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Sign involutions on para-abelian varieties

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Abstract

We study the so-called sign involutions on twisted forms of abelian varieties, and show that such a sign involution exists if and only if the class in the Weil–Châtelet group is annihilated by two. If these equivalent conditions hold, we prove that the Picard scheme of the quotient is étale and contains no points of finite order. In dimension one, such quotients are Brauer–Severi curves, and we analyze the ensuing embeddings of the genus-one curve into twisted forms of Hirzebruch surfaces and weighted projective spaces.
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Keywords: Para-abelian varieties; Quotients by group actions; Picard schemes

0. Introduction

Recall that an *abelian variety* A over a ground field k is a group scheme that is proper, smooth, and connected. As a non-trivial consequence, the group law is commutative, such that A comes with a canonical automorphism $x \mapsto -x$, the *sign involution*. Note that over the field $k = \mathbb{C}$ of complex numbers, the abelian varieties correspond to complex tori $\mathbb{C}^g/\Lambda$, where Λ is a full lattice admitting a polarization. An excellent exposition of the theory was given by Mumford [23].

Abelian varieties play a fundamental role in algebraic geometry, since they are basic building blocks for algebraic groups. In particular, for every proper scheme X the Picard group, viewed as a group scheme, contains a maximal abelian subvariety $A = \mathrm{Pic}^{\alpha}_{X/k} \subset \mathrm{Pic}_{X/k}$, which encodes

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crucial geometric information ([18], Section 7 and [33], Section 3). For smooth curves X, these are the *jacobian varieties*. Abelian varieties are also important objects in arithmetic geometry, where the ground field could be a number field or a function field. Geometric and arithmetic aspects are strongly interrelated: In fibrations $f : Y \to B$ of proper schemes, one has to understand the generic fiber $X = f^{-1}(\eta)$ as a scheme over the function field $k(B)$ of the base.

The sign involution $\sigma(x) = -x$ on abelian varieties A plays an important role, because it gives rise to the notion of symmetric sheaves. Furthermore, one can form the quotient A/G for the corresponding group $G = \{\pm 1\}$ of order two. In dimension $g = 1$ this gives the projective line, whereas for $g = 2$ we get Kummer surfaces, a fascinating topic going back to the 19th century. In characteristic $p \neq 2$ Kummer surfaces are K3 surfaces with rational double points. The case $p = 2$ requires extra attention, because than A/G may also be a rational surface with an elliptic singularity [14,35]. This is a prime example of a *wild quotient singularity* (see for example [19,20] for more on this topic). To our best knowledge, no resolution of singularities is known in dimension $g \geq 3$.

In this paper we study various aspects of sign involutions, both of arithmetic and geometric nature. Our first goal is to investigate the *existence of sign involutions* σ on twisted forms X of abelian varieties A, over general ground fields k of arbitrary characteristic $p \geq 0$. These σ are involutions on X that become a sign involutions with respect to a suitable group law that arises on some base-change. These varieties are usually introduced as torsors over some abelian variety. The following alternative point of view, developed in [18,33], is most suitable: A *para-abelian variety* is a proper scheme X such that $X \otimes k'$ admits the structure of an abelian variety, for some field extension $k \subset k'$. It then turns out that the subgroup scheme $A \subset \text{Aut}_{X/k}$ that acts trivially on the numerically trivial part $\text{Pic}^{\tau}_{X/k}$ is an abelian variety, and that the canonical A-action on X is free and transitive. In turn, one may view the scheme X as a torsor with respect to the abelian variety A (the traditional point of view), and obtains a class $[X]$ in the *Weil–Châtelet group* $H^1(k, A)$. Our first main result relates these cohomology classes with the kernel $A[2]$ for the multiplication-by-two map and the existence of sign involutions on X:

Theorem (*See Theorem 1.2*). *Let X be a para-abelian variety. Then the following are equivalent:*

(i) *There is a sign involution $\sigma : X \to X$.*
(ii) *We have $2 \cdot [X] = 0$ in the* Weil–Châtelet group $H^1(k, A)$.
(iii) *There is a torsor P with respect to $H = A[2]$ such that $X \simeq P \wedge^H A$.*

Here $P \wedge^H A$ denotes the quotient of $P \times A$ by the diagonal H-action, usually called *contracted product* or *associated fiber bundle*. The main idea for the above result is to introduce the *scheme of sign involutions* $\text{Inv}^{\text{sgn}}_{X/k} \subset \text{Aut}_{X/k}$, analyze the effect of the conjugacy action on this subscheme, and derive consequences using the general machinery of twisted forms and non-abelian cohomology.

We now turn to more geometric aspects: Given an abelian variety A with its standard sign involution $\sigma(x) = -x$, one can form the quotient $B = A/G$ with respect to the cyclic group $G = \{e, \sigma\}$ of order two. This brings us into the realm of *geometric invariant theory*: Locally, the quotient arises from the ring of invariants in suitable coordinate rings for the abelian variety. In characteristic two, not much seems to be known on the resulting proper normal scheme, and it would be highly interesting to construct and understand a resolution of singularities. Our second main result, which is concerned with the numerically trivial part $\text{Pic}^{\tau}_{B/k}$ of the Picard scheme, which could shed some light on the problem:

Theorem (*See Theorem 2.1*)**.** *In the above situation, the group scheme* $\mathrm{Pic}^{\tau}_{B/k}$ *is trivial.*

This relies on Grothendieck's two spectral sequences abutting to equivariant cohomology groups [9]. The result is not difficult in the tame case $p \neq 2$, but requires a careful analysis in the wild case $p = 2$. Also note that the statement immediately carries over to para-abelian varieties. In dimension $g = 1$ the para-abelian varieties X are usually called *genus-one curves*; we like to call them *para-elliptic curves*. These play an important role in the geometry and arithmetic of elliptic surfaces, in particular for bielliptic surfaces, which also go by the name of hyperelliptic surfaces. The above result shows that the quotient by any sign involution is a *Brauer–Severi curve*, that is, a twisted form of $\mathbb{P}^1$.

Our third main result deals with the converse situation: Suppose there is a degree-two morphism $f : X \to B$ from a para-elliptic curve X to some Brauer–Severi curve B. Then the projectivization $S = \mathbb{P}(\mathscr{E})$ of the rank-two sheaf $\mathscr{E} = f_*(\mathscr{O}_X)$ is a twisted form of a Hirzebruch surface with invariant $e = 2$, and comes with a contraction to a normal surface S', having a unique singularity, which is often factorial. The geometry of the situation is as follows:

Theorem (*See Section 3*)**.** *Assumptions as above. Then* $f : X \to B$ *is the quotient by some sign involution* σ *on the para-elliptic curve* X*, and the latter embeds into both surfaces* S *and* S' *as an anti-canonical curve. Moreover,* S' *is the anti-canonical model of* S*, and also a twisted form of the weighted projective space* $\mathbb{P}(1, 1, 2)$*.*

We also show that if there are two different sign involutions $\sigma_1 \neq \sigma_2$, the ensuing diagonal map gives an embedding $X \subset B_1 \times B_2$ into a product of Brauer–Severi curves. Such products were studied by Kollár [15] and Hogadi [13]. Again X becomes an anti-canonical curve, and it turns out that $B_1 \times B_2$ embeds into $\mathbb{P}^3$ if and only if the factors are isomorphic.

The paper is structured as follows: In Section 1 we recall the theory of para-abelian varieties X, introduces the scheme of sign involutions $\mathrm{Inv}^{\mathrm{sgn}}_{X/k} \subset \mathrm{Aut}_{X/k}$, analyze the conjugacy action, and establish the link between sign involutions, cohomology classes, and structure reductions. Section 2 is devoted to the Picard scheme of the quotient $B = A/G$ of an abelian variety A of arbitrary dimension $g \geq 0$ by a sign involution. In Section 3 we consider the case $g = 1$, and unravel the geometry attached to degree-two maps $X \to B$ from a para-elliptic curve X to a Brauer–Severi curve B.

1. The scheme of sign involutions

Let k be a ground field of characteristic $p \geq 0$, and X be a proper scheme. Then the group scheme $\mathrm{Aut}_{X/k}$ is locally of finite type, and the connected component $\mathrm{Aut}^0_{X/k}$ of the neutral element $e = \mathrm{id}_X$ is of finite type ([21], Theorem 3.7). By the Yoneda Lemma, the map $\sigma \mapsto \sigma^2$ defines a morphism of the scheme $\mathrm{Aut}_{X/k}$ to itself, which usually disrespects the group law. The *scheme of involutions* $\mathrm{Inv}_{X/k}$ is defined via a cartesian diagram

$$\begin{array}{ccc} \mathrm{Inv}_{X/k} & \longrightarrow & \mathrm{Aut}_{X/k} \\ \downarrow & & \downarrow{\scriptstyle \sigma\mapsto\sigma^2} \\ \mathrm{Spec}(k) & \xrightarrow[e]{} & \mathrm{Aut}_{X/k}. \end{array}$$

It contains the neutral element and is stable under the inverse map $\sigma \mapsto \sigma^{-1}$, but otherwise carries no further structure in general.

Now suppose that X can be endowed with the structure of an abelian variety. Recall that for each rational point $x_0 \in X$, there is a unique group law that turns X into an abelian variety, with origin $0 = x_0$. Fix such a datum, and write A for the abelian variety obtained by endowing X with the ensuing group law. Note that A can also be regarded as the pair (X, x_0). The automorphism group scheme becomes a semidirect product

$$\mathrm{Aut}_{X/k} = A \rtimes \mathrm{Aut}_{A/k},$$

where the normal subgroup on the left acts on X by translations $x \mapsto a + x$. The cokernel $\mathrm{Aut}_{A/k}$ on the right is an étale group scheme with countably many points, acting on A in the canonical way. Its rational points are the automorphisms $\sigma : X \to X$ fixing the origin x_0. It contains a canonical element, namely the *standard sign involution* $x \mapsto -x$. This defines a morphism $(-1) : \mathrm{Spec}(k) \to \mathrm{Aut}_{A/k}$. Its fiber with respect to the canonical projection $A \rtimes \mathrm{Aut}_{A/k} \to \mathrm{Aut}_{A/k}$ is denoted by $A \otimes \kappa(-1)$.

Lemma 1.1. *The closed subscheme $A \otimes \kappa(-1) \subset \mathrm{Aut}_{X/k}$ is invariant under the conjugacy action of $\mathrm{Aut}_{X/k}$, lies inside $\mathrm{Inv}_{X/k}$, and does not depend on the choice of the origin $x_0 \in X$.*

Proof. Let $x, a, b \in A(R)$ and $\varphi \in \mathrm{Aut}_{A/k}(R)$ be R-valued points, for some k-algebra R. Then $x \mapsto a - x$ is some R-valued point of $A \otimes \kappa(-1)$. Conjugation by (b, id) is

$$x \longmapsto -b + x \longmapsto a - (-b + x) \longmapsto (a + 2b) - x, \tag{1}$$

whereas conjugation by $(0, \varphi)$ takes the form

$$x \longmapsto \varphi^{-1}(x) \longmapsto a - \varphi^{-1}(x) \longmapsto \varphi(a) - x.$$

Both are R-valued points of $A\otimes\kappa(-1)$. Furthermore, the composition $x \mapsto a-x \mapsto a-(a-x)$ is the identity. With the Yoneda Lemma, we see that $A \otimes \kappa(-1)$ is invariant under conjugacy, and must be contained in $\mathrm{Inv}_{X/k}$.

Now let $a_0 \in X$ be another origin. The ensuing new group law and negation are given by

$$x \oplus y = x + y - x_0' \quad \text{and} \quad \ominus x = -x + 2a_0,$$

and thus $a \ominus x = (a + a_0) - x$. This shows that the closed subscheme $A \otimes \kappa(-1) \subset \mathrm{Aut}_{X/k}$ does not depend on the choice of origin. □

Recall that a proper scheme X is called a *para-abelian variety* if there is a field extension $k \subset k'$ such that the base-change $X' = X \otimes k'$ admits the structure of an abelian variety. This notation was introduced and studied by Laurent and the third author [18]. According to loc. cit., Proposition 5.2, the closed subscheme $A \subset \mathrm{Aut}_{X/k}$ that acts trivial on $\mathrm{Pic}^{\tau}_{X/k}$ is an abelian variety, and the canonical A-action on X is free and transitive. The resulting class

$$[X] \in H^1(k, A)$$

in the Weil–Châtelet group is called the *cohomology class* of the para-abelian variety. Note that since A is smooth, the étale and fppf topology yield the same cohomology groups ([11], Theorem 11.7). Consequently, the class $[X]$ has some finite order; this number is usually called *period* $\mathrm{per}(X) \geq 1$.

Conversely, if H is any commutative group scheme, with a torsor P and a homomorphism $H \to A$, we get a para-abelian variety $X = P \wedge^H X_0$. The latter denotes the quotient of $P \times X_0$ by the diagonal action $h \cdot (p, x) = (h \cdot p, h + x)$, and X_0 is the underlying scheme of the abelian variety A. By construction, this X is a twisted form of X_0.

Recall that the *index* $\text{ind}(X) \geq 1$ is the greatest common divisor of the degrees $[\kappa(a) : k]$ for the closed points $a \in X$. This is indeed the index for the image of the degree map $\text{CH}_0(X) \to \mathbb{Z}$ on the Chow group of zero-cycles. Note that in dimension one this can also be seen as the degree map on the Picard group. According to [17], Proposition 5 the divisibility property $\text{per}(X) \mid \text{ind}(X)$ holds, and both numbers have the same prime factors.

As explained in [34], Section 3, the group scheme $\text{Aut}_{X/k}$ is a twisted form of $\text{Aut}_{X_0/k}$ with respect to the conjugacy action. In turn, the conjugacy-invariant closed subscheme $A \otimes \kappa(-1) \subset \text{Aut}_{X_0/k}$ becomes a closed subscheme

$$\text{Inv}^{\text{sgn}}_{X/k} \subset \text{Aut}_{X/k},$$

which we call the *scheme of sign involutions*. Any automorphism $\sigma : X \to X$ belonging to $\text{Inv}^{\text{sgn}}_{X/k}$ is called a *sign involution*.

Theorem 1.2. *For each para-abelian variety X of dimension $g \geq 0$, the following three conditions are equivalent:*

(i) *There is a sign involution $\sigma : X \to X$.*
(ii) *We have $2 \cdot [X] = 0$ in the* Weil–Châtelet group $H^1(k, A)$.
(iii) *There is an torsor P with respect to $H = A[2]$ such that $X \simeq P \wedge^H A$.*

It these conditions hold we have the divisibility property $\text{ind}(X) \mid 4^g$.

Proof. We start with some general observations: The first projection

$$\text{Aut}_{X_0/k} = A \rtimes \text{Aut}_{A/k} \longrightarrow A$$

identifies the scheme of sign involutions $Z_0 = \text{Inv}^{\text{sgn}}_{X_0/k} = A \otimes \kappa(-1)$ with a copy of $X_0 = A$. According to (1), the kernel for the conjugacy homomorphism $A \to \text{Aut}_{Z_0/k}$ is $A[2]$, so this factors over multiplication-by-two map $A \xrightarrow{2} A$. It is now convenient to write $X = T \wedge^A X_0$ for some A-torsor T. Note that since the X_0 is the trivial A-torsor, one actually has $T = X$. What is important now is that the scheme of sign involutions $Z = \text{Inv}^{\text{sgn}}_{X/k}$ coincides with $Z = T \wedge^A Z_0$, and the latter is the quotient of $T \times Z_0$ by the A-action $a \cdot (t, z_0) = (a + t, 2a + z_0)$.

This quotient can be computed as successive quotients, first for the action of $H = A[2]$ and then for the induced action of $A/A[2]$. The group H acts trivially on the second factor, hence $H\backslash(T \times X_0) = (H\backslash T) \times X_0$. In light of the short exact sequence

$$0 \longrightarrow H \longrightarrow A \xrightarrow{2} A \longrightarrow 0, \tag{2}$$

we may regard $\bar{T} = H\backslash T$ as the A-torsor induced from T with respect to $A \xrightarrow{2} A$. In other words $Z = \bar{T} \wedge^{\bar{A}} Z_0$, where we write $\bar{A} = A/H = A$ to indicate the nature of the action. By construction, the $\bar{A}$-action on Z_0 is free and transitive, so the projection $\bar{T} \otimes \kappa(-1) \to Z$ is an isomorphism. We conclude that there is a rational point $\sigma \in Z$ if and only if the torsor $\bar{T}$ is trivial.

From the short exact sequence (2) we get a long exact sequence

$$H^0(k, A) \xrightarrow{2} H^0(k, A) \longrightarrow H^1(k, H) \longrightarrow H^1(k, A) \xrightarrow{2} H^1(k, A).$$

It follows that the element $[X] = [T]$ in $H^1(k, A)$ is annihilated by two if and only if there is an H-torsor P such that $X \simeq P \wedge^H X_0$, giving the equivalence of (ii) and (iii). Similarly, we see that $[X] = [T]$ is annihilated by two if and only if $\bar{T}$ is trivial. Together with the previous paragraph this gives the equivalence of (i) and (ii).

It remains to verify the divisibility property of the index. This is just a special case of general fact: Suppose X has period $n \geq 1$. From the long exact sequence for the multiplication-by-n map we see that the quotient of X by $A[n]$ contains a rational point, so its fiber $Z \subset X$ is a torsor with respect to $A[n]$. According to [23], page 147 the kernel $A[n]$ is finite of length $l = n^{2g}$. Clearly, the torsor Z has the same length, hence X contains a zero-cycle of degree n^{2g}. Now if (ii) holds, we have $n \mid 2$, and thus $\mathrm{ind}(X) \mid 4^g$. □

Recall that for each $m \geq 1$ there is an identification $H^1(k, \mu_m) = k^\times / k^{\times m}$. Suppose now that k contains a primitive mth root of unity, such that $\mu_n \simeq (\mathbb{Z}/m\mathbb{Z})_k$. Let us recall the following result of Lang and Tate ([17], Theorem 8): Assume that the ground field k, the abelian variety A, and the integer $m \geq 0$ satisfies the following conditions: The $\mathbb{Z}/m\mathbb{Z}$-module $k^\times / k^{\times m}$ contains a free module of infinite rank, the quotient $A(k)/mA(k)$ is finite, and $A(k)$ contains an element of order m. Then the Weil–Châtelet group $H^1(k, A)$ contains infinitely many elements X whose period and index equals m. Note that for global fields k, the first two conditions are automatic, and the third can be obtained after a finite extension, provided the abelian variety has dimension $g \geq 1$ and the characteristic exponent $p \geq 1$ of k is prime to m.

2. The Picard scheme of the quotient

Let A be an abelian variety, with its standard sign involution $\sigma(x) = -x$. Write $G \subset \mathrm{Aut}(A)$ the corresponding subgroup of order two. The quotient $B = A/G$ is a projective scheme that is geometrically integral and geometrically normal, with $h^0(\mathscr{O}_B) = 1$. Following [7], Section 2, we write $\mathrm{Sing}(B/k)$ for the *locus of non-smoothness*. In contrast to the *locus of non-regularity* $\mathrm{Sing}(B)$, it comes with a scheme structure, defined via Fitting ideals for Kähler differentials.

Let $\mathrm{Pic}^\tau_{B/k}$ be the open-and-closed subgroup scheme inside the Picard scheme comprising numerically trivial invertible sheaves. Its Lie algebra is $H^1(B, \mathscr{O}_B)$, and the group scheme of connected components is the torsion part of the Néron–Severi group scheme. It therefore encodes important information on B. In dimension two, $B = A/G$ yields the classical Kummer surfaces, which give rise to K3 surfaces, and in characteristic $p = 2$ also to rational surfaces [14,35]. In both cases the tau-part of the Picard scheme vanishes. This generalizes to higher dimensions:

Theorem 2.1. *The group scheme* $\mathrm{Pic}^\tau_{B/k}$ *is trivial. Moreover,* $\mathrm{Sing}(B/k)$ *is finite, and is contained in the image of the fixed scheme* $A^\sigma = A[2]$.

Proof. It suffices to treat the case that k is algebraically closed. Write $q : A \to B$ for the quotient map, let $U \subset A$ be the complement of the fixed scheme $A^\sigma = A[2]$, and $V = q(U)$ be its image. The induced map $q : U \to V$ is a G-torsor, in particular smooth. According to [10], Theorem 17.11.1 the smoothness of U ensures the smoothness of V. Thus $\mathrm{Sing}(B/k)$ is contained in the image of $A[2]$, and is therefore finite.

The structure sheaf $\mathscr{O}_A$ has a G-linearization, and thus comes with *equivariant cohomology groups* $H^i(A, G, \mathscr{O}_A)$, and likewise we have $H^i(A, G, \mathscr{O}_A^\times)$. According to [9], Section 5.2, for every abelian sheaf F on A endowed with a G-linearization there are two spectral sequences

$$E_2^{rs} = H^r(G, H^s(A, F)) \quad \text{and} \quad E_2^{rs} = H^r(B, \underline{H}^s(G, F)), \tag{3}$$

both with equivariant cohomology $H^{r+s}(A, G, F)$ as abutment. For $F = \mathscr{O}_A^\times$ this gives two exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{Pic}(B) \longrightarrow H^1(A, G, \mathscr{O}_A^\times) \rightarrow H^0(B, P) \rightarrow H^2(B, \mathscr{O}_B^\times) \\ \| \\ 0 \rightarrow H^1(G, k^\times) \rightarrow H^1(A, G, \mathscr{O}_A^\times) \longrightarrow \mathrm{Pic}(A)^G \longrightarrow H^2(G, k^\times), \end{array} \tag{4}$$

where the abelian sheaf $P = \underline{H}^1(G, \mathscr{O}_A^\times)$ is supported by the singular locus of B, and the composition $\mathrm{Pic}(B) \rightarrow H^1(A, G, \mathscr{O}_A^\times) \rightarrow \mathrm{Pic}(A)^G$ is given by pullback of invertible sheaves. Recall that the cohomology groups for the cyclic group $G = \{e, \sigma\}$ are given by

$$H^{2j+1}(G, M) = \frac{\mathrm{Ker}(\sigma + \mathrm{id})}{\mathrm{Im}(\sigma - \mathrm{id})} \quad \text{and} \quad H^{2j+2}(G, M) = \frac{\mathrm{Ker}(\sigma - \mathrm{id})}{\mathrm{Im}(\sigma + \mathrm{id})},$$

for any G-module M. It follows that $H^2(G, k^\times)$ vanishes, because G acts trivially on $k^\times$, and $k^\times = k^{\times 2}$, whereas $H^1(G, k^\times) = \mu_2(k) = \{\pm 1\}$. According to (4), the kernel for $\mathrm{Pic}(B) \rightarrow \mathrm{Pic}(A)$ is the intersection of $\mathrm{Pic}(B) \cap H^1(G, k^\times)$ inside the equivariant cohomology group. Furthermore, the image of $\mathrm{Pic}^\tau(B) \rightarrow \mathrm{Pic}(A)$ is contained in

$$\mathrm{Pic}^\tau(A) \cap \mathrm{Pic}(A)^G = A(k) \cap \mathrm{Pic}(A)^G = A(k)[2] = \mathrm{Pic}(A)[2].$$

This already shows that the group scheme $\mathrm{Pic}^\tau_{B/k}$ must be finite. It also settles the case of dimension $g = 1$: Then B is a normal curve with finite Picard scheme. The latter is smooth, according to [22], Section 27 because $H^2(B, \mathscr{O}_B) = 0$. Consequently $B = \mathbb{P}^1$, and thus $\mathrm{Pic}^\tau_{B/k} = 0$.

From now on, we assume that we are in dimension $g \geq 2$. At each $a \in A[2]$, the induced G-action on the local ring $\mathscr{O}_{A,a}$ is ramified only at the origin. It follows that the local ring at the image $b \in B$ is singular, and that the finite degree-two extension $\mathscr{O}_{B,b} \subset \mathscr{O}_{A,a}$ is not flat: the arguments in [19], last paragraph in the proof for Proposition 3.2, hold true for the action of our group G of order two in characteristic $p \geq 0$. Consequently, the quotient map $q : A \rightarrow B$ induces a bijection between $A[2]$ and $\mathrm{Sing}(B)$. Furthermore, the short exact sequence $0 \rightarrow \mathscr{O}_B \rightarrow q_*(\mathscr{O}_A) \rightarrow \mathscr{F} \rightarrow 0$ defines a coherent sheaf $\mathscr{F}$ that is invertible on the open set $V = \mathrm{Reg}(B)$, but not at the points $b \in \mathrm{Sing}(B)$.

We claim that the canonical map $\mathrm{Pic}(B) \rightarrow \mathrm{Pic}(A)^G$ is injective. Equivalently, the intersection $\mathrm{Pic}(B) \cap H^1(G, k^\times)$ inside $H^1(A, G, \mathscr{O}_A^\times)$ is trivial. The group $H^1(G, k^\times) = \mu_2(k)$ vanishes in characteristic two, so only the case $p \neq 2$ requires attention. Then the *trace map* $q_*(\mathscr{O}_A) \rightarrow \mathscr{O}_B$, which sends a local section viewed as an $\mathscr{O}_B$-linear homothety to its trace, gives a splitting $q_*(\mathscr{O}_A) = \mathscr{O}_B \oplus \mathscr{F}$, thus $\mathscr{F}$ satisfies Serre's Condition (S_2). The canonical identification $\mathscr{F}_V \otimes \mathscr{F}_V^\vee = \mathscr{O}_V$ yields an element in $\Gamma(V, q_*(\mathscr{O}_A) \otimes \mathscr{F}^\vee) = \Gamma(U, q^*(\mathscr{F}^\vee))$ without zeros, and it follows that the invertible sheaf $\mathscr{F}|V$ becomes trivial on U. Using the diagram (4) for the quotient $V = U/G$ instead of $B = A/G$, we conclude that $\mathscr{F}|V$ generates the kernel of $\mathrm{Pic}(V) \rightarrow \mathrm{Pic}(U)$. Seeking a contradiction, we now assume that there is a non-trivial invertible sheaf $\mathscr{L}$ on B that becomes trivial on A, we therefore must have $\mathscr{L}|V = \mathscr{F}|V$. Using that both $\mathscr{L}$ and $\mathscr{F}$ satisfy Serre's Condition (S_2) together with [12], Theorem 1.12 we infer that $\mathscr{L} = \mathscr{F}$, contradicting that $\mathscr{F}$ is not invertible. This establishes our claim. In turn, the canonical map $\mathrm{Pic}^\tau(B) \rightarrow \mathrm{Pic}(A)[2]$ becomes an inclusion $\mathrm{Pic}^\tau(B) \subset \mathrm{Pic}(A)[2]$.

We next check that for $p \neq 2$ the finite group scheme $\mathrm{Pic}^\tau_{B/k}$ is reduced. Equivalently, its Lie algebra $H^1(B, \mathscr{O}_B)$ vanishes. To see this, consider the spectral sequences (3) with the additive sheaf $\mathscr{O}_A$ instead the multiplicative sheaf $\mathscr{O}_A^\times$. For $i \geq 1$, the vector spaces $H^i(G, k)$

are annihilated by the group order $|G| = 2$. For $p \neq 2$ they consequently vanish, and we obtain inclusions

$$H^1(B, \mathscr{O}_B) \subset H^1(A, G, \mathscr{O}_A) \subset H^1(A, \mathscr{O}_A)^G.$$

Moreover, the term on the right also vanishes because G acts via the sign involution on the cohomology group, according to [26], proof of Proposition 2.3). This establishes the claim.

To proceed we use the fact that for any finite commutative group scheme N the isomorphism classes of N-torsors $B' \to B$ correspond to homomorphisms of group schemes $N^* \to \mathrm{Pic}_{B/k}$, where $N^* = \underline{\mathrm{Hom}}(N, \mathbb{G}_m)$ denotes the *Cartier dual* (see [25], Proposition 6.2.1, and also the discussion in [31], Section 4).

The constant group scheme $N = (\mathbb{Z}/2\mathbb{Z})_k$ has Cartier dual $N^* = \mu_2$. Suppose we have an inclusion $\mu_2 \subset \mathrm{Pic}^\tau_{B/k}$ such that the composite map $\mu_2 \to \mathrm{Pic}^\tau_{A/k}$ remains a monomorphism. The corresponding N-torsor $B' \to B$ thus induces a non-trivial N-torsor $A' \to A$. According to the Serre–Lang Theorem ([23], page 167), there is a unique structure of an abelian variety for A' so that $A' \to A$ is a homomorphism. This gives an embedding $N \subset A'$ defined by a 2-division point $a' \in A'$. The composite $A' \to B$ is the quotient for the action of $N \rtimes \{\pm 1\}$. Since this semidirect product is actually a direct product, the projection $A' \to B'$ must be the quotient by $G = \{\pm 1\}$. Now choose a closed point $x' \in A'$ with $2x' = a'$. It follows that the orbit $G \cdot x' = \{\pm x'\}$, viewed as a rational point on B', is fixed by the N-action, contradiction. This settles the case $p \neq 2$: Then $\mu_2 = (\mathbb{Z}/2\mathbb{Z})_k$, and we see that $\mathrm{Pic}^\tau(B) \subset \mathrm{Pic}(A)[2]$ is trivial. We already saw in the previous paragraph that $\mathrm{Pic}^\tau_{B/k}$ is reduced, and infer that it must be trivial.

It remains to treat the case $p = 2$, where the arguments in some sense run parallel to the preceding paragraph. At each $a \in A[2]$, the local ring at the image $b \in B$ is singular, with $\mathrm{depth}(\mathscr{O}_{B,b}) = 2$, according to [19], Proposition 3.2. Note that this is in stark contrast to the situation $p \neq 2$, when such rings of invariants are Cohen–Macaulay. Again we consider the short exact sequence $0 \to \mathscr{O}_B \to q_*(\mathscr{O}_A) \to \mathscr{F} \to 0$ of coherent sheaves on B. For the images $b \in B$ of the $a \in A[2]$, the short exact sequence of local cohomology

$$H^0_b(B, q_*(\mathscr{O}_A)) \longrightarrow H^0_b(B, \mathscr{F}) \longrightarrow H^1_b(B, \mathscr{O}_B),$$

reveals that $H^0_b(B, \mathscr{F}) = 0$, in other words, $\mathscr{F}$ is torsion-free. The trace map $q_*(\mathscr{O}_A) \to \mathscr{O}_B$ vanishes on the subsheaf $\mathscr{O}_B \subset q_*(\mathscr{O}_A)$ since we are in characteristic two. The induced map $\mathscr{F} \to \mathscr{O}_B$ is bijective on the locus where $\mathscr{F}$ is invertible, which one easily sees by a local computation. This gives an inclusion $\mathscr{F} \subset \mathscr{O}_B$. Using that $\mathscr{F}$ is not invertible we infer $H^0(B, \mathscr{F}) = 0$. The exact sequence

$$H^0(B, \mathscr{F}) \longrightarrow H^1(B, \mathscr{O}_B) \longrightarrow H^1(A, \mathscr{O}_A)$$

ensures that the map on the right is injective. On the other hand, its kernel is the Lie algebra for the kernel of $\mathrm{Pic}^\tau_{B/k} \to \mathrm{Pic}_{A/k}[2]$. It follows that this map is actually a closed embedding $\mathrm{Pic}^\tau_{B/k} \subset \mathrm{Pic}_{A/k}[2]$.

Now we use that the Lie algebra of any group scheme in characteristic $p > 0$ carries as additional structure the p-map $x \mapsto x^{[p]}$ and becomes a *restricted Lie algebra* (see [34], Section 1 for more details). Suppose $H^1(B, \mathscr{O}_B) \neq 0$. Then there is a p-closed vector $x \neq 0$, in other words $x^{[p]}$ is a multiple of x. The case $x^{[p]} \neq 0$ yields an inclusion of $\mu_p \subset B$ where the composite map $\mu_p \to A$ is injective. We saw above that this is impossible. In turn we must have $x^{[p]} = 0$. This gives an inclusion of $N^* = \alpha_p$ into B where the composite map $\alpha_p \to A$ remains injective. The Cartier dual is $N = \alpha_p$. Thus we get a non-trivial α_p-torsor $B' \to B$ for

α_p whose base-change $A' \to A$ remains non-trivial. A similar situation with $N^* = (\mathbb{Z}/2\mathbb{Z})_k$ and $N = \mu_p$ arises if there is a point of order two on $\mathrm{Pic}_{B/k}$. In both cases the discussion in [26], beginning of Section 2 shows that A' has the structure of an abelian variety so that the projection $A' \to A$ is a homomorphism, and we get an inclusion $N \subset A'$. The composition $A' \to B$ is the quotient by the group scheme $N \rtimes \{\pm 1\}$. Again this is actually a direct product. In the cartesian diagram

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

the vertical maps are quotients by the action of the infinitesimal group scheme N, and the horizontal maps are quotients by $G = \{\pm 1\}$. Fix some $a' \in A'[2]$, with image $b' \in \mathrm{Sing}(B')$, and consider the ring of invariants $\mathscr{O}_{B',b'} \subset \mathscr{O}_{A',a'}$. According to [19], Lemma 3.3 no element $f \in \mathfrak{m}_{a'} \smallsetminus \mathfrak{m}_{a'}^2$ is G-invariant. It follows that the infinitesimal neighborhood $\mathrm{Spec}(\mathscr{O}_{a'}/\mathfrak{m}_{a'}^2)$ maps to $Z' = \mathrm{Spec}(\mathscr{O}_{b'}/\mathfrak{m}_{b'})$, and therefore the same holds for the orbit $N \cdot \{a'\}$. In light of the above commutative diagram, the N-action on B' is not free, contradiction. $\square$

The result immediately carries over to para-abelian varieties, because the formation of both the quotient $B = A/G$ and the Picard scheme $\mathrm{Pic}_{B/k}$ commutes with ground field extensions. The para-abelian varieties X of dimension $g = 1$ are usually called *genus-one curves*. Throughout, we shall prefer the term *para-elliptic curves*. These are twisted forms of elliptic curves. The moduli stack of such curves was studied by the second author [5]. Recall that the *Brauer–Severi varieties* Y are twisted forms of projective space $\mathbb{P}^n$, for some $n \geq 0$. For more details we refer to [2]. In case $n = 1$ we also say that Y is a *Brauer–Severi curve*.

Corollary 2.2. *Assumption as in the proposition, and suppose additionally* $g = 1$. *Then the corresponding quotient* $B = X/G$ *is a Brauer–Severi curve.*

Proof. The scheme B is geometrically normal and of dimension one, hence smooth. According to the theorem, the Picard scheme is discrete. It follows that the tangent space $H^1(B, \mathscr{O}_B)$ vanishes. If there is a rational point $a \in X$, the resulting invertible sheaf $\mathscr{L} = \mathscr{O}_B(a)$ is very ample, with $h^0(\mathscr{L}) = 2$, and we obtain an isomorphism $B \to \mathbb{P}^1$. $\square$

In dimension $g = 2$ and characteristic $p \neq 2$, the quotient $B = A/\{\pm 1\}$ is called a *Kummer surface*, and is a K3 surface with rational double points. For $p = 2$, the quotient B is either a K3 surface with rational double points, or a rational surface with an elliptic singularity. This was discovered by Shioda [35], see also [14,16,29,30]. The formation of such quotients is studied by the first author [4]. Little seems to be known on the quotient in higher dimensions, in particular in characteristic two, compare Schilson's investigation [27,28].

3. Morphisms to Brauer–Severi curves

Let X be a para-elliptic curve over a ground field k. If there is a sign involution $\sigma : X \to X$, the quotient B by the corresponding group of order two is a Brauer–Severi curve, according to Corollary 2.2. In this section we conversely assume that our para-elliptic curve X admits a morphism $f : X \to B$ of degree two to some Brauer–Severi curve B, and derive several geometric consequences.

First note that the corresponding function field extension $k(B) \subset k(X)$ has degree two. It must be separable, because X and B are smooth of different genus. So this is a Galois extension, and the Galois group G is cyclic of order two. Let $\sigma \in G$ be the generator.

Proposition 3.1. *The automorphism $\sigma : X \to X$ is a sign involution.*

Proof. It suffices to treat the case that k is algebraically closed. The action is not free, because $\chi(\mathscr{O}_X) = 0 \neq 2 = |G| \cdot \chi(\mathscr{O}_B)$. Choose a fixed point $x_0 \in X$, and regard $E = (X, x_0)$ as an elliptic curve. If $\mathrm{Aut}(E)$ is cyclic, there is a unique element of order two, and we infer that σ equals the sign involution. Suppose now that $\mathrm{Aut}(E)$ is non-cyclic. According to [6], Proposition 5.9 this group is either the semi-direct product $\mathbb{Z}/3\mathbb{Z} \rtimes \mu_4(k)$ in characteristic $p = 3$, or $Q \rtimes \mu_3(k)$ in characteristic $p = 2$, where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ denotes the quaternion group. In these groups, the respective elements $(0, -1)$ and $(-1, 1)$ are the only ones of order two, and we again conclude that σ coincides with the sign involution. □

Proposition 3.2. *The cokernel for the inclusion $\mathscr{O}_B \subset f_*(\mathscr{O}_X)$ is isomorphic to ω_B, and the resulting extension $0 \to \mathscr{O}_B \to f_*(\mathscr{O}_X) \to \omega_B \to 0$ of coherent sheaves splits.*

Proof. The sheaf $f_*(\mathscr{O}_X)$ has rank two and is torsion-free, hence is locally free. The inclusion of $\mathscr{O}_B$ is locally a direct summand, so the cokernel $\mathscr{L}$ is invertible. We have $0 = \chi(\mathscr{O}_X) = \chi(\mathscr{O}_B) + \chi(\mathscr{L}) = 2 + \deg(\mathscr{L})$ and conclude $\deg(\mathscr{L}) = -2$. Since $\deg : \mathrm{Pic}(B) \to \mathbb{Z}$ is injective, this gives $\mathscr{L} \simeq \omega_B$. The extension yields a class in $\mathrm{Ext}^1(\omega_B, \mathscr{O}_B) = H^1(X, \omega_B^{\otimes -1})$, which vanishes by Serre Duality. So the extension splits. □

Choose a splitting and set $\mathscr{E} = f_*(\mathscr{O}_X) = \mathscr{O}_B \oplus \omega_B$. The smooth surface

$$S = \mathbb{P}(\mathscr{E}) = \mathrm{Proj}(\mathrm{Sym}^\bullet \mathscr{E})$$

is a twisted form of the Hirzebruch surface $S_0 = \mathbb{P}(\mathscr{E}_0)$, where $\mathscr{E}_0 = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2)$. Let us call S the *twisted Hirzebruch surface* attached to the Brauer–Severi curve B. Since $f : X \to B$ is affine, the invertible sheaf $\mathscr{O}_X$ is relatively very ample, and we get a closed embedding $X \subset S$. By abuse of notation we also write $f : S \to B$ for the extension of our original morphism on X.

Recall that each invertible quotient $\mathscr{E} \to \mathscr{N}$ defines a section $s : B \to S$, whose image D has self-intersection $D^2 = \deg(\mathscr{N}) - \deg(\mathscr{N}')$, where $\mathscr{N}' \subset \mathscr{E}$ is the kernel. For more details we refer to [8], Section 6. In particular, $\mathrm{pr}_1 : \mathscr{E} \to \mathscr{O}_B$ yields a curve $D \subset S$ with $D^2 = 2$, whereas $\mathrm{pr}_2 : \mathscr{E} \to \omega_B$ gives some $E \subset S$ with $E^2 = -2$, and the two sections are disjoint. The Adjunction Formula gives $(\omega_S \cdot D) = -4$ and $(\omega_S \cdot E) = 0$. Hence $\omega_S = f^*(\omega_B^{\otimes 2}) \otimes \mathscr{O}_S(-2E)$, because both sides have the same intersection numbers with D and E. In particular $c_1^2 = (\omega_S \cdot \omega_S) = -8 \cdot \deg(\omega_B) + 4 \cdot E^2 = 8$. Setting

$$\omega_S^{\otimes 1/2} = f^*(\omega_B) \otimes \mathscr{O}_S(-E),$$

we get an invertible sheaf whose square is isomorphic to the dualizing sheaf. In other words, the surface S comes with a canonical *theta characteristic*, or *spin structure*, compare [3,24].

Proposition 3.3. *The dual sheaf $\mathscr{L} = \omega_S^{\otimes -1/2}$ is globally generated with $h^0(\mathscr{L}) = 4$. The image of the resulting $r : S \to \mathbb{P}^3$ is an integral normal surface $S' \subset \mathbb{P}^3$ of degree two, and the induced morphism $r : S \to S'$ is the contraction of E. Moreover, the image $a = r(E)$ is a rational point, the local ring $\mathscr{O}_{S',a}$ is singular, and the restriction $r|X$ is a closed embedding.*

Proof. Our sheaf has intersection numbers $(\mathscr{L}\cdot\mathscr{L})=2$ and $(\mathscr{L}\cdot E)=0$. Serre Duality gives $h^2(\mathscr{L})=h^0(\omega_S^{\otimes 3/2})=0$, and Riemann–Roch yields

$$h^0(\mathscr{L})\geq\chi(\mathscr{L})=\frac{c_1^2/4+c_1^2/2}{2}+\chi(\mathscr{O}_S)=(2+4)/2+1=4.$$

The base locus $\mathrm{Bs}(\mathscr{L})$ is contained in E, because $\omega_B^{\otimes -1}$ is globally generated. The short exact sequence $0\to f^*(\omega_B^{\otimes -1})\to\mathscr{L}\to\mathscr{L}|E\to 0$ yields an exact sequence

$$0\longrightarrow H^0(S,f^*(\omega_B^{\otimes -1}))\longrightarrow H^0(S,\mathscr{L})\longrightarrow H^0(E,\mathscr{O}_E),$$

consequently $h^0(\mathscr{L})\leq h^0(\omega_B^{\otimes -1})+h^0(\mathscr{O}_E)=4$. This ensures $h^0(\mathscr{L})=4$, and that $\mathscr{L}$ is globally generated.

In turn, our spin structure yields a morphism $r:S\to\mathbb{P}^3$ with $r^*(\mathscr{O}_{\mathbb{P}^3}(1))=\omega_S^{\otimes -1/2}$. It therefore contracts E. Moreover, the image $S'\subset\mathbb{P}^3$ is integral and two-dimensional, of some degree $n\geq 1$. This image is not a plane, because the morphism is defined by the complete linear system $H^0(S,\mathscr{L})$. From $2=(\mathscr{L}\cdot\mathscr{L})=\deg(S/S')\cdot n$ we infer that $S\to S'$ is birational and $n=2$. The Adjunction Formula gives $\omega_{S'}=\mathscr{O}_{S'}(-2)$, consequently $r^*(\omega_{S'})=\omega_S$. It follows that the birational morphism $r:S\to S'$ is in Stein factorization. Since $\mathrm{Pic}(S)$ has rank two, the exceptional divisor is irreducible, whence must coincide with E.

The image $a=r(E)$ is a rational point, because $h^0(\mathscr{O}_E)=1$. The local ring $\mathscr{O}_{S',a}$ must be singular, because otherwise $S=\mathrm{Bl}_a(S')$, such that $E=r^{-1}(a)$ must be a projective line with $E^2=-1$, contradiction.

It remains to verify that the curves $X,E\subset S$ are disjoint. Since $\deg(X/B)=2$ we have $\omega_S=\mathscr{O}_S(-X)\otimes f^*(\mathscr{N})$ for some invertible sheaf $\mathscr{N}$ on B. The Adjunction Formula gives

$$0=(\omega_S\cdot X)+X^2=-X^2+2\deg(\mathscr{N})+X^2.$$

Consequently $\mathscr{N}$ is trivial, and $\omega_S=\mathscr{O}_S(-X)$. This gives $X^2=c_1^2=8$, and furthermore $(X\cdot E)=-(\omega_S\cdot E)=0$. Thus the integral curves X and E must be disjoint, hence $r|X$ is a closed embedding. □

Note that the local ring $\mathscr{O}_{S',a}$ is factorial provided that $B\not\simeq\mathbb{P}^1$. The above also shows that the image $S'=r(S)$ can also be viewed as the *anti-canonical model* $P(S,-K_S)$ of the scheme S, which is defined as the homogeneous spectrum of the *anti-canonical ring* $R(S,-K_S)=\bigoplus_{t\geq 0}H^0(S,\omega_S^{\otimes t})$.

Recall that the *weighted projective space* $\mathbb{P}(d_0,\ldots,d_n)$ is the homogeneous spectrum of $k[U_0,\ldots,U_n]$, where the generators have degrees $d_i=\deg(U_i)$. The case $d_0=\cdots=d_n=1$ gives back the standard projective space $\mathbb{P}^n$. Let us say that a closed subscheme of a Gorenstein surface is an *anti-canonical curve* if its sheaf of ideals is isomorphic to the dualizing sheaf.

Proposition 3.4. *The anti-canonical model $S'=P(S,-K_S)$ is a twisted form of the weighted projective space $\mathbb{P}(1,1,2)$. Moreover, $X\subset S$ and the resulting inclusion $X\subset S'$ are anti-canonical curves.*

Proof. It suffices to treat the case that k is algebraically closed. We claim that S' is defined inside $\mathbb{P}^3=\mathrm{Proj}\,k[T_0,\ldots,T_3]$ by the equation $T_0^2-T_1T_2=0$, for a suitable choice of homogeneous coordinates. The main challenge is the case $p=2$: According to [1], Satz 2 our quadric $X\subset\mathbb{P}^3$ must be defined by an equation of the form

$$\sum_{i=1}^{r}(\alpha_iX_i^2+X_iY_i+\gamma_iY_i^2)+\sum_{j=1}^{s}\delta_jZ_j^2=0,$$

with $1 \leq 2r + s \leq 4$, and non-zero coefficients δ_j. Since k is algebraically closed, we can make a change of variables and achieve $\delta_j = 1$, and furthermore $\alpha_i = \gamma_i = 0$. For $s \geq 1$ the coordinate change $Z_1 = Z'_1 + \cdots + Z'_s$ reduces us to the case $s = 1$. One now immediately sees that only for $r = s = 1$ the quadric $S' \subset \mathbb{P}^3$ is normal and singular, and setting $T_0 = Z_1$ and $T_1 = X_1$ and $T_2 = Y_1$ gives the claim. For $p \neq 2$ our quadric can be defined by an equation of the form $\sum_{j=0}^{3} \delta_j Z_j^2 = 0$, and one argues similarly.

Consider the graded ring $A = k[U_0, U_1, U_2]$ with weights $(1, 1, 2)$. The Veronese subring $A^{(2)}$ is generated by the homogeneous elements $U_0U_1, U_0^2, U_1^2, U_2$, which satisfy the relation $(U_0U_1)^2 = U_0^2 \cdot U_1^2$. This gives a surjection

$$k[T_0, T_1, T_2, T_3]/(T_0^2 - T_1T_2) \longrightarrow A^{(2)},$$

defined by the assignments $T_0 \mapsto U_0U_1$ and $T_1 \mapsto U_0^2$ and $T_2 \mapsto U_1^2$ and $T_3 \mapsto U_2$. Both rings are integral of dimension three. Using Krull's Principal Ideal Theorem, we infer that the above surjection is bijective. The homogeneous spectrum of $A^{(2)}$ coincides with $\mathbb{P}(1, 1, 2) = \text{Proj}(A)$, and by the above also with S'.

We already saw in the previous proof that $\omega_S = \mathscr{O}_S(-X)$, hence $X \subset S$ is an anti-canonical curve. From the Theorem of Formal functions one infers $f_*(\omega_S)$ is invertible, and this ensures that the direct image coincides with $\omega_{S'}$. Using $X \cap E = \varnothing$ we infer $\omega_{S'} = \mathscr{O}_{S'}(-X)$. □

Now suppose that we have *two* morphism $B_1 \xleftarrow{f_1} X \xrightarrow{f_2} B_2$ to Brauer–Severi curves, with $\deg(X/B_i) = 2$. According to Proposition 3.1, they come from sign involutions σ_1 and σ_2, respectively.

Proposition 3.5. *If $\sigma_1 \neq \sigma_2$, the diagonal morphism $i : X \to B_1 \times B_2$ is a closed embedding, and its image is an anti-canonical curve.*

Proof. Let $A \subset \text{Aut}_{X/k}$ be the subgroup scheme that fixes $\text{Pic}^\tau_{X/k}$. As discussed in Section 1, this is an elliptic curve, and the action on the para-elliptic curve X is free and transitive. Moreover, the dual abelian variety is identified with $\text{Pic}^0_{X/k}$. But note that the principal polarization stemming from the origin also gives $A = \text{Pic}^0_{X/k}$. We saw in the proof of Lemma 1.1 that the two rational points $\sigma_1, \sigma_2 \in \text{Inv}^{\text{sgn}}_{X/k}$ differ by the action of some non-zero $a \in A(k)$. In other words, $\sigma_2(x) = a + \sigma_1(x)$. It follows that there is no rational point $x \in X$ with $\sigma_1(x) = \sigma_2(x)$. In particular, the fixed schemes X^{σ_1} and X^{σ_2} are disjoint.

To proceed, we assume that k is algebraically closed. Let $x \in X$ be a closed point and write $y = i(x) = (b_1, b_2)$. The inverse image $i^{-1}(y)$ is the intersection of the fibers $f_1^{-1}(b_1) \cap f_2^{-1}(b_2)$. This is just the spectrum of $\kappa(x)$, by the previous paragraph. According to [10], Corollary 18.12.6 the finite morphism $i : X \to B_1 \times B_2$ is a closed embedding.

By construction, we have $\deg(X/B_1) = \deg(X/B_2) = 2$. Set $V = B_1 \times B_2$. Its Picard scheme $\text{Pic}_{V/k}$ can seen as the Galois module $\text{Pic}(V \otimes k^{\text{sep}}) = \mathbb{Z} \times \mathbb{Z}$, compare the discussion in [32], Section 1. Obviously, the elements $(2, 0)$ and $(0, 2)$ are fixed by $\text{Gal}(k^{\text{sep}}/k)$, hence the whole Galois action is trivial, and thus $\text{Pic}_{V/k} = (\mathbb{Z} \times \mathbb{Z})_k$ is a constant group scheme. The dualizing sheaf $\omega_V = \text{pr}_1^*(\omega_{B_1}) \otimes \text{pr}_2^*(\omega_{B_2})$ has class $(2, 2)$, and we infer $\omega_V = \mathscr{O}_S(-X)$. □

Note that ω_V is anti-ample, so the smooth surface $V = B_1 \times B_2$ coincides with its anti-canonical model $P(V, -K_V)$. Products of Brauer–Severi curves were studied by Kollár [15] and Hogadi [13]. Let us close this paper with the following observation:

Proposition 3.6. *The surface $V = B_1 \times B_2$ admits an embedding into $\mathbb{P}^3$ if and only if $B_1 \simeq B_2$.*

Proof. The Picard scheme is given by $\mathrm{Pic}_{V/k} = (\mathbb{Z}\times\mathbb{Z})_k$. The classes $(-2, 0)$ and $(0, -2)$ come from the preimages of the invertible sheaves on B_1 and B_2, and thus belong to the subgroup $\mathrm{Pic}(V) \subset \mathrm{Pic}_{V/k}(k)$.

Suppose we have $V \subset \mathbb{P}^3$, and write $d \geq 1$ for its degree. From $\omega_V = \mathscr{O}_V(d-4)$ we get $8 = (\omega_V \cdot \omega_V) = d(d-4)^2$, and thus $d = 2$. In particular, V admits the spin structure $\omega_V^{\otimes 1/2} = \mathscr{O}_V(-1)$. The dual sheaf $\mathscr{L} = \mathscr{O}_V(1)$ has $h^0(\mathscr{L}) = 4$, which easily follows from the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3}(-1) \to \mathscr{O}_{\mathbb{P}^3}(1) \to \mathscr{L} \to 0$. Choose some non-zero global section $s \neq 0$ from $\mathscr{L}$, and let $D \subset V$ the resulting effective Cartier divisor. Suppose D is reducible. Since $\deg(D) = 2$ we see that there are two components. Since $\mathscr{L}$ has class $(1, 1)$ in $\mathrm{Pic}_{V/k}(k)$, it follows that $D = D_1 + D_2$, where the summands are preimages of rational points on B_1 and B_2, respectively. Thus both Brauer–Severi curves are copies of $\mathbb{P}^1$. Suppose now that D is irreducible. Then $\deg(D/B_i) = 1$, so the morphism $D \to B_i$ are birational. By Zariski's Main Theorem, it must be an isomorphism, and therefore $B_1 \simeq B_2$.

Conversely, suppose there is an isomorphism $h : B_1 \to B_2$. Its graph defines an effective Cartier divisor $D \subset B_1 \times B_2$ with class $(1, 1) \in \mathrm{Pic}_{V/k}(k)$. Set $\mathscr{L} = \mathscr{O}_V(D)$. Passing to the algebraic closure of k, we get $\mathscr{L} = \mathrm{pr}_1^*(\mathscr{O}_{\mathbb{P}^1}(1)) \otimes \mathrm{pr}_2^*(\mathscr{O}_{\mathbb{P}^1}(1))$, and compute $h^0(\mathscr{L}) = 4$. Moreover, $\mathscr{L}$ is very ample, and thus defines a closed embedding $X \subset \mathbb{P}^3$. □

Given a sign involution $\sigma : X \to X$ and a non-zero rational point $a \in A(k)$, we get another sign involution $x \mapsto a + \sigma(x)$. We see that the situation $B_1 \stackrel{f_1}{\leftarrow} X \stackrel{f_2}{\rightarrow} B_2$ with $\sigma_1 \neq \sigma_2$ appears if and only if the set $\mathrm{Inv}^{\mathrm{sgn}}_{X/k}(k)$ is non-empty and the group $A(k)$ is non-trivial.

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