}^n$, namely $U = \{u_1, \ldots, u_r\}$, then $conv(U)$ is called a *polytope*. In this case, it can be described as:

$$conv(U) = \left\{ \sum_{i=1}^{r} \lambda_i u_i \mid \forall i, \;\; 0 \leq \lambda_i \leq 1 \;\; and \;\; \sum_{i=1}^{r} \lambda_i = 1 \right\}.$$

Notation 4.2.5. Both polytopes and polyhedra are usually denoted by P, and their distinction will be clear from the context.

Notation 4.2.6. Dropping the condition $\forall i, \;\; 0 \leq \lambda_i \leq 1$ in Definition 4.2.4, we obtain the *affine hull (affine span)* of the set U which is denoted by $\mathrm{aff}(U)$.
More generally, affine subspaces which are translates of vector(also called linear) subspaces can be described as the affine hull of a finite set of points.

Definition 4.2.7 ([17], Section 2.3)**.** A *face* F of a polyhedron P is a set of the form

$$\{x \in P \mid \forall y \in P, \; v \cdot x \leq v \cdot y\},$$

where v is a linear functional in $(\mathbb{R}^n)^\vee$. In this case, we say F is determined by v, and we use the notation $F = \mathrm{face_v(P)}$. All faces of P, except for P itself are called proper faces.

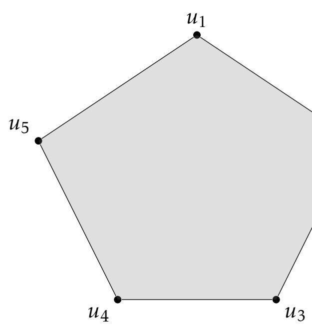


Figure 4.1: The convex hull of the set $U = \{u_1, u_2, u_3, u_4, u_5\}$ defines a polytope .

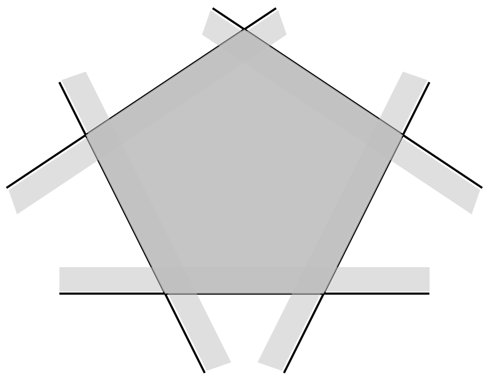


Figure 4.2: The polytope of Figure 4.1 is viewed as the intersection of finitely many half spaces.

The *dimension of a face F* is defined to be the dimension of the affine span of F. In particular, the *dimension of a polyhedron* equals to the dimension of its affine span. The faces of dimension 0 are called vertices and the set of all vertices of P is denoted by V_P. The faces of dimension 1 are called *edges*. A face that is not included in any other proper face is called a *facet*. Additionally, we consider that $\varnothing$ is a face of dimension -1. Note that faces are topologically closed sets.

Remark 4.2.8. Since a polytope is a bounded polyhedron, it is meaningful to talk about a face of a polytope. Moreover, a polytope P is a face of itself, which is of maximal dimension.

Example 4.2.9. The polytope $P = conv(U)$ illustrated in Figure 4.1 is of dimension two. It has five vertices $u_1,\dots,u_5$, and five edges which are the segments connecting these vertices.

In [17], the Newton polytope associated with a Laurent polynomial is discussed. In the following definition, we extend this concept to the difference case.

Definition 4.2.10. Let $f(x) = \sum\limits_{u(\sigma)\in(\mathbb{Z}[\sigma])^n} c_{u(\sigma)}x^{u(\sigma)}$ be a Laurent difference polynomial. We can associate a polytope to $f(x)$. Assume ρ is the scaling exponent of σ, which is transcendental. For any exponent $u(\sigma)$ appearing in f, consider $u(\rho) \in \mathbb{R}^n$. Then this polytope is defined as follows:

$$P = conv(U) \text{ where } U = \left\{u(\rho) \mid c_{u(\sigma)} \neq 0\right\}.$$

This polytope is called the *Newton polytope* associated with f.

Proposition 4.2.11 ([24], Proposition 2.3)**.** *Let $P \subseteq \mathbb{R}^n$ be a polytope, and F be a face of P. Then F is also a polytope, and the set of its vertices is $V_F := V_P \cap F$. More generally, the faces of F are those faces of P which are contained in F.*

Proof. The reader can find the proof in Proposition 2.3 of [24]. □

Definition 4.2.12 ([24], Definition 7.9)**.** Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be two polytopes. Then an affine map

$$\begin{aligned} \pi : \mathbb{R}^n &\longrightarrow \mathbb{R}^m, \\ x &\longmapsto Ax - z, \end{aligned}$$

with $A \in M_{m\times n}(\mathbb{R})$ and $z \in \mathbb{R}^m$, is called a *projection of polytopes*, provided we have $\pi(P) = Q$.

Lemma 4.2.13 ([24], Lemma 7.10). Let $\pi : P \longrightarrow Q$ be a projection of polytopes, and F be a face of Q. Then the preimage of F, $\pi^{-1}(F)$, is a face of P.

Proof. The proof can be found in [24], Lemma 7.10. □

Definition 4.2.14 ([24], Section 2.3)**.** Let F be a face of a polyhedron (or a polytope) P, and $\dim(F) = d$ for some $d \leq \dim(P)$. The *relative interior* of F, denoted by $\operatorname{relint}(F)$, is the set of all the points on F that are not on any face of dimension less than d.
It can be shown that if F is full dimensional this definition coincide with the topological interior of F as a set of points. In this case, we write $\operatorname{int}(F)$ instead of $\operatorname{relint}(F)$.

Definition 4.2.15 ([17], Section 2.3)**.** A finite collection Σ of polyhedra (all in the same $\mathbb{R}^n$) is called a *polyhedral complex*, if it satisfies two following conditions:

1. For each $P \in \Sigma$, any face F of P is also an element of Σ;

2. for any two elements $P, P' \in \Sigma$, $P \cap P'$ is a face of both P and P'.

Each polyhedron appearing in a polyhedral complex Σ is called a *cell* of Σ.
Note that, $\varnothing$ is also a cell of Σ.
The *dimension of a polyhedral complex* is defined to be the maximum of the dimensions of its cells.
A cell of a polyhedral complex Σ is called a *facet* of Σ, if it is not a face of another cell with higher dimension. We say Σ is *pure*, if all its facets are of the same dimension. In Figure 4.3, we have a non example for this concept. In this polyhedral complex P and P' are two facets of dimension two, while P'' is a facet of dimension one. So, it is not pure.

Below, we extend Definition 2.3.2 of [17] to the difference case.

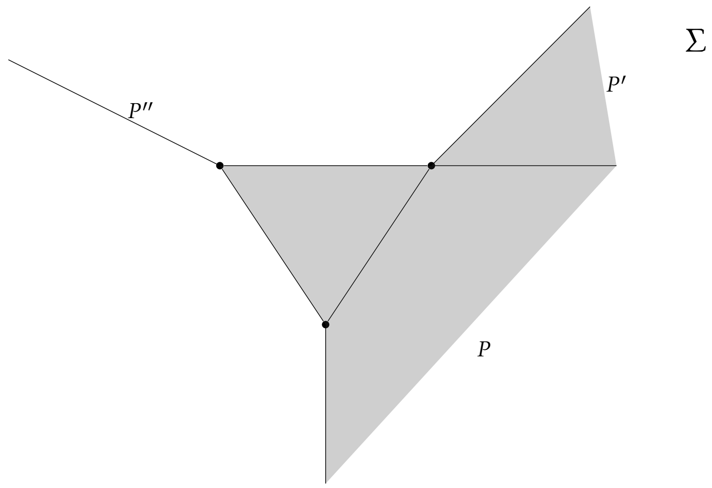


Figure 4.3: Σ is a polyhedral complex of dimension two which is not pure.

Definition 4.2.16. A polyhedron $P = \{x \in \mathbb{R}^n \mid \mathcal{A} \cdot x \leq \mathcal{B}\}$ is called a $(\Gamma, \mathbb{Q}(\rho))$-*polyhedron*, with ρ being a positive transcendental real number, if $\mathcal{A} \in M_{d\times n}(\mathbb{Q}(\rho))$ and $\mathcal{B} \in \Gamma^d$.
Σ is called a $(\Gamma, \mathbb{Q}(\rho))$-*polyhedral complex*, if each of its cells is a $(\Gamma, \mathbb{Q}(\rho))$-polyhedron.

Definition 4.2.17 ([17], Section 2.3)**.** For a polyhedral complex $\Sigma \subset \mathbb{R}^n$, its *support* is denoted by $|\Sigma|$, and is defined as follows:

$$|\Sigma| = \{x \in \mathbb{R}^n \mid \text{There is a cell in } \Sigma \text{ which contains } x\}.$$

Definition 4.2.18 ([17], page 63)**.** Suppose r vectors $u_1, \dots, u_r$ in $\mathbb{R}^n$ are given. A vector $w = (w_1, \dots, w_r) \in \mathbb{R}^r$ is called a *weight vector*. We consider the two following polytopes:

$$P = conv\,\{u_i \mid 1 \leq i \leq r\} \subseteq \mathbb{R}^n, \tag{4.2.1}$$

$$P_w = conv\,\{(u_i, w_i) \mid 1 \leq i \leq r\} \subseteq \mathbb{R}^{n+1}. \tag{4.2.2}$$

We say F is a *lower face* of P_w if $\mathrm{face_v(P_w)} = F$ for some $v = (v_1, \dots, v_{n+1}) \in (\mathbb{R}^{n+1})^\vee$, with v_{n+1} positive.

Suppose $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates. Using this map, if we project all lower faces of P_w to $\mathbb{R}^n$, we obtain the following polyhedral complex:

$$\Sigma_w = \{\pi(F) \mid F \text{ is a lower face of } P_w\},$$

for which we have $|\Sigma_w| = P$. This polyhedral complex gives a subdivision of P. This is called a *regular subdivision* of $u_1,\ldots,u_r$(or a *regular subdivision* of P) induced by w. See Figure 4.4.

Remark 4.2.19. The projection map defined in Definition 4.2.18 is a projection of polytopes from the face F to $\pi(F)$.

Remark 4.2.20. Let v be a vector in $\mathbb{R}^n$. Then for the vector $(v,1) \in \mathbb{R}^{n+1}$, we can find a nonempty lower face F in P_w such that $\text{face}_{(v,1)}(P_w) = F$.

Proof. Assume $\phi_v : P_w \to \mathbb{R}$ is defined by $\phi_v(x) = (v,1)\cdot x$. Clearly ϕ_v is a continuous map. As P_w is closed and bounded, it is compact. Hence, ϕ_v obtains its minimum on P_w. In other words, there exists a nonempty subset F of P_w such that for all x in F, $(v,1)\cdot x$ obtains the minimum. This means

$$F = \{x \in P_w \mid (v,1)\cdot x \leq (v,1)\cdot y \ \forall y \in P_w\}.$$

By definition, F is a face determined by $(v,1)$ which is a vector with positive last coordinate, so F is a lower face. □

Modifying Definition 2.3.3 from [17], we present the following two definitions.

Definition 4.2.21. Let $P \subseteq \mathbb{R}^n$ be a polytope. To each face F of P, we associate the cone $\mathcal{N}_P(F)$ defined as follows:

$$\mathcal{N}_P(F) = \{v \in (\mathbb{R}^n)^\vee \mid \text{face}_v(P) \supseteq F\}.$$

It is called the *closed normal cone* associated to F.

The *open normal cone* associated to F is also defined as

$$\mathring{\mathcal{N}}_P(F) = \{v \in (\mathbb{R}^n)^\vee \mid \text{face}_v(P) = F\}.$$

Obviously, we have $\mathring{\mathcal{N}}_P(F) \subseteq \mathcal{N}_P(F)$.

Definition 4.2.22. Using the previous definition, we define the *normal fan* of a polytope P, which is

$$\mathcal{N}_P = \{\mathcal{N}_P(F) \mid F \text{ is a face of } P\}.$$

Note that $\mathcal{N}_P$ is a polyhedral complex.
The following definition is extracted from the proof of Proposition 3.1.6 from [17].

Definition 4.2.23. Assume $P \subseteq \mathbb{R}^n$, $P_w \subseteq \mathbb{R}^{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are as defined in Definition 4.2.18. We write $\tilde{\pi}$ for the restriction of π to the following set:

$$\{(v,1) \mid v \in \mathbb{R}^n\}.$$

We associate the following set to each lower face F of P_w:

$$\tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right) = \left\{v \in \mathbb{R}^n \mid (v,1) \in \mathring{\mathcal{N}}_P(F)\right\}.$$

The collection of all such sets form a polyhedral complex which is called the *dual complex*, and is denoted by Σ_w°. In other words, we have

$$\Sigma_w^\circ = \left\{\tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right) \mid F \text{ is a lower face of } P_w\right\}.$$

Remark 4.2.24. Given the assumptions of Definition 4.2.23, the dual complex Σ_w° can intuitively be understood as the complex obtained by intersecting the normal fan of P_w with the hyperplane defined by fixing the last coordinate to be one. See Figure 4.5.

Definition 4.2.25 ([17], page 98)**.** Let Σ be a polyhedral complex of dimension n. For $k < n$, the *k-skeleton* is the polyhedral complex consisting of all cells of Σ with dimension at most k.

In the following definition we present some terminology about posets.

Definition 4.2.26. ([24], Definition 2.5)**.**

- By a *poset* S, we mean a partially ordered set.
- A totally ordered subset of a poset S is called a *chain* in S. If a chain has n elements, its *length* is defined to be $n-1$. For a chain with infinitely many elements, the length is defined to be ∞.
- A poset S is called *bounded*, if it has a unique minimal element which is usually denoted by $\hat{0}$ and a unique maximal element which is usually denoted by $\hat{1}$.

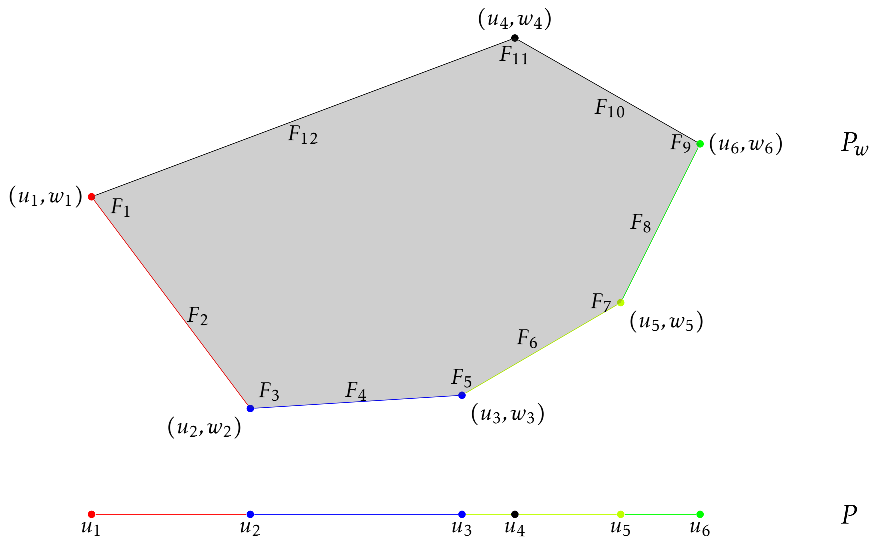


Figure 4.4: The polytope P is lifted to P_w by using the weight vector $w = (w_1, \ldots w_6)$. The lower faces of P_w are $F_1, \ldots, F_9$. The projection of these faces yields a subdivision of P, which are shown in the same colors as the faces.

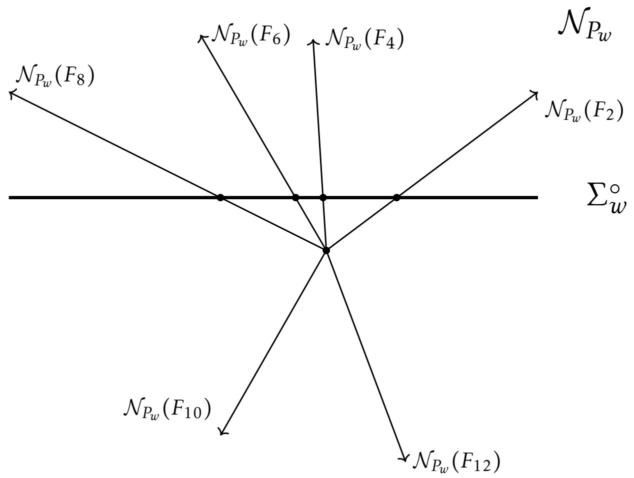


Figure 4.5: The normal fan $\mathcal{N}_{P_w}$ of P_w from Figure 4.4 is drawn in xy-plane. The dual complex Σ_w° is obtained from intersection of $\mathcal{N}_{P_w}$ and $y = 1$.

- A finite poset is called *graded*, if it is bounded and all its maximal chains are of the same length.
- For a graded poset S and an element a of S, the *rank* of a is defined to be the length of the maximal chain starting with $\hat{0}$ and ending with a. It is denoted by $r(a)$.
- A *lattice* is a bounded poset S such that for any two elements $a, b \in S$, there is an infimum (denoted by $a \wedge b$) and also a supremum (denoted by $a \vee b$) in S.

Definition 4.2.27 ([24], Definition 2.6)**.** Let $P \subseteq \mathbb{R}^n$ be a polytope. The *face lattice* of P, denoted by $L(P)$, is the poset of all faces of P where the partial order is inclusion.

Theorem 4.2.28 ([24], Theorem 2.7)**.** *If P is a polytope, then its face lattice, L(P), is a graded lattice of length* $\dim(P) + 1$*, where the length of a graded lattice means the length of its maximal chains. Moreover, if F is a face of P, then we have* $r(F) = \dim(F) + 1$.

Proof. The proof can be found in [24], Theorem 2.7. □

Definition 4.2.29. Let Σ be a polyhedral complex. Consider the poset of all its cells, where the partial order is inclusion and denote it by $L(\Sigma)$. Then $(L(\Sigma), \subseteq)$ is called the *cell poset* of Σ.

Note that, this poset does not necessarily have a unique maximal element; in this case, it is not a bounded poset. To turn such cell posets into a bounded poset, we add an artificial maximal element, $\hat{1}$. Define $\hat{L}(\Sigma) = L(\Sigma) \cup \hat{1}$, then $\big(\hat{L}(\Sigma), \subseteq\big)$ is a bounded poset. Thus, for a pure polyhedral complex, $\big(\hat{L}(\Sigma), \subseteq\big)$ is a graded poset.

Remark 4.2.30. Let Σ be a pure polyhedral complex. Then for any cell F of the graded poset $\big(\hat{L}(\Sigma), \subseteq\big)$, the rank of F is defined, and we have $r(F) = \dim(F) + 1$.

Below, we state two lemmas which are well-known in polyhedral geometry. Since we could not find a reference for their proof, we provide a proof for each here.

Lemma 4.2.31. Let $P \subseteq \mathbb{R}^n$ be a full dimensional polytope and F be one of its faces. For $\mathcal{N}_P(F)$, we define

$$\dim(\mathcal{N}_P(F)) := \dim(\operatorname{aff}(\mathcal{N}_P(F))).$$

Then we have $\dim(F) + \dim(\mathcal{N}_P(F)) = n$.

Proof. In this proof, by $L_0(P)$ we mean the face lattice of P excluding $\varnothing$. Define the following map

$$\begin{aligned} \phi : L_0(P) &\longrightarrow \mathcal{N}_P, \\ F &\longmapsto \mathcal{N}_P(F). \end{aligned}$$

Firstly, we prove that ϕ is an anti-isomorphism, meaning that it is one-to-one and for any two faces F and F' in $L_0(P)$, if $F \subset F'$ then $\mathcal{N}_P(F) \supset \mathcal{N}_P(F')$. From the definition, $\mathcal{N}_P(F')$ can be written as

$$\mathcal{N}_P(F') = \{v \mid \operatorname{face}_v(P) \supseteq F'\} = \{v \mid \forall x' \in F' \ \forall y \in P \ v \cdot x' \leq v \cdot y\}. \tag{4.2.3}$$

Suppose $u \in \mathcal{N}_P(F')$ and $F \subset F'$, from 4.2.3, this means that for any $x \in F$ and for any $y \in P$, we have $u \cdot x \leq u \cdot y$. In other words, we have $u \in \{v \mid \forall x \in F \ \forall y \in P \ v \cdot x \leq v \cdot y\}$ and similar to 4.2.3, this means that $u \in \mathcal{N}_P(F)$. Hence, $\mathcal{N}_P(F) \supseteq \mathcal{N}_P(F')$.

What remains is to prove that ϕ is one-to-one. Assume for $F \neq F'$, we have $\mathcal{N}_P(F) = \mathcal{N}_P(F')$. Take an element v of $\mathring{\mathcal{N}}_P(F)$. From the definition, this means $F = \operatorname{face}_{\mathrm{v}}(\mathrm{P})$. By assumption, $\mathcal{N}_P(F) = \mathcal{N}_P(F')$ and from 4.2.21 we have $\mathring{\mathcal{N}}_P(F) \subseteq \mathcal{N}_P(F) = \mathcal{N}_P(F')$. This implies $v \in \mathcal{N}_P(F')$ meaning that $F' \subseteq \operatorname{face}_{\mathrm{v}}(\mathrm{P}) = F$. Similarly, by taking an element v' of $\mathcal{N}(F')$, we can deduce $F \subseteq F'$. Hence, $F = F'$. Therefore, ϕ is one to one.

From Theorem 4.2.28, we know that for a given polytope P, its face lattice $L(P)$ is a graded lattice of length $\dim(P) + 1$. This means that all the maximal chains in $L(P)$ have the same length, which is $\dim(P) + 1$. As we have assumed that $L_0(P)$ is the face lattice excluding $\varnothing$, we have reduced the length of all maximal chains by one. Therefore, all maximal chains of $L(P)$ have length $\dim(P)$.

Let $F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset F_n = P$ be a maximal chain in $L_0(P)$. Let F_i and F_{i+1} be two successive elements of this chain. From Theorem 4.2.28, we

know that $r(F_i) = \dim(F_i) + 1$ and $r(F_{i+1}) = \dim(F_{i+1}) + 1$. As this chain is assumed to be maximal, this means that there is no other face $\hat{F}_i$ such that $F_i \subset \hat{F}_i \subset F_{i+1}$. Therefore, we can deduce that $r(F_{i+1}) = r(F_i) + 1$, which gives $\dim(F_{i+1}) - \dim(F_i) = 1$. In this proof, we supposed that the empty face is excluded from $L(P)$. Hence, F_0 is a face of dimension 0, which is a vertex. More precisely, this maximal chain of length $\dim(P) = n$, starts with a face of dimension 0, and ends to a face of dimension n. This means that for any natural number i, $0 \le i \le n$, we can find a face in this chain, namely F_i, which is of dimension i.
Apply the map ϕ on this maximal chain. We obtain a chain

$$\mathcal{N}_P(F_0) \supset \mathcal{N}_P(F_1) \supset \cdots \supset \mathcal{N}_P(F_i) \supset \cdots \supset \mathcal{N}_P(F_n),$$

which is of length $\dim(P)$.
For $\mathcal{N}_P(F_i)$ and $\mathcal{N}_P(F_{i+1})$ which are two successive elements of this chain, $\mathcal{N}_P(F_i) \supset \mathcal{N}_P(F_{i+1})$ means $\mathcal{N}_P(F_{i+1})$ is a proper face of the cone $\mathcal{N}_P(F_i)$. Therefore, we have $\dim \mathcal{N}_P(F_i) > \dim \mathcal{N}_P(F_{i+1})$. As the length of this chain is $\dim(P)$, it implies $\dim(\mathcal{N}_P(F_i)) = \dim(P) - i$. Hence, for any i, we have $\dim(F_i) + \dim(\mathcal{N}_P(F_i)) = i + \dim(P) - i = \dim(P) = n$.

Note that for F, an arbitrarily given face of P, F is necessarily contained in such a maximal chain. Otherwise, as we have $F \subset P$, this is a part of a chain that is not contained in any maximal chain of the described form. Therefore, this chain is also maximal but not of the length $\dim(P)$, but this can not happen because $L_0(P)$ is graded. Finally, this means that for any face F of P, we have $\dim(F) + \dim(\mathcal{N}_P(F)) = n$. □

Lemma 4.2.32. Let Σ be a polyhedral complex whose support is $\mathbb{R}^n$. Then Σ is pure of dimension n.

Proof. Let $\mathcal{A}$ be the set of all cells of Σ of dimension n. In other words, we have

$$\mathcal{A} = \{F \mid F \text{ is a cell in } \Sigma \text{ such that } \dim(F) = n\}.$$

First, we want to prove that $\bigcup_{F \in \mathcal{A}} int(F)$ is dense in $\mathbb{R}^n$, which means $\overline{\bigcup_{F \in \mathcal{A}} int(F)} = \mathbb{R}^n$. Equivalently, we want to prove that the topological interior of $\mathbb{R}^n \setminus \bigcup_{F \in \mathcal{A}} int(F) = |\Sigma| \setminus \bigcup_{F \in \mathcal{A}} int(F)$ is empty.

If we define

$$\mathcal{B} = \{F \mid F \text{ is a cell in } \Sigma \text{ such that } \dim(F) < n\},$$

we have

$$|\Sigma| \setminus \bigcup_{F \in \mathcal{A}} int(F) = \bigcup_{F \in \mathcal{B}} F.$$

As $\mathcal{B}$ consists of cells of dimension less than n, $\bigcup_{F \in \mathcal{B}} F$ can not contain a n-ball. Therefore the interior of $\bigcup_{F \in \mathcal{B}} F$ is empty.

Hence, $\overline{\bigcup_{F \in \mathcal{A}} int(F)} = \mathbb{R}^n$. As cells are polyhedra and therefore are topologically closed sets, we have

$$\mathbb{R}^n = \overline{\bigcup_{F \in \mathcal{A}} int(F)} \subseteq \bigcup_{F \in \mathcal{A}} \overline{int(F)} \subseteq \bigcup_{F \in \mathcal{A}} \overline{F} = \bigcup_{F \in \mathcal{A}} F \subseteq \mathbb{R}^n.$$

So, we have

$$\bigcup_{F \in \mathcal{A}} F = \mathbb{R}^n. \tag{4.2.4}$$

Finally, we choose arbitrarily a facet G of Σ, and we prove $\dim(G) = n$. To prove this, we assume the opposite, meaning that $\dim(G) < n$. Choose $x \in G$ such that $x \in \text{relint}(G)$ or in other words, x is not on a proper face of G. As x is also a point in $\mathbb{R}^n$, 4.2.4 implies that there is a cell F of dimension n such that $x \in F$. Therefore, we have $G \cap F \neq \varnothing$. From the definition of a polyhedral complex, $G \cap F$ is a face of both G and F. As x was chosen from an improper face of G, we have $G \cap F = G$. This means, $G \subseteq F$, and this contradicts the assumption that G is a facet. Hence, $\dim(G) = n$, and Σ is pure of dimension n. □

4.3 Combinatorics

In this section, we describe the combinatorics of a difference tropical hypersurface. This is done in Proposition 4.3.1. The proof is similar to the classical case, as in Proposition 3.1.6 of [17]. All polyhedral geometry needed to understand this result is presented in Section 4.2.

Proposition 4.3.1. *If* $f = \sum_{u(\sigma)} c_{u(\sigma)} x^{u(\sigma)}$ *is a Laurent difference polynomial, then its associated difference tropical hypersurface,* $\text{trop}(V(f))$*, is the support*

of a pure $(\Gamma, \mathbb{Q}(\rho))$*–polyhedral complex of dimension* $(n-1)$*.*
More precisely, it is the $(n-1)$*-skeleton of the polyhedral complex dual to the regular subdivision* Σ_{val} *of* P*, where* P *is defined as in Definition 4.2.10, and* val *is the weight vector given by* $v\left(c_{u(\sigma)}\right)$ *for* $c_{u(\sigma)} \neq 0$*.*

Proof. We begin the proof by describing P_{val}, and the dual complex Σ°_{val}. In the first step, we prove Claim 4.3.2 which allows us to show that $\text{trop}(V(f))$ is the $(n-1)$-skeleton of the dual complex. Subsequently, we establish Claim 4.3.3, and having this claim in hand, we demonstrate that $\text{trop}(V(f))$ is a $(\Gamma, \mathbb{Q}(\rho))$–polyhedral complex. Finally, we show that $|\Sigma^{\circ}_{val}| = \mathbb{R}^n$ guaranteeing that $\text{trop}(V(f))$ is pure.

Based on Definition 4.1.4, we know that $\text{trop}(V(f))$ is the set of all points for which the minimum in $\text{trop}(f)$ is attained at least twice.
Considering the Newton polytope of f, we define

$$P_{val} := conv\left\{\left(u(\rho), v(c_{u(\sigma)})\right) : c_{u(\sigma)} \neq 0\right\} \subset \mathbb{R}^{n+1}.$$

Suppose $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is defined by $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$. As we discussed in Definition 4.2.18, if we consider $\pi(F)$ for all lower faces F of P_{val}, we obtain the regular subdivision of P induced by $v\left(c_{u(\sigma)}\right)$ for $c_{u(\sigma)} \neq 0$.
Taking $\tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$ for all lower faces F of P_{val}, we obtain the polyhedral complex dual to the regular subdivision of P induced by $v\left(c_{u(\sigma)}\right)$ for $c_{u(\sigma)} \neq 0$.
Suppose $v = (v_1, \ldots, v_n, 1) \in \mathring{\mathcal{N}}_P(F)$. Then we have

$$\text{in}_{\pi(v)}(f) = \sum_{\substack{u(\sigma): v(c_{u(\sigma)}) + u(\rho)\cdot(v_1,\ldots,v_n) \\ = \text{trop}(f)(v_1,\ldots,v_n)}} \overline{t^{v(c_{u(\sigma)})} \cdot c_{u(\sigma)}} x^{u(\sigma)}. \tag{4.3.1}$$

Note that, we can write

$$v\left(c_{u(\sigma)}\right) + u(\rho) \cdot (v_1, \ldots, v_n) = 1 \cdot v\left(c_{u(\sigma)}\right) + u(\rho) \cdot (v_1, \ldots, v_n) = v \cdot \left(u(\rho), v\left(c_{u(\sigma)}\right)\right).$$

So we have

$$\text{in}_{\pi(v)}(f) = \sum_{\substack{u(\sigma): v\cdot(u(\rho), v(c_{u(\sigma)})) \\ = \text{trop}(f)(\pi(v))}} \overline{t^{-v(c_{u(\sigma)})} \cdot c_{u(\sigma)}} x^{u(\sigma)}. \tag{4.3.2}$$

Claim 4.3.2. *Let $u(\sigma)$ be an exponent appearing in* $\mathrm{in}_{\pi(v)}(f)$. *Then $u(\rho)$ is in $\pi(F)$. Moreover, for each vertex $u(\rho)$ of $\pi(F)$, $u(\sigma)$ is an exponent appearing in* $\mathrm{in}_{\pi(v)}(f)$.

Proof. First, we suppose $u_0(\sigma)$ is an exponent appearing in $\mathrm{in}_{\pi(v)}(f)$, then we prove that $u_0(\rho)$ is in $\pi(F)$.
As $u_0(\sigma)$ is an exponent appearing in $\mathrm{in}_{\pi(v)}(f)$, this means

$$v\cdot\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) = \mathrm{trop}(f)(\pi(v)).$$

In other words, $v\cdot\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right)$ obtains the minimum among all tropical monomials of the form $v\left(c_{u(\sigma)}\right) + u(\rho)\cdot\pi(v)$.
Assume $y \in P_{val} = conv\left\{\left(u(\rho), v(c_{u(\sigma)})\right) : c_{u(\sigma)} \neq 0\right\}$, so y can be written as:

$$y = \sum_{u(\sigma)} \lambda_{u(\sigma)} \left(u(\rho), v\left(c_{u(\sigma)}\right)\right),$$

such that $\forall u(\sigma),\quad 0 \le \lambda_{u(\sigma)} \le 1$ and $\sum\limits_{u(\sigma)} \lambda_{u(\sigma)} = 1$.
We also have

$$v\cdot y = \sum_{u(\sigma)} \lambda_{u(\sigma)} v\cdot\left(u(\rho), v\left(c_{u(\sigma)}\right)\right) \tag{4.3.3}$$

$$= \lambda_{u_0(\sigma)} v\cdot\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) + \sum_{u(\sigma)\neq u_0(\sigma)} \lambda_{u(\sigma)} v\cdot\left(u(\rho), v\left(c_{u(\sigma)}\right)\right). \tag{4.3.4}$$

Since $u_0(\sigma)$ is one of the exponents appearing in $\mathrm{in}_{\pi(v)}(f)$, we have

$$v\cdot\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) \le v\cdot\left(u(\rho), v(c_{u(\sigma)})\right),$$

so (4.3.4) can be written as

$$\lambda_{u_0(\sigma)} v\cdot\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) + \sum_{u(\sigma)\neq u_0(\sigma)} \lambda_{u(\sigma)} \left(v\cdot\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) + \alpha_{u(\sigma)}\right),$$

where

$$\alpha_{u(\sigma)} = v\cdot\left(u(\rho), v(c_{u(\sigma)})\right) - v\cdot\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) \ge 0.$$

As

$$\sum_{u(\sigma)} \lambda_{u(\sigma)} = 1,$$

we have

$$v \cdot y = v \cdot \left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) + \sum_{u(\sigma) \neq u_0(\sigma)} \lambda_{u(\sigma)} \alpha_{u(\sigma)}.$$

On the other hand, for all $u(\sigma)$, we have $\lambda_{u(\sigma)} \geq 0$. Hence, this gives

$$v \cdot y \geq v \cdot \left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right). \tag{4.3.5}$$

As $\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) \in P_{val}$ and $v \in \mathring{\mathcal{N}}_P(F)$, (4.3.5) implies $\left(u_0(\rho), v\left(c_{u_0(\sigma)}\right)\right) \in F$. Therefore $u_0(\rho) \in \pi(F)$.

Now, assume a vertex of $\pi(F)$ is given. We prove that there is an exponent in $\mathrm{in}_{\pi(v)}(f)$ associated to this vertex.

From Definition 4.2.18, a vertex of $\pi(F)$ is of the form $u(\rho)$. Consider the restriction of the projection map on the face F. Hence, $\pi\big|_F : F \longrightarrow \pi(F)$ is a projection of polytopes. Therefore, by Lemma 4.2.13, the preimage of $u(\rho)$ is a face of F; in fact it is a vertex of F.

To see this, assume $\pi\big|_F^{-1}(u(\rho))$ is a face G such that $\dim(G) \geq 1$. As G is a face of F and F is a face of P_{val}, from Proposition 4.2.11, we have $G = conv(\mu)$ where $\mu = G \cap U$ for U being the set of vertices of P_{val}. As $\dim(G) \geq 1$, it contains at least two vertices, each of which is a vertex of P_{val}. Suppose $\left(u_i(\rho), v\left(c_{u_i(\sigma)}\right)\right)$ and $\left(u_j(\rho), v\left(c_{u_j(\sigma)}\right)\right)$ are two distinct vertices of G, so we have

$$u(\rho) = \pi\big|_F \left(u_i(\rho), v\left(c_{u_i(\sigma)}\right)\right) = u_i(\rho) = \pi\big|_F \left(u_j(\rho), v\left(c_{u_j(\sigma)}\right)\right) = u_j(\rho),$$

and $u_i(\rho) = u_j(\rho)$ contradicts the assumption that ρ is transcendental. Hence, $\dim(G) = 0$ which means G is a vertex of F and by definition we have $G = \left(u(\rho), v(c_{u(\sigma)})\right)$.

Since $v = (v_1, \ldots, v_n, 1) \in \mathring{\mathcal{N}}_P(F)$, for all $y \in P_{val}$, we have $v \cdot y \geq v \cdot \left(u(\rho), v\left(c_{u(\sigma)}\right)\right)$. More specifically, for each vertex $\left(u'(\rho), v\left(c_{u'(\sigma)}\right)\right) \in P_{val}$, we have

$$\begin{aligned} & v \cdot \left(u(\rho), v\left(c_{u(\sigma)}\right)\right) \leq v \cdot \left(u'(\rho), v\left(c_{u'(\sigma)}\right)\right) \\ \Longleftrightarrow\ & \pi(v) \cdot u(\rho) + v\left(c_{u(\sigma)}\right) \leq \pi(v) \cdot u'(\rho) + v\left(c_{u'(\sigma)}\right). \end{aligned}$$

This last inequality means, in $\mathrm{trop}(f)(\pi(v))$ the minimum is achieved at $u(\sigma)$ and equivalently $u(\sigma)$ appears as an exponent in $\mathrm{in}_{\pi(v)}(f)$. ∎

Now, by using Claim 4.3.2, we want to prove that $\mathrm{trop}(V(f))$ is the $(n-1)$–skeleton of the dual complex Σ°_{val}. Let F be a nonvertex lower face of P_{val}, and $w \in \tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$. As F is a nonvertex face, $\pi(F)$ has more than one vertex. From Claim 4.3.2, we know each of these vertices corresponds to an exponent in $\mathrm{in}_w(f)$. This means $\mathrm{in}_w(f)$ is not a monomial, or in other words from Lemma 4.1.7 , we have $w \in \mathrm{trop}(V(f))$.
Now, choose $w \in \mathrm{trop}(V(f))$. By Remark 4.2.20, we can set $F = \mathrm{face}_{(\mathrm{w},1)}(\mathrm{P}_{\mathrm{val}})$. This means that $w \in \tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$. As we assumed, $w \in \mathrm{trop}(V(f))$, equivalently from Lemma 4.1.7, this means $\mathrm{in}_w(f)$ is not a monomial. So, at least two different exponents appear in $\mathrm{in}_w(f)$, each of which corresponds to a point on $\pi(F)$. Hence, $\pi(F)$ has more that one vertex, and therefore F is a face with more than one vertex.
Finally, this gives, $w \in \mathrm{trop}(V(f))$ if and only if $w \in \tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$ where F is a face with more than one vertex. As F has at least two vertices, we have $\dim(F) \geq 1$, and by Lemma 4.2.31, this means $\dim\left(\mathcal{N}_p(F)\right) \leq n-1$. We know that $\mathring{\mathcal{N}}_P(F) \subseteq \mathcal{N}_p(F)$. This implies that $\dim\left(\mathring{\mathcal{N}}_P(F)\right) \leq \dim\left(\mathcal{N}_p(F)\right) \leq n-1$, meaning that both $\mathring{\mathcal{N}}_P(F)$ and $\tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$ are not full dimensional. Hence, $w \in \mathrm{trop}(V(f))$ if and only if $\tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$ is a cell in the $(n-1)$–skeleton of the dual complex which contains w.
Thus, $\mathrm{trop}(V(f))$ is the $(n-1)$–skeleton of the dual complex. What remains is to prove that it is a pure $(\Gamma, \mathbb{Q}(\rho))$–polyhedral complex.

Suppose $v \in \mathbb{R}^{n+1}$ is given arbitrarily. We define

$$m(v) := \inf\{v \cdot y \mid y \in P_{val}\}.$$

Now for $m(v)$, we prove the following claim.

Claim 4.3.3. *Let $V_{P_{val}}$ be the set of vertices of P_{val}. Then for $m(v)$ which is defined above, we have*

$$m(v) = \min\left\{v \cdot u \mid u \in V_{P_{val}}\right\}.$$

Proof. Put $m := \min\left\{v \cdot u \mid u \in V_{P_{val}}\right\}$. Since $V_{P_{val}}$ is included in P_{val}, we have $m(v) \leq m$.
Now, we prove $m \leq m(v)$. Given $y \in P_{val}$ arbitrarily. From the definition of

P_{val}, we have

$$y = \sum_{u \in V_{P_{val}}} \lambda_u \cdot u \text{ where } \forall u,\ 0 \le \lambda_u \le 1 \text{ and } \sum_{u \in V_{P_{val}}} \lambda_u = 1.$$

This gives

$$v \cdot y = v \cdot \sum_{u \in V_{P_{val}}} \lambda_u \cdot u = \sum_{u \in V_{P_{val}}} \lambda_u (v \cdot u) \ge \sum_{u \in V_{P_{val}}} \lambda_u m = m.$$

Thus, $\forall y \in P_{val}\ v \cdot y \ge m$, which means $m(v) \ge m$. Hence, $m(v) = m$. ∎

By using Claim 4.3.3, we prove that this $(n-1)$–skeleton is a $(\Gamma, \mathbb{Q}(\rho))$–polyhedral complex.

Let F be a lower face of P_{val}. For $v \in \mathring{\mathcal{N}}_P(F)$, we have

$$\forall x \in F\ v \cdot x \le v \cdot y\ \forall y \in P_{val} \Rightarrow \forall x \in F\ v \cdot x = m(v) = m. \tag{4.3.6}$$

If we denote $F \cap V_{P_{val}}$ by V_F (4.3.6) means, for any vertex $x \in V_F$, we have $v \cdot x = m$. In other words, $\mathring{\mathcal{N}}_P(F)$ can be written as:

$$\mathring{\mathcal{N}}_P(F) = \left\{ v \mid \forall x \in V_F\ \ \forall y \in V_{P_{val}}\ v \cdot x \le v \cdot y \right\}.$$

Therefore, $\tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$, which is a cell in the $(n-1)$–skeleton of the dual complex, can be written as:

$$\begin{aligned} \tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right) &= \left\{ w \in \mathbb{R}^n \mid (w,1) \in \mathring{\mathcal{N}}_P(F) \right\} \\ &= \left\{ w \in \mathbb{R}^n \mid \forall x \in V_F\ \ \forall y \in V_{P_{val}}\ (w,1) \cdot x \le (w,1) \cdot y \right\}. \end{aligned} \tag{4.3.7}$$

Note that in (4.3.7), x and y are both vertices of P_{val}. Therefore, for some i and j, they are of the following form:

$$\begin{aligned} x &= \left(u_i(\rho), v\left(c_{u_i(\sigma)}\right)\right), \\ y &= \left(u_j(\rho), v\left(c_{u_j(\sigma)}\right)\right). \end{aligned} \tag{4.3.8}$$

Using (4.3.8), we can rewrite the inequality that appeared in (4.3.7) as follows:

$$\begin{aligned} (w,1) \cdot x \le (w,1) \cdot y &\Longleftrightarrow w \cdot u_i(\rho) + v\left(c_{u_i(\sigma)}\right) \le w \cdot u_j(\rho) + v\left(c_{u_j(\sigma)}\right) \\ &\Longleftrightarrow v\left(c_{u_i(\sigma)}\right) - v\left(c_{u_j(\sigma)}\right) \le w \cdot \left(u_j(\rho) - u_i(\rho)\right). \end{aligned}$$

As $v\left(c_{u_i(\sigma)}\right) - v\left(c_{u_j(\sigma)}\right) \in \Gamma$ and $\left(u_j(\rho) - u_i(\rho)\right) \in \mathbb{Q}(\rho)^n$, we conclude $\tilde{\pi}\left(\mathring{\mathcal{N}}_P(F)\right)$ is a $(\Gamma, \mathbb{Q}(\rho))$–polyhedron. Therefore, the $(n-1)$–skeleton of the dual complex is a $(\Gamma, \mathbb{Q}(\rho))$–polyhedral complex. To complete the proof, we show that it is also pure.

To do so, arbitrarily choose a facet F of this $(n-1)$–skeleton; we prove that F is of dimension $n-1$. Assume the opposite. In fact, we assume $\dim(F) = m < n-1$.

Suppose $v \in \mathbb{R}^n$ is given. From Remark 4.2.20, we can set $G = \text{face}_{(\text{v},1)}(\text{P}_{\text{val}})$. In other words $v \in \tilde{\pi}\left(\mathring{\mathcal{N}}_p(G)\right)$ which is a cell of the dual complex Σ°_{val}. Therefore, $v \in |\Sigma^{\circ}_{val}| \subseteq \mathbb{R}^n$. Hence, $|\Sigma^{\circ}_{val}| = \mathbb{R}^n$. From Lemma 4.2.32, Σ°_{val} is pure of dimension n, meaning that, there exists a facet F' in Σ°_{val} that is of dimension n and contains F.

Consider the cell poset of Σ°_{val}, and the rank function on the elements of $\hat{L}(\Sigma^{\circ}_{val})$. We have

$$r(F') = \dim(F') + 1 = n + 1 \text{ and } r(F) = \dim(F) + 1 = m + 1 < n.$$

Suppose $\varnothing \subset \cdots \subset F \subset \cdots \subset F'$ is a maximal chain. Note that, $r(F) = m + 1 < n + 1 = r(F')$. Since we have $m < n-1$, there exists a cell F'' in this chain such that $m + 1 < r(F'') < n + 1$. This gives $\dim(F'') < n$. Thus, F'' is a cell in the $(n-1)$-skeleton which contains F. But this is a contradiction, because F is assumed to be a facet in the $(n-1)$-skeleton. Hence, $dim(F) = n-1$, and the $(n-1)$-skeleton is pure. Thus, $\text{trop}(V(f))$ is a pure $(\Gamma, \mathbb{Q}(\rho))$-polyhedral complex.

□

Chapter 5

Difference Newton Lemma

This section provides an essential tool for proving the Difference Kapranov Theorem: the Difference Newton Lemma. For the notation consult 2.0.20. It states

Theorem 5.0.1. *(Difference Newton Lemma) Let K be a multiplicative valued difference field of characteristic zero which is spherically complete. Assume* **k**, *the difference residue field of K, is an ACFA and of characteristic zero. Suppose the difference value group Γ is a subgroup of $\mathbb{R}$ that is a $\mathbb{Q}(\rho)$-module, where ρ, the scaling exponent of σ, is transcendental. Given $f \in K_\sigma[x]$ is not constant and suppose $b \in K$ such that $f(b) \neq 0$.*
We define $\varepsilon := \max_{\substack{J \\ |J|\geq 1}} \varepsilon_J$, where

$$\varepsilon_J := \frac{1}{|J|_\rho}\big(v(f(b)) - v(f_{(J)}(b))\big).$$

Then there exists a root $a \in K$ of f such that $v(a-b) = \varepsilon$.

From this section on, we keep these assumptions on K.

To prove the Difference Newton Lemma, firstly, we need some definitions. The main reference for them is [6]. Note that these definitions are well-known and are defined in a more general context. Therefore, the assumptions on the field K stated in Theorem 5.0.1 are not needed for these definitions.

Definition 5.0.2. Let K be a valued field, and $\left(a_\rho\right)$ be a sequence of elements in K. The sequence $\left(a_\rho\right)$ is called *well-indexed* if the set of indices is well-ordered without maximal element.

A well-indexed sequence $\left(a_\rho\right)$ is called *pseudoconvergent* to a point a if and only if

$$\exists \rho_0 \ \ \forall \delta, \rho : \ \delta > \rho > \rho_0 \longrightarrow v(a - a_\delta) > v(a - a_\rho).$$

In this case, we use the notation $a_\rho \rightsquigarrow a$, and a is called a *pseudolimit* of this sequence.

Definition 5.0.3. A well-indexed sequence is called *pseudocauchy* if and only if

$$\exists \rho_0 \ \ \forall \tau, \delta, \rho : \ \tau > \delta > \rho > \rho_0 \longrightarrow v(a_\tau - a_\delta) > v(a_\delta - a_\rho).$$

Lemma 5.0.4 states that if a point b is not a root of a difference polynomial f, we can find a point a in its neighbourhood which is a better estimation of a possible root. Our approach to proving this lemma is inspired by the proof of Lemma 5.4 from [21].

Lemma 5.0.4. Suppose $f \in K_\sigma[x]$ is a nonconstant difference polynomial. Let b be an element of K which is not a root of f. We define $\varepsilon := \max\limits_{\substack{J \\ |J| \geq 1}} \varepsilon_J$ where

$$\varepsilon_J := \frac{1}{|J|_\rho} \big(v(f(b)) - v(f_{(J)}(b)) \big).$$

Then

1. there exists $a \in K$ such that $v(a - b) = \varepsilon$ and $v(f(a)) > v(f(b))$.

2. for any point $a \in K$ with the properties in (1), we have $\varepsilon < \varepsilon'$ where $\varepsilon' := \max\limits_{\substack{J \\ |J| \geq 1}} \varepsilon'_J$ and $\varepsilon'_J := \frac{1}{|J|_\rho} \big(v(f(a)) - v(f_{(J)}(a)) \big)$.

Proof. 1. Note that ε is an element of the difference value group Γ. Fix any $a' \in K$ such that $v(a' - b) = \varepsilon$.

Assume I is a multi-index for which $|I| \geq 1$.

Then we have

$$\begin{aligned} v\left(\frac{f_{(I)}(b)\boldsymbol{\sigma}^I(a'-b)}{f(b)}\right) &= v(f_{(I)}(b)) + v\left(\boldsymbol{\sigma}^I(a'-b)\right) - v(f(b)) \\ &= v(f_{(I)}(b)) + |I|_\rho v(a'-b) - v(f(b)) \qquad (5.0.1) \\ &= -|I|_\rho \varepsilon_I + |I|_\rho \varepsilon \\ &= |I|_\rho(\varepsilon - \varepsilon_I), \end{aligned}$$

which is nonnegative. Obviously, if J is a multi-index for which $|J| \geq 1$ and $\varepsilon_J = \varepsilon$, then we have

$$v\left(\frac{f_{(J)}(b)\boldsymbol{\sigma}^J(a'-b)}{f(b)}\right) = |J|_\rho(\varepsilon - \varepsilon_J) = 0,$$

which means $\frac{f_{(J)}(b)\sigma^J(a'-b)}{f(b)} \notin \mathcal{M}$, with $\mathcal{M}$ being the maximal ideal of the valuation ring.

Hence, we can define the following nonconstant difference polynomial in $\mathbf{k}_\sigma[x]$:

$$\phi(x) := 1 + \sum_{\substack{I \\ |I|\geq 1}} \overline{\left(\frac{f_{(I)}(b)\boldsymbol{\sigma}^I(a'-b)}{f(b)}\right)} \bar{\boldsymbol{\sigma}}^I(x).$$

Since $\mathbf{k}$ is an ACFA, and $\phi(x)$ is not monomial, by Lemma 2.0.28, $\phi(x)$ has a nonzero root $\bar{u}$. This means that if u is a lift of $\bar{u}$ in the valuation ring then $v(u) = 0$.

Define $a := (a'-b)u + b$. We have

$$v(a-b) = v((a'-b)u) = v(a'-b) + v(u) = \varepsilon + 0 = \varepsilon.$$

Consider the Taylor expansion of $f(a)$ around the point b. It is

$$f(a) = f(b) + \sum_{\substack{I \\ |I|\geq 1}} f_{(I)}(b)\boldsymbol{\sigma}^I(a-b),$$

from which we have

$$\frac{f(a)}{f(b)} = 1 + \sum_{\substack{I \\ |I|\geq 1}} \frac{f_{(I)}(b)\boldsymbol{\sigma}^I(a-b)}{f(b)}. \qquad (5.0.2)$$

Similar to what we did in (5.0.1), we can see that

$$\forall I \ |I| \geq 1 \ : \ v\left(\frac{f_{(I)}(b)\sigma^I(a-b)}{f(b)}\right) \geq 0.$$

Taking the residue of both sides in (5.0.2), and substituting $a = (a' - b)u + b$ gives

$$\overline{\frac{f(a)}{f(b)}} = 1 + \sum_{\substack{I \\ |I|\geq 1}} \overline{\frac{f_{(I)}(b)\sigma^I(a-b)}{f(b)}} = 1 + \sum_{\substack{I \\ |I|\geq 1}} \overline{\frac{f_{(I)}(b)\sigma^I(a'-b)}{f(b)}}\bar{\sigma}^I(\bar{u}) = \phi(\bar{u}) = 0.$$

This implies $v(f(a)) > v(f(b))$.

2. The proof of this part mainly consists of proof by contradiction. For this purpose, we need two technical steps which enable us to obtain the contradiction.

   - **Step 1**: We assume $\varepsilon = \varepsilon_I$ where I is the maximal multi-index, with respect to lexicographical order, for which ε_I attains the maximum. Then we prove $v(f_{(I)}(a)) = v(f_{(I)}(b))$.
   - **Step 2**: If for some multi-index J, we have $\varepsilon = \varepsilon_J$, then we have $\min_I \left(|I|_\rho \varepsilon + v(f_{(I)}(b))\right) = |J|_\rho \varepsilon + v(f_{(J)}(b))$.

   Step 1: Let I is as assumed above. Then $v(f_{(I)}(a)) = v(f_{(I)}(b))$.

   Proof. For an arbitrary nonzero multi-index $L \in \mathbb{N}^{n+1}$, we have $I <_{lex} I + L$. As we have assumed I is the maximal multi-index for which $\varepsilon = \varepsilon_I$, we then have $\varepsilon_{I+L} < \varepsilon_I = \varepsilon$. This means

$$\begin{aligned}
&\frac{1}{|I+L|_\rho}\left(v(f(b)) - v(f_{(I+L)}(b)\right) < \frac{1}{|I|_\rho}\left(v(f(b)) - v(f_{(I)}(b))\right) \\
\Leftrightarrow & v(f(b)) - v(f_{(I+L)}(b) < \frac{|I+L|_\rho}{|I|_\rho}\left(v(f(b)) - v(f_{(I)}(b))\right) \\
\Leftrightarrow & -v(f_{(I+L)}(b)) < \left(1 + \frac{|L|_\rho}{|I|_\rho}\right)\left(v(f(b)) - v(f_{(I)}(b))\right) - v(f(b)) \qquad (5.0.3) \\
\Leftrightarrow & -v(f_{(I+L)}(b)) < -v(f_{(I)}(b)) + |L|_\rho \varepsilon_I \\
\Leftrightarrow & v\left(f_{(I+L)}(b)\sigma^L(a-b)\right) > v(f_{(I)}(b)). \qquad (5.0.4)
\end{aligned}$$

Consider the Taylor expansion of $f_{(I)}(a)$ around the point b. It is

$$f_{(I)}(a) = f_{(I)}(b) + \sum_{L \neq 0} f_{(I)(L)}(b)\boldsymbol{\sigma}^L(a-b).$$

This gives

$$v(f_{(I)}(a)) \geq \min_{L \neq 0} \left\{ v(f_{(I)}(b)), v(f_{(I)(L)}(b)\boldsymbol{\sigma}^L(a-b)) \right\}.$$

Clearly, we have $v(f_{(I)(L)}(b)\boldsymbol{\sigma}^L(a-b)) = v(f_{(I+L)}(b)\boldsymbol{\sigma}^L(a-b))$. Thus, (5.0.4) implies

$$v(f_{(I)}(a)) \geq \min \left\{ v(f_{(I)}(b)), v(f_{(I+L)}(b)\boldsymbol{\sigma}^L(a-b)) \right\} = v(f_{(I)}(b)).$$

which gives $v(f_{(I)}(a)) = v(f_{(I)}(b))$. ∎

Step 2: If for a multi-index J, we have $\varepsilon = \varepsilon_J$, then we have

$$\min_I \left(|I|_\rho \varepsilon + v(f_{(I)}(b)) \right) = |J|_\rho \varepsilon + v(f_{(J)}(b)).$$

Proof. Suppose I is arbitrarily chosen. We want to show that

$$|J|_\rho \varepsilon + v(f_{(J)}(b)) \leq |I|_\rho \varepsilon + v(f_{(I)}(b)).$$

We have

$$\begin{aligned} \varepsilon_I \leq \varepsilon = \varepsilon_J &\Rightarrow |I|_\rho \varepsilon_I + v(f_{(I)}(b)) \leq |I|_\rho \varepsilon + v(f_{(I)}(b)) \\ &\Rightarrow v(f(b)) \leq |I|_\rho \varepsilon + v(f_{(I)}(b)) \\ &\Rightarrow |J|_\rho \varepsilon_J + v(f_{(J)}(b)) \leq |I|_\rho \varepsilon + v(f_{(I)}(b)). \end{aligned}$$

From $\varepsilon_J = \varepsilon$, it follows that

$$\min_I \left(|I|_\rho \varepsilon + v(f_{(I)}(b)) \right) = |J|_\rho \varepsilon + v(f_{(J)}(b)).$$

∎

Going back to the main statement, we want to prove $\varepsilon < \varepsilon'$. We prove this by contradiction. Suppose $\varepsilon \geq \varepsilon'$, where $\varepsilon = \varepsilon_I$ (and I is the maximal multi-index attaining the maximum). Also assume $\varepsilon' = \varepsilon'_J$. Thus, we have

$$\varepsilon \geq \varepsilon' \Rightarrow |I|_\rho \varepsilon + v(f_{(I)}(b)) \geq |I|_\rho \varepsilon' + v(f_{(I)}(b)).$$

From what we proved in step 1, for such I we have $v(f_{(I)}(b)) = v(f_{(I)}(a))$. Thus, $|I|_\rho \varepsilon + v(f_{(I)}(b)) \geq |I|_\rho \varepsilon' + v(f_{(I)}(a))$. Using Step 2 and the assumption that $\varepsilon = \varepsilon_I$, we can write

$$\begin{aligned} v(f(b)) &\overset{\varepsilon=\varepsilon_I}{=} |I|_\rho \varepsilon + v(f_{(I)}(b)) \\ &\overset{Step1}{\geq} |I|_\rho \varepsilon' + v(f_{(I)}(a) \\ &\overset{Step2}{\geq} |J|_\rho \varepsilon' + v(f_{(J)}(a)) \\ &\overset{\varepsilon'=\varepsilon'_J}{=} v(f(a)). \end{aligned}$$

This gives $v(f(b)) \geq v(f((a))$ which contradicts the condition in part (1). Hence $\varepsilon < \varepsilon'$.

□

As we proved in Lemma 5.0.4, if b is not a root of f, we can find a better estimation of a possible root around it. In Proposition 5.0.5, by using Lemma 5.0.4, we build a pc-sequence. Assuming that the field K is spherically complete, this pc-sequence has necessarily a pseudolimit which is a root of f. This implies the main result of this section which is Theorem 5.0.1. We prove Proposition 5.0.5 by similar techniques to those in [21], Lemma 5.6.

Proposition 5.0.5. *Let $f \in K_\sigma[x]$ be a nonconstant difference polynomial and assume $b \in K$ is not a root of f. Define $\varepsilon := \max_{\substack{J \\ |J|\geq 1}} \varepsilon_J$ where*

$$\varepsilon_J := \frac{1}{|J|_\rho}\big(v(f(b)) - v(f_{(J)}(b))\big).$$

Similarly, for a point a_η, we define $\varepsilon_{(\eta,J)}$ and also $\varepsilon_\eta := \max_{\substack{J \\ |J|\geq 1}} \varepsilon_{(\eta,J)}$.

Then there exists a pc-sequence $\{a_\eta\}$ in K with the following properties:

1. *$a_0 = b$ and $\{a_\eta\}$ has a pseudolimit $a \in K$, such that $f(a) = 0$ and $v(a - b) = \varepsilon$;*

2. *$\{v(f(a_\eta))\}$ is strictly increasing;*

3. *$v(a_{\eta'} - a_\eta) = \varepsilon_\eta$ whenever $\eta < \eta'$;*

4. *For any $\eta < \eta'$ we have $\varepsilon_\eta < \varepsilon_{\eta'}$.*

Proof. We prove this by contradiction. Suppose no such pc-sequence exists. Assume $a_0 = b$. For some ordinal $\lambda > 0$, by using Lemma 5.0.4, we inductively construct a sequence $\{a_\eta\}_{\eta<\lambda}$ such that for all η, a_η is not a root of f, and it satisfies properties (2),(3) and (4). Then we use transfinite recursion to extend this sequence to an arbitrarily long sequence which arises a contradiction.
From properties (2), (3) and (4), we conclude that for all $\eta < \eta' < \eta'' < \lambda$ we have

$$v(a_\eta - a_{\eta'}) = \varepsilon_\eta < \varepsilon_{\eta'} = v(a_{\eta'} - a_{\eta''}),$$

which means $\{a_\eta\}_{\eta<\lambda}$ is a pc-sequence. We consider two different possible cases for λ:

(i) λ is a successor ordinal, which means it can be written as $\lambda = \mu + 1$. By Lemma 5.0.4, for $f \in K_\sigma[x]$ and $a_\mu \in K$, there exists $a_\lambda \in K$ such that

- $v(a_\lambda - a_\mu) = \varepsilon_\mu$ and $v(f(a_\lambda)) > v(f(a_\mu))$;
- $\varepsilon_\mu < \varepsilon_\lambda$.

If a_λ is a root of f, these two properties imply $\{v(a_\lambda - a_\eta)\}$ is eventually increasing, which means a_λ is a pseudolimit of $\{a_\eta\}_{\eta<\lambda}$ and we are done. Otherwise, we extend the sequence to $\{a_\eta\}_{\eta<\lambda+1}$ with the same properties.

(ii) Let λ is a limit ordinal. As K is spherically complete, $\{a_\eta\}_{\eta<\lambda}$ as a pc-sequence has a pseudolimit a_λ. Assume a_λ is not a root of f. We want to check whether a_λ has the properties (2),(3) and (4). If so, we extend the sequence to $\{a_\eta\}_{\eta<\lambda+1}$.
We start with checking (3). Since $a_\eta \rightsquigarrow a_\lambda$, by definition we have

$$\exists \eta_0 \text{ such that } \forall\ \eta', \eta;\ \eta' > \eta > \eta_0 \Rightarrow v(a_\lambda - a_{\eta'}) > v(a_\lambda - a_\eta). \quad (5.0.5)$$

Thus, we have

$$\begin{aligned}\lambda > \eta+1 > \eta > \eta_0 \Rightarrow v(a_{\eta+1}-a_\eta) &= v(a_{\eta+1}-a_\lambda+a_\lambda-a_\eta)\\ &\geq \min\{v(a_{\eta+1}-a_\lambda), v(a_\eta-a_\lambda)\}\\ &= v(a_\eta-a_\lambda).\end{aligned} \tag{5.0.6}$$

Since the inequality in (5.0.5) is strict, therefore $v(a_{\eta+1}-a_\lambda) \neq v(a_\eta - a_\lambda)$. Then, we have $v(a_\eta-a_\lambda) = v(a_{\eta+1}-a_\eta) = \varepsilon_\eta$.
If γ is such that $\gamma \leq \eta_0 < \eta$, we can write

$$\begin{aligned}v(a_\lambda-a_\gamma) = v(a_\lambda-a_\eta+a_\eta-a_\gamma) &\geq \min\{v(a_\lambda-a_\eta), v(a_\eta-a_\gamma)\}\\ &= \min\{\varepsilon_\eta, \varepsilon_\gamma\} \qquad (5.0.7)\\ &= \varepsilon_\gamma.\end{aligned}$$

Therefore, (5.0.6) and (5.0.7) imply that a_λ satisfies property (3) for the sequence.
We continue by checking (2). Consider the Taylor expansion of $f(a_\lambda)$ around the point a_η. This is

$$f(a_\lambda) = f(a_\eta) + \sum_{\substack{J\\|J|\geq 1}} f_{(J)}(a_\eta)\boldsymbol{\sigma}^J(a_\lambda-a_\eta).$$

By definition, $\varepsilon_\eta = \max\limits_{\substack{J\\|J|\geq 1}} \varepsilon_{(\eta,J)}$ where $\varepsilon_{(\eta,J)} := \frac{1}{|J|_\rho}\big(v(f(a_\eta)) - v(f_{(J)}(a_\eta))\big)$.
For an arbitrary (η, J) we have

$$\varepsilon_{(\eta,J)} \leq \varepsilon_\eta \Rightarrow \frac{1}{|J|_\rho}\big(v(f(a_\eta) - v(f_{(J)}(a_\eta))\big) \leq \varepsilon_\eta.$$

This gives

$$\begin{aligned}v(f(a_\eta)) &\leq |J|_\rho \varepsilon_\eta + v(f_{(J)}(a_\eta))\\ &= |J|_\rho v(a_{\eta+1}-a_\eta) + v(f_{(J)}(a_\eta))\\ &= |J|_\rho v(a_\lambda-a_\eta) + v(f_{(J)}(a_\eta))\\ &= v(\boldsymbol{\sigma}^J(a_\lambda-a_\eta)\cdot f_{(J)}(a_\eta)).\end{aligned}$$

In the Taylor expansion of $f(a_\lambda)$, take the valuation of both sides. This yields

$$v(f(a_\lambda)) \geq \min_{\substack{J\\|J|\geq 1}}\{v(f(a_\eta)), v(f_{(J)}(a_\eta)\boldsymbol{\sigma}^J(a_\lambda-a_\eta))\} = v(f(a_\eta)). \tag{5.0.8}$$

But the equality is impossible. To see this, suppose $v(f(a_\lambda)) = v(f(a_\eta))$ for some η. As λ is a limit ordinal, we have $\lambda > \eta + 1 > \eta$. From the second property of the sequence, this means

$$v(f(a_\lambda)) = v(f(a_\eta)) < v(f(a_{\eta+1})),$$

and this contradicts (5.0.8). Hence $v(f(a_\lambda)) > v(f(a_\eta))$.

Finally, we check (4). Apply Lemma 5.0.4 for the difference polynomial f, a nonroot a_η and the point a_λ which satisfies the properties of the first part. Thus, the second part of this lemma implies $\varepsilon_\eta < \varepsilon_\lambda$. Therefore, a_λ has all the properties of the sequence $\{a_\eta\}_{\eta<\lambda}$ which enables us to add a_λ to the sequence.

This means, we can build an arbitrarily long sequence $\{a_\eta\}_{\eta<|K|^+}$ such that for all η, a_η is not a root of f, and it satisfies properties (2),(3) and (4). This is a contradiction since all a_η are distinct elements in K. Hence, for some λ, a_λ is a root of f.

□

Chapter 6

The Difference Kapranov Theorem

This chapter consists of two sections. We will see the final result, the Difference Kapranov Theorem, in the second section. In the first one, we prove Proposition 6.1.9 which is the main ingredient to prove the Difference Kapranov Theorem.

By extending the ideas from Theorem 3.1.3 and Proposition 3.1.5 of [17] to our context, we prove the Difference Kapranov Theorem and Proposition 6.1.9.

The general assumption of this section is that K is a multiplicative valued difference field of characteristic zero, and is spherically complete. A particular setting is given by Example 2.0.10. We also assume that the valuation has a splitting and the scaling exponent ρ of σ is transcendental. Besides, we assume that the difference value group Γ is a subgroup of $\mathbb{R}$ that is a $\mathbb{Q}(\rho)$-module. The difference residue field of K is also assumed to be an ACFA, and of characteristic zero.

6.1 Lifting Roots

We gradually work towards the goal of this section, which is Proposition 6.1.9 . First, we will prove a simpler version of this proposition for the case where f is a difference polynomial in one variable.

Lemma 6.1.1. Suppose $f \in K_\sigma[x]$ is a difference polynomial. Assume $w \in \Gamma$ and $\mathrm{in}_w(f)$ is not a monomial. Let $\bar{\alpha}$ be a nonzero root of $\mathrm{in}_w(f)$ in the difference residue field $\mathbf{k}$. Then f has a root $a \in K$ such that $v(a) = w$ and $\overline{t^{-w}a} = \bar{\alpha}$.

Proof. Choose α as a representative of $\bar{\alpha}$ and let $b = t^w\alpha \in K$. Then we have $v(b) = v(t^w\alpha) = v(t^w) + v(\alpha) = w + 0 = w$. Besides $\overline{t^{-w}b} = \overline{t^{-w} \cdot t^w\alpha} = \bar{\alpha}$. Applying Theorem 5.0.1, for $f \in K_\sigma[x]$ and $b \in K$, there exists a root $a \in K$ such that $v(a-b) = \varepsilon$ (where ε is as defined in the same theorem). It suffices to prove that this root a satisfies the desired properties.

Claim 6.1.2. *Let $a \in K$ be a root of f which is obtained by applying Theorem 5.0.1, and ε be as defined in this theorem. If $\varepsilon > w$, then this root satisfies the following properties:*

- $v(a) = w$;
- $\overline{t^{-w}a} = \bar{\alpha}$.

Proof. We write

$$v(a) = v(a - b + b) \geq \min\{v(a-b), v(b)\} = \min\{\varepsilon, w\} = w.$$

Since the minimum is attained uniquely, we have $v(a) = w$.
Moreover, we have

$$\begin{aligned} \varepsilon - w > 0 &\Rightarrow v(a-b) + v(t^{-w}) > 0 \\ &\Leftrightarrow v\left(t^{-w}a - t^{-w}b\right) > 0 \\ &\Leftrightarrow \overline{t^{-w}a} = \overline{t^{-w}b} = \bar{\alpha}. \end{aligned}$$

This means both conditions are satisfied by the root a. ∎

Now, it suffices to prove that $\varepsilon > w$.
As $f \in K_\sigma[x]$, it is of the form $f(x) = \sum_{J\in\Lambda} c_J\sigma^J(x)$ where Λ is a finite subset

of $\mathbb{N}^{n+1}$. The tropicalization of f at w is

$$\begin{aligned}\operatorname{trop}(f)(w) &= \min_{J\in\Lambda}\{v(c_J) + J\sigma_\Gamma(w)\}\\ &= \min_{J\in\Lambda}\{v(c_J) + (j_0, j_1, \dots, j_n)\cdot(w, \rho\cdot w, \dots, \rho^n\cdot w)\}\\ &= \min_{J\in\Lambda}\left\{v(c_J) + |J|_\rho\cdot w\right\}.\end{aligned}$$

By the assumptions, $\operatorname{in}_w(f)$ is not a monomial. Suppose $\Delta\subseteq\Lambda$ consists of all those multi-indices whose corresponding monomials appear in $\operatorname{in}_w(f)$. Then Δ has more than one element.
From the definition of $\operatorname{in}_w(f)$, for any $I\in\Delta$, we can write

$$\operatorname{trop}(f)(w) = \min_{J\in\Lambda}\left\{v(c_J) + |J|_\rho\cdot w\right\} = v(c_I) + |I|_\rho\cdot w. \tag{6.1.1}$$

On the other hand, we have

$$f(b) = \sum_{J\in\Lambda} c_J\sigma^J(b).$$

This implies

$$v(f(b)) \geq \min_{J\in\Lambda}\left\{v(c_J) + v(\sigma^J(b))\right\} = \min_{J\in\Lambda}\left\{v(c_J) + |J|_\rho\cdot w\right\}. \tag{6.1.2}$$

What we obtained in (6.1.2) and (6.1.1) together gives

$$\forall I\in\Delta,\ \ v(f(b)) \geq \min_{J\in\Lambda}\left\{v(c_J) + |J|_\rho\cdot w\right\} = v(c_I) + |I|_\rho\cdot w. \tag{6.1.3}$$

By $J\geq I$, we mean, for all r such that $0\leq r\leq n$, we have $j_r\geq i_r$. Also considering Definition 2.0.25, then for any $I \in \Delta$, we have

$$\begin{aligned}
f_{(I)}(b) = P_{(I)}(\boldsymbol{\sigma}(b)) &= \frac{\partial^{|I|} P(b, \sigma(b), \dots, \sigma^n(b))}{\partial x_0^{i_0} \partial x_1^{i_1} \dots \partial x_n^{i_n}} \cdot \frac{1}{i_0! i_1! \dots i_n!} \\
&= \left. \frac{\partial^{|I|} \sum_{J \in \Lambda} c_J \boldsymbol{x}^J}{\partial x_0^{i_0} \partial x_1^{i_1} \dots \partial x_n^{i_n}} \cdot \frac{1}{i_0! i_1! \dots i_n!} \right|_{\boldsymbol{x}=\boldsymbol{\sigma}(b)} \\
&= \sum_{J \geq I} c_J \left. \frac{\partial^{|I|} \boldsymbol{x}^J}{\partial x_0^{i_0} \partial x_1^{i_1} \dots \partial x_n^{i_n}} \cdot \frac{1}{i_0! i_1! \dots i_n!} \right|_{\boldsymbol{x}=\boldsymbol{\sigma}(b)} \\
&= \sum_{J \geq I} c_J \left. \left(\frac{J!}{(J-I)!} \cdot \boldsymbol{x}^{(J-I)} \cdot \frac{1}{I!} \right) \right|_{\boldsymbol{x}=\boldsymbol{\sigma}(b)} \\
&= \sum_{J \geq I} c_J \left. \left(\binom{J}{I} \cdot \boldsymbol{x}^{(J-I)} \right) \right|_{\boldsymbol{x}=\boldsymbol{\sigma}(b)} .
\end{aligned}$$

Here by $\binom{J}{I}$ we mean $\binom{j_0}{i_0}\binom{j_1}{i_1}\cdots\binom{j_n}{i_n}$. Denoting this coefficient by α_J we have

$$f_{(I)}(b) = \sum_{J \geq I} c_J \cdot \alpha_J \cdot \boldsymbol{\sigma}^{(J-I)}(b).$$

Let I_m be the maximal multi-index in Δ with respect to lexicographical order.

Then we have

$$f_{(I_m)}(b) = \sum_{J \geq I_m} c_J \cdot \alpha_J \cdot \boldsymbol{\sigma}^{(J-I_m)}(b). \tag{6.1.4}$$

Since I_m is the greatest element in Δ, there is no element of Δ appearing in $f_{(I_m)}(b)$. Since α_J is a natural number, we have $v(\alpha_J) = 0$. Therefore, (6.1.4) and (6.1.3) imply

$$\begin{aligned}
v(f_{(I_m)}(b)) &\geq \min_{J \geq I_m} \left\{ v(c_J) + v(\alpha_J) + |J - I_m|_\rho v(b) \right\} \\
&= \min_{J \geq I_m} \left\{ v(c_J) + |J|_\rho w \right\} - |I_m|_\rho w \\
&= v(c_{I_m}) + |I_m|_\rho w - |I_m|_\rho w = v(c_{I_m}).
\end{aligned}$$

On the right side of the above inequality, the minimum is attained only once. Hence,

$$v(f_{(I_m)}(b)) = v(c_{I_m}). \tag{6.1.5}$$

As $I_m \in \Delta$, (6.1.3) gives

$$v(f(b)) \geq v(c_{I_m}) + |I_m|_\rho w. \tag{6.1.6}$$

From (6.1.5) together with (6.1.6), we have

$$v(f(b)) - v(f_{(I_m)}(b)) \geq |I_m|_\rho \cdot w.$$

Multiply both sides by $\dfrac{1}{|I_m|_\rho}$ and consider the definition of ε_{I_m}. Thus, we have

$$\varepsilon_{I_m} = \frac{1}{|I_m|_\rho}\left(v(f(b)) - v(f_{(I_m)}(b))\right) \geq w.$$

Therefore, we obtain

$$\varepsilon = \max_{\substack{J \\ |J|\geq 1}} \varepsilon_J \geq \varepsilon_{I_m} \geq w.$$

Thus far, we have $\varepsilon \geq w$. We show that this equality can not happen. We prove this in two steps.

- **Step 1**: We consider

$$\mathrm{in}_w(f)(\bar{\alpha}) = \sum_{I\in\Delta} \overline{t^{-v(c_I)} c_I} \bar{\boldsymbol{\sigma}}^I(\bar{\alpha}). \tag{6.1.7}$$

  From the assumptions, $\bar{\alpha}$ is a root of $\mathrm{in}_w(f)$, and (6.1.7) is zero.
  On the other hand, we consider $f(b) = \sum_{J\in\Lambda} c_J \boldsymbol{\sigma}^J(b)$, and divide the both sides of this equation by $t^{v\left(c_{I_m}\boldsymbol{\sigma}^{I_m}(b)\right)}$. Since (6.1.6) implies that $v\left(\dfrac{f(b)}{t^{v\left(c_{I_m}\boldsymbol{\sigma}^{I_m}(b)\right)}}\right)$ is non negative, we can consider the residue of both sides. This gives

$$\overline{\frac{f(b)}{t^{v\left(c_{I_m}\boldsymbol{\sigma}^{I_m}(b)\right)}}} = \overline{\sum_{J\in\Lambda} t^{-v\left(c_{I_m}\boldsymbol{\sigma}^{I_m}(b)\right)} c_J \boldsymbol{\sigma}^J(b)}. \tag{6.1.8}$$

  In this step, we mainly prove that

$$\overline{\frac{f(b)}{t^{v\left(c_{I_m}\boldsymbol{\sigma}^{I_m}(b)\right)}}} = \mathrm{in}_w(f)(\bar{\alpha}) = 0. \tag{6.1.9}$$

- **Step 2**: In this final step, we use (6.1.9) to show that $\varepsilon > w$.

Step 1: We prove (6.1.9).

Proof. Suppose $J \in \Lambda$. Then the valuation of the J-th summand appearing on the right side of (6.1.8), is

$$\begin{aligned} v\left(t^{-v\left(c_{I_m}\sigma^{I_m}(b)\right)}c_J\sigma^J(b)\right) &= -v\left(c_{I_m}\sigma^{I_m}(b)\right) + v(c_J) + v(\sigma^J(b)) \\ &= -v(c_{I_m}) - |I_m|_\rho w + v(c_J) + |J|_\rho w. \end{aligned}$$

As we discussed before, $v(c_{I_m}) + |I_m|_\rho w$ attains the minimum. This means $v\left(t^{-v\left(c_{I_m}\sigma^{I_m}(b)\right)}c_J\sigma^J(b)\right)$ is nonnegative.
Also note that for all $J \in \Lambda \setminus \Delta$, we have

$$v\left(t^{-v\left(c_{I_m}\sigma^{I_m}(b)\right)}c_J\sigma^J(b)\right) > 0.$$

Hence, (6.1.8) can be written as:

$$\overline{\frac{f(b)}{t^{v\left(c_{I_m}\sigma^{I_m}(b)\right)}}} = \sum_{I\in\Delta} \overline{t^{-v\left(c_{I_m}\sigma^{I_m}(b)\right)}c_I\sigma^I(b)}. \tag{6.1.10}$$

We want to show that each summand appearing on the right side of (6.1.10) coincides with the corresponding summand appearing in $\mathrm{in}_w(f)(\bar{\alpha})$.
Suppose $I \in \Delta$ and $\overline{t^{-v\left(c_{I_m}\sigma^{I_m}(b)\right)}c_I\sigma^I(b)}$ is the corresponding summand in (6.1.10). From the definition of σ , it can be written as:

$$\begin{aligned} \overline{t^{-v\left(c_{I_m}\sigma^{I_m}(b)\right)}c_I\sigma^I(b)} \cdot \frac{\sigma^I(\bar{\alpha})}{\sigma^I(\bar{\alpha})} &= \frac{\overline{t^{-v\left(c_{I_m}\sigma^{I_m}(b)\right)}c_I\sigma^I(b)}}{\sigma^I(t^{-w}b)} \cdot \sigma^I(\bar{\alpha}) \\ &= \overline{t^{-v(c_{I_m})-|I_m|_\rho w+|I|_\rho w}c_I} \cdot \sigma^I(\bar{\alpha}). \end{aligned} \tag{6.1.11}$$

Since $v(c_{I_m}) + |I_m|_\rho w = v(c_I) + |I|_\rho w$, (6.1.11) equals $\overline{t^{-v(c_I)}c_I}\sigma^I(\bar{\alpha})$ which is the I-th term of $\mathrm{in}_w(f)(\bar{\alpha})$. This yields

$$\overline{\frac{f(b)}{t^{v\left(c_{I_m}\sigma^{I_m}(b)\right)}}} = \mathrm{in}_w(f)(\bar{\alpha}) = 0.$$

∎

Step 2: We prove $\varepsilon > w$.

Proof. From Step 1, we have $\overline{\frac{f(b)}{t^{v(c_{I_m}\sigma^{I_m}(b))}}} = 0$. This means that

$$v\left(\frac{f(b)}{t^{v(c_{I_m}\sigma^{I_m}(b))}}\right) > 0.$$

Equivalently, this gives

$$v(f(b)) > v\left(c_{I_m}\sigma^{I_m}(b)\right) = v(c_{I_m}) + |I_m|_\rho w. \tag{6.1.12}$$

From the equality in (6.1.5), (6.1.12) can be written as:

$$v(f(b)) - v(f_{(I_m)}(b)) > |I_m|_\rho w.$$

Dividing both sides of this inequality by $|I_m|_\rho$ gives $\varepsilon_{I_m} > w$. Therefore $\varepsilon > w$. ∎

□

In Lemma 6.1.3, we generalize Lemma 6.1.1 to a Laurent difference polynomial in one variable.

Lemma 6.1.3. Suppose $f \in K_\sigma[x^{\pm 1}]$ is a Laurent difference polynomial. Assume $w \in \Gamma$ and $\text{in}_w(f)$ is not a monomial. Let $\bar{\alpha}$ be a nonzero root of $\text{in}_w(f)$ in the difference residue field $\mathbf{k}$. Then f has a root $a \in K$ such that $v(a) = w$ and $\overline{t^{-w}a} = \bar{\alpha}$.

Proof. If $f \in K_\sigma[x^{\pm 1}]$, from Remark 2.0.23, there exists $g(x) \in K_\sigma[x]$, such that

$$g(x) = f(x) \cdot \boldsymbol{\sigma}^{J_{\max}}(x).$$

By Lemma 4.1.9, we have

$$\text{in}_w(g) = \text{in}_w(f \cdot \boldsymbol{\sigma}^{J_{\max}}) = \text{in}_w(f) \cdot \text{in}_w(\boldsymbol{\sigma}^{J_{\max}}).$$

By the assumptions $\text{in}_w(f)$ is not a monomial. Hence, $\text{in}_w(g)$ is not a monomial. Moreover,

$$\text{in}_w(g)(\bar{\alpha}) = \text{in}_w(f)(\bar{\alpha}) \cdot \text{in}_w(\boldsymbol{\sigma}^{J_{\max}})(\bar{\alpha}) = 0,$$

which means $\bar{\alpha}$ is a root of $\text{in}_w(g)$. Therefore, we can apply Lemma 6.1.1 to obtain a root $a \in K$ for g such that

- $v(a) = w$ and
- $\overline{t^{-w}a} = \bar{\alpha}$.

Hence, $f(a)\boldsymbol{\sigma}^{J_{\max}}(a) = g(a) = 0$. As $\bar{\alpha} \in \mathbf{k}^*$ and $\overline{t^{-w}a} = \bar{\alpha}$, we have $a \neq 0$. This means $\boldsymbol{\sigma}^{J_{\max}}(a) \neq 0$ which implies $f(a) = 0$. We also have $v(a) = w$ and $\overline{t^{-w}a} = \bar{\alpha}$. □

Moving another step forward to obtain Proposition 6.1.9, we prove a similar statement in Proposition 6.1.4 for a Laurent difference polynomial f in n variables, with an extra assumption on f.

Proposition 6.1.4. *Let $f \in K_\sigma[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be a Laurent difference polynomial with different σ-powers of x_n in its different monomials (see Notation 2.0.16). Assume $\underline{w} = (w_1,\dots,w_n) \in \Gamma^n$ such that $in_{\underline{w}}(f)$ is not a monomial. Suppose $\bar{\alpha} = (\bar{\alpha}_1,\dots,\bar{\alpha}_n)$ is a root of $in_{\underline{w}}(f)$ in $(\mathbf{k}^*)^n$. Then there exists an element $y = (y_1,\dots,y_n)$ in $(K^*)^n$ that is a root of f, and satisfies the following conditions:*

- *$v(y) = \underline{w}$,*
- *for all i, $1 \le i \le n$ we have $\overline{t^{-w_i} \cdot y_i} = \bar{\alpha}_i$.*

Proof. First, we consider the case $n = 1$. By Lemma 6.1.3, there exists a root y for f which satisfies the desired conditions. The difference polynomial f can be regarded as a difference polynomial in one variable x_n with coefficients in $K_\sigma[x_1^{\pm 1},\dots,x_{n-1}^{\pm 1}]$.
To see this better, we use the following notation.

Notation 6.1.5. Given $\underline{u}(\sigma) := (u_1(\sigma),\dots,u_{n-1}(\sigma),u_n(\sigma)) \in (\mathbb{Z}[\sigma])^n$. Set $u(\sigma) := (u_1(\sigma),\dots,u_{n-1}(\sigma))$. This means that for a monomial $c_{\underline{u}(\sigma)}x^{u(\sigma)}x_n^{u_n(\sigma)}$, we have $c_{\underline{u}(\sigma)} \in K$, and by $x^{u(\sigma)}$, we mean $x_1^{u_1(\sigma)} \cdots x_{n-1}^{u_{n-1}(\sigma)}$.

By using this notation each monomial of f takes the form $c_{\underline{u}(\sigma)}x^{u(\sigma)}x_n^{u_n(\sigma)}$. We regard f in this manner, and we use it to define a one-variable Laurent difference polynomial g. Subsequently, we apply the case $n = 1$ to find a root for g. This root ultimately yields a root of f that satisfies the desired conditions.
Suppose for each i, $1 \le i \le n-1$, w_i and $\bar{\alpha}_i \neq 0$ are given. Choose α_i to be

a representative of $\bar{\alpha}_i$. Define $y_i = t^{w_i}\alpha_i$. Then, for each i, $1 \le i \le n-1$, y_i satisfies the following conditions:

- $v(y_i) = w_i$,
- $\overline{t^{-w_i} y_i} = \bar{\alpha}_i$.

Note that, $\alpha_i \neq 0$, and clearly y_i is nonzero.
Since, for each i, we have $y_i \in K^*$, and all monomials in f have different σ-powers of x_n, $g(x_n) = f(y_1, \dots, y_{n-1}, x_n)$ is a nonzero Laurent difference polynomial. Now we will find y_n such that $y = (y_1, \dots, y_n)$ is a root of f satisfying the intended conditions.
Using Notation 6.1.5, $g(x_n)$ can be written as:

$$g(x_n) = \sum_{\underline{u}(\sigma)} d_{\underline{u}(\sigma)} x_n^{u_n(\sigma)},$$

where $d_{\underline{u}(\sigma)} = c_{\underline{u}(\sigma)} y^{u(\sigma)}$ and $u_n(\sigma) \in \mathbb{Z}[\sigma]$ which is of the form $u_n(\sigma) = \sum_{j_n=1}^{m_n} a_{j_n}\sigma^{j_n}$. So the tropicalization of each monomial of $g(x_n)$ is as follows:

$$\begin{aligned}
\operatorname{trop}\left(d_{\underline{u}(\sigma)} x_n^{u_n(\sigma)}\right)(w_n) &= v(d_{\underline{u}(\sigma)}) + \operatorname{trop}\left(\prod_{j_n=1}^{m_n} \left(\sigma^{j_n}(x_n)\right)^{a_{j_n}}\right)(w_n) \\
&= v(d_{\underline{u}(\sigma)}) + \sum_{j_n=1}^{m_n} a_{j_n}\sigma_\Gamma^{j_n}(w_n) \\
&= v(c_{\underline{u}(\sigma)}) + v(y^{u(\sigma)}) + \sum_{j_n=1}^{m_n} a_{j_n}\rho^{j_n} \cdot w_n. \\
&= v(c_{\underline{u}(\sigma)}) + v(y^{u(\sigma)}) + u_n(\rho) \cdot w_n.
\end{aligned} \tag{6.1.13}$$

As $u(\sigma) \in (\mathbb{Z}[\sigma])^{n-1}$, and for all i with $1 \le i \le n-1$, we have $v(y_i) = w_i$, we can write

$$v(y^{u(\sigma)}) = v(\prod_{i=1}^{n-1} y_i^{u_i(\sigma)}) = \sum_{i=1}^{n-1}\sum_{j_i=1}^{m_i} \rho^{j_i} v(y_i^{a_{j_i}}) = \sum_{i=1}^{n-1}\sum_{j_i=1}^{m_i} \rho^{j_i} \cdot a_{j_i} \cdot w_i = \sum_{i=1}^{n-1} u_i(\rho) \cdot w_i.$$

Hence, from (6.1.13), we have

$$\operatorname{trop}\left(d_{\underline{u}(\sigma)} x_n^{u_n(\sigma)}\right)(w_n) = v(c_{\underline{u}(\sigma)}) + \sum_{i=1}^{n} u_i(\rho) \cdot w_i.$$

Suppose for $\underline{w} = (w_1, \ldots, w_{n-1}, w_n)$, for all i with $1 \leq i \leq n-1$, we have $w_i = v(y_i)$, and w_n is a variable. Then

$$\operatorname{trop}\left(d_{\underline{u}(\sigma)} x_n^{u_n(\sigma)}\right)(w_n) = v(c_{\underline{u}(\sigma)}) + \underline{u}(\rho) \cdot \underline{w}, \tag{6.1.14}$$

which is a monomial of $\operatorname{trop}(g)(w_n)$.
On the other hand, consider the tropicalization of a monomial $c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)}$ in $f(x_1, \ldots, x_{n-1}, x_n) = \sum c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)}$, where for each i, we have $u_i(\sigma) \in \mathbb{Z}[\sigma]$, i.e $u_i(\sigma) = \sum_{j_i=1}^{m_i} a_{j_i} \sigma^{j_i}$. It is of the following form:

$$\begin{aligned} \operatorname{trop}\left(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)}\right)(\underline{w}) &= v(c_{\underline{u}(\sigma)}) + \sum_{i=1}^{n} \operatorname{trop}\left(x_i^{u_i(\sigma)}\right)(w_i) \\ &= v(c_{\underline{u}(\sigma)}) + \sum_{i=1}^{n} \sum_{j_i=1}^{m_i} a_{j_i} \sigma_\Gamma^{j_i}(w_i) \\ &= v(c_{\underline{u}(\sigma)}) + \underline{u}(\rho) \cdot \underline{w}. \end{aligned} \tag{6.1.15}$$

The equations in (6.1.14) and (6.1.15) together imply

$$\operatorname{trop}\left(d_{\underline{u}(\sigma)} x_n^{u_n(\sigma)}\right)(w_n) = \operatorname{trop}\left(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)}\right)(\underline{w}).$$

So in general, $\operatorname{trop}(g)(w_n) = \operatorname{trop}(f)(\underline{w})$.
Moreover, we have

$$\begin{aligned} in_{w_n}(g) &= \sum_{\underline{u}(\sigma): v(d_{\underline{u}(\sigma)}) + u_n(\rho)\cdot w_n = \operatorname{trop}(g)(w_n)} \overline{t^{-v(d_{\underline{u}(\sigma)})} d_{\underline{u}(\sigma)}} \cdot x_n^{u_n(\sigma)} \\ &= \sum_{\underline{u}(\sigma): v(c_{\underline{u}(\sigma)}) + u(\rho)\cdot w + u_n(\rho)\cdot w_n = \operatorname{trop}(g)(w_n)} \overline{t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)} t^{-u(\rho)\cdot w} y^{u(\sigma)}} \cdot x_n^{u_n(\sigma)}, \end{aligned}$$

where by w we mean $(w_1, \ldots, w_{n-1})$.
Since $v\left(t^{-u(\rho)\cdot w} y^{u(\sigma)}\right) = 0$, and also $v(t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)}) = 0$, we have

$$
\begin{aligned}
in_{w_n}(g) &= \sum_{\underline{u}(\sigma):v(c_{\underline{u}(\sigma)})+\underline{u}(\rho)\cdot\underline{w}=\operatorname{trop}(f)(\underline{w})} \overline{t^{-v(c_{\underline{u}(\sigma)})}c_{\underline{u}(\sigma)}}\ \overline{t^{-u(\rho)\cdot w}y^{u(\sigma)}}\cdot x_n^{u_n(\sigma)} \\
&= \sum_{\underline{u}(\sigma):v(c_{\underline{u}(\sigma)})+\underline{u}(\rho)\cdot\underline{w}=\operatorname{trop}(f)(\underline{w})} \overline{t^{-v(c_{\underline{u}(\sigma)})}c_{\underline{u}(\sigma)}}\ \overline{\prod_{i=1}^{n-1} t^{-u_i(\rho).w_i}y_i^{u_i(\sigma)}}\cdot x_n^{u_n(\sigma)} \\
&= \sum_{\underline{u}(\sigma):v(c_{\underline{u}(\sigma)})+\underline{u}(\rho)\cdot\underline{w}=\operatorname{trop}(f)(\underline{w})} \overline{t^{-v(c_{\underline{u}(\sigma)})}c_{\underline{u}(\sigma)}}\ \overline{\prod_{i=1}^{n-1} (t^{-w_i})^{u_i(\sigma)}y_i^{u_i(\sigma)}}\cdot x_n^{u_n(\sigma)} \\
&= \sum_{\underline{u}(\sigma):v(c_{\underline{u}(\sigma)})+\underline{u}(\rho)\cdot\underline{w}=\operatorname{trop}(f)(\underline{w})} \overline{t^{-v(c_{\underline{u}(\sigma)})}c_{\underline{u}(\sigma)}}\ \overline{\prod_{i=1}^{n-1} (t^{-w_i}y_i)^{u_i(\sigma)}}\cdot x_n^{u_n(\sigma)}.
\end{aligned}
\tag{6.1.16}
$$

As for all $i,\ 1 \le i \le n-1$, $v(y_i) = w_i$, we have

$$
\begin{aligned}
v\left((t^{-w_i}y_i)^{u_i(\sigma)}\right) &= v\left(\prod_{j_i=1}^{m_i} \sigma^{j_i}\left((t^{-w_i}y_i)^{a_{j_i}}\right)\right) \\
&= \sum_{j_i=1}^{m_i} v\left(\sigma^{j_i}\left((t^{-w_i}y_i)^{a_{j_i}}\right)\right) \\
&= \sum_{j_i=1}^{m_i} \rho^{j_i} a_{j_i} v(t^{-w_i}y_i) = 0,
\end{aligned}
$$

that allows us to write

$$
\overline{\prod_{i=1}^{n-1} (t^{-w_i}y_i)^{u_i(\sigma)}} = \prod_{i=1}^{n-1} \overline{(t^{-w_i}y_i)^{u_i(\sigma)}}. \tag{6.1.17}
$$

The automorphism σ on K induces an automorphism $\bar{\sigma}$ on the residue field, such that for all $\bar{x} \in \mathbf{k}$, we have $\bar{\sigma}(\bar{x}) = \overline{\sigma(x)}$. By abuse of notation, we denote $\bar{\sigma}$ also by σ. Hence (6.1.17) can be written as:

$$
\prod_{i=1}^{n-1} \overline{(t^{-w_i}y_i)^{u_i(\sigma)}} = \prod_{i=1}^{n-1} \left(\overline{t^{-w_i}y_i}\right)^{u_i(\sigma)}.
$$

Finally, (6.1.16) can be written as:

$$
\begin{aligned}
in_{w_n}(g)(x_n) &= \sum_{\underline{u}(\sigma):v(c_{\underline{u}(\sigma)})+\underline{u}(\rho)\cdot\underline{w}=\mathrm{trop}(f)(\underline{w})} \overline{t^{-v(c_{\underline{u}(\sigma)})}c_{\underline{u}(\sigma)}} \prod_{i=1}^{n-1}(\overline{t^{-w_i}y_i})^{u_i(\sigma)}\cdot x_n^{u_n(\sigma)} \\
&= \sum_{\underline{u}(\sigma):v(c_{\underline{u}(\sigma)})+\underline{u}(\rho)\cdot\underline{w}=\mathrm{trop}(f)(\underline{w})} \overline{t^{-v(c_{\underline{u}(\sigma)})}c_{\underline{u}(\sigma)}} \prod_{i=1}^{n-1}\bar{\alpha}_i^{u_i(\sigma)}\cdot x_n^{u_n(\sigma)} \\
&= in_{\underline{w}}(f)(\bar{\alpha}_1,\ldots,\bar{\alpha}_{n-1},x_n).
\end{aligned}
\tag{6.1.18}
$$

By the assumptions, $\bar{\alpha}$ is a root of $in_{\underline{w}}(f)$. This means

$$in_{w_n}(g)(\bar{\alpha}_n) = in_{\underline{w}}(f)(\bar{\alpha}_1,\ldots,\bar{\alpha}_{n-1},\bar{\alpha}_n) = 0.$$

Note that f is a Laurent difference polynomial with different σ-powers of x_n in its different monomials. From the definition of the initial forms, as $in_{\underline{w}}(f)$ is not a monomial, therefore $in_{\underline{w}}(f)(\bar{\alpha}_1,\ldots,\bar{\alpha}_{n-1},x_n)$ is not a monomial. From (6.1.18), it is implied that $in_{w_n}(g)$ is not a monomial.
To sum up, we have g which is a Laurent difference polynomial in one variable x_n and $w_n \in \Gamma$ is such that $in_{w_n}(g)$ is not a monomial. We also know that $\bar{\alpha}_n \in \mathbf{k}^*$ is a root of $in_{w_n}(g)$. From the case $n = 1$, there exists a point $y_n \in K^*$ such that

- $0 = g(y_n) = f(y_1,\ldots,y_{n-1},y_n)$,
- $v(y_n) = w_n$,
- $\overline{t^{-w_n}y_n} = \bar{\alpha}_n$.

Hence, $(y_1,\ldots,y_n)$ is a root of f which satisfies the desired properties of the statement. □

In Proposition 6.1.4, we imposed a condition on f. Lemma 6.1.6 shows that this does not lead to loss of generality. More precisely, even if f is an arbitrary Laurent difference polynomial, associated to f, one can find a Laurent difference polynomial g with the desired condition.

Lemma 6.1.6. Let f be a Laurent difference polynomial in $K_\sigma[x_1^{\pm1},\ldots,x_n^{\pm1}]$. For a natural number l, we define the following automorphism:

$$\phi_l^* : K_\sigma[x_1^{\pm1},\ldots,x_n^{\pm1}] \longrightarrow K_\sigma[x_1^{\pm1},\ldots,x_n^{\pm1}],$$

such that, for all i with $1 \le i \le n-1$, we have $\phi_l^*(x_i) := x_i x_n^{l^i}$. We also define $\phi_l^*(x_n) := x_n$, and $\phi_l^*(x^\sigma) := (\phi_l^*(x))^\sigma$, which means that $\phi_l^*(f(x_1,\ldots,x_n)) := f\left(x_1 x_n^{l^1},\ldots,x_{n-1}x_n^{l^{n-1}},x_n\right)$. Then, for a large enough l, $g := \phi_l^*(f)$ is a Laurent difference polynomial with different σ-powers of x_n in its different monomials.

Proof. Using Notation 6.1.5, for a monomial $x^{u(\sigma)}x_n^{u_n(\sigma)}$, we have

$$\begin{aligned}\phi_l^*\left(x^{u(\sigma)}x_n^{u_n(\sigma)}\right) &= \phi_l^*\left(x^{u(\sigma)}\right)\phi_l^*\left(x_n^{u_n(\sigma)}\right)\\ &= \prod_{i=1}^{n-1}\left(x_i x_n^{l^i}\right)^{u_i(\sigma)} x_n^{u_n(\sigma)}\\ &= x^{u(\sigma)}\prod_{i=1}^{n-1} x_n^{u_i(\sigma)l^i}\cdot x_n^{u_n(\sigma)}\\ &= x^{u(\sigma)}x_n^{u_n(\sigma)+\sum_{i=1}^{n-1}u_i(\sigma)l^i}.\end{aligned}$$

Suppose $x^{u(\sigma)}x_n^{u_n(\sigma)}$ and $x^{u'(\sigma)}x_n^{u'_n(\sigma)}$ are two monomials. We have

$$\phi_l^*\left(x^{u(\sigma)}x_n^{u_n(\sigma)}\right) = x^{u(\sigma)}x_n^{u_n(\sigma)+\sum_{i=1}^{n-1}u_i(\sigma)l^i},$$

and also

$$\phi_l^*\left(x^{u'(\sigma)}x_n^{u'_n(\sigma)}\right) = x^{u'(\sigma)}x_n^{u_n(\sigma)+\sum_{i=1}^{n-1}u'_i(\sigma)l^i}.$$

Consider the following equality

$$u_n(\sigma)+\sum_{i=1}^{n-1}u_i(\sigma)l^i = u_n(\sigma)+\sum_{i=1}^{n-1}u'_i(\sigma)l^i.$$

This gives

$$\sum_{i=1}^{n-1}\left(u_i(\sigma)-u'_i(\sigma)\right)l^i = 0.$$

The element $\sum_{i=1}^{n-1}\left(u_i(\sigma)-u'_i(\sigma)\right)l^i$ of $\mathbb{Z}[\sigma][l]$ has finitely many roots in $\mathbb{N}$. If we choose l to be a natural number greater than all of them, then the image of these two monomials under ϕ_l^* would be two monomials with different σ-powers of x_n.

Hence, for a large enough natural number l, ϕ_l^* maps f to a Laurent difference polynomial with different σ-powers of x_n in its different monomials. □

Lemma 6.1.7. Let f be a Laurent difference polynomial in $K_\sigma[x_1^{\pm 1},\dots,x_n^{\pm 1}]$. Suppose $\underline{w} = (w_1,\dots,w_n) \in \Gamma^n$, and $\bar{\alpha} = (\bar{\alpha}_1,\dots,\bar{\alpha}_n) \in (\mathbf{k}^*)^n$ are given. Assume for a large enough natural number l, ϕ_l^* is defined as in Lemma 6.1.6. If $y' = (y_1',\dots,y_n')$ is a root of $g = \phi_l^*(f)$ with the following properties:

- $\forall i,\ 1 \le i \le n-1 : v(y_i') = w_i - l^i w_n := w_i'$ and $\overline{t^{-w_i + l^i w_n} y_i'} = \bar{\alpha}_i \bar{\alpha}_n^{-l^i}$,
- $v(y_n') = w_n$ and $\overline{t^{-w_n} y_n'} = \bar{\alpha}_n$.

Then there exists a root $y = (y_1,\dots,y_n)$ for f satisfying the following conditions:

- $v(y) = \underline{w}$,
- $\forall i,\ 1 \le i \le n : \overline{t^{-w_i} y_i} = \bar{\alpha}_i$.

Proof. Let $y' = (y_1',\dots,y_n')$ be a root of $\phi_l^*(f)$ such that $y_1',\dots,y_n' \in K^*$, and it satisfies the following conditions:

- $\forall i,\ 1 \le i \le n-1 : v(y_i') = w_i' = w_i - l^i w_n$ and $\overline{t^{-w_i + l^i w_n} y_i'} = \bar{\alpha}_i \bar{\alpha}_n^{-l^i}$,
- $v(y_n') = w_n$ and $\overline{t^{-w_n} y_n'} = \bar{\alpha}_n$.

Define $y = (y_1,\dots,y_n)$ as follows:

$$\forall i,\ 1 \le i \le n-1\ \ y_i = y_i' {y_n'}^{l^i} \ \text{and}\ \ y_n = y_n'.$$

Since $\phi_l^*(f)(y') = 0$, from the definition of ϕ_l^*, we have

$$0 = \phi_l^*(f)(y_1',\dots,y_n') = f\left(y_1' {y_n'}^{l},\dots,y_{n-1}' {y_n'}^{l^{n-1}}, y_n'\right) = f(y),$$

which means y is a root of f. For the root y, we have

$$\begin{aligned} v(y) &= (v(y_1),\dots,v(y_{n-1}),v(y_n)) \\ &= (v(y_1') + l^1 v(y_n'),\dots,v(y_{n-1}') + l^{n-1} v(y_n'), v(y_n')) = \underline{w}. \end{aligned} \tag{6.1.19}$$

We assumed for the root y' of $\phi_l^*(f)$ that

$$\overline{t^{-w_i + l^i w_n} y_i'} = \bar{\alpha}_i \bar{\alpha}_n^{-l_i} \text{ for all } i,\ 1 \le i \le n-1.$$

So for all i with $1 \le i \le n-1$, by an easy computation, we obtain

$$\bar{\alpha}_i = \overline{t^{-w_i + l^i w_n - w_n l^i} y_i' y_n'^{\,l^i}} = \overline{t^{-w_i} y_i}. \tag{6.1.20}$$

By (6.1.19) and (6.1.20), the root y of f has the intended properties. □

Lemma 6.1.8. Let f be a Laurent difference polynomial in $K_\sigma[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Suppose $\underline{w} = (w_1, \ldots, w_n) \in \Gamma^n$, and $\bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_n) \in (\mathbf{k}^*)^n$ are given. Assume that $in_{\underline{w}} f$ is not a monomial and we have $in_{\underline{w}} f(\bar{\alpha}) = 0$.
If for a large enough natural number l, we define ϕ_l^* as in Lemma 6.1.6, then for $g = \phi_l^*(f)$, we have $in_{\underline{w}'}(g)$ is not a monomial, where $\underline{w}' = (w_1 - l^1 w_n, \ldots, w_{n-1} - l^{(n-1)} w_n, w_n)$, and $\bar{\alpha}' = (\bar{\alpha}_1 \bar{\alpha}_n^{-l^1}, \ldots, \bar{\alpha}_{n-1} \bar{\alpha}_n^{-l^{(n-1)}}, \bar{\alpha}_n)$ is a root of $in_{\underline{w}'}(g)$.

Proof. We start by proving that $\operatorname{trop}(f)(\underline{w}) = \operatorname{trop}\big(\phi_l^*(f)\big)(\underline{w}')$. To do so, we look at the tropicalization of each monomial in each of them. The tropicalization of the monomial $c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)}$ of f is

$$\operatorname{trop}(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)})(\underline{w}) = v(c_{\underline{u}(\sigma)}) + \underline{u}(\rho) \cdot \underline{w},$$

and the tropicalization of the corresponding monomial in $\phi_l^*(f)$ at the point $\underline{w}'$ is

$$\begin{aligned}
&\operatorname{trop}\Big(\phi_l^*\Big(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)}\Big)\Big)(\underline{w}') = \operatorname{trop}\Big(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma) + \sum_{i=1}^{n-1} u_i(\sigma) l^i}\Big)(\underline{w}') \\
&= v(c_{\underline{u}(\sigma)}) + \underline{u}(\rho) \cdot \underline{w}' + \sum_{i=1}^{n-1} l^i \sum_{j_i=1}^{m_i} a_{j_i} \rho^{j_i} w_n' \\
&= v(c_{\underline{u}(\sigma)}) + \underline{u}(\rho) \cdot \underline{w}' + \sum_{i=1}^{n-1} l^i u_i(\rho) \cdot w_n' \\
&= v(c_{\underline{u}(\sigma)}) + (u_1(\rho), \ldots, u_{n-1}(\rho), u_n(\rho))(w_1 - l^1 w_n, \ldots, w_{n-1} - l^{(n-1)} w_n, w_n) \\
&+ \sum_{i=1}^{n-1} l^i u_i(\rho) \cdot w_n \\
&= v(c_{\underline{u}(\sigma)}) + \sum_{i=1}^{n-1} u_i(\rho) w_i - l^i u_i(\rho) w_n + u_n(\rho) w_n + \sum_{i=1}^{n-1} l^i u_i(\rho) \cdot w_n \\
&= v(c_{\underline{u}(\sigma)}) + \underline{u}(\rho) \cdot \underline{w} = \operatorname{trop}(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)})(\underline{w}).
\end{aligned}$$

Hence, we have

$$\mathrm{trop}(f)(\underline{w}) = \mathrm{trop}\left(\phi_l^*(f)\right)(\underline{w}').$$

Suppose $in_{\underline{w}}(f)$ is not a monomial. Let

$$\overline{t^{-v(c_{\underline{u}}(\sigma))} \cdot c_{\underline{u}(\sigma)}} x^{u(\sigma)} x_n^{u_n(\sigma)}$$

be one of its monomials. From the definition of $in_{\underline{w}}(f)$, it follows that

$$\mathrm{trop}\left(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)}\right)(\underline{w})$$

attains the minimum. Since $\mathrm{trop}(f)(\underline{w}) = \mathrm{trop}\left(\phi_l^*(f)\right)(\underline{w}')$, we conclude that,

$$\mathrm{trop}\left(\phi_l^*(c_{\underline{u}(\sigma)} x^{u(\sigma)} x_n^{u_n(\sigma)})\right)(\underline{w}')$$

attains the minimum in $\mathrm{trop}\left(\phi_l^*(f)\right)(\underline{w}')$. Therefore, the corresponding monomial appears in $in_{\underline{w}'}\left(\phi_l^*(f)\right)$. Thus, if $in_{\underline{w}}(f)$ has more than one monomial, so does $in_{\underline{w}'}\left(\phi_l^*(f)\right)$. More precisely, this corresponding monomial in $in_{\underline{w}'}\left(\phi_l^*(f)\right)$ is

$$\overline{t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)}} \phi_l^*(x^{u(\sigma)} x_n^{u_n(\sigma)}) = \overline{t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)}} x^{u(\sigma)} x_n^{u_n(\sigma)+\sum_{i=1}^{n-1} u_i(\sigma) l^i}.$$

Assume $\bar{\alpha}$ is a root of $in_{\underline{w}}(f)$. We have

$$\begin{aligned}
&\left(\overline{t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)}} x^{u(\sigma)} x_n^{u_n(\sigma)+\sum_{i=1}^{n-1} u_i(\sigma) l^i}\right)(\bar{\alpha}') \\
&= \overline{t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)}} \bar{\alpha}_1^{u_1(\sigma)} \bar{\alpha}_n^{-u_1(\sigma) l^1} \cdots \bar{\alpha}_{n-1}^{u_{n-1}(\sigma)} \bar{\alpha}_n^{-u_{n-1}(\sigma) l^{(n-1)}} \bar{\alpha}_n^{u_n(\sigma)+\sum_{i=1}^{n-1} u_i(\sigma) l^i} \\
&= \overline{t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)}} \bar{\alpha}_1^{u_1(\sigma)} \cdots \bar{\alpha}_{n-1}^{u_{n-1}(\sigma)} \bar{\alpha}_n^{-\sum_{i=1}^{n-1} u_i(\sigma) l^i} \bar{\alpha}_n^{u_n(\sigma)+\sum_{i=1}^{n-1} u_i(\sigma) l^i} \\
&= \left(\overline{t^{-v(c_{\underline{u}(\sigma)})} c_{\underline{u}(\sigma)}} x^{u(\sigma)} x_n^{u_n(\sigma)}\right)(\bar{\alpha}).
\end{aligned}$$

Hence, we have $in_{\underline{w}'}(\phi_l^*(f))(\bar{\alpha}') = in_{\underline{w}}(f)(\bar{\alpha}) = 0$ which means $\bar{\alpha}'$ is a root of $in_{\underline{w}'}(\phi_l^*(f))$. □

By combining Proposition 6.1.4 with the three preceding lemmas, we derive the main result of this section, which is presented in the following proposition.

Proposition 6.1.9. *Let $f \in K_\sigma[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be a Laurent difference polynomial, and $\underline{w} = (w_1,\dots,w_n) \in \Gamma^n$ such that $in_{\underline{w}}(f)$ is not a monomial. Suppose $\bar{\alpha}$ is a root of $in_{\underline{w}}(f)$ in $(\mathbf{k}^*)^n$. Then there exists an element y in $(K^*)^n$ which is a root of f, and satisfies the following conditions:*

- $v(y) = \underline{w}$,
- $\forall i,\ 1 \le i \le n :\ \overline{t^{-w_i} \cdot y_i} = \bar{\alpha}_i$.

Proof. By Lemma 6.1.6, there exists a Laurent difference polynomial g corresponding to f with different σ-powers of $x_n$in its different monomials. Lemma 6.1.8 guarantees that the assumptions of Proposition 6.1.4 hold for the Laurent difference polynomial g, $\underline{w}'$ and $\bar{\alpha}'$ (where $\underline{w}'$ and $\bar{\alpha}'$ are defined as in Lemma 6.1.8). Hence, from Proposition 6.1.4, we find a root y' for g which satisfies the following conditions:

- $v(y') = \underline{w}'$,
- $\forall i,\ 1 \le i \le n :\ \overline{t^{-w'_i}.y'_i} = \bar{\alpha}'_i$.

Finally, Lemma 6.1.7 implies that corresponding to y', there exist a root y for f which satisfies the desired conditions.

□

6.2 The Difference Kapranov Theorem

In this section by using Proposition 6.1.9, we prove the difference version of Kapranov's Theorem, which is one of the main results of this thesis.

Theorem 6.2.1. *(Difference Kapranov Theorem)*

Let K be a multiplicative valued difference field of characteristic zero, which is spherically complete. Assume its difference residue field is an ACFA of characteristic zero and the scaling exponent ρ is transcendental.
Let the difference value group Γ be a subgroup of $\mathbb{R}$ that is a $\mathbb{Q}(\rho)$-module.
Suppose $f \in K_\sigma\left[x_1^{\pm 1},\dots,x_n^{\pm 1}\right]$ is a Laurent difference polynomial.The following sets coincide:

1. $\mathrm{trop}(V(f)) \subseteq \mathbb{R}^n$ *which is the difference tropical hypersurface associated to* f*;*

2. *the set of all the points* $w \in \mathbb{R}^n$ *for which the initial form* $in_w(f)$ *is not a monomial;*

3. *the closure of the set* $A = \{(v(y_1),\dots,v(y_n)) : (y_1,\dots,y_n) \in V(f)\}$ *in* $\mathbb{R}^n$*.*

Proof. $(1) = (2)$: As it is defined in Definition 4.1.4, $\mathrm{trop}(V(f))$ is the set of all tropical roots of f. Therefore by Lemma 4.1.7, the sets in (1) and (2) are equal.

$(3) \subseteq (1)$: From Proposition 4.3.1, we know that $\mathrm{trop}(V(f))$ is the support of a polyhedral complex, or equivalently it is the union of some polyhedra. As f has finitely many monomials, $\mathrm{trop}(V(f))$ is the union of finitely many polyhedra, each of which is closed so $\mathrm{trop}(V(f))$ is closed.

Let $(v(y_1),\dots,v(y_n)) \in A$. From the definition of A, $y = (y_1,\dots,y_n) \in (K^*)^n$ is a root of f. This means for any monomial $c_{u(\sigma)}x^{u(\sigma)}$ of f, where $c_{u(\sigma)} \neq 0$, we have $v(c_{u(\sigma)}y^{u(\sigma)}) < v(f(y)) = v(0) = \infty$; in fact

$$v\left(\sum_{u(\sigma)\in(\mathbb{Z}[\sigma])^n} c_{u(\sigma)}y^{u(\sigma)}\right) \neq \min_{\substack{u(\sigma)\in(\mathbb{Z}[\sigma])^n \\ c_{u(\sigma)}\neq 0}} \left\{v(c_{u(\sigma)}) + u(\rho)\cdot v(y)\right\}.$$

Therefore, there exist two indices $u(\sigma)$ and $u'(\sigma)$ for which we have

$$v(c_{u(\sigma)}) + u(\rho)\cdot v(y) = v(c_{u'(\sigma)}) + u'(\rho)\cdot v(y),$$

and they achieve the minimum, or in other words $\mathrm{trop}(f)$ attains its minimum in $v(y)$ at least twice. This means $v(y) \in \mathrm{trop}(V(f))$, so $\{v(y) : y \in V(f)\} \subseteq \mathrm{trop}(V(f))$. As $\mathrm{trop}(V(f))$ is closed, we have $\overline{\{v(y) : y \in V(f)\}} \subseteq \mathrm{trop}(V(f))$. It follows that the set in (3) is contained in the set in (1).

$(1) \subseteq (3)$: Let $w \in \mathrm{trop}(V(f)) \cap \Gamma^n$. Since $\mathrm{trop}(f)$ attains its minimum at w at least twice, we deduce from the equality of the sets in (1) and (2) that $in_w(f)$ is not a monomial. Besides, $in_w(f)$ is a Laurent difference polynomial with coefficients in an ACFA field. Therefore, by Theorem 2.0.42 $in_w(f)$ has a root $\bar{\alpha} \in (\mathbf{k}^*)^n$. So by Proposition 6.1.9, there exists $y \in (K^*)^n$ such that $f(y) = 0$, or equivalently $y \in V(f)$ and $v(y) = w$.

This means that $\operatorname{trop}(V(f)) \cap \Gamma^n \subseteq A$. Moreover, $\operatorname{trop}(V(f)) \cap \Gamma^n$ is dense in $\operatorname{trop}(V(f))$. To see this, consider Proposition 4.3.1. From this result, $\operatorname{trop}(V(f))$ is the support of a pure $(\Gamma, \mathbb{Q}(\rho))$–polyhedral complex of dimension $(n-1)$. Suppose P is a facet of this polyhedral complex. So it is a $(\Gamma, \mathbb{Q}(\rho))$–polyhedron of dimension $(n-1)$. Define the projection map π on P, which takes each point to its first $(n-1)$ coordinates. This map is also bijective. Since the interior of $\pi(p)$ in $\mathbb{R}^{n-1}$ is nonempty, we have $\overline{\pi(p)^\circ} = \pi(p)$. From this, it is not difficult to show that $\pi(p) \cap \Gamma^{n-1}$ is dense in $\pi(p)$. Using the bijection, $P \cap \Gamma^n$ is also dense in P. Consequently $\operatorname{trop}(V(f)) \cap \Gamma^n$ is dense in $\operatorname{trop}(V(f))$. Thus, we have

$$\operatorname{trop}(V(f)) = \overline{\operatorname{trop}(V(f)) \cap \Gamma^n} \subseteq \overline{A}.$$

Hence, $\operatorname{trop}(V(f))$ is included in the set in (3) and this implies the equality of the sets in (1) and (3). $\square$

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