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On positive association of absolute-valued and squared multivariate Gaussians beyond MTP_2

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ABSTRACT

We show that positively associated squared (and absolute-valued) multivariate normally distributed random vectors need not be multivariate totally positive of order 2 (MTP_2) for $p \geq 3$. This result disproves Theorem 1 in Eisenbaum (2014, Ann. Probab.) and the conjecture that positive association of squared multivariate normals is equivalent to MTP_2 and infinite divisibility of squared multivariate normals. Among others, we show that there exist absolute-valued multivariate normals which are conditionally increasing in sequence (CIS) (or weakly CIS (WCIS)) and hence positively associated but not MTP_2. Moreover, we show that there exist absolute-valued multivariate normals which are positively associated but not CIS. As a by-product, we obtain necessary conditions for CIS and WCIS of absolute normals. We illustrate these conditions in some examples. With respect to implications and applications of our results, we show PA beyond MTP_2 for some related multivariate distributions (chi-square, t, skew normal) and refer to possible conservative multiple test procedures and conservative simultaneous confidence bounds. Finally, we obtain the validity of the strong form of Gaussian product inequalities beyond MTP_2.

1. Introduction

We are concerned with the question which squared (and absolute-valued) Gaussian random vectors are positively associated (PA). The notion of positive association of random variables was introduced by Esary, Proschan and Walkup in 1967, see [11]. In 2000, Joseph Glaz stated that it is still an open problem to characterize when $|X| = (|X_1|, \dots, |X_p|)^{\mathsf{T}}$ is PA, if $X = (X_1, \dots, X_p)^{\mathsf{T}}$ has a multivariate normal distribution, see the remark following Theorem 2.2.2 in [14]. In 2014, Theorem 1 in [9] claimed that a squared centered Gaussian vector is PA if and only if (iff) it is infinitely divisible. Unfortunately, we found that the proof fails and all our attempts to deliver a correct proof failed, too. Finally, doubt came up that positive association is as restrictive as infinite divisibility and MTP_2. In this note we show that there exist positively associated absolute-valued (or squared) multivariate normally distributed random vectors which are not multivariate totally positive of order 2 (MTP_2). Hence, there exist positively associated squared centered Gaussians which are not infinitely divisible. Consequently, Theorem 1 in [9] is false. Based on our findings, N. Eisenbaum published an Erratum, see [10].

The paper is organized as follows. In Section 1, we first introduce some dependence concepts relevant for this paper. Then we give a brief summary of previous results and facts related to these concepts. In Section 3 the main results of this paper are presented

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in Theorem 1, Lemma 1, Theorem 2 and Theorem 3. The key step is Lemma 1. It states that for $p = 3$ there exists absolute-valued multivariate normals $|X| = (|X_1|, |X_2|, |X_3|)^\top$ which are conditionally increasing in sequence (CIS) but not MTP_2 iff the underlying covariance matrix satisfies a certain condition. This lemma is proved in Section 4. Theorem 1 is a consequence of Lemma 1 and gives a condition on the covariance matrix of X for $p = 3$ such that $|X|$ is PA but not MTP_2. Based on Theorems 1, 2 and its short (constructive) proof yields positive definite covariance matrices for any $p \geq 4$ such that the corresponding $|X|$ is PA but not MTP_2. Finally, Theorem 3 tells us that for $p \geq 4$ the existence of a CIS sequence is not necessary for $|X|$ to be PA. Section 5 gives some necessary conditions which may serve as a quick check whether CIS or WCIS is possible. In this context, we briefly discuss three examples for $p \geq 4$. In the concluding remarks in Section 6 we briefly discuss some implications and applications with respect to the class $\mathcal{S}_p$ (say) of covariance matrices for which $|X|$ is PA. Thereby we discuss PA beyond MTP_2 with respect to $\mathcal{S}_p$ for some related distributions, that is, a specific multivariate chi-square distribution, a specific multivariate t-distribution and some multivariate skew normal distributions. Moreover, we refer to possible applications with respect to conservative multiple test procedures and conservative simultaneous confidence bounds. Finally, we obtain the validity of the strong form of Gaussian product inequalities beyond MTP_2.

2. Definitions and previous results

We first introduce some concepts of positive dependence. Let $\mathbb{N}_p = \{1, \dots, p\}$ for $p \in \mathbb{N}$. A vector $X = (X_1, \dots, X_p)^\top$ of real-valued random variables is said to be

- multivariate totally positive of order 2 (MTP_2) if it has a probability density f (say) with respect to a product of σ-finite measures on $\mathbb{R}$ that satisfies

$$f(x \vee y) f(x \wedge y) \geq f(x) f(y) \text{ for all } x, y \in \mathbb{R}^p$$

with $x \wedge y = (\min(x_1, y_1), \dots, \min(x_p, y_p))^\top$, $x \vee y = (\max(x_1, y_1), \dots, \max(x_p, y_p))^\top$.
- conditionally increasing in sequence (CIS) if

$$\mathbb{P}(X_{i+1} > x_{i+1} | X_1 = x_1, \dots, X_i = x_i)$$

is non-decreasing in $(x_1, \dots, x_i)^\top$ for all $i \in \mathbb{N}_{p-1}$ and $x_p \in \mathbb{R}$.
- weakly conditionally increasing in sequence (WCIS) if

$$\mathrm{E}(h(X_{i+1}, \dots, X_p) | X_1 = x_1, \dots, X_{i-1} = x_{i-1}, X_i = x_i^*)$$

is non-decreasing in x_i^* for all $(x_1, \dots, x_{i-1})^\top$ and all non-decreasing (measurable) functions $h : \mathbb{R}^{p-i} \to \mathbb{R}$ for $i \in \mathbb{N}_{p-1}$.
- positively associated (PA) if

$$\mathrm{Cov}[f(X), g(X)] \geq 0$$

for all pairs (f, g) of componentwise non-decreasing (measurable) functions (or for all pairs (f, g) of componentwise non-increasing (measurable) functions) f and g for which $\mathbb{E}[f(X)]$, $\mathbb{E}[g(X)]$ and $\mathbb{E}[f(X)g(X)]$ are finite.
- setwise (or strongly) positive lower orthant dependent (SPLOD) if

$$\mathbb{P}(X_i \leq c_i \ \forall i \in \mathbb{N}_p) \geq \prod_{r=1}^{s} \mathbb{P}(X_j \leq c_j \ \forall j \in J_r) \tag{1}$$

for all $c_i \in \mathbb{R}$, $i \in \mathbb{N}_p$, and, for all choices of $s \in \{2, \dots, p\}$ pairwise disjoint non-empty sets $J_r \subset \mathbb{N}_p$ with $\sum_{r=1}^{s} J_r = \mathbb{N}_p$. For $J_r = \{r\}$ $\forall r$, (1) means PLOD.
- setwise (or strongly) positive upper orthant dependent (SPUOD) if

$$\mathbb{P}(X_i \geq c_i \ \forall i \in \mathbb{N}_p) \geq \prod_{r=1}^{s} \mathbb{P}(X_j \geq c_j \ \forall j \in J_r) \tag{2}$$

for all $c_i \in \mathbb{R}$, $i \in \mathbb{N}_p$, and, for all choices of $s \in \{2, \dots, p\}$ pairwise disjoint non-empty sets $J_r \subset \mathbb{N}_p$ with $\sum_{r=1}^{s} J_r = \mathbb{N}_p$. For $J_r = \{r\}$ $\forall r$, (2) means PUOD.
- one-two dependent (OTD) if the one-two inequality applies, that is, if

$$\mathbb{P}\Big(\bigcap_{i=1}^{p} \{c_i \leq X_i \leq d_i\}\Big) \leq \mathbb{P}\Big(\bigcap_{i=1}^{p} \{X_i \geq c_i\}\Big) \mathbb{P}\Big(\bigcap_{i=1}^{p} \{X_i \leq d_i\}\Big)$$

for all $c_i \leq d_i$, $i \in \mathbb{N}_p$.
- (strictly) positively correlated ((S)PC) if $\mathrm{Cov}(X_i, X_j) \geq (>)\ 0$ for all $i, j \in \mathbb{N}_p$ with $i \neq j$.

The following implications are well-known, that is,

(a) $\mathrm{MTP}_2 \Rightarrow \mathrm{CIS} \Rightarrow \mathrm{WCIS} \Rightarrow \mathrm{PA} \Rightarrow \mathrm{SPUOD}, \mathrm{SPLOD}$,
(b) $\mathrm{SPUOD} \Rightarrow \mathrm{PUOD} \Rightarrow \mathrm{PC}$,
(c) $\mathrm{SPLOD} \Rightarrow \mathrm{PLOD} \Rightarrow \mathrm{PC}$,

where the first implication in (a) requires positive densities throughout the domain of definition. Moreover, we have PA ⇒ OTD ⇒ PLOD, PUOD, see the recent paper [13].

In this note, we focus on centered multivariate normally distributed random variables $X = (X_1, \dots, X_p)^\top$ with some covariance matrix Σ (in symbols, $X \sim N_p(0, \Sigma)$). Thereby, Σ is said to be irreducible if there does not exist any partitioning (X_J, X_K) of X such that X_J, X_K are stochastically independent. In connection with MTP_2 it is always assumed that Σ is positive definite. The most celebrated results with respect to characterizations of MTP_2 of $|X|$ (and hence X^2) were obtained around 1980. Important references are [6,19–22], and, [31]. We briefly list the most important facts. Thereby, let $\mathcal{D}_p$ denote the set of all p-dimensional signature matrices, that is, diagonal matrices D with diagonal entries ± 1.

(a) $|X|$ is MTP_2 iff there exists a $D \in \mathcal{D}_p$ such that $D\Sigma^{-1}D$ is an M-matrix (that is, all off-diagonal elements of $D\Sigma^{-1}D$ are non-positive).
(b) $|X|$ is MTP_2 iff X is PPC, that is, iff all partial covariances $\sigma_{ij\cdot} = \mathrm{Cov}(X_i, X_j | X_k, k \in I_{ij})$ with $I_{ij} = \mathbb{N}_p \setminus \{i, j\}$, $i < j$, are non-negative.
(c) $|X|$ is MTP_2 iff X^2 is infinitely divisible, see [4].
(d) If $|X|$ is MTP_2, then $|X|$ is PA.
(e) X is MTP_2 iff Σ^{-1} is an M-matrix.
(f) Already 1971, Šidák (see pp. 171–172 in [35]) showed that $|X|$ is not always PUOD even if X is SPC, see also [36], Example 2.3.1, pp. 27–28. Consequently, SPC of X is not sufficient for PA of $|X|$. This example already contradicts the (false) conjecture formulated much later by Evans in 1991 (see [12]) that PA of X^2 is equivalent to PC of DX for some $D \in \mathcal{D}_p$.
(g) If X is MTP_2 and if Σ is irreducible, then X is SPC. Surprisingly, this important fact is rarely mentioned in connection with MTP_2 in the statistical literature. It is an immediate consequence of a well-known result on M-matrices, that is, an irreducible M-matrix is strictly inverse positive, see, e.g., [5], Theorem 2.7 on p. 141.
(h) The validity of the famous Gaussian correlation conjecture (proved by Thomas Royen, see [30]) implies that $|X|$ is always SPLOD. More striking, SPLOD of $|X|$ is equivalent to the validity of the Gaussian correlation conjecture, see, e.g., [34].
(i) In 1982, Pitt proved that X itself is PA iff X is PC, see [28]. Extensions of this result can be found in [15]. At the same time, based on *lengthy, straightforward calculations* (unpublished), E. Bølviken gave a general counterexample for $p = 3$ showing that PC of DX for some $D \in \mathcal{D}_p$ is necessary for SPUOD (and hence for PA) of $|X|$, see [6]. Some speculations in [6] suggest that Bølviken had some doubt that SPUOD of $|X|$ is possible without requiring that all partial covariances are non-negative.
(j) We were able to sharpen Bølviken's result and to extend it to all $p \geq 3$: If Σ is positive definite and irreducible, then SPUOD (and especially PA) of $|X|$ implies SPC of DX for some $D \in \mathcal{D}_p$. We will report on this elsewhere.
(k) It seems unknown yet whether PC of DX for some $D \in \mathcal{D}_p$ is necessary for PUOD of $|X|$.
(ℓ) Theorem 3.1 in [16] together with a weak convergence argument for degenerate cases yields the following. Let $Z_i \sim N(0, \sigma_i^2)$, $\sigma_i \geq 0$, $i \in \mathbb{N}_p$, be independently distributed, and, let $X = (X_1, \dots, X_p)^\top$ be independent of $Z = (Z_1, \dots, Z_p)^\top$. If $|X|$ is associated, then $|X + Z|$ is associated. We refer to this fact as Jogdeo's PA construction.
(m) For $p = 2$, $|X|$ is always PA, and, if Σ is positive definite, $|X|$ is MTP_2.

Finally, we note that absolute-valued multivariate normal distributions are often denoted as folded multivariate normal distributions as well as multivariate folded normal distributions, see, e.g., [18] and the references therein. Supposing that $X \sim N_p(0, \Sigma)$ with positive definite Σ, a continuous version of the probability density function of $|X| = (|X_1|, \dots, |X_p|)^\top$ on $[0, \infty)^p$ in terms of on $\Sigma^{-1} = (k_{ij})$ is given by

$$f(y_1, \dots, y_p) = (2\pi)^{-p/2} |\Sigma|^{-1/2} \sum_{\delta \in \Delta_p} \exp\left(-\frac{1}{2} \sum_{i=1}^{p} k_{ii} y_i^2 - \sum_{1 \leq i < j \leq p} k_{ij} \delta_i \delta_j y_i y_j \right)$$

with $\Delta_p = \{(\delta_1, \dots, \delta_p)^\top : \delta_i \in \{0, 1\}, i \in \mathbb{N}_p\}$.

3. The main results

Firstly, we are concerned with the study of non-degenerate absolute-valued, trivariate normal random vectors which are not MTP_2. According to Theorem 3.1 in [1] such random vectors are characterized by the positivity of the product of the three upper off-diagonal elements of the inverse covariance matrices of their underlying normal random vectors. Hence, it is easily verified that in order to treat properties of these absolute-valued random vectors like CIS, WCIS, PA etc., it suffices to restrict attention to covariance matrices as considered in the following theorem.

Theorem 1. *Let $X \sim N_3(0, \Sigma)$ with Σ positive definite such that*

$$\Sigma^{-1} = \begin{bmatrix} 1 & k_{12} & -k_{13} \\ k_{12} & 1 & -k_{23} \\ -k_{13} & -k_{23} & 1 \end{bmatrix}$$

and $0 < k_{12}, k_{13}, k_{23} < 1$. Let $\tau = k_{12}/(k_{13}k_{23})$. If

$$0 < \tau \leq 1/2, \tag{3}$$

then $|X|$ and X^2 are PA but not MTP_2.

Theorem 1 is a consequence of the following lemma.

Lemma 1. *Under the general assumptions of Theorem 1, $(|X_1|,|X_2|,|X_3|)^\top$ is CIS iff (3) applies.*

Remark 1. Under the general assumptions of Theorem 1 we have that Σ is SPC iff $0<\tau<1$, and, Σ is positive definite iff $(1/k_{13}^2-1)(1/k_{23}^2-1)>(1-\tau)^2$. It is an open question whether $|X|$ and X^2 are PA for some $\tau\in(1/2,1)$.

Remark 2. Theorem 1 can be reformulated in terms of the entries of the original covariance matrix Σ. To this end, let $X\sim N_3(0,\Sigma)$ with Σ positive definite such that

$$\Sigma=\begin{bmatrix}1 & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & 1 & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & 1\end{bmatrix}$$

with $0<\sigma_{12},\sigma_{13},\sigma_{23}<1$ and $\sigma_{12}<\sigma_{13}\sigma_{23}$. Then $(|X_1|,|X_2|,|X_3|)^\top$ is CIS iff

$$2(\sigma_{12}-\sigma_{13}\sigma_{23})(1-\sigma_{12}^2)+(\sigma_{13}-\sigma_{12}\sigma_{23})(\sigma_{23}-\sigma_{13}\sigma_{12})\ge 0. \tag{4}$$

It can be shown that for given values of $\sigma_{13},\sigma_{23}\in(0,1)$ there exist threshold values $a_0\in[0,\sigma_{13}\sigma_{23})$ and $a_1\in(a_0,\sigma_{13}\sigma_{23})$ such that Σ is positive definite for all $\sigma_{12}\in(a_0,\sigma_{13}\sigma_{23})$ and condition (4) holds for all $\sigma_{12}\in[a_1,\sigma_{13}\sigma_{23})$, whereas for all $\sigma_{12}\in(0,a_1)$ condition (4) is not fulfilled. For example, for $\sigma_{13}=\sigma_{23}=7/10$ we get $a_1=(\sqrt{42009}-53)/400=0.3799\ldots$. In the latter case, $|X|$ is CIS but not MTP_2 for $\sigma_{12}\in[a_0,0.49)$.

Remark 3. Cohen and Sackrowitz, see [7], showed that CIS $\Rightarrow$ WCIS $\Rightarrow$ PA. They also showed that very simple criteria in terms of conditional means suffice to characterize CIS or WCIS of $X\sim N_p(0,\Sigma)$. For $p=3$, it can be shown that CIS can be replaced by WCIS in Lemma 1 because the argumentation following (6) in Section 4 also applies for WCIS. For $p\ge 4$ we are not aware of any simple criteria characterizing CIS or WCIS of $|X|$.

Mainly based on Theorem 1 we have the following two general results for $p\ge 4$.

Theorem 2. *For any $p\ge 4$ there exists a positive definite SPC covariance matrix Σ such that $|X|$, assuming $X=(X_1,\ldots,X_p)^\top\sim N_p(0,\Sigma)$, is PA but not MTP_2.*

Proof. The assertion follows immediately by combining Theorem 1 and Jogdeo's PA construction. First, let $(X_1,X_2,X_3)^\top\sim N_3(0,\Sigma)$ with Σ positive definite and SPC such that $(|X_1|,|X_2|,|X_3|)^\top$ is PA but not MTP_2. Note that the p-variate vector $(|X_1|,|X_2|,|X_3|,\ldots,|X_3|)^\top$ is PA, too. For $p\ge 4$, let $(X_1,X_2,X_3)^\top$, $Z_j\sim N(0,1)$, $j\in\{4,\ldots,p\}$, be independently distributed and define $X_j=X_3+Z_j$ for $j\in\{4,\ldots,p\}$. Then Jogdeo's PA construction yields that $(|X_1|,\ldots,|X_p|)^\top$ is PA. But $(|X_1|,|X_2|,|X_3|)^\top$ is not MTP_2, hence $(|X_1|,\ldots,|X_p|)^\top$ is not MTP_2. Clearly, the covariance matrix of $(X_1,\ldots,X_p)^\top$ is SPC and positive definite. □

Theorem 3. *For any $p\ge 4$ there exists a positive definite SPC covariance matrix Σ such that $|X|$, assuming $X=(X_1,\ldots,X_p)^\top\sim N_p(0,\Sigma)$, is PA but no permutation of $(|X_1|,\ldots,|X_p|)^\top$ is CIS.*

Proof. Let $Y=(Y_1,Y_2,Y_3)^\top\sim N_3(0,\Gamma)$ with

$$\Gamma=\begin{bmatrix}1 & 0.4 & 0.7\\ 0.4 & 1 & 0.7\\ 0.7 & 0.7 & 1\end{bmatrix}.$$

Theorem 1 yields that $|Y|$ is PA but not MTP_2. Now let $p\ge 4$ and Y, $Z_j\sim N(0,1/10)$, $j\in\mathbb{N}_p$, be independently distributed and define $Y_j=Y_3$, $j\in\{4,\ldots,p\}$, and, $X_j=\sqrt{10/11}(Y_j+Z_j)$ for $j\in\mathbb{N}_p$. Then Jogdeo's PA construction yields that $(|X_1|,\ldots,|X_p|)^\top$ is PA. The covariance matrix $\Sigma=(\sigma_{ij})$ is a correlation matrix with entries $\sigma_{12}=4/11$, $\sigma_{ij}=7/11$ for $j\in\{3,\ldots,p\}$, $i\in\mathbb{N}_2$, and, $\sigma_{ij}=10/11$ for $j\in\{i+1,\ldots,p-1\}$, $i\in\{3,\ldots,p\}$. For example, for $p=4$ we obtain

$$\mathrm{Cov}(X)=\frac{1}{11}\begin{bmatrix}11 & 4 & 7 & 7\\ 4 & 11 & 7 & 7\\ 7 & 7 & 11 & 10\\ 7 & 7 & 10 & 11\end{bmatrix}.$$

Replacing $4/11, 7/11, 10/11$ by a,b,c in the correlation matrix Σ of $(X_1,\ldots,X_p)^\top$ and setting

$$d_p=|\Sigma|=(1-c)^{p-3}(1-a)((p-3)ac-(2p-4)b^2+a+(p-3)c+1)$$

and $B = (b_{ij}) = d_p \Sigma^{-1}/(1-c)^{p-4}$ we get

$$\begin{array}{rcll} b_{11} = b_{22} & = & (1-c)[1+(p-3)c-(p-2)b^2] & = (61p-111)/1331, \\ b_{12} = b_{21} & = & (1-c)[(p-2)b^2-(p-3)ac-a] & = (9p-22)/1331, \\ b_{ii} & = & -((p-4)c+1)a^2-2(p-3)b^2(1-a)+(p-4)c+1 & = (364p-987)/1331, \quad i \in \{3,\dots,p\}, \\ b_{ij} = b_{ji} & = & -(1-c)(b(1-a)) & = -49/1331, \quad i \in \mathbb{N}_2,\ j \in \{3,\dots,p\}, \\ b_{ij} = b_{ji} & = & -(1-a)(c(1+a)-2b^2) & = -364/1331, \quad i,j \in \{3,\dots,p\},\ i<j. \end{array}$$

Obviously, $b_{12} > 0$ for $p \geq 3$. Hence, Σ^{-1} is not an M-matrix and $(|X_1|,\dots,|X_p|)^\top$ is not MTP$_2$.

Now let $(|X_{i_1}|,\dots,|X_{i_p}|)^\top$ denote a permutation of $(|X_1|,\dots,|X_p|)^\top$. We consider two cases, that is, (I) $i_p \notin \mathbb{N}_2$, and, $i_p \in \mathbb{N}_2$. By symmetry considerations, it suffices to consider $(|X_1|,\dots,|X_p|)^\top$ in case (I) and $(|X_2|,\dots,|X_p|,|X_1|)^\top$ in case (II). If these two configurations are not CIS, then all permutations are not CIS. We show that in both cases the CIS condition (8) developed in Section 5 is violated. In case (I) we obtain

$$\operatorname{sgn}(b_{1p}b_{2p}(2b_{12}b_{pp} - b_{1p}b_{2p})) = \frac{6552}{1771561}p^2 - \frac{33782}{1771561}p + \frac{41027}{1771561} > 0,\ p \geq 4,$$

where $\operatorname{sgn}(\cdot)$ denotes the signum function. Hence, the CIS condition (8) developed in Section 5 is violated. Consequently, $(|X_1|,\dots,|X_p|)^\top$ is not CIS for $p \geq 4$. In case (II), let $\tilde{\Sigma}$ denote the covariance matrix of $(X_2,\dots,X_p,X_1)^\top$ and let $\tilde{B} = (\tilde{b}_{ij}) = d_p\tilde{\Sigma}^{-1}/(1-c)^{p-4}$. Note that $\tilde{B}$ can be obtained as a permutation of B. In this case we obtain

$$\operatorname{sgn}(\tilde{b}_{1p}\tilde{b}_{2p}(2\tilde{b}_{12}\tilde{b}_{pp} - \tilde{b}_{1p}\tilde{b}_{2p})) = \operatorname{sgn}(b_{21}b_{31}(2b_{23}b_{11} - b_{21}b_{31})) = \frac{5537}{1771561}p - \frac{9800}{1771561} > 0,\ p \geq 4.$$

Hence again, the CIS condition (8) developed in Section 5 is violated. Consequently, $(|X_2|,\dots,|X_p|,|X_1|)^\top$ is not CIS for $p \geq 4$. This completes the proof. $\square$

Remark 4. It remains an open problem whether CIS in Theorem 3 can be replaced by WCIS. Application of the Quick WCIS Check (see Lemma 4 in Section 5) yields that $|X| = (|X_1|,\dots,|X_4|)^\top$ in the proof of Theorem 3 passes this check, that is, (10) is satisfied in this case. We found no numerical evidence that $|X|$ is not WCIS. The general problem with WCIS is that it is typically as difficult to prove as PA in terms of the corresponding definitions. We also tried some other starting PA covariance matrices for $p = 3$ but found no example where the construction in the proof of Theorem 3 leads to 4-dimensional $|X|$ such that the Quick WCIS Check rejects WCIS for all permutations.

Remark 5. The construction in the proof of Theorem 3 does not work for $p = 3$. Starting with a two-dimensional $Y \sim N_2(0,\Gamma)$ with Γ positive definite and SPC yields that $|Y|$ is MTP$_2$. Assume that Y and $Z_j \sim N(0,\sigma_j^2)$, $j \in \mathbb{N}_3$, are independently distributed with $\sigma_3^2 > 0$ and define $Y_3 = Y_2$ and $X_j = Y_j + Z_j$ for $j \in \mathbb{N}_3$. Then the inverse of the covariance matrix of $X = (X_1,X_2,X_3)^\top$ is an M-matrix, that is, $|X|$ is MTP$_2$ and hence all permutations of $|X|$ are CIS.

4. Proof of Lemma 1

We found that the notational effort simplifies by considering

$$(U_1,U_2,Z)^\top = (-k_{13}X_1, -k_{23}X_2, X_3)^\top.$$

Thereby, CIS of $(|X_1|,|X_2|,|X_3|)^\top$ is equivalent to CIS of $(|U_1|,|U_2|,|Z|)^\top$. Noting that the inverse covariance matrix $(\tilde{k}_{ij})$ (say) of $(U_1,U_2,Z)^\top$ has entries $\tilde{k}_{11} = 1/k_{13}^2$, $\tilde{k}_{22} = 1/k_{23}^2$, $\tilde{k}_{13} = \tilde{k}_{23} = \tilde{k}_{33} = 1$ and $\tilde{k}_{12} = \tau = k_{12}/(k_{13}k_{23})$, elementary calculus yields that a (the only) continuous version of the conditional density of $|Z|$ given $|U_1| = u_1 \geq 0, |U_2| = u_2 \geq 0$ is given by

$$f(z|u_1,u_2;\tau) = (2\pi)^{-1/2}\exp(-(u_1^2+u_2^2+z^2)/2)\ \frac{\cosh((u_1+u_2)z)\exp(-\tau u_1u_2) + \cosh((u_1-u_2)z)\exp(\tau u_1u_2)}{\cosh((1-\tau)u_1u_2)}$$

for $z \geq 0$. The corresponding conditional cumulative distribution function $F(z|u_1,u_2;\tau)$ (say) can be expressed as a mixture of two folded normal distributions, that is,

$$F(z|u_1,u_2;\tau) = \kappa\mathbb{P}(|Y-(u_1+u_2)| \leq z) + (1-\kappa)\mathbb{P}(|Y-(u_1-u_2)| \leq z)$$

with $Y \sim \mathrm{N}(0,1)$ and

$$\kappa \equiv \kappa(u_1u_2;\tau) = \frac{\exp(2(1-\tau)u_1u_2)}{(1+\exp(2(1-\tau)u_1u_2))}.$$

For $\tau \in (0,1)$ we observe that $\kappa(t;\tau)$ is increasing in $t \geq 0$ with values in $[1/2,1)$. Setting $\overline{F}(z|u_1,u_2;\tau) = 1 - F(z|u_1,u_2;\tau)$ we get

$$\overline{F}(z|u_1,u_2;\tau) = \kappa\mathbb{P}(|Y-(u_1+u_2)| > z) + (1-\kappa)\mathbb{P}(|Y-(u_1-u_2)| > z).$$

Noting that $(|U_1|,|U_2|)^\top$ is MTP$_2$ and hence CIS, $(|U_1|,|U_2|,|Z|)^\top$ is CIS iff $\overline{F}(z|u_1,u_2;\tau)$ is non-decreasing in u_1 and u_2. In view of $\overline{F}(z|u_1,u_2;\tau) = \overline{F}(z|u_2,u_1;\tau)$, it is no loss of generality to assume $u_1 \geq u_2$. Altogether, it suffices to show that

$$\overline{F}(z|v_1,v_2;\tau) \geq \overline{F}(z|u_1,u_2;\tau) \text{ for all } v_i > u_i > 0,\ i \in \mathbb{N}_2, v_1 > v_2, u_1 > u_2. \tag{5}$$

Once (5) is proved, continuity and symmetry arguments yield

$$\overline{F}(z|v_1,v_2;\tau) \geq \overline{F}(z|u_1,u_2;\tau) \text{ for all } v_i \geq u_i \geq 0,\ i \in \mathbb{N}_2.$$

In order to show that (3) is necessary for $(|U_1|,|U_2|,|Z|)^{\mathsf{T}}$ to be CIS, we take a first look at the likelihood ratio function defined by

$$LR(z) \equiv LR(z|u_1,u_2,v_1,v_2;\tau) = \frac{f(z|v_1,v_2;\tau)}{f(z|u_1,u_2;\tau)}$$

for $0 \leq u_1 \leq v_1, 0 \leq u_2 \leq v_2,\ z \geq 0$. A necessary condition for $(|U_1|,|U_2|,|Z|)^{\mathsf{T}}$ to be CIS is that

$$LR(0|u_1,0,u_1,v_2;\tau) \leq 1 \text{ for all } u_1,v_2 > 0. \tag{6}$$

Noting that

$$LR(0|u_1,0,u_1,v_2;\tau) = \exp(-v_2^2/2)\frac{\cosh(\tau u_1 v_2)}{\cosh((1-\tau)u_1 v_2)},$$

a little analysis yields that $\lim_{u_1\to\infty} LR(0|u_1,0,u_1,v_2;\tau) = \infty$ for $v_2 > 0$ and $\tau \in (1/2,\infty)$. Hence, $(|U_1|,|U_2|,|Z|)^{\mathsf{T}}$ can only be CIS if $0 < \tau \leq 1/2$, that is, if (3) is satisfied.

We now show that $0 < \tau \leq 1/2$ implies CIS of $(|U_1|,|U_2|,|Z|)^{\mathsf{T}}$. We consider two cases, that is, (I) $v_1 - v_2 \geq u_1 - u_2 > 0$, and, (II) $0 < v_1 - v_2 < u_1 - u_2$.

We prove case (I) via suitable inequalities for $\overline{F}$. In case (II), we apply a rule of signs of Laguerre for the likelihood ratio function LR.

Case (I). Assuming $v_1 - v_2 \geq u_1 - u_2 > 0,\ v_i > u_i > 0,\ i \in \mathbb{N}_2$, we have to show that

$$\overline{F}(z|v_1,v_2;\tau) \geq \overline{F}(z|u_1,u_2;\tau).$$

Set $s = u_1u_2$ and $t = v_1v_2$, which yields $s < t$. Consider all pairs $y_1, y_2 > 0$ with $y_1y_2 = t$ and $y_1 > u_1,\ y_2 > u_2$, that is, all pairs $y_1, y_2 > 0$ with $y_1 \in (u_1, u_1t/s) = (u_1, t/u_2),\ y_2 = t/y_1$. In this setting we get

$$\overline{F}(z|y_1,y_2;\tau) = \kappa(t;\tau)\mathbb{P}(|Y-(y_1+y_2)| > z) + (1-\kappa(t;\tau))\mathbb{P}(|Y-(y_1-y_2)| > z).$$

Note that κ is fixed for different values of y_1, y_2 with $y_1y_2 = t$. Moreover $y_1 \pm y_2 = y_1 \pm t/y_1$ is non-decreasing in $y_1 \geq \sqrt{t}$. This immediately yields that $\overline{F}(z|y_1,t/y_1;\tau)$ is non-decreasing in $y_1 \geq \sqrt{t}$. Moreover, there exists a $w_1 \in (\sqrt{t}, v_1]$ such that $w_1 - w_2 = u_1 - u_2$ with $w_2 = t/w_1 \geq v_2$. It follows that

$$\overline{F}(z|v_1,v_2;\tau) \geq \overline{F}(z|w_1,w_2;\tau).$$

Noting that $\kappa(w_1w_2;\tau) > \kappa(u_1u_2;\tau)$, $\mathbb{P}(|Y-(w_1+w_2)| > z) > \mathbb{P}(|Y-(u_1+u_2)| > z)$, and, $\mathbb{P}(|Y-(w_1-w_2)| > z) = \mathbb{P}(|Y-(u_1-u_2)| > z)$, we immediately get

$$\overline{F}(z|w_1,w_2;\tau) \geq \overline{F}(z|u_1,u_2;\tau).$$

This completes the proof of case (I).

Case (II). Assuming $0 < v_1 - v_2 < u_1 - u_2,\ v_i > u_i > 0,\ i \in \mathbb{N}_2$, we prove that $LR(z|u_1,u_2,v_1,v_2;\tau) - 1$ has exactly one sign change on $(0,\infty)$. This property together with $LR(0|u_1,u_2,v_1,v_2;\tau) < 1$ and $\lim_{z\to\infty} LR(z) = \infty$ (which can easily be verified) then implies (5).

Setting

$$\begin{aligned}
D &= \exp(-((v_1^2-u_1^2)+(v_2^2-u_2^2))/2),\ B = \frac{\cosh((1-\tau)v_1v_2)}{\cosh((1-\tau)u_1u_2)},\\
C_1(z) &= \cosh((v_1+v_2)z)\exp(-\tau v_1v_2) + \cosh((v_1-v_2)z)\exp(\tau v_1v_2),\\
C_2(z) &= \cosh((u_1+u_2)z)\exp(-\tau u_1u_2) + \cosh((u_1-u_2)z)\exp(\tau u_1u_2),
\end{aligned}$$

we get

$$LR(z) = \frac{DC_1(z)}{BC_2(z)}$$

with $D < 1 < B$. By noting that $\cosh(ax)/\cosh(bx)$ is strictly increasing in $x > 0$ for $a > b > 0$, we obtain

$$\begin{aligned}
LR(0|u_1,u_2,v_1,v_2;\tau) &= \frac{D\cosh(\tau v_1v_2)}{B\cosh(\tau u_1u_2)} = D\frac{\cosh(\tau v_1v_2)/\cosh(\tau u_1u_2)}{\cosh((1-\tau)v_1v_2)/\cosh((1-\tau)u_1u_2)}\\
&< 1 \quad \text{for all } \tau \in (0,1/2].
\end{aligned}$$

Setting $G(z) = DC_1(z) - BC_2(z)$, we observe that $LR(z) = (<,>)\ 1$ iff $G(z) = (<,>)\ 0$. Hence, it suffices to study the sign changes of G. With

$$\begin{array}{llll}
\alpha_1 = v_1+v_2, & \alpha_2 = u_1+u_2, & \alpha_3 = u_1-u_2, & \alpha_4 = v_1-v_2,\\
A_1 = D\exp(-\tau v_1v_2), & A_2 = -B\exp(-\tau u_1u_2), & A_3 = -B\exp(\tau u_1u_2), & A_4 = D\exp(\tau v_1v_2),
\end{array}$$

$G(z)$ can be expressed as

$$G(z) = \sum_{i=1}^{4} A_i \cosh(\alpha_i z).$$

We now apply the following rule of signs, see Théorème I in [24], p. 125 and Problem 85 in [29], Part 5, Chapter 1, §6, p. 49 and p. 226.

Lemma 2 (*A Rule of Signs of Laguerre (1883)*). *Suppose that $F(x) = \sum_{i=0}^{\infty} a_i x^i \in (0,\infty)$ for $x \in (0,\rho)$ for some $0 < \rho \le \infty$ with $a_i \ge 0$ for all $i \in \mathbb{N} \cup \{0\}$ and $a_i > 0$ for infinitely many $i \in \mathbb{N} \cup \{0\}$. Let $1 \ge \beta_1 > \cdots > \beta_n > 0$, $A_i \in \mathbb{R} \setminus \{0\}$, $i \in \mathbb{N}_n$, $S_m = \sum_{i=1}^{m} A_i$, $i \in \mathbb{N}_m$. Then the number of roots in $(0,\rho)$ of the equation $\sum_{i=1}^{n} A_i F(\beta_i x) = 0$ is bounded by the number of sign changes of S_m, $m \in \mathbb{N}_n$.*

Setting $F(z) = \cosh(\alpha_1 z)$, $\beta_i = \alpha_i/\alpha_1$, $i \in \mathbb{N}_4$, we get

$$G(z) = \sum_{i=1}^{4} A_i \cosh(\alpha_i z) = \sum_{i=1}^{4} A_i F(\beta_i z)$$

with $1 = \beta_1 > \beta_2 > \beta_3 > \beta_4 > 0$, and, $S_1 = A_1 > 0$. In view of $D < 1 < B$ it is easy to check that $S_2 = A_1 + A_2 < 0$, which then yields $S_3 = A_1 + A_2 + A_3 < 0$. Moreover, $S_4 = A_1 + A_2 + A_3 + A_4 < 0$ is equivalent to $LR(0) < 1$, which is true. Consequently, the pattern of signs of S_m, $m \in \mathbb{N}_4$, is $+---$, that is, we have exactly one sign change. This completes the proof of case (II).

Hence, (3) implies CIS of $(|U_1|, |U_2|, |Z|)^\top$. Altogether, this completes the proof of Lemma 1. □

Remark 6. Although there exist hundreds of papers on rule of signs, we found no further references except [29] where Laguerre's rule of signs (Lemma 2) appears. In Laguerre's original work [24] there is no restriction on β_1, and it seems implicitly assumed that $\rho = \infty$ and the power series defining $F(x)$ is non-terminating. Thanks are due to Stewart A. Levin for an English version of Laguerre's original work [24] including footnotes with minor corrections, see [26].

5. Quick checks for the possibility of CIS and WCIS

For any positive definite matrix $B = (b_{ij})$ we define $\mathrm{Cor}(B) = \left(b_{ij}/(b_{ii}b_{jj})^{1/2}\right)$. Let $X \sim N_p(0,\Sigma)$ and let Σ be positive definite. Suppose we want to check whether there is a chance that $(|X_1|, \ldots, |X_p|)^\top$ is CIS. Set $A = (a_{ij}) = \mathrm{Cor}(\Sigma^{-1})$ and consider

$$LR(z) \equiv LR(z|u_1, \ldots, u_{p-1}, v_1, \ldots, v_{p-1}) = \frac{f(z|v_1, \ldots, v_{p-1})}{f(z|u_1, \ldots, u_{p-1})}$$

for $0 \le u_i \le v_i$, $i \in \mathbb{N}_{p-1}$ and $z \ge 0$. Thereby, $f(z|\cdot)$ denotes the continuous version of the conditional density.

Remark 7. In order to compute the continuous version of the conditional density $f(x_p|x_1, \ldots, x_{p-1})$, it is worth noting that $D = (d_{ij}) = (\Sigma_{\{1,\ldots,p-1\}})^{-1}$ has a very simple form in terms of $C = (c_{ij}) = \Sigma^{-1}$, see Lemma 2.2 in [6]. Assuming $\Sigma^{-1} = \mathrm{Cor}(\Sigma^{-1})$, we obtain

$$d_{ii} = 1 - c_{ip}^2,\ i \in \mathbb{N}_{p-1},\ d_{ij} = d_{ji} = c_{ij} - c_{ip}c_{jp},\ i,j \in \mathbb{N}_{p-1},\ i < j.$$

Results of this type can also be found in [17]. Moreover, Jacobi's complementary minor formula yields $|\Sigma^{-1}| = |(\Sigma_{\{1,\ldots,p-1\}})^{-1}|$.

If we can show that

$$LR(0|u_1, \ldots, u_{p-1}, v_1, \ldots, v_{p-1}) > 1$$

for some choice of $u_1, \ldots, u_{p-1}, v_1, \ldots, v_{p-1}$, then $(|X_1|, \ldots, |X_p|)^\top$ is not CIS. Let $i, j \in \mathbb{N}_{p-1}$, $i \ne j$, and let $u_i = v_i > 0$, $u_j = 0$ and $v_j > 0$, $u_r = v_r = 0$ for all $r \in \mathbb{N}_{p-1} \setminus \{i,j\}$. In this setting we get (see Remark 7)

$$LR \equiv LR(0|u_1, \ldots, u_{p-1}, v_1, \ldots, v_{p-1}) = \exp(-a_{jp}^2 v_j^2/2)\frac{\cosh(a_{ij}u_i v_j)}{\cosh((a_{ij} - a_{ip}a_{jp})u_i v_j)}.$$

For $u_i \to \infty$, LR tends to infinity if $|a_{ij}| > |a_{ij} - a_{ip}a_{jp}|$. Hence, noting that $|x| \le |x-y|$ iff $\mathrm{sgn}(y(x-y/2)) \le 0$ for $x, y \in \mathbb{R}$, the condition

$$\max_{1 \le i<j \le p-1} \mathrm{sgn}(a_{ip}a_{jp}(2a_{ij} - a_{ip}a_{jp})) \le 0 \tag{7}$$

is necessary for CIS of $(|X_1|, \ldots, |X_p|)^\top$. Setting $B = (b_{ij}) = \Sigma^{-1}$, (7) is equivalent to

$$\max_{1 \le i<j \le p-1} \mathrm{sgn}(b_{ip}b_{jp}(2b_{ij}b_{pp} - b_{ip}b_{jp})) \le 0. \tag{8}$$

In other words, if

$$\mathrm{sgn}(b_{ip}b_{jp}(2b_{ij}b_{pp} - b_{ip}b_{jp})) > 0 \text{ for some } i < j,\ i,j \in \mathbb{N}_{p-1},$$

then

$$\mathbb{P}(|X_p| > x_p | |X_1| = x_1, \dots, |X_{p-1}| = x_{p-1})$$

is not non-decreasing in $(x_1, \dots, x_{p-1})^\top$ for some $x_p \geq 0$.

Noting that CIS of $|X| = (|X_1|, \dots, |X_p|)^\top$ implies CIS of $(|X_1|, \dots, |X_{r-1}|, |X_k|)^\top$ for all $k \in \{r, \dots, p\}$ and all $r \in \{3, \dots, p\}$ (see Remark 2.8 in [7]), the following lemma is obvious.

Lemma 3 (*Quick CIS Check*). *Let $p \geq 3$ and $X \sim N_p(0, \Sigma)$ with Σ positive definite. Setting $B^{(rk)} = (b_{ij}^{(rk)}) = (\Sigma_{\{1,\dots,r-1,k\}})^{-1}$ for $k \in \{r, \dots, p\}$ and $r \in \{3, \dots, p\}$, the condition*

$$\max_{3 \leq r \leq p} \max_{r \leq k \leq p} \max_{1 \leq i < j \leq r-1} \operatorname{sgn}\left(b_{ik}^{(rk)} b_{jk}^{(rk)} (2 b_{ij}^{(rk)} b_{kk}^{(rk)} - b_{ik}^{(rk)} b_{jk}^{(rk)})\right) \leq 0 \tag{9}$$

is necessary for CIS of $|X| = (|X_1|, \dots, |X_p|)^\top$.

Similarly, we get the following necessary condition for WCIS.

Lemma 4 (*Quick WCIS Check*). *Under the assumptions of Lemma 3 the condition*

$$\max_{3 \leq r \leq p} \max_{r \leq k \leq p} \max_{1 \leq i < r-1} \operatorname{sgn}\left(b_{ik}^{(rk)} b_{r-1,k}^{(rk)} (2 b_{i,r-1}^{(rk)} b_{kk}^{(rk)} - b_{ik}^{(rk)} b_{r-1,k}^{(rk)})\right) \leq 0 \tag{10}$$

is necessary for WCIS of $|X| = (|X_1|, \dots, |X_p|)^\top$.

Remark 8. It turns out that neither the CIS condition (9) is sufficient for CIS of $|X|$ nor the WCIS condition (10) is sufficient for WCIS of $|X|$, see Example 1 below. One may generate further necessary conditions for CIS/WCIS by studying for example the limit of $LR(0) \equiv LR(0|u_1, \dots, u_{p-1}, v_1, \dots, v_{p-1})$ for $u_i, v_i \in \{0, w\}$, $u_i \leq v_i$ for $w \to \infty$ (for WCIS we need $u_{p-1} < v_{p-1}$ and $u_i = v_i$ for $i \in \mathbb{N}_{p-2}$). Instead of choosing $u_i, v_i \in \{0, w\}$ for all $i \in \mathbb{N}_{p-1}$, one may set for example $u_j = 0, v_j = v > 0$ for some j. If LR tends to infinity for some appropriate configuration, then $|X|$ is not CIS/WCIS. However, it seems difficult to develop manageable formulas for such limits.

Remark 9. For $p = 3$ we have that CIS (WCIS) of $|X|$ implies CIS (WCIS) of X if Σ is positive definite and SPC. For $p \geq 4$ we were able to show that already WCIS of $|X|$ implies CIS of X if Σ is positive definite and SPC. We will report on this elsewhere.

We conclude this section with three examples.

Example 1. Let $X \sim \mathrm{N}_4(0, \Sigma)$ with

$$\Sigma = \begin{bmatrix} 1 & 0.7 & 0.4 & 0.1 \\ 0.7 & 1 & 0.7 & 0.2 \\ 0.4 & 0.7 & 1 & 0.3 \\ 0.1 & 0.2 & 0.3 & 1 \end{bmatrix}.$$

First we note that for all $(|X_i|, |X_j|, |X_k|)^\top$ with $i, j, k \in \mathbb{N}_4$, $i < j < k$, there exists a permutation which is CIS. Hence, these four three-dimensional subvectors are PA according to Theorem 1. Moreover, (8) is satisfied for $p = 4$ but (10) fails. Hence, $(|X_1|, \dots, |X_4|)^\top$ is neither WCIS nor CIS. Moreover, (9) and (10) are satisfied for exactly two permutations $(|X_i|, |X_j|, |X_r|, |X_s|)^\top$ of $(|X_1|, \dots, |X_4|)^\top$, that is, for $(i, j, r, s) \in \{(1, 4, 3, 2), (4, 1, 3, 2)\}$. In both cases we found that, e.g., $LR(0|5, 5, 0, 5, 5, 1) > 1$ and even $\lim_{w \to \infty} LR(0|w, w, 0, w, w, 1) = \infty$, see Remark 8. Hence, the remaining two permutations in question are also neither WCIS nor CIS.

This example illustrates that neither the CIS condition (9) is sufficient for CIS of $|X|$ nor the WCIS condition (10) is sufficient for WCIS of $|X|$. It also illustrates that limiting considerations as outlined in Remark 8 may lead to further necessary conditions for CIS/WCIS.

Clearly, it would be nice to have a general example for $p \geq 4$ with positive definite SPC covariance matrix Σ where some permutation of $|X| = (|X_1|, \dots, |X_p|)^\top$ is CIS (or WCIS) but not MTP_2. We leave this issue for future investigation. We give two examples for general $p \geq 4$ with positive definite SPC covariance matrices Σ where it seems at least possible that $|X|$ is CIS (or WCIS) but not MTP_2.

Example 2. Let $X \sim \mathrm{N}_p(0, \Sigma)$, $p \geq 4$, $\Sigma = (\sigma_{ij})$ with $\sigma_{ii} = 1$, $i \in \mathbb{N}_p$, $\sigma_{12} = \sigma_{21} = \kappa$, $\sigma_{ij} = \rho$ for all $i \neq j$ with $\{i, j\} \neq \{1, 2\}$, and, $\rho, \kappa \in (0, 1)$ such that Σ is positive definite. Then Σ^{-1} is an M-matrix iff

$$\kappa \geq \frac{(p-2)\rho^2}{(p-3)\rho + 1} = \kappa^*(p, \rho) \equiv \kappa^* \text{ (say)}.$$

We note that $|\Sigma| > 0$ iff $1 > \kappa > \max(0, 2\kappa^*(p, \rho) - 1)$. The lower M-matrix bound κ^* for κ is less than ρ but tends to ρ for $p \to \infty$. This may illustrate the rigid regime of the MTP_2 concept and it seems counter-intuitive that $|X|$ shall be not PA for all admissible values of $\kappa < \kappa^*$.

Set $A = (a_{ij}) = \Sigma^{-1}$. Note that $a_{12} > 0$ for $\kappa < \kappa^*$. The question is which values of $\kappa < \kappa^*$ yield CIS or WCIS of $(|X_1|, \ldots, |X_p|)^\top$. A look at $p = 4$ with $\rho = 7/10$ yields $\kappa^* = 49/85 = 0.5764\ldots$. Remember that $(|X_1|, |X_2|, |X_3|)^\top$ is CIS for $\kappa \geq 0.3799\ldots$.

Setting $d_p = |\Sigma|$ and $(b_{ij}) = d_p\Sigma^{-1}$ we get

$$
\begin{aligned}
d_p &= (1-\rho)^{p-3}(1-\kappa)[(p-3)\rho(1+\kappa) - 2(p-2)\rho^2 + \kappa + 1],\\
b_{11} = b_{22} &= (1-\rho)^{p-2}(1+(p-2)\rho),\ b_{12} = (1-\rho)^{p-3}[(p-2)\rho^2 - (p-3)\kappa\rho - \kappa],\\
b_{1i} = b_{2i} &= -(1-\rho)^{p-3}(1-\kappa)\rho,\ b_{ii} = d_{p-1},\ i \in \{3,\ldots,p\},\\
b_{ij} &= -(1-\rho)^{p-4}(1-\kappa)\rho(1+\kappa-2\rho),\ i,j \in \{3,\ldots,p\},\ i<j,\ a_{ij} = b_{ij}/d_p,\ i,j \in \mathbb{N}_p.
\end{aligned}
$$

Herewith, we have all ingredients for checking the basic CIS condition (8). Numerical investigations suggest that the most critical case appears in the CIS condition (8) for $(i,j) = (1,2)$. Noting that $a_{1p}, a_{2p} < 0$, it suffices to consider the inequality $2a_{12}a_{pp} - a_{1p}a_{2p} \leq 0$. This gives a lower CIS bound $\underline{\kappa}$ (say) for κ, which unfortunately tends to ρ. For $p = 4$ with $\rho = 7/10$ we obtain the lower CIS bound $\underline{\kappa} = 2039/6800 + 3\sqrt{299049}/6800 = 0.5411\ldots$.

For $p \geq 4$ and $\rho = 7/10$, it seems that the Quick WCIS Check (10) does not reject WCIS of $(|X_1|, \ldots, |X_p|)^\top$ for $\kappa \geq 4/10$ while WCIS is rejected for all $\kappa < 4/10$. Noting that $a_{ij} > 0$ iff $\kappa < 4/10$ for $i,j \in \{3,\ldots,p\}$, $i < j$, a little analysis yields that, e.g., $\mathrm{sgn}(a_{1p}a_{p-1,p}(2a_{1,p-1}a_{p,p} - a_{1p}a_{p-1,p})) > 0$ iff $\kappa < 4/10$. Therefore, the WCIS condition (10) is violated for $\kappa < 4/10$ (choose $(r,k,i) = (p,p,1)$ in (10)).

Example 3. Let $X \sim \mathrm{N}_p(0,\Sigma)$, $p \geq 4$, $\Sigma = (\sigma_{ij})$ with $\sigma_{ii} = 1$, $i \in \mathbb{N}_p$, $\sigma_{ip} = \sigma_{pi} = \kappa$, $i \in \mathbb{N}_{p-1}$, and $\sigma_{ij} = \rho$ for all $i \neq j$ otherwise, and, $\rho, \kappa \in (0,1)$ such that Σ is positive definite. Note that $|\Sigma| > 0$ iff $1 > \rho > \max(0, \kappa^2 - (1-\kappa^2)/(p-2)) = \rho^{**}(say)$. Obviously, ρ^{**} tends to κ^2 for $p \to \infty$. Moreover, Σ^{-1} is an M-matrix iff

$$\rho \geq \kappa^2 = \rho^* \text{ (say)}.$$

Hence, for large p there is not much room for CIS/WCIS beyond MTP_2.

Setting $A = (a_{ij}) = \Sigma^{-1}$, we get that and $a_{ij} > 0$ for $\rho < \rho^*$, $i,j \in \mathbb{N}_{p-1}$, $i \neq j$. The question is whether there exist values of $\rho \in (\rho^{**}, \rho^*)$ yielding CIS or WCIS of $(|X_1|, \ldots, |X_p|)^\top$. A look at $p = 4$ with $\kappa = 7/10$ yields $\rho^* = 49/100$. Remember that $(|X_1|, |X_2|, |X_3|)^\top$ is CIS/WCIS for $\rho \geq 0.3799\ldots$.

Setting $d_p = |\Sigma|$ and $(b_{ij}) = d_p\Sigma^{-1}$ we get

$$
\begin{aligned}
d_p &= (1-\rho)^{p-2}\left(1+(p-2)\rho-(p-1)\kappa^2\right),\ b_{ii} = d_{p-1},\ i \in \mathbb{N}_{p-1},\\
b_{pp} &= (1-\rho)^{p-2}(1+(p-2)\rho),\ b_{ip} = b_{pi} = -\kappa(1-\rho)^{p-2},\ i \in \mathbb{N}_{p-1},\\
b_{ij} = b_{ji} &= (\kappa^2-\rho)(1-\rho)^{p-3},\ i,j \in \mathbb{N}_{p-1},\ i<j,\ a_{ij} = b_{ij}/d_p,\ i,j \in \mathbb{N}_p.
\end{aligned}
$$

Numerical investigations suggest that the CIS condition (8) as well as the WCIS condition (10) are fulfilled iff

$$\mathrm{sgn}(a_{1p}a_{p-1,p}(2a_{1,p-1}a_{p,p} - a_{1p}a_{p-1,p})) \leq 0.$$

This gives the lower CIS/WCIS bound $\underline{\rho}$ (say) for ρ given by

$$\underline{\rho} = \frac{1}{4(p-2)}\left(2\kappa^2 p - 3\kappa^2 - 2 + \sqrt{4\kappa^4p^2 - 12\kappa^4 p + 9\kappa^4 - 4\kappa^2 + 4}\right),$$

which unfortunately tends to $\rho^* = \kappa^2$ for $p \to \infty$. For $p = 4$ with $\kappa = 7/10$ we obtain the lower CIS/WCIS bound $\underline{\rho} = 0.4107\ldots < \rho^* = 49/100$.

6. Concluding remarks

Theorem 1 together with Theorem 2 show that there is at least some free space beyond the celebrated but tiny MTP_2 world for $|X|$ to be PA (and hence SPUOD, PUOD, OTD) for $p \geq 3$. More formally, setting $\mathcal{S}_p = \{\Sigma : \exists X \sim N_p(0,\Sigma) \text{ and } |X| \text{ is PA}\}$ and $\mathcal{M}_p = \{\Sigma : \exists X \sim N_p(0,\Sigma) \text{ and } |X| \text{ is } \mathrm{MTP}_2\}$, we proved that $\mathcal{M}_p$ is a strict subset of $\mathcal{S}_p$ and presented explicit positive definite covariance matrices $\Sigma \in \mathcal{S}_p \setminus \mathcal{M}_p$. However, at present there seems to be no evidence for any serious conjecture concerning a simple description of the complete class of covariance matrices $\mathcal{S}_p$ yielding PA of squared (and absolute-valued) Gaussians. The same issue appears for, e.g., CIS, WCIS, SPUOD, PUOD, OTD and some other concepts not discussed here (e.g., weak PA, weak OTD). Hence, there are various Gaussian correlation puzzles left for future research.

Following the advice of the referee we briefly discuss some implications and applications with respect to $\mathcal{S}_p$. We first show that our main results can be applied to obtain PA beyond MTP_2 for some closely related multivariate distributions. We start with a look at specific multivariate chi-square distributions. If $(X_{1i}, \ldots, X_{pi})^\top \sim N_p(0,\Sigma)$, $i \in \mathbb{N}_n$, are independently distributed, then the distribution of $V = (V_1, \ldots, V_p)^\top = (\sum_{i=1}^n X_{1i}^2, \ldots, \sum_{i=1}^n X_{pi}^2)^\top$ is said to be a multivariate chi-square (or Wishart chi-square) distribution. The distribution of V corresponds to the distribution of the diagonal elements of a Wishart distributed random $p \times p$ matrix $Q \sim W_p(n,\Sigma)$ (say). Up to now it was only known that V is PA if $\Sigma \in \mathcal{M}_p$. However, elementary rules for PA variables immediately yield that V is PA whenever $\Sigma \in \mathcal{S}_p$. Next we consider a specific multivariate t-distribution. Let $X \sim N_p(0,\Sigma)$ and let $S = (\chi^2/\nu)^{1/2}$, where χ^2 is chi-square distributed with ν degrees of freedom. Suppose that X and S are independently distributed and set $T = (X_1/S, \ldots, X_p/S)^\top$. A characterization of PA of T itself is a difficult issue. For example, if $\Sigma = I_p$, then T is not PA and even more striking, T is not positive quadrant dependent for $p = 2$, see [33]. Hence, T is neither PLOD nor PUOD. However, again elementary rules for PA variables yield that $|T|$ is PA whenever $\Sigma \in \mathcal{S}_p$. We note that K. Jogdeo proved PA of $|T|$ for a subclass of $\mathcal{S}_p$, see Section 5 in [16].

The validity of the Gaussian correlation conjecture implies that $|X|$ and $|T|$ are always SPLOD and hence always PLOD. In applications, the product-type bounds in the PLOD inequalities for $|X|$ and $|T|$ yield conservative multiple test procedures and conservative simultaneous confidence bounds. If $\Sigma \in \mathcal{S}_p$, we get in addition that $|X|$, V and $|T|$ are SPUOD and hence PUOD. We are not aware of any applications of the PUOD inequalities with respect to $|T|$. However, PUOD of V yields conservative $(1-\alpha)$ upper confidence bounds for all variances σ_{ii} (say) via $\mathbb{P}(V_i/\sigma_{ii} \geq c_i\ \forall i) \geq \prod_{i=1}^p \mathbb{P}(V_i/\sigma_{ii} \geq c_i) = 1-\alpha$. We note that C.G. Khatri (see [23]) derived various PUOD inequalities and related confidence bounds by assuming that the covariance matrix $\Sigma = (\sigma_{ij})$ has *structure l*, that is, $\sigma_{ii} > 0$ for all $i \in \mathbb{N}_p$ and $\sigma_{ij} = \alpha_i\alpha_j(\sigma_{ii}\sigma_{ii})^{1/2}$ for $i,j \in \mathbb{N}_p$, $i \neq j$, with $|\alpha_i| \leq 1$ for $i \in \mathbb{N}_p$. All results in Khatri's paper [23] based on the *structure l* assumption remain valid for $\Sigma \in \mathcal{S}_p$. Note that if Σ is positive definite and has *structure l*, then $D\Sigma^{-1}D$ is an M-matrix for $D \in \mathcal{D}_p$ with diagonal elements d_i satisfying $d_i = -1$ for all $\alpha_i < 0$, $i \in \mathbb{N}_p$.

A referee mentioned that in the last twenty-five years there has been an immense interest in multivariate skewed distributions and asked for possible applications of our results in that area of research. However, we were surprised that positive dependence properties of multivariate skewed distributions seem to be rarely treated in the literature. In what follows, we restrict attention to skewed normal distributions with location parameter 0. A p-variate random vector Z has a generalized skew normal distribution $GSN_p(\Sigma, \pi)$ (see, e.g., [27]) if the underlying pdf is given by $2\phi_p(z,\Sigma)\pi(z)$ with $z \in \mathbb{R}^p$, Σ a $p \times p$ covariance matrix, $\phi_p(z,\Sigma)$ pdf of the $N_p(0,\Sigma)$ distribution and skewing function π satisfying $0 \leq \pi(z) \leq 1$ and $\pi(z) = 1 - \pi(-z)$ for all $z \in \mathbb{R}^p$. If $\pi(z) = \Phi(\sum_i \alpha_i z_i)$ with $\alpha = (\alpha_1, \ldots, \alpha_p)^\top \in \mathbb{R}^p$ and with Φ denoting the cdf of the standard normal distribution, then Z is said to have a skew normal distribution $SN_p(\Sigma, \alpha)$ (see, e.g., [2]). A striking property of a (generalized) skew normal vector Z is that $f(Z)$ and $f(X)$ (with $X \sim N_p(0,\Sigma)$) have the same distribution for all even functions $f : \mathbb{R}^p \to \mathbb{R}^p$ (referred to as modulation invariance or perturbation invariance). Hence, $|Z|$ is distributed as $|X|$ and Z^2 is distributed as X^2. Consequently, $\Sigma \in \mathcal{S}_p$ implies that $|Z|$ (Z^2) is PA. We found no references where PA of Z, $|Z|$, (Z^2) is explicitly treated. However, we found one paper where a sufficient condition (that is, Σ^{-1} is M-matrix and α has at most two non-zero components, which then must have opposite signs) for MTP_2 (affiliation) of $Z \sim SN_p(\Sigma, \alpha)$ is given, see [37]. Moreover, it is mentioned in [37] that affiliation of Z implies PA and PLOD of Z. In order to find a sufficient condition for PA of a SN distribution, the following stochastic representation is helpful (see, e.g., [3], p. 128-9). If

$$Z_j = (1-\delta_j^2)^{1/2} W_j + \delta_j |W_0|, j \in \mathbb{N}_p, \tag{11}$$

with $W = (W_1, \ldots, W_p)^\top \sim N_p(0, \Psi)$ and $W_0 \sim N(0,1)$ independently distributed, Ψ a positive definite correlation matrix and $\delta_i \in (-1,1)$ for $i \in \mathbb{N}_p$, then $Z = (Z_1, \ldots, Z_p)^\top$ has a $SN_p(\Sigma, \alpha)$ distribution where Σ and α are determined in terms of Ψ and the δ_i's. From the representation (11) we easily obtain that Z is PA if W is PA and $\delta_i \geq 0$ for all $i \in \mathbb{N}_p$.

A referee suggested that $|Z|$ might be PA while $Z \sim SN_p(\Sigma, \alpha)$ is not PA. This is in fact true. Helpful formulas for the computation of $\mathrm{Cov}(Z)$ can be found in Section 5.1 in [3]. Choosing $\Sigma = I_3 \in \mathcal{S}_3$ and $\alpha = (1,1,1)^\top$ yields that all off-diagonal elements of $\mathrm{Cov}(Z)$ are $-(2\pi)^{-1}$. Hence, $|Z|$ is PA while Z is not PA and no signature of Z is PA. If we replace W_0 by W_{0i} in (11) with $(W_{01}, \ldots, W_{0p})^\top \sim N_p(0, I_p)$ and W independently distributed (see [25,32] for this model), then the resulting Z is PA if W is PA and $\delta_i \geq 0$ for all $i \in \mathbb{N}_p$. However, in this case it seems much more difficult to give a sufficient condition for PA of $|Z|$. All in all it seems worth studying PA (and other notions of positive/negative dependence) of all the proposed skew multivariate distributions in a systematic manner. Clearly, this is beyond the scope of this paper.

Finally, we note that our results imply the validity of some moment inequalities beyond MTP_2 including the strong form of Gaussian product inequalities (GPI), that is,

$$\mathbb{E}\left(\prod_{i\in\mathbb{N}_p} |X_i|^{\alpha_i}\right) \geq \mathbb{E}\left(\prod_{i\in J} |X_i|^{\alpha_i}\right) \mathbb{E}\left(\prod_{i\in\mathbb{N}_p\setminus J} |X_i|^{\alpha_i}\right) \text{ for all } \emptyset \neq J \subset \mathbb{N}_p \text{ and all } \alpha_i > 0,\ i \in \mathbb{N}_p, \tag{12}$$

see the recent paper [8] for further details and further references on the validity of (12) for special cases. For example, it is still an open question whether (12) is valid for $X \sim N_p(0,\Sigma)$ with DX PC for some $D \in D_p$. Clearly, PA (and, a fortiori, MTP_2) of $|X|$ implies the validity of (12) . Hence, our results imply the validity of (12) beyond MTP_2 for some $\Sigma \in \mathcal{S}_p$ where $|X|$ is not MTP_2. Moreover, we also get the validity of (12) beyond MTP_2 for some related multivariate t-distributions and multivariate chi-square distributions.

CRediT authorship contribution statement

Helmut Finner: Conceptualization, Methodology, Writing. **Markus Roters:** Conceptualization, Methodology, Writing.

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