Stable Lévy Processes

Parabolic Fractal Geometry
&
Applications to Random Schrödinger Operators

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Part 1. **Introduction**

1. Overview of the Thesis

The *Brownian Motion* is the most investigated stochastic process. The Scottish botanist Robert Brown observed that pollen on a drop of water exhibit a non-smooth dynamic which is characteristic of the graph of Brownian Motion, see [8]. Therefore this phenomenon exists in the real world by observation. In 1905, Albert Einstein explained diffusion by means of Brownian Motion, see [14]. It took 18 years until Norbert Wiener mathematically rigorously proved the existence of Brownian motion, see [45]. In this thesis we deal with a generalisation of the Brownian motion as a stochastic process on the path space, i.e. isotropic stable Lévy processes which are introduced in the first part.

In the second part, our aim is to analyse isotropic stable Lévy processes plus (arbitrary) Borel measurable drift functions by methods from fractal geometry. In particular, we determine formulas for the Hausdorff dimension of the graph and the range of an isotropic stable Lévy process plus drift. Stochastic processes plus drift occur very naturally in the context of partial differential equations. They describe the world from a macroscopic point of view, whereas stochastics pays attention to individual particle movements, i.e. the microscopic world. Both disciplines can be rigorously translated into each other by yielding the same dynamics. For example the incompressible Navier-Stokes equations

$$\dot{v} + (v \cdot \nabla)v - \mu\Delta v + \nabla p = 0, \quad \text{div } v = 0.$$

can be written as the stochastic Euler-Lagrange system

$$v(y,t) = \mathbb{E}\mathsf{P}[\nabla Y(y,t) \cdot v_0(Y(y,t))],$$

$$X(x,t) = x + \int_0^t v(x,s)ds + \sqrt{2\mu}B_t =: y.$$

Here, $Y := X^{-1}$ denotes the spatial inverse of the molecule motion X, the operator P is the Helmholtz projection and B_t is the d-dimensional Brownian motion, see [11]. Now, the motion X consists of Brownian motion plus drift term.

Starting with Brownian motion, in the past decades much effort has been made to explicitly calculate the Hausdorff dimension of the range and the graph of stable

Lévy processes with an even more general self-similarity relation than (2.1), e.g. see [6, 7, 38, 20, 24, 3, 27, 43] in chronological order or the excellent review article [47]. Only recently, Peres and Sousi started to deal with Hausdorff dimension results of self-similar processes with an additional drift function by considering Brownian motion [36] and fractional Brownian motion [37]. We will follow the method in [37] to prove corresponding results for isotropic stable Lévy processes. The restriction to isotropic stable Lévy processes is due to rotational symmetry which is needed in the proof method. Compared to the method in [37] we have to overcome with some additional issues:

(1) An isotropic α-stable Lévy process for $\alpha \in (0, 2)$ is a pure jump process. Hence we cannot use Hölder continuity of the sample paths to derive upper bounds for the Hausdorff dimension as in case of fractional Brownian motion in [37].
(2) The Hurst index $H = 1/\alpha$ of an isotropic α-stable Lévy process is restricted to $H \geq 1/2$, whereas $H \in (0, 1)$ for fractional Brownian motion. It will turn out that the case $H \geq 1$ needs different arguments than the blueprint given for $H \in (0, 1)$ in [37].
(3) The tail of the probability density of an isotropic α-stable Lévy process falls off as a power function, see (2.3), whereas the normal density of fractional Brownian motion decreases exponentially fast.

In the third part we treat problems in the spectral theory of fractional random Schrödinger operators via isotropic stable Lévy processes. In particular we deal with the fractional Schrödinger equation with random potential

$$i \cdot \dot{\psi} = (-\Delta)^{\alpha/2}[\psi] + V_\omega \cdot \psi,$$

containing the Riesz fractional Laplacian plus some multiplicative random potential in the continuous setting. This equation describes phenomena in semiconductors. The Riesz fractional Laplacian is the generator of an isotropic stable Lévy process and we use its stochastic properties in order to prove certain results for the Integrated Density of States (IDS) of the operator. We follow Nakao's work [29] who developed the technique for the genuine Laplacian plus Poissonian and Gaussian potentials. In [31], Ōkura generalised his ideas to nonlocal Schrödinger operators with Poissonian potentials. We complement his work by dealing with fractional random Schrödinger operators with Gaussian potentials. In particular, we prove the existence of Lifshitz tails. Further, we analyse the asymptotics at the right end of the spectrum even for arbitrary stationary random potentials that fulfil a mild condition. These asymptotics mainly rely on the stationarity and the self-similarity of stable Lévy processes.

In Section 2 we define isotropic stable Lévy processes and its analytical analogue, the fractional Laplacian. We introduce a generalised version of the genuine Hausdorff dimension which is called the α-parabolic Hausdorff dimension in Section 3. We also give a priori upper and lower bounds for the α-parabolic Hausdorff dimension in terms of the genuine Hausdorff dimension. It turns out that covers by α-parabolic cylinders are optimal for treating self-similar processes, since their distinct non-linear scaling between time and space geometrically matches the self-similarity of the processes. We provide explicit formulas for the Hausdorff dimension of the graph and the range of an isotropic α-stable Lévy process plus Borel measurable drift function in Section 4 and defer the proofs to Sections 5–7. In sum the α-parabolic Hausdorff dimension of the drift term f alone contributes to the Hausdorff dimension of $X + f$. We derive new formulas and estimates for the α-parabolic Hausdorff dimension of constant functions and Hölder continuous functions in Section 8.
In Section 9 we introduce the model of the fractional random Schrödinger operator with Gaussian potential. In Section 10 we prove the existence of the IDS for the fractional Schrödinger operator with Gaussian potential. Then we analyse its asymptotics at the left end of the spectrum as $\lambda \to -\infty$. In Section 11 we prove Lifshitz tails for the fractional random Schrödinger operator with Gaussian potential, i.e. exponential decay of the IDS at the left end of the spectrum. Finally, in Section 12, we analyse the asymptotics at the right end of the spectrum as $\lambda \to +\infty$ for arbitrary stationary random potentials that satisfy some mild condition.

Stable Lévy processes are self-similar but discontinuous. So they act as test processes for self-similarity. Peres and Sousi worked with the Hölder continuity of the fractional Brownian motion and we can show that the self-similarity is more fundamental than the continuity in this context. In addition even the asymptotics of the quantum operators relay on self-similarity and not on continuity, as shown in our part three. Hence, we conjecture that here, self-similarity is a more fundamental concept than continuity.

2. Stable Lévy Processes and the Fractional Laplacian

Let $X = (X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$ which is a stochastic process on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the following properties:

(i) The process $\mathbb{P}$-almost surely starts in $0 \in \mathbb{R}^d$.
(ii) X possesses independent increments, i.e. for any $0 \leq t_0 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
(iii) X has stationary increments, i.e. the distribution of $X_{t+h} - X_t \overset{\mathrm{d}}{=} X_h$ does not depend on t, where the symbol $\overset{\mathrm{d}}{=}$ denotes equality in distribution.
(iv) X is stochastically continuous, i.e. $\mathbb{P}(|X_{t+h} - X_t| > \varepsilon) \to 0$ as $h \to 0$ for any $t \geq 0$ and $\varepsilon > 0$.

Additionally assuming self-similarity, the Lévy process is called stable. In this paper we only deal with the special case of an isotropic α-stable Lévy process in which case the self-similarity is given by

$$(X_{c\cdot t})_{t\geq 0} \overset{\mathrm{fd}}{=} (c^{1/\alpha} \cdot X_t)_{t\geq 0} \quad \text{for all } c > 0, \tag{2.1}$$

where $\overset{\mathrm{fd}}{=}$ denotes equality of all finite-dimensional distributions which characterise the stochastic processes in law. In this case the Hurst index $H = 1/\alpha$ is restricted to $H \geq \frac{1}{2}$, i.e. $\alpha \in (0, 2]$ and the isotropic α-stable Lévy process is also uniquely determined by the characteristic function $\mathbb{E}\big[e^{i\langle \xi, X_t\rangle}\big] = e^{-t\,C\cdot\|\xi\|^\alpha}$ with Lévy exponent $\Psi(\xi) = C \cdot \|\xi\|^\alpha$ for some constant $C > 0$. In case of $\alpha = 2$ we obtain Brownian motion. For details on stable Lévy processes we refer to the monograph [39].
The integrability of $\exp(-t\,C \cdot ||\xi||^\alpha)$ ensures the applicability of the Fourier inversion formula. Therefore, for any $t > 0$ the random variable X_t possesses the continuous density function

$$x \mapsto p(t, x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle} e^{-t\Psi(\xi)}\,\mathrm{d}\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle} e^{-t\,C\cdot\|\xi\|^\alpha}\,\mathrm{d}\xi$$

which for $\alpha \in (0, 2)$ cannot be expressed in simple terms but belongs to $C^\infty(\mathbb{R}^d)$ with all derivatives in $L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$; see [39]. Further, from the self-similarity property (2.1) it easily follows that

$$p(t, x) = t^{-d/\alpha} \cdot p\left(1, \frac{x}{t^{1/\alpha}}\right) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{2.2}$$

Thus we define $p(x) := p(1,x)$ as the density at time $t = 1$ and by Theorem 2.1 in [5] we have the tail estimate

$$p(x) \in \mathcal{O}\big(\|x\|^{-d-\alpha}\big) \quad \text{as } \|x\| \to \infty. \tag{2.3}$$

This density is bounded and rotationally symmetric, i.e. writing $x = ry$ with $r = \|x\| > 0$ and $y = x/\|x\| \in S^{d-1}$ the density $p(x) = p(ry)$ does not depend on y and due to unimodality, see [39], $r \mapsto p(ry)$ is non-increasing.

We also introduce the fractional Brownian motion.

Definition 2.1 (Gaussian Process, fractional Brownian motion)**.** A *Gaussian process* is a stochastic process on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that every finite collection of its random variables is multivariate normally distributed.
Let $H \in (0,1]$. The *fractional Brownian motion* $B^H = \big(B_t^H\big)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is a centered Gaussian process with covariance function

$$\mathbb{E}\big[B_t^H \cdot B_s^H\big] := \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big).$$

In case of $H = 1/2$ we obtain the Brownian motion.

The analytic analogon of an α-stable Lévy process is the (Riesz) fractional Laplacian. In $\mathbb{R}^d$ there exist plenty (almost) equivalent definitions of the fractional Laplacian, see [18], [23] or [25]. We use a pseudo-differential approach which is most suitable for our needs. Throughout this text we define the Fourier transform by

$$\mathscr{F}[\psi](\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \psi(x)\,\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}^d$$

acting on $L^1(\mathbb{R}^d)$. For non-negative ψ with $\|\psi\|_1 = 1$ it coincides with the characteristic function of a random variable Y with probability density ψ up to a constant, since

$$\mathbb{E}\left[e^{i\langle \xi, Y\rangle}\right] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle}\psi(x)\,\mathrm{d}x = (2\pi)^{d/2}\mathscr{F}[\psi](\xi).$$

The Fourier transform translates differential operators into polynomials and vice versa. Extending this to fractional powers, for fixed $\alpha \in (0,2)$ we define the fractional Laplacian $(-\Delta)^{\alpha/2}$ as

$$(-\Delta)^{\alpha/2}[\psi](x) := \mathscr{F}^{-1}\big[||\xi||^{\alpha} \cdot \mathscr{F}[\psi]\big](x), \tag{2.4}$$

where $||\xi||$ denotes the Euclidean norm of $\xi \in \mathbb{R}^d$, with domain

$$\mathscr{D}\left((-\Delta)^{\alpha/2}\right) := \left\{\psi \in L^2(\mathbb{R}^d) : (-\Delta)^{\alpha/2}[\psi] \in L^2(\mathbb{R}^d)\right\}.$$

The fractional Laplacian $(-\Delta)^{\alpha/2}$ generates a strongly continuous semigroup

$$T_t[\psi](x) := e^{-t(-\Delta)^{\alpha/2}}[\psi](x) \tag{2.5}$$

which solves the fractional heat equation with initial value v_0, i.e.

$$\dot{v}(t,x) = -(-\Delta)^{\alpha/2}[v](t,x), \quad v(0,x) = v_0(x). \tag{2.6}$$

The strongly continuous semigroup can also be expressed by its Markov transition kernel $p(t,x,y)$ given by the Lebesgue density function $p(t,x-y)$ of an isotropic α-stable Lévy process $X = (X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. The connection between the analytic, potential-theoretic and stochastic perspective is given by

$$T_t[\psi](x) = \int_{\mathbb{R}^d} p(t,x,y) \cdot \psi(y)\,\mathrm{d}y = \mathbb{E}_x[\psi(X_t)], \tag{2.7}$$

where $\mathbb{E}_x$ denotes the expected value with respect to $\mathbb{P}_x := \mathbb{P}(\,\cdot\,|X_0 = x)$. Hence, from an analytic perspective the fractional Laplacian is the negative generator of the strongly continuous semigroup and the transition kernel is its Green's function solution.

In part two we are interested in the fractal path behaviour $t \mapsto X_t + f(t)$ of an isotropic α-stable Lévy process plus Borel measurable drift f. The corresponding density functions $x \mapsto q(t,x)$ have the characteristic function

$$\begin{aligned}\widehat{q}(t,\xi) &= \mathbb{E}\left[e^{i\langle \xi, X_t + f(t)\rangle}\right] \\ &= \exp\left(-Ct \cdot ||\xi||^\alpha + i \cdot \langle \xi, f(t)\rangle\right)\end{aligned}$$

for $t > 0$ with derivative

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{q}(t,\xi) &= \left(-C \cdot ||\xi||^\alpha + i \cdot \langle \xi, \dot{f}(t)\rangle\right) \cdot \widehat{q}(t,\xi) \\ &= -C \cdot \widehat{(-\Delta)^{\alpha/2}}[q](t,\xi) - \langle \widehat{\nabla[q]}(t,\xi), \dot{f}(t)\rangle\end{aligned}$$

provided thet f is differentiable. Formal Fourier inversion leads to the fractional heat equation with drift

$$\dot{q} = -C \cdot (-\Delta)^{\alpha/2}[q] - \langle \nabla[q], \dot{f} \rangle$$

as the corresponding macroscopic flow. Our aim is to analyse the pathwise solutions by means of fractal geometry.

In part three we are interested in the fractional random Schrödinger operator

$$H_\omega[\psi] := (-\Delta)^{\alpha/2}[\psi] + V_\omega \cdot \psi. \tag{2.8}$$

The corresponding evolution semigroup has a probabilistic interpretation by means of the Feynman-Kac formula

$$e^{-tH_\omega}[\psi](x) = \mathbb{E}_x\left[e^{-\int_0^t V_\omega(X_s)\,\mathrm{d}s}\,\psi(X_t)\right]$$

as laid out in [12]. Now we want to restrict the fractional Laplacian $(-\Delta)^{\alpha/2}$ to a bounded open box $\Lambda \subset \mathbb{R}^d$ containing the origin. Since the Fourier transform is only defined in the whole space, the pseudo-differential approach fails. Exterior conditions are necessary, since X exits Λ almost surely with a jump into $\Lambda^{\complement} = \mathbb{R} \setminus \Lambda$ and does not touch $\partial\Lambda$. We choose zero Dirichlet conditions on the exterior $\Lambda^{\complement}$ corresponding to the first exit time $\tau_\Lambda := \inf\{t \geq 0 : X_t \in \Lambda^{\complement}\}$. In analogy to (2.5) and (2.7) we define the restricted fractional Laplacian $(-\Delta)_\Lambda^{\alpha/2}$ by its evolution semigroup

$$e^{-t(-\Delta)_\Lambda^{\alpha/2}}[\psi](x) = \int_\Lambda p_\Lambda(t,x,y)\psi(y)\,\mathrm{d}y = \mathbb{E}_x[\psi(X_t)\mathbb{1}_{\{t<\tau_\Lambda\}}] \tag{2.9}$$

for $\psi \in L^2(\Lambda)$ and the kernel is given by

$$p_\Lambda(t,x,y) = p(t,x,y)\,\mathbb{E}_{x,y}^{0,t}[\mathbb{1}_{\{t<\tau_\Lambda\}}] \quad \text{for all } x,y \in \Lambda \text{ and } t > 0,$$

where $\mathbb{E}_{x,y}^{0,t}$ denotes expectation with respect to $\mathbb{P}(\,\cdot\,|X_0 = x, X_t = y)$. For the construction of this α-stable bridge measure we refer to [10]. This yields again a probabilistic interpretation of the restricted fractional random Schrödinger operator $H_{\omega,\Lambda}$ for our Gaussian or Poissonian random potential by means of the Feynman-Kac formula

$$e^{-tH_{\omega,\Lambda}}[\psi](x) = \mathbb{E}_x\left[e^{-\int_0^t V_\omega(X_s)\,\mathrm{d}s}\,\psi(X_t)\,\mathbb{1}_{\{t<\tau_\Lambda\}}\right]$$

with $\psi \in L^2(\Lambda)$; see [12]. Since this operator is Hilbert-Schmidt due to $|\Lambda| < \infty$ and boundedness of the kernel

$$p_{\omega,\Lambda}(t,x,y) = p(t,x,y)\,\mathbb{E}^{0,t}_{x,y}\left[e^{-\int_0^t V_\omega(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{t<\tau_\Lambda\}}\right]$$

for all $x, y \in \Lambda$ and $t > 0$, this indeed results in a spectrum of the form

$$\sigma(H_{\omega,\Lambda}) = \left\{\lambda^{(1)}_{\omega,\Lambda} \leq \lambda^{(2)}_{\omega,\Lambda} \leq \dots\right\}.$$

for the restricted operator $H_{\omega,\Lambda}$. This is our starting point for investigations of the spectral theory of fractional random Schrödinger operators in part three of the thesis.

Part 2. Parabolic Fractal Geometry of Lévy Processes with Drift

3. On the Parabolic Hausdorff Dimension

We introduce the *α-parabolic Hausdorff dimension* and examine some of its properties from a measure theoretical point of view. This nonlinear fractal dimension inheres a distinct non-linear scaling between time and space. Hence it is usefull for the study of self-similiar stochastic processes like isotropic α-stable Lévy processes and the fractional Brownian motion.
We follow the measure theoretical arguments of Taylor and Watson in [42] who introduced their *parabolic Hausdorff dimension* in order to determine polar sets for the heat equation. In [37] Peres and Sousi applied their *H-parabolic Hausdorff dimension* to the fractional Brownian motion with drift. In this way they were able to calculate the Hausdorff dimension of this process solely in terms of the Hurst index H and the drift function. We apply their ideas to the graph and range of isotropic stable Lévy processes plus measurable drift function. For that purpose we use a novel version of their H-parabolic Hausdorff dimension which coincides with Taylor and Watson's parabolic Hausdorff dimension in the Gaussian case.

First we remark that the notion of *diameter* is the most fundamental concept for the following measure theoretical objects. In a metric space (X, d) it is defined for any set – regardless of how irregular it might be – per

$$|\cdot| : \mathcal{P}(X) \to [0, \infty], \; |A| = \sup_{x,y \in A} d(x, y).$$

The diameter enables us to assign a real value to objects like fractals, Suslin sets or even non-measurable sets with regard to their overall extent. But this alone does not give much information about the geometric microstructure or even the outer shape of our object. Instead we will cover an object by many small sets in order to involve its distinct geometry. Smallness is achieved by letting the radii of the covering sets tend to zero.
One easy way to do so is to cover our object with open balls, defined by the metric, which form together with the whole space and the empty set a topology. The union of these balls forms one big cover of our object. We can always obtain a cover from open sets – in the worst case we just take the whole space itself as the trivial cover. For getting information about the metric details of our covered object we want to

sum up the diameters of all of its covering sets which leads to series. Uncountable covers contain no metric information about our object since uncountable sums of positive numbers always diverge. Therefore we want our metric space in addition to be Lindelöf, i.e. each open cover of our set possesses a countable subcover. In our metric space this is equivalent to separability, i.e. there exists a countable dense subset. Separability in addition with completeness further ensures that we can choose either open or closed sets as covering sets: The diameter of a closed ball is determined by points sitting on its boundary. Such a point can be approximated by a Cauchy sequence. The existence of such a countable sequence is due to separability and the existence of the limit which equals the boundary point follows from completeness. For the same reason we can approximate closed sets by open sets and vice versa. For our purposes it is enough that there exists at least one metric on our separable space which induces a complete topology. Thus we could work on a *Polish space*, i.e. a separable completely metrisable topological space.

But we restrict our considerations to subsets $A = (t, x) \subseteq \mathbb{R}^{n+d}$. For $n = 1$ we interpret $\mathbb{R}_+ = [0, \infty)$ as time which is mapped to the d-dimensional Euclidean space. In the sequel we mostly work on $\mathbb{R}^{1+d}$ in order to treat stochastic processes, but the general case $\mathbb{R}^{n+d}$ could be interesting for stochastic fields. For real functions f, g the symbol $f \lesssim g$ denotes the existence of a constant $C \in (0, \infty)$ not depending on the variables such that $f \leq C \cdot g$ and $f \asymp g$ is short for $f \lesssim g$ and $g \lesssim f$

Definition 3.1 (Hausdorff {measure, dimension}). Let $A \subseteq \mathbb{R}^d$ be an arbitrary set and $\beta \in [0, \infty]$. We define the *β-Hausdorff (outer) measure of A* as the set function

$$\mathcal{H}^\beta : \mathcal{P}\big(\mathbb{R}^d\big) \to [0, \infty],$$

$$\mathcal{H}^\beta(A) := \lim_{\delta \downarrow 0} \inf \left\{ \sum_{k=1}^\infty |A_k|^\beta : A \subseteq \bigcup_{k=1}^\infty A_k,\ |A_k| \leq \delta \right\}.$$

If $\beta \leq \gamma$ and $\mathcal{H}^\beta(A) = 0$, then also $\mathcal{H}^\gamma(A) = 0$. Thus we can define the *Hausdorff dimension of A* as

$$\dim A := \inf \big\{ \beta > 0 : \mathcal{H}^\beta(A) = 0 \big\} = \sup \big\{ \beta > 0 : \mathcal{H}^\beta(A) = \infty \big\}.$$

A priori the Hausdorff *measure* is an *outer measure*. But the term *measure* usually refers to a *Borel measure*. However it can be shown that the Hausdorff *outer measure* is also a *metric outer measure* and thus in particular a *Borel measure*, see, e.g., Chapter 1 in Falconer's book [15]. The same folklore could be applied to our parabolic Hausdorff *outer measure* but we see no use in the restriction to Borel sets at this point.

Now we recall a well-known fact: The shapes of the covering sets in Definition 3.1 are, in some sense, irrelevant for the calculation of the Hausdorff dimension.

Remark 3.2. Let $k \in \mathbb{N}$. Any bounded set $A_k \subseteq \mathbb{R}^d$ can be covered by a d-dimensional hyperball with radius $|A_k/2|$ which can be inscribed into an isodimensional hypercube with sidelength $|A_k|$. We write

$$A_k \subseteq B_{|A_k|/2} \subset \square_{|A_k|}$$

where

$$|A_k| = \left|B_{|A_k|/2}\right| < \left|\square_{|A_k|}\right| = \sqrt{d}\ |A_k|.$$

This results in

$$\begin{aligned}
&\mathcal{H}^\beta(A) \\
&\quad= \lim_{\delta\downarrow 0} \inf_{\substack{A\subseteq \cup_{k\in\mathbb{N}} A_k \\ |A_k|\leq\delta}} \sum_{k=1}^{\infty} |A_k|^\beta \\
&\quad= \lim_{\delta\downarrow 0} \inf_{\substack{A\subseteq \cup_{k\in\mathbb{N}} B_{|A_k|/2} \\ B_{|A_k|/2}\leq\delta}} \sum_{k=1}^{\infty} \left|B_{|A_k|/2}\right|^\beta \\
&\quad\leq \lim_{\delta\downarrow 0} \inf_{\substack{A\subseteq \cup_{k\in\mathbb{N}} \square_{|A_k|} \\ |\square_{|A_k|}|\leq\sqrt{d}\cdot\delta}} \sum_{k=1}^{\infty} \left|\square_{|A_k|}\right|^\beta \\
&\quad= \sqrt{d}^{\,\beta} \cdot \mathcal{H}^\beta(A).
\end{aligned}$$

Both the left-hand side and the right-hand side become zero iff the β-Hausdorff measure vanishes. We say that the Hausdorff measures induced by bounded sets, hyperballs and hyperrectangles are *comparable* and thus yield the same Hausdorff dimension. We relate comparable measures by the symbol $\asymp$. Hence in order to determine the Hausdorff dimension of A it is immaterial what exact shape the covering sets have – as long as their diameter stays the same up to a uniform constant positive factor.

We next define the *parabolic Hausdorff outer measure*, a restriction of the ordinary Hausdorff measure. Only hyperrectangles with a certain proportion are permitted as covering sets.

Definition 3.3 (α-Parabolic {Hausdorff measure, cylinders, dimension})**.** Let $A \subseteq \mathbb{R}^{n+d}$ be an any set and $\alpha, \beta \in (0,\infty)$. The *$\alpha$-parabolic β-Hausdorff (outer) measure* of A is defined as the set function

$$\mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta} : \mathcal{P}\big(\mathbb{R}^{n+d}\big) \to [0,\infty],$$

$$\mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A) := \lim_{\delta \downarrow 0} \inf \left\{ \sum_{k=1}^{\infty} |\mathsf{P}_k|^{\beta} : A \subseteq \bigcup_{n=1}^{\infty} \mathsf{P}_k,\ \mathsf{P}_k \in \mathscr{P}^{\alpha},\ |\mathsf{P}_k| \le \delta \right\},$$

where the *α-parabolic cylinders* $(\mathsf{P}_k)_{k\in\mathbb{N}}$ are contained in

$$\mathscr{P}^{\alpha} := \left\{ \prod_{i=1}^{n} [t_i, t_i + c] \times \prod_{j=1}^{d} [x_j, x_j + c^{1/\alpha}],\ t_i,\ x_j \in \mathbb{R},\ c \in (0,1] \right\}. \tag{3.1}$$

We define the *α-parabolic Hausdorff dimension* of A as

$$\mathscr{P}^{\alpha}\text{-}\dim A := \inf\big\{\beta > 0 : \mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A) = 0\big\} = \sup\ \big\{\beta > 0 : \mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A) = \infty\big\}.$$

The case $\alpha = 1$ equals the genuine Hausdorff dimension which is simply denoted by the symbol dim.

Let us compare the α-parabolic Hausdorff dimension to other parabolic Hausdorff dimensions appearing in the literature. Taylor and Watson introduced the *parabolic Hausdorff dimension* $\mathscr{P}$-dim in [42] in order to determine polar sets for the heat equation. They defined the *parabolic Hausdorff measure* and *parabolic Hausdorff dimension* in the same way we did in Definition 3.3 for $n = 1$ and $\alpha = 2$ but they use parabolic cylinders of the form

$$\big[t, t + r^2\big] \times \prod_{j=1}^{d} \big[x_j, x_j + r\big],\ t,\ x_i \in \mathbb{R},\ r \in (0,1].$$

This makes the name "parabolic" in case of $\alpha = 2$ clear. Taylor and Watson's parabolic Hausdorff dimension coincides with our parabolic Hausdorff dimension via the substitution

$$r := \sqrt{c}.$$

On the contrary, for $H \in (0,1]$ Peres and Sousi used in [37] a slightly different construction. They calculated the Hausdorff dimension of the graph and the range of $B^H = \left(B_t^H\right)_{t\geq 0}$, the fractional Brownian motion of Hurst index $H \in (0,1]$, plus Borel measurable drift function. Instead, we treat isotropic α-stable Lévy processes plus Borel measurable drift function. Let the symbol $\overset{\mathrm{d}}{=}$ denote equality in *d*istribution and the symbol $\overset{\mathrm{fd}}{=}$ denote equality of all *f*inite-dimensional *d*istributions which characterise stochastic processes in law. An isotropic α-stable Lévy process inheres a certain self-similarity between time and space, i.e.

$$\left(X_{c\cdot t}\right)_{t\geq 0} \overset{\mathrm{fd}}{=} \left(c^{1/\alpha} \cdot X_t\right)_{t\geq 0}, \qquad \text{for all } c > 0,$$

for $\alpha \in (0,2]$. Hence the distribution of X_t for any fixed $t > 0$ can be derived from $X_1 \overset{\mathrm{d}}{=} t^{-1/\alpha} \cdot X_t$ by rescaling.
Now, the fractional Brownian motion B^H follows the self-similarity rule

$$\left(B_{c\cdot t}^H\right)_{t\geq 0} \overset{\mathrm{fd}}{=} \left(c^H \cdot B_t^H\right)_{t\geq 0}, \qquad \text{for all } c > 0.$$

Thus we may replace its self-similarity parameter which is the Hurst index $H \in (0,1]$ by $1/\alpha$. Peres and Sousi defined the *H-parabolic Hausdorff dimension* in terms of the $1/H$-parabolic cylinders $\mathscr{P}^{1/H}$ from Equation (3.1) in Definition 3.3. They used the *H-parabolic β-Hausdorff content*

$$\Psi_H^\beta(A) := \inf\left\{ \sum_{k=1}^\infty c_k^\beta,\ A \subseteq \bigcup_{k=1}^\infty \mathsf{P}_k,\ \mathsf{P}_k \in \mathscr{P}^{1/H} \right\}$$

for the definition of the *H-parabolic Hausdorff dimension* $\dim_{\Psi,H}$ which reads as follows

$$\dim_{\Psi,H} A := \inf\left\{\beta : \Psi_H^\beta(A) = 0\right\} = \sup\left\{\beta : \Psi_H^\beta(A) > 0\right\}.$$

The *H-parabolic β-Hausdorff content* differs from our *α-parabolic β-Hausdorff measure* in two ways: On the one hand the diameters of the covering sets do not explicitly tend to zero; on the other hand c_k^β instead of $|\mathsf{P}_k|^\beta \asymp \left(c_k^{1/\alpha}\right)^\beta$ is added up over k. The first difference has no influence on the induced dimensions, whereas the latter makes the dimension differ by a factor of $\alpha = 1/H$.

Proposition 3.4. *Let $A \subseteq \mathbb{R}^{1+d}$ be an arbitrary set, $\alpha = 1/H \in [1, \infty)$ and $\beta \in (0, \infty)$. Then one has*

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\beta(A) = 0 \quad \Leftrightarrow \quad \Psi_H^{\beta/H}(A) = 0. \tag{3.2}$$

Thus the induced fractal dimensions are related by

$$\mathscr{P}^\alpha\text{-}\dim A = (\dim_{\Psi,H} A)/H, \tag{3.3}$$

i.e. Peres and Sousi's H-parabolic Hausdorff dimension differs from our α-parabolic Hausdorff dimension by a constant factor $\alpha = 1/H$.

Proof. For the comparison of the dimensions we only have to proof the claim in Equation (3.2). Then the Equation (3.3) immediately follows.

First we introduce the auxiliary *α-parabolic β-Hausdorff content* and relate it to the *H-parabolic β-Hausdorff content*. Let it be defined as

$$\Phi_\alpha^\beta(A) := \inf \left\{ \sum_{k=1}^{\infty} |\mathsf{P}_k|^\beta, \ A \subseteq \bigcup_{k=1}^{\infty} \mathsf{P}_k, \ \mathsf{P}_k \in \mathscr{P}^\alpha \right\}.$$

Since for $\alpha \geq 1$ and $\mathsf{P}_k = [t_k, t_k + c_k] \times \prod_{j=1}^d [x_{j,k}, x_{j,k} + c_k^{1/\alpha}] \in \mathscr{P}^\alpha$ we have

$$|\mathsf{P}_k|^\beta \asymp c_k^{\beta/\alpha} = c_k^{H\cdot\beta}$$

one has

$$\Phi_\alpha^\beta(A) = \Psi_H^{\beta/H}(A). \tag{3.4}$$

Next we follow Proposition 4.9 in [28] in order to proof

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\beta(A) = 0 \quad \Leftrightarrow \quad \Phi_\alpha^\beta(A) = 0 \tag{3.5}$$

which together with Equation (3.4) shows (3.2).

" $\Rightarrow$ ": For the *if* part of (3.5) we show the contraposition

$$\Phi_\alpha^\beta(A) > 0 \quad \Rightarrow \quad \mathscr{P}^\alpha\text{-}\mathcal{H}^\beta(A) > 0.$$

But this is clear since $\mathscr{P}^\alpha$-$\mathcal{H}^\beta$ is derived from Φ_α^β by adding an extra condition on the size of the covering sets.

" $\Leftarrow$ ": For the *only if* part of (3.5) let $\Phi_\alpha^\beta(A) = 0$. Then for every $\delta > 0$ there exists a cover $\big(\widetilde{\mathsf{P}}_k\big)_{k\in\mathbb{N}} \subseteq \mathscr{P}^\alpha$ of A such that

$$\sum_{k=1}^{\infty} \big|\widetilde{\mathsf{P}}_k\big|^\beta \leq \delta.$$

Hence $\big|\widetilde{\mathsf{P}}_k\big| \leq \delta^{1/\beta}$ for every $k \in \mathbb{N}$. Therefore

$$\begin{aligned}
&\mathscr{P}^\alpha\text{-}\mathcal{H}^\beta(A)\\
&= \lim_{\delta\downarrow 0}\ \inf\left\{ \sum_{k=1}^{\infty} |\mathsf{P}_k|^\beta : A \subseteq \bigcup_{k=1}^{\infty} \mathsf{P}_k,\ \mathsf{P}_k \in \mathscr{P}^\alpha,\ |\mathsf{P}_k| \leq \delta \right\}\\
&= \lim_{\delta\downarrow 0}\ \inf\left\{ \sum_{k=1}^{\infty} |\mathsf{P}_k|^\beta : A \subseteq \bigcup_{k=1}^{\infty} \mathsf{P}_k,\ \mathsf{P}_k \in \mathscr{P}^\alpha,\ |\mathsf{P}_k| \leq \delta^{1/\beta} \right\}\\
&\leq \lim_{\delta\downarrow 0} \sum_{k=1}^{\infty} \big|\widetilde{\mathsf{P}}_k\big|^\beta \leq \lim_{\delta\downarrow 0}\ \delta = 0,
\end{aligned}$$

as desired. □

The α-parabolic Hausdorff dimension fulfils the following countable stability property which easily follows from monotonicity and σ-subadditivity of the α-parabolic Hausdorff measure as argued for the genuine Hausdorff dimension on page 29 in [15].

Proposition 3.5 (Countable stability property)**.** *Let* $(A_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^{n+d}$ *be a family of arbitrary sets. The* α*-parabolic Hausdorff dimension fulfils the* countable stability property

$$\mathscr{P}^\alpha\text{-}\dim \bigcup_{k=1}^{\infty} A_k = \sup_{k\in\mathbb{N}}\ \mathscr{P}^\alpha\text{-}\dim A_k$$

for every $\alpha \in (0,\infty)$*.*

Proof. We proof lower and upper bounds.

" $\geq$ ": Since the α-parabolic Hausdorff dimension is defined via an outer measure which is σ-subadditive, $A' \supseteq A$ implies $\mathscr{P}^\alpha\text{-}\dim A' \geq \mathscr{P}^\alpha\text{-}\dim A$. Hence

$$\mathscr{P}^\alpha\text{-}\dim \bigcup_{k=1}^{\infty} A_k \geq \sup_{k\in\mathbb{N}}\ \mathscr{P}^\alpha\text{-}\dim A_k.$$

" $\leq$ ": To see the converse, for each $k \in \mathbb{N}$ we use an α-parabolic cover $(\mathsf{P}_{k,l})_{l\in\mathbb{N}} \subseteq \mathscr{P}^\alpha$ of the set A_k with $|\mathsf{P}_{k,l}| \leq \delta$. Then one has

$$\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty}\bigcup_{l=1}^{\infty} \mathsf{P}_{k,l}$$

and we get

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\beta\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |\mathsf{P}_{k,l}|^\beta.$$

If for each $k \in \mathbb{N}$ we take the infimum over such covers of A_k we get by letting $\delta \downarrow 0$

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\beta\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mathscr{P}^\alpha\text{-}\mathcal{H}^\beta(A_k).$$

Now, for all $\beta > \sup_{k\in\mathbb{N}} \mathscr{P}^\alpha\text{-}\dim A_k$ we have $\mathscr{P}^\alpha\text{-}\mathcal{H}^\beta(A_k) = 0$ for each $k \in \mathbb{N}$ and thus $\mathscr{P}^\alpha\text{-}\mathcal{H}^\beta\big(\bigcup_{k=1}^\infty A_k\big) = 0$. This shows that $\mathscr{P}^\alpha\text{-}\dim\big(\bigcup_{k=1}^\infty A_k\big) \leq \beta$ and since $\beta > \sup_{k\in\mathbb{N}} \mathscr{P}^\alpha\text{-}\dim A_k$ is arbitrary the claim follows. $\square$

We derive the following a priori estimates for the Hausdorff dimension in terms of the parabolic Hausdorff dimension.

Theorem 3.6. *Let $A \subseteq \mathbb{R}^{n+d}$ be any set. Let $\phi_\alpha = \mathscr{P}^\alpha\text{-}\dim A$. Then one has*

$$\dim A \leq \begin{cases} \phi_\alpha \ \wedge \ \alpha \cdot \phi_\alpha + (1-\alpha)\cdot n, & \alpha \in (0,1], \\ \phi_\alpha \ \wedge \ \frac{1}{\alpha} \cdot \phi_\alpha + \big(1 - \frac{1}{\alpha}\big)\cdot d, & \alpha \in [1,\infty) \end{cases} \tag{3.6}$$

and

$$\dim A \geq \begin{cases} \phi_\alpha + \big(1 - \frac{1}{\alpha}\big)\cdot d, & \alpha \in (0,1], \\ \phi_\alpha + (1-\alpha)\cdot n, & \alpha \in [1,\infty). \end{cases} \tag{3.7}$$

Proof. (i) Let $\alpha \in (0,\infty)$. By the definition of the α-parabolic β-Hausdorff measure there are only coverings by $\mathscr{P}^\alpha$-sets permitted. So besides $\mathscr{P}^\alpha$ there could exist more efficient covers of A with respect to their shape. Therefore

$$
\begin{aligned}
&\mathcal{H}^{\beta}(A)\\
&\quad = \lim_{\delta\downarrow 0}\ \inf\left\{\sum_{k=1}^{\infty}|A_k|^{\beta} : A\subseteq\bigcup_{k=1}^{\infty}A_k,\ |A_k|\le\delta\right\}\\
&\quad \le \lim_{\delta\downarrow 0}\ \inf\left\{\sum_{k=1}^{\infty}|A_k|^{\beta} : A\subseteq\bigcup_{k=1}^{\infty}A_k,\ A_k\in\mathscr{P}^{\alpha},\ |A_k|\le\delta\right\}\\
&\quad = \mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A).
\end{aligned}
$$

Hence

$$\dim A = \inf\left\{\beta :\ \mathcal{H}^{\beta}=0\right\}\le\inf\left\{\beta :\ \mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A)=0\right\}=\phi_{\alpha}$$

always holds.

(ii) Let $\alpha\in(0,1]$ and $\varepsilon>0$ be arbitrary. If $\beta>\alpha\cdot\phi_{\alpha}+(1-\alpha)\cdot n$, then

$$\frac{\beta}{\alpha}+\left(1-\frac{1}{\alpha}\right)\cdot n>\phi_{\alpha}.$$

We can cover A by the α-parabolic cylinders

$$\left(\mathsf{P}_{c_k}\right)_{k\in\mathbb{N}}=\left(\prod_{i=1}^{n}\left[t_{i,k},\ t_{i,k}+c_k\right]\times\prod_{j=1}^{d}\left[x_{j,k},\ x_{j,k}+c_k^{1/\alpha}\right]\right)_{k\in\mathbb{N}}\subseteq\mathscr{P}^{\alpha}$$

with $\left|\mathsf{P}_{c_k}\right|\asymp c_k\le 1$ for every $k\in\mathbb{N}$ such that

$$\sum_{k=1}^{\infty}\left|\mathsf{P}_{c_k}\right|^{\beta/\alpha+(1-1/\alpha)\cdot n}\le\varepsilon.$$

Each P_{c_k} can be covered by $\left\lceil c_k^{1-1/\alpha}\right\rceil^{n}$ hypercubes $\square_{c_k^{1/\alpha}}$ with sidelength $c_k^{1/\alpha}$. Hence

$$
\begin{aligned}
&\mathcal{H}^{\beta}(A) \\
&\quad \le \sum_{k=1}^{\infty} \left\lceil c_k^{1-1/\alpha} \right\rceil^n \cdot \left| \Box_{c_k^{1/\alpha}} \right|^{\beta} \\
&\quad \lesssim \sum_{k=1}^{\infty} c_k^{\beta/\alpha+(1-1/\alpha)\cdot n} \\
&\quad \lesssim \sum_{k=1}^{\infty} \left| \mathsf{P}_{c_k} \right|^{\beta/\alpha+(1-1/\alpha)\cdot n} \\
&\quad \le \varepsilon.
\end{aligned}
$$

Since $\beta > \alpha \cdot \phi_\alpha + (1-\alpha) \cdot n$ is arbitrary we have

$$\dim A \le \alpha \cdot \phi_\alpha + (1-\alpha) \cdot n,$$

as claimed in (3.6).

(iii) Let $\alpha \in [1, \infty)$ and $\varepsilon > 0$ be arbitrary. If $\beta > 1/\alpha \cdot \phi_\alpha + (1 - 1/\alpha) \cdot d$, then

$$\alpha\beta + (1-\alpha) \cdot d > \phi_\alpha.$$

We can cover A by the α-parabolic cylinders

$$\left(\mathsf{P}_{c_k^{1/\alpha}}\right)_{k\in\mathbb{N}} = \left(\prod_{i=1}^{n} \left[t_{i,k},\, t_{i,k} + c_k\right] \times \prod_{j=1}^{d} \left[x_{j,k},\, x_{j,k} + c_k^{1/\alpha}\right] \right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^{\alpha}$$

with $|\mathsf{P}_{c_k^{1/\alpha}}| \asymp c_k^{1/\alpha} \le 1$ for every $k \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \left| \mathsf{P}_{c_k^{1/\alpha}} \right|^{\alpha\beta+(1-\alpha)\cdot d} \le \varepsilon.$$

Each $\mathsf{P}_{c_k^{1/\alpha}}$ can be covered by $\left\lceil c_k^{1/\alpha-1} \right\rceil^d$ hypercubes $\Box_{c_k}$ with sidelength c_k. Then

$$
\begin{aligned}
&\mathcal{H}^{\beta}(A) \\
&\quad \leq \sum_{k=1}^{\infty} \left\lceil c_k^{1/\alpha-1} \right\rceil^{d} \cdot \left|\square_{c_k}\right|^{\beta} \\
&\quad \lesssim \sum_{k=1}^{\infty} \left(c_k^{1/\alpha}\right)^{\alpha\beta+(1-\alpha)\cdot d} \\
&\quad \lesssim \sum_{k=1}^{\infty} \left|\mathsf{P}_{c_k^{1/\alpha}}\right|^{\alpha\beta+(1-\alpha)\cdot d} \\
&\quad \leq \varepsilon.
\end{aligned}
$$

Since $\beta > 1/\alpha \cdot \phi_\alpha + (1-1/\alpha) \cdot d$ is arbitrary we have

$$
\dim A \leq \frac{1}{\alpha} \cdot \phi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d,
$$

as claimed in (3.6).

(iv) Let $\alpha \in (0,1]$. Further, let $\beta > \dim A$ and $\varepsilon > 0$ be arbitrary. Then we can cover A with hypercubes

$$
\left(\square_{c_k}\right)_{k\in\mathbb{N}} = \left(\prod_{i=1}^{n} \left[t_{i,k},\, t_{i,k} + c_k\right] \times \prod_{j=1}^{d} \left[x_{j,k},\, x_{j,k} + c_k\right] \right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^1
$$

of sidelength $c_k \leq 1$ for every $k \in \mathbb{N}$ such that

$$
\sum_{k=1}^{\infty} \left|\square_{c_k}\right|^{\beta} \leq \varepsilon.
$$

Each $\square_{c_k}$ can be covered by $\left\lceil c_k^{1-1/\alpha} \right\rceil^{d}$ α-parabolic cylinders

$$
\left(\mathsf{P}_{c_k}\right)_{k\in\mathbb{N}} = \left(\prod_{i=1}^{n} \left[t_{i,k},\, t_{i,k} + c_k\right] \times \prod_{j=1}^{d} \left[x_{j,k},\, x_{j,k} + c_k^{1/\alpha}\right] \right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^{\alpha}
$$

with $|P_{c_k}| \asymp c_k$. By choosing $\gamma = \beta + (1/\alpha - 1) \cdot d$ one has

$$
\begin{aligned}
&\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(A)\\
&\quad\le \sum_{k=1}^\infty \left\lceil c_k^{1-1/\alpha}\right\rceil^d \cdot \left|\mathsf{P}_{c_k}\right|^\gamma\\
&\quad\lesssim \sum_{k=1}^\infty c_k^{(1-1/\alpha)d+\gamma}\\
&\quad= \sum_{k=1}^\infty c_k^\beta\\
&\quad\lesssim \sum_{k=1}^\infty \left|\square_{c_k}\right|^\beta\\
&\quad\le \varepsilon.
\end{aligned}
$$

Since $\beta > \dim A$ is arbitrary, one has

$$\mathscr{P}^\alpha\text{-}\dim A \le \dim A + \left(\frac{1}{\alpha} - 1\right) \cdot d,$$

as claimed in (3.7).

(v) Let $\alpha \in [1, \infty)$. Further, let $\beta > \dim A$ and $\varepsilon > 0$ be arbitrary. Then we can cover A with hypercubes

$$\left(\square_{c_k}\right)_{k\in\mathbb{N}} = \left(\prod_{i=1}^n \left[t_{i,k},\, t_{i,k} + c_k\right] \times \prod_{j=1}^d \left[x_{j,k},\, x_{j,k} + c_k\right]\right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^1$$

of sidelength $c_k \le 1$ for every $k \in \mathbb{N}$ such that

$$\sum_{k=1}^\infty \left|\square_{c_k}\right|^\beta \le \varepsilon.$$

Each $\square_{c_k}$ can be covered by $\left\lceil c_k^{1-\alpha}\right\rceil^n$ α-parabolic cylinders

$$\left(\mathsf{P}_{c_k}\right)_{k\in\mathbb{N}} = \left(\prod_{i=1}^n \left[t_{i,k},\, t_{i,k} + c_k^\alpha\right] \times \prod_{j=1}^d \left[x_{j,k},\, x_{j,k} + c_k\right]\right) \subseteq \mathscr{P}^\alpha$$

with $|P_{c_k}| \asymp c_k$. By choosing $\gamma = \beta + (\alpha - 1) \cdot n$ one has

$$
\begin{aligned}
&\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(A) \\
&\quad \le \sum_{k=1}^{\infty} \left\lceil c_k^{1-\alpha} \right\rceil^n \cdot \left|\mathsf{P}_{c_k}\right|^\gamma \\
&\quad \lesssim \sum_{k=1}^{\infty} c_k^{(1-\alpha)\cdot n+\gamma} \\
&\quad = \sum_{k=1}^{\infty} c_k^{\beta} \\
&\quad \lesssim \sum_{k=1}^{\infty} \left|\square_{c_k}\right|^\beta \\
&\quad \le \varepsilon.
\end{aligned}
$$

Since $\beta > \dim A$ is arbitrary, one has

$$\mathscr{P}^\alpha\text{-}\dim A \le \dim A + (\alpha - 1) \cdot n,$$

as claimed in (3.7), and the theorem is proven. $\square$

4. Main Results: Formulas for the Hausdorff Dimension

So far our considerations regarding the parabolic Hausdorff dimension were purely of geometric nature. Now we will apply it to stochastic processes. It is easy to translate the results from Theorem 1.2 in [37] into both Taylor and Watson's and our language and to generalise them to arbitrary Borel sets $T \subseteq \mathbb{R}_+$ via Proposition 3.4 and the countable stability property given by Proposition 3.5.

Theorem 4.1 (Peres & Sousi, 2012)**.** *Let $B^H = \big(B_t^H\big)_{t\geq 0}$ be a fractional Brownian motion in $\mathbb{R}^d$ of Hurst index $1/\alpha = H \in (0,1]$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $T \subseteq \mathbb{R}_+$ be a Borel set and $f : T \to \mathbb{R}^d$ be a Borel measurable function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^\alpha\text{-}\mathcal{G}_T(f)$ of the graph of f over T. Then one $\mathbb{P}$-almost surely has*

$$\dim \mathcal{G}_T\big(B^H + f\big) = \varphi_\alpha \ \wedge \ \frac{1}{\alpha} \cdot \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d.$$

Further one $\mathbb{P}$-almost surely has

$$\dim \mathcal{R}_T\big(B^H + f\big) = \varphi_\alpha \ \wedge \ d$$

for the range of $B^H + f$ over T.

We unite the cogitations of the following sections and derive a formula for the Hausdorff dimension of the graph $\mathcal{G}_T(X + f) = \{(t, X_t + f(t)) : t \in T\}$ of an isotropic stable Lévy process X plus Borel measurable drift function f.

Theorem 4.2 (Hausdorff dimension of the graph.)**.** *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0, 2]$. Let $X = (X_t)_{t\geq 0} \subseteq \mathbb{R}^d$ be an isotropic α-stable Lévy process on $\mathbb{R}^d$. Further, let $f : T \to \mathbb{R}^d$ be a Borel measurable function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ of the graph of f over T where $\varphi_1 = \dim \mathcal{G}_T(f)$ denotes the genuine Hausdorff dimension. Then one $\mathbb{P}$-almost surely has*

$$\dim \mathcal{G}_T(X + f) = \begin{cases} \varphi_1, & \alpha \in (0,1], \\ \varphi_\alpha \ \wedge \ \frac{1}{\alpha} \cdot \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d, & \alpha \in [1,2]. \end{cases}$$

Proof. The Gaussian case where $\alpha = 2$ follows from Theorem 4.1. The other cases will follow by the combination of Corollary 5.3 with Theorem 6.11 for drift functions f with $||f(t) - f(s)|| \leq 1$ for all $s, t \in T$. For the treatment of arbitrary drift functions we choose a countable sequence $\big(x_n\big)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$, where each coordinate of x_n belongs to $\mathbb{Z}/2$ such that the closed balls with radius $1/2$ and center x_n are a cover of the range of f on T, i.e.

$$\mathcal{R}_T(f) \subseteq \bigcup_{n\in\mathbb{N}} B_{x_n}\left(\frac{1}{2}\right).$$

Define the Borel sets

$$T_n := \left\{t \in T : f(t) \in B_{x_n}\left(\frac{1}{2}\right)\right\},\ n \in \mathbb{N},$$

then

$$\bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(X+f) = \mathcal{G}_T(X+f)$$

and

$$\bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(f) = \mathcal{G}_T(f).$$

For $\alpha \in (0,1]$ Proposition 3.5 yields

$$\begin{aligned}
&\dim \mathcal{G}_T(X+f)\\
&= \dim \bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(X+f)\\
&= \sup_{n\in\mathbb{N}}\ \dim \mathcal{G}_{T_n}(X+f)\\
&= \sup_{n\in\mathbb{N}}\ \dim \mathcal{G}_{T_n}(f)\\
&= \dim \bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(f)\\
&= \dim \mathcal{G}_T(f).
\end{aligned}$$

For $\alpha \in [1,2)$ and $\varphi_\alpha < d$ it yields

$$\begin{aligned}
&\dim \mathcal{G}_T(X+f)\\
&= \dim \bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(X+f)\\
&= \sup_{n\in\mathbb{N}}\ \dim \mathcal{G}_{T_n}(X+f)\\
&= \sup_{n\in\mathbb{N}}\ \mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_{T_n}(f)
\end{aligned}$$

$$
\begin{aligned}
&= \mathscr{P}^{\alpha}\text{- dim} \bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(f) \\
&= \mathscr{P}^{\alpha}\text{- dim}\, \mathcal{G}_T(f).
\end{aligned}
$$

For $\alpha \in [1, 2)$ and $\varphi_\alpha \geq d$ it yields

$$
\begin{aligned}
&\dim \mathcal{G}_T(X + f) \\
&\quad = \dim \bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(X + f) \\
&\quad = \sup_{n\in\mathbb{N}}\ \dim \mathcal{G}_{T_n}(X + f) \\
&\quad = \sup_{n\in\mathbb{N}}\ \big(1/\alpha \cdot \mathscr{P}^{\alpha}\text{- dim}\, \mathcal{G}_{T_n}(f) + (1 - 1/\alpha) \cdot d\big) \\
&\quad = 1/\alpha \cdot \mathscr{P}^{\alpha}\text{- dim} \bigcup_{n\in\mathbb{N}} \mathcal{G}_{T_n}(f) + (1 - 1/\alpha) \cdot d \\
&\quad = 1/\alpha \cdot \mathscr{P}^{\alpha}\text{- dim}\, \mathcal{G}_T(f) + (1 - 1/\alpha) \cdot d
\end{aligned}
$$

which proofs the claim. □

The formula for the Hausdorff dimension of the range $\mathcal{R}_T(X+f) = \{X_t+f(t) : t \in T\}$ of an isotropic stable Lévy process X plus Borel measurable drift function f reads as follows.

Theorem 4.3 (Hausdorff dimension of the range.)**.** *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0, 2]$. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process on $\mathbb{R}^d$ and $f : T \to \mathbb{R}^d$ be a Borel measurable function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^{\alpha}\text{- dim}\, \mathcal{G}_T(f)$ of the graph of f over T. Then one $\mathbb{P}$-almost surely has*

$$
\dim \mathcal{R}_T(X + f) = \begin{cases} \alpha \cdot \varphi_\alpha \ \wedge\ d, & \alpha \in (0, 1], \\ \varphi_\alpha \ \wedge\ d, & \alpha \in [1, 2]. \end{cases}
$$

Proof. The Gaussian case where $\alpha = 2$ follows from Theorem 4.1. The other cases follow by the combination of Lemma 7.1 and Lemma 7.4 for drift functions f with $||f(t) - f(s)|| \leq 1$ for all $s, t \in T$. For arbitrary drift functions we can use the countable stability from Proposition 3.5 analogously to the proof of Theorem 4.2. □

Our main theorems imply an improvement of the a priori estimates from Theorem 3.6 in case of $\alpha \in (0,1]$ for the Hausdorff dimension of the graph of f over T.

Corollary 4.4. *Let $T \subseteq \mathbb{R}_+$ be a Borel set and let $f : T \to \mathbb{R}^d$ be a Borel measurable function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ of the graph of f over T where φ_1 denotes the genuine Hausdorff dimension. In case of $\alpha \in (0,1]$ one has*

$$\varphi_1 \geq \alpha \cdot \varphi_\alpha \ \vee \ \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d.$$

Proof. For $\alpha \in (0,1]$ the combination of Theorem 4.2 and Theorem 4.3 directly yields

$$\varphi_1 = \dim \mathcal{G}_T(f) \geq \dim \mathcal{G}_T(X+f) \geq \dim \mathcal{R}_T(X+f) = \alpha \cdot \varphi_\alpha \ \wedge \ d.$$

Further we have

$$\alpha \cdot \varphi_\alpha \geq \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d \quad \text{if and only if} \quad \alpha \cdot \varphi_\alpha \leq d$$

and

$$d \geq \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d \quad \text{if and only if} \quad \alpha \cdot \varphi_\alpha \leq d$$

which proves the claim. □

5. Graph: Upper Bound via Geometric Measure Theory

We calculate an upper bound for the Hausdorff dimension of the graph of an isotropic stable Lévy process X plus drift function by means of an efficient covering.

Theorem 5.1. *Let $T \subseteq \mathbb{R}_+$ be any set and $\alpha \in (0,2]$. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process on $\mathbb{R}^d$ and $f : T \to \mathbb{R}^d$ be any function. Further let $\varphi_\alpha = \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ be the α-parabolic Hausdorff dimension of the graph of f over T. For $\alpha \in (0,1]$ one $\mathbb{P}$-almost surely has*

$$\dim \mathcal{G}_T(X+f) \leq \dim \mathcal{G}_T(f) = \varphi_1,$$

and for $\alpha \in [1,2]$ one $\mathbb{P}$-almost surely has

$$\mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(X+f) \leq \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f) = \varphi_\alpha.$$

Proof. (i) Let $\alpha \in (0,1]$, $\beta = \varphi_1$ and let δ, $\varepsilon > 0$ be arbitrary. Then $\mathcal{G}_T(f)$ can be covered by hypercubes

$$\left(\square_{c_k}\right)_{k\in\mathbb{N}} = \left(\left[t_k,\, t_k + c_k\right] \times \prod_{i=1}^{d} \left[x_{i,k},\, x_{i,k} + c_k\right]\right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^1$$

such that

$$\sum_{k=1}^{\infty} \left|\square_{c_k}\right|^{\beta+\delta} \lesssim \sum_{k=1}^{\infty} c_k^{\beta+\delta} \leq \varepsilon.$$

Let $M_k(\omega)$ be the random number of a fixed 2^d-nested collection of hypercubes with sidelength $c_k^{1/\alpha}$ that the path $t \mapsto X_t(\omega)$ hits at some time $t \in [t_k,\, t_k + c_k]$. Let $\cup_{k\in\mathbb{N}} \mathsf{P}_{c_k} \supseteq \mathcal{G}_T(X(\omega))$ with

$$\left(\mathsf{P}_{c_k}(\omega)\right)_{k\in\mathbb{N}} = \left(\left[t_k,\, t_k + c_k\right] \times \bigcup_{j=1}^{M_k(\omega)} \prod_{i=1}^{d} \left[\xi_{i,j,k}(\omega),\, \xi_{i,j,k}(\omega) + c_k^{1/\alpha}\right]\right)_{k\in\mathbb{N}}$$

being a corresponding random parabolic cover of the graph of this path. Then for all $t \in [t_k,\, t_k + c_k]$ there exists $j \in \{1, \ldots, M_k(\omega)\}$ such that for the i-th component of $X + f$ we have

$$\begin{aligned} \xi_{i,j,k}(\omega) + x_{i,k} &\leq X_t^{(i)}(\omega) + f_{(i)}(t) \\ &\leq \xi_{i,j,k}(\omega) + x_{i,k} + c_k^{1/\alpha} + c_k \\ &\leq \xi_{i,j,k}(\omega) + x_{i,k} + 2c_k. \end{aligned}$$

Hence we obtain a random cover $\cup_{k\in\mathbb{N}} \widetilde{\square}_{c_k}(\omega) \supseteq \mathcal{G}_T(X(\omega)+f)$ where

$$\widetilde{\square}_{c_k}(\omega) = \big[t_k,\, t_k+c_k\big] \times \bigcup_{j=1}^{M_k(\omega)} \prod_{i=1}^{d} \Big[\xi_{i,j,k}(\omega)+x_{i,k},\, \xi_{i,j,k}(\omega)+x_{i,k}+c_k\Big]$$
$$\cup \ \Big[\xi_{i,j,k}(\omega)+x_{i,k}+c_k,\, \xi_{i,j,k}(\omega)+x_{i,k}+2c_k\Big]$$

This is a union of $M_k(\omega)\cdot 2^d$ sets with

$$\big|\widetilde{\square}_{c_k}\big| \asymp c_k.$$

An application of Pruitt and Taylor's covering Lemma 6.1 in [38] and Lemma 3.4 in [27] shows that for all $\delta'>0$ one has

$$\mathbb{E}[M_k] \lesssim \frac{c_k}{\mathbb{E}\Big[T\Big(c_k^{1/\alpha}/3, c_k\Big)\Big]} \lesssim c_k^{-\delta'/\alpha},$$

where $T\Big(c_k^{1/\alpha}/3, c_k\Big)$ is the sojourn time of the process $(X_t)_{t\in[0,c_k]}$ in a ball of radius $c_k^{1/\alpha}/3$ centred at the origin. Hence we get for $\varepsilon' = \delta + \delta'/\alpha > 0$

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{H}^{\beta+\varepsilon'}(\mathcal{G}_T(X+f))\Big] \\
&\le \mathbb{E}\left[\sum_{k=1}^{\infty} |\widetilde{\square}_{c_k}|^{\beta+\varepsilon'}\right] \\
&\lesssim \sum_{k=1}^{\infty} \mathbb{E}[M_k(\omega)]\cdot c_k^{\beta+\varepsilon'} \\
&\lesssim \sum_{k=1}^{\infty} c_k^{\beta+\delta} \\
&\le \varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ are arbitrary, we get for all $\alpha\in(0,1]$ and $\varepsilon'>0$

$$\mathbb{E}\Big[\mathcal{H}^{\beta+\varepsilon'}(\mathcal{G}_T(X+f))\Big] = 0$$

which implies

$$\mathcal{H}^{\beta+\varepsilon'}(\mathcal{G}_T(X+f)) = 0, \quad \mathbb{P}\text{-almost surely.}$$

Hence

$$\mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(X+f) \le \beta + \varepsilon', \quad \mathbb{P}\text{-almost surely.}$$

Since $\varepsilon' > 0$ is arbitrary we finally get

$$\dim \mathcal{G}_T(X+f) \le \beta = \varphi_1, \quad \mathbb{P}\text{-almost surely,}$$

as claimed.

(ii) Let $\alpha \in [1,2]$, $\beta = \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ and let $\varepsilon, \delta > 0$ be arbitrary. Then $\mathcal{G}_T(f)$ can be covered by α-parabolic cylinders

$$\left(\mathsf{P}_{c_k^{1/\alpha}}\right)_{k\in\mathbb{N}} = \left(\left[t_k,\, t_k + c_k\right] \times \prod_{i=1}^d \left[x_{i,k},\, x_{i,k} + c_k^{1/\alpha}\right]\right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^\alpha$$

such that

$$\sum_{k=1}^\infty \left|\mathsf{P}_{c_k^{1/\alpha}}\right|^{\beta+\delta} \lesssim \sum_{k=1}^\infty c_k^{(\beta+\delta)/\alpha} \le \varepsilon.$$

Let $M_k(\omega)$ be the random number of a fixed 2^d-nested collection of hypercubes with sidelength $c_k^{1/\alpha}$ that the path $t \mapsto X_t(\omega)$ hits at some time $t \in [t_k,\, t_k + c_k]$. As in *(i)*, we obtain a random parabolic cover $\cup_{k\in\mathbb{N}} \widetilde{\mathsf{P}}_{c_k^{1/\alpha}}(\omega) \supseteq \mathcal{G}_T(X(\omega)+f)$ where

$$\begin{aligned}\widetilde{\mathsf{P}}_{c_k^{1/\alpha}}(\omega) = \left[t_k,\, t_k + c_k\right] \times \bigcup_{j=1}^{M_k(\omega)} \prod_{i=1}^d &\left[\xi_{i,j,k}(\omega) + x_{i,k},\, \xi_{i,j,k}(\omega) + x_{i,k} + c_k^{1/\alpha}\right] \\ &\cup \left[\xi_{i,j,k}(\omega) + x_{i,k} + c_k^{1/\alpha},\, \xi_{i,j,k}(\omega) + x_{i,k} + 2c_k^{1/\alpha}\right].\end{aligned}$$

This is a union of $M_k(\omega) \cdot 2^d$ sets with diameter

$$|\widetilde{\mathsf{P}}_{c_k^{1/\alpha}}(\omega)| \lesssim c_k^{1/\alpha}.$$

An application of Pruitt and Taylor's covering Lemma 6.1 in [38] and Lemma 3.4 in [27] show that

$$\mathbb{E}[M_k] \le \frac{c_k}{\mathbb{E}\left[T\left(c_k^{1/\alpha}/3, c_k\right)\right]} \lesssim c_k^{-\delta'/\alpha},$$

where $T\Big(c_k^{1/\alpha}/3, c_k\Big)$ denotes the sojourn time of the process $(X_t)_{t\in[0,c_k]}$ in a ball of radius $c_k^{1/\alpha}/3$ centred at the origin. Hence we get for $\varepsilon' = \delta + \delta' > 0$ with the same calculations as above

$$\begin{aligned}
&\mathbb{E}\Big[\mathscr{P}^\alpha\text{-}\mathcal{H}^{\beta+\varepsilon'}(\mathcal{G}_T(X+f))\Big] \\
&\lesssim \sum_{k=1}^\infty \mathbb{E}[M_k]\cdot c_k^{(\beta+\varepsilon')/\alpha} \\
&\lesssim \sum_{k=1}^\infty c_k^{(\beta+\varepsilon'-\delta')/\alpha} \\
&= \sum_{k=1}^\infty c_k^{(\beta+\delta)/\alpha} \\
&\le \varepsilon.
\end{aligned}$$

Since $\varepsilon, \varepsilon' > 0$ are arbitrary, as in *(i)* we finally get

$$\mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(X+f) \le \beta = \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f),\ \mathbb{P}\text{-almost surely},$$

as claimed. □

Remark 5.2. The reason why the parabolic scaling is just the right one in order to treat stable Lévy processes lies in the self-similarity of X. This is reflected in the interplay between the number $M_{c^{1/\alpha}}(s)$ of hypercubes with sidelength $c^{1/\alpha}$ hit by $(X_t)_{t\in[0,s]}$ and its sojourn time. Since by Lemma 6.1 in [38]

$$\mathbb{E}\big[M_{c^{1/\alpha}}(s)\big] \lesssim \frac{s}{\mathbb{E}\Big[T_{c_k^{1/\alpha}/3}(s)\Big]}$$

and by Lemma 3.4 in [27]

$$\mathbb{E}\Big[T_{c_k^{1/\alpha}/3}(s)\Big] \gtrsim c^{1+\delta'/\alpha}$$

we have to choose $s = c$ in order get rid of the c in the denominator. This is exactly the proportion of self-similarity of X.

Finally, we get an upper bound for the Hausdorff dimension of the graph of $X + f$.

Corollary 5.3. *Let $T \subseteq \mathbb{R}_+$ be any set and $\alpha \in (0, 2)$. Let $X = (X_t)_{t \geq 0}$ be an isotropic α-stable Lévy process on $\mathbb{R}^d$ and $f : T \to \mathbb{R}^d$ be any function. Define $\varphi_\alpha := \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ where $\varphi_1 = \dim \mathcal{G}_T(f)$ denotes the genuine Hausdorff dimension. Then one $\mathbb{P}$-almost surely has*

$$\dim \mathcal{G}_T(X + f) \leq \begin{cases} \varphi_1, & \alpha \in (0, 1], \\ \varphi_\alpha \ \wedge \ \frac{1}{\alpha} \cdot \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d, & \alpha \in [1, 2). \end{cases}$$

Proof. For $\alpha \in (0, 1]$ this follows directly from Theorem 5.1.

For $\alpha \in [1, 2]$ we have by (3.6) and Theorem 5.1

$$\begin{aligned} &\dim \mathcal{G}_T(X + f) \\ &\quad \leq \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(X + f) \wedge \frac{1}{\alpha} \cdot \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(X + f) + \left(1 - \frac{1}{\alpha}\right) \cdot d \\ &\quad \leq \varphi_\alpha \wedge \frac{1}{\alpha} \cdot \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d, \end{aligned}$$

as claimed. $\square$

6. Graph: Lower Bound via Potential Theory

Next we want to calculate a lower bound for the Hausdorff dimension of isotropic stable Lévy processes with drift. This will be accomplished by the energy method, cf. Section 4.3 in [28]. This method makes use of the Lebesgue integral. Hence for the first time we have to impose restrictions on the domain $T \subseteq \mathbb{R}_+$ and the drift function $f : T \to \mathbb{R}^d$ with regard to their measurability. Since the upper bound of the parabolic Hausdorff dimension holds for any set $A \subseteq \mathbb{R}^{n+d}$ and any drift function f we conjecture that we could also find a lower bound for this very general setting. We believe that we may overcome the difficulties with respect to measurability by the following remark.

Remark 6.1. We could presume a transitive $\in$-model of **ZFC** and the existence of a strongly inaccessible cardinal. This is equivalent to the axiom of the existence of a Grothendieck universe, i.e. a set which is closed under taking power sets, see [1]. Solovay showed in [40] that under these assumptions there exists a transitive $\in$-model of **ZF** such that every subset of the reals is Lebesgue measurable. In other words: In Solovay's model the axiom of choice is required to produce geometrically absurd and hence non-evident sets which are non-Lebesgue measurable and therefore we may replace it by postulating a number which is far away from our accessible universe. Loosely speaking we could pass our measure theoretical problems of the Banach-Tarski type on to the philosophers and logicians. As a consequence we would not have to restrict our sets in any ways with regard to Lebesgue measurability. Further in Remark 1.5 of [40] Solovay conjectures that in his model all real sets are Choquet capacitable. Hence in Solovay's model the energy method should work for all real sets $A \subseteq \mathbb{R}^{n+d}$ and drift functions $f : T \to \mathbb{R}^d$ by endowing the reals with the σ-algebra of its power set which would be equal to the σ-algebra of the Lebesgue measurable sets.

But we have to restrict our considerations to the work which has been done so far under the aspect of utility for our applications. The σ-algebra on T should fit its standard topology. Hence it is natural to choose T from the Borel σ-algebra, i.e. the smallest σ-algebra containing the open sets. Borel sets are indeed adequate for handling the graph of a Borel measurable function:

Proposition 6.2. *The graph of a Borel measurable function over a Borel set again is a Borel set, i.e. if $f : \mathbb{R}^n \to \mathbb{R}^d$ is a Borel measurable function then one has $\mathcal{G}_T(f) \in \mathscr{B}(\mathbb{R}^n) \otimes \mathscr{B}(\mathbb{R}^d)$ for every $T \in \mathscr{B}(\mathbb{R}^n)$.*

Proof. The function $g : \mathbb{R}^{n+d} \to \mathbb{R}_+,\ g(t,x) = ||f(t) - x||$ is Borel measurable since it is composed of Borel measurable functions. Singletons are closed in any Hausdorff space and thus they belong to $\mathscr{B}(\mathbb{R}_+)$. Hence $g^{-1}(\{0\}) = \mathcal{G}_T(f)$ is a Borel set. □

In contrast the image of a Borel set under a Borel measurable function does not have to be a Borel set, see e.g. [21]. But it is the (continuous) projection of the graph and thus an analytic set which is equivalent to be a Suslin set. Suslin sets can be described by a certain Suslin operator consisting of unions and cuts of closed sets whereas Borel sets are in addition stable under taking complements. Therefore the Suslin sets contain the Borel sets. Furthermore Suslin sets are stable under mappings by Borel measurable functions. We introduce some notions from potential theory.

Definition 6.3 (Difference kernel, energy, capacity)**.** Let $K : \mathbb{R}^{n+d} \to [0,\infty]$ be a Lebesgue measurable function which is called the *difference kernel*, $A \subseteq \mathbb{R}^{n+d}$ be a Suslin set and μ be a probability measure supported on A, i.e. $\mu \in \mathcal{M}^1(A)$. The *K-energy of a probability measure* μ is defined to be

$$\mathcal{E}_K(\mu) := \int_A \int_A K(t-s, x-y)\, \mathrm{d}\mu(t,x)\, \mathrm{d}\mu(s,y)$$

and the *equilibrium value of* A is defined as

$$\mathcal{E}_K* := \inf_{\mu \in \mathcal{M}^1(A)} \mathcal{E}_K(\mu).$$

We define the *K-capacity of* A as

$$\mathrm{Cap}_K(A) := \frac{1}{\mathcal{E}_K*}.$$

Whenever the kernel has the form

$$K(t,x) = ||(t,x)||^{-\beta},$$

we write $\mathcal{E}_\beta(\mu)$ for $\mathcal{E}_K(\mu)$ and $\mathrm{Cap}_\beta(A)$ for $\mathrm{Cap}_K(A)$ and we refer to them as the *β-energy of a probability measure* μ and the *Riesz β-capacity of* A, respectively. Note that the norm $||\cdot||$ above does not have to be the Euclidean one since all norms are equivalent in $\mathbb{R}^{n+d}$.

We can always use the Riesz potential in order to calculate a lower bound for the parabolic Hausdorff dimension.

Theorem 6.4 (Frostman's Theorem)**.** *Let $\alpha > 0$. For any Suslin set $A \subseteq \mathbb{R}^{n+d}$ one has*

$$\mathscr{P}^\alpha\text{-}\dim A \geq \dim A = \sup\{\beta :\ \mathrm{Cap}_\beta(A) > 0\}.$$

Proof. This follows from Theorem 3.6 together with [9] or Appendix B of [4]. □

From Frostman's Theorem 6.4 we see the following correspondence between the β-Hausdorff outer measure $\mathcal{H}^\beta(A)$, the Riesz β-capacity $\mathrm{Cap}_\beta(A)$, and the β-energy $\mathcal{E}_\beta(\mu) \geq \mathcal{E}_\beta(\mu_*)$ of any probability measure $\mu \in \mathcal{M}^1(A)$ supported on a Suslin set $A \subseteq \mathbb{R}^{n+d}$:

	$\mathcal{H}^\beta(A)$	$\mathrm{Cap}_\beta(A)$	$\mathcal{E}_\beta(\mu)$
$\beta \in [0, \dim A)$	$= \infty$	> 0	$< \infty$
$\beta \in (\dim A, \infty)$	$= 0$	$= 0$	$= \infty$

In our case the set A from Definition 6.3 is just $\mathcal{G}_T(X + f)$. The next lemma shows that we can work with an energy integral where the stable process X is transformed into the kernel.

Lemma 6.5 (Kernel transformation)**.** *Let $T \subseteq \mathbb{R}_+$ be a Borel set. Let $X = (X_t)_{t\geq 0}$ be a stochastic process with stationary increments on $\mathbb{R}^d$ and $f : \mathbb{R}_+ \to \mathbb{R}^d$ be a Borel measurable function. Define the difference kernel*

$$K^\beta(t, x) := \mathbb{E}\big[||(t, \mathrm{sign}(t) \cdot X_{|t|}(\omega) + x)||^{-\beta}\big].$$

From

$$\mathrm{Cap}_{K^\beta}(\mathcal{G}_T(f)) > 0$$

follows

$$\mathrm{Cap}_\beta(\mathcal{G}_T(X(\omega) + f)) > 0,\ \mathbb{P}\text{-almost surely.}$$

Hence $\mathcal{E}_{K^\beta}(\mu) < \infty$ for some probability measure $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$ implies

$$\dim \mathcal{G}_T(X + f) \geq \beta,\ \mathbb{P}\text{-almost surely.}$$

Proof. For every $\omega \in \Omega$, the pathwise bijection

$$(t, f(t)) \in \mathcal{G}_T(f)$$

if and only if

$$(t, X_t(\omega) + f(t)) \in \mathcal{G}_T(X_t(\omega) + f),$$

yields the existence of some random probability measure $\nu_\omega \in \mathcal{M}^1(\mathcal{G}_T(X(\omega)+f))$ with $\nu_\omega(\widetilde{A}_\omega) = \mu(A)$ for all Borel sets $A \subseteq \mathcal{G}_T(f)$ where $\widetilde{A}_\omega := \{(t, x+X_t(\omega)) : (t,x) \in A\}$. Therefore, Tonelli's theorem and the stationarity of the increments of X yield

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{E}_\beta(\nu_\omega)\big] \\
= \;& \mathbb{E}\Big[\int_{\mathcal{G}_T(X(\omega)+f)} \int_{\mathcal{G}_T(X(\omega)+f)} ||(t-s, x-y)||^{-\beta} \,\mathrm{d}\nu_\omega(t,x)\,\mathrm{d}\nu_\omega(s,y)\Big] \\
= \;& \mathbb{E}\Big[\int_{\mathcal{G}_T(f)} \int_{\mathcal{G}_T(f)} ||(t-s, x+X_t(\omega)-(y+X_s(\omega)))||^{-\beta} \,\mathrm{d}\mu(t,x)\,\mathrm{d}\mu(s,y)\Big] \\
= \;& \int_{\mathcal{G}_T(f)} \int_{\mathcal{G}_T(f)} \mathbb{E}\big[||(t-s, X_t(\omega)-X_s(\omega)+x-y)||^{-\beta}\big] \,\mathrm{d}\mu(t,x)\,\mathrm{d}\mu(s,y) \\
= \;& \int_{\mathcal{G}_T(f)} \int_{\mathcal{G}_T(f)} \mathbb{E}\big[||(t-s, \mathrm{sign}(t-s)\cdot X_{|t-s|}(\omega)+x-y)||^{-\beta}\big] \,\mathrm{d}\mu(t,x)\,\mathrm{d}\mu(s,y) \\
= \;& \mathcal{E}_{K^\beta}(\mu)
\end{aligned}$$

By assumption, there exists $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$ such that $\mathcal{E}_{K^\beta}(\mu) < \infty$, therefore $\mathcal{E}_\beta(\nu_\omega) < \infty$, $\mathbb{P}$-almost surely, and the rest of the claim follows by Frostman's Theorem 6.4. □

Since we are concerned with parabolic Hausdorff dimensions, the following parabolic version of Frostman's Lemma provides the suitable candidate for the probability measure μ which was used in the proof of Lemma 6.5.

Theorem 6.6 (Frostman's Lemma for parabolic cylinders)**.** *Let $A \subseteq \mathbb{R}^{1+d}$ be a Borel set with $\mathcal{P}^\alpha$-$\mathcal{H}^\beta(A) > 0$, then there exists a probability measure $\mu \in \mathcal{M}^1(\mathbb{R}^{1+d})$ with*

$\mu(A) = 1$ *such that we have*

$$\mu\left(\left[t,\, t+c\right] \times \prod_{i=1}^{d} \left[x_i,\, x_i + c^{1/\alpha}\right]\right) \lesssim \begin{cases} c^{\beta}, & \alpha \in (0,1], \\ c^{\beta/\alpha}, & \alpha \in [1,\infty) \end{cases}$$

for every $c \in (0,1]$ *and* $t, x_1, \ldots, x_d \in \mathbb{R}$.

The result in case of $\alpha = 1$ is a classical, see e.g. Theorem 4.30 in [28] or Theorem 8.8 in [26]. We follow the elegant graph theoretical approach in [28] in order to show the α-parabolic version of Frostman's Lemma. First, we need some terminology from graph theory:

Definition 6.7. A *tree* $T = (V, E)$ is a connected graph. It consists of at most countably many *vertices* V, a *root* $\varrho \in V$ and a set of *directed edges* $E \subseteq V \times V$ with the following properties:

(i) For each vertex $v \in V \setminus \{\varrho\}$ the set $\{u \in V : (u,v) \in E\}$ consist of exactly one element $\bar{v}$ which is called the *parent* of v. The root ϱ has no parent, i.e. $\{u \in V : (u,\varrho) \in E\} = \emptyset$.

(ii) For each vertex $v \in V$ there exits a unique self-avoiding path from ϱ to v. Its *order* $|v|$ is the number of required edges. Therefore

$$|\varrho| = 0$$

and

$$|v| = n \in \mathbb{N}$$

if and only if there exists

$$v_0 = \varrho, v_1, \ldots, v_{n-1} \in V, v_n = v$$

with

$$(v_k, v_{k+1}) \in E,\ \forall\ k = 0, \ldots, n-1.$$

(iii) For each vertex $v \in V$ the set of its *children* $\{w \in V : (v,w) \in E\}$ is finite.

The *order* $|e|$ of an edge $e = (v, w)$ is the order $|v|$ of its initial vertex v. Each (infinitely long) self-avoiding path started in the root is called *ray.* The set of all rays of a tree T is denoted by ∂T.
A mapping $C : E \to [0, \infty)$ is called *capacity.* A *flow* of *strength* $s > 0$ with capacity C is a mapping $\vartheta : E \to [0, s]$ with the following properties:

(i) $\sum_{w \in V : \bar{w} = \varrho} \vartheta(\varrho, w) = s$.

(ii) $\vartheta(\bar{v}, v) = \sum_{w \in V : \bar{w} = v} \vartheta(v, w), \ \forall v \in V \setminus \{\varrho\}$, i.e. the flow into and out of each vertex other than the root is conserved.

(iii) $\vartheta(e) \leq C(e), \ \forall e \in E$, i.e. the flow through the edge e is bounded by its capacity.

A set $\Pi \subseteq E$ is called a *cutset* if every ray includes an edge from Π.

We need the following result of graph theory, the max-flow min-cut theorem from Theorem 12.36 in [28].

Theorem 6.8 (Max-flow min-cut theorem)**.**

$$\max\{\textit{strength}(\vartheta) : \vartheta \ \textit{a flow with capacity } C\} = \inf\left\{\sum_{e \in \Pi} C(e) : \Pi \ \textit{a cutset}\right\}.$$

Now we are able to proof the parabolic version of Frostman's Lemma.

Proof of Theorem of 6.6. Let $A \subseteq \mathbb{R}^{1+d}$ be a Borel set with $\mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A) > 0$. Due to the countable stability property of $\mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta}$ from Proposition 3.5 we can assume, without loss of generality, that A is contained in a compact hypercube in $\mathbb{R}^{1+d}$. Moreover, by translation and rescaling we can assume $A \subseteq [0, 1]^{1+d}$.
Each α-parabolic cylinder with length c in time direction can be covered by $\lceil 2^{1/\alpha} \rceil^d$ α-parabolic cylinders with sidelength $c/2$ in time direction. They can be chosen disjoint by removing some hypersurfaces. We construct a tree $T = (V, E)$ with root $\varrho := [0, 1]^{1+d}$ whose children are the $\lceil 2^{1/\alpha} \rceil^d$ α-parabolic cylinders with half length in time direction who have a non-empty intersection with A. A ray is then a sequence of α-parabolic cylinders each of which posses half the length in time direction per generation.

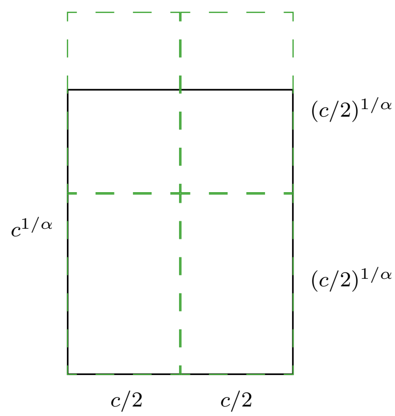

FIGURE 1. Parabolic covers with half length in time.

There exists a canonical mapping $\Phi : \partial T \to A$ which maps sequences of nested α-parabolic cylinders to their intersection. For every $x \in A$ there exists a unique ray for which each α-parabolic cylinder involved contains x. Hence Φ is bijective. We assign the capacity

$$C(e) := \left(2^{-2n} + d \cdot 2^{-2n/\alpha}\right)^{\beta/2} \tag{6.1}$$

to every edge e of order $|e| = n$. The capacity corresponds to the diameter (taken to the power of β) of a dyadic α-parabolic cylinder

$$\mathsf{P}_n := [t, t + 2^{-n}] \times \prod_{i=1}^{d} \left[x_i, x_i + 2^{-n/\alpha}\right] \in \mathscr{P}^\alpha$$

associated with the initial vertex of an edge of order n in the cutset. We now associate to every cutset Π a covering of A, consisting of those cylinders associated with the initial vertices of the edges in the cutset. To see that the resulting collection of cylinders is indeed a covering, let $\xi = (\xi_n)_{n\in\mathbb{N}}$ be a ray. As Π is a cutset, it contains exactly one of the edges ξ_n in this ray, and the cylinder associated with the initial vertex of this edge contains the point $\Phi(\xi)$.

Since Φ is surjective we indeed cover the entire set $\Phi(\partial T) = A$. Hence for $\delta > 0$ one has by (3.5)

$$\begin{aligned}
&\inf\left\{\sum_{e\in\Pi} C(e) : \Pi \text{ a cutset}\right\} \\
&\geq \inf\left\{\sum_{n=1}^{\infty} |\mathsf{P}_n|^{\beta} : A \subseteq \bigcup_{n\in\mathbb{N}} \mathsf{P}_n,\ \mathsf{P}_n \in \mathscr{P}^{\alpha}\right\} \\
&= \Phi_{\alpha}^{\beta} > 0.
\end{aligned}$$

According to the max-flow min-cut theorem there exists a flow ϑ of positive maximal strength $s \in (0, \infty)$ with $\vartheta(e) \leq C(e)$ for each edge e of the tree.
Next, we construct the probability measure $\mu \in \mathcal{M}^1(A)$. Given an edge $e \in E$ we associate a set $T(e) \subseteq \partial T$ consisting of all rays containing the edge e. Set

$$\widetilde{\nu}(T(e)) := \vartheta(e).$$

The set $\mathscr{R} = \{\bigcup_{i=1}^{n} T(e_i) : e_1, \ldots, e_n \text{ edges },\ n \in \mathbb{N}_0\}$ forms a ring over ∂T. Since ϑ is a flow, our $\widetilde{\nu}$ together with $\widetilde{\nu}(\emptyset) = 0$ constitutes a pre-measure on $\mathscr{R}$. According to Carathéodory's extension theorem we can extend $\widetilde{\nu}$ to a measure ν on $\sigma(\mathscr{R})$.
The mapping $\Phi : \partial T \to A$ is $\sigma(\mathscr{R})-\mathscr{B}(A)$-measurable. This is because for each cylinder $Q \subseteq [0,1]^{1+d}$ which possesses corner points in the dyadic rationals the preimage $\Phi^{-1}(A \cap Q)$ lies in $\sigma(\mathscr{R})$ and further $\mathscr{B}(A)$ is generated by finite unions of such $A \cap Q$. Hence the pushforward measure $\mu := \Phi(\nu)$ is a Borel measure on A. One has

$$\mu(A) = \nu\big(\Phi^{-1}(A)\big) = \nu(\partial T) = \text{strength}(\vartheta) = s \in (0, \infty).$$

Hence μ can be normed to a probability measure on $(A, \mathscr{B}(A))$ and it can be extended to a probability measure on $\big(\mathbb{R}^{1+d}, \mathscr{B}\big(\mathbb{R}^{1+d}\big)\big)$ by letting $\mu(\mathbb{R}^{1+d} \backslash A) = 0$.
For a dyadic α-parabolic cylinder P_n which is associated with the initial vertex of an edge e of order n Equation (6.1) yields

$$\mu(\mathsf{P}_n) = \nu\big(\Phi^{-1}(\mathsf{P}_n)\big) = \widetilde{\nu}(T(e)) = \vartheta(e) \leq C(e) \asymp \begin{cases} 2^{-n\beta}, & \alpha \in (0,1], \\ 2^{-n\beta/\alpha}, & \alpha \in [1, \infty). \end{cases} \tag{6.2}$$

Now, let

$$\mathsf{P} := [t,\, t+c] \times \prod_{i=1}^{d} \left[x_i,\, x_i + c^{1/\alpha}\right] \in \mathscr{P}^\alpha.$$

(i) Let $\alpha \in (0,1]$. Choose $n \in \mathbb{N}$ such that

$$2^{-n} \le |\mathsf{P}| \le 2^{-(n-1)}.$$

Then B can be covered by $2 \cdot \lceil 2^{1/\alpha} \rceil^d$ α-parabolic cylinders P_n with sidelength 2^{-n} in time which are associated to an edge of order n. From Equation (6.2) it follows that

$$\mu(\mathsf{P}) \le 2 \cdot \lceil 2^{1/\alpha} \rceil^d \cdot \mu(\mathsf{P}_n) \lesssim 2^{-n\beta} \lesssim |\mathsf{P}|^\beta \lesssim c^\beta.$$

(ii) Let $\alpha \in [1,\infty)$. Choose $n \in \mathbb{N}$ such that

$$2^{-n/\alpha} < |\mathsf{P}| \le 2^{1/\alpha} \cdot 2^{-n/\alpha}.$$

Then we can cover P by 2^{d+1} α-parabolic cylinders P_n with sidelength 2^{-n} in time associated to an edge of order n. With Equation (6.2) it follows that

$$\mu(\mathsf{P}) \le 2^{d+1} \cdot \mu(\mathsf{P}_n) \lesssim 2^{-n\beta/\alpha} \lesssim |\mathsf{P}|^\beta \lesssim c^{\beta/\alpha},$$

as claimed. □

Let us inspect the difference kernel $K^\beta(t,x) = \mathbb{E}\big[||(\tau, \operatorname{sign}(t) \cdot X_{|t|} + x)||^{-\beta}\big]$ more precisely. In view of Lemma 6.5 we want to show the finiteness of some energy integral

$$\mathcal{E}_{K^\beta}(\mu) = \int\int_{\mathcal{G}_T(f)\times\mathcal{G}_T(f)} K^\beta(t-s, f(t)-f(s))\,\mathrm{d}\mu(s,x)\,\mathrm{d}\mu(t,y).$$

Therefore, we have to estimate the expression

$$\mathbb{E}\big[||(t-s, \operatorname{sign}(t-s) \cdot X_{|t-s|} + f(t) - f(s))||^{-\beta}\big].$$

We define the increments $\tau = t-s$ and $\delta = f(t) - f(s)$. Then the object becomes

$$\mathbb{E}\big[||(\tau, \operatorname{sign}(\tau) \cdot X_{|\tau|} + \delta)||^{-\beta}\big].$$

Hence there are three essential objects involved which can generate convergence of the whole expression for fixed $\beta > 0$: τ, $X_{|\tau|}$ and δ. We compare these objects by their magnitude and investigate which part will be the dominant one. Then the fractal dimension of the whole object is determined by the rate of convergence of the dominant part(s) as $\tau \downarrow 0$ or $\delta \downarrow 0$, respectively. Inspired by Lemma 2.5 in [37], we want to give a priori estimates for the difference kernel K^β. The following lemma is a refinement of (2.7) in [37].

Lemma 6.9. *Let $t \in \mathbb{R}$ be fixed and $h : \mathbb{R}^d \to \mathbb{R},\ h(x) = \|(t,x)\|^{-\beta} = (t^2 + ||x||^2)^{-\beta/2}$. Then h is rotationally symmetric and the mapping $r \mapsto h(r \cdot y)$ is non-increasing for $r = ||x||$ and does not not depend on $y = x/||x|| \in S^{d-1}$. Further, let $p : \mathbb{R}^d \to \mathbb{R}$ be a rotationally symmetric function such that also $r \mapsto p(r \cdot y)$ is non-increasing for $r = ||x||$ and $y = x/||x|| \in S^{d-1}$. Then for all $u \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} h(x+u) \cdot p(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^d} h(x) \cdot p(x)\,\mathrm{d}x,$$

provided that the integrals exist.

Proof. The first part is obvious. Further, by monotocity we have

$$\begin{aligned} &\int_{\mathbb{R}^d} h(x+u) \cdot p(x)\,\mathrm{d}x \\ &= \int_{\{||x||<||x+u||\}} \underbrace{h(x+u)}_{\leq h(x)} \cdot\, p(x)\,\mathrm{d}x + \int_{\{||x||\geq||x+u||\}} h(x+u) \cdot \underbrace{p(x)}_{\leq p(x+u)}\,\mathrm{d}x \\ &\leq\ 2\int_{\mathbb{R}^d} h(x) \cdot p(x)\,\mathrm{d}x, \end{aligned}$$

as claimed. □

Inspired by Lemma 2.5 in [37], we give a priori estimates for the difference kernel $K^\beta = \mathbb{E}\big[||(t, \mathrm{sign}(t) \cdot X_{|t|}(\omega) + x)||^{-\beta}\big]$ from Lemma 6.5.

Lemma 6.10 (Kernel estimates)**.** *Let $\alpha \in (0,2)$ and $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$. Let $\beta \geq 0$ and $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^d$ be such that $|\tau| \in (0,1]$, $||\delta|| \in [0,1]$. Then appropriate choices for estimating the difference kernel from Lemma 6.5,*

$$K^\beta(\tau,\delta) := \mathbb{E}\big[||(\tau, \mathrm{sign}(\tau) \cdot X_{|\tau|} + \delta)||^{-\beta}\big]$$

are

$$K^\beta(\tau,\delta) \lesssim \begin{cases} |\tau|^{-\beta}, & \\ |\tau|^{-\beta/\alpha}, & \text{for } \beta < d, \\ |\tau|^{(1-1/\alpha)d-\beta}, & \text{for } \beta > d. \end{cases} \tag{6.3}$$

and one has

$$K^\beta(\tau,\delta) \lesssim \begin{cases} ||\delta||^{-\beta}, & \textit{for } \alpha \in (0,1],\ |\tau| \le ||\delta|| \\ ||\delta||^{-\beta}, & \textit{for } \alpha \in [1,2),\ \beta \le d, |\tau| \le ||\delta||^\alpha, \\ ||\delta||^{(\alpha-1)d-\alpha\beta}, & \textit{for } \alpha \in [1,2),\ \beta > d, |\tau| \le ||\delta||^\alpha. \end{cases} \tag{6.4}$$

Proof. Let $p(x)$ denote the density function of $X_1 \overset{\mathrm{d}}{=} 1/|\tau|^{1/\alpha} X_{|\tau|}$. We define rescaled increments $\widetilde{\tau} := \tau/|\tau|^{1/\alpha}$ and $\widetilde{\delta} := \delta/|\tau|^{1/\alpha}$. Trivial estimation always yields

$$\mathbb{E}\big[||(\tau, \operatorname{sign}(\tau)\cdot X_{|\tau|} + \delta)||^{-\beta}\big] \le |\tau|^{-\beta}.$$

The self-similarity of the stable Lévy process and Lemma 6.9 yield

$$\begin{aligned} &\mathbb{E}\big[||(\tau, \operatorname{sign}(\tau)\cdot X_{|\tau|} + \delta)||^{-\beta}\big] \\ &= |\tau|^{-\beta/\alpha} \int_{\mathbb{R}^d} ||(\widetilde{\tau}, \operatorname{sign}(\tau)\cdot x + \widetilde{\delta})||^{-\beta} \cdot p(x)\,\mathrm{d}x \\ &= |\tau|^{-\beta/\alpha} \int_{\mathbb{R}^d} ||(|\widetilde{\tau}|, x + \operatorname{sign}(\tau)\cdot \widetilde{\delta})||^{-\beta} \cdot p(x)\,\mathrm{d}x \\ &\lesssim |\tau|^{-\beta/\alpha} \int_{\mathbb{R}^d} ||(|\widetilde{\tau}|, x)||^{-\beta} \cdot p(x)\,\mathrm{d}x. \end{aligned} \tag{6.5}$$

Let $\beta < d$. Then by (6.5) we get

$$\begin{aligned} &\mathbb{E}\big[||(\tau, \operatorname{sign}(\tau)\cdot X_{|\tau|} + \delta)||^{-\beta}\big] \\ &\lesssim |\tau|^{-\beta/\alpha} \int_{\mathbb{R}^d} ||x||^{-\beta} \cdot p(x)\,\mathrm{d}x \\ &\lesssim |\tau|^{-\beta/\alpha} \cdot \mathbb{E}\big[||X_1||^{-\beta}\big] \lesssim |\tau|^{-\beta/\alpha}, \end{aligned}$$

since negative moments of order $\beta < d$ exist; see Lemma 3.1 in [3].

Let $\beta > d$. Then by (6.5) one has using the volume of a ball with radius $\widetilde{\tau}$

$$\begin{aligned} &\mathbb{E}\big[||(\tau, \operatorname{sign}(\tau)\cdot X_{|\tau|} + \delta)||^{-\beta}\big] \\ &\lesssim |\tau|^{-\beta/\alpha} \int_{\mathbb{R}^d} ||(|\widetilde{\tau}|, x)||^{-\beta} \cdot p(x)\,\mathrm{d}x \\ &\le |\tau|^{-\beta/\alpha} \bigg(\int_{\{||x|| < |\widetilde{\tau}|\}} |\widetilde{\tau}|^{-\beta} \cdot p(x)\,\mathrm{d}x + \int_{\{||x|| \ge |\widetilde{\tau}|\}} ||x||^{-\beta} \cdot p(x)\,\mathrm{d}x \bigg) \end{aligned}$$

$$\leq |\tau|^{-\beta/\alpha}\Big(|\widetilde{\tau}|^{d-\beta} + \int_{\{|\widetilde{\tau}|\leq ||x||\leq 1\}} ||x||^{-\beta}\,\mathrm{d}x + \int_{\{||x||>1\}} p(x)\,\mathrm{d}x\Big)$$

$$\leq |\tau|^{-\beta/\alpha}\Big(|\widetilde{\tau}|^{d-\beta} + \int_{|\widetilde{\tau}|}^{1}\int_{S^{d-1}} ||ry||^{-\beta}\cdot r^{d-1}\,\mathrm{d}y\,\mathrm{d}r + 1\Big)$$

$$\lesssim |\tau|^{-\beta/\alpha}\Big(|\widetilde{\tau}|^{d-\beta} + \int_{|\widetilde{\tau}|}^{1} r^{d-\beta-1}\,\mathrm{d}r\Big)$$

$$\lesssim |\tau|^{-\beta/\alpha}\cdot|\widetilde{\tau}|^{d-\beta}$$

$$= |\tau|^{-\beta/\alpha}\cdot|\tau|^{(1-1/\alpha)(d-\beta)}$$

$$= |\tau|^{(1-1/\alpha)d-\beta}.$$

This proves (6.3). To prove (6.4) consider the event $||x|| \leq |\widetilde{\delta}|/2$ which yields

$$||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}|| \geq \big|||x|| - ||\widetilde{\delta}||\big| = ||\widetilde{\delta}|| - ||x|| \geq \frac{1}{2}\cdot||\widetilde{\delta}||.$$

Thus for the estimates in (6.4) we have

$$\mathbb{E}\big[||(\tau, \mathrm{sign}(\tau)\cdot X_{|\tau|} + \delta)||^{-\beta}\big]$$

$$= |\tau|^{-\beta/\alpha}\int_{\mathbb{R}^d} ||(\widetilde{\tau}, \mathrm{sign}(\tau)\cdot x + \widetilde{\delta})||^{-\beta}\cdot p(x)\,\mathrm{d}x$$

$$\lesssim ||\delta||^{-\beta} + \underbrace{|\tau|^{-\beta/\alpha}\int_{\{||x||\geq||\widetilde{\delta}||/2,\,||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\geq|\widetilde{\tau}|\}} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta}\cdot p(x)\,\mathrm{d}x}_{=:I_1}$$

$$+ \underbrace{|\tau|^{-\beta/\alpha}\int_{\{||x||\geq||\widetilde{\delta}||/2,\,||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\leq|\widetilde{\tau}|\}} \widetilde{\tau}^{-\beta}\cdot p(x)\,\mathrm{d}x}_{=:I_2}.$$

Now,

$$I_1 = |\tau|^{-\beta/\alpha}\int_{\{||x||\geq||\widetilde{\delta}||/2,\,||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\geq|\widetilde{\tau}|\}} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta}\cdot p(x)\,\mathrm{d}x$$

$$= |\tau|^{-\beta/\alpha}\int_{\{||x||\geq||\widetilde{\delta}||/2,\,||x+\widetilde{\delta}||\geq|\widetilde{\tau}|,\,||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\geq||\widetilde{\delta}||\}} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta}\cdot p(x)\,\mathrm{d}x$$

$$+ \ |\tau|^{-\beta/\alpha} \int_{\{||x||\geq||\widetilde{\delta}||/2,\, ||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\geq|\widetilde{\tau}|,\, ||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\leq||\widetilde{\delta}||\}} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta} \cdot p(x)\,\mathrm{d}x$$

$$\leq ||\delta||^{-\beta} + \underbrace{|\tau|^{-\beta/\alpha} \int_{\{||x||\geq||\widetilde{\delta}||/2,\, ||\widetilde{\delta}||\geq||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\geq|\widetilde{\tau}|\}} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta} \cdot p(x)\,\mathrm{d}x}_{=:I_3}\,.$$

By using (2.3) we further have

$$\begin{aligned} I_3 &= |\tau|^{-\beta/\alpha} \int_{\{||x||\geq||\widetilde{\delta}||/2,\, ||\widetilde{\delta}||\geq||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\geq|\widetilde{\tau}|\}} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta} \cdot p(x)\,\mathrm{d}x \\ &\lesssim |\tau|^{-\beta/\alpha} \cdot ||\widetilde{\delta}||^{-d-\alpha} \int_{\{||\widetilde{\delta}||\geq||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\geq|\widetilde{\tau}|\}} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta}\,\mathrm{d}x \qquad (6.6) \\ &= |\tau|^{-\beta/\alpha} \cdot ||\widetilde{\delta}||^{-d-\alpha} \int_{|\widetilde{\tau}|}^{||\widetilde{\delta}||} r^{d-\beta-1}\,\mathrm{d}r. \end{aligned}$$

For $\alpha \in [1,2)$, $\beta < d$ and $|\tau| \leq ||\delta||^{\alpha}$ by (6.6) we get

$$I_3 \lesssim |\tau|^{-\beta/\alpha} \cdot ||\widetilde{\delta}||^{-\alpha-\beta} = |\tau| \cdot ||\delta||^{-\alpha-\beta} \lesssim ||\delta||^{-\beta},$$

whereas for $\alpha \in (0,1]$ and $|\tau| \leq ||\delta||$ one has

$$I_3 \lesssim |\tau| \cdot ||\delta||^{-\alpha-\beta} \leq ||\delta||^{1-\alpha} \cdot ||\delta||^{-\beta} \lesssim ||\delta||^{-\beta}.$$

For $\alpha \in [1,2)$, $\beta > d$ and $|\tau| \leq ||\delta||^{\alpha}$ by (6.6) one has

$$\begin{aligned} I_3 &\lesssim |\tau|^{-\beta/\alpha} \cdot ||\widetilde{\delta}||^{-d-\alpha} \int_{\widetilde{\tau}}^{\infty} r^{-\beta} \cdot r^{d-1}\,\mathrm{d}r \\ &\lesssim |\tau|^{-\beta/\alpha} \cdot ||\widetilde{\delta}||^{-d-\alpha} \cdot \widetilde{\tau}^{\,d-\beta} \\ &= |\tau|^{d+1-\beta} \cdot ||\delta||^{-d-\alpha} \\ &\leq ||\delta||^{\alpha(d+1-\beta)} \cdot ||\delta||^{-d-\alpha} \\ &= ||\delta||^{(\alpha-1)d-\alpha\beta}. \end{aligned}$$

Finally, by using (2.3) we get

$$\begin{aligned} I_2 &= |\tau|^{-\beta/\alpha} \int_{\{||x|| \geq ||\widetilde{\delta}||/2,\, ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}|| \leq |\widetilde{\tau}|\}} |\widetilde{\tau}|^{-\beta} \cdot p(x)\,\mathrm{d}x \\ &\lesssim |\tau|^{-\beta/\alpha} \cdot |\widetilde{\tau}|^{-\beta} \int_{||x + \mathrm{sign}(\tau)\cdot \widetilde{\delta}|| \leq |\widetilde{\tau}|\}} ||x||^{-d-\alpha}\,\mathrm{d}x \\ &\lesssim |\tau|^{-\beta/\alpha} \cdot |\widetilde{\tau}|^{-\beta} \cdot ||\widetilde{\delta}||^{-d-\alpha} \cdot |\widetilde{\tau}|^{d} \\ &= |\tau|^{d-\beta+1} \cdot ||\delta||^{-d-\alpha} \end{aligned}$$

using the volume of a ball with radius $|\widetilde{\tau}|$ and center $-\mathrm{sign}(\tau)\cdot\widetilde{\delta}$. Now, $\alpha \in (0,1]$, $\beta < d$ and $|\tau| \leq ||\delta||$ result in

$$I_2 \lesssim |\tau|^{d-\beta+1} \cdot ||\delta||^{-d-\alpha} \leq ||\delta||^{1-\alpha-\beta} \leq ||\delta||^{-\beta}.$$

If $\alpha \in [1,2)$, $\beta \leq d$ and $|\tau| \leq ||\delta||^{\alpha}$ one has

$$I_2 \lesssim |\tau|^{d-\beta+1} \cdot ||\delta||^{-d-\alpha} \leq ||\delta||^{(\alpha-1)\cdot d - \alpha\beta} \leq ||\delta||^{-\beta}.$$

If $\alpha \in [1,2)$, $\beta \geq d$ and $|\tau| \leq ||\delta||^{\alpha}$ one has

$$I_2 \lesssim ||\delta||^{(\alpha-1)\cdot d-\alpha\beta} \quad \text{and} \quad ||\delta||^{-\beta} \leq ||\delta||^{(\alpha-1)\cdot d - \alpha\beta}$$

which concludes the proof. □

Now, we are able to calculate the lower bound for the Hausdorff dimension of the graph of $X+f$.

Theorem 6.11 (Energy estimates)**.** *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0,2)$. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$ and $f : T \to \{y \in \mathbb{R}^d : \|y-x\| \leq \frac{1}{2}\}$ for fixed $x \in \mathbb{R}^d$ be a Borel measurable function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f)$ where $\varphi_1 = \dim \mathcal{G}_T(f)$ denotes the genuine Hausdorff dimension. Then one $\mathbb{P}$-almost surely has*

$$\dim \mathcal{G}_T(X+f) \geq \begin{cases} \varphi_1, & \alpha \in (0,1], \\ \varphi_\alpha \;\wedge\; \frac{1}{\alpha}\cdot\varphi_\alpha + \left(1-\frac{1}{\alpha}\right)\cdot d, & \alpha \in [1,2]. \end{cases} \tag{6.7}$$

Proof. We define the increments $\tau := t-s$ and $\delta := f(t) - f(s)$ with $||\delta|| \in [0,1]$ and consider the difference kernel $K^{\beta}(t,x) = \mathbb{E}\big[||(t, \mathrm{sign}(t)\cdot X_{|t|} + x)||^{-\beta}\big]$. We prove that $\mathcal{E}_{K^\beta}(\mu) < \infty$ holds for $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$ from the parabolic version of Frostman's lemma in Theorem 6.6 and for every β less than the right-hand side of (6.7). Then the claim follows due to Lemma 6.5. For the energy integral we have

$$\begin{aligned}\mathcal{E}_{K^\beta}(\mu) &= \int\int_{\mathcal{G}_T(f)\times\mathcal{G}_T(f)} K^\beta(t-s, f(t)-f(s))\,\mathrm{d}\mu(s,x)\,\mathrm{d}\mu(t,y)\\ &\le \int\int_{\{|t-s|\in(0,1]\}} K^\beta(\tau,\delta)\,\mathrm{d}\mu\,\mathrm{d}\mu + \int\int_{\{|t-s|\in(1,\infty)\}} |t-s|^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu \qquad (6.8)\\ &\lesssim \int\int_{\{|\tau|\in(0,1]\}} \mathbb{E}\big[||(\tau, \mathrm{sign}(\tau)\cdot X_{|\tau|}(\omega)+\delta)||^{-\beta}\big]\,\mathrm{d}\mu\,\mathrm{d}\mu\end{aligned}$$

in all cases.

(i) We begin with the case $\alpha \in (0,1]$ and $\beta = \varphi_1 - 2\varepsilon$ for some arbitrary $\varepsilon > 0$. Due to Lemma 6.10 we have

$$\mathcal{E}_{K^\beta}(\mu) \lesssim \underbrace{\int\int_{\{|\tau|\in(0,\,1],\,||\delta||\in[0,\,|\tau|]\}} |\tau|^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\,I_1} + \underbrace{\int\int_{\{|\tau|\in(0,\,1],\,||\delta||\in(|\tau|,\,1]\}} ||\delta||^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\,I_2}.$$

We get

$$I_1 \lesssim \sum_{k=1}^{\infty} 2^{k\beta}\cdot\mu\otimes\mu\left\{(\tau,\delta):\ |\tau|\in\left(2^{-k},\,2\cdot2^{-k}\right],\,||\delta||\in\left[0, 2\cdot 2^{-k}\right]\right\}.$$

Further,

$$I_2 \lesssim \sum_{k=1}^{\infty} 2^{k\beta}\cdot\mu\otimes\mu\left\{(\tau,\delta):\ |\tau|\in\left(0,\,2\cdot2^{-k}\right];\ ||\delta||\in\left(2^{-k},\,2\cdot 2^{-k}\right]\right\}.$$

Now we have to calculate the expressions $\mu\otimes\mu\{\cdot\}$ for I_1 and I_2. For each $k\in\mathbb{N}$ we tile $\mathbb{R}_+\times\mathbb{R}^d$ by disjoint hypercubes of size $2^{-k}\times\cdots\times2^{-k}$ and denote the collection of such hypercubes by $\mathcal{D}_k$. For every $c\in(0,1]$, $\gamma=\varphi_1-\varepsilon$ and $\alpha\in(0,1]$ Frostman's Lemma 6.6 yields

$$\mu\left([t,\,t+c]\times\prod_{i=1}^{d}\,[x_i,\,x_i+c]\right)\lesssim c^\gamma,$$

in particular we have $\mu(Q')\lesssim 2^{-k\gamma}$ for each $Q'\in(\mathcal{D}_k)_{k\in\mathbb{N}}$.

In order to estimate I_1 we define the following relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two hypercubes Q, Q' of the same generation we write $Q\sim Q'$ if there exists $(s,x)\in Q$ and $(t,y)\in Q'$

such that $|\tau| \in \left(2^{-k},\, 2 \cdot 2^{-k}\right]$ and $||\delta|| = ||y - x|| \in \left[0,\, 2 \cdot 2^{-k}\right]$. Thus

$$I_1 \lesssim \sum_{k=1}^{\infty} 2^{k\beta} \sum_{\substack{Q, Q' \in \mathcal{D}_k \\ Q \sim Q'}} \mu \otimes \mu \left(Q \times Q'\right)$$

$$= \sum_{k=1}^{\infty} 2^{k\beta} \sum_{\substack{Q, Q' \in \mathcal{D}_k \\ Q \sim Q'}} \mu(Q) \cdot \mu(Q').$$

The number of hypercubes related to some fixed Q via $\sim$ is bounded by a universal constant not depending on k and Q. Hence

$$I_1 \lesssim \sum_{k=1}^{\infty} 2^{k\beta} \sum_{Q \in \mathcal{D}_k} \sum_{Q \sim Q'} \mu(Q) \cdot \mu(Q')$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta} \sum_{Q \in \mathcal{D}_k} \sum_{Q' \sim Q} \mu(Q) \cdot 2^{-k\gamma}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta} \sum_{Q \in \mathcal{D}_k} \mu(Q) \cdot 2^{-k\gamma}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k(\beta - \gamma)} \cdot \sum_{Q \in \mathcal{D}_k} \mu(Q).$$

Note that $\sum_{Q \in \mathcal{D}_k} \mu(Q) = \mu(\cup_{Q \in \mathcal{D}_k} Q) = \mu\left(\mathbb{R}_+ \times \mathbb{R}^d\right) = 1$ and we conclude

$$I_1 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta - \gamma)} = \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty,$$

since $\beta = \varphi_1 - 2\varepsilon$ and $\gamma = \varphi_1 - \varepsilon$.

For the estimation of I_2 we define a novel relation on $(\mathcal{D}_k)_{k \in \mathbb{N}}$: For two hypercubes Q, Q' of the same generation we write $Q \approx Q'$ if there exists $(s, x) \in Q$ and $(t, y) \in Q'$ such that $|\tau| \in \left(0,\, 2 \cdot 2^{-k}\right]$ and $||\delta|| \in \left(2^{-k},\, 2 \cdot 2^{-k}\right]$. Thus

$$I_2 \lesssim \sum_{k=1}^{\infty} 2^{k\beta} \sum_{\substack{Q, Q' \in \mathcal{D}_k \\ Q \approx Q'}} \mu(Q) \cdot \mu(Q').$$

Now the number of hypercubes related to some fixed Q via $\approx$ is bounded by a universal constant. Hence the same calculation as for I_1 yields

$$I_2 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\gamma)} = \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty$$

since $\beta = \varphi_1 - 2\varepsilon$ and $\gamma = \varphi_1 - \varepsilon$.

(ii) Now we treat the case $\alpha \in [1,2)$ and $\varphi_\alpha \le d$. Let $\beta = \varphi_\alpha - 2\alpha \cdot \varepsilon < d$ for some arbitrary $\varepsilon > 0$. Due to Lemma 6.10 we have

$$\mathcal{E}_{K^\beta}(\mu)$$

$$\lesssim \underbrace{\int\int_{\{|\tau|\in(0,1]\ ||\delta||\in[0,|\tau|^{1/\alpha}]\}} |\tau|^{-\beta/\alpha}, \,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\ I_3} + \underbrace{\int\int_{\{|\tau|\in(0,1],\,||\delta||\in(|\tau|^{1/\alpha},1]\}} ||\delta||^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\ I_4}.$$

We get

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \cdot \mu\otimes\mu\left\{(\tau,\delta):\ |\tau| \in \left(2^{-k},\, 2\cdot 2^{-k}\right],\, ||\delta|| \in \left[0, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Further,

$$I_4 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \cdot \mu\otimes\mu\left\{(\tau,\delta):\ |\tau| \in \left(0,\, 2\cdot 2^{-k}\right];\ ||\delta|| \in \left(2^{-k/\alpha},\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Now we have to calculate the expressions $\mu\otimes\mu\{\cdot\}$ for I_3 and I_4. For each $k \in \mathbb{N}$ we tile $\mathbb{R}_+ \times \mathbb{R}^d$ by disjoint α-parabolic cylinders of size $2^{-k} \times 2^{-k/\alpha} \times \cdots \times 2^{-k/\alpha}$ and again denote the collection of such cylinders by $\mathcal{D}_k$. For every $c \in (0,1]$, $\gamma = \varphi_\alpha - \alpha\cdot\varepsilon$ and $\alpha \in [1,2)$ Frostman's lemma 6.6 yields

$$\mu\left([t,\, t+c] \times \prod_{i=1}^{d} \left[x_i,\, x_i + c^{1/\alpha}\right]\right) \lesssim c^{\gamma/\alpha},$$

in particular we have $\mu(Q') \lesssim 2^{-k\gamma/\alpha}$ for each $Q' \in \mathcal{D}_k$.

In order to estimate I_3 we define the following relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two cylinders Q, Q' of the same generation we write $Q \sim Q'$ if there exists $(s,x) \in Q$ and $(t,y) \in Q'$

such that $|\tau| \in \big(2^{-k},\, 2 \cdot 2^{-k}\big]$ and $||\delta|| \in \big[0,\, 2^{1/\alpha} \cdot 2^{-k/\alpha}\big]$. Thus

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q' \in \mathcal{D}_k \\ Q \sim Q'}} \mu \otimes \mu\,(Q \times Q')$$

$$= \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q' \in \mathcal{D}_k \\ Q \sim Q'}} \mu(Q) \cdot \mu(Q').$$

Now we fix some cylinder Q. The number of cylinders related to Q via $\sim$ is bounded by a universal constant. Hence

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q \in \mathcal{D}_k} \sum_{Q \sim Q'} \mu(Q) \cdot \mu(Q')$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q \in \mathcal{D}_k} \sum_{Q' \sim Q} \mu(Q) \cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q \in \mathcal{D}_k} \mu(Q) \cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\gamma)/\alpha} \cdot \sum_{Q \in \mathcal{D}_k} \mu(Q).$$

Note that $\sum_{Q \in \mathcal{D}_k} \mu(Q) = \mu(\cup_{Q \in \mathcal{D}_k} Q) = \mu\big(\mathbb{R}_+ \times \mathbb{R}^d\big) = 1$ and we conclude

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\gamma)/\alpha} \leq \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty,$$

since $\beta = \varphi_\alpha - 2\alpha \cdot \varepsilon$ and $\gamma = \varphi_\alpha - \alpha \cdot \varepsilon$.

For the estimation of I_4 we define a novel relation on $(\mathcal{D}_k)_{k \in \mathbb{N}}$: For two cylinders Q, Q' of the same generation we write $Q \approx Q'$ if there exists $(s,x) \in Q$ and $(t,y) \in Q'$ such that $|\tau| \in \big(0,\, 2 \cdot 2^{-k}\big]$ and $||\delta|| \in \big(2^{-k/\alpha},\, 2^{1/\alpha} \cdot 2^{-k/\alpha}\big]$. Thus

$$I_4 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q' \in \mathcal{D}_k \\ Q \approx Q'}} \mu(Q) \cdot \mu(Q').$$

Now the number of cylinders related to some fixed Q via $\approx$ is bounded by a universal constant. Hence the same calculation as for I_3 yields

$$I_4 \lesssim \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty.$$

(iii) Finally, we treat the case $\alpha \in [1,2)$ and $\varphi_\alpha > d$. Let $\beta = (1-\frac{1}{\alpha})\cdot d + \frac{1}{\alpha}\cdot\varphi_\alpha - 2\varepsilon > d$ for for sufficiently small $\varepsilon > 0$. Due to Lemma 6.10 we have

$$\mathcal{E}_{K^\beta}(\mu) \lesssim \underbrace{\int\int_{\{\tau\in(0,1],\, ||\delta||\in[0,\,\tau^{1/\alpha}]\}} \tau^{(1-1/\alpha)d-\beta},\, \mathrm{d}\mu\,\mathrm{d}\mu}_{=:\, I_5}$$

$$+ \underbrace{\int\int_{\{\tau\in(0,1],\, ||\delta||\in(\tau^{1/\alpha},\,1]\}} ||\delta||^{(\alpha-1)d-\alpha\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\, I_6}.$$

We get

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \cdot \mu\otimes\mu\left\{(\tau,\delta):\ |\tau| \in \left(2^{-k},\, 2\cdot 2^{-k}\right],\, ||\delta|| \in \left[0,\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Further,

$$I_6 \lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \cdot \mu\otimes\mu\left\{(\tau,\delta):\ |\tau| \in \left(0,\, 2\cdot 2^{-k}\right];\ ||\delta|| \in \left(2^{-k/\alpha},\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Now we have to calculate the expressions $\mu\otimes\mu\{\cdot\}$ for I_5 and I_6. For each $k \in \mathbb{N}$ we tile $\mathbb{R}_+ \times \mathbb{R}^d$ by disjoint α-parabolic cylinders of size $2^{-k} \times 2^{-k/\alpha} \times \cdots \times 2^{-k/\alpha}$ and again denote the collection of such cylinders by $\mathcal{D}_k$. For every $c \in (0,1]$, $\gamma = \varphi_\alpha - \alpha\cdot\varepsilon$ and $\alpha \in [1,2)$ Frostman's Lemma 6.6 yields

$$\mu\left([t,\, t+c] \times \prod_{i=1}^{d},\, \left[x_i,\, x_i + c^{1/\alpha}\right]\right) \lesssim c^{\gamma/\alpha}$$

in particular we have $\mu(Q') \lesssim 2^{-k\gamma/\alpha}$ for each $Q' \in \mathcal{D}_k$.

In order to estimate I_5 we define the following relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two cylinders Q, Q' of the same generation we write $Q \sim Q'$ if there exists $(s,x) \in Q$ and $(t,y) \in Q'$ such that $|\tau| \in \left(2^{-k},\, 2\cdot 2^{-k}\right]$ and $||\delta|| \in \left[0,\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]$.

Thus

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\sim Q'}} \mu\otimes\mu\,(Q\times Q')$$

$$= \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\sim Q'}} \mu(Q)\cdot\mu(Q').$$

Now we fix some cylinder Q. The number of cylinders related to Q via $\sim$ is bounded by a universal constant. Hence

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \sum_{Q\in\mathcal{D}_k} \sum_{Q\sim Q'} \mu(Q)\cdot\mu(Q')$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \sum_{Q\in\mathcal{D}_k} \sum_{Q'\sim Q} \mu(Q)\cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \sum_{Q\in\mathcal{D}_k} \mu(Q)\cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]-k\gamma/\alpha}\cdot \sum_{Q\in\mathcal{D}_k} \mu(Q).$$

Note that $\sum_{Q\in\mathcal{D}_k}\mu(Q)=\mu(\cup_{Q\in\mathcal{D}_k}Q)=\mu\big(\mathbb{R}_+\times\mathbb{R}^d\big)=1$ and we conclude

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty,$$

since $\beta=(1-1/\alpha)\cdot d+\varphi_\alpha/\alpha-2\varepsilon$ and $\gamma=\varphi_\alpha-\alpha\cdot\varepsilon$.

For the estimation of I_6 we define a novel relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two cylinders Q,Q' of the same generation we write $Q\approx Q'$ if there exists $(s,x)\in Q$ and $(t,y)\in Q'$ such that $|\tau|\in\big(0,\,2\cdot 2^{-k}\big]$ and $||\delta||\in\big(2^{-k/\alpha},\,2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$. Thus

$$I_6 \lesssim \sum_{k=1}^{\infty} 2^{-k[(1-1/\alpha)d-\beta]} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\approx Q'}} \mu(Q)\cdot\mu(Q').$$

Now the number of cylinders related to some fixed Q via $\approx$ is bounded by a universal constant. Hence the same calculation as for I_5 yields

$$I_6 \lesssim \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty$$

and the theorem is proven. $\square$

7. Range: Upper and Lower Bounds

We give upper and lower bounds for the Hausdorff dimension of the range of a stable Lévy process with drift.

Theorem 7.1. *Let $T \subseteq \mathbb{R}_+$ be any set and $\alpha \in (0,2)$. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$ and $f : T \to \mathbb{R}^d$ be any function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$. Then one $\mathbb{P}$-almost surely has*

$$\dim \mathcal{R}_T(X+f) \leq \begin{cases} \alpha \cdot \varphi_\alpha \ \wedge \ d, & \alpha \in (0,1], \\ \varphi_\alpha \ \wedge \ d, & \alpha \in [1,2), \end{cases} \tag{7.1}$$

where $\mathcal{R}_T(X+f)$ denotes the range of $X+f$ over T.

Proof. The Gaussian case follows from the proof of Theorem 1.2 in [37] and Proposition 3.4. Since the Hausdorff dimension of the range never exceeds the topological dimension of the space a function maps to we always have

$$\dim \mathcal{R}_T(X+f) \leq d.$$

In case of $\alpha \in [1,2)$ the claim directly follows from Theorem 3.6 and Theorem 5.1 which yield

$$\dim \mathcal{R}_T(X+f) \leq \dim \mathcal{G}_T(X+f) \leq \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(X+f) \leq \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f) = \varphi_\alpha.$$

Let $\alpha \in (0,1]$, $\beta = \alpha \cdot \varphi_\alpha$ and let $\delta, \varepsilon > 0$ be arbitrary. Then $\mathcal{G}_T(f)$ can be covered by α-parabolic cylinders

$$\left(\mathsf{P}_{c_k}\right)_{k\in\mathbb{N}} = \left([t_k,\, t_k + c_k] \times \prod_{i=1}^{d} \left[x_{i,k},\, x_{i,k} + c_k^{1/\alpha}\right]\right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^\alpha$$

such that

$$\sum_{k=1}^{\infty} |\mathsf{P}_{c_k}|^{(\beta+\delta)/\alpha} \lesssim \sum_{k=1}^{\infty} c_k^{(\beta+\delta)/\alpha} \leq \varepsilon.$$

Let $M_k(\omega)$ be the random number of a fixed 2^d-nested collection of hypercubes with sidelength $c_k^{1/\alpha}$ that the path $t \mapsto X_t(\omega)$ hits at some time $t \in [t_k,\, t_k + c_k]$.

Let $\cup_{k\in\mathbb{N}} \mathsf{P}'_{c_k}(\omega) \supseteq \mathcal{G}_T(X(\omega))$ with

$$\big(\mathsf{P}'_{c_k}(\omega)\big)_{k\in\mathbb{N}} = \left([t_k,\, t_k + c_k] \times \bigcup_{j=1}^{M_k(\omega)} \prod_{i=1}^{d} \left[\xi_{i,j,k}(\omega),\, \xi_{i,j,k}(\omega) + c_k^{1/\alpha}\right]\right)_{k\in\mathbb{N}}$$

being a corresponding cover of the graph of this path. Then for all $t \in [t_k,\, t_k + c_k]$ there exists $j \in \{1, \ldots, M_k(\omega)\}$ such that for the i-th component of $X + f$ we have

$$\xi_{i,j,k}(\omega) + x_{i,k} \leq X_t^{(i)}(\omega) + f_{(i)}(t) \leq \xi_{i,j,k}(\omega) + x_{i,k} + 2c_k^{1/\alpha}.$$

Hence we obtain a random cover $\cup_{k\in\mathbb{N}} \widetilde{\mathsf{P}}_{c_k}(\omega) \supseteq \mathcal{G}_T(X(\omega) + f)$ where

$$\widetilde{\mathsf{P}}_{c_k}(\omega) = [t_k,\, t_k + c_k] \times \bigcup_{j=1}^{M_k(\omega)} \prod_{i=1}^{d} \left[\xi_{i,j,k}(\omega) + x_{i,k},\, \xi_{i,j,k}(\omega) + x_{i,k} + c_k^{1/\alpha}\right]$$
$$\cup \ \left[\xi_{i,j,k}(\omega) + x_{i,k} + c_k^{1/\alpha},\, \xi_{i,j,k}(\omega) + x_{i,k} + 2c_k^{1/\alpha}\right].$$

By projection we get the random cover $\bigcup_{k=1}^{\infty} \square_{c_k^{1/\alpha}}(\omega) \supseteq \mathcal{R}_T(X(\omega) + f)$ of the range with

$$\square_{c_k^{1/\alpha}}(\omega) = \bigcup_{k=1}^{M_k(\omega)} \prod_{i=1}^{d} \left[\xi_{i,j,k}(\omega) + x_{i,k}, \xi_{i,j,k}(\omega) + x_{i,k} + c_k^{1/\alpha}\right]$$
$$\cup \ \left[\xi_{i,j,k}(\omega) + x_{i,k} + c_k^{1/\alpha}, \xi_{i,j,k}(\omega) + x_{i,k} + 2c_k^{1/\alpha}\right].$$

This is a union of $M_k(\omega) \cdot 2^d$ hypercubes with diameter $\sqrt{d} \cdot c_k^{1/\alpha}$. An application of Pruitt and Taylor's covering Lemma 6.1 in [38] shows that

$$\mathbb{E}[M_k] \lesssim \frac{c_k}{\mathbb{E}\Big[T\Big(c_k^{1/\alpha}/3, c_k\Big)\Big]},$$

where $T\big(c_k^{1/\alpha}/3, c_k\big)$ denotes the sojourn time of the process $(X_t)_{t\in[0,c_k]}$ in a ball of radius $c_k^{1/\alpha}/3$ centerd at the origin. By Lemma 3.4 in [27] we have for all $\delta' > 0$

$$\mathbb{E}\Big[T\Big(c_k^{1/\alpha}/3, c_k\Big)\Big] \gtrsim c_k^{1+\delta'/\alpha}.$$

Hence we get for $\varepsilon' = \delta + \delta' > 0$

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{H}^{\beta+\varepsilon'}(\mathcal{R}_T(X+f))\Big] \\
&\quad \le \mathbb{E}\Bigg[\sum_{k=1}^{\infty} \big|\square_{c_k^{1/\alpha}}\big|^{\beta+\varepsilon'}\Bigg] \\
&\quad \lesssim \mathbb{E}\Bigg[\sum_{k=1}^{\infty} M_k(\omega)\cdot 2^d \cdot c_k^{(\beta+\varepsilon')/\alpha}\Bigg] \\
&\quad \lesssim \sum_{k=1}^{\infty} \mathbb{E}[M_k(\omega)]\cdot c_k^{(\beta+\varepsilon')/\alpha} \\
&\quad \lesssim \sum_{k=1}^{\infty} \frac{c_k}{\mathbb{E}\Big[T\Big(c_k^{1/\alpha}/3, c_k\Big)\Big]}\, c_k^{(\beta+\varepsilon')/\alpha} \\
&\quad \lesssim \sum_{k=1}^{\infty} c_k^{(\beta+\varepsilon'-\delta')/\alpha} \\
&\quad = \sum_{k=1}^{\infty} c_k^{(\beta+\delta)/\alpha} \\
&\quad \le \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get for all $\alpha \in (0,1]$

$$\mathbb{E}\Big[\mathcal{H}^{\beta+\varepsilon'}(\mathcal{R}_T(X+f))\Big] = 0.$$

Therefore

$$\mathcal{H}^{\beta+\varepsilon'}(\mathcal{R}_T(X+f)) = 0$$

holds $\mathbb{P}$-almost surely. Hence

$$\dim \mathcal{R}_T(X+f) \le \beta + \varepsilon'$$

$\mathbb{P}$-almost surely. Since $\varepsilon' > 0$ is also arbitrary we finally get

$$\dim \mathcal{R}_T(X+f) \le \beta = \alpha \cdot \varphi_\alpha,$$

$\mathbb{P}$-almost surely, as claimed. □

The lower bound is obtained by the energy method. In our case the set A from Definition 6.3 is just $\mathcal{R}_T(X+f)$. The next lemma shows that we can work again with an energy integral where the stable process X is transformed into the kernel.

Lemma 7.2 (Kernel transformation)**.** *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0,2)$. Let $X = (X_t)_{t\geq 0}$ be an isotropic stable Lévy process in $\mathbb{R}^d$ and $f : \mathbb{R}_+ \to \mathbb{R}^d$ be a Borel measurable function. Define the difference kernel*

$$\kappa^\beta(t,x) := \mathbb{E}\big[||\mathrm{sign}(t)\cdot X_{|t|}(\omega) + x||^{-\beta}\big].$$

Then

$$\mathrm{Cap}_{\kappa^\beta}(\mathcal{G}_T(f)) > 0$$

implies

$$\mathrm{Cap}_\beta(\mathcal{R}_T(X(\omega)+f)) > 0, \ \mathbb{P}\text{-almost surely.}$$

Hence $\mathcal{E}_{\kappa^\beta}(\mu) < \infty$ for some probability measure $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$ implies

$$\dim \mathcal{R}_T(X+f) \geq \beta, \ \mathbb{P}\text{-almost surely.}$$

Proof. Let $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$ and π_t denote the projection onto the time component, i.e. $\pi_t(t,f(t)) = t$. Define the probability measure $\nu \in \mathcal{M}^1(\mathbb{R}_+)$ as the pushforward measure

$$\nu(A) = \mu\big(\pi_t^{-1}(A)\big)$$

for Borel sets $A \subseteq \mathbb{R}_+$ and further the random probability measure

$$\widetilde{\mu}_\omega(R) = \nu((X(\omega)+f)^{-1}(R))$$

for every Borel set $R \subseteq \mathbb{R}^d$. Then Tonelli's theorem and the stationarity of the increments of X yield

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{E}_\beta(\widetilde{\mu}_\omega)\big] \\
&\quad= \mathbb{E}\bigg[\int_{\mathcal{R}_T(X(\omega)+f)}\int_{\mathcal{R}_T(X(\omega)+f)} ||x-y||^{-\beta}\,\mathrm{d}\widetilde{\mu}_\omega(x)\,\mathrm{d}\widetilde{\mu}_\omega(y)\bigg] \\
&\quad= \mathbb{E}\bigg[\int_T\int_T ||X_t(\omega)+f(t)-(X_s(\omega)+f(s))||^{-\beta}\,\mathrm{d}\nu(t)\,\mathrm{d}\nu(s)\bigg] \\
&\quad= \int_{\mathcal{G}_T(f)}\int_{\mathcal{G}_T(f)} \mathbb{E}\big[||X_t(\omega)-X_s(\omega)+x-y||^{-\beta}\big]\,\mathrm{d}\mu(t,x)\,\mathrm{d}\mu(s,y)
\end{aligned}$$

$$= \int_{\mathcal{G}_T(f)} \int_{\mathcal{G}_T(f)} \mathbb{E}\big[||\mathrm{sign}(t-s)\cdot X_{|t-s|}(\omega) + x - y||^{-\beta}\big]\,\mathrm{d}\mu(t,x)\,\mathrm{d}\mu(s,y)$$

$$= \mathcal{E}_{\kappa^\beta}(\mu).$$

By assumption, there exists $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$ such that $\mathcal{E}_{\kappa^\beta}(\mu) < \infty$ holds. From that one $\mathbb{P}$-almost surely has $\mathcal{E}_\beta(\widetilde{\mu}_\omega) < \infty$ and the final statement immediately follows by Frostman's Theorem 6.4 since the range of a Borel set under a Borel measurable function is a Suslin set, see Section 11 in [21]. □

We make use of new kernel estimates.

Lemma 7.3 (Kernel estimates.)**.** *Let $\alpha \in (0,2)$ and $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$. Let $\beta \in (0,d)$ and $\tau \in \mathbb{R}$, $\delta \in \mathbb{R}^d$ be such that $|\tau| \in (0,1]$, $||\delta|| \in [0,1]$. Then appropriate choices for estimating the difference kernel $\kappa^\beta(\tau,\delta) = \mathbb{E}\big[||\mathrm{sign}(t)\cdot X_{|t|}(\omega) + x||^{-\beta}\big]$ are*

$$\kappa^\beta(\tau,\delta) \lesssim |\tau|^{-\beta/\alpha},\ \textit{for } \beta < d$$

and one has

$$\kappa^\beta(\tau,\delta) \lesssim ||\delta||^{-\beta} \textit{ for } \beta < \delta \textit{ and } |\tau| \leq ||\delta||^\alpha.$$

Proof. Let $p(x)$ denote the density function of $X_1 \stackrel{\mathrm{d}}{=} |\tau|^{-1/\alpha}X_{|\tau|}$. We define the rescaled increment $\widetilde{\delta} := \delta/|\tau|^{1/\alpha}$. The self-similarity of the stable Lévy process and Lemma 6.9 yield

$$\begin{aligned}
&\mathbb{E}\big[||\mathrm{sign}(\tau)\cdot X_{|\tau|} + \delta||^{-\beta}\big] \\
&\quad = |\tau|^{-\beta/\alpha}\int_{\mathbb{R}^d} ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}||^{-\beta}\cdot p(x)\,\mathrm{d}x \\
&\quad = |\tau|^{-\beta/\alpha}\int_{\mathbb{R}^d} ||x + \mathrm{sign}(\tau)\cdot \widetilde{\delta}||^{-\beta}\cdot p(x)\,\mathrm{d}x \\
&\quad \lesssim |\tau|^{-\beta/\alpha}\int_{\mathbb{R}^d} ||x||^{-\beta}\cdot p(x)\,\mathrm{d}x \lesssim |\tau|^{-\beta/\alpha}\cdot \mathbb{E}\big[||X_1||^{-\beta}\big] \lesssim |\tau|^{-\beta/\alpha},
\end{aligned}$$

since negative moments of order $\beta < d$ exist; see Lemma 3.1 in [3].

Now consider the region $||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}|| \leq ||\widetilde{\delta}||/2$ which yields

$$||x|| = ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta} - \widetilde{\delta}|| \geq \big|||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}|| - ||\widetilde{\delta}||\big| = ||\widetilde{\delta}|| - ||\mathrm{sign}(\tau)\cdot x + \widetilde{\delta}|| \geq \frac{1}{2}\cdot||\widetilde{\delta}||.$$

Thus $\beta < d$ and $\tau \leq ||\delta||^\alpha$ lead to

$$\begin{aligned}
&\mathbb{E}\big[||\mathrm{sign}(\tau)\cdot X_{|\tau|}+\delta||^{-\beta}\big] \\
&\quad = |\tau|^{-\beta/\alpha}\int_{\mathbb{R}^d} ||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||^{-\beta}\cdot p(x)\,\mathrm{d}x \\
&\quad \lesssim ||\delta||^{-\beta} + |\tau|^{-\beta/\alpha}\int_{\{||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||\leq||\widetilde{\delta}||/2\}} ||\mathrm{sign}(\tau)\cdot x+\widetilde{\delta}||^{-\beta}\cdot p(x)\,\mathrm{d}x \\
&\quad \lesssim ||\delta||^{-\beta} + |\tau|^{-\beta/\alpha}\cdot||\widetilde{\delta}||^{-d-\alpha}\int_0^{||\widetilde{\delta}||} r^{d-\beta-1}\,\mathrm{d}r = ||\delta||^{-\beta} + |\tau|^{-\beta/\alpha}\cdot||\widetilde{\delta}||^{-\alpha-\beta} \\
&\quad = ||\delta||^{-\beta} + |\tau|\cdot||\delta||^{-\alpha-\beta} \lesssim ||\delta||^{-\beta},
\end{aligned}$$

where we have used (2.3) to estimate the tail-densities. □

We will proof the lower bounds for the range of isotropic stable Lévy processes with drift by the same methods as for the graph.

Theorem 7.4 (Lower bound for the range)**.** *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0,2)$. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$ and $f : T \to \{y \in \mathbb{R}^d : \|y - x\| \leq \frac{1}{2}\}$ for fixed $x \in \mathbb{R}^d$ be a Borel measurable function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$. Then one $\mathbb{P}$-almost surely has*

$$\dim \mathcal{R}_T(X+f) \geq \begin{cases} \alpha\cdot\varphi_\alpha \;\wedge\; d, & \alpha\in(0,1], \\ \varphi_\alpha \;\wedge\; d, & \alpha\in[1,2). \end{cases} \tag{7.2}$$

Proof. We define the increments $\tau = t - s$ and $\delta = f(t) - f(s)$ with $||\delta|| \in [0,1]$ and consider the difference kernel

$$\kappa^\beta(t,x) = \mathbb{E}\big[||\mathrm{sign}(t)\cdot X_{|t|}(\omega)+x||^{-\beta}\big].$$

We prove that $\mathcal{E}_{\kappa^\beta}(\mu) < \infty$ holds for $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$ from the parabolic version of Frostman's Lemma in Theorem 6.6 for every β less than the right side of Equation (7.2). Then the claim follows due to Lemma 7.2. A priori our Lemma 7.3 yields

$$
\begin{aligned}
&\mathcal{E}_{\kappa^\beta}(\mu)\\
&\quad = \int\int_{\mathcal{G}_T(f)\times\mathcal{G}_T(f)} \kappa^\beta(t-s, f(t)-f(s))\,\mathrm{d}\mu(s,x)\,\mathrm{d}\mu(t,y)\\
&\quad \le \int\int_{\{|\tau|\in(0,1],\,||\delta||\in[0,1]\}} \kappa^\beta(\tau,\delta)\,\mathrm{d}\mu\,\mathrm{d}\mu + \int\int_{\{|\tau|\in(1,\infty),\,||\delta||\in[0,1]\}} |\tau|^{-\beta/\alpha}\,\mathrm{d}\mu\,\mathrm{d}\mu\\
&\quad \lesssim \int\int_{\{|\tau|\in(0,1],\,||\delta||\in[0,1]\}} \mathbb{E}\big[||\mathrm{sign}(\tau)\cdot X_{|\tau|}(\omega)+\delta||^{-\beta}\big]\,\mathrm{d}\mu\,\mathrm{d}\mu
\end{aligned}
$$

in all cases.

(i) We begin with the case $\alpha \in (0,1]$ and $\alpha \cdot \varphi_\alpha \le d$, then $\beta = \alpha\cdot\varphi_\alpha - 2\alpha\cdot\varepsilon < d$ for some arbitrary $\varepsilon > 0$. Due to Lemma 7.3 we have

$$
\mathcal{E}_{\kappa^\beta}(\mu) \lesssim \underbrace{\int\int_{\{|\tau|\in(0,1],\,||\delta||\in[0,|\tau|^{1/\alpha}]\}} |\tau|^{-\beta/\alpha}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\ I_1} + \underbrace{\int\int_{\{|\tau|\in(0,1],\,||\delta||\in(|\tau|^{1/\alpha},1]\}} ||\delta||^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\ I_2}.
$$

We get

$$
I_1 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha}\cdot\mu\otimes\mu\left\{(\tau,\delta):\ |\tau|\in\left(2^{-k},\,2\cdot 2^{-k}\right],\,||\delta||\in\left[0, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.
$$

Further,

$$
I_2 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha}\cdot\mu\otimes\mu\left\{(\tau,\delta):\ |\tau|\in\left(0,\,2\cdot 2^{-k}\right];\ ||\delta||\in\left(2^{-k/\alpha},\,2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.
$$

Now we have to calculate the expressions $\mu\otimes\mu\{\cdot\}$ for I_1 and I_2. For each $k\in\mathbb{N}$, we tile $\mathbb{R}_+\times\mathbb{R}^d$ by disjoint α-parabolic cylinders $\mathcal{D}_k$ of dimension $2^{-k}\times 2^{-k/\alpha}\times\cdots\times 2^{-k/\alpha}$. For every $c\in(0,1]$, $\gamma = \varphi_\alpha - \varepsilon$ and $\alpha\in(0,1]$ Frostman's Lemma 6.6 yields

$$
\mu\left([t,\,t+c]\times\prod_{i=1}^{d}\left[x_i,\,x_i+c^{1/\alpha}\right]\right) \lesssim c^\gamma,
$$

in particular we have $\mu(Q') \lesssim 2^{-k\gamma}$ for each $Q'\in\mathcal{D}_k$.

In order to estimate I_1 we define the following relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two α-parabolic cylinders Q, Q' of the same generation we write $Q \sim Q'$ if there exists $(s,x) \in Q$ and $(t,y) \in Q'$ such that $|\tau| \in \big(2^{-k},\, 2\cdot 2^{-k}\big]$ and $||\delta|| \in \big[0, 2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$. Thus

$$I_1 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\sim Q'}} \mu\otimes\mu\,(Q\times Q')$$

$$= \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\sim Q'}} \mu(Q)\cdot\mu(Q').$$

Now we fix some α-parabolic cylinder Q. The number of α-parabolic cylinders related to Q via $\sim$ is bounded by a universal constant not depending on k and Q. Hence

$$I_1 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q\sim Q'} \mu(Q)\cdot\mu(Q')$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q'\sim Q} \mu(Q)\cdot 2^{-k\gamma}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \mu(Q)\cdot 2^{-k\gamma}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\alpha\cdot\gamma)/\alpha}\cdot \sum_{Q\in\mathcal{D}_k} \mu(Q).$$

Note that $\sum_{Q\in\mathcal{D}_k} = \mu(\cup_{Q\in\mathcal{D}_k} Q) = \mu\big(\mathbb{R}_+\times\mathbb{R}^d\big) = 1$ and we conclude

$$I_1 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\alpha\cdot\gamma)/\alpha} = \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty,$$

since $\beta = \alpha\cdot\varphi_\alpha - 2\alpha\cdot\varepsilon$ and $\gamma = \varphi_\alpha - \varepsilon$.

For the estimation of I_2 we define a novel relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two hypercubes Q, Q' of the same generation we write $Q \approx Q'$ if there exists $(s,x)\in Q$ and $(t,y)\in Q'$ such that $|\tau| \in \big(0,\, 2\cdot 2^{-k}\big]$ and $||\delta|| \in \big(2^{-k},\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$.

Thus

$$I_2 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k \\ Q\approx Q'}} \mu(Q)\cdot\mu(Q').$$

Now the number of hypercubes related to some fixed Q via $\approx$ is bounded by a universal constant. Hence the same calculation as for I_1 yields

$$I_2 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\alpha\cdot\gamma)/\alpha} = \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty$$

as claimed.

(ii) Now we treat the case $\alpha \in [1,2)$ and $\beta = \varphi_\alpha - 2\varepsilon < d$ for some arbitrary $\varepsilon > 0$. Due to Lemma 7.3 we have

$$\mathcal{E}_{\kappa^\beta}(\mu) \lesssim \underbrace{\int\int_{\{|\tau|\in(0,1],\,||\delta||\in[0,|\tau|^{1/\alpha}]\}} |\tau|^{-\beta/\alpha}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\,I_3} + \underbrace{\int\int_{\{|\tau|\in(0,1],\,||\delta||\in(|\tau|^{1/\alpha},1]\}} ||\delta||^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\,I_4}.$$

We get

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha}\cdot\mu\otimes\mu\left\{(\tau,\delta):\ |\tau|\in\left(2^{-k},\,2\cdot 2^{-k}\right],\ ||\delta||\in\left[0,2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Further,

$$I_4 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha}\cdot\mu\otimes\mu\left\{(\tau,\delta):\ |\tau|\in\left(0,\,2\cdot 2^{-k}\right];\ ||\delta||\in\left(2^{-k/\alpha},\,2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Now we have to calculate the expressions $\mu\otimes\mu\{\cdot\}$ for I_3 and I_4. For each $k\in\mathbb{N}$ we tile $\mathbb{R}_+\times\mathbb{R}^d$ by disjoint α-parabolic cylinders $\mathcal{D}_k$ of dimension $2^{-k}\times 2^{-k/\alpha}\times\cdots\times 2^{-k/\alpha}$. For every $c\in(0,1]$, $\gamma=\varphi_\alpha-\varepsilon$ and $\alpha\in[1,2)$ Frostman's Lemma 6.6 yields

$$\mu\left([t,\,t+c]\times\prod_{i=1}^{d}\left[x_i,\,x_i+c^{1/\alpha}\right]\right) \lesssim c^{\gamma/\alpha}$$

in particular we have $\mu(Q') \lesssim 2^{-k\gamma/\alpha}$ for each $Q'\in\mathcal{D}_k$.

In order to estimate I_3 we define the following relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two cylinders Q, Q' of the same generation we write $Q\sim Q'$ if there exists $(s,x)\in Q$ and $(t,y)\in Q'$

such that $|\tau| \in \big(2^{-k},\, 2\cdot 2^{-k}\big]$ and $||\delta|| \in \big[0, 2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$. Thus

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\sim Q'}} \mu\otimes\mu\,(Q\times Q')$$

$$= \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\sim Q'}} \mu(Q)\cdot\mu(Q').$$

Now we fix some cylinder Q. The number of cylinders related to Q via $\sim$ is bounded by a universal constant. Hence

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q\sim Q'} \mu(Q)\cdot\mu(Q')$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q'\sim Q} \mu(Q)\cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \mu(Q)\cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\gamma)/\alpha}\cdot \sum_{Q\in\mathcal{D}_k} \mu(Q).$$

Note that $\sum_{Q\in\mathcal{D}_k} = \mu(\cup_{Q\in\mathcal{D}_k} Q) = \mu\big(\mathbb{R}_+\times\mathbb{R}^d\big) = 1$ and we conclude

$$I_3 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\gamma)/\alpha} \le \sum_{k=1}^{\infty} 2^{-k\varepsilon/\alpha} < \infty,$$

since $\beta = \varphi_\alpha - 2\varepsilon$ and $\gamma = \varphi_\alpha - \varepsilon$.

For the estimation of I_4 we define a novel relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two cylinders Q, Q' of the same generation we write $Q \approx Q'$ if there exists $(s,x)\in Q$ and $(t,y)\in Q'$ such that $|\tau| \in \big(0,\, 2\cdot 2^{-k}\big]$ and $||\delta|| \in \big(2^{-k/\alpha},\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$. Thus

$$I_4 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k\\ Q\approx Q'}} \mu(Q)\cdot\mu(Q').$$

Now the number of cylinders related to some fixed Q via $\approx$ is bounded by a universal constant. Hence the same calculation as for I_3 yields

$$I_4 \lesssim \sum_{k=1}^{\infty} 2^{-k\varepsilon/\alpha} < \infty.$$

(iii) Next, we treat the case $\alpha \in (0,1]$, $\alpha \cdot \varphi_\alpha \in [d, d+1]$. Let $\beta = d - 2\varepsilon$ for some arbitrary $\varepsilon > 0$. Due to Lemma 7.3 we have

$$\mathcal{E}_{\kappa^\beta}(\mu) \lesssim \underbrace{\int\int_{\{|\tau|\in(0,1],\,||\delta||\in[0,|\tau|^{1/\alpha}]\}} |\tau|^{-\beta/\alpha}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\ I_5} + \underbrace{\int\int_{\{|\tau|\in(0,1],\,||\delta||\in(|\tau|^{1/\alpha},1]\}} ||\delta||^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu}_{=:\ I_6}.$$

We get

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \cdot \mu \otimes \mu \left\{(\tau,\delta) :\ |\tau| \in \left(2^{-k},\, 2\cdot 2^{-k}\right],\, ||\delta|| \in \left[0, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Further,

$$I_6 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \cdot \mu \otimes \mu \left\{(\tau,\delta) :\ |\tau| \in \left(0,\, 2\cdot 2^{-k}\right];\ ||\delta|| \in (2^{-k/\alpha},\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]\right\}.$$

Now we have to calculate the expressions $\mu\otimes\mu\{\cdot\}$ for I_5 and I_6. For every $k \in \mathbb{N}$ we tile $\mathbb{R}_+ \times \mathbb{R}^d$ with α-parabolic cylinders $\mathcal{D}_k$ of dimension $2^{-k} \times 2^{-k/\alpha} \times \cdots \times 2^{-k/\alpha}$. For every $c \in (0,1]$, $\gamma = d/\alpha - \varepsilon/\alpha < \varphi_\alpha$ and $\alpha \in [1,2)$ Frostman's Lemma 6.6 yields

$$\mu\left([t,\, t+c] \times \prod_{i=1}^{d} \left[x_i,\, x_i + c^{1/\alpha}\right]\right) \lesssim c^\gamma$$

in particular we have $\mu(Q') \lesssim 2^{-k\gamma}$ for each $Q' \in \mathcal{D}_k$.

In order to estimate I_5 we define the following relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two α-parabolic cylinders Q, Q' of the same generation we write $Q \sim Q'$ if there exists $(s,x) \in Q$ and $(t,y) \in Q'$ such that $|\tau| \in \left(2^{-k},\, 2\cdot 2^{-k}\right]$ and $||\delta|| \in \left[0, 2^{1/\alpha}\cdot 2^{-k/\alpha}\right]$.

Thus

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k \\ Q\sim Q'}} \mu\otimes\mu\,(Q\times Q')$$

$$= \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k \\ Q\sim Q'}} \mu(Q)\cdot\mu(Q').$$

Now we fix some cylinder Q. The number of cylinders related to Q via $\sim$ is bounded by a universal constant. Hence

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q\sim Q'} \mu(Q)\cdot\mu(Q')$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q'\sim Q} \mu(Q)\cdot 2^{-k\gamma}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \mu(Q)\cdot 2^{-k\gamma}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k(\beta/\alpha-\gamma)}\cdot \sum_{Q\in\mathcal{D}_k} \mu(Q).$$

Note that $\sum_{Q\in\mathcal{D}_k} = \mu(\cup_{Q\in\mathcal{D}_k} Q) = \mu\big(\mathbb{R}_+\times\mathbb{R}^d\big) = 1$ and we conclude

$$I_5 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta/\alpha-\gamma)} \le \sum_{k=1}^{\infty} 2^{-k\varepsilon/\alpha} < \infty,$$

since $\beta = d-2\varepsilon$ and $\gamma = d/\alpha - \varepsilon/\alpha$.

For the estimation of I_6 we define a novel relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two cylinders Q, Q' of the same generation we write $Q\approx Q'$ if there exists $(s,x)\in Q$ and $(t,y)\in Q'$ such that $|\tau|\in\big(0,\,2\cdot 2^{-k}\big]$ and $||\delta||\in\big(2^{-k/\alpha},\,2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$.

Thus

$$I_6 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q' \in \mathcal{D}_k \\ Q \approx Q'}} \mu(Q) \cdot \mu(Q').$$

Now the number of cylinders related to some fixed Q via $\approx$ is bounded by a universal constant. Hence the same calculation as for I_5 yields

$$I_6 \lesssim \sum_{k=1}^{\infty} 2^{-k\varepsilon/\alpha} < \infty,$$

as claimed.

(iv) Finally, we treat the case $\alpha \in [1, 2)$, $\varphi_\alpha \in [d, d+1]$. Let $\beta = d - 2\varepsilon$ for some arbitrary $\varepsilon > 0$. Due to Lemma 7.3 we have

$$\mathcal{E}_{\kappa^\beta}(\mu) \lesssim \underbrace{\int\int_{\{|\tau| \in (0,1],\, ||\delta|| \in [0, |\tau|^{1/\alpha}]\}} |\tau|^{-\beta/\alpha} \,\mathrm{d}\mu \,\mathrm{d}\mu}_{=:\ I_7} + \underbrace{\int\int_{\{|\tau| \in (0,1],\, ||\delta|| \in (|\tau|^{1/\alpha}, 1]\}} ||\delta||^{-\beta} \,\mathrm{d}\mu \,\mathrm{d}\mu}_{=:\ I_8}.$$

We get

$$I_7 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \cdot \mu \otimes \mu \left\{(\tau, \delta) :\ |\tau| \in \left(2^{-k},\, 2 \cdot 2^{-k}\right],\, ||\delta|| \in \left[0, 2^{1/\alpha} \cdot 2^{-k/\alpha}\right]\right\}.$$

Further,

$$I_8 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \cdot \mu \otimes \mu \left\{(\tau, \delta) :\ |\tau| \in \left(0,\, 2 \cdot 2^{-k}\right];\ ||\delta|| \in \left(2^{-k/\alpha},\, 2^{1/\alpha} \cdot 2^{-k/\alpha}\right]\right\}.$$

Now we have to calculate the expressions $\mu \otimes \mu\{\cdot\}$ for I_7 and I_8. For every $k \in \mathbb{N}$ we tile $\mathbb{R}_+ \times \mathbb{R}^d$ with α-parabolic cylinders $\mathcal{D}_k$ of dimension $2^{-k} \times 2^{-k/\alpha} \times \cdots \times 2^{-k/\alpha}$. For every $c \in (0, 1]$, $\gamma = d - \varepsilon < \varphi_\alpha \in [d, d+1]$ and $\alpha \in [1, 2)$ Frostman's Lemma 6.6 yields

$$\mu\left([t,\, t+c] \times \prod_{i=1}^{d} \left[x_i,\, x_i + c^{1/\alpha}\right]\right) \lesssim c^{\gamma/\alpha},$$

in particular we have $\mu(Q') \lesssim 2^{-k\gamma/\alpha}$ for each $Q' \in \mathcal{D}_k$.

In order to estimate I_7 we define the following relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two α-parabolic cylinders Q, Q' of the same generation we write $Q \sim Q'$ if there exists $(s,x) \in Q$ and $(t,y) \in Q'$ such that $|\tau| \in \big(2^{-k},\, 2\cdot 2^{-k}\big]$ and $||\delta|| \in \big[0, 2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$. Thus

$$I_7 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k \\ Q\sim Q'}} \mu\otimes\mu\,(Q\times Q')$$

$$= \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k \\ Q\sim Q'}} \mu(Q)\cdot\mu(Q').$$

Now we fix some cylinder Q. The number of cylinders related to Q via $\sim$ is bounded by a universal constant. Hence

$$I_7 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q\sim Q'} \mu(Q)\cdot\mu(Q')$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \sum_{Q'\sim Q} \mu(Q)\cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{Q\in\mathcal{D}_k} \mu(Q)\cdot 2^{-k\gamma/\alpha}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\gamma)/\alpha}\cdot \sum_{Q\in\mathcal{D}_k} \mu(Q).$$

Note that $\sum_{Q\in\mathcal{D}_k} = \mu(\cup_{Q\in\mathcal{D}_k} Q) = \mu\big(\mathbb{R}_+\times\mathbb{R}^d\big) = 1$ and we conclude

$$I_7 \lesssim \sum_{k=1}^{\infty} 2^{k(\beta-\gamma)/\alpha} \le \sum_{k=1}^{\infty} 2^{-k\varepsilon/\alpha} < \infty,$$

since $\beta = d - 2\varepsilon$ and $\gamma = d - \varepsilon$.

For the estimation of I_8 we define a novel relation on $(\mathcal{D}_k)_{k\in\mathbb{N}}$: For two cylinders Q, Q' of the same generation we write $Q \approx Q'$ if there exists $(s,x) \in Q$ and $(t,y) \in Q'$ such that $|\tau| \in \big(0,\, 2\cdot 2^{-k}\big]$ and $||\delta|| \in \big(2^{-k/\alpha},\, 2^{1/\alpha}\cdot 2^{-k/\alpha}\big]$. Thus

$$I_8 \lesssim \sum_{k=1}^{\infty} 2^{k\beta/\alpha} \sum_{\substack{Q,Q'\in\mathcal{D}_k \\ Q\approx Q'}} \mu(Q)\cdot\mu(Q').$$

Now the number of cylinders related to some fixed Q via $\approx$ is bounded. Hence the same calculation as for I_7 yields

$$I_8 \lesssim \sum_{k=1}^{\infty} 2^{-k\varepsilon/\alpha} < \infty,$$

and the lemma is proven. $\square$

8. Estimates for the Parabolic Hausdorff Dimension

So far we calculated the Hausdorff dimension of a stable Lévy process plus deterministic drift function in terms of the parabolic dimension of the drift alone. Now, we want to give estimates for the α-parabolic Hausdorff dimension itself. We begin with the parabolic Hausdorff dimension of a constant function.

Lemma 8.1. *Let $T \subseteq \mathbb{R}^n$ be any set and $\alpha \in (0, \infty)$. Define the constant function $f_C : T \mapsto C \in \mathbb{R}^d$. Then one has*

$$\mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f_C) \leq (\alpha \vee 1) \cdot \dim T$$

Proof. Without loss of generality, let $f_C = f_0 \equiv 0 \in \mathbb{R}^d$.

(i) Let $\alpha \in (0, 1]$, $\beta = \dim T$ and let δ, $\varepsilon > 0$ be arbitrary. Then there exists a cover $\cup_{k \in \mathbb{N}} T_k \supseteq T$ with

$$T_k = \prod_{i=1}^n [t_{i,k},\, t_{i,k} + c_k]$$

and $c_k \leq 1$ such that

$$\sum_{k=1}^\infty |T_k|^{\beta+\delta} \lesssim \sum_{k=1}^\infty c_k^{\beta+\delta} \leq \varepsilon.$$

Now, $\mathcal{G}_T(f_0)$ can be covered by α-parabolic cylinders

$$\left(\mathsf{P}_{c_k}\right)_{k \in \mathbb{N}} = \left(\prod_{i=1}^n [t_{i,k},\, t_{i,k} + c_k] \times \prod_{j=1}^d \, [0, c_k^{1/\alpha}] \right)_{k \in \mathbb{N}} \subseteq \mathscr{P}^\alpha.$$

Note that $|\mathsf{P}_{c_k}| \asymp c_k$. Hence

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^{\beta+\delta}(\mathcal{G}_T(f_0)) \leq \sum_{k=1}^\infty |\mathsf{P}_{c_k}|^{\beta+\delta} \lesssim \sum_{k=1}^\infty c_k^{\beta+\delta} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, for all $\delta > 0$ we have

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^{\beta+\delta}(\mathcal{G}_T(f_0)) = 0$$

and therefore one has

$$\mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f_0) \leq \beta + \delta.$$

Since $\delta > 0$ is also arbitrary, we obtain

$$\mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f_0) \leq \beta = \dim T.$$

(ii) Let $\alpha \in [1, \infty)$, $\beta = \alpha \cdot \dim T$ and let δ, $\varepsilon > 0$ be arbitrary. Then there exists a cover $\cup_{k\in\mathbb{N}} T_k \supseteq T$ with

$$T_k = \prod_{i=1}^{n} [t_{i,k},\, t_{i,k} + c_k]$$

and $c_k \leq 1$ such that

$$\sum_{k=1}^{\infty} |T_k|^{(\beta+\delta)/\alpha} = \sum_{k=1}^{\infty} c_k^{(\beta+\delta)/\alpha} \leq \varepsilon.$$

Then $\mathcal{G}_T(f_0)$ can be covered by α-parabolic cylinders

$$\left(\mathsf{P}_{c_k^{1/\alpha}}\right)_{k\in\mathbb{N}} = \left(\prod_{i=1}^{n} [t_{i,k},\, t_{i,k} + c_k] \times \prod_{j=1}^{d} [0, c_k^{1/\alpha}]\right)_{k\in\mathbb{N}} \subseteq \mathscr{P}^{\alpha}.$$

Since $\left|\mathsf{P}_{c_k^{1/\alpha}}\right| \asymp c_k^{1/\alpha}$ it follows that

$$\mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta+\delta}(\mathcal{G}_T(f_0)) \leq \sum_{k=1}^{\infty} \left|\mathsf{P}_{c_k^{1/\alpha}}\right|^{\beta+\delta} \lesssim \sum_{k=1}^{\infty} \left(c_k^{1/\alpha}\right)^{\beta+\delta} \lesssim \sum_{k=1}^{\infty} c_k^{(\beta+\delta)/\alpha} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, for all $\delta > 0$ we have

$$\mathscr{P}^{\alpha}\text{-}\mathcal{H}^{\beta+\delta}(\mathcal{G}_T(f_0)) = 0$$

and therefore one has

$$\mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f_0) \leq \beta + \delta.$$

Since $\delta > 0$ is also arbitrary, we obtain

$$\mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f_0) \leq \beta = \alpha \cdot \dim T,$$

as desired. □

We can calculate the α-parabolic Hausdorff dimension of an isotropic α-stable Lévy process itself. This shows that α-parabolic covers are the most efficient coverings for this self-similar process.

Theorem 8.2. *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0,2]$. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process. One $\mathbb{P}$-almost surely has*

$$\mathscr{P}^\alpha\text{- dim}\, \mathcal{G}_T(X) = (\alpha \vee 1) \cdot \dim T.$$

Proof. By Theorem 3.2 of [43], Theorem 3.6, Theorem 5.1 and Lemma 8.1, for $\alpha \cdot \dim T \geq 1$, i.e. $\alpha \in [1,2]$, and $f_0 \equiv 0 \in \mathbb{R}^d$ one $\mathbb{P}$-almost surely has

$$\begin{aligned}
&\dim T + 1 - 1/\alpha \\
&\quad = \dim \mathcal{G}_T(X) \\
&\quad \leq 1/\alpha \cdot \mathscr{P}^\alpha\text{- dim}\, \mathcal{G}_T(X) + 1 - 1/\alpha \\
&\quad \leq 1/\alpha \cdot \mathscr{P}^\alpha\text{- dim}\, \mathcal{G}_T(f_0) + 1 - 1/\alpha \\
&\quad \leq \dim T + 1 - 1/\alpha.
\end{aligned}$$

In the other cases, Theorem 3.2 of [43] and the same theorems as above $\mathbb{P}$-almost surely yield

$$\begin{aligned}
&(\alpha \vee 1) \cdot \dim T \\
&\quad = \dim \mathcal{G}_T(X) \\
&\quad \leq \mathscr{P}^\alpha\text{- dim}\, \mathcal{G}_T(X) \\
&\quad \leq \mathscr{P}^\alpha\text{- dim}\, \mathcal{G}_T(f_0) \\
&\quad \leq (\alpha \vee 1) \cdot \dim T
\end{aligned}$$

and the claim follows. □

Remark 8.3. We can also calculate the Hausdorff dimension of the graph of the fractional Brownian motion. Let $B^H = \left(B_t^H\right)_{t\geq 0}$ be a fractional Brownian motion in $\mathbb{R}^d$ of Hurst index $1/\alpha = H \in (0,1]$. One $\mathbb{P}$-almost surely has

$$\mathscr{P}^\alpha\text{- dim}\, \mathcal{G}_T\left(B^H\right) = \frac{\dim T}{H} = \alpha \cdot \dim T.$$

This follows from Theorem 2.1 in [46], Proposition 3.4, Theorem 3.6, Lemma 2.2 in [37] and Lemma 8.1 for $\alpha \cdot \dim T \leq d$ and $f_0 \equiv 0 \in \mathbb{R}^d$ which $\mathbb{P}'$-almost surely yield

$$\begin{aligned}
&\alpha \cdot \dim T \\
&\quad = \dim \mathcal{G}_T\big(B^H\big) \\
&\quad \leq \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T\big(B^H\big) \\
&\quad = \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f_0) \\
&\quad \leq \alpha \cdot \dim T.
\end{aligned}$$

In the other cases the same theorems $\mathbb{P}$-almost surely yield

$$\begin{aligned}
&\dim T + (1 - 1/\alpha) \cdot d \\
&\quad = \dim \mathcal{G}_T\big(B^H\big) \\
&\quad \leq \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T\big(B^H\big)/\alpha + (1 - 1/\alpha) \cdot d \\
&\quad = \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f_0)/\alpha + (1 - 1/\alpha) \cdot d \\
&\quad \leq \dim T + (1 - 1/\alpha) \cdot d
\end{aligned}$$

and the claim follows.

The calculations in the proof of the previous theorem show that equality holds in Lemma 8.1 for $n = 1$.

Corollary 8.4. *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0, \infty)$. Define the constant function $f_C : T \mapsto C \in \mathbb{R}^d$. Then*

$$\mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f_C) = (\alpha \vee 1) \cdot \dim T$$

holds.

Proof. Without loss of generality, let $f_C \equiv 0 \in \mathbb{R}^d$. The claim follows by the calculations in the proof of Theorem 8.2. □

As a consequence, we recover a well-known result for the range of an isotropic α-stable Lévy process; see [6] and Theorem 3.1 in [27]. Note that it makes no sense to talk about the α-parabolic Hausdorff dimension of the range of a function since this notion of dimension always relies on the scaling between time and space.

Theorem 8.5. *Let $T \subseteq \mathbb{R}_+$ be a Borel set and $\alpha \in (0,2]$. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. One $\mathbb{P}$-almost surely has*

$$\dim \mathcal{R}_T(X) = \alpha \cdot \dim T \wedge d.$$

Proof. From Theorem 4.3 and Corollary 8.4 follows

$$\dim \mathcal{R}_T(X) = (\alpha \wedge 1) \cdot \mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f_0) \wedge d = \alpha \cdot \dim T \wedge d,$$

as claimed. □

We can also give some a priori estimates for the α-parabolic Hausdorff dimension of the graph of a function in terms of the genuine Hausdorff dimension.

Theorem 8.6. *Let $T \subseteq \mathbb{R}^n$ be any set and $f : T \to \mathbb{R}^d$ be any function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f)$ where $\varphi_1 = \dim \mathcal{G}_T(f)$ denotes the genuine Hausdorff dimension of the grapf of f over T. Then one has*

$$\varphi_\alpha \leq \begin{cases} \varphi_1 + \left(\frac{1}{\alpha} - 1\right) \cdot d \ \wedge \ n + d, & \alpha \in (0,1], \\ \varphi_1 + (\alpha - 1) \cdot n \ \wedge \ n + d, & \alpha \in [1, \infty) \end{cases}$$

and

$$\varphi_\alpha \geq \begin{cases} \varphi_1 \ \vee \ \frac{1}{\alpha} \cdot \varphi_1 + \left(1 - \frac{1}{\alpha}\right) \cdot n, & \alpha \in (0,1], \\ \varphi_1 \ \vee \ \alpha \cdot \varphi_1 + (1 - \alpha) \cdot d, & \alpha \in [1, \infty). \end{cases}$$

Further, if $T \subseteq \mathbb{R}_+$ is a Borel set and $f : T \to \mathbb{R}^d$ is a Borel measurable function, then the sharper estimate

$$\varphi_\alpha \leq \frac{1}{\alpha} \cdot \varphi_1 \ \wedge \ \varphi_1 + \left(\frac{1}{\alpha} - 1\right) \cdot d \ \wedge \ d + 1, \quad \alpha \in (0,1]$$

holds.

Proof. This follows immediately by Lemma 3.6 and Corollary 4.4 and the fact that the Hausdorff dimension never exceeds the toplogical dimension of the whole space. □

Next we calculate some bounds for the parabolic Hausdorff dimension of β-Hölder continuous functions. These are functions $f : \mathbb{R}^n \supseteq T \to \mathbb{R}^d$ that fulfil

$$||f(t) - f(s)|| \leq C \cdot ||t - s||^\beta \tag{8.1}$$

for all $s, t \in T$ and some $\beta \in (0,1]$, $C > 0$, denoted as $f \in C^\beta(T, \mathbb{R}^d)$. In case of $\alpha = 1$, the following theorem restates a classical result; e.g., see §10, Theorem 6 in [22].

Theorem 8.7. *Let $T \subseteq \mathbb{R}^n$ be any set, $\alpha \in (0,\infty)$, $\beta \in (0,1]$ and $f \in C^\beta\big(T, \mathbb{R}^d\big)$ be a β-Hölder continuous function. Define the α-parabolic Hausdorff dimension $\varphi_\alpha := \mathscr{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ of the graph of f over T. Then the estimates*

$$\varphi_\alpha \leq \begin{cases} \dim T + d \cdot \left(\frac{1}{\alpha} - \beta\right) \ \wedge \ \frac{\dim T}{\alpha\beta} \ \wedge \ n + d, & \alpha \in (0,1], \\ \alpha \cdot \dim T + d \cdot (1 - \alpha\beta) \ \wedge \ \frac{\dim T}{\beta} \ \wedge \ n + d, & \alpha \in \left[1, \frac{1}{\beta}\right], \\ \alpha \cdot \dim T \ \wedge \ \frac{1}{\beta} \cdot (\dim T - 1) + \alpha \ \wedge \ n + d, & \alpha \in \left[\frac{1}{\beta}, \infty\right) \end{cases}$$

hold.

Proof. Let $\tau > \dim T$ and $\varepsilon > 0$ be arbitrary. Then we can cover T by hypercubes $(\mathsf{T}_k)_{k\in\mathbb{N}}$ with diameter $|T_k| < 1$ such that $\sum_{k=1}^\infty |\mathsf{T}_k|^\tau < \varepsilon$. Since $f \in C^\beta\big(T, \mathbb{R}^d\big)$, we can cover $\mathcal{G}_T(f)$ by $(\mathsf{B}_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^{n+d}$ where

$$\mathsf{B}_k := \mathsf{T}_k \times \prod_{j=1}^d \left[x_{j,k},\ x_{j,k} + C \cdot |\mathsf{T}_k|^\beta\right]$$

for every $k \in \mathbb{N}$. Note that, without loss of generality, we may assume $C \geq 1$ for the constant in (8.1).

(i) Let $\alpha \in (0,1]$. On the one hand, for every $k \in \mathbb{N}$ we can cover B_k by (several) α-parabolic cylinders with sidelength $|\mathsf{T}_k|$ in the time domain. Since $K \cdot |\mathsf{T}_k|^{1/\alpha} \geq C \cdot |\mathsf{T}_k|^\beta$ iff $K \geq C \cdot |\mathsf{T}_k|^{\beta - 1/\alpha}$ we find a cover

$$\mathcal{G}_T(f) \subseteq \bigcup_{k=1}^\infty \bigcup_{l=1}^{\left\lceil C\cdot|\mathsf{T}_k|^{\beta-1/\alpha}\right\rceil^d} \mathsf{T}_k \times \square_{|\mathsf{T}_k|^{1/\alpha},\, l}$$

with $\mathsf{T}_k \times \square_{|\mathsf{T}_k|^{1/\alpha},l} \in \mathscr{P}^\alpha$ for every $k, l \in \mathbb{N}$. Now, for $\gamma = \tau + d \cdot (1/\alpha - \beta)$ we have

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(\mathcal{G}_T(f)) \lesssim \sum_{k=1}^{\infty} |\mathsf{T}_k|^{d\cdot(\beta-1/\alpha)+\gamma} = \sum_{k=1}^{\infty} |T_k|^\tau < \varepsilon.$$

Since $\tau > \dim T$ is arbitrary, this results in

$$\varphi_\alpha \leq \dim T + d \cdot (1/\alpha - \beta).$$

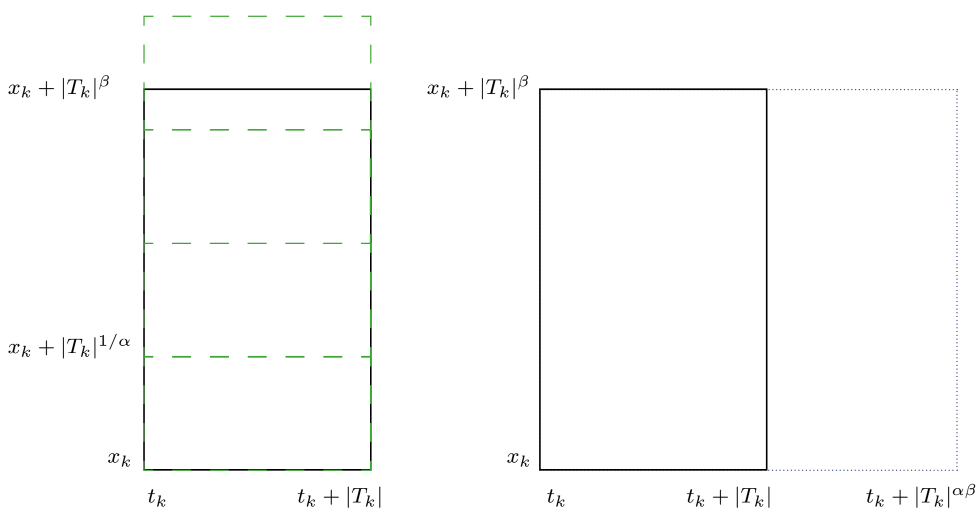

FIGURE 2. Two possibilities to cover B_k with sets from $\mathcal{P}^\alpha$ in case (i).

On the other hand, see the right picture in Figure 2, for every $k \in \mathbb{N}$ we can cover B_k by a single α-parabolic cylinder with sidelength $C^\alpha \cdot |\mathsf{T}_k|^{\alpha\beta}$ in the time domain. Then $\mathcal{G}_T(f) \subseteq \cup_{k\in\mathbb{N}} \mathsf{P}_{|\mathsf{T}_k|^{\alpha\beta}}$ with

$$\mathsf{P}_{|\mathsf{T}_k|^{\alpha\beta}} := \prod_{i=1}^{n} \left[t_{i,k}, t_{i,k} + C^\alpha \cdot |\mathsf{T}_k|^{\alpha\beta}\right] \times \prod_{j=1}^{d} \left[x_{j,k},\, x_{j,k} + C \cdot |\mathsf{T}_k|^{\beta}\right] \in \mathscr{P}^\alpha.$$

Now, for $\gamma = \tau/(\alpha\beta)$ we have

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(\mathcal{G}_T(f)) \lesssim \sum_{k=1}^{\infty} |\mathsf{T}_k|^{\alpha\beta\cdot\gamma} < \varepsilon.$$

Since $\tau > \dim T$ is arbitrary this results in

$$\varphi_\alpha \leq \frac{\dim T}{\alpha\beta}.$$

(ii) Let $\alpha \in [1, 1/\beta]$. On the one hand, for every $k \in \mathbb{N}$ we can cover B_k by (several) α-parabolic cylinders with sidelength $|\mathsf{T}_k|$ in time. Since $K \cdot |\mathsf{T}_k|^{1/\alpha} \geq C \cdot |\mathsf{T}_k|^\beta$ iff $K \geq |\mathsf{T}_k|^{\beta - 1/\alpha}$ we find a cover

$$\mathcal{G}_T(f) \subseteq \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{\left\lceil C\cdot|\mathsf{T}_k|^{\beta-1/\alpha}\right\rceil^d} \mathsf{T}_k \times \square_{|\mathsf{T}_k|^{1/\alpha},\, l}.$$

with $\mathsf{T}_k \times \square_{|\mathsf{T}_k|^{1/\alpha},\, l} \in \mathscr{P}^\alpha$ for every $k, l \in \mathbb{N}$. Now, for $\gamma = \alpha \cdot \tau + d \cdot (1 - \alpha\beta)$ we have

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(\mathcal{G}_T(f)) \lesssim \sum_{k=1}^{\infty} |\mathsf{T}_k|^{d\cdot(\beta - 1/\alpha) + \gamma/\alpha} = \sum_{k=1}^{\infty} |\mathsf{T}_k|^\tau < \varepsilon.$$

Since $\tau > \dim T$ is arbitrary, this results in

$$\varphi_\alpha \leq \alpha \cdot \dim T + d \cdot (1 - \alpha\beta).$$

On the other hand, for every $k \in \mathbb{N}$ we can cover B_k by a single α-parabolic cylinder with sidelength $|\mathsf{T}_k|^{\alpha\beta}$ in time. Then $\mathcal{G}_T(f) \subseteq \cup_{k\in\mathbb{N}} \mathsf{P}_{|\mathsf{T}_k|^{\alpha\beta}}$ with

$$\mathsf{P}_{|\mathsf{T}_k|^{\alpha\beta}} := \prod_{i=1}^{n} \left[t_{i,k}, t_{i,k} + |\mathsf{T}_k|^{\alpha\beta}\right] \times \prod_{j=1}^{d} \left[x_{j,k},\, x_{j,k} + |\mathsf{T}_k|^\beta\right] \in \mathscr{P}^\alpha.$$

Now, for $\gamma = \tau/\beta$ we have

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(\mathcal{G}_T(f)) \lesssim \sum_{k=1}^{\infty} |\mathsf{T}_k|^{\beta\cdot\gamma} = \sum_{k=1}^{\infty} |\mathsf{T}_k|^\tau < \varepsilon.$$

Since $\tau > \dim T$ is arbitrary, this results in

$$\varphi_\alpha \leq \dim T/\beta.$$

(iii) Let $\alpha \in [1/\beta, \infty)$. On the one hand, see the right picture in Figure 3, for every $k \in \mathbb{N}$ we can cover B_k by (several) α-parabolic cylinders with sidelength $|\mathsf{T}_k|^{\alpha\beta}$ in the time domain. Since $K \cdot |\mathsf{T}_k|^{\alpha\beta} \geq C \cdot |\mathsf{T}_k|$ iff $K \geq C \cdot |\mathsf{T}_k|^{1-\alpha\beta}$ we find a cover

$$\mathcal{G}_T(f) \subseteq \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{\left\lceil C\cdot|\mathsf{T}_k|^{1-\alpha\beta}\right\rceil} T_{k,l} \times \square_{|T_{k,l}|^{1/\alpha}}.$$

with $T_{k,l} \times \square_{|T_{k,l}|^{1/\alpha}} \in \mathscr{P}^\alpha$ for every $k, l \in \mathbb{N}$. Now, for $\gamma = (\tau + \alpha\beta - 1)/\beta$ we have

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(\mathcal{G}_T(f)) \lesssim \sum_{k=1}^{\infty} |\mathsf{T}_k|^{1-\alpha\beta+\gamma\beta} = \sum_{k=1}^{\infty} |\mathsf{T}_k|^\tau < \varepsilon.$$

Since $\tau > \dim T$ is arbitrary, this results in

$$\varphi_\alpha \leq \frac{1}{\beta} \cdot (\dim T - 1) + \alpha.$$

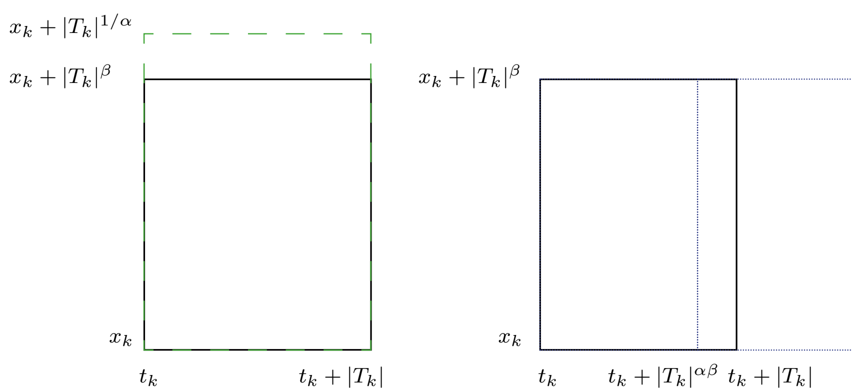

FIGURE 3. Two possibilities to cover B_k with sets from $\mathcal{P}^\alpha$ in case (iii).

On the other hand, see the left picture in Figure 3, for every $k \in \mathbb{N}$ we can cover B_k by a single α-parabolic cylinder with length $C^\alpha \cdot |\mathsf{T}_k|$ in the time domain. Therefore we find a cover

$$\mathcal{G}_T(f) \subseteq \bigcup_{k=1}^{\infty} \mathsf{T}'_k \times \square_{|\mathsf{T}_k|^{1/\alpha}}.$$

with $\mathsf{T}'_k \times \square_{|\mathsf{T}_k|^{1/\alpha}} \in \mathscr{P}^\alpha$ for every $k \in \mathbb{N}$. Now, for $\gamma \geq \alpha \cdot \tau$ we have

$$\mathscr{P}^\alpha\text{-}\mathcal{H}^\gamma(\mathcal{G}_T(f)) \lesssim \sum_{k=1}^{\infty} |\mathsf{T}_k|^{\gamma/\alpha} < \varepsilon.$$

Since $\tau > \dim T$ is arbitrary, this results in

$$\varphi_\alpha \leq \alpha \cdot \dim T.$$

The rest follows from the fact that the Hausdorff dimension never exceeds the topological dimension of the whole space. $\square$

Let us inspect the most important case where $n = 1$ and $\alpha = 2$, i.e. we aim to get a bound for the graph of Brownian motion plus β-Hölder continuous drift function over T according to its regularity β.

Corollary 8.8. *Let $T \subseteq \mathbb{R}_+$ be any set. Let $B = (B_t)_{t\geq 0}$ denote the Brownian motion in $\mathbb{R}^d$ and let $f \in C^\beta\big(T, \mathbb{R}^d\big)$ for some $\beta \in (0,1]$. One $\mathbb{P}$-almost surely has*

$$\dim \mathcal{G}_T(B+f) \leq \begin{cases} d + \frac{1}{2}, & \beta \leq \frac{\dim T}{d} - \frac{1}{2d}, \\ \dim T + d\cdot(1-\beta), & \frac{\dim T}{d} - \frac{1}{2d} \leq \beta \leq \frac{\dim T}{d} \ \wedge \ \frac{1}{2}, \\ \frac{\dim T}{\beta}, & \frac{\dim T}{d} \leq \beta \leq \frac{1}{2}, \\ 2\cdot \dim T \wedge \dim T + \frac{d}{2}, & \frac{1}{2} \leq \beta \end{cases}$$

for the graph of $B+f$ over T. Moreover, one $\mathbb{P}$-almost surely has

$$\dim \mathcal{R}_T(B+f) \leq \begin{cases} \frac{\dim T}{\beta}, & \frac{\dim T}{d} \leq \beta \leq \frac{1}{2}, \\ 2\cdot \dim T \wedge d, & \frac{\dim T}{d} \leq \frac{1}{2} \leq \beta, \\ d, & \textit{else} \end{cases}$$

for the range of $B+f$ over T.

Proof. Let $\varphi_2 := \mathscr{P}^2\text{-}\dim \mathcal{G}_T(f)$ denote the parabolic Hausdorff dimension of the graph of f over T. Corollary 5.3 $\mathbb{P}$-almost surely yields

$$\dim \mathcal{G}_T(B+f) \leq \varphi_2 \ \wedge \ \frac{\varphi_2 + d}{2}$$

and Theorem 7.1 $\mathbb{P}$-almost surely yields

$$\dim \mathcal{R}_T(X+f) \leq \varphi_2 \ \wedge \ d.$$

Our Theorem 8.7 yields

$$\varphi_2 \leq \begin{cases} 2\cdot \dim T + d\cdot(1-2\beta) \ \wedge \ d+1, & \beta \leq \frac{\dim T}{d} \ \wedge \ \frac{1}{2}, \\ \frac{\dim T}{\beta}, & \frac{\dim T}{d} \leq \beta \leq \frac{1}{2}, \\ 2\cdot \dim T, & \frac{1}{2} \leq \beta \end{cases}$$

and the claim follows. □

We cite the classical result for the Hausdorff dimension of the range of a Hölder continuous function. Again, note that it makes no sense to talk about the α-parabolic Hausdorff dimension of the range of a function since this notion of dimension always relies on the scaling between time and space.

Theorem 8.9. *Let $\beta \in (0,1]$, $T \subseteq \mathbb{R}^n$ be any set and let $f \in C^\beta(T,\mathbb{R}^d)$ be a β-Hölder continuous function. Then one has*

$$\dim \mathcal{R}_T(f) \leq \frac{1}{\beta} \cdot \dim T \ \wedge \ d$$

for the range of f over T.

Proof. The result corresponds to §10, Theorem 6 in [22]. It is formulated for compact sets T but its geometrical proof also works for arbitrary sets. □

Finally, we give an example for the applicability of our results. We consider the fractional heat equation with initial condition, i.e.

$$\dot{u} = -(-\Delta)^{\alpha/2}[u],$$

$$u|_{t=0} = u_0 \in C^\beta.$$

It is well known, see [2], that for an isotropic α-stable process $X = (X_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, its solution can be represented by

$$u(t,x) = \mathbb{E}_x[u_0(X_t)] = \mathbb{E}[u_0(X_t + x)]. \tag{8.2}$$

Since the expected value in (8.2) averages the paths of $u_0(X_t + x)$ to a smooth macroscopic flow, it makes no sense to analyse the whole solution by means of fractal geometry. But we can analyse the pathwise solutions of a similar non-averaged model with methods from parabolic fractal geometry.

Theorem 8.10. *Let $\alpha \in (0,2]$, $\beta \in (0,1]$ and $T \subseteq \mathbb{R}_+$ be any set. Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$ and $u_0 \in C^\beta$ be a β-Hölder continuous function. Define the constant function $f_x : T \to \{x\} \in \mathbb{R}^d$ and consider the non-averaged system*

$$u(t,x;\omega) = u_0(X_t(\omega) + f_x(t)),$$

$$u|_{t=0} = u_0 \in C^\beta.$$

Then one $\mathbb{P}$-almost surely has

$$\dim \mathcal{G}_T(u) \leq \alpha \cdot \dim T + d \cdot (1-\beta) \ \wedge \ \frac{\alpha}{\beta} \cdot \dim T \ \wedge \ d+1$$

and

$$\dim \mathcal{R}_T(u) \leq \frac{\alpha}{\beta} \cdot \dim T \ \wedge \ d$$

for the graph and range of u over T.

Proof. According to Theorem 8.7, Theorem 7.1 and Lemma 8.1 one $\mathbb{P}$-almost surely has

$$\begin{aligned}
&\dim \mathcal{G}_T(u) \\
&\quad = \dim \mathcal{G}_T(u_0(X+f_x)) \\
&\quad = \dim \mathcal{G}_{\mathcal{R}_T(X+f_x)}(u_0) \\
&\quad \leq \dim \mathcal{R}_T(X+f_x) + d \cdot (1-\beta) \ \wedge \ \frac{1}{\beta} \cdot \dim \mathcal{R}_T(X+f_x) \\
&\quad \leq (\alpha \wedge 1) \cdot \mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f_x) + d \cdot (1-\beta) \ \wedge \ \frac{\alpha \wedge 1}{\beta} \cdot \mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f_x) \\
&\quad \leq (\alpha \wedge 1) \cdot (\alpha \vee 1) \cdot \dim T + d \cdot (1-\beta) \ \wedge \ \frac{(\alpha \wedge 1) \cdot (\alpha \vee 1)}{\beta} \cdot \dim T \\
&\quad = \alpha \cdot \dim T + d \cdot (1-\beta) \ \wedge \ \frac{\alpha}{\beta} \cdot \dim T.
\end{aligned}$$

The rest of the claim follows from the fact that the Hausdorff dimension never exceeds the topological dimension of the space.

According to Theorem 8.9, Theorem 7.1 and Lemma 8.1 one $\mathbb{P}$-almost surely has

$$\begin{aligned}
&\dim \mathcal{R}_T(u) \\
&\quad = \dim \mathcal{R}_T(u_0(X+f_x)) \\
&\quad = \dim \mathcal{R}_{\mathcal{R}_T(X+f_x)}(u_0) \\
&\quad \leq \frac{1}{\beta} \cdot \dim \mathcal{R}_T(X+f_x) \ \wedge \ d
\end{aligned}$$

$$= \frac{\alpha \wedge 1}{\beta} \cdot \mathscr{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f_x) \ \wedge \ d$$

$$\leq \frac{(\alpha \wedge 1) \cdot (\alpha \vee 1)}{\beta} \cdot \dim T \ \wedge \ d$$

$$= \frac{\alpha}{\beta} \cdot \dim T \ \wedge \ d,$$

as claimed. □

Part 3. Spectral Theory of Nonlocal Random Schrödinger Operators

9. Introduction of the Model

For fixed $\alpha \in (0,2)$ we will show some spectral properties of the fractional random Schrödinger operator

$$H_\omega[\psi] := (-\Delta)^{\alpha/2}[\psi] + V_\omega \cdot \psi \tag{9.1}$$

acting on suitably regular functions ψ on $\mathbb{R}^d$, where V_ω is either a Gaussian or Poissonian random potential. The fractional Laplacian $(-\Delta)^{\alpha/2}$ was introduced in Section 2 and reduces to the negative Laplacian $-\Delta$ when $\alpha = 2$ but in contrast to the ordinary Laplacian, the fractional Laplacian is a nonlocal operator. We are interested in the spectrum of the operator H_ω; in particular, we want to calculate the average number of energies λ_ω per volume up to a certain level $\lambda \in \mathbb{R}$. For that purpose we restrict the operator H_ω to a box Λ and choose zero Dirichlet boundary conditions. This results in a countable number of energies $\lambda_{\omega,\Lambda}$ bounded from below such that the spectrum of the restricted operator $H_{\omega,\Lambda}$ can be ordered as

$$\sigma(H_{\omega,\Lambda}) = \left\{ \lambda_{\omega,\Lambda}^{(1)} \leq \lambda_{\omega,\Lambda}^{(2)} \leq \dots \right\}. \tag{9.2}$$

Define the normalized eigenvalue counting function of $H_{\omega,\Lambda}$ by

$$N_{\omega,\Lambda}(\lambda) := \frac{1}{|\Lambda|} \sum_{k=1}^{\infty} \mathbb{1}_{\{\lambda_{\omega,\Lambda}^{(k)} \leq \lambda\}},$$

where $|\Lambda|$ denotes the volume of the box. Enlarging the box to the whole $\mathbb{R}^d$ denoted by $|\Lambda| \to \infty$ and taking expectations this results in the so-called integrated density of states (IDS)

$$N(\lambda) := \lim_{|\Lambda| \to \infty} \mathbb{E}[N_{\omega,\Lambda}(\lambda)].$$

Nakao proved in [29] the existence of the IDS for random Schrödinger operators with Poissonian and Gaussian random potentials. He works in the setting where $\alpha = 2$, i.e. the free part is the classical Laplacian. Further he proves the asymptotics of the IDS at the left and right end of the spectrum of random Schrödinger operators both for Poissonian and Gaussian potentials. Thus we can exclude the case $\alpha = 2$ in our considerations. Nakao's work is based on Pastur [33] and the important work of Donsker and Varadhan [13] on the Wiener sausage.

Throughout this part we assume that the Gaussian potential $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ is a real-valued stationary centered Gaussian random field on a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Then V_ω is determined by its covariance function $c(x) := \mathbb{E}[V_\omega(0)V_\omega(x)]$ which is assumed to be strictly positive at the origin. A more rigorous model can be found in Section 2 of [16].

Ōkura gereneralized Nakao's work [31] to a larger class of nonlocal operators with random potential including the fractional Laplacian. In detail he proves the existence of the IDS for operators generated by symmetric Lévy processes whose Lévy exponent fulfils some mild exponential integrability condition plus some stationary potential whose negative part is exponentially r-integrable for some $r > 2$, see Theorem 10.1 below. These potentials subsume the Poissonian and Gaussian case and Ōkura determines the asymptotics at the left end of the spectrum for Poissonian potentials. He leaves the case of Gaussian potentials and the asymptotics for Poissonian potentials at high energies open. These cases are the subject of this part.

In Section 10 we prove the existence of the IDS for the fractional Schrödinger operator with Gaussian potential based on Ōkura's general result from [31]. Then we analyze its asymptotics at the left end of the spectrum as $\lambda \to -\infty$.

In Section 11 we follow the idea of Nakao [29] in proving Lifshitz tails for the fractional random Schrödinger operator with Gaussian potential, i.e. exponential decay of the IDS at the left end of the spectrum. For that purpose we use a technique developed by Pastur [34] which was generalized by Ōkura [31]. For a treatment in the discrete setting, see [19].

Finally, we analyse the asymptotics at the right end of the spectrum as $\lambda \to +\infty$ for arbitrary stationary random potentials that satisfy some mild condition which implies

$$\mathbb{E}\left[e^{-tV_\omega(0)}\right] < \infty. \tag{9.3}$$

In [32] a similar result was proven in the Gaussian case under this assumption. We have to add another condition, viz.

$$\lim_{t\downarrow 0} \frac{\mathcal{L}^-[N](t)}{t^{-d/\alpha}} = 0,$$

in order to fit a suitable Tauber theorem. Both Gaussian and Poissonian potentials fulfil these conditions.

We give stochastic proofs instead of Nakao's functional analytic ones in [29]. They mainly do not rely on external theorems and are self-contained in the text. It turns out that the asymptotics at the left end of the spectrum does not depend on $\alpha \in (0,2)$ and thus is completely determined by the random potential, whereas the asymptotics at the right end of the spectrum only depends on $\alpha \in (0,2)$ in case of both Gaussian and Poissonian potentials. Hence the potential is the dominant part at low energies whereas the fractional Laplacian is leading at high energies.

10. Existence of the IDS for Gaussian Potentials

First of all we make sure that the IDS actually exists in case of our fractional random Schrödinger operators with Gaussian potential. We will apply the following general existence theorem of Ōkura [31] which further proves a representation of the Laplace-Stieltjes transform

$$\mathcal{L}[N](t) := \int_{\mathbb{R}} e^{-\lambda t}\, \mathrm{d}N(\lambda). \tag{10.1}$$

In the following $\mathbb{E} \times \mathbb{E}^{t,0}_{0,0}$ denotes the expected value with respect to $\mathbb{P} \times \mathbb{P}^{t,0}_{0,0}$. Here, $\mathbb{P}$ denotes the probability measure of the random potential. Further, $\mathbb{P}^{t,0}_{0,0}$ denotes the probability measure of the $(0,0;t,0)$-pinned conditional process of the Lévy process X; see §2 in [31].

Theorem 10.1 ([31], Theorem 5.1)**.** *Let L be the generator of a d-dimensional symmetric Lévy process $X = (X_t)_{t\geq 0}$ on a probability space $(\Omega', \mathscr{A}', \mathbb{P}')$ with Lévy exponent $\Psi(\xi)$ and let $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ be a stationary random field over a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Suppose that the following two conditions are satisfied:*

$$e^{-t\sqrt{\Psi(\xi)}} \in L^1(\mathbb{R}^d) \qquad \text{for every } t > 0 \tag{10.2}$$

and there exists a constant $r > 2$ such that

$$\exp\left(\int_0^t V_\omega^-(X_s(\eta))\,\mathrm{d}s\right) \in L^r\big(\mathbb{P}(\mathrm{d}\omega) \otimes \mathbb{P}'_0(\mathrm{d}\eta)\big) \qquad \text{for every } t > 0, \tag{10.3}$$

where $V_\omega^- := \max\{-V_\omega, 0\}$. Then the IDS for the operator $H_\omega := -L + V_\omega$ exists as a right-continuous nondecreasing function $N(\lambda)$ on $\mathbb{R}$ with $\lim_{\lambda\downarrow-\infty} N(\lambda) = 0$ such that for every continuity point λ of N we have

$$\lim_{|\Lambda|\to\infty} \mathbb{E}[N_{\omega,\Lambda}(\lambda)] = N(\lambda).$$

Moreover, for every $t > 0$ we have

$$\mathcal{L}[N](t) = p(t,0)\ \mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}^{t,0}_{0,0}(\mathrm{d}\eta)\left[e^{-\int_0^t V_\omega(X_s(\eta))\,\mathrm{d}s}\right]. \tag{10.4}$$

Note that condition (10.2) implies $\exp(-t\Psi(\xi)) \in L^1(\mathbb{R}^d)$ for every $t > 0$ and thus (2) and stationarity of the increments guarantee the existence of $p(t,0)$ in (10.4).

Since our isotropic α-stable process X and our Gaussian random field V_ω are within the setting of Theorem 10.1, as an application we get:

Corollary 10.2. *For fixed $\alpha \in (0,2)$ consider the fractional random Schrödinger operator H_ω in (9.1) with Gaussian potential V_ω as above. Then the IDS of H_ω exists as a nondecreasing càdlàg function with $\lim_{\lambda\downarrow-\infty} N(\lambda) = 0$. Further, for every $t > 0$ its Laplace-Stieltjes transform is represented by*

$$\mathcal{L}[N](t) = p(t,0)\ \mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}^{t,0}_{0,0}(\mathrm{d}\eta)\left[e^{-\int_0^t V_\omega(X_s(\eta))\,\mathrm{d}s}\right]. \tag{10.5}$$

Proof. We only have to check the conditions (10.2) and (10.3) of Theorem 10.1 in our model. For the isotropic α-stable process X we have $\Psi(\xi) = |\xi|^\alpha$ and thus

$$\int_{\mathbb{R}^d} \left| e^{-t\sqrt{\Psi(\xi)}} \right| \mathrm{d}\xi = \int_{\mathbb{R}^d} e^{-t|\xi|^{\alpha/2}}\,\mathrm{d}\xi < \infty \quad \text{for every } t > 0$$

shows that condition (10.2) is satisfied.

In order to check condition (10.3), note that by Jensen's inequality for the normalized Lebesgue measure on $[0,t]$ and monotone convergence we have

$$\begin{aligned}
&\exp\left(r\int_0^t V_\omega^-(X_s(\eta))\,\mathrm{d}s\right)\\
&= \sum_{k=0}^\infty \frac{(rt)^k}{k!}\left(\int_0^t V_\omega^-(X_s(\eta))\,\frac{\mathrm{d}s}{t}\right)^k\\
&\le \sum_{k=0}^\infty \frac{(rt)^k}{k!}\int_0^t \left(V_\omega^-(X_s(\eta))\right)^k\,\frac{\mathrm{d}s}{t}\\
&= \frac{1}{t}\int_0^t \exp\left(rt\,V_\omega^-(X_s(\eta))\right)\,\mathrm{d}s.
\end{aligned}$$

By the Tonelli-Fubini theorem and stationarity $\mathbb{P}_{V_\omega(x)} = \mathcal{N}_{0,c(0)}$ of the centered Gaussian field we further get for every $t > 0$ and arbitrary $r > 2$

$$\begin{aligned}
&\mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}_0(\mathrm{d}\eta)\left[\exp\left(r\int_0^t V_\omega^-(X_s(\eta))\,\mathrm{d}s\right)\right]\\
&\le \frac{1}{t}\int_0^t\int_{\mathbb{R}^d}\int_\Omega \exp\left(rt\,V_\omega^-(x)\right)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}\mathbb{P}_{X_s|X_0=0}(x)\,\mathrm{d}s\\
&= \frac{1}{t}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}} \exp(rt\max\{-y,0\})\,\mathrm{d}\mathcal{N}_{0,c(0)}(y)\,\mathrm{d}\mathbb{P}_{X_s|X_0=0}(x)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \exp(rt\max\{-y,0\})\,\mathrm{d}\mathcal{N}_{0,c(0)}(y) \\
&= \int_0^\infty 1\,\mathrm{d}\mathcal{N}_{0,c(0)}(y) + \int_{-\infty}^0 \exp(-rty)\,\mathrm{d}\mathcal{N}_{0,c(0)}(y) \\
&= \frac{1}{2} + \int_0^\infty \exp(rty)\,\mathrm{d}\mathcal{N}_{0,c(0)}(y) < \infty,
\end{aligned}
$$

since Gaussian random variables have finite exponential moments. Thus condition (10.3) is fulfilled and a direct application of Theorem 10.1 concludes the proof. □

11. Lifshitz Tails of the IDS for Gaussian Potentials

In this section we derive the precise asymptotics of the IDS with Gaussian potential at the left end of the spectrum, i.e. the decay of the IDS as $\lambda \to -\infty$. For this purpose, we use the following result of Ōkura [31] on lower and upper bounds for the Laplace-Stieltjes transform of the IDS. Thereafter we derive the asymptotics of the IDS itself by using a Tauberian theorem stated in [17] which translates the behavior of the Laplace-Stieltjes transform of the IDS as $t \to +\infty$ to the behavior of the IDS itself as $\lambda \to -\infty$. In the following let $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ denote the Dirichlet form of a symmetric Lévy process with Lévy exponent $\Psi(\xi)$ given by

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \Psi(\xi)\left|\mathscr{F}[f](\xi)\right|^2 \mathrm{d}\xi \tag{11.1}$$

with domain

$$\mathscr{D}(\mathcal{E}) = \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \Psi(\xi)\left|\mathscr{F}[f](\xi)\right|^2 \mathrm{d}\xi < \infty \right\}.$$

Ōkura's theorem states:

Theorem 11.1 ([31], Theorem 7.1)**.** *Let $X = (X_t)_{t\geq 0}$ be a d-dimensional symmetric Lévy process on a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ with Lévy exponent $\Psi(\xi)$ satisfying $e^{-t\Psi(\xi)} \in L^1(\mathbb{R}^d)$ for all $t > 0$ and let $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ be the Dirichlet form of X. Further, let $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ be a stationary random field defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that*

$$\mathbb{E}[\exp(-tV_\omega(0))] < \infty \quad \textit{for all } t > 0. \tag{11.2}$$

Then for all $f \in \mathscr{D}(\mathcal{E})$ with $||f||_2 = 1$ and $t > 0$ we have

$$\begin{aligned} ||f||_1^{-2} e^{-t\mathcal{E}(f,f)-\Phi_t(f)} &\leq p(t,0)\,\mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}_{0,0}^{t,0}(\mathrm{d}\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\,\mathrm{d}s\right)\right] \\ &\leq p(t,0)\,\mathbb{E}\left[e^{-tV_\omega(0)}\right], \end{aligned} \tag{11.3}$$

where

$$\Phi_t(f) = -\ln \mathbb{E}\left[\exp\left(-t\int_{\mathbb{R}^d} V_\omega(x)\,|f(x)|^2\,\mathrm{d}x\right)\right].$$

Note that in our case we have $\Psi(\xi) = |\xi|^\alpha$ for the isotropic α-stable Lévy process such that $\exp(-t\Psi(\xi)) \in L^1(\mathbb{R}^d)$ for all $t > 0$ and condition (11.2) is automatically fulfilled for our Gaussian potential, since exponential moments of Gaussian random variables exist. According to Corollary 10.2 the expression in the middle of the inequality (11.3) is exactly the Laplace-Stieltjes transform of the IDS. This enables us to prove:

Lemma 11.2. *For the Laplace-Stieltjes transform of the IDS of the fractional random Schrödinger operator H_ω with Gaussian potential from Corollary 10.2 we have*

$$\lim_{t\to+\infty} \frac{\ln \mathcal{L}[N](t)}{t^2} = \frac{c(0)}{2}. \tag{11.4}$$

Proof. Upper bound: Since $p(t,0) \leq 1$ for t large enough and for a centered Gaussian random variable Y we have $\mathbb{E}[\exp(Y)] = \exp(\frac{1}{2}\,\mathbb{E}[Y^2])$ it follows from (11.3) that for large t we have

$$\begin{aligned}\ln \mathcal{L}[N](t) &\leq \ln p(t,0) + \ln \mathbb{E}[\exp(-tV_\omega(0))] \\ &\leq \ln \mathbb{E}[\exp(-tV_\omega(0))] = \frac{1}{2}\,\mathbb{E}[(-tV_\omega(0))^2] = \frac{c(0)}{2}\,t^2\end{aligned}$$

which yields

$$\limsup_{t\to\infty} \frac{\ln \mathcal{L}[N](t)}{t^2} \leq \frac{c(0)}{2}. \tag{11.5}$$

Lower bound: We choose a test function $\psi \in \mathscr{D}(\mathcal{E})$ with $||\psi||_2 = 1$. Following the argumentation in Theorem 9.3 of [35] let us define $R = R(t) := t^{-\frac{1}{2}+\beta}$ for some $\beta \in (0, \frac{1}{2})$ and $\psi_R(x) := R^{-d/2}\psi(R^{-1}x)$. Plugging this function into (11.3) we get

$$\begin{aligned}\ln \mathcal{L}[N](t) &\geq \ln ||\psi_R||_1^{-2} - t\,\mathcal{E}(\psi_R,\psi_R) + \ln \mathbb{E}\left[\exp\left(-t\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,\mathrm{d}x\right)\right] \\ &=: (I) - (II) + (III)\end{aligned}$$

and consider these three parts separately.

First part (I): By a change of variables $y = R^{-1}x$ we get

$$
\begin{aligned}
\ln ||\psi_R||_1^{-2} &= -2 \ln \int_{\mathbb{R}^d} |R^{-d/2}\psi(R^{-1}x)| \,\mathrm{d}x \\
&= -2 \ln \left(R^{-d/2} \int_{\mathbb{R}^d} R^d |\psi(y)| \,\mathrm{d}y \right) \\
&= -2 \ln \left(R^{d/2} ||\psi||_1 \right) \\
&= -d \ln t^{-\frac{1}{2}+\beta} - 2 \ln ||\psi||_1 \\
&= d(\tfrac{1}{2} - \beta) \ln t - 2 \ln ||\psi||_1.
\end{aligned} \tag{11.6}
$$

Second part (II): By (11.1) and a change of variables $\eta = R\xi$ we obtain

$$
\begin{aligned}
t\,\mathcal{E}(\psi_R, \psi_R) &= t \int_{\mathbb{R}^d} |\xi|^\alpha |\mathscr{F}[\psi_R(\cdot)](\xi)|^2 \,\mathrm{d}\xi \\
&= t \int_{\mathbb{R}^d} |\xi|^\alpha |\mathscr{F}[R^{-d/2}\psi(R^{-1}\cdot)](\xi)|^2 \,\mathrm{d}\xi \\
&= t \int_{\mathbb{R}^d} |\xi|^\alpha R^{-d} |R^d \mathscr{F}[\psi](R\xi)|^2 \,\mathrm{d}\xi \\
&= t \int_{\mathbb{R}^d} |\xi|^\alpha R^d |\mathscr{F}[\psi](R\xi)|^2 \,\mathrm{d}\xi \\
&= t \int_{\mathbb{R}^d} |R^{-1}\eta|^\alpha |\mathscr{F}[\psi](\eta)|^2 \,\mathrm{d}\eta \\
&= t\, R^{-\alpha} \mathcal{E}(\psi, \psi) \\
&= t^{\frac{\alpha}{2} - \alpha\beta + 1} \mathcal{E}(\psi, \psi).
\end{aligned} \tag{11.7}
$$

Third part (III): First note that $Y(\omega) = -t\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,\mathrm{d}x$ is a centered Gaussian random variable and thus $\mathbb{E}[\exp(Y)] = \exp(\frac{1}{2}\mathbb{E}[Y^2])$ which yields

$$
\begin{aligned}
&\ln \mathbb{E}\left[\exp\left(-t\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,\mathrm{d}x\right)\right] \\
&\quad = \frac{t^2}{2}\,\mathbb{E}\left[\left(\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,\mathrm{d}x\right)^2\right] \\
&\quad = \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{E}[V_\omega(x)V_\omega(y)]\,|\psi_R(x)|^2|\psi_R(y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)|R^{-\frac{d}{2}}\psi(R^{-1}x)|^2|R^{-\frac{d}{2}}\psi(R^{-1}y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)R^{-2d}|\psi(R^{-1}x)|^2|\psi(R^{-1}y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(R(u-v))\,|\psi(u)|^2|\psi(v)|^2\,du\,dv.
\end{aligned}
\tag{11.8}
$$

Alltogether, by (11.6), (11.7) and (11.8) we get the lower bound

$$
\begin{aligned}
\ln\mathcal{L}[N](t) \geq{}& d(\tfrac{1}{2}-\beta)\ln t - 2\ln||\psi||_1 - t^{\frac{\alpha}{2}-\alpha\beta+1}\mathcal{E}(\psi,\psi) \\
&+ \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c\left(t^{\beta-\frac{1}{2}}(u-v)\right)\,|\psi(u)|^2|\psi(v)|^2\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}
$$

Since $\beta \in (0,\frac{1}{2})$, c is continuous at the origin and $c(x) \leq c(0)$ by the Cauchy-Schwarz inequality, using dominated convergence we finally get

$$
\liminf_{t\to\infty} \frac{\ln\mathcal{L}[N](t)}{t^2} \geq \frac{c(0)}{2}
\tag{11.9}
$$

which together with (11.5) concludes the proof. □

Now we apply the following Tauberian theorem which was first stated by Fukushima, Nagai and Nakao in [17] and later proven by Nagai in [30].

Theorem 11.3 ([30], Corollary 2)**.** *Let $f(\lambda)$ be a non-decreasing function on $\mathbb{R}$ such that $f(-\infty) = 0$ and let $\mathcal{L}[f](t) < \infty$ be its Laplace-Stieltjes transform. Then we have*

$$f(\lambda) \sim e^{-A|\lambda|^{\beta}} \quad \textit{as } \lambda \downarrow -\infty \qquad \Longleftrightarrow \qquad \mathcal{L}[f](t) \sim e^{Bt^{\gamma}} \quad \textit{as } t \uparrow +\infty \tag{11.10}$$

for constants $A, B > 0$ *and* $\beta, \gamma > 1$ *fulfilling*

$$\gamma = \frac{\beta}{\beta - 1} \qquad \Longleftrightarrow \qquad \beta = \frac{\gamma}{\gamma - 1} \tag{11.11}$$

and

$$B = (\beta - 1)\beta^{\beta/(1-\beta)} A^{1/(1-\beta)} \qquad \Longleftrightarrow \qquad A = (\gamma - 1)\gamma^{\gamma/(1-\gamma)} B^{1/(1-\gamma)}. \tag{11.12}$$

A direct application of Theorem 11.3 to the situation of Lemma 11.2 shows the occurrence of Lifshitz tails:

Theorem 11.4. *For the IDS of the fractional random Schrödinger operator* H_ω *from Corollary 10.2 we have*

$$\lim_{\lambda \downarrow -\infty} \frac{\ln N(\lambda)}{\lambda^2} = -\frac{1}{2c(0)}. \tag{11.13}$$

Proof. In view of (11.4) the right-hand side of (11.10) is fulfilled with $\gamma = 2$ and $B = c(0)/2$. This yields $\beta = 2$ and $A = (2c(0))^{-1}$ by (11.11) and (11.12) from which the assertion easily follows by (11.10). □

12. Asymptotic Behaviour of the IDS at $+\infty$ for Random Potentials

In this section we derive the precise asymptotics of the IDS at the right end of the spectrum, i.e. as $\lambda \to +\infty$. We decompose the Laplace-Stieltjes transform of the IDS into the positive and negative unilateral Laplace-Stieltjes transforms given by

$$\mathcal{L}^+[N](t) = \int_0^\infty e^{-\lambda t}\,\mathrm{d}N(\lambda) \quad \text{and} \quad \mathcal{L}^-[N](t) = \int_{-\infty}^0 e^{-\lambda t}\,\mathrm{d}N(\lambda).$$

In the proof of our next theorem we will make use of the following Tauberian theorem of the Hardy-Littlewood type.

Theorem 12.1 ([44], Theorem 4.6)**.** *Let f be a function on $\mathbb{R}_+$ such that $\mathcal{L}^+[f](t)$ exists for every $t > 0$ and for some constants $K, A, \gamma > 0$ the function $\lambda \mapsto f(\lambda) + K\lambda^\gamma$ is non-decreasing on $\mathbb{R}_+$ and fulfils*

$$\lim_{t \downarrow 0} \frac{\mathcal{L}^+[f](t)}{t^{-\gamma}} = A. \tag{12.1}$$

Then we have

$$\lim_{\lambda \to \infty} \frac{f(\lambda)}{\lambda^\gamma} = \frac{A}{\Gamma(\gamma + 1)}. \tag{12.2}$$

We prove an asymptotic result for general random potentials.

Theorem 12.2. *In the situation of Theorem 10.1, let $V_\omega = (V_\omega(x))_{x \in \mathbb{R}^d}$ be a stationary random field which fulfils the condition (10.3). If*

$$\lim_{t \downarrow 0} \frac{\mathcal{L}^-[N](t)}{t^{-d/\alpha}} = 0,$$

then one has the asymptotics

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/\alpha}} = \frac{p(1,0)}{\Gamma(\frac{d}{\alpha} + 1)}. \tag{12.3}$$

Proof. The condition (10.3) implies

$$\exp\left(\int_0^t V_\omega^-(X_s(\eta))\,\mathrm{d}s\right) \in L^1\big(\mathbb{P}(\mathrm{d}\omega) \otimes \mathbb{P}_0'(\mathrm{d}\eta)\big) \quad \text{for every } t > 0,$$

since $L^q \subseteq L^p$ for $1 \le p \le q$ and for finite measures. One has

$$\mathbb{E}(\mathrm{d}\omega)\big[e^{-tV_\omega(0)}\big] \le \mathbb{E}(\mathrm{d}\omega)\big[e^{tV_\omega^-(0)}\big] < \infty \quad \text{for every } t > 0.$$

Hence

$$\lim_{t\downarrow 0} \mathbb{E}(\mathrm{d}\omega)\big[e^{-tV_\omega(0)}\big] = 1.$$

Now, Equation (10.4), Jensen's inequality for the normalized Lebesgue measure and Tonelli's theorem result in

$$\begin{aligned}
&\frac{\mathcal{L}[N](t)}{p(t,0)} \\
&\quad= \mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}^{t,0}_{0,0}(\mathrm{d}\eta)\left[e^{-\int_0^t V_\omega(X_s(\eta))\,\mathrm{d}s}\right] \\
&\quad= \mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}^{t,0}_{0,0}(\mathrm{d}\eta)\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(-\int_0^t V_\omega(X_s(\eta))\,\frac{\mathrm{d}s}{t}\right)^k\right] \\
&\quad\le \mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}^{t,0}_{0,0}(\mathrm{d}\eta)\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^t \left(-V_\omega(X_s(\eta))\right)^k\,\frac{\mathrm{d}s}{t}\right] \\
&\quad= \mathbb{E}(\mathrm{d}\omega) \times \mathbb{E}^{t,0}_{0,0}(\mathrm{d}\eta)\left[\frac{1}{t}\int_0^t \exp\left(-t\,V_\omega(X_s(\eta))\right)\,\mathrm{d}s\right] \\
&\quad= \frac{1}{t}\int_0^t\int_{\mathbb{R}^d}\mathbb{E}(d\omega)\left[e^{-tV_\omega(x)}\right]\mathrm{d}\mathbb{P}_{X_s|X_0=0}(x)\,\mathrm{d}s \\
&\quad= \frac{1}{t}\int_0^t\int_{\mathbb{R}^d}\mathbb{E}(d\omega)\left[e^{-tV_\omega(0)}\right]\mathrm{d}\mathbb{P}_{X_s|X_0=0}(x)\,\mathrm{d}s \\
&\quad= \frac{1}{t}\int_0^t\mathbb{E}(d\omega)\left[e^{-tV_\omega(0)}\right]\mathrm{d}s \\
&\quad= \mathbb{E}(d\omega)\left[e^{-tV_\omega(0)}\right] \\
&\quad\to 1 \quad \text{as } t\downarrow 0.
\end{aligned}$$

Hence

$$\lim_{t\downarrow 0}\frac{\mathcal{L}^+[N](t)}{t^{-d/\alpha}} = \lim_{t\downarrow 0}\frac{\mathcal{L}[N](t)}{t^{-d/\alpha}} = p(1,0)$$

holds due to the self-similarity of p, see Equation (2.2). Now we are able to make use of the Tauberian Theorem 12.1. Its conditions are fulfilled for $f = N|_{[0,\infty)}$ and constants $\gamma = d/\alpha$, $A = p(1,0)$ and arbitrary $K > 0$. A direct application of Theorem 12.1 yields (12.3) and the theorem is proven. □

The theorem directly yields the asymptotics for Gaussian potentials at high energies.

Corollary 12.3. *The IDS of the fractional random Schrödinger operator H_ω with Gaussian potential defined in Corollary 10.2 exhibits the asymptotics*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/\alpha}} = \frac{p(1,0)}{\Gamma(\frac{d}{\alpha}+1)}. \tag{12.4}$$

Proof. According to Theorem 12.2 we only need to show that

$$\lim_{t\downarrow 0} \frac{\mathcal{L}^{-}[N](t)}{t^{-d/\alpha}} = 0.$$

Since by Theorem 11.4 we have $N(\lambda) \sim \exp(-\frac{\lambda^2}{2c(0)})$ as $\lambda \downarrow -\infty$ by partial integration for Riemann-Stieltjes integrals we get for every $t > 0$

$$\begin{aligned} 0 \le \frac{\mathcal{L}^{-}[N](t)}{t^{-d/\alpha}} &= t^{d/\alpha} \int_{-\infty}^{0} e^{-\lambda t}\,\mathrm{d}N(\lambda) \\ &= t^{d/\alpha}\left[e^{-\lambda t}N(\lambda)\right]_{\lambda=-\infty}^{0} + t^{d/\alpha+1}\int_{-\infty}^{0} e^{-\lambda t}N(\lambda)\,\mathrm{d}\lambda. \\ &= t^{d/\alpha}N(0) + t^{d/\alpha+1}\int_{-\infty}^{0} e^{-\lambda t}N(\lambda)\,\mathrm{d}\lambda. \end{aligned}$$

Choose $R > 0$ such that by Theorem 11.4 we have

$$N(\lambda) \le 2\exp(-\tfrac{\lambda^2}{2c(0)}) \quad \text{for all } \lambda \le -R$$

and choose $t > 0$ sufficiently small such that $t/R \le (4c(0))^{-1}$ then we get

$$\begin{aligned} 0 &\le \frac{\mathcal{L}^{-}[N](t)}{t^{-d/\alpha}} \\ &\le t^{d/\alpha}N(0) + 2t^{d/\alpha+1}\int_{-\infty}^{-R} e^{-\lambda t}\exp(-\tfrac{\lambda^2}{2c(0)})\,\mathrm{d}\lambda + t^{d/\alpha+1}\int_{-R}^{0} e^{-\lambda t}N(\lambda)\,\mathrm{d}\lambda \\ &\le t^{d/\alpha}N(0) + 2t^{d/\alpha+1}\int_{-\infty}^{-R} \exp(-\lambda^2(\tfrac{1}{2c(0)} - \tfrac{t}{R}))\,\mathrm{d}\lambda + t^{d/\alpha+1}N(0)\int_{-R}^{0} e^{-\lambda t}\,\mathrm{d}\lambda \\ &\le t^{d/\alpha}N(0) + 2t^{d/\alpha+1}\int_{-\infty}^{-R} \exp(-\tfrac{\lambda^2}{4c(0)})\,\mathrm{d}\lambda + t^{d/\alpha}N(0)(e^{Rt}-1) \to 0 \end{aligned}$$

as $t \downarrow 0$ as desired. □

Due to Theorem 12.2, in case of a fractional random Schrödinger operator with Poissonian potential we derive the same asymptotics at $+\infty$ as in the case of Gaussian potentials. The Poisson potential is given by convolution of a shape function with respect to a Poisson random measure.

Corollary 12.4. *For fixed $\alpha \in (0,2)$ consider the fractional random Schrödinger operator H_ω in (9.1) with Poissonian potential $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ given by*

$$V_\omega(x) = \int_{\mathbb{R}^d} \varphi(x-y)\ \mathrm{d}\mathsf{P}_\omega(\mathrm{d}y),$$

where P_ω denotes a Poisson random measure with Lebesgue intensity on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\varphi \in L^1(\mathbb{R}^d)$ is a nonnegative continuous function.
Then the IDS of the operator H_ω exhibits the asymptotics

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/\alpha}} = \frac{p(1,0)}{\Gamma(\frac{d}{\alpha}+1)}. \tag{12.5}$$

Proof. Ōkura showed the existence of the integrated density of states for fractional Schrödinger operators with Poissonian potential in Theorem 6.1 of [31]. By the fact that the spectrum of H_ω is contained in the positive real line we get

$$\lim_{t\downarrow 0} \frac{\mathcal{L}^-[N](t)}{t^{-d/\alpha}} = 0.$$

Now, the claim follows by Theorem 12.2. □

Corollary 12.5. *Under the conditions of Corollary 12.4, in dimension $d = 1$ we have for all $\alpha \in (0,2)$*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{1/\alpha}} = \frac{1}{\pi}. \tag{12.6}$$

Proof. In dimension $d = 1$ we can describe the constant $p(1,0)$ in (12.4) explicitly. For $\alpha \neq 1$ equations (14.30) and (14.33) in [39] yield $p(1,x) = \pi^{-1}\Gamma(\frac{1}{\alpha}+1) + \mathcal{O}(x)$ as $x \to 0$ and for $\alpha = 1$ the symmetric Cauchy density fulfils $p(1,0) = \pi^{-1} = \pi^{-1}\Gamma(2)$. Plugging this into (12.5) proves the claim. □

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Eidesstattliche Versicherung

Ich, Leonard Jobst Eberhard Pleschberger, versichere an Eidesstatt, dass die vorliegende Dissertation von mir selbstständig und ohne unzulässige fremde Hilfe unter Beachtung der "Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf" erstellt worden ist.

Düsseldorf, 21. Dezember 2023