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Abstract
We consider average- and min-based altruistic hedonic games and study the problem of verifying popular and strictly popular coalition structures. While strict popularity verification has been shown to be coNP-complete in min-based altruistic hedonic games, this problem has been open for equal- and altruistic-treatment average-based altruistic hedonic games. We solve these two open cases of strict popularity verification and then provide the first complexity results for popularity verification in (average- and min-based) altruistic hedonic games, where we cover all three degrees of altruism.

Keywords Coalition formation · Hedonic game · Altruism · Cooperative game theory · Popularity

1 Introduction

Much work has been done in recent years to study *hedonic games*, coalition formation games where players express their preferences over those coalitions that contain them. From a higher perspective, this line of research is closely related to coalition structure generation [2], multiagent team formation [3], and ad hoc teamwork research [4], topics that the multiagent community at large has paid a lot of attention to.

Drèze & Greenberg [5] were the first to propose hedonic games and Bogomolnaia & Jackson [6]; Banerjee et al. [7] formally defined and investigated them. One major focus in hedonic games research is on how to represent the agents' preferences in a plausible, expressive, and succinct way. For example, the preferences of the agents are often represented via cardinal values. In such encodings, the agents assign an individual value to

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each other agent. When evaluating coalitions, these values are then aggregated to an overall utility. For example, in additively separable hedonic games (ASHGs) [6], an agent's utility for a coalition is the sum of the values that this agent assigns to its member. In fractional hedonic games [8], the utility is aggregated by taking the average of these values. Other representations assume that there are only a few categories associated with an agent's preference. In the friends-and-enemies encoding [9], a special case of ASHGs, an agent categorizes the other agents into friends and enemies, thus assigning only two distinct values to them.

In all these classical models, the agents only consider their own valuations when comparing two coalitions or two coalition structures (i.e., partitions of all players into coalitions). They are assumed to act egoistically, trying to maximize only their own personal utilities and not caring about others. In contrast to that, recent research has taken another approach and investigated models that integrate social and altruistic components into hedonic games.[1] This research is motivated by the fact that social, unselfish behavior might actually be beneficial and even essential to the success (e.g., essential to survive) of a social group (compare, e.g., the investigations of Hare and Woods [12] about the advantage of social behavior for biological species).

Along these lines, Nguyen et al. [13] introduced *altruistic hedonic games* (AHGs) that are based on the friend-and-enemy encoding [9]. In these games, the agents categorize the other agents into friends and enemies, and they take into account their friends' valuations when comparing coalition structures. There are multiple types of AHGs (presented by Kerkmann et al. [11] in a more recent journal version of [13]) that differ in the way how agents aggregate their friends' valuations and to which degree they act altruistically. Kerkmann et al. [11] present average- and min-based aggregations and three degrees of altruism, namely selfish-first treatment, equal treatment, and altruistic treatment. These three degrees of altruism differ in the order in which the agents take their own and their friends' preferences into account: *Selfish-first treatment* means that agents first look at their own preferences, and only in case of a tie between two coalitions, they ask their friends for their preferences; *equal treatment* means that agents look at their own and their friends' preferences at the same time when making their decision; and *altruistic treatment* means that agents first ask their friends which of two coalitions they prefer, and only in case of a tie, they decide according to their own preferences. Given that altruistic behavior is an essential part of our lives and our decision-making, this is a very natural model of preferences in hedonic games. On the one hand, Kerkmann et al. [11] consider axiomatic properties of the six resulting models of AHGs and, on the other hand, they investigate various solution concepts for them, such as Nash stability, perfectness, and core stability.

Nash stability means that no player prefers to move to another coalition than the one assigned by the current coalition structure, and *perfectness* means that all players weakly prefer their assigned coalition to every other coalition containing them. Both concepts model the incentives and deviation behavior of *single* players. In contrast, *core stability* is a concept focusing on the incentives of *groups* of players: It means for a coalition structure that it is *not blocked* by any coalition, i.e., for no nonempty group of players does it hold that they all would like to leave their currently assigned coalition and form a new one on their own (while all other players remain in their assigned coalition).

[1] For a broader and more comprehensive treatise of altruism in both cooperative and noncooperative game theory, we refer to the survey by Rothe [10] and to the work of Kerkmann et al. [11, Section 2].

Now, to motivate (strict) popularity, let us take this even a level higher: Instead of looking at which other coalitions players may want to join (thus deviating from their currently assigned coalition), let them look at entire coalition structures and compare all possible ones, one by one.[2] Suppose that the current coalition structure wins each such head-to-head contest by a weak majority (or even a majority) of players. This would give a very strong argument in favor of the current coalition structure: It then is *(strictly) popular*. Our main research question is how hard it is to verify whether a given coalition structure in a given AHG is (strictly) popular. Note that Nguyen et al. [13]; Wiechers & Rothe [14] have already studied strict popularity in both average- and min-based AHGs; however, they leave some questions on the complexity of the verification problem open. Below, we discuss them in more detail.

Outline: our contribution We continue the study of average- and min-based AHGs under all three degrees of altruism, focusing on the notions of *popularity* and *strict popularity*. For these notions, we look at entire coalition structures and ask—similarly to the notion of (weak) Condorcet winner in voting—whether (a weak majority or even) a majority of players prefer a given coalition structure to every other coalition structure. We study the complexity of verifying (strictly) popular coalition structures in AHGs and of deciding whether such coalition structures exist. While strict popularity verification is known to be coNP-complete for all three degrees of min-based AHGs [14] and also for selfish-first average-based AHGs [13], its complexity remained open for the other two degrees of average-based altruism: equal treatment and altruistic treatment. We solve these two missing cases via technically rather involved constructions in Sect. 3.

In Sect. 4, we turn to the notion of popularity, which we consider to be even more natural than strict popularity. By definition, the latter is a more demanding notion; therefore, popular coalitions structures are more likely to exist than strictly popular coalitions structures. Note that, just as is known for the notion of Condorcet winner in voting, if a strictly popular coalition structure exists, it must be unique. So, innocent ties can lead to the nonexistence of strictly popular coalition structures. In contrast, there can be more than one popular coalition structure. On the other hand, for a popular coalition structure to *not* exist, a top cycle of coalition structures is required each of which dominates the next one in the cycle. We provide the first complexity results for popularity verification in AHGs, covering all three degrees of altruism and both aggregation methods. We show that the problem is coNP-complete for all six models.

Having closed the two open problems for strict popularity verification in these six models and having established all six complexity results for popularity verification, in Sect. 5 we briefly discuss the related problem of whether (strictly) popular coalition strutures exist in AHGs, and we conclude our work and give some future work directions in Sect. 6.

Related work

To put our work into context within the field of multiagent systems, let us first take a higher perspective and briefly discuss some closely related work on generating coalition structures [2], multiagent team formation [3], and ad hoc teamwork [4]. These lines of research have been intensively pursued in multiagent systems for at least two decades. For eample, Gaston & Desjardins [15] consider multiagent systems as complex

[2] Of course, there are extremely many coalition structures (for n players, their number is given by the n-th Bell number, which rapidly grows with n), and thus extremely many comparisons to make. Yet, remember that there are ways to compactly represent hedonic games and that a player's preference on two coalition structures only depends on the coalitions containing the player.

networks (such as supply chains and sensor networks) of autonomous but interdependent agents that interact with each other based on environmental knowledge, cognitive capabilities, and resource and communication constraints. For such multiagent networks, they develop distributed, dynamic network adaptation mechanisms for discovering effective network structures; that is, in the context of dynamic team formation, they propose strategies for agent-organized networks and evaluate how effective they are for improving organizational performance.

Guo & Lim [16] study negotiation support systems and team negotiations from a coalition formation perspective. In particular, they discuss how cultural diversity can be viewed as an antecedent to coalition formation. Barrett et al. [17] present the first empirical evaluation of ad hoc teamwork in the pursuit domain. More specifically, they evaluate various effective algorithms for online behavior generation of a single ad hoc team agent that has to collaborate with a number of possible teammates. Marcolino et al. [3] investigate whether the diversity within a team can be more important than the strength of its individual members when multiagent teams are forming. They propose a model to address this question and show that diversity can outperform a uniform team of individually strong members; specifically, they provide necessary conditions for this to happen. Further, they propose optimal voting rules for a diverse team to form; they provide experiments on synthetic data showing that both diversity and strength contribute to the performance of a team; and they experimentally study how useful their model is when applied to a key challenge in AI research—Computer Go.

Leibo et al. [18] look at multiagent team formation from a game-theoretical perspective and the related social dilemmas, such as whether it is better for agents to cooperate or to defect. Specifically, they introduce sequential social dilemmas that share the mixed incentive structure of social dilemmas in matrix games, while at the same time requiring agents to learn policies that implement their strategic intentions in team formation. They experimentally analyze the dynamics of policies learned by multiple selfish independent learning agents in two Markov games they introduce. On the other hand, Bachrach et al. [19] propose a framework that can be used to train agents to negotiate and form teams via deep reinforcement learning. Their method is completely based on experience rather than making any assumptions on the specific negotiation protocol used. They evaluate their approach for various team-formation negotiation environments and show that their agents outperform hand-crafted bots. Specifically, they obtain negotiation outcomes that are consistent with fair solutions predicted by cooperative game theory. They also study the influence of the physical location of agents on the negotiation outcomes.

For an excellent overview, we refer to Rahwan et al. [2] who comprehensively survey the known algorithmic approaches (e.g., dynamic programming and anytime algorithms) to the computationally challenging problem of coalition formation and generating coalition structures. In particular, they focus on techniques specifically designed for various compact representation schemes for coalitional games. Finally, Mirsky et al. [4] survey the progress made for ad hoc teamwork—the problem of designing agents able to collaborate with new teammates without prior coordination. They also list a number of important open questions and challenges in the field of ad hoc teamwork.

Next, we turn to some previous work related to our specific research problem: Unlike the above-mentioned work that often is experimental and considers coalition formation in specific multiagent systems, we provide theoretical results pinpointing the computational complexity of verifying popularity and strict popularity in altruistic hedonic games. For a general background on hedonic games, we refer to the book chapters by Aziz & Savani [20]; Bullinger et al. [21] and the survey by Woeginger [22].

Inspired by the work of Nguyen et al. [13] on AHGs, there is quite some follow-up research on altruism in hedonic games. Schlueter & Goldsmith [23] generalize AHGs to *"super AHGs,"* using ideas of the *"social distance games"* due to Brânzei & Larson [24]. Bullinger & Kober [25] introduce the related notion of *loyalty in hedonic games*, where agents are *"loyal"* to all agents that they assign a positive value to. Kerkmann & Rothe [26] apply the original model of AHGs to *coalition formation games* in general. In their model, agents expand their altruistic behavior also to friends outside of their own coalitions. Their results have been extended by Kerkmann et al. [27], and some related open question regarding core stability—one of the most central solution concept for hedonic games—have been solved by Hoffjan et al. [28]. For an overview of various other notions of altruism in cooperative and noncooperative game theory, we refer to the survey by Rothe [10].

The notion of popularity was first proposed by Gärdenfors [29] in the context of marriage games. For hedonic games, popularity and strict popularity were later studied by, e.g., Aziz et al. [30], Brandt & Bullinger [31]; Kerkmann et al. [32]. Aziz et al. [30] study popularity in the context of *additively separable hedonic games* (ASHGs). They show that verifying popularity is coNP-complete and checking the existence of popular coalition structures is an NP-hard problem for ASHGs. Brandt & Bullinger [31] continue this study and also investigate strict popularity in ASHGs. They show that also strict popularity verification is coNP-complete in these games and that checking popularity existence is coNP-hard and NP-hard for symmetric ASHGs. They conjecture that checking the existence of popular coalition structures in symmetric ASHGs may be a Σ_2^p-complete problem. Kerkmann et al. [32] study popularity and strict popularity in the context of so-called *FEN-hedonic games* where agents divide the other players into friends, enemies, and neutral players. Also for these games, all verification problems are coNP-complete.

2 Preliminaries

We use the notation $[m] = \{1, \ldots, m\}$ for any integer $m \geq 0$. We consider a set $N = [n]$ of n *players* (or *agents*), where subsets of N are called *coalitions*. For any player $i \in N$, $\mathcal{N}^i = \{C \subseteq N \mid i \in C\}$ denotes the set of coalitions containing i. A *coalition structure* is a partition $\Gamma = \{C_1, \ldots, C_k\}$ of the players into coalitions (i.e., $\bigcup_{i=1}^{k} C_i = N$ and $C_i \cap C_j = \emptyset$ for all $i, j \in [k]$ with $i \neq j$), where the coalition containing player i is denoted by $\Gamma(i)$. $\mathcal{C}_N$ is the set of all coalition structures for a set of agents N.

A *coalition formation game* is a pair $(N, \succeq)$, where N is a set of agents, $\succeq = (\succeq_1, \ldots, \succeq_n)$ is a profile of preferences, and every preference $\succeq_i \subseteq \mathcal{C}_N \times \mathcal{C}_N$ is a complete weak order over all coalition structures for N. For any two coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, we say that agent i *weakly prefers* Γ to Δ if $\Gamma \succeq_i \Delta$; that i *prefers* Γ to Δ ($\Gamma \succ_i \Delta$) if $\Gamma \succeq_i \Delta$ but not $\Delta \succeq_i \Gamma$; and that i is *indifferent between* Γ and Δ ($\Gamma \sim_i \Delta$) if $\Gamma \succeq_i \Delta$ and $\Delta \succeq_i \Gamma$.

A *hedonic game* is a coalition formation game $(N, \succeq)$ where the preference $\succeq_i$ of any agent $i \in N$ only depends on the coalitions containing i. This means that i is indifferent between any two coalition structures $\Gamma, \Delta \in \mathcal{C}_N$ as long as i's coalition in them is the same, i.e., $\Gamma(i) = \Delta(i)$ implies $\Gamma \sim_i \Delta$. Agent i's preference can then be represented by a complete weak order over the set $\mathcal{N}^i$ of coalitions containing i. For $A, B \in \mathcal{N}^i$, we say that player i *weakly prefers* A to B if $A \succeq_i B$, and analogously for (strict) preference and indifference.

2.1 Altruistic hedonic games

Nguyen et al. [13] used the friends-and-enemies encoding by Dimitrov et al. [9] when first introducing altruistic hedonic games (AHGs). Under this encoding, each player i partitions the other players into a set of friends F_i and a set of enemies E_i, and assigns the following friend-oriented value to a coalition $A \in \mathcal{N}^i$:

$$v_i(A) = n|A \cap F_i| - |A \cap E_i|.$$

The friendship relations, which are assumed to be mutual, can then be represented by a *network of friends*, an undirected graph where two players are connected by an edge if and only if they are friends of each other.

Nguyen et al. [13] introduced altruism into an agent's preference by incorporating the average of their friends' valuations (of the friends that are in the same coalition) into their utility. Wiechers & Rothe [14] vary this model by considering the minimum instead. For any $A \in \mathcal{N}^i$, we use:

$$\mathrm{avg}_i^F(A) = \sum_{a \in A \cap F_i} \frac{v_a(A)}{|A \cap F_i|}; \quad \mathrm{avg}_i^{F+}(A) = \sum_{a \in (A \cap F_i) \cup \{i\}} \frac{v_a(A)}{|(A \cap F_i) \cup \{i\}|}; \tag{1}$$

$$\min_i^F(A) = \min_{a \in A \cap F_i} v_a(A); \quad \min_i^{F+}(A) = \min_{a \in (A \cap F_i) \cup \{i\}} v_a(A), \tag{2}$$

where the minimum of the empty set is defined as zero. We also define these values for coalition structures $\Gamma \in \mathcal{C}_N$, e.g., by $\mathrm{avg}_i^F(\Gamma) = \mathrm{avg}_i^F(\Gamma(i))$. The three *degrees of altruism*, introduced by Nguyen et al. [13], are the following. For a constant $M \geq n^5$, agent i's

- *selfish-first (SF) preference* is defined by $A \succeq_i^{SF} B \iff u_i^{SF}(A) \geq u_i^{SF}(B)$, with the SF utility function $u_i^{SF}(A) = M \cdot v_i(A) + \mathrm{avg}_i^F(A)$;
- *equal-treatment (EQ) preference* is defined by $A \succeq_i^{EQ} B \iff u_i^{EQ}(A) \geq u_i^{EQ}(B)$, with the EQ utility function $u_i^{EQ}(A) = \mathrm{avg}_i^{F+}(A)$; and
- *altruistic-treatment (AL) preference* is defined by $A \succeq_i^{AL} B \iff u_i^{AL}(A) \geq u_i^{AL}(B)$, with the AL utility function $u_i^{AL}(A) = v_i(A) + M \cdot \mathrm{avg}_i^F(A)$.

As mentioned in the Introduction, these three degrees very naturally model altruistic behavior by taking the order in which agents look at their own or their friends' preferences into account.

The constant factor $M \geq n^5$ ensures that the SF preference is determined by first looking at the agent's own valuation for their coalition while the AL preference is determined by first looking at the agent's friends' valuations for it (see [13, Theorems 1 & 2]). The min-based altruistic preferences are defined analogously, using the minimum according to (2) instead of the average (1). They will be denoted by $\succeq^{minSF}$, $\succeq^{minEQ}$, and $\succeq^{minAL}$.

A pair $(N, \succeq)$, where $\succeq$ is a profile of preferences defined by one of the average-based degrees of altruism, is called an *altruistic hedonic game* (AHG) with *average-based altruistic preferences* $\succeq$. A game $(N, \succeq^{\min})$ with *min-based altruistic preferences* $\succeq^{\min}$ is said to be a *min-based altruistic hedonic game* (MBAHG). Based on the degree of altruism, we call, say, an AHG with SF preferences an *SF AHG*, etc.

2.2 Popularity

We now define popularity, which is based on the pairwise comparison of coalition structures. For a hedonic game $(N, \succeq)$ and two coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, let $\#_{\Gamma > \Delta} = |\{i \in N \mid \Gamma \succ_i \Delta\}|$ be the number of players that prefer Γ to Δ. A coalition structure $\Gamma \in \mathcal{C}_N$ is *popular* (respectively, *strictly popular*) if, for every other coalition structure $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$, it holds that $\#_{\Gamma > \Delta} \geq \#_{\Delta > \Gamma}$ (respectively, $\#_{\Gamma > \Delta} > \#_{\Delta > \Gamma}$). Define the problems:

P-Verification	
Given:	A hedonic game $(N, \succeq)$ and a coalition structure Γ.
Question:	Is Γ popular in $(N, \succeq)$?

P-Existence	
Given:	A hedonic game $(N, \succeq)$.
Question:	Is there a popular coalition structure in $(N, \succeq)$?

The strict variants of these two problems, SP-Verification and SP-Existence, are defined analogously.

In the following two sections, we will solve the two missing cases of Nguyen et al. [13] by showing that SP-Verification is coNP-complete for EQ and AL AHGs, and we will also show that P-Verification is coNP-complete as well for all three degrees of altruism in AHGs and MBAHGs.

It is easy to see that all these verification problems are in coNP (cf. Nguyen et al. [13, Theorem 12]). To show their coNP-hardness, we reduce from the complement of the following NP-complete problem [33, 34]:

Restricted Exact Cover by 3-Sets (RX3C)	
Given:	A set $B = \{1, \ldots, 3k\}$ (for some integer $k \geq 2$) and a collection $\mathscr{S} = \{S_1, \ldots, S_{3k}\}$ of 3-element subsets of B, where each element of B occurs in exactly three sets in $\mathscr{S}$.
Question:	Does there exist an exact cover of B in $\mathscr{S}$, i.e., a subset $\mathscr{S}' \subseteq \mathscr{S}$ of size k such that every element of B occurs in exactly one set in $\mathscr{S}'$?

Specifically, to prove coNP-hardness of (strict) popularity verification, we construct from an RX3C instance $(B, \mathscr{S})$ the network of friends of a hedonic game $(N, \succeq)$ and a coalition structure Γ and show that Γ is *not* (strictly) popular under the considered model if and only if there exists an exact cover of B in $\mathscr{S}$.

3 Verifying strict popularity in AHGs

We start with strict popularity. While Wiechers and Rothe [14] showed that SP-Verification is coNP-complete for all three degrees of altruism in MBAHGs, Nguyen et al. [13] showed the same result only for SF AHGs. We solve the two missing cases (i.e., for EQ and AL) in Theorems 1 and 2. In their proofs, we will use the following two observations. The first observation says that, under EQ and AL, a player i prefers adding a friend's friend k to their current coalition, provided that k is not i's own friend.

Observation 1 For any $D \in \mathcal{N}^i$, $j \in F_i \cap D$, and $k \in (F_j \setminus F_i) \setminus D$, it holds that $D \cup \{k\} \succ_i^{EQ} D$ and $D \cup \{k\} \succ_i^{AL} D$.

Proof It holds that

$$\begin{aligned} u_i^{EQ}(D \cup \{k\}) = \mathrm{avg}_i^{F+}(D \cup \{k\}) &= \sum_{a \in (D \cap F_i) \cup \{i\}} \frac{v_a(D \cup \{k\})}{|(D \cap F_i) \cup \{i\}|} \\ &= \frac{1}{|(D \cap F_i) \cup \{i\}|} \cdot \Big(v_i(D \cup \{k\}) + v_j(D \cup \{k\}) + \sum_{a \in (D \cap F_i) \setminus \{j\}} v_a(D \cup \{k\}) \Big) \\ &\geq \frac{1}{|(D \cap F_i) \cup \{i\}|} \cdot \Big(v_i(D) - 1 + v_j(D) + n + \sum_{a \in (D \cap F_i) \setminus \{j\}} (v_a(D) - 1) \Big) \\ &= \frac{1}{|(D \cap F_i) \cup \{i\}|} \cdot \Big(\sum_{a \in (D \cap F_i) \cup \{i\}} v_a(D) + n - |D \cap F_i| \Big) \\ &> \mathrm{avg}_i^{F+}(D) = u_i^{EQ}(D). \end{aligned}$$

By similar transformations of equations, we obtain that

$$\mathrm{avg}_i^{F}(D \cup \{k\}) > \mathrm{avg}_i^{F}(D).$$

Thus $D \cup \{k\} \succ_i^{EQ} D$ and $D \cup \{k\} \succ_i^{AL} D$. □

By means of Observation 1, we obtain the following.

Observation 2 If player $i \in N$ has only one friend j (i.e., $F_i = \{j\}$), then $C = \{j\} \cup F_j$ is i's unique most preferred coalition under EQ and AL.

Proof The proof is the same for $\succeq_i^{EQ}$ and $\succeq_i^{AL}$. We will simply use u_i for which either u_i^{EQ} or u_i^{AL} can be substituted. Assume that $D \neq C$ is one of i's most preferred coalitions. Then $u_i(D) \geq u_i(C)$. It is obvious that $D \subseteq C$ because every player in $N \setminus C$ is an enemy of i's and j's and can thus only decrease i's utility. Further, since j is i's only friend, it is clear that $j \in D$ (otherwise, we would have $u_i(D) \leq 0 < u_i(C)$). Then, by Observation 1, it follows that D contains all friends of j's. Hence, $D = C$, which is a contradiction. □

We are now ready to solve the two problems that Nguyen et al. [13] left open regarding the complexity of SP-Verification, namely for EQ AHGs and AL AHGs. We start with the former.

Theorem 1 *SP-Verification is coNP-complete for EQ AHGs.*

Proof Given an instance of $(B, \mathscr{S})$ of RX3C, with $B = \{1, \ldots, 3k\}$ and $\mathscr{S} = \{S_1, \ldots, S_{3k}\}$, we define the set of players $N = P \cup A \cup \bigcup_{S \in \mathscr{S}} Q_S$ with

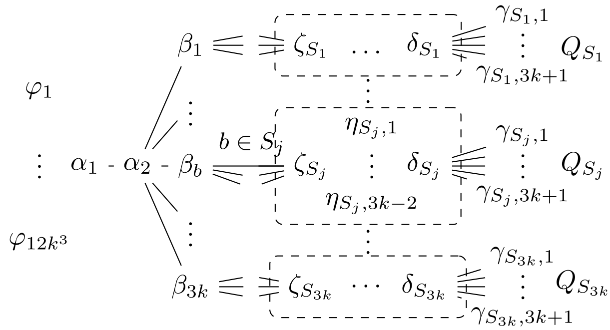

Fig. 1 Network of friends in the proof of Theorem 1. A dashed rectangle indicates that all players inside are friends of each other

$$P = \{\varphi_1, \dots, \varphi_{12k^3}\}, \quad A = \{\alpha_1, \alpha_2\} \cup \{\beta_b \mid b \in B\}, \quad \text{and}$$
$$Q_S = \{\zeta_S, \eta_{S,j}, \delta_S, \gamma_{S,\ell} \mid j \in [3k-2], \ell \in [3k+1]\} \quad \text{for every } S \in \mathscr{S}.$$

We then construct the network of friends shown in Fig. 1 and define the coalition structure $\Gamma = \{\{\varphi_1\}, \dots, \{\varphi_{12k^3}\}, A, Q_{S_1}, \dots, Q_{S_{3k}}\}$. It holds that

$$n = |N| = 12k^3 + 2 + 3k + 3k(6k+1) = 12k^3 + 18k^2 + 6k + 2.$$

Specifically, the friendship relationships are as follows:

- α_2 is friends with α_1 and every $\beta_b, b \in B$.
- For $S \in \mathscr{S}$, ζ_S is friends with the three β_b with $b \in S$.
- For $S \in \mathscr{S}$, all players in $\{\zeta_S, \eta_{S,j}, \delta_S \mid j \in [3k-2]\}$ are friends of each other.
- For $S \in \mathscr{S}$, δ_S is friends with every $\gamma_{S,\ell}, \ell \in [3k+1]$.

The idea of this proof is to show that there can be a coalition structure Δ that is equally popular as Γ if and only if there is an exact cover for B.

We start by stating some useful claims. The first two claims are direct consequences of Observation 2 and the third claim is obvious, as the φ-players do not have any friends.

Claim 1 *α_1 prefers A to every other coalition.*

Claim 2 *For every $S \in \mathscr{S}$ and $\ell \in [3k+1]$, $\gamma_{S,\ell}$ prefers Q_S to every other coalition.*

Claim 3 *For $h \in [12k^3]$, φ_h prefers $\{\varphi_h\}$ to every other coalition.*

To complete the proof, we further need the following two claims. The proof of Claim 4 is technically rather involved and is perhaps the key ingredient in this proof of the theorem.

Claim 4 *For $S \in \mathscr{S}$, ζ_S prefers Q_S to every other coalition.*

Proof of Claim 4 It holds that

$$\begin{aligned}
u_{\zeta_S}^{EQ}(Q_S) =& \frac{v_{\zeta_S}(Q_S) + (3k-2)v_{\eta_{S,1}}(Q_S) + v_{\delta_S}(Q_S)}{3k} \\
=& \frac{(3k-1)(n(3k-1)-(3k+1)) + n(6k)}{3k} \\
=& \frac{n(9k^2+1) - (9k^2-1)}{3k} \\
=& n\left(3k + \frac{1}{3k}\right) - \left(3k - \frac{1}{3k}\right).
\end{aligned}$$

Hence, ζ_S and their friends have more than $3k$ friends in Q_S on average. Now, assume that there is a coalition $D \neq Q_S$ that ζ_S weakly prefers to every other coalition. It is clear that

$$D \subseteq Q_S \cup \{\beta_b \mid b \in S\} \cup \{\alpha_2\} \cup \{\zeta_{S'} | S' \in \mathscr{S}, S \cap S' \neq \emptyset\}$$

because all other players are enemies of ζ_S's and of all their friends.

Assume that there is some β_b in D. We will show that ζ_S prefers $D \setminus \{\beta_b\}$ to D, which is a contradiction. It holds that ζ_S prefers $D \setminus \{\beta_b\}$ to D if and only if $u_{\zeta_S}^{EQ}(D \setminus \{\beta_b\}) > u_{\zeta_S}^{EQ}(D)$. Let

$$\begin{aligned}
x =& |D \cap F_{\zeta_S}|, \\
t =& \sum_{a \in ((D \setminus \{\beta_b\}) \cap F_{\zeta_S}) \cup \{\zeta_S\}} v_a(D \setminus \{\beta_b\}), \\
v =& \sum_{a \in (D \cap F_{\zeta_S}) \cup \{\zeta_S\}} v_a(D), \text{ and} \\
w =& v - t.
\end{aligned}$$

Then

$$\begin{aligned}
u_{\zeta_S}^{EQ}(D \setminus \{\beta_b\}) > u_{\zeta_S}^{EQ}(D) \Leftrightarrow& \frac{v-w}{x} - \frac{v}{x+1} > 0 \\
\Leftrightarrow& \frac{(x+1)(v-w) - xv}{x(x+1)} > 0 \\
\Leftrightarrow& (x+1)(v-w) - xv > 0 \\
\Leftrightarrow& xv + v - (x+1)w - xv > 0 \\
\Leftrightarrow& v - (x+1)w > 0 \\
\Leftrightarrow& \frac{v}{x+1} > w \\
\Leftrightarrow& u_{\zeta_S}^{EQ}(D) > w.
\end{aligned}$$

For w, we have

$$\begin{aligned}
w = & \sum_{a \in (D \cap F_{\zeta_S}) \cup \{\zeta_S\}} \Big(v_a(D) \Big) - t \\
= & \sum_{a \in (D \setminus \{\beta_b\}) \cap F_{\zeta_S}} \Big(v_a(D) \Big) + v_{\beta_b}(D) + v_{\zeta_S}(D) - t \\
= & \sum_{a \in (D \setminus \{\beta_b\}) \cap F_{\zeta_S}} \Big(v_a(D \setminus \{\beta_b\}) - 1 \Big) + v_{\beta_b}(D) + v_{\zeta_S}(D \setminus \{\beta_b\}) + n - t \\
= & \sum_{a \in ((D \setminus \{\beta_b\}) \cap F_{\zeta_S}) \cup \{\zeta_S\}} \Big(v_a(D \setminus \{\beta_b\}) \Big) - |(D \setminus \{\beta_b\}) \cap F_{\zeta_S}| \\
& \qquad + v_{\beta_b}(D) + n - t \\
= & - |(D \setminus \{\beta_b\}) \cap F_{\zeta_S}| + v_{\beta_b}(D) + n \\
\leq & - |(D \setminus \{\beta_b\}) \cap F_{\zeta_S}| + 4n + n \\
< & 5n \; < \; u^{EQ}_{\zeta_S}(Q_S) \; \leq \; u^{EQ}_{\zeta_S}(D).
\end{aligned}$$

Hence, ζ_S prefers $D \setminus \{\beta_b\}$ to D; a contradiction that implies that there is no β_b in D. It follows that $D \subseteq Q_S$.

Since ζ_S has at most $3k - 1$ friends in D and

$$u^{EQ}_{\zeta_S}(D) \geq u^{EQ}_{\zeta_S}(Q_S) = n(3k + \frac{1}{3k}) - (3k - \frac{1}{3k}),$$

there has to be at least one friend of ζ_S in D who has at least $3k + 1$ friends in D. The only player for which this is possible is δ_S, so $\delta_S \in D$. By Observation 1, it holds that $\gamma_{S,1}, \dots, \gamma_{S,3k+1} \in D$. Thus $\{\zeta_S, \delta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1}\} \subseteq D \subseteq Q_S$.

Now, let $y = |D \cap \{\eta_{S,1}, \dots, \eta_{S,3k-2}\}|$ be the number of η-players in D. Then

$$\begin{aligned}
u^{EQ}_{\zeta_S}(D) = & \frac{v_{\zeta_S}(D) + y \cdot v_{\eta_{S,j}}(D) + v_{\delta_S}(D)}{y+2} \\
= & \frac{(y+1) \cdot (n(y+1) - (3k+1)) + n(y + 1 + 3k + 1)}{y+2} \\
= & n\left(\frac{y^2 + 3y + 3k + 3}{y+2} \right) - \frac{(y+1)(3k+1)}{y+2}.
\end{aligned}$$

We know that $u^{EQ}_{\zeta_S}(D) \geq u^{EQ}_{\zeta_S}(Q_S)$ holds, for which we get the following equivalences:

$$
\begin{aligned}
& u^{EQ}_{\zeta_S}(D) \geq u^{EQ}_{\zeta_S}(Q_S) \\
\Leftrightarrow & n\left(\frac{y^2+3y+3k+3}{y+2}\right) - \frac{(y+1)(3k+1)}{y+2} \geq n\left(\frac{9k^2+1}{3k}\right) - \frac{9k^2-1}{3k} \\
\Leftrightarrow & 0 \geq n\left(\frac{9k^2+1}{3k} - \frac{y^2+3y+3k+3}{y+2}\right) - \left(\frac{(3k-1)(3k+1)}{3k} - \frac{(y+1)(3k+1)}{y+2}\right) \\
\Leftrightarrow & 0 \geq n\left(\frac{(9k^2+1)(y+2)-(y^2+3y+3k+3)(3k)}{(3k)(y+2)}\right) \\
& \quad - \left(\frac{(3k-1)(3k+1)(y+2)-(y+1)(3k+1)(3k)}{(3k)(y+2)}\right) \\
\Leftrightarrow & 0 \geq n\left(\frac{9k^2y+y+18k^2+2-3ky^2-9ky-9k^2-9k}{(3k)(y+2)}\right) \\
& \quad - \left(\frac{(3k+1)\Big((3k-1)(y+2)-(y+1)(3k)\Big)}{(3k)(y+2)}\right) \\
\Leftrightarrow & 0 \geq n\left(\frac{(3k-y-2)(3ky+3k-1)}{(3k)(y+2)}\right) - \left(\frac{(3k+1)(3k-y-2)}{(3k)(y+2)}\right).
\end{aligned} \tag{3}
$$

This implies

$$0 \geq \frac{(3k-y-2)(3ky+3k-1)}{(3k)(y+2)}.$$

For a contradiction, assume that $0 < \frac{(3k-y-2)(3ky+3k-1)}{(3k)(y+2)}$. Since $3ky + 3k - 1 > 0$, $3k > 0$, and $y + 2 > 0$, it follows that $3k - y - 2 > 0$, i.e., $y < 3k - 2$. Then, for $0 \leq y < 3k - 2$ and $k \geq 2$, the minimum of $\frac{(3k-y-2)(3ky+3k-1)}{(3k)(y+2)}$ is reached for $y = 3k - 3$, namely $\frac{9k^2-6k-1}{3k(3k-1)}$. But even for the minimum we have $n\left(\frac{9k^2-6k-1}{3k(3k-1)}\right) - \frac{(3k+1)(3k-y-2)}{(3k)(y+2)} > 0$, which is a contradiction to Equation (3).

Since $3ky + 3k - 1 > 0$, $3k > 0$, and $y + 2 > 0$, we have $3k - y - 2 \leq 0$. Thus $y \geq 3k - 2$. Hence, all η-players are in D, so $D = Q_S$. This is a contradiction and completes the

proof. □ Claim 4

Claim 5 *If β_b with $b \in B$ prefers Δ to Γ, then $\zeta_S \in \Delta(\beta_b)$ for some $S \in \mathscr{S}$ with $b \in S$.*

Proof of Claim 5 Assume that there is no ζ_S with $b \in S$ in $\Delta(\beta_b)$. Then α_2 is β_b's only remaining friend that could be in $\Delta(\beta_b)$. By Observation 1, β_b gets the most utility from Δ if $\Delta(\beta_b) = A$. This means that β_b does not prefer Δ to Γ. □ Claim 5

Now, using these claims, we will show that there is an exact cover of B if and only if Γ is not strictly popular under EQ preferences.

Only if: Assume there exists an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ of B. Then, for the coalition structure

$$\Delta = \left\{ \{\varphi_1\}, \ldots, \{\varphi_{12k^3}\}, A \cup \bigcup_{S \in \mathscr{S}'} Q_S\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}' \right\},$$

we can show that Δ and Γ are equally popular under EQ preferences: All players in Q_S with $S \in \mathscr{S} \setminus \mathscr{S}'$ and all $\varphi_h, h \in [12k^3]$, are obviously indifferent between Γ and Δ. By Claims 1, 2, and 4, α_1, $\gamma_{S,\ell}$, and ζ_S, with $S \in \mathscr{S}'$ and $\ell \in [3k+1]$, prefer Γ to Δ. The remaining players prefer Δ to Γ: For α_2, it holds that

$$\begin{aligned}
u^{EQ}_{\alpha_2}(\Delta) =& \frac{1}{3k+2} \cdot \Big(\sum_{a \in \{\alpha_1, \alpha_2, \beta_1, \ldots, \beta_{3k}\}} v_a(\Delta) \Big) \\
=& \frac{1}{3k+2} \cdot \Big(v_{\alpha_1}(\Gamma) - k(6k+1) + v_{\alpha_2}(\Gamma) - k(6k+1) \\
&+ \sum_{b \in [3k]} \big(v_{\beta_b}(\Gamma) + n - k(6k+1) + 1 \big) \Big) \\
=& \frac{1}{3k+2} \cdot \Big(\sum_{a \in \{\alpha_1, \alpha_2, \beta_1, \ldots, \beta_{3k}\}} v_a(\Gamma) \Big) \\
&+ \frac{1}{3k+2} \cdot \Big(-(3k+2)k(6k+1) + 3k(n+1) \Big) \\
=& u^{EQ}_{\alpha_2}(\Gamma) + \frac{k}{3k+2} \cdot \Big(-(3k+2)(6k+1) + 3(n+1) \Big) \\
=& u^{EQ}_{\alpha_2}(\Gamma) + \frac{k}{3k+2} \cdot \Big(36k^3 + 36k^2 + 3k + 7 \Big) > u^{EQ}_{\alpha_2}(\Gamma).
\end{aligned}$$

For any $b \in [3k]$, β_b is part of exactly one $S \in \mathscr{S}'$. Thus there is exactly one ζ_S in $\Delta(\beta_b)$ that is β_b's friend, and we have

$$\begin{aligned}
u^{EQ}_{\beta_b}(\Delta) =& \frac{1}{3}\Big(v_{\beta_b}(\Delta) + v_{\alpha_2}(\Delta) + v_{\zeta_S}(\Delta) \Big) = \frac{1}{3}\Big(2n - (3k + k(6k+1) - 1) \\
&+ n(3k+1) - k(6k+1) + n(3k+2) - (k(6k+1) - 1) \Big) \\
=& \frac{1}{3}\Big(n(6k+5) - (3k(6k+1) + 3k - 2) \Big) \\
=& n(2k + \frac{5}{3}) - (k(6k+2) - \frac{2}{3}) \\
>& n\Big(\frac{3}{2} \cdot k + 1 \Big) - \frac{3}{2} \cdot k \\
=& \frac{1}{2}\big(n - 3k + n(3k+1) \big) = \frac{1}{2}\big(v_{\beta_b}(\Gamma) + v_{\alpha_2}(\Gamma) \big) = u^{EQ}_{\beta_b}(\Gamma).
\end{aligned}$$

For $\eta_{S,j}$ and δ_S with $S \in \mathscr{S}'$ and $j \in [3k-2]$, we can similarly calculate that $u^{EQ}_{\eta_{S,j}}(\Delta) > u^{EQ}_{\eta_{S,j}}(\Gamma)$ and $u^{EQ}_{\delta_S}(\Delta) > u^{EQ}_{\delta_S}(\Gamma)$. In more detail, for $\eta_{S,j}$ with $S \in \mathscr{S}'$ and $j \in [3k-2]$, we have

$$
\begin{aligned}
u^{EQ}_{\eta_{S,j}}(\Delta) =& \frac{1}{3k}\Big(v_{\zeta_S}(\Delta) + (3k-2)v_{\eta_{S,j}}(\Delta) + v_{\delta_S}(\Delta)\Big) \\
=& \frac{1}{3k}\Big(v_{\zeta_S}(\Gamma) + 3n - (3k-1) - (k-1)(6k+1) \\
&+ (3k-2)\big(v_{\eta_{S,j}}(\Gamma) - (3k+2) - (k-1)(6k+1)\big) \\
&+ v_{\delta_S}(\Gamma) - (3k+2) - (k-1)(6k+1)\Big) \\
=& \frac{v_{\zeta_S}(\Gamma) + (3k-2)v_{\eta_{S,j}}(\Gamma) + v_{\delta_S}(\Gamma)}{3k} \\
&+ \frac{1}{3k}\Big(3n + 3 - (3k)\big(3k + 2 + (k-1)(6k+1)\big)\Big) \\
=& u^{EQ}_{\eta_{S,j}}(\Gamma) + 6k^2 + 20k + 5 + {}^{3}/_{k} \\
>& u^{EQ}_{\eta_{S,j}}(\Gamma).
\end{aligned}
$$

Further, for δ_S with $S \in \mathscr{S}'$, we have

$$
\begin{aligned}
u^{EQ}_{\delta_S}(\Delta) =& \frac{v_{\zeta_S}(\Delta) + (3k-2)v_{\eta_{S,j}}(\Delta) + v_{\delta_S}(\Delta) + (3k+1)v_{\gamma_{S,\ell}}(\Delta)}{6k+1} \\
=& \frac{v_{\zeta_S}(\Gamma) + (3k-2)v_{\eta_{S,j}}(\Gamma) + v_{\delta_S}(\Gamma) + (3k+1)v_{\gamma_{S,\ell}}(\Gamma)}{6k+1} \\
&+ \frac{3n + 3 - (6k+1)\big(3k + 2 + (k-1)(6k+1)\big)}{6k+1} \\
=& u^{EQ}_{\delta_S}(\Gamma) + \frac{60k^2 + 14k + 8}{6k+1} \\
>& u^{EQ}_{\delta_S}(\Gamma).
\end{aligned}
$$

Overall, we have

$$
\begin{aligned}
\#_{\Delta \succ \Gamma} &= |\{\alpha_2, \beta_1, \dots, \beta_{3k}\} \cup \{\eta_{S,j}, \delta_S \mid S \in \mathscr{S}', j \in [3k-2]\}| \\
&= 1 + 3k + k(3k-1) = 1 + k(3k+2) \\
&= |\{\alpha_1\} \cup \{\zeta_S, \gamma_{S,\ell} \mid S \in \mathscr{S}', \ell \in [3k+1]\}| = \#_{\Gamma \succ \Delta}.
\end{aligned}
$$

Hence, Γ is not strictly popular.

If: Assume that Γ is not strictly popular under EQ, i.e., there is a coalition structure $\Delta \neq \Gamma$ with $\#_{\Delta \succ \Gamma} \geq \#_{\Gamma \succ \Delta}$. Let $k' = |\{S \in \mathscr{S} \mid Q_S \notin \Delta\}|$ be the number of sets Q_S that are not a coalition in Δ. Then, by Claims 2 and 4, all $\gamma_{S,\ell}$ and ζ_S from these k' sets Q_S prefer Γ to Δ. Further, no φ_h can ever prefer Δ to Γ, and all players in the $3k - k'$ sets $Q_S \in \Delta$ are indifferent between Γ and Δ.

First, observe that $k' \geq 1$. If $k' = 0$ then, for every $S \in \mathscr{S}$, Q_S is a coalition in Δ. Then, by Claim 5, no β_b prefers Δ to Γ and, obviously, β_b can only be indifferent between Γ and Δ if $\Delta(\beta_b) = A$. It follows that $A \in \Delta$ because otherwise all β_b would prefer Γ to Δ and there would thus be more players who prefer Γ to Δ than vice versa. However, this means that $\Delta = \Gamma$, which is a contradiction.

Second, observe that A is not a coalition in Δ. If this were the case, all players in A were indifferent between Γ and Δ. Then

$$\#_{\Gamma \succ \Delta} \geq |\{\zeta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1} \mid S \in \mathscr{S}'\}| = k' \cdot (3k+2) \quad \text{and}$$
$$\#_{\Delta \succ \Gamma} \leq |\{\eta_{S,1}, \dots, \eta_{S,3k-2}, \delta_S \mid S \in \mathscr{S}'\}| = k' \cdot (3k-1).$$

With $k' \geq 1$, this contradicts $\#_{\Delta \succ \Gamma} \geq \#_{\Gamma \succ \Delta}$.

Third, observe that $k' \leq k$. For a contradiction, assume that $k' > k$. Since $A \notin \Delta$, we know by Claim 1 that α_1 prefers Γ to Δ. So, we have

$$\#_{\Gamma \succ \Delta} \geq |\{\alpha_1\} \cup \{\zeta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1} \mid S \in \mathscr{S}'\}| = 1 + k' \cdot (3k+2) \quad \text{and}$$
$$\#_{\Delta \succ \Gamma} \leq |\{\alpha_2, \beta_1, \dots, \beta_{3k}\} \cup \{\eta_{S,1}, \dots, \eta_{S,3k-2}, \delta_S \mid S \in \mathscr{S}'\}|$$
$$= 1 + 3k + k' \cdot (3k-1).$$

But this implies the following contradiction to $\#_{\Delta \succ \Gamma} \geq \#_{\Gamma \succ \Delta}$:

$$\#_{\Delta \succ \Gamma} \leq 1 + 3k + k' \cdot (3k-1) < 1 + 3k' + k' \cdot (3k-1) = 1 + k' \cdot (3k+2) = \#_{\Gamma \succ \Delta}.$$

Finally, observe that $k' \geq k$. For a contradiction, assume that $k' < k$. Because of Claim 5 we then know that at most $3k'$ β-players prefer Δ to Γ. The remaining $3k - 3k'$ β_b do not have any ζ_S with $b \in S$ in their coalitions in Δ. Together with $A \notin \Delta$, it follows that these $3k - 3k'$ β-players prefer Γ to Δ. Hence,

$$\#_{\Gamma \succ \Delta} \geq 1 + 3k - 3k' + k' \cdot (3k+2) = 1 + 3k + k' \cdot (3k-1)$$
$$> 1 + 3k' + k' \cdot (3k-1) \geq \#_{\Delta \succ \Gamma}.$$

This again contradicts our assumption $\#_{\Delta \succ \Gamma} \geq \#_{\Gamma \succ \Delta}$.

Since we have $k' = k$, for $\#_{\Delta \succ \Gamma} \geq \#_{\Gamma \succ \Delta}$ to hold, every β_b needs to prefer Δ to Γ. By Claim 5, this is only possible if every β_b has some ζ_S with $b \in S$ in their coalition in Δ. This implies that $\{S \in \mathscr{S} \mid Q_S \notin \Delta\}$ is an exact cover of B. □

For strict popularity in AL AHGs, we can use the same construction but have to modify our arguments appropriately.

Theorem 2 *SP-Verification is coNP-complete for AL AHGs.*

Proof Consider the construction from the proof of Theorem 1, again with the network of friends shown in Fig. 1. Only some details in the proof of correctness are different when considering AL instead of EQ. We again start our proof by stating some claims. Claims 1, 2, 3, and 5 from the proof of Theorem 1 also hold for AL preferences:

- α_1 prefers A to every other coalition;
- for every $S \in \mathscr{S}$ and $\ell \in [3k+1]$, $\gamma_{S,\ell}$ prefers Q_S to every other coalition;
- for every $h \in [12k^3]$, φ_h prefers $\{\varphi_h\}$ to every other coalition; and
- if β_b prefers Δ to Γ, then $\zeta_S \in \Delta(\beta_b)$ for some $S \in \mathscr{S}$ with $b \in S$.

In addition, we now have the following claim.

Claim 6 *For $S \in \mathscr{S}$, ζ_S prefers $\{\zeta_S, \delta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1}\}$ and every coalition $\{\zeta_S, \delta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1}\} \setminus \{\gamma_{S,\ell}\}$, $\ell \in [3k+1]$, to Q_S, and ζ_S prefers Q_S to every other coalition.*

Proof of Claim 6 For $C = \{\zeta_S, \delta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1}\}$, it holds that $\mathrm{avg}^F_{\zeta_S}(C) = v_{\delta_S}(C) = n(3k+2)$; for $C_\ell = C \setminus \{\gamma_{S,\ell}\}$ with $\ell \in [3k+1]$, it holds that $\mathrm{avg}^F_{\zeta_S}(C_\ell) = v_{\delta_S}(C_\ell) = n(3k+1)$; and for Q_S, we have

$$\begin{aligned}\mathrm{avg}^F_{\zeta_S}(Q_S) =& \frac{(3k-2)v_{\eta_{S,j}}(Q_S) + v_{\delta_S}(Q_S)}{3k-1} \\ =& \frac{(3k-2)(n(3k-1)-(3k+1)) + n(6k)}{3k-1} \\ =& \frac{n(9k^2-3k+2)-(9k^2-3k-2)}{3k-1} \\ =& n\left(3k + \frac{2}{3k-1}\right) - \left(3k - \frac{2}{3k-1}\right).\end{aligned}$$

Thus ζ_S prefers C to every C_ℓ, $\ell \in [3k+1]$, and every C_ℓ to Q_S.

Now, let D with $D \neq C$ and $D \neq C_\ell$ for $\ell \in [3k+1]$ be a coalition that ζ_S weakly prefers to every coalition except for C and C_ℓ. We will show that $D = Q_S$. Similarly as in the proof Claim 4, it follows that $\{\zeta_S, \delta_S\} \subseteq D \subseteq Q_S$. (We omit the details because this proof is very similar.) Now, let $x = |D \cap \{\gamma_{S,1}, \dots, \gamma_{S,3k+1}\}|$ be the number of γ-players in D and let $y = |D \cap \{\eta_{S,1}, \dots, \eta_{S,3k-2}\}|$ be the number of η-players in D.

First, assume $y = 0$. Then $\mathrm{avg}^F_{\zeta_S}(D) = v_{\delta_S}(D) = n(x+1)$. Since ζ_S weakly prefers D to Q_S, we know that $\mathrm{avg}^F_{\zeta_S}(D) \geq \mathrm{avg}^F_{\zeta_S}(Q_S)$, i.e., $n(x+1) \geq n\left(3k + \frac{2}{3k-1}\right) - \left(3k - \frac{2}{3k-1}\right)$. This implies $x \geq 3k$. This is a contradiction because it implies that $D = C$ or $D = C_\ell$ for some $\ell \in [3k+1]$. Thus we have $y \geq 1$.

By Observation 1, $\{\gamma_{S,1}, \dots, \gamma_{S,3k+1}\} \subseteq D$; otherwise, ζ_S would prefer $D' = D \cup \{\gamma_{S,1}, \dots, \gamma_{S,3k+1}\}$ to D. This would be a contradiction to ζ_S weakly preferring D to every coalition except for C and C_ℓ. (Note that $D' \neq C$ and $D' \neq C_\ell$ because of $y \geq 1$.) It holds that

$$\begin{aligned}\mathrm{avg}^F_{\zeta_S}(D) &= \frac{y v_{\eta_{S,j}}(D) + v_{\delta_S}(D)}{y+1} \\ &= \frac{y\big(n(y+1)-(3k+1)\big) + n(y+1+3k+1)}{y+1} \\ &= n\left(\frac{y^2+2y+3k+2}{y+1}\right) - \frac{y(3k+1)}{y+1}.\end{aligned}$$

Now, rearranging $\mathrm{avg}^F_{\zeta_S}(D) \geq \mathrm{avg}^F_{\zeta_S}(Q_S)$, the difference

$$n\left(\frac{(3k-y-2)(3ky-y-2)}{(y+1)(3k-1)}\right) - \frac{(3k-y-2)(3k+1)}{(y+1)(3k-1)}$$

cannot be positive. It follows that

$$0 \geq \frac{(3k-y-2)(3ky-y-2)}{(y+1)(3k-1)}.$$

Since $k \geq 2$ and $y \geq 1$, this implies that $0 \geq 3k - y - 2$, i.e., $y \geq 3k-2$. Hence, we have $D = Q_S$. □ Claim 6

We now show that there is an exact cover of B if and only if Γ is not strictly popular under AL preferences.

Only if: Assume that there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ of B. As in the proof of Theorem 1, consider the coalition structure

$$\Delta = \left\{ \{\varphi_1\}, \dots, \{\varphi_{12k^3}\}, A \cup \bigcup_{S \in \mathscr{S}'} Q_S\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}' \right\}.$$

We will show that Δ and Γ are equally popular under AL preferences.

Omitting the detailed calculations, for $S \in \mathscr{S}'$ and $j \in [3k-2]$, we have

$$\begin{aligned}
\mathrm{avg}^F_{\alpha_2}(\Delta) =& \mathrm{avg}^F_{\alpha_2}(\Gamma) + \frac{3k(n+1) - (3k+1)k(6k+1)}{3k+1};\\
\mathrm{avg}^F_{\eta_{S,j}}(\Delta) =& \mathrm{avg}^F_{\eta_{S,j}}(\Gamma) + \frac{3n+3-(3k-1)\big(3k+2+(k-1)(6k+1)\big)}{3k-1};\\
\mathrm{avg}^F_{\delta_S}(\Delta) =& \mathrm{avg}^F_{\delta_S}(\Gamma) + \frac{3n+3-6k\big(3k+2+(k-1)(6k+1)\big)}{6k}.
\end{aligned}$$

Also, using the preceding claims, it then follows that

$$\begin{aligned}
\#_{\Delta \succ \Gamma} &= |\{\alpha_2, \beta_1, \dots, \beta_{3k}\} \cup \{\eta_{S,1}, \dots, \eta_{S,3k-2}, \delta_S \mid S \in \mathscr{S}'\}|\\
&= 1 + 3k + k(3k-1) = 1 + k(3k+2)\\
&= |\{\alpha_1\} \cup \{\zeta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1} \mid S \in \mathscr{S}'\}| = \#_{\Gamma \succ \Delta}.
\end{aligned}$$

Hence, Γ is not strictly popular.

If: Assume that Γ is not strictly popular, i.e., that there is a coalition structure $\Delta \neq \Gamma$ with $\#_{\Delta \succ \Gamma} \geq \#_{\Gamma \succ \Delta}$.

First, note that there is no $S \in \mathscr{S}$ with $\{\zeta_S, \delta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1}\}$ in Δ or with $\{\zeta_S, \delta_S, \gamma_{S,1}, \dots, \gamma_{S,3k+1}\} \setminus \{\gamma_{S,\ell}\}$ in Δ for any $\ell \in [3k+1]$: Indeed, for the sake of contradiction, assume that there is such an $S \in \mathscr{S}$. Then ζ_S prefers Δ to Γ; and $\eta_{S,j}$, δ_S, and $\gamma_{S,\ell}$, $j \in [3k-2]$ and $\ell \in [3k+1]$, prefer Γ to Δ. For ζ_S and $\gamma_{S,\ell}$, this follows from the preceding claims. For $\eta_{S,j}$ and δ_S, this can be shown by direct calculations. In more detail, for δ_S, we have

$$\begin{aligned}
\text{avg}^F_{\delta_S}(\Gamma) &= \frac{v_{\zeta_S}(\Gamma) + (3k-2)v_{\eta_{S,j}}(\Gamma) + (3k+1)v_{\gamma_{S,\ell}}(\Gamma)}{6k} \\
&= \frac{(3k-1)(n(3k-1)-(3k+1)) + (3k+1)(n-(6k-1))}{6k} \\
&= \frac{n(9k^2-3k+2) - (27k^2+3k-2)}{6k} \\
&= n(\frac{3}{2}\cdot k - \frac{1}{2} + \frac{1}{3k}) - (\frac{27}{6}\cdot k + \frac{1}{2} - \frac{1}{3k}); \\
\text{avg}^F_{\delta_S}(C) &= \frac{v_{\zeta_S}(C) + (3k+1)v_{\gamma_{S,\ell}}(C)}{3k+2} \\
&= \frac{n-(3k+1) + (3k+1)(n-(3k+1))}{3k+2} \\
&= n-(3k+1) \\
&< \text{avg}^F_{\delta_S}(\Gamma); \text{ and} \\
\text{avg}^F_{\delta_S}(C_\ell) &= \frac{v_{\zeta_S}(C_\ell) + (3k)v_{\gamma_{S,\ell}}(C_\ell)}{3k+1} = \frac{n-3k+(3k)(n-3k)}{3k+1} = n-3k \\
&< \text{avg}^F_{\delta_S}(\Gamma).
\end{aligned}$$

Further, for any $\eta_{S,j}$ with $j \in [3k-2]$, we have

$$\text{avg}^F_{\eta_{S,j}}(\Gamma) = \text{avg}^F_{\zeta_S}(\Gamma) = n\left(3k + \frac{2}{3k-1}\right) - \left(3k - \frac{2}{3k-1}\right).$$

If C or C_ℓ is in Δ, then the best coalition that could form for $\eta_{S,j}$ is $\{\eta_{S,1}, \ldots, \eta_{S,3k-2}\}$. Hence,

$$\text{avg}^F_{\eta_{S,j}}(\Delta) \leq \frac{(3k-3)(n(3k-3))}{3k-3} = n(3k-3) < \text{avg}^F_{\eta_{S,j}}(\Gamma).$$

It follows that all δ_S and $\eta_{S,j}$ with $j \in [3k-2]$ prefer Γ to Δ.

For $T \in \mathscr{S}$, let $\#^{\text{in } Q_T}_{\Delta \succ \Gamma}$ denote the number of players in Q_T that prefer Δ to Γ, and let $\#^{\text{in } Q_T}_{\Gamma \succ \Delta}$ denote the number of players in Q_T that prefer Γ to Δ. Then $\#^{\text{in } Q_S}_{\Delta \succ \Gamma} = 1$ and $\#^{\text{in } Q_S}_{\Gamma \succ \Delta} = 6k$. For all other $S' \in \mathscr{S}$, it holds that $\#^{\text{in } Q_{S'}}_{\Delta \succ \Gamma} \leq 3k$ and $\#^{\text{in } Q_{S'}}_{\Gamma \succ \Delta} \geq 3k+1$ if $Q_{S'} \notin \Delta$; and $\#^{\text{in } Q_{S'}}_{\Delta \succ \Gamma} = \#^{\text{in } Q_{S'}}_{\Gamma \succ \Delta} = 0$ if $Q_{S'} \in \Delta$. This means that, in $Q_{S'}$, at least as many players prefer Γ to Δ as the other way around. Then only the players in $\{\alpha_2, \beta_1, \ldots, \beta_{3k}\}$ could prefer Δ to Γ. However, since $\#^{\text{in } Q_S}_{\Gamma \succ \Delta} = 6k$, this means that $\#_{\Gamma \succ \Delta} > \#_{\Delta \succ \Gamma}$, a contradiction.

The remainder of the proof proceeds identically to the *If*-part in the proof of Theorem 1. □

From Theorems 1 and 2, we get the following corollary.

Corollary 1 *SP-Existence is coNP-hard for EQ and AL AHGs.*

Proof We use the same reduction as in the proof of Theorem 1 but do not give any coalition structure as a part of the instance. It holds that there exists a strictly popular coalition structure for the defined game if and only if there is no exact cover of B. The correctness of this equivalence follows from the proofs of Theorems 1 and 2. Indeed, Γ as defined in the proof of Theorem 1 is strictly popular under EQ and AL preferences if there is no exact cover. If, on the other hand, there does exist an exact cover, then Δ as defined in the proof

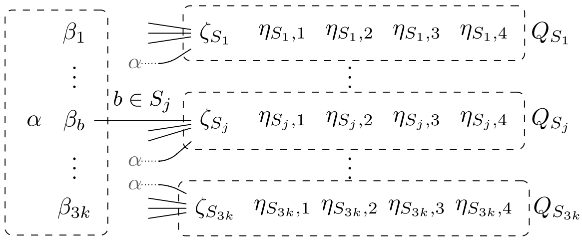

Fig. 2 Network of friends in the proof of Theorem 3. A dashed rectangle indicates that all players inside are friends of each other

of Theorem 1 is as popular as Γ while there is still no coalition structure that is more popular than Γ. Hence, no strictly popular coalition structure can exist in this case. □

Corollary 1 establishes a lower bound on the complexity of SP-Existence. Yet, we suspect that this bound is not tight (i.e., SP-Existence is not in coNP) but SP-Existence might even be hard for the complexity class Σ_2^p.

4 Verifying popularity in AHGs and MBAHGs

Now, we provide the first complexity results for P-Verification in AHGs and MBAHGs, and we cover for both all three degrees of altruism. As mentioned earlier, Nguyen et al. [13, Theorem 12] showed that SP-Verification is coNP-complete for SF AHGs and Wiechers and Rothe [14, Theorem 4] showed the same result for SF MBAHGs. We modify their proofs to establish the same results for P-Verification.

Theorem 3 *P-Verification is coNP-complete for SF AHGs and SF MBAHGs.*

Proof The proof of this theorem, which is the same for SF AHGs and SF MBAHGs, is inspired by the proofs of Nguyen et al. [13, Theorem 12] and Wiechers and Rothe [14, Theorem 4] for SP-Verification. Given an instance $(B, \mathscr{S})$ of RX3C, with $B = \{1, \dots, 3k\}$ and $\mathscr{S} = \{S_1, \dots, S_{3k}\}$, we construct the network of friends shown in Fig. 2 with the set of players $N = \{\alpha\} \cup \{\beta_b \mid b \in B\} \cup \bigcup_{S\in\mathscr{S}} Q_S$, where $Q_S = \{\zeta_S, \eta_{S,j} \mid j \in [4]\}$ for $S \in \mathscr{S}$, and we define the coalition structure $\Gamma = \{\{\alpha, \beta_1, \dots, \beta_{3k}\}, Q_{S_1}, \dots, Q_{S_{3k}}\}$. Specifically, the friendship relationships are:

- All players in $\{\alpha\} \cup \{\beta_b \mid b \in B\}$ are friends.
- For $S \in \mathscr{S}$, ζ_S is friends with α and all β_b with $b \in S$.
- For $S \in \mathscr{S}$, all players in Q_S are friends of each other.

We show that Γ is not popular under SF preferences if and only if there exists an exact cover of B in $\mathscr{S}$.

If: Assume that there exists an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ of B. Then the coalition structure

$$\Delta = \left\{ \{\alpha, \beta_1, \dots, \beta_{3k}\} \cup \bigcup_{S\in\mathscr{S}'} Q_S\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}'\} \right\}$$

is more popular than Γ:

$$\#_{\Delta \succ \Gamma} = |\{\alpha, \beta_1, \dots, \beta_{3k}\} \cup \{\zeta_S \mid S \in \mathscr{S}\}| = 1 + 3k + k$$
$$> 4k = |\{\eta_{S,j} \mid S \in \mathscr{S}, j \in [4]\}| = \#_{\Gamma \succ \Delta}.$$

Hence, Γ is not popular.

Only if: Assume that Γ is not popular, so there is a coalition structure $\Delta \neq \Gamma$ with $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$. First observe, for any $S \in \mathscr{S}$ and $j \in [4]$, that Q_S is $\eta_{S,j}$'s unique most preferred coalition, as it contains all of their friends and none of their enemies. Thus $\eta_{S,j}$ prefers Γ to Δ if $Q_S \notin \Delta$, and is indifferent between them if $Q_S \in \Delta$.

Now, let $k' = |\{S \in \mathscr{S} \mid Q_S \notin \Delta\}|$ be the number of sets Q_S that are not a coalition in Δ.

Assume that $k' > k$. Then

$$\#_{\Gamma \succ \Delta} \geq |\{\eta_{S,j} \mid Q_S \notin \Delta, j \in [4]\}| = 4k' \text{ and}$$
$$\#_{\Delta \succ \Gamma} \leq |\{\alpha, \beta_1, \dots, \beta_{3k}\} \cup \{\zeta_S \mid Q_S \notin \Delta\}| = 3k + 1 + k' < 4k' + 1.$$

Since $\#_{\Delta \succ \Gamma}$ is integral, this implies $\#_{\Gamma \succ \Delta} \geq 4k' \geq \#_{\Delta \succ \Gamma}$, a contradiction. Hence, $k' \leq k$.

Next, assume that $k' < k$. For any $b \in B$, observe that $\Gamma(\beta_b) = \{\alpha, \beta_1, \dots, \beta_{3k}\}$ is a clique. Hence, β_b can only prefer Δ to Γ if there are at least $3k + 1$ of β_b's friends in $\Delta(\beta_b)$, i.e., there is at least one ζ_S with $b \in S$ in $\Delta(\beta_b)$. Since there are k' ζ_S available (with $Q_S \notin \Delta$), there thus are at most $3k'$ β-players who prefer Δ to Γ. All other β-players (at least $3k - 3k'$) prefer Γ to Δ. Note that they are not indifferent between the two coalition structures: They would only be indifferent if $\{\alpha, \beta_1, \dots, \beta_{3k}\} \in \Delta$. However, this is not possible as it would imply that Δ is not more popular than Γ. We now have

$$\begin{aligned}
\#_{\Gamma \succ \Delta} \geq& |\{\eta_{S,j} \mid Q_S \notin \Delta, j \in [4]\} \cup \{\beta_b \mid \text{there is no } Q_S \notin \Delta \text{ with } b \in S\}| \\
=& 4k' + 3k - 3k' = 3k + k' > 4k' \text{ and} \\
\#_{\Delta \succ \Gamma} \leq& |\{\alpha\} \cup \{\beta_b \mid \text{there is an } Q_S \notin \Delta \text{ with } b \in S\} \cup \{\zeta_S \mid Q_S \notin \Delta\}| \\
=& 1 + 3k' + k' = 4k' + 1.
\end{aligned}$$

Since $\#_{\Gamma \succ \Delta}$ is integral, this implies $\#_{\Gamma \succ \Delta} \geq 4k' + 1 \geq \#_{\Delta \succ \Gamma}$, which is a contradiction. Thus we have $k' = k$.

Now, since exactly $4k$ η-players prefers Γ to Δ and because of $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$, $4k + 1$ players need to prefer Δ to Γ. Thus α, all β_b with $b \in B$, and all ζ_S with $Q_S \notin \Delta$ prefer Δ to Γ. As observed earlier, this means that every β_b has some ζ_S with $b \in S$ in their coalition in Δ. Since there are $3k$ β-players and k ζ_S with $Q_S \notin \Delta$, this implies that $\{S \in \mathscr{S} \mid Q_S \notin \Delta\}$ is an exact cover of B. □

Since the altruistic tie-breaker is never used in the construction of Theorem 3, we get the following corollary.

Corollary 2 *P-Verification is coNP-complete for friend-oriented hedonic games.*

With a slight adaptation of the construction in the proof of Theorem 1 we can show the following theorem.

Theorem 4 *P-Verification is coNP-complete for EQ AHGs and AL AHGs.*

Proof Consider the same construction as in the proof of Theorem 1 but delete player α_1 who under EQ and AL preferred Γ to the equally popular coalition structure Δ. Then

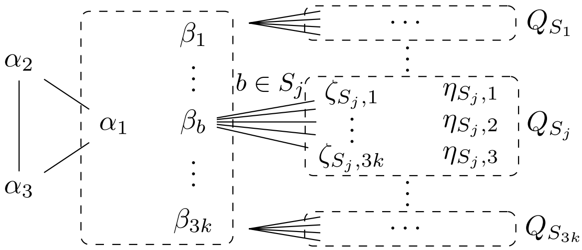

Fig. 3 Network of friends in the proof of Theorem 5. A dashed rectangle indicates that all players inside are friends of each other

$\Gamma' = \{\{\varphi_1\}, \dots, \{\varphi_{12k^3}\}, A \setminus \{\alpha_1\}, Q_{S_1}, \dots, Q_{S_{3k}}\}$ is not popular if and only if there is an exact cover of B.[3] The proof is analogous to the proofs of Theorems 1 and 2. □

Wiechers and Rothe [14] showed that SP-Verification is coNP-complete for EQ MBAHGs. We substantially modify their proof to establish the same result for P-Verification.

Theorem 5 *P-Verification is coNP-complete for EQ MBAHGs.*

Proof The proof of this theorem is inspired by proofs of Wiechers and Rothe [14, Theorem 4] and Kerkmann and Rothe [26, Theorem 7]. Given an instance of $(B, \mathscr{S})$ of RX3C, with $B = \{1, \dots, 3k\}$ and $\mathscr{S} = \{S_1, \dots, S_{3k}\}$, we construct the network of friends shown in Fig. 3 with the set of players $N = \{\alpha_1, \alpha_2, \alpha_3\} \cup \{\beta_b \mid b \in B\} \cup \bigcup_{S \in \mathscr{S}} Q_S$, where $Q_S = \{\zeta_{S,\ell}, \eta_{S,j} \mid \ell \in [3k], j \in [3]\}$ for $S \in \mathscr{S}$, and we define the coalition structure

$$\Gamma = \{\{\alpha_2, \alpha_3\}, \{\alpha_1, \beta_1, \dots, \beta_{3k}\}, Q_{S_1}, \dots, Q_{S_{3k}}\}.$$

The friendship relationships are as follows:

- α_1, α_2, and α_3 are friends of each other.
- All players in $\{\alpha_1\} \cup \{\beta_b \mid b \in B\}$ are friends.
- For $S \in \mathscr{S}$ and $\ell \in [3k]$, $\zeta_{S,\ell}$ is friends with the three β_b with $b \in S$.
- For $S \in \mathscr{S}$, all players in Q_S are friends of each other.

We show that Γ is not popular under min-based EQ preferences if and only if there exists an exact cover of B in $\mathscr{S}$.

If: Assume that there exists an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ of B. Then, for the coalition structure

$$\begin{aligned}\Delta = &\{\{\alpha_1, \alpha_2, \alpha_3\}\} \cup \{\{\beta_b \mid b \in S\} \cup \{\zeta_{S,1}, \dots, \zeta_{S,3k}\} \mid S \in \mathscr{S}'\} \cup \\ &\{\{\eta_{S,1}, \eta_{S,2}, \eta_{S,3}\} \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}'\},\end{aligned}$$

it holds that

[3] Specifically, $\Delta' = \{\{\varphi_1\} \dots, \{\varphi_{12k^3}\}, (A \setminus \{\alpha_1\}) \cup \bigcup_{S \in \mathscr{S}'} Q_S\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}'\}$ is more popular than Γ' (by one player) if there is an exact cover $\mathscr{S}'$ for B.

- α_2 and α_3 prefer Δ to Γ, as they are in a clique of size three in Δ but in a clique of size two in Γ;
- all β_b with $b \in B$ prefer Δ to Γ, as they are in a clique of size $3k+3$ in Δ but in a clique of size $3k+1$ in Γ;
- α_1 prefers Γ to Δ, as α_1 is in a clique of size $3k+1$ in Γ but in a clique of size three in Δ;
- all $\eta_{S,j}$ with $S \in \mathscr{S}$ and $j \in [3]$ prefer Γ to Δ, as they are in a clique of size $3k+3$ in Γ but in a clique of size three in Δ; and
- all remaining players are indifferent between Γ and Δ, as they are in cliques of the same size in both coalition structures.

So, we have

$$\begin{aligned}\#_{\Delta \succ \Gamma} &= |\{\alpha_2, \alpha_3, \beta_1, \dots, \beta_{3k}\}| = 3k+2 > 3k+1 = 3|\mathscr{S}'|+1 \\ &= |\{\alpha_1\} \cup \{\eta_{S,1}, \eta_{S,2}, \eta_{S,3} \mid S \in \mathscr{S}'\}| = \#_{\Gamma \succ \Delta}.\end{aligned}$$

Hence, Γ is not popular.

Only if: Assume that there is a coalition structure $\Delta \neq \Gamma$ with $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$. Then the following four claims hold from which we will be able to conclude that there exists an exact cover of B.

Claim 7 *For $S \in \mathscr{S}$ and $j \in [3]$, $\eta_{S,j}$ prefers Q_S to every other coalition.*

Proof of Claim 7 Since Q_S is a clique of size $3k+3$, it holds that $u^{minEQ}_{\eta_{S,j}}(Q_S) = n(3k+2)$. As every $\eta_{S,j}$ has only $3k+2$ friends in total, Q_S is the only clique of size $3k+3$ that can reach this utility for $\eta_{S,j}$. Every other coalition $C \in \mathcal{N}_{\eta_{S,j}}$ either contains fewer friends or more enemies of $\eta_{S,j}$'s than Q_S, which leads to a decrease in utility for $\eta_{S,j}$. □ Claim 7

Claim 8 *For $S \in \mathscr{S}$ and $\ell \in [3k]$, $\zeta_{S,\ell}$ has exactly two most preferred coalitions: Q_S and $\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$.*

Proof of Claim 8 Since the coalitions

$$A = Q_S \quad \text{and} \quad B = \{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$$

are cliques of size $3k+3$, it holds that $u^{minEQ}_{\zeta_{S,\ell}}(A) = u^{minEQ}_{\zeta_{S,\ell}}(B) = n(3k+2)$. For a contradiction, assume that there is another coalition C with $u^{minEQ}_{\zeta_{S,\ell}}(C) \geq n(3k+2)$.

In case of $u^{minEQ}_{\zeta_{S,\ell}}(C) = n(3k+2)$, C would be a clique of size $3k+3$. However, there are no other cliques of size $3k+3$ containing $\zeta_{S,\ell}$ besides A and B.

In case of $u^{minEQ}_{\zeta_{S,\ell}}(C) > n(3k+2)$, $\zeta_{S,\ell}$ and all their friends each need to have at least $3k+3$ friends in C. Each η-player has only $3k+2$ friends in total and thus cannot be part of C. However, without the η-players, $\zeta_{S,\ell}$ has only $3k+2$ friends in total. Thus we have a contradiction. □ Claim 8

Claim 9 $\{\alpha_1, \beta_1, \dots, \beta_{3k}\}$ *is α_1's unique most preferred coalition.*

Proof of Claim 9 For coalition $A = \{\alpha_1, \beta_1, \dots, \beta_{3k}\}$, it holds that $u^{minEQ}_{\alpha_1}(A) = n3k$. If there were another coalition $B \neq A$ with $u^{minEQ}_{\alpha_1}(B) \geq u^{minEQ}_{\alpha_1}(A) = n3k$, α_1 would have at least $3k$

friends in B and all these friends would also have at least $3k$ friends in B. Since α_2 and α_3 have only two friends in total, it holds that $\alpha_2 \notin B$ and $\alpha_3 \notin B$. However, α_1's remaining $3k$ friends are β-players, which implies $A = \{\alpha_1, \beta_1, \dots, \beta_{3k}\} \subseteq B$. Since any additional ζ- or η-player in B would contradict $u^{minEQ}_{\alpha_1}(B) \geq n3k$, we get $A = B$, which also is a contradiction. □ Claim 9

Note that Claims 7, 8, and 9 imply that there is no coalition structure that any of $\eta_{S,j}$, $\zeta_{S,\ell}$, or α_1 prefers to Γ. The β-players, however, do prefer some coalition structures to Γ, and the following claim says which coalitions of such preferred structures they are in.

Claim 10 *For any $b \in B$, if β_b prefers Δ to Γ, then $\Delta(\beta_b) = \{\beta_a \mid a \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$ for some $S \in \mathscr{S}$ with $b \in S$.*

Proof of Claim 10 Assume that $\Delta \succ^{minEQ}_{\beta_b} \Gamma$. Since $\Gamma(\beta_b) = \{\alpha_1, \beta_1, \dots, \beta_{3k}\}$ is a clique, it follows that β_b has a friend in $\Delta(\beta_b)$ that is not in $\Gamma(\beta_b)$. Hence, there is a $\zeta_{S,\ell}$ with $b \in S$ in $\Delta(\beta_b)$.

Now assume that $\Delta(\beta_b) \neq \{\beta_a \mid a \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$. Then $\zeta_{S,\ell} \in \Delta(\beta_b)$ together with Claim 8 implies that all $\zeta_{S,\ell'}$ with $\ell' \in [3k]$ prefer Γ to Δ. Further, Claim 7 implies that $\eta_{S,j}$, $j \in [3]$, prefer Γ to Δ. Hence, we have $\#_{\Gamma \succ \Delta} \geq 3k + 3$. From Claims 7, 8, and 9 we know that no η, ζ, or α_1 can prefer Δ to Γ. Hence, $\#_{\Delta \succ \Gamma} \leq |\{\alpha_2, \alpha_3, \beta_1, \dots, \beta_{3k}\}| = 3k + 2$. We get $\#_{\Gamma \succ \Delta} > \#_{\Delta \succ \Gamma}$, which is a contradiction. □ Claim 10

Now, since $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$, there is a player $i \in N$ who prefers Δ to Γ. By the previous claims, this player i can only be either a β-player or one of α_2 and α_3. Accordingly, we distinguish the following two cases, and in the first case (that i is a β-player) we will show that there exists an exact cover of B, while the second case (that i is either α_2 or α_3) must reduce to the first case.

Case 1 $i = \beta_c$ for some $c \in B$. Let $k' = |\{S \in \mathscr{S} \mid Q_S \notin \Delta\}|$. Then, by Claim 7, there are $3k'$ η-players who prefer Γ to Δ and the remaining η-players are indifferent between Γ and Δ. Since β_c prefers Δ to Γ, we know by Claim 10 that $\Delta(\beta_c) = \{\beta_a \mid a \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$ for some $S \in \mathscr{S}$ with $c \in S$. Thus, by Claim 9, α_1 prefers Γ to Δ.

We will now see that $k' = k$.

First, assume that $k' > k$. Then

$$\#_{\Gamma \succ \Delta} \geq |\{\alpha_1\} \cup \{\eta_{S,j} \mid Q_S \notin \Delta, j \in [3]\}| = 3k' + 1 > 3k + 1 \quad \text{and}$$
$$\#_{\Delta \succ \Gamma} \leq |\{\alpha_2, \alpha_3, \beta_1, \dots, \beta_{3k}\}| = 3k + 2.$$

Hence, $\#_{\Gamma \succ \Delta} \geq 3k + 2 \geq \#_{\Delta \succ \Gamma}$, which contradicts $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$.

Second, assume that $k' < k$. Per one $Q_S \notin \Delta$, there are at most three β_b with $b \in S$ who prefer Δ to Γ (see Claim 10). Hence,

$$\#_{\Delta \succ \Gamma} \leq |\{\alpha_2, \alpha_3\} \cup \{\beta_b \mid b \in S, Q_S \notin \Delta\}| = 3k' + 2 < 3k + 2.$$

All remaining $3k - 3k'$ players β_b do not have any $\zeta_{S,\ell}$ with $b \in S$ in $\Delta(\beta_b)$ and thus prefer Γ to Δ. We get

Table 1 Utilities of the players $\alpha_1, \alpha_2, \alpha_3, \beta_b$ for $b \in B$, and $\zeta_{S,\ell}$ and $\eta_{S,j}$ for $S \in \mathscr{S}', \ell \in [3k]$, and $j \in [3]$ in the proof of Theorem 6

Player i	$u_i^{minAL}(\Gamma)$		$u_i^{minAL}(\Delta)$
α_1	$M \cdot n3k + n3k$	$>$	$M \cdot 2n + 2n$
α_2, α_3	$M \cdot n + n$	$<$	$M \cdot 2n + 2n$
β_b	$M \cdot n3k + n3k$	$<$	$M \cdot n(3k+2) + n(3k+2)$
$\zeta_{S,\ell}$	$M \cdot n(3k+2) + n(3k+2)$	$=$	$M \cdot n(3k+2) + n(3k+2)$
$\eta_{S,j}$	$M \cdot n(3k+2) + n(3k+2)$	$>$	$M \cdot 2n + 2n$

$$\begin{aligned}\#_{\Gamma>\Delta} \geq & |\{\alpha_1\} \cup \{\eta_{S,j} \mid Q_S \notin \Delta, j \in [3]\} \cup \\ & \{\beta_b \mid Q_S \in \Delta \text{ for all } S \in \mathscr{S} \text{ with } b \in S\}| \\ \geq & 1 + 3k' + 3k - 3k' = 3k + 1.\end{aligned}$$

Hence, $\#_{\Gamma>\Delta} \geq 3k + 1 \geq \#_{\Delta>\Gamma}$, which again is a contradiction.

It follows that $k' = k$ and thus

$$\#_{\Gamma>\Delta} \geq |\{\alpha_1\} \cup \{\eta_{S,j} \mid Q_S \notin \Delta, j \in [3]\}| = 3k + 1.$$

Hence, since $\#_{\Delta>\Gamma} > \#_{\Gamma>\Delta}$, there are at least $3k+2$ players preferring Δ to Γ, which can only be α_2, α_3, and all β_b, $b \in B$. Then, by Claim 10, every β_b is in a coalition $\Delta(\beta_b) = \{\beta_a \mid a \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$ for some $S \in \mathscr{S}$ with $b \in S$. This implies that $\{S \in \mathscr{S} \mid Q_S \notin \Delta\}$ is an exact cover of B.

Case 2: $i = \alpha_2$ or $i = \alpha_3$. Since α_2 or α_3 prefer Δ to Γ, it follows that $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq \Delta(\alpha_2)$. Then, considering only the α-players, we have $\#_{\Delta>\Gamma} \geq 2$ and $\#_{\Gamma>\Delta} \geq 1$. If at least one β_b prefers Δ to Γ, we are in Case 1 and an exact cover of B is already implied. Hence, assume that there is no β_b that prefers Δ to Γ. Then $\#_{\Delta>\Gamma} = 2$ because, by Claims 7 and 8, no other player can prefer Δ to Γ. With $\#_{\Delta>\Gamma} > \#_{\Gamma>\Delta}$, it follows that no player β_b, $\zeta_{S,\ell}$, nor $\eta_{S,j}$ prefers Γ to Δ. Hence, by Claim 7, $Q_S \in \Delta$ for every $S \in \mathscr{S}$. However, this implies that all β_b prefer Γ to Δ, which is a contradiction. □

Wiechers and Rothe [14, Theorem 4] showed that SP-Verification is coNP-complete for AL MBAHGs. We extensively modify their proof to establish the same result for P-Verification.

Theorem 6 *P-Verification is coNP-complete for AL MBAHGs.*

Proof We use the same set of players and network of friends as in the proof of Theorem 5 that is shown in Fig. 3. We again consider coalition structure $\Gamma = \{\{\alpha_1, \beta_1, \ldots, \beta_{3k}\}, \{\alpha_2, \alpha_3\}, Q_{S_1}, \ldots, Q_{S_{3k}}\}$ and show that Γ is not popular under AL preferences if and only if there exists an exact cover of B in $\mathscr{S}$.

If: Assume that there exists an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ of B. Then the coalition structure

$$\begin{aligned}\Delta = & \{\{\alpha_1, \alpha_2, \alpha_3\}\} \cup \{\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\} \mid S \in \mathscr{S}'\} \cup \\ & \{\{\eta_{S,1}, \eta_{S,2}, \eta_{S,3}\} \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}'\}\end{aligned}$$

is more popular than Γ: All players in Q_S with $S \in \mathscr{S} \setminus \mathscr{S}'$ are obviously indifferent between Γ and Δ because their coalitions stay the same. The utilities of the other players are shown in Table 1.

Hence, Γ is not popular because

$$\begin{aligned}\#_{\Delta \succ \Gamma} &= |\{\alpha_2, \alpha_3, \beta_1, \dots, \beta_{3k}\}| = 3k + 2 > 3k + 1 = 3|\mathscr{S}'| + 1 \\ &= |\{\alpha_1\} \cup \{\eta_{S,j} \mid S \in \mathscr{S}', j \in [3]\}| = \#_{\Gamma \succ \Delta}.\end{aligned}$$

Only if: Assume that there is a coalition structure $\Delta \neq \Gamma$ with $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$. Then we can iteratively show and use the following claims.

Claim 11 *For any $S \in \mathscr{S}$ and $\ell \in [3k]$, if $\zeta_{S,\ell}$ prefers Δ to Γ, then $\Delta(\zeta_{S,\ell})$ contains no $\eta_{S,j}$ with $j \in [3]$, no $\zeta_{S,\ell'}$ with $\ell' \in [3k]$ and $\ell' \neq \ell$, at least one β_b with $b \in S$, and $3k + 2$ other friends of β_b's.*

Proof of Claim 11 Assume that $\zeta_{S,\ell}$ prefers Δ to Γ and let $\Delta(\zeta_{S,\ell}) = D$. As $\Gamma(\zeta_{S,\ell}) = Q_S$ is a clique of size $3k + 3$, we have $u^{minAL}_{\zeta_{S,\ell}}(\Delta) > u^{minAL}_{\zeta_{S,\ell}}(\Gamma) = n(3k + 2) + Mn(3k + 2)$. Thus D contains at least one friend of $\zeta_{S,\ell}$'s and every friend of $\zeta_{S,\ell}$'s in D has at least $3k + 3$ friends in D. Since the players $\eta_{S,j}$, $j \in [3]$, each have only $3k + 2$ friends in total, they cannot be part of D. By omitting these players, all $\zeta_{S,\ell'}$, $\ell' \neq \ell$, only have $3k + 2$ friends left and cannot be part of D either. Hence, D contains at least one β_b with $b \in S$, and $3k + 2$ other friends of β_b's. □ Claim 11

Claim 12 *For any $S \in \mathscr{S}$ and $\ell \in [3k]$, if $\zeta_{S,\ell}$ prefers Δ to Γ then at least $3k$ other players in Q_S prefer Γ to Δ.*

Proof of Claim 12 Assume that $\zeta_{S,\ell}$ prefers Δ to Γ. Since there are only three β_b with $b \in S$, we know by Claim 11 that at most two other $\zeta_{S,\ell'}$ with $\ell' \in [3k]$ and $\ell' \neq \ell$ can prefer Δ to Γ at the same time. All other players from Q_S obviously prefer Γ to Δ because they can only stay among themselves in Δ. Thus at least $3k + 3 - 3 = 3k$ players in Q_S prefer Γ to Δ. □ Claim 12

Claim 13 *For any $S \in \mathscr{S}$ and $\ell \in [3k]$, there are exactly two coalitions $A \subseteq N$ with $v^{minAL}_{\zeta_{S,\ell}}(A) = Mn(3k + 2) + n(3k + 2)$, namely Q_S and $\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$.*

Proof of Claim 13 Since Q_S and $\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$ are cliques of size $3k + 3$, the statement is clearly true for them. Every other coalition C with the same valuation would also have to be a clique of size $3k + 3$ containing $\zeta_{S,\ell}$. However, such a clique C does not exist in the given network of friends displayed in Fig. 3. □ Claim 13

Claim 14 *For every $\eta_{S,j}$ with $S \in \mathscr{S}$ and $j \in [3]$, there is no coalition that is in a tie with Q_S.*

Proof of Claim 14 Let $C \subseteq N$ be a coalition containing $\eta_{S,j}$ and satisfying $u^{minAL}_{\eta_{S,j}}(C) = u^{minAL}_{\eta_{S,j}}(Q_S) = M(3k + 2) + n(3k + 2)$. Then it has to contain exactly $3k + 2$ friends of $\eta_{S,j}$'s who are all friends with each other. Since $\eta_{S,j}$ has exactly $3k + 2$ friends, this is clearly determined as Q_S. □ Claim 14

Claim 15 *For any $S \in \mathscr{S}$ and $j \in [3]$, if $\eta_{S,j}$ prefers Δ to Γ then all other players in Q_S prefer Γ to Δ.*

Proof of Claim 15 Assume that $\eta_{S,j}$ prefers Δ to Γ and let $\Delta(\eta_{S,j}) = D$. Then every friend of $\eta_{S,j}$'s in D has at least $3k+3$ friends in D. Thus $\eta_{S,j'} \notin D$ for $j' \in [3]$, $j' \neq j$, since they only have $3k+2$ friends in total. Now there only remain the players $\zeta_{S,\ell}$, $\ell \in [3k]$, which (after omitting the $\eta_{S,j'}$) have exactly $3k+3$ friends left. Thus $\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\} \subseteq D$. As $\eta_{S,j}$ has fewer friends in D than in Q_S, it follows that every $\zeta_{S,\ell}$, $\ell \in [3k]$, prefers Q_S to D. Since the $\eta_{S,j'}$ with $j' \in [3]$, $j' \neq j$, only have themselves left as friends, they clearly also prefer Γ to Δ. □ Claim 15

Claim 16 *If α_2 or α_3 prefer Δ to Γ then α_1 prefers Γ to Δ.*

Proof of Claim 16 Assume that α_2 prefers Δ to Γ. Then α_2 has at least one friend in $\Delta(\alpha_2)$ and every friend of α_2's in $\Delta(\alpha_2)$ has at least two friends in $\Delta(\alpha_2)$. Hence, $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq \Delta(\alpha_2)$ or $\{\alpha_1, \alpha_2, \beta_b\} \subseteq \Delta(\alpha_2)$ for some $b \in B$. In both cases, $u_{\alpha_1}^{minAL}(\Delta) \leq M \cdot v_{\alpha_2}(\Delta) + v_{\alpha_1}(\Delta) \leq M \cdot 2n + v_{\alpha_1}(\Delta) < M \cdot n3k + n3k = u_{\alpha_1}^{minAL}(\Gamma)$. Thus α_1 prefers Γ to Δ. Due to symmetry, the same arguments work if α_3 prefers Δ to Γ. □ Claim 16

Claim 17 *No $\zeta_{S,\ell}$ with $S \in \mathscr{S}$ and $\ell \in [3k]$ prefers Δ to Γ.*

Proof of Claim 17 Assume that some $\zeta_{S,\ell}$ prefers Δ to Γ. Then, by Claim 12, $3k$ players from Q_S prefer Γ to Δ. Further, by Claim 11, $\Delta(\zeta_{S,\ell})$ does not contain any other player from Q_S but does contain a β_b with $b \in S$ and $3k+2$ friends of β_b that are not in Q_S. Then

$$\begin{aligned} u_{\beta_b}^{minAL}(\Delta) &\leq M \cdot v_{\zeta_{S,\ell}}(\Delta) + v_{\beta_b}(\Delta) \leq M \cdot 3n + v_{\beta_b}(\Delta) \\ &< M \cdot n3k + n3k = u_{\beta_b}^{minAL}(\Gamma). \end{aligned}$$

Hence, β_b prefers Γ to Δ. Summing up, we have $\#_{\Gamma \succ \Delta} \geq 3k+1$. With $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$, this implies $\#_{\Delta \succ \Gamma} \geq 3k+2$. Thus there have to be $3k+1$ players, besides $\zeta_{S,\ell}$, who prefer Δ to Γ. Since there are only $3k-1$ β-players left who might prefer Δ to Γ, there have to be at least two other players who prefer Δ to Γ. Because of Claim 16, there can only be two α -players who prefer Δ to Γ if there is also one α-player who prefers Γ to Δ. Hence, in any case, there has to be at least one additional player i of the form $i = \zeta_{S',\ell'}$ or $i = \eta_{S',j'}$ who prefers Δ to Γ. If $i = \zeta_{S,\ell'}$ for some $\ell' \in [3k]$, $\ell' \neq \ell$, then with the same arguments as for $\zeta_{S,\ell}$ there has to be an additional $\beta_{b'}$ who prefers Γ to Δ. If i is from another $Q_{S'}$, $S' \neq S$, then again, by Claim 12 or 15, at least $3k$ further players prefer Γ to Δ. Both cases again imply that there have to be some more ζ- and η-players who prefer Δ to Γ. Inductively, it follows that there are more players who prefer Γ to Δ than vise versa. This is a contradiction. □ Claim 17

Claim 18 *No $\eta_{S,j}$ with $S \in \mathscr{S}$ and $j \in [3]$ prefers Δ to Γ.*

Proof of Claim 18 Assume that some $\eta_{S,j}$ prefers Δ to Γ. Then, by Claim 15, the other $3k+2$ players in Q_S prefer Γ to Δ. Hence, $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$ implies $\#_{\Delta \succ \Gamma} \geq 3k+3$. Since no $\zeta_{S,\ell}$ prefers Δ to Γ by Claim 17 and since not all β- and all α-players can prefer Δ to Γ at the same time (see Claim 16), there is another player $\eta_{S',j'}$ with $S' \neq S$, $j' \in [3]$, who prefers Δ to Γ. However, this again implies that $3k+2$ players from $Q_{S'}$ prefer Γ to Δ. Inductively, there are always more players who prefer Γ to Δ than vise versa, which is a contradiction. □ Claim 18

Claim 19 *α_1 prefers Γ to Δ.*

Proof of Claim 19 First, if α_1 prefers Δ to Γ, then by Claim 16, α_2 and α_3 do not prefer Δ to Γ. Moreover,

$$u_{\alpha_1}^{minAL}(\Delta) > u_{\alpha_1}^{minAL}(\Gamma) = M \cdot n3k + n3k,$$

which means that all friends of α_1's in $\Delta(\alpha_1)$ have at least $3k+1$ friends in $\Delta(\alpha_1)$. Clearly, $\alpha_2 \notin \Delta(\alpha_1)$ and $\alpha_3 \notin \Delta(\alpha_1)$ but there is at least one β_b in $\Delta(\alpha_1)$. Since this β_b needs $3k+1$ friends in $\Delta(\alpha_1)$, there is at least one $\zeta_{S,\ell}$ with $b \in S$ in $\Delta(\alpha_1)$. With Claims 13, 14, 17, and 18, it follows that all $3k+3$ players from Q_S prefer Γ to Δ. Hence,

$$\#_{\Gamma \succ \Delta} \geq |Q_S| = 3k+3 \quad \text{and}$$
$$\#_{\Delta \succ \Gamma} \leq |\{\alpha_1\} \cup \{\beta_1, \dots, \beta_{3k}\}| = 3k+1,$$

contradicting $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$.

Second, if α_1 is indifferent between Γ and Δ, then $u_{\alpha_1}^{minAL}(\Delta) = M \cdot n3k + n3k$, which means that α_1 has exactly $3k$ friends in $\Delta(\alpha_1)$ and all these friends have exactly $3k$ friends in $\Delta(\alpha_1)$. This implies $\Delta(\alpha_1) = \{\alpha_1, \beta_1, \dots, \beta_{3k}\}$. However, this is a contradiction because there is no player left who could prefer Δ to Γ. □ Claim 19

Claim 20 *For every $S \in \mathscr{S}$, either $Q_S \in \Delta$ or $\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\} \in \Delta$.*

Proof of Claim 20 Assume that the statement does not hold for some $S \in \mathscr{S}$. Then, by Claims 13 and 14, no player in Q_S is indifferent between Γ and Δ. By Claims 17 and 18, no player in Q_S prefers Δ to Γ. Thus all $3k+3$ players from Q_S prefer Γ to Δ. Hence,

$$\#_{\Gamma \succ \Delta} \geq |Q_S| = 3k+3 \quad \text{and}$$
$$\#_{\Delta \succ \Gamma} \leq |\{\alpha_2, \alpha_3, \beta_1, \dots, \beta_{3k}\}| = 3k+2,$$

which is a contradiction to $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$. □ Claim 20

Now, we use all these claims to show that the existence of Δ implies the existence of an exact cover of B. Let $k' = |\{S \in \mathscr{S} \mid Q_S \notin \Delta\}|$ (or, equivalently, $k' = |\{S \in \mathscr{S} \mid \{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\} \in \Delta\}|$). It is clear that $k' \geq 1$ because otherwise Δ could not be more popular than Γ. We show that $k' = k$.

First, assume that $k' > k$. Then, by the preceding claims, we have

$$\#_{\Gamma \succ \Delta} \geq |\{\eta_{S,j} \mid Q_S \notin \Delta, j \in [3]\} \cup \{\alpha_1\}| = 3k'+1 > 3k+1 \quad \text{and}$$
$$\#_{\Delta \succ \Gamma} \leq |\{\alpha_2, \alpha_3, \beta_1, \dots, \beta_{3k}\}| = 3k+2.$$

This contradicts $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$.

Second, assume that $k' < k$. All $3k'$ β-players that are in one of the k' coalitions of the form $\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$ prefer Δ to Γ. However, all other $3k - 3k'$ β-players have no ζ -players in their coalitions and thus prefer Γ to Δ. Hence,

$$\#_{\Gamma \succ \Delta} \geq 3k'+1+(3k-3k') = 3k+1 > 3k'+1 \quad \text{and}$$
$$\#_{\Delta \succ \Gamma} \leq 3k'+2.$$

This again contradicts $\#_{\Delta \succ \Gamma} > \#_{\Gamma \succ \Delta}$.

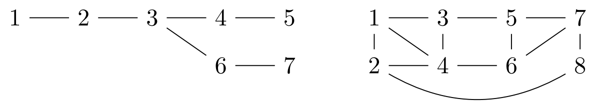

Fig. 4 Networks of friends in Example 1 (left) and Example 2 (right)

Thus $k' = k$. Now, since there are $k' = k$ sets $S \in \mathscr{S}$ such that each $\{\beta_b \mid b \in S\} \cup \{\zeta_{S,\ell} \mid \ell \in [3k]\}$ contains three distinct β_b, $\{S \in \mathscr{S} \mid Q_S \notin \Delta\}$ is an exact cover of B of size k. □

5 Do popular and strictly popular coalition structures exist in AHGs and MBAHGs?

Finally, we turn to P-Existence. Note that we cannot simply modify the preceding theorems in order to show the hardness of P-Existence (similarly to how we used Theorems 1 and 2 to obtain Corollary 1) because, obviously, a tie between two most popular coalition structures would not suffice to show the nonexistence of a popular coalition structure. However, for both AHGs and MBAHGs and all three degrees of altruism, we now provide examples (which were verified by brute force) where no popular coalition structures exist, and we suspect that P-Existence is hard for all six considered models.

Example 1 Under all three degrees of altruism in AHGs, there is no popular coalition structure for the left network of friends in Fig. 4.

Example 2 Under all three degrees of altruism in MBAHGs, there is no popular coalition structure for the right network of friends in Fig. 4.

6 Conclusions and future research

We have solved the two remaining open problems regarding the complexity of strict popularity verification in AHGs, namely for equal treatment (Theorem 1) and altruistic treatment (Theorem 2). The proofs of these results required new ideas and are technically demanding. The corresponding results for MBAHGs have already been established by Wiechers and Rothe [14, Theorem 4]. In addition, we have provided the first complexity results for popularity verification in AHGs and MBAHGs, covering for both all three degrees of altruism (Theorems 3, 4, 5, and 6). Hence, the complexity of popularity verification and strict popularity verification is now settled in AHGs and MBAHGs; the picture is complete.

Moreover, we have seen that our hardness result for popularity verification (Theorem 3) extends to friend-oriented hedonic games. Additionally, we get some implications for classes of hedonic games that generalize AHGs. For instance, since the "super AHGs" by Schlueter and Goldsmith [23] generalize SF AHGs, all our hardness results for SF AHGs extend to this class as well. Also, all our results for EQ MBAHGs carry over to the "loyal variant of symmetric friend-oriented hedonic games" by Bullinger and Kober [25].

For future research, we propose the consideration of restricted classes of AHGs. It would be interesting to find restrictions on the underlying network of friends that guarantee that (strictly) popular coalition structures can be verified in polynomial time. Another important direction for future research is to study the (strict) popularity existence problems. While we have established coNP-hardness, e.g., for strict popularity existence in EQ and AL AHGs (and this is also known for SF AHGS [13]), it is not known whether this bound is tight. We suspect that this is not the case and these problems might in fact be Σ_2^p -complete. Another idea for future research are altruistic hedonic games in which agents may dynamically change their degree of altruism. In such a model, the agents' degree of altruism might depend on the well-being of others. For instance, they might act more altruistically when others are doing worse than themselves, while they are more selfish when others are doing better than themselves. Also, their degree of altruism might depend on the global level of welfare. While a global well-being might not evoke a strong degree of altruism among agents, a severe suffering of their friends might do so.

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Declarations

Conflict of interest While conducting this work, the second author has been on the following editorial boards of scientific journals: *Annals of Mathematics and Artificial Intelligence* (AMAI), Associate Editor, since 01/2020, *Journal of Artificial Intelligence Research* (JAIR), Associate Editor, 09/2017–08/2023, and *Journal of Universal Computer Science* (J.UCS), Editorial Board, since 01/2005.

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