Groups acting on rooted trees, the generalised Magnus property and zeta functions of groups

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Zusammenfassung

Diese kumulative Dissertation besteht aus drei Teilen, welche unterschiedliche Aspekte der Theorie der unendlichen Gruppen behandeln. Es werden sowohl strukturelle als auch asymptotische Eigenschaften verschiedener Klassen von Gruppen untersucht.

Im ersten Teil werden Gruppen mit einer treuen Wirkung auf einen regulären Baum mit Wurzel behandelt. Dabei wird eine neuartige Konstruktion eingeführt, welche von der Definition der Basilica-Gruppe inspiriert ist. Viele interessante Eigenschaften von Gruppen des genannten Typs bleiben unter der Konstruktion erhalten; außerdem läßt sich die Hausdorff-Dimension der entstehenden Gruppen unter bestimmten Voraussetzungen konkret berechnen. Es folgt eine Betrachtung spezieller Klassen von Gruppen von Automorphismen gewurzelter Bäume. Zwei Kriterien für Periodizität konstant-spinaler Gruppen werden bewiesen, die Konjugationsklassen von poly-spinalen Gruppen werden bestimmt und die Automorphismengruppen aller multi-GGS-Gruppen berechnet. Den Schlußpunkt bildet eine Beschreibung der abgeleiteten Reihen aller GGS-Gruppen.

Der zweite Teil widmet sich der Magnus-Eigenschaft. Eine Gruppe hat diese, wenn alle Elemente, welche denselben Normalteiler erzeugen, entweder konjugiert oder invers-konjugiert zueinander sind. Zunächst untersuchen wir die Magnus- und einige verwandte Eigenschaften innerhalb der Klassen der endlichen und der kristallographischen Gruppen. Danach studieren wir freie polynilpotente Gruppen, und ermitteln, wann eine solche Gruppe die Magnus-Eigenschaft besitzt.

Der letzte Teil besteht aus einer Untersuchung der Darstellungszetafunktionen bestimmter Untergruppen und bestimmter Erweiterungen der Gruppe $\mathrm{SL}_2^1(\mathbb{Z}_p)$, der ersten Hauptkongruenzuntergruppe der speziellen linearen Gruppe von Grad 2 auf den p-adischen Ganzzahlen $\mathbb{Z}_p$. Es wird gezeigt, daß die untersuchten Zetafunktionen die Zetafunktion von $\mathrm{SL}_2^1(\mathbb{Z}_p)$ als Faktor besitzen.

Kapitel 1 bis 6 sowie 8 und 9 sind je als für sich stehende Aufsätze angelegt und teilweise in Zusammenarbeit mit verschiendenen Koautoren verfaßt. Kapitel 1 ist eine Gemeinschaftsarbeit mit Karthika Rajeev und erschien in „*Groups, Geometry and Dynamics* “, [125]. Kapitel 4 ist in Zusammenarbeit mit Anitha Thillaisundaram entstanden und kann auf dem arXiv gefunden werden [126]. Kapitel 8 ist eine Gemeinschaftsarbeit mit Benjamin Klopsch und Luis Mendonça, welche in „Forum Mathematicum“ erschienen ist [95]. Kapitel 9 ist in Zusammenarbeit mit Margherita Piccolo entstanden. Kapitel 2 erschien im „Journal of Algebra “ [124], Kapitel 3 wird in „Proceedings of the Edinburgh Mathematical Society“ und Kapital 5 in „Journal of Algebra and its Applications“ erscheinen. Kapitel 6 ist als Vorversion auf dem arXiv zu finden, siehe [122].

Abstract

This dissertation is cumulative; it consists of three parts that deal with different aspects of infinite group theory. We study both structural and asymptotic aspects of various classes of groups.

In the first part, we consider groups with a faithful action on a regular rooted tree. We introduce a new construction that is inspired by the definition of the Basilica group. Many interesting properties of groups of the kind considered are preserved under the construction; furthermore, under some conditions the Hausdorff dimension of the resulting groups can be computed explicitly. We continue with a study of certain classes of groups of automorphisms of rooted trees. We prove two criteria for constant spinal groups to be periodic, we study the conjugacy classes of poly-spinal groups, and we calculate the automorphism groups of multi-GGS-groups. Finally, we give a description of the derived series of all GGS-groups.

The second part deals with the Magnus property. A group is said to possess this property, if elements generating the same normal subgroup are either conjugate, or inverse-conjugate to each other. First we consider the Magnus property and akin properties within the class of finite groups and the class of crystallographic groups. Then we turn our attention to free polynilpotent groups, and find out under which additional conditions such groups do possess the Magnus property.

The last part is a study of the representation zeta function of certain subgroups and certain extensions of the group $\mathrm{SL}_2^1(\mathbb{Z}_p)$, the first principal congruence subgroup of the special linear group of degree 2 over the p-adic integers $\mathbb{Z}_p$. We show that the zeta functions under consideration have the zeta function of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ as a factor.

Chapter 1 to 6 as well as 8 and 9 are written as stand-alone research articles. Chapter 1 is written in collaboration with Karthika Rajeev and was published under the title 'On the Basilica operation' in '*Groups, Geometry and Dynamics*' [125]. Chapter 2 is a joint work with Anitha Thillaisundaram and can be found on the arXiv [126] (`arXiv:2201.03266`). Chapter 8 is written in collaboration with Benjamin Klopsch and Luis Mendonça and published in 'Forum Mathematicum' [95]. Chapter 9 is a collective work with Margherita Piccolo. Chapters 2 was published in the 'Journal of Algebra' [124], Chapter 3 will be published in the 'Proceedings of the Edinburgh Mathematical Society', and Chapter 5 will be published in the 'Journal of Algebra and its Applications'. Chapter 6 can be found as a preprint on the arXiv [122] (`arXiv:2208.14975`).

Contents

Introduction

This thesis is concerned with various problems in group theory. It naturally divides into three parts of quite different length. All three parts deal almost exclusively with infinite groups, and their asymptotic and structural properties. Most of the chapters are written in the form of academic articles intended for publication. Some of them are written in collaboration with other researchers. The main results of each chapter are stated in an introductory section at the respective chapter's opening. In this main introduction we will only refer to a selection of the results in this thesis. Let us briefly describe the organisation of the text, before we describe its mathematical contents.

Organisation. — The first part, entitled 'Groups acting on rooted trees', is a collection of six research articles intended for publication. All of them are available on the arXiv. They are entitled 'On the Basilica operation' [125], which has appeared in 'Groups, Geometry and Dynamics' and is written in collaboration with the author's academic sibling Karthika Rajeev, 'Two periodicity conditions for spinal groups' [124], which has appeared in the 'Journal of Algebra', 'Groups of small period growth' [123], which will appear in the 'Proceedings of the Edinburgh Mathematical Society', 'Conjugacy classes of polyspinal groups' [126], written in collaboration with Anitha Thillaisundaram, 'The automorphism group of a multi-GGS-group' [121], to appear in 'Journal of Algebra and its Applications', and 'On the derived series of GGS-groups' [122].

The second part, 'The Magnus property and its generalisations', is formed by an exposition of the Magnus property, containing many new results, and a research article written in collaboration with the author's adviser, Benjamin Klopsch, and with Luis Mendonça, titled 'Free polynilpotent groups and the Magnus property', published by 'Forum Mathematicum' [95].

The third part is a document written in collaboration with another academic sibling, Margherita Piccolo, on the theory of representation zeta functions of p-adic analytic groups. It is written in the form of an academic article, but is yet unpublished.

For the convenience of the reader, all references are collected in a shared bibliography.

The subject matter. — The first, and by far most extensive, of the three parts of this thesis is concerned with groups acting faithfully on regular rooted trees, and thereby inheriting some of the structural properties of the corresponding trees. The origins of the subject date back to the 1980's, when Aleshin [3], Grigorchuk [65], Gupta and Sidki [77], and Neumann [114] discovered groups of this kind with remarkable properties. The examples of Aleshin, Grigorchuk and of Gupta and Sidki are finitely generated, infinite, and periodic; hence solutions of the so-called General Burnside problem posed by Burnside [27], asking whether such groups exist. These groups were not the first examples with these properties, but their construction, and the proof that they are in fact periodic, is far less involved than the construction used for the first such examples, the Golod–Shafarechich groups [58,59]. Furthermore, Grigorchuk soon discovered that his example is a group of intermediate word growth [66]. This group, nowadays called the 'Grigorchuk group', was the first group known to have this property, answering a long-standing question of Milnor [28]. Until today all examples of groups of intermediate growth are based on Grigorchuk's construction in some way, although Nekrasheyvich's celebrated construction of simple groups of intermediate growth [111] translates these ideas away from automorphisms of rooted trees.

In addition to the striking examples of groups discussed above, certain groups acting on rooted trees appear as one of the cases in the trichotomy developed by Wilson [153,154] and Grigorchuk [68] for just-infinite groups. To be precise, all just-infinite groups (i.e. infinite groups such that all their proper quotients are finite) that are neither simple nor hereditarily just-infinite (i.e. bequeath this property to their subgroups) are branch groups. The latter are certain well-behaved groups of automorphisms of rooted trees, whose subgroup lattice is similar to the one of the full automorphism group of the tree. All examples mentioned in the first place are branch groups, and in fact just-infinite. However, not all branch groups are just-infinite, and not all groups acting on rooted trees are branch.

While branch groups are defined by certain algebraic properties (they can be defined without any reference to the action on an underlying tree, cf. [155]), there is another interesting class of groups acting on regular rooted trees based on a certain kind of dynamical behaviour. Every regular rooted tree has strong self-similarity properties: given any of its vertices, the set of vertices 'below' it (i.e. the set of vertices whose unique minimal path from the root to the vertex in question passes through the initially fixed vertex) forms an isomorphic copy of the full tree. Identifying certain subtrees, one obtains a self-map on the automorphism group of the tree assigning to an automorphism its section on a subtree. This map is in general not a group homomorphism, but 'almost': it is a homomorphism on a finite index subgroup. A group of automorphisms of a rooted tree is called self-similar if it is closed under all possible sections. This gives a very rich class of groups that is connected with the theory of automata; indeed, self-similar groups are precisely the groups generated by finite-state automata, see [110]. A finite-state automaton is a finite collection Ω of states, together with an alphabet X and a transition function $\tau : \Omega \times X \to \Omega$ and an output function $\omega : \Omega \times X \to X$, that describe the behaviour of the automaton, given the current state of the machine and the input given, in terms of a new state of the machine and an output. Automata are usually depicted as directed graphs.

An m-regular rooted tree is isomorphic to the Cayley graph of the free monoid on any set X of cardinality $m \in \mathbb{N}_+$. Thus, we may identify the vertices of the tree with finite words in an alphabet X. Given any automata on the alphabet X, we can define an action on the tree by

$$(x_n \dots x_0)q = (x_n \dots x_1)(q, x_0)\tau(x_0)\omega,$$

where q is a state of the automaton. The state $(q, x_0)\tau$ corresponds to the section of the automorphism induced by q at x_0. For example, Fig. 1 depicts the automaton containing the four states (i.e. automorphisms) that generate the Grigorchuk group.

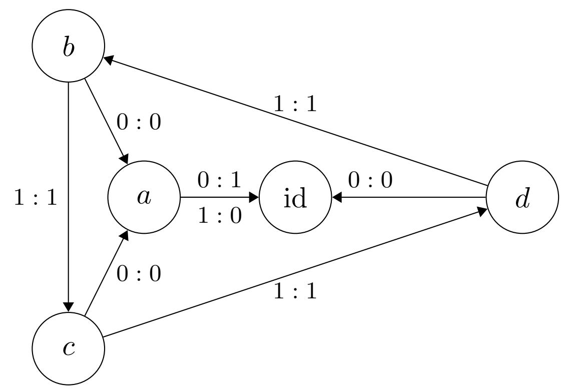


Figure 1: The finite-state automaton generating the first Grigorchuk group. The alphabet is the set $\{0, 1\}$. The state a defines a rooted automorphism, the states b, c and d define directed ones.

In many ways, the extraordinary groups mentioned above still shape the development of the field. Diverse classes of generalisations have been defined, and it is an active area of research to determine the exact conditions under which these generalisations display the same exotic properties as the original examples that served as an inspiration. Therefore, we quickly sketch the specific classes of groups that we will examine.

A *spinal* group is a group of automorphisms generated by a finite group of rooted automorphisms – i.e. automorphisms that rigidly permute the subtrees below the vertices of the first layer – and a finite group of directed automorphisms, i.e. automorphisms that act simultaneously by rooted automorphisms on the subtrees below vertices on a fixed infinite ray inside the tree (a 'spine'), cf. Fig. 2. This class of groups was the first that allowed for a unified theory of the examples of Grigorchuk and of Gupta and Sidki. They were defined by Bartholdi and Šunić in [19], and expanded on in [13].

A *polyspinal* group is a group generated from finitely many spinal groups that may have different spines. This very broad class of groups is considered in Chapter 4. Noteworthy examples of non-spinal polyspinal groups are the extended Gupta–Sidki-groups ('EGS'-groups) defined by Pervova, which are the first example of branch groups without the congruence subgroup property, cf. [119].

We call a spinal group such that the action of the directed generators is equal on every

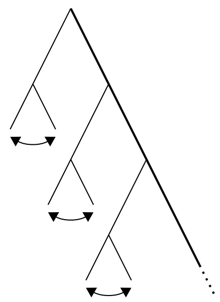


Figure 2: A directed automorphism acting on the binary tree.

level a constant spinal group, or short a CS *group*. A CS group on a regular rooted tree with prime degree p such that the rooted group acts as a p-cycle is called a multi-GGS-group. This well-studied class contains the examples of Gupta and Sidki.

We now introduce the group that is central to Chapter 1 and motivate its construction. This chapter was written in collaboration with Karthika Rajeev and has been accepted for publication in the journal 'Groups, Geometry and Dynamics'.

The construction of a group of intermediate growth had immediate consequences in various areas of group theory. For example, it had been an open problem if all amenable groups are in fact elementary amenable, i.e. groups in the smallest class containing all finite and abelian groups and being closed under quotients, subgroups, extensions and direct limits, cf. [31]. However the class of elementary amenable groups contains no groups of intermediate word growth. At the same time, all groups of sub-exponential word growth are amenable.

Due to the flexibility of the class of groups acting on rooted trees, one can achieve even more. Knowing that groups of intermediate word growth exist, one wonders if those groups are (in a sense) the only non-elementary amenable but amenable groups. Define the class of subexponentially amenable groups to be the smallest class being closed under the same operations as above, but containing all groups of subexponential word growth. Naturally, one asks if all amenable groups are subexponentially amenable, and again, the answer is no, with a counterexample constructed as a group of automorphisms of the rooted binary tree. This is the so-called Basilica group, introduced by Grigorchuk and Żuk in [72]. While it is in many ways different from the other examples mentioned above – it is not a branch group, but a so-called weakly branch group – the same techniques and concepts can be applied to understand its structure.

Comparing the automaton defining the Basilica group with the automaton defining the binary odometer – that is the (dense) embedding of the integers into the copy of the p-adic integers naturally included in the automorphism group of the binary rooted tree – we notice some similarities. Indeed, one can obtain the Basilica automaton by 'delaying' the odometer automaton by adding additional, non-acting states.

Based on this observation, we define the *Basilica operation*. On the level of automata, a simple ‘instructive diagram’ is shown in Fig. 3. A purely algebraic definition can be found in Definition 1.2.6 on Page 13.

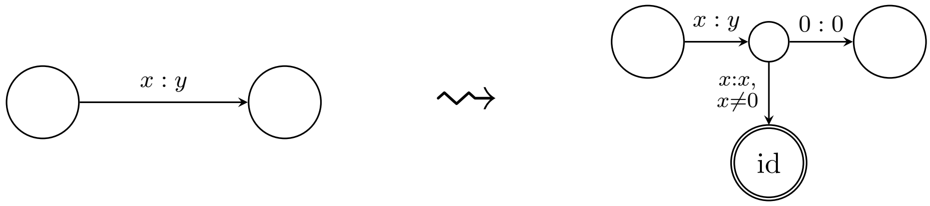


Figure 3: The replacement rule defining the Basilica operation. For every transition of the automaton, one adds a number of intermediate states.

Many desirable properties of the Basilica group turn out to be consequences of certain properties of the odometer, and, generally, the Basilica operation preserves many properties of the original group, including being self-similar, fractal, contracting, or weakly branch, see Theorem 1.1.1 on Page 5. Furthermore, under certain circumstances one can describe the layer stabilisers rather explicitly; this description can be found in Theorem 1.1.4 on Page 6. The description of the layer stabilisers is usually a very difficult task; consequently, there are only few groups for which explicit calculations are known. Using our description of the layer stabilisers, we portray the exact relationship between the Hausdorff dimension of a group and the Hausdorff dimension of its Basilica groups.

The second half of the chapter deals with groups more closely resembling the original Basilica group, being obtained by means of the Basilica operation from certain free abelian groups. We describe the new groups in detail, showing that their algebraic properties are diverging from the original example, and by giving a recursive presentation. Using these structural descriptions, we prove that these ‘generalised Basilica groups’ have a weak (‘local’) version of the congruence subgroup property for groups acting on rooted trees.

Surprisingly, the Basilica construction does not only relate the odometer to the Basilica group, but also relates spinal groups to other spinal groups. This allows us to apply techniques developed to understand the Basilica group to the study of spinal groups. Indeed, we give a formula for the Hausdorff dimensions of certain spinal groups. The GGS-groups (see [48]), the first and second Grigorchuk groups (see [69, 115]) and certain generalisations of the first Grigorchuk group (see [141, 144]) are the only spinal groups for which this invariant was previously computed.

It is an open question if periodicity is a property that is invariant under the Basilica operation. If this was the case, one would have a strong tool for finding new infinite finitely generated periodic groups. Our partial answer to this problem leads on to Chapter 2, where we investigate conditions that ensure that a constant spinal group is periodic. Since the examples of Gupta and Sidki are CS groups, one expects a fruitful outcome of searching for periodic groups within the class of CS groups. Indeed, one finds many such groups, but also groups with elements of infinite order. It is an interesting problem to describe the exact transission line between the two phenomena.

Our considerations are not without precendent. Gupta and Sidki [78] developed a sufficient criterion for certain groups to be periodic. In [68], Grigorchuk gave an elegant sufficient and necessary condition for when a GGS-group acting on a p-regular rooted tree is periodic, which was expanded by Vovkivsky [151], and later by Bartholdi [9], who introduced criteria formulated in terms of the dynamics of certain maps of the finite rooted group.

Generalising all previous examples, we develop two conditions, given in Theorem 2.1.1 and Theorem 2.1.2 on Page 59, that imply that a CS group is periodic. These form the main results of Chapter 2. Both theorems are stated in terms of finite dynamical systems, and reduce the problem to a finite computation. Previously, proofs of periodicity heavily relied on the rooted group being abelian or at least acting regularly. Our refined methods show that the condition that the rooted group is abelian can perhaps not be dropped entirely, but can be replaced by weaker, localised versions of being abelian. Our proofs rely on the concept of stabilised sections, that is a variant of the section map which is specialised for proofs of periodicity, and on the consideration of the local geometry of the rooted tree: We consider every set of children of a given vertex not only as a set, but with some structure preserved by the local actions of our groups. This proves to be useful both for developing our conditions for periodicity, and for constructing explicit examples.

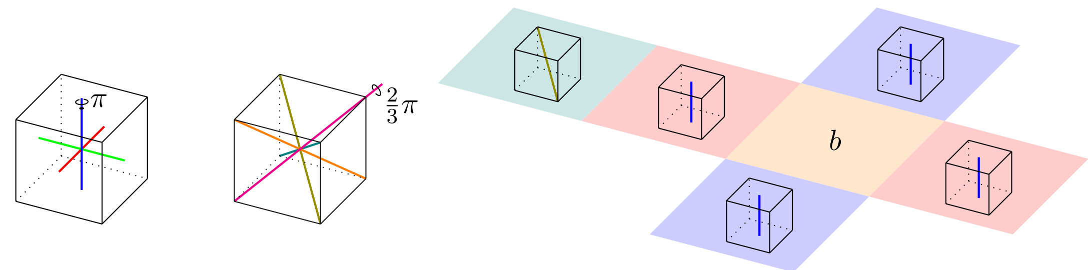


Figure 4: The rooted (left and middle) generators and the directed (right) generator b of a periodic constant spinal group with non-regular local action. The local action is that of Alt(4) on the cube.

Using these methods, we construct the first infinite finitely generated periodic groups acting locally non-regularly on a rooted tree; cf. Fig. 4. Furthermore, returning to the topic of Chapter 1, we prove that the Basilica operation preserves periodicity within the class of constant spinal groups.

As mentioned above, the Grigorchuk group was not only an early example of a group of Burnside type, but also the first example of a group of intermediate word growth. This is not a coincidence, as both properties can be obtained from similar inequations on the word length of group elements and their sections. Indeed, by-passing some technicalities, inequations of the type

$$\ell(g|_x) < \ell(g),$$

where ℓ denotes some length function, g a group element, and $|_x$ the section map at a vertex x of the tree, imply periodicity; while inequalities of the type

$$\sum_{x \in X^n} \ell(g|_x) < \ell(g),$$

where the sum ranges over all vertices of a layer X^n of the tree, imply intermediate word growth.

There is another concept of growth connected to periodicity, which is the subject of Chapter 3. Given a finitely generated group G, the period growth of G is the rate of growth of the sequence $(a_n)_{n\in\mathbb{N}}$, where a_n is defined as the maximal order of an element contained in balls within the Cayley graph of G, centered around the identity and of radius n. If G is not periodic, the sequence $(a_n)_{n\in\mathbb{N}}$ is an eventually constant sequence equal to ∞, from some point on. If G has finite exponent, the sequence is equal to this exponent from some point on. Finally, the period growth of a periodic group measures how close it is to being of finite exponent.

The celebrated solution of the Restricted Burnside problem by Zel'manov [159, 160] implies that finitely generated, infinite, residually finite groups of bounded exponent do not exist. A group is called residually finite if every non-trivial element can be distinguished from the trivial element in a finite quotient.

Thus finitely generated, infinite, residually finite groups cannot have bounded period growth. It is a natural question how close such a group may be to having bounded exponent, i.e. how small their period growth may be. For the main result of Chapter 3, Theorem 3.1.1 on Page 84, we construct a family of groups acting on rooted trees (such groups are always residually finite) whose period growth is bounded from above by an iterated logarithm function. This is achieved by proving a stronger version of the first kind of inequality described above. In Fig. 5, we depict one of the resulting generators.

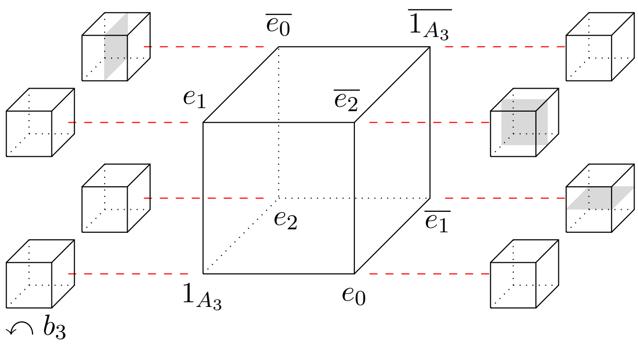


Figure 5: A pictoral description of the action of a directed generator of a group with small period growth. The idea is to lift the strong inequality between the exponent and the diameter of the elementary abelian 2-group, acting on a cube of arbitrary finite dimension, to an inequality of the type described above.

The construction of groups of very small period growth also allows us to answer a question of Bradford on the possible lawlessness growth of a group. This kind of growth is defined similarly to period growth, but using a different invariant measuring the extent to which a group is lawless, i.e. not fulfilling any group identity, on growing finite subsets. Our groups are lawless, since they are weakly branch, and have, due to some inversion, lawlessness growth equivalent to that of a tower of exponentials.

In Chapter 4, that is written in collaboration with Anitha Thillaisundaram, we examine the following question. When are two polyspinal groups isomorphic? When are they conjugate within the automorphism group of the full tree? Within the wide class of polyspinal groups, only vague answers seem realistically possible. However, restricting to multi-EGS- or further to multi-GGS-groups, we obtain descriptions of the conjugacy classes of such groups. Furthermore, in these cases the conjugacy classes coincide with the isomorphism classes.

Coming from isomorphisms between different multi-GGS-groups, we consider the automorphism groups of multi-GGS-groups in Chapter 5. Automorphisms groups of trees acting on rooted trees are, typically, relatively rigid objects, and the multi-GGS-groups are no exeption. The original group G embeds as the group of inner automorphisms into $\mathrm{Aut}(G)$, because multi-GGS-groups are branch groups (with one exception) and branch groups are centreless.

While there are many theoretical results on the rigidity of branch or self-similar groups, only for very few examples the automorphism group has been computed explicitly. Within the class of multi-GGS-groups, these examples are the three one-dimensional groups acting on the ternary tree. We compute the automorphism groups of all multi-GGS-groups excluding the constant GGS-group. The result is given in Theorem 5.1.1 on Page 114. Most interesting is the subgroup of automorphisms of order coprime to the degree of the tree (which in case of multi-GGS-groups, is an odd prime). Its structure is described in linear terms based on the defining subspace of the group. We also find that the usual dichotomy between regular and symmetric multi-GGS-groups is present; a corollary of our result is that the outer automorphism group of an multi-GGS-group is finite if and only if the group is non-regular.

Our methods of proof are not entirely new, but rather a combination of techniques developed by Sidki [138] and by the author in [120]. However, the subject is delicate, and a careful analysis of the structure of a multi-GGS-group is necessary so that, along the way, we prove some statements on multi-GGS-groups that are interesting in their own right, e.g. Proposition 5.4.1 on Page 125.

In the last chapter of the first part, Chapter 6, we continue our study of multi-GGS-groups, but restrict ourselves to the one-dimensional case (the GGS-groups proper). While many aspects of GGS-groups relating to their action on the tree are well-known, most of their purely algebraic invariants are still mysterious. In Chapter 6, we calculate the index of the members of the derived series of a given GGS-group, and we describe these members recursively. The results are stated in Theorem 6.1.2 on Page 136. Apart from the case of the Gupta–Sidki 3-group, where similar results were found by Vieira [149] using different methods, the results are entirely new.

Every GGS-group is defined by a one-dimensional subspace in $\mathbb{F}_p^{p-1}$, or equivalently, by a corresponding basis vector $\mathbf{e}$. Our formula for the index of the n^{th} derived subgroup depends only on the degree of the tree, the vector $\mathbf{e}$ and its first two vectors of (entrywise) differences. Thus, in this regard the derived series behaves even more regularly than the series of layer stabilisers that were computed in [48].

To obtain our formula, we refine the methods of examining GGS-groups by connecting their quotients to circulant vector subspaces, i.e. subspaces of a vector space invariant under cyclic permutation of its basis elements. We furthermore introduce the series of iterated local laws that establishes an 'algebraic analogue' of the series of layer stabilisers, and we prove that, in the case of GGS-groups, it coincides with the derived series.

The second part of the thesis is not concerned with a specific class, but with the following property of groups. A group has the Magnus property if all pairs of elements generating the same normal subgroup are either conjugate or inverse-conjugate to each other; i.e. if there are at most two conjugacy classes of elements generating the same given normal subgroup. Magnus proved that free groups have this property [107], as an application of his famous Freiheitssatz. In recent years, various other groups were proven to posses the Magnus property; for example, all fundamental groups of closed compact surfaces have the property [23, 46].

Groups with the Magnus property have a well-behaved family of one-relator quotients, i.e. quotients that arise by adding a single relation to a presentation of the group. For free groups, this gives rise to the fruitful theory of one-relator groups.

The study of groups with the Magnus property is hampered by the fact that the Magnus property is neither inherited by normal subgroups, nor by quotients, nor by direct products. On the other hand, every countable torsion-free group is contained in a group with the Magnus property, using the classic Higman–Neumann–Neumann embedding theory [82], making groups with the Magnus property (in a sense) abundant.

In Chapter 7, we pursue two goals. First, we extend and modify the definition of the Magnus property, allowing both for new directions where the extended, very strong property fails and for relative versions useful for establishing (or disproving) the original Magnus property for a given group. Second, we consider two kinds of groups that are 'small' in comparison to the groups that are known to have the Magnus property, and search within these classes for groups with the Magnus property and for obstructions for having the property. The classes of interest are the class of finite groups and the class of crystallographic groups.

We prove that every finite group with the Magnus property is a soluble $\{2, 3, 5, 7\}$-group, and that every finite group with a 'strong' version of the Magnus property is a $\{2, 3\}$-group.

These results build on an existing theory of finite groups that are rational, i.e. groups where all generators of a given cyclic subgroup are conjugate. We also prove that, surprisingly, the class of finite groups with the Magnus property is closed under quotients. As mentioned above, this is by far not the case for arbitrary groups, as the classic example of free groups demonstrates.

We then sketch the connexion between the Magnus property for a group and the property of an associated quotient of the associated group ring to possess only trivial units. We prove a variation of Higman's classic result on periodic groups with group ring having only trivial units. Using these results, we prove that every crystallographic group with the Magnus property has a point group that is built from certain 'atoms' coming from a finite list of finite groups, thus restricting the possible point groups to $\{2, 3\}$-groups. The precise formulation can be found in Theorem 7.6.1 on Page 195. On the other hand, we construct some new crystallographic groups with the Magnus property.

Chapter 8, written in collaboration with Benjamin Klopsch and Luis Mendonça, is motivated by the observation that both free and free abelian groups possess the Magnus property. We deal with the following question: What other relatively free groups possess the Magnus property? The main result of this chapter, Theorem 8.1.1 on Page 208, gives a complete answer for free polynilpotent groups, these groups are precisely the quotients of abstract free groups by verbal subgroups generated by iterated commutator words $\gamma_{(c_1+1,\ldots,c_\ell+1)}$ for a tuple $(c_1, \ldots, c_\ell) \in \mathbb{N}^\ell$, which are given by the ordinary lower central words for tuples of length 1, and are otherwise recursively defined by

$$\gamma_{(c_1+1,\ldots,c_\ell+1)} = \gamma_{c_\ell+1}(\gamma_{(c_1+1,\ldots,c_{\ell-1}+1)}),$$

with the usual convention that separate sets of variables are used for repeated occurrences of the inner word. In particular, free nilpotent (the case $\ell = 1$) and free soluble (the case $(1, \ldots, 1)$) groups are polynilpotent. We prove that a free polynilpotent group has the Magnus property if and only if it is nilpotent of class 1 or 2, involving the development of new methods to disprove and to establish the Magnus–property.

In contrast, we prove that all torsion-free nilpotent groups of class 2 have the Magnus property, and we provide examples of finitely generated nilpotent groups of arbitrary class with the Magnus property as well as a torsion-free class-3 group with the Magnus property. Finally, we construct torsion-free nilpotent groups of arbitrary class with the Magnus property, using a crafty ultra-product construction. The question if there exist finitely generated torsion-free nilpotent groups of arbitrary class that possess the Magnus property, however, remains a challenging open question.

At the end of the second part of this thesis, we collect some computer programs in GAP [52] that we have used for the preparation of Chapter 7.

The last and third part of the thesis, written in collaboration with Margherita Piccolo, is concerned with the study of representation zeta functions of groups. Let G be a topological group, and let r_n be the number of n-dimensional irreducible continuous representations of G. For many infinite groups, these numbers are not finite, the easiest example being the cyclic group of infinite order equipped with the discrete topology, having infinitely many one-dimensional representations. Groups for which all r_n are finite are called representation rigid.

The growth rate of the sequence $(r_n)_{n \in \mathbb{N}_+}$ is in a sense a sibling of the subgroup growth of a group. In both cases we approximate the full group by considering larger and larger quotients, in contrast to the word growth or the period growth discussed in Chapter 3 that approximate a group using larger and larger subsets.

Following Grunewald, Segal and Smith [76], we introduce the following Dirichlet generating function called the representation zeta function of G that is defined by

$$s \mapsto \zeta_G(s) = \sum_{n \in \mathbb{N}_+} r_n n^{-s},$$

for a complex variable $s \in \mathbb{C}$. It is a well-known fact that the degree of polynomial growth of $(r_n)_{n \in \mathbb{N}}$ is equal to the abscissa of convergence of the series giving ζ_G.

The theory of representation growth and representation zeta functions is mostly developed for the classes of nilpotent groups [150] and for arithmetic groups. Given an arithmetic group with the congruence subgroup property, Larsen and Lubotzky [98] establish an Euler product decomposition of the representation zeta function. We are interested in the local factors, which are representation zeta function of p-adic analytic groups.

There exits a well-developed theory describing these functions. By [21], a compact p-adic analytic group (indeed, any finitely generated profinite group) is representation rigid if and only if every open subgroup has finite abelianisation. Jaikin proved in [89], using model-theoretic methods, that the representation zeta function of such a group (for odd primes) is close to a rational function in p^{-s}. In [7], Avni, Klopsch, Onn and Voll establish functional equalities for the representation zeta functions of certain p-adic analytic groups; they make use of p-adic integration techniques and the Kirillov orbit method, see also [61, 84], that allows to construct a bijection between irreducible representations and orbits under the so-called co-adjoint action.

Every compact p-adic analytic group contains a uniform pro-p-group as a finite-index subgroup. Indeed, compact p-adic analytic groups are characterised by this property, cf. [39]. Using the Lazard correspondence between uniform pro-p-groups and uniform $\mathbb{Z}_p$-Lie-lattices and the Kirillov orbit method, one can describe the representation zeta function of a uniform pro-p-group as a p-adic integral.

Unfortunately, the explicit calculation of the integrals arising from concrete examples is a very hard task. It is our aim to provide new examples that might be useful in the further development of the theory. We base our work on the tools described above. Motivated by the easiest example, the first principal congruence subgroup of the special linear group over the p-adic integers, $\mathrm{SL}_2^1(\mathbb{Z}_p)$, we prove that certain finite index subgroups of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ have, up to a constant factor, the same representation zeta function as the full group. Using this astonishing fact, we prove the main result in Chapter 9: all semidirect products $\mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes V$ of the group with an abelian module $V \cong \mathbb{Z}_p^n$ allow for their zeta functions to be written

$$\zeta_{\mathrm{SL}_2^1(\mathbb{Z}_p)\ltimes V}(s) = \zeta_{\mathrm{SL}_2^1(\mathbb{Z}_p)}(s) \cdot \zeta_{\mathrm{SL}_2^1(\mathbb{Z}_p)}^{\mathrm{SL}_2^1(\mathbb{Z}_p)\ltimes V}(s-1),$$

where the second factor on the right hand side is the zeta function associated to the representation induced from the trivial representation on V. The latter is the Dirichlet generating function associated to the sequence of numbers of n-dimensional irreducible components of the induced representation; see Theorem 9.3.6 on Page 250 for a precise statement. The theory of these zeta functions was recently developed by Kionke and Klopsch in [92].

Using our description of the zeta function of such a semidirect product, we calculate explicitly the representation zeta functions of $\mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2$ with the natural action, $\mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes (\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2)$ with the diagonal action, and $\mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes \mathrm{Sym}^2(\mathbb{Z}_p^2)$ with the symmetric square action.

Statement on the author's contribution to shared research. — I declare that the research, the writing and the ideas for the articles [95, 125, 126] that are included in this thesis as Chapters 1, 4, 8, and for the development of the content of Chapter 9, were shared equally among myself and the respective coauthors. A detailed account of contributions follows below.

The initial idea for the work carried out in Chapter 1 was based on a calculation of K. Rajeev of the level stabiliser of special cases of what we later called 'generalised Basilica groups'. We ventured to find the 'minimal' necessary axioms, which led the author to develop Definition 1.2.3. The research was carried out in many discussions, circulating manuscripts and in constant exchange of ideas. The heart of the paper, Theorem 1.1.4 and its proof, its extensions and applications, was achieved with equal parts of contribution. The earlier sections dealing with the general case of Basilica groups were chiefly established by the author; while the later sections studying the class of generalised Basilica groups were mostly done by K. Rajeev. However, both authors had a part in every section of the paper.

The investigation of the conjugacy classes of multi-GGS-groups in Chapter 4 was suggested by A. Thillaisundaram after a talk by the author on the results of his Diploma thesis dealing with the same question in the case of GGS-groups. During various research visits of A. Thillaisundaram at the Heinrich-Heine-Universität Düsseldorf the results of Chapter 4 were produced in joint work.

The contents of Chapter 8 were in their foundations established during meetings of the three contributors, B. Klopsch, L. Mendonça and the author. All three wrote and proved parts of the resulting paper, these contributions were discussed and subsequently improved by all three contributors. The author's main contributions are the strategy of the proof in the case of the free nilpotent groups and the construction of the counterexample within the wreath product $\mathbb{Z} \wr \mathbb{Z}$.

The main result of Chapter 9, Theorem 9.3.6, and the foundations of its proof were jointly developed by M. Piccolo and the author based on preliminary calculations of the author (that were made obsolete by the theorem). Theorem 9.3.1 and its proof were mainly contributed by the author. The examples and calculations in the latter half of the chapter were produced in joint work of the contributors.

I have shared the above explanations with my respective co-authors and they have confirmed that they happily agree with the relevant statements to be included in my thesis in this form.

Part I:

Groups acting on rooted trees

Chapter 1

On the Basilica operation

Written in collaboration with Karthika Rajeev.

Abstract. Inspired by the Basilica group $\mathcal{B}$, we describe a general construction which allows us to associate to any group of automorphisms $G \leq \mathrm{Aut}(T)$ of a rooted tree T a family of Basilica groups $\mathrm{Bas}_s(G), s \in \mathbb{N}_+$. For the dyadic odometer $\mathcal{O}_2$, one has $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$. We study which properties of groups acting on rooted trees are preserved under this operation. Introducing some techniques for handling $\mathrm{Bas}_s(G)$, in case G fulfils some branching conditions, we are able to calculate the Hausdorff dimension of the Basilica groups associated to certain GGS-groups and of generalisations of the odometer, $\mathcal{O}_m^d$. Furthermore, we study the structure of groups of type $\mathrm{Bas}_s(\mathcal{O}_m^d)$ and prove an analogue of the congruence subgroup property in the case $m = p$, a prime.

1.1 — Introduction

Groups acting on rooted trees play an important role in various areas of group theory, for example in the study of groups of intermediate growth, just infinite groups and groups related to the Burnside problem. Over the years, many groups of automorphisms of rooted trees have been defined and studied. Often they can be regarded as generalisations of early constructions to wider families of groups with similar properties.

In this paper, we consider an operation on the subgroups of the automorphism group $\mathrm{Aut}(T)$ of a rooted tree T with degree $m \geq 2$. It is inspired by the *Basilica group* $\mathcal{B}$, a group acting on the binary rooted tree, which was introduced by Grigorchuk and Żuk in [73] and [72]. The Basilica group $\mathcal{B}$ is a particularly interesting example in its own right: it is a self-similar torsion-free weakly branch group, just-(non-soluble) and of exponential word growth. It was the first group known to be not sub-exponentially amenable [73], but amenable [14,20]. Furthermore, it is the iterated monodromy group of $z^2 - 1$ [72,127], and it has the 2-congruence subgroup property [55].

The Basilica group $\mathcal{B}$ is usually defined as the group generated by two automorphisms

$$a = (b, \mathrm{id}) \quad \text{and} \quad b = (0\ 1)(a, \mathrm{id}),$$

acting on the binary rooted tree (in [73] the elements are defined with id on the left, which is merely notational). We point out the similarities between these two generators and the single automorphism generating the *dyadic odometer*. The latter provides an embedding of the infinite cyclic group into the automorphism group $\mathrm{Aut}(T)$ of the binary rooted tree T, given by

$$c = (0\ 1)(c, \mathrm{id}).$$

We can regard b as a delayed version of c, that takes an intermediate step acting as a, before returning to itself. Considering the automata defining the generators of both groups (cf. Fig. 1.2), the relationship is even more apparent. We obtain the automaton defining b from the automaton defining c by replacing every edge that does not point to the state of the trivial element with an edge pointing to a new state, which in turn points to the old state upon reading 0 and to the state of the trivial element upon reading any other letter. See Fig. 1.1 for an illustration of this replacement rule.

Figure 1.1: Replacement rule for edges.

The same can be done for any automorphism of T and any number s of intermediate states. For any group of automorphisms G, this operation yields a new group of tree automorphisms defined by the automaton with s intermediate steps, which we call $\mathrm{Bas}_s(G)$, the s^{th} Basilica group of G. A precise, algebraic definition that does not refer to automata will be given in Definition 1.2.3. Fig. 1.2 depicts for example the automaton defining $\mathrm{Bas}_8(\mathcal{O}_2)$, while Fig. 1.3 depicts the automaton defining the generators of the Gupta–Sidki 3-group $\ddot{\Gamma}$ and the corresponding automaton obtained by the operation Bas_2.

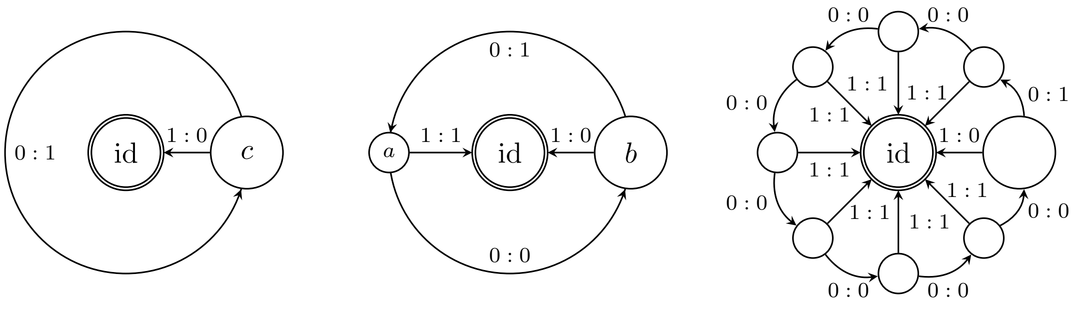
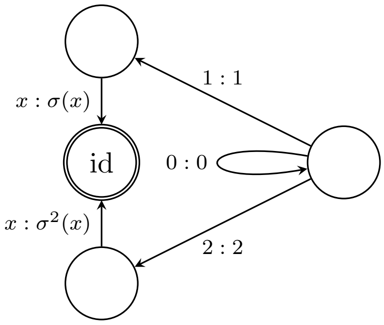


Figure 1.2: Automata for the dyadic odometer $\mathcal{O}_2$, the Basilica group $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$, and $\mathrm{Bas}_8(\mathcal{O}_2)$.



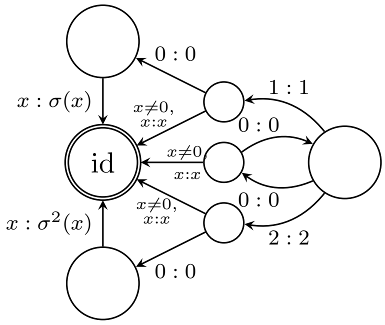


Figure 1.3: Automata for the Gupta–Sidki 3-group $\ddot{\Gamma}$ and $\mathrm{Bas}_2(\ddot{\Gamma})$, where σ is a cyclic permutation.

We prove that many of the desirable properties of the original Basilica group $\mathcal{B}$ are a consequence of the fact that the binary odometer $\mathcal{O}_2$ has those properties and that the properties are preserved under the Basilica operation. We summarise results of this kind for the general Basilica operation in the following theorem.

Theorem 1.1.1. *Let G be a group of automorphisms of a regular rooted tree. Let $\mathcal{P}$ be a property from the list below. Then, if G has $\mathcal{P}$, the s^{th} Basilica group $\mathrm{Bas}_s(G)$ of G has $\mathcal{P}$ for all $s \in \mathbb{N}_+$.*

(i) *spherically transitive*
(ii) *self-similar*
(iii) *(strongly) fractal*
(iv) *contracting*
(v) *weakly branch*
(vi) *generated by finite-state bounded automorphisms*

As a consequence we derive conditions for $\mathrm{Bas}_s(G)$ to have solvable word problem and to be amenable. Furthermore, we provide a condition for $\mathrm{Bas}_s(G)$ to be a weakly regular branch group given that G satisfies a group law. This enables us to construct a weakly regular branch group branching over a prescribed verbal subgroup.

The class of spinal groups, defined in [19], is another important class of groups acting on T; it contains the Grigorchuk group and all GGS-groups, see Definition 1.3.7. It is not true that the Basilica operation preserves being spinal, however groups obtained from spinal groups act as spinal groups on another tree $\delta_s T$, obtained by deleting layers from T.

Theorem 1.1.2. *Let G be a spinal group (resp. a* GGS*-group) acting on T. Then $\mathrm{Bas}_s\, G$ is a spinal (resp. a* GGS*-group) acting on $\delta_s T$ for all $s \in \mathbb{N}_+$.*

In contrast to Theorem 1.1.1, the exponential word growth of the original Basilica group $\mathcal{B}$ is not a general feature of groups obtained by the Basilica operation. In fact, the situation appears to be chaotic, for which we provide some examples, see Proposition 1.3.17 and Proposition 1.3.18.

Next we turn our attention to a class of groups G whose Basilica groups $\mathrm{Bas}_s(G)$ more closely resemble the original Basilica group. For this, we introduce the concept of the group G being *s-split* (see Definition 1.4.1). An s-split group decomposes by definition as a semi-direct product, algebraically modelling the property that the image of a delayed automorphism can be detected by observing the layers on which it has trivial labels. We prove that all abelian groups acting locally regular are s-split for all $s \in \mathbb{N}_+$, and that conversely, all s-split groups acting spherically transitive are abelian. Furthermore we obtain the following.

Theorem 1.1.3. *Let $s > 1$ and let G be an s-split self-similar group of automorphisms of a regular rooted tree acting spherically transitively. If G is torsion-free, then $\mathrm{Bas}_s(G)$ is torsion-free. Furthermore $\mathrm{Bas}_s(G)^{\mathrm{ab}} \cong G^s$.*

The $(s-1)^{th}$ *splitting kernel* K_{s-1} is a normal subgroup of G measuring the failure G to be s-split. A rigorous definition is found in Definition 1.4.1. If G is weakly regular branch over K_{s-1} (allowing K_{s-1} to be trivial, hence including s-split groups), we obtain a strong structural description of the layer stabilisers of $\mathrm{Bas}_s(G)$. The maps β_i are the algebraic analogues of the various added steps delaying an automorphism, defined in Definition 1.2.2.

Theorem 1.1.4. *Let G be a self-similar and very strongly fractal group of automorphisms of a regular rooted tree. Assume that G is weakly regular branch over K_{s-1}. Let $n \in \mathbb{N}_0$. Write $n = sq + r$ with $q \geq 0$ and $0 \leq r \leq s-1$. Then, for all $s > 1$,*

$$\mathrm{St}_{\mathrm{Bas}_s(G)}(n) = \langle \beta_i(\mathrm{St}_G(q+1)), \beta_j(\mathrm{St}_G(q)) \mid 0 \leq i < r \leq j < s \rangle^{\mathrm{Bas}_s(G)}.$$

This description allows us to provide an exact relationship between the Hausdorff dimension of a group G fulfilling the conditions of Theorem 1.1.4 and the related Basilica groups $\mathrm{Bas}_s(G)$. The precise description makes use of the *series of obstructions* of G, a tailor-made technical construction, see Section 1.4.2 for details. Observing this series, we prove that the Hausdorff dimension of $\mathrm{Bas}_s(G)$ is bounded below by the Hausdorff dimension of G for all $s > 1$.

Corollary 1.1.5. *Let $G \leq \mathrm{Aut}\, T$ be very strongly fractal, self-similar, weakly regular branch over K_{s-1}, with $\dim_{\mathrm{H}} G < 1$. Then for all $s > 1$*

$$\dim_{\mathrm{H}} G < \dim_{\mathrm{H}} \mathrm{Bas}_s(G).$$

Here we define the Hausdorff dimension of $G \leq \Gamma$ as the Hausdorff dimension of its closure in Γ, where Γ is the subgroup of all automorphisms acting locally by a power of a fixed m-cycle. This subgroup is isomorphic to

$$\Gamma \cong \varprojlim_{n \in \mathbb{N}_+} \mathrm{C}_m \wr \overset{n}{\cdots} \wr \mathrm{C}_m.$$

If $m = p$, a prime, then Γ is a Sylow pro-p subgroup of $\mathrm{Aut}(T)$. The notion of Hausdorff dimension in the profinite setting as above was initially studied by Abercrombie [1] and subsequently by Barnea and Shalev [8]. It is analogous to the Hausdorff dimension defined as usual over $\mathbb{R}$.

In the second half of this paper we restrict our attention to the class of *generalised Basilica groups* $\mathrm{Bas}_s(\mathcal{O}_m^d)$, for d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$, defined by applying Bas_s to the free abelian group of rank d with a self-similar action derived from the m-adic odometer. We remark that the above generalisation of the original Basilica group $\mathcal{B}$ is different from the one given in [15], but it includes the class of p-Basilica groups, where p is a prime, studied recently in [38]. For every odd prime p, we obtain the p-Basilica group by setting $d = 1, m = p$ and $s = 2$ in $\mathrm{Bas}_s(\mathcal{O}_m^d)$. Our construction also includes special cases, $d = 1$ and $m = s = p$, studied by Hanna Sasse in her master's thesis supervised by Benjamin Klopsch. We record the properties of the generalised Basilica groups in the following theorem.

Theorem 1.1.6. *Let d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$. Let $B = \mathrm{Bas}_s(\mathcal{O}_m^d)$ be a generalised Basilica group. The following assertions hold:*

(i) *B acts spherically transitively on the corresponding m-regular rooted tree,*
(ii) *B is self-similar and strongly fractal,*
(iii) *B is contracting, and has solvable word problem,*
(iv) *The group $\mathcal{O}_m^d$ is s-split, and $B^{\mathrm{ab}} \cong \mathbb{Z}^{ds}$,*
(v) *B is torsion-free,*
(vi) *B is weakly regular branch over its commutator subgroup,*
(vii) *B has exponential word growth.*

Theorem 1.1.6(i) to Theorem 1.1.6(vi) are obtained by direct application of Theorem 1.1.1 and Theorem 1.1.3. The proof of Theorem 1.1.6(vii) is analogous to that of the original Basilica group $\mathcal{B}$ and can easily be generalised from [73, Proposition 4]. Nevertheless, one can prove Theorem 1.1.6 directly by considering the action of the group on the corresponding rooted tree, see [135].

We explicitly compute the Hausdorff dimension of $\mathrm{Bas}_s(\mathcal{O}_m^d)$, which turns out to be independent of the rank d of the free abelian group $\mathcal{O}_m^d$:

Theorem 1.1.7. *For all d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$*

$$\dim_{\mathrm{H}}(\mathrm{Bas}_s(\mathcal{O}_m^d)) = \frac{m(m^{s-1} - 1)}{m^s - 1}.$$

The above equality agrees with the formula of the Hausdorff dimension of p-Basilica groups given by [38], and also with the Hausdorff dimension of the original Basilica group $\mathcal{B}$ given in [11].

Theorem 1.1.8. *Let d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$. The generalised Basilica group* $\mathrm{Bas}_s(\mathcal{O}_m^d)$ *admits an L-presentation*

$$L = \langle Y \mid Q \mid \Phi \mid R\rangle$$

where the data Y, Q, R and Φ are specified in Section 1.6.

The concrete L-presentation requires unwieldy notation, whence it is not given here. It is analogous to the L-presentation of the original Basilica group $\mathcal{B}$ given in [73]. The name L-presentation stands as a tribute to Igor Lysionok who obtained such a presentation for the Grigorchuk group in [105]. It is now known that, every finitely generated, contracting, regular branch group admits a finite L-presentation but it is not finitely presentable (cf. [10]). Unfortunately, this result is not applicable to generalised Basilica groups as they are merely weakly branch. Also, the L-presentation of the generalised Basilica group is not finite as the set of relations is infinite. Nonetheless, akin to [73, Proposition 11], we can introduce a set of endomorphisms of the free group on the set of generators of the generalised Basilica group and obtain a finite L-presentation, see Definition 1.6.1, as defined in [10].

Using the concrete L-presentation of a generalised Basilica group, we obtain the following structural result.

Theorem 1.1.9. *Let d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$ and let B be the generalised Basilica group* $\mathrm{Bas}_s(\mathcal{O}_m^d)$. *We have:*

(i) For $s = 2$, the quotient group $\gamma_2(B)/\gamma_3(B) \cong \mathbb{Z}^{d^2}$,
(ii) For $s > 2$, the quotient group $\gamma_2(B)/\gamma_3(B) \cong \mathrm{C}_m^{ds-2} \times \mathrm{C}_{m^2}$.

This implies that the quotients $\gamma_i(B)/\gamma_{i+1}(B)$ of consecutive terms of the lower central series of a generalised Basilica group for $s > 2$ are finite for all $i \geq 2$, whereas a similar behaviour happens for the original Basilica group $\mathcal{B}$ from $i \geq 3$, see [12] for details.

For a group G of automorphisms of an m-regular rooted tree, we say that G has the congruence subgroup property (CSP) if every subgroup of finite index in G contains some layer stabiliser in G. The congruence subgroup property of branch groups has been studied comprehensively over the years, see [18, 49, 54]. The generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ does not have the CSP as its abelianisation is isomorphic to $\mathbb{Z}^{ds}$ (Theorem 1.1.6). However, the quotients of $\mathrm{Bas}_s(\mathcal{O}_m^d)$ by the layer stabilisers are isomorphic to subgroups of $C_m \wr \overset{n}{\cdots} \wr C_m$, for suitable $n \in \mathbb{N}_0$. If $m = p$, a prime, then these quotients are, in particular, finite p-groups. The class of all finite p-groups is a well-behaved class, i.e., it is closed under taking subgroups, quotients, extensions and direct limits. In light of this, we prove that $\mathrm{Bas}_s(\mathcal{O}_p^d)$ has the p-congruence subgroup property (p-CSP), a weaker version of CSP introduced by Garrido and Uria-Albizuri in [55]. The group G has the p-CSP if every subgroup of index a power of p in G contains some layer stabiliser in G. In [55] one finds a sufficient condition for a weakly branch group to have the p-CSP and it is also proved that the original Basilica group $\mathcal{B}$ has the 2-CSP. This argument is generalised by Fernandez-Alcober, Di Domenico, Noce and Thillaisundaram to see that the p-Basilica groups have the p-CSP. We further generalise these result.

Theorem 1.1.10. *For all d, $s \in \mathbb{N}_+$ with $s > 2$, and all primes p, the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_p^d)$ has the p-congruence subgroup property.*

Even though we follow the same strategy as in [55], the arguments differ significantly because of Theorem 1.1.9. Here we make use of Theorem 1.1.4 to obtain a normal generating set for the layer stabilisers of the generalised Basilica groups (Theorem 1.5.1). We remark that the result of Fernandez-Alcober, Di Domenico, Noce and Thillaisundaram on p-Basilica groups can be generalised to all $d \geq 2$ with additional work.

The organisation of the paper is as follows: In Section 1.2, we introduce the basic theory of groups acting on rooted trees and give the formal definition of the Basilica operation, together with important examples. The proofs of Theorem 1.1.1 and Theorem 1.1.2 are given in Section 1.3. Theorem 1.1.3 and related results for s-split groups are contained in Section 1.4, as well as the proofs of Theorem 1.1.4 and Theorem 1.1.7. Section 1.6 contains the proof of Theorem 1.1.8, while Section 1.7 and Section 1.8 contain the proofs of Theorem 1.1.9 and Theorem 1.1.10.

Acknowledgements. — We carried out the work for this paper as PhD students under the supervision of Benjamin Klopsch at the Heinrich-Heine-Universität Düsseldorf. We thank him for introducing us to the subject and are grateful for his comments on earlier versions. We wish to express our thanks to Hanna Sasse for providing access to her Master thesis. We thank Alejandra Garrido and Anitha Thillaisundaram for helpful discussions, and we are grateful to the anonymous referee for the valuable suggestions.

1.2 — Preliminaries and main definitions

For any two integers i, j, let $[i, j]$ denote the interval in $\mathbb{Z}$. From here on, $T_m = T$ denotes the m-regular rooted tree for an arbitrary but fixed integer $m > 1$. The vertices of T are identified with the elements of the free monoid X^* on $X = [0, m-1]$ by labeling the vertices from left-to-right. We denote the empty word by ϵ. For $n \in \mathbb{N}_0$, the n^{th} *layer of* T is the set X^n of vertices represented by words of length n.

Every (graph) automorphism of T fixes ϵ and moreover maps the n^{th} layer to itself for all $n \in \mathbb{N}_0$. The action of the full group of automorphisms $\mathrm{Aut}(T)$ on each layer is transitive. A subgroup of $\mathrm{Aut}(T)$ with this property is called *spherically transitive.* The stabiliser of a word u under the action of a group G of automorphisms of T is denoted by $\mathrm{st}_G(u)$ and the intersection of all stabilisers of words of length n is called the n^{th} *layer stabiliser*, denoted $\mathrm{St}_G(n)$.

Let $a \in \operatorname{Aut}(T)$ and let u, v be words. Since layers are invariant under a, the equation

$$a(uv) = a(u)a|_u(v)$$

defines a unique automorphism $a|_u$ of T called the *section of a at u*. This automorphism can be thought of as the automorphism induced by a by identifying the subtrees of T rooted at the vertices u and $a(u)$ with the tree T. If G is a group of automorphisms, $G|_u$ will denote the set of all sections of group elements at u. The restriction of the action of the section $a|_u$ to $X^1 = X$ is called the *label of a at u* and it will be written as $a|^u$.

The following holds for all words u, v and all automorphisms a, b:

$$(a|_u)|_v = a|_{uv},$$
$$(ab)|_u = a|_{b(u)}b|_u.$$

The analogous identities hold for the labels $a|^u$, so the action of a on any word $x_0 \dots x_{n-1}$ of length n is given by

$$a(x_0 \dots x_{n-1}) = a|^\epsilon(x_0)a|_{x_0}(x_1 \dots x_{n-1}) = a|^\epsilon(x_0)a|^{x_0}(x_1) \dots a|^{x_0 \dots x_{n-2}}(x_{n-1}).$$

Hence every automorphism a is completely described by the label map $X^* \to \operatorname{Sym}(X)$, $u \mapsto a|^u$, called the *portrait of a*.

For $n \in \mathbb{N}_0$, the isomorphim

$$\psi_n : \operatorname{St}(n) \to (\operatorname{Aut}(T))^{m^n}, \ g \mapsto (g|_x)_{x \in X^n},$$

is called the *n^{th} layer section decomposition.* We will shorten the notation of big tuples arising for example in this way by writing g^{*k} for a sequence of k identical entries g in a tuple, implicitly ordering the vertices lexicographically.

We can uniquely describe an automorphism $g \in \operatorname{Aut}(T)$ by its label at ϵ and the first layer section decomposition of $(g|^\epsilon)^{-1}g$, i.e. by

$$g = g|^\epsilon \ (g|_x)_{x \in X}.$$

Let $H \leq \operatorname{Sym}(X)$ be any subgroup of the symmetric group on X. Then denote by $\Gamma(H)$ the subgroup of $\operatorname{Aut}(T)$ defined as

$$\Gamma(H) = \langle a \in \operatorname{Aut}(T) \mid \forall u \in T, a|^u \in H \rangle.$$

If H is a Sylow-p subgroup of $\operatorname{Sym}(X)$, then $\Gamma(H)$ is a Sylow-pro-p subgroup of $\operatorname{Aut}(T)$. We further fix $\sigma = (0\ 1\ \dots\ m-1) \in \operatorname{Sym}(X)$ and write Γ for $\Gamma(\langle \sigma \rangle)$.

A group $G \leq \operatorname{Aut}(T)$ is called *self-similar* if it is closed under taking sections at every vertex, i.e. if $G|_v \subseteq G$ for all $v \in T$. Self-similar groups correspond to certain automata modelling the behaviour of the section map: there is a state for every element $g \in G$, and an arrow $g \to g|_x$ labelled $x : g(x)$ for every $x \in X$ (for details see [110]).

We follow [147] in the terminology for the first three of the following self-referential properties, and add a fourth one: A group $G \leq \mathrm{Aut}(T)$ acting spherically transitively is called

(i) *fractal* if $\mathrm{st}_G(u)|_u = G$ for all $u \in T$.
(ii) *strongly fractal* if $\mathrm{St}_G(1)|_x = G$ for all $x \in X$.
(iii) *super strongly fractal* if $\mathrm{St}_G(n)|_u = G$ for all $n \in \mathbb{N}_0$ and $u \in X^n$.
(iv) *very strongly fractal* if $\mathrm{St}_G(n+1)|_x = \mathrm{St}_G(n)$ for all $n \in \mathbb{N}_0$ and $x \in X$.

Notice that for every group H acting regularly on X and $G \leq \Gamma(H)$ the properties (1) and (2) coincide. The following lemma will be of great use.

Lemma 1.2.1. *Let $G \leq \mathrm{Aut}(T)$ be fractal and self-similar, and let $x, y \in X$. For every $g \in G$ there exists an element $\tilde{g} \in G$ such that $\tilde{g}(x) = y$ and $\tilde{g}|_x = g$. Furthermore, if $H \leq G$ is any subgroup of G such that $H \times \{\mathrm{id}\} \times \cdots \times \{\mathrm{id}\} \leq \psi_1(K)$ for some normal subgroup $K \trianglelefteq G$, then $(H^G)^m \leq \psi_1(K)$.*

Proof. Since G is fractal, it is spherically transitive and in particular it is transitive on the first layer of T. Hence there exists some element $h \in G$ mapping x to y. Also because G is fractal and $h|_x \in G$ by self-similarity, there is some element $k \in \mathrm{st}_G(x)$ such that $k|_x = (h|_x)^{-1}g$. Now $\tilde{g} = hk$ fulfils both $\tilde{g}(x) = y$ and $\tilde{g}|_x = h|_x k|_x = g$.

Assume further that $H \leq G$ and $H \times \{\mathrm{id}\} \times \cdots \times \{\mathrm{id}\} \leq \psi_1(K)$ for $K \trianglelefteq G$. Let $g \in G$. Choose an element $\tilde{g} \in G$ such that $\tilde{g}(x) = 0$ and $\tilde{g}|_x = g$. Then for every $h \in H$

$$(\mathrm{id}^{*x}, h^g, \mathrm{id}^{*(m-x-1)}) = \psi_1((\tilde{g})^{-1}\psi_1^{-1}(h, \mathrm{id}, \dots, \mathrm{id})\tilde{g}) \in \psi_1((\tilde{g})^{-1}K\tilde{g}) = \psi_1(K). \qquad \square$$

From this point on, we fix a positive integer s.

Definition 1.2.2. There is a set of s interdependent monomorphims $\beta_i^s : \mathrm{Aut}(T) \to \mathrm{Aut}(T)$ defined by

$$\beta_i^s(g) = (\beta_{i-1}^s(g), \mathrm{id}, \dots, \mathrm{id}) \qquad \text{for } i \in [1, s-1],$$
$$\beta_0^s(g) = g|^\epsilon(\beta_{s-1}^s(g|_0), \dots, \beta_{s-1}^s(g|_{m-1})).$$

We adopt the convention that the subscript for these maps is taken modulo s, whence $\beta_i^s(g)|_x \in \beta_{i-1}^s(\mathrm{Aut}(T))$ for all $i \in [0, s-1]$ and $g \in \mathrm{Aut}(T)$. Whenever there is no reason for confusion, we drop the superscript s.

Definition 1.2.3. Let $G \leq \mathrm{Aut}(T)$. The s^{th} *Basilica group of* G is defined as

$$\mathrm{Bas}_s(G) = \langle \beta_i^s(g) \mid g \in G, i \in [0, s-1] \rangle.$$

Clearly, for $s = 1$ the homomorphism β_0^1 is the identity map and $\mathrm{Bas}_1(G) = G$. In the case of a self-similar group G, the s^{th} Basilica group of G can be equivalently defined as the self-similar closure of the group $\beta_0^s(G)$, i.e. the smallest self-similar group containing $\beta_0^s(G)$. If G is finitely generated by $g_1, \dots, g_r$, then $\mathrm{Bas}_s(G)$ is generated by $\beta_i^s(g_j)$ with $i \in [0, s-1]$ and $j \in [1, r]$.

The operation Bas_s is multiplicative in s, i.e. for $s, t \in \mathbb{N}_+$ and $G \leq \mathrm{Aut}(T)$ we have $\mathrm{Bas}_s \mathrm{Bas}_t(G) = \mathrm{Bas}_{st}(G)$. This is a consequence of

$$\beta_i^s(\beta_j^t(g)) = \beta_{i+sj}^{st}(g),$$

which is an easy consequence of Definition 1.2.2.

We now describe the monomorphisms β_i^s for $i \in [0, s-1]$ in terms of their portraits. We define a map $\omega_i : T \to T$. For every $k \in \mathbb{N}_0$ and every vertex $u \in X^k$, write $u = x_0 \dots x_{k-1} \in X^k$, and define

$$\omega_i(u) := 0^i \prod_{j=0}^{k-2} \left(x_j 0^{s-1}\right) x_{k-1}.$$

Writing $\omega_i(T)$ for the subgraph of T induced by the image of ω_i, with edges inherited from paths in T, we again obtain an m-regular rooted tree.

Lemma 1.2.4. *Let $g \in \mathrm{Aut}(T)$ and $i \in [0, s-1]$. Then the portrait of $\beta_i^s(g)$ is given by*

$$\beta_i^s(g)|^u = \begin{cases} g|^v, & \text{if } u = \omega_i(v), \\ \mathrm{id}, & \text{if } u \notin \omega_i(T). \end{cases}$$

In particular $\mathrm{Bas}_s(G) \leq \Gamma(H)$, if $G \leq \Gamma(H)$ for some $H \leq \mathrm{Sym}(X)$.

Proof. First suppose that $u = \omega_i(v)$ for $v = x_0 \dots x_{k-1}$. From Definition 1.2.2 follows

$$\beta_i^s(g)|^{\omega_i(x_0 \dots x_{k-1})} = \beta_0^s(g)|^{\omega_0(x_0 \dots x_{k-1})} = \beta_{s-1}^s(g|_{x_0})|^{\omega_{s-1}(x_1 \dots x_{k-1})},$$

and iteration establishes $\beta_i^s(g)|^u = g|^v$. Now, if $u = u_0 \dots u_{k-1} \notin \omega_i(T)$, there is some minimal number $n \not\equiv_s i$ such that $u_n \neq 0$. Thus $u = \omega_i(v) 0^t u_n \dots u_{k-1}$ for $n \equiv_s t < i$ and some vertex v, hence

$$\beta_i^s(g)|^u = \beta_i^s(g|_v)|^{0^t u_n \dots u_k} = \beta_{i-t}^s(g|_v)|^{u_n \dots u_k} = \mathrm{id}. \qquad \square$$

It is interesting to compare the effect of the Basilica operation with another method of deriving new self-similar groups from given ones described by Nekrashevych.

Proposition 1.2.5. *[110, Proposition 2.3.9] Let $G \leq \mathrm{Aut}(T)$ be a group and let d be a* *positive integer. There is a set of d injective endomorphisms of* $\mathrm{Aut}(T)$ *given by*

$$\pi_0(g) := g|^{\epsilon}\ (\pi_{d-1}(g|_x))_{x\in X},$$
$$\pi_i(g) := (\pi_{i-1}(g))_{x\in X} \text{ for } i \in [1, d-1].$$

The group $D_d(G) := \langle \pi_i(G) \mid i \in [0, d-1]\rangle$ is isomorphic to the direct product G^d.

We combine both constructions to define a class of groups very closely resembling the original Basilica group $\mathcal{B}$.

Definition 1.2.6. Let d, m, $s \in \mathbb{N}_+$ with $m \geq 2$. The *m-adic odometer* $\mathcal{O}_m$ is the infinite cyclic group generated by

$$a = \sigma(a, \mathrm{id}, \dots, \mathrm{id}),$$

where σ is the m-cycle $(m-1\ m-2\ \dots\ 1\ 0)$. Write $\mathcal{O}_m^d$ for $D_d(\mathcal{O}_m)$, the d-fold direct product of $\mathcal{O}_m$ embedded into $\mathrm{Aut}(T)$ by the construction described in Proposition 1.2.5. We call the group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ the *generalised Basilica group.*

Clearly, $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$ is the original Basilica group introduced by Grigorchuk and Żuk in [73].

For illustration we depict explicitly the automaton defining the self-similar action of the dyadic odometer $\mathcal{O}_2$, the automaton defining the action of $D_8(\mathcal{O}_2)$ described above and the automaton defining $\mathrm{Bas}_8(\mathcal{O}_2)$ in Fig. 1.4.

We shall prove in the following (cf. Section 1.6, Section 1.7, Section 1.8) that generalised Basilica groups resemble the original Basilica group in many ways, justifying the terminology.

Proposition 1.2.7. *Let* $\mathrm{Aut}_{\mathrm{fin}}(T)$ *be the group of all* finitary automorphisms, *i.e. the group generated by all automorphisms $g_{\tau,v}$ for $v \in T$, $\tau \in \mathrm{Sym}(X)$ that have label τ at v and trivial label everywhere else. For any $s \in \mathbb{N}_+$*

$$\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T)) = \mathrm{Aut}_{\mathrm{fin}}(T).$$

On the other hand $\mathrm{Bas}_s(\mathrm{Aut}(T))$ *is not of finite index in* $\mathrm{Aut}(T)$ *for all $s > 1$.*

Proof. Define for every $n \in \mathbb{N}_0$ a map $\mu_n : \mathrm{Aut}(T) \to \mathbb{N}_0$ by

$$\mu_n(g) = |\{u \in X^n \mid g|_u \neq \mathrm{id}\}|.$$

Lemma 1.2.4 shows that $g_{\tau,v} = \beta_i(g_{\tau,\omega_i^{-1}(v)}) \in \mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T))$ for every $v \in \bigcup_{i=0}^{s-1} \omega_i(T)$. Conjugation with suitable elements produces all other generators, hence $\mathrm{Aut}_{\mathrm{fin}}(T)$ is contained in $\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T))$. On the other hand, $\sum_{n\in\mathbb{N}_0} \mu_n(g) < \infty$ for any $g \in \mathrm{Aut}_{\mathrm{fin}}(T)$, implying that the same holds for all generators (and hence, all elements) of $\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T))$. Thus, $\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T)) = \mathrm{Aut}_{\mathrm{fin}}(T)$.

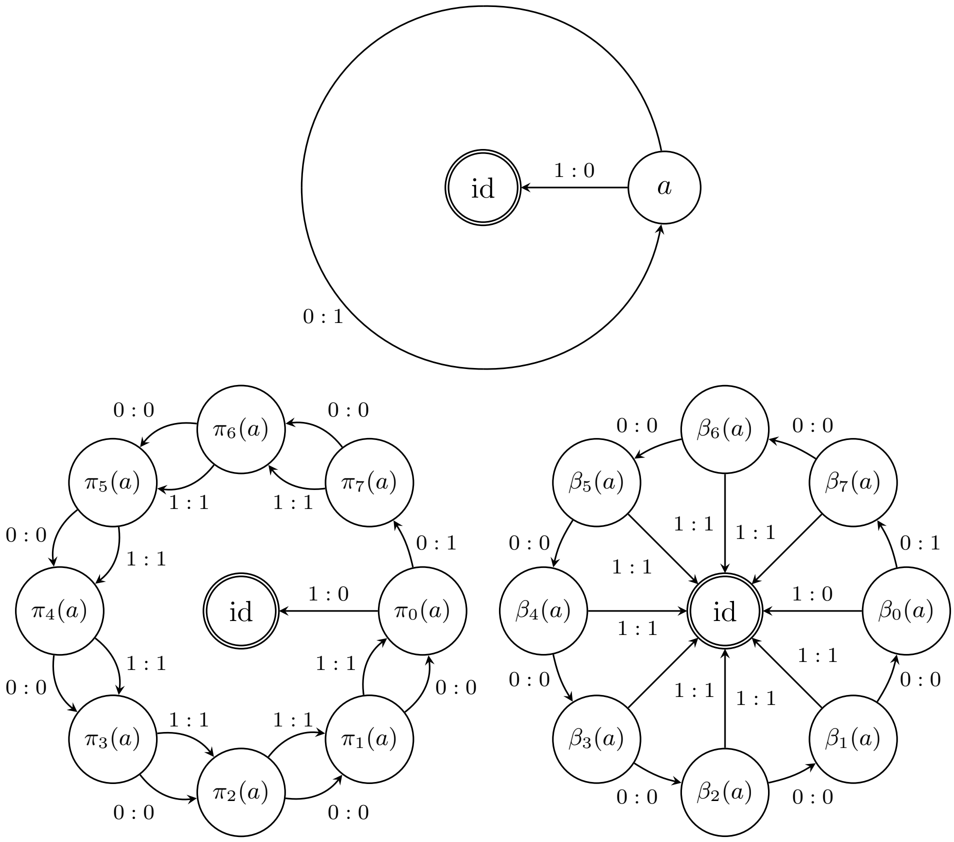


Figure 1.4: The automata defining the generators of $\mathcal{O}_2$, $D_8(\mathcal{O}_2)$ and $\mathrm{Bas}_8(\mathcal{O}_2)$.

For any $g \in \mathrm{Aut}(T)$ we have $\mu_n(g) \leq |X^n| = m^n$. But for all generators $\beta_i(g)$ of $\mathrm{Bas}_s(\mathrm{Aut}(T))$ the stronger inequality $\mu_{sn+i}(\beta_i(g)) \leq m^n$ holds, since $\beta_i(g)$ has trivial label at all vertices outside of $\omega_i(T)$. Let $g \in \mathrm{Aut}(T)$ and $q(g) \in \mathbb{Q}_+$ be the infimum of all numbers r such that

$$\limsup_{n\to\infty} \frac{\mu_{sn}(g)}{m^{(1+r)n}} = \infty.$$

Then g cannot be in $\mathrm{Bas}_s(\mathrm{Aut}(T))$, since the inequality $\mu_n(ab) \leq \mu_n(a) + \mu_n(b)$ for $a, b \in \mathrm{Aut}(T)$ implies that it cannot be a finite product of the generators of $\mathrm{Bas}_s(\mathrm{Aut}(T))$. By the same reason, all elements with different $q(g)$ are in different cosets. Since $q(\mathrm{Aut}(T)) = (0, s-1) \cap \mathbb{Q}$, the second statement follows. □

Question 1.2.8. *In view of Proposition 1.2.7 and the original Basilica group $\mathcal{B}$ it seems plausible that the operation* Bas_s *makes (in some vague sense) big groups smaller and small groups bigger. Let $H \leq \mathrm{Sym}(X)$ be a transitive subgroup. Write $\Gamma_{\mathrm{fin}}(H) = \mathrm{Aut}_{\mathrm{fin}}(T) \cap \Gamma(H)$. Replacing $\mathrm{Aut}_{\mathrm{fin}}(T)$ with $\Gamma_{\mathrm{fin}}(H)$ in the proof of Proposition 1.2.7 we obtain the equation $\mathrm{Bas}_s(\Gamma_{\mathrm{fin}}(H)) = \Gamma_{\mathrm{fin}}(H)$.*

Is there a group G not of the form $\Gamma_{\mathrm{fin}}(H)$ such that $\mathrm{Bas}_s(G) = G$?

1.3 — Properties inherited by Basilica groups

We recall our standing assumptions: m and s are positive integers with $m \neq 1$, $X = [0, m-1]$, and T the m-regular rooted tree. The subscript of the maps β_i^s is taken modulo s, and we will drop the superscript s from now on.

1.3.1. Self-similarity and fractalness. —

Lemma 1.3.1. *Let $G \leq \mathrm{Aut}(T)$ act spherically transitively on T. Then $\mathrm{Bas}_s(G)$ acts spherically transitively on T.*

Proof. It is enough to prove that for any number $n = qs + r \in \mathbb{N}_+$ with $r \in [0, s-1]$ and $q \geq 0$, and $y \in X$ there is an element $b \in \mathrm{Bas}_s(G)$ such that $b(0^n 0) = 0^n y$. Let $g \in G$ be such that $g(0^q 0) = 0^q y$ and observe that $\beta_r(g)$ stabilises 0^n. By Lemma 1.2.4 it follows

$$\beta_r(g)(0^n 0) = 0^n \beta_0(g|_{0^q})(0) = 0^n y.$$ □

Lemma 1.3.2. *Let $G \leq \mathrm{Aut}(T)$ be self-similar. Then $\mathrm{Bas}_s(G) \leq \mathrm{Aut}(T)$ is self-similar.*

Proof. We check that $\beta_i(g)|_v$ is a member of $\mathrm{Bas}_s(G)$ for all $v \in T$. This holds by Definition 1.2.2 for words v of length 1, and follows from $g|_x|_y = g|_{xy}$ by induction for words of any length. □

Lemma 1.3.3. *Let $G \leq \mathrm{Aut}(T)$ be self-similar, and fractal (resp. strongly fractal). Then*

(i) The group $B = \mathrm{Bas}_s(G) \leq \mathrm{Aut}(T)$ is fractal (resp. strongly fractal).

(ii) For all $b \in B$ there is an element $c \in \mathrm{st}_B(0)$ (resp. $c \in \mathrm{St}_B(1)$) such that $c|_0 = b$ and $c|_x \in \beta_{s-1}(G)$ for all $x \in [1, m-1]$.

Proof. Lemma 1.3.1 shows that B acts spherically transitively, and by Lemma 1.3.2 the group B is self-similar. First suppose that G is fractal. Since the statement (ii) implies the statement (i), it is enough to prove (ii).

Observe that

$$H = \{g \in \mathrm{st}_B(0) \mid g|_x \in \beta_{s-1}(G) \text{ for all } x \in [1, m-1]\}$$

is a subgroup since $h(x) \neq 0$ and $(gh)|_x = g|_{h(x)}h|_x \in \beta_{s-1}(G)$ for all $g, h \in H, x \in [1, m-1]$. Thus, it is enough to show that $\beta_i(G) \leq H|_0$ for all $i \in [0, s-1]$.

It is easy to see that $\beta_i(G) \leq H$ for $i \neq 0$, hence since $\beta_i(G)|_0 = \beta_{i-1}(G)$ we have $\beta_i(G) \leq H|_0$ for $i \neq s-1$. But also $\beta_0(\mathrm{st}_G(0)) \leq H$. Note that, since G is fractal, we have $\mathrm{st}_G(0)|_0 = G$. Hence $\beta_{s-1}(G) \leq \beta_0(\mathrm{st}_G(0))|_0 \leq H|_0$.

If G is strongly fractal, we may replace H by its intersection with $\mathrm{St}_B(1)$ and $\mathrm{st}_G(0)$ by $\mathrm{St}_G(1)$ to obtain a proof for the analogous statement. □

Lemma 1.3.1, Lemma 1.3.3 and Lemma 1.3.2 yield proofs for the statements (1), (2) and (3) of Theorem 1.1.1.

1.3.2. Amenability. — The original Basilica group $\mathcal{B}$ was the first example of an amenable, but not subexponentially amenable group. This had been conjectured already in [73], where non-subexponentially amenability of $\mathcal{B}$ was proven. Amenability was proven by Bartholdi and Virág in [20]. Later, Bartholdi, Kaimanovich and Nekrashevych proved that all groups generated from bounded finite-state automorphisms are amenable [14], which includes $\mathcal{B}$. We recall the relevant definitions and then apply the result of Bartholdi, Kaimanovich and Nekrashevych to a wider class of groups produced by the Basilica operation.

Definition 1.3.4. An automorphism $f \in \mathrm{Aut}(T)$ is called

(i) *finite-state* if the set $\{f|_u \mid u \in T\}$ is finite, and
(ii) *bounded* if the sequence $\mu_n(f) := |\{u \in X^n \mid f|_u \neq \mathrm{id}\}|$ is bounded.

Proposition 1.3.5. *Let $G \leq \mathrm{Aut}(T)$ be generated from finite-state bounded automorphisms. Then $\mathrm{Bas}_s(G)$ is also generated from finite-state bounded automorphisms.*

Proof. It is enough to prove that for every finite-state bounded $f \in \mathrm{Aut}(T)$ and $i \in [0, s-1]$ the element $\beta_i(f)$ is again finite-state and bounded. Notice that all sections of f are of the form $\beta_j(f|_u)$ for some $u \in T$, hence there are only finitely many candidates and $\beta_i(f)$ is finite-state. Moreover, by Definition 1.2.2 $\mu_n(\beta_i(f)) = \mu_{\lfloor \frac{n-i}{s} \rfloor}(f)$, bounding $\mu_n(\beta_i(f))$. □

This proves statement (6) of Theorem 1.1.1, and we use [14] to conclude:

Corollary 1.3.6. *Let $G \leq \mathrm{Aut}(T)$ be generated by finite-state bounded automorphisms. Then $\mathrm{Bas}_s(G)$ is amenable.*

1.3.3. Spinal groups. — A well-known class of subgroups of $\mathrm{Aut}(T)$ containing most known branch groups is the class of *spinal groups*, containing both the first and the second Grigorchuk group, and all GGS-*groups*. We use, with modifications for GGS-groups, the definition given in [13].

Definition 1.3.7 (Definition 2.1 of [13])**.** Let $R \leq \mathrm{Sym}(X)$, let D be a finite group and let

$$\omega = (\omega_{i,j})_{i\in\mathbb{N}_+, j\in[1,m-1]}$$

be a family of homomorphisms $\omega_{i,j} : D \to \mathrm{Sym}(X)$. Identify R with $\{r(\mathrm{id},\dots,\mathrm{id}) \mid r \in R\} \leq \mathrm{Aut}(T)$ and identify each $d \in D$ with the automorphism of T given by

$$d|^w := \begin{cases} \omega_{i,j}(d) & \text{if } w = 0^{i-1}j \text{ for } i \in \mathbb{N}_+, j \in [1, m-1], \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Suppose that the following holds:

(i) The group R and all groups $\langle \omega_{n,j}(D) \mid j \in [1, m-1]\rangle$, for $n \in \mathbb{N}_+$, act transitively on X.

(ii) For all $n \in \mathbb{N}_+$,

$$\bigcap_{i=n}^{\infty} \bigcap_{j=1}^{m-1} \ker \omega_{i,j} = 1.$$

Then $\langle R, D\rangle \leq \mathrm{Aut}(T)$ is called the *spinal group acting on T with defining triple* (R, D, ω). The spinal group with defining triple (R, D, ω) is called a GGS-*group acting on T* if $\omega_{n,j} = \omega_{k,j}$ for all $n, k \in \mathbb{N}_+$ and $j \in [1, m-1]$.

We now describe the Basilica groups of spinal groups. For this, we record the following lemma.

Lemma 1.3.8. *Let $i, j \in [0, s-1]$ with $i \neq j$. Denote by $\mathrm{st}(\overline{0})$ the stabiliser of the infinite ray $\overline{0} := \{0^i \mid i \in \mathbb{N}_0\}$ in $\mathrm{Aut}(T)$ (a so-called parabolic subgroup). Then*

$$[\beta_i(\mathrm{st}(\overline{0})), \beta_j(\mathrm{st}(\overline{0}))] = \mathrm{id}\,.$$

Proof. We prove that for all $g_0, g_1 \in \mathrm{st}(\overline{0})$ the images $b_0 = \beta_i(g_0)$ and $b_1 = \beta_j(g_1)$ commute, using the fact that $\mathrm{st}(\overline{0})|_0 = \mathrm{st}(\overline{0})$. Assume without loss of generality that either $j > i > 0$ or $i = 0$. In the first case both b_0 and b_1 stabilise the i^{th} layer, we can consider

$$\psi_i([b_0, b_1]) = ([b_0|_{0^i}, b_1|_{0^i}], \mathrm{id}^{*(m^i-1)}) = ([\beta_0(g_0), \beta_{j-i}(g_1)], \mathrm{id}^{*(m^i-1)}),$$

and thus reduce to the second case. Suppose now that $i = 0$. Since the only non-trivial first layer section of b_1 is at the vertex 0 and by assumption b_0 fixes this vertex,

$$\psi_1([b_0, b_1]) = ([b_0|_0, b_1|_0], \mathrm{id}^{*(m-1)}).$$

Since $b_0|_0, b_1|_0 \in \mathrm{st}(\overline{0})$, we conclude by infinite descent that $[b_0, b_1]$ fixes all vertices outside the ray $\overline{0}$, thus acts trivially on the entire tree T. □

The elements $d \in D$ of a spinal group defined by (R, D, ω) can be characterised by the fact that they stabilise the infinite ray (or "spine") $\overline{0}$ and $d|^x \neq \mathrm{id}$ implies that x has distance precisely 1 from $\overline{0}$. Therefore it is easy to see that a Basilica group $B = \mathrm{Bas}_s(G)$ of a spinal group G acting on T cannot act as a spinal group on T, as the elements $\beta_i^s(d)$ have non-trivial labels at vertices of distance s from the ray $\overline{0}$. However, the group B acts as a spinal group on a tree obtained from T by deletion of layers.

Motivated from Example 1.3.10 and 1.3.11 below, we introduce the following notations.

There is an injection $\iota_s : (X^s)^* \to X^*$ given by

$$(x_{0,0} \cdots x_{0,s-1}) \cdots (x_{n-1,0} \cdots x_{n-1,s-1}) \mapsto x_{0,0} \cdots x_{n-1,s-1},$$

whose image is the union $\bigcup_{n\in\mathbb{N}_0} X^{sn}$. The restriction map induces an injection

$$\iota_s^* : \mathrm{Aut}(X^*) \to \mathrm{Aut}((X^s)^*),$$

and clearly the image $\iota_s^*(\mathrm{Aut}(T))$ is

$$\Gamma(\mathrm{Sym}(X) \wr \cdots \wr \mathrm{Sym}(X)) \leq \mathrm{Aut}((X^s)^*),$$

where the permutational wreath product is iterated s times. Recall that $\Gamma(H)$ for a permutation group G denotes the subgroup of $\mathrm{Aut}(T)$ with every local action a member of H. Define for $i \in [0, s-1]$

$$\begin{aligned} \tau_i : \mathrm{Sym}(X) &\to \mathrm{Sym}(X) \wr \cdots \wr \mathrm{Sym}(X) \\ \rho &\mapsto \iota_s^*(g_{\rho,0^i})|^\epsilon, \end{aligned}$$

where $g_{\rho,0^i}$ is the automorphism with $g|^{0^i} = \rho$ and $g|^x = \mathrm{id}$ everywhere else. It is easy to see that for every transitive permutation group $H \leq \mathrm{Sym}(X)$ the group $\langle \tau_k(H) \mid k \in [0, s-1] \rangle$ is isomorphic to the s-fold iterated permutational wreath product $H \wr \cdots \wr H$.

Now given a family of homomorphisms $(\omega_{i,j} : D \to \mathrm{Sym}(X))_{i\in\mathbb{N}_+, j\in X\setminus\{0\}}$ we define a new family $\tilde{\omega} = (\tilde{\omega}_{i,j} : D^s \to \mathrm{Sym}(X^s))_{i\in\mathbb{N}_+, j\in X^s\setminus\{0^s\}}$ by

$$\tilde{\omega}_{n,j} = \begin{cases} \tau_i \circ \omega_{n,x} \circ \pi_i, & \text{if } j = 0^i x 0^{s-i-1} \text{ for some } x \in [1, m-1] \text{ and } i \in [0, s-1], \\ d \mapsto \mathrm{id}, d \in D^s, & \text{otherwise,} \end{cases}$$

where $\pi_i : D^s \to D$ denotes the projection to the $(i+1)^{\text{th}}$ factor.

Proposition 1.3.9. *Let G be the spinal group on T with defining triple (R, D, ω). Then $\iota_s^*(\mathrm{Bas}_s(G))$ is the spinal group on $(X^s)^*$ with defining triple $(R \wr \cdots \wr R, D^s, \tilde{\omega})$, by the action of $\mathrm{Bas}_s(G)$ on the m^s-regular tree $\delta_s T$ defined by the deletion of layers.*

If furthermore G is a GGS-group on T, $\iota_s^*(\mathrm{Bas}_s(G))$ is a GGS-group on $(X^s)^*$.

Proof. First consider the elements of the form $\beta_k(a)$, for $a \in R$, $k \in [0, s-1]$. On $(X^s)^*$ this element acts as $\tau_k(a)$. Since R is transitive, the images of R generate $R \wr \cdots \wr R$, and the first entry of the defining triple is described.

We deal in a similar way with the sections $\beta_i(d|_{0^k y})$ of a directed element for every $d \in D, i \in [0, s-1], k \in \mathbb{N}_0, y \in X \setminus \{0\}$. To obtain the first section decomposition of the action of $\beta_i(d|_{0^k})$ on $\delta_s T$ (which stabilises the first layer) we have to take sections of $\beta_i(d|_{0^k})$ at words $x = x_0 \dots x_{s-1}$ of length s in T. Now by Lemma 1.2.4,

$$\beta_i(d|_{0^k})|_x = \begin{cases} \beta_i(d|_{0^{k+1}}) & \text{if } x = 0^s, \\ \beta_i(\omega_{k+1,x_i}(d)) = \tau_i \omega_{k+1,x_i}(d) & \text{if } x = 0^i x_i 0^{s-i-1}, x_i \neq 0, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

By Lemma 1.3.8 all pairs $\beta_i(d_1), \beta_j(d_2)$ with $d_1, d_2 \in D$, $i, j \in [0, s-1]$ and $i \neq j$ commute. We identify $\beta_i(D)$ with the $(i+1)^{\text{th}}$ direct factor of D^s. Thus $\mathrm{Bas}_s(G)$ is generated by $R \wr \cdots \wr R$ and $\langle \beta_i(D) \mid i \in [0, s-1] \rangle \cong D^s$, where $(\mathrm{id}, \dots, \mathrm{id}, d_i, \mathrm{id}, \dots, \mathrm{id}) \in D^s$ acts on $\delta_s T$ by

$$(\mathrm{id}, \dots, \mathrm{id}, d_i, \mathrm{id}, \dots, \mathrm{id})|_{0^{ks} x} = \beta_i(d|_{0^k})|_x,$$

thus, the elements of D^s are defined by the family $\tilde{\omega}$ of homomorphisms.

It remains to establish the two defining properties of spinal groups. Property (1) holds by the observation that

$$\langle \tilde{\omega}_{i,j}(D^s) \mid j \in [1, m^s - 1] \rangle$$

acts as $\langle \tau_k(\omega_{i,j}(D)) \mid j \in [1, m-1], k \in [0, s-1] \rangle$, hence $\langle \tilde{\omega}_{i,j}(D^s) \mid j \in [1, m^s - 1] \rangle$ acts as the s-fold wreath product of $\langle \omega_{i,j}(D) \mid j \in [1, m-1] \rangle$, in particular, transitively on the first layer of $\delta_s T$.

For (2) consider

$$\ker \tilde{\omega}_{n,j} = \begin{cases} \ker(\omega_{n,x} \circ \pi_i), & \text{if } j = 0^i x 0^{s-i-1}, \text{ for some } x \in [1, m-1], i \in [0, s-1] \\ D^s, & \text{else,} \end{cases}$$

hence

$$\bigcap_{j \in X^s \setminus \{0^s\}} \ker \tilde{\omega}_{n,j} = \left(\bigcap_{j \in X \setminus \{0\}} \ker \omega_{n,j} \right) \times \cdots \times \left(\bigcap_{j \in X \setminus \{0\}} \ker \omega_{n,j} \right).$$

Therefore we see that since (2) holds for G, (2) holds for $\mathrm{Bas}_s(G)$.

The statement regarding GGS-groups follows directly from the description of the defining triple of $\mathrm{Bas}_s(G)$. □

Proposition 1.3.9 yields Theorem 1.1.2.

Example 1.3.10. One of the eponymous examples of a GGS-group is the family of the Gupta–Sidki p-groups acting on the p-adic tree. In the language of spinal groups they are defined by the triple

$$(\langle\sigma\rangle, \langle\sigma\rangle, (\sigma \mapsto \sigma, \sigma \mapsto \sigma^{-1}, \sigma \mapsto \mathrm{id}, \dots, \sigma \mapsto \mathrm{id})_{i\in\mathbb{N}_+}),$$

or in usual notation by the generators $a = \sigma(\mathrm{id}, \dots, \mathrm{id}), b = (b, a, a^{-1}, \mathrm{id}, \dots, \mathrm{id})$. We can describe the generators of the second Basilica group of the Gupta–Sidki 3-group $\ddot{\Gamma}$ by

$$\begin{aligned} \beta_0^2(a) &= \sigma(\mathrm{id}, \mathrm{id}, \mathrm{id}) = a & \beta_0^2(b) &= (\beta_1^2(b), \beta_1^2(a), \beta_1^2(a^{-1})), \\ \beta_1^2(a) &= (a, \mathrm{id}, \mathrm{id}) & \beta_1^2(b) &= (\beta_0^2(b), \mathrm{id}, \mathrm{id}). \end{aligned}$$

The automaton describing these generators is given explicitly in Fig. 1.3. By ordering X^2 reverse lexicographically, the action of the generators on $(X^2)^*$ is

$$\begin{aligned} \beta_0^2(a) &= (00\ 10\ 20)(01\ 11\ 21)(02\ 12\ 22) \\ \beta_0^2(b) &= (\beta_0^2(b), \beta_0^2(a), \beta_0^2(a)^{-1}, \mathrm{id}, \dots, \mathrm{id}) \\ \beta_1^2(a) &= (00\ 01\ 02) \\ \beta_1^2(b) &= (\beta_1^2(b), \mathrm{id}, \mathrm{id}, \beta_1^2(a), \mathrm{id}, \mathrm{id}, \beta_1^2(a)^{-1}, \mathrm{id}, \mathrm{id}). \end{aligned}$$

Example 1.3.11. The first Grigorchuk group $\mathcal{G}$ is the spinal group acting on the binary tree defined by $\mathrm{C}_2, \mathrm{C}_2^2$ and the sequence $\omega_{i,1}$ of (the three) monomorphisms $\mathrm{C}_2 \to \mathrm{C}_2^2$, where $\omega_{i,1} = \omega_{j,1}$ holds if and only if $i \equiv_3 j$. Writing a for the non-trivial rooted element and b, c, d for the non-trivial directed elements, one has the descriptions

$$a = (0\ 1)(\mathrm{id}, \mathrm{id}), \quad b = (c, a), \quad c = (d, a), \quad d = bc = (b, \mathrm{id}).$$

By Proposition 1.3.9 $\mathrm{Bas}_2(\mathcal{G})$ is a spinal group on the 4-adic tree $(X^2)^*$, generated by the elements

$$\begin{aligned} \alpha &:= \beta_0^2(a) &&= (0\ 2)(1\ 3), & \mathrm{A} &:= \beta_1^2(a) &&= (0\ 1), \\ \beta &:= \beta_0^2(b) &&= (\kappa, \alpha, \mathrm{id}, \mathrm{id}), & \mathrm{B} &:= \beta_1^2(b) &&= (\mathrm{K}, \mathrm{id}, \mathrm{A}, \mathrm{id}), \\ \kappa &:= \beta_0^2(c) &&= (\delta, \alpha, \mathrm{id}, \mathrm{id}), & \mathrm{K} &:= \beta_1^2(c) &&= (\Delta, \mathrm{id}, \mathrm{A}, \mathrm{id}), \\ \delta &:= \beta\kappa, & && \Delta &:= \mathrm{BK}, \end{aligned}$$

where we identify $[0, 3]$ with X^2 by the reverse lexicographic ordering.

1.3.4. Contracting groups. — For this subsection, we fix a self-similar group $G \leq \mathrm{Aut}(T)$ and a generating set S of G, which yields a natural generating set $\bigcup_{i\in[0,s-1]} \beta_i(S)$ for $B := \mathrm{Bas}_s(G)$.

The group $G \leq \mathrm{Aut}(T)$ is said to be *contracting*, if there exists a finite set $\mathcal{N} \subset G$ (called a *nucleus* of G) such that for all $g \in G$ there is an integer $k(g)$ such that $g|_v \in \mathcal{N}$ for all $v \in T$ with $|v| > k(g)$, where $|\cdot|$ denotes the word norm.

In this section we prove that a contracting group G has contracting Basilica groups $B = \mathrm{Bas}_s(G)$, considering the natural generating set for B. For this we define yet another length function, the *syllable length*, denoted by $\mathrm{syl}(b)$, of an element $b \in B$ as the word length w.r.t. the infinite generating set $\bigcup_{i\in[0,s-1]} \beta_i(G)$, i.e. as

$$\mathrm{syl}(b) := \min\{\ell \in \mathbb{N}_0 \mid b = \prod_{j=0}^{\ell-1} \beta_{i_j}(g_j),\ \text{with suitable } i_j \in [0,s-1], g_j \in G\},$$

where $\prod_{j=0}^{\ell-1} \beta_{i_j}(g_j)$ is a word representing b in B with respect to the generating set $\{\beta_i(g) \mid i \in [0,s-1], g \in G\}$. Consequently, we will call a non-trivial element of the given generating set a *syllable* and the corresponding index i its *type*. Since for every non-trivial element $b \in \beta_i(G)$ there is some $u \in X^{ns+i}$ for some $n \in \mathbb{N}_0$ such that $b|^u \neq \mathrm{id}$, while there is no $u \in T \setminus \bigcup_{n\in\mathbb{N}_0} X^{ns+i}$ such that $b|^u \neq \mathrm{id}$, the type of a syllable is unique. Since all sections of a syllable are either trivial or a syllable itself, the syllable length of a section of b is at most $\mathrm{syl}(b)$.

We further define for every $g \in \mathrm{Aut}(T)$,

$$\mathfrak{r}(g) := \begin{cases} \min\{n \in \mathbb{N}_0 \mid g|^{0^n}(0) \neq 0\} & \text{if } g \text{ does not stabilise } \overline{0} = \{0^n \mid n \in \mathbb{N}_0\}, \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 1.3.12. *Let* $r \in \mathbb{N}_0$. *Define*

$$\begin{aligned} D_r := \{\beta_{a_1}(h_1)\beta_{a_2}(h_2)\beta_{a_3}(h_3) \mid h_1, h_2, h_3 \in G\backslash\{1\}, \\ a_1, a_2, a_3 \in [0, s-1] \text{ such that } a_1 \neq a_2 \neq a_3, \\ \mathfrak{r}(\beta_{a_2}(h_2)) = r\}. \end{aligned}$$

Then $\mathrm{syl}(c|_u) < 3$ *for* $c \in D_r$ *and all* u *with* $|u| > r$.

Proof. Let $c = \beta_{a_1}(h_1)\beta_{a_2}(h_2)\beta_{a_3}(h_3) \in D_r$, where $a_1, a_2, a_3, h_1, h_2, h_3$ satisfy the conditions stated above. We use induction on r. First consider the case $r = 0$. From

$\beta_{a_2}(h_2)(0) \neq 0$ we deduce that $a_2 = 0$. Calculate, for $x \in [0, m-1]$,

$$c|_x = \begin{cases} \beta_{s-1}(h_2|_0)\beta_{a_3-1}(h_3) & \text{if } x = 0, \\ \beta_{a_1-1}(h_1)\beta_{s-1}(h_2|_x) & \text{if } x = h_2^{-1}(0), \\ \beta_{s-1}(h_2|_x) & \text{otherwise.} \end{cases}$$

This shows that $c|_x$ and, by recursion, $c|_u$ for all u with $|u| \geq 1$ have syllable length at most 2. Now we assume that $r > 0$. We may reduce to the case that $0 \in \{a_1, a_2, a_3\}$. If $0 \notin \{a_1, a_2, a_3\}$, $c|_0 \in D_{r-1}$ and $c|_x = \mathrm{id}$ for all $x \in X, x \neq 0$. Therefore, by induction $\mathrm{syl}(c|_{xu}) < 3$ for $x \in X$ and $|u| > r-1$, hence $\mathrm{syl}(x|_u) < 3$ for all $|u| > r$.

If $a_3 = 0 \neq a_1$, respectively, $a_1 = 0 \neq a_3$, we have

$$c|_x = \begin{cases} \beta_{a_1-1}(h_1)\beta_{a_2-1}(h_2)\beta_{s-1}(h_3|_x) \in D_{r-1} & \text{if } x = h_3^{-1}(0), \\ \beta_{s-1}(h_3|_x) & \text{otherwise,} \end{cases}$$

respectively,

$$c|_x = \begin{cases} \beta_{s-1}(h_1|_0)\beta_{a_2-1}(h_2)\beta_{a_3-1}(h_3) \in D_{r-1} & \text{if } x = 0, \\ \beta_{s-1}(h_1|_x) & \text{otherwise.} \end{cases}$$

Finally, if $a_1 = a_3 = 0$, we find

$$c|_x = \begin{cases} \beta_{s-1}(h_1|_0)\beta_{a_2-1}(h_2)\beta_{s-1}(h_3|_x) \in D_{r-1} & \text{if } x = h_3^{-1}(0), \\ \beta_{s-1}((h_1h_3)|_x) & \text{otherwise.} \end{cases}$$

In all three cases all but at most one section have length < 3 and the remaining section is contained in D_{r-1}, hence by induction $\mathrm{syl}(c|_{xu}) < 3$ for all $x \in X, |u| > r-1$.

The case $a_2 = 0$ remains. Now $r > 0$ implies $h_2^{-1}(0) = 0$ and we have $\mathfrak{r}(\beta_{s-1}(h_2|_0)) = r-1$. Thus

$$c|_x = \begin{cases} \beta_{a_1-1}(h_1)\beta_{s-1}(h_2|_0)\beta_{a_3-1}(h_3) \in D_{r-1} & \text{if } x = 0, \\ \beta_{s-1}(h_2|_x) & \text{otherwise.} \end{cases}$$

Hence we conclude that $\mathrm{syl}(c|_{xu}) < 3$ for all u with $|u| \geq 1$ by induction as before. □

Lemma 1.3.13. *For every element $b \in B$ with* $\mathrm{syl}(b) > s+1$ *there is a number $r \in \mathbb{N}_0$ such that for all sections $b|_u$ with $|u| > r$,*

$$\mathrm{syl}(b|_u) < \mathrm{syl}(b).$$

Proof. Let $b \in B$ be an element with $\mathrm{syl}(b) > s+1$. If b is minimally represented by a word w, it suffices to prove that there is a subword of w representing an element which has a reduction of the syllable length upon taking sections.

Since $\mathrm{syl}(b) > s+1$ there must be at least one syllable type appearing twice, and there is a subword of w that can be written in the form

$$\beta_i(\tilde{g}_1)b_0\beta_i(\tilde{g}_2)b_1 \text{ or } b_1\beta_i(\tilde{g}_1)b_0\beta_i(\tilde{g}_2),$$

where b_0, b_1 are non-trivial and contain neither two syllables of the same type nor a syllable of type i. Passing to the inverse if necessary we restrict to the first case.

Under the assumption of w being minimal it is impossible that both b_0 and $\beta_i(\tilde{g}_2)$ fix the infinite ray $\overline{0}$, since if they did, they would commute by Lemma 1.3.8, and consequently it would be possible to reduce the number of syllables.

Thus there are syllables in $b_0\beta_i(\tilde{g}_2)$ that do not stabilise the ray $\overline{0}$. Among these we choose k such that $r := \mathfrak{r}(\beta_{j_k}(g_k))$ is minimal.

Apply Lemma 1.3.12 to the subword $\beta_{j_{k-1}}(g_{k-1})\beta_{j_k}(g_k)\beta_{j_{k+1}}(g_{k+1})$ of $\beta_i(\tilde{g}_1)b_0\beta_i(\tilde{g}_2)b_1$ consisting only of the syllable $\beta_{j_k}(g_k)$ and its direct neighbours, and obtain for all $u \in T, |u| > r$

$$\mathrm{syl}(b|_u) < \mathrm{syl}(b).$$ □

Although interesting in its own right we use Lemma 1.3.13 solely to prove the following proposition.

Proposition 1.3.14. *Let $G \leq \mathrm{Aut}(T)$ be contracting. Then $B = \mathrm{Bas}_s(G)$ is contracting.*

Proof. Let $\mathcal{N}(G)$ be a nucleus of G. Define

$$\mathcal{N}(B) := \left\{ \prod_{i=0}^{\ell} \beta_{j_i}(g_i) \mid \ell \leq s+1, j_i \in [0, s-1], g_i \in \mathcal{N}(G) \right\}.$$

Since $\mathcal{N}(G)$ is a finite set, $\mathcal{N}(B)$ is finite as well. We will prove that it is a nucleus of B. Let $b \in B$. If $\mathrm{syl}(b) > s+1$, by Lemma 1.3.13 there is a layer, from which onwards all sections of b have syllable length $s+1$ or smaller.

Hence we can assume, that $\mathrm{syl}(b) \leq s+1$. Write $b = \prod_{i=0}^{\mathrm{syl}(b)-1} \beta_{j_i}(g_i)$. Since G is contracting, for every g_i there is a number $k(g_i)$ such that $g_i|_u \in \mathcal{N}(G)$ for all $|u| \geq k(g_i)$. Set $K := \max\{k(g_i) \mid i \in [0, \mathrm{syl}(b)-1]\}$, and observe that for u with $|u| \geq sK$ the section $b|_u$ is a product of at most $\mathrm{syl}(b) \leq s+1$ syllables of the form $\beta_i(g)$ with $g \in \mathcal{N}(G)$. Thus $b|_u$ is in $\mathcal{N}(B)$ and B is contracting. □

Proposition 1.3.14 proves statement (4) of Theorem 1.1.1.

As a consequence, the word problem for Basilica groups of self-similar and contracting groups is solvable, since it is solvable for self-similar and contracting groups [110, Proposition 2.13.8].

Corollary 1.3.15. *Let G be self-similar and contracting. Then $\mathrm{Bas}_s(G)$ has solvable word problem.*

Question 1.3.16. *Let $G \leq \mathrm{Aut}(T)$ be contracting. The fact that $\mathrm{Bas}_s(G)$ is contracting implies the existence of constants $\lambda < 1, L, C \in \mathbb{R}_+$ such that for every $g \in G$, $u \in X^n$ with $n > L$ it holds*

$$|g|_u| < \lambda |g| + C.$$

In [73] one set of constants is given for the original Basilica group $\mathcal{B}$, namely $\lambda = \frac{2}{3}$ and $L = C = 1$.

Is there a general formula for the above constants valid for all contracting groups and their Basilica groups, yielding $\lambda = \frac{2}{3}$ for $\mathcal{B}$?

1.3.5. Word growth. — We now provide some examples of the possible growth types of Basilica groups. It is known that the original Basilica group $\mathcal{B}$ has exponential word growth, cf. [73, Proposition 4]. The same proof as the one given there also shows that $\mathrm{Bas}_2(\mathcal{O}_m)$ is of exponential growth for all $m \geq 2$. This, however, is not a general phenomenon.

Proposition 1.3.17. *Let $a = (0\ 1)(a, \mathrm{id})$ be the generator of the dyadic odometer acting on the binary rooted tree. Then $\mathrm{Bas}_s(\langle(\mathrm{id}, a)\rangle)$ is a free abelian group of rank s, and is of polynomial growth in particular.*

Proof. The element (id, a) stabilises the ray $\overline{0}$, thus by Lemma 1.3.8 we have

$$[\beta_i(\langle(\mathrm{id}, a)\rangle), \beta_j(\langle(\mathrm{id}, a)\rangle)] = \mathrm{id}$$

for distinct i, $j \in [0, s-1]$. Also $\beta_i(\langle(\mathrm{id}, a)\rangle) \cong \mathbb{Z}$ for all $i \in [0, s-1]$. □

As another example, we prove that there is a group of intermediate word growth such that its second Basilica group has exponential word growth.

Proposition 1.3.18. *Let $G = \langle a = (1\ 2\ 3), b = (a, 1, b)\rangle$ be the Fabrykowski–Gupta group [43] acting on the ternary rooted tree, which is of intermediate growth according to [16]. Then there exists an element $f \in \mathrm{Aut}(T)$ such that the group $\mathrm{Bas}_2(G^f)$ is of exponential growth.*

Proof. The Fabrykowski-Gupta group is a GGS-group. In contrast to the Gupta–Sidki 3-group it is not periodic: an example for an element of infinite order is ab, for which the relation

$$(ab)^3 = (ab, ba, ba)$$

holds. In view of the decomposition it is clear that ab acts spherically transitively on T and thus by a result of Gawron, Nekrashevych and Sushchansky [56] it is $\mathrm{Aut}(T)$-conjugate to the 3-adic odometer group. Let $f \in \mathrm{Aut}(T)$ be an element such that $(ab)^f = (1\ 2\ 3)((ab)^f, 1, 1)$. Then the subgroup generated by $\beta_0((ab)^f)$ and $\beta_1((ab)^f)$ in $\mathrm{Bas}_2(G^f)$ is isomorphic to the generalised Basilica group $\mathrm{Bas}_2(\mathcal{O}_3)$, which is of exponential growth by following the proof of [73, Proposition 4] (which is the same result for $\mathcal{B}$) replacing the 2-cycle with a 3-cycle corresponding to $a|^{\epsilon}$. □

The same idea can be used to obtain the following proposition.

Proposition 1.3.19. *Let $G \leq \mathrm{Aut}(T)$ be a group containing an element acting spherically transitively on T. Then there is an $\mathrm{Aut}(T)$-conjugate G^f of G such that $\mathrm{Bas}_s(G^f)$ has exponential word growth.*

1.3.6. Weakly branch groups. — For every vertex $v \in T$ the *rigid vertex stabiliser of v in G* is the subgroup of all elements that fix all vertices outside the subtree rooted at v. For every $n \in \mathbb{N}_0$ the n^{th} *rigid layer stabiliser* $\mathrm{Rist}_G(n)$ is the normal subgroup generated by all rigid vertex stabilisers of n^{th} layer vertices. A group $G \leq \mathrm{Aut}(T)$ is called a *weakly branch group*, if G acts spherically transitively and all rigid layer stabilisers $\mathrm{Rist}_G(n)$ are non-trivial. If there is a subgroup $H \leq G$ such that $\psi_1(\mathrm{St}_H(1)) \geq H \times \cdots \times H$, the group G is said to be *weakly regular branch over H*. Clearly, a group that is weakly regular branch group over a non-trivial subgroup is a weakly branch group.

From Lemma 1.2.4, it follows that elements of the rigid layer stabilisers of G translate to elements of rigid layer stabilisers of $\mathrm{Bas}_s(G)$.

Lemma 1.3.20. *Let $n = qs + r \in \mathbb{N}_0$, with $r \in [0, s-1]$ and $q \geq 0$. Let $B = \mathrm{Bas}_s(G)$ for $G \leq \mathrm{Aut}(T)$. Then $\mathrm{Rist}_B(n)$ contains $\beta_i(\mathrm{Rist}_G(q+1))$ and $\beta_j(\mathrm{Rist}_G(q))$ for $0 \leq i < r$ and for $r \leq j < s$.*

We immediately obtain the following proposition.

Proposition 1.3.21. *Let $G \leq \mathrm{Aut}(T)$ be a weakly branch group. Then $B := \mathrm{Bas}_s(G)$ is again weakly branch.*

This proves the statement (5) of Theorem 1.1.1.

The group $\mathrm{Bas}_s(G)$ can be weakly branch even when G is not weakly branch. We recall that for any group G and an abstract word ω on k letters, the set of ω-elements and the verbal subgroup associated to ω are

$$G_\omega := \{\omega(h_0, \dots, h_{k-1}) \mid h_0, \dots, h_{k-1} \in G\} \text{ and } \omega(G) := \langle G_\omega \rangle \text{ respectively.}$$

Proposition 1.3.22. *Let $G \leq \mathrm{Aut}(T)$ be a self-similar strongly fractal group and let $B := \mathrm{Bas}_s(G)$. Let ω be a law in G, i.e. a word ω such that $\omega(G) = 1$, but let ω not be a law in B. Then B is weakly regular branch over $\omega(B)$.*

Proof. Let $b = \omega(b_0, \dots, b_{k-1}) \neq \mathrm{id}$ with $b_i \in B$ for $i \in [0, k-1]$. By Lemma 1.3.3 there are elements $c_i \in \mathrm{St}_B(1)$ such that $c_i|_0 = b_i$ and $c_i|_x \in \beta_{s-1}(G)$ for all $x \in X \setminus \{0\}$.

For every $x \in X$, let $d_x \in B$ be an element such that $d_x|_x = \mathrm{id}$ and $d_x(x) = 0$ (cf. Lemma 1.2.1). Then $c_i^{d_x}$ stabilises the first layer and has sections $c_i^{d_x}|_x = b_i$ and $c_i^{d_x}|_y = (c_i|_{d_x(y)})^{d_x|_y} \in \beta_{s-1}(G)^{d_x|_y}$ for $y \neq x$.

Since $c_i^{d_x}$ stabilises the first layer, the section maps are homomorphisms and

$$\omega(c_0^{d_x},\ldots,c_{k-1}^{d_x})|_y = \omega(c_0^{d_x}|_y,\ldots,c_{k-1}^{d_x}|_y) = \begin{cases} b, & \text{if } y = x \\ \mathrm{id} & \text{else,} \end{cases}$$

because in the second case we are evaluating ω in a group isomorphic to G. This shows that $B_\omega \times \cdots \times B_\omega$ is geometrically contained in B_ω, and thus the same holds for the verbal subgroups that are generated by these sets. □

We point out that, if ω is a law in B, then B cannot be weakly branch as it satisfies an identity. Proposition 1.3.22 allows to obtain examples of groups that are weakly branch over some prescribed verbal subgroup. We provide an easy example:

Example 1.3.23. The group $D := \langle \sigma, b\rangle$, with $\sigma = (0\ 1)$ and $b = (b, \sigma)$, acting on the binary tree is isomorphic to the infinite dihedral group (hence metabelian). It is self-similar and strongly fractal. Considering

$$[[\beta_1(\sigma),\beta_0(\sigma)],[\beta_0(\sigma),\beta_0(\sigma b)]] = ([\beta_0(\sigma),\beta_1(\sigma b)],[\beta_0(\sigma),\beta_1(b^{-1}\sigma)]) \neq \mathrm{id},$$

we see that the second Basilica $\mathrm{Bas}_2(D)$ is not metabelian, and thus it is weakly branch over the second derived subgroup of $\mathrm{Bas}_2(D)$.

1.4 — Split groups, layer stabilisers and Hausdorff dimension

The subgroup $\beta_i(G) \leq \mathrm{Bas}_s(G)$, for $i \in [0, s-1]$, has the property that its elements have non-trivial portrait only at vertices at levels $n \equiv_s i$ for $n \in \mathbb{N}_0$.

We consider an algebraic analogue of this property that will be used to determine the structure of the stabilisers of $\mathrm{Bas}_s(G)$.

Definition 1.4.1. Let $G \leq \mathrm{Aut}(T)$ and $B := \mathrm{Bas}_s(G)$. Define:

$$S_i := \langle \beta_j(G) \mid j \neq i\rangle \leq B \text{ and } N_i := (S_i)^B \trianglelefteq B.$$

We write $\phi_i : B \to B/N_i$ for the canonical epimorphism with kernel N_i. The quotient B/N_i is isomorphic to the quotient of G by the normal subgroup $K_i := \beta_i^{-1}(\beta_i(G) \cap N_i)$. We call K_i the i^{th} *splitting kernel* of G. The group G is called *s-split* if its s^{th} Basilica group B is a split extension of N_i by $\beta_i(G)$ for all $i \in [0, s-1]$, or equivalently if all splitting kernels of G are trivial.

Proposition 1.4.2. *Let $G \leq \mathrm{Aut}(T)$ be a group that does not stabilise the vertex* 0. *Then $\beta_i([G,G]) \leq N_i$ for $i \in [1, s-1]$. In particular, an s-split group (for $s > 1$) is abelian.*

Proof. Let $g, h \in G$, $k \in G\backslash \mathrm{st}(0)$ and let $i \in [1, s-1]$. Write $\gamma = \beta_{i-1}(g), \eta = \beta_{i-1}(h), \overline{\gamma} =$

$\beta_i(g), \overline{\eta} = \beta_i(h)$ and $\kappa = \beta_0(k)$. Then

$$\begin{aligned}\kappa^{-1}(\kappa)^{\overline{\gamma}^{-1}}(\kappa^{-1})^{\overline{\gamma}^{-1}\overline{\eta}}(\kappa)^{\overline{\eta}}|_x &= \kappa^{-1}|_{\kappa(x)}\overline{\gamma}|_{\kappa(x)}\kappa|_x(\overline{\gamma}^{-1}\overline{\eta}^{-1}\overline{\gamma})|_x\kappa^{-1}|_{\kappa(x)}\overline{\gamma}^{-1}|_{\kappa(x)}\kappa|_x\overline{\eta}|_x \\ &= \kappa|_x^{-1}\overline{\gamma}|_{\kappa(x)}\kappa|_x(\overline{\gamma}^{-1}\overline{\eta}^{-1}\overline{\gamma})|_x\kappa|_x^{-1}\overline{\gamma}|_{\kappa(x)}^{-1}\kappa|_x\overline{\eta}|_x \\ &= \begin{cases}[\gamma,\eta] & \text{if } x = 0,\\ \mathrm{id} & \text{otherwise.}\end{cases}\end{aligned}$$

Thus $\kappa^{-1}(\kappa)^{\overline{\gamma}^{-1}}(\kappa^{-1})^{\overline{\gamma}^{-1}\overline{\eta}}(\kappa)^{\overline{\eta}} = ([\gamma,\eta], \mathrm{id}, \dots, \mathrm{id}) = [\overline{\gamma}, \overline{\eta}]$ is an element of $N_i \cap \beta_i(G)$. □

We remark that $[G,G] \leq K_0$ does not necessarily hold. For example, consider a group G such that $[G,G] \not\leq \mathrm{St}_G(1)$. Since $N_0 \leq \mathrm{St}_{\mathrm{Bas}_s(G)}(1)$, the zero$^{\text{th}}$ splitting kernel cannot contain $[G,G]$.

Definition 1.4.3. We call a subgroup H of a group G *non-absorbing in* G if for all $h_0, \dots, h_{m-1} \in H$ such that $\psi_1^{-1}(h_0, \dots, h_{m-1}) \in G$, implies $\psi_1^{-1}(h_0, \dots, h_{m-1}) \in H$. If G is weakly branch over H, then H is non-absorbing in G.

Proposition 1.4.4. *Let $G \leq \mathrm{Aut}(T)$ be self-similar and such that $G|^\epsilon$ acts regularly on X. Assume that $[G,G]$ is non-absorbing in G. Then for $i \in [1, s-1]$ we have $K_i = [G,G]$, and $K_0 \leq [G,G]$. In particular, if G is abelian, it is s-split for all $s \in \mathbb{N}_+$.*

Proof. The inclusion $[G,G] \leq K_i$ for $i \in [1, s-1]$ is proven in Proposition 1.4.2. Thus we prove $K_i \leq [G,G]$ for $i \in [0, s-1]$.

Set $B \coloneqq \mathrm{Bas}_s(G)$ and define $\mathcal{N} \coloneqq \bigcup_{i=0}^{s-1}(\beta_i(G) \cap N_i)$. We employ the decomposition in syllables, cf. Section 1.3.4. For every $b \in \mathcal{N}$ there is an index $i \in [0, s-1]$ such that b can be written both as an element of the image of some β_i and a word in N_i, i.e.

$$b = \beta_i(g_0) = \prod_{j=1}^{\ell(b)} (h_j)^{\beta_i(g_j)} \tag{$*$}$$

for suitable $\ell(b) \in \mathbb{N}_0$, $g_j \in G$ and $h_j \in S_i$. The minimal possible value of $\ell(b)$ is called the *restricted syllable length*, and from here onwards we use the symbol ℓ for this invariant. Write $\mathcal{C} = \bigcup_{i=0}^{s-1} \beta_i([G,G])$ (notice that this a union of subsets with pairwise trivial intersection), and define

$$\mathcal{M} \coloneqq \{b \in \mathcal{N} \setminus \mathcal{C} \mid \ell(b) \leq \ell(c) \text{ for all } c \in \mathcal{N} \setminus \mathcal{C}\},$$

the set of all non-commutator elements with minimal restricted syllable length.

We shall prove that for every $b \in \mathcal{M}$ there exists a first level vertex $x_i \in X$ such that:

(i) $b|_{x_i} \in \mathcal{M}$ and
(ii) $b|_x = \mathrm{id}$ for all $x \in X \setminus \{x_i\}$.

Furthermore we prove that

(iii) $b \in \mathrm{St}_B(1)$, i.e. $\mathcal{M} \leq \mathrm{St}_B(1)$.

Every subset $\mathcal{M} \subseteq \mathrm{Aut}(T)$ with these properties is empty. Indeed, if $b \in \mathcal{M}$, there is some vertex $u \in T$ such that $b|^u \neq \mathrm{id}$, since b is not trivial. But by properties (i) and (ii) $b|_u$ is either trivial or a member of $\mathcal{M}$, hence by property (ii) stabilises the first layer, a contradiction.

But if $\mathcal{M}$ is empty, $\mathcal{N}$ is contained in $\mathcal{C}$, hence all splitting kernels are subgroups of $[G,G]$, finishing the proof.

Assume that there is some $b \in \mathcal{M}$. We fix the decomposition and the type given by $(*)$, but write ℓ for $\ell(b)$ to shorten the notation.

We first observe that $\ell \neq 1$. If $\ell = 1$, we have $\beta_i(g_0) = h_1^{\beta_i(g_1)}$, consequently $h_1 \in \beta_i(G) \cap S_i$. But $h_1|^u = \mathrm{id}$ for all u with $|u| \equiv_s i$, while $\beta_i(G)|^u = \{\mathrm{id}\}$ for $u \notin \omega_i(T)$ by Lemma 1.2.4. Thus $h_1 = \mathrm{id} = b \notin \mathcal{M}$, which is a contradiction.

We split the proof of statements (1) to (3) into two cases: $i = 0$ and $i \neq 0$.

Case $i = 0$: Since $N_0 \leq \mathrm{St}_B(1)$, statement (iii) is fulfilled. We have $S_0|_0 = S_{s-1}$ and $S_0|_x = \{\mathrm{id}\}$ for $x \in X \setminus \{0\}$. Also $\beta_0(G)|_x \leq \beta_{s-1}(G|_x)$ for $x \in X$, hence $N_0|_x \leq N_{s-1}$. Thus all sections $b|_x$ are members of $\beta_{s-1}(G) \cap N_{s-1} \subseteq \mathcal{N}$.

The first layer sections of b are given by

$$b|_x = \beta_{s-1}(g_0|_x) = \prod_{j \in L_x} (h_j|_0)^{\beta_{s-1}(g_j|_x)}, \qquad \text{for } x \in X,$$

where $L_x = \{j \mid 1 \leq j \leq \ell \text{ and } g_j(x) = 0\}$. The sum $\sum_{x \in X} |L_x|$ equals ℓ. By the minimality of ℓ, either all sections of b are contained in $\beta_{s-1}([G,G])$, or there is some $x_i \in X$ such that $\ell(b|_{x_i}) = |L_{x_i}| = \ell$. In the first case, since $[G,G]$ is non-absorbing in G, this implies $b \in \beta_0([G,G])$, a contradiction. In the second case, $L_x = \emptyset$ for $x \neq x_i$, i.e. $b|_x = \mathrm{id}$ for $x \neq x_i$. This proves statement (ii). Furthermore, if $b|_{x_i} \notin \mathcal{M}$, it is contained in $\beta_{s-1}([G,G])$. Since $[G,G]$ is non-absorbing over G, this implies $b \in \beta_0([G,G])$. Thus $b|_{x_i} \in \mathcal{M}$, and statement (i) is true.

Case $i \neq 0$: Recall that $b|_x = \beta_i(g_0)|_x = \mathrm{id}$ for $x \neq 0$. This is statement (ii) with $x_i = 0$.

We consider the first layer sections of b. For $x \in X$ and $1 \leq j \leq \ell$,

$$h_j^{\beta_i(g_j)}|_x = \begin{cases} (h_j|_x)^{\beta_{i-1}(g_j)} & \text{if } x = 0 \text{ and } h_j \in \mathrm{st}_B(0), \\ h_j|_x \beta_{i-1}(g_j) & \text{if } x = 0 \text{ and } h_j \notin \mathrm{st}_B(0), \\ \beta_{i-1}(g_j^{-1}) h_j|_x & \text{if } h_j \notin \mathrm{st}_B(0) \text{ and } x = h_j^{-1}(0), \\ h_j|_x & \text{otherwise.} \end{cases} \qquad (\dagger)$$

Since $G|^\epsilon$ acts regularly, $\mathrm{st}_B(0) = \mathrm{St}_B(1)$. We divide the long product in $(*)$ into segments that stabilise the first layer: Let $x \in X$, and consider the subsequence $(j_x^{(k)})_{k \in [1,t_x]}$ of $[1,\ell]$

consisting of all indices $j_x^{(k)}$ such that $(\prod_{j=j_x^{(k)}}^{\ell} h_j)(x) = 0$. Clearly $\sum_{x\in X} t_x = \ell$.

Set $j_x^{(0)} = 1$ and $j_x^{(t_x+1)} = \ell + 1$. Then $\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} h_j \in \mathrm{St}_B(1)$ for all $k \in [1, t_x]$, and one may write

$$b = \prod_{k=0}^{t_x} \prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} (h_j)^{\beta_i(g_j)}. \tag{‡}$$

We now make another case distinction.

Subcase $t_x = \ell$ for some $x \in X \setminus \{0\}$: We will prove that this case cannot occur. The equation $t_x = \ell$ implies $h_\ell(x) = 0$ and $h_j \in \mathrm{St}_B(1)$ for all $j \in [1, \ell - 1]$. We may assume $g_\ell = \mathrm{id}$, by passing to a conjugate if necessary. Looking at the second and fourth case of (†), we obtain

$$\beta_{i-1}(g_0) = b|_0 = \prod_{j=1}^{\ell-1} (h_j|_{h_\ell(0)}) \cdot h_\ell|_0 \in N_{i-1}.$$

Thus $\beta_{i-1}(g_0)$ is an element of $\mathcal{N}$ of restricted syllable length at most 1, hence trivial. Consequently g_0 and b are trivial, a contradiction.

Subcase $t_0 = \ell$: This implies $h_j \in \mathrm{St}_B(1)$ for all $j \in [1, \ell]$, and statement (iii) holds. By the first case of (†)

$$b|_0 = \prod_{j=1}^{\ell(b)} h_j|_0{}^{\beta_{i-1}(g_j)} \in N_{i-1} \cap \beta_{i-1}(G),$$

which is of restricted syllable length at most ℓ. As we previously argued in the case $i = 0$, we have $b|_0 \notin \beta_{i-1}([G, G])$ and consequently statement (i) holds, since otherwise $b \in \beta_i([G, G])$ because $[G, G]$ is non-absorbing over G.

Subcase $t_x < \ell$ for all $x \in X$: We shall prove that this case cannot occur. Combining (‡) with (†) for $x \in X$ we calculate

$$b|_x = \prod_{k=0}^{t_x-1} \left(\left(\Pi_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} (h_j)^{\beta_i(g_j)} \right) |_0 \right) \left(\Pi_{j=j_x^{(t_x)}}^{\ell(b)} (h_j)^{\beta_i(g_j)} \right) |_x$$

and for $k \in [1, t_x - 1]$

$$\begin{aligned} \prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} (h_j)^{\beta_i(g_j)}|_0 &= \beta_{i-1}(g_{j_x^{(k)}}^{-1})(\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} h_j|_{\prod_{i=j+1}^{j_x^{(k+1)}-1} h_i(0)})\beta_{i-1}(g_{j_x^{(k+1)}-1}) \\ &= \beta_{i-1}(g_{j_x^{(k)}}^{-1} g_{j_x^{(k+1)}-1})(\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} h_j|_{\prod_{i=j+1}^{j_x^{(k+1)}-1} h_i(0)})^{\beta_{i-1}(g_{j_x^{(k+1)}-1})}. \end{aligned}$$

Consequently, every segment $\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1}(h_j)^{\beta_i(g_j)}$ of b contributes at most one syllable of N_{i-1} and a member of $\beta_{i-1}(G)$ to $b|_x$. We obtain

$$b|_x \equiv_{N_{i-1}} \begin{cases} \beta_{i-1}\left(g_1^{-1}\prod_{k=1}^{t_x}\left(g_{j_x^{(k)}-1}g_{j_x^{(k)}}^{-1}\right)g_\ell\right) & \text{if } x=0,\\ \beta_{i-1}\left(\qquad\prod_{k=1}^{t_x}\left(g_{j_x^{(k)}-1}g_{j_x^{(k)}}^{-1}\right)\qquad\right) & \text{otherwise.}\end{cases}$$

Write $b|_x = \beta_{i-1}(f_x)n_x$ with $n_x \in N_i$ and f_x equal to the corresponding product in G in the last equation. Since the subsequences form a partition, every $\beta_{i-1}(g_{j_x^{(k)}})$ and its inverse appear in precisely one section of b, and we have

$$\prod_{x\in X} b|_x \equiv_{N_{i-1}} \prod_{x\in X}\beta_{i-1}(f_x) \equiv_{\beta_{i-1}([G,G])} \prod_{j=1}^{\ell}\beta_{i-1}(g_jg_j^{-1}) = 1.$$

Now we look at n_x. Since every segment $\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1}(h_j)^{\beta_i(g_j)}$ contributes at most one syllable, and $h_j \notin \mathrm{St}_B(1)$ for some $j \in [1,\ell]$, we have $\ell(n_x) \le t_x < \ell$. Also $\beta_{i-1}(f_x)n_x = b|_x = \mathrm{id}$ for $x \neq 0$, hence $n_x = \beta_{i-1}(f_x^{-1}) \in \mathcal{N}$. By minimality, $f_x \in [G,G]$. Then also $f_0 \equiv_{[G,G]} \prod_{x\in X} f_x \equiv_{[G,G]} \mathrm{id}$, and $\beta_{i-1}(f_0^{-1}g_0) = \beta_{i-1}(f_0^{-1})b|_0 = n_0 \in \mathcal{N}$. Again, by minimality, $f_0^{-1}g_0 \in [G,G]$, thus $g_0 \in [G,G]$, a contradiction.

This completes the proof. □

Example 1.4.5. Let $\mathrm{Bas}_s(\mathcal{O}_m^d)$ be a generalised Basilica group (cf. Definition 1.2.6). Since $\mathcal{O}_m^d$ is free abelian and self-similar, and $\mathcal{O}_m^d|^\epsilon$ is cyclic of order m, by Proposition 1.4.4, the group $\mathcal{O}_m^d$ is s-split.

Question 1.4.6. *Motivated by the small gap between Proposition 1.4.4 and 1.4.2 we ask: Is every abelian group $G \le \mathrm{Aut}(T)$ acting spherically transitive s-split for all $s > 1$?*

Corollary 1.4.7. *Let $G \le \mathrm{Aut}(T)$ be a self-similar s-split group. Then the abelianisation $\mathrm{Bas}_s(G)$ is*

$$\mathrm{Bas}_s(G)^{\mathrm{ab}} \cong G^s.$$

Proof. Consider the normal subgroup $H := \langle[\beta_i(G),\beta_j(G)] \mid i,j \in [0,s-1], i \neq j\rangle^{\mathrm{Bas}_s(G)}$ and observe that $H \le N_i$ for all $i \in [0,s-1]$. We obtain an epimorphism $G^s \to \mathrm{Bas}_s(G)/H$, mapping the i^{th} component of G^s to $\beta_i(G)(H)$, for $i \in [0,s-1]$. This map is also injective. Let $\prod_{i\in[0,s-1]}\beta_i(g_i) \equiv_H \prod_{i\in[0,s-1]}\beta_i(h_i)$ for some $g_i, h_i \in G$. Then for all $x \in X$

$$\beta_x(g_xh_x^{-1}) \equiv_H \prod_{i\in[0,s-1]\setminus\{x\}} \beta_i(g_i^{-1}h_i) \in N_x$$

and $\beta_x(g_xh_x^{-1}) \in N_x$. Since G is s-split, this implies $g_x = h_x$. Thus $\mathrm{Bas}_s(G)/H \cong G^s$. But from Proposition 1.4.2 G is abelian and consequently $H = [\mathrm{Bas}_s(G),\mathrm{Bas}_s(G)]$. □

Proposition 1.4.8. *Let $G \leq \mathrm{Aut}(T)$ be a torsion-free self-similar group such that the quotient G/K with $K = \beta_0^{-1}(\beta_0(G) \cap N_0)$ is again torsion-free. Then $\mathrm{Bas}_s(G)$ is torsion-free.*

Proof. Let $b \in \mathrm{Bas}_s(G)$ be a torsion element. Since G/K is torsion-free, we obtain $b \in \ker \phi_0 = N_0 \leq \mathrm{St}_{\mathrm{Bas}_s(G)}(1)$. Thus the first layer sections of b are again torsion elements of $\mathrm{Bas}_s(G)$, because $\mathrm{Bas}_s(G)$ is self-similar by Lemma 1.3.2. Hence an iteration of the argument yields $b = \mathrm{id}$. □

Question 1.4.9. *On the other end of the spectrum, the group* $\mathrm{Bas}_2(\mathcal{G})$ *(cf. Example 1.3.11) is periodic as is* $\mathcal{G}$*, which can be proven analogous to [13, Theorem 6.1], and the second Basilica groups of the periodic Gupta-Sidki-p-groups (cf. Example 1.3.10) are periodic by [?]. Motivated by this observation we ask:*

Is there a periodic group $G \leq \mathrm{Aut}(T)$ acting spherically transitive such that $\mathrm{Bas}_s(G)$ is not periodic for some $s \in \mathbb{N}_+$?

Proposition 1.4.8 and Corollary 1.4.7 prove Theorem 1.1.3.

1.4.1. Layer stabilisers. — For an s-split group $G \leq \mathrm{Aut}(T)$ the s^{th} Basilica decomposes as $\mathrm{Bas}_s(G) = N_i \rtimes \beta_i(G)$. Recall from Definition 1.4.1 that ϕ_i denotes the map to $\mathrm{Bas}_s(G)/N_i$, identified with the quotient G/K_i, such that $\phi_i(n\beta_i(g)) = gK_i$ for all $g \in G, n \in N_i$.

Lemma 1.4.10. *Let $G \leq \mathrm{Aut}(T)$ be a strongly fractal group and let $B = \mathrm{Bas}_s(G)$. Let $b_0, \dots, b_{m-1} \in B$. Then $\psi_1^{-1}(b_0, \dots, b_{m-1})$ is an element of $\mathrm{St}_B(1)$ if and only if there is an element $g \in \mathrm{St}_G(1)$ such that for all $x \in X$*

$$\phi_{s-1}(b_x) = g|_x K_{s-1}.$$

Proof. If there is some element $g \in \mathrm{St}_G(1)$ of the required form, clearly

$$\beta_0(g) \equiv_{\psi_1^{-1}(N_{s-1}^m)} (b_0, \dots, b_{m-1}).$$

Now we claim that $\psi_1(N_0) \geq N_{s-1}^m$. Let

$$b = \prod_{j=0}^{\ell-1} h_j^{\beta_{s-1}(g_j)} \in N_{s-1},$$

with $h_j \in S_{s-1}$. Then there are elements $\hat{h}_j = (h_j, \mathrm{id}, \dots, \mathrm{id}) \in S_0$ by the definition of S_{s-1}. Furthermore, since G is strongly fractal, there are elements $\hat{g}_j \in \mathrm{St}_G(1)$ such that $\beta_0(\hat{g}_j)|_0 = \beta_{s-1}(g_j)$, yielding

$$\prod_{j=0}^{\ell-1} \hat{h}_j^{\beta_0(\hat{g}_j)} = (b, \mathrm{id}, \dots, \mathrm{id}).$$

Since G acts spherically transitively, the claim follows by Lemma 1.2.1. Thus there is an element in $N_0\beta_0(g) \leq \mathrm{St}_B(1)$ with sections $(b_0, \dots, b_{m-1})$.

Let now $b = \psi_1^{-1}(b_0, \dots, b_{m-1}) \in \mathrm{St}_B(1)$. Then b decomposes as a product $n\beta_0(g)$ with $n \in N_0$ and $g \in \mathrm{St}_G(1)$. This implies, for any $x \in X$,

$$\phi_{s-1}(b_x) = \phi_{s-1}((n\beta_0(g))|_x) = \phi_{s-1}(\beta_{s-1}(g|_x)) = g|_x K_{s-1}. \qquad \square$$

Lemma 1.4.11. *Let G be fractal and self-similar and let $B = \mathrm{Bas}_s(G)$. Let $n \in \mathbb{N}_0$.*

(i) $\psi_1(\beta_i(\mathrm{St}_G(n))^B) = (\beta_{i-1}(\mathrm{St}_G(n))^B)^m$ *for all* $i \neq 0$.

Assuming further that G is very strongly fractal,

(ii) $\psi_1([\beta_0(\mathrm{St}_G(n+1)), N_0]^B) = ([\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m$.

Proof. (i) The inclusion $\psi_1(\beta_i(\mathrm{St}_G(n))^B) \leq (\beta_{i-1}(\mathrm{St}_G(n))^B)^m$ is obvious. We prove the other direction. Let $g \in \mathrm{St}_G(n)$ and $b \in B$. Since B is fractal by Lemma 1.3.3, there is an element $c \in \mathrm{st}_B(0)$ such that $c|_0 = b$. Now

$$(\beta_i(g))^c = (\beta_{i-1}(g), \mathrm{id}, \dots, \mathrm{id})^c = (\beta_{i-1}(g)^b, \mathrm{id}, \dots, \mathrm{id}),$$

yielding statement (i), by Lemma 1.2.1.

(ii) The inclusion $\psi_1([\beta_0(\mathrm{St}_G(n+1)), N_0]^B) \leq ([\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m$ follows directly from $N_0|_x \leq N_{s-1}$ and $\beta_0(\mathrm{St}_G(n+1))|_x \leq \beta_{s-1}(\mathrm{St}_G(n))$, where $x \in X$. Thanks to Lemma 1.2.1, for the other inclusion it is enough to prove that $([\beta_{s-1}(g), k], \mathrm{id}, \dots, \mathrm{id})$ is contained in $\psi_1([\beta_0(\mathrm{St}_G(n+1)), N_0]^B)$ for all $g \in \mathrm{St}_G(n)$ and $k \in N_{s-1}$. Let

$$k = \prod_{j=0}^{\ell} (\beta_{i_j}(k_j))^{\beta_{s-1}(k'_j)} \in N_{s-1}.$$

Since G is strongly fractal there are elements $t_j \in \mathrm{St}_G(1)$ such that $t_j|_0 = k'_j$. Furthermore, since G is very strongly fractal there is an element $h \in \mathrm{St}_G(n+1)$ such that $h|_0 = g$. Then

$$[\beta_0(h), \prod_{j=0}^{\ell} (\beta_{i_j+1}(k_j))^{\beta_0(t_j)}] \in [\beta_0(\mathrm{St}_G(n+1)), N_0]^B$$

and

$$\begin{aligned}
[\beta_0(h), \prod_{j=0}^{\ell} (\beta_{i_j+1}(k_j))^{\beta_0(t_j)}]|_x &= [(\beta_0(h))|_x, \prod_{j=0}^{\ell} ((\beta_{i_j+1}(k_j))|_x)^{(\beta_0(t_j))|_x}] \\
&= \begin{cases} [\beta_{s-1}(g), k] & \text{if } x = 0, \\ [\beta_{s-1}(h|_x), \prod_{j=0}^{\ell} \mathrm{id}^{\beta_{s-1}(t_j|_x)}] = \mathrm{id} & \text{otherwise.} \end{cases}
\end{aligned}$$

□

Proof of Theorem 1.1.4. Let $B = \mathrm{Bas}_s(G)$. For any $n \in \mathbb{N}_0$, write $n = sq + r$ with $q \geq 0$ and $r \in [0, s-1]$. We have to prove

$$\mathrm{St}_B(n) = \langle \beta_i(\mathrm{St}_G(q+1)), \beta_j(\mathrm{St}_G(q)) \mid 0 \leq i < r \leq j < s \rangle^B.$$

For convenience, we will denote the right-hand side of this equation by H_n. It is clear that $H_n \leq \mathrm{St}_B(n)$ for all $n \in \mathbb{N}_0$. It remains to establish the other inclusion. For $n = 0$ the statement is clearly true, so we proceed by induction and assume that the statement is true for some fixed $n = sq + r$ with $q \geq 0$ and $r \in [0, s-1]$. Define

$$J \coloneqq \langle \beta_i(\mathrm{St}_G(q+1)), \beta_j(\mathrm{St}_G(q)), [\beta_{s-1}(\mathrm{St}_G(q)), N_{s-1}]^B \mid 0 \leq i \leq r-1 < j < s-1 \rangle^B,$$

and observe that by Lemma 1.4.11 we find $J^m \leq \psi_1(H_{n+1})$, which yields

$$(\mathrm{St}_B(n))^m / \psi_1(H_{n+1}) = (\beta_{s-1}(\mathrm{St}_G(q)))^m \psi_1(H_{n+1}) / \psi_1(H_{n+1}).$$

Hence for every $g \in \mathrm{St}_B(n+1)$, there are elements $t_0, \ldots, t_{m-1} \in \mathrm{St}_G(q)$ such that

$$\psi_1(g) \equiv_{\psi_1(H_{n+1})} (\beta_{s-1}(t_0), \ldots, \beta_{s-1}(t_{m-1})).$$

Since $\phi_{s-1}\beta_{s-1}(t_x) = t_x K_{s-1}$ for all $x \in X$, $g \in \mathrm{St}_B(1)$ and $H_{n+1} \leq \mathrm{St}_B(1)$, by Lemma 1.4.10 there are elements $k_0, \ldots, k_{m-1} \in K_{s-1}$ and $h \in \mathrm{St}_G(1)$ such that

$$\psi_1^{-1}(h|_0 k_0, \ldots, h|_{m-1} k_{m-1}) = \psi_1^{-1}(t_0, \ldots, t_{m-1}).$$

Define $\widetilde{h} = h\psi_1^{-1}(k_0, \ldots, k_{m-1})$. Now G is weakly regular branch over K_{s-1}, hence $\psi_1^{-1}(K_{s-1}^m) \leq \mathrm{St}_{K_{s-1}}(1)$, and consequently $\widetilde{h} \in \mathrm{St}_G(1)$. But $\widetilde{h}|_x = t_x \in \mathrm{St}_G(q)$ for $x \in X$, whence $\widetilde{h} \in \mathrm{St}_G(q+1)$ and

$$(\beta_{s-1}(t_0), \ldots, \beta_{s-1}(t_{m-1})) = \psi_1(\beta_0(\widetilde{h})) \in \psi_1(\beta_0(\mathrm{St}_G(q+1))) \leq \psi_1(H_{n+1}),$$

implying $g \in H_{n+1}$. This completes the proof. □

1.4.2. Hausdorff dimension. — We remind the reader that Γ is defined as the subgroup of $\mathrm{Aut}(T)$ consisting of all automorphisms whose labels are elements of $\langle \sigma \rangle$, with σ being a fixed m-cycle in $\mathrm{Sym}(X)$.

Definition 1.4.12. Let $G \leq \Gamma$. The *Hausdorff dimension of G relative to Γ* is defined by

$$\dim_{\mathrm{H}} G \coloneqq \liminf_{n\to\infty} \frac{\log_m |G/\mathrm{St}_G(n)|}{\log_m |\Gamma/\mathrm{St}_\Gamma(n)|} = (m-1) \liminf_{n\to\infty} \frac{\log_m |G/\mathrm{St}_G(n)|}{m^n}.$$

This relates to the usual definition of Hausdorff dimension over arbitrary spaces by taking the closure, i.e. using this definition, the group G has the same Hausdorff dimension as its closure $\overline{G}$ in Γ, cf. [8]. We drop the base m in $\log_m$ from now on. Denote the quotient $\mathrm{St}_G(n)/\mathrm{St}_G(n+1)$ by $L_G(n)$. The integer series (for $n > 0$) obtained by

$$o_G(n) := \log(|L_G(n-1)|^m) - \log|L_G(n)|$$

is called the *series of obstructions of* G. We set $o_G(0) = -1$ for convenience.

The series of obstructions of a group G determines its Hausdorff dimension, precisely how we will see in Lemma 1.4.13. Nevertheless, one might wonder why it is necessary to define this seemingly impractial invariant. We will demonstrate in Proposition 1.4.16 that it is (to some degree) preserved under $G \mapsto \mathrm{Bas}_s(G)$. Furthermore, many well-studied subgroups of Γ have a well-behaved series of obstructions. For example, it is easy to see that Γ itself has

$$\begin{aligned} o_\Gamma(n) &= \log|\wr^n \mathrm{C}_m / \wr^{n-1} \mathrm{C}_m|^m - \log|\wr^{n+1} \mathrm{C}_m / \wr^n \mathrm{C}_m| \\ &= m \log m^{m^n} - \log m^{m^{n+1}} = 0, \end{aligned}$$

for $n \in \mathbb{N}_+$, where $\wr^n A$ is the n-times iterated wreath product of A, with the convention that $\wr^0 A$ is the trivial group. On the other hand, since the layer stabiliser of $\mathcal{O}_m^d$ are the subgroups generated by $\langle \pi_0(a)^{m^{k+1}}, \dots, \pi_{l-1}(a)^{m^{k+1}}, \pi_l(a)^{m^k}, \dots, \pi_{d-1}(a)^{m^k} \rangle$, the quotients $L_{\mathcal{O}_m^d}(n)$ are all cyclic of order m, and

$$o_{\mathcal{O}_m^d}(n) = m - 1.$$

A Gupta–Sidki p-group G has precisely two terms unequal to 0, a consequence of $\mathrm{St}_G(n) = \mathrm{St}_G(n-1)^p$ for $n \geq 3$, cf. [48]. Similarly, the series of obstructions of the Grigorchuk group has only one non-zero term.

Lemma 1.4.13. *Let $G \leq \Gamma$ act spherically transitive. Then*

$$\dim_{\mathrm{H}} G = 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_G(i).$$

Proof. By definition $\log|L_G(0)| = 1$ and $\log|L_G(n)| = m\log|L_G(n-1)| - o_G(n)$ for $n \geq 1$. An inductive argument yields

$$\log|G/\mathrm{St}_G(n+1)| = \log|G/\mathrm{St}_G(n)| - \sum_{k=0}^{n} m^{n-k} o_G(k) \quad = -\sum_{k=0}^{n} \frac{m^{k+1}-1}{m-1} o_G(n-k).$$

This gives

$$\begin{aligned}\liminf_{n\to\infty} \frac{(m-1)}{m^{n+1}} \log\left|\frac{G}{\mathrm{St}_G(n+1)}\right| &= -\limsup_{n\to\infty} \sum_{i=0}^{n} (m^{i-n} - m^{-(n+1)}) o_G(n-i) \\ &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_G(i).\end{aligned}$$ □

Lemma 1.4.14. *Let $G \leq \Gamma$ be self-similar. Then for all $n > 0$*

$$o_G(n) = \log[\mathrm{St}_G(n-1)^m : \psi_1(\mathrm{St}_G(n))] - \log[\mathrm{St}_G(n)^m : \psi_1(\mathrm{St}_G(n+1))].$$

Proof. We have, for $n > 0$,

$$\begin{aligned}\left|\frac{\mathrm{St}_G(n-1)^m}{\psi_1(\mathrm{St}_G(n))}\right| &= \frac{|\,\mathrm{St}_G(n-1)^m/\psi_1(\mathrm{St}_G(n+1))|}{|L_G(n)|} \\ &= \frac{|L_G(n-1)|^m}{|L_G(n)|}\left|\frac{\mathrm{St}_G(n)^m}{\psi_1(\mathrm{St}_G(n+1))}\right|,\end{aligned}$$

hence

$$o_G(n) = \log[\mathrm{St}_G(n-1)^m : \psi_1(\mathrm{St}_G(n))] - \log[\mathrm{St}_G(n)^m : \psi_1(\mathrm{St}_G(n+1))].$$ □

Lemma 1.4.15. *Let G be very strongly fractal, self-similar and weakly regular branch over the splitting kernel K_{s-1}. Then for all $\ell, n \in \mathbb{N}_+$*

$$\begin{aligned}&\psi_1(\beta_0(\mathrm{St}_G(\ell+1)) \cap [\beta_0(\mathrm{St}_G(n+1)), N_0]^B) \\ &= (\beta_{s-1}(\mathrm{St}_G(\ell)) \cap [\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m.\end{aligned}$$

Proof. The left-hand set is clearly contained in the right-hand set. We prove the other inclusion. Let $(b_0, \ldots, b_{m-1}) \in (\beta_{s-1}(\mathrm{St}_G(\ell)) \cap [\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m$. By Lemma 1.4.11(ii) there exists $b \in [\beta_0(\mathrm{St}_G(n+1)), N_0]^B \leq \mathrm{St}_B(1)$ such that $\psi_1(b) = (b_0, \ldots, b_{m-1})$. It remains to prove that $b \in \beta_0(\mathrm{St}_G(\ell+1))$.

Since the set $\beta_{s-1}(\mathrm{St}_G(\ell)) \cap [\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B$ is contained in $\beta_{s-1}(\mathrm{St}_{K_{s-1}}(1))$ and since G weakly regular branch over K_{s-1}, there is an element $g \in K_{s-1}$ such that

$$\psi_1(g) = (\beta_{s-1}^{-1}(b_0), \ldots, \beta_{s-1}^{-1}(b_{m-1})) \in \mathrm{St}_G(\ell)^m.$$

Consequently, $\psi_1(\beta_0(g)) = (b_0, \ldots, b_{m-1}) = \psi_1(b)$, and $b = \beta_0(g)$ is a member of the set $\psi_1(\beta_0(\mathrm{St}_G(\ell+1)) \cap [\beta_0(\mathrm{St}_G(n+1)), N_0]^B)$. □

Proposition 1.4.16. *Let $G \leq \Gamma$ be very strongly fractal, self-similar and weakly regular branch over the splitting kernel K_{s-1}. Then the series of obstructions for $B = \mathrm{Bas}_s(G)$*

fulfils

$$o_B(n) = \begin{cases} 0 & \textit{if } n \not\equiv_s 0, \\ o_G(\frac{n}{s}) & \textit{otherwise.} \end{cases}$$

Proof. Consider first the case $n \equiv_s k \neq 0$. By Theorem 1.1.4 the quotient $L_B(n)$ is normally generated in B by images of elements of $\beta_k(\mathrm{St}_G(\lfloor n/s \rfloor))$. Similarly the images of $\beta_{k-1}(\mathrm{St}_G(\lfloor n/s \rfloor))$ are the normal generators of $L_B(n-1)$. Thus Lemma 1.4.11(i) shows that $o_B(n) = 0$.

Now consider the case $n = qs$. To shorten the notation, we abbreviate

$$R_q := \beta_0(\mathrm{St}_G(q)) \text{ for } q \in \mathbb{N}_0 \text{ and}$$
$$T_q := \beta_{s-1}(\mathrm{St}_G(q)) \text{ for } q \in \mathbb{N}_0.$$

Define the normal subgroups

$$U = \langle \mathrm{St}_B(n+1) \cup [R_q, N_0]^B \rangle \trianglelefteq B \quad \text{and}$$
$$V = \langle \mathrm{St}_B(n) \cup [T_{q-1}, N_{s-1}]^B \rangle \trianglelefteq B.$$

Using Theorem 1.1.4, we see that U and V, respectively, are normally generated by the sets

$$R_{q+1} \cup \bigcup_{i=1}^{s-1} (\beta_i(\mathrm{St}_G(q))) \cup [R_q, N_0] \quad \text{and} \quad T_q \cup \bigcup_{i=0}^{s-2} (\beta_i(\mathrm{St}_G(q))) \cup [T_{q-1}, N_{s-1}].$$

Let $g \in \mathrm{St}_G(q+1)$ and $b \in B$. We write $b = \beta_0(g_b) n_b$ for $g_b \in G$ and $n_b \in N_0$. Then

$$\beta_0(g)^b = \beta_0(g^{g_b})^{n_b} = \beta_0(g^{g_b})[\beta_0(g^{g_b}), n_b] \in R_{q+1}[R_{q+1}, N_0].$$

Consequently, we drop the conjugates of R_{q+1} in our generating set for U, and write

$$U = \langle R_{q+1} \cup \bigcup_{i=1}^{s-1} \left(\beta_i(\mathrm{St}_G(q))^B\right) \cup [R_q, N_0]^B \rangle.$$

Similarly, the subgroup V is generated by

$$T_q \cup \bigcup_{i=0}^{s-2} \left(\beta_i(\mathrm{St}_G(q))^B\right) \cup [T_{q-1}, N_{s-1}]^B.$$

Using Theorem 1.1.4, it is now easy to see that

$$\mathrm{St}_B(n)/U \cong R_q/(R_q \cap U).$$

Since $\beta_i(\mathrm{St}_G(q)) \le \mathrm{St}_B(n+1)$ for $i \neq 0$, we see that the intersection

$$\langle \beta_1(\mathrm{St}_G(q)) \cup \dots \cup \beta_{s-2}(\mathrm{St}_G(q)) \cup T_q \rangle^B \cap R_q \le R_{q+1}$$

is contained in R_{q+1}. We conclude

$$R_q \cap U = R_q \cap R_{q+1}[R_q, N_0]^B.$$

Now

$$R_q \cap R_{q+1}[R_q, N_0]^B = R_{q+1}(R_q \cap [R_q, N_0]^B)$$

and

$$[R_q \cap R_{q+1}[R_q, N_0]^B : R_{q+1}] = [R_q \cap [R_q, N_0]^B : R_{q+1} \cap [R_q, N_0]^B].$$

Consequently, the order of $\mathrm{St}_B(n)/U$ equals

$$|L_G(q)| \cdot [R_q \cap [R_q, N_0]^B : R_{q+1} \cap [R_q, N_0]^B]^{-1}.$$

A similar computation shows that the order of $\mathrm{St}_B(n-1)/V$ is

$$|L_G(q-1)| \cdot [T_{q-1} \cap [T_{q-1}, N_{s-1}]^B : T_q \cap [T_{q-1}, N_{s-1}]^B]^{-1}.$$

We now apply Lemma 1.4.15 in the cases $\ell = n = q-1$ and $\ell = n+1 = q$, i.e. we have

$$\psi_1(R_q \cap [R_q, N_0]^B) = (T_{q-1} \cap [T_{q-1}, N_{s-1}]^B)^m \text{ and}$$
$$\psi_1(R_{q+1} \cap [R_q, N_0]^B) = (T_q \cap [T_{q-1}, N_{s-1}]^B)^m.$$

We see that the second factor in the formula for the order of $\mathrm{St}_B(n)/U$ is the m^{th} power of the corresponding factor for $\mathrm{St}_B(n-1)/V$, and obtain

$$\frac{|\mathrm{St}_B(n-1)/V|^m}{|\mathrm{St}_B(n)/U|} = \frac{|L_G(q-1)|^m}{|L_G(q)|} = m^{o_G(q)}.$$

Now we compare V^m and $\psi_1(U)$. By Lemma 1.4.11(i) and (ii), $\psi_1(U)$ is generated by

$$\psi_1(R_{q+1}) \cup \bigcup_{i=0}^{s-2} \big(((\beta_i(\mathrm{St}_G(q)))^B)^m\big) \cup ([T_{q-1}, N_{s-1}]^B)^m.$$

We define yet another subgroup

$$W = \langle \bigcup_{i=0}^{s-2} \big(((\beta_i(\mathrm{St}_G(q)))^B)^m\big) \cup ([T_{q-1}, N_{s-1}]^B)^m \rangle \le \psi_1(U) \le B^m.$$

Evidently $W \trianglelefteq B^m$, $W \leq N_{s-1}^m$, and $W \trianglelefteq \psi_1(U) \leq V^m$. We have

$$\begin{aligned}\psi_1(U)/W &\cong \psi_1(R_{q+1})/(\psi_1(R_{q+1}) \cap W) \text{ and}\\ V^m/W &\cong T_q^m/(T_q^m \cap W).\end{aligned}$$

The two divisors are equal: Clearly $\psi_1(R_{q+1}) \cap W$ is contained in $T_q^m \cap W$. Let

$$(\beta_{s-1}(g_0), \dots, \beta_{s-1}(g_{m-1})) \in T_q^m \cap W \leq (T_q \cap N_{s-1})^m.$$

Since $T_q \cap N_{s-1} \leq \beta_{s-1}(K_{s-1})$, the elements $g_0, \dots, g_{m-1}$ are members of $K_{s-1} \cap \mathrm{St}_G(q)$. Now since G is weakly regular branch over K_{s-1}, there is an element $k \in K_{s-1} \cap \mathrm{St}_G(q+1)$ such that $\psi_1(k) = (g_0, \dots, g_{m-1})$, and consequently $\beta_0(k) \in R_{q+1}$ fulfils

$$\psi_1(\beta_0(k)) = (\beta_{s-1}(g_0), \dots, \beta_{s-1}(g_{m-1})) \in \psi_1(R_{q+1}) \cap W.$$

We compute

$$\begin{aligned}[V^m : \psi_1(U)] &= [V^m/W : \psi_1(U)/W]\\ &= [T_q^m : \psi_1(R_{q+1})]\\ &= [(\beta_{s-1} \times \dots \times \beta_{s-1})(\mathrm{St}_G(q)^m) : (\beta_{s-1} \times \dots \times \beta_{s-1})(\psi_1(\mathrm{St}_G(q+1)))]\\ &= [\mathrm{St}_G(q)^m : \psi_1(\mathrm{St}_G(q+1))].\end{aligned}$$

This implies

$$\begin{aligned}[\mathrm{St}_B(n-1)^m : \psi_1(\mathrm{St}_B(n))] &= [\mathrm{St}_B(n-1)^m/\psi_1(U) : \psi_1(\mathrm{St}_B(n))/\psi_1(U)]\\ &= \frac{[\mathrm{St}_B(n-1)^m : V^m][V^m : \psi_1(U)]}{[\psi_1(\mathrm{St}_B(n)) : \psi_1(U)]}\\ &= \frac{|L_G(q-1)|^m}{|L_G(q)|} \cdot [\mathrm{St}_G(q)^m : \psi_1(\mathrm{St}_G(q+1))].\end{aligned}$$

Since $o_B(k) = 0$ for $k \not\equiv_s 0$, by Lemma 1.4.14,

$$\log[\mathrm{St}_B(n)^m : \psi_1(\mathrm{St}_B(n+1))] = \log[\mathrm{St}_B(n+s-1)^m : \psi_1(\mathrm{St}_B(n+s)],$$

hence

$$\begin{aligned}o_B(n) =& \log[\mathrm{St}_B(n-1)^m : \psi_1(\mathrm{St}_B(n))] - \log[\mathrm{St}_B(n+s-1)^m : \psi_1(\mathrm{St}_B(n+s))]\\ =& o_G(q) + \log\left|\frac{\mathrm{St}_G(q)^m}{\psi_1(\mathrm{St}_G(q+1))}\right| - o_G(q+1) - \log\left|\frac{\mathrm{St}_G(q+1)^m}{\psi_1(\mathrm{St}_G(q+2))}\right|\\ =& o_G(q) - o_G(q+1) + o_G(q+1)\\ =& o_G(q).\end{aligned}$$

□

Proof of Corollary 1.1.5. By Lemma 1.4.13 and Proposition 1.4.16

$$\begin{aligned}\dim_{\mathrm{H}} G &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_G(i) \quad \text{and} \\ \dim_{\mathrm{H}} \mathrm{Bas}_s(G) &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_{\mathrm{Bas}_s(G)}(i) \\ &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-si} - m^{-(sn+1)}) o_G(i).\end{aligned}$$

We prove $m^{-i} - m^{-(n+1)} > m^{-si} - m^{-(sn+1)}$, equivalently $m^{sn+1-i} + 1 > m^{s(n-i)+1} + m^{(s-1)n}$. This is a consequence of $sn+1-i-(s(n-i)+1) = (s-1)i \geq 1$ and $sn+1-i-(s-1)n = n-i+1 \geq 1$, with equality precisely when $i = 1, s = 2$, resp. $n = i$. Therefore at least one of the differences is greater than 1, and the limit of $\sum_{i=1}^{n}(m^{-si} - m^{-(sn+1)})o_G(i)$ is strictly greater than the limit of $\sum_{i=1}^{n}(m^{-i} - m^{-(n+1)})o_G(i)$. The statement follows. □

Example 1.4.17. Let $G \leq \mathrm{Aut}(T_p)$, p a prime, be a GGS-group defined by the triple $(\mathsf{C}_p, \mathsf{C}_p, \omega)$, cf. Definition 1.3.7, where C_p denotes the cyclic group of order p acting regularly on X. To be a GGS-group means $\omega_i = \omega_j$ for $i, j \in \mathbb{N}_0$, thus we write ω for ω_1. This is a $(p-1)$-tuple of endomorphisms of C_p. Every such endomorphism is a power map, hence we may identify ω with an element $(e_1, \dots, e_{p-1})$ of $\mathbb{F}_p^{p-1}$. Assume that

$$e_1 + \cdots + e_{p-1} \equiv_p 0 \tag{$\star$}$$

and that there is some $i \in [1, p-1]$

$$e_i \neq e_{p-i}. \tag{$\diamond$}$$

In [48] the order of the congruence quotients $G/\mathrm{St}_G(n)$ is explicitly calculated in terms of the rank t of the circulant matrix associated to the vector $(0, e_1, \dots, e_{p-1})$, i.e. the matrix with rows being all cyclic permutations of the given vector. Under our assumptions $(\star)$ and $(\diamond)$, for all $n \in \mathbb{N}_+$

$$\log_p(G/\mathrm{St}_G(n+1)) = tp^{n-1} + 1,$$

and $\log_p(G/\mathrm{St}_G(1)) = 1$. Additionaly, $(\star)$ is equivalent to $t < p$. By Lemma 1.4.14, for $n > 2$,

$$\begin{aligned}o_G(n) &= p \cdot \log_p(|L_G(n-1)|) - \log_p(|L_G(n)|) \\ &= p \cdot \log_p \frac{p^{t\cdot p^{n-2}+1}}{p^{t\cdot p^{n-3}+1}} - \log \frac{p^{t\cdot p^{n-1}+1}}{p^{t\cdot p^{n-2}+1}} = 0\end{aligned}$$

and

$$o_G(2) = p \cdot \log \frac{p^{t+1}}{p} - \log \frac{p^{t \cdot p+1}}{p^{t+1}} = tp - t(p-1) = t \quad \text{and}$$
$$o_G(1) = p \cdot \log p - \log \frac{p^{t+1}}{p} = p - t.$$

Consequently, $\dim_{\mathrm{H}} G = t(p-1)/p^2$ (cf. [48] for a more general formula).

We aim to apply Proposition 1.4.16. Condition ($\diamond$) is equivalent to G being weakly regular branch (in fact, regular branch) over $[G, G]$, by [48, Lemma 3.4]. More precisely, we have

$$\psi_1([\mathrm{St}_G(1), \mathrm{St}_G(1)]) = [G, G]^p.$$

By Proposition 1.4.4 this implies that $K_{s-1} = [G, G]$. We now prove that G is very strongly fractal. It is easy to see that $\mathrm{St}_G(1)|_x = G$ for all $x \in X$, and by [48, Lemma 3.3] $\psi_1(\mathrm{St}_G(n)) = \mathrm{St}_G(n-1)^p$ for all $n \geq 3$. Thus it remains to check if $\mathrm{St}_G(2)|_x = \mathrm{St}_G(1)$ for all $x \in X$. By the fact that $[\mathrm{St}_G(1), \mathrm{St}_G(1)]|_x = [G, G]$ for all $x \in X$ and $[\mathrm{St}_G(2) : [\mathrm{St}_G(1), \mathrm{St}_G(1)]] = p^{p-t} \geq p$ (see again [48]), we see that $\mathrm{St}_G(2)$ contains an element g such that $\psi_1(g) \in \mathrm{St}_G(1)^p \setminus [G, G]^p$. Hence at least for one $x \in X$

$$\mathrm{St}_G(1) \geq \mathrm{St}_G(2)|_x > [G, G].$$

But since $[\mathrm{St}_G(1) : [G, G]] = p$ by [48, Theorem 2.1], this implies $\mathrm{St}_G(2)|_x = \mathrm{St}_G(1)$, and since G is spherically transitive, this holds for all $x \in X$, and G is very strongly fractal. We remark that by [147, Proposition 5.1] the condition ($\star$) alone implies that G is super strongly fractal, but our argument additionally needs ($\diamond$), since otherwise $[[G, G]^p : \psi_1([\mathrm{St}_G(1), \mathrm{St}_G(1)])] = p$ (cf. [48, Lemma 3.5]).

Now we may apply Proposition 1.4.16 to calculate the Hausdorff dimension of $\mathrm{Bas}_s(G)$:

$$o_{\mathrm{Bas}_s(G)}(s) = p - t \quad \text{and} \quad o_{\mathrm{Bas}_s(G)}(2s) = t$$

and $o_{\mathrm{Bas}_s(G)}(n) = 0$ for all other values $n \in \mathbb{N}_+$, hence

$$\begin{aligned} \dim_{\mathrm{H}} \mathrm{Bas}_s(G) &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{p^i} - \frac{1}{p^{n+1}} \right) o_{\mathrm{Bas}_s(G)}(i) \\ &= 1 - \limsup_{n\to\infty} \left(\frac{p-t}{p^s} + \frac{t}{p^{2s}} - \frac{p-t+t}{p^{n+1}} \right) \\ &= 1 - \left(\frac{p-t}{p^s} + \frac{t}{p^{2s}} \right) = \frac{p^{s-1}-1}{p^{s-1}} + \frac{t(p^s-1)}{p^{2s}}. \end{aligned}$$

1.5 — The generalised Basilica groups

Let d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$. In the subsequent Section 1.5, Section 1.6, Section 1.7 and Section 1.8 we study the generalised Basilica groups, $\mathrm{Bas}_s(\mathcal{O}_m^d)$, where $\mathcal{O}_m^d = D_d(\mathcal{O}_m) = \langle \pi_i(a) \mid i \in [0, d-1] \rangle$ (cf. Proposition 1.2.5 and Definition 1.2.6). For convenience, we use the following notation for the generators of $\mathrm{Bas}_s(\mathcal{O}_m^d)$: let $i \in [0, d-1]$ and $j \in [0, s-1]$, and

$$\begin{aligned} a_{i,j} &:= \beta_j(\pi_i(a)) &&= (a_{i,j-1}, \mathrm{id}, \ldots, \mathrm{id}), && \text{for } j \neq 0 \\ a_{i,0} &:= \beta_0(\pi_i(a)) &&= (a_{i-1,s-1}, \ldots, a_{i-1,s-1}), && \text{for } i \neq 0 \\ a_{0,0} &:= \beta_0(\pi_0(a)) &&= \sigma(a_{d-1,s-1}, \mathrm{id}, \ldots, \mathrm{id}), \end{aligned}$$

where σ is the m-cycle $(0\ 1\ \ldots\ m-1)$. For any fixed j, the elements $a_{i,j}$ commute and are of infinite order.

Now we prove Theorem 1.1.6, which is obtained as corollaries of results from Section 1.3 and Section 1.4.

Proof of Theorem 1.1.6. The statements (i) and (ii) follow directly from Lemma 1.3.1, Lemma 1.3.2 and Lemma 1.3.3. Proposition 1.3.5 together with Corollary 1.3.6 imply the statement (ii). The statement (iii) is a consequence of Proposition 1.3.14 and Corollary 1.3.15. Thanks to Proposition 1.4.4, the group $\mathcal{O}_m^d$ is s-split. Therefore the statements (iv), (v) and (vi) follow from Corollary 1.4.7, Proposition 1.4.8 and Proposition 1.3.22. The proof of (vii) can easily be generalised from [73, Proposition 4]. For the special case $\mathrm{Bas}_p(\mathcal{O}_p)$, where p is a prime, see [135]. □

We use Theorem 1.1.4 to provide a normal generating set for the layer stabilisers of the generalised Basilica groups. This description of layer stabilisers is crucial in proving the p-congruence subgroup property of the generalised Basilica groups (see Section 1.8).

Theorem 1.5.1. *Let $n \in \mathbb{N}_0$. Write $n = sq + r$ with $r \in [0, s-1]$ and $q = dk + l \geq 0$ with $l \in [0, d-1]$. Then the n^{th} layer stabiliser of $B = \mathrm{Bas}_s(\mathcal{O}_m^d)$ is given by*

$$\mathrm{St}_B(n) = \langle a_{i,j}^{m^{k+1}}, a_{i',j'}^{m^k} \mid 0 \leq is + j \leq ls + r - 1 < i's + j' \leq ds - 1 \rangle^B.$$

Proof. Let a be the generator of the m-adic odometer $\mathcal{O}_m$. Set $G = D_d(\mathcal{O}_m) \cong \mathbb{Z}^d$. For every $i \in [0, d-1]$, denote by $a_i = \pi_i(a)$ the generators of G. Since powers of the elements $a_0, \ldots, a_{d-1}$ act on vertices of disjoint levels of the m-regular rooted tree T and they commute with each other, we have

$$\mathrm{St}_G(q) = \langle a_0^{m^{k+1}}, \ldots, a_{l-1}^{m^{k+1}}, a_l^{m^k}, \ldots, a_{d-1}^{m^k} \rangle.$$

Now observe that for every vertex $x \in X$, $i \in [0, d]$ and $k \in \mathbb{N}_0$,

$$a_i^{m^k}|_x = a_{i-1}^{m^k}$$
$$a_0^{m^k}|_x = a_{d-1}^{m^{k-1}}.$$

Therefore $\mathrm{St}_G(q)|_x = \mathrm{St}_G(q-1)$ and hence G is very strongly fractal. A straightforward calculation using Theorem 1.1.4 yields the result. □

Using the description of the layer stabilisers of G, we obtain Theorem 1.1.7 as a direct application of Lemma 1.4.13 and Proposition 1.4.16.

Proof of Theorem 1.1.7. The series of obstructions of $G = \mathcal{O}_m^d$ is constant $m-1$ for all $n \in \mathbb{N}_+$, signifying Hausdorff-dimension 0 (cf. Lemma 1.4.13). We have seen in the proof of Theorem 1.5.1 that $\mathrm{Bas}_s(G)$ is very strongly fractal. Therefore, by Proposition 1.4.16 one has $o_{\mathrm{Bas}_s(G)}(qs) = m-1$ for all $q \in \mathbb{N}_+$ and $o_{\mathrm{Bas}_s(G)}(n) = 0$ for all other levels.

According to Lemma 1.4.13 it is

$$\begin{aligned}
\dim_{\mathrm{H}} \mathrm{Bas}_s(G) &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_{\mathrm{Bas}_s(G)}(i) \\
&= 1 - (m-1) \limsup_{n\to\infty} \left(m^{-s} \frac{1 - m^{-s\lfloor n/s \rfloor}}{1 - m^{-s}} - \lfloor n/s \rfloor m^{-(n+1)} \right) \\
&= 1 - (m-1) \frac{m^{-s}}{1 - m^{-s}} \\
&= \frac{m^s - m}{m^s - 1}.
\end{aligned}$$

In particular, the Hausdorff dimension is independent of d. □

1.6 — An L-presentation for the generalised Basilica group

Let d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$. In this section we will provide a concrete L-presentation for the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$, hence proving Theorem 1.1.8. We will later use this presentation to prove that all generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_p^d)$ with p a prime have the p-congruence subgroup property.

Definition 1.6.1. [10, Definition 1.2] An L-presentation (or an endomorphic presentation) is an expression of the form

$$L = \langle Y \mid Q \mid \Phi \mid R \rangle,$$

where Y is an alphabet, $Q, R \subset F_Y$ are sets of reduced words in the free group F_Y on Y and Φ is a set of endomorphisms of F_Y. The expression L gives rise to a group G_L defined as

$$G_L = F_Y/\langle Q \cup \langle \Phi \rangle(R)\rangle^{F_Y},$$

where $\langle \Phi \rangle(R)$ denotes the union of the images of R under every endomorphism in the monoid $\langle \Phi \rangle$ generated from Φ. An L-presentation is finite if Y, Q, Φ, R are finite.

We now set out to prove Theorem 1.1.8. To do this, we follow the strategy from [73] which is motivated from [67]: let

$$Y = \{a_{i,j} \mid i \in [0, d-1], j \in [0, s-1]\}. \tag{1.1}$$

For convenience, we do not distinguish notationally between the generators of $\mathrm{Bas}_s(\mathcal{O}_m^d)$ and the free generators for the presentation. Observe that for a fixed j the generators $a_{i,j}$ and $a_{i',j}$ of $\mathrm{Bas}_s(\mathcal{O}_m^d)$ commute for all $i, i' \in [0, d-1]$. Write

$$Q = \{[a_{i,j}, a_{i',j}] \mid i, i' \in [0, d-1],\ j \in [0, s-1]\} \subseteq F_Y \tag{1.2}$$

and denote by F the quotient of F_Y by the normal closure of Q in F_Y. We identify F with a free product of free abelian groups

$$F = *_{j\in[0,s-1]}\langle a_{i,j} \mid i \in [0, d-1]\rangle \cong \mathbb{Z}^d * \cdots * \mathbb{Z}^d.$$

The group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ is a quotient of F. Let $\mathrm{proj} : F \to \mathrm{Bas}_s(\mathcal{O}_m^d)$ be the canonical epimorphism. Now observe that the subgroup

$$\Delta = \langle a_{i,j}^{a_{0,0}^k},\ a_{0,0}^m \mid (i,j) \in [0, d-1] \times [0, s-1]\backslash\{(0,0)\},\ k \in [0, m-1]\rangle, \tag{1.3}$$

is normal of index m in F and it is the full preimage of $\mathrm{St}_{\mathrm{Bas}_s(\mathcal{O}_m^d)}(1)$ under the epimorphism proj (cf. Theorem 1.5.1). We define a homomorphism $\Psi : \Delta \to F^m$ modelling the process of taking sections as follows:

$$\begin{aligned}
\Psi(a_{0,0}^m) &= (a_{d-1,s-1}, \dots, a_{d-1,s-1}) && =: z_0, \\
\Psi(a_{i,0}^{a_{0,0}^k}) = \Psi(a_{i,0}) &= (a_{i-1,s-1}, \dots, a_{i-1,s-1}) && =: z_i \quad \text{for } i \neq 0, \\
\Psi(a_{i,j}^{a_{0,0}^k}) &= (\mathrm{id}^{*k}, a_{i,j-1}, \mathrm{id}^{*(m-k-1)}) && =: x_{i,j,k} \quad \text{for } j \neq 0, \\
\Psi(a_{i,j}^{a_{0,0}^{-k}}) &= (\mathrm{id}^{*(m-k)}, a_{i,j-1}^{a_{d-1,s-1}^{-1}}, \mathrm{id}^{*(k-1)}),
\end{aligned}$$

where the ranges of i, j and k are as in (1.3). Clearly, $\ker(\Psi) \leq \ker(\mathrm{proj})$. Define

$$\alpha(v,k) = a_{0,0}^{mv_0+k} a_{1,0}^{v_1} \cdots a_{d-1,0}^{v_{d-1}} \text{ for } v = (v_0, \dots, v_{d-1}) \in \mathbb{Z}^d \text{ and } k \in [0, m-1], \tag{1.4}$$

$$R = \{[a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \mid i, i' \in [0, d-1],\ j, j' \in [1, s-1],\ k \in [1, m-1],\ v \in \mathbb{Z}^d\}, \tag{1.5}$$

where by abuse of notation we interpret $\alpha(v,k)$ and $r \in R$ both as elements of F_Y and their images in F. We will prove in Proposition 1.6.3 that the kernel of Ψ is normally generated from the image of R in F, implying that the set R belongs to the set of defining relators of $\mathrm{Bas}_s(\mathcal{O}_m^d)$. By definition of the elements $a_{i,j}$, we may obtain the elements of the set R as vertex sections. To incorporate these elements to the set of defining relators we introduce the following endomorphism of F_Y defined as

$$\Phi : \begin{cases} a_{i,j} & \mapsto a_{i,j+1} \text{ for } j \neq s-1, \\ a_{i,s-1} & \mapsto a_{i+1,0} \text{ for } i \neq d-1, \\ a_{d-1,s-1} & \mapsto a_{0,0}^m, \end{cases} \tag{1.6}$$

where $i \in [0, d-1]$ and $j \in [0, s-1]$.

Theorem 1.6.2. *The generalised Basilica group admits the L-presentation*

$$L = \langle Y \mid Q \mid \Phi \mid R \rangle$$

where Y, Q, R *and* Φ *are given by* (1.1), (1.2), (1.5) *and* (1.6).

Observe that for any $g \in Q$ and $r \in \mathbb{N}_0$, it holds that $\Phi^r(g) \in \langle Q^{F_Y} \rangle$. Considering the presentation defining F we may assume that Φ is an endomorphism of F and that R is a subset of F. To prove Theorem 1.6.2, it is enough to show that $\ker(\Psi) = \langle R^F \rangle$ and $\ker(\mathrm{proj}) = \bigcup_{r \in \mathbb{N}_0} \Phi^r(R)$. We will obtain the first part from Proposition 1.6.3 and the latter from Lemma 1.6.5 to Lemma 1.6.7.

Proposition 1.6.3. *Let* $\tilde{\Delta}$ *be the image of* Δ *under* Ψ. *Let* z^v *be the product* $z_0^{v_0} \cdots z_{d-1}^{v_{d-1}}$ *for every* $v = (v_0, \ldots, v_{d-1}) \in \mathbb{Z}^d$. *Then* $\tilde{\Delta}$ *admits the presentation*

$$\langle\, \mathcal{S} \mid \mathcal{R} \,\rangle$$

where $\mathcal{S} = \{x_{i,j,k},\ z_i \mid i \in [0, d-1],\ j \in [1, s-1],\ k \in [0, m-1]\}$ *and*

$$\mathcal{R} = \left\langle \begin{array}{c} [x_{i,j,k}, x_{i',j,k}],\ [x_{i,j,k}, x_{i',j',k'}^{z^v}], \\ [z_i, z_{i'}] \end{array} \middle| \begin{array}{l} i, i' \in [0, d-1],\ j, j' \in [1, s-1], \\ k, k' \in [0, m-1] \text{ with } k \neq k',\ v \in \mathbb{Z}^d \end{array} \right\rangle.$$

As a consequence, we obtain that

$$\ker(\Psi) = \langle \{ [a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \mid i, i' \in [0, d-1],\ j, j' \in [1, s-1],\ k \in [1, m-1],\ v \in \mathbb{Z}^d \} \rangle^F,$$

where $\alpha(v, k)$ *is given by* (1.4).

Proof. Let $A = \langle a_{i,j} \mid i \in [0, d-1],\ j \in [0, s-2] \rangle^F$ and $Z = \langle z_0, \ldots, z_{d-1} \rangle \cong \mathbb{Z}^d$ be subgroups of F and $\tilde{\Delta}$ respectively. Notice that $\tilde{\Delta}$ is a sub-direct product of m copies of F and the elements $x_{i,j,k}$ and $x_{i',j',k'}$ commute if $k \neq k'$ or if $k = k'$ and $j = j'$. It follows from the definition of Ψ that

$$A^m = \left\langle\ x_{i,j,k}\ \middle|\ i \in [0, d-1],\ j \in [1, s-1],\ k \in [0, m-1]\ \right\rangle^{\tilde{\Delta}} \leq \tilde{\Delta}.$$

Hence $\tilde{\Delta} = A^m Z$, yielding $\tilde{\Delta} = A^m \rtimes Z$. Now, since F is a free product of free abelian groups, the group A is freely generated from the elements of the form

$$a_{i,j}^{a_{d-1,s-1}^{v_0} a_{0,s-1}^{v_1} \cdots a_{d-2,s-1}^{v_{d-1}}},$$

where $v_i \in \mathbb{Z}$, $i \in [0, d-1]$ and $j \in [0, s-2]$. Therefore, the group A^m is generated from the elements

$$x_{i,j,k}^{z^v} = (\mathrm{id}^{*k}, a_{i,j-1}^{a_{d-1,s-1}^{v_0} a_{0,s-1}^{v_1} \cdots a_{d-2,s-1}^{v_{d-1}}}, \mathrm{id}^{*(m-k-1)}),$$

where $i \in [0, s-1]$, $j \in [1, s-1]$, $k \in [0, m-1]$ and

$$z^v = z_0^{v_0} \cdots z_{d-1}^{v_{d-1}} = (a_{d-1,s-1}^{v_0} a_{0,s-1}^{v_1} \cdots a_{d-2,s-1}^{v_{d-1}}, \ldots, a_{d-1,s-1}^{v_0} a_{0,s-1}^{v_1} \cdots a_{d-2,s-1}^{v_{d-1}}),$$

with $v_i \in \mathbb{Z}$. We obtain a presentation of A^m as

$$\left\langle\ x_{i,j,k}^{z^v}\ \middle|\ \begin{array}{l} [x_{i,j,k}, x_{i',j,k}] = [x_{i,j,k}^{z^v}, x_{i',j',k'}^{z^{v'}}] = \mathrm{id},\ i, i' \in [0, d-1],\ j, j' \in [1, s-1], \\ k, k' \in [0, m-1] \text{ with } k \neq k',\ v, v' \in \mathbb{Z}^d \end{array} \right\rangle.$$

Hence $\tilde{\Delta}$, being a semi-direct product, admits the presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$, since conjugating an element $x_{i,j,k}$ by z_i does not yield a new relation. Therefore, the kernel of Ψ is normally generated from the preimage of the set of defining relators for $\tilde{\Delta}$. Notice that the preimages of the elements $[z_i, z_{i'}]$ and $[x_{i,j,k}, x_{i',j,k}]$ are trivial in Δ. Hence,

$$\ker(\Psi) = \left\langle\ [a_{i,j}^{\alpha(v,k)}, a_{i',j'}^{\alpha(v',k')}]\ \middle|\ \begin{array}{l} i, i' \in [0, d-1],\ j, j' \in [1, s-1], \\ k, k' \in [0, m-1] \text{ with } k \neq k',\ v, v' \in \mathbb{Z}^d \end{array} \right\rangle^{\Delta}.$$

Indeed, $\ker(\Psi)$ is normal in F. Given $v \in \mathbb{Z}^d$ and $k \in [0, m-1]$, define

$$\begin{aligned} \underline{v} &= (\lfloor (mv_0 + k + 1)/m \rfloor, v_1, \ldots, v_{d-1}) \in \mathbb{Z}^d \quad \text{and} \\ \underline{k} &= k + 1 \pmod{m} \in [0, m-1]. \end{aligned}$$

Then

$$\begin{aligned} \alpha(v, k) a_{0,0} &= a_{0,0}^{mv_0+k+1} a_{1,0}^{v_1} \cdots a_{d-1,0}^{v_{d-1}} = \alpha(\underline{v}, \underline{k}) \\ \alpha(v', k') a_{0,0} &= a_{0,0}^{mv_0'+k'+1} a_{1,0}^{v_1'} \cdots a_{d-1,0}^{v_{d-1}'} = \alpha(\underline{v}', \underline{k}') \end{aligned}$$

implies

$$[a_{i,j}^{\alpha(v,k)}, a_{i',j'}^{\alpha(v',k')}]^{a_{0,0}} = [a_{i,j}^{\alpha(v,k)a_{0,0}}, a_{i',j'}^{\alpha(v',k')a_{0,0}}] = [a_{i,j}^{\alpha(\underline{v},\underline{k})}, a_{i',j'}^{\alpha(\underline{v}',\underline{k}')}] \in \ker(\Psi).$$

A similar calculation shows $[a_{i,j}^{\alpha(v,k)}, a_{i',j'}^{\alpha(v',k')}]^{a_{0,0}^{-1}} \in \ker(\Psi)$. We get

$$\ker(\Psi) = \left\langle \; [a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \;\middle|\; i, i' \in [0, d-1],\, j, j' \in [1, s-1],\, k \in [1, m-1],\, v \in \mathbb{Z}^d \; \right\rangle^F. \qquad \square$$

Notation 1.6.4. *Let $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$, $k \in [1, m-1]$, $v \in \mathbb{Z}^d$ and $n \in \mathbb{N}_0$. Define*

$$\Omega_0 := \ker(\Psi), \qquad \Omega_n := \Psi^{-1}(\Omega_{n-1}^m) \text{ for } n \geq 1,$$
$$\tau_{v,k}(i, j, i', j') := [a_{i,j}, a_{i',j'}^{\alpha(v,k)}], \qquad X_n := \langle \Phi^r(\tau_{v,k}(i, j, i', j')) \mid r \in [0, n] \rangle^F,$$

where $\alpha(v, k)$ is given by (1.4). Denote further by Ω the kernel of the epimorphism proj : $F \to \mathrm{Bas}_s(\mathcal{O}_m^d)$. *We will prove $\Omega_n = X_n$ and $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$, proving Theorem 1.6.2.*

Lemma 1.6.5. *For $w \in F'$ the identity $\Psi(\Phi(w)^{a_{0,0}^k}) = (\mathrm{id}^{*k}, w, \mathrm{id}^{*(m-k-1)})$ holds for every $k \in [0, m-1]$.*

Proof. Observe from the definition of Φ that

$$\Phi(F) = \langle a_{i,j}, a_{0,0}^m \mid (i, j) \in [0, d-1] \times [0, s-1] \backslash \{(0, 0)\} \rangle \leq \Delta.$$

Then by direct calculation using the definition of the homomorphism Ψ and Φ we get the desired identity. $\square$

Lemma 1.6.6. *The equality $\Omega_n = X_n$ holds for all $n \in \mathbb{N}_0$.*

Proof. It follows from Proposition 1.6.3 that $\Omega_0 = \ker(\Psi) = X_0$. The proof proceeds by induction on n. Since $\Phi(F) \leq \Delta$, for every $r \in \mathbb{N}_0$, we have $\Phi^r(\Delta) \leq \Delta$. Hence $X_n \leq \Delta$ for all $n \in \mathbb{N}_0$. Assume for some $n \geq 1$ that $\Omega_{n-1} = X_{n-1}$. We will prove that

$$\Psi(X_n) = \Omega_{n-1}^m = \Psi(\Omega_n).$$

Let $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$, $k \in [1, m-1]$, $r \in [1, n]$ and $v \in \mathbb{Z}^d$. For every $\Phi^r(\tau_{v,k}(i, j, i', j')) \in X_n$ and for every $\ell \in [0, m-1]$, since $\Phi^{r-1}(\tau_{v,k}(i, j, i', j')) \in F'$, we obtain from Lemma 1.6.5 that

$$\Psi((\Phi^r(\tau_{v,k}(i, j, i', j')))^{a_{0,0}^{\ell}}) = (\mathrm{id}^{*\ell}, \Phi^{r-1}(\tau_{v,k}(i, j, i', j')), \mathrm{id}^{*(m-\ell-1)}).$$

Since $\tilde{\Delta}$ is a sub-direct product of m copies of F and X_{n-1} is normally generated from the elements of the form $\Phi^{r-1}(\tau_{v,k}(i, j, i', j'))$, we obtain that $\Psi(X_n) = \Omega_{n-1}^m = \Psi(\Omega_n)$.

But since $\ker(\Psi|_{\Omega_n}) = \ker(\Psi) \cap \Omega_n = \Omega_0 = X_0 = \ker(\Psi) \cap X_n = \ker(\Psi|_{X_n})$, we get $\Omega_n = X_n$, and the result follows by induction. □

Lemma 1.6.7. *We have* $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$.

Proof. Write B for $\mathrm{Bas}_s(\mathcal{O}_m^d)$ and recall that $\mathrm{proj} : F \to B$ is the canonical epimorphism. Notice that $\mathrm{St}_B(1)$ is a quotient of Δ and further $\Omega_0 = \ker(\Psi) \leq \ker(\mathrm{proj}) = \Omega$. Proceeding by induction on n, we will prove that $\bigcup_{n=0}^{\infty} \Omega_n \leq \Omega$. Assume that $\Omega_{n-1} \leq \Omega$ for some $n \geq 1$. Let $w \in \Omega_n$ and let w_k be the k^{th} component of $\Psi(w)$. Then $w_k \in \Omega_{n-1}$ for all $k \in [0, m-1]$. Then the first layer sections of $\mathrm{proj}(w) \in \mathrm{St}_B(1)$ act trivially on the subtrees hanging from the vertices of level one of the m-regular rooted tree. Hence $\mathrm{proj}(w)$ acts trivially and $\mathrm{proj}(w) = \mathrm{id}$ in B. It follows by induction that $\Omega_n \leq \Omega$ for all $n \in \mathbb{N}_0$. Since $\Omega_{n-1} \leq \Omega_n$ for all $n \in \mathbb{N}_+$, we obtain $\bigcup_{n=0}^{\infty} \Omega_n \leq \Omega$.

Now, to see the converse choose an arbitrary element $w \in F$ such that $\mathrm{proj}(w) = \mathrm{id}$ in B. Then by Theorem 1.5.1 $\mathrm{proj}(w) \in \mathrm{St}_B(1)$ and hence $w \in \Delta$. Denote by w_k the k^{th} component of $\Psi(w)$. Then $\mathrm{proj}(w) = \mathrm{id}$ if and only if $\mathrm{proj}(w_k) = \mathrm{id}$ for all $k \in [0, m-1]$, implying that $w_k \in \Delta$ for all $k \in [0, m-1]$. Now repeat this process of taking sections by replacing w with w_k. This process is equivalent to the algorithm solving the word problem for B, cf. [73, Proposition 5]. Thanks to Corollary 1.3.15, the word problem for B is solvable and hence this process terminates in a finite number of steps. This implies the existence of an element $n \in \mathbb{N}_0$ such that $w \in \Omega_n$, completing the proof. □

To conclude this section, we want to point out that akin to [73, Proposition 11], one can introduce a set of d endomorphisms, each corresponding to a generator $a_{i,0}$, and obtain a finite L-presentation for $\mathrm{Bas}_s(\mathcal{O}_m^d)$. We omit the proof of Theorem 1.6.8 below due to its technicality, but a rigorous proof can be found in the Ph.D. dissertation of the second author.

Theorem 1.6.8. *The group* $\mathrm{Bas}_s(\mathcal{O}_m^d)$ *admits the following L-presentation:*

$$\left\langle \begin{array}{c} a_{i,j} \\ i \in [0, d-1] \\ j \in [0, s-1] \end{array} \middle| \begin{array}{c} [a_{i,j}, a_{i',j}] \\ i, i' \in [0, d-1] \\ j \in [0, s-1] \end{array} \middle| \; \Phi, \Theta_0, \ldots, \Theta_{d-1} \; \middle| \begin{array}{c} [a_{i,j}, a_{i',j'}^{\alpha(v,k)}], i, i' \in [0, d-1], \\ j, j' \in [1, s-1], \, k \in [1, m-1] \\ v \in \{0\} \times \{0,1\}^{d-1} \end{array} \right\rangle$$

where $\alpha(v,k)$ *and* Φ *are given by* (1.4) *and* (1.6)*, respectively, and* $\Theta_{i'}$ *are endomorphisms of the free group on the set of generators defined as*

$$\Theta_{i'} : \begin{cases} a_{i,j} \mapsto a_{i,j} a_{i,j}^{a_{i',0}} \text{ for } j \neq 0, i' \neq 0, \\ a_{i,j} \mapsto a_{i,j} a_{i,j}^{a_{0,0}^m} \text{ for } j \neq 0, i' = 0, \\ a_{i,0} \mapsto a_{i,0}. \end{cases}$$

1.7 — Structural properties of the generalised Basilica groups

Let d, m, $s \in \mathbb{N}_+$ with m, $s \geq 2$. Here we prove some structural properties of the generalised Basilica groups $\mathrm{Bas}_2(\mathcal{O}_m^d)$. These result reflect a significant structural dissimilarity between $\mathrm{Bas}_2(\mathcal{O}_m^d)$ and $\mathrm{Bas}_s(\mathcal{O}_m^d)$ for $s > 2$. This structural dissimilarity plays a vital role when we consider the p-congruence subgroup property of the generalised Basilica groups, see Fig. 1.5, which is treated in Section 1.8.

For convenience, we omit the subscript from ψ_1 and identify an element $g \in \mathrm{St}_B(1)$ with its image under the map ψ_1.

Proposition 1.7.1. *Let B be the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$. Then $\psi^{-1}((B')^m)$ is a subgroup of B' and*

$$\begin{aligned} B'/\psi^{-1}((B')^m) &= \Big\langle \; c_{i,j,k}\, \psi^{-1}((B')^m) \;\Big|\; i \in [0, d-1], j \in [1, s-1],\, k \in [1, m-1] \; \Big\rangle \\ &\cong \mathbb{Z}^{d(m-1)(s-1)}, \end{aligned}$$

where $c_{i,j,k} = [a_{i,j}, a_{0,0}^k]$. In particular, it holds that $\psi^{-1}((B')^m) \geq B''$.

Proof. Notice that $B' = \langle [a_{i,j}, a_{i',j'}] \mid i, i' \in [0, d-1],\, j, j' \in [0, s-1] \rangle^B$. For $i, i' \in [0, d-1]$ and $j, j' \in [1, s-1]$, we have $[a_{i,j}, a_{i',j}] = \mathrm{id}$ and for $j \neq j'$

$$\begin{aligned} [a_{i,j}, a_{i',j'}] &= ([a_{i,j-1}, a_{i',j'-1}], \mathrm{id}^{*(m-1)}) \\ [a_{i,j}, a_{i',0}] &= ([a_{i,j-1}, a_{i'-1,s-1}], \mathrm{id}^{*(m-1)}) \text{ for } i' \neq 0, \\ [a_{i,j}, a_{0,0}^m] &= ([a_{i,j-1}, a_{d-1,s-1}], \mathrm{id}^{*(m-1)}). \end{aligned}$$

Therefore, we obtain

$$\langle [a_{i,j}, a_{i',j'}] \mid i, i' \in [0, d-1],\, j, j' \in [0, s-1] \rangle \times \{\mathrm{id}\} \times \cdots \times \{\mathrm{id}\} \leq \psi(B'),$$

yielding that $(B')^m \leq \psi(B')$ by Lemma 1.2.1.

Now, recall our definition $c_{i,j,k} = [a_{i,j}, a_{0,0}^k]$ and

$$C = \langle c_{i,j,k} \mid i \in [0, d-1],\, j \in [1, s-1],\, k \in [1, m-1] \rangle.$$

We claim that $B'/\psi^{-1}((B')^m) = \overline{C}$, where $\overline{C}$ denotes the image of C in the quotient group. For convenience, we will write the equivalence $\equiv_{\psi^{-1}((B')^m)}$ without the subscript. Observe that, for $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$ and $k \in [1, m-1]$,

$$[a_{i,j}, a_{i',j'}] \equiv \mathrm{id}, \qquad [a_{i,j}, a_{i',0}] \equiv \mathrm{id} \;\text{ for } i' \neq 0, \qquad [a_{i,j}, a_{0,0}] = c_{i,j,1},$$

and

$$c_{i,j,k} = [a_{i,j}, a_{0,0}^k] \equiv (a_{i,j-1}^{-1}, \mathrm{id}^{*(k-1)}, a_{i,j-1}, \mathrm{id}^{*(m-k-1)}).$$

Therefore, to prove the claim, it suffices to show that $\overline{C}$ is normal in $B/\psi^{-1}((B')^m)$. Let $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$ and $k \in [1, m-1]$. An easy calculation yields

$$c_{i,j,k}^{a_{i',j'}^{\pm 1}} \equiv c_{i,j,k} \qquad \text{and} \qquad c_{i,j,k}^{a_{i',0}^{\pm 1}} \equiv c_{i,j,k} \text{ for } i' \neq 0.$$

Furthermore,

$$\begin{aligned}
c_{i,j,k}^{a_{0,0}} &\equiv (\mathrm{id}, a_{i,j-1}^{-1}, \mathrm{id}^{*(k-1)}, a_{i,j-1}, \mathrm{id}^{*(m-k-2)}) &&\equiv c_{i,j,1}^{-1} c_{i,j,k+1} && \text{if } k \neq m-1,\\
c_{i,j,k}^{a_{0,0}} &\equiv (a_{i,j-1}, a_{i,j-1}^{-1}, \mathrm{id}^{*(m-2)}) &&\equiv c_{i,j,1}^{-1} && \text{if } k = m-1,\\
c_{i,j,k}^{a_{0,0}^{-1}} &\equiv (\mathrm{id}^{*(k-1)}, a_{i,j-1}, \mathrm{id}^{*(m-k-1)}, a_{i,j-1}^{-1}) &&\equiv \begin{cases} c_{i,j,m-1}^{-1} c_{i,j,k-1} & \text{if } k \neq 1,\\ c_{i,j,m-1}^{-1} & \text{if } k = 1, \end{cases}
\end{aligned}$$

implying that $B'/\psi^{-1}((B')^m) = \overline{C}$. Observe that, for a fixed $i \in [0, d-1]$ and $j \in [1, s-1]$,

$$\mathbb{Z}^{m-1} \cong \{(a_{i,j-1}^{x_1}, \dots, a_{i,j-1}^{x_m}) \mid x_r \in \mathbb{Z}, \sum_{r=1}^{m} x_r = 0\} = \langle \overline{c}_{i,j,k} \mid k \in [1, m-1] \rangle \leq \overline{C}.$$

Since $B/B' \cong \mathbb{Z}^{ds}$ (Theorem 1.1.6(iv)), this yields

$$B'/\psi^{-1}((B')^m) = \overline{C} = \prod_{(i,j) \in [0,d-1] \times [1,s-1]} \langle \overline{c}_{i,j,k} \mid k \in [1, m-1] \rangle \cong \mathbb{Z}^{d(m-1)(s-1)}.$$

□

Now we prove Theorem 1.1.9. In addition, we provide a generating set for the quotient group $\gamma_2(\mathrm{Bas}_s(\mathcal{O}_m^d))/\gamma_3(\mathrm{Bas}_s(\mathcal{O}_m^d))$.

Theorem 1.7.2. *Let B be the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$. We have:*

(i) For $s = 2$, $B'/\gamma_3(B) = \langle [a_{i,0}, a_{i',1}]\, \gamma_3(B) \mid i, i' \in [0, d-1] \rangle \cong \mathbb{Z}^{d^2}$.

(ii) For $s > 2$, the quotient group $B'/\gamma_3(B) \cong \mathrm{C}_m^{ds-2} \times \mathrm{C}_{m^2}$. Moreover, it is generated from the set

$$\{[a_{i,j}, a_{0,0}]\, \gamma_3(B), [a_{0,1}, a_{i',0}]\, \gamma_3(B) \mid i \in [0, d-1],\ i' \in [1, d-1],\ j \in [1, s-1]\}.$$

Proof. (i) We use Theorem 1.6.2 to obtain a presentation for $B/\gamma_3(B)$. Take Y, Q, Φ and R as given in Theorem 1.6.2 and set $Q' = Q \cup \gamma_3(F_Y)$, where F_Y is the free group on Y. If $s = 2$, the set R becomes

$$R = \{[a_{i,1}, a_{i',1}^{\alpha(v,k)}] \mid i, i' \in [0, d-1],\ k \in [1, m-1],\ v \in \mathbb{Z}^d\}$$

and for every $[a_{i,1}, a_{i',1}^{\alpha(v,k)}] \in R$,

$$[a_{i,1}, a_{i',1}^{\alpha(v,k)}] \equiv_{\gamma_3(F_Y)} [a_{i,1}, a_{i',1}] \in \langle Q' \rangle^{F_Y},$$

where $\alpha(v,k)$ is given by (1.4). Since $\langle Q'\rangle$ is invariant under Φ, the presentation $\langle\, Y \mid Q' \,\rangle$ defines the group $B/\gamma_3(B)$, yielding that

$$B'/\gamma_3(B) = \langle [a_{i,0}, a_{i',1}] \mid i,i' \in [0,d-1]\rangle \cong \mathbb{Z}^{d^2}.$$

(ii) Consider again Y, Q, Φ and R as given in Theorem 1.6.2 and $Q' = Q \cup \gamma_3(F_Y)$. First observe that the element

$$[a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \equiv_{\gamma_3(F_Y)} [a_{i,j}, a_{i',j'}]$$

belongs to $\langle Q'\rangle^{F_Y}$ if and only if $j = j'$. Setting

$$S = \{[a_{i,j}, a_{i',j'}] \mid i,i' \in [0,d-1],\ j,j' \in [1,s-1] \text{ with } j \neq j'\} \subseteq F_Y,$$

we notice that the group $B/\gamma_3(B)$ admits the L-presentation $\langle\, Y \mid Q' \mid \Phi \mid S \,\rangle$. Now, define

$$T = \left\{ \begin{array}{c|c} [a_{i,j}, a_{i',0}], [a_{i'',1}, a_{i',0}], & i \in [0,d-1], \\ [a_{i,j}, a_{0,0}]^m, [a_{i',1}, a_{0,0}]^m, [a_{0,1}, a_{i',0}]^m, & i',i'' \in [1,d-1], \\ [a_{0,1}, a_{0,0}]^{m^2} & j \in [2,s-1] \end{array} \right\}$$

and $N = Q' \cup S \cup T$ as subsets of F_Y. We claim that $\Phi^r(S) \subseteq N^{F_Y}$ for all $r \in \mathbb{N}_0$, and hence the presentation $\langle\, Y \mid N \,\rangle$ defines the group $B/\gamma_3(B)$. Therefore, the commutator subgroup of $B/\gamma_3(B)$ is generated from the set

$$\left\{ \begin{array}{c|c} [a_{i,j}, a_{0,0}], [a_{i',1}, a_{0,0}], & i \in [0,d-1],\ i' \in [1,d-1], \\ [a_{0,1}, a_{i',0}], [a_{0,1}, a_{0,0}] & j \in [2,s-1] \end{array} \right\},$$

yielding that:

$$B'/\gamma_3(B) \cong \mathrm{C}_m^{d(s-2)} \times \mathrm{C}_m^{d-1} \times \mathrm{C}_m^{d-1} \times \mathrm{C}_{m^2} = \mathrm{C}_m^{ds-2} \times \mathrm{C}_{m^2}.$$

Now, let $i,i' \in [0,d-1]$. Observe first that, for $j,j' \in [1,s-2]$,

$$\Phi([a_{i,j}, a_{i',j'}]) = [a_{i,j+1}, a_{i',j'+1}] \in S.$$

To prove the claim, it is enough to consider the elements of the form $\Phi^r([a_{i,j}, a_{i',j'}])$ with either j or j', but not both, equal to $s-1$. Without loss of generality suppose that $1 \le j \le s-2$ and $j' = s-1$. Since $\gamma_3(F_Y) \le N^{F_Y}$, we work modulo $\gamma_3(F_Y)$. We have

$$\Phi([a_{i,j}, a_{i',s-1}]) \equiv \begin{cases} [a_{i,j+1}, a_{i'+1,0}]^m & \text{if } i' = d-1 \\ [a_{i,j+1}, a_{i'+1,0}] & \text{otherwise.} \end{cases}$$

For convenience, the images of $\Phi^2([a_{i,j}, a_{i',s-1}])$ and $\Phi^3([a_{i,j}, a_{i',s-1}])$ are given in the tabular form, see Table 1.1 and Table 1.2.

		$j \neq s-2$	$j = s-2$
$i' \neq d-1$	$i \neq d-1$	$[a_{i,j+2}, a_{i'+1,1}]$	$[a_{i+1,0}, a_{i'+1,1}]$
	$i = d-1$		$[a_{0,0}, a_{i'+1,1}]^m$
$i' = d-1$	$i \neq d-1$	$[a_{i,j+2}, a_{0,1}]^m$	$[a_{i+1,0}, a_{0,1}]^m$
	$i = d-1$		$[a_{0,0}, a_{0,1}]^{m^2}$

Table 1.1: Images of $\Phi^2([a_{i,j}, a_{i',s-1}])$.

		$j \notin \{s-3, s-2\}$	$j = s-2$	$j = s-3$
$i' \neq d-1$	$i \neq d-1$	$[a_{i,j+3}, a_{i'+1,2}]$	$[a_{i+1,1}, a_{i'+1,2}]$	$[a_{i+1,0}, a_{i'+1,2}]$
	$i = d-1$		$[a_{0,1}, a_{i'+1,2}]^m$	$[a_{0,0}, a_{i'+1,2}]^m$
$i' = d-1$	$i \neq d-1$	$[a_{i,j+3}, a_{0,2}]^m$	$[a_{i+1,1}, a_{0,2}]^m$	$[a_{i+1,0}, a_{0,2}]^m$
	$i = d-1$		$[a_{0,1}, a_{0,2}]^{m^2}$	$[a_{0,0}, a_{0,2}]^{m^2}$

Table 1.2: Images of $\Phi^3([a_{i,j}, a_{i',s-1}])$.

Observe that the element $\Phi^r([a_{i,j}, a_{i',s-1}]) \in N^{F_Y}$ for $r \in [1,3]$. By iterating the process we see that $\Phi^r([a_{i,j}, a_{i',j'}]) \in N^{F_Y}$, for all $r \in \mathbb{N}_0$ and $[a_{i,j}, a_{i',j'}] \in S$. $\square$

Lemma 1.7.3. *Let B be the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$. The following assertions hold:*

(i) For $s = 2$, $B'' = \psi^{-1}(\gamma_3(B)^m)$.
(ii) For $s > 2$, $B'' \gneq \psi^{-1}(\gamma_3(B)^m)$.

Proof. We first prove that $\gamma_3(B)^m \leq \psi(B'')$ for all $s \geq 2$. From Lemma 1.2.1, since

$$\gamma_3(B) = \langle [[a_{i_1,j_1}, a_{i_2,j_2}], a_{i_3,j_3}] \mid i_1, i_2, i_3 \in [0, d-1],\, j_1, j_2, j_3 \in [0, s-1] \rangle^B,$$

and B is self-similar and fractal (Theorem 1.1.6(ii)), it is enough to prove that the set

$$\{([[a_{i_1,j_1}, a_{i_2,j_2}], a_{i_3,j_3}], \mathrm{id}^{*(m-1)}) \mid i_1, i_2, i_3 \in [0, d-1],\, j_1, j_2, j_3 \in [0, s-1]\} \qquad (*)$$

is contained in $\psi(B'')$. Let $i_1, i_2, i_3 \in [0, d-1]$ and $j_1, j_2, j_3 \in [0, s-1]$. We split the proof into four cases.

Case 1: $j_1 = j_2 = j_3 = s-1$. Clearly, $[[a_{i_1,s-1}, a_{i_2,s-1}], a_{i_3,s-1}] = \mathrm{id}$.

Case 2: $j_3 \neq s-1$. In light of Proposition 1.7.1, the elements $([a_{i_1,j_1}, a_{i_2,j_2}], \mathrm{id}^{*(m-1)})$ and $(a_{i_3,j_3}, a_{i_3,j_3}^{-1}, \mathrm{id}^{*(m-2)}) = [a_{i_3,j_3+1}, a_{0,0}]^{-1}$ belong to $\psi(B')$, implying that

$$([[a_{i_1,j_1}, a_{i_2,j_2}], a_{i_3,j_3}], \mathrm{id}^{*(m-1)}) \in \psi(B'').$$

Now, observe from Proposition 1.7.1 that $\psi(B'') \geq (B'')^m$. Therefore, if there exist $g = (g_0, \dots, g_{m-1}), h = (h_0, \dots, h_{m-1}) \in B$ such that $g_i \equiv_{B''} h_i$ for all $i \in [0, m-1]$ then $g \equiv_{\psi(B'')} h$.

Case 3: $j_3 = s-1$, $j_1 \neq s-1$ and $j_2 \neq s-1$. Now, from the Hall–Witt identity (see [131, p. 123]), we can easily derive that

$$[[y,x],z][[z,y],x][[x,z],y] \equiv_{B''} [[y,x],z^y][[z,y],x^z][[x,z],y^x] = \mathrm{id},$$

for all $x, y, z \in B$. Setting $x = a_{i_1,j_1}$, $y = a_{i_2,j_2}$ and $z = a_{i_3,j_3}$, we get that the element

$$([[y,x],z], \mathrm{id}^{*(m-1)})^{-1} \equiv_{\psi(B'')} ([[z,y],x][[x,z],y], \mathrm{id}^{*(m-1)})$$

belongs to $\psi(B'')$, as the right-hand side product belongs to $\psi(B'')$ by Case 2.

Case 4: $j_3 = s-1 = j_1$, $j_2 \neq s-1$ or $j_3 = s-1 = j_2$, $j_1 \neq s-1$. Notice that

$$[[a_{i_1,j_1}, a_{i_2,s-1}], a_{i_3,s-1}] \equiv_{B''} [[a_{i_2,s-1}, a_{i_1,j_1}], a_{i_3,s-1}]^{-1},$$

thus, it is enough to consider the first case. We claim that, for every $j \in [0, s-1]$, it holds $[[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}] \equiv_{B''} \mathrm{id}$. Then by taking the j_2th projection of the element $[[a_{i_1,s-1}, a_{i_2,j_2}], a_{i_3,s-1}]$ we obtain,

$$\begin{aligned}\psi_{j_2}([[a_{i_1,s-1}, a_{i_2,j_2}], a_{i_3,s-1}]) &= ([[a_{i_1,(s-1-j_2)}, a_{i_2,0}], a_{i_3,(s-1-j_2)}], \mathrm{id}^{*(m^{j_2}-1)}) \\ &\equiv_{\psi_{j_2}(B'')} \mathrm{id},\end{aligned}$$

implying $[[a_{i_1,s-1}, a_{i_2,j_2}], a_{i_3,s-1}] \equiv_{B''} \mathrm{id}$, and hence $(*)$ follows.

If $i_2 = 0$ or $j = 0$, it is then immediate that $[[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}] = \mathrm{id}$. Assume that $i_2 \neq 0$ and $j \neq 0$. From the presentation of B given in Theorem 1.6.2, we have

$$[[a_{i_1,j}, \alpha(v,k)], a_{i_3,j}] = [a_{i_1,j}^{-1} a_{i_1,j}^{\alpha(v,k)}, a_{i_3,j}] = [a_{i_1,j}^{-1}, a_{i_3,j}]^{a_{i_1,j}^{\alpha(v,k)}} [a_{i_1,j}^{\alpha(v,k)}, a_{i_3,j}] = \mathrm{id},$$

where $\alpha(v,k)$ is given by (1.4). Now, by setting $v = (0^{*(i_2-1)}, 1, 0^{*(m-i_2-1)})$ and $k = 1$, we get $\alpha(v,k) = a_{0,0}a_{i_2,0}$ and consequently

$$\begin{aligned}\mathrm{id} &= [[a_{i_1,j}, a_{0,0}a_{i_2,0}], a_{i_3,j}] = [[a_{i_1,j}, a_{i_2,0}][a_{i_1,j}, a_{0,0}]^{a_{i_2,0}}, a_{i_3,j}] \\ &\equiv_{B''} [[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}][[a_{i_1,j}, a_{0,0}]^{a_{i_2,0}}, a_{i_3,j}] \equiv_{B''} [[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}].\end{aligned}$$

Next we prove (i). Assume that $s = 2$ and notice that it suffices to prove that $B'/\psi^{-1}(\gamma_3(B)^m)$ is abelian. We use the fact that the commutator subgroup can be described by $B' = \langle [a_{i_1,1}, a_{i_2,0}] \mid i_1, i_2 \in [0, d-1] \rangle^B$ as $s = 2$.

Looking at the section decomposition of these generators,

$$[a_{i_1,1}, a_{i_2,0}] = ([a_{i_1,0}, a_{i_2-1,1}], \mathrm{id}^{*(m-1)}) \quad \text{for } i_2 \neq 0, \text{ and}$$
$$[a_{i_1,1}, a_{0,0}] = (a_{i_1,0}^{-1}, a_{i_1,0}, \mathrm{id}^{*(m-2)}),$$

we immediately see that they commute modulo $\gamma_3(B)^m$. Thus, $B'/\psi^{-1}(\gamma_3(B)^m)$ is abelian.

(ii) The inclusion $\psi^{-1}(\gamma_3(B)^m) \leq B''$ has been already proven above. We prove that $\psi^{-1}(\gamma_3(B)^m)$ is a proper subgroup of B'', by showing that $B'/\psi^{-1}(\gamma_3(B)^m)$ is non-abelian. Suppose to the contrary $B'/\psi^{-1}(\gamma_3(B)^m)$ is abelian. Then, for every $i \in [0, d-1]$ and $j \in [2, s-1]$

$$\mathrm{id} \equiv_{\psi^{-1}(\gamma_3(B)^m)} [[a_{i,j}, a_{0,0}], [a_{0,1}, a_{0,0}]] = ([a_{i,j-1}^{-1}, a_{0,0}^{-1}], [a_{i,j-1}, a_{0,0}], \mathrm{id}^{*(m-2)}).$$

This implies $[a_{i,j-1}, a_{0,0}] \equiv_{\gamma_3(B)} \mathrm{id}$, which is a contradiction to Theorem 1.7.2(ii). □

1.8 — Congruence properties of the generalised Basilica groups

Here we prove that the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_p^d)$ has the p-CSP for d, $s \in \mathbb{N}_+$ with $s > 2$ and p a prime. We follow the strategy from [55], where it is proved that the original Basilica group $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$ has the 2-congruence subgroup property. However, on account of Theorem 1.7.2 and Lemma 1.7.3, our reasoning must be different, and we will use Theorem 1.5.1.

Let G be a subgroup of the automorphism group of the p-regular rooted tree T and let $\mathcal{C}$ be the class of all finite p-groups.

Definition 1.8.1 (Definition 5 of [55])**.** A subgroup G of $\mathrm{Aut}(T)$ has the p-congruence subgroup property (p-CSP) if every normal subgroup $N \trianglelefteq G$ satisfying $G/N \in \mathcal{C}$ contains some layer stabiliser in G. The group G has the p-CSP modulo a normal subgroup $M \trianglelefteq G$ if every normal subgroup $N \trianglelefteq G$ satisfying $G/N \in \mathcal{C}$ and $M \leq N$ contains some layer stabiliser in G.

By setting $\mathcal{C}$ as the class of all finite p-groups in [55, Lemma 6], we obtain the following result:

Lemma 1.8.2. *Let G be a subgroup of* $\mathrm{Aut}(T)$ *and $N \trianglelefteq M \trianglelefteq G$. If G has the p-CSP modulo M and M has the p-CSP modulo N then G has the p-CSP modulo N.*

Let d, $s \in \mathbb{N}_+$ with $s > 2$ and let p be a prime. Set $B = \mathrm{Bas}_s(\mathcal{O}_p^d)$. From Theorem 1.1.6(vi) B is weakly regular branch over its commutator subgroup B' and from Lemma 1.7.3

$$B' \geq \gamma_3(B) \geq B'' > \psi^{-1}(\gamma_3(B)^p).$$

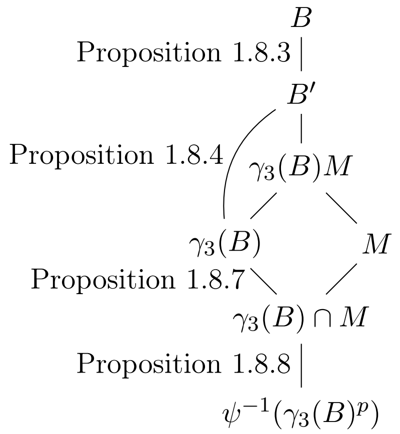


Figure 1.5: The steps of the proof of Theorem 1.1.10, where $M := \psi^{-1}((B')^p)$

We will prove that

(i) B has the p-CSP modulo $\gamma_3(B)$, and,
(ii) $\gamma_3(G)$ has the p-CSP modulo $\psi^{-1}(\gamma_3(B)^p)$.

Then Theorem 1.1.10 follows by a direct application of [55, Theorem 1]. We an application of Lemma 1.8.2 to Proposition 1.8.3 and Proposition 1.8.4 we will obtain step (1). Similarly, by applying Lemma 1.8.2 to Proposition 1.8.7 and Proposition 1.8.8 yields step (2). Now, set $M := \psi^{-1}((B')^p)$ and $N := \psi^{-1}(\gamma_3(B)^p)$. Considering Proposition 1.7.1, Theorem 1.7.2 and Lemma 1.7.3, we summarise the proof of Theorem 1.1.10 in Fig. 1.5.

Proposition 1.8.3. *The group B has the p-CSP modulo B'.*

Proof. Set $b_{is+j} = a_{i,j}$ for all $i \in [0, d-1]$ and $j \in [0, s-1]$. Define, for $r \in [0, ds-1]$, $A_r = \langle b_r, \dots, b_{ds-1}\rangle B'$ and set $A_{ds} = B'$. We will prove that A_r has the p-CSP modulo A_{r+1} for all $r \in [0, ds-1]$. Then the result follows from the Lemma 1.8.2.

Clearly, $A_r/A_{r+1}\,\mathrm{St}_{A_r}(n) \in \mathcal{C}$ and by Theorem 1.1.6(iv) we have $A_r/A_{r+1} = \langle b_r\rangle \cong \mathbb{Z}$. In $\mathbb{Z}$, the subgroups of index a power of p are totally ordered, whence it suffices to prove that $|A_r : A_{r+1}\,\mathrm{St}_{A_r}(n)|$ tends to infinity when n tends to infinity. In fact, we prove that $b_r^{p^n} \notin A_{r+1}\,\mathrm{St}_{A_r}(nds + r + 1)$ for $n \in \mathbb{N}_0$. Assume to the contrary that $b_r^{p^n} \in A_{r+1}\,\mathrm{St}_{A_r}(nds + r + 1)$. In particular, $b_r^{p^n} \in A_{r+1}\,\mathrm{St}_B(nds + r + 1)$. Thanks to Theorem 1.5.1, we have $\mathrm{St}_B(nds+r+1) = \langle b_0^{p^{n+1}}, \dots, b_r^{p^{n+1}}, b_{r+1}^{p^n}, \dots, b_{ds-1}^{p^n}\rangle^B$. Thus, there exists $x_0, \dots, x_{ds-1} \in \mathbb{Z}$ such that

$$b_r^{p^n} \equiv_{B'} b_0^{x_0 p^{n+1}} \cdots b_r^{x_r p^{n+1}} b_{r+1}^{x_{r+1}} \cdots b_{ds-1}^{x_{ds-1}},$$

contradicting Theorem 1.1.6(iv). □

Proposition 1.8.4. *The group B' has the p-CSP modulo $\gamma_3(B)$.*

Proof. Notice from Theorem 1.7.2(ii) that $\gamma_3(B)$ is a subgroup of index a power of p in B' and hence it suffices to prove that $\mathrm{St}_{B'}(n)$ is contained in $\gamma_3(B)$ for some n, equivalently $|B'/\gamma_3(G)\,\mathrm{St}_{B'}(n)| = |B'/\gamma_3(B)|$. Observe that,

$$B'/\gamma_3(B)\,\mathrm{St}_{B'}(n) \cong B'\,\mathrm{St}_B(n)/\gamma_3(B)\,\mathrm{St}_B(n).$$

Now, in light of Theorem 1.7.2(ii), we choose $n \in \mathbb{N}_+$ such that the set

$$\{[a_{i,j}, a_{0,0}] \mid i \in [0, d-1],\ j \in [1, s-1]\} \cup \{[a_{0,1}, a_{i',0}] \mid i' \in [1, d-1]\} \cup \{[a_{0,1}, a_{0,0}]^p\},$$

has trivial intersection with $\mathrm{St}_B(n)$. One can easily compute from the description of the stabilisers in Theorem 1.5.1 that $n = ds+2$ is the smallest number with this property. We construct a group H which admits an epimorphism from the group $B/\gamma_3(B)\,\mathrm{St}_B(ds+2)$ and see that commutator subgroup H' has the desired size.

Now fix $n = ds+2$ and set $\Gamma = B/\gamma_3(B)\,\mathrm{St}_B(n)$. Again from Theorem 1.5.1 we have $\mathrm{St}_B(n) = \langle b_0^{p^2}, b_1^{p^2}, b_2^p, \ldots, b_{ds-1}^p\rangle^B$, where $b_{is+j} = a_{i,j}$ as in the proof of Proposition 1.8.3. By a straightforward calculation using the presentation of $B/\gamma_3(B)$, given in the proof of Theorem 1.7.2(ii), we obtain the following presentation for Γ:

$$\langle\ \mathcal{S} \mid \mathcal{R}\ \rangle, \tag{1.7}$$

where $\mathcal{S} = \{b_r \mid r \in [0, ds-1]\}$ and

$$\mathcal{R} = \left\langle \begin{array}{l|l} b_0^{p^2}, b_1^{p^2}, b_t^p, [b_t, b_{t'}], & t, t' \in [2, ds-1] \\ {[b_1, b_{t''}]}, & t'' \in [2, ds-1],\ \text{not a multiple of } s \\ {[b_0, b_{is}]}, \gamma_3(F) & i \in [1, d-1] \end{array} \right\rangle,$$

where F is the free group on the set of generators of Γ.

Let R be the ring $\mathbb{Z}/p^2\mathbb{Z}$. Let $\mathrm{UT}_{ds+1}(R) \le \mathrm{GL}_{ds+1}(R)$ be the group of all upper triangular matrices over R with entries 1 along the diagonal. Denote by $E_{i,j}(\ell)$ the element of $\mathrm{UT}_{ds+1}(R)$ with the entry $\ell \in R$ at the position (i,j). For $i \in [1, d(s-1)-1]$ and $j \in [1, d-1]$, define

$$\begin{aligned} x_i &= E_{i,ds-1}(p), & y_j &= E_{d(s-1)+j,ds}(p), \\ y &= E_{ds-1,ds}(1), & z &= E_{ds,ds+1}(1), \end{aligned}$$

and define $\mathcal{H}$ to be the subgroup of $\mathrm{UT}_{ds+1}(R)$ generated by the set $\{x_i, y_j, y, z\}$. By abuse of notation denote the image of the set of generators of $\mathcal{H}$ in the quotient group $\mathcal{H}/\gamma_3(\mathcal{H})$ by the same symbols and set $H = \mathcal{H}/\gamma_3(\mathcal{H})$. By an easy computation, we obtain

$$x_i^p = y_j^p = y^{p^2} = z^{p^2} = [x_i, x_{i'}] = [y_j, y_{j'}] = [y, y_j] = [x_i, y_j] = [x_i, z] = \mathrm{id},$$

for all $i, i' \in [1, d(s-1)-1]$ and $j, j' \in [1, d-1]$. Now, fix a bijection α from the set $\{b_r \mid r \in [2, ds-1] \setminus \{s, 2s, \dots, (d-1)s\}$ to the set $\{x_i \mid i \in [1, d(s-1)-1]\}$. Define a map φ from the set of generators of Γ to the set of generators of H by

$$\begin{aligned} \varphi(b_0) &= y & \varphi(b_1) &= z \\ \varphi(b_{js}) &= y_j \text{ for } j \in [1, d-1] & \varphi(b_r) &= \alpha(b_r), \text{ otherwise.} \end{aligned}$$

Then φ extends to an epimorphism $\Gamma \to H$, since as seen above, $\varphi(b_r)$ satisfies all the relations of the given presentation (1.7) of the group Γ. Furthermore, observe that the commutator subgroup of H is generated by the union of the sets

$$\{\ [x_i, y] \mid i \in [1, d(s-1)-1]\ \} \cup \{\ [y_j, z] \mid j \in [1, d-1]\ \} \cup \{\ [y, z]\ \}.$$

Hence,

$$|\Gamma'| \geq |\varphi(\Gamma')| = |H'| = p^{d(s-1)-1} p^{d-1} p^2 = p^{ds}.$$

Indeed $|\Gamma'| \leq |B'/\gamma_3(B)| = p^{ds}$, and thus $|\Gamma'| = p^{ds}$, completing the proof. □

We now need two general lemmata.

Lemma 1.8.5. *Let $H \leq \mathrm{Aut}(T)$ and $L, K \trianglelefteq H$ with $L \leq K$ and let $\mathcal{C}$ be the class of all finite p-groups. Assume further that $H/K \in \mathcal{C}$ and H/L is abelian. If H has the p-CSP modulo L, then K has the p-CSP property modulo L.*

Proof. Let $\tilde{K}$ be a normal subgroup of K satisfying $L \leq \tilde{K}$ and $K/\tilde{K} \in \mathcal{C}$. Since H/L is abelian, $\tilde{K}/L$ is normal in H/L and hence $\tilde{K}$ is normal in H. Also notice that $H/\tilde{K} \in \mathcal{C}$. As H has the p-CSP there exists $n \in \mathbb{N}_0$ such that $\mathrm{St}_H(n) \leq \tilde{K}$. In particular $\mathrm{St}_K(n) = \mathrm{St}_H(n) \cap K \leq \mathrm{St}_H(n) \leq \tilde{K}$, completing the proof. □

Lemma 1.8.6. *Let $H \leq \mathrm{Aut}(T)$ and $L, K \trianglelefteq H$. If KL has the p-CSP modulo L, then K has the p-CSP property modulo $K \cap L$.*

Proof. Choose $\tilde{K} \trianglelefteq K$ with $K \cap L \leq \tilde{K}$ and $K/\tilde{K} \in \mathcal{C}$. Then, $\tilde{K}L \trianglelefteq KL$ and $KL/\tilde{K}L \cong K/\tilde{K} \in \mathcal{C}$. As KL has the p-CSP property modulo L, it holds that $\mathrm{St}_{KL}(n) \leq \tilde{K}L$ for some n. Thus, $\mathrm{St}_K(n) = \mathrm{St}_{KL}(n) \cap K \leq \tilde{K}L \cap K = \tilde{K}$. □

Proposition 1.8.7. *The group $\gamma_3(B)$ has the p-CSP modulo $\gamma_3(B) \cap M$.*

Proof. We prove that $\gamma_3(B)M$ has the p-CSP modulo M. Then by Lemma 1.8.6 we obtain the result. It follows from Proposition 1.7.1 and Theorem 1.7.2(ii) that B'/M is abelian and that $B'/\gamma_3(B)M \in \mathcal{C}$, respectively. Thanks to Lemma 1.8.5, it is enough to prove that B' has the p-CSP modulo M.

Let $i \in [0, d-1], j \in [1, s-1]$ and $k \in [1, p-1]$. Define $c_{i(s-1)+j} := b_{is+j} := a_{i,j}$. Set $t = i(s-1)+j$ and $r = is+j$ and note that c_t is a relabeling of the elements b_r (defined in the proof of Proposition 1.8.3) by excluding the elements of the form b_{is} for $i \in [0, d-1]$. From Proposition 1.7.1, we have

$$B'/M = \langle\ [a_{i,j}, a_{0,0}^k] \mid i \in [0, d-1],\ j \in [1, s-1],\ k \in [1, p-1]\ \rangle.$$

Set $\ell = (k-1)(ds-d) + t$ and $e_\ell = [c_t, a_{0,0}^k]$. Then,

$$\psi(e_\ell) = \psi([c_t, a_{0,0}^k]) = \psi([b_r, a_{0,0}^k]) = (b_{r-1}^{-1}, \mathrm{id}^{*(k-1)}, b_{r-1}, \mathrm{id}^{*(p-k-1)}).$$

For $\ell \in [1, (p-1)(ds-d)]$, set $M_\ell = \langle e_\ell, \dots, e_{(p-1)(ds-d)}\rangle M$ and $M_{(p-1)(ds-d)+1} = M$. It follows from Theorem 1.1.6(iv) that $M_\ell / M_{\ell+1} = \langle e_\ell \rangle \cong \mathbb{Z}$. We will prove that $|M_\ell : M_{\ell+1} \operatorname{St}_{M_\ell}(n)|$ tends to infinity as n tends to infinity. Assume to the contrary that there are $n, n' \in \mathbb{N}_+$ such that for all $\tilde{n} \geq n'$, $e_\ell^{p^n} \in M_{\ell+1} \operatorname{St}_{M_\ell}(\tilde{n})$. There exist integers $x_{\ell+1}, \dots, x_{(p-1)(ds-d)} \in \mathbb{Z}$ such that

$$e_\ell^{p^n} e_{\ell+1}^{x_1} \cdots e_{(p-1)(ds-d)}^{x_{(p-1)(ds-d)}} \in M \operatorname{St}_{M_\ell}(\tilde{n}) \leq M \operatorname{St}_B(\tilde{n}),$$

hence

$$\psi(e_\ell^{p^n} e_{\ell+1}^{x_1} \cdots e_{(p-1)(ds-d)}^{x_{(p-1)(ds-d)}}) \in (B')^p \cdot (\operatorname{St}_B(\tilde{n}-1))^p.$$

Consider the k^{th} coordinate, $x b_{r-1}^{p^n} \in B' \operatorname{St}_B(\tilde{n}-1)$, where x is a product of elements of the form $b_{r'}$ such that $r' > r-1$. Then $x \in A_r$, where A_r is defined as in the proof of Proposition 1.8.3. This implies $b_{r-1}^{p^n} \in A_r \operatorname{St}_B(\tilde{n})$ for all $\tilde{n} \geq n'-1$, which contradicts Proposition 1.8.3. □

Proposition 1.8.8. *The group $\gamma_3(B) \cap M$ has the p-CSP modulo N.*

Proof. It is straightforward from Theorem 1.7.2(ii) that the group M/N is a finite abelian and $M/N \in \mathcal{C}$. By Lemma 1.8.5, it suffices to prove that M has the p-CSP modulo N. From Proposition 1.8.4, it follows that $\operatorname{St}_{B'}(n) \leq \gamma_3(B)$ for some n. Therefore,

$$\psi(\operatorname{St}_M(n+1)) \leq (\operatorname{St}_{B'}(n))^p \leq \gamma_3(B)^p,$$

and hence $\operatorname{St}_M(n+1) \leq \psi^{-1}((\operatorname{St}_{B'}(n))^p) \leq N$. □

Proof of Theorem 1.1.10. By applying Lemma 1.8.2 to Proposition 1.8.3 and Proposition 1.8.4 we obtain that the group B has the p-CSP modulo $\gamma_3(B)$. Further application of Lemma 1.8.2 to Proposition 1.8.7 and Proposition 1.8.8 yields that $\gamma_3(G)$ has the p-CSP modulo N. Now, the result follows by [55, Theorem 1]. □

Chapter 2

Two periodicity conditions for spinal groups

Abstract. A constant spinal group is a subgroup of the automorphism group of a regular rooted tree, generated by a group of rooted automorphisms A and a group of directed automorphisms B whose action on a subtree is equal to the global action. We provide two conditions in terms of certain dynamical systems determined by A and B for constant spinal groups to be periodic, generalising previous results on Grigorchuk–Gupta–Sidki groups and other related constructions. This allows us to provide various new examples of finitely generated infinite periodic groups.

2.1 — Introduction

Spinal groups are subgroups of the automorphism group of a regular rooted tree, that are generated by a group of rooted automorphisms A and a group of directed automorphisms B. Their definition is motivated by early constructions of Grigorchuk [65] and Gupta and Sidki [77], and they provide examples of groups of intermediate word growth, of just-infinite groups, and of finitely generated infinite periodic groups. All spinal groups are infinite, and it is easy to recognise when they are finitely generated, but it is a complicated task to obtain conditions in terms of the defining data, i.e. the rooted and directed groups, that ensure that a spinal group is periodic. In this paper, we establish two such conditions.

The subclasses of spinal groups for which periodicity conditions are known fall in two categories. They are either generalisations of the (first) Grigorchuk group, or of the Gupta–Sidki p-groups. We shall concentrate on the latter case, and consider what we will call constant spinal groups (in short CS groups), i.e. spinal groups with all directed automorphisms b referring directly to themselves, hence allowing a wreath decomposition

$$b = (b, a_1, \dots, a_{m-1}),$$

where the elements a_i are rooted. The class of CS groups includes all groups defined by special decoration functions described in [78], all (multi-)GGS-groups, and the generalised GGS-groups of Bartholdi [9]. Within these subclasses, some conditions for periodicity are known, coming in three strands: a precise criterion for generalised GGS-groups with abelian rooted group given by Bartholdi [9], adapted from Vovkivski's criterion for (some) GGS-groups [151], which is related to a variant for multi-GGS-groups studied by Alexoudas, Klopsch and Thillaisundaram [4]. On the other strand we have a sufficient condition for groups defined by special decoration functions formulated by Gupta and Sidki [78]. Furthermore, criteria for general subgroups of the automorphism group of a binary tree to be periodic were given by Sidki in [139]. We develop the first two strands and provide two conditions for periodicity, that can be easily applied also to groups with non-cyclic directed groups and non-regular action of the rooted group. Furthermore, our results extend to non-locally finite trees and infinite rooted groups.

This is a previously unobserved phenomenon and allows us to give new examples of finitely generated infinite periodic groups. In particular, we show that the group of automorphisms of a regular rooted tree acting locally like a prescribed transitive permutation group H allows infinite finitely generated periodic groups that replicate the action of H on all vertices for many (but not all) permutation groups H.

We state our conditions in the form of certain dynamical systems on the power set of the rooted group. The system Λ_b – depending on a directed automorphism b – models the stabilised sections of automorphisms ba, for $a \in A$, which are themselves a new tool tailor-made to observe periodic elements, see Definition 2.2.2. The system Σ_S models the possible local actions of automorphisms that do not reduce a certain length under taking stabilised sections. See Section 2.2.4 for a precise definition of the dynamical systems Λ_b and Σ_S. We hope that these tools prove to be useful for further considerations of periodic groups within $\mathrm{Aut}(X^*)$. Let us now state both conditions as theorems.

Theorem 2.1.1. *Let $G = \langle A \cup B \rangle$ be a stable and strongly orbitwise-abelian* CS *group with periodic rooted group A such that either (i) A is of finite exponent, or (ii) the directed group B has finite support. Assume that B is abelian and periodic. If the dynamical system Λ_b is eventually trivial for all $b \in B$, then G is periodic.*

Theorem 2.1.2. *Let $G = \langle A \cup B \rangle$ be a* CS *group with periodic rooted group A such that either (i) A is of finite exponent, or (ii) the directed group B has finite support. Let S be a generating set for B. If the dynamical system Σ_S is eventually trivial, then G is periodic.*

For a CS group to be strongly orbitwise-abelian the sections of directed elements along certain orbits of the rooted group must commute, generalising the case when the rooted group is abelian. Stability is a related homogeneity condition also trivially fulfilled by CS groups with abelian rooted group, for precise definitions see Definition 2.2.12 and 2.2.13. However, Theorem 2.1.1 applies to many more groups, cf. Example 2.3.5.

In general, the dynamical systems Λ_b are more likely to be eventually trivial than the systems Σ_S, which justifies the additional conditions of Theorem 2.1.1.

All previously mentioned conditions for periodicity of subclasses of CS groups can be derived from the two theorems above; see Section 2.3.2 and Remark 2.3.6. Although

Theorem 2.1.1 and Theorem 2.1.2 give only sufficient conditions for periodicity, in the case of an abelian rooted group the sufficient and necessary criterion of Bartholdi may be extended to a broader class of groups; cf. Corollary 2.3.2.

Recently, Rajeev and the author [125] described a method of constructing, given a group $G \leq \mathrm{Aut}(X^*)$ and a positive integer s, a new group $\mathrm{Bas}_s(G) \leq \mathrm{Aut}(X^*)$ called the s^{th} Basilica group of G, based on the Basilica group defined by Grigorchuk and Żuk [73]. It is a remarkable feature that the Basilica groups of CS groups are CS groups. We prove that Basilica groups of CS groups satisfying the conditions of Theorem 2.1.2 again satisfy Theorem 2.1.2, which is relevant to the question if Basilica groups of periodic groups are periodic in general. Combined with our good understanding of periodic CS groups in the abelian case, this provides a wealth of examples of periodic CS groups with non-regular rooted action.

Both Theorem 2.1.1 and 2.1.2 only provide sufficient conditions, and, in addition to a family of groups that are periodic in accordance with Theorem 2.1.2 (Example 2.3.8), we give an example of a periodic CS group that satisfies neither the condition of Theorem 2.1.1 nor those of Theorem 2.1.2.

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2.2 — Preliminaries and CS groups

2.2.1. Automorphisms of rooted trees. — Let X be a non-empty set, $0 \in X$ an element and X^* the free monoid over X, which we identify with its Cayley graph, i.e. a regular rooted tree of valency $|X| + 1$. We call X the *alphabet*, its members *letters* and 0 the *distinguished letter*. Observe that we do not restrict to finite sets. Denote $X \setminus \{0\}$ by $\dot{X}$. If X is a group, we require that the distinguished letter equals the neutral element. We write ϵ for root, i.e. the empty word. The *n^{th} layer of X^** is the set X^n of vertices represented by words of length n, or, equivalently, of vertices of distance n to the root.

A *(rooted tree) automorphism* of X^* is a graph automorphism fixing the root ϵ. The invariance of the root is a feature of all graph automorphisms if X is finite. Since the root is fixed, every rooted tree automorphism maps the n^{th} layer to itself. The action of the full group of rooted tree automorphisms $\mathrm{Aut}(X^*)$ is transitive on each layer. A subgroup of $\mathrm{Aut}(X^*)$ with this property is called *spherically transitive*. The stabiliser of a word u under the action of a group $G \leq \mathrm{Aut}(X^*)$ is denoted by $\mathrm{st}_G(u)$, and the intersection of all stabilisers of words of length n is called the *n^{th} layer stabiliser*, denoted $\mathrm{St}_G(n)$.

Let a be an automorphism of X^* and let u, v be words. We denote *concatenation*, i.e.

the monoid multiplication in X^*, by $\star$. Since layers are invariant under a, the equation

$$(u \star v).a = u.a \star v.(a|_u)$$

defines a unique automorphism $a|_u$ of X^* called the *section of a at u*.

Consequently, any automorphism a can be decomposed into the sections prescribing the action at the subtrees of the first layer, and $a|^\epsilon$, the action of a on the first layer $X^1 = X$. We adopt the convention that an X-indexed family of automorphisms $(a_x)_{x\in X}$ is identified with the automorphism having section a_x at x and $(a_x)_{x\in X}|^\epsilon = \mathrm{id}$. Hence for any a we write

$$a = (a|_x)_{x\in X}\; a|^\epsilon.$$

Conversely, any family $(a|_x)_{x\in X}$ of automorphisms together a permutation $a|^\epsilon \in \mathrm{Sym}(X)$ defines a unique automorphism of X^*. For all words $u, v \in X^*$ and automorphisms $a, b \in \mathrm{Aut}(X^*)$ we have

$$(a|_u)|_v = a|_{u\star v}, \quad (ab)|_u = a|_u b|_{u.a}, \quad a^{-1}|_u = (a|_{u.a^{-1}})^{-1}.$$

Let $H \le \mathrm{Sym}(X)$ be a subgroup. By the computation rules above, we see that the set of automorphisms a such that $a|_u|^\epsilon \in H$ for all $u \in X^*$ is a subgroup, denoted $\Gamma(H)$ and called the *H-labeled subgroup of* $\mathrm{Aut}(X^*)$.

By convention, permutations of X act on X^* by permuting the first layer subtrees, i.e. we consider $\mathrm{Sym}(X)$ to be embedded into $\mathrm{Aut}(X^*)$ by $\rho \mapsto (\mathrm{id})_{x\in X}\,\rho$. Automorphisms of this kind are called *rooted*. Consequently, we think of $\mathrm{Aut}(X^*)$ as acting from the right, however, we conjugate from the left, writing ${}^a b = aba^{-1}$. The purpose of this is twofold: calculation of sections of conjugates involve fewer inversions, i.e. for $g, h \in \mathrm{Aut}(X^*)$ and $u \in X^*$ we have $({}^h g)|_x = {}^{h|_x}(g|_{x.h})$, and furthermore, since on some occasions group elements or vertices will be represented with integers, we hope to better distinguish between powers and conjugation.

2.2.2. Constant spinal groups. — We now define the family of constant spinal groups, which includes all (multi-)GGS-groups and the more general groups defined in [78] by a special decorating function. Both name and definition are derived from the family of spinal groups defined by Bartholdi and Šunik [13, 19], which we shall not recall in detail, but we remark that the defining data of a general spinal group contains a certain sequence of homomorphisms, and the constant spinal groups are those where the defining sequence is constant.

Definition 2.2.1. Let $A \le \mathrm{Sym}(X)$ be a transitive permutation group, embedded into $\mathrm{Aut}(X^*)$ as rooted automorphisms, and let $B \le \mathrm{St}(1)$ be a subgroup such that

$$\langle B|_x \mid x \in \dot{X}\rangle = A \quad \text{and} \quad b|_0 = b \;\text{ for all } b \in B.$$

We call the group $G = \langle A, B\rangle$ a *constant spinal group*, or in short a CS *group*. The groups A and B are referred to as the *rooted group* and the *directed group of* G, respectively. We say that B has *finite support* if for all $b \in B$ and all almost all $x \in X$ we have $b|_x = \mathrm{id}$.

A GGS-group is most commonly defined by a so-called *defining vector*: Let $m \in \mathbb{N}_{\geq 2}$, let $X = \{0, \dots, m-1\}$, let $a \in \mathrm{Sym}(X)$ be an m-cycle, and $\underline{e} = (e_1, \dots, e_{m-1})$ a vector in the free module $(\mathbb{Z}/m\mathbb{Z})^{m-1}$ such that $\langle e_i \mid i \in \dot{X}\rangle = \mathbb{Z}/m\mathbb{Z}$. Define an automorphism $b = (b, a^{e_1}, \dots, a^{e_{m-1}})$. Then $\langle a, b\rangle$ is the GGS-group with defining vector $\underline{e}$. In the terminology of CS groups, this is the CS group with rooted group $\langle a\rangle$ and directed group $\langle b\rangle$.

For comparison with previous results on periodicity of spinal groups, we give a small dictionary that describes certain well-studied subfamilies of spinal groups in the terminology of CS groups.

- GGS-groups are CS groups whose rooted group and directed group are cyclic.
- The multi-GGS-groups of [4] are CS groups with cyclic rooted group.
- The generalised GGS-groups of [9] are CS groups with cyclic directed group.
- The groups of [78] defined by a special decorating function are CS groups with rooted group acting regularly and cyclic directed group.
- Neither the so-called $\mathcal{G}$ groups of [19] nor the EGS groups of [119] (both are classes of spinal groups where periodicity criteria are known) are CS groups. This is clear for their usual embedding into $\mathrm{Aut}(X^*)$, and follows from [126] for their isomorphism classes.

It is evident that all CS groups are *self-similar*, i.e. that for all $x \in X$ and $g \in G$ the section $g|_x$ is contained in G. Furthermore they are *fractal*, i.e. $\mathrm{St}_G(1)|_x = G$ for all $x \in X$: observe that ${}^a b|_x = b|_{x.a}$ with $a \in A, b \in B$ produces a generating set for A, as well as for B, using the transitivity of A. This implies that G is spherically transitive. Clearly G is contained in the A-labelled subgroup of $\mathrm{Aut}(X^*)$. Since $\mathrm{St}_G(1)$ is a proper subgroup with a surjection to the full group, G is infinite.

Every CS group G is a homomorphic image of the free product $A * B$. Thus every element $g \in G$ can be written in the form

$$g = ({}^{a_0}b_0)({}^{a_1}b_1)\dots({}^{a_{n-1}}b_{n-1})a_n, \qquad a_i \in A, b_i \in B, n \in \mathbb{N}.$$

The minimum of all possible numbers n in such words is called the *syllable length* of $g \in G$, denoted $\mathrm{syl}(g)$. This is the weighted word length with respect to the generating set $A \cup B$ with elements in A having weight 0. Consequently, we refer to the elements ${}^A B$ as *syllables*. Observe that all syllables stabilise the first layer, hence $g|^\epsilon = a_n$.

The common strategy for proving that a CS group is periodic, starting from the proofs for the examples of Grigorchuk and Gupta and Sidki, is to establish a contraction of the syllable length upon taking sections. We refine the usual statements concerning this contraction by employing the new concept of stablised sections.

Definition 2.2.2. Let $v \in X^*$ be a vertex and $g \in \operatorname{Aut}(X^*)$. Let $\ell_g(v)$ be the length of the orbit of v under g. If $\ell_g(v) < \infty$, define the *stablised section of g at v* by

$$g\|_v := g^{\ell_g(v)}|_v.$$

The usefulness of stabilised sections for establishing periodicity of groups of automorphisms of X^* stems from the fact that powers of g stabilising the n^{th} layer can be described by certain stabilised sections. We record this and two other useful facts in a lemma.

Lemma 2.2.3. *Let $g \in \operatorname{Aut}(X^*)$ and let $n, m \in \mathbb{N}$ such that $g^n \in \operatorname{St}(m)$. Then*

(i) $g^n|_v = (g\|_v)^{\frac{n}{\ell_g(v)}}$ *for all $v \in X^m$.*

Let $u, v \in X^*$ such that $\ell_g(u \star v)$ is finite. We have:

(ii) $\ell_g(u \star v) = \ell_g(u) \cdot \ell_{g\|_u}(v)$.
(iii) $g\|_{u\star v} = g\|_u\|_v$.

Proof. Since v is stabilised by g^n, the number n is a multiple of $\ell_g(v)$. Thus

$$g^n|_v = g^{\ell_g(v)\cdot\frac{n}{\ell_g(v)}}|_v = (g\|_v)^{\frac{n}{\ell_g(v)}}.$$

For (ii), observe that $\ell_g(u \star v)$ must be a multiple of $\ell_g(u)$, and calculate

$$(u \star v).g^{\ell_g(u)} = u.g^{\ell_g(u)} \star v.(g^{\ell_g(u)}|_u) = u \star v.(g\|_u).$$

Using the equation again, we prove (iii):

$$g\|_{u\star v} = g^{\ell_g(u)\ell_{g\|_u}(v)}|_{u\star v} = (g^{\ell_g(u)}|_u)^{\ell_{g\|_u}(v)}|_v = g\|_u\|_v. \qquad \square$$

We shall use Lemma 2.2.3 regularly and without reference.

Lemma 2.2.4. *Let G be a* CS *group and let $g \in G$. We have:*

(i) $\operatorname{syl}(g|_x) \leq \frac{1}{2}(\operatorname{syl}(g) + 1)$ *for all $x \in X$.*
(ii) $\sum_{x\in X} \operatorname{syl}(g|_x) \leq \operatorname{syl}(g)$.
(iii) $\operatorname{syl}(g\|_x) \leq \operatorname{syl}(g)$ *for all $x \in X$ such that $\ell_g(x) < \infty$.*
(iv) *If $g|^\epsilon$ has finite order, then $g\|_x \in A$ for almost all $x \in X$.*

Proof. (i) Consider an element $g = {}^{a_0}b_0{}^{a_1}b_1 \ldots {}^{a_{n-1}}b_{n-1}a_n \in G$, where $n = \operatorname{syl}(g)$. Taking the section at $x \in X$, one obtains

$$g|_x = b_0|_{x.a_0} b_1|_{x.a_1} \ldots b_{n-1}|_{x.a_{n-1}}. \qquad (*)$$

For each $i \in \{0, \dots, n-1\}$ the section $b_i|_{x.a_i}$ is an element of $G \setminus A$ if and only if $x.a_i = 0$. In this case it is a member of B. If two consecutive syllables both yield an element of B, they reduce to a single syllable. Hence at most every second syllable of g produces a syllable of $g|_x$, yielding the inequality.

(ii) This is a consequence of the fact that every syllable produces exactly one letter from B in all of its sections.

(iii) For any section of $g^{\ell_g(x)}$ we have

$$g^{\ell_g(x)}|_x = g|_x g|_{x.g} \cdots g|_{x.g^{\ell_g(x)-1}}. \tag{**}$$

But since $\{x, x.g, \dots, x.g^{\ell_g(x)-1}\}$ is the orbit of x under g, all sections are taken at different positions in X and the assertion follows from (ii).

(iv) If $g|^\epsilon$ has finite order, every $\langle g \rangle$-orbit in X is finite. By (ii) there are only finitely many sections outside A, i.e. there are only finitely many $x \in X$ such that $g|_x \notin A$. Thus only finitely many stabilised sections of g can have positive syllable length. □

Since $b\|_0 = b|_0 = b$ holds for all $b \in B$, the inequality in (iii) of the previous lemma cannot be strict for all non-trivial elements of a given CS group. However, for some CS groups it is possible to obtain a strict inequality excluding a controllable 'error' set, on which our conditions for G to be periodic rest.

2.2.3. Generalities on periodic CS groups. — Since we do not restrict to finite alphabets X, we need two lemmata to control possible problems for infinite X.

Lemma 2.2.5. *Let $H \leq \mathrm{Sym}(X)$ be periodic and $\Gamma(H) \leq \mathrm{Aut}(X^*)$ the H-labelled subgroup. Then $\Gamma(H)/\mathrm{St}_{\Gamma(H)}(n)$ is periodic for all $n \in \mathbb{N}$.*

Proof. The group $\Gamma(H)/\mathrm{St}_{\Gamma(H)}(n)$ is isomorphic to the n-fold iterated unrestricted wreath product $H \wr \dots \wr H$, which is periodic. □

Lemma 2.2.6. *Let G be a CS group such that the directed group B has finite support, let $n \in \mathbb{N}$. Then for every $g \in G$ the set $\{g|_v \mid v \in X^n\}$ is finite.*

Proof. This is an immediate consequence of the composition rule for sections. □

Proposition 2.2.7. *Let G be a CS group with periodic rooted group A. Let either (i) A be of finite exponent, or (ii) let the directed group B have finite support. Let $T \subseteq G$ be a set of elements of finite order. If for every $g \in G \setminus A$ there is some $m \in \mathbb{N}$ such that for all $v \in X^m$ we have*

$$\mathrm{syl}(g\|_v) < \mathrm{syl}(g) \quad \text{or} \quad g\|_v \in T, \tag{†}$$

the group G is periodic.

Proof. We use induction on the syllable length. Rooted elements are of finite order, thus we consider only g with $\mathrm{syl}(g) \geq 1$. Let m be the integer such that (†) holds for g and X^n. Since G is contained in $\Gamma(A)$, the order of the action of g on X^m is a finite number $k \in \mathbb{N}$ by Lemma 2.2.5, thus

$$g^k = ((g\|_v)^{\frac{k}{\ell_g(v)}})_{v \in X^m}.$$

By assumption, either $g\|_v$ is of shorter syllable length than g, or it is a member of T, hence of finite order. By induction $g\|_v$ is of finite order. Also, by Lemma 2.2.4(4) at most finitely many stabilised sections have positive syllable length.

In case (i), let k' be the least common multiple of the order of these sections and the exponent of A, in case (ii) let k' be the least common multiple of the orders of all (finitely many by Lemma 2.2.6) non-trivial first layer sections of $g\|_v$. In both cases $(g\|_v)^{k'} = \mathrm{id}$ for all $v \in X^n$, hence $(g^k)^{k'} = \mathrm{id}$. ☐

We remark that the set T has to include the directed group B as a subset, since $b|_0 = b$. Therefore the existence of an error set T implies that B is a periodic group.

2.2.4. Some dynamical systems. — We use Proposition 2.2.7 to establish our two dynamical conditions for periodicity. It is beneficial to reformulate this proposition using the language of atomic dynamical systems.

Definition 2.2.8. Let A be a set and let $\phi : \mathcal{P}(A) \to \mathcal{P}(A)$ be a self-map of the powerset of A. The dynamical system $(\mathcal{P}(A), \phi)$ is called *atomic* if

$$\phi(S) = \bigcup_{s \in S} \phi(\{s\})$$

for all $S \in \mathcal{P}(A)$. Clearly, an atomic dynamical system is defined by its action on singletons. It is called *eventually* T if for all $a \in A$ there is an integer $n \in \mathbb{N}$ such that for all $m > n$ we have $\phi^m(\{a\}) \subset T$. If A is a group, the system is *eventually trivial* if it is eventually $\{1_A\}$.

Now we may restate Proposition 2.2.7. A CS group G with periodic rooted and direct group fulfilling condition (i) or (ii) as stated in the proposition is periodic if the dynamical system $(\mathcal{P}(G), \|_X)$ is eventually $A \cup T$, where the self-map is atomic and hence defined by

$$\|_X : \mathcal{P}(G) \to \mathcal{P}(G), \quad \{g\} \mapsto \{g\|_x \mid x \in X\}.$$

We want to obtain, at least incase of G having a finite rooted group, similar conditions on finite dynamical systems, which we shall now define. First, we need some terminology.

Let $x, y \in X$ and let $A \leq \mathrm{Sym}(X)$ be a group acting transitively on X. Write $\mathrm{mp}_A(x, y)$ for the set of elements $a \in A$ such that $x.a = y$, and fix an element $e_{x \mapsto y} \in \mathrm{mp}_A(x, y)$. Then

$$\mathrm{mp}_A(x, y) = e_{x \mapsto y} \cdot \mathrm{st}_A(y) = \mathrm{st}_A(x) \cdot e_{x \mapsto y}.$$

By convention, set $e_{0\mapsto 0} = 1_A$. The sets $\mathrm{mp}_A(0,x)$ with $x \in X$ are right cosets of the point stabiliser $\mathrm{st}_A(0)$, hence they form a partition of A, which we use to define certain subsets of the set of conjugates of a given element.

Definition 2.2.9. Let $G = \langle A \cup B\rangle$ be a CS group. For $a \in A$ and $x \in X$, we define

$$\mathfrak{C}(a,x) := {}^{\mathrm{mp}_A(0,x)}a \subseteq A \quad \text{and} \quad \mathfrak{X}(a,x) := \bigcup_{c\in\mathfrak{C}(a,x)} \mathrm{orb}_c(0) \setminus \{0\} \subseteq \dot{X}.$$

For every x in the orbit $\mathrm{orb}_a(0)$ of 0 under a we have $a \in \mathfrak{C}(a,x)$. If A acts regularly, we identify X with A such that $e_{0\to x} = x$, hence $\mathfrak{C}(a,x) = \{{}^x a\}$ and $\mathfrak{X}(a,x) = \mathrm{orb}_{{}^x a}(0)\setminus\{0\} = 0.(\langle {}^x a\rangle \setminus \{1_A\})$. We now define the first dynamical system.

Definition 2.2.10. Let $G = \langle A \cup B\rangle$ be a CS group, let $S \subseteq B$, let $a \in A$ and let $x \in X$. Define a map

$$\sigma_S(a,x) = \langle \prod_{c'\in\langle c\rangle\setminus \mathrm{st}_A(0)} b|_{0.c'} \mid c \in \mathfrak{C}(a,x), b \in S\rangle \cdot \langle b|_y \mid y \in \mathfrak{X}(a,x), b \in S\rangle' \subseteq A. \qquad (\P)$$

Notice that the order of the products generating the left side of the definition can be chosen arbitrarily, using the derived subgroup on the right side. Now the dynamical system $(\mathcal{P}(A), \Sigma_S)$ is the atomic system defined by

$$\Sigma_S : \mathcal{P}(A) \to \mathcal{P}(A), \quad \Sigma_S(\{a\}) = \bigcup_{x\in X} \sigma_S(a,x).$$

Our second dynamical system makes use of the stabilised section map $\|_0$.

Definition 2.2.11. Let $G = \langle A \cup B\rangle$ be a CS group with periodic rooted group A. For every $b \in B$ we define a map $\lambda_b : A \to A$ by

$$\lambda_b(a) := \prod_{i=1}^{\ell_a(0)-1} b|_{0.a^i} \quad \text{so that} \quad (ba)\|_0 = b\lambda_b(a).$$

Let $x \in X$. Then we define

$$\lambda_b(a,x) = \lambda_b({}^{e_{0\mapsto x}}a),$$

and an atomic dynamical system $(\mathcal{P}(A), \Lambda_b)$ given by the map

$$\Lambda_b(\{a\}) = \{\lambda_b(a,x) \mid x \in X\}.$$

Clearly, $\lambda_b(a,0) = \lambda_b(a)$ for all $a \in A, b \in B$. If the group A is abelian, $\lambda_b(a,x) = \lambda_b(a)$ for all $a \in A, b \in B$ and $x \in X$. Consequently, we may regard Λ_b as a map of type $A \to A$ in the abelian case, and write $\Lambda_b(a) = \lambda_b(a)$.

Definition 2.2.12. A CS group G is called *stable* if for all $a \in A, x \in X$ and $b \in B$

$$\lambda_b(c') = \lambda_b(c'') \text{ for all } c', c'' \in \mathfrak{C}(a, x).$$

Let G be a stable CS group, and let $a, a' \in A, b \in B$. Then $\lambda_b({}^{a'}a) \in \Lambda_b(a)$, since $\lambda_b({}^{a'}a) = \lambda_b({}^{e_{0 \mapsto 0.a'}}a)$.

2.2.5. (Strongly) orbitwise-abelian CS groups. — Reviewing the dynamical systems on $\mathcal{P}(A)$ defined by Σ_S and Λ_b with $S \subseteq B, b \in B$, we see that (heuristically), the systems Λ_b are more likely to be eventually trivial, since $|\Lambda_b(a)| \leq |X|$, while the case $\Sigma_S(a) = \sigma_S(a, x) = A$ does frequently (and naturally, cf. Section 2.4.1) occur. The drawback of the dynamical systems Λ_b is that we need some (weak) form of abelianess within A to conclude the periodicity of G, even if all Λ_b are eventually trivial.

Definition 2.2.13. A CS group is called *orbitwise-abelian* if for all $a \in A$

$$\langle b|_{0.c} \mid c \in \langle a \rangle \setminus \mathrm{st}_A(0), b \in B \rangle \leq A$$

is abelian. It is called *strongly orbitwise-abelian* if

$$\langle b|_y \mid y \in \mathfrak{X}(a, x), b \in B \rangle \leq A$$

is abelian for all $a \in A$ and $x \in X$.

Any strongly orbitwise-abelian group is orbitwise-abelian, since $a \in \mathfrak{C}(a, 0) = {}^{\mathrm{st}_A(0)}a$, hence $0.c \in \mathfrak{X}(a, 0)$ for all $c \in \langle a \rangle \setminus \mathrm{st}_A(0)$. Notice that for a strongly orbitwise-abelian group, the derived group in the Definition 2.2.10 of Σ_S is trivial. We also gain better control over the interplay between the dynamical systems Λ_b for various $b \in B$.

Lemma 2.2.14. *Let G be a orbitwise-abelian* CS *group with periodic rooted group. Then the map $b \mapsto \lambda_b : B \to A^A$ is a group homomorphism.*

Proof. Let $b_1, b_2 \in B$. Then for all $a \in A$,

$$\begin{aligned}\lambda_{b_1 b_2}(a) = \prod_{i=1}^{\ell_a(0)-1} (b_1 b_2)|_{0.a^i} &= \prod_{i=1}^{\ell_a(0)-1} b_1|_{0.a^i} b_2|_{0.a^i} \\ &= \prod_{i=1}^{\ell_a(0)-1} b_1|_{0.a^i} \prod_{i=1}^{\ell_a(0)-1} b_2|_{0.a^i} = \lambda_{b_1}(a)\lambda_{b_2}(a). \square\end{aligned}$$

The map $b \mapsto \lambda_b$ is not necessarily injective, even if G is orbitwise-abelian. Take for example a GGS-p-group $G = \langle a, b \rangle$ with defining vector $\underline{e}$ on a p-adic rooted tree such that

$\sum_{i=1}^{p-1} e_i \equiv_p 0$. On p-adic trees, these are precisely the periodic GGS-groups. We have

$$\lambda_b(a^j) = \prod_{i=1}^{p-1} b|_{0.a^{ji}} = a^{\sum_{i=1}^{p-1} e_i} = 1_A$$

for all $j \in \{0, \dots, p-1\}$. Thus the image of B under $b \mapsto \lambda_b$ is trivial.

2.3 — Proofs and discussion of the theorems

2.3.1. Proof of Theorem 2.1.1. — It is useful to consider the next lemma separately, as it will also be used in the proof of Theorem 2.1.2.

Lemma 2.3.1. *Let $G = \langle A \cup B \rangle$ be a* CS *group, let $g = {}^{a_0}b_0 \dots {}^{a_{n-1}}b_{n-1}a_n \in G$ and let $x \in X$ such that $n = \mathrm{syl}(g\|_x) = \mathrm{syl}(g)$. Then there are integers $j(i)$, for all $i \in \{0, \dots, n-1\}$, such that*

$$g\|_x = \prod_{j=0}^{\ell-1} \prod_{i=0}^{n-1} b_i|_{0.{}^{a_i^{-1}}(a_n^{j+j(i)})}$$

where $\ell = \ell_g(x)$ and ${}^{a_i^{-1}}a_n \in \mathfrak{C}(a_n, x)$.

Proof. We compute the stabilised section at x, using the equations $(*)$ and $(**)$;

$$\begin{aligned} g\|_x = \left(\prod_{j=0}^{\ell-1} \left(\left(\prod_{i=0}^{n-1} ({}^{a_i}b_i) \right) a_n \right) \right) \Big|_x &= \prod_{j=0}^{\ell-1} \prod_{i=0}^{n-1} ({}^{a_i}b_i|_{x.a_n^j}) \\ = \quad & b_0\,|_{x.a_0} \cdot b_1\,|_{x.a_1} \quad \cdots \quad \cdot\, b_{n-1}\,|_{x.a_{n-1}} \\ & \cdot\, b_0\,|_{x.a_na_0} \cdot b_1\,|_{x.a_na_1} \quad \cdots \quad \cdot\, b_{n-1}\,|_{x.a_na_{n-1}} \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ & \cdot\, b_0\,|_{x.a_n^{\ell-1}a_0} \cdot b_1\,|_{x.a_n^{\ell-1}a_1} \quad \cdots \quad \cdot\, b_{n-1}\,|_{x.a_n^{\ell-1}a_{n-1}}. \end{aligned}$$

Notice that in every column, we are taking the sections of a fixed element $b_i \in B$ at the vertices from the shifted orbit $\mathrm{orb}_{a_n}(x).a_i$. Consequently all sections are taken at different vertices, and there is at most one directed element in every column of the product. Since there are only n columns, by $\mathrm{syl}(g) = \mathrm{syl}(g\|_x)$ every column must contain precisely one directed element, so for all i we have $0 \in \mathrm{orb}_{a_n}(x).a_i$, hence there is some integer $j(i)$ such that $x.a_n^{-j(i)}a_i = 0$. Thus $0.a_i^{-1}a_n^{j(i)} = x$, and ${}^{a_i^{-1}}a_n = {}^{a_i^{-1}a_n^{j(i)}}a_n \in \mathfrak{C}(a_n, x)$. Lastly we notice that $x.a_n^j a_i = 0.{}^{a_i^{-1}}(a_n^{j+j(i)})$, thus

$$g\|_x = \prod_{j=0}^{\ell-1} \prod_{i=0}^{n-1} b_i|_{x.a_n^j a_i} = \prod_{j=0}^{\ell-1} \prod_{i=0}^{n-1} b_i|_{0.{}^{a_i^{-1}}(a_n^{j+j(i)})}. \qquad \square$$

Proof of Theorem 2.1.1. We establish condition (†) of Proposition 2.2.7, for $T = B$.

Let $x \in X$ and let $g \in G$ be of syllable length $n > 0$ and let

$$g = {}^{a_0}b_0 \dots {}^{a_{n-1}}b_{n-1}a_n \tag{§}$$

such that $n = \operatorname{syl}(g\|_x) = \operatorname{syl}(g)$. If such an element does not exist outside of B, (†) is established.

By Lemma 2.3.1 the elements ${}^{a_i^{-1}}a_n$ are members of $\mathfrak{C}(a_n, x)$ for $i \in \{0, \dots, n-1\}$. Now $\operatorname{orb}_{a_n}(x).a_i = \operatorname{orb}_{{}^{a_i^{-1}}a_n}(0)$, hence we write $\ell \coloneqq \ell_g(x) = \ell_{a_n}(x) = \ell_{{}^{a_i^{-1}}a_n}(0)$. Using the notation of Lemma 2.3.1 we have

$$g\|_x = \prod_{j=0}^{\ell-1}\prod_{i=0}^{n-1} b_i|_{0.{}^{a_i^{-1}}(a_n^{j+j(i)})} \equiv_{\operatorname{St}_G(1)} \prod_{j=0}^{\ell-1} \prod_{\substack{i=0\\ j+j(i)\neq\ell}}^{n-1} b_i|_{0.({}^{a_i^{-1}}a_n)^{j+j(i)}}. \tag{§§}$$

Since G is strongly orbitwise-abelian we may reorder the product, calculating:

$$\begin{aligned}
g\|_x &\equiv_{\operatorname{St}_G(1)} \prod_{i=0}^{n-1} \prod_{\substack{j=0\\ j+j(i)\neq\ell}}^{\ell-1} b_i|_{0.({}^{a_i^{-1}}a_n)^{j+j(i)}} \\
&= \prod_{i=0}^{n-1}\prod_{j=1}^{\ell-1} b_i|_{0.({}^{a_i^{-1}}a_n)^{j}} && \text{(shifting by } j(i)) \\
&= \prod_{i=0}^{n-1} \lambda_{b_i}({}^{a_i^{-1}}a_n) && (\text{since } \ell = \ell_{{}^{a_i^{-1}}a_n}(0)) \\
&= \prod_{i=0}^{n-1} \lambda_{b_i}(a_n, x) && (\text{since } G \text{ is stable and } {}^{a_i^{-1}}a_n \in \mathfrak{C}(a_n, x)) \\
&= \lambda_{\prod_{i=0}^{n-1} b_i}(a_n, x). && \text{(by Lemma 2.2.14)}
\end{aligned}$$

We write $b \coloneqq \prod_{i=0}^{n-1} b_i$ to shorten the notation. The B-symbols in the middle term of (§§) representing $g\|_x$ are precisely the B-symbols occurring in the word (§) representing g, thus there exists some permutation σ of $\{0, \dots, n-1\}$ and some $\hat{a}_i \in A$ for $i \in \{0, \dots, n-1\}$ such that

$$g\|_x = {}^{\hat{a}_0}b_{0.\sigma} \dots {}^{\hat{a}_{n-1}}b_{(n-1).\sigma}\, \lambda_b(a_n, x).$$

Since B is abelian, $\prod_{i=0}^{n-1} b_{i.\sigma} = b$ and we may iterate our calculation for elements $v = x_0 \star \dots \star x_{k-1} \in X^k$, $k \in \mathbb{N}$, such that $\operatorname{syl}(g\|_v) = \operatorname{syl}(g)$. Then

$$g\|_v \bmod \operatorname{St}(1) \in \lambda_b(\dots(\lambda_b(\lambda_b(a_n, x_0), x_1), \dots), x_{k-1}) \subseteq \Lambda_b^k(a_n).$$

Since Λ_b is eventually trivial we conclude that there exist some $k \in \mathbb{N}$ such that for all $v \in X^k$ either $\operatorname{syl}(g\|_v) < \operatorname{syl}(g)$ or $g\|_v \in \operatorname{St}_G(1)$. To obtain (†) it is enough to consider all v such that the second case holds. For these, by Lemma 2.2.4(1) either $\operatorname{syl}(g\|_{v\star x}) < \operatorname{syl}(g)$ or $g\|_{v\star x} \in B$ holds, since $g\|_{v\star x} = g\|_v|_x$. □

2.3.2. Discussion of Theorem 2.1.1. — Clearly, all CS groups with abelian rooted group are strongly orbitwise-abelian. Furthermore, if A is abelian, it must act regularly on X, hence $\mathfrak{C}(a,x) = \{{}^x a\} = \{a\}$. Thus G is stable, and $\lambda_b(a,x) = \lambda_b(a)$ for all $a \in A, b \in B$ and $x \in X$. We record the statement of Theorem 2.1.1 in this case as a corollary, which turns out to be a version of a result of Bartholdi [9, Theorem 13.4], with the additional feature that non-cyclic directed groups are allowed. We remark that [9, Theorem 13.4] does not explicitly require that the rooted group is abelian; this appears to be an oversight, see Example 2.3.4. Using a similar construction to the one in [9], we are able to prove that the condition is not only sufficient but necessary.

Corollary 2.3.2. *Let $G = \langle A \cup B\rangle$ be a* CS *group with abelian and periodic rooted group A, and such that either (i) A is of finite exponent, or (ii) B has finite support. Then G is periodic if and only if the dynamical system (A, λ_b) is eventually trivial for all $b \in B$.*

Proof. Since A is abelian, B is abelian as well. Assuming that (i) or (ii) holds, B is also periodic. Now, identifying the singletons of $\mathcal{P}(A)$ with A, we have $\Lambda_b|_A = \lambda_b$, and the 'if'-direction follows from Theorem 2.1.1.

For the other implication, assume that there are $a \in A$ and $b \in B$ such that $\lambda_b^n(a) \neq 1_A$ for all $n \in \mathbb{N}$. Consider the orbit length of elements of the form 0^{*n} under ba. From $\ell_{ba}(0^{*n}) = \ell_{ba}(0) \cdot \ell_{(ba)\|_0}(0^{\star n-1})$ we conclude that either the orbit lengths are unbounded and ba has infinite order, or that there is some $m \in \mathbb{N}$ such that $\ell_{(ba)\|_{0^{\star n}}}(0) = 1$ for $n > m$. But

$$(ba)\|_0 = b\lambda_b(a),$$

hence $\ell_{(ba)\|_{0^{*n}}}(0) = \ell_{\lambda_b^n(a)}(0) > 1$ for all $n \in \mathbb{N}$. Thus, ba has infinite order. □

Corollary 2.3.2 may be reformulated to say: A CS group with abelian rooted group satisfying either (i) or (ii) is periodic if and only if the elements $ba \in B \cdot A$ are of finite order. Motivated by this characterisation, we ask if similar bounds exist for nilpotent and for soluble groups:

Question 2.3.3. *For any group G, let $I(G)$ be the set of all elements of infinite order. If G is a* CS *group, write $B_G(n)$ for the set of elements of syllable length at most n. Let $\mathcal{C}(c)$ be the set of all non-periodic (isomorphism classes of)* CS *groups $G = \langle A \cup B\rangle$ with rooted group nilpotent of class c and satisfying (i) or (ii). Define a function $f_{\mathrm{nil}} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by*

$$f_{\mathrm{nil}}(c) = \max_{G \in \mathcal{C}(c)} \min\{n \in \mathbb{N} \mid I(G) \cap B_G(n) \neq \emptyset\}.$$

Is $f_{\mathrm{nil}}(c) < \infty$ for all $c \in \mathbb{N}$? One can define an analogous function $_{\mathrm{sol}}$ for soluble groups. Is $f_{\mathrm{sol}}(l) < \infty$ for all $l \in \mathbb{N}$?

Theorem 2.1.1 says that $f_{\mathrm{nil}}(1) = f_{\mathrm{sol}}(1) = 1$. We now provide an example of a non-periodic CS group where the dynamical systems defined by λ_b are all eventually trivial,

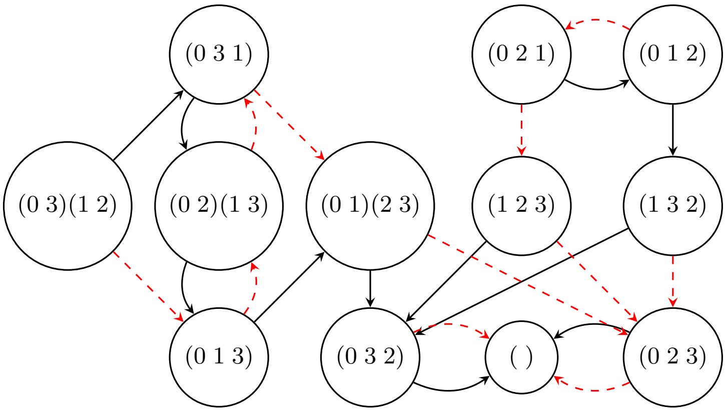

Figure 2.1: The maps λ_b and λ_{b^2} of the group in Example 2.3.4. Dashed red arrows represent the latter, solid black arrows the former.

i.e. all elements of syllable length less than two are of finite order. The rooted group is isomorphic to the alternating group on four elements acting naturally, implying $f_{\mathrm{sol}}(2) \geq 2$.

Example 2.3.4 (Corollary 2.3.2 cannot be extended to CS groups with non-abelian rooted group)**.** Define $s_1 = (0\ 3\ 2), s_2 = (0\ 1\ 3), s_3 = s_2^{-1}$ and $A = \langle s_1, s_2, s_3\rangle \cong \mathrm{A}_4$. Set $b = (b, s_1, s_2, s_3)$ and $B = \langle b\rangle$. Clearly b has order 3. The maps λ_b and λ_{b^2} are shown in Fig. 2.1, and they are eventually trivial. Clearly (i) and (ii) are satisfied.

We now produce an element of infinite order. Let

$$g = s_3 s_1 b s_3 b = (0\ 2)(1\ 3)b(0\ 3\ 1)b.$$

Then

$$g\|_0 = g^3|_0 = b|_2 b|_2 b|_0 b|_3 b|_1 b|_0 = s_3 b s_3 s_1 b$$
$$g\|_{0\star 0} = (s_3 b s_3 s_1 b)^3|_0 = b|_3 b|_1 b|_0 b|_2 b|_2 b|_0 = g.$$

Thus g is not of finite order.

Aside from CS groups with abelian rooted groups, Theorem 2.1.1 applies to far more CS groups. We give an exemplary construction.

Example 2.3.5 (A non-abelian strongly orbitwise-abelian and stable CS group)**.** Let $\mathrm{Sym}(4)$ be the symmetric group on four letters. The set X of transpositions in $\mathrm{Sym}(4)$ is a conjugacy class of $\mathrm{Sym}(4)$, and the alternating group $A = \mathrm{Alt}(4)$ acts faithfully and transitively on X by conjugation. Equivalently, the alternating group acts on the faces of a cube by rotations in the following way. The double transpositions rotate the cube by π along an axis defined by the centre points of opposing sides, and the three-cycles rotate it by $\frac{2}{3}\pi$ along an axis going through two opposing corners, see Section 2.3.2.

We fix the transposition $(1\,2)$, resp. the top face of the cube, as the distinguished letter. Clearly $\mathrm{st}_A(1\,2) = \langle(1\,2)(3\,4)\rangle$ is of order two. We write V for the Klein four group

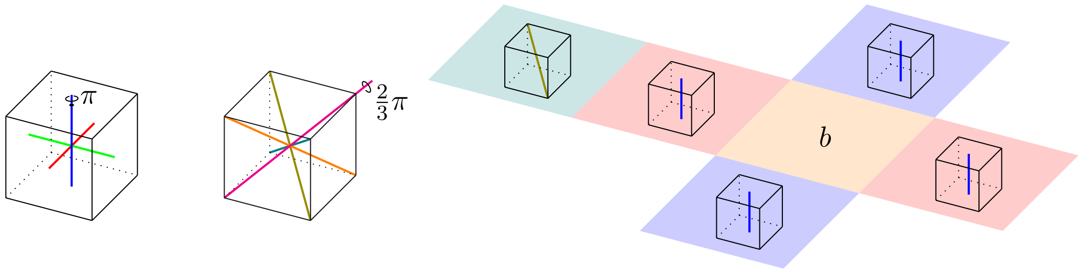

Figure 2.2: The action of $\mathrm{Alt}(4)$ on the cube X and the action of b on X^* described in Example 2.3.5.

consisting of the double transpositions. Define an automorphism b by

$$b = ((1\,2) : b,\ (3\,4) : (1\,2\,3),\ x : (1\,2)(3\,4) \text{ for } x \in X \setminus \{(1\,2), (3\,4)\}).$$

Compare again with Section 2.3.2. The CS group G defined by A and $B = \langle b \rangle$ is stable, strongly orbitwise-abelian and the dynamical system defined by $\Lambda_{b'}$ for any $b' \in B$ is eventually trivial, hence periodic by Theorem 2.1.1. Our discussion will show that minor variations in the definition of b lead to the same conclusion, e.g. one may replace $b|_{(3\,4)}$ by an arbitrary three-cycle, or define the sections of b on the blue (resp. the red) faces to be trivial. We omit more careful calculations that show, again using Theorem 2.1.1, that there are further examples of CS groups with rooted group A. We first show that G is strongly orbitwise-abelian. Let $a \in V$. Since V is normal, $\mathfrak{C}(a, x) = {}^{\mathrm{mp}_A((1\,2),x)}a \subset V$, and since it is abelian, it is a singleton set. Now the image of $(1\,2)$ under an element of V is either $(1\,2)$ or $(3\,4)$ (the bottom side of the cube). Consequently

$$\mathfrak{X}(a, x) \in \{\{(1\,2)\}, \{(3\,4)\}\}$$

for all $a \in V, x \in X$. Now let $a \in A$ be an element of order three. The set T of such elements is a normal subset, hence $\mathfrak{C}(a, x) \subseteq T$. Now T acts (as a set) non-transitively on X, since $(3\,4) \notin T.(1\,2)$. Viewing at the cube, it is apparent that the bottom and top faces cannot be interchanged by an element of T. At the same time, no element of T has any fixed points. Consequently,

$$\mathfrak{X}(a, x) \in \{\{(1\,3), (1\,4)\}, \{(2\,3), (2\,4)\}\}$$

for all $a \in T, x \in X$. These are sets of opposing sides. A calculation shows that $\mathfrak{X}(a, x)$, with a ranging over A, takes all values

$$\{\{(1\,2)\}, \{(3\,4)\}, \{(1\,3), (1\,4)\}, \{(2\,3), (2\,4)\}\}.$$

These correspond to the colouring in Section 2.3.2. Clearly the sections within such a set commute, hence G is orbitwise-abelian.

Since $\mathfrak{C}(a,x)$ is a singleton for $a \in V$, the group G being stable is equivalent to λ_b taking the same values for all $a \in \mathfrak{C}(t,x)$, $t \in T$, $x \in X$, $b' \in B$. The two elements in any $\mathfrak{C}(t,x)$ are $\mathrm{st}_A(1\,2)$-conjugate, hence

$$\mathfrak{C}(t,x) \in \{\{(1\,2\,3),(1\,4\,2)\},\{(1\,3\,2),(1\,2\,4)\},\{(1\,3\,4),(2\,4\,3)\},\{(1\,4\,3),(2\,3\,4)\}\}$$

and all these values are taken. Any such pair corresponds to two orbits of $(1\,2)$ going around opposite corners of the top face, i.e. a red and a blue face. The product over the sections is hence $((1\,2)(3\,4))^2 = 1_A$ in all cases. Thus G is stable, and furthermore for all $t \in T$

$$\Lambda_b(t) = \Lambda_{b^2}(t) = \{1_A\}.$$

Finally, the conjugate of $(1\,2)$ under a double transposition is either $(1\,2)$ or $(3\,4)$. Hence

$$\Lambda_b(a) = \{b|_{(1\,2)}, b|_{(3\,4)}\} = \{1_A,(1\,2\,3)\} \quad \text{and} \quad \Lambda_{b^2}(a) = \{1_A,(1\,3\,2)\}$$

for all $a \in V$, and by $\Lambda_b^2(a) = \Lambda_{b^2}^2(a) = \{1_A\}$ we see that $\Lambda_{b'}$ is eventually trivial for all $b' \in B$. Thus G is periodic by Theorem 2.1.1.

2.3.3. Proof and discussion of Theorem 2.1.2. — If we drop the assertion that G is strongly orbitwise-abelian, the argument used in the proof of Theorem 2.1.1 to reduce the set of candidates of infinite order to elements g with $\mathrm{syl}(g) \leq 1$ fails in the general case, as illustrated by Example 2.3.4. But we may still consider the "coarser" dynamical system defined by the maps Σ_B. This gives us a broadly applicable sufficient condition for periodicity, that does not require B to be abelian. Also, one does not need to check the full system Σ_B, which may be replaced with Σ_S, for a generating set $S \subseteq B$.

Proof of Theorem 2.1.2. We aim to apply Proposition 2.2.7 with $T = B$. Let

$$g = ({}^{a_0}b_0)\dots({}^{a_{n-1}}b_{n-1})a_n \in G \setminus A$$

be an element of syllable length $n > 0$ and $x \in X$ such that $\mathrm{syl}(g\|_x) = n$. Write $\ell := \ell_g(x)$. By Lemma 2.3.1 we find that ${}^{a_i^{-1}}a_n \in \mathfrak{C}(a_n,x)$ for all $i \in \{0,\dots,n-1\}$ and there are integers $j(i)$ such that

$$g\|_x = \prod_{j=0}^{\ell-1}\prod_{i=0}^{n-1} b_i|_{0.a_i^{-1}a_n^{j+j(i)}}.$$

Decompose all b_i as a product of generators in S, writing $b_i = \prod_{m=0}^{n_i-1} b_{i,i_m}$. Then calculating modulo $\mathrm{St}(1)$ and

$$N := \langle b|_y \mid y \in \mathfrak{X}(a_n,x), b \in S\rangle' \leq A'$$

we find

$$g\|_x \equiv_{\langle N,\mathrm{St}(1)\rangle} \prod_{i=1}^{n-1}\prod_{m=0}^{n_i-1}\left(\prod_{j=1}^{\ell-1} b_{i,i_m}|_{0.{}^{a_i^{-1}}a_n^j}\right).$$

Thus $g\|_x|^\epsilon \in \sigma_S(a_n, x)$.

Let $k \in \mathbb{N}$ be such that $\Sigma_S^k(a_n) = \{1_A\}$, and let $v \in X^k$. If $\mathrm{syl}(g\|_v) = n$, all intermediate stabilised sections must have syllable length n and we obtain $g\|_v|^\epsilon \in \Sigma_S^k(a_n) = \{1_A\}$ by the calculation above. Let $y \in X$ be any letter. Then

$$\mathrm{syl}(g\|_{vy}) = \mathrm{syl}(g\|_v|_y) < \mathrm{syl}(g\|_v) = n,$$

or $g\|_{vy} \in B$ by Lemma 2.2.4(1). Thus, by Proposition 2.2.7, G is periodic. □

Remark 2.3.6. *The special case of a* CS *group where the rooted group A acts regularly (on itself with the distinguished letter 1_A) was first considered by Gupta and Sidki [77, 78]. They considered automorphisms defined by decorating functions, roughly corresponding to directed elements, and gave a list of five conditions for such a group $G = \langle A \cup \{\delta\}\rangle$ with periodic rooted group A and an automorphism δ defined by a decorating function (i.e. a* CS *group with cyclic directed group) to be periodic, in this case, the decorating function is called periodicity preserving. Translated into the language of* CS *groups, the five conditions are:*

(i) *The element δ is directed.*
(ii) *The group G is a* CS *group.*
(iii) *The element δ has finite support.*
(iv) *The group G is orbitwise-abelian.*
(v) *For all $a \in A \setminus \{1_A\}$ we have $\prod_{a' \in \langle a\rangle\setminus\{1_A\}} \delta|_{a'} = 1_A$.*

Note that by the fourth condition, the product in the fifth condition is well-defined.

We argue that the conditions of Theorem 2.1.2 are implied by the five conditions of Gupta and Sidki. Using Theorem 2.1.2, we see from (i), (ii) and (iii) that we only need to prove that Σ_S is eventually trivial for some generating set S of B. Naturally, we set $S = \{\delta\}$.

Since A acts regularly, $\mathfrak{C}(a, x) = \{{}^x a\}$ and $\mathfrak{X}(a, x) = \mathrm{orb}_{{}^x a}(0) \setminus \{0\}$. For all $a \in A$ and non-trivial $x \in A$, we thus have

$$\langle \prod_{c' \in \langle c\rangle \setminus \mathrm{st}_A(0)} \delta|_{c'} \mid c \in \mathfrak{C}(a, x)\rangle = \langle \prod_{a' \in \langle {}^x a\rangle\setminus\{1_A\}} \delta|_{a'}\rangle \overset{(v)}{=} \{1_A\},$$

and the group in Definition 2.2.10 of $\sigma_S(a, x)$ that is generated by products is trivial. Since A acts regularly, (iv) implies that G is strongly orbitwise-abelian, hence the derived subgroup *in the definition of $\sigma_S(a,x)$ is trivial as well. Thus the dynamical system defined by*

$$\Sigma_S(a) = \bigcup_{x\in X} \sigma_S(a,x) = \bigcup_{x\in X} \{1_A\}$$

is eventually (or rather, immediately) trivial. Consequently, the conditions of Gupta and Sidki appear as a special case of Theorem 2.1.2.

Gupta and Sidki proved that nearly all (precisely all non-dihedral, non-cyclic) finitely generated periodic groups allow a periodicity preserving function. In case of a finite group A generated by a set of non-involutions N, one may define such a function $\beta : A\backslash\{1_A\} \to A$ by

$$\beta(x) = \begin{cases} x & \text{if } x \in S, \\ 1_A & \text{otherwise.} \end{cases}$$

It is an interesting question which finite permutation groups may occur as the local action (equivalently, as rooted subgroup) of a periodic CS group, cf. Section 2.4.1. In combination with Corollary 2.4.3, which we will discuss later, this shows that every group of the form $A^s \curvearrowleft A^{\wr s}$ for A a non-dihedral, non-cyclic finite group allows a periodic CS group that features $A^{\wr s}$ (with this action) as its rooted subgroup. It is not true that every transitive permutation group allows a periodic CS group.

Even if we restrict to the regular case, Theorem 2.1.2 is very flexible. As a demonstration, we prove the following corollary.

Corollary 2.3.7. *Let A be a finite periodic group that decomposes as a semi-direct product $A = N \rtimes \langle g\rangle$ with a cyclic quotient of order greater than two. There exists a finitely generated infinite periodic group generated by two isomorphic copies of A.*

Proof. Assume that $A = N \rtimes \langle g\rangle$ and let $S \subseteq A$ be a generating system containing g such that $S \setminus \{g\}$ generates N. We construct a group $B \in \operatorname{Aut}(A^*)$ isomorphic to A by defining

$$b_s|_g = s, \quad b_s|_{g^{-1}} = \begin{cases} 1_A & \text{if } s \neq g, \\ g^{-1} & \text{if } s = g, \end{cases}, \quad b_s|_x = 1_A, \quad \text{for } x \in A \setminus \{1_A, g, g^{-1}\},$$

for every $s \in S$. Set $B = \langle b_s \mid s \in S\rangle$. This group is isomorphic to the group generated by

$$\{(s, 1_A) \mid s \in S \setminus \{g\}\} \cup \{(g, g^{-1})\}.$$

But since $N = \langle S \setminus \{g\}\rangle$, this group is isomorphic to A. We now prove that the CS group defined by A and B, with A acting regularly on itself, is periodic. As B is finite, by Theorem 2.1.2 it is sufficient to prove that the dynamical system defined by Σ_{S_B} is eventually trivial, where we set $S_B = \{b_s \mid s \in S\}$.

Let $x, y \in A$. We calculate $\sigma_{S_B}(x,y)$. First observe that $\mathfrak{C}(x,y) = \{{}^y x\}$, since A acts regularly. There are two cases, depending if $a \in \langle {}^y x\rangle$. In case $g \in \langle {}^y x\rangle$, also g^{-1} is in

$\langle {}^y x \rangle$, and the products generating the subgroup on the left in the equation (¶) defining $\sigma_{S_B}(x,y)$ in Definition 2.2.10 are $s \in S \setminus \{g\}$ and 1_A, while the generators of the subgroup on the right are the members of S. Then,

$$\sigma_{S_B}(x,y) = N \cdot A' = N.$$

In case $g \notin \langle {}^y x \rangle$, all sections $b_s|_z$ for $z \in \langle {}^y x \rangle \setminus \mathrm{st}_A(0)$ and $s \in S$ are trivial, hence

$$\sigma_{S_B}(x,y) = \{1_A\}.$$

Thus for any $x \in A$, we have $\Sigma_{S_B}(x) \leq N$.

Now assume $x \in N$. Since N is normal, all conjugates ${}^y x$ are again in N, and $g \notin \langle {}^y x \rangle$. Thus

$$\Sigma_{S_B}(x) = \{1_A\},$$

and Σ_{S_B} is eventually trivial. □

Using Theorem 2.1.2, it is also not too difficult to find examples of periodic CS groups where the action of the rooted group is not regular, a previously unrecorded phenomenon.

Example 2.3.8. We define a family of CS groups satisfying the condition of Theorem 2.1.2. Let p be an odd prime and identify $X = \{0, \dots, 2p-1\}$ with the vertices of the regular $2p$-gon. We fix the rotation $r = (0\,1\,\dots\,2p-1)$ and the reflexion $s = (1\,2p-1)(2\,2p-2)\dots(p-1\,p+1)$ in the axis through 0 and p as the generators of the rooted group $A \cong D_{2p}$, the dihedral group of order $4p$.

Define an automorphism b by

$$b|_x = \begin{cases} r & \text{if } x \equiv_2 1 \text{ and } x \neq p, \qquad 1_A \quad \text{if } x = p, \\ s & \text{if } x \equiv_2 0 \text{ and } x \neq 0, \qquad b \quad \text{if } x = 0. \end{cases}$$

See Fig. 2.3 for an illustration of this automorphism. Clearly, the first layer sections of b generates A. By considering separately the elements of order 2, p and $2p$ in A, we show that the CS group defined by A and $\langle b \rangle$ gives rise to an eventually trivial dynamical system $\Sigma_{\{b\}}$ and is thus periodic.

Let $a \in A$ be an element of order p, i.e. $a \in \langle r^2 \rangle$. The orbit of 0 under any such element is the set of even elements in X. Excluding 0, all sections at even elements are equal to s, hence for all $x \in X$

$$\sigma_{\{b\}}(a,x) = \langle s^{p-1} \rangle \cdot \langle s \rangle' = \{1_A\}.$$

Let $a \in A$ be an element of order $2p$, i.e. of the form r^i for some odd i that is not a multiple of p. The orbit of 0 under any conjugate of a is the full set X, and we find

$$\sigma_{\{b\}}(a,x) = \langle r^{p-1} \cdot s^{p-1} \rangle \cdot \langle s, r \rangle' = \langle r^2 \rangle$$

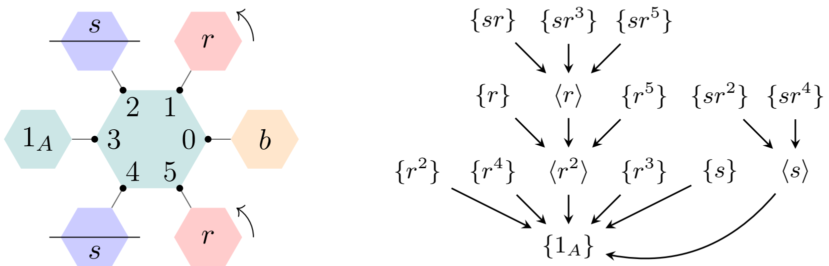

Figure 2.3: The generator b of Example 2.3.8 for $p = 3$ and the part of the dynamical system defined by Σ_b reached by singletons.

for any $x \in X$, hence $\Sigma_{\{b\}}(a) = \langle r^2 \rangle$, which becomes trivial after another step as we saw above.

Let $a \in A$ be an element of order 2. Then either $a = sr^i$ for some $i \not\equiv_{2p} 0$, $a = s$ or $a = r^p$. In the third case, the element r^p is central, hence $\mathfrak{C}(r^p, x) = \{r^p\}$ for all $x \in X$ and $\mathfrak{X}(r^p, x) = \{p\}$, thus $\Sigma_{\{b\}}(r^p) = \sigma_{\{b\}}(r^p, x) = \{1_A\}$. In the second case, the element s fixes the distinguished letter and we also find $\Sigma_{\{b\}}(s) = \{1_A\}$. In the first case, $\mathfrak{C}(sr^i, x) = \{sr^i, sr^{i-2}\}$ and

$$\sigma_{\{b\}}(sr^i, x) = \begin{cases} \langle r \rangle \cdot \langle r \rangle' = \langle r \rangle & \text{if } i \equiv_2 1, \\ \langle s \rangle \cdot \langle s \rangle' = \langle s \rangle & \text{if } i \equiv_2 0, \end{cases}$$

since $0.sr^i = i \equiv_2 0$ if and only if $0.sr^{-i} = 2p - i \equiv_2 0$. Thus the system $\Sigma_{\{b\}}$ is eventually trivial.

Obviously there are many variants of this construction, e.g. replacing s with another involution of A.

Furthermore, using similar arguments, periodic CS groups with rooted group isomorphic to dihedral groups of even exponent acting naturally may easily be found.

Question 2.3.9. *As described by Grigorchuk in [68], a* GGS*-group on a p-adic rooted regular tree is periodic if and only if its defining vector $\underline{e}$ satisfies the equation $\sum_{i=1}^{p-1} e_i \equiv_p 0$. Hence for* CS *groups with rooted group C_p, there is a criterion for periodicity that can be stated rather easily.*

Is there a similar characterisation for the periodic groups among the CS *groups with rooted group D_{2p}?*

2.4 — Applications and examples

In the previous section we have seen that for nearly all finitely generated periodic groups acting regularly on themselves, we find directed groups such that the resulting CS group is periodic. Now we will give an example how to construct CS groups acting locally non-regularly, that fulfil the conditions of Theorem 2.1.2, and at the same time explore its limitations.

2.4.1. Rooted groups where Theorem 2.1.1 and 2.1.2 cannot be applied. — We make the following observation. Let G be a CS group with rooted group $A \leq \mathrm{Sym}(X)$ that is perfect and contains an element $a \in A$ acting transitively on X (i.e. an $|X|$-cycle). Clearly G is not orbitwise-abelian, so Theorem 2.1.1 cannot be applied to G. Furthermore, let $S \subseteq B$ be any generating set for B. Then the sections $b|_x$ for $x \in \dot{X}$ and $b \in S$ generate A, and since $\sigma_S(a, 0)$ contains the derived subgroup of the group generated by these sections, we have $\sigma_S^k(a, 0) = A$ for all $k > 0$. Consequently, G does not fulfil the conditions of Theorem 2.1.2.

In view of this limitation, we ask:

Question 2.4.1. *Are there any* CS *groups based on a rooted group* $A \cong \mathrm{Sym}(X)$ *acting naturally on* X *that are periodic?*

In regards to this question we point out that it is possible to exclude certain special cases. It is well known that the only CS group on a two element set is infinite dihedral. An extensive brute-force computer calculation conducted by the author using GAP [52] show that there is no CS group with rooted group Sym(3) acting naturally.

We consider another special case, and write $X = \{0, \dots, m-1\}$. Assume that there is a generating set S of B such that the collection of sections $b|_x$ for $b \in S$ and $x \in \dot{X}$ contains the cycle $a_m = (0\,1\,\dots\,m-1)$ exactly once, the transposition $a_2 = (0\;1)$ exactly twice, and nothing more. Then G cannot be periodic:

Let $b \in S$ be the unique element with a section equal to a_m. There are three cases: Either b has (aside from the sections a_m and b) no, one, or two sections equal to a_2. If there are none, $a_m b$ is of infinite order, since $\lambda_b(a_m) = a_m$. If there are two, the element $\lambda_b(a_m)$ is a product of a_2, a_2 and a_m, hence either equal to a_m or to ${}^{a_2}a_m$. In the first subcase $a_m b$ is of infinite order. In the second subcase, $\lambda_b({}^{a_2}a_m)$ is again equal to a_m or ${}^{a_2}a_m$. Thus either λ_b or λ_b^2 have a fixed point, and $a_m b$ is of infinite order. Lastly, if there is precisely one section of b equal to a_2, there is another generator $b' \in S$ with exactly one section equal to a_2. Now $\lambda_{bb'}(a_m)$ is either a_m or ${}^{a_2}a_m$, and we may argue as before.

Clearly, similar methods can be applied to exclude other generating sets of $\mathrm{Sym}(X)$, which feature sparse section decompositions.

2.4.2. Basilica groups of GGS-groups. — We now turn our attention to the s^{th} *Basilica groups* of CS groups resulting from the Basilica construction introduced in [125]. To every group of tree automorphisms H one can associate a family of Basilica groups, all sharing certain properties with H. Most famously, the second Basilica group of the dyadic odometer is the (classical) Basilica group defined by Grigorchuk and Żuk [73], while the Basilica groups of spinal groups are again spinal, although on another tree;

cf. [125, Proposition 3.9]. We prove that if a CS group G satisfies the conditions of Theorem 2.1.2, its Basilica groups do so as well, providing many examples of periodic CS groups with non-regular rooted action. In particular, this is also of interest in the context of Basilica groups, as it is not known if Basilica groups of periodic groups are again periodic.

Instead of recalling the general definition of the s^{th} Basilica group, we provide an ad hoc description for CS groups.

Definition 2.4.2. Let $G = \langle A \cup B \rangle$ be a CS group acting on the tree Y and $s \in \mathbb{N}$. For $i \in \{0, \ldots, s-1\}$ and for every $a \in A$ define an element $a_i \in A^{\wr s}$ by the following action on $X = Y^s$:

$$(y_0 \star \cdots \star y_{s-1}).a_i := \begin{cases} y_0 \star \cdots \star y_{i-1} \star y_i.a \star 0 \star \cdots \star 0 & \text{if } y_0 = \cdots = y_{i-1} = 0 \\ y_0 \star \cdots \star y_{s-1} & \text{otherwise.} \end{cases}$$

Then $\langle a_i \mid a \in A, i \in \{0, \ldots, s-1\} \rangle \cong A^{\wr s}$ acts transitively on X. For every $i \in \{0, \ldots, s-1\}$ and $b \in B$ define an automorphism of X^* by

$$b_i|_u = \begin{cases} a_i & \text{if } u = 0^{\star(i)} \star y \star 0^{\star(s-1-i)} \text{ and } a = b|_y, \\ \text{id} & \text{otherwise.} \end{cases}$$

Defining $A_i = \{a_i \mid a \in A\}$ and B_i equivalently, we define the *s^{th} Basilica group* $\mathrm{Bas}_s(G)$ as the CS group on the tree X^* defined by the rooted group $\langle \bigcup_{i=0}^{s-1} A_i \rangle$ and the directed group $\langle \bigcup_{i=0}^{s-1} B_i \rangle$.

It is a general fact that the directed group of a Basilica group of a CS group (or, more generally, any spinal group) G is the s-fold direct product of the directed group of G, cf. [125, Proposition 3.9]. Let $T \subset G$ be a generating set of G. Then the set

$$\mathbb{T} = \{t_i \mid i \in \{0, \ldots, s-1\}, t \in T\}$$

is a generating set for $\mathrm{Bas}_s(G)$.

Note that $\mathrm{Bas}_s(G)$ is a CS group on the tree X^*, where the alphabet X itself is the finite rooted tree Y^s. To distinguish the two tree-structures we omit the symbol $\star$ for elements in the finite tree from here on.

Corollary 2.4.3. *Let G be a* CS *group satisfying the conditions of Theorem 2.1.2 for the generating system $T \subseteq G$. Then* $\mathrm{Bas}_s(G)$ *is periodic for all $s \in \mathbb{N}$.*

Proof. Since the rooted group of $\mathrm{Bas}_S(G)$ is isomorphic to $A^{\wr s}$ and the directed group of $\mathrm{Bas}_S(G)$ is isomorphic to B^s, conditions (i) and (ii) transfer to B^s, and by Theorem 2.1.2 it is enough to prove that $\Sigma_{\mathbb{T}}$ is eventually trivial.

The rooted group $A^{\wr s}$ acts on the finite rooted regular tree $X = Y^s$. Write $S(i)$ for the i^{th} level stabiliser of $A^{\wr s}$, with is isomorphic to $(A^{\wr s-i})^{|Y|^i}$, and set $S(s) = \{1_{A^{\wr s}}\}$. Clearly $A_j \subseteq S(i)$ if and only if $j \geq i$.

Let $k \in \mathbb{N}$ be such that $\Sigma_T^k(A) = \{1_A\}$. We shall prove that $\Sigma_{\mathbb{T}}^{s\cdot k}(A^{\wr s})$ is trivial by proving that

$$\Sigma_{\mathbb{T}}^k(S(i)) \subseteq S(i+1)$$

for all $i \in \{0, \dots, s-1\}$. Thus let $g \in S(i)$ and let $x = y_0 \dots y_{s-1} \in Y^s$. To calculate

$$\sigma_{\mathbb{T}}(g,x) = \langle \prod_{c' \in \langle c \rangle \backslash \mathrm{st}_{A^{\wr s}}(0^s)} t|_{0^s.c'} \mid c \in \mathfrak{C}(g,x), t \in \mathbb{T}\rangle \cdot \langle t|_{x'} \mid x' \in \mathfrak{X}(g,x), t \in \mathbb{T}\rangle'.$$

we first notice that $\mathfrak{C}(g,x)$ is a subset of the conjugates of g, hence since $S(i)$ is normal, $\mathfrak{C}(g,x) \subseteq S(i)$. Thus for any $c \in \mathfrak{C}(g,x)$ the orbit of the distinguished letter is of the form

$$\mathrm{orb}_c(0^s) \subseteq 0^i Y^{s-i},$$

and by the definition of t_j for $t \in T$, $j \in \{0, \dots, s-1\}$, the sections of t_j along $\mathrm{orb}_c(0^s) \backslash \{0^s\}$ are trivial if $j < i$. Now $B_j|_{x'} \subseteq B_j \cup A_j \subseteq S(i)$ for all $x' \in X$ implies $\sigma_{\mathbb{T}}(g,x) \subseteq S(i)$.

We now calculate modulo $S(i+1)$. Consequently we may ignore all sections of elements t_j with $t \in T$ and $j \neq i$, and get

$$\begin{aligned}\sigma_{\mathbb{T}}(g,x) &\equiv_{S(i+1)} \langle \prod t_i|_{0^s.c} \mid c \in \mathfrak{C}(g,x), t \in T\rangle \cdot \langle t_i|_{x'} \mid x' \in \mathfrak{X}(g,x), t \in T\rangle' \\ &\equiv_{S(i+1)} \sigma_T(g \bmod S(i+1), y_0 \dots y_i).\end{aligned}$$

Thus $\Sigma_{\mathbb{T}}^k(S(i)) \subseteq S(i+1)$, and $\Sigma_{\mathbb{T}}$ is eventually trivial. □

2.4.3. A periodic CS group satisfying neither Theorem 2.1.1 nor 2.1.2. — At last, we demonstrate that, in contrast to Corollary 2.3.2 for abelian rooted groups, neither Theorem 2.1.1 nor Theorem 2.1.2 nor the union of their scopes provide necessary conditions for periodicity. Indeed, we construct a periodic CS group that is subject to neither of the sets of conditions.

Let $X = \{0,1,2,3\}$ be the alphabet and let $A \leq \mathrm{Sym}(X)$ be generated by $s = (1\,3)$ and $r = (0\,1\,2\,3)$. The group A is isomorphic to the dihedral group D_4 of order 8. Define $b \in \mathrm{St}(1)$ by

$$b = (0 : b,\ 1 : s,\ 2 : s,\ 3 : sr),$$

and $B = \langle b \rangle$. Let G be the CS group defined by A and B.

Clearly, G is not orbitwise-abelian, since the rotation r acts transitively on X. Thus G is in particular not strongly orbitwise-abelian and does not satisfy the hypothesis of Theorem 2.1.1.

To prove that G does not satisfy the hypothesis of Theorem 2.1.2, we have to consider all generating sets of B. But B is cyclic of order two, whence it suffices to determine $\Sigma_{\{b\}}$. The point stabiliser of 0 in A is generated by s, hence the set $\mathfrak{C}(a,x)$ is of cardinality at most two for all $a \in A, x \in X$. For our argument, it suffices to calculate

$$\mathfrak{C}(r^{\pm 1}, x) = \{r^{\pm 1}\} \quad \text{for all } x \in X \quad \text{and} \quad \mathfrak{C}(sr, 0) = \{sr, sr^3\}.$$

This yields

$$\sigma_{\{b\}}(r^{\pm 1}, x) = \langle sr \rangle \cdot \langle s, sr \rangle' = \langle sr \rangle \times \langle r^2 \rangle,$$

for each $x \in X$, hence $\Sigma_{\{b\}}(r^{\pm 1}) = \langle sr \rangle \times \langle r^2 \rangle$, and

$$\sigma_{\{b\}}(sr, 0) = \langle s, sr \rangle \cdot \langle s, sr \rangle' = A,$$

whence $sr \in \Sigma_{\{b\}}(sr)$. Thus $\Sigma_{\{b\}}$ is not eventually trivial.

We now prove that G is periodic. To achieve this, we use Proposition 2.2.7 with the set

$$T = B \cup {}^G A \cup {}^B (\langle bsr \rangle \cdot \{1_A, bs\}).$$

A standard computation shows that the elements of T have finite order: The first two sets in the union clearly consist of elements of finite order. For the third subset, first calculate

$$\begin{aligned} (bs)^2 &= (1_A, r, 1_A, r^3), && \text{hence } \operatorname{ord}(bs) = 8, \text{ and} \\ (bsr)^2 &= (bs, (bs)^{-1}, r, r^3), && \text{hence } \operatorname{ord}(bsr) = 16. \end{aligned}$$

The first equation also implies that $(bs)^{2n} r^{\pm 1}$ is of finite order for all $n \in \mathbb{N}$. Similarly

$$\begin{aligned} ((bs)^{2n+1} r^2)^2 &= ((b, sr^{-n}, s, sr^{n+1}) sr^2)^2 \\ &= (bs, 1_A, (bs)^{-1}, 1_A) \end{aligned}$$

has finite order for all $n \in \mathbb{N}$. Using this, we see that for arbitrary $n \in \mathbb{N}$

$$\begin{aligned} ((bsr)^{2n} bs)^2 &= \Big(((bs)^n, (bs)^{-n}, r^n, r^{-n})\, bs \Big)^2 \\ &= \Big(((bs)^n b, (bs)^{-n} s, sr^{-n}, sr^{n+1})\, s \Big)^2 \\ &= (1_A, (bs)^{-n} r^{n+1}, 1_A, ((bs)^{-n} r^{n+1})^{-1}) \end{aligned}$$

is of finite order. Finally,

$$\begin{aligned} ((bsr)^{2n+1} bs)^4 &= ((bs, (bs)^{-1}, r, r^3)^n b ({}^{sr} b) r^3)^4 \\ &= (((bs)^{n+1}, (bs)^{-n-1}, r^{n+1}, r^{-n-1}) r^3)^4 = 1_A. \end{aligned}$$

Thus the elements of $\langle bsr \rangle \cdot \{1_A, bs\}$ are of finite order, and T is a valid choice for (†).

Now let $g = {}^{a_0}b\, {}^{a_1}b \dots {}^{a_{n-1}}b\, a_n \in G$ be an element of syllable length $n \in \mathbb{N}$. If $x \in X$ is a letter such that $\operatorname{syl}(g\|_x) = n$, we have $g\|_x|^\epsilon \in \Sigma_{\{b\}}(a_n)$, as we have seen in the proof of Theorem 2.1.2. Thus if a_n is of order four, i.e. $a_n \in \{r, r^3\}$, our computation above shows that $g\|_x|^\epsilon \in \Sigma_{\{b\}}(r^{\pm 1}) = \langle sr \rangle \times \langle r^2 \rangle$ is of order two. Thus it is enough to prove that for every g such that $g|^\epsilon$ has order two, there is a number $k \in \mathbb{N}$ such that $g\|_u \in T$ or $\mathrm{syl}(g\|_u) < n$ for all $u \in X^k$.

Assume that a_n is of order two and that $\mathrm{syl}(g\|_x) = n$. Then we calculate $g\|_x$ as we have done in Lemma 2.3.1. If n is even, we see that either

$$\begin{aligned} g\|_x = \quad & b \cdot c_0 \cdot b \cdots c_{\frac{n}{2}} \cdot b \\ & \cdot c_{\frac{n}{2}+1} \cdot b \cdot c_{\frac{n}{2}+2} \cdots b \cdot c_{n-1}, \end{aligned}$$

where $c_i \in \{b|_x \mid x \in \dot{X}\} = \{s, sr\}$ for $i \in \{0, \ldots, n-1\}$, or $g\|_x$ is conjugate to such an expression. If n is odd, we obtain

$$\begin{aligned} g\|_x = \quad & b \cdot c_0 \cdot b \cdots b \cdot c_{\frac{n+1}{2}} \\ & \cdot c_{\frac{n+1}{2}+1} \cdot b \cdot c_{\frac{n+1}{2}+2} \cdots c_{n-1} \cdot b, \end{aligned}$$

with $c_i \in \{s, sr\}$ for $i \in \{0, \ldots, n-1\}$. In this case, the decomposition beginning with a c_i-letter is of syllable length $n-2$, since the two instances of b in the middle of the word cancel each other. Assume that there is a letter c_i, for any $i \neq n-1$ in case n is even, and $i \notin \{(n+1)/2, (n+1)/2+1\}$ if n is odd, such that $c_i = s$. Then there are elements $\hat{a}_i \in A$ for $i \in \{0, \ldots, n\}$ such that we may represent

$$g\|_x = {}^{\hat{a}_0}b\,{}^{\hat{a}_1}b \ldots {}^{\hat{a}_{n-1}}b\,\hat{a}_n$$

and $\hat{a}_{i+1} = \hat{a}_i s$. Assume that $\mathrm{syl}(g\|_{x\star y}) = n$ for some letter $y \in X$. According to Lemma 2.3.1, we calculate

$$g\|_{x\star y} = \prod_{j=0}^{\ell_{g\|_x}(y)} \prod_{i=0}^{n-1} b|_{x_1.\hat{a}_n^j \hat{a}_i}.$$

Since $\mathrm{syl}(g\|_{x\star y}) = n$ there is some $j \in \{0, \ldots, \ell_{g\|_x}(y)\}$ such that $y.\hat{a}_n^j a_i = 0$. But then $y.\hat{a}_n^j a_{i+1} = y.\hat{a}_n^j a_i s = 0.s = 0$, and two of the sections that evaluate to b cancel each other. Thus, $\mathrm{syl}(g\|_{x\star y}) \leq n-2$, which is a contradiction. Thus either the syllable length of g reduces when taking stabilised sections at words in X^2, or

$$g\|_x \in \{(bsr)^{n-1}b \cdot \{s, sr\} \mid n \text{ even}\} \cup \{(bsr)^{\frac{n-1}{2}} b \cdot \{1_A, r^3\} \cdot (bsr)^{\frac{n-1}{2}} b \mid n \text{ odd}\}.$$

But this set is contained in T, hence the conditions of Proposition 2.2.7 are satisfied and G is periodic.

Chapter 3

Groups of small period growth

Abstract. We construct finitely generated groups of small period growth, i.e. groups where the maximum order of an element of word length n grows very slowly in n. This answers a question of Bradford related to the lawlessness growth of groups and is connected to an approximative version of the restricted Burnside problem.

3.1 — Introduction

In this paper we provide an affirmative answer to the following question posed by H. Bradford at the "New Trends around Profinite Groups" conference in Levico Terme, 2021.

Q1 Is there a lawless finitely generated p-group of sublinear period growth?

Let G be a group generated by a finite set S. For any $n \in \mathbb{N}$ write $B_G^S(n)$ for the set of elements in G of word length at most n (with respect to S). The *period growth function* $\pi_G^S : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ *of G with respect to S*, first considered by Grigorchuk [66], is defined by

$$\pi_G^S(n) = \max\{\mathrm{ord}(g) \mid g \in B_G^S(n)\}.$$

Grigorchuk proved that the growth type of π_G^S is independent of the choice of S. Consequently, **Q1** is well-posed and we drop the superscript S in statements regarding the growth type of the period growth function of a group.

Bradford's question was motivated by an application to *lawlessness growth*, cf. [26, Example 2.7 & Question 10.2]. The lawlessness growth of a lawless group measures the minimal word length of witnesses to the non-triviality of the verbal subgroup $w(G)$ for group words w of increasing length. Since elements of order m do not satisfy any power words of length smaller than m, there is a connexion to the period growth of G. In fact, an example of a lawless p-group, p being some prime, with the properties required by **Q1** has super-linear lawlessness growth. For a detailed study on lawlessness growth, we refer to [26].

Clearly a group with the properties demanded in **Q1** is infinite, since it is lawless, and it is periodic, since otherwise there exists some $n_0 \in \mathbb{N}$ such that $\pi_G^S(n) = \infty$ for all $n \geq n_0$.

Little is known regarding the period growth of finitely generated infinite periodic groups. Grigorchuk proved that the (first) Grigorchuk group $\mathcal{G}$ fulfils $\pi_{\mathcal{G}} \precsim n^9$, where, given two non-decreasing functions $f, g : \mathbb{N} \to \mathbb{R}_{>0}$, we write $f \precsim g$ if $\limsup_{n\to\infty} f(n)/g(n) < \infty$. This bound was improved by Bartholdi and Šunić [19] to $n^{3/2}$, also extending the result to certain generalisations of $\mathcal{G}$. In [26, Remark 5.7] Bradford constructs a Golod–Shafarevich p-group of at most linear period growth. We remark that the standard proof that the Gupta–Sidki 3-group Γ_3 is periodic yields $\pi_{\Gamma_3} \precsim n^{1/\log_3(4/3)}$.

To state our main result, we need to define some functions growing very slowly. The *tetration function* $\operatorname{tetr}_k : \mathbb{N} \to \mathbb{N}$ *with base* k is defined recursively by $\operatorname{tetr}_k(0) = 1$ and $\operatorname{tetr}_k(n+1) = k^{\operatorname{tetr}_k(n)}$ for $n \in \mathbb{N}$. We define a left-inverse non-decreasing function by $\operatorname{slog}_k(n) = \max\{l \in \mathbb{N} \mid \operatorname{tetr}_k(l) \leq n\}$.

Now we may state our main result.

Theorem 3.1.1. *There exists a* 4*-generated infinite residually finite periodic* 2*-group* G *such that*

$$\pi_G \precsim \exp_8 \circ \operatorname{slog}_2 .$$

In particular, the function π_G grows slower than any iterated logarithm. Theorem 3.1.1 gives an affirmative answer to **Q1**.

The group we construct to prove Theorem 3.1.1 is realised as a group of automorphisms of a spherically homogeneous locally finite rooted tree, whose valency is unbounded. In the theory of automorphisms of rooted trees it is often interesting to obtain examples acting on regular trees, i.e. locally finite trees where all vertices (except the root vertex) have the same valency. On our way to prove Theorem 3.1.1, we obtain a family of groups of slow (albeit far faster than the growth described in Theorem 3.1.1) period growth that act on regular rooted trees without additional work.

Theorem 3.1.2. *Let* $\epsilon > 0$*. There exists a finitely generated infinite residually finite periodic* 2*-group* G_ϵ *acting on a regular rooted tree (depending on* ϵ*) such that*

$$\pi_{G_\epsilon} \precsim n^\epsilon.$$

We stress the fact that the groups we construct are residually finite. This is important in the context of the following approximative variant of the restricted Burnside problem. The restricted Burnside problem may be formulated as: Are residually finite groups with bounded period growth function finite? Thus, considering groups with slow but not bounded period growth as the next best thing to groups of finite exponent, we ask:

Q2 Among all m-generated infinite residually finite p-groups G, what are the minimal growth types of π_G?

By Zel'manovs [159, 160] solution to the restricted Burnside problem, the finite residual $\operatorname{res} \mathrm{B}(m, n)$ of the free Burnside group of rank m and exponent n is a finite group for all values of m and n. Define

$$\mathrm{zel}_m(n) = \max\{k \in \mathbb{N} \mid |\mathrm{res}\ \mathrm{B}(m,k)| \leq n\}.$$

Since **Q2** excludes finite groups, this function yields a lower bound for the period growth function of any m-generated residually finite infinite p-group. The best known lower bound for $\mathrm{zel}_m(n)$ is due to Groves and Vaughan-Lee [74], who prove that

$$\mathrm{zel}_m(n^{(4^n)}) \geq \mathrm{slog}_m(n).$$

Theorem 3.1.1 provides a group whose period growth comes close to the best known upper bound for zel_m,

$$\mathrm{zel}_m(2^{2^{\cdot^{\cdot^{\cdot^{2^m}}}}}) \leq 2^k,$$

with k appearances of the number 2 in the tower on the left side, which is due to Newman, whose argument is given in [148].

Organisation. — After some preliminary definitions, we first prove Theorem 3.1.2, and then use the groups constructed for this purpose as a model for the more involved construction of the group we use to prove Theorem 3.1.1. We then establish that all the groups constructed are lawless and thus constitute examples of groups with fast lawlessness growth. We end with some open questions related to the subject.

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3.2 — Groups of automorphisms of rooted trees

Let G be a group generated by a set S. We write $||\cdot||_S : G \to \mathbb{N}$ for the word length function of G with respect to S and $B_G^S(n)$ for the set of elements of G of length n with respect to S. For two integers $l, u \in \mathbb{Z}$, we denote by $[l,u]$ and $[l,u)$ the set of integer numbers within the corresponding intervals.

Let $(X_n)_{n\in\mathbb{N}_+}$ be a sequence of finite non-empty sets. The *(spherically homogeneous) rooted tree of type* $(X_n)_{n\in\mathbb{N}_+}$ is the tree T with finite strings $x_1 \dots x_k$, $x_i \in X_i$ for $i \in [1,k]$, as vertices and edges between strings that only differ by one letter. The empty string is called the root of the tree. Every vertex of distance k for some fixed $k \in \mathbb{N}$ from the root is a string of length k, which has valency $|X_{k+1}|+1$. The set $\mathcal{L}_T(k)$ of vertices of distance k to the root is called the k^{th} layer of the tree. We identify the first layer with the set X_1. Every vertex $u \in \mathcal{L}_T(k)$ is the root of a rooted subtree T_u of type $(X_n)_{n\geq k}$. We may compose strings in the following way: if $v \in \mathcal{L}_T(k)$ and $u \in T_v$, then the concatenation vu is a vertex of T.

If the sequence $(X_n)_{n\in\mathbb{N}_+}$ is constant, we call the corresponding tree *regular*. In this case, all subtrees T_u for $u \in T$ are isomorphic.

A *(tree) automorphism* of T is a (graph) automorphism of T fixing the root. Such a map must also leave the layers of T invariant. Let $v \in T$ and $u \in T_v$ be two vertices, and $a \in \mathrm{Aut}(T)$ an automorphism of T. Then the equation

$$(vu).a = (v.a)(u.(a|_v))$$

defines a unique automorphism $a|_v$ of T_v called the *section of a at v*.

Any automorphism a can be decomposed into its sections prescribing the action at the subtrees of the first layer, and $a|^\epsilon$, the action of a on the first layer $\mathcal{L}_T(1) = X_1$. We adopt the convention that an X_1-indexed family $(x : a_x)_{x\in X_1}$ of automorphisms $a_x \in \mathrm{Aut}(T_x)$ is identified with the automorphism having section a_x at x which stabilises the first layer. Hence for any $a \in \mathrm{Aut}(T)$ we write

$$a = (x : a|_x)_{x\in X_1} a|^\epsilon.$$

We record some important equalities for sections. Let $a \in \mathrm{Aut}(T), u \in T$ and $v \in T_u$. Then

$$(a|_u)|_v = a|_{uv}, \quad (ab)|_u = a|_u b|_{u.a}, \quad a^{-1}|_u = (a|_{u.a^{-1}})^{-1}.$$

We call an automorphism *rooted* if all its first layer sections are trivial, i.e. if it permutes the set of subtrees $\{T_x \mid x \in X_1\}$. The subgroup of rooted automorphisms is isomorphic to $\mathrm{Sym}(X_1)$.

Let $G \leq \mathrm{Aut}(T)$ be a group of automorphisms. The (pointwise) stabiliser of the k^{th} layer of T in G is denoted $\mathrm{St}_G(k)$ and called the *k^{th} layer stabiliser*. All layer stabilisers are normal subgroups of finite index in G. Their intersection is trivial, hence the group G is residually finite. The group G is called *spherically transitive* if it acts transitively on every layer $\mathcal{L}_T(k)$.

The *k^{th} rigid layer stabiliser* $\mathrm{Rist}_G(k)$ of a spherically transitive group G for some $k \in \mathbb{N}$ is the product of all (equivalently, the normal closure of a) *rigid vertex stabiliser* $\mathrm{rist}_G(u) = \{g \in G \mid g|_v = \mathrm{id} \text{ for } v \in T \setminus T_u\}$, where $u \in \mathcal{L}_T(k)$. A spherically transitive group G is *weakly branch* if $\mathrm{Rist}_G(k)$ is non-trivial for all $k \in \mathbb{N}$. Every weakly branch group is lawless, cf. [2].

If T is regular, a group $G \leq \mathrm{Aut}(T)$ is called *self-similar* if for all $u \in T$ the image of the section map $G|_u$ is contained in G. It is called *fractal* if $\mathrm{st}_G(x)|_x = G$ for all $x \in \mathcal{L}_T(1)$. The group G is called *weakly regular branch* if it contains a non-trivial subgroup $H \leq G$ such that $\mathrm{rist}_H(x) \geq H$ for all $x \in \mathcal{L}_T(1)$. Every weakly regular branch group is weakly branch.

Since we aim to provide examples of periodic groups, we need the following criterion for periodicity, which is adopted from the methods developed by Grigorchuk, Gupta and Sidki, cf. [66,77]. Since our criterion is adapted to a more general situation, we give a short proof.

Proposition 3.2.1. *Let $G \leq \operatorname{Aut}(T)$ be a group, let π be a set of primes and let $n \in \mathbb{N}$ be a positive integer, such that $G|_u/\operatorname{St}_{G|_u}(n)$ is a π-group for $u \in T$. For every vertex $u \in T$, let $||\cdot||_u : G|_u \to \mathbb{N}$ be a length function such that $||g||_u \leq 1$ implies that g is a π-element.*

If for all vertices $u, v \in T$ such that $v = uw$ for some string w of length n, and all $g \in G|_u$ we have

$$||g|_w||_v < ||g||_u / \exp(G|_u/\operatorname{St}_{G|_u}(n)), \tag{$\star$}$$

then G is a π-group.

Proof. Let $g \in G|_u$ for some $u \in \mathcal{L}_T(k)$ and $k \in \mathbb{N}$. We prove that the order of g is finite and divisible by primes in π only. The statement then is obtained by considering $u = \epsilon$. We use induction on $\ell = ||g||_u$. If $\ell \leq 1$, the element is a π-element by assumption. If $\ell > 1$, write $q = \exp(G|_u/\operatorname{St}_{G|_u}(n))$. By assumption, q is only divisible by primes in π. Now g^q stabilises the n^{th} layer, hence $g^q = (x : g^q|_x)_{x \in \mathcal{L}_{T_u}(n)}$ and $\operatorname{ord}(g) | q \cdot \operatorname{lcm}\{\operatorname{ord}(g^q|_x) \mid x \in \mathcal{L}_{T_u}(n)\}$. Using ($\star$) we obtain

$$||g^q|_x||_{ux} < ||g^q||_u / q \leq ||g||_u = \ell$$

for all $x \in \mathcal{L}_{T_u}(n)$. Thus, by induction $\operatorname{ord}(g^q|_x)$ is finite and divisible by primes in π only, and consequently the same holds for g. □

3·3 — Layerwise length reduction and the proof of Theorem 3.1.2

We construct a family of groups K_r, indexed by the positive integers, acting on regular rooted trees $T^{(r)}$ whose type depends on r. Fix a positive integer r, and write $A_r = C_2^r$ for the elementary abelian 2-group of rank r. Also fix a (minimal) generating set $E_r = \{e_i \mid i \in [0, r)\}$. Let $T^{(r)}$ be the regular rooted tree of type $(A_r)_{n \in \mathbb{N}_+}$. We now construct K_r as a group of automorphisms of $T^{(r)}$, using a construction much in spirit of the Gupta–Sidki p-groups or the second Grigorchuk group. In fact, K_r is a (constant) spinal group in the terminology of [**?**, 13].

View the group A_r as rooted automorphisms of $T^{(r)}$ by embedding A_r into $\operatorname{Sym}(A_r)$ via its right multiplication action. Notice that we may see an element $a \in A_r$ both a as vertex of $T^{(r)}$ and an automorphism acting on $T^{(r)}$. We fix a translation map of A_r, given by $a \mapsto \overline{a} := (\prod_{i=0}^{r-1} e_i)a$. Therefore $||\overline{e_i}||_{E_r} = r - 1$ for all $i \in [0, r)$.

Define $b_r \in \operatorname{Aut}(T^{(r)})$ by

$$b_r = (1_{A_r} : b_r;\ \overline{e_i} : e_i \text{ for } i \in [0, r);\ * : \mathrm{id}),$$

where $*$ stands for every element of A_r not referred to elsewhere in the tuple. Fig. 3.1

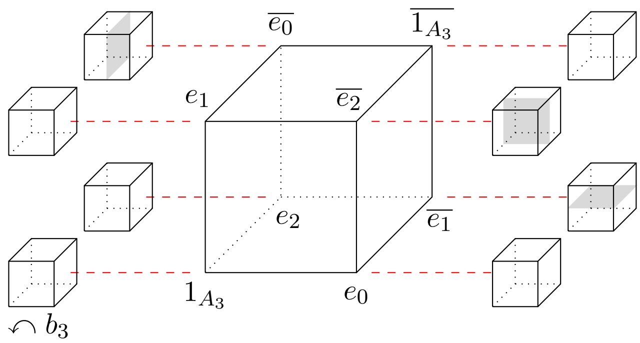

Figure 3.1: The action of the generator b_3 of P_3 on the first two layers of $T^{(3)}$.

depicts the case $r = 3$ as an example. Notice that $b_r \in \mathrm{St}(1)$ is an involution. We define

$$K_r = \langle A_r \cup \{b_r\}\rangle.$$

This is a group generated by $r + 1$ involutions. For $r = 1$ we obtain a group isomorphic to the infinite dihedral group, and also P_2 contains elements of infinite order, but for $r > 2$ all groups K_r are periodic by [**?**, Theorem A]. We do not need to rely on this result, since the bounds establishing slow period growth also show that K_r is periodic for $r > 4$. Since we are mostly interested in K_r for big r, this suffices for our purposes.

We fix two generating sets for K_r,

$$\mathbb{E}_r = E_r \cup \{b_r\} \quad \text{and} \quad \mathbb{S}_r = A_r \cup b_r^{A_r},$$

and establish some basic properties of the groups K_r.

Lemma 3.3.1. *Let $r \in \mathbb{N}_+$ be a positive integer. The group K_r is self-similar, fractal and spherically transitive. In particular, it is infinite.*

Proof. The rooted group A_r acts transitively on the first layer. Since rooted elements have trivial sections, self-similarity follows from the fact that all sections of b_r are in $\mathbb{E}_r \subset K_r$. In fact, all elements of $\mathbb{E}_r$ appear as sections of $b_r \in \mathrm{St}_{K_r}(1)$. Conjugating by rooted elements, we may achieve any section if b_r at any first layer vertex, thus K_r is fractal. By the transitivity of A_r the group K_r acts transitively on the second layer, and inductively, K_r is spherically transitive. □

Now we come to the core of our argument for establishing slow period growth. We prove an inequality between the length of an element and its sections at vertices of the second layer, using that the automorphism b_r has short sections with respect to $\mathbb{E}_r$, but the only conjugates in b_r^A aside from b_r which have non-trivial section at the vertex 1_{A_r} are big with respect to $\mathbb{E}_r$. In preparation for the proof of Theorem 3.1.1, we prove this inequality for a more general class of groups than just those of the form K_r. Therefore we need the following technical definition. Let $r \in \mathbb{N}_+$, and let $\tilde{T}$ be a rooted tree of type $(X_n)_{n\in\mathbb{N}_+}$ such that $X_1 = X_2 = A_r$. An element $b \in \mathrm{St}_{\mathrm{Aut}(\tilde{T})}(1)$ is said to *two-layer resemble* b_r if the following three conditions hold:

(i) $b|_x = b_r|_x$ for $x \in A_r \setminus \{1_{A_r}\}$,
(ii) $b|_{1_{A_r}} \in \mathrm{St}(1)$,
(iii) $b|_{1_{A_r}x} = b_r|_{1_{A_r}x}$ for $x \in A_r \setminus \{1_{A_r}\}$.

A group $\mathcal{G} \leq \mathrm{Aut}(\tilde{T})$ is said to *two-layer resemble* K_r *with respect to* b if it is generated by a set $\mathcal{E} = E_r \cup \langle b \rangle$, where b is an automorphism that two-layer resembles b_r.

Clearly b_r two-layer resembles itself. Notice that the coset $b_r\,\mathrm{St}(2)$ contains many elements that do not two-layer resemble b_r, since the first (and second) layer sections of an element in $\mathrm{St}(2)$ do not need to be rooted. In fact, if the trees $\tilde{T}$ and $T^{(r)}$ coincide, the set of elements that two-layer resembles b_r is equal to the coset $b_r \cdot \mathrm{rist}_{\mathrm{Aut}(T)}(1_{A_r}1_{A_r})$.

Lemma 3.3.2. *Let* $\mathcal{G} \leq \mathrm{Aut}(\tilde{T})$ *be a group that two-layer resembles* K_r *with respect to* $b \in \mathrm{Aut}(\tilde{T})$. *Write* $\mathcal{S} = A_r \cup \langle b \rangle^{A_r}$ *and* $\mathcal{S}'' = A_r \cup \langle b|_{1_{A_r}1_{A_r}} \rangle^{A_r}$. *Then for all* $g \in \mathcal{G}$ *and* $u \in \mathcal{L}_{\tilde{T}}(2)$ *we have*

$$||g|_u||_{\mathcal{S}''} \leq \lceil\, ||g||_{\mathcal{S}}/r \rceil\,.$$

Proof. The reader less interested in the technicalities may consider this proof in its application to the example $b = b_r$, reading $\mathcal{G} = K_r$, $\mathcal{E} = \mathcal{E}' = \mathbb{E}_r$ and $\mathcal{S} = \mathcal{S}'' = \mathbb{S}_r$, avoiding some of the cumbersome notation necessary to deal with the more delicate construction proving Theorem 3.1.1.

It is sufficient to prove $||g|_u||_{\mathcal{S}''} \leq 1$ for all $g \in B^{\mathcal{S}}_{\mathcal{G}}(r)$. From this one derives the desired inequality by splitting a minimal $\mathcal{S}$-word representing g into pieces of length at most r.

We may write

$$g = (b^{n_1})^{a_1} \dots (b^{n_{k-1}})^{a_{k-1}} a_k$$

for some $a_i \in A_r$ and $n_i \in \mathbb{Z}$ with $i \in [1, k]$. For any $x \in A_r$ we have

$$g|_x = (b^{n_1})^{a_1}|_x \dots (b^{n_{k-1}})^{a_{k-1}}|_x,$$

hence the section at x equals product of k elements of the form $(b^n)^a|_x = b^n|_{xa} = (b|_{xa})^n$ with $n \in \mathbb{Z}, a \in A_r$. If $x \neq a$, this section is equal to $(b_r|_{xa})^n$, by the first property of automorphisms two-layer resembling b_r. Otherwise we obtain $(b|_{1_{A_r}})^n$. Thus, writing $\mathcal{E}' = E_r \cup \langle b|_{1_{A_r}} \rangle$, we see that $||g|_x||_{\mathcal{E}'} \leq k \leq r$.

Now we look at $g|_{xy}$ for $y \in A_r$. Write $\underline{b}$ for $b|_{1_{A_r}}$. There are elements $a_{i,j} \in E_r$ and $n_i \in \mathbb{Z}$ with $i \in [1, k]$ and $j \in [1, m_i]$ for some $m_i \leq r$ such that

$$g|_x = a_{1,1} \dots a_{1,m_1} \underline{b}^{n_1} a_{2,1} \dots a_{2,m_2} \underline{b}^{n_2} \dots a_{k-1,1} \dots a_{k-1,m_{k_1}} \underline{b}^{n_{k-1}} a_{k,1} \dots a_{k,m_k}$$

and $k - 1 + \sum_{i=1}^{k} m_i \leq r$, since every element on the right side is obtained as a section of an expression $(b|_{xa})^n$.

Writing $\widehat{a_i} = a_{i,1} \dots a_{i,m_i}$ for $i \in [1,k]$ we obtain

$$g|_x = (\underline{b}^{n_1})^{\widehat{a_1}} (\underline{b}^{n_2})^{\widehat{a_1}\widehat{a_2}} \dots (\underline{b}^{n_{k-1}})^{\widehat{a_1} \dots \widehat{a_{k-1}}} \widehat{a_1} \dots \widehat{a_k}. \qquad (*)$$

Using the second and the third property of automorphisms two-layer resembling b_r we find, for all $n \in \mathbb{Z}$ and $a \in A_r \setminus \{y\}$,

$$(\underline{b}^n)^a|_y = \underline{b}^n|_{ya} = (\underline{b}|_{1_{A_r}(ya)})^n = (b_r|_{ya})^n,$$

hence the only generators of form $(\underline{b}^n)^a$ with non-trivial section at 1_{A_r} are powers of either $\underline{b}^y$ or $\underline{b}^{y\overline{e_i}}$ for some $i \in [0,r)$. Thus, calculating $g|_{xy}$, we might ignore all others. Assume that $(*)$ contains no generator of type $\underline{b}^y$. Then $g|_{xy}$ is a product of elements in A_r, hence of $\mathcal{S}''$-length 1. Similarly, if it does not contain a generator $\underline{b}^{y\overline{e_i}}$, it is a power of $b|_{1_{A_r}1_{A_r}}$ and also of length 1. Consequently, we have to exclude the case that both types appear in $(*)$. Assume for contradiction that this is the case. Without loss of generality we may suppose that $\underline{b}^y$ appears first. Then there exist $\ell_0, \ell_1 \in [1,k]$ such that

$$\widehat{a_1} \dots \widehat{a_{\ell_0}} = y \quad \text{and} \quad \widehat{a_1} \dots \widehat{a_{\ell_1}} = y\overline{e_i}.$$

Thus, in $(*)$, left of the $\ell_0{}^{\text{th}}$ $\underline{b}$-symbol (which is associated to the generator $\underline{b}^y$) there appear at least $||y||_{\mathcal{E}'}$ letters from $\mathcal{E}'$. Between the $\ell_0{}^{\text{th}}$ and the $\ell_1{}^{\text{th}}$ $\underline{b}$-symbols appear at least $||y\overline{e_i}||_{\mathcal{E}'} \geq r - 1 - ||y||_{\mathcal{E}'}$ letters. Thus, also counting the at least two $\underline{b}$-symbols, we obtain

$$r \geq ||g|_x||_{\mathcal{E}'} \geq ||y||_{E_r} + ||y\overline{e_i}||_{E_r} + 2 \geq r + 1,$$

a contradiction. □

Applying the lemma to $b = b_r$ and $G = K_r$ (and using the self-similarity of K_r) we obtain the following inequality.

Lemma 3.3.3. *Let $g \in K_r$ be an element and let $u \in \mathcal{L}_{T^{(r)}}(2)$. Then*

$$||g|_u||_{\mathbb{S}_r} \leq \lceil ||g||_{\mathbb{S}_r}/r \rceil .$$

Proof of Theorem 3.1.2. Notice that Proposition 3.2.1, Lemma 3.3.1 and Lemma 3.3.3 show that K_r is an infinite 2-group in case $r > 4$.

We prove $\pi_{K_r}^{\mathbb{S}_r}(n) \leq n^{1/(\log_4(r)-1)}$ for every $n \in \mathbb{N}$ and $r > 4$. Clearly, choosing some big integer r, this proves the theorem.

Let $g \in K_r$ be an element. Write $n = ||g||_{\mathbb{S}_r}$. Since A_r is a group of exponent two, $g^2 \in \mathrm{St}_{K_r}(1)$ and $g^4 \in \mathrm{St}_{K_r}(2)$. Consequently, the order of g^4 is the least common multiple of the orders of $g^4|_u$ for $u \in \mathcal{L}_{T^{(r)}}(2)$, which equals, since K_r is a 2-group, the maximum of their orders, i.e.

$$\mathrm{ord}(g) \leq 4 \cdot \max\{\mathrm{ord}(g^4|_u) \mid u \in \mathcal{L}_{T^{(r)}}(2)\}.$$

In view of Lemma 3.3.3, we see $||g^4|_u||_{\mathbb{S}_r} \leq \lceil \frac{4n}{r} \rceil$, so for $n \geq r$

$$\pi_{K_r}^{\mathbb{S}_r}(n) \leq 4 \cdot \pi_{K_r}^{\mathbb{S}_r}(\lceil 4n/r \rceil),$$

hence, using that K_r is generated by involutions,

$$\pi_{K_r}^{\mathbb{S}_r}((\tfrac{r}{4})^k) \leq 4^k \pi_{K_r}^{\mathbb{S}_r}(1) = 2 \cdot 4^k.$$

This implies

$$\pi_{K_r}^{\mathbb{S}_r} \precsim \exp_4 \circ \log_{\frac{r}{4}} \sim n^{1/(\log_4(r)-1)}. \qquad \square$$

3.4 — Growing valency and the proof of Theorem 3.1.1

We now construct a group G with the properties described in Theorem 3.1.1. To achieve this we take the generators b_r of the groups K_r constructed in the previous chapter and build a single automorphism d acting on a rooted tree with unbounded valency, that resembles some b_{r_0} for two layers (where the valency is $2^{r_0}+1$), then uses one layer to increase the valency to $2^{r_1}+1$ for some $r_1 > r_0$, then resembles b_{r_1} for two layers &c. This will allow us to use the reduction formulas for the b_r, but with (rapidly) increasing r.

The slowest period growth (using this construction) will be achieved if one arranges the sequence $(r_n)_{n\in\mathbb{N}}$ to grow as fast as possible. For this there is a natural upper bound. We want the sections of d at a given layer of valency $r_{n+1}+1$ to generate an elementary abelian 2-group acting on the layer below, but can use no more than $2^{r_n}-1$ sections as generators. Hence the maximum possible increase of valency is given by the following function $f : \mathbb{N} \to \mathbb{N}$. Let $f(0) = 3$ and $f(k+1) = 2^{f(k)} - 1$ for $k \in \mathbb{N}$. Since we aim to increase the valency of our tree on every third layer, we also introduce $f_3(k) = f(\lfloor k/3 \rfloor)$, a function that takes every value of f thrice. These functions grow very quickly.

Lemma 3.4.1. *For all $k \in \mathbb{N}$ we have $f(k) \geq \mathrm{tetr}_2(k)$.*

Proof. We use induction on k for the statement $f(k) - 1 \geq \mathrm{tetr}_2(k)$. Clearly $f(0) - 1 = 2 \geq 1 = \mathrm{tetr}_2(0)$. Now for all $k > 0$

$$f(k+1) - 1 = 2^{f(k)} - 2 \geq 2^{f(k)-1} \geq 2^{\mathrm{tetr}_2(k)} \geq \mathrm{tetr}_2(k+1). \qquad \square$$

Recall from the previous chapter that A_r denotes a copy of the elementary abelian 2-group with an (ordered) basis $E_r = \{e_0, \dots, e_{r-1}\}$. We now fix some enumeration (which may depend on r) $\{a_i \mid i \in [0, 2^r)\} = A_r$ for these groups, such that a_0 is the trivial element. Also recall also the translation map $a \mapsto \overline{a}^{(r)} = a \prod_{i=0}^{r-1} e_i$ defined in the previous chapter. We introduce the superscript to make precise within which group we are translating.

Now we define T as the rooted tree of type $(A_{f_3(k)})_{k\in\mathbb{N}}$. For any $k \equiv_3 0$ excluding $k = 0$, the k^{th}, $(k+1)^{\text{st}}$ and $(k+2)^{\text{nd}}$ layers of T have valency $2^{f_3(k)} + 1$. Write T_k for the (isomorphism class) of any subtree of T_u for some $u \in \mathcal{L}_T(k)$, i.e. $T_0 = T$ and T_k of type $(A_{f3(l)})_{l \geq k}$.

Again we view the group $A_{f3(k)}$ as rooted automorphisms by their right multiplication action. Define a sequence of automorphisms $d_n \in \mathrm{Aut}(T_k)$ for $k \in \mathbb{N}$ by

$$\begin{aligned} d_k &= (1_{A_{f3(k)}} : d_{k+1};\ \overline{e_i}^{(f3(k))} : e_i;\ * : \mathrm{id}) && \text{for } k \equiv_3 0, 1 \text{ and} \\ d_k &= (1_{A_{f3(k)}} : d_{k+1};\ a_i : e_i \text{ for } i \in [1, 2^{f3(k)})) && \text{for } k \equiv_3 2. \end{aligned}$$

Finally, we define $G_k = \langle A_{f3(k)} \cup \{d_k\} \rangle \leq \mathrm{Aut}(T_k)$, and write G for G_0.

Note that among the sections of d_k are all the elements of E_{k+1}. Using this, we see that, for every $v \in T$ of length k, we have $G|_v = G_k$, and G acts spherically transitively on T.

For $k \in \mathbb{N}$, define $\mathrm{S}_k = A_{f3(k)} \cup \{d_k\}^{A_{f3(k)}}$ and $\mathrm{E}_k = E_{f3(k)} \cup \{d_k\}$, filling the rôles of $\mathbb{S}_r$ and $\mathbb{E}_r$ of Section 3.3. Both are generating sets for G_k. Note that $d_k^2 = 1$, hence both sets consist of involutions.

Lemma 3.4.2. *Let $k \in \mathbb{N}$ be a positive integer such that $k \equiv_3 0$ and $g \in G_k$ an element. Then for all $v \in \mathcal{L}_{T_k}(2)$ we have*

$$||g|_v||_{\mathrm{S}_{k+2}} \leq \left\lceil \frac{||g||_{\mathrm{S}_n}}{f(k/3)} \right\rceil.$$

Proof. We apply Lemma 3.3.2. This is possible since by definition d_k two-layer resembles $b_{f(k/3)}$. Notice that $\mathcal{S} = \mathrm{S}_k$ and $\mathcal{S}'' = \mathrm{S}_{k+2}$. □

Lemma 3.4.3. *Let $k \in \mathbb{N}$ and let $g \in G_k$. Then for all $x \in \mathcal{L}_{T_k}(1)$*

$$||g^2|_x||_{\mathrm{S}_{k+1}} \leq ||g||_{\mathrm{S}_k} + 1.$$

Proof. Since $\langle d_k \rangle^{A_{f3(k)}}$ is closed under conjugation with $A_{f3(k)}$, we may write $g = d_k^{a_1} \dots d_k^{a_l}$ for $l = ||g||_{\mathrm{S}_n}$, for some $a_i \in A_{f3(k)}$ for $i \in [1, l]$. Then $g|_x = d_k^{a_1}|_x \dots d_k^{a_l}|_x$. Now at most every second expression $d_k^{a_i}|_x$ can evaluate to d_k. Otherwise there is some i such that $a_i = a_{i+1} = x$, which implies

$$g = d_k^{a_1} \dots d_k^{a_{i-1}} d_k^u d_k^u d_k^{a_{i+2}} \dots d_k^{a_l} = d_k^{a_1} \dots d_k^{a_{i-1}} d_k^{a_{i+2}} \dots d_k^{a_l}.$$

But then $||g||_{\mathrm{S}_n} \leq l - 2$, a contradiction. Hence there are at most $\lceil l/2 \rceil$ symbols d_k in the product $d_k^{a_1}|_x \dots d_k^{a_l}|_x$, and we have $||g|_x||_{\mathrm{S}_k} \leq \lceil \frac{1}{2} ||g||_{\mathrm{S}_k} \rceil$. Now

$$||g^2|_x||_{\mathrm{S}_{k+1}} = ||g|_x g|_{x.g}||_{\mathrm{S}_{k+1}} \leq ||g|_x||_{\mathrm{S}_{k+1}} ||g|_{x.g}||_{\mathrm{S}_{k+1}} \leq 2\lceil \tfrac{1}{2} ||g||_{\mathrm{S}_k} \rceil \leq ||g||_{\mathrm{S}_k} + 1. \qquad \square$$

Lemma 3.4.4. *Let $k \equiv_3 0$ and let $g \in G_k$. Then for all $u \in \mathcal{L}_{T_k}(3)$*

$$||g|_u||_{\mathrm{S}_{k+3}} \leq \left\lceil \frac{4 \cdot ||g||_{\mathrm{S}_k}}{f(k/3)} \right\rceil + 1.$$

Proof. Since $A_{f_3(k)}$ and $A_{f_3(k+1)}$ are of exponent two, we have $g^4 \in \mathrm{St}_{G_k}(2)$. Hence $g^8|_u = (g^4|_{u_1u_2})^2|_{u_3}$, where $u = u_1u_2u_3$. Now

$$\begin{aligned} ||g^8|_u||_{\mathrm{S}_{k+3}} &= ||(g^4|_{u_1u_2})^2|_{u_3}||_{\mathrm{S}_{k+3}} \\ &\leq ||g^4|_{u_1u_2}||_{\mathrm{S}_{k+2}} + 1 && \text{(by Lemma 3.4.3)} \\ &\leq \left\lceil \frac{||g^4||_{\mathrm{S}_k}}{f(k/3)} \right\rceil + 1 && \text{(by Lemma 3.4.2)} \\ &\leq \left\lceil \frac{4 \cdot ||g||_{\mathrm{S}_k}}{f(k/3)} \right\rceil + 1. \end{aligned}$$

□

Lemma 3.4.5. *The group G is a 2-group.*

Proof. This follows from Proposition 3.2.1 and Lemma 3.4.2. Using the notation of Proposition 3.2.1, let $n = 10$. Since $G|_u / \mathrm{St}_{G|_u}(1)$ is an elementary abelian 2-group for all $u \in T_0$, we see that $\exp(G|_u / \mathrm{St}_{G|_u}(n)) \leq 2^n$. Now, irregardless of the value of k modulo 3, taking the 10^{th} section of some $g \in G_k$ allows us to invoke Lemma 3.4.2 at least three times. Hence for all $w \in \mathcal{L}_{T_k}(10)$

$$||g|_w||_{\mathrm{S}_{k+10}} \leq \frac{||g||_{\mathrm{S}_k}}{f(0)f(1)f(2)} = \frac{||g||_{\mathrm{S}_k}}{3 \cdot 7 \cdot 127} < \frac{||g||_{\mathrm{S}_k}}{2^{10}}$$

and we conclude that G is a 2-group. □

Proof of Theorem 3.1.1. Let $n, k \in \mathbb{N}$ with $k \equiv_3 0$, and let $g \in B_{G_k}^{\mathrm{S}_k}(n)$. Since $\exp(A_l) = 2$ for all $l \in \mathbb{N}$, the 2^3-power of g fixes the third layer of T_n, hence

$$\mathrm{ord}(g^8) \leq 8 \cdot \max\{\mathrm{ord}(g^8|_v) \mid v \in \mathcal{L}_{T_k}(3)\}.$$

Now Lemma 3.4.4 implies

$$\pi_{G_k}^{\mathrm{S}_k}(n) \leq 8 \cdot \pi_{G_{k+3}}^{\mathrm{S}_{k+3}} \left(\left\lceil \tfrac{4 \cdot n}{f(k/3)} \right\rceil + 1 \right).$$

Writing $v_k(n) = \lceil 4 \cdot n / f(k/3) \rceil + 1$ and

$$u(n) = \min\{l \in \mathbb{N} \mid v_l(v_{l-1}(\dots(v_0(n))\dots)) = 2\}$$

we find

$$\pi_G^{\mathrm{S}}(u(n)) \leq 8^n \cdot \pi_{G_{3n}}^{\mathrm{S}_{3n}}(2).$$

Now, using the same argument as before, we see that $\pi_{G_{3n}}^{\mathrm{S}_{3n}}(2) \leq 4$ by Lemma 3.4.2. Thus, deriving $\mathrm{tetr}_2 \precsim u(n)$ from Lemma 3.4.1, we obtain

$$\pi_G \precsim \exp_8 \circ \operatorname{slog}_2 . \qquad \square$$

3.5 — Lawlessness growth

Let G be a lawless group generated by a finite set S. By the definition of lawlessness, the image of the word map $w(G^m)$ is non-trivial for every reduced word $w \in F_m \setminus \{1\}$ in m letters, $m \in \mathbb{N}$. We may define the *complexity of w in G with respect to S* by

$$\chi_G^S(w) = \min \left\{ \sum_{i=1}^{m} ||g_i||_S \mid \underline{g} = (g_i)_{i=1}^m \in G^m, w(\underline{g}) \neq 1 \right\} \in \mathbb{N}.$$

Now the *lawlessness growth function* $\mathcal{A}_G^S : \mathbb{N} \to \mathbb{N}$ of G with respect to S is defined by

$$\mathcal{A}_G^S(n) = \max\{\chi_G^S(w) \mid w \in F_m \setminus \{1\} \text{ with } ||w|| \leq n\}.$$

This definition is due to Bradford, first given in [26], where he proves the independence of the growth type from the choice of generating set and establishes a connexion to the period growth in the case of periodic p-groups.

Proposition 3.5.1. *[26] Let G be a finitely generated lawless periodic p-group for some prime p and $f : \mathbb{N} \to \mathbb{N}$ some function. Then $\pi_G^S(n) \leq f(n)$ implies $\mathcal{A}_G^S(f(n)) \geq n$.*

Using this, we give examples of groups with large lawlessness growth (cf. [26, Question 10.2]) by proving that the groups constructed in the previous sections are in fact lawless. As a consequence of Theorem 3.1.1 and Proposition 3.5.1 we obtain the following corollary.

Corollary 3.5.2. *There is a finitely generated lawless group G such that*

$$\mathcal{A}_G^S \succsim \mathrm{tetr}_2 \circ \log_8 .$$

It remains to prove that the group G of Theorem 3.1.1 is lawless. We prove that it is weakly branch, which is sufficient by [2]. Our proof is technical, but also establishes that the groups K_r are weakly branch for all integers $r > 5$. To avoid some obstacles appearing for small valencies, we look at G_6 instead of $G = G_0$, for which the proof of Theorem 3.1.1 works verbatim, except for the number of generators. Thus, in the remainder of this section, we write G for G_6 and define the function f prescribing the valencies of the tree upon which G acts by $f(0) = 127$ and $f(n+1) = 2^{f(n)} - 1$ for $n > 0$.

Lemma 3.5.3. *Let $r \in \mathbb{N}_{>5}$ and let $\mathcal{G} \leq \operatorname{Aut}(\widetilde{T})$ be a group that two-layer resembles K_r*

with respect to b. Define

$$N = \langle [b, e_i, e_j] \mid i, j \in [0, r), i \neq j \rangle^{\mathcal{G}} \leq \operatorname{Aut}(\widetilde{T}), \quad \textit{and}$$
$$\underline{N} = \langle [b|_{1_{A_r}}, e_i, e_j] \mid i, j \in [0, r), i \neq j \rangle^{\mathcal{G}|_{1_{A_r}}} \leq \operatorname{Aut}(\widetilde{T}|_{1_{A_r}})$$

Then for every $x \in \mathcal{L}_{\widetilde{T}}(1)$ *we have* $\operatorname{rist}_N(x) \geq \underline{N}$.

Proof. Write $c_{i,j} = [b, e_i, e_j]$ for the (normal) generators of N. Clearly $N \leq \operatorname{St}_{\mathcal{G}}(1)$. We compute

$$c_{i,j}|_x = \begin{cases} b|_{1_{A_r}} & \text{if } x \in \{1_{A_r}, e_i, e_j, e_i e_j\}, \\ e_t & \text{if } x \in \{\overline{e_t}, \overline{e_t e_i}, \overline{e_t e_j}, \overline{e_t e_i e_j}\} \text{ and } t \in [0, r) \setminus \{i, j\}, \\ e_i e_j & \text{if } x \in \{\overline{1_{A_r}}, \overline{e_i e_j}, \overline{e_i}, \overline{e_j}\}, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Let i, j, k, m, n be pairwise distinct elements of $[0, r)$ (here we need $r > 4$). We look at $[c_{i,j}, c_{m,n}^{\overline{e_k}}]$. Since both $c_{i,j}$ and $c_{m,n}^{\overline{e_k}}$ are in $\operatorname{St}(1)$, taking the commutator commutes with taking sections. All sections except $b|_{1_{A_r}}$ commute, so we have $[c_{i,j}, c_{m,n}^{\overline{e_k}}]|_x = \mathrm{id}$ for all $x \notin \{1_{A_r}, e_i, e_j, e_i e_j, \overline{e_k}, \overline{e_k e_m}, \overline{e_k e_n}, \overline{e_k e_m e_n}\}$. Since $r > 5$, all these vertices are distinct. Furthermore, for the remaining cases we calculate

$$[c_{i,j}, c_{m,n}^{\overline{e_k}}]|_x = \begin{cases} [b|_{1_{A_r}}, e_k] & \text{if } x = 1_{A_r}, \\ [e_k, b|_{1_{A_r}}] & \text{if } x = \overline{e_k}, \\ [b|_{1_{A_r}}, \mathrm{id}] = \mathrm{id} & \text{if } x \in \{e_i, e_j, e_i e_j\}, \\ [\mathrm{id}, b|_{1_{A_r}}] = \mathrm{id} & \text{if } x \in \{\overline{e_k e_m}, \overline{e_k e_n}, \overline{e_k e_m e_n}\}. \end{cases}$$

Now let $l \in [0, r) \setminus \{i, j, k\}$. Then $c_{i,j}^{\overline{e_l}}|_{1_{A_r}} = e_l$ and $c_{i,j}^{\overline{e_l}}|_{\overline{e_k}} = c_{i,j}|_{e_k e_l} = \mathrm{id}$. Consequently

$$[c_{i,j}, c_{m,n}^{\overline{e_k}}, c_{i,j}^{\overline{e_l}}]|_x = \begin{cases} [b|_{1_{A_r}}, e_k, e_l] & \text{if } x = 1_{A_r}, \\ \mathrm{id} & \text{else,} \end{cases}$$

thus $\operatorname{rist}_N(1_{A_r}) \geq \langle [b|_{1_{A_r}}, e_i, e_j] \mid i, j \in [0, r), i \neq j \rangle$. Since $\{b^{\overline{e_i}}|_{1_{A_r}} \mid i \in [0, r)\} \cup \{b|_{1_{A_r}}\}$ generates $\mathcal{G}|_{1_{A_r}}$, for every $g \in \mathcal{G}|_{1_{A_r}}$ we find an element $\widehat{g} \in \operatorname{St}_{\mathcal{G}}(1)$ such that $\widehat{g}|_{1_{A_r}} = g$. Conjugating with these elements, we find $\operatorname{rist}_N(1_{A_r}) \geq \underline{N}$. Since $\mathcal{G}$ acts transitively on the first layer, all rigid vertex stabilisers are conjugate, and we obtain the result. □

Proposition 3.5.4. *Let* $r \in \mathbb{N}_{>5}$. *Then* K_r *is weakly regular branch, hence lawless.*

Proof. This follows directly from Lemma 3.5.3, since the two normal subgroups $N, \underline{N}$ are equal in the case of K_r. □

Lemma 3.5.5. *Let* $k \in \mathbb{N}$ *and* $x \in \mathcal{L}_{T_k}(1)$. *Then* $\operatorname{St}_{G_k}(1)|_x \geq G_{k+1}$.

Proof. Observe $\mathrm{E}_{k+1} = \{d_k|_x \mid x \in \mathcal{L}_{T_k}(1)\}$ and that G_k acts transitively on $L_{T_k}(1)$. $\square$

Proposition 3.5.6. *The group G is a weakly branch group, hence a lawless group.*

Proof. Let $k \in \mathbb{N}$ be an integer such that $k \equiv_3 0$. We adopt the following notation to better distinguish between the generators of $A_{f_3(k)}$ and $A_{f_3(k+3)}$. If $a = e_{i_0} \dots e_{i_t}$ is a non-trivial element of $A_{f_3(k)}$, we write $\underline{e}_{i_0 \dots i_t}$ for the generator $d_{k+2}|_a$ of $A_{f_3(k+3)}$. Each element of $E_{f_3(k+3)}$ appears in this way. Define

$$N_k = \langle [d_k, e_i, e_j] \mid i, j \in [0, f_3(k)), i \neq j \rangle^{G_k}, \quad \text{and}$$
$$M_k = \left\langle [[d_k, a_1], [d_k, a_2]^g] \;\middle|\; \begin{array}{l} g \in G_k, a_1 = \underline{e}_j \underline{e}_{ij} \underline{e}_l \underline{e}_{il}, a_2 = \underline{e}_n \underline{e}_{mn} \underline{e}_s \underline{e}_{ms}, \\ i, j, l, m, n, s \in [0, f_3(k-1)) \text{ pairwise distinct} \end{array} \right\rangle^{G_k}.$$

The group G_k two-layer resembles $P_{f_3(k)}$, thus Lemma 3.5.3 implies $\mathrm{rist}_{N_{k+1}}(u) \geq N_{k+2}$ for $u \in \mathcal{L}_{T_{k+1}}(1)$. We show that

$$\mathrm{rist}_{M_k}(w) \geq N_{k+1} \text{ for } k > 0, \text{ and} \tag{†}$$
$$\mathrm{rist}_{N_{k+2}}(v) \geq M_{k+3}. \tag{‡}$$

Using this, we see that for all $u \in \mathcal{L}_T(l)$

$$\mathrm{rist}_G(u) \geq \begin{cases} M_l & \text{if } l \equiv_3 0, \\ N_l & \text{otherwise.} \end{cases}$$

Since N_l and M_l are non-trivial for all $l \in \mathbb{N}$, this shows that G is a weakly branch group.

In both cases it is enough to show that the normal generators of N_{k+1}, resp. M_{k+3}, are contained in the rigid vertex stabiliser of $1_{A_{f_3(k+1)}}$, resp. $1_{A_{f_3(k+3)}}$. Using Lemma 3.5.5, we find the full normal subgroup within the rigid vertex stabiliser of $1_{A_{f_3(k)}}$, and since G_k acts spherically transitive, all rigid vertex stabilisers of the same layer are conjugate.

We first prove (†). Let $k > 0$. Let $a_1, a_2 \in B^{E_{f_3(k)}}_{A_{f_3(k)}}(4)$ such that $[[d_k, a_1], [d_k, a_2]]$ is a normal generator of M_k. Calculate

$$[d_k, a_1]|_x = d_k d_k^{a_1}|_x = \begin{cases} d_{k+1} & \text{if } x \in \{1_{A_{f_3(k)}}, a_1\}, \\ e_t & \text{if } x \in \{\overline{e_t}, \overline{e_t} a_1\}, \text{ for some } t \in [0, f_3(k)), \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

We want to compute $[[d_k, a_1], [d_k, a_2]^{\overline{e}_s}]$ for arbitrary $s \in [0, f_3(k))$. The set of vertices where this element might have non-trivial sections is $\{1_{A_{f_3(k)}}, a_1, \overline{e_s}, \overline{e_s} a_2\}$.

We now prove that the sections $[d_k, a_1]|_{\overline{e_s}a_2}$ and $[d_k, a_2]^{\overline{e_s}}|_{a_1}$ are trivial, i.e. that

$$\overline{e_s}a_2 \notin \{1_{A_{f_3(k)}}, a_1, \overline{e_t}, \overline{e_t}a_1 \mid t \in [0, f_3(k)), \quad \text{and}$$
$$\overline{e_s}a_1 \notin \{1_{A_{f_3(k)}}, a_2, \overline{e_t}, \overline{e_t}a_2 \mid t \in [0, f_3(k)).$$

Now $||\overline{e_s}a_2||_{A_{f_3(k)}} \geq f_3(k)-5$, hence $\overline{e_s}a_2$ is neither trivial nor equal to a_1 of length 4. Here we use that $f_3(k) \geq f(0) > 9$. Finally $\overline{e_t}a_1 = \overline{e_s}a_2$ implies $a_1e_s = a_2e_t$, which contradicts the definition of a_1 and a_2. This proves the first, and by analogy the second, non-inclusion statement above.

Thus, we find

$$[[d_k, a_1], [d_k, a_2]^{\overline{e}_s}]|_x = \begin{cases} [d_{k+1}, e_s] & \text{if } x = 1_{A_{f_3(k)}}, \\ [e_s, d_{k+1}] & \text{if } x = \overline{e_s}, \\ \text{id} & \text{otherwise.} \end{cases}$$

For every $q \in [0, f_3(k)) \setminus \{s\}$ we obtain

$$h = [[d_k, a_1], [d_k, a_2]^{\overline{e}_s}, [d_k, a_1]^{\overline{e}_q}] \in \mathrm{rist}_{M_k}(1_{A_{f_3(k)}}),$$

such that $h|_{1_{A_{f_3(k)}}} = [d_{k+1}, e_s, e_q]$. This concludes the proof of (†).

We now prove (‡). Write $c_{i,j}$ for the element $[d_{k+2}, e_i, e_j] \in N_{k+2}$, where $i, j \in [0, f_3(k+2))$ are two distinct integers. Observe that

$$c_{i,j}|_{1_{A_{f_3(k+2)}}} = d_{k+3}\underline{e}_i\underline{e}_j\underline{e}_{ij}$$

and that $c_{i,j}|_u \in A_{f_3(k+3)}$ for all $u \in \mathcal{L}_{T_{k+2}}(1)$ except the (distinct) vertices $1_{A_{f_3(k+2)}}$, e_i, e_j and e_ie_j. Thus, for $l \in [0, f_3(k+2)) \setminus \{i, j\}$, we compute

$$[c_{i,j}, c_{i,l}]|_x = \begin{cases} [d_{k+3}\underline{e}_i\underline{e}_j\underline{e}_{ij}, d_{k+3}\underline{e}_i\underline{e}_l\underline{e}_{il}] & \text{if } x = 1_{A_{f_3(k+2)}}, \\ \text{possibly non-trivial} & \text{if } x \in \{1_{A_{f_3(k)}}, e_i, e_j, e_l, e_ie_j, e_ie_l\}, \\ \text{id} & \text{otherwise.} \end{cases}$$

By Lemma 3.5.5 there is an element $\widehat{g}_0 \in \mathrm{St}_{G_{k+2}}(1)$ such that $\widehat{g}_0|_{1_{A_{f_3(k+2)}}} = \underline{e}_i\underline{e}_j\underline{e}_{ij}$. Now

$$[c_{i,j}, c_{i,l}]^{\widehat{g}_0}|_{1_{A_{f_3(k+2)}}} = [d_{k+3}\underline{e}_i\underline{e}_j\underline{e}_{ij}, d_{k+3}\underline{e}_i\underline{e}_l\underline{e}_{il}]^{\underline{e}_i\underline{e}_j\underline{e}_{ij}} = [d_{k+3}, \underline{e}_j\underline{e}_l\underline{e}_{ij}\underline{e}_{il}],$$

and the set of vertices x such that $[c_{i,j}, c_{i,l}]^{g_0}|_x$ is possibly non-trivial is, as for $[c_{i,j}, c_{i,l}]$, the set $\{1_{A_{f_3(k)}}, e_i, e_j, e_l, e_ie_j, e_ie_l\}$.

Let $g \in G_{k+3}$. There is an element $\widehat{g}_1 \in \mathrm{St}_{G_k}(1)$ such that $\widehat{g}_1|_{1_{A_{f_3(k)}}} = g$. We conclude that for three pairwise distinct integers $m, n, s \in [0, f_3(k+2)) \setminus \{i, j, l\}$ (which is possible since the minimum value of f_3 greater then 5)

$$[[c_{i,j}, c_{i,l}], [c_{m,n}, c_{m,s}]^{\widehat{g}}]|_{1_{A_{f_3(k)}}} = [[d_{k+2}, \underline{e}_j\underline{e}_{ij}\underline{e}_l\underline{e}_{il}], [d_{k+2}, \underline{e}_n\underline{e}_{mn}\underline{e}_s\underline{e}_{ms}]^g],$$

while all other sections are trivial, hence $\mathrm{rist}_{N_k}(1_A) \geq M_{k+1}$. □

3.6 — Open questions and related concepts

In [19], the authors refer to an unpublished text of Leonov [101], where he establishes a connexion between the word growth and the period growth of the Grigorchuk group. It seems plausible that there is such a connexion: slow word growth makes for few elements of a given length, hence for a smaller set of candidates that might have big order. Consequently, we pose the following refinement of the question of Bradford.

Q3 Is there an infinite finitely generated residually finite periodic group of exponential word growth and sublinear period growth?

To answer this, it would be sufficient to prove that the groups constructed in Theorem 3.1.1 and Theorem 3.1.2 are of exponential growth, but we doubt that this is true. In view of the numerical relation between the word and period growth in the Grigorchuk group, we think that the groups G and G_ϵ are interesting candidates for groups of slow intermediate word growth. Thus we ask:

Q4 Of what growth type is the word growth of G and of G_ϵ?

Chapter 4

Conjugacy of polyspinal groups

Written in collaboration with Anitha Thillaisundaram.

Abstract. Spinal groups and multi-GGS groups are both generalisations of the well-known Grigorchuk-Gupta-Sidki (GGS-)groups. Here we give a necessary condition for spinal groups to be conjugate, and we establish a necessary and sufficient condition for multi-GGS groups to be conjugate. We also introduce a natural common generalisation of both classes, which we call polyspinal groups. Our results enable us to give an alternative negative answer to a question of Bartholdi, Grigorchuk and Šunić, on whether every finitely generated branch group is isomorphic to a weakly branch spinal group.

4.1 — Introduction

Let $m \in \mathbb{N}_{\geq 2}$ and let $T = T_m$ be the m-adic tree. Groups acting on m-adic trees have received quite a bit of attention, especially in the case when m is a prime. The interest in these groups is largely due to their nice structure, their importance in the theory of just infinite groups, and the fact that many such groups have exotic algebraic properties; we refer the reader to [13] for a good introduction.

The (first) Grigorchuk group [65] was the first notable group acting on an m-adic tree that was constructed, and it continues to play a central role in the subject. It is a 3-generated infinite periodic group acting on the binary rooted tree T_2 with many interesting properties. Its three generators include the rooted automorphism a which swaps the two maximal subtrees of T_2, and two directed automorphisms β and γ, both of which stabilise the rightmost infinite ray of the tree. They are defined recursively as follows:

$$\beta = (a, \gamma), \quad \gamma = (a, \delta),$$

where for x and y automorphisms of T_2, the notation (x, y) indicates the independent actions on the respective maximal subtrees, and

$$\delta = (1, \beta).$$

Soon after the Grigorchuk group was defined, other similar constructions followed, including the well-studied classes of Grigorchuk-Gupta-Sidki (GGS-)groups and Šunić groups (also called siblings of the Grigorchuk group).

These constructions were generalised to a natural class of so-called *spinal groups*, which are generated by a group R of rooted automorphisms (which can also be thought of as permutations of the first layer subtrees) and a group D of directed automorphisms. Both are defined by restricting the area of the tree where the automorphisms act non-trivially; rooted automorphisms act only at the root of the tree, while directed automorphisms only act on a 1-sphere around a constant path. We refer the reader to Section 4.2.2 for the precise definition. To any group D of directed automorphisms, there is an associated sequence of homomorphisms $\omega = (\omega_n)_{n\in\mathbb{N}}$ from D to $\mathrm{Sym}(X)$, prescribing the action of D at the different levels of the tree. This sequence fully determines the directed automorphisms. We write $\sigma^n R$ for the group generated by the images of the associated rooted automorphisms at level n. Note also that we can identify the vertices of the m-adic tree T with the elements of the free monoid X^*, for the alphabet $X = \{0, 1, \ldots, m-1\}$; see Section 4.2 for precise details.

Recently, Petschick [120] identified the conjugacy classes of GGS-groups within the automorphism group of their respective trees. In this paper, we give a condition for spinal groups to be conjugate in the same sense.

Theorem 4.1.1 (A necessary condition for spinal groups to be conjugate)**.** *Let G and $\widetilde{G}$ be spinal groups with defining data R, ω, respectively $\widetilde{R}$, $\widetilde{\omega}$, and let $f \in \mathrm{Aut}\, T$ be such that $\widetilde{G}^f = G$. Then there is an integer N and an isomorphism $\iota\colon D \to \widetilde{D}$, such that for all $n \geq N$ there is:*

(i) *an inner automorphism ϕ_n of $\mathrm{St}_{\mathrm{Sym}(X)}(0)$ such that $\phi_n(\sigma^n\widetilde{R}) = \sigma^n R$,*
(ii) *a tuple of inner automorphisms $\rho_n \in (\mathrm{Inn}(\sigma^n\widetilde{R}))^{X\setminus\{0\}}$,*
(iii) *an automorphism α_n of $\mathrm{Sym}(X)^{X\setminus\{0\}}$ permuting the direct factors by an element $\alpha' \in \mathrm{St}_{\mathrm{Sym}(X)}(0)$,*

such that

$$\omega_n = \phi_n^{X\setminus\{0\}} \circ \rho_n \circ \alpha_n \circ \widetilde{\omega}_n \circ \iota.$$

Furthermore, the inner automorphism ϕ_n is induced by $f|^{0^n}$ and the automorphism α_n is induced by $f|^{0^{n-1}}$.

A subclass of spinal groups, resembling the GGS-groups more closely, have received increased attention in recent times. These are called *multi-GGS groups*, which are generated by a rooted automorphism permuting the maximal subtrees cyclically, and the sequences defining their directed generators are constant, but – in contrast to the GGS-groups – they have possibly more than one directed generator. Up to relabelling the vertices of the tree, all multi-GGS groups on a fixed tree have the same rooted group $A = \langle(0\ 1\ \cdots\ m-1)\rangle \cong C_m$. These groups were first defined in [4], where they were originally called multi-edge spinal groups. The term multi-GGS groups is preferred, as the original name can be easily confused with the term *multispinal group*, which represents a more generalised family of groups generated by rooted and tree-like automorphisms, as defined in [143]. We also consider the class of *multi-EGS groups*, which allows for directed generators associated to different constant paths, and includes branch groups that do not have the congruence subgroup property. To deal with both the classes of multi-EGS and spinal groups simultaneously, we introduce a common natural generalisation, which we call *polyspinal groups*; see Section 4.2.2 for details.

For multi-GGS groups, we are able to provide a necessary and sufficient condition for the groups to be conjugate.

Theorem 4.1.2 (A condition for multi-GGS groups to be conjugate)**.** *Let G and $\widetilde{G}$ be multi-GGS groups with defining data ω and $\widetilde{\omega}$ respectively. There is an element $f \in \operatorname{Aut} T$ such that $\widetilde{G}^f = G$ if and only if there exists*

(i) *an automorphism α of A^{m-1} permuting the direct factors by an element of*

$$\mathrm{N}_{\mathrm{Sym}(X)}(A) \cap \mathrm{St}_{\mathrm{Sym}(X)}(0)$$

and

(ii) *an isomorphism $\iota \colon D \to \widetilde{D}$ such that*

$$\omega = \alpha \circ \widetilde{\omega} \circ \iota.$$

It was asked by Bartholdi, Grigorchuk and Šunić [13, Question 4] whether every finitely generated branch group is isomorphic to a spinal group. The answer to this question is known to be negative since (poly-)spinal groups are amenable [90, Proposition 23], whereas there exist finitely generated branch groups that are not amenable [140]. Recall that a group is *amenable* if it admits a left-invariant finitely-additive measure. One could then pose the question as to whether every finitely generated amenable branch group is isomorphic to a spinal group. Using the above results, we give a negative answer to this question within the class of weakly branch groups. We refer the reader to Section 4.2 for the definitions of weakly branch and branch groups.

Theorem 4.1.3. *There exists a finitely generated amenable branch group $G \leq \operatorname{Aut} T_3$ such that G is not isomorphic to any spinal group $S \leq \operatorname{Aut} T_3$. Also, if G is isomorphic to a spinal group $S \leq \operatorname{Aut} \widetilde{T}$ on a different tree $\widetilde{T}$, then S is not weakly branch with respect to its embedding into $\operatorname{Aut} \widetilde{T}$.*

Organisation. Section 4.2 consists of background material on groups acting on the m-adic tree and the definitions of weakly branch, branch, spinal, and polyspinal groups. In Section 4.3 we prove Theorem 4.1.1 and 4.1.2, and in Section 4.4 we prove Theorem 4.1.3.

4.2 — Preliminaries

For two integers $l, u \in \mathbb{Z}$, we denote by $[l, u]$ and $[l, u)$ the set of integers within the respective intervals.

Let $m \in \mathbb{N}_{\geq 2}$ and let $T = T_m$ be the m-adic tree, that is, a rooted tree where all vertices have m children. Using the alphabet $X = [0, m)$, we identify T with the Cayley graph of the free monoid X^* with respect to X, by identifying the root of the tree with the empty word. Thereby we establish a natural length function on T. We will routinely refer to vertices of the tree as words, using the said identification. The words u of length $|u| = n$, (i.e. vertices of distance n from the root) are called the n^{th} level vertices and constitute the n^{th} *layer* of the tree.

By T_u we denote the full rooted subtree of T that has its root at a vertex u and includes all vertices having u as a prefix. For any two vertices u and v, the map induced by replacing the prefix u by v, yields an isomorphism between the subtrees T_u and T_v.

Every $f \in \operatorname{Aut} T$ fixes the root, and the orbits of $\operatorname{Aut} T$ on the vertices of the tree T are precisely its layers. For $u \in X^*$ and $x \in X$ we have $f(ux) = f(u)x'$, for $x' \in X$ uniquely determined by u and f. This induces a permutation $f|^u$ of X which satisfies

$$f(ux) = f(u)f|^u(x).$$

The permutation $f|^u \in \operatorname{Sym}(X)$ is called the *label* of f at u, and the collection of all labels of f constitutes the *portrait* of f. There is a one-to-one correspondence between automorphisms of T and portraits. We say that an automorphism $f \in \operatorname{Aut} T$ has *constant portrait* induced by a permutation τ of X if all labels of f equal τ; this automorphism is denoted by $\kappa(\tau)$.

The automorphism f is *rooted* if $f|^\omega = 1$ for ω unequal to the root. Rooted automorphisms can be thought of as both elements of $\operatorname{Aut} T$ and $\operatorname{Sym}(X)$.

Let $x \in X$ be a letter. We write $\overline{x}$ for the infinite simple rooted path $(x^n)_{n \in \mathbb{N}_0}$. The automorphism f is *directed*, with directed path $\overline{x}$ for some $x \in X$, if the support $\{\omega \mid f|^\omega \neq 1\}$ of its labelling is infinite and marks only vertices at distance 1 from the set of vertices corresponding to the path $\overline{x}$.

More generally, for f an automorphism of T, since the layers are invariant under f, for $u, v \in X^*$, the equation

$$f(uv) = f(u)f|_u(v)$$

defines a unique automorphism $f|_u$ of T called the *section of f at u*. This automorphism can be viewed as the automorphism of T induced by f upon identifying the rooted subtrees of T at the vertices u and $f(u)$ with the tree T. For G a subgroup of $\operatorname{Aut} T$, we will denote the set of all sections of group elements at u by $G|_u$.

The action of $\operatorname{Aut} T$ on T will be on the left. We observe that, for any $u, v \in X^*$ and automorphisms $f, g \in \operatorname{Aut} T$, we have

$$\begin{aligned} (f|_u)|_v &= f|_{uv}, \\ (fg)|_u &= f|_{g(u)} g|_u, \\ f^{-1}|_u &= (f|_{f^{-1}(u)})^{-1}. \end{aligned}$$

The corresponding equations also hold for the labels $f|^u$.

4.2.1. Subgroups of $\operatorname{Aut} T$**.** — Let G be a subgroup of $\operatorname{Aut} T$ acting *spherically transitively*, that is, transitively on every layer of T. The *vertex stabiliser* $\operatorname{st}_G(u)$ is the subgroup consisting of elements in G that fix the vertex u. For $n \in \mathbb{N}$, the n^{th} *level stabiliser* $\operatorname{St}_G(n) = \bigcap_{|u|=n} \operatorname{st}_G(u)$ is the subgroup consisting of automorphisms that fix all vertices at level n.

We also write $\operatorname{st}_G(u)|_u$ for the restriction of the vertex stabiliser $\operatorname{st}_G(u)$ to the subtree T_u rooted at a vertex u. Since G acts spherically transitively, the vertex stabilisers at every level are conjugate under G. The group G is *fractal* if $\operatorname{st}_G(u)|_u$ coincides with $G|_u$ for all vertices u.

Recall the cyclic subgroup A_m of $\operatorname{Sym}(X)$ generated by $(0\ 1\ \cdots\ m-1)$. We denote by Γ the subgroup of all automorphisms, whose labels are all contained in A_m. In other words, the group Γ is the inverse limit of n-fold iterated wreath products of A_m:

$$\Gamma = \varprojlim_{n \in \mathbb{N}} A_m \wr \overset{n}{\cdots} \wr A_m.$$

A group $G \leq \operatorname{Aut} T$ is called *reducing with respect to* $(\mathcal{N}_n)_{n\in\mathbb{N}}$, if there is a sequence of finite sets $\mathcal{N}_n \subset \operatorname{Aut} T$ such that for all $g \in G$ there is a positive integer N such that $g|_u \in \mathcal{N}_m$ for all $u \in X^m$ with $m \geq N$. The sequence $(\mathcal{N}_n)_{n\in\mathbb{N}}$ is called the *nuclear sequence of* G. If there is some $k \in \mathbb{N}$ such that the sequence $(\mathcal{N}_n)_{n\geq k}$ is constant, then G is called *contracting* and the set $\mathcal{N} = \mathcal{N}_k$ is called the *nucleus* of G.

Let $u \in T$ be a vertex. The *rigid vertex stabiliser of* u is the subgroup $\operatorname{rst}(u) \leq \operatorname{Aut} T$ consisting of automorphisms whose portrait is trivial outside of T_u, and the n^{th} *rigid level stabiliser* $\operatorname{Rist}(n)$ is the product of all rigid vertex stabilisers of vertices at the n^{th} level. A group $G \leq \operatorname{Aut} T$ is called *weakly branch* if it acts spherically transitively and $G \cap \operatorname{Rist}(n)$ is non-trivial for all $n \in \mathbb{N}$. It is called *branch* if it is weakly branch and $G \cap \operatorname{Rist}(n)$ is of finite index in G for all $n \in \mathbb{N}$.

4.2.2. Polyspinal groups. — Spinal groups were first introduced by Bartholdi and Šunić in [19] as a common generalisation of the Grigorchuk group and the class of GGS-groups. A more general definition was formulated by Bartholdi, Grigorchuk and Šunić in [13]. Below we define a natural generalisation of spinal groups acting on the m-adic tree T.

Let $(a_n)_{n\in\mathbb{N}}$ be any sequence. We denote the shift operator $(a_n)_{n\in\mathbb{N}} \mapsto (a_{n+1})_{n\in\mathbb{N}}$ by σ.

A directed automorphism f is described by a sequence of $(m-1)$-tuples of permutations of X and a path $\overline{x}$. More generally, given a non-trivial group D, a directed path $\overline{x}$ and a sequence $\omega = (\omega_n)_{n\in\mathbb{N}}$ of homomorphisms $\omega_n\colon D \to \mathrm{Sym}(X)^{X\setminus\{x\}}$, we (recursively) define a tree automorphism for every $d \in D$ by

$$d_{\omega,x}|_y = \begin{cases} d_{\sigma\omega,x} & \text{if } y = x,\\ \pi_y\omega_1(d) & \text{otherwise.}\end{cases}$$

Here π_y denotes the projection to the y^{th} component, for $y \in X$. Write $D_{\omega,x} = \{d_{\omega,x} \mid d \in D\}$ for the set of all such automorphisms. Since $d_{\omega,x}$ fixes $\overline{x}$, we have $d'_{\omega,x}d''_{\omega,x} = (d'd'')_{\omega,x}$ for all $d', d'' \in D$, and hence a homomorphism $D \to D_{\omega,x}$ with kernel

$$\bigcap_{n\in\mathbb{N}} \ker(\omega_n).$$

All sections $D_{\omega,x}|_{x^k}$ are isomorphic to D if and only if $\bigcap_{n\geq k}\ker(\omega_n) = 1$ for all $k \in \mathbb{N}$. In this case we call $D_{\omega,x}$ a *directed group defined by* $\overline{x}$ *and* ω and drop the indices, identifying it with D.

Let D be a directed group. For every $n \in \mathbb{N}$ we define the n^{th} rooted companion group

$$\sigma^n R(D) = \langle d|^v \mid |v| = n\rangle = \langle d|^{x^{n-1}y} \mid y \in X\rangle.$$

To shorten the notation, we write

$$\begin{aligned} \sigma^n d &= d|_{x^n} && \text{for } d \in D,\ n \in \mathbb{N},\\ \sigma^n D &= D_{\sigma^n\omega,x} && \text{for } n \in \mathbb{N}.\end{aligned}$$

Definition 4.2.1. Let R be a group of rooted automorphisms acting transitively on X, and for some $r \in [1, m]$, let $x^{(0)}, \dots, x^{(r-1)}$ be r distinct elements in X. Let $D^{(0)}, \dots, D^{(r-1)}$ be directed groups defined by the constant paths given by $x^{(0)}, \dots, x^{(r-1)}$ and $\omega^{(0)}, \dots, \omega^{(r-1)}$ respectively, where the latter are sequences of homomorphisms $\omega_n^{(i)}\colon D^{(i)} \to \mathrm{Sym}(X)^{X\setminus\{x^{(i)}\}}$ such that

(i) the groups $\sigma^n R(D^{(i)})$ for $i \in [0, r)$ act transitively on X for all $n \in \mathbb{N}$, and
(ii) for all $i \in [0, r)$ and $k \in \mathbb{N}$

$$\bigcap_{n\geq k} \ker(\omega_n^{(i)}) = 1.$$

Then

$$G = \langle R, D^{(i)} \mid i \in [0, r)\rangle$$

is called the *polyspinal* group with data R, $\omega^{(0)}, \dots, \omega^{(r-1)}$, and $x^{(0)}, \dots, x^{(r-1)}$. If $r = 1$

we drop the superscript (0), and call G the *spinal group with data* R, ω.

Remark 4.2.2. *The choice of the path* $\overline{x}$ *does not matter in the case of spinal groups, which is why it is omitted from the defining data. More generally, one easily defines* directed automorphisms along ℓ, *an arbitrary infinite simple rooted path in* T, *and, with this in mind, one can more generally define a spinal group to be equipped with an arbitrary directed path. However, by conjugating by an appropriate element* $f \in \operatorname{Aut} T$, *we can always assume that* $\ell = \overline{0}$. *The same cannot be said about polyspinal groups with more than one arbitrary, non-constant, directed path; compare [50, Lemma 2.3]. We will not consider this more general case here.*

For any $n \in \mathbb{N}$, the n^{th} shifted companion of G,

$$\sigma^n G = \langle \sigma^n R, \sigma^n D^{(i)} \mid i \in [0, r) \rangle,$$

where $\sigma^n R := \langle \sigma^n R(D^{(i)}) \mid i \in [0, r) \rangle$, is again a polyspinal group.

Definition 4.2.3. We record some previously studied special cases:

- If $\sigma^n R$ is equal to the group $A = \langle (0\ 1\ \cdots\ m-1) \rangle$ for all n, all $D^{(i)}$ are necessarily direct products of cyclic groups of order m. In this case, we drop the rooted group from the defining data. Assume further that the sequences $\omega^{(i)}$ are all constant. In this case, one calls G a *multi-EGS group*; compare [96]. Clearly, all multi-EGS groups are subgroups of Γ.
- A group that is both spinal and a multi-EGS group is called a *multi-GGS group*; compare [4].
- A multi-GGS group such that the unique non-trivial $D^{(0)}$ is cyclic is called a *GGS-group.*

Lemma 4.2.4. *Let* G *be a polyspinal group with defining data* R, $\omega^{(0)}, \dots, \omega^{(r-1)}$, *and* $x^{(0)}, \dots, x^{(r-1)}$. *Then* G *is reducing with respect to*

$$\left(\sigma^n R \cup \bigcup_{i=0}^{r-1} \sigma^n D^{(i)} \right)_{n \in \mathbb{N}}.$$

Proof. Every element $g \in G$ can be represented by a word of the form $\left(\prod_{j=0}^{l-1} d_j^{r_j}\right) r_l$ for some $l \in \mathbb{N}$, with $r_j \in R$ and $d_j \in D^{(i_j)}$ with $i_j \in [0, r)$, for $j \in [0, l]$. We further assume that $i_j \neq i_{j+1}$ if $r_j = r_{j+1}$, for some $j \in [0, l)$. An element of the form $d_j^{r_j}$ is called a *syllable*, and consequently $l = \operatorname{syl}(g)$ is called the *syllable length* of g. It is enough to prove $\operatorname{syl}(g|_{xy}) < \operatorname{syl}(g)$ for all $g \in G$ with $\operatorname{syl}(g) > 1$ and $xy \in X^2$.

Let $d_j^{r_j} d_{j+1}^{r_{j+1}}$ be two neighbouring syllables. Then

$$(d_j^{r_j} d_{j+1}^{r_{j+1}})|_x = d_j|_{r_j(x)}\, d_{j+1}|_{r_{j+1}(x)}$$

has syllable length one or zero if $r_j^{-1}(x^{(i_j)}) \neq r_{j+1}^{-1}(x^{(i_{j+1})})$. Otherwise, for the element $x = r_j^{-1}(x^{(i_j)}) = r_{j+1}^{-1}(x^{(i_{j+1})})$ it is $(d_j^{r_j} d_{j+1}^{r_{j+1}})|_x = (\sigma d_j)(\sigma d_{j+1})$. If $i_j = i_{j+1}$, this is again of syllable length 1. Hence we may assume $i_j \neq i_{j+1}$. But then

$$(d_j^{r_j} d_{j+1}^{r_{j+1}})|_{xy} = \big((\sigma d_j)(\sigma d_{j+1})\big)|_y = \sigma d_j|_y \, \sigma d_{j+1}|_y$$

has syllable length at most 1.

We have proven that upon taking sections at vertices of level 2 at most every second syllable may contribute a syllable to the section. Hence $\mathrm{syl}(g|_{xy}) < \mathrm{syl}(g)$ if $\mathrm{syl}(g) > 1$. □

Lemma 4.2.5. *Polyspinal groups are fractal.*

Proof. Since $\sigma G = G|_u$, where u is any first-level vertex, is again a polyspinal group, it suffices to show that $\mathrm{st}_G(u)|_u$ equals σG. The result follows from the definition of σG and upon considering $\langle D^{(0)}, \dots, D^{(r-1)} \rangle^G|_u$. □

4.3 — Conditions to be conjugate

4.3.1. Necessary conditions for spinal groups to be conjugate. — Here, let G, resp. $\widetilde{G}$, denote polyspinal groups with defining data R, $\omega^{(0)}, \dots, \omega^{(r-1)}$, $x^{(0)}, \dots, x^{(r-1)}$, resp. $\widetilde{R}$, $\widetilde{\omega}^{(0)}, \dots, \widetilde{\omega}^{(\widetilde{r}-1)}$, $\widetilde{x}^{(0)}, \dots, \widetilde{x}^{(\widetilde{r}-1)}$. To be consistent, we write $\sigma^n \widetilde{R}$ for the rooted generators of $\sigma^n \widetilde{G}$.

Lemma 4.3.1. *Let G and $\widetilde{G}$ be polyspinal groups that are conjugate via $f \in \mathrm{Aut}\, T$; that is, $G^f = \widetilde{G}$. Then the right coset of $\sigma^n G$ defined by $f|_u$ is equal to the right coset of $\sigma^n G$ defined by $f|_v$ for all $n \in \mathbb{N}$ and $u, v \in X^n$. In particular, the right coset of $\sigma^n R$ defined by $f|^u$ is equal to the right coset of $\sigma^n R$ defined by $f|^v$.*

Proof. Let $n \in \mathbb{N}$ and $u, v \in X^n$. Since $\widetilde{G}$ acts spherically transitively, there is an element $g' \in \widetilde{G}$ such that $g'(u) = v$. Now $g'|_u \in \sigma^n \widetilde{G}$, and since $\mathrm{st}_{\widetilde{G}}(u)|_u = \sigma^n \widetilde{G}$ there is an element $g'' \in \mathrm{st}_{\widetilde{G}}(u)$ such that $g''|_u = (g'|_u)^{-1}$. Thus $g = g'g''$ maps u to v and $g|_u = 1$. Let $h \in G$ be such that $h^f = g$. Then, recalling that the action on the tree is on the left, we have $hf(u) = f(f^{-1}hf)(u) = f(v)$. Thus,

$$1 = g|_u = (h^f)|_u = f^{-1}|_{hf(u)} h|_{f(u)} f|_u = f^{-1}|_{f(v)} h|_{f(u)} f|_u = f|_v^{-1} h|_{f(u)} f|_u.$$

Restricting to the label at the vertex u yields the second statement. □

Lemma 4.3.2. *Let $G = \langle R, D \rangle$ be a spinal group directed along $\overline{0}$ and $H \leq \mathrm{Aut}\, T$ be reducing with respect to $(\mathcal{N}_n)_{n \in \mathbb{N}}$. If there is some $f \in \mathrm{Aut}\, T$ such that $H^f = G$, then there is a positive integer N such that for all $n \geq N$,*

$$\sigma^n D \subseteq \mathcal{N}_n^{f|_{0^n}}.$$

Proof. For every $d \in D$ there is some element $h(d) \in H$ such that $(h(d))^f = d$. Since H is reducing, there is a positive integer N such that for all $d \in D$, we have $h(d)|_u \in \mathcal{N}_n$ for all $u \in X^n$ with $n \geq N$. Since d fixes $\overline{0}$, the conjugate $h(d)$ must fix $f(\overline{0})$. Thus

$$\sigma^n d = d|_{0^n} = (h(d))^f|_{0^n} = (h(d)|_{f(0^n)})^{f|_{0^n}} \in \mathcal{N}_n^{f|_{0^n}}. \qquad \square$$

Lemma 4.3.3. *Let D and $\widetilde{D}$ be two directed groups defined by $\overline{x}, \omega$ and $\overline{y}, \widetilde{\omega}$ respectively. If there exists an automorphism $f \in \operatorname{Aut} T$ such that $\widetilde{D}^f = D$, then there exists an isomorphism $\iota\colon D \to \widetilde{D}$ such that for all $n \in \mathbb{N}$,*

$$\omega_n = (c(f|^{x^{n-1}z}))_{z \in X\setminus\{x\}} \circ p(f|^{x^{n-1}}) \circ \widetilde{\omega}_n \circ \iota,$$

where $c(\alpha)$ is the inner automorphism induced by α and $p(\alpha)$ is the relabelling of the components of the direct product $\operatorname{Sym}(X)^{X\setminus\{y\}}$ by α for any $\alpha \in \operatorname{Sym}(X)$, i.e. the k^{th} component of $(\operatorname{Sym}(X)^{X\setminus\{y\}})^{p(\alpha)} = \operatorname{Sym}(X)^{X\setminus\{\alpha^{-1}(y)\}}$ is the $\alpha(k)^{th}$ component of $\operatorname{Sym}(X)^{X\setminus\{y\}}$.

Proof. Denote by ι the isomorphism induced by $D = \widetilde{D}^f$. Since all elements of D have labels only at distance 1 from $\overline{x}$, respectively all elements of $\widetilde{D}$ have labels only at distance 1 from $\overline{y}$, we have $f(\overline{x}) = \overline{y}$. For any $d \in D$, $z \in X$ and $n \in \mathbb{N}$, it follows that

$$\left.\begin{matrix} \pi_z\omega_n(d) & \text{for } z \neq x \\ \sigma^n d & \text{for } z = x \end{matrix}\right\} = \sigma^{n-1} d|_z$$
$$= d|_{x^{n-1}z}$$
$$= (\iota(d))^f|_{x^{n-1}z}$$
$$= (\sigma^{n-1}\iota(d))^{f|_{x^{n-1}}}|_z$$
$$= \begin{cases} \left(\pi_{f|^{x^{n-1}}(z)}\widetilde{\omega}_n(\iota(d))\right)^{f|_{x^{n-1}z}} & \text{for } f|^{x^{n-1}}(z) \neq y \Leftrightarrow z \neq x, \\ (\sigma^n \iota(d))^{f|_{x^{n-1}z}} & \text{for } f|^{x^{n-1}}(z) = y \Leftrightarrow z = x. \end{cases}$$

Hence the result. $\square$

Proof of Theorem 4.1.1. By Lemma 4.2.4 and Lemma 4.3.2 there is $N \in \mathbb{N}$ such that

$$\sigma^n D \subseteq (\sigma^n \widetilde{D})^{f|_{0^n}} \cup (\sigma^n \widetilde{R})^{f|_{0^n}},$$

for all $n \geq N$. However since $\sigma^n D \leq \operatorname{St}(1)$ for all such n, one obtains $\sigma^n D \leq (\sigma^n \widetilde{D})^{f|_{0^n}}$. By symmetry (possibly increasing N) we have equality for all $n \geq N$. By Lemma 4.3.3 we have

$$\omega_n = (c(f|^{0^{n-1}x}))_{x \in X\setminus\{0\}} \circ p(f|^{0^{n-1}}) \circ \widetilde{\omega}_n \circ \iota$$

for some isomorphism $\iota\colon D \to \widetilde{D}$ and all $n \geq N$. By Lemma 4.3.1 we may write $f|^{0^{n-1}x} =$

$r_x f|^{0^n}$ for some $r_x \in \sigma^n \widetilde{R}$. For all $x \in X \setminus \{0\}$, it follows that

$$\pi_x \omega_n(d) = (\pi_{f|^{0^{n-1}}(x)} \widetilde{\omega}_n(\iota(d)))^{f|^{0^{n-1}x}} = (\pi_{f|^{0^{n-1}}(x)} \widetilde{\omega}_n(\iota(d)))^{r_x f|^{0^n}}, \tag{4.1}$$

implying $(\sigma^n \widetilde{R})^{f|^{0^n}} = \sigma^n R$. Thus, for every $n \geq N$, we choose

- the inner automorphism of $\mathrm{St}_{\mathrm{Sym}(X)}(0)$ induced by $f|^{0^n}$ as ϕ_n,
- the inner automorphisms of $\sigma^n \widetilde{R}$ induced by the r_x as ρ_n, and
- the automorphism $p(f|^{0^{n-1}})$ induced by the permutation $f|^{0^{n-1}}$ as α_n.

Then (4.1) implies the statement. □

4.3.2. Multi-EGS and multi-GGS groups. — Recall the group Γ, which is the inverse limit of n-fold iterated wreath products of $A = \langle (0\ 1\ \cdots\ m-1) \rangle$.

Lemma 4.3.4. *Let $g, h \in \Gamma$ be directed elements along $\overline{x}$, and $f \in \mathrm{Aut}\, T$ such that $g^f \in \Gamma$ is directed along $\overline{y}$, for some $x, y \in X$. Then h^f is directed along $\overline{y}$.*

Proof. Without loss of generality one can consider the case $x = y = 0$. Since $g^f|_0$ is directed, the element f stabilises 0. For any $x \in X \setminus \{0\}$ there is $n \in \mathbb{Z}/m\mathbb{Z}$ such that

$$g^f|_x = (g|_{f(x)})^{f|_x} = ((0\,1 \cdots m-1)^n)^{f|_x}.$$

Since g^f is directed and a member of Γ,

$$g^f|_x \in A,$$

hence $f|_x$ normalises A. Thus

$$h^f|_x = \begin{cases} (h|_0)^{f|_0} & \text{if } x = 0, \\ (h|_{f(x)})^{f|_x} \in A & \text{otherwise.} \end{cases}$$

Repeating this argument for $g|_0, h|_0$ and $f|_0$ shows that h^f fixes $\overline{0}$ and has non-trivial labels only at vertices of distance 1 to this ray. Thus h^f is directed along $\overline{0}$, as required. □

We observe that the above result cannot be generalised from Γ to $\mathrm{Aut}\, T$, since an element $f \in \mathrm{Aut}\, T$ which normalises one subgroup of $\mathrm{Sym}(X)$ may not normalise all of $\mathrm{Sym}(X)$. For example, if $f = (1, \overset{m-1}{\ldots}, 1, f')$, for some $f' \in \mathrm{Aut}\, T$, then f normalises the subgroup $\langle (1\,2\ \cdots\ m-1) \rangle$, but f does not normalise $\langle (m-1\ \ m) \rangle$ unless $f' = 1$.

Recall that for a multi-EGS group G, for $i \in X$ the directed groups $D^{(i)}$ are direct products of cyclic groups of order m and the rooted group is equal to A.

Proposition 4.3.5. *Let G and $\widetilde{G}$ be multi-EGS groups defined by ω and $\widetilde{\omega}$, respectively, such that $\widetilde{G}^f = G$ for an element $f \in \mathrm{Aut}\, T$. Then for $i \in [0, r)$, there exist*

- *automorphisms* $\alpha^{(i)}$ *of* $A^{X\setminus\{\widetilde{x}^{(i)}\}}$ *permuting the direct factors by an element of the group* $\mathrm{N}_{\mathrm{Sym}(X)}(A)\cap\mathrm{St}_{\mathrm{Sym}(X)}(\widetilde{x}^{(i)})$,
- *a map* $\theta\colon[0,r)\to[0,\widetilde{r})$ *such that* $\operatorname{rank}D^{(i)}=\operatorname{rank}\widetilde{D}^{(\theta(i))}$, *and*
- *isomorphisms* $\iota^{(\theta(i))}\colon D^{(i)}\to\widetilde{D}^{(\theta(i))}$ *such that*

$$\omega^{(i)}=\alpha^{(\theta(i))}\circ\widetilde{\omega}^{(\theta(i))}\circ\iota^{(\theta(i))}.$$

Proof. Let $i\in[0,r)$. Then the group generated by A and $D^{(i)}$ is a multi-GGS group. Write x for $x^{(i)}$, for $x\in X$. By Lemma 4.3.2 and Lemma 4.2.4, recalling that $\sigma D^{(i)}=D^{(i)}$ and $\sigma R=R=A$ for a multi-EGS group, there is some $n\in\mathbb{N}$ such that

$$D^{(i)}\subseteq\big(A\cup\bigcup_{j=0}^{\widetilde{r}}\widetilde{D}^{(j)}\big)^{f|_{x^n}}.$$

Since $D^{(i)}$ stabilises the first layer, in fact

$$D^{(i)}\subseteq\big(\bigcup_{j=0}^{\widetilde{r}}\widetilde{D}^{(j)}\big)^{f|_{x^n}}.$$

Let $1\neq d\in D^{(i)}$ and $1\neq e\in\widetilde{D}^{(j)}$, for some $j\in[0,\widetilde{r})$, be such that $e^{f|_{x^n}}=d$. Then

$$d=d|_x=(e^{f|_{x^n}})|_x=(e|_{f|^{x^n}(x)})^{f|_{x^{n+1}}}.$$

Now write y for $\widetilde{x}^{(j)}$. Since there is only one non-trivial first layer section of e stabilising the first layer, this implies $y=f|^{x^n}(x)$. Defining $\theta(i)=j$ we have

$$D^{(i)}\leq(\widetilde{D}^{(\theta(i))})^{f|_{x^n}}.$$

Now let $e\in\widetilde{D}^{(\theta(i))}$. By [120, Lemma 3.3] and the fact that multi-EGS groups are fractal, we obtain $e^{f|_{x^n}}\in G$. However by Lemma 4.3.4 it follows that since there are elements $e'\in\widetilde{D}^{(\theta(i))}$ such that $e'^{f|_{x^n}}\in D^{(i)}$ is $\overline{x}$-spinal, the element $e^{f|_{x^n}}$ is $\overline{x}$-spinal. Hence by Lemma 4.2.4 there is a positive integer $k(e)$ such that $(e^{f|_{x^n}})|_{x^{k(e)}}\in D^{(i)}$. Thus

$$(e^{f|_{x^n}})|_{x^{k(e)}}=(e|_{f|^{x^n}(x^{k(e)})})^{f|_{x^{n+k(e)}}}\in\mathrm{St}_G(1)\setminus\{1\},$$

hence $f|^{x^n}(x^{k(e)})=y^{k(e)}$ and $(e^{f|_{x^n}})|_{x^{k(e)}}=e^{f|_{x^{n+k(e)}}}\in D^{(i)}$.

Set $k_{\max}=\max\{k(e)\mid e\in\widetilde{D}^{(\theta(i))}\}$ to obtain

$$(\widetilde{D}^{(\theta(i))})^{f|_{x^{n+k_{\max}}}}\leq D^{(i)}.$$

Hence $D^{(i)}=(\widetilde{D}^{(\theta(i))})^{f|_{x^{n+k_{\max}}}}$. Taking further sections it is clear that

$$D^{(i)}=(\widetilde{D}^{(\theta(i))})^{f|_{x^k}}$$

for all $k \geq n + k_{\max}$.

Since all directed groups involved are abelian and both rooted groups are equal and cyclic, we have by Lemma 4.3.3 that for all $i \in [0, r)$,

$$\omega^{(i)} = (\phi^{(\theta(i))})^{X\setminus\{x\}} \circ \alpha^{(\theta(i))} \circ \widetilde{\omega}^{(\theta(i))} \circ \iota^{(\theta(i))}$$

with $\alpha^{(\theta(i))}$ as desired, and $\phi^{(\theta(i))} \in \mathrm{N}_{\mathrm{Sym}(X)}(A) \cap \mathrm{St}(x) \cong (\mathbb{Z}/m\mathbb{Z})^{\times}$. Hence, writing $c = (0\ 1\ \cdots\ m-1)$, we have $\phi^{(\theta(i))}(c) = c^{k^{(\theta(i))}}$ for some $k^{(\theta(i))}$ coprime to m. Therefore, replacing $\iota^{(\theta(i))}$ with

$$\hat{\iota}^{(\theta(i))} = \mu_{k^{(\theta(i))}} \circ \iota^{(\theta(i))} \colon D^{(i)} \to \widetilde{D}^{(\theta(i))}$$

where $\mu_{k^{(\theta(i))}}(\iota^{(\theta(i))}(d)) = (\iota^{(\theta(i))}(d))^{k^{(\theta(i))}}$, we obtain

$$\omega^{(i)} = \alpha^{(\theta(i))} \circ \widetilde{\omega}^{(\theta(i))} \circ \hat{\iota}^{(\theta(i))},$$

as required. □

Lemma 4.3.6. *Let G be a multi-GGS group defined by ω, and let $\iota \in \mathrm{Aut}(D)$, and $\alpha \in \mathrm{N}_{\mathrm{Sym}(X)}(A) \cap \mathrm{St}_{\mathrm{Sym}(X)}(0)$. Then the multi-GGS group $\widetilde{G}$ defined by $p(\alpha) \circ \omega \circ \iota$ is conjugate to G.*

Proof. The multi-GGS group defined by $\omega \circ \iota$ is equal to G, as for any $d \in D$,

$$\iota(d)_\omega = d_{\omega\circ\iota}$$

and vice versa.

Let $\kappa = \kappa(\alpha)$ be the automorphism with constant portrait α. Then

$$d^\kappa = (d^\kappa, (d|_{1^\alpha})^\kappa, \dots, (d|_{(m-1)^\alpha})^\kappa)$$

for all $d \in D$, hence $d^\kappa = d_{p(\alpha)\circ\omega\circ\hat{\iota}}$, using the notation $\hat{\iota}$ from the previous lemma. Since κ commutes with rooted automorphisms, we have $G^\kappa = \widetilde{G}$. □

Proof of Theorem 4.1.2. The necessity of the condition is a direct consequence of Proposition 4.3.5. The sufficiency follows from Lemma 4.3.6. □

4.4 — Finitely generated non-spinal branch groups

We now prove that one of the (finitely generated branch) *Extended Gupta–Sidki groups* defined by Pervova [119] is not isomorphic to a weakly branch spinal group. The attentive reader will notice that this by far is not the only example for a group with this property within the class of polyspinal groups.

Let $a = (0\ 1\ 2)$ be rooted and define $b = (b, a, a^2), c = (a^2, c, a)$. The group $G = \langle a, b, c\rangle$ is a polyspinal group with defining data $R = \langle a\rangle$ and $\omega_n^{(0)} = (1 : x \mapsto x,\ 2 : x \mapsto x^{-1})$,

$\omega_n^{(1)} = (0 : x \mapsto x^{-1}, 2 : x \mapsto x)$ for all $n \in \mathbb{N}$, where the letter in front of a colon signifies the component of the homomorphism after it. Both $\langle a, b\rangle$ and $\langle a, c\rangle$ are isomorphic to the Gupta–Sidki 3-group. We record some of the results of [119] in a lemma.

Lemma 4.4.1. *The group G is a just infinite torsion branch group.*

We now show the following.

Lemma 4.4.2. *The group G is not conjugate to any spinal group $S \leq \operatorname{Aut} T_3$.*

Proof. Assume for contradiction that $G^f = \widetilde{G} = \langle R, D\rangle$ is a spinal group, for some $f \in \operatorname{Aut} T_3$. Clearly, the rooted group of $\widetilde{G}$ must be cyclic of order 3. Following our usual strategy, by Lemma 4.3.2 and Lemma 4.2.4 we find $n \in \mathbb{N}$ such that

$$D_{\sigma^n\omega} \subseteq (\langle a\rangle \cup \langle b\rangle \cup \langle c\rangle)^{f|_{0^n}}.$$

Since $D_{\sigma^n\omega}$ stabilises the first layer, we have $D_{\sigma^n\omega} \cap \langle a\rangle^{f|_{0^n}} = 1$. Let $d_{\sigma^n\omega} \in D_{\sigma^n\omega}$ equal $(c^i)^{f|_{0^n}}$ for some $i \in \mathbb{F}_3$. Then, recalling that $c \in \mathrm{St}(1)$, we obtain

$$d_{\sigma^{n+1}\omega} = d_{\sigma^n\omega}|_0 = (c^i)^{f|_{0^n}}|_0 = (f|_{0^n})^{-1}|_{f|_{0^n}(0)}\, c^i\, |_{f|_{0^n}(0)}\, f|_{0^{n+1}} = (c^i\, |_{f|_{0^n}(0)})^{f|_{0^{n+1}}}.$$

As $d_{\sigma^{n+1}\omega} \in \mathrm{St}(1)$, it follows that $f|_{0^n}(0) = 1$. Repeating the argument with some $e_{\sigma^n\omega} \in D_{\sigma^n\omega}$, which equals $(b^j)^{f|_{0^n}}$ for some $j \in \mathbb{F}_3$, we see that

$$e_{\sigma^{n+1}\omega} = e_{\sigma^n\omega}|_0 = (b^j)^{f|_{0^n}}|_0 = (b^j\, |_1)^{f|_{0^{n+1}}} = (a^j)^{f|_{0^{n+1}}} \in \mathrm{St}(1),$$

hence $j = 0$ and $e_{\sigma^n\omega} = 1$. It follows $D_{\sigma^n\omega} \leq \langle c\rangle^{f|_{0^n}}$. Thus $D \cong C_3$ and $\widetilde{G}$ is generated by two elements. But since $G/G' \cong C_3^3$ the group G cannot be two-generated. This is a contradiction. □

Proof of Theorem 4.1.3. By Lemma 4.4.1, the group G is branch, torsion, and just infinite. From [14], it follows that G is amenable. If $S \leq \operatorname{Aut} T_3$ is isomorphic to G, it is conjugate to G by [71, Corollary 1(a)] and [96, Proof of Corollary 3.8]. However by Lemma 4.4.2, this is impossible.

If $S \leq \operatorname{Aut} \widetilde{T}$ is weakly branch (with respect to its embedding into $\operatorname{Aut} \widetilde{T}$) and isomorphic to G, it is just infinite and hence branch. Thus, by [71, Theorem 2] we have $\widetilde{T} \cong T_3$ and the first assertion implies the second. □

We remark that all known spinal groups that are not weakly branch act on the binary tree such that the sequence of companion groups stabilises as the infinite dihedral group; compare [116, Proposition 3.4]. Therefore we ask: "Is every involution-free spinal group weakly branch with respect to its natural embedding?"

It is natural to update [13, Question 4] to "Does every finitely generated amenable branch group admit an embedding into some $\operatorname{Aut} T$ as a branch polyspinal group?". To conclude this paper, we will make some remarks concerning the candidate presented in [13], which is the perfect regular branch group defined by Peter Neumann [114]. Neumann's group G acts on the 6-adic tree T_6 and is defined as follows. Consider the alternating group $\mathrm{Alt}(6)$ acting on $X = \{1, \dots, 6\}$. For each pair (r, x) with $x \in X$ and $r \in \mathrm{St}_{\mathrm{Alt}(6)}(x)$, we recursively define $b_{(r,x)} \in \mathrm{Aut}\, T_6$ as

$$b_{(r,x)} = (1, \dots, 1, b_{(r,x)}, 1, \dots, 1)r,$$

where the section $b_{(r,x)}$ appears in position x. Then Neumann's group G is

$$G = \langle b_{(r,x)} \mid x \in X,\, r \in \mathrm{St}_{\mathrm{Alt}(6)}(x) \rangle.$$

For further details, see [13, Section 1.6.6].

Proposition 4.4.3. *Neumann's group $G \leq \mathrm{Aut}\, T$ is not conjugate to any spinal group $S \leq \mathrm{Aut}\, T$.*

Proof. Assume for contradiction that G^f is a spinal group with data R, ω. Then by Lemma 4.3.2 there is a number $n \in \mathbb{N}$ such that

$$\sigma^n D \subseteq \mathcal{N}^{f|_{0^n}},$$

where $\mathcal{N} = \{b_{(r,x)} \mid x \in X,\, r \in \mathrm{St}_{\mathrm{Alt}(6)}(x)\}$. As the elements of $\sigma^n D$ stabilise the first layer, we arrive at a contradiction since $\mathcal{N}$ consists of elements that do not belong to $\mathrm{St}(1)$. □

By a rigidity result of Lavreniuk and Nekrashevych [99, Section 8] the automorphism group of Neumann's group G coincides with its normaliser in $\mathrm{Aut}\, T$, but further results allowing us to reduce any isomorphism to a subgroup of $\mathrm{Aut}\, T$ (or even $\mathrm{Aut}\, \widetilde{T}$) would be necessary to negatively answer the updated question.

Chapter 5

The automorphism group of a multi-GGS-group

Abstract. A multi-GGS-group is a group of automorphisms of a regular rooted tree, generalising the Gupta–Sidki p-groups. We compute the automorphism groups of all non-constant multi-GGS-groups.

5.1 — Introduction

The family of Grigorchuk–Gupta–Sikdi-groups, hereafter abbreviated 'GGS-groups', is best known as a source of groups with exotic properties, e.g. just-infinite groups or infinite finitely generated periodic groups. It generalises earlier examples constructed by its three namesakes in the 80's, see [65, 77]. These groups are defined as groups of automorphisms of a p-regular rooted tree X^*, for an odd prime p. They are two-genereated, and one of the generators is defined according to a one-dimensional subspace $\mathbf{E} \subseteq \mathbb{F}_p^{p-1}$. Allowing $\mathbf{E}$ to be more than one-dimensional yields a natural generalisation, these 'higher-dimensional' GGS-groups are called *multi*-GGS-*groups* or *multi-edge spinal groups*. We prefer the first term.

In many regards, multi-GGS-groups do not differ overly much from their one-dimensional counterparts, e.g. they are periodic under similar conditions, see [4, Theorem 3.2], they possess the congruence subgroup property, see [55], and they allow similar branching structures. Their virtue, aside from extending the list of subgroups of $\mathrm{Aut}(X^*)$ with remarkable properties, lies therein that many conditions on GGS-groups, when generalised to the higher-dimensional counterparts, reveal themselves as linear conditions. In this sense, multi-GGS-groups are the more natural class.

We are concerned with the computation of the automorphism groups of a given multi-GGS-group. The automorphisms of groups acting on rooted trees have been investigated before, e.g. in [17, 99]. In general, such groups are quite rigid objects, and their automorphisms are induced by homeomorphisms of the tree. Indeed, in many cases all automorphisms are actually induced by automorphisms of the tree, cf. [71,99], and for some specific classes the automorphism groups can be uniformly computed, see [17]. More generally, the (abstract) commensurator of groups acting on rooted trees has been investigated, cf. [132]; this is the group of 'almost automorphisms', i.e. automorphisms between two finite-index subgroups.

However, there are only few explicit computations of the automorphism group of GGS- and related groups. Sidki computed the automorphism group of the Gupta–Sidki 3-group in [138], and building on the approach for this group, the automorphism groups of the first Grigorchuk group [70], the Fabrikowski–Gupta and the constant GGS-group on the ternary tree [138] have been computed. The first and the last two examples give a complete list of GGS-groups acting on the ternary tree.

We now state our main result.

Theorem 5.1.1. *Let G be a non-constant multi-*GGS*-group, and let U be the maximal subgroup of $\mathbb{F}_p^\times$ such that $\mathbf{E}$ – seen as a matrix consisting of its (row) basis vectors – is invariant under the permutation action induced by U by reordering the columns according to multiplication, and W the maximal subgroup of $\mathbb{F}_p^\times$ of elements λ such that $\mathbf{E} \subseteq \mathrm{Eig}_\lambda(u)$ for some $u \in U$. Then the following holds.*

(i) *If G is regular, then*

$$\mathrm{Aut}(G) \cong (G \rtimes \prod_{\infty} \mathrm{C}_p) \rtimes (U \times W).$$

(ii) *If G is symmetric, then*

$$\mathrm{Aut}(G) \cong (G \rtimes \mathrm{C}_p) \rtimes (U \times W).$$

The definitions of 'regular' and 'symmetric' can be found in Section 5.2.2. The slightly obscure definitions of U and W are made more transparent in Section 5.5. We can immediately derive the following corollary.

Corollary 5.1.2. *Let G be a non-constant multi-*GGS*-group. Then the following statements hold.*

(i) *The outer automorphism group of G is finite if and only if G is a symmetric* GGS-*group.*
(ii) *The outer automorphism group of G is non-trivial.*
(iii) *The automorphism group of G contains elements of order coprime to p if and only if $\mathbf{E}$ is invariant under a permutation induced by multiplication in $\mathbb{F}_p$.*
(iv) *The automorphism group of G is a p-group if and only if G is periodic and $\mathbf{E}$ is not invariant under any permutation induced by multiplication in $\mathbb{F}_p$.*

We also explicitly compute the automorphism group for a selection of examples, e.g. all Gupta–Sidki p-groups, see Section 5.6.

Our proof combines the methods developed by Sidki in [138] (cf. [17] for a sketch of the strategy used in Sidki's paper) with techniques used by the author to determine the isomorphism classes of GGS-groups in [120]. This, together with a theorem on the rigidity of branch groups of Grigorchuk and Wilson [71], allows to reduce the complexity of the computations. On the other hand, the inclusion of the symmetric GGS-groups complicates some of the arguments.

5.2 — Higher dimensional Grigorchuk–Gupta–Sidki-groups

5.2.1. Regular rooted trees and their automorphisms. — Let p be an odd prime, and denote by X the set $\{0, \dots, p-1\}$. The Cayley graph X^* of the free monoid on X is a p-regular rooted tree. We think of the vertices of X^* as words in X. The root of the tree is the empty word $\varnothing$. We write X^n for the set of all words of length n, called the *n-th layer of X^**, and we identify X and X^1.

Every (graph) automorphism $g \in \operatorname{Aut}(X^*)$ necessarily fixes the root, since it has a smaller valency than every other vertex. Consequently, every automorphism g leaves all layers X^n invariant. We write $\operatorname{St}(n)$ for the (setwise) stabilisier of X^n, and $\operatorname{St}_G(n)$ for its intersection with some subgroup $G \leq \operatorname{Aut}(X^*)$. We call a group $G \leq \operatorname{Aut}(X^*)$ *spherically transitive* if it acts transitively on all layers X^n.

The group $\operatorname{Aut}(X^*)$ inherits the self-similar structure of X^*, and decomposes as a wreath product

$$\operatorname{Aut}(X^*) \cong \operatorname{Aut}(X^*) \wr_X \operatorname{Sym}(X).$$

We deduce that $\operatorname{Aut}(X^*) \cong \operatorname{Aut}(X^*) \wr_{X^n} (\operatorname{Sym}(X) \wr \times \overset{p^n}{\dots} \times \wr \operatorname{Sym}(X))$, for every $n \in \mathbb{N}_0$, where the finite iterated wreath product acts on X^n as on the leaves of the the finite rooted p-regular tree with n layers. The base group of the n^{th} such wreath product decomposition is equal to the n^{th} layer stabiliser. We denote the induced isomorphism $\operatorname{St}(n) \to \operatorname{Aut}(X^*) \times \overset{p^n}{\dots} \times \operatorname{Aut}(X^*)$ by ψ_n. For $v \in X^n$, we denote the projection to the v^{th} component of the base group by $|_v \colon \operatorname{Aut}(X^*) \to \operatorname{Aut}(X^*)$, this so-called *section map* is a group homomorphism on the pointwise stabiliser $\operatorname{stab}(v)$ of v. We call a subgroup $G \leq \operatorname{Aut}(X^*)$ *self-similar* if $G|_v \subseteq G$ for all $v \in X^*$, and we call it *fractal* if $\operatorname{St}_G(1)|_x \leq G$ for all $x \in X$.

The image of an element $g \in \operatorname{Aut}(X^*)$ in $\operatorname{Sym}(X)$ under the quotient by $\operatorname{St}(1)$ is denoted $g|^\varnothing$, and we write $g|^v = g|_v|^\varnothing$, for any $v \in X^*$, for the *label of g at v*. Any automorphism is uniquely determined by the collection of its labels.

We fix an embedding $\operatorname{rt} \colon \operatorname{Sym}(X) \to \operatorname{Aut}(X^*)$ by $\operatorname{rt}(\sigma)|^\varnothing = \sigma$ and $\operatorname{rt}(\sigma)|^v = 1$ for all $v \in X^* \smallsetminus \{\varnothing\}$. We call the elements $\operatorname{rt}\operatorname{Sym}(X)$ *rooted automorphisms*.

Let $\Gamma \leq \operatorname{Sym}(X)$ be a permutation group. We define the *Γ-labelled subgroup of* $\operatorname{Aut}(X^*)$ by

$$\operatorname{lab}(\Gamma) = \{g \in \operatorname{Aut}(X^*) \mid g|^v \in \Gamma \text{ for all } v \in X^*\}.$$

It is a well-known fact that if Γ is of order p, the subgroup $\operatorname{lab}(\Gamma)$ is a Sylow pro-p subgroup of $\operatorname{Aut}(X^*)$.

Let $(x_i)_{i \in \mathbb{N}_0}$ be a sequence of elements $x_i \in X$. The words $\{x_0 \cdots x_k \mid k \in \mathbb{N}_0\}$ form a half-infinite *ray* R in X^* (or, equivalenty, a point of the boundary). Write $\overline{x}$ for the ray associated to the constant sequence $(x)_{i\in\mathbb{N}_0}$. An *R-directed automorphism* is an automorphism g fixing R such that for all $v \in X^*$ either v connected by an edge to an element of R, or

$$g|^v = \mathrm{id}\,.$$

A spherically transitive group $G \leq \mathrm{Aut}(X^*)$ is *regular branch over* K, for a finite-index subgroup $K \leq G$, if K contains $\psi_1^{-1}(K \times .\overset{p}{.}. \times K)$ as a subgroup of finite index.

5.2.2. Multi-GGS-groups. — Fix the permutation $\sigma = (0\,1\,\ldots\,p{-}1)$. Write $a = \mathrm{rt}(\sigma)$ and $A = \langle a \rangle$, as well as $\Sigma = \langle \sigma \rangle$. Let $\mathbf{E}$ be an r-dimensional subspace of $\mathbb{F}_p^{p-1}$, for $r > 0$. Choose an ordered basis $(\mathbf{b}_1, \ldots, \mathbf{b}_r)$ of $\mathbf{E}$, and denote the standard basis of $\mathbb{F}_p^r$ by $(\mathbf{s}_1, \ldots, \mathbf{s}_r)$. Let $E \in \mathrm{Mat}(r, p-1; \mathbb{F}_p)$ be the matrix with the basis elements as rows. The columns of E are denoted $\mathbf{e}_i$ for $i \in \{1, \ldots, p-1\}$; thinking of $\mathbf{E}$ as a subspace of $\{0\} \times \mathbb{F}_p^{p-1} \leq \mathbb{F}_p^p$, we will also write $\mathbf{e}_0$ for the zero column vector of length r. Define, for all $j \in \{1, \ldots, r\}$, the $\overline{0}$-directed automorphisms

$$b^{\mathbf{s}_j} := \psi_1^{-1}(b^{\mathbf{s}_j}, a^{\mathbf{s}_j\cdot\mathbf{e}_1}, \ldots, a^{\mathbf{s}_j\cdot\mathbf{e}_{p-1}}).$$

Since a has order p, we may extend this definition to arbitrary vectors $\mathbf{n} \in \mathbb{F}_p^r$, such that $\psi_1(b^{\mathbf{n}}) = (b^{\mathbf{n}}, a^{\mathbf{n}\cdot E})$, where for any $\mathbf{m} = (m_1, \ldots, m_{p-1}) \in \mathbb{F}_p^{p-1}$ we set $a^{\mathbf{m}}$ to be the tuple $(a^{m_1}, \ldots, a^{m_{p-1}})$ (and tuples are combined appropriately). The associated map $b^{\bullet}\colon \mathbb{F}_p^r \to \mathrm{Aut}(X^*)$ is an injective group homomorphism. We write B for the image $b^{\mathbb{F}_p^r}$.

Definition 5.2.1. The *multi-*GGS-*group associated to* $\mathbf{E}$ is the group $G_{\mathbf{E}}$ of automorphisms generated by the set

$$A \cup B.$$

The subgroup A (shared by all multi-GGS-groups) is called the *rooted group*, and the subgroup B is called the *directed group*. The generating set in the definition is clearly not minimal; a minimal generating set is given by $\{a\} \cup \{b^{\mathbf{s}_j} \mid j \in \{1, \ldots, r\}\}$.

If the dimension r of $\mathbf{E}$ is 1, one usually speaks of a GGS-group rather than a multi-GGS-group. In this case, abusing notation, we write b for $b^{\mathbf{s}_1}$.

Depending on the space $\mathbf{E}$, we distinguish three classes of multi-GGS-groups:

(i) If $\mathbf{E} = \{(\lambda, \ldots, \lambda) \in \mathbb{F}_p^{p-1} \mid \lambda \in \mathbb{F}_p\}$, we call $G_{\mathbf{E}}$ the *constant* GGS-*group*. This special case behaves very differently to all other multi-GGS-groups; we will, for the most part, exclude it from our considerations.
(ii) If $r = 1$, the space $\mathbf{E}$ is contained in $\{(\lambda_1, \ldots, \lambda_{p-1}) \in \mathbb{F}_p^{p-1} \mid \lambda_i = \lambda_{p-i}$ for all $i \in \{1, \ldots, p-1\}\}$, and $G_{\mathbf{E}}$ is not the constant GGS-group, we call $G_{\mathbf{E}}$ a *symmetric* GGS-group.
(iii) If $G_{\mathbf{E}}$ is neither constant nor symmetric, we call it a *regular* multi-GGS-group.

We record some of the key properties of multi-GGS-groups that have been established in the literature, cf. [96, Proposition 3.3] & [55, Lemma 2] for statement (i), [96, Proposition 4.3 and Proposition 3.2] for statements (ii) and (iii), [48, Lemma 3.5] for (iv), [49, Theorem C] for (v), and [4, Proposition 3.1] for (vi).

Theorem 5.2.2. *Let $G = G_{\mathbf{E}}$ be a multi-*GGS*-group. Then the following statements hold.*

(i) *If G is regular, it is regular branch over the derived subgroup G', and the equality $\psi_1(\mathrm{St}_G(1)') = G' \times \overset{p}{\ldots} \times G'$ holds.*
(ii) *The abelisation of G is an elementary abelian p-group of rank $r+1$.*
(iii) *If G is not constant, it is regular branch over $\gamma_3(G)$, such that $\psi_1(\gamma_3(\mathrm{St}_G(1))) = \gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G)$.*
(iv) *If G is a symmetric* GGS*-group, the intersection $\psi_1(\mathrm{St}_G(1)) \cap G' \times \overset{p}{\ldots} \times G'$ fulfils*

$$\psi_1(\mathrm{St}_G(1)) \cap (G' \times \overset{p}{\ldots} \times G') = \{(g_0, \ldots, g_{p-1}) \mid \textstyle\prod_{i=0}^{p-1} g_i \in \gamma_3(G)\},$$

and thus is of index p in $G' \times \overset{p}{\ldots} \times G'$.
(v) *Every multi-*GGS*-group is self-similar and fractal.*

It is fruitful to introduce the following overgroup to deal with the special case of symmetric GGS-groups.

Definition 5.2.3. Let G be a multi-GGS-group. Set $\underline{c} = \psi_1^{-1}([b^{\mathbf{s}_1}, a], \mathrm{id}, \ldots, \mathrm{id})$. The *regularisation G_{reg} of G* is the group

$$G_{\mathrm{reg}} = \langle G \cup \{\underline{c}\} \rangle .$$

We record the following lemma on the regularisation of a multi-GGS-group.

Lemma 5.2.4. *Let G be a non-constant multi-*GGS*-group. Then the following statements hold.*

(i) *$G_{\mathrm{reg}} = G$ if and only if G is regular.*
(ii) *If G is symmetric, then $|G_{\mathrm{reg}} : G| = p$.*
(iii) *The derived subgroups of G and G_{reg} are equal.*

The first two statements are immediate consequences of Theorem 5.2.2. Also the last statement follows, in view of

$$\psi_1([b^{\mathbf{s}_1}, \underline{c}]) = ([b^{\mathbf{s}_1}, [b, a]], \mathrm{id}, \ldots, \mathrm{id}) \in \gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G),$$

and of

$$\psi_1([a, \underline{c}]) = ([b, a], \mathrm{id}, \ldots, \mathrm{id}, [b, a]^{-1}) \in \{(g_0, \ldots, g_{p-1}) \mid \textstyle\prod_{i=0}^{p-1} g_i \in \gamma_3(G)\},$$

from Theorem 5.2.2 (iii) and (iv).

5.2.3. Constructions within $\mathrm{Aut}(X^*)$**.** — We introduce some notation. Let $g \in \mathrm{Aut}(X^*)$ and $n \in \mathbb{N}_0$. We define the n^{th} *diagonal of* g as the element

$$\kappa_n(g) = \psi_n^{-1}(g, \times\overset{p^n}{\ldots}\times, g).$$

Analogously, for any subset $G \subseteq \mathrm{Aut}(X^*)$ we define $\kappa_n(G) = \{\kappa_n(g) \mid g \in G\}$. Note that if G is a group, the set $\kappa_n(G)$ is a group isomorphic to G.

Definition 5.2.5. Let $S \subseteq \mathrm{Aut}(X^*)$ be a set of tree automorphisms. The *diagonal closure of* S is the set

$$\overline{S} = \left\{ \prod_{i=0}^{\infty} \kappa_i(s_i) \;\middle|\; s_i \in S \text{ for } i \in \mathbb{N}_0 \right\}.$$

Since the n^{th} factor of the infinite product is contained in $\mathrm{St}(n)$, the product is defined as $\mathrm{Aut}(X^*)$ is closed in the (profinite) topology induced by the layer stabilisers. Note that the diagonal closure is in general not a subgroup, even if $S \leq \mathrm{Aut}(X^*)$ is one.

Definition 5.2.6. Let $S = \mathrm{rt}(\Sigma)$ be a group of rooted automorphisms of $\mathrm{Aut}(X^*)$. The group

$$\kappa_\infty(S) = \langle \kappa_n(s) \mid n \in \mathbb{N}_0, s \in S \rangle$$

is called the *group of layerwise constant labels in* Σ.

It is easy to see that $\kappa_\infty(S) \cong \prod_\omega S$, and $\overline{\kappa_\infty(S)} = \overline{S}$.

5.2.4. Coordinates for multi-GGS-groups. — We first establish the following lemma, that allows us to uniquely describe elements of the first layer stabiliser in terms of 'coordinates'. To be precise, we construct an isomorphism

$$\mathrm{St}_G(1) \cong (G' \times \overset{p}{\ldots} \times G') \rtimes B.$$

This uses the fact that, also modulo $\psi_1^{-1}(G' \times \overset{p}{\ldots} \times G')$, the labels $g|^x$ at first layer vertices of an element $g \in \mathrm{St}_G(1)$ are completely determined by the image of g in G/G'. Recall that $\mathbf{e}_i$ is the i^{th} column of E, and that $\mathbf{e}_0$ denotes the zero column vector of length r.

Lemma 5.2.7. *Let G be a non-constant multi-*GGS*-group. Let $g_0, \ldots, g_{p-1} \in G$ be a collection of elements of G. Then*

$$\psi_1^{-1}(g_0, \ldots, g_{p-1}) \in \mathrm{St}_{G_{\mathrm{reg}}}(1)$$

if and only if there exist $\mathbf{n}_k \in \mathbb{F}_p^r$ and $y_k \in G'$ for $k \in \{0, \ldots, p-1\}$ such that

$$g_k = a^{s_k} b^{\mathbf{n_k}} y_k, \quad \textit{where} \quad s_k = \sum_{i=0}^{p-1} \mathbf{n}_i \cdot \mathbf{e}_{k-i}.$$

Proof. We first prove that every element with its sections determined by a collection of vectors and elements of the commutator subgroup defines an element of the regularisation. Fix some $\mathbf{n}_k \in \mathbb{F}_p^r$ and $y_k \in G'$ for all $k \in X$. Then the element

$$g = b^{\mathbf{n}_0}(b^{\mathbf{n}_1})^{a^{p-1}} \dots (b^{\mathbf{n}_{p-1}})^a \in \mathrm{St}_G(1)$$

fulfils

$$\begin{aligned} g|_k &= a^{\mathbf{n}_0 \cdot \mathbf{e}_k} a^{\mathbf{n}_1 \cdot \mathbf{e}_{k-1}} \dots a^{\mathbf{n}_{k-1} \cdot \mathbf{e}_1} b^{\mathbf{n}_k} a^{\mathbf{n}_{k+1} \cdot \mathbf{e}_{p-1}} \dots a^{\mathbf{n}_{p-1} \cdot \mathbf{e}_{k+1}} \\ &= a^{s_k} b^{\mathbf{n}_k} \tilde{y}_k \end{aligned}$$

for some $\tilde{y}_k \in G'$. We have

$$\mathrm{St}_{G_{\mathrm{reg}}}(1) \geq \langle \mathrm{St}_G(1)' \cup \{\underline{c}\} \rangle^G = \psi_1^{-1}(G' \times \dots \times G'),$$

which follows directly from Theorem 5.2.2 (i) for regular G. For symmetric G, by Theorem 5.2.2 (iv), it is enough to show that $\langle \underline{c} \rangle^G = \psi_1^{-1}(G' \times \dots \times G')$. Clearly $[b, a]$ normally generates G'. The conjugates of $\underline{c}$ have only one non-trivial section, which is equal to $[b, a]$. The statement follows, since G is fractal.

Thus the element $y = \psi_1^{-1}(\tilde{y}_0^{-1} y_0, \dots, \tilde{y}_{p-1}^{-1} y_{p-1})$ is contained in G_{reg}, so the element

$$gy = \psi_1^{-1}(a^{s_0} b^{\mathbf{n}_0} y_0, \dots, a^{s_{p-1}} b^{\mathbf{n}_{p-1}} y_{p-1}) \in G_{\mathrm{reg}}$$

has the prescribed sections.

Now let $g \in \mathrm{St}_{G_{\mathrm{reg}}}(1)$. Up to $\psi_1^{-1}(G' \times \overset{p}{\dots} \times G')$, i.e. up to the choice of $y_k \in G'$ for $k \in \{0, \dots, p-1\}$, we may calculate modulo the subgroup $L \coloneqq \langle \mathrm{St}_G(1)' \cup \{\underline{c}\} \rangle^G \leq G_{\mathrm{reg}}$. Thus there are $\mathbf{n}_k \in \mathbb{F}_p^r$ for $k \in X$ such that

$$g \equiv_L b^{\mathbf{n}_0}(b^{\mathbf{n}_1})^{a^{p-1}} \dots (b^{\mathbf{n}_{p-1}})^a.$$

Taking sections as we did above shows that $g|_k \equiv_{G'} a^{s_k} b^{\mathbf{n}_k}$. □

Given $g \in \mathrm{St}_{G_{\mathrm{reg}}}(1)$, we call the vectors $\mathbf{n}_k$ introduced in Lemma 5.2.7 the *B-coordinates of g*, and the collection of elements $y_k \in G'$ the *L-coordinates of g*. The elements s_k (since they are fixed by the B-coordinates) are called the *forced A-coordinates of g*.

5.2.5. Strategy for the proof of Theorem 5.1.1. — By [71, Theorem 1] and [96, Proposition 3.7], the automorphism group of G coincides with the normaliser of G in $\mathrm{Aut}(X^*)$. Hence we compute this normaliser $\mathrm{N}(G)$. In general, for any $H \leq \mathrm{Aut}(X^*)$, we denote by $\mathrm{N}(H)$ (without subscript) the normaliser of H in $\mathrm{Aut}(X^*)$.

The normaliser of Σ in $\mathrm{Sym}(X)$ has the form $\mathrm{N}_{\mathrm{Sym}(X)}(\Sigma) \cong \Sigma \rtimes \Delta$, where $\Delta \cong \mathbb{F}_p^\times$ with the multiplication action on $\Sigma \cong \mathbb{F}_p$. Heuristically, the automorphism group of a multi-GGS-group G allows for a similar decomposition. Since G is contained in $\mathrm{lab}(\Sigma)$, its normaliser is contained in $\mathrm{lab}(\mathrm{N}_{\mathrm{Sym}(X)}(\Sigma))$, which decomposes as described above. The normaliser of G within $\mathrm{lab}(\Sigma)$ is not identical to G, but turns out to be closely related. Apart from G being symmetric or not, the structure of $\mathbf{E}$ only comes into play when considering the normaliser of G in $\mathrm{lab}(\Delta)$.

We first consider the normaliser of G in an appropriate closure within $\mathrm{lab}(\Sigma)$. Then we prove that the full normaliser splits as a semidirect product of the normaliser of G within said closure, and normaliser of G within an appropriate subgroup of $\mathrm{lab}(\Delta)$. At last, we compute the normaliser of G within $\mathrm{lab}(\Delta)$, and combine our results.

5·3 — The normaliser in $\overline{G_{\mathrm{reg}}}$

We begin our study of elements normalising G. Adating the strategy of Sidki in [138], we start not with the normaliser in the full automorphism group, but rather in the group $\overline{G_{\mathrm{reg}}} \geq G$. This a natural candidate, since it contains the normaliser of the rooted group A (cf. Lemma 5.3.2) and the group G itself.

Lemma 5.3.1. *Let G be a non-constant multi-*GGS*-group. Then*

$$\kappa_1(G_{\mathrm{reg}}) \leq \kappa_\infty(A) \cdot G_{\mathrm{reg}}.$$

Proof. We check that the generators of $\kappa_1(G_{\mathrm{reg}})$ are contained in the group on the right hand side. Clearly $\kappa_1(a) \in \kappa_\infty(A)$.

To see that $\kappa_1(b^{\mathbf{s}_j})$ is contained in $\kappa_\infty(A) \cdot G_{\mathrm{reg}}$ for a given $j \in \{1, \dots, r\}$, we use Lemma 5.2.7. We have no choice for the set of B-coordinates; since $\kappa_1(b^{\mathbf{s}_j})|_x = b^{\mathbf{s}_j}$ for all $x \in X$ they are all equal to $\mathbf{s}_j$. Thus we compute the forced A-coordinates

$$s_k = \sum_{i=0}^{p-1} \mathbf{n}_i \cdot \mathbf{e}_{k-i} = \sum_{i=0}^{p-1} \mathbf{s}_j \cdot \mathbf{e}_{k-i} = \sum_{i=0}^{p-1} e_{j,k-i},$$

where $e_{j,k-i}$ is the respective entry of E. Consequently, all forced A-coordinates are equal to some fixed $s \in \mathbb{F}_p$ and independent of k. Hence

$$\kappa_1(a^s b^{\mathbf{s}_j}) \in G_{\mathrm{reg}}.$$

Since we have already established that $\kappa_1(a) \in \kappa_\infty(A)$, this implies that $\kappa_1(b^{\mathbf{s}_j})$ is contained in $\kappa_\infty(A) \cdot G_{\mathrm{reg}}$ for all $j \in \{1, \dots, r\}$.

Finally, in the case that G is symmetric, we have $[\kappa_1(a), b] = \underline{c}$, and hence

$$\psi_1([\kappa_2(a), \kappa_1(b)]) = \kappa_1([\kappa_1(a), b]) = \kappa_1(\underline{c}). \qquad \square$$

The problem to determine the normaliser is easily solved for the rooted group A. To determine the normaliser of B is significantly harder.

Lemma 5.3.2. *The centraliser and the normaliser of the rooted group A are given by*

$$\mathrm{C}(A) = \kappa_1(\mathrm{Aut}(X^*)) \rtimes A, \quad \textit{and}$$
$$\mathrm{N}(A) = \kappa_1(\mathrm{Aut}(X^*)) \rtimes \mathrm{rt}(\mathrm{N}_{\mathrm{Sym}(X)}(\Sigma)).$$

Proof. Given $x \in X$, we have $a^g|_x = \mathrm{id}$ if and only if $g|_x = g|_{x+1}$, hence we have $a^g|_x = a^j|_x$ for some $j \in \{1, \dots, p-1\}$ if and only if $g|_0 = g|_x$ for all $x \in X$. The image of a under conjugation with g now only depends on $g|^{\varnothing}$, hence we only need to observe $\mathrm{C}_{\mathrm{Sym}(X)}(\Sigma) = \Sigma$. ☐

Lemma 5.3.3. *Let G be a non-constant multi-*GGS*-group. Then* $\mathrm{N}(B) \leq \mathrm{stab}(0)$*, the point stabiliser of the vertex $0 \in X$, and*

$$\mathrm{N}(B)|_0 \leq \mathrm{N}(B) \qquad \textit{and} \qquad \mathrm{C}(B)|_0 \leq \mathrm{C}(B).$$

Proof. Let $g \in \mathrm{N}(B)$. Then there is some $\mathbf{n} \in \mathbb{F}_p^r \smallsetminus \{0\}$ such that $(b^{\mathbf{s}_1})^g = b^{\mathbf{n}}$. If $0^{g^{-1}} = x \neq 0$, we see that

$$b^{\mathbf{n}} = b^{\mathbf{n}}|_0 = (b^{\mathbf{s}_1})^g|_0 = (a^{\mathbf{b}_{i,x}})^{g|_0}.$$

But a rooted automorphism cannot be conjugate to a directed automorphism. Thus $g \in \mathrm{stab}(0)$. Similarly, we find

$$b^{\mathbf{n}} = (b^{\mathbf{s}_1})^g|_0 = (b^{\mathbf{s}_1}|_0)^{g|_0} = (b^{\mathbf{s}_1})^{g|_0}.$$

This shows (allowing $\mathbf{n} = \mathbf{s}_1$) both other statements. ☐

For the next lemma we introduce the (word) length function $\ell \colon G \to \mathbb{N}_0$, with respect to the generating set $A \cup B$, i.e. the mapping

$$\ell(g) = \min\{n \in \mathbb{N}_0 \mid g \text{ can be written as a product of length } n \text{ in } A \cup B\}.$$

It is well-known that this length function is *contracting*, i.e. that $\ell(g|_x) \leq g$ for $x \in X$. We need some finer analysis to establish a strict inequality in certain cases. Note that the strictness of the inequality above, for a more general class of self-similar groups G, is related to G being a periodic group.

Lemma 5.3.4. *Let G be a non-constant multi-*GGS*-group, and let $g \in G$ be an element with $\ell(g) > 1$. Then there is some $i \in X \smallsetminus \{0\}$ such that $\ell(g|_0 g|_i^{-1}) < \ell(g)$.*

Proof. Write $g = a^{i_0} b^{\mathbf{n}_0} \dots a^{i_{n-1}} b^{\mathbf{n}_{m-1}} a^{i_m}$, where $m \in \mathbb{N}_0$, $\mathbf{n}_k \in \mathbb{F}_p^\times$, $i_k \in \mathbb{Z} \smallsetminus \{0\}$ for $k \in \{0, \dots, m-1\}$, and $i_m \in \mathbb{Z}$. Passing to a conjugate if necessary, every $g \in G$ can be written in this way. Taking sections, we see that

$$g|_k = b^{\mathbf{n}_0}|_{k-i_0} (b^{\mathbf{n}_1})|_{k-i_0-i_1} \cdots (b^{\mathbf{n}_{m-1}})|_{k-\sum_{t=0}^{m-1} i_t}$$

for any $k \in X$, hence $\ell(g|_k) \leq m$. Since every B-letter contributes at most one B-letter to one of the sections, we have $\sum_{k=0}^{p-1} \ell(g|_k) \leq \ell(g) + p - 1$. Assume that $\ell(g|_0 g|_1^{-1}) \geq \ell(g)$. Then $\ell(g|_0) + \ell(g|_1) \geq \ell(g)$, hence $\ell(g|_0) = \ell(g|_1) = m$. This can only be the case if every B-letter contributes its only section that is contained in B either to $g|_0$ and $g|_1$, i.e. $g|_k \in A$ for all other $k \in X$. Thus, if $m > 2$, we have

$$\ell(g|_0 g|_k^{-1}) \leq \ell(g|_0) + \ell(g|_k) \leq m + 1 \leq 2m - 1 < \ell(g),$$

If $m = 2$, we see that $g = a^{i_0} b^{\mathbf{n}_0}$, implying $g|_0 \in A$. Since there is at least a second section contained in A, the result follows. □

Lemma 5.3.5. *Let G be a non-constant multi-*GGS*-group. Then*

$$\mathrm{N}_{\mathrm{C}(A)}(G) \cap (\mathrm{C}(B) \cdot G) \subseteq \kappa_\infty(A) \cdot G_{\mathrm{reg}}.$$

Proof. Let $g \in \mathrm{N}_{\mathrm{C}(A)}(G) \cap (\mathrm{C}(B) \cdot G)$ and $h \in G$ be an element of minimal length such that we may write $g = g'h$ for some $g' \in \mathrm{C}(B)$. The proof uses induction on the length of h.

First assume that h has length one, i.e. $h \in A \cup B$. If h is in B, we find that $g \in \mathrm{C}(B)$. Thus h centralises G, but it is well-known that the centraliser of a branch group in $\mathrm{Aut}(X^*)$ is trivial; hence $g = \mathrm{id}$. If h is a power of a, the same holds for gh^{-1}, hence $g \in A \leq G_{\mathrm{reg}}$.

Now we assume that $\ell(h) > 1$. By Lemma 5.3.2 we may write $g = \kappa_1(g|_0)a^k$ for some $k \in \mathbb{Z}$, yielding for any $\mathbf{n} \in \mathbb{F}_p^r$

$$\begin{aligned}((b^{\mathbf{n}})^{g|_0}, (a^{\mathbf{n}\cdot\mathbf{e}_1})^{g|_0}, \dots, (a^{\mathbf{n}\cdot\mathbf{e}_{p-1}})^{g|_0})^{a^k} &= \psi_1((b^{\mathbf{n}})^g) = \psi_1((b^{\mathbf{n}})^h) \\ &= ((b^{\mathbf{n}})^{h|_0}, (a^{\mathbf{n}\cdot\mathbf{e}_1})^{h|_1}, \dots, (a^{\mathbf{n}\cdot\mathbf{e}_{p-1}})^{h|_{p-1}})^{h|^\varnothing}.\end{aligned}$$

Since a and $b^{\mathbf{n}}$ are not conjugate in $\mathrm{Aut}(X^*)$, this shows that $g|^\varnothing = h|^\varnothing$ and $a^{g|_0} = a^{h|_i}$ for all $i \in X \smallsetminus \{0\}$. Thus $g|_0 h|_i^{-1}$ centralises A, and by Lemma 5.3.3 we find $(b^{\mathbf{n}})^{g|_0 h|_i^{-1}} = (b^{\mathbf{n}})^{h|_0 h|_i^{-1}} \in B^G$, hence $g|_0 h|_i^{-1}$ normalises G. By Lemma 5.3.4, there is some $i \in X \smallsetminus \{0\}$ such that $\ell(h|_0 h|_i^{-1}) < \ell(h)$, so by induction we see that $g|_0 h|_i^{-1} \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}$. Since $h|_i \in G$, we have $g|_0 \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}$, and

$$g = \kappa_1(g|_0)a^k = a^k \kappa_1(g|_0) \in A \cdot \kappa_1(\kappa_\infty(A) \cdot G_{\mathrm{reg}}) = \kappa_\infty(A) \cdot \kappa_1(G_{\mathrm{reg}}).$$

Now Lemma 5.3.1 yields $g \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}$. □

With a little care, we can use the same idea to extend the result to G_{reg}.

Lemma 5.3.6. *Let G be a non-constant multi-*GGS*-group. Then*

$$\mathrm{N}_{\mathrm{C}(a)}(G) \cap (\mathrm{C}(b) \cdot G_{\mathrm{reg}}) \leq \kappa_\infty(A) \cdot G_{\mathrm{reg}}.$$

Proof. In view of the previous lemma, we may restrict to symmetric G. Let $g \in \mathrm{N}_{\mathrm{C}(A)}(G) \cap (\mathrm{C}(B) \cdot G_{\mathrm{reg}})$ and choose $g' \in \mathrm{C}(B)$, $h \in G$ and $j \in \mathbb{Z}$, such that $g = g' \underline{c}^j h$. Write $g = \kappa_1(g|_0)a^k$ for some $k \in \mathbb{Z}$, and calculate,

$$\begin{aligned}(b^{g|_0}, (a^{\mathbf{e}_1})^{g|_0}, \dots, (a^{\mathbf{e}_{p-1}})^{g|_0})^{a^k} &= \psi_1(b^g) = \psi_1(b^{\underline{c}^j h}) \\ &= (b^{\underline{c}^j|_0 h|_0}, (a^{\mathbf{e}_1})^{h|_1}, \dots, (a^{\mathbf{e}_{p-1}})^{h|_{p-1}})^{h|^{\varnothing}}.\end{aligned}$$

As we did in the proof of Lemma 5.3.5, we may conclude that $h|^{\varnothing} = a^k$. Consequently, for all $i \in X \smallsetminus \{0\}$, the element $g|_0 h|_j^{-1}$ centralises a. By Lemma 5.3.3, the element $g|_0$ is in $\mathrm{C}(B) \cdot \underline{c}^j|_0 h|_0$. Since $\underline{c}^j|_0 = [b, a]^j \in G$, this implies that $g|_0 h|_i^{-1}$, for all $i \in X \smallsetminus \{0\}$, is an element in $\mathrm{N}_{\mathrm{C}(A)}(G) \cap (\mathrm{C}(B) \cdot G)$. By Lemma 5.3.5, the element $g|_0 h|_i^{-1}$, and consequently also $g|_0$ is contained in $\kappa_\infty(A) \cdot G_{\mathrm{reg}}$. Finally, by Lemma 5.3.1, we find $g = \kappa_1(g|_0) g|^{\varnothing} \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}$. □

Lemma 5.3.7. *Let G be a non-constant multi-*GGS*-group. Then*

$$\mathrm{N}_{\mathrm{lab}(\Sigma)}(A) \leq \mathrm{C}(A) \quad \textit{and} \quad \mathrm{N}_{\mathrm{lab}(\Sigma)}(B) \leq \mathrm{C}(B).$$

Proof. We use the description of $\mathrm{N}(A)$ given in Lemma 5.3.2. Let $g \in \mathrm{N}_{\mathrm{lab}(\Sigma)}(A)$. For all $h \in \mathrm{lab}(\Sigma)$, we have $h|^{\varnothing} \in \langle \sigma \rangle$. Thus we see that $a^{\mathrm{rt}\, \kappa_1(g|_0) g|^{\varnothing}} = a^{\mathrm{rt}\, g|^{\varnothing}} = a$.

Now let $g \in \mathrm{N}_{\mathrm{lab}(\Sigma)}(B)$, let $\mathbf{n} \in \mathbb{F}_p^r$ be arbitrary and let $\mathbf{m} \in \mathbb{F}_p^r$ be such that $(b^{\mathbf{n}})^g = b^{\mathbf{m}}$. Then

$$(b^{\mathbf{m}}, a^{\mathbf{m}\cdot\mathbf{e}_1}, \dots, a^{\mathbf{m}\cdot\mathbf{e}_{p-1}}) = b^{\mathbf{m}} = (b^{\mathbf{n}})^g = ((b^{\mathbf{n}})^{g|_0}, (a^{\mathbf{n}\cdot\mathbf{e}_1})^{g|_1}, \dots, (a^{\mathbf{n}\cdot\mathbf{e}_{p-1}})^{g|_{p-1}})^{g|^{\varnothing}}.$$

The label $g|^{\varnothing}$ is a power of $a|^{\varnothing}$. Since $b^{\mathbf{n}}$ and a are not conjugate in $\mathrm{Aut}(X^*)$, the element $g|^{\varnothing}$ must stabilise the vertex 0, thus it is trivial. Varying $\mathbf{n}$, we see that $g|_i$ normalises A for all $i \in X \smallsetminus \{0\}$ for which $\mathbf{e}_i \neq \mathbf{0}$. Now since $\mathrm{lab}(\Sigma)$ is self-similar, this implies $g|_i \in \mathrm{C}(A)$ by the first part of this lemma, hence $a^{\mathbf{n}\cdot\mathbf{e}_i} = a^{\mathbf{m}\cdot\mathbf{e}_i}$ for all $i \in X \smallsetminus \{0\}$. Thus $b^{\mathbf{m}} = b^{\mathbf{n}}$. Since $g|_0 \in \mathrm{N}_{\mathrm{lab}(\Sigma)}(B)$ by Lemma 5.3.3, we can argue in the same way for $g|_0$, hence $g \in \mathrm{C}(B)$. □

Lemma 5.3.8. *Let G be a non-constant multi-*GGS*-group. Then*

$$\mathrm{N}_{\overline{G_{\mathrm{reg}}}}(G) \subseteq \kappa_\infty(A) \cdot G_{\mathrm{reg}}.$$

Proof. Let $g \in \mathrm{N}_{\overline{G_{\mathrm{reg}}}}(G)$. There is a sequence $(g_i)_{i \in \mathbb{N}_0}$ with $g_i \in G_{\mathrm{reg}}$ such that

$$g = \prod_{i=0}^{\infty} \kappa_i(g_i).$$

Write h_n for the partial product $\prod_{i=0}^{n} \kappa_i(g_i)$. By Lemma 5.3.1, we find $h_n \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}$

for all $n \in \mathbb{N}_0$. We may write

$$g|_{0^n} = h_n|_{0^n} \left(\prod_{i=n+1}^{\infty} \kappa_i(g_i) \right) |_{0^n.h_n} = h_n|_{0^n} \prod_{i=1}^{\infty} \kappa_i(g_{i+n}).$$

In view of Lemma 5.3.2, we conclude that $h_n|_{0^n}^{-1} g|_{0^n} \in \mathrm{C}(A)$. There is nothing special about 0^n; indeed, we see that $h_n|_v^{-1} g|_v = h_n|_{0^n}^{-1} g|_{0^n}$ for all $v \in X^n$. By [120, Lemma 3.4], there exists an integer $n \in \mathbb{N}_0$ such that $g|_{0^n} \in \mathrm{N}(B)$. By Lemma 5.3.7 $g|_{0^n} \in \mathrm{C}(B)$. Consequently

$$B^{g|_{0^n}^{-1} h_n|_{0^n}} = B^{h_n|_{0^n}}.$$

Since $h_n|_v \in G_{\mathrm{reg}}$ for all $v \in X^n$, we may use Lemma 5.3.6 and obtain $g|_{0^n}^{-1} h_n|_{0^n} \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}$. Using Lemma 5.3.1 again, we find $\kappa_n(h_n|_{0^n}^{-1} g|_{0^n}) \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}$, and moreover

$$g = h_n \psi_n^{-1}(h_n|_{0^n}^{-1} g|_{0^n}, \dots, h_n|_{(p-1)^n}^{-1} g|_{(p-1)^n}) = h_n \kappa_n(h_n|_{0^n}^{-1} g|_{0^n}) \in \kappa_\infty(A) \cdot G_{\mathrm{reg}}. \qquad \square$$

Lemma 5.3.9. *Let G be a non-constant multi-GGS-group. Write G_{lay} for the product set $\kappa_\infty(A) \cdot G_{\mathrm{reg}}$.*

(i) *If G is regular, then we have $\kappa_\infty(A) \le \mathrm{N}_{\mathrm{Aut}(X^*)}(G)$, hence G_{lay} acquires the structure of a semidirect product.*
(ii) *If G is symmetric, we find $\mathrm{N}_{\kappa_\infty(A)}(G) = A$.*
(iii) *Then*

$$\mathrm{N}_{G_{\mathrm{lay}}}(G) = \begin{cases} G_{\mathrm{lay}} & \text{if } G \text{ is regular,} \\ G_{\mathrm{reg}} & \text{if } G \text{ is symmetric.} \end{cases}$$

Proof. Let $n \in \mathbb{N}_0$. Clearly $a^{\kappa_n(a)} = a$, and for all $j \in \{1, \dots, r\}$

$$\begin{aligned} [b^{\mathbf{s}_j}, \kappa_n(a)] &= \psi_1^{-1}([b^{\mathbf{s}_j}, \kappa_{n-1}(a)], [a^{\mathbf{b}_{j,1}}, \kappa_{n-1}(a)], \dots, [a^{\mathbf{b}_{j,p-1}}, \kappa_{n-1}(a)]) \\ &= \psi_n^{-1}([b^{\mathbf{s}_j}, a], \mathrm{id}, \dots, \mathrm{id}) \in \psi_n^{-1}(G' \times \dots \times G') \le G. \end{aligned}$$

This shows (i), and it also shows that $\kappa_n(a)$ does not normalise a symmetric GGS-group G for $n > 0$, since $\psi_1^{-1}([b, a], \mathrm{id}, \dots, \mathrm{id}) \notin G$. Thus (ii) is proven.

Statement (iii) is a consequence of (i) in case G is regular, and an immediate consequence of Lemma 5.2.4 (iii) in case G is symmetric. $\square$

Proposition 5.3.10. *Let G be a non-constant multi-GGS-group. Then*

$$\mathrm{N}_{\overline{G_{\mathrm{reg}}}}(G) = \begin{cases} G \rtimes \kappa_\infty(A), & \text{if } G \text{ is regular,} \\ G_{\mathrm{reg}}, & \text{if } G \text{ is symmetric.} \end{cases}$$

Proof. Assume that G is regular. By Lemma 5.3.9, the set $\kappa_\infty(A) \cdot G_{\mathrm{reg}}$ is a group. In view of Lemma 5.3.8 and $G = G_{\mathrm{reg}}$, this proves the first case. If G is symmetric, the result follows from Lemma 5.3.8 and Lemma 5.3.9. □

5.4 — The normaliser as a product

We now prove that the normaliser of G in $\mathrm{Aut}(X^*)$ decomposes as a semi-direct product. To begin with, we prove the following generalisation of [138, 2.2.5(i)], which is an interesting proposition in its own right.

Proposition 5.4.1. *Let G be a non-constant multi-*GGS*-group. Every element of G that has order p is either contained in $\mathrm{St}_G(1)$ or is conjugate to a power of a in G_{reg}.*

Proof. We have to prove that, given $g \in \mathrm{St}_G(1)$ and $i \in \mathbb{Z}$, every element $a^i g$ of order p may be written $(a^h)^i$ for some $h \in G_{\mathrm{reg}}$. Passing to an appropriate power of a, we may assume that $i = 1$. From $(ag)^p = 1$ we derive the equations

$$\mathrm{id} = (ag)^p|_0 = g|_0 \cdots g|_{p-1}, \text{ resp.}$$
$$g|_{p-1} = g|_{p-2}^{-1} \cdots g|_0^{-1}.$$

Since $g \in \mathrm{St}_G(1)$, by Lemma 5.2.7 there exists a set of B-coordinates $\mathbf{n}_k \in \mathbb{F}_p^r$ and a set of L-coordinates $y_k \in G'$ uniquely describing g. Reformulated in these B-coordinates, the condition above reads

$$\sum_{i=0}^{p-2} \mathbf{n}_i = -\mathbf{n}_{p-1}.$$

Given some integer $s \in \mathbb{Z}$, we define an element

$$h_s = \psi_1^{-1}(a^s, a^s g|_0, a^s g|_0 g|_1, \dots, a^s g|_0 \cdots g|_{p-2}) \in \psi_1^{-1}(G \times \overset{p}{\dots} \times G).$$

Since

$$\begin{aligned} a^{h_s}|_k = h_s|_k^{-1} h_s|_{k+1} &= (a^s \prod_{i=0}^{k-1} g|_i)^{-1} a^s \prod_{i=0}^{k} g|_i \\ &= \begin{cases} g|_k, & \text{if } k \neq p-1, \\ (\prod_{i=0}^{p-2} g|_i)^{-1} = g|_{p-1}, & \text{if } k = p-1, \end{cases} \end{aligned}$$

the conjugate a^{h_s} is equal to ag. It remains to prove that $h_s \in G_{\mathrm{reg}}$ for some $s \in \mathbb{Z}$. If it is contained in G_{reg}, the element h_s has the B-coordinates $\mathbf{h}_k = \sum_{i=0}^{k-1} \mathbf{n}_i$ (and some commutators z_k that we shall not need to specify). We have to prove that the corresponding forced A-coordinates $\tilde{s}_k = \sum_{i=0}^{p-1} \mathbf{h}_i \cdot \mathbf{e}_{k-i}$ are equal to the actual a-exponents of the corresponding sections of h. Since it is enough to show that $h_s \in G_{\mathrm{reg}}$ for one s, we fix $s = \tilde{s}_0$, so that the proposed equality holds in the first component by definition.

A quick calculation shows that, for all $k \in X \smallsetminus \{0\}$,

$$\begin{aligned}\tilde{s}_k - \tilde{s}_{k-1} &= \sum_{i=0}^{p-1} \mathbf{h}_i \cdot \mathbf{e}_{k-i} - \sum_{i=0}^{p-1} \mathbf{h}_i \cdot \mathbf{e}_{k-1-i} \\ &= \sum_{i=0}^{p-1} (\mathbf{h}_i - \mathbf{h}_{i-1}) \cdot \mathbf{e}_{k-i} = \sum_{i=0}^{p-1} \mathbf{n}_{i-1} \cdot \mathbf{e}_{k-i} = s_{k-1},\end{aligned}$$

and consequently the a-exponent of $h|_k$ is equal to

$$s + \sum_{i=0}^{k-1} s_i = \tilde{s}_0 + \sum_{i=0}^{k-1} s_i = \tilde{s}_k,$$

for all $k \in X$. But the values s_i are the forced A-coordinates of g, hence, comparing with the definition of h_s, we see that the forced A-coordinates of $\mathbf{h}_k$, for $k \in X$, and the actual a-exponents of $h|_k$ coincide. Hence $h \in G_{\mathrm{reg}}$. □

Notice that for a symmetric GGS-group, we do have to pass to G_{reg} to make this statement true: take the element $d = ([b,a],[b,a]^{-1}, \mathrm{id}, \dots, \mathrm{id}) \in G$. Clearly $a^{[b,a]} = d$, but assume for contradiction that there is another element $h \in \mathrm{St}_G(1)$ such that $a^{h^{-1}} = d$. Then $\underline{c}h$ centralises a, hence

$$[b,a]h|_0 = (\underline{c}h)|_0 = (\underline{c}h)|_i = h|_i,$$

for all $i \in \mathbb{F}_p^{\times}$. Counting the powers of $[b,a]$ in $h|_k$ as in Theorem 5.2.2 (iv), we see that if $h|_1 \equiv_{\gamma_3(G)} a^s(b)^n([b,a])^v$ the sum of the $[b,a]$-exponents over all sections mod $\gamma_3(G)$ equals $v - 1 + (p-1)v \equiv_p p-1$, contradicting Theorem 5.2.2 (iv). Thus there is no such $h \in \mathrm{St}_G(1)$.

Recall that the group Δ is $\mathrm{N}_{\mathrm{Sym}(X)}(\Sigma) \cap \mathrm{stab}_{\mathrm{Sym}(X)}(0)$. Set $D = \mathrm{rt}(\Delta)$, i.e. the group of rooted automorphisms normalising but not centralising a.

Lemma 5.4.2. *Let G be a non-constant multi-GGS-group. Then*

$$\mathrm{N}(G) \subseteq \overline{G_{\mathrm{reg}}} \cdot \overline{D}.$$

Proof. Let $g_0 \in \mathrm{N}(G)$. Let $k \in \mathbb{Z}$ be such that $(a^{g_0})^k|^{\varnothing} = a$. By Proposition 5.4.1 there exists an element $h_0 \in G_{\mathrm{reg}}$ such that $(a^{g_0})^k = a^{h_0}$. Consequently $h_0^{-1}g_0 \in \mathrm{N}(A)$. Using Lemma 5.3.2 and the fact that $\mathrm{N}(G)$ is self-similar, cf. [120, Lemma 3.3], we may write

$$h_0^{-1}g_0 = \kappa_1((h_0^{-1}g_0)|_0)\,\mathrm{rt}((h_0^{-1}g_0)|^{\varnothing})$$

for $h_0^{-1}g_0|_0 = g_1 \in \mathrm{N}(G)$. Since $g_0h_0^{-1}|^{\varnothing}$ normalises σ, we may write $g_0h_0^{-1}|^{\varnothing} = a^{k_0}d_0$ for some $d_0 \in D$ and $k_0 \in \mathbb{Z}$, so that we obtain the equation

$$g_0 = h_0 \kappa_1(g_1) a^{k_0} d_0 = h_0 a^{k_0} \kappa_1(g_1) d_0,$$

using the fact that $\kappa_1(\mathrm{Aut}(X^*))$ normalises a for the second equality. Repeating the procedure for g_1, we obtain $g_2 \in \mathrm{N}(G), h_1 \in G_{\mathrm{reg}}, d_1 \in D$ and $k_1 \in \mathbb{Z}$ such that

$$\begin{aligned} g_0 &= h_0 a^{k_0} \kappa_1(h_1 a^{k_1} \kappa_1(g_2) d_1) d_0 \\ &= h_0 a^{k_0} \kappa_1(h_1 a^{k_1}) \kappa_2(g_2) \kappa_1(d_1) d_0 \\ &= h_0 a^{k_0} \kappa_1(h_1 a^{k_1}) \kappa_2(g_2) d_0 \kappa_1(d_1). \end{aligned}$$

In the last step we have used the fact that $\overline{D}$ is abelian. Going on, we obtain a sequence of products

$$t_n = \prod_{i=0}^{n} \kappa_i(h_i a^{k_i}) \prod_{i=0}^{n} \kappa_{n-i}(d_i) = \prod_{i=0}^{n} \kappa_i(h_i a^{k_i}) \prod_{i=0}^{n} \kappa_i(d_i),$$

such that $t_n \equiv_{\mathrm{St}(n+1)} g_0$, i.e. that are converging to g_0 in the topology induced by the layer stabilisers. Since both $\overline{D}$ and $\overline{G_{\mathrm{reg}}}$ are closed sets, the corresponding limits are well-defined. We obtain

$$g_0 = \prod_{i=0}^{\infty} \kappa_i(h_i a^{k_i}) \prod_{i=0}^{\infty} \kappa_i(d_i).$$

This shows $g_0 \in \overline{G_{\mathrm{reg}}} \cdot \overline{D}$. □

Lemma 5.4.3. *Let G be a non-constant multi-*GGS*-group and let $g \in G$ be an element directed along $\overline{0}$. Then $g \in B$.*

Proof. Consider that, since $G \leq \mathrm{lab}(\Sigma)$, there are $(x_1, \dots, x_{p-1}) \in \mathbb{F}_p^{p-1}$ such that

$$\psi_1(g) = (g|_0, a^{x_1}, \dots, a^{x_{p-1}}).$$

Since directed elements stabilise the first layer, there exist B-coordinates $\mathbf{n}_0, \dots, \mathbf{n}_{p-1}$ and $y_0, \dots, y_{p-1} \in G'$ for g. The equation above shows that $\mathbf{n}_1 = \dots = \mathbf{n}_{p-1} = \mathbf{0}$ and $y_1 = \dots = y_{p-1} = \mathrm{id}$. Thus the forced A-coordinate at 0 fulfils

$$s_0 = \sum_{i=0}^{p-1} \mathbf{n}_i \cdot \mathbf{c}_{p-i} = 0,$$

hence $g|_0 = a^{s_0} b^{\mathbf{n}_0} = b^{\mathbf{n}_0} \in B$, and in consequence $g = b^{\mathbf{n}_0} \psi_1^{-1}(y_0, \mathrm{id}, \dots, \mathrm{id})$. Since the set of elements directed along $\overline{0}$ forms a subgroup, the element $\psi_1^{-1}(y_0, \mathrm{id}, \dots, \mathrm{id})$, and consequently also y_0 is directed along $\overline{0}$. We can argue as above for y_0, but since $y_0 \in G'$, by Theorem 5.2.2 (ii), the sum $\sum_{i=0}^{p-1} \mathbf{n}_i = \mathbf{0}$. Thus $\mathbf{n}_0 = \mathbf{0}$, and, chasing down the spine, we find $y_0 = \mathrm{id}$. Thus $g \in B$. □

Lemma 5.4.4. *Let G be a non-constant multi-*GGS*-group, and let $h \in \overline{G_{\text{reg}}}$. Then*

$$a^h \equiv_{G'} a.$$

Proof. Let $(g_i)_{i\in\mathbb{N}_0}$ be a sequence of elements $g_i \in G_{\text{reg}}$ for $i \in \mathbb{N}_0$ such that

$$h = \prod_{i=0}^{\infty} \kappa_i(g_i) = g_0 \prod_{i=1}^{\infty} \kappa_i(g_i) = g_0\kappa_1 \left(\prod_{i=0}^{\infty} \kappa_i(g_{i+1}) \right).$$

By Lemma 5.3.2, the element $\kappa_1\left(\prod_{i=0}^{\infty} \kappa_i(g_{i+1})\right)$ centralises a. Thus it is sufficient to consider $h = g_0$. The statement now follows from Lemma 5.2.4 (iii). □

Lemma 5.4.5. *Let G be a non-constant multi-*GGS*-group. Then*

$$\mathrm{N}(G) = \mathrm{N}_{\overline{G_{\text{reg}}}}(G) \rtimes \mathrm{N}_{\overline{D}}(G)$$

Proof. Assume that $\mathrm{N}(G)$ is equal to the product set $\mathrm{N}_{\overline{G_{\text{reg}}}}(G) \cdot \mathrm{N}_{\overline{D}}(G)$. By Proposition 5.3.10, $\mathrm{N}_{\overline{G_{\text{reg}}}}(G)$ is equal to $G \rtimes \kappa_\infty(A)$ or to G_{reg}. Both groups are normalised by $\mathrm{N}_{\overline{D}}(G)$, the first one since $\overline{D}$ normalises $\kappa_\infty(A)$, and the second one since for every $d_0\kappa_1(d_1)$ with $d_0 \in D$ and $d_1 \in \overline{D}$,

$$\underline{c}^{d_0\kappa_1(d_1)} = \psi_1^{-1}([b,a]^{d_1}, \mathrm{id}, \dots, \mathrm{id})^{d_0} \in \psi_1^{-1}(G' \times \overset{p}{\dots} \times G') \leq G_{\text{reg}}.$$

Thus the product set is in fact a semidirect product. It remains to show the equality $\mathrm{N}(G) = \mathrm{N}_{\overline{G_{\text{reg}}}}(G) \cdot \mathrm{N}_{\overline{D}}(G)$.

By Lemma 5.4.2, we may write $g \in \mathrm{N}(G)$ as a product $g = h' \cdot d$ with $h' \in \overline{G_{\text{reg}}}$ and $d \in \overline{D}$. Clearly $\overline{D}$ normalises A. Thus it is enough to prove: For all $\mathbf{n} \in \mathbb{F}_p^r$ such that $(b^{\mathbf{n}})^d \notin G$, then $h'd \notin \mathrm{N}(G)$ for all $h' \in \overline{G_{\text{reg}}}$. We to prove that $(h'd)^{-1} \notin \mathrm{N}(G)$. Since $\overline{D}$ is a group, we may replace d by its inverse. Write h for h'^{-1}. Notice that

$$h = \kappa_1(h_1)h_0$$

for some $h_1 \in \overline{G_{\text{reg}}}$ and $h_0 \in G_{\text{reg}}$. Since G_{reg} normalises G, we may assume that $h_0 = \mathrm{id}$, and thus $h|^{\varnothing} = \mathrm{id}$. Let $d = \prod_{i=0}^{\infty} \kappa_i(d_i)$ for a sequence $(d_i)_{i\in\mathbb{N}_0}$ of elements $d_i \in D$ such that, for all $i \in \mathbb{N}_0$, we have $a^{d_i} = a^{j_i}$ for some $j_i \in \mathbb{Z}$. Then, for all $\mathbf{n} \in \mathbb{F}_p^r$,

$$\begin{aligned}\psi_1((b^{\mathbf{n}})^d) &= ((b^{\mathbf{n}})^{d|_0}, (a^{\mathbf{n}\cdot\mathbf{e}_1})^{d|_0}, \dots, (a^{\mathbf{n}\cdot\mathbf{e}_{p-1}})^{d|_0})^{d_0} \\ &= ((b^{\mathbf{n}})^{d|_0}, a^{j_1\cdot\mathbf{n}\cdot\mathbf{e}_{1.d_0}}, \dots, a^{j_1\cdot\mathbf{n}\cdot\mathbf{e}_{(p-1).d_0}}).\end{aligned}$$

Write $x_{1,i} = j_1 \cdot \mathbf{n} \cdot \mathbf{e}_{i.d_0}$ for all $i \in \{1, \dots, p-1\}$. Since $\overline{D}$ is self-similar, we see that $(b^{\mathbf{n}})^d$ is directed along $\overline{0}$, and we write $\mathbf{x}_k = (x_{k,1}, \dots x_{k,p-1})$ for the A-exponents of the sections at $i \in \{1, \dots, p-1\}$ of $(b^{\mathbf{n}})^d|_{0^{k-1}}$.

Now, using Lemma 5.4.3, we see that if $(b^{\mathbf{n}})^d \in G$, then actually $(b^{\mathbf{n}})^d \in B$. Assume that the $\overline{0}$-directed element $(b^{\mathbf{n}})^d$ is not a member of B. Then there are two possibilities:

(i) there exists some $k \in \mathbb{N}_0$ such that $\mathbf{x}_k$ is not contained in the row space of E, or,

(ii) if for all $k \in \mathbb{N}_0$ the vector $\mathbf{x}$ is contained in the row space of E, but there exists some $k \in \mathbb{N}_0$ such that $\mathbf{x}_k \neq \mathbf{x}_{k+1}$.

In both cases, we may assume that $k = 0$, since $\overline{G_{\text{reg}}}$ and $\overline{D}$ are self-similar.

Given $(b^{\mathbf{n}})^d$, we compute the conjugate by dh,

$$\begin{aligned}\psi_1((b^{\mathbf{n}})^{dh}) &= (b^{\mathbf{n}})^d|_0, a^{x_1}, \dots, a^{x_{p-1}})^h \\ &= (((b^{\mathbf{n}})^d|_0)^{h|_0}, (a^{x_1})^{h|_1}, \dots, (a^{x_{p-1}})^{h|_{p-1}}).\end{aligned}$$

Since $\overline{G_{\text{reg}}}^{-1}$ is self-similar, we may apply Lemma 5.4.4, and we find

$$\psi_1((b^{\mathbf{n}})^{dh}) \equiv_{G' \times \dots \times G'} (((b^{\mathbf{n}})^d|_0)^{h|_0} \bmod G', a^{x_1}, \dots, a^{x_{p-1}}). \tag{$*$}$$

Assume that we are in case (i), i.e. that $\mathbf{x}_0$ is not contained in the row space of E. Then, by $(*)$ and Lemma 5.2.7, also $(b^{\mathbf{n}})^{dh} \notin G$.

Assume that we are in case (ii), i.e. that $\mathbf{x}_0 \neq \mathbf{x}_1$, but both represent the forced a-exponents of an element $b^{\mathbf{m}_0}$ and $b^{\mathbf{m}_1}$, respectively. Thus by $(*)$ and Lemma 5.2.7,

$$\begin{aligned}(b^{\mathbf{n}})^{dh} &\equiv_{\psi_1^{-1}(G' \times \dots \times G')} (b^{\mathbf{m}_0}) \quad \text{and} \\ (b^{\mathbf{n}})^{dh}|_0 &\equiv_{\psi_1^{-1}(G' \times \dots \times G')} (b^{\mathbf{m}_1})^{h|^0}.\end{aligned}$$

Thus

$$(b^{\mathbf{m}_1})^{h|^0} \equiv (b^{\mathbf{n}})^{dh}|_0 \equiv b^{\mathbf{m}_0}|_0 \equiv b^{\mathbf{m}_0} \bmod \psi_1^{-1}(G' \times \dots \times G').$$

Since $\psi_1^{-1}(G' \times \dots \times G') \cap G = \mathrm{St}_G(1)'$, this implies $(b^{\mathbf{m}_1})^{a^k} \equiv_{\mathrm{St}_G(1)'} b^{\mathbf{m}_0}$ for some $k \in \mathbb{Z}$, hence $b^{\mathbf{m}_1} = b^{\mathbf{m}_0}$ and $\mathbf{m}_0 = \mathbf{m}_1$. But then $\mathbf{x}_0 = \mathbf{x}_1$, a contradiction. □

5·5 — Elements normalising G with labels in Δ

Recall that the permutation group $\Delta = \langle \delta \rangle$ is isomorphic to $\mathbb{F}_p^\times$. The rooted automorphism $d = \mathrm{rt}\,\delta$ acts in two different ways on $G = \mathrm{St}_G(1) \rtimes A$. It raises a to a power, i.e. it acts my multiplication on the exponent of a; and it acts on an element of $g \in \mathrm{St}_G(1)$ by permuting the tuple $\psi_1(g)$, i.e. by multiplication of the indices of said tuple. Note that the vertex 0 is fixed by δ.

Recall that B is isomorphic to $\mathbf{E} \leq \mathbb{F}_p^{p-1}$. We now show that B is normalised by every normaliser of G in $\overline{D}$.

Lemma 5.5.1. *Let G be a non-constant multi-*GGS*-group. Then* $\mathrm{N}_{\overline{D}}(G) = \mathrm{N}_{\overline{D}}(B)$.

Proof. Since $\overline{D} \leq \mathrm{N}(A)$, the inclusion $\mathrm{N}_{\overline{D}}(G) \geq \mathrm{N}_{\overline{D}}(B)$ is obvious. We now prove the other inclusion. Let $g \in \mathrm{N}_{\overline{D}}(G)$. By [120, Lemma 3.4], there exists an integer $k \in \mathbb{N}_0$

such that $g|_{0^k}$ normalises B. Thus it is enough to prove that if $g|_0$ normalises B, also g normalises B.

Assume $g|_0 \in \mathrm{N}(B)$, and let $\mathbf{m}$ and $\mathbf{n} \in \mathbb{F}_p^r$ be such that $(b^{\mathbf{n}})^{g|_0} = b^{\mathbf{m}}$. We may write $g = \kappa_1(g|_0)g|^{\varnothing}$, where $g|^{\varnothing}$ normalises $a|^{\varnothing}$. Hence there exists $j \in \mathbb{Z}$ such that $a^{g|_0} = a^j$. We calculate

$$\begin{aligned}(b^{\mathbf{n}})^{-1}(b^{\mathbf{m}})^g &= (b^{\mathbf{n}})^{-1}\psi_1^{-1}((b^{\mathbf{m}})^{g|_0}, a^{j\cdot\mathbf{m}\cdot\mathbf{e}_1}, \dots, a^{j\cdot\mathbf{m}\cdot\mathbf{e}_{p_1}})^{g|^{\varnothing}} \\ &= (\mathrm{id}, a^{-\mathbf{n}\cdot\mathbf{e}_1 + j\cdot\mathbf{m}\cdot\mathbf{e}_{1.g|^{\varnothing}}}, \dots, a^{-\mathbf{n}\cdot\mathbf{e}_{p-1}+j\cdot\mathbf{m}\cdot\mathbf{e}_{(p-1).g|^{\varnothing}}}).\end{aligned}$$

We see that the commutator coordinates of $(b^{\mathbf{n}})^{-1}(b^{\mathbf{m}})^g$ are trivial, the B-coordinates are all zero, and hence the forced A-coordinates are also $s_k = \sum_{i=0}^{p-1}\mathbf{n}_i \cdot \mathbf{c}_{k-i} = 0$. Thus $b^{\mathbf{n}} = b^{\mathbf{m}}$. □

Thus, we may restrict our attention to the group B. It is fruitful to consider B as a subgroup of the directed subgroup $b^{\mathbb{F}_p^{p-1}}$ of the multi-GGS-group associated to the full space $\mathbb{F}_p^{p-1}$ (with standard basis), which is, by the previous lemma, also invariant under $\mathrm{N}_{\overline{D}}(G)$. Write $\mu\colon D \to \mathbb{F}_p^{\times}$ for the isomorphism induced by $\alpha^{\delta} = \alpha^{\delta^{\mu}}$, where the second operation is taking the power, and define a map $P_{\bullet}\colon D \to \mathrm{GL}_p(p-1)$ such that P_d is the permutation matrix corresponding to the permutation $d|^{\varnothing} = \delta$. Let $g = \prod_{i=0}^{\infty}\kappa_i(d_i) \in \overline{D}$ for a sequence $(d_i)_{i\in\mathbb{N}_0}$ of elements in D. Then, for all $j \in \{1, \dots, p-1\} = \mathbb{F}_p^{\times}$,

$$\begin{aligned}\psi_1((b^{\mathbf{s}_j})^g) &= ((b^{\mathbf{s}_j})^{g|_0}, \mathrm{id}, \dots, \mathrm{id}, a^{d_1}, \mathrm{id}, \dots, \mathrm{id})^{d_0} \\ &= \psi_1((b^{\mathbf{s}_j})^{g|_0}, \mathrm{id}, \dots, \mathrm{id}, (d_1)^{\mu}a, \mathrm{id}, \dots, \mathrm{id}),\end{aligned}$$

where the non-trivial entries (in the second line) are the positions 0 and $(d_1)^{\mu}j$. If g normalises G, it must normalise B, and the conjugate $(b^{\mathbf{s}_j})^g$ is determined by the exponents of the sections at the positions in $\{1, \dots, p-1\}$. Thus

$$(b^{\mathbf{s}_j})^g = b^{(d_1)^{\mu}\mathbf{s}_{(d_0)^{\mu}j}}.$$

In other words, the action of $g \in \mathrm{N}_{\overline{D}}(G)$ induces, via the isomorphism $b^{\bullet}$, the linear map

$$(d_1)^{\mu}P_{d_0} \qquad (\dagger)$$

on $\mathbb{F}_p^{p-1}$. Returning to the directed group B, we see that every $g \in \mathrm{N}_{\overline{D}}(G)$ must be such that P_{d_0} leaves $\mathbf{E}$ invariant. Hence we define

$$U := \{u \in D \mid \mathbf{E}P_u = \mathbf{E}\} = \mathrm{stab}_D(\mathbf{E}).$$

Furthermore, we define the subgroup

$$\begin{aligned} V :=& \{v \in U \mid \text{ for all } \mathbf{e} \in \mathbf{E} \text{ there exists } \lambda \in \mathbb{F}_p^\times \text{ such that } \mathbf{e}P_v = \lambda\mathbf{e}\} \\ =& \{v \in U \mid \mathbf{E} \subseteq \mathrm{Eig}_\lambda(P_v) \text{ for some } \lambda \in \mathbb{F}_p^\times\}. \end{aligned}$$

Since $V \leq D$ is cyclic, the element $\lambda \in \mathbb{F}_p^\times$ generating a maximal subgroup is uniquely determined. Finally, define the subgroup

$$W := \langle \lambda \rangle.$$

Proposition 5.5.2. *Let G be a non-constant multi-*GGS*-group. Let U and W be defined as above. Then*

$$\mathrm{N}_{\overline{D}}(G) \cong U \times W.$$

Proof. By (†), the action of a given element $g = \prod_{i=0}^\infty \kappa_i(d_i)$ is determined by d_0 and d_1. Furthermore, since $\mathbf{E}$ must be invariant under P_{d_0}, we see that necessarily $d_0 \in U$. Since $\mathrm{N}(G)$, by [120, Lemma 3.3], and $\overline{D}$ are self-similar, we find, for all $k \in \mathbb{N}_0$,

$$g|_{0^k} = \prod_{i=0}^\infty \kappa_i(d_{i+k}) \in \mathrm{N}_{\overline{D}}(G).$$

Since, for all $\mathbf{n} \in \mathbb{F}_p^r$ and $g \in \mathrm{N}_{\overline{D}}(G)$,

$$(b^{\mathbf{n}})^g = (b^{\mathbf{n}})^g|_0 = (b^{\mathbf{n}}|_0)^{g|_0} = (b^{\mathbf{n}})^{g|_0},$$

we see that the action induced on $\mathbb{F}_p^{p-1}$ by all elements $g|_{0^k}$, for $k \in \mathbb{N}_0$, are equal, i.e. that the following equalities of matrices hold,

$$(d_{k+1})^\mu P_{d_k} = (d_1)^\mu P_{d_0}, \qquad \text{hence} \qquad \mathrm{I}_{p-1} = (d_{k+1}d_{k+2}^{-1})^\mu P_{d_k d_{k+1}^{-1}},$$

where I_{p-1} is the identity matrix. Recall that, for all $k \in \mathbb{N}_0$, the matrix $P_{d_k d_{k+1}^{-1}}$ acts either does not act as a scalar on $\mathbf{E}$, hence there is no d_{k+1} fulfilling the equation above, or it acts as some scalar λ^i, for some $i \in \mathbb{Z}$. Thus, every difference $d_k d_{k+1}^{-1}$ must be an element of W, otherwise, g cannot be normalising G.

On the other hand, for $d_0 \in U$ and $d_1 \in d_0^{-1}W$ there is a unique sequence $(d_i)_{i\in\mathbb{N}_0}$ that defines an element of $\mathrm{N}_{\overline{D}}(G)$, since

$$\begin{aligned} d_{k+2} &= d_{k+1}(d_k d_{k+1}^{-1})^r \\ &= d_{k+1}(d_k((d_{k-1}d_k^{-1})^r)^{-1}d_k^{-1})^r \\ &= d_{k+1}((d_{k-1}d_k^{-1})^{-1})^{r^2} \\ &= d_{k+1}((d_0 d_1^{-1})^{(-1)^k})^{r^{k+1}}. \end{aligned}$$

Thus $\mathrm{N}_{\overline{D}}(G) \cong U \times W$. □

In particular, if $r = 1$, every linear map leaving $\mathbf{E}$ invariant is a scalar multiplication, i.e. the subgroups U and V coincide. Clearly, if W is the trivial group, the only elements of $\mathrm{N}_M(G)$ are defined by the constant sequences. More generally, the sequence $(d_i)_{i\in\mathbb{N}_0}$ defining the normalising element with given d_0 and d_1 is periodic with periodicity prescribed by the order of λ.

Now all ingredients are ready for the proof of our main theorem.

Proof of Theorem 5.1.1. By [71, Theorem 1] and [96, Proposition 3.7], the automorphism group of G coincides with the normaliser of G in $\mathrm{Aut}(X^*)$. By Lemma 5.4.5, this normaliser is the semidirect product

$$\mathrm{N}_{\overline{G_{\mathrm{reg}}}}(G) \rtimes \mathrm{N}_{\overline{D}}(G).$$

These two groups were computed in Proposition 5.3.10 and Proposition 5.5.2. □

5.6 — Examples

To illustrate the definitions of U, V and W we compute some explicit examples.

Example 5.6.1. Let G be the GGS-group acting on the 5-adic tree with $\mathbf{E}$ generated by $(1, 2, 2, 1)$. Clearly G is symmetric. For every symmetric GGS-group, the space $\mathbf{E}$ is by definition invariant under the permutation induced by $-1 \in \mathbb{F}_p^\times$. In fact, it always acts trivially, hence $-1 \in W$. In our case, this is the only non-trivial permutation leaving $\mathbf{E}$ invariant, since

$$(1, 2, 2, 1)P_{x\mapsto 2x} = (2, 1, 1, 2) = (1, 2, 2, 1)P_{x\mapsto 3x},$$

while $(2, 1, 1, 2) \notin \mathbf{E}$. Thus

$$\mathrm{Aut}(G) = (G \rtimes \langle \underline{c} \rangle) \rtimes \langle \textstyle\prod_{i=0}^{\infty} \kappa_i(x \mapsto -x) \rangle \cong (G \rtimes \mathrm{C}_5) \rtimes \mathrm{C}_2.$$

The group G and the group defined by $(1, 4, 4, 1)$ are the multi-GGS-groups with the smallest possible outer automorphism group.

Example 5.6.2. Let G be the (regular) GGS-group acting on the p-adic tree with E generated by $\mathbf{b} = (1, 2, \ldots, p-1)$. Let $(\lambda_1, \ldots, \lambda_{p-1})$ be the image of $\mathbf{b}$ under P_d, for $d \in D$. Since

$$\lambda_i = \mathbf{b}_{d^{-1}i} = (d^{-1})^{\mu} i = (d^{-1})^{\mu} \mathbf{b}_i,$$

we see that $\mathbf{b}P_d = (d^{-1})^{\mu}\mathbf{b}$. Thus $W = V = U = D$, and the automorphism group is ‘maximal’,

$$\mathrm{Aut}(G) = (G \rtimes \kappa_\infty(A)) \rtimes (\mathbb{F}_p^\times)^2.$$

Example 5.6.3. The distinction between the subgroups U, V and W is not superficial. Consider the vector $\mathbf{b}_1 = (1, 2, 11, 3, 12, 10, 10, 12, 3, 11, 2, 1) \in \mathbb{F}_{13}^{12}$. An easy calculation shows that

$$\mathbf{b}_1 P_{x\mapsto 5x} = (12, 11, 2, 10, 1, 3, 3, 1, 10, 2, 11, 12) = -\mathbf{b}_1,$$

while $\mathbf{b}_1 P_{x\mapsto 3x}$ is not a multiple of $\mathbf{b}_1$. Set $\mathbf{b}_2 = \mathbf{b}_1 P_{x\mapsto 3x}$ and $\mathbf{b}_3 = \mathbf{b}_1 P_{x\mapsto 9x}$ and let $\mathbf{E}$ be the space spanned by $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$. Since $\mathbb{F}_{13}^{\times}$ is generated by 3 and 5, the space $\mathbf{E}$ is invariant under all permutations induced by index-multiplication, i.e. $U = D$. But only the multiplication by the multiples of 5 act by scalar multiplication of $\mathbf{E}$, hence $V = \langle x \mapsto 5x \rangle$. The corresponding scalars are 1 and 12, hence W is of order 2.

Example 5.6.4. Let G_p be a Gupta–Sidki p-group, i.e. the GGS-group with $\mathbf{E}$ spanned by $\mathbf{b} = (1, -1, 0, \dots, 0) \in \mathbb{F}_p^{p-1}$. All Gupta–Sidki p-groups are regular. Let $n \in N$, and consider $\mathbf{b}P_n$. Since the projection to the last $p-3$ coordinates of $\mathbf{E}$ is trivial, the index 1 must be mapped to 1 or 2 under n, and the same holds for 2. This is only possible if $n = 1$ or $n = 2$ and $2 \cdot 2 \equiv_p 1$, hence in case $p = 3$. Otherwise U is trivial. If $p = 3$, the group W is equal to U, since the non-trivial permutation induced by the index multiplication by 2 is equal to pointwise multiplication by 2. This recovers the result of [138], where the automorphism group of G_3 was first computed. Interestingly, this example is the 'odd one out', having automorphisms of order 2.

In conclusion, we found

$$\mathrm{Aut}(G_p) = \begin{cases} (G_p \rtimes \kappa_\infty(A)) \rtimes \mathrm{C}_2^2 & \text{if } p = 3, \\ G_p \rtimes \kappa_\infty(A) & \text{otherwise.} \end{cases}$$

Example 5.6.5. Let $G_{\mathbb{F}_p^{p-1}}$ be the multi-GGS-group defined by the full space $\mathbb{F}_p^{p-1}$. This group is regular, and every permutation P_n leaves $\mathbb{F}_p^{p-1}$ invariant. On the other hand, no non-trival permutation acts on the full space as a multiplication. Thus

$$\begin{aligned} \mathrm{Aut}(G_{\mathbb{F}_p^{p-1}}) &= (G_{\mathbb{F}_p^{p-1}} \rtimes \kappa_\infty(A)) \rtimes \{\textstyle\prod_{i=0}^{\infty} \kappa_i(d') \mid d' \in D\} \\ &\cong (G_{\mathbb{F}_p^{p-1}} \rtimes \prod_{\omega} \mathrm{C}_p) \rtimes \mathrm{C}_{p-1}. \end{aligned}$$

Chapter 6

The derived series of GGS-groups

Abstract. Given a GGS-group G with non-constant defining tuple over a prime-regular rooted tree, we calculate the indices $|G : G^{(n)}|$ and describe the structure of the higher derived subgroups $G^{(n)}$ for all $n \in \mathbb{N}$. We find that the values $|G : G^{(n)}|$ depend only mildly on the structure of the defining tuple. In the course of our proof, we refine the understanding of the structure of subgroups of small index in a GGS-group.

6.1 — Introduction

The class of groups of automorphisms of regular rooted trees provides many examples with interesting asymptotic and structural properties. One particularly well-studied case is the family of Grigorchuk–Gupta–Sidki-groups (usually abbreviated as 'GGS-groups'). It contains at least one group of intermediate growth [43] and many finitely generated infinite periodic groups, cf. [77]. GGS-groups are groups of automorphisms of the p-regular rooted tree, for an odd prime p, and generalise the Gupta–Sidki p-groups. They are easily defined by using a non-zero element $\mathbf{e}$ of $\mathbb{F}_p^{p-1}$ as 'input data', and many of their properties can be read off the element $\mathbf{e}$: One can determine whether the corresponding GGS-group is periodic or contains elements of infinite order, if the group is just-infinite, cf. [151], whether it is a branch group, cf. [48], or if it has the congruence subgroup property, cf. [49]. Furthermore, one may compute its Hausdorff dimension, cf. [48], or decide if two GGS-groups are isomorphic, cf. [120], just by considering the defining tuples. Most of these results require subtle insights into the structure of a general GGS-group, and some involve heavy computation. Many of the results extend to larger classes of groups, cf. for example [**?**, 4, 19], but have been established first for GGS-groups, making the class of GGS-groups a playground for establishing new techniques.

However, other questions remain open; in contrast to the features related to the action on the tree, many purely algebraic properties of GGS-groups is not well-understood. In this work, we describe the derived series $(G^{(n)})_{n\in\mathbb{N}}$ of all GGS-groups, excluding those that arising from constant tuples, i.e. elements of the form $(\lambda, \lambda, \dots, \lambda) \in \mathbb{F}_p^{p-1}$ for some $\lambda \in \mathbb{F}_p^{\times}$. A description of the derived series has previously been obtained for the special case of the

Gupta–Sidki 3-group $\ddot{\Gamma}$ by Vieira in [149], along with some results concerning the lower central series of $\ddot{\Gamma}$. The proof, however, does not carry over to general GGS-groups.

We now state our main result.

Theorem 6.1.1. *Let p be an odd prime and let G be a* GGS*-group with non-constant defining tuple* $\mathbf{e}$*. Denote by* $\mathbf{e}'$ *the tuple of differences between the entries of* $\mathbf{e}$*, and by* $\mathbf{e}''$ *the tuple of differences of* $\mathbf{e}'$*. Then*

$$\log_p |G : G^{(n)}| = \begin{cases} p^{n-2}(p + \mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'')) - \frac{p^{n-1}-1}{p-1}\,\mathrm{sym}(\mathbf{e}) + 1 & \text{if } n \geq 2, \\ 2 & \text{for } n = 1, \end{cases}$$

where

$$\mathrm{sym}(\mathbf{d}) = \begin{cases} 1 & \text{if } \mathbf{d} \text{ is symmetric,} \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathrm{con}(\mathbf{d}) = \begin{cases} 1 & \text{if } \mathbf{d} \text{ is constant,} \\ 0 & \text{otherwise.} \end{cases}$$

A tuple is called *symmetric* if its i^{th} entry is equal to its i^{th}-to-last entry.

It is no surprise that the vector $\mathbf{e}' \in \mathbb{F}_p^{p-2}$ of differences between neighbouring entries in $\mathbf{e}$ is associated to the determination of the structure of the derived subgroups, since it describes the sections of the commutator $[b, a]$ of the two generators of a GGS-group. Interestingly, the indices of the derived subgroups do not depend on the higher iterates of the differences.

It is worthwhile to compare our result with the main result of [48], where the indices of the congruence subgroups, i.e. the stabilisers $\mathrm{St}_G(n)$ of elements of a distance $n \in \mathbb{N}$ from the root of the tree, are computed to be

$$\log_p |G : \mathrm{St}_G(n)| = \begin{cases} tp^{n-2} + \frac{p^{n-2}-1}{p-1}\,\mathrm{sym}(\mathbf{e}) + 1 & \text{if } n \geq 2, \\ 1 & \text{for } n = 1, \end{cases}$$

where t is the rank of a certain matrix associated to $\mathbf{e}$, and might take values in $\{2, \dots p\}$. In particular, the number of configurations of the indices $|G : \mathrm{St}_G(n)|$ grows linearly with p. In comparison, the indices of the derived series are more uniform and depend only on three (interconnected) binary invariants of $\mathbf{e}$; i.e. the indices of the derived subgroups of any GGS-groups (aside from the dependency on the prime p itself), fall in precisely four distinct classes.

From a group-theoretic standpoint, it is an inherently interesting problem to determine the derived series of a given group. This is especially true since GGS-groups are hypoabelian, i.e. the intersection of all members of the derived series is trivial; hence every element of G appears as a non-trivial element in some quotient $G^{(n)}/G^{(n+1)}$. Furthermore, the derived series fulfils the analogue of the congruence subgroup property: all finite index subgroups contain some derived subgroup. This is an immediate consequence of the congruence subgroup property of GGS-groups, that was established in [49], and the fact that the n^{th} derived subgroup $G^{(n)}$ is contained in the n^{th} level stabiliser $\mathrm{St}_G(n)$. The analogy to the congruence subgroups goes further. We prove the following theorem.

Theorem 6.1.2. *Let G be a* GGS*-group with defining tuple* $\mathbf{e}$ *and let $n \in \mathbb{N}_{\geq 3}$. Then*

$$\psi(G^{(n)}) = G^{(n-1)} \times \overset{p}{\ldots} \times G^{(n-1)}.$$

If $\mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'') - \mathrm{sym}(\mathbf{e}) = 0$*, the same holds for $n = 2$.*

Note that a similar statement is true for the congruence subgroups of GGS-groups. To make the connexion between congruence and derived subgroups more transparent, we introduce the *series of iterated local laws*. It is a descending series $(L_n)_{n\in\mathbb{N}}$ of normal subgroups of a group G acting on a rooted tree, such that $L_n \leq \mathrm{St}_G(n)$ for all $n \in \mathbb{N}$, and is formed by the elements that have to stabilise vertices of a certain distance by virtue of fulfilling certain algebraic equations in G. In this sense, it is the 'algebraic analogue' of the sequence of layer stabilisers. See Definition 6.2.2 for a precise definition.

As a corollary to Theorem 6.1.1, we prove that the series of iterated local laws and the derived series coincide for all GGS-groups defined by a non-constant vector $\mathbf{e}$, thus explaining, at least heuristically, the similarities mentioned above; see Corollary 6.3.4.

The paper is organised in the following way. After establishing our notation, we prove some structural results on GGS-groups. Then we prove Proposition 6.3.1, in which we compute the index of the second derived subgroup in the full group. This is the main technical step. Afterwards, we proceed to derive our other results. Being aware of the multitude of subgroups appearing, we point the reader to Fig. 6.1, which depicts the relevant portion of the top of the subgroup lattice of a GGS-groups.

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6.2 — On Grigorchuk–Gupta–Sidki-groups

We begin with some generalities. We fix an odd prime p. Given a group G and two elements g, h, we use the following conventions for conjugation and the commutator

$$g^h = h^{-1}gh \quad \text{and} \quad [g,h] = g^{-1}h^{-1}gh = (h^{-1})^g h.$$

6.2.1. Groups of automorphisms of regular rooted trees. — Write X for the set $\{0, \ldots, p-1\}$. We will sometimes identify X with the set underlying $\mathbb{F}_p$. We write X^* for the Cayley graph of the free monoid on X, which is a rooted p-regular tree, i.e. a loop-free graph in which all but one vertex have valency $p+1$, and the remaining vertex $\varnothing$, called the *root* of the tree, has valency p. The vertices of X^* are the set of finite sequences in X. We write X^n for the set of all vertices of a given length $n \in \mathbb{N}$, and call this set the n^{th} *level of* X^*. The root $\varnothing$ has length 0.

Any (graph) automorphism $g \in \operatorname{Aut}(X^*)$ necessarily fixes $\varnothing$, since it has fewer neighbours than every other vertex, and must consequently leave the levels X^n invariant for all $n \in \mathbb{N}$. We write $\operatorname{St}(n)$ for the stabiliser of X^n, and $\operatorname{St}_G(n)$ for its intersection with some subgroup $G \leq \operatorname{Aut}(X^*)$. Let u and v be vertices of X^*. We write u^g for the image of u under g. Since levels are invariant under g, the equation

$$(uv)^g = u^g v^{g|_u}$$

defines uniquely a map $|_u : \operatorname{Aut}(X^*) \to \operatorname{Aut}(X^*)$, called the *section map at* u. The image is consequently called *the section of* g *at* u. Using these images, any tree automorphism g can be decomposed into the sections prescribing the action at the subtrees of the first level, and the *action of* g *at the root* $g|^{\varnothing} \in \operatorname{Sym}(X)$, which is just the action of g on the first level $X = X^1$. In particular, the map

$$\begin{aligned} \psi : \operatorname{St}(1) &\to \operatorname{Aut}(X^*) \times \overset{p}{\ldots} \times \operatorname{Aut}(X^*) \\ g &\mapsto (\diamond : g|_\diamond) \end{aligned}$$

is a group isomorphism. Here we adopt the convention that the expression

$$(i_0 : a_0, \ldots, i_k : a_k, \diamond : a_\diamond)$$

denotes the tuple indexed by X, with the object a_0 at position i_0, the object a_1 at position i_1 and so forth, and the object $a_\diamond$ (maybe varying in $\diamond$) at all other positions $\diamond \in X \smallsetminus \{i_m \mid m = 0, \ldots, k\}$. The symbol $\diamond$ will be reserved for this use. An automorphism with at most one trivial section is called *rooted*, rooted automorphisms must necessarily permute the subtrees $\{xX^* \mid x \in X\}$ of the first level and can be identified with permutations of X.

We record some equations for sections. Let u and v be vertices of X^* and g and h be any automorphisms, then

$$(g|_u)|_v = g|_{uv}, \quad (gh)|_u = g|_u h|_{u^g}, \quad g^{-1}|_u = (g|_{u^{g^{-1}}})^{-1}.$$

A subgroup $G \leq \operatorname{Aut}(X^*)$ is called *self-similar*, if for all vertices $u \in X^*$, the image of the section map $|_u : g \mapsto g|_u$ is contained in G. A self-similar group G is called *contracting*, if there exists a finite set $\mathcal{N} \subseteq G$, such that for all $g \in G$ there exists some $n \in \mathbb{N}$ such that for all $m \geq n$ and all $v \in X^m$ the section $g|_v$ is an element of the finite set $\mathcal{N}$. For a contracting group, there is a unique minimal set $\mathcal{N}$ with this property, which is called the *nucleus of* G. A group $G \leq \mathrm{Aut}(X^*)$ is called *fractal*, if for every $g \in G$ and every $x \in X$ there is an element $\widehat{g}$ stabilising the vertex x such that $\widehat{g}|_x = g$. A group $G \leq \mathrm{Aut}(X^*)$ is called *spherically transitive* if it acts transitively on every level X^n.

A self-similar group $G \leq \mathrm{Aut}(X^*)$ is called a *regular branch group*, if it is spherically transitive, and if there is a finite index subgroup $K \leq G$ such that

$$K \times \overset{p}{\ldots} \times K \leq \psi(K).$$

A standard technique for establishing that a group is regular branch is given by the following lemma, cf. [48, Proposition 2.18].

Proposition 6.2.1. *Let $G \leq \mathrm{Aut}(X^*)$ be a spherically transitive fractal group, $H \leq G$ a subgroup and let $S \subseteq G$ be a subset. If $\{(0 : s,\ \diamond : \mathrm{id}) \mid s \in S\}$ is contained in $\psi(H)$, then*

$$\langle S\rangle^G \times \overset{|X|}{\ldots} \times \langle S\rangle^G \leq \psi(H^G).$$

We now come to the precise definition of the series of iterated local laws for a spherically transitive group $G \leq \mathrm{Aut}(X^*)$. Consider the set $\{(g|_u)|^{\varnothing} \mid g \in G, u \in X^*\} \subseteq \mathrm{Sym}(X)$. By the algebra for sections above, it is easily seen that this set forms a subgroup $P(G) \leq \mathrm{Sym}(X)$ of the symmetric group on X. Given a subgroup $H \leq \mathrm{Sym}(X)$, we may define a corresponding subgroup of $\mathrm{Aut}(X^*)$ by

$$\Lambda(H) = \{g \in \mathrm{Aut}(X^*) \mid (g|_u)|^{\varnothing} \in H \text{ for all } u \in X^*\}.$$

If H is a p-group, it is necessarily cyclic, and $\Lambda(H)$ is a Sylow pro-p-subgroup of $\mathrm{Aut}(X^*)$.

Let $R = R(H)$ be the collection of all group laws of H, i.e. elements of the free group F_∞ on infinitely many generators that evaluate to the trivial element for all assignments of the generators to elements of H. Given any group K, we may consider the subgroup $L_R(K)$ generated by all verbal subgroups of K corresponding to elements in R, i.e. by all images of R under any assignment of the generators of F_∞ to elements of K.

Returning to the subgroup $P(G)$ defined by a group of tree automorphisms, we see that $L_{R(P(G))}(G)$ is contained in the first level stabiliser, since $L_{R(P(G))}(G|^{\varnothing})$ is trivial by construction; note that the map $|^{\varnothing} : G \to P(G) \leq \mathrm{Sym}(X)$ is a (not necessarily surjective) homomorphism. Consider the iterates R_n of R, that are recursively defined by

$$R_n = \left\{ s(r_1, \dots, r_n) \;\middle|\; \begin{array}{l} s \in R_{n-1}, r_i \in R \text{ for } i \in \{1, \dots, n\}, \\ s \text{ an element involving } n \text{ generators, and} \\ r_1, \dots, r_n \text{ share no generator of } F_\infty \end{array} \right\},$$

for $n > 1$ and $R_1 = R$, where by the expression $s(r_1, \dots, r_n)$ we mean the element of F_∞ obtained by replacing the n generators occurring in s by the elements r_1 to r_n. By the same argument as above,

$$L_{R_n(P(G))}(G) \leq \mathrm{St}_G(n).$$

Definition 6.2.2. Given a spherically transitive group of $G \leq \mathrm{Aut}(X^*)$, we write $L_n(G)$ for $L_{R_n(P(G))}(G)$, and we call $(L_n(G))_{n\in\mathbb{N}}$ the *series of iterated local laws.*

We consider this series in the case of GGS-groups. It will be apparent from the definition that every GGS-group G acts locally by permutations from a cyclic group of order p, i.e. it fulfils

$$P(G) = \langle\, (0\,1\,\ldots\,p-1)\,\rangle,$$

The laws of such a group are generated by the commutators and p^{th} powers of generators in F_∞. Consequently, the group $L_1(G)$ is the subgroup generated by G' and the p^{th} powers in G. We shall prove that $L_n(G)$ is in fact equal to $G^{(n)}$.

6.2.2. GGS-groups and their defining tuples. — Let $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a non-*constant* tuple, i.e. such that there are at least two different entries. We call the group $G_{\mathbf{e}}$ generated by the rooted automorphism $a = (0\ 1 \ldots p-1)$ and the automorphism defined by

$$b = \psi^{-1}(0 : b,\ \diamond : a^{e_\diamond})$$

the GGS-*group defined by* $\mathbf{e}$, and we call $\mathbf{e}$ the *defining tuple of* $G_{\mathbf{e}}$.

Note that we exclude all constant tuples (in particular the zero tuple). The groups defined by constant non-zero tuples in the same fashion as above are usually also referred to as GGS-groups. Furthermore, groups defined by the same construction using elements of $\mathbb{Z}/m\mathbb{Z}$ whose entries are set-wise coprime may also be referred to as GGS-groups. In general, the structure of these groups is much less understood than in the case we consider here. Even for prime powers $m = p^n$, the situation is much more involved, see for example [37], where the branching structures for these groups have been computed.

We consider the derived subgroups of a GGS-group $G = G_{\mathbf{e}}$. Since G is two-generated, the first derived subgroup is normally generated by the commutator $c = [b, a]$, whose action on the tree is given by

$$\begin{aligned}\psi([b,a]) &= \psi(b^{-1})\psi(b^a)\\ &= (0 : b^{-1},\ \diamond : a^{-\mathbf{e}_\diamond})(1 : b,\ \diamond : a^{\mathbf{e}_{\diamond-1}})\\ &= (0 : b^{-1}a^{\mathbf{e}_{p-1}},\ 1 : a^{-\mathbf{e}_1}b,\ \diamond : a^{\mathbf{e}_{\diamond-1}-\mathbf{e}_\diamond}).\end{aligned}$$

This signifies the importance of the *first difference tuple of* $\mathbf{e}$, which we define as

$$\mathbf{e}' = (e'_2, e'_3, \ldots, e'_{p-2}, e'_{p-1}) \in \mathbb{F}_p^{p-2},$$

where $e'_i = e_{i-1} - e_i$ for all $i \in \{2, \ldots, p-1\}$. We shall see that the index of the second derived subgroup in G depends furthermore on the *second difference tuple of* $\mathbf{e}$, given by

$$\mathbf{e}'' = (e''_3, e''_4, \dots, e''_{p-2}, e''_{p-1}) \in \mathbb{F}_p^{p-3},$$

where $\mathbf{e}''_i = \mathbf{e}_{i-2} - 2\mathbf{e}_{i-1} + \mathbf{e}_i = \mathbf{e}'_{i-1} - \mathbf{e}'_i$ for all $i \in \{3, \dots, p-1\}$. In case $p = 3$ the tuple $\mathbf{e}''$ is the empty tuple. Clearly, we have described the beginning of an iterative procedure, but surprisingly, the indices of the higher derived subgroup do not depend on 'higher' difference tuples.

In the following, we consider the elements of the vector space $\mathbb{F}_p^p$ and more generally of direct products of groups $G \times \overset{p}{\dots} \times G$ as indexed by the set X. The choice of indexing for the defining tuple and its differences we have made above is for the following reason. Let $\log_a : \langle a \rangle \to \mathbb{F}_p$ be the map assigning the power i to any a^i, and $\theta : G \to \mathbb{F}_p^p$ the map $g \mapsto (\log_a(g|_0)^\varnothing, \dots, \log_a(g|_{p-1})^\varnothing)$ assigning to g its local actions under the first layer vertices. Then, by definition,

$$\theta(b) = (0, e_1, \dots, e_{p-1}).$$

Thus we think of the defining tuple as an 'incomplete element' of $\mathbb{F}_p^p$, and the element above as its full counterpart; similarly we think of $\mathbf{e}'$ and $\mathbf{e}''$ as the 'tails' of regularly formed elements of

$$\theta(c) = (e_{p-1}, -e_1, e'_2, e'_3 \dots, e'_{p-1})$$

and

$$\theta([c, a]) = (e_{p-2} - 2e_{p-1}, e_1 + e_{p-1}, -2e_1 + e_2, e''_3, e''_4, \dots, e''_{p_1})$$

respectively.

We call $\mathbf{e}$, resp. $\mathbf{e}''$, *symmetric* if and only if

$$\begin{aligned} e_i &= e_{p-i} && \text{for all } i \in \{1, \dots, p-1\}, \text{ resp.} \\ e''_{i+1} &= e''_{p+1-i} && \text{for all } i \in \{2, \dots, p-2\}. \end{aligned}$$

These are clearly linear conditions. We define two linear subspaces $S = \ker(M)$ and $\ddot{S} = \ker(\ddot{M})$ of $\mathbb{F}_p^{p-1}$ as the kernels of the two linear maps given by the matrices

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ & 1 & 0 & \dots & 0 & -1 & \\ & & \ddots & & \iddots & & \\ & & & 1 \quad -1 & & & \end{pmatrix}^{\intercal},$$

in $\mathrm{Mat}(p-1, \frac{p-1}{2}; \mathbb{F}_p)$ and

$$\ddot{M} = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \dots & \dots & 0 & 0 & -1 & 2 & -1 \\ & 1 & -2 & 1 & 0 & \dots & \dots & 0 & -1 & 2 & -1 & \\ & & & \ddots & & & & & \iddots & & & \\ & & & 1 & -2 & 1 & -1 & 2 & 1 & & & \\ & & & & 1 & -3 & 3 & -1 & & & & \end{pmatrix}^{\mathsf{T}}$$

in $\mathrm{Mat}(p-1, \frac{p-3}{2}; \mathbb{F}_p)$. Clearly $\mathbf{e}$ is symmetric if and only if $\mathbf{e}$ is an element of S, and we have $\mathbf{e} \in \ddot{S}$ if and only if $\mathbf{e}''$ is symmetric. Using this description, the following lemma becomes apparent.

Lemma 6.2.3. *Let* $\mathbf{e}$ *be symmetric. Then* $\mathbf{e}''$ *is symmetric.*

Proof. It can be easily seen that the subspace generated by the columns (displayed above as rows) of $\ddot{M}$ is contained in the subspace generated by the columns of M, i.e. that $S \subseteq \ddot{S}$. Thus all symmetric $\mathbf{e}$ yield symmetric $\mathbf{e}''$. □

Of course, if a vector is constant, all its difference tuples are trivial, hence in particular constant and symmetric.

We can also see that the containment of Lemma 6.2.3 is proper and that $\operatorname{codim}_{\ddot{S}} S = 1$. For further computations, we simplify the basis given by the columns of $\ddot{M}$ using Gauß-Jordan elimination, and obtain

$$\ddot{N} = \begin{pmatrix} 1 & 0 & \dots & 0 & 2 & -2 & 0 & \dots & 0 & -1 \\ & 1 & \dots & 0 & 4 & -4 & 0 & \dots & -1 & \\ & & \ddots & & \vdots & \vdots & & \iddots & & \\ & & & 1 & -3 & 3 & -1 & & & \end{pmatrix}^{\mathsf{T}}.$$

Lemma 6.2.4. *Let* $\mathbf{e} \in \mathbb{F}_p^{p-1}$. *If the second difference tuple* $\mathbf{e}''$ *is symmetric, then*

$$2(\mathbf{e}_{p-1} - \mathbf{e}_1) + (\mathbf{e}_2 - \mathbf{e}_{p-2}) = 0.$$

Proof. Since $\mathbf{e}''$ is symmetric, the vector $\mathbf{e}$ is contained in $\ddot{S}$. But the given linear equation is a linear combination of the first two columns of $\ddot{N}$, from whence the equality follows. □

For the use in formulas, we define the shorthand notation

$$\mathrm{con}(\mathbf{d}) = \begin{cases} 1 & \text{if } \mathbf{d} \text{ is constant,} \\ 1 & \text{otherwise.} \end{cases} \quad \text{and} \quad \mathrm{sym}(\mathbf{d}) = \begin{cases} 1 & \text{if } \mathbf{d} \text{ is symmetric,} \\ 1 & \text{otherwise.} \end{cases}$$

Aside from the first and second difference tuples and the defining tuple itself, all cyclic shifts of $\mathbf{e}$ (under the action of a) influence the structure of the GGS-group $G_{\mathbf{e}}$. Fernández-

Alcober and Zugadi-Reizabal [48] demonstrated that the index of the level stabilisers in a GGS-group depends only on whether the defining tuple is symmetric, and the rank of the *circulant matrix* $\mathrm{Circ}(0\ \mathbf{e}) \in \mathrm{Mat}(p, p; \mathbb{F}_p)$ associated to the 'full version' $\theta(b)$ of the defining tuple $(0\ \mathbf{e})$, that is the matrix whose rows are the cyclic shifts of the vector $\theta(b)$.

For the computation of the indices of the level stabilisers and the derived subgroups, we use the König-Rados theorem, cf. e.g. [97, § 134] or [88], which solves the problem of determining the rank of a circulant matrix over a prime field.

Theorem 6.2.5 (König-Rados)**.** *Let p be a prime and $\mathbf{d} \in \mathbb{F}_p^n$ a vector. Then*

$$\mathrm{rk}\,\mathrm{Circ}(\mathbf{d}) = n - m,$$

where m is the multiplicity of 1 as a root of the polynomial $E_{\mathbf{d}} = \sum_{i=0}^{n-1} d_i X^i$. In particular, $\mathrm{rk}\,\mathrm{Circ}(\mathbf{d}) = n$ if and only if $\sum_{i=0}^{n-1} d_i \neq 0$.

For our purposes, we make a more general definition. Let V be a finite-dimensional vector space over a finite field, let $\mathcal{B}$ be a basis for V, and let C be the linear map that cyclically permutes the basis elements. Given a subset $M \subseteq V$, we denote by $\mathrm{Circ}(M)$ the smallest C-invariant subspace containing M, the *circulant space of M*. The notational conflict with the definition of the circulant matrix given above can be ignored; the circulant space is just the row space of the circulant matrix. We will often make no a distinction between the two.

There are not many C-invariant subspaces. This is no surprise, since C defines the regular representation of a group of order p, which is the sum of p one-dimensional irreducible sub-representations. In fact, there is a unique (full) flag of C-invariant subspaces in V, which we record in the following proposition.

Proposition 6.2.6. *Let V be an $\mathbb{F}_p$-vector space of dimension $n \in \mathbb{N}$, with basis $\mathcal{B}$. Then the set of circulant spaces of V has cardinality $n+1$ and forms a full flag of V. In particular, for any $M \subseteq V$,*

$$\mathrm{Circ}(M) = \bigcup_{m \in M} \mathrm{Circ}(m).$$

Proof. We prove that, for every $i \in \{0, \ldots, n\}$, the set

$$\mathrm{Circ}_i(V) = \{\mathbf{d} \in V \mid \mathrm{rk}\,\mathrm{Circ}(\mathbf{d}) \leq i\}$$

is an i-dimensional C-invariant subspace of V. If this is true, the circulant space associated to any $\mathbf{d} \in \mathrm{Circ}_i(V) \smallsetminus \mathrm{Circ}_{i-1}(V)$ is in fact equal to $\mathrm{Circ}_i(V)$; since it is the minimal invariant subspace containing $\mathbf{d}$, it is contained in $\mathrm{Circ}_i(V)$, and by the choice of $\mathbf{d}$ it has the same dimension as $\mathrm{Circ}_i(V)$. Now for any subset $M \subseteq V$, the circulant space is equal to the smallest invariant subspace containing all $\mathrm{Circ}(\mathbf{m})$ for $\mathbf{m} \in M$. It is easy to see that these spaces are linearly ordered, hence $\mathrm{Circ}(M)$ is equal to the maximal subspace of the form $\mathrm{Circ}(\mathbf{m})$.

It remains to prove the claim. Fix $i \in \{0, \dots, n\}$ and $\mathbf{d} \in \mathrm{Circ}_i(V)$. Clearly the image of $\mathbf{d}$ under C defines the same circulant space, hence $\mathrm{Circ}_i(V)$ is invariant. It remains to prove that it is an i-dimensional subspace.

To achieve this, we review some combinatorics of polynomials. Let $Q = Q^{(n)} = \sum_{k=0}^{n-1} q_k X^k \in \mathbb{F}_p[X]$ be a polynomial of degree $n-1$. In view of Theorem 6.2.5, we conduct Euclidean division by $(X-1)$, and write $Q^{(n)} = (X-1)Q^{(n-1)} + R_n(Q)$, for a polynomial $Q^{(n-1)}$ and a constant $R_n(Q) \in \mathbb{F}_p$. Iterating this, we write $Q^{(i)} = (X-1)Q^{(i-1)} + R_i(Q)$ for $i \in \{0, \dots, n-1\}$. Clearly $\deg Q^{(i)} = i-1$. The coefficients of $Q^{(i)}$ and the value of $R_i(Q)$ can be calculated in terms of the starting polynomial $Q^{(n)}$. This we now perform. (Alternatively, we could compute the coefficients of $Q^{(n)}$ as a polynomial in $X-1$). Indeed, it is easy to check that

$$Q^{(n-1)} = \sum_{k=0}^{n-2} \sum_{\ell=k+1}^{n-1} q_\ell X^k \qquad \text{and} \qquad R_n(Q) = \sum_{k=0}^{n-1} q_k.$$

Thus, both the coefficient of X^k in $Q^{(i)}$ and the value $R_i(Q)$ are weighted sums (i.e. positive $\mathbb{F}_p$-linear combinations) of the coefficients of Q. Write $\kappa(i,j,k)$ for the multiplicity of q_j in the coefficient of X^k in $Q^{(i)}$. The equation above shows that

$$\kappa(i,j,k) = \sum_{\ell=k+1}^{n-1} \kappa(i+1, j, \ell).$$

Consequently, for $i < n$, we find the familiar (ignoring j) recursion formula

$$\kappa(i,j,k) = \kappa(i,j,k+1) + \kappa(i+1,j,k+1),$$

using that $\kappa(i,j,n) = 0$ for all $i,j \in \{0,\dots,n\}$. Since $\kappa(n-1,j,k)$ is 1 for $j > k$, and is equal to 0 otherwise, we find

$$\kappa(i,j,j+i-n) = 1 \qquad \text{and} \qquad \kappa(i,j,k) = 0 \qquad \text{for } k > j+i-n,$$

and obtain the equality

$$\kappa(i,j,k) = \binom{j-k-1}{n-i-1},$$

where we agree on $\binom{r}{s} = 0$ for $r < s$. The remainder $R_i(Q)$ is equal to the sum of all coefficients of $Q^{(i)}$. Since $\deg Q^{(i)} = i-1$, we have to calculate the sum

$$R_i(Q) = \sum_{k=0}^{i-1} \sum_{j=0}^{n-1} \kappa(i,j,k) q_j = \sum_{j=0}^{n-1} q_j \sum_{k=0}^{i-1} \binom{j-k-1}{n-i-1}.$$

We may ignore all summands with $j < n-i$, since then $j-k-1 < n-i-1$ for all $k \in \{0,\dots,i-1\}$, and the binomial coefficient is zero. Likewise we may ignore all cases where $k > j + i - n$. It remains to use 'Stifel's law'. We find

$$R_i(Q) = \sum_{j=n-i}^{n-1} q_j \sum_{k=0}^{j+i-n} \binom{j-k-1}{n-i-1} = \sum_{j=n-i}^{n-1} q_j \sum_{k=n-i-1}^{j-1} \binom{k}{n-i-1} = \sum_{j=i}^{n} q_j \binom{j}{n-i}.$$

Coming back to our circulant spaces, Theorem 6.2.5 tells us that the condition $\mathbf{d} \in \mathrm{Circ}_i(V)$, for any i, translates to $R_j(E_\mathrm{d}) = 0$ for all $j \in \{i+1, \dots, n\}$, and $R_i(E_\mathrm{d}) \neq 0$. By our computations, the map $R_j \colon V \to \mathbb{F}_p$ (for any $j \in \{1, n\}$) assigning to an element $\mathbf{d} = (d_0, \dots, d_{n-1})$, represented in the basis $\mathcal{B}$, the value of $R_j(E_\mathrm{d})$ as given above is $\mathbb{F}_p$-linear. Define $R \colon V \to \mathbb{F}_p^p$ by $\mathbf{d}R = (\mathbf{d}R_1, \dots, \mathbf{d}R_n)$. This map is, due to $\mathcal{B}$ and the standard basis, represented by the matrix

$$\begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \cdots & \binom{n-2}{0} & \binom{n-1}{0} \\ & \binom{1}{1} & \cdots & \binom{n-2}{1} & \binom{n-1}{1} \\ & & \ddots & & \vdots \\ & & & \binom{n-2}{n-2} & \binom{n-1}{n-2} \\ 0 & & & & \binom{n-1}{n-1} \end{pmatrix},$$

a right-justified Pascal triangle. The subspace $\mathrm{Circ}_i(V)$ is the kernel of the composition $R \circ \pi_{\leq i}$, where $\pi_{\leq n-i}$ denotes the projection to the first $n-i$ coordinates. Since R has full rank, the image under this map has dimension $n-i$, whence the kernel has dimension i. Thus $\dim \mathrm{Circ}_i(V) = i$. □

It is not true that every defining tuple gives rise to a unique GGS-group. In particular, multiples of a given $\mathbf{e}$ define the same group (as a subgroup of $\mathrm{Aut}(X^*)$). Furthermore, certain reorderings of the entries give isomorphic groups, which helps us to reduce the difficulty of our computations. We use the following characterisation.

Theorem 6.2.7. *[120] Let G and H be two* GGS*-groups over the p-regular tree defined by $\mathbf{e}$ and $\mathbf{d}$, respectively. Then the following two statements are equivalent:*

(i) $G \cong H$;
(ii) *there exist $\lambda, \mu \in \mathbb{F}_p^\times$ such that $\mathbf{e}_i = \mu \cdot \mathbf{d}_{\lambda \cdot i}$ for all $i \in \{1, \dots, p-1\}$.*

This allows us to choose defining tuples with desirable properties.

Corollary 6.2.8. *Let G be a* GGS*-group. Then*

(i) *there is an* GGS*-group $G_\mathbf{e}$ isomorphic to G such that $\mathbf{e}_1 = 1$, and*
(ii) *there is an* GGS*-group $G_\mathbf{e}$ isomorphic to G such that $\mathbf{e'}_i = 1$ for some $i \in \{1, \dots, p-1\}$.*

6.2.3. Properties and structure of GGS-groups. — We shall fix some further notation. Recall that, working with a given GGS-group G, we shall denote the rooted generator a, the directed generator b, and we write c for the commutator $[a^{-1}, b]$. Furthermore we shall use the following shorthand notation for the conjugates of c,

$$c_i = c^{a^i} = [b^{a^i}, a].$$

In particular, $c_0 = c$. We now describe the sections of c. We will use this computation often and without constant reference. The sections of c_i are the sections of c_0 cyclically shifted; in general, for any $g \in \mathrm{Aut}(X^*)$ and for any $i \in \mathbb{Z}$, the first level sections of g^{a^i} are the sections of g, permuted by the inverse a^{-i}, since the sections of a are trivial and

$$g^{a^i}|_j = a^{-i}|_j g|_{j^{a^i}} a^i|_{j^{a^i}} = g|_{j-i},$$

i.e. $\psi(g^{a^i}) = \psi(g)^{a^{-i}}$, with a^{-i} acting as a permutation of the index set X.

It is well-known that all GGS-groups posses strong 'self-similarity' properties, which is one of the reasons making this class of groups an interesting object to study. We collect some statements into a lemma, for proofs see e.g. [120].

Lemma 6.2.9 (Fractality properties of GGS-groups)**.** *Let G be a* GGS*-group. Then G is self-similar, fractal and contracting with nucleus $\langle a\rangle \cup \langle b\rangle$.*

Furthermore, every GGS-group is a regular branch group, which is not true for the analogous of GGS-groups defined by constant tuples.

Theorem 6.2.10 (Branching properties of GGS-groups, cf. [48])**.** *Let G be a* GGS*-group with defining tuple* $\mathbf{e}$*. Then*

(i) $\psi(\gamma_3(\mathrm{St}_G(1))) = \gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G)$*, and*
(ii) $[G' \times \overset{p}{\ldots} \times G' : \psi(\mathrm{St}_G(1)')] = p^{\delta(\mathbf{e})}$.

Also $\mathrm{St}_G(2) \leq \gamma_3(G)$. In particular, G is regular branch over $\gamma_3(G)$, and it is regular branch over G' if $\mathbf{e}$ is non-symmetric.

This allows the application of the following lemma of Šunić, which provides the analogue of Theorem 6.1.2 for level stabilisers.

Lemma 6.2.11. *[144, Lemma 10] Let G be a regular branch group over $K \leq G$, such that $\mathrm{St}_G(n) \leq K$ for some $n \in \mathbb{N}$. Then for all $m \geq n$*

$$\psi(\mathrm{St}_G(m+1)) = \mathrm{St}_G(m) \times \overset{p}{\ldots} \times \mathrm{St}_G(m).$$

For GGS-groups in particular, the value of n is 2.

We now begin with a study of certain small quotients of GGS-groups which will play a role in the determination of the derived series. Most of these results are known, but using the next lemma, we can give new and short proofs that have the benefit of being easily generalised to larger families of groups, as they for the most part do not involve p-group methods.

Lemma 6.2.12. *Let G be a* GGS*-group, let $g \in G$, and let $w(a,b)$ be a word in the letters a and b that evaluates to g in G. Then clearly $w(1,b)$ evaluates to an element in $\langle b \rangle \cong \mathbb{F}_p$. The map*

$$\varepsilon : G \to \mathbb{F}_p, \quad g \to w(1,b)$$

that assigns to g is b-exponent sum *is a well-defined homomorphism. In other words, the quotient of G by the normal closure of $\langle a \rangle$ is a group of order p.*

Proof. Rewrite $w(a,b)$ into a word of the form

$$b^{a^{k_0}\ell_0} b^{a^{k_1}\ell_1} \ldots b^{a^{k_{n-1}}\ell_{n-1}} a^{k_n},$$

with $\ell_i, k_i \in \mathbb{F}_p$ for $i \in \{0, \ldots, n\}$ and $k_i \neq k_{i+1}$ for $i \in \{0, \ldots, n-2\}$. We may also assume $n \geq 1$, since the statement is clearly true otherwise. Let x be any letter of X. Since all conjugates of b fix x, a word representing $w(a,b)|_x$ is

$$b|_{x-k_0}^{\ell_0} b|_{x-k_1}^{\ell_1} \ldots b|_{x-k_{n-1}}^{\ell_{n-1}}.$$

Look at a length-2 subword $b|_{x-k_j}^{\ell_j} b|_{x-k_{j+1}}^{\ell_{j+1}}$. Since $k_j \neq k_{j+1}$, at least one of the two sections is a power of a, since only one section of b is not. We collect all resulting elements of $\langle a \rangle$ and combine consecutive powers of b (which does not change the b-exponent sum) and obtain a word $w_x(a,b)$ with fewer syllables representing $w(a,b)|_x$, such that $\sum_{x=0}^{p-1} w_x(1,b) = w(1,b)$. Since G is contracting (which is an consequence of the same argument), for every $w(a,b)$ there is a level X^n for $n \in \mathbb{N}$, such that all sections of $w(a,b)$ at vertices $v \in X^n$ are in the nucleus of G, hence powers of a or b. Clearly, all sections are minimal words, and the sum of their b-exponent sums is the b-exponent sum of $w(a,b)$. Thus this value does not depend on the word representing a certain element. Consequently ε is well-defined and a homomorphism. □

Using this lemma, we are able to quickly recover some well-known facts about certain small quotients of GGS-groups.

Lemma 6.2.13. *Let G be a* GGS*-group. Then*

(i) *G/G' is isomorphic to an elementary abelian p-group of rank* 2*,*
(ii) *$G/\gamma_3(G)$ is isomorphic to the Heisenberg group over $\mathbb{F}_p$,*
(iii) *$G/\operatorname{St}(1)'$ has order p^{p+1}, and*
(iv) *the subgroup generated by the p^{th} powers of G is contained in G'.*

Proof. (i): The quotient G/G' is generated by the images of a and b, hence a quotient of an elementary abelian p-group of rank 2. Clearly b is not a power of a, and a is not a stabiliser of the first level, hence not in the commutator subgroup. Since the b-exponent sum of any element in the commutator subgroup is 0 mod p, the generator b is not in G' either by Lemma 6.2.12.

(ii): The centre of $G/\gamma_3(G)$ is generated by $c = [b, a]$. Since $c^p \equiv_{\gamma_3(G)} [b, a^p] = 1$, the group $G/\gamma_3(G)$ is isomorphic to the Heisenberg group over $\mathbb{F}_p$.

(iii): The the first level stabiliser is generated by the conjugates of b by powers of a. Since the position of the section equal to b is different for every different power of a, it is generated by p elements of order p, all of b-exponent sum 1. Thus, as in (i), the quotient $\mathrm{St}_G(1)/\mathrm{St}_G(1)'$ is an elementary abelian p-group of rank p. Since $G/\mathrm{St}_G(1)$ is cyclic of order p, the result follows.

(vi): Lastly, it is enough to prove that b is not a p^{th} power. But the b-exponent sum of a p^{th} power is 0 mod p, hence b is not a p^{th} power by Lemma 6.2.12. □

Proposition 6.2.14. *Let G be a* GGS*-group. Then*

$$[\mathrm{St}_G(1)', G'] = \gamma_3(\mathrm{St}_G(1)).$$

In particular, $\gamma_3(\mathrm{St}_G(1)) \leq G''$.

Proof. Recall from Theorem 6.2.10 that

$$\psi(\gamma_3(\mathrm{St}_G(1))) = \gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G).$$

The inclusion $[\mathrm{St}_G(1)', G'] \leq \gamma_3(\mathrm{St}_G(1))$ holds true for any group. It remains to prove the other inclusion. Using Proposition 6.2.1, it is enough to prove that $(0 : [c, b],\ \diamond : \mathrm{id})$ and $(0 : [c, a],\ \diamond : \mathrm{id}) \in \psi([\mathrm{St}_G(1)', G'])$, since $\gamma_3(G)$ is normally generated by $[c, b]$ and $[c, a]$. We distinguish two cases.

Case 1: The defining tuple is non-symmetric. Then by Theorem 6.2.10(ii)

$$\psi(\mathrm{St}_G(1)') = G' \times \overset{p}{\ldots} \times G',$$

and $(0 : c,\ \diamond : \mathrm{id}) \in \psi(\mathrm{St}_G(1)')$. By Corollary 6.2.8(ii) we may assume $\mathbf{e}'_i = 1$ for some $i \in \{2, \ldots, p-1\}$. Consequently $c|_i = a$, hence

$$c_{p-i}|_0 = c|_i = a,$$
$$(c_{p-i}^{\mathbf{e}_{p-1}} c^{-1})|_0 = a^{\mathbf{e}_{p-1}} a^{-\mathbf{e}_{p-1}} b = b.$$

Since c_{p-i} and c are elements of G', we obtain that

$$[(0 : c,\ \diamond : \mathrm{id}), \psi(c_{p-i})] = (0 : [c, a],\ \diamond : \mathrm{id}),$$
$$[(0 : c,\ \diamond : \mathrm{id}), \psi(c_{p-i}^{\mathbf{e}_{p-1}} c^{-1})] = (0 : [c, b],\ \diamond : \mathrm{id})$$

are both elements of $\psi([\mathrm{St}_G(1)', G'])$.

Case 2: The defining tuple is symmetric. Since it is not constant, the prime p is necessarily greater then 3. By Corollary 6.2.8(i), we may assume $\mathbf{e}_1 = \mathbf{e}_{p-1} = 1$. Observe that

$$\psi([b, b^a]) = [\psi(b), \psi(b^a)] = (0 : [b, a],\ 1 : [a, b],\ \diamond : \mathrm{id}) = (0 : c,\ 1 : c^{-1},\ \diamond : \mathrm{id}).$$

Now let $j \in \mathbb{F}_p^\times \smallsetminus \{1, p-1\}$, which exists since $p > 3$. Then

$$\psi([a^{2j}, b]) = (0 : a^{-\mathbf{e}_{p-2j}} b,\ 2j : b^{-1} a^{\mathbf{e}_{2j}},\ \diamond : a^{\mathbf{e}_\diamond - \mathbf{e}_{\diamond - 2j}}),$$

in particular $[a^{2j}, b]|_j = a^{\mathbf{e}_j - \mathbf{e}_{p-j}} = \mathrm{id}$, since $\mathbf{e}$ is symmetric. Thus

$$[a^{2j}, b]^{a^{1-j}}|_1 = [a^{2j}, b]|_j = \mathrm{id}, \quad \text{and}$$
$$[a^{2j}, b]^{a^{1-j}}|_0 = [a^{2j}, b]|_{j-1} = a^{\mathbf{e}_{j-1} - \mathbf{e}_{p-j-1}},$$

since $j \notin \{1, p-1\}$. If there is such a j with $\mathbf{e}_{j-1} \neq \mathbf{e}_{p-j-1}$, let $i \in \mathbb{Z}$ be such that $i \equiv_p (\mathbf{e}_{j-1} - \mathbf{e}_{p-j-1})^{-1}$, and observe that

$$\begin{aligned}\psi([[b^a, b], [a^{2j}, b]^i]) &= [(0 : c,\ 1 : c^{-1},\ \diamond : \mathrm{id}), (0 : a,\ 1 : \mathrm{id},\ \diamond : [a^{2j}, b]^i)|_\diamond] \\ &= (0 : [c, a],\ \diamond : \mathrm{id}).\end{aligned}$$

But such an element j must always exist. Assume otherwise, for contradiction. Then for any $j \notin \{1, p-1\}$

$$\mathbf{e}_j = \mathbf{e}_{(1+j)-1} = \mathbf{e}_{p-(1+j)-1} = \mathbf{e}_{j+2}.$$

But since every element of $\mathbb{F}_p^\times$ is a multiple of 2, this implies that $\mathbf{e}$ is constant, a possibility that we have excluded.

It remains to show $(0 : [c, b],\ \diamond : \mathrm{id}) \in \psi([\mathrm{St}_G(1)', G'])$. If $\mathbf{e}_{p-2} \neq 0$, consider the element $g = [a^{2j}, b]^{i\mathbf{e}_{p-2}} [a^2, b]$, which fulfils

$$g|_0 = a^{\mathbf{e}_{p-2}} a^{-\mathbf{e}_{p-2}} b \quad \text{and} \quad g|_1 = a^{\mathbf{e}_1 - \mathbf{e}_{p-1}} = \mathrm{id},$$

using the fact that $\mathbf{e}$ is assumed to be symmetric. If otherwise $\mathbf{e}_{p-2} = 0$, set $g = [a^2, b]$, which has the sections

$$g|_0 = a^{-\mathbf{e}_{p-2}} b = b, \quad \text{and} \quad g|_1 = a^{\mathbf{e}_1 - \mathbf{e}_{p-1}} = \mathrm{id}\,.$$

In both cases

$$\psi([[b^a, b], g]) = (0 : [c, b],\ \diamond : \mathrm{id}). \qquad \square$$

Using Proposition 6.2.14, we may derive a nice corollary.

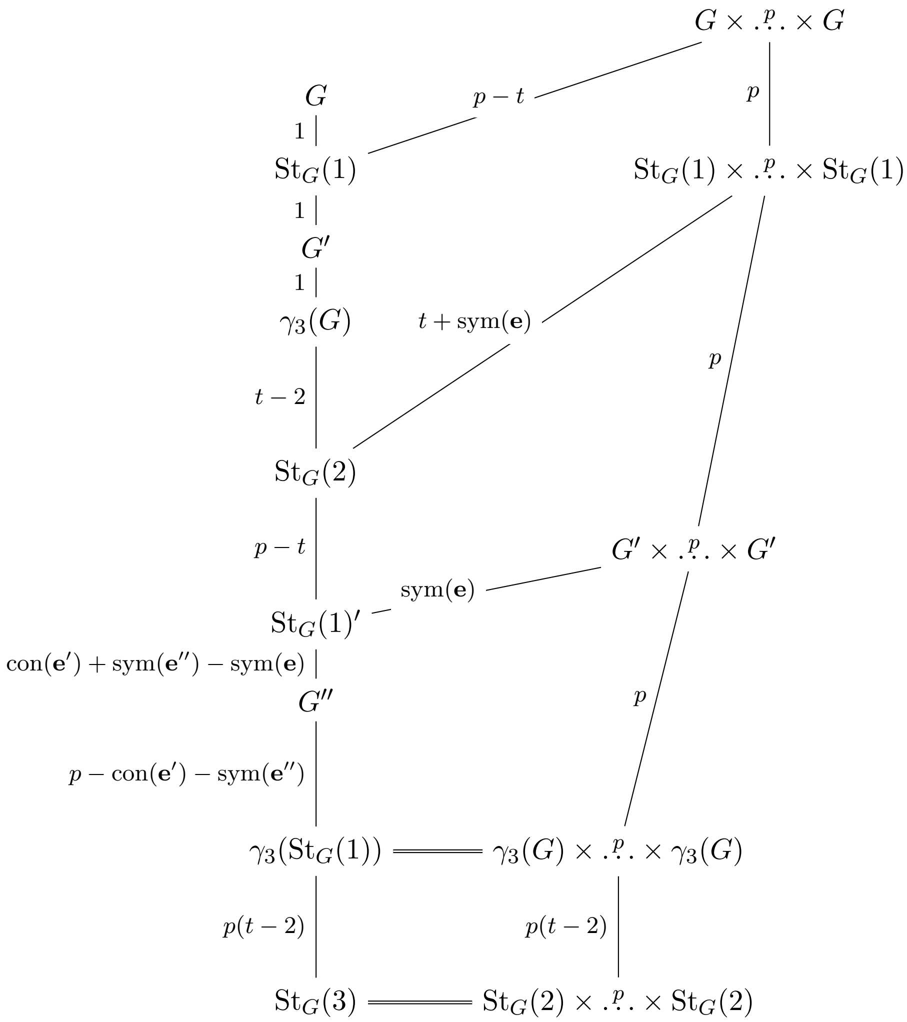

Figure 6.1: Part of the top of the subgroup lattice of a GGS-group, with some supergroups added. Passage from the left to the right side signifies the application of ψ. All indices are logarithmic.

Corollary 6.2.15. *Let G be a* GGS-*group. Then G is branch over G'', independent on the value of* $\mathrm{sym}(\mathbf{e})$.

Proof. Using Theorem 6.2.10(i) and Proposition 6.2.14 we find the inclusion

$$\begin{aligned} G'' \times .\overset{p}{.}. \times G'' \le \gamma_3(G) \times .\overset{p}{.}. \times \gamma_3(G) &= \psi(\gamma_3(\mathrm{St}(1))) \\ &= \psi([\mathrm{St}(1)', G']) \le \psi(G''). \end{aligned}$$

□

6.3 — The derived series of GGS-groups

The main difficulty for the computation of $|G : G^{(n)}|$ for $n > 1$ is the calculation of $|G : G''|$. We now begin with this, applying the theory of circulant spaces sketched in

Proposition 6.2.6. Recall that we can compute the circulant space generated by some vector $\mathbf{d} \in \mathbb{F}_p^n$ by counting the number of trailing zeros in the image of $\mathbf{d}$ under the linear map R, and that the last three components may be computed as

$$\mathbf{d}R_n = \sum_{i=0}^{n-1} d_i, \quad \mathbf{d}R_{n-1} = \sum_{i=0}^{n-1} i d_i, \quad \text{and} \quad \mathbf{d}R_{n-2} = \sum_{i=0}^{n-1} \binom{i}{2} d_i.$$

Proposition 6.3.1. *Let G be a* GGS*-group with defining tuple* $\mathbf{e}$*. Then*

$$\log_p |G : G''| = p + 1 + \mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'') - \mathrm{sym}(\mathbf{e}).$$

Proof. Since $G' \leq \mathrm{St}_G(1)$, the second derived subgroup G'' is contained in $\mathrm{St}_G(1)'$, which is in turn contained in $G' \times \overset{p}{\ldots} \times G'$ by Theorem 6.2.10. By Proposition 6.2.14 and Theorem 6.2.10 we find

$$\psi(G'') \geq \psi(\gamma_3(\mathrm{St}(1))) = \gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G),$$

so that G'' is wedged in between the subgroups $\psi^{-1}(\gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G))$ and $\psi^{-1}(G' \times \overset{p}{\ldots} \times G')$. Since the quotient $G'/\gamma_3(G)$ is cyclic of order p by Lemma 6.2.13(i) and (ii), we may identify the quotient

$$(G' \times \overset{p}{\ldots} \times G')/(\gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G)) \cong G'/\gamma_3(G) \times \overset{p}{\ldots} \times G'/\gamma_3(G)$$

with a p-dimensional $\mathbb{F}_p$-vector space V, such that $G'' \cdot (\gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G))$ represents a sub-vector space W of V. We calculate the dimension of this subspace. Knowing it, we can easily deduce the index of the second derived subgroup. Recall that $\log_p[G' \times \overset{p}{\ldots} \times G' : \mathrm{St}(1)'] = \mathrm{sym}(\mathbf{e})$ by Theorem 6.2.10, and has that $\mathrm{St}(1)'$ has p-logarithmic index $p+1$ in G by Lemma 6.2.13. Thus

$$\begin{aligned} \log_p |G : G''| &= \log_p |G : \mathrm{St}_G(1)'| + \log_p [\mathrm{St}_G(1)' : G''] \\ &= \log_p |G : \mathrm{St}_G(1)'| + \log_p |G' \times \overset{p}{\ldots} \times G' : G''| - \log_p |G' \times \overset{p}{\ldots} \times G' : \mathrm{St}_G(1)'| \\ &= p + 1 + \mathrm{codim}_V(W) - \mathrm{sym}(\mathbf{e}). \end{aligned}$$

Note that it is a consequence of [48, Lemma 3.5] (and indeed, of the calculation of $[b, b^a]$ at the beginning of 'Case 2' in the proof of Proposition 6.2.14) that the subspace U represented by $\mathrm{St}(1)'$ in V is equal to $\mathrm{Circ}_{p-1}(V)$ if $\mathbf{e}$ is symmetric. We will not use this fact here, but find it helpful to keep in mind.

To compute the dimension of W, we calculate the images of a set of generators of W in V. Since G' is generated by the elements $\{c^g \mid g \in G\}$, the second derived subgroup G'' is generated by elements of the form

$$[c^{g_1}, c^{g_2}]^{g_3} = [c, c^{g_2 g_1^{-1}}]^{g_1 g_3}$$

for some g_1, g_2 and $g_3 \in G$, i.e. a generating set is given by

$$\{[c, c^g]^h \mid g, h \in G\}.$$

Since we calculate modulo $\gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G)$, we may restrict the choices of g and h significantly. Indeed, writing $g = a^i s$ and $h = a^j t$ for some $i, j \in F_p$ and $s, t \in \mathrm{St}(1)$, we see that, for any $x \in X$ (and writing $y = x^{a^j}$ and $z = y^{a^i} = x^{a^{i+j}}$), and using the fact that the section map is a homomorphism on St(1)

$$[c, c^{a^i s}]^{a^j t}|_x = [c, c^{a^i s}]^t|_y = [c|_y, c^{a^i s}|_y]^{t|_y} = [c|_y, (c|_z)^{s|_z}]^{t|_y} \equiv_{\gamma_3(G)} [c|_y, c|_z].$$

Thus s and t play no role modulo $\gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G)$, and we may restrict to the generating set

$$\{[c, c^{a^{j-i}}]^{a^{-j}} \mid i, j \in \mathbb{F}_p\} = \{[c_i, c_j] \mid i, j \in \mathbb{F}_p\}.$$

We do also know how a acts (via the conjugation action on $\mathrm{St}(1)'$) on the vector space U. To describe this action, we choose a basis; since $G'/\gamma_3(G)$ is generated by the image $\overline{c}$ of c, we decide to consider this abelian quotient as a group written additively, i.e. $c^i \gamma_3(G) = i\overline{c}$ for any $i \in \mathbb{F}_p$. Our basis is hence the standard basis with respect to $\overline{c}$

$$((\overline{c}, 0, \ldots, 0), (0, \overline{c}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \overline{c}))$$

Now conjugation by a (on any element s of the stabiliser) cyclically permutes the entries of the tuple $\psi_1(s)$. This corresponds to the linear map cyclically permuting the basis elements, hence extends naturally to an action on the full space V. Since W is the subspace representing G'', a normal subgroup, it is invariant under this action, i.e. it is a circulant space. Notice that $[c_i, c_j]^{a^{-k}} = [c_{i+k}, c_{j+k}]$. Furthermore

$$[c, c_i]^{-1} = [c_i, c] = [c^{a^i}, c] = [c, c_{p-i}]^{a^i},$$

thus the set $\{[c, c_i] \mid i \in \{1, \ldots, (p-1)/2\}\}$ normally generates G''. Equivalently, the images under θ (the map $\mathrm{St}(1) \ni g \mapsto (\log_a(g|_0), \ldots, \log_a(g|_{p-1}))$ defined at the beginning of Section 6.2.2)

$$\mathbf{d_i} = \theta([c, c_i])$$

generate W as a cyclically invariant subspace, i.e.

$$W = \mathrm{Circ}\left(\{\mathbf{d_i} \mid i \in \{1, \ldots, (p-1)/2\}\}\right).$$

By Proposition 6.2.6, it is enough to compute the rank of $\mathrm{Circ}(\mathbf{d_i})$ for every i. So let us compute the images of $[c, c_i]$ under θ. We begin with the cases $i \in \{2, \ldots, (p-1)/2\}$, where we find

$$\psi([c,c_i]) = \begin{pmatrix} 0: & [b^{-1}a^{\mathbf{e}_{p-1}}, a^{\mathbf{e}'_{p-i}}], & 1: & [a^{-\mathbf{e}_1}b, a^{\mathbf{e}'_{p-i+1}}] \\ i: & [a^{\mathbf{e}'_i}, b^{-1}a^{\mathbf{e}_{p-1}}]), & i+1: & [a^{\mathbf{e}'_{i+1}}, a^{-\mathbf{e}_1}b] \\ \diamond: & \mathrm{id} & & \end{pmatrix}$$

$$\equiv \begin{pmatrix} 0: & [b,a]^{-\mathbf{e}'_{p-i}}, & 1: & [b,a]^{\mathbf{e}'_{p-i+1}}, \\ i: & [b,a]^{\mathbf{e}'_i}, & i+1: & [b,a]^{-\mathbf{e}'_{i+1}}, \\ \diamond: & \mathrm{id} & & \end{pmatrix} \pmod{\gamma_3(G)\times \overset{p}{\ldots} \times \gamma_3(G)}.$$

By our choice of basis for V, we find

$$\mathbf{d_i} = \theta([c,c_i]) = (0: -\mathbf{e}'_{p-i},\ 1: \mathbf{e}'_{p-i+1},\ i: \mathbf{e}'_i,\ i+1: -\mathbf{e}'_{i+1}). \qquad (*_i)$$

The last generator $[c,c_1]$ has a slightly different form,

$$\begin{aligned}\psi([c,c_1]) &= (0: [b^{-1}a^{\mathbf{e}_{p-1}}, a^{\mathbf{e}'_{p-1}}]),\ 1: [a^{-\mathbf{e}_1}b, b^{-1}a^{\mathbf{e}_{p-1}}],\ 2: [a^{\mathbf{e}'_2}, a^{-\mathbf{e}_1}b],\ \diamond: \mathrm{id}) \\ &\equiv (0: c^{-\mathbf{e}'_{p-1}},\ 1: c^{\mathbf{e}_{p-1}-\mathbf{e}_1},\ 2: c^{-\mathbf{e}'_2},\ \diamond: \mathrm{id}) \pmod{\gamma_3(G)\times \overset{p}{\ldots} \times \gamma_3(G)},\end{aligned}$$

and gives rise to the vector

$$\mathbf{d_1} = \theta([c,c_1]) = (0: -\mathbf{e}'_{p-1},\ 1: \mathbf{e}_{p-1}-\mathbf{e}_1,\ 2: -\mathbf{e}'_2,\ \diamond: 0). \qquad (*_1)$$

Using these descriptions, we are able to compute some of the circulant spaces associated to $\mathbf{d_i}$ by applying the maps R_p, R_{p-1} and R_{p-2} to $\mathbf{d_i}$. To do so, we make a case distinction.

Case 1: If the second differences vector $\mathbf{e}''$ is not symmetric, there is some index $i \in \{3,\ldots,p-1\}$ such that $\mathbf{e}''_{i+1} \neq \mathbf{e}''_{p-i+1}$. In this case, the sum of the components of the corresponding vector $\mathbf{d_i}$ is

$$\mathbf{d_i}R_p = (\mathbf{e}'_{p-i} - \mathbf{e}'_{p-i+1}) - (\mathbf{e}'_i - \mathbf{e}'_{i+1}) = \mathbf{e}''_{p-i+1} - \mathbf{e}''_{i+1} \neq 0,$$

and by Theorem 6.2.5 the rank of the circulant matrix $\mathrm{Circ}(\mathbf{d_i}) = p$, since the sum of the coefficients of $\mathbf{d_i}$ is not 0. The subspace W is equal to the full space V, and we find

$$\log_p |G:G''| = \log_p |G:G'\times \overset{p}{\ldots} \times G'|.$$

Assume that the first differences vector $\mathbf{e}'$ is constant. Then $\mathbf{e}''$ is constant 0, hence symmetric, a contradiction. Also, by Lemma 6.2.3, the asymmetricality of $\mathbf{e}''$ forces $\mathbf{e}$ to not be symmetric, hence $G'\times \overset{p}{\ldots} \times G'$ is in fact equal to $\mathrm{St}(1)'$ and a subgroup of G. We conclude that the index of the second derived subgroup fulfils

$$\log_p |G:G''| = p+1 = p+1+\underbrace{\mathrm{con}(\mathbf{e}')}_{=0}+\underbrace{\mathrm{sym}(\mathbf{e}'')}_{=0}-\underbrace{\mathrm{sym}(\mathbf{e})}_{=0}.$$

Case 2: Now assume that the defining tuple $\mathbf{e}''$ is symmetric. Using the description of the generators given in the first case, we see that $\mathrm{Circ}(\theta([c, c_i])) \subseteq \mathrm{Circ}_{p-1}(V)$. We furthermore need to consider the generator $[c, c_1]$, that played no role in the previous case. Since

$$\mathbf{d_1} R_p = -\mathbf{e}'_{p-1} + \mathbf{e}_{p-1} - \mathbf{e}_1 - \mathbf{e}'_2 = 2(\mathbf{e}_{p-1} - \mathbf{e}_1) + (\mathbf{e}_2 - \mathbf{e}_{p-2}) = 0$$

by Lemma 6.2.4, the sum of the coefficients of $[c, c_1]$ vanishes, and $\mathrm{Circ}([c, c_1])$ is contained in $\mathrm{Circ}_{p-1}(V)$ as well. Consequently, W is a subspace of $\mathrm{Circ}_{p-1}(V)$. We can go on and reduce the possible size of $\mathrm{Circ}([c, c_1])$ further. Consider the image of the cyclic shift $[c, c_1]^{a^{-1}}$ under θ,

$$\widehat{\mathbf{d_1}} = \theta([c, c_1]^{a^{-1}}) = (\mathbf{e}_{p-1} - \mathbf{e}_1, -\mathbf{e}'_1, 0, \dots, 0, -\mathbf{e}'_{p-2}).$$

Of course, this vector generates the same circulant space. But clearly

$$\widehat{\mathbf{d_1}} R_1 = 0 \cdot (\mathbf{e}_{p-1} - \mathbf{e}_1) - \mathbf{e}'_1 - (p-1)\mathbf{e}'_{p-2} = -\mathbf{e}'_1 + \mathbf{e}'_{p-2} = 0,$$

since $\mathbf{e}'$ is symmetric. Hence $\mathrm{Circ}(\mathbf{d_1}) \subseteq \mathrm{Circ}_{p-2}(V)$.

Contrary to the situation in the previous case, the symmetry of $\mathbf{e}''$ does not force $\mathbf{e}$ or $\mathbf{e}'$ to be symmetric (resp. constant). Therefore, we have to make another case distinction.

Subcase 2.1: Assume additionally that $\mathbf{e}'$ is not constant, i.e. that $\mathbf{e}''$ is not zero. We want to show that $\mathrm{Circ}(\mathbf{d_i}) = \mathrm{Circ}_{p-2}(V)$ for some $i \in \{1, \dots, (p-1)/2\}$. Notice that for all $i \in \{2, \dots, p-1\}$ we find

$$\mathbf{e}'_{i+1} - \mathbf{e}'_{p-i+1} = \mathbf{e}''_{i+1} + \mathbf{e}'_{i+2} - \mathbf{e}'_{p-i} - \mathbf{e}''_{p-i+1} = \mathbf{e}'_{i+2} - \mathbf{e}'_{p-i},$$

since $\mathbf{e}''$ is symmetric, and consequently

$$\mathbf{e}'_{i+1} - \mathbf{e}'_{p-i+1} = \mathbf{e}'_{(p-1)/2} - \mathbf{e}'_{(p+1)/2} = \mathbf{e}''_{(p+1)/2}. \tag{†}$$

Building on this calculation, we can compute

$$\begin{aligned}\mathbf{d_i} R_1 &= \binom{0}{1} \cdot -\mathbf{e}'_{p-i} + \binom{1}{1}\mathbf{e}'_{p-i+1} + \binom{i}{1}\mathbf{e}'_i - \binom{i+1}{1}\mathbf{e}'_{i+1} \\ &= \mathbf{e}'_{p-i+1} - \mathbf{e}'_{i+1} + i(\mathbf{e}'_i - \mathbf{e}'_{i+1}) \\ &= i\mathbf{e}''_{i+1} - \mathbf{e}''_{(p+1)/2}.\end{aligned}$$

This is not 0 if $\mathbf{e}''_{(p+1)/2} = 0$, since there must exist some $i \in \{3, \dots, (p-3)/2\}$ such that $\mathbf{e}''_i \neq 0$, because $\mathbf{e}''$ is by assumption symmetric and non-zero. If otherwise $\mathbf{e}''_{(p+1)/2} \neq 0$, for $i = (p-1)/2$ the equation becomes

$$\mathbf{d_i} R_{p-1} = i\mathbf{e}''_{i+1} - \mathbf{e}''_{(p+1)/2} = \frac{p-3}{2}\mathbf{e}''_{(p+1)/2} \neq 0.$$

Thus $\mathrm{Circ}_{p-1}(V) = \mathrm{Circ}(\mathbf{d_i}) \subseteq W \subseteq \mathrm{Circ}_{p-1}$, and we can verify that

$$\begin{aligned}\log_p |G : G''| &= p + 1 + \mathrm{codim}_V(W) - \mathrm{sym}(\mathbf{e}) \\ &= p + 2 - \mathrm{sym}(\mathbf{e}) \\ &= p + 1 + \underbrace{\mathrm{con}(\mathbf{e}')}_{=0} + \underbrace{\mathrm{sym}(\mathbf{e}'')}_{=1} - \mathrm{sym}(\mathbf{e}).\end{aligned}$$

Subcase 2.2: Finally, assume, in addition to $\mathbf{e}''$ being symmetric, that $\mathbf{e}'$ is constant. Write $k \in \mathbb{F}_p^\times$ for the constant value of $\mathbf{e}'$. Note that $k \neq 0$, since otherwise $\mathbf{e}'$ is zero and $\mathbf{e}$ constant.

As we have argued above, W is contained in $\mathrm{Circ}_{p-1}(V)$. In view of the formula we want to establish, we aim to prove that $W = \mathrm{Circ}_{p-2}(V)$. We first show that $\mathrm{Circ}_{p-2}(V) \subseteq W$, by computing

$$\mathbf{d_1} R_{p-2} = -\binom{0}{2}\mathbf{e}'_{p-2} + \binom{1}{2}(\mathbf{e}_{p-1} - \mathbf{e}_1) - \binom{2}{2}\mathbf{e}'_1 = -\mathbf{e}'_1 = -k \neq 0,$$

hence $\mathrm{Circ}_{p-2}(V) = \mathrm{Circ}(\mathbf{y}_1) \subseteq W$.

It remains to show that $\mathrm{Circ}(\mathbf{d_i}) \subseteq \mathrm{Circ}_{p-2}(V)$ for all $i \in \{2, \ldots, \frac{p-1}{2}\}$, i.e. to show that $\mathbf{d_i} R_1 = 0$. We calculate

$$\begin{aligned}\mathbf{d_i} R_{p-1} &= \binom{0}{1} \cdot -\mathbf{e}'_{p-i} + \binom{1}{1}\mathbf{e}'_{p-i+1} + \binom{i}{1}\mathbf{e}'_i - \binom{i+1}{1}\mathbf{e}'_{i+1} \\ &= k + ik - (i+1)k = 0\end{aligned}$$

Thus $W = \mathrm{Circ}_{p-2}(V)$ and

$$\log_p |G : G''| = p + 3 - \mathrm{sym}(\mathbf{e}) = p + 1 + \underbrace{\mathrm{con}(\mathbf{e}')}_{=1} + \underbrace{\mathrm{sym}(\mathbf{e}'')}_{=1} - \mathrm{sym}(\mathbf{e}),$$

which concludes the proof. □

In fact, our proof gives more than just the index, but allows for some structural description of G''. We record this in the following corollary.

Corollary 6.3.2. *Let G be a* GGS*-group. Write V for the vector space $G'/\gamma_3(G) \times \overset{p}{\ldots} \times G'/\gamma_3(G)$. Write π for the quotient map $G \to G/\gamma_3(\mathrm{St}(1))$. Then*

$$\psi(G'') = \left\{(g_1, \ldots, g_p) \in G' \times \overset{p}{\ldots} \times G' \mid (\pi(g_1), \ldots, \pi(g_p)) \in \mathrm{Circ}_{p-i}(V)\right\},$$

where $i = \mathrm{con}(\mathbf{e}) + \mathrm{sym}(\mathbf{e}'') - \mathrm{sym}(\mathbf{e})$.

To describe the series of iterated local laws, we record the following lemma.

Lemma 6.3.3. *The group of p^{th} powers of G' is contained in $\gamma_3(\mathrm{St}_G(1))$. In particular, it is contained in G''.*

Proof. The group G' is geometrically contained in the direct product $G \times \overset{p}{\ldots} \times G$. Now $G/\gamma_3(G)$ is isomorphic to the Heisenberg group over $\mathbb{F}_p$, hence of exponent p. Consequently $(G')^p \le \psi^{-1}(\gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G)) = \gamma_3(\mathrm{St}_G(1))$. □

Now that we have established the value of $|G : G''|$, it remains derive our main results.

Proof of Theorem 6.1.2. Assume that the given equation holds true for some $n \in \mathbb{N}$. Then we find

$$\psi(G^{(n+1)}) = \psi(G^{(n)})' = (G^{(n-1)} \times \overset{p}{\ldots} \times G^{(n-1)})' = G^{(n)} \times \overset{p}{\ldots} \times G^{(n)},$$

since $G^{(n)} \le \mathrm{St}_G(1)$. Thus by induction it is enough to consider the case $n = 3$, or the case $n = 2$, respectively.

First assume that $\mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'') - \mathrm{sym}(\mathbf{e}) = 0$. By Theorem 6.2.10 we find $\psi(\mathrm{St}_G(1)') = G' \times \overset{p}{\ldots} \times G'$, and by Proposition 6.3.1 $\log_p [G : G''] = p+1$. By Lemma 6.2.13(iii), we also have $\log_p |G : \mathrm{St}_G(1)'| = p + 1$, and since $G'' \le \mathrm{St}_G(1)'$, the subgroups G'' and $\mathrm{St}(1)' = \psi^{-1}(G' \times \overset{p}{\ldots} \times G')$ coincide. Hence the equation holds for $n = 2$.

Now we drop the assumption on the defining tuple, and prove the desired equation for $n = 3$. Since G'' is normally generated by the elements $\{[c, c^g] \mid g \in G\}$, we have to prove that $(0 : [c, c^g],\ \diamond : \mathrm{id}) \in \psi(G^{(3)})$ for any $g \in G$, then an application of Proposition 6.2.1 concludes the proof.

Let $g \in G$ be arbitrary. Since G is fractal, we may find $\widehat{g} \in \mathrm{St}(1)$ such that $\widehat{g}|_0 = g$. Furthermore, we know by Corollary 6.3.2 that we find

$$h = (0 : c,\ 1 : c^{-2},\ 2 : c,\ \diamond : \mathrm{id}) \in \psi(G'').$$

Thus

$$[h, h^{\widehat{g}a^{p-2}}] = (\diamond : [h|_\diamond, h^{\widehat{g}}|_{\diamond+2}]) = (0 : [c, c^g],\ \diamond : \mathrm{id}) \in \psi(G^{(3)}).$$ □

Proof of Theorem 6.1.1. Using Theorem 6.1.2, Proposition 6.3.1 and Theorem 6.2.10, we obtain

$$\begin{aligned}\log_p |G'' : G^{(3)}| &= \log_p |G' \times \overset{p}{\ldots} \times G' : G'' \times \overset{p}{\ldots} \times G''| - \log_p |G' \times \overset{p}{\ldots} \times G' : \psi(G'')| \\ &= \log_p(|G' : G''|^p) - \mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'') \\ &= (p-1)(p + \mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'')) - p\,\mathrm{sym}(\mathbf{e})).\end{aligned}$$

Using Theorem 6.1.2 yet again, we find for $n > 2$

$$\begin{aligned}\log_p |G^{(n-1)} : G^{(n)}| &= \log_p |G^{(n-2)} \times \overset{p}{\cdots} \times G^{(n-2)} : G^{(n-1)} \times \overset{p}{\cdots} \times G^{(n-1)}| \\ &= p \log_p |G^{(n-2)} : G^{(n-1)}|,\end{aligned}$$

and consequently

$$\begin{aligned}\log_p |G'' : G^{(n)}| &= \sum_{i=2}^{n-1} \log_p |G^{(i)} : G^{(i+1)}| \\ &= \frac{p^{n-2}-1}{p-1} \log_p |G'' : G^{(3)}| \\ &= (p^{n-2}-1)(p + \mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'')) - \frac{p^{n-1}-p}{p-1} \mathrm{sym}(\mathbf{e}),\end{aligned}$$

hence by Proposition 6.3.1

$$\log_p |G : G^{(n)}| = p^{n-2}(p + \mathrm{con}(\mathbf{e}') + \mathrm{sym}(\mathbf{e}'')) - \frac{p^{n-1}-1}{p-1} \mathrm{sym}(\mathbf{e}) + 1. \qquad \square$$

Corollary 6.3.4. *Let G be a* GGS-*group defined by a non-constant vector. Then the series of iterated local laws $(L_n(G))_{n\in\mathbb{N}}$ coincides with the derived series $(G^{(n)})_{n\in\mathbb{N}}$.*

Proof. Let G be a GGS-group. We have to show $G^{(n)} = L_n(G)$ for all $n \in \mathbb{N}$. It is enough to show that all p^{th} powers in $G^{(n)}$ are contained in $G^{(n+1)}$ for all $n \in \mathbb{N}$; for in this case all words involving a p^{th} power give rise to verbal subgroups that are contained in the $(n+1)^{\text{st}}$ derived subgroup.

By Lemma 6.2.13(iv) and Lemma 6.3.3 this holds for $n \in \{1, 2\}$. Hence we assume that $n > 2$. Then, by Theorem 6.1.2, the group $\psi(G^{(n)})$ is contained in $G^{(n-1)} \times \overset{p}{\cdots} \times G^{(n-1)}$. Using induction, we see that

$$\begin{aligned}\psi(G^{(n)})^p &= (G^{(n-1)} \times \overset{p}{\cdots} \times G^{(n-1)})^p \\ &\leq ((G^{(n-1)})^p \times \overset{p}{\cdots} \times (G^{(n-1)})^p) \\ &\leq (G^{(n)} \times \overset{p}{\cdots} \times G^{(n)}).\end{aligned} \qquad \square$$

Part II:

The Magnus property and its generalisations

Chapter 7

The Generalised Magnus property for finite and crystallographic groups

7.1 — Introduction

In 1930, Magnus published the proof of his 'Freiheitssatz', which has since been fundamental for the study of one-relator groups. A *one-relator group* G is a group given by a presentation $\langle X \mid \{r\}\rangle$, with X a set of generators and $\{r\}$ a set consisting of a single relation r, i.e. a reduced word in the free group F_X on the set X. In other words, one-relator groups are quotients of the form

$$G = F_X/\langle r\rangle^{F_X},$$

where we denote the normal closure of a subgroup $H \leq G$ in G by H^G. We notice immediately that the group G does not depend on the element r on its own, but on the normal subgroup generated by it; in particular, if r and r' are conjugate elements of F_X, the one-relator groups defined by r and r' are equal. This allows to restrict the choice of relations r to elements that are of minimal length in a given generating set. The appropriate length function is the *cyclically reduced length* of a word. An element $r \in F_X$, for any given set X, is called *cyclically reduced* if its usual word-length with respect to X is minimal among all conjugates of r. Equivalently, a word $r = x_1 \dots x_n$ given by n letters in $X^{\pm 1} = X \cup X^{-1}$ is cyclically reduced if and only if all cyclic shifts of r are reduced words, i.e. if the word

$$x_{(i+1 \bmod n)} x_{(i+2 \bmod n)} \cdots x_{(i \bmod n)}$$

is reduced for every $i \in \mathbb{Z}$. It is clear that this property is shared by all cyclic shifts. Since cyclic shifts are always conjugate to one another, by virtue of

$$(x_1 \dots x_i)^{-1} (x_1 \dots x_n) (x_1 \dots x_i) = \prod_{j=1}^{n} x_{(i+j \bmod n)},$$

there may exist (in general) more than one cyclically reduced word in any given conjugacy class. In fact, the converse of the above is true as well: all cyclically reduced words in a conjugacy class of F_X are cyclic shifts of each other. The word length of these cyclically reduced elements is, by definition, the cyclically reduced length of all elements in the conjugacy class.

We can now formulate Magnus' Freiheitssatz in the following way.

Theorem 7.1.1 (Magnus' Freiheitssatz, [107]). *Let X be a set and let $r \in F_X$ be a cyclically reduced word. Let $G = \langle X \mid \{r\} \rangle$ be the corresponding one-relator group and let $x \in X$ be a generator. If $r \notin F_{X \smallsetminus \{x\}}$, where we use the natural embedding of a free group on a subset $S \subseteq T$ into the free group on the superset T, then the subgroup of G generated by the image of $X \smallsetminus \{x\}$ in G is a free group on the image of $X \smallsetminus \{x\}$ in G.*

This theorem can be seen as an analogue of the rank–nullity theorem in linear algebra, in the sense that the addition of a single relation r to a free object reduces the rank precisely by one.

We have already noted that a one-relator-group $G = \langle X \mid \{r\} \rangle$ is determined by the normal subgroup generated by r rather than r itself. The Freiheitssatz suggests that it is helpful to choose a cyclically reduced word in the conjugacy class of r, which is unique up to cyclic permutation (in particular up to finitely many choices). It is natural to question if one may improve the choice of r further by considering another conjugacy class of F_X included in $\langle r \rangle^{F_X}$. The answer is 'no', and a direct consequence of the following theorem of Magnus.

Theorem 7.1.2 (Magnus property for free groups, [107]). *Let r and s be elements of F_X for some set X. Then the two following statements are equivalent:*

(i) $\langle r \rangle^{F_X} = \langle s \rangle^{F_X}$, *and*
(ii) *the elements r and s are conjugate or inverse conjugate in F_X, i.e. there exists $g \in F_X$ such that $r^g = s$ or $r^g = s^{-1}$.*

The proof of this theorem uses the Freiheitssatz, but both in the classical form by Magnus [107] and in more modern formulations [108, 109] it is quite technical, whence we omit it here.

We say that a group G has the *Magnus property* if it satisfies the analog of Theorem 7.1.2. (In Definition 7.2.1 we shall give a more general definition.) Motivated by Magnus' result, there have been various groups for which the Magnus property or its logical negation have been established [23, 24, 46, 47]. One of the main difficulties in establishing both positive and negative results is the fact that the class $\mathcal{M}$ of all groups with the Magnus property is badly behaved with respect to basic group-theoretic constructions. In fact:

- The class $\mathcal{M}$ is not closed under quotients. This is quite obvious, since every group is the quotient of a free group, hence, by Theorem 7.1.2 a group in $\mathcal{M}$. Thus, it suffices to construct a group that does not have the Magnus property. We shall see a plethora of examples in this chapter.

- The class $\mathcal{M}$ is not closed under subgroups. We give a basic example. Let $G = C_7 \rtimes C_3 = \langle a, t \mid a^7 = t^3 = 1, a^t = a^2 \rangle$ be the non-abelian group of order 21. This group has the Magnus property; indeed, the group has precisely three normal subgroups, the only non-trivial being $H = C_7 = \langle a \rangle$. The (non-trivial) orbits of H under the conjugation by $C_3 = \langle t \rangle$ are of size 3, hence every element generating H is conjugate either to a or a^{-1}. Clearly, the trivial element is the only element normally generating the trivial subgroup. Every other element normally generates the full group G. But these elements fall into precisely two conjugacy classes, since $t^a = a^3 t$, and consequently

$$t^G = t\langle a \rangle \quad \text{and} \quad (t^{-1})^G = t^{-1}\langle a \rangle.$$

On the other hand, the group H does not possess the Magnus property. The elements t and t^2 both generate H, but since H is an abelian group, it suffices to notice that they are neither equal nor inverse to each other.

- The argument we have just used to prove that C_7 does not possess the Magnus property can be easily extended to characterise the abelian groups within $\mathcal{M}$. Of course, in an abelian group normal subgroups are just subgroups, and being conjugate or inverse conjugate is equivalent to being equal or inverse. Thus, the Magnus property for abelian groups is equivalent to the statement that every cyclic subgroup has at most two generators. Given a cyclic group of finite order n, the number of generators is equal to the value of the Euler ϕ-function at n. There are only five integers n such that $\phi(n) \leq 2$, namely $1, 2, 3, 4$ and 6. Additionally, an infinite cyclic group also has precisely two generators. We conclude that an abelian group has the Magnus property if and only if all elements have order $1, 2, 3, 4, 6$ or ∞.

  Now let G be a group, not necessarily abelian, with the Magnus property. Then the centre of G also has the Magnus property; indeed, if $g, h \in Z(G)$, then the normal subgroups generated by g and h both in $Z(G)$ and in G are just the subgroups generated by these elements, and since g and h are central, they are equal or inverse to each other if they are conjugate or inverse conjugate in G. But the conjugation action of G on its centre is trivial, hence the two elements are either equal or inverse to another.

  Thus, we see that the class $\mathcal{M}$ is closed under taking the centre. Together with the classification of abelian groups within $\mathcal{M}$, this excludes many groups from $\mathcal{M}$.

- The class $\mathcal{M}$ is not closed under direct products. Indeed, we have seen that C_3 and C_4 are in $\mathcal{M}$, but since $C_3 \times C_4 \cong C_{12}$, their direct product is not in $\mathcal{M}$. One might think that this behaviour is restricted to torsion groups, however, the failure of $\mathcal{M}$ to be closed under products is more severe. In [94], Klopsch and Kuckuck construct an example of a direct product of two finitely generated, torsion-free, residually finite groups with the Magnus property that itself does not possess the Magnus property.

> However, they also provide conditions when direct products are again in $\mathcal{M}$. For example, the class of residually finite-p groups with the Magnus property, for an odd prime p, is closed under direct products. This shows that, in particular, direct products of free groups possess the Magnus property, cf. [47] for another proof of this fact.

However, the full class $\mathcal{M}$ does also possess certain closure properties, which are model-theoretic in nature. To make this precise, we have to introduce some terminology. A *first-order sentence (in the language of groups)* is an expression built from the symbols $\vee, \wedge, \neg, \implies, \forall, \exists, \doteq$, the brackets (and), $\circ$ (signifying the group multiplication), $^{-1}$ (signifying the inverse element), 1 (signifying the neutral element), and an arbitrary finite set of variable symbols such that the expression parses as a validly formulated mathematical statement. For a precise definition, see for instance [129]. Let G be a group and φ a first-order sentence. We say that G models φ, symbolically $G \models \varphi$, if the mathematical statement of φ is true in G. For example, if $\varphi = \forall g \forall h\ gh \doteq hg$, then every abelian group models φ.

We say that two groups G and H are *elementarily equivalent* if they have the same first-order theory, i.e. if for every first-order-sentence φ

$$G \models \varphi \quad \text{if and only if} \quad H \models \varphi.$$

We can express 'having the Magnus property' as a collection of (infinitely many) first-order sentences. Since first-order logic cannot directly express subsets, we have to work around the fact that the Magnus property concerns the subgroup normally generated by a single element. This is done in the following way. Given some element g of a group G, every element of $\langle g \rangle^G$ can be expressed as

$$(g^{\varepsilon_1})^{k_1} \dots (g^{\varepsilon_n})^{k_n} \tag{†}$$

for some $n \in \mathbb{N}$, and $\varepsilon_i \in \{-1, 1\}$ and $k_i \in G$ for $i \in \{1, \dots, n\}$. Thus, for every pair of non-negative integers $n, m \in \mathbb{N}$ and every pair of maps $\varepsilon \in \{-1,1\}^{\{1,\dots,n\}}, \delta \in \{-1,1\}^{\{1,\dots,m\}}$ we define a first-order sentence

$$\begin{aligned} \mu_{n,m,\varepsilon,\delta} := \forall g \quad \forall h \quad & \forall k_1 \dots \forall k_n \quad \forall \ell_1 \dots \forall \ell_m \\ & \left(g \doteq k_1^{-1} h^{\varepsilon(1)} k_1 \dots k_n^{-1} h^{\varepsilon(n)} k_n \ \wedge \ h \doteq \ell_1^{-1} g^{\delta(1)} \ell_1 \dots \ell_m^{-1} g^{\delta(m)} \ell_m \right) \\ & \implies \exists t \quad \left(g \doteq h^t \vee g^{-1} \doteq h^t \right). \end{aligned}$$

Now the collection of all such sentences

$$M = \{\mu_{n,m,\varepsilon,\delta} \mid n, m \in \mathbb{N}, \varepsilon \in \{-1,1\}^{\{1,\dots,n\}}, \delta \in \{-1,1\}^{\{1,\dots,m\}}\}$$

describes the Magnus property. Indeed, if $G \models \mu$ for all $\mu \in M$, then every pair of elements g and $h \in G$ generating the same normal subgroup does this according to some pair of expressions of the form (†). Thus, the statement left of the $\implies$-symbol in the first-order sentence $\mu_{n,m,\varepsilon,\delta}$ corresponding to this pair of expressions is true, and since the sentence $\mu_{n,m,\varepsilon,\delta}$ is true in G, the elements g and h are conjugate or inverse conjugate. We conclude that the class $\mathcal{M}$ is closed under elementary equivalence.

This has immediate consequences. We have seen that free groups have the Magnus property, and the groups that are elementary equivalent to a (non-abelian) free group have been identified by Kharlampovich and Myasnikov [91], and by Sela [136] independently, solving the so-called Tarski problem (posed in [145]) of describing these groups. The groups elementary equivalent to free groups are certain hyperbolic groups, including for example all fundamental groups of compact orientable surfaces of genus $g \geq 2$ and all fundamental groups of compact non-orientable surfaces of genus $g \geq 4$. There exist alternative, algebraic proofs for the fact that a surface group of this kind has the Magnus property, cf. [23].

The class $\mathcal{M}$ enjoys another nice property: all countable torsion-free groups have a supergroup in $\mathcal{M}$. This is a consequence of the following classical result of Higman, Neumann and Neumann [82]: every countable torsion-free group can be embedded into a group with precisely two conjugacy classes. By a theorem of Osin [117], this supergroup with the same property can be chosen such that it is even finitely generated. In any group with precisely two conjugacy classes, all non-trivial elements are conjugate, whence the group possesses the Magnus property.

We quickly summarise some further results on the Magnus property.

- In contrast to the abstract case, free pro-p groups do not possess the Magnus property; this was first noticed by Gildenhuys [57].
- A group G is called *locally indicable* if every finitely generated subgroup $H \leq G$ is either trivial or possesses the infinite cyclic group C_∞ as a quotient. This property appears naturally in the study of systems of equations over a group, cf. [85], or in the theory of free products with amalgamation, cf. [86]. Building on a version of the Freiheitssatz established by Howie in [85], Edjvet proved in [42] that the Magnus property is preserved under free products within the class of locally indicable groups.
- By a result of Feldkamp [46], the results on fundamental groups of closed surfaces mentioned above can be extended to all compact closed surfaces, irregardless of the genus of the surface, using a criterion for certain one-relator groups to have the Magnus property.
- In contrast, Bogopolski [23] constructs countably many non-isomorphic torsion-free hyperbolic one-relator groups without the Magnus property.

Note that most of these results concern 'big' – e.g. free, hyperbolic, one-relators – groups. Our aim is to make some explorations in different directions, and consider the Magnus property for 'small' groups: be they finite, nilpotent, soluble, or virtually free abelian. It is not futile to search for the Magnus property among such groups; recall that the non-abelian finite group $C_7 \rtimes C_3$ is in $\mathcal{M}$, as we have seen above. We consider another example.

Example 7.1.3. The infinite dihedral group $D_\infty = \langle a, b \mid a^b = a^{-1}, b^2 = 1\rangle$ has the Magnus property.

Indeed, all elements of D_∞ may be written in the form $a^n b^\epsilon$, for some $n \in \mathbb{Z}$ and $\epsilon \in \{0, 1\}$. The conjugacy class of an element of the form a^n (such that $\epsilon = 0$) has cardinality 2, since a centralises a^n, and $(a^n)^b = a^{-n}$. Thus, the normal subgroup generated by a^n is just $\langle a^n\rangle$, an infinite cyclic group with two generators that are inverse to each other. Consider an element of the form $a^n b$. Its conjugacy class is infinite, since $(a^n b)^{a^m} = a^{n-2m}b$ for all $m \in \mathbb{Z}$. We find that the conjugacy class equals $\{a^{n+2m}b \mid m \in \mathbb{Z}\}$. If n is an even number, the element normally generates the subgroup $\langle a^2, b\rangle$, otherwise, it generates the distinct subgroup $\langle a^2, ab\rangle$. Therefore, all elements in different conjugacy classes generate different normal subgroups, and D_∞ is a member of $\mathcal{M}$. Indeed, we did not need the 'inverse-conjugate'-part of the Magnus property, and proved that D_∞ has what we will later call the 'strong Magnus property'.

For the remainder of this chapter, we proceed as follows. As a first step, we will revisit the definition of the Magnus property to generalise it. We identify four main variants, and provide some examples for the *characteristic Magnus property*. This is the content of Section 7.2. In Section 7.3, we consider finite groups with different Magnus properties, and prove the following two theorems, cf. Theorem 7.3.5 and Theorem 7.3.7,

Theorem. *Let G be a finite group with the 'strong Magnus property'. Then G is a $\{2, 3\}$-group.*

Theorem. *Let $G \in \mathcal{M}$ be a finite group. Then G is a $\{2, 3, 5, 7\}$-group.*

Afterwards, we explore the relation of the Magnus property to group rings and their units in Section 7.4 and Section 7.5. As a part of this investigation, we prove a variant of Higman's theorem on group rings with only trivial units [83], see Theorem 7.5.10 for the precise statement. Finally, in Section 7.6 we apply these results to prove severe restrictions for the structure of crystallographic groups with the Magnus property. A crystallographic group is a finite extension of a free abelian group of finite rank by some group Γ acting faithfully on the free abelian group, e.g. the infinite dihedral group. We prove the following theorem, cf. Theorem 7.6.1,

Theorem. *Let G be a crystallographic group with point group Γ. If G has the Magnus property, then Γ is isomorphic to a group in the following list,*

(i) *direct products $C_2^n \times C_3^m$ or $C_2^n \times C_4^m$ for n and $m \in \mathbb{N}_0$,*

(ii) *a subdirect product* Δ *of copies of* C_4 *and* Q_8 *such that* Δ *is a group of exponent* 4 *with all involutions central,*
(iii) *a subdirect product* Δ *of copies of* $\mathrm{Dic}_{4\cdot 3}$ *such that* Δ *is a group with all elements of order at most* 6 *and with all involutions central,*
(iv) *a subdirect product* Δ *of copies of* $\mathrm{SL}(2,3)$ *such that* Δ *is a group with all elements of order at most* 6 *and with all involutions central.*

In particular, Γ *is a* $\{2,3\}$*-group. If* $|\Gamma|$ *is odd, it is an elementary abelian* 3*-group.*

7.2 — The generalised Magnus properties

We review the definition of the Magnus property, trying to obtain a more general perspective. It is our aim to produce a more general concept, that will allow us both to

- formulate weaker and stronger forms of the Magnus property,
- introduce a relative version of the Magnus property, that we shall employ in Section 7.6 to achieve restrictions on the possible holonomy groups of crystallographic groups with the (usual) Magnus property.

We want to replace the concepts 'normal subgroup' and '(inverse-)conjugate' with something more flexible. To do so, we make a preliminary definition, in spirit of the theory of operator groups used e.g. for a unified treatment of normal and characteristic series of a group, cf. [131]. A *pseudo-operator group* is a pair (G,Ω) consisting of a group G and a monoid Ω with respect to concatenation, called the *operator domain*, consisting of mappings of G. We require the identity mapping id to be an element in Ω. Often, we will restrict to the case where Ω is a group, furthermore, the set Ω will often be a set of semi-endomorphisms, where a *semi-endomorphism of a group* G is a self-map $\omega\colon G\to G$ such that for all g and $h\in G$ we have $(ghg)^\omega = g^\omega h^\omega g^\omega$. Clearly all endomorphisms and anti-endomorphisms (i.e. homomorphisms from G into its opposite group G^{op}) are semi-endomorphisms.

Just as for operator groups, we say that a subgroup $H\leq G$ is Ω-invariant if $H^\omega\subseteq H$ for all $\omega\in\Omega$, and we define $\langle X\rangle^\Omega$ to be the minimal Ω-invariant subgroup of G containing X, for any subset $X\subseteq G$. Using this, we give our definition.

Definition 7.2.1 (Generalised Magnus properties)**.** Let (G,Ω) be a pseudo-operator group. We say G has the *Magnus property with respect to* Ω, written $(\mathrm{MP})_\Omega$, if for every pair of elements g and $h\in G$ the following two statements are equivalent:

(i) The minimal Ω-invariant subgroups $\langle g\rangle^\Omega$ and $\langle h\rangle^\Omega$ are equal.
(ii) There is an element $\omega\in\Omega$ such that $g^\omega = h$.

Using the terminology of the above definition, we say that g and h are Ω*-conjugate* if there is $\omega\in\Omega$ such that $g^\omega=h$. Although the two statements that are now required to be equivalent are more symmetric than before (since we do not allow for being 'inverse Ω-conjugate'), we can recover the Magnus property in the traditional sense (as discussed in the previous section). Write $\mathrm{Inn}(G)$ for the group of inner automorphisms of a group G, and $\mathrm{AInn}(G)$ for the group of inner and anti-automorphisms $\langle\, \mathrm{Inn}(G) \cup \{x \mapsto x^{-1}\}\rangle$. The Magnus property with respect to $\mathrm{AInn}(G)$ is precisely the traditional Magnus property, since every subgroup of G is $\mathrm{AInn}(G)$-invariant if and only if it is normal, and two elements g and h are $\mathrm{AInn}(G)$-conjugate if and only if they are conjugate or inverse-conjugate. In this specific case, we drop the subscript $\mathrm{AInn}(G)$ and simply write (MP).

Definition 7.2.1 is quite flexible, and invites the consideration of various new kinds of Magnus properties. We first explore the limits of the definition, to return to more familiar grounds afterwards.

7.2.1. Extreme cases. —

Trivial operator domain

A group G has the property $(\mathrm{MP})_{\{\mathrm{id}\}}$ if all subgroups $H \leq G$ have a unique generator. This is only possible if every element has order 1 or 2, hence if G is an elementary abelian 2-group. Conversely, these are in fact $(\mathrm{MP})_{\{\mathrm{id}\}}$ groups.

Operator domain $\Omega = \langle x \mapsto x^{-1}\rangle$

This coincides with the usual Magnus property for abelian groups. For any group, all subgroups are Ω-invariant, and every Ω-conjugacy class has at most two elements, hence the argument we have used above to restrict the possible orders for element of abelian groups with the usual Magnus property applies also to arbitrary groups with (the respective property) $(\mathrm{MP})_\Omega$. Thus, a group with $(\mathrm{MP})_\Omega$ has elements only of order $1, 2, 3, 4$ or 6. If such a group is finitely generated, it is finite, by the solution of the Burnside problem for exponents $2, 3, 4$ and 6, see [27, 79, 134]. On the contrary, all groups with the given restriction on the element orders possesses $(\mathrm{MP})_\Omega$.

Operator domain $\Omega = \{x \mapsto x^n \mid n \in \mathbb{Z}\}$

Again, all subgroups are Ω-invariant. Since all elements of a given cyclic subgroup are Ω-conjugate, all groups possess $(\mathrm{MP})_\Omega$.

7.2.2. Reasonable cases. —

The strong version, $\Omega = \mathrm{Inn}(G)$

In this case we speak about the *strong Magnus property.* The class of groups with $(\mathrm{MP})_\Omega$ is much better behaved than $\mathcal{M}$. It is, for example, closed under direct products: Let G and H have $(\mathrm{MP})_\Omega$ (with the respective Ω for each group). Let (g, h) and (g', h') be two elements of the direct product $G \times H$ such that $\langle (g,h) \rangle^{G\times H} = \langle (g',h') \rangle^{G\times H}$. This implies $\langle g \rangle^G = \langle g' \rangle^G$ and $\langle h \rangle^H = \langle h' \rangle^H$, by looking at the projections of the normal subgroup generated by either (g,h) or (g',h'). Now since G and H have $(\mathrm{MP})_\Omega$, there exist $c_g \in G$ and $c_h \in H$ such that $g^{c_g} = g'$ and $h^{c_h} = h'$. But then, clearly, $(g,h)^{(c_g,c_h)} = (g',h')$.

The weak version, $\Omega = \langle\, \mathrm{Inn}(G) \cup \{x \mapsto x^n \mid n \in \mathbb{Z}\}\,\rangle_{\mathrm{Monoid}}$

Here we speak about the *weak Magnus property.* In a sense, this is the most symmetric condition: The Ω-invariant subgroups are precisely the normal subgroups, i.e. the subsets closed under conjugation, inversion, and powers. We have, in comparison with the usual Magnus property, enlarged Ω such that we are allowed to use all these operations once to Ω-conjugate generators, while before we were missing the power-operation.

The weak Magnus property does not exhibit the restriction to certain kinds of allowed orders. Now, clearly, all abelian groups have $(\mathrm{MP})_\Omega$. In general, the problems arising from torsion elements are eliminated if one replaces the Magnus property with this weak version.

The characteristic version, $\Omega = \langle \mathrm{Aut}(G) \cup \{x \mapsto x^{-1}\}\rangle$

In this straight-forward variation, we consider characteristic subgroups and demand that generators are automorphic or inverse-automorphic. Note that the group Ω is sometimes called $\mathrm{AAut}(G)$, the group of automorphisms and anti-automorphisms, i.e. bijections $\alpha : G \to G$ with $(gh)^\alpha = h^\alpha g^\alpha$. It is easy to see that $\mathrm{AAut}(G) = \mathrm{Aut}(G) \times \langle x \mapsto x^{-1}\rangle$.

We note that groups with $(\mathrm{MP})_{\mathrm{AAut}(G)}$ actually exist. For example, let K be any field. Then the automorphism group of the additive group contains the multiplicative group $K^\times$, which acts transitively on $K \setminus \{0\}$. Consequently $(K,+)$ has $(\mathrm{MP})_{\mathrm{AAut}}$. We shall see a generalisation of this statement in Proposition 7.2.3.

In the next subsection, we shall consider the characteristic Magnus property in more detail.

7.2.3. The characteristic Magnus property.

— The following result shows that the classical result of Magnus on free groups does not generalise to the characteristic case.

Proposition 7.2.2. *The free group F_X on an finite set X of cardinality $|X| > 1$ does not have* $(\mathrm{MP})_{\mathrm{AAut}}$.

Proof. Let $x, y \in X$ be two distinct generators. Let $H = \langle x, y\rangle$ be the free subgroup on these generators. By a lemma of Neumann [113, Lemma 3.1], the elements x^3 and x^3y^3 have the same characteristic closure in H; hence they also have the same characteristic closure in F. Thus, it is enough to prove that x^3 and x^3y^3 are not automorphic to another in F. The general problem to decide whether two elements of a free group are automorphic to another can be solved algorithmically, using an algorithm by Whitehead [152], in the form of a certain finite generating set W of $\mathrm{Aut}(F)$. Applying the automorphisms in

W to an element $u \in F$, either the cyclically reduced length decreases under at least one automorphism, or u is of minimal cyclically reduced length within its automorphic class. If two elements u and $v \in F$ have minimal cyclically reduced length, they can be transformed into each other using automorphisms from W that do not increase the cyclically reduced length (i.e. every intermediate element has the same length), or they are not automorphic. In particular, if the minimal cyclically reduced length of u and v is different, they are not automorphic.

Using the description of Whitehead's algorithm in [104], every automorphism $w \in W$ for F is given by one of the two following kinds:

(i) the automorphism w permutes the symmetrises generating set $X \cup X^{-1}$,
(ii) the automorphism w maps each generator z to either $z, za, a^{-1}z$ or z^a, for some fixed $a \in X \cup X^{-1}$, such that w is indeed an automorphism. Note that, since w is an automorphism, this forces $a^w = a$.

Clearly no automorphism in W can reduce the length of x^3. Also none of them reduces the length of x^3y^3: The automorphisms induced by permutations clearly fix the length of this word, so we consider an automorphism of the second kind. If $a \notin \{x, y, x^{-1}, y^{-1}\}$, the only possible cancellations involve the generator a, hence the length cannot decrease. Thus, by symmetry, we may assume $a \in \{x, x^{-1}\}$. The resulting automorphisms fix x and map y to $yx, xy, x^{-1}y, yx^{-1}, y^x$ or $y^{(x^{-1})}$, respectively. If only x (and not its inverse) appears in the image of y, the image of x^3y^3 is clearly of greater length. Thus, it remains to check

$$\begin{aligned}
&x^3(x^{-1}y)^3 = x^2(yx^{-1})^2y && \text{of cyclically reduced length 7,}\\
&x^3(yx^{-1})^3 && \text{of cyclically reduced length 7,}\\
&x^3(y^x)^3 = x^3(y^3)^x && \text{of cyclically reduced length 6,}\\
&x^3(y^{(x^{-1})})^3 = x^3(y^3)^{x^{-1}} && \text{of cyclically reduced length 6.}
\end{aligned}$$

As a consequence, the elements x^3 and x^3y^3 are not automorphic. By symmetry, also x^3 and $y^{-3}x^{-3}$ are not automorphic, hence F does not possess the characteristic Magnus property. □

With the failure of the characteristic Magnus property for non-abelian free groups established, it is natural to try to expand the example of the additive group of a field that we have considered above.

Proposition 7.2.3. *Let R be an Euclidean domain and let M be a free R-module of finite rank. Then M does possess the equivalent of the characteristic Magnus property for R-modules, i.e. for all $\mathbf{v}$ and $\mathbf{w} \in M$ such that the $\mathrm{GL}(M)$-invariant R-submodules generated by $\mathbf{v}$ and $\mathbf{w}$ coincide, there exists a linear map $A \in \mathrm{GL}(M)$ such that $\mathbf{v}^A = \mathbf{w}$. In particular, free abelian groups of finite rank possess $(\mathrm{MP})_{\mathrm{AAut}}$ and $(\mathrm{MP})_{\mathrm{Aut}}$.*

Proof. The second assertion follows directly from the first, due to the fact that the automorphism group of a free d-dimensional $\mathbb{Z}$-module is the general linear group $\mathrm{GL}_d(\mathbb{Z})$, or $\mathrm{GL}_d(K)$, respectively. Since every module is abelian, anti-automorphisms coincide, and so do $(\mathrm{MP})_{\mathrm{Aut}}$ and $(\mathrm{MP})_{\mathrm{AAut}}$.

Let now R and M be as described. For simplicity, we choose a basis and write $M = R^d$. The orbits of $\mathrm{GL}_d(R)$ on R^d are in correspondence with the associativity classes of the elements of R. Indeed, two elements $\mathbf{v} = (v_1, \dots, v_d)$ and $\mathbf{w} = (w_1, \dots, w_d) \in R^d$ are automorphic if and only if the greatest common divisors of the entries coincide up to a unit, i.e.

$$\gcd(v_1, \dots, v_d) = \gcd(w_1, \dots, w_d).$$

This is just the 1-by-d-dimensional case of the Smith normal form for Euclidean domains. Consequently, all characteristic R-submodules are of the form rR^d for some element $r \in R$. In particular, for every $\mathbf{v} = (v_1, \dots, v_d) \in R^d$, we find $\langle \mathbf{v} \rangle^{\mathrm{GL}_d(R)} = \gcd(v_1, \dots, v_d)R^d$. But by the statement above, all elements with the same greatest common divisor of their entries are $\mathrm{GL}_d(R)$-conjugate. Consequently, R^d has $(\mathrm{MP})_{\mathrm{GL}_d(R)}$. □

In Chapter 8, torsion-free nilpotent groups with the Magnus property are considered. Theorem 8.1.1 states that all torsion-free nilpotent groups of class 2 possesses the Magnus property. The equivalent statement is not true for the characteristic Magnus property. We consider the free nilpotent group of rank 2 and class 2, which is isomorphic to the discrete Heisenberg group $\mathbf{H}_3(\mathbb{Z})$ of upper-triangular integer matrices with all diagonal entries equal to 1, i.e.

$$\mathbf{H}_3(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix} \right) \,\middle|\, a, b, c \in \mathbb{Z} \right\}.$$

For better readability, we write (a, b, c) for matrices of this form, such that the multiplication takes the form

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab').$$

The centre and the derived subgroup of $\mathbf{H}_3(\mathbb{Z})$ coincide and are equal to the set of elements $\{(0, 0, c) \mid c \in \mathbb{Z}\}$.

Proposition 7.2.4. *The group* $\mathbf{H}_3(\mathbb{Z})$ *does not have* $(\mathrm{MP})_{\mathrm{AAut}}$.

Proof. We need a description of the automorphism group of $\mathbf{H}_3(\mathbb{Z})$. By a classic result of Andreadakis [5], the quotient map $F_{x,y} \to F_{x,y}/\gamma_3(F_{x,y}) \cong \mathbf{H}_3(\mathbb{Z})$, for the free group $F_{x,y}$ on x, y, induces a surjective homomorphism

$$\mathrm{Aut}(F_{x,y}) \to \mathrm{Aut}(\mathbf{H}_3(\mathbb{Z})).$$

Since $\mathrm{Aut}(F_{x,y})$ is generated by the three automorphisms (the 'Nielsen moves') with the following action on the generators,

$$\left\{\begin{array}{ll} x & \mapsto y \\ y & \mapsto x \end{array}\right\}, \quad \left\{\begin{array}{ll} x & \mapsto x^{-1} \\ y & \mapsto y \end{array}\right\}, \quad \left\{\begin{array}{ll} x & \mapsto xy \\ y & \mapsto y \end{array}\right\},$$

the automorphism group of the Heisenberg group is generated by the following automorphisms,

$$\begin{aligned} (a,b,c) &\mapsto (b,a,-c-ab), \\ (a,b,c) &\mapsto (-a,b,-c), \\ (a,b,c) &\mapsto (a,a+b,c+a(a+1)/2), \end{aligned}$$

where we have identified the image of x with $(1,0,0)$ and the image of y with $(0,1,0)$. Abusing notation, we denote these three automorphisms by their induced automorphism on $\mathbf{H}_3(\mathbb{Z})/\mathbf{H}_3(\mathbb{Z})'$, i.e. by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

respectively.

Let $H \leq \mathbf{H}_3(\mathbb{Z})$ be a characteristic subgroup. Then, since the map $\mathrm{Aut}(F_{\{x,y\}}) \to \mathrm{GL}_2(\mathbb{Z})$ is surjective, the quotient $H/H \cap \mathrm{Z}(\mathbf{H}_3(\mathbb{Z}))$ is a characteristic subgroup in $\mathbb{Z}^2$, hence of the form $n\,\mathbb{Z}^2$ for some $n \in \mathbb{Z}$, cf. Proposition 7.2.3. Given an element $(a,b,c) \in \mathbf{H}_3(\mathbb{Z})$, the respective n is the greatest common divisor of a and b. Since $\mathrm{Aut}(\mathbf{H}_3(\mathbb{Z}))$ acts on the first two entries of an element as $\mathrm{GL}_2(\mathbb{Z})$, there exists some $x \in \mathbb{Z}$ such that $(n,0,x)$ is in the characteristic closure U of (a,b,c). The computation

$$[(n,0,x),(0,1,0)] = [(n,0,0),(0,1,0)] = (0,0,n)$$

shows that we may change the last entry of an element of U by any $m \in n\,\mathbb{Z}$ without leaving U. On the other hand, a quick examination of the automorphisms given above shows that if (a,b,c) and (a',b',c') are automorphic, then

$$c \equiv_n c' \quad \text{or} \quad c \equiv_n -c'. \tag{$*$}$$

Consider the element $(5,0,c) \in \mathbb{H}_3(\mathbb{Z})$ for $c \in \mathbb{Z}$, and compute first

$$(5,0,c)^{-1} = (-5,0,-c)$$

and, building on that,

$$\begin{aligned}\Big((-5,0,-c)^{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}(5,0,c)\Big)^{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}(0,0,-10) &= ((0,-5,c)(5,0,c))^{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}(0,0,-10)\\ &= (5,-5,2c)^{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}(0,0,-10)\\ &= (5,0,2c+10)(0,0,-10) = (5,0,2c) \in U.\end{aligned}$$

We conclude that $(5,0,2)$ is an element of the characteristic closure of $(5,0,1)$, that $(5,0,4)$ is in the closure of $(5,0,2)$, &c. Since 2 generates the multiplicative group $\mathbb{F}_5^\times$, all these closures coincide, hence $(5,0,1)$ and $(5,0,2)$ have the same characteristic closure in $\mathbf{H}_3(\mathbb{Z})$. But by $(*)$, neither these elements nor their inverses are automorphic. Thus, $\mathbf{H}_3(\mathbb{Z})$ does not possess the characteristic Magnus property. $\square$

7·3 — Finite groups with the (strong) Magnus property

Most groups for which the Magnus property has been established are 'big' groups, e.g. they are free, hyperbolic, have only two conjugacy classes &c., which is a natural consequence of the genesis of the Magnus property as a property of free groups. We take a different approach and consider 'small' groups with and without the Magnus property, for different interpretations of the word 'small'. It turns out that the intersection of $\mathcal{M}$ (and related classes $\mathcal{M}_\Omega$ of groups satisfying some generalised Magnus property $(\mathrm{MP})_\Omega$) with certain classes of groups is much better behaved. We make this precise in the following proposition (which is a version of Proposition 8.2.4).

Proposition 7.3.1. *Let (G,Ω) be a pseudo-operator group such that the operator domain Ω is a group and $\mathrm{Inn}(G) \le \Omega \le \mathrm{AAut}(G)$. Let $N \le G$ be a Ω-invariant subgroup, and let $\Omega_{G/N} = \{\omega N \mid \omega \in \Omega_G\}$ with $(gN)^{\omega N} = g^\omega N$. Assume that, for all $g \in G \smallsetminus N$, the set*

$$\{\langle gn\rangle^\Omega \mid n \in N\}$$

has the minimal condition. If G has $(\mathrm{MP})_\Omega$, then G/N has $(\mathrm{MP})_{\Omega_{G/N}}$.

Proof. First note that the operators ωN are well-defined since $\omega \in \mathrm{AAut}(G)$.

Let $g,h \in G$ be such that $\langle gN\rangle^{\Omega_N} = \langle hN\rangle^{\Omega_{G/N}}$. If $g \in N$, this directly implies $h \in N$, hence $gN = hN$. Therefore we may assume that $g \notin N$. Let $v_1,\dots,v_k,w_1,\dots,w_\ell \in \Omega_{G/N}$ such that

$$\prod_{i=1}^{k}(gN)^{v_i} = hN \quad \text{and} \quad \prod_{j=1}^{\ell}(hN)^{w_j} = gN.$$

Now by our first assumption, for all $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,\ell\}$ for all collections of $\widehat{v_i} \in \Omega$ and $\widehat{w_j} \in \Omega$ we find that $(gN)^{v_i} = g^{\widehat{v_i}}N$ and $(gN)^{w_j} = g^{\widehat{w_j}}N$. Thus, we may rewrite the two equations above to obtain

$$\prod_{i=1}^{k} g^{\widehat{v_i}} N) = hN \quad \text{and} \quad \prod_{i=j}^{\ell} h^{\widehat{w_j}} N = gN.$$

Using our assumption relating to the minimal condition, choose $n_{\min} \in N$ such that $\langle gn_{\min}\rangle^{\Omega}$ is minimal among $\{\langle gn\rangle^{\Omega} \mid n \in N\}$. Consider

$$h' := \prod_{i=1}^{k} (gn_{\min})^{\widehat{v_i}} \quad \text{and} \quad g' := \prod_{j=1}^{\ell} (h')^{\widehat{w_j}}.$$

Since N is Ω-invariant and $\mathrm{Inn}(G) \leq \Omega$, we find $h' \in hN$ and $g' \in gN$. By definition, we have $\langle g'\rangle^{\Omega} \leq \langle h'\rangle^{\Omega} \leq \langle gn_{\min}\rangle^{\Omega}$, hence by minimality we have $\langle g'\rangle^{\Omega} = \langle h'\rangle^{\Omega}$. Now since G does possess $(\mathrm{MP})_{\Omega}$, there exists $v \in \Omega$ such that $g' = (h')^{v}$. By our first assumption, see we see that

$$(hN)^{v^q} = (h'N)^{v^q} = (h')^{v} N = g'N = gN. \qquad \square$$

The conditions on Ω, G, and N seem confusingly complex, but it turns out that they are satisfied in the main variants of the Magnus property.

- For the usual Magnus property, every pair of a group G and a normal subgroup N fulfil the conditions on the assignment, where $q : \mathrm{AInn}(G) \to \mathrm{AInn}(G/N)$ turns out to be the natural map on the inner automorphisms, and assigns the inversion map of G to the inversion map of G/N. The minimal condition is, of course, not always given. However, if one restricts to the class of groups satisfying the minimal condition on normal subgroups, or, even further, to the class of finite groups, we find that (MP) is inherited by quotients.
- The situation is the same for the strong Magnus property. We point out that the strong Magnus property $(\mathrm{MP})_{\mathrm{Inn}}$ defines a relatively nice class of finite groups: it is closed under direct products and quotients, but not under (normal) subgroups; e.g. the dihedral group of order 12 has the strong Magnus property, while its normal subgroup C_6 does not.
- It is not true in general that a surjection $G \to G/N$ from a group G to a quotient of G by a characteristic subgroup N induces a surjection $\mathrm{Aut}(G) \to \mathrm{Aut}(G/N)$, even though it induces a homomorphism. Thus, even for finite groups the class $\mathcal{M}_{\mathrm{Aut}}$ is not closed under quotients; consider for example the group C_6^2, which has the characteristic Magnus property due to an argument similar to Proposition 7.2.3, while its quotient $\mathrm{C}_2 \times \mathrm{C}_6$ does not have it.

7.3.1. Finite groups, rational groups, and the strong Magnus property. — We restrict ourselves to finite groups for now. Perhaps surprisingly, the strong and the usual Magnus property imply certain representation-theoretic properties. It is a classical result that the values of a character $\chi\colon G \to \mathbb{C}$ of a group, i.e. the composition of a (finite-dimensional) representation $G \to \mathrm{GL}(V)$ with the trace map, are algebraic integers, i.e. roots of some monic integer polynomial. Naturally, one wonders if there are situations in which this restriction is even more tight. A finite group G is called *rational*, if all character values are rational numbers. Since all rational algebraic integers are in fact usual integers, all character values of a rational group are integers.

This character-theoretic statement can be translated into a purely group-theoretical one. Indeed, a finite group G is rational if and only if, for all $g \in G$, all generators of $\langle g \rangle$ are conjugate in G. This is a classic result, for a proof see [87, Satz 13.7]. Now if a group G has the strong Magnus property, it must necessarily be rational, since all elements generating the same subgroup clearly also generate the same normal subgroup and hence are conjugate.

The class of rational groups has been the subject of study for long. We mention some results regarding their classification.

- (Feit and Seitz, [45]) Let G be a finite rational group. Then there are precisely five non-abelian non-alternating simple groups that may appear as a composition factor in G, namely $\mathrm{PSp}(4,3), \mathrm{Sp}(6,2), \mathrm{SO}^{+}(8,2), \mathrm{PSL}(3,4)$ and $\mathrm{PSU}(4,3)$. This result uses the classification of finite simple groups.
- (Hegedűs, [80], building on Gow [64]) Let G be a finite soluble rational group. Then it is a $\{2,3,5\}$-group and the Sylow 5-subgroup of G is elementary abelian and normal in G.

The finite groups with the strong Magnus property are subject to even stronger restrictions. Indeed, already the usual Magnus property implies solvability within the class of finite groups.

Proposition 7.3.2. *Let* (G,Ω) *be pseudo-operator group with* $(\mathrm{MP})_\Omega$. *Assume that the group* G *is finite, and that the operator domain is either* $\mathrm{Inn}(G)$ *or* $\mathrm{AInn}(G)$. *Then* G *is soluble.*

Proof. Let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ be a composition series of G. If all composition factors are abelian, we are done. Without loss of generality (using Proposition 7.3.1), we may assume, for contradiction, that G_1 is a non-abelian simple group. Let g and $h \in G_1$ be non-trivial elements. Then since Ω includes the inner automorphisms, the Ω-closures of g and h in G contain the respective normal closures in G_1 (and are generated by them as Ω-invariant subgroups). But since G_1 is simple, the normal closures coincide with G_1, hence $\langle g\rangle^\Omega = G_1^\Omega = \langle h\rangle^\Omega$. Since p-groups are simple if and only if they are of order p, there are at least two distinct prime numbers p and q dividing the order of G_1. Take for g and h elements of order p and q, respectively. Then g cannot be Ω-conjugate to h, since Ω is the group generated by inner automorphisms, which preserve order, and maybe the inversion map, which also preserves order. But G has $(\mathrm{MP})_\Omega$, a contradiction. Thus G is soluble. □

To proceed, we need the following well-known lemma on the *socle* $\operatorname{soc}(G)$ *of the finite group* G, i.e. the subgroup generated by all minimal normal subgroups.

Lemma 7.3.3. *Let G be a finite group. Then the following statements hold:*

(i) *There exist a collection of minimal subgroups $M_1, \dots, M_n$ of G such that* $\operatorname{soc}(G) = M_1 \times \cdots \times M_n$.
(ii) *If* $\operatorname{soc}(G)$ *is an elementary abelian p-group of rank d, the group* $G/\operatorname{C}_G(\operatorname{soc}(G))$ *is isomorphic to a subgroup of* $\operatorname{GL}_d(p)$.

For a proof, see for example [40, Theorem 4.3A] for the first part. The second part is an immediate consequence of the fact that the automorphism group of an elementary abelian p-group of rank d is isomorphic to $\operatorname{GL}_d(p)$. Furthermore, we need the following lemma.

Lemma 7.3.4. *Let G be a finite soluble group and let p be a prime dividing the order of G. Then there exists a quotient Q of G such that* $\operatorname{soc}(Q)$ *is a Sylow p-subgroup of Q and* $\operatorname{soc}(Q)$ *is an elementary abelian p-group.*

Proof. If the order of $G/\operatorname{soc}(G)$ is divisible by p, we consider this quotient instead of G, until $|G/\operatorname{soc}(G)|$ is no longer divisible by p. Necessarily, p must divide the order of $\operatorname{soc}(G)$. Since G is soluble, its minimal normal subgroups are elementary abelian q-groups for some primes q. Thus, by Lemma 7.3.3(i), $\operatorname{soc}(G)$ is a direct product of elementary abelian q-groups, hence $\operatorname{soc}(G) = M \times E$ for some elementary abelian p-group E, and some normal subgroup $M \trianglelefteq G$. Consider G/M. Either $\operatorname{soc}(G/M)$ is an elementary abelian p-group, or we repeat the process, until it is. The resulting quotient is fulfils the conditions of the lemma. □

Theorem 7.3.5. *Let $G \in \mathcal{M}_{\mathrm{Inn}}$ be a finite group. Then G is a $\{2,3\}$-group.*

Proof. Let M be the Sylow 5-subgroup of G. We want to show that $M = 1$. By Proposition 7.3.2, G is soluble, by Proposition 7.3.1 all quotients of G have the strong Magnus property, hence, using Lemma 7.3.4 we may assume that $M = \operatorname{soc}(G)$ and that M is an elementary abelian 5-group, the latter is generally true for finite soluble rational groups due to Hegedűs' theorem [80, Theorem 1].

Let $d \in \mathbb{N}$ be the rank of M. Since M is minimal, every non-trivial element of M generates the same normal subgroup. The strong Magnus property of G therefore asserts that the conjugation action of G on M, and since M is abelian, of G/M on M, is transitive on $M \setminus \{1\}$. Thus

$$(5^d - 1) \mid |G/M|.$$

Now $|G/M|$ is a $\{2,3\}$-number, hence also $5^d - 1$ is a $\{2,3\}$-number, which severely limits our choices for d. Note that if $d = ef$ is a product of integers, the number $5^d - 1$ can be factorised as follows:

$$5^d - 1 = (5^e - 1)(1 + 5^e + \cdots + 5^{e(f-1)}).$$

Consequently, we only have to consider the prime powers $d = p^n$ for which $5^d - 1$ is a $\{2,3\}$-number. Let p be an odd prime. Then

$$5^p - 1 \equiv_3 (-1)^p - 1 \equiv_3 1 \not\equiv_3 0 \quad \text{and} \quad 5^p - 1 \equiv_8 5 - 1 \not\equiv_8 0.$$

Thus, neither 3 nor 8 divides the $\{2,3\}$-number $|G/M|$, and $5^d - 1 < 8$ is false for d an odd prime. Similarly, if $d = 2^2$ we find

$$5^4 - 1 = 624 = 2^4 \cdot 3 \cdot 13.$$

Thus, the only possible values for d are 1 and 2. Assume that $d = 1$. By Lemma 7.3.3(ii), the group $G/\operatorname{C}_G(M)$ is a subgroup of $\operatorname{GL}(1,5) \cong \operatorname{C}_4$ acting transitively on M. Hence G has C_4 as a quotient, but this group does not have the strong Magnus property, hence G does not possess the strong Magnus property as well. Therefore we may suppose that $d \neq 1$.

Now assume that $d = 2$. By Lemma 7.3.3(ii), the group $H = G/\operatorname{C}_G(M)$ is (isomorphic to) a subgroup of $\operatorname{GL}(2,5)$ acting transitively on $M \cong \mathbb{F}_5^2$. Furthermore, $5^2 - 1 = 24$ divides the order of H. The order of $\operatorname{GL}(2,5)$ is $480 = 24 \cdot 20$. Since H is a $\{2,3\}$-group, it must have order $24, 48$ or 96.

If $|H| = 96 = \frac{480}{5}$, the group H contains a Sylow 2-subgroup, hence a conjugate of H contains the centre of $\operatorname{GL}(2,5)$. But the centre is the group of scalars, hence cyclic of order 4, and since the centre of a group with the strong Magnus property has the strong Magnus property, the group H cannot possess $(\mathrm{MP})_{\mathrm{Inn}}$. It remains to consider the orders 24 and 48.

There are six isomorphism classes of subgroups of $\operatorname{GL}_2(5)$ with these orders, namely

$$\operatorname{C}_{24}, \operatorname{C}_3 \rtimes \operatorname{C}_8, \operatorname{C}_4 \times \operatorname{Sym}_3, \operatorname{SL}(2,3), (\operatorname{C}_4 \circ \operatorname{D}_4) \rtimes \operatorname{C}_3, \operatorname{C}_8 \rtimes \operatorname{Sym}_3 .$$

The only cyclic groups with the strong Magnus property are those of order 1 or 2. All groups in the list above have a quotient of order 3, hence no group in the list above has the strong Magnus property. A computer program using GAP [52] computing both the list above and verifying the fact that no group in this list has the strong Magnus property, is attached as the appendix Section 8.5 at the end of this part of the thesis. □

Note that a 3-group cannot possess the strong Magnus property, since the non-trivial central elements that such a group possesses are not conjugate to their inverses. However, further computer experiments using the program mentioned above suggests that the problem of classifying the 2-groups (or the non-2-groups) with the strong Magnus property is difficult.

7.3.2. Finite groups and the usual Magnus property. — Most of the arguments in the previous sections, if they did not already apply to the usual Magnus property, can be carried over into this less restrictive context. Indeed, there is a weaker version of rationality for finite groups. A (finite) group G is called *inverse semi-rational*, if for every $g \in G$ all generators of $\langle g \rangle$ are conjugate or inverse-conjugate to g, and a group is called *semi-rational* if all generators of every cyclic subgroup are contained in at most two conjugacy classes, which interestingly does not imply inverse semi-rationality as long as G is not of odd order. Both concepts were introduced and studied by Chillag and Dolfi in [30]. Clearly, every group with the usual Magnus property is inverse semi-rational. We have already seen that every finite group in $\mathcal{M}$ is soluble, wherefore the following characterisation of soluble inverse semi-rational groups, also obtained by Chillag and Dolfi, is of interest for us.

Theorem 7.3.6 (Chillag and Dolfi, Theorem 2, [30])**.** *Let G be a inverse semi-rational finite soluble group. Then G is a $\{2, 3, 5, 7, 13\}$-group.*

It is not a mistake that the prime 11 is omitted. Building on this theorem, we proceed to restrict the possible primes dividing the order of a finite group with the Magnus property.

Theorem 7.3.7. *Let G be a finite group with the usual Magnus property. Then G is a $\{2, 3, 5, 7\}$-group.*

Proof. We proceed similarly as in the proof of Theorem 7.3.5. In view of Theorem 7.3.6, we only have to exclude the prime 13 as a divisor. Using Proposition 7.3.1, Lemma 7.3.4 and Proposition 7.3.2 we may restrict to the case where the socle $\mathrm{soc}(G) = M$ is an elementary abelian Sylow 13-group.

Since every non-trivial element of M normally generates M and G has the Magnus property, the quotient G/M acts semi-transitively on M, such that there are at most two orbits. If there are two orbits, the elements of one of the orbits are precisely the inverses of the other orbit. By the orbit-stabiliser theorem,

$$(13^d - 1)/2 \mid |G/M|,$$

a $\{2, 3, 5, 7\}$-number. As in Theorem 7.3.5, we may restrict d by considering certain prime powers. For any odd prime p,

$$(13^p-1)\equiv_8(5^p-1)\equiv_8 4$$
$$(13^p-1)\equiv_9(4^p-1)\equiv_9\begin{cases}0 & \text{if } p=3,\\ 3 & \text{if } p\equiv_3 1,\\ 6 & \text{if } p\equiv_3 2,\end{cases}$$
$$(13^p-1)\equiv_5(3^p-1)\equiv_5\begin{cases}2 & \text{if } p\equiv_4 1,\\ 1 & \text{if } p\equiv_4 3,\end{cases}$$
$$(13^p-1)\equiv_7((-1)^p-1)\equiv_7 5,$$

furthermore

$$17\mid(13^4-1)/2 \quad\text{and}\quad 61\mid(13^3-1)/2.$$

Thus, the only candidates for d are 1 and 2. In case $d=1$, by Lemma 7.3.3 the group $H=G/M$ is isomorphic to a subgroup of $\mathrm{GL}(1,13)\cong\mathrm{C}_{12}$ acting transitively on $M\cong\mathbb{F}_{13}$, but the only such subgroup is C_{12}, which does not have the Magnus property. This contradicts Proposition 7.3.1.

If $d=2$, the group $H=G/M$ is a $\{2,3,5,7\}$-subgroup of $\mathrm{GL}(2,13)$, a group of order $2^5\cdot3^2\cdot7\cdot13$, such that $(13^2-1)/2=2^2\cdot3\cdot7$ divides $|H|$. The only possible orders are

$$|H|\in\{84,168,252,336,504,672\}.$$

There are seven isomorphism classes of subgroups of $\mathrm{GL}_2(13)$ with these orders, namely

$$\mathrm{C}_3\times\mathrm{Dic}_{4\cdot7},\mathrm{C}_{84},\mathrm{C}_6\times\mathrm{D}_{2\cdot7},\mathrm{C}_3\times\mathrm{C}_7\rtimes\mathrm{C}_8,\mathrm{C}_{168},\mathrm{C}_{12}\times\mathrm{D}_{2\cdot7},\mathrm{C}_3\times\mathrm{C}_8\rtimes\mathrm{D}_{2\cdot7}.$$

The only cyclic groups with the Magnus property are those of order $1,2,3,4$ or 6. Furthermore, the dihedral group $\langle a,t\mid a^7=t^2=1,a^t=a^{-1}\rangle$ of order 14 does not have the Magnus property, since a and a^3 generate the same normal subgroup $\langle a\rangle$, but $a^{\mathrm{D}_{2\cdot7}}=\{a,a^6\}$. Since $\mathrm{D}_{2\cdot7}$ is a quotient of the dicyclic group $\mathrm{Dic}_{4\cdot7}$, no group in the list above has the Magnus property. A computer program using GAP [52] that computes both the list above and the fact that no subgroup in this list has the Magnus property, is attached as appendix Section 8.5.

Note that we have already seen that there is a group of order 21 with the Magnus property. The primes 2 and 5 appear as the orders of cyclic groups with the Magnus property. Hence all primes mentioned in the theorem appear; however, by [30, Theorem 3], all finite inverse semi-rational groups of odd order are $\{3,7\}$-groups. Consequently, not all combinations of the primes may arise as divisors of a finite group with the Magnus property. □

7.4 — Near-rings and trivial units

We now show that a generalised Magnus property associated to a sufficiently well-structured assignment Ω can be reformulated as a (somewhat) ring-theoretic statement and introduce a relative version of the usual Magnus property. Both apply, in particular, to the case of crystallographic groups which we consider in Section 7.6.

A set R with two binary operations $+$ ('addition') and $\cdot$ ('multiplication') is called a *near-ring*, if the following hold:

(i) R is a group with respect to $+$. The neutral element is denoted 0. (Note that, although it is natural to write this operation additively, we do not assume this group to be abelian.)

(ii) R is a semi-group with respect to $\cdot$. The neutral element is denoted 1 and is required to be distinct from 0.

(iii) The law of *left-distributivity* holds, for all $r, s, t \in R$ we find

$$r \cdot (s + t) = r \cdot s + r \cdot t.$$

As usual, we shall omit the symbol $\cdot$ where it does not lead to confusion. Near-rings and their structure have been an area for study for quite some time, cf. [102] for an overview, but the most well-known special case is the classification of *near-fields* by Zassenhaus [157]. A near-ring is called a near-field if the additional axioms

(4) The neutral elements of the operations $+$ and $\cdot$ are not equal, and

(5) $R \setminus \{0\}$ is a group with respect to the operation $\cdot$,

hold. It is an interesting fact that the additive group of a near-field is automatically abelian [112].

The classical example of a near-ring is the set $M(G) = \{\alpha \mid \alpha : G \to G\}$ of all self-maps of a group G, with the addition being the pointwise group-operation of G, and the multiplication being the composition of maps. All near-rings that we consider in this thesis are of this form.

Definition 7.4.1. Let (G, Ω) be a pseudo-operator group. The sub-near-ring of $M(G)$ generated by all maps in Ω,

$$G[\Omega] = \langle \Omega \rangle_{M(G)},$$

is called the *action near-ring of G with respect to Ω.*

The connexion to the (generalised) Magnus property becomes apparent in the following statement.

Lemma 7.4.2. *Let (G, Ω) be a pseudo-operator group such that the inversion map $x \mapsto x^{-1}$ is contained in Ω. Then*

$$\langle g \rangle^{\Omega} = g^{G[\Omega]} = \{g^r \mid r \in G[\Omega]\}.$$

Proof. Let $r \in G[\Omega]$ be an element. Since $G[\Omega]$ is generated by the multiplicative submonoid $\Omega \leq M(G)$, we may write $r = \sum_{i=1}^{n} \omega_i$ for some $\omega_i \in \Omega$, for $i \in \{1, \ldots, n\}$. Note that the additive inverse of an element ω is realised as the product of ω and the inversion map. Computing the image of g, we find

$$g^r = \prod_{i=1}^{n} g^{\omega_i} \in \langle g \rangle^{\Omega}.$$

Since we may choose $n \in \mathbb{N}$ and the elements ω_i arbitrary, we may represent every element of $\langle g \rangle^{\Omega}$ in this way. Note that the neutral element id of Ω is the identity function such that powers of g are images under multiples of id. □

We need some further notation. Let $g \in G$ be an element of a group G and R a sub-near-ring of $M(G)$. We call an element $r \in R$ a *g-unit* if there exists $s \in M(G)$ such that $g.(rs) = g$. We say that two elements r and $s \in R$ are *g-equivalent* if $g.r = g.s$. This equivalence relation is induced by the right ideal of functions mapping g to the group identity, where by a *right ideal of a near-ring* R we mean a normal subgroup I of the additive group of R such that $IR \subseteq R$. We call a (proper) unit of the action near-ring $G[\Omega]$ a *trivial unit* if it is an element of the generating set Ω of $G[\Omega]$.

Lemma 7.4.3. *Let (G, Ω) be a pseudo-operator group such that Ω contains the inversion map. The following two statements are equivalent:*

(i) *G has* $(\mathrm{MP})_{\Omega}$.
(ii) *For all $g \in G$, every g-unit of $G[\Omega]$ is g-equivalent to a trivial unit.*

Proof. We prove by contraposition that (i) implies (ii). Let $g \in G$ and let $u \in G[\Omega]$ be a g-unit that is not g-equivalent to a trivial unit. Then the sets $g^{G[\Omega]}$ and $(g^u)^{G[\Omega]}$ are equal, hence by Lemma 7.4.2 the elements g and g^u generate the same Ω-invariant subgroup. But since u is not g-equivalent to a trivial unit, there is no $\omega \in \Omega$ such that $g^{\omega} = g^u$. Thus, G does not possess $(\mathrm{MP})_{\Omega}$.

Now assume (ii), i.e. that every g-unit of $G[\Omega]$ is g-equivalent to a trivial unit. Let g and h be two elements of G generating the same Ω-invariant subgroup. Then, by Lemma 7.4.2, there exist u and $v \in G[\Omega]$ such that $g^u = h$ and $h^v = g$. Consequently $g^{uv} = g$, and u is a g-unit, hence $u \in \Omega$. Thus, g and h are Ω-conjugate and G has $(\mathrm{MP})_{\Omega}$. □

The situation becomes more familiar if we consider abelian groups G and Ω such that Ω is a group of automorphisms. Then $G[\Omega]$ is in fact a ring. Indeed, it is a homomorphic image of the integral group ring over Ω, under the map that evaluates the abstract group element $\omega \in \Omega(G)$ as the concrete function on G. Under these circumstances, the property (ii) of Lemma 7.4.3 resembles the property of a group algebra $R[H]$ of a group H over a ring R to only have *trivial units*, i.e. that $R[H]^{\times} = R^{\times} H$. We will develop this connexion further in Section 7.5.

The examples of generalised Magnus properties we have considered up to now are, however, not very interesting within the class of abelian groups. But there is another natural example where the above interpretation proves very useful.

Definition 7.4.4. Let G be a group and $N \trianglelefteq G$ be a normal subgroup. Write $\overline{G/\mathrm{Z}(N)} = G/\mathrm{Z}(N) \rtimes \langle x \mapsto x^{-1} \rangle$, where we see $G/\mathrm{Z}(N)$ as a group of automorphisms of N, acting by conjugation. The generalised Magnus property $(\mathrm{MP})_{\overline{G/\mathrm{Z}(N)}}$ is called the *relative Magnus property of N with respect to G.*

Clearly, for every group G, the full group has the relative Magnus property with respect to itself if and only if it has the (usual) Magnus property, while the trivial subgroup of any group G has the relative Magnus property with respect to G. The true importance of the relative Magnus property, however, lies in the following observation.

Lemma 7.4.5. *Let G be a group and let K and $N \trianglelefteq G$ be two normal subgroups of G such that $K \leq N$. If N has $(\mathrm{MP})_{\overline{G/\mathrm{Z}(N)}}$, then K has $(\mathrm{MP})_{\overline{G/\mathrm{Z}(K)}}$.*

Proof. Let k_0 and $k_1 \in K$ be two elements that generate the same normal subgroup in G. Since $K \leq N$, these are two elements of N that generate the same normal subgroup in G, hence they are G-conjugate or inverse G-conjugate. □

7·5 — Units in rings of integral representations

The similarity between Lemma 7.4.3 and the condition of an integral group ring to have only trivial units has already been mentioned. In this section, we will expand on this connexion in a special case. In Section 7.6 we shall apply the results of this section to narrow down the groups with the Magnus property in the class of crystallographic groups.

The *group algebra* $R[G]$ of a group G over a ring R is the set of formal R-linear combinations $\sum_{g\in G} r_g G$, i.e. the set of maps $G \to R$ with finite support. The group G is clearly (isomorphic to) a subgroup of the unit group of $R[G]$, more generally, all elements of the form εg, with $\varepsilon \in R^\times$ and $g \in G$ are units. The subgroup $R^\times G \leq R[G]^\times$ is called the *subgroup of trivial units.* The question in which cases this subgroup is proper has been investigated thoroughly for a long time, beginning with the classic result of Higman [83] on the trivial units of integral group rings.

Theorem 7.5.1. *(Higman's thesis, Theorem 9, as cited in [133], and [83, Theorem 3]) Let Γ be a finitely generated torsion group, i.e. a group such that all elements of Γ have finite order, and let R be the integer ring of a finite field extension of $\mathbb{Q}$. The group algebra $R[\Gamma]$ has only trivial units precisely in the following cases:*

(i) $\Gamma \cong \mathrm{C}_2^n$ *and* $K = \mathbb{Q}(\sqrt{-d})$, *for* $n \in \mathbb{N}_0$ *and* $d \in \mathbb{N}_0$,
(ii) $\Gamma \cong \mathrm{C}_2^n \times \mathrm{C}_3^m$ *and* $K = \mathbb{Q}$ *or* $K = \mathbb{Q}(\omega)$, *the third cyclotomic field, for* n *and* $m \in \mathbb{N}_0$,
(iii) $\Gamma \cong \mathrm{C}_2^n \times \mathrm{C}_4^m$ *and* $K = \mathbb{Q}$ *or* $K = \mathbb{Q}(i)$, *the fourth cyclotomic field, for* n *and* $m \in \mathbb{N}_0$,

(iv) $\Gamma \cong \mathrm{Q}_8 \times \mathrm{C}_2^n$ *and* $K = \mathbb{Q}$, *for* $n \in \mathbb{N}_0$.

If $R[\Gamma]$ *has non-trivial units and* Γ *is abelian, the unit group* $R[\Gamma]^\times$ *is infinite.*

This result has been generalised in various directions, e.g. by considering, for Γ a finite group, integral domains R of characteristic 0 in which no prime divisor of the order of Γ is invertible. Such rings are called Γ-*adapted*, where the list of groups Γ allowing $R[\Gamma]$ to have only trivial units depends on the unit group of R, but is otherwise quite similar to Higman's list given above [130].

In the case that the ring R is a field, Higman conjectured in his thesis that if Γ is a torsion-free group, all units are trivial. This conjecture (often called the 'unit conjecture') has been disproven recently by Gardam [53].

Motivated by Lemma 7.4.3, we go in another direction. Instead of considering different rings than Higman did, we consider certain homomorphic images of group algebras. We make the following definition.

Definition 7.5.2. Let Γ be a group, let R be a commutative unital ring, and let $\rho\colon \Gamma \to \mathrm{GL}_d(R)$ be a faithful R-representation of Γ. The R-algebra $R[\Gamma]_\rho$ generated by the image of Γ under ρ in $\mathrm{Mat}_d(R)$ is called the *R-algebra of the representation ρ*. In case $R = \mathbb{Z}$, we speak about the *ring of the representation ρ*.

Concretely, the algebra $R[\Gamma]_\rho$ is the R-span of the matrices in the image Γ^ρ. It is a quotient of the group ring $R[\Gamma]$, and it is in fact isomorphic to the group ring if ρ is the *regular representation of* Γ, i.e. the representation given by the action of Γ on the group ring itself, which we record as a fact.

Lemma 7.5.3. *Let R be a commutative unital ring, let Γ be a finite group and let $\rho\colon \Gamma \to \mathrm{GL}_{|\Gamma|}(R)$ denote the* regular representation of Γ*, i.e. the representation induced by the right-multiplication action on Γ seen as the basis of the free R-module $R[\Gamma]$. Then the algebra of the representation ρ is equal to the group ring of Γ over R.*

We may still speak about trivial units, since Γ naturally embeds into $R[\Gamma]_\rho$ via ρ, all units in the product set $R^\times\Gamma^\rho$ are called *trivial*. For brevity, we write $\mathrm{Tr}(R, \Gamma)$ for the set $R^\times\Gamma^\rho$ of trivial units. However, it might be that the images of element in Γ only differ by units of R. Consider, for example, the one-dimensional integral representation of C_2 with image $\{1, -1\}$; the ring of this representation is $\mathbb{Z}$, and the trivial units are precisely the units of $\mathbb{Z}$. The same phenomenon may occur whenever Γ can be embedded into $R^\times$.

Of course, every ring generated (as a ring) by its units has 'trivial units' with respect to its unit group. Therefore we only speak about trivial units of an algebra of a representation to make clear what subgroup of units we mean.

Our goal is to obtain some information about the possible combinations of groups, representations and rings yielding algebras with only trivial units. Note that if R itself has a finite unit group and Γ is a finite group, the group of trivial units is finite; if furthermore R

is commutative and Γ abelian, to have only trivial units implies having a finite abelian unit group. The characterisation of abelian groups that are the unit group of a commutative ring is known as 'Fuchs' problem' [51], which is still open in general, see [36] for a solution for finite groups and integral domains.

There is another description of the algebra of a representation ρ. It is precisely the action near-ring of the natural action of Γ, or equivalently $\mathrm{Tr}(R[\Gamma]_\rho)$, on $G = R^d$ via ρ. In particular, the following lemma connects the study of trivial units to the relative Magnus property.

Lemma 7.5.4. *Let Γ be a finite group, let R be a torsion-free infinite principal ideal domain with finite unit group $R^\times$, let $\rho\colon \Gamma \to \mathrm{GL}_d(R)$ be a faithful R-representation and let V be a $R[\Gamma]_\rho$-submodule of R^d. Denote by ρ_V the sub-representation of ρ induced by the action on V, and denote by Γ_V the quotient of Γ by the kernel of ρ_V. We regard ρ_V as a faithful R-representation of Γ_V. If R^d has $(\mathrm{MP})_{\mathrm{Tr}(R,\Gamma)}$, then $R[\Gamma_V]_{\rho_V}$ has only trivial units.*

Proof. We first provide a useful description of the Γ-submodules generated by elements $\mathbf{v} \in V \leq R^d$. The Γ-invariant submodule generated by $\mathbf{v}$ is equal to the set of images under maps in $R[\Gamma_V]_{\rho_V}$. Indeed, by Lemma 7.4.2,

$$\langle \mathbf{v} \rangle_{R-\mathrm{mod}}^{\Gamma^\rho} = \mathbf{v}^{R[\Gamma]_\rho}.$$

Let $r \in R[\Gamma]_\rho$. Then r may be (not necessarily uniquely) written as a linear combination of the form

$$r = \sum_{\gamma \in \Gamma} r_\gamma \gamma^\rho,$$

for some $r_\gamma \in R$. But by construction, the action of $\gamma \in \Gamma$ on $\mathbf{v} \in V$ depends only on the image of γ in Γ_V, and the image of $\mathbf{v}$ under this action is contained in V, since V is Γ^ρ-invariant. Thus, r acts on $\mathbf{v}$ as the element

$$\sum_{\sigma \in \Gamma_V} \Big(\sum_{\gamma \in \sigma N} r_\gamma \Big) \sigma^{\rho_V} \in R[\Gamma_V]_{\rho_V},$$

where N denotes the kernel of the map $\Gamma \to \Gamma_V$. Clearly every element of $R[\Gamma_V]_{\rho_V}$ is of the above form for a suitable choice of coefficients. Thus

$$\langle \mathbf{v} \rangle^{\Gamma^\rho} = \mathbf{v}^{R[\Gamma]_\rho} = \mathbf{v}^{R[\Gamma_V]_{\rho_V}}.$$

We now prove the statement of the lemma using contraposition. Let $u \in R[\Gamma_V]_{\rho_V}$ be a non-trivial unit. To prove the lemma, it is sufficient to show that there exists an element $\mathbf{v} \in V$ such that $\mathbf{v}$ and $\mathbf{v}^u$ are not $\mathrm{Tr}(R,\Gamma)$-conjugate, i.e. that $\mathbf{v}^u \notin \mathbf{v}^{R^\times \Gamma^\rho}$. Indeed, by our considerations above, the Γ^ρ-invariant submodules generated by $\mathbf{v}$ and $\mathbf{v}^u$ are equal;

$$\langle \mathbf{v} \rangle^{\Gamma^\rho} = \mathbf{v}^{R[\Gamma_V]\rho_V} = \mathbf{v}^{uR[\Gamma_V]\rho_V} = \langle \mathbf{v}^u \rangle^{\Gamma^\rho}.$$

Thus R^d does not possess $(\mathrm{MP})_{\mathrm{Tr}(R,\Gamma)}$. Thus, we now prove that there exists such an element. For every $\tau \in \mathrm{Tr}(R,\Gamma)$, define

$$V_\tau = \{\mathbf{w} \in V \mid \mathbf{w}^u = \mathbf{w}^\tau\} = \ker(u - \tau).$$

Since the abelian group underlying V_τ is a subgroup of the torsion-free group $R^d \cong \mathbb{Z}^{md}$, it is isomorphic to $\mathbb{Z}^k$ for some $k \in \{0, \dots, md\}$. If no $\mathbf{v} \in V$ exists such that u does act on $\mathbf{v}$ unlike any trivial unit τ, we find

$$V = \bigcup_{\tau \in \mathrm{Tr}(R,\Gamma)} V_\tau.$$

Since this is a finite union, at least one of the submodules, say $V_{\varepsilon\gamma}$ for some fixed $\varepsilon\gamma \in \mathrm{Tr}(R,\Gamma)$, must be of rank md. Thus $[V : V_\gamma]$ is finite, hence for every $\mathbf{v} \in V$ there exists some integer $k \in \mathbb{N}$ such that $k\mathbf{v} \in V_\gamma$. Thus

$$k\mathbf{v}^u = (k\mathbf{v})^u = (k\mathbf{v})^{\varepsilon\gamma} = k\mathbf{v}^{\varepsilon\gamma}.$$

Since R is a principal ideal domain, we find that $V_{\varepsilon\gamma} = V$. Both u and $\varepsilon\gamma$ are R-linear maps of V, and they act identically on V, hence they are equal. This is a contradiction, since u was assumed to not be a trivial unit. Thus, there exists some $\mathbf{v} \in V$ as desired, and R^d does not have $(\mathrm{MP})_{R^\times \Gamma^\rho}$. $\square$

Indeed, the two conditions of Lemma 7.5.4 are equivalent for certain rings R, which we shall see in Proposition 7.5.8.

We now restrict ourselves to the rings that by Higman's Theorem 7.5.1 allow group rings with finite unit group; i.e. $\mathbb{Z}$ and the rings of algebraic integers of imaginary quadratic number fields. In view of Lemma 7.5.4, we furthermore require our ring to be principal ideal domains; thus we restrict to a finite number of possible rings. We shall write $\mathcal{R}$ for these rings. In particular, every ring $\mathcal{R}$ embeds into $\mathbb{C}$. In this situation, we are able to proceed in parallel to Higman's proof of Theorem 7.5.1.

Lemma 7.5.5. *Let Γ be finite group, let $\rho\colon \Gamma \to \mathrm{GL}_d(\mathcal{R})$ be a faithful $\mathcal{R}$-representation and let $\Delta \leq \Gamma$ be a subgroup. If $\mathcal{R}[\Delta]_{\rho|_\Delta}$ has infinitely many non-trivial units, then $\mathcal{R}[\Gamma]_\rho$ has infinitely many non-trivial units.*

Proof. The algebra $\mathcal{R}[\Delta]_{\rho|_\Delta}$ is an $\mathcal{R}$-algebra generated by a subset of the generators of $\mathcal{R}[\Gamma]_\rho$, hence is a subalgebra. Clearly, every unit of a subalgebra is a unit of the full algebra. By assumption, we find an infinitude of units in $\mathcal{R}[\Gamma]_\rho$. But since both $\mathcal{R}^\times$ and

Γ are finite, there are only finitely many trivial units. Thus, there are infinitely many non-trivial units. □

Using this lemma, we restrict the possible element orders of a finite group with a faithful $\mathcal{R}$-representation affording an algebra with only trivial units.

Lemma 7.5.6. *Let Γ be a finite group and let $\rho\colon \Gamma \to \mathrm{GL}_d(\mathcal{R})$ be a faithful $\mathcal{R}$-representation. Then either the algebra $\mathcal{R}[\Gamma]_\rho$ has non-trivial units, or every element of Γ has order $1, 2, 3, 4$ or 6.*

Proof. It is enough to consider elements of prime power order, thus let $\gamma \in \Gamma$ be of order p^n, for some integer $n \in \mathbb{N}$ and some prime p. Write H for the subgroup generated by γ. Consider the algebra of the representation $\mathcal{R}[H]_{\rho|_H}$. We may view this $\mathcal{R}$-algebra as a finite-dimensional $\mathbb{Z}$-algebra. The $\mathbb{Z}$-span of the image of H under ρ is a subring of $\mathcal{R}[H]_{\rho|_H}$. By Lemma 7.5.5, it is enough to show that this subring has infinitely many units in case $p^n \notin \{2, 3, 4\}$. Since the order of γ is finite, the matrix γ^ρ is diagonalisable over $\mathbb{C}$, i.e. it is conjugate by a matrix $P \in \mathrm{GL}_d(\mathbb{C})$ to a diagonal matrix

$$\mathrm{Diag}(\zeta_1, \dots, \zeta_d),$$

whose entries are all $(p^n)^{\text{th}}$ roots of unity, and such that ζ_1 is a primitive root. Thus the $\mathbb{Z}$-span of H^ρ is isomorphic to the $\mathbb{Z}$-span of D, which is given by

$$\{(c, c^{\pi_2}, \dots, c^{\pi_d}) \mid c \in \mathbb{Z}[\zeta_1]\},$$

where $\pi_i\colon \mathbb{Z}[\zeta_1] \to \mathbb{Z}[\zeta_i]$, for $i \in \{2, \dots, d\}$, is the surjective homomorphism defined by sending ζ_1 to ζ_i. If $c \in \mathbb{Z}[\zeta_1]$ is a unit, all its homomorphic images are units, and consequently $(c, c^{\pi_2}, \dots, c^{\pi_d})$ is a unit. Thus, if there are infinitely many units in $\mathbb{Z}[\zeta_1]$, there are infinitely many units in $\mathcal{R}[H]_{\rho|_H}$. The ring $\mathbb{Z}[\zeta_1]$ is isomorphic to the ring of the $(p^n)^{\text{th}}$ cyclotomic integers. By Dirichlet's unit theorem (or by Theorem 7.5.1), the only rings of cyclotomic integers with finite unit group are the first, the second, the third, the fourth and the sixth. Thus, $\mathrm{ord}(\gamma)$ is an integer in $\{1, 2, 3, 4, 6\}$. □

Lemma 7.5.7. *Let Γ be a finite group, let $\rho\colon \Gamma \to \mathrm{GL}_d(\mathcal{R})$ be a faithful $\mathcal{R}$-representation such that $\mathcal{R}[\Gamma]_\rho$ has only trivial units. If Γ contains a non-trivial element γ of prime order p such that γ^ρ has an eigenvalue 1, then the $\mathcal{R}$-submodule $\mathrm{Eig}_{\gamma^\rho}(1)$ is Γ-invariant and there exists an $\mathcal{R}[\Gamma]_\rho$-submodule V of $\mathcal{R}^d$ such that the direct sum $\mathrm{Eig}_{\gamma^\rho}(1) \oplus V$ is of finite index in $\mathcal{R}^d$.*

Proof. Let γ be as described, and write $A = \gamma^\rho$. Note that in Lemma 7.5.6 we have shown that we may assume $p \in \{2, 3\}$, but we do not need this here.

We consider the $\langle A \rangle$-invariant $\mathcal{R}$-submodule $V = \ker \Phi(A)$, where Φ is the p^{th} cyclotomic polynomial. Since A has order p, it possesses a primitive p^{th} root of unity as an eigenvalue. Thus, one of the eigenvalues of $\Phi(A)$ is 0, hence $\det \Phi(A) = 0$ and the $\mathcal{R}$-submodule V is not zero. At the same time, $A|_V$ is a root of the polynomial Φ, so all its eigenvalues are primitive p^{th} roots of unity.

The ring $\mathcal{R}$ is contained in $\mathbb{C}$. Clearly $\mathbb{Q} \otimes \mathcal{R}^d$ decomposes as a direct sum $\mathrm{Eig}_A(1) \oplus \ker \Phi(A)$, both as submodules of $\mathbb{Q} \otimes \mathcal{R}^d$, since for every $\mathbf{v} \in \mathrm{Eig}_A(1)$ we have

$$\mathbf{v}^{\Phi(A)} = \mathbf{v}^{A^{p-1}+\cdots+A+\mathrm{I}} = p\mathbf{v},$$

hence the image of $\Phi(A)$ is precisely $p\,\mathrm{Eig}_A(1)$. Thus, the direct sum, as a sum of $\mathcal{R}$-submodules, is of dimension d and hence a finite index submodule.

Let $\delta \in \Gamma$ be an element. Write $B = \delta^\rho$, and denote the identity matrix by I. Consider the elements

$$X = \Phi(A) \cdot B \cdot (\mathrm{I} - A) \in \mathcal{R}[\Gamma]_\rho \text{ and}$$
$$Y = (\mathrm{I} - A) \cdot B \cdot \Phi(A) \in \mathcal{R}[\Gamma]_\rho.$$

It is easy to see that

$$X^2 = \Phi(A) \cdot B \cdot (A^p - \mathrm{I}) \cdot B \cdot (\mathrm{I} - A) = 0,$$

since A has order p. In the same way we find that $Y^2 = 0$. This implies that

$$(\mathrm{I} + X)(\mathrm{I} - X) = \mathrm{I} - X^2 = \mathrm{I} \quad \text{and} \quad (\mathrm{I} + Y)(\mathrm{I} - Y) = \mathrm{I} - Y^2 = \mathrm{I},$$

i.e. $(\mathrm{I} + X), (\mathrm{I} - X), (\mathrm{I} + Y)$ and $(\mathrm{I} - Y)$ are units of $\mathcal{R}[\Gamma]_\rho$. By the assumption on the ring of the representation ρ, these units are trivial. Since $X^2 = Y^2 = 0$, for any $n \in \mathbb{N}$ we find that

$$(1 + X)^n = 1 + nX \quad \text{and} \quad (1 + Y)^n = 1 + nY.$$

Since $1 + X$ is a trivial unit, it is of finite order, say n, hence $nX = 0$. But since $\mathcal{R}$ is a integral domain, we find $X = 0$, and similarly $Y = 0$.

Let $\mathbf{v} \in \mathrm{Eig}_A(1)$. We find

$$0 = \mathbf{v}^X = \mathbf{v}^{\Phi(A) \cdot B \cdot (\mathrm{I} - A)} = p\mathbf{v}^B - p\mathbf{v}^{BA},$$

hence (since p is not a zero-divisor) $\mathbf{v}^B = \mathbf{v}^{BA}$, and thus $\mathbf{v}^B \in \mathrm{Eig}_A(1)$. Now let $\mathbf{w} \in \ker \Phi(A)$. On $\ker \Phi(A)$, the matrix $\mathrm{I} - A$ has only primitive p^{th} roots of unity as eigenvalues. Thus, $\mathrm{I} - A$ is invertible on this submodule and its inverse leaves $\ker \Phi(A)$ invariant. Therefore $0 = \mathbf{w}^Y$ (i.e. the invariance of $\ker \Phi(A)$ under Y) implies $\mathbf{w}^B \in \ker \Phi(A)$. All in all, B leaves both $\mathrm{Eig}_A(1)$ and $\ker \Phi(A)$ invariant, hence both submodules are in fact Γ^ρ-invariant. □

Using the same technique, one can drop the assertion that γ is of prime order. Then, by

Lemma 7.5.6, either γ is of order 4 or of order 6. Instead of the factorisation $A^p - 1 = \Phi(A) \cdot (A-1)$, use the factorisation $A^4 - 1 = \Phi(A) \cdot (A^2+1)$ resp. $A^6 - 1 = \Phi(A) \cdot (A^4 + A^3 - A - 1)$ (where Φ is the respective cyclotomic polynomial), and proceed as in the proof above.

Note that there is in general no hope to decompose the full module R^d, instead of a finite index submodule, into $\mathrm{Eig}_A(1)$ and some complement, for some arbitrary non-trivial $A \in \Gamma^\rho$. For example, the group ring of a cyclic group $\langle g \rangle$ of order p over the integers is indecomposable, cf. [81], but any element of the form $(n, \ldots, n)$ is an eigenvector of g with eigenvalue 1.

We can now show the converse to Lemma 7.5.4.

Proposition 7.5.8. *Let Γ be a finite group, and let $\rho\colon \Gamma \to \mathrm{GL}_d(\mathcal{R})$ be a faithful $\mathcal{R}$-representation. The following conditions are equivalent:*

- (i) *$\mathcal{R}^d$ has $(\mathrm{MP})_{\mathrm{Tr}(\mathcal{R},\Gamma)}$, and*
- (ii) *for every $\mathcal{R}[\Gamma]_\rho$-submodule V of $\mathcal{R}^d$, writing Γ_V for the quotient of Γ by the kernel of its action on V and ρ_V for the corresponding representation, the ring $\mathcal{R}[\Gamma_V]_{\rho_V}$ has only trivial units,*
- (iii) *for all r and $s \in \mathcal{R}[\Gamma]_\rho$ such that the product rs has an eigenvalue 1, both r and s act on the corresponding eigenmodule as trivial units.*

Proof. The statement that (i) implies (ii) is the one of Lemma 7.5.4; thus it is true for more general rings. Assuming (ii), we show (iii): Let r and s be as required. By Lemma 7.5.7, the eigenmodule $V = \mathrm{Eig}_{rs}(1)$ is invariant, hence the corresponding ring has only trivial units. But on V, the product rs acts as the identity, hence r and s are units in the corresponding ring, and therefore trivial units in it. Since Γ_V acts as Γ on V, both r and s act as a trivial unit of the full ring of the representation ρ. This shows (iii). That (iii) implies (i) is the direction (ii) implies (i) in Lemma 7.4.3; the elements r and s are $\mathbf{v}$-units for all $\mathbf{v} \in \mathrm{Eig}_{rs}(1)$, and since both act on this submodule as a trivial unit, they are $\mathbf{v}$-equivalent to one. $\square$

Using the methods applied in Lemma 7.5.7 (that are adapted from Higman) we may show more.

Lemma 7.5.9. *Let Γ be a finite group and let $\rho\colon \Gamma \to \mathrm{GL}_d(\mathcal{R})$ be a faithful $\mathcal{R}$-representation such that $\mathcal{R}[\Gamma]_\rho$ has only trivial units. Then all involutions of Γ are central.*

Proof. We proceed similarly to the previous proof. Let $\gamma \in \Gamma$ be an involution and $\delta \in \Gamma$ any element. Set $A = \gamma^\rho$ and $B = \delta^\rho$. Define, as in the proof of Lemma 7.5.7,

$$X = (\mathrm{I} + A) \cdot B \cdot (\mathrm{I} - A) \text{ and } Y = (\mathrm{I} - A) \cdot B \cdot (\mathrm{I} + A).$$

Arguing as before, the elements X and Y are both equal to 0, hence B leaves both $\mathrm{Eig}_A(1)$ and $\ker(A+1) = \mathrm{Eig}_A(-1)$ invariant, and their direct sum is of finite index in V. On both submodules A acts by the scalar matrices I and $-\mathrm{I}$, hence it is central among all elements leaving of submodules invariant. Thus, A and B commute. Since $B \in \Gamma^\rho$ was chosen arbitrarily, γ is central in Γ. □

Lemma 7.5.7 shows that, if a non-trivial element γ of Γ acts as a matrix with an eigenvalue 1, there exists an invariant submodule of $\mathcal{R}^d$ of smaller dimension. The fact that the dimension decreases is important; since in any case, $\mathcal{R}^d$ has invariant (proper) submodules of dimension d, namely the modules rR^d for some non-unit $r \in \mathcal{R}$. Consequently, we call an $\mathcal{R}$-representation ρ *irreducible*, if no invariant proper submodule has smaller dimension over $\mathcal{R}$ than ρ. Since R is not a field (of adapted characteristic), this is not equivalent to ρ to being *decomposable*, i.e. there exist invariant submodules U and V such that $\mathcal{R}^d = U \oplus V$. The given notion of irreducibility fits naturally with the usual irreducibility over a field, in fact, the $\mathcal{R}$-representation ρ is irreducible if and only if the corresponding representation is irreducible in the usual sense, cf. [34, Theorem 73.9].

Notice that Lemma 7.5.7 states that (given a non-trivial element with eigenvalue 1) the module R^d is not irreducible and a finite index submodule is decomposable. Using this, we are able to describe the groups giving rise to algebras of representations ρ such that ρ is irreducible.

Theorem 7.5.10. *Let Γ be a finite group, let $\mathcal{R}$ be either $\mathbb{Z}$ or the ring of integers of a imaginary quadratic extension of $\mathbb{Q}$, and let $\rho\colon \Gamma \to \mathrm{GL}_d(\mathcal{R})$ be a faithful irreducible $\mathcal{R}$-representation. If $\mathcal{R}[\Gamma]_\rho$ has only trivial units, the group Γ is isomorphic to one of the following groups*

$$\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_6, \mathrm{Q}_8, \mathrm{Dic}_{4\cdot 3}, \mathrm{SL}(2,3).$$

Proof. We have done most of the work already. By Lemma 7.5.7, all non-trivial elements $\gamma^\rho \in \Gamma^\rho$ have no eigenvalue 1, hence are fixed-point-free. Furthermore, by Lemma 7.5.6, every element of Γ has order $1, 2, 3, 4$ or 6. It is a classical result that groups affording a fixed-point-free representation over a field have cyclic Sylow p-subgroups for odd primes p, and Sylow 2-subgroups that are either cyclic or generalised quaternion groups [87, V 8.12b].

A *generalised quaternion group* $\mathrm{Q}_{4\cdot 2^n}$ *of order* $4 \cdot 2^n$, for a positive integer $n \in \mathbb{N}$, is the group given by the presentation

$$\mathrm{Q}_{4\cdot 2^n} = \langle i, j \mid j^{2^n} = i^2, i^4 = 1, j^i = j^{-1} \rangle.$$

For $n = 1$, this is the usual quaternion group. It is easy to see that the exponent of $\mathrm{Q}_{4\cdot 2^n}$ is at least 2^{n+1}, hence by our order restrictions, only the case $n = 1$ is possible. Thus, the Sylow 2-subgroups of Γ are cyclic of order 1, 2 or 4, or isomorphic to the quaternion group. Using Lemma 7.5.6 again, we find that the Sylow 3-subgroup of Γ is trivial or cyclic of order 3, and there are no other primes dividing $|\Gamma|$. Thus, we see that $|\Gamma|$ divides $2^3 \cdot 3 = 24$.

By Lemma 7.5.9, every involution γ of Γ is central. Viewing R as a subring of $\mathbb{C}$ as we have done before, all matrices γ^ρ that are images of involutions γ can by diagonalised simultaneously over $\mathbb{C}$, and are therefore uniquely determined by the dimension of the eigenspaces for the only possible eigenvalues, 1 and -1. But all non-trivial γ act fixed-point-free, hence all matrices γ^ρ do only have eigenvalues equal to -1. Thus, Γ contains a unique involution.

We now check the possible candidates for Γ, i.e. the groups of order $1, 2, 3, 4, 6, 8, 12$ or 24, and exclude the following groups:

Order	Isomorphism class	(one) reason for exclusion
4	C_2^2	Sylow 2-subgroup not cyclic or quaternion
6	Sym_3	three involutions
8	C_8	exponent 8
	$C_2 \times C_4$	Sylow 2-subgroup not cyclic or quaternion
	C_2^3	Sylow 2-subgroup not cyclic or quaternion
	$D_{2\cdot 2}$	four involutions
12	C_{12}	elements of order > 6
	Alt_4	Sylow 2-subgroup not cyclic or quaternion
	$D_{2\cdot 3}$	six involutions
	$C_2 \times C_6$	Sylow 2-subgroup not cyclic or quaternion
24	C_{24}	exponent 24
	Sym_4	Sylow 2-subgroup not cyclic or quaternion
	$D_{2\cdot 6}$	twelve involutions
	Dic_{24}	elements of order > 6
	$C_3 \rtimes D_{2\cdot 2}$	Sylow 2-subgroup not cyclic or quaternion
	$C_3 \rtimes C_8$	Sylow 2-subgroup not cyclic or quaternion
	$C_2 \times \mathrm{Alt}_4$	Sylow 2-subgroup not cyclic or quaternion
	$C_4 \times \mathrm{Sym}_3$	elements of order > 6
	$C_3 \times D_{2\cdot 2}$	elements of order > 6
	$C_2^2 \times \mathrm{Sym}_3$	Sylow 2-subgroup not cyclic or quaternion
	$C_3 \times Q_8$	elements of order > 6
	$C_2 \times \mathrm{Dic}_{4\cdot 3}$	elements of order > 6
	$C_2 \times C_{12}$	elements of order > 6
	$C_2^2 \times C_6$	Sylow 2-subgroup not cyclic or quaternion.

The only remaining groups of the possible orders are the ones given in the statement of the theorem. □

We now show that the list in Theorem 7.5.10 does not contain superfluous entries in the case of $R = \mathbb{Z}$.

Note that all the groups appearing in Higman's theorem Theorem 7.5.1 afford a faithful representation such that the associated ring has only trivial units, since the ring of the regular representation is isomorphic to the integral group ring. Thus, we are only concerned with the groups $\mathrm{Dic}_{4\cdot 3}$ and $\mathrm{SL}(2,3)$.

7.5.1. $\mathrm{Dic}_{4\cdot 3}$ as a group of trivial units. — We begin with an example of a faithful integral representation of $\mathrm{Dic}_{4\cdot 3}$ such that the ring of the representation has only trivial units. The group $\mathrm{Dic}_{4\cdot 3}$ is given by the following presentation

$$\langle s,t \mid s^6 = 1, t^2 = s^3, s^t = s^{-1}\rangle.$$

Consider the matrices

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The homomorphism given by $s \mapsto BA^2$ and $t \mapsto A$ yields a faithful integral representation of $\mathrm{Dic}_{4\cdot 3}$. Equivalently, we may use the isomorphism induced by identifying the ring generated by $\left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with the ring of Eisenstein integers, and consider the representation of $\mathrm{Dic}_{4\cdot 3}$ over $\mathbb{Z}(\omega)$ (where ω denotes a primitive third root of unity) given by the group matrices generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}(\omega)) \quad \text{and} \quad B = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}(\omega)).$$

A quick calculation shows that A and B indeed generate (multiplicatively) the group $\mathrm{Dic}_{4\cdot 3}$ of order 12. The twelve corresponding matrices, however, are neither linearly independent over $\mathbb{Z}$, nor over $\mathbb{Z}(\omega)$. Indeed, we find the equalities

$$\begin{aligned} A^2 &= -\mathrm{I}, & A^3 &= -A, & B^2 &= -(\mathrm{I}+B), & BA^2 &= -B, \\ BA^3 &= -BA, & B^2A &= -BA + A^3, & B^2A^2 &= \mathrm{I}+B, & B^2A^3 &= A^3 - BA, \end{aligned}$$

implying that the ring of the representation over $\mathbb{Z}$ is generated (as a ring) by A, B and BA only. Any sum of those generators (and the identity matrix I) is of the form

$$M(n,m,l,k) = \begin{pmatrix} n + m\omega & -(l + k\omega) \\ l + k\omega^2 & n + m\omega^2 \end{pmatrix}$$

for some integers n, m, l and $k \in \mathbb{Z}$, and since the basis elements generate a finite group, this set is indeed closed under multiplication, and hence equal to the full ring.

Note that over the ring $\mathbb{Z}(\omega)$, we find that $\omega\,\mathrm{I}, \omega A, \omega B$ and ωBA are not of the form above, and the $\mathbb{Z}(\omega)$-algebra of the corresponding representation is 8-dimensional over $\mathbb{Z}$.

The determinant of an element of the ring of the representation may be computed as

$$\begin{aligned}\det M(n,m,l,k) &= n^2+m^2+nm(\omega+\omega^2)+l^2+k^2+kl(\omega+\omega^2)\\ &= n^2+m^2-nm+l^2+k^2-kl.\end{aligned}$$

The only unit this expression may equal is the identity. If this is the case, multiplying by 2 and completing the square, we see

$$2 = m^2+n^2+l^2+k^2+(m-n)^2+(k-l)^2.$$

Since all numbers in this equation are positive, it is clear that there are only twelve solutions; eight by setting one of the variables to be ±1 (and the others to be 0), as well as $m=n=\pm1, k=l=0$ and $m=n=0, k=l=\pm1$. All these solutions correspond to elements of the image of $\mathrm{Dic}_{4\cdot3}$ in $R^\times$. There is no need for checking this by hand, since all twelve elements of $\mathrm{Dic}_{4\cdot3}$ are automatically units.

Indeed, we have proven more: The only element of the ring of ρ with an eigenvalue 1 is the identity matrix. Thus, by Proposition 7.5.8, the fact that the ring has only trivial units already implies $(\mathrm{MP})_{\mathrm{Dic}_{4\cdot3}}$ for $\mathbb{Z}^4$.

7.5.2. $\mathrm{SL}(2,3)$ **as a group of trivial units.** — The group $\mathrm{SL}(2,3)$ affords a representation with a ring with only trivial units as well. We preceed similarly to the case of $\mathrm{Dic}_{4\cdot3}$. The matrices

$$S=\begin{pmatrix}-1&1&1&-1\\-1&0&1&0\\0&0&0&-1\\0&0&1&-1\end{pmatrix}\quad\text{and}\quad T=\begin{pmatrix}-1&1&1&0\\-1&0&0&1\\-1&1&1&-1\\-1&0&1&0\end{pmatrix}$$

generate a group of integer matrices isomorphic to $\mathrm{SL}(2,3)$. One quickly checks the following identities

$$\mathrm{I}+S=-S^2,\quad T^2=-\mathrm{I}\quad\text{and}\quad TS=-(\mathrm{I}+T+ST).$$

Since $\mathrm{SL}(2,3)$ is generated by S and T, the linearly independent set $\{\mathrm{I},S,T,ST\}$ generates a submodule containing $\mathrm{SL}(2,3)$ in $\mathrm{GL}_4(\mathbb{Z})$. As before, we switch to representing our matrices over the ring of Eisenstein integers, obtaining

$$S=\begin{pmatrix}\omega&-\omega\\0&\omega^2\end{pmatrix},\quad T=\begin{pmatrix}\omega&1\\\omega&-\omega\end{pmatrix}\quad\text{and}\quad ST=\begin{pmatrix}0&-1\\1&-1\end{pmatrix}.$$

We see that every element of the submodule generated by these three matrices and the identity matrix is of the form

$$M(n,m,k,l) = \begin{pmatrix} n+\omega(m+k) & (k-l)-\omega m \\ l+\omega k & (n-l)-\omega k+\omega^2 m \end{pmatrix},$$

for integers n, m, k and $l \in \mathbb{Z}$. Computing the determinant of such a matrix, we find

$$\begin{aligned} 2\det M(n,m,k,l) &= 2(n^2+m^2+l^2+k^2+mk-nm-nl-kl) \\ &= (n-m)^2+(k-l)^2+(m+k)^2+(n-l)^2. \end{aligned}$$

This expression is equal to 2 in precisely 24 cases; namely if one of the variables is equal to ± 1, and in the cases

$$\begin{aligned} &m=-k=\pm 1 \quad \text{and} \quad n=l=0, \quad \text{or} \quad n=m=\pm 1 \quad \text{and} \quad k=l=0, \\ &k=l=\pm 1 \quad \text{and} \quad n=m=0, \quad \text{or} \quad n=l=\pm 1 \quad \text{and} \quad m=k=0, \\ &m=-k=n=\pm 1 \quad \text{and} \quad l=0, \quad \text{or} \quad n=m=l=\pm 1 \quad \text{and} \quad k=0, \\ &m=-k=l=\pm 1 \quad \text{and} \quad n=0, \quad \text{or} \quad n=k=l=\pm 1 \quad \text{and} \quad m=0. \end{aligned}$$

Thus, there are no more units other than the trivial units in $\mathrm{SL}(2,3)$, and we see that all groups in the list given in Theorem 7.5.10 actually appear in the case of $R = \mathbb{Z}$.

7.6 — Crystallographic groups

Let E denote the d-dimensional Euclidean space $\mathbb{R}^d$, and $\mathrm{Isom}(E)$ the group of isometries of E. Introducing the topology induced by the point-wise convergence on E makes $\mathrm{Isom}(E)$ into a topological group. A subgroup G of $\mathrm{Isom}(E)$ is called a *crystallographic group* if it is discrete and co-compact. The name derives from the three-dimensional case, where these groups are of use in the classification of crystals in mineralogy. These groups have been studied for a long time. In the theory of crystallographic groups, the theorems of Bieberbach, cf. [22], and Zassenhaus, cf. [158], are essential. These show that there are only finitely many isomorphism classes of crystallographic groups of a given dimension, and give the following algebraic characterisation of crystallographic groups: a group G is crystallographic of dimension d if and only if it contains a free abelian group of rank d as a maximal abelian normal subgroup of finite index. In other words, the group G fits into a short exact sequence

$$1 \to \mathbb{Z}^d \to G \to \Gamma \to 1,$$

where Γ is a finite group acting faithfully on $\mathbb{Z}^d$. All isomorphisms between such groups are realised by affine transformations. The group Γ is called the *point group* or the *holonomy group* of G, and the maximal abelian normal subgroup is called the *translation group*.

Every crystallographic group G gives rise to a faithful action of the point group Γ on $\mathbb{Z}^d$, hence to a faithful integral representation $\rho\colon \Gamma \to \mathrm{GL}_d(\mathbb{Z})$. Given some crystallographic group G, we shall always denote this representation by ρ.

We have already encountered crystallographic groups with the Magnus property. Of course, all finitely generated free abelian groups are crystallographic (with trivial point group) and have (MP). The infinite dihedral group is crystallographic and has the Magnus property by Example 7.1.3. Furthermore, the group

$$\mathbb{Z} \rtimes \mathbb{Z} = \langle a, b \mid a^b = a^{-1} \rangle$$

is both a two-dimensional crystallographic group with point group of order 2 and the fundamental group of the Klein bottle, hence it has the Magnus property (as all fundamental groups of compact surfaces do). The two crystallographic groups with the Magnus property given in [94] are generalisations of the Klein bottle group of dimensions 3 and 8 and with point groups C_2^2 and C_3^2, respectively. They are given as the members of an infinite family of crystallographic groups of dimension $p^2 - 1$ and with point group C_p^2, for any prime p. As a corollary of Theorem 7.6.1, that we will see shortly, the groups associated to the primes 2 and 3 are the only groups with the Magnus property within this infinite family. We shall see some further examples (and non-examples) later on.

It is not unreasonable to expect to find groups with the Magnus property within the class of crystallographic groups, since their conjugacy classes are relatively tame. Indeed, the conjugacy classes within the maximal abelian normal subgroup $\mathbb{Z}^d$ are all finite, while the conjugacy classes outside $\mathbb{Z}^d$ are big in the following sense. If $\overline{a} \in \Gamma$ is the image of some element a in a crystallographic group G, the conjugacy class of its inverse a^{-1} contains $\mathrm{Img}(\overline{a}^\rho - \mathrm{I})a^{-1}$, where $\overline{a}^\rho$ is the matrix describing the action on $\mathbb{Z}^d$, since for all $\mathbf{w} \in \mathbb{Z}^d$

$$(a^{-1})^{\mathbf{w}} = -\mathbf{w}\mathbf{w}^{\overline{a}^\rho} a^{-1} = \mathbf{w}^{\overline{a}^\rho - \mathrm{I}} a^{-1}.$$

Thus, the conjugacy classes outside the maximal abelian normal subgroup are 'submodule sized'. For an in-depth consideration of the normal subgroup structure of crystallographic groups cf. [41].

Our aim is to apply the results of Section 7.5 to obtain restrictions to the possible point groups of crystallographic groups with the Magnus property. To describe these, we make use of *subdirect products*. A subdirect product of a collection of groups $G_1, \dots, G_n$ is a subgroup G of the direct product $G_1 \times \cdots \times G_n$ such that the projection maps $\pi_i : G \to G_i$ to the i^{th} component are surjective for all $i \in \{1, \dots, n\}$. The reason that subdirect products appear in our description below is quite natural. Assume that $\mathbb{Z}^d$ decomposes into a direct sum of Γ^ρ-invariant submodules. Then Γ^ρ is $\mathrm{GL}_d(\mathbb{Z})$-conjugate to a group of block diagonal matrices

$$\left\{ \begin{pmatrix} \gamma^{\rho_1} & & & 0 \\ & \gamma^{\rho_2} & & \\ & & \ddots & \\ 0 & & & \gamma^{\rho_n} \end{pmatrix} \,\middle|\, \gamma \in \Gamma \right\},$$

for some integral representations $\rho_i : \Gamma \to \mathrm{GL}_{d_i}(\mathbb{Z})$ and some positive integers d_i, for $i \in \{1, \ldots, n\}$ such that $\sum_{i=1}^{n} d_i = d$. Then Γ is (isomorphic to) a subdirect product of the groups $\Gamma^{\rho_1}, \ldots, \Gamma^{\rho_n}$.

Note that if ρ decomposes, the crystallographic group G inducing this representation does not need to decompose as a subdirect product. However, every crystallographic group, say of dimension d and with point group Γ, is contained as a finite index subgroup in a crystallographic group that is split, i.e. isomorphic to a semidirect product $\mathbb{Z}^d \rtimes \Gamma$; this group decomposes as a subdirect product like the associated integral representation, cf. [44]. In general, every finite index (and every normal) subgroup of a crystallographic group is again crystallographic, cf. [44]. However, since the Magnus property is not inherited by subgroups (be they normal or of finite index or not), this is only of limited use, and we shall have to fall back onto our considerations of trivial units in certain rings. The unit groups of integral group rings of crystallographic groups have been examined, and those with only trivial units have been characterised topologically, cf. [35].

We remark that all our considerations for the exclusion of certain point groups only depend on the integral representation ρ. We do not consider the actual extension. It is not without merits to restrict to considering coarser classes of crystallographic groups and apply the theory of integral representations, cf. [128], although it can never suffice: There are crystallographic groups with isomorphic point groups and the same integral representation of the point group that are not isomorphic.

For example, there are 17 isomorphism classes of plane (two-dimensional) crystallographic groups; these are sometimes called 'wallpaper groups'. Among these, four classes have a point group of order 2. For a more detailed overview on these groups, see e.g. [33].

We now state the last theorem we prove in the chapter.

Theorem 7.6.1. *Let G be a crystallographic group with point group Γ. If G has the Magnus property, then Γ is isomorphic to a group in the following list,*

(i) *direct products $\mathrm{C}_2^n \times \mathrm{C}_3^m$ or $\mathrm{C}_2^n \times \mathrm{C}_4^m$ for n and $m \in \mathbb{N}_0$,*
(ii) *a subdirect product Δ of copies of C_4 and Q_8 such that Δ is a group of exponent 4 with all involutions central,*
(iii) *a subdirect product Δ of copies of $\mathrm{Dic}_{4\cdot 3}$ such that Δ is a group with all elements of order at most 6 and with all involutions central,*
(iv) *a subdirect product Δ of copies of $\mathrm{SL}(2,3)$ such that Δ is a group with all elements of order at most 6 and with all involutions central.*

In particular, Γ is a $\{2,3\}$-group. If $|\Gamma|$ is odd, it is an elementary abelian 3-group.

Note that the groups of point (ii) fall in the well-understood class of finitely generated (hence, by a result of Sanov [134], finite) groups of exponent 4, cf. [146].

Proof of Theorem 7.6.1. If the group G has the Magnus property, the translation subgroup $\mathbb{Z}^d$ has the relative Magnus property $(\mathrm{MP})_{\overline{\Gamma}}$, where $\overline{\Gamma} = \Gamma \ltimes \langle x \mapsto x^{-1} \rangle$ and Γ acts by conjugation. We do not consider the structure of the actual crystallographic group from this point on, and only use properties of the representation ρ induced by the action of the point group Γ. Let ρ_1 be an irreducible subrepresentation of ρ. As a $\mathbb{Q}$-representation, $\sigma = (\rho)_{\mathbb{Q}}$ decomposes into a direct sum $\sigma_1 \oplus \cdots \oplus \sigma_n$ of irreducible $\mathbb{Q}$-representations. Let $\mathcal{B}$ a basis of $\mathbb{Q}^d$ respecting this decomposition. Multiplying the basis elements in every given summand with the least common multiple of their denominators, we obtain a basis for a finite-index $\mathbb{Z}$-submodule of $\mathbb{Z}^d$ that decomposes into n irreducible Γ^{ρ}-invariant submodules, each affording a $\mathbb{Z}$-representation $\mathbb{Q}$-equivalent to one of the σ_i. Any invariant submodule of $\mathbb{Z}^d$ still has $(\mathrm{MP})_{\overline{\Gamma}}$, since the Γ-action (and the inversion) on the submodule is the same as for the full module. Thus, we may assume that ρ decomposes as a sum of irreducible subrepresentations.

Write V_i for the submodule on which σ_i acts, and Γ_i for the quotient $\Gamma / \ker \sigma_i$ for $i \in \{1, \ldots, n\}$. The group Γ is a subdirect product of $\Gamma_1 \times \cdots \times \Gamma_n$.

By Lemma 7.5.4, applied to the submodule V_i of $\mathbb{Z}^d$ for $i \in \{1, \ldots, n\}$, the ring of the representation σ_i has no non-trivial units. Thus, by Theorem 7.5.10, the group Γ_i is isomorphic to a group in the list below,

$$\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_6, \mathrm{Q}_8, \mathrm{Dic}_{4\cdot 3}, \mathrm{SL}(2,3).$$

If $n = 1$, we are done. Otherwise, a subdirect product Γ of the family $\{\Gamma_i \mid i \in \{1, \ldots, n\}\}$ of groups is naturally a subdirect product of the family $\{\Gamma_i \mid i \in \{1, \ldots, n-2\}\} \cup \{S\}$, where S is the image of Γ under the projection map

$$\pi_{n-1} \times \pi_n : \Gamma_1 \times \cdots \times \Gamma_n \to \Gamma_{n-1} \times \Gamma_n$$

onto the last two factors. The submodule $V_{n-2} \times V_{n-1}$ inherits the property $(\mathrm{MP})_S$, and hence the ring of the representation $\sigma_{n-2} \oplus \sigma_{n-1}$ has only trivial units. In the determination of groups that are subdirect products of groups in the list above, we may thus consider subdirect products of two groups in the list first. Call these groups G and H. If there are no subdirect products of G and H with all elements of order 6 or lower, by Lemma 7.5.6 the groups G and H cannot both appear in the list of groups $\Gamma_1, \ldots, \Gamma_n$.

Notice first that C_6 is the direct product of C_2 and C_3, hence in particular a subdirect product. We may thus omit it from the list above.

We record the list of isomorphism types of subdirect products with the maximal order of any element not exceeding 6 of the two groups G and H in the following table. Since this is clearly a symmetric table, we omit the lower half for better readability.

	C_2	C_3	C_4	Q_8	$Dic_{4\cdot 3}$	$SL(2,3)$
C_2	C_2 C_2^2	C_6	C_4 $C_2 \times C_4$	Q_8 $C_2 \times Q_8$	$Dic_{4\cdot 3}$ $C_2 \times Dic_{4\cdot 3}$	$C_2 \times SL(2,3)$
C_3		C_3, C_3^2	none	none	none	$SL(2,3)$
C_4			C_4, C_4^2 $C_2 \times C_4$	$C_4 \rtimes C_4$, $C_4 \times Q_8$	$Dic_{4\cdot 3}$ $C_2 \times Dic_{4\cdot 3}$	none
Q_8				Q_8, Q_8^2, $C_2 \times Q_8$, $C_2 \rtimes Q_8$	none	none
$Dic_{4\cdot 3}$					$Dic_{4\cdot 3}$ $C_2 \times Dic_{4\cdot 3}$ $C_3 \rtimes Dic_{4\cdot 3}$ $C_2 \times (C_3 \rtimes Dic_{4\cdot 3})$	none
$SL(2,3)$						$SL(2,3)$ $C_2 \times SL(2,3)$, $Q_3 \rtimes SL(2,3)$

All semidirect products are with respect to the action of a cyclic quotient of order 2 by $x \mapsto x^{-1}$, except the action of $SL(2,3)$ on Q_8, which factors over the quotient by the quaternion group inside $SL(2,3)$. A computer program using GAP [52] that produces and verifies this table is attached to this chapter, see Section 8.5.

We see that certain combinations of groups cannot appear within $\Gamma_1, \dots, \Gamma_n$ at the same time, e.g. $SL(2,3)$ and $Dic_{4\cdot 3}$. Furthermore, some combinations are redundant, e.g. all subdirect products involving C_2 and $SL(2,3)$ are can be achieved without using a copy of C_2.

Checking carefully, we find all possible groups covered by those given in the list in the theorem. Note that $C_2^n \times C_4^m$ (for $m > 0$) is a subdirect product of $C_4^{\max(n,m)}$. Using Lemma 7.5.9 and yet again Lemma 7.5.6, we obtain the additional restrictions to the subdirect products in the list. □

7.6.1. Examples of crystallographic groups with and without the Magnus property. — The remainder of this section is devoted to a partial inverse of Theorem 7.6.1. We are able to exhibit crystallographic groups with the Magnus property and all point groups of (i) in the list in the theorem. Unfortunately, we cannot achieve the same for the other points. However, in our efforts to make sure that the list of groups in Theorem 7.5.10 is minimal in case of $R = \mathbb{Z}$, we have established some integral representations ρ such that the ring of ρ has no non-trivial units. Crystallographic groups with point groups acting according to those representations are the natural examples to consider for finding groups belonging to the later points of Theorem 7.6.1. We exclude these natural, low-dimensional candidates, by proving that the corresponding groups do not have the Magnus property.

Groups with the Magnus property

We aim to prove the following proposition.

Proposition 7.6.2. *All groups in (i) of the list in Theorem 7.6.1 appear as point groups of crystallographic groups with the Magnus property.*

It is not surprising that we need to consider direct products to achieve the proof of Proposition 7.6.2. While the class of groups with the Magnus property is not closed under direct products, one can describe precisely how it fails. We use the following two results of Klopsch and Kuckuck.

Lemma 7.6.3. *[94, Lemma 2.3] Let G and H be two groups with the Magnus property. Then the direct product $G \times H$ has the Magnus property if and only if, for every element $(g, h) \in G \times H$, one of the following statements holds true:*

(i) *g and g^{-1} are G-conjugate,*
(ii) *h and h^{-1} are H-conjugate,*
(iii) *(g, h) and (g, h^{-1}) do not generate the same normal subgroup in $G \times H$.*

Theorem 7.6.4. *[94, Theorem 1.1] Let p be an odd prime, and let G and H be residually finite-p groups. If G and H have the Magnus property, then their direct product $G \times H$ has the Magnus property.*

With these tools, we shall construct the sought-for groups out of the following examples with cyclic point groups.

Example 7.6.5. Let $G = \mathbb{Z}^2 \rtimes_A \mathrm{C}_4$, with C_4 generated by an element a acting on $\mathbb{Z}^2$ by the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The group G has the Magnus property.

Proof. It is easy to see that the ring of the representation $a \mapsto A$ is generated by I and A, hence every element has the form

$$\begin{pmatrix} n & -m \\ m & n \end{pmatrix}$$

for some m and $n \in \mathbb{Z}$. The determinant of a matrix of this form is $n^2 + m^2$, consequently, the ring only has trivial units. Furthermore, the only matrix with eigenvalue 1 within this ring is the identity matrix. By Proposition 7.5.8, the group $\mathbb{Z}^2$ has $(\mathrm{MP})_{\mathrm{C}_4}$. Thus, every two elements in the maximal normal abelian subgroup that generate the same normal subgroup are conjugate or inverse-conjugate. It remains to consider the elements with non-trivial image in the quotient C_4.

Let $\mathbf{v}$ and $\mathbf{w} \in \mathbb{Z}^2$. For all $i \in \{1,2,3\}$ we have

$$(\mathbf{v}a^{-i})^{\mathbf{w}} = \mathbf{v}\mathbf{w}^{A^i - \mathrm{I}} a^{-i}.$$

The images of $\mathbb{Z}^2$ under the linear maps

$$A^i - \mathrm{I} = \begin{cases} \left(\begin{smallmatrix} -1 & -1 \\ 1 & -1 \end{smallmatrix}\right) & \text{if } i = 1, \\ \left(\begin{smallmatrix} -2 & 0 \\ 0 & -2 \end{smallmatrix}\right) & \text{if } i = 2, \\ \left(\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right) & \text{if } i = 3. \end{cases}$$

are equal to

$$V = \{(x,y) \in \mathbb{Z}^2 \mid x + y \in 2\,\mathbb{Z}\}, \text{ for } i \neq 2, \qquad \text{and} \qquad W = 2\,\mathbb{Z}^2 \text{ for } i = 2,$$

respectively. Thus, the conjugacy classes of elements of the form $\mathbf{v}a^i$, for some $\mathbf{v} \in \mathbb{Z}^2$, contain $(\mathbf{v} + V)a^i$ for $i \neq 2$, and $(\mathbf{v} + W)a^2$ otherwise. Let $(x,y) \in V$. Then $(x,y)^A \in V$, since

$$y - x = x + y - 2x = 2(n - x)$$

for some $n \in \mathbb{Z}$. Also, if $(x,y) \notin V$, then $(x,y)^A \notin V$. The same holds for A^2, A^3 and W in any combination. The action of A on the quotient $\mathbb{Z}^2/V$ is hence trivial, and the action of A on the quotient $\mathbb{Z}^2/W$ is given by

$$A \equiv_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the conjugacy classes in G are precisely the following: The conjugacy classes $\mathbf{v}^{\mathrm{C}_4}$ of finite size within $\mathbb{Z}^2$, and the classes

$$\begin{aligned}
Va, &\text{ normally generating} \quad V\mathrm{C}_4, \\
((1,0) + V)a, &\text{ normally generating} \quad G, \\
Va^3, &\text{ normally generating} \quad V\mathrm{C}_4, \\
((1,0) + V)a^3, &\text{ normally generating} \quad G, \\
Wa^2, &\text{ normally generating} \quad W\langle a^2 \rangle, \\
(1,1) + Wa^2, &\text{ normally generating} \quad V\langle a^2 \rangle, \\
\{(0,1),(1,0)\} + Wa^2, &\text{ normally generating} \quad \mathbb{Z}^2\langle a^2 \rangle.
\end{aligned}$$

The first and third, and the second and fourth classes are inverses of each other, respectively. Thus, G has the Magnus property. □

Example 7.6.6. The group $G = \mathbb{Z}^2 \rtimes_A \mathrm{C}_3$, acting by

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

has the Magnus property.

Proof. The matrices A and A^2 fulfil $A^2 + A + \mathrm{I} = 0$. Thus, the ring of the associated representation is generated by I and A, and every element of said ring has the form and the determinant

$$\left|\begin{pmatrix} n-m & m \\ -m & n \end{pmatrix}\right| = n^2 - nm + m^2.$$

Thus, there are only the six trivial units corresponding to $n = \pm 1$ and $m = 0$ (the matrices $\pm\mathrm{I}$), to $m = \pm 1$ and $n = 0$ (the matrices A and $-A$), and $n = m = \pm 1$ (the matrices A^2 and $-A^2$). Furthermore, there are no non-trivial elements with eigenvalue 1, hence $\mathbb{Z}^3$ has $(\mathrm{MP})_{\mathrm{C}_3}$.

We proceed as in the previous examples. The images under

$$A - \mathrm{I} = \begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad A^2 - \mathrm{I} = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix}$$

are equal to the submodule $V = \{(x, y) \in \mathbb{Z}^2 \mid x + y \equiv_3 0\}$. But this submodule is not invariant, and (as in the previous example), its invariant closure is $\mathbb{Z}^2$. Hence the infinite conjugacy classes of G are

$$\mathbb{Z}^2\{a\} \quad \text{and} \quad \mathbb{Z}^2\{a^2\},$$

which are inverse to each other and generate the full group. Thus, G has the Magnus property. □

Finally, although we shall not need it for the proof of Proposition 7.6.2, we provide an example of a crystallographic group with the Magnus property, and with cyclic point group of order 3, such that the action of the ring of the associated representation ρ contains, in contrast to the previous examples, non-trivial elements with eigenvalue 1.

Example 7.6.7. The group $G = \mathbb{Z}^3 \rtimes_A \mathrm{C}_3$, acting by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

has the Magnus property.

Proof. The ring of the associated representation is clearly isomorphic to the group ring of C_3, which has only trivial units by Theorem 7.5.1. However, we have to work around the problem that there are non-trivial elements with eigenvalue 1. Indeed, the determinant of an element $m\,\mathrm{I} + nA + kA^2$ is equal to

$$\begin{vmatrix} m & n & k \\ k & m & n \\ n & k & m \end{vmatrix} = (m+n+k)((m-n)^2 + (m-k)^2 + (n-k)^2).$$

By substitution of $m-1$ for m, we see that the element has an eigenvalue 1 if and only if either $m = 1-(n+k)$ or $n = k = m-1$. By Proposition 7.5.8, these are the only elements we need to consider.

Assume $n = k = m-1$, i.e. we consider matrices of the form

$$\begin{pmatrix} l+1 & l & l \\ l & l+1 & l \\ l & l & l+1, \end{pmatrix} \quad \text{for } l \in \mathbb{Z}\,.$$

Clearly the eigenmodule $\mathrm{Eig}_A(1)$ is equal to the diagonal submodule $D = \{(c,c,c) \mid c \in \mathbb{Z}\}$. The action of A on D is trivial, hence the invariant submodule generated by an element $(b,b,b) \in D$ is equal to bD. Consequently, all generators, i.e. (b,b,b) and $-(b,b,b)$, are inverses to each other.

Now assume $m = 1-(n+k)$. Then the 1-eigenmodule of the matrix $m\,\mathrm{I} + nA + kA^2$ is the kernel of

$$\begin{pmatrix} -n-k & n & k \\ k & -n-k & n \\ n & k & -n-k \end{pmatrix},$$

which is equal to the invariant submodule $V = \{(x,y,z) \in \mathbb{Z}^3 \mid x+y+z = 0\}$. Choose the basis $((1,-1,0),(0,1,-1))$ for V, then A acts on V by

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

In Example 7.6.6 we have seen that the ring associated to this representation has only trivial units and that there are no non-trivial elements with eigenvalue 1. Thus, two elements of V generating the same invariant submodule are conjugate or inverse-conjugate under the action of A. Thus, $\mathbb{Z}^3$ has $(\mathrm{MP})_{C_3}$.

Consider the matrices

$$A - \mathrm{I} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad A^2 - \mathrm{I} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

and their image W. Clearly W is A-invariant. Let $\mathbb{Z}\times\{0\}^2 = \{(c,0,0) \mid c \in \mathbb{Z}\}$ be a set of representatives for the quotient $\mathbb{Z}^3/D$. By construction A acts trivially on the quotient. Thus, the infinite conjugacy classes in G are

$$((c,0,0)+D)\{a\}, \text{ normally generating } \quad \{(x,y,z)\in\mathbb{Z}^3 \mid x+y+z \equiv_c 0\}\mathrm{C}_3,$$

and the same for a^2, for all $c \in \mathbb{Z}$. Since a and a^2 are inverses, the group G has the Magnus property. □

Proof of Proposition 7.6.2. The class of crystallographic groups is closed under direct products. The point group of the direct product of two crystallographic groups is the direct product of the respective point groups. Thus, to find crystallographic groups with the Magnus property and point groups of type (1), it remains to establish the Magnus property for the direct products of the crystallographic groups with the Magnus groups exhibited in Example 7.1.3, Example 7.6.5 and Example 7.6.6.

Let G be a group such that all elements $g \in G$ are conjugate to their inverse. Such a group is called *ambivalent*. Assume that G has the Magnus property. Lemma 7.6.3 clearly implies that the direct product $G \times H$ with another group, that also does possess the Magnus property, has the Magnus property. The infinite dihedral group is an ambivalent crystallographic group with the Magnus property. Thus, the factors of type C_2^n are dealt with.

The group $\mathbb{Z}^2 \rtimes \mathrm{C}_3$ from Example 7.6.6 is a residually finite-3 group; by Theorem 7.6.4 the direct products of such groups have the Magnus property.

It remains to consider powers of C_4. Consider the group $\mathbb{Z}^2 \rtimes \mathrm{C}_4$ as defined in Example 7.6.5. Let $(\mathbf{v}a^i, \mathbf{w}a^j) \in (\mathbb{Z}^2 \rtimes \mathrm{C}_4)^2$ be an element of the direct product, for some $\mathbf{v}$ and $\mathbf{w} \in \mathbb{Z}^2$ and $i, j \in \{0,1,2,3\}$. We aim to use Lemma 7.6.3. If i or j is equal to 0 or 2, the element $(\mathbf{v}a^i, \mathbf{w}a^j)$ is conjugate to its inverse, since

$$\mathbf{v}^{-1} = -\mathbf{v} = \mathbf{v}^{a^2} \quad \text{and} \quad (\mathbf{v}a^2)^{-1} = a^2(-\mathbf{v}) = \mathbf{v}a^2.$$

Thus, without loss of generality, let $i = j = 1$. Consider the image of the normal subgroup generated by $(\mathbf{v}a, \mathbf{w}a)$ and $(\mathbf{v}a, (\mathbf{w}a)^{-1})$ in the quotient C_4^2, i.e. the subgroup generated by the images (a,a) and (a,a^{-1}). Since these do not coincide, the normal closures in the crystallographic group do not coincide, and by Lemma 7.6.3, the direct product has the Magnus property. Arguing similarly, one sees that direct products of finitely many factors of $\mathbb{Z}^2 \rtimes \mathrm{C}_4$ have the Magnus property. Thus, we can realise all point groups of (1). □

Point group Q_8 and the Magnus property

The representation $\varphi\colon \mathrm{Q}_8 \to \mathrm{GL}_4(\mathbb{Z})$ given by

$$i \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

describes the smallest-dimensional fixed point-free faithful integral representation of the quaternion group Q_8. The only crystallographic group that has a point group isomorphic to Q_8 acting via φ on $\mathbb{Z}^4$ is the semidirect product $G = \mathbb{Z}^4 \rtimes_\varphi \mathrm{Q}_8$, which may be checked using the Cryst-package for Gap. Using the methods of the previous section, one can check that the ring of the representation φ has indeed only trivial units.

Write $c = i^2$ for the central involution. Notice that $c^\varphi = -\mathrm{I}$. Let $\mathbf{v}$ and $\mathbf{w} \in \mathbb{Z}^4$ and $g \in \mathrm{Q}_8$. Calculate

$$(\mathbf{v}c)^{\mathbf{w}g} = \mathbf{v}^g \mathbf{w}^{(c-1)g} c^g = \mathbf{v}^g \mathbf{w}^{-2g} c.$$

Choosing $\mathbf{w}$ as the pre-images of a basis of $\mathbb{Z}^4$ under the respective transformations g, we see that the conjugacy class of $\mathbf{v}c$ is equal to the union $\bigcup_{g \in \mathrm{Q}_8} \mathbf{v}^g 2\,\mathbb{Z}^4\, c$. Since we want to show the existence of an element having the same normal closure as $\mathbf{v}c$ but being not conjugate to it, we might calculate entry wise modulo 2 from here on. Choose $\mathbf{v} = (1, 0, 0, 0)$ and consider

$$(\mathbf{v}c)^i (\mathbf{v}c)^j (\mathbf{v}c) \equiv_2 \mathbf{v}^{i+j+1} c.$$

Since

$$i^\varphi + j^\varphi + \mathrm{I} \equiv_2 \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

is invertible, we have $\langle \mathbf{v}c \rangle^G = \langle \mathbf{v}^{i+j+1} c \rangle^G$. But $\mathbf{v}^{i+j+1} = (1, 1, 1, 0)$, while $\mathbf{v}^g$ has precisely one non-zero entry for all $g \in \mathrm{Q}_8$. Thus, G does not have the Magnus property.

Point group $\mathrm{Dic}_{4\cdot 3}$ and the Magnus property

We prove that the four-dimensional integral representation of $\mathrm{Dic}_{4\cdot 3}$, as described by the matrices A and B in Section 7.5.1, does not afford a crystallographic group with the Magnus property. We write a and b for the corresponding generators of $\mathrm{Dic}_{4\cdot 3}$, and φ for the representation defined by $a^\varphi = A$ and $b^\varphi = B$. We have seen that the ring of the representation φ has only trivial units. But it does not give rise to a crystallographic group with the Magnus property. The only crystallographic group with the point group $\mathrm{Dic}_{4\cdot 3}$ and action prescribed by φ is the semidirect product $G = \mathbb{Z}^4 \rtimes_\varphi \mathrm{Dic}_{4\cdot 3}$; this can be checked using the Cryst-package for Gap.

We will explicitly produce two elements generating the same normal subgroup that are neither conjugate nor inverse-conjugate. Since we established in Section 7.5.1 that $\mathbb{Z}^4$ has $(\mathrm{MP})_{\mathrm{Dic}_{4\cdot3}}$, these elements must have non-trivial image in $\mathrm{Dic}_{4\cdot3}$. Let

$$\mathbf{v} = (0,0,0,1) \quad \text{and} \quad \tilde{\mathbf{v}} = (1,-1,2,-1) \in \mathbb{Z}^4 .$$

We claim that $\mathbf{v}a$ and $\tilde{\mathbf{v}}a$ are the promised pair of elements. Let us first check that they are neither conjugate nor inverse-conjugate in G.

Notice that a is not conjugate to its inverse in $\mathrm{Dic}_{4\cdot3}$. Thus, $\mathbf{v}a$ and $(\tilde{\mathbf{v}}a)^{-1}$ cannot be conjugate in G as well. We now describe the intersection of the conjugacy class of $\mathbf{v}a$ with $\mathbb{Z}^4\, a$. To do so, calculate the centraliser of a in $\mathrm{Dic}_{4\cdot3}$, which is $\langle a\rangle$. Then consider the conjugates

$$(\mathbf{v}a)^{\mathbf{w}a^i} = \left(\mathbf{v}^{A^i} + \mathbf{w}^{(A^3-\mathrm{I})A^i}\right) a,$$

for $\mathbf{w} \in \mathbb{Z}^4$ and $i \in \{0,\dots,3\}$. This allows us to give the following description of the relevant part of the conjugacy class,

$$\begin{aligned}(\mathbf{v}a)^G \cap \mathbb{Z}^4\, a &= \left\{(\mathbf{v}a)^{\mathbf{w}a^i} \mid \mathbf{w}\in\mathbb{Z}^4, i \in \{0,1,2,3\}\right\} \\ &= \left(\left\{\mathbf{v}^{A^i} \mid i \in \{0,1,2,3\}\right\} + (\mathbb{Z}^4)^{A^3-\mathrm{I}}\right) a.\end{aligned}$$

Consider the entry wise reduction of the $\mathbb{Z}^4$-part of the elements of this intersection. Since

$$A^3 - \mathrm{I} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \equiv_2 \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

and $A^2 = -\mathrm{I}$, we find that

$$(\mathbf{v}a)^G \cap \mathbb{Z}^4\, a \equiv_2 \left(\left\{\mathbf{v} \bmod 2, \mathbf{v}^A \bmod 2\right\} + \{(x,y,x,y) \mid x,y \in \mathbb{F}_2\}\right) a.$$

But $\tilde{\mathbf{v}} - \mathbf{v} \equiv_2 (1,1,0,0)$ and $\tilde{\mathbf{v}} - \mathbf{v}^A \equiv_2 (1,0,0,1)$. Thus, the elements $\tilde{\mathbf{v}}a$ and $\mathbf{v}a$ are not conjugate in G. It remains to prove that they generate the same normal subgroup. For this, we will show that $\tilde{\mathbf{v}}a \in \langle \mathbf{v}a\rangle^G$ and vice versa.

We use the following identity in $\mathrm{Dic}_{4\cdot3}$ to our advantage;

$$a^b a^3 a^{b^2} = a.$$

We compute, for all $\mathbf{w} \in \mathbb{Z}^4$,

$$(\mathbf{v}a)^{\mathbf{w}b}(\mathbf{v}a)^3(\mathbf{v}a)^{b^2} = (0,0,2,0)\mathbf{w}^{(A^3-\mathrm{I})B} a^b a^3 a^{b^2}.$$

For $\mathbf{w} = (1, 0, 0, 0)$, we obtain

$$(0, 0, 2, 0)\mathbf{w}^{(A^3-\mathrm{I})B} a^b a^3 a^{b^2} = \tilde{\mathbf{v}} a^b a^3 a^{b^2} = \tilde{\mathbf{v}} a.$$

On the other hand, we find that, using the same equation, but with $\mathbf{w} = (-2, -2, 2, 0)$,

$$(\tilde{\mathbf{v}} a)^{\mathbf{w} b} (\tilde{\mathbf{v}} a)^3 (\tilde{\mathbf{v}} a)^{b^2} = \mathbf{v} a^b a^3 a^{b^2} = \mathbf{v} a.$$

Point groups $\mathrm{SL}(2,3)$ and the Magnus property

At last, we consider the integral representation φ of $\mathrm{SL}(2,3)$ that gives rise to a ring with only trivial units in Section 7.5.2. Write s and t for the generators of $\mathrm{SL}(2,3)$, such that $s^\varphi = S$ and $t^\varphi = T$, using the matrices given in Section 7.5. As for the point groups Q_8 and $\mathrm{Dic}_{4\cdot 3}$, there are no crystallographic groups with point group $\mathrm{SL}(2,3)$, the action induced by φ, and the Magnus property. However, to see this, we have to use different methods, as there are four distinct crystallographic groups with this prescribed action; these can again be found using the Cryst-package for Gap. Therefore it is not enough to consider the semidirect product, which would be easy to deal with: it has a retract isomorphic to $\mathrm{SL}(2,3)$, which does not possess the Magnus property. A group G with a retract R that does not have (MP) can not have (MP) either, since every pair of elements generating the same normal subgroup in R generates the same normal subgroup in G, but elements of R are only conjugate in G if they are conjugate in R.

Let G be a crystallographic group with point group $\mathrm{SL}(2,3)$, acting by φ on a normal subgroup isomorphic to $\mathbb{Z}^4$. Let $r \in \mathrm{SL}(2,3)$ be an element of order 6, say $r = st^3$. Then

$$R := r^\varphi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Let $X = \{x_g \mid g \in \mathrm{SL}(2,3)\}$ be a transversal for $\mathrm{SL}(2,3)$ in G.

Observe that the finite group $\mathrm{SL}(2,3)$ does not have the Magnus property, since both r and r^2 normally generate the full group $\mathrm{SL}(2,3)$, but are of different order and thus not conjugate. As a consequence, the lifts x_r and x_{r^2} are not conjugate in G (even though they might have the same order). We prove that both lifts normally generate the whole group G. For any element $g \in \mathrm{SL}(2,3)$ that normally generates $\mathrm{SL}(2,3)$, it is enough to show that $\mathbb{Z}^4 x_g \subseteq \langle x_g \rangle^G$ to obtain that x_g normally generates G. Indeed, let $h \in \mathrm{SL}(2,3)$ be another element. There exist elements $c_i \in \mathrm{SL}(2,3)$ for $i \in \{0, \ldots, n-1\}$ such that

$$h = \prod_{i=0}^{n-1} g^{c_i}.$$

Consequently $x_h = \mathbf{u}_h \prod_{i=0}^{n-1} (x_g)^{x_{c_i}}$ for some $\mathbf{u}_h \in \mathbb{Z}^4$. But since $\mathbf{u}_h x_g$ and x_g are contained in $\langle x_g \rangle^G$, so is $\mathbf{u}_h$, and hence x_h.

For any power r^i of r and any $\mathbf{v}$ and $\mathbf{w} \in \mathbb{Z}^4$ we have

$$(\mathbf{v}x_{r^i})^{\mathbf{w}} = \mathbf{v}\mathbf{w}^{R^{-i}-\mathrm{I}} x_{r^i}.$$

In case $i = 1$ we find $R^5 - \mathrm{I} = ST$, an invertible matrix. Hence $\mathbb{Z}^4\, x_r \subseteq (\mathbf{v}x_r)^G$, and since r normally generates $\mathrm{SL}(2,3)$, we have $\langle x_r \rangle^G = G$.

In case $i = 3$ we obtain, in the same way, $2\,\mathbb{Z}^4\, x_{r^3} \subseteq (\mathbf{v}x_{r^3})^G$. Since $\langle x_{r^2} \rangle^G$ contains an element of the form $\mathbf{v}x_{r^3}$, we have $2\,\mathbb{Z}^4\, x_{r^2} \subseteq \langle x_{r^2} \rangle^G$. But

$$R^4 - \mathrm{I} = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix} \equiv_2 (R^5 - \mathrm{I})^2$$

is invertible modulo 2, hence $\mathbb{Z}^4\, x_{r^2} \subseteq \langle x_{r^2} \rangle^G$ and thus $\langle x_{r^2} \rangle^G = G$.

Chapter 8

Free polynilpotent groups and the Magnus property

Written in collaboration with Benjamin Klopsch and Luis Mendonça.

Abstract. Motivated by a classic result for free groups, one says that a group G has the Magnus property if the following holds: whenever two elements generate the same normal subgroup of G, they are conjugate or inverse-conjugate in G.

It is a natural problem to find out which relatively free groups display the Magnus property. We prove that a free polynilpotent group of any given class row has the Magnus property if and only if it is nilpotent of class at most 2. For this purpose we explore the Magnus property more generally in soluble groups, and we produce new techniques, both for establishing and for disproving the property. We also prove that a free centre-by-(polynilpotent of given class row) group has the Magnus property if and only if it is nilpotent of class at most 2.

On the way, we display 2-generated nilpotent groups (with non-trivial torsion) of any prescribed nilpotency class with the Magnus property. Similar examples of finitely generated, torsion-free nilpotent groups are hard to come by, but we construct a 4-generated, torsion-free, class-3 nilpotent group of Hirsch length 9 with the Magnus property. Furthermore, using a weak variant of the Magnus property and an ultraproduct construction, we establish the existence of metabelian, torsion-free, nilpotent groups of any prescribed nilpotency class with the Magnus property.

8.1 — Introduction

A group G has the *Magnus property* if the following holds: whenever $g, h \in G$ generate the same normal subgroup $\langle g \rangle^G = \langle h \rangle^G$, the element g is already conjugate in G to h or to h^{-1}. Magnus [107] established this property for free groups, using his "Freiheitssatz". The Magnus property is a first-order property, in the sense of model theory; consequently all groups with the same elementary theory as free groups have the Magnus property. During the last two decades, the Magnus property has been explored and established for various classes of groups using different techniques, e.g., for fundamental groups of closed surfaces [24], direct products of free groups KK16,Fel21, and certain amalgamated products [46].

Groups with the Magnus property are typically torsion-free and 'big', for instance, in the sense that they do not satisfy any non-trivial law, viz. any non-trivial identical relation. Even so free abelian groups possess the Magnus property, for obvious reasons, and certain crystallographic groups with the Magnus property were manufactured in [94]. In conjunction with Magnus' classic result, this prompts a natural question for relatively free groups, viz. $\mathcal{V}$-free groups for any given variety $\mathcal{V}$ of groups.

Problem A. Let $\mathcal{V}$ be a variety of groups, viz. the class of all groups satisfying each one of a given set of laws. Which $\mathcal{V}$-free groups have the Magnus property?

By basic considerations, it is enough to settle the question for relatively free groups of finite rank, viz. on finitely many free generators, because for each variety of groups $\mathcal{V}$ there is a precise cut-off point $\delta_{\mathcal{V}} \in \mathbb{N}_0 \cup \{\infty\}$ for the ranks of $\mathcal{V}$-free groups with the Magnus property; see Corollary 8.2.3. As mentioned above, for the variety $\mathcal{U}$ of all groups and for the variety $\mathcal{A}$ of abelian groups, we know that every free group and every $\mathcal{A}$-free group has the Magnus property: hence $\delta_{\mathcal{U}} = \delta_{\mathcal{A}} = \infty$. Likewise, it is easy to see that the variety $\mathcal{A}_m$ of abelian groups of exponent m (that is, m or dividing m) has $\delta_{\mathcal{A}_m} = \infty$ if $m \in \{1, 2, 3, 4, 6\}$, and $\delta_{\mathcal{A}_m} = 0$ otherwise.

Perhaps it is natural to concentrate first on varieties of exponent zero, viz. varieties $\mathcal{V}$ such that x^m is not a universal law in $\mathcal{V}$-groups, for any $m \in \mathbb{N}$. Prominent examples of this kind are the varieties $\mathcal{N}_{\mathbf{c}}$ of all *polynilpotent groups* of class row $\mathbf{c}$, for any given length $l \in \mathbb{N}$ and class tuple $\mathbf{c} = (c_1, \ldots, c_l) \in \mathbb{N}^l$. We recall that a group G belongs to $\mathcal{N}_{\mathbf{c}}$ if the term $\gamma_{(c_1+1,\ldots,c_l+1)}(G)$ of its *iterated lower central series* vanishes; here $\gamma_{(1)}(G) = \gamma_1(G) = G$ and inductively we set

$$\gamma_{(c_1+1,\ldots,c_l+1)}(G) = \gamma_{c_l+1}\big(\gamma_{(c_1+1,\ldots,c_{l-1}+1)}(G)\big), \qquad \text{for } l > 1,$$

where $\gamma_{(c_1+1)}(G) = \gamma_{c_1+1}(G) = [\gamma_{c_1}(G), G]$ is the $(c_1+1)^{\text{st}}$ term of the ordinary lower central series of G. For instance, for $l = 1$ and $\mathbf{c} = (c)$ the variety $\mathcal{N}_{\mathbf{c}}$ consists of all nilpotent groups of class at most c; for $l \in \mathbb{N}$ and $\mathbf{c} = (1, \ldots, 1) \in \mathbb{N}^l$, the variety $\mathcal{N}_{\mathbf{c}}$ consists of all soluble groups of derived length at most l. For free polynilpotent groups, we resolve Problem A completely.

Theorem 8.1.1. *Let G be an $\mathcal{N}_{\mathbf{c}}$-free group of rank d, i.e., a free polynilpotent group of class row $\mathbf{c}$ that is freely generated by d elements, where $d, l \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{N}^l$.*

Then G has the Magnus property if and only if G is nilpotent of class at most 2; equivalently, if and only if $d = 1$ or $\mathbf{c} \in \{(1), (2)\}$.

The proof uses the notion of basic witness pairs for not having the Magnus property; which are defined in Lemma 8.2.6. The starting point of the proof is that the restricted wreath product $C_\infty \wr C_\infty$ admits such witness pairs; see Proposition 8.4.1. Almost as a by-product, we obtain the following similar result, for further varieties.

Theorem 8.1.2. *Let G be a free centre-by-$\mathcal{N}_{\mathbf{c}}$ group of rank d, where $d, l \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{N}^l$. Then G has the Magnus property if and only if G is nilpotent of class at most 2; equivalently, if and only if $d = 1$ or $\mathbf{c} = (1)$.*

Since the Magnus property is a first-order property, one may wonder about groups with the same elementary theory as free polynilpotent groups. Groups that are elementarily equivalent to free nilpotent groups were considered in MS09,MS11. We remark that $\mathcal{N}_{\mathbf{c}}$-free groups are torsion-free, while free centre-by-$\mathcal{N}_{\mathbf{c}}$ groups may involve central torsion of exponent 2; compare with Kuz82,Sto89.

In order to prove Theorems 8.1.1 and 8.1.2 we explore the Magnus property in more general groups, and we produce new techniques, both for establishing and for disproving the property. In Proposition 8.2.4 we provide a useful sufficient criterion under which the Magnus property passes to factor groups; for instance, if G has the Magnus property and $N \trianglelefteq G$ is finite then G/N inherits the Magnus property. In Proposition 8.3.2 we see that every torsion-free, class-2 nilpotent group has the Magnus property. Example 8.3.8 provides an explicit family of finitely generated, nilpotent groups (with non-trivial 3-torsion) of any prescribed nilpotency class that possess the Magnus property. In contrast it appears much harder to capture finitely generated, nilpotent groups of prescribed nilpotency class $c \geq 3$ with the Magnus property that are torsion-free. In Example 8.3.11 we construct explicitly a 4-generated, torsion-free, class-3 nilpotent group of Hirsch length 9 with the Magnus property. This can be seen as a very first step toward tackling the following problem which suggests itself.

Problem B. Are there finitely generated, torsion-free, nilpotent groups with the Magnus property of any prescribed nilpotency class? Characterise or even classify finitely generated, torsion-free, nilpotent groups with the Magnus property.

Currently, we seem to be far from solving this problem, but we can establish a related and somewhat surprising result, using a weak variant of the Magnus property and an ultraproduct construction.

Theorem 8.1.3. *For every $c \in \mathbb{N}$ there exists a countable, metabelian, torsion-free, nilpotent group with the Magnus property that has nilpotency class precisely c.*

In this context we remark that, by a classical embedding theorem of Higman, Neumann and Neumann [82], every countable torsion-free group G can be embedded into a countable torsion-free group $\mathcal{G}$ with only two conjugacy classes; such a group $\mathcal{G}$ has the Magnus property. Using small cancellation techniques, Osin [117] showed that one can even arrange for $\mathcal{G}$ to be finitely generated. However, the structure of such groups $\mathcal{G}$, which arise as inductive limits, can be very different from the one of the input group G, and they are far from the groups we produce for Theorem 8.1.3.

Notation. Let G be a group. For $x, y \in G$ we write $x^y = y^{-1}xy$ and $[x, y] = x^{-1}x^y$. Throughout, we use left-normed commutators; for instance, we write $[x, y, z] = [[x, y], z]$. A similar convention applies to iterated Lie commutators in associated Lie rings. For $X \subseteq G$ we denote by $\langle X\rangle^G = \langle x^g \mid x \in X, g \in G\rangle$ the normal closure of X in G, viz. the smallest normal subgroup containing X. For a singleton $X = \{x\}$ we use the shorter notation $\langle x\rangle^G$.

We denote by $\mathrm{Z}(G)$ the centre of G, and we write $\mathrm{Z}_i(G)$, $i \in \mathbb{N}_0$, for the terms of the upper central series of G. The iterated lower central series and, in particular, the lower central series $\gamma_i(G)$, $i \in \mathbb{N}$, were already discussed above.

Suppose that $\mathcal{V}$ is a non-trivial variety of groups. The *rank* of a $\mathcal{V}$-free group G is the cardinality of a $\mathcal{V}$-free generating set for G. We use the term sparingly and no confusion with other common notions of rank, such as Prüfer rank should arise.

For $m \in \mathbb{N} \cup \{\infty\}$ we write C_m to denote a cyclic group of order m.

8.2 — Preliminaries and auxiliary results

We recall that the *Magnus property* is a first-order property in the sense of model theory; indeed, sometimes it is useful to rephrase it for a group G as follows:

$$\begin{aligned} \forall k, l \in \mathbb{N}_0 \quad \forall g, h \in G \qquad \forall m_1, \ldots, m_k \in \{1, -1\} \quad \forall v_1, \ldots, v_k \in G \\ \forall n_1, \ldots, n_l \in \{1, -1\} \quad \forall w_1, \ldots, w_l \in G : \end{aligned} \qquad (\mathsf{MP})$$
$$\Big(h = \prod\nolimits_{i=1}^{k} (g^{m_i})^{v_i} \ \wedge\ g = \prod\nolimits_{j=1}^{l} (h^{n_j})^{w_j}\Big) \implies \Big(\exists v \in G: \quad g^v = h \ \vee\ g^v = h^{-1}\Big),$$

where the quantifier over the integers k, l can be eliminated by passing to a countable collection of sentences in the first-order language of groups. For short we say that G is an MP-*group* if G has the Magnus property.[†]

We recall that, if $\mathcal{P}$ is any property of groups, then a group G is *locally a $\mathcal{P}$-group* if each finite subset of G is contained in a $\mathcal{P}$-subgroup of G. If $\mathcal{P}$ is inherited by subgroups, this is equivalent to the requirement that each finitely generated subgroup of G has $\mathcal{P}$. The proof of the following lemma is routine, using (MP).

Lemma 8.2.1. *Every locally* MP*-group is an* MP*-group.*

Of course, the Magnus property does not generally pass from a group to its subgroups or quotients. Nevertheless there are interesting situations, where this happens. We record a simple, but useful observation.

[†]The terms "M-group" and "Magnus group" are unfortunately already in use with other meanings. For lack of better alternatives, we have settled for "MP-group".

Lemma 8.2.2. *Let G be a group, and suppose that $G = H \ltimes N$ splits over a normal subgroup $N \trianglelefteq G$. If G is an* MP*-group then so is H.*

For each variety of groups $\mathcal{V}$ we set

$$\delta_{\mathcal{V}} = \sup\{d \in \mathbb{N}_0 \mid \mathcal{V}\text{-free groups of rank } d \text{ are MP-groups}\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Lemmas 8.2.1 and 8.2.2 already have a useful consequence for relatively free groups.

Corollary 8.2.3. *Let $\mathcal{V}$ be a variety of groups. If $\delta_{\mathcal{V}} = \infty$, then every $\mathcal{V}$-free group is an* MP*-group. If $\delta_{\mathcal{V}} < \infty$, then a $\mathcal{V}$-free group is an* MP*-group if and only if it has rank at most $\delta_{\mathcal{V}}$.*

The next result is less obvious, if not surprising; in particular, it provides a powerful handle to deal with free nilpotent and, more generally, free polynilpotent groups.

Proposition 8.2.4. *Let G be an* MP*-group, and let $N \trianglelefteq G$ such that for each $g \in G \smallsetminus N$ the $\subseteq$-partially ordered set of normal subgroups*

$$\Omega_{gN} = \left\{\langle gz\rangle^G \mid z \in N\right\}$$

satisfies the minimal condition. Then G/N is an MP*-group.*

Proof. Let $g, h \in G$ such that their images in G/N have the same normal closure, in other words such that $\langle g\rangle^G \equiv_N \langle h\rangle^G$. If $g \equiv_N 1$, also $h \equiv_N 1$, and they are conjugate to one another modulo N. Now suppose that $g \not\equiv_N 1$. Choose $k, l \in \mathbb{N}$, $m_1, \ldots, m_k, n_1, \ldots, n_l \in \mathbb{Z}$ and $v_1, \ldots, v_k, w_1, \ldots, w_l \in G$ such that

$$\prod\nolimits_{i=1}^{k} (g^{m_i})^{v_i} \equiv_N h \qquad \text{and} \qquad \prod\nolimits_{j=1}^{l} (h^{n_j})^{w_j} \equiv_N g.$$

Since Ω_{gN} satisfies the minimal condition, we find $g_{\min} \in gN$ such that $\langle g_{\min}\rangle^G$ is $\subseteq$-minimal among all subgroups of the form $\langle y\rangle^G$ for $y \in gN$. Consider

$$h_0 = \prod\nolimits_{i=1}^{k} \left(g_{\min}^{m_i}\right)^{v_i} \equiv_N h \qquad \text{and} \qquad g_0 = \prod\nolimits_{j=1}^{l} \left(h_0^{n_j}\right)^{w_j} \equiv_N g.$$

These elements satisfy $\langle g_0\rangle^G \subseteq \langle h_0\rangle^G \subseteq \langle g_{\min}\rangle^G$, hence, by the minimal choice of $g_{\min}$, we conclude that $\langle g_0\rangle^G = \langle h_0\rangle^G$. Since G has the Magnus property, there exists $v \in G$ such that $g_0^v = h_0$ or $g_0^v = h_0^{-1}$, hence

$$g^v \equiv_N g_0^v \equiv_N h_0 \equiv_N = h \quad \text{or, similarly,} \quad g^v \equiv_N h^{-1}. \qquad \square$$

We record some immediate consequences, which are quite remarkable.

Corollary 8.2.5. *Let G be an* MP*-group, $\Omega_G = \{\langle g\rangle^G \mid g \in G\}$, and let $N \trianglelefteq G$.*

(1) *If Ω_G satisfies the minimal condition, then G/N is an* MP*-group.*
(2) *If N is finite, then G/N is an* MP*-group.*

In particular, if G is a finitely generated nilpotent group with the Magnus property then G modulo its torsion subgroup $\tau(G)$ is a finitely generated, torsion-free nilpotent group with the Magnus property; in contrast, the finite group $\tau(G)$ does not in general inherit the Magnus property; compare with Example 8.3.8.

The following sufficient criterion turns out to be useful for rejecting the Magnus property and gives rise to the notion of basic witness pairs which is to play a key role.

Lemma 8.2.6. *Let G be a group, and let $g \in G$ and $v \in [G,G] \smallsetminus \{[g,w] \mid w \in G\}$ be such that $g^2 \not\equiv_{[G,G]} 1$ and $\langle g \rangle^G = \langle gv \rangle^G$. Then G is not an* MP*-group.*

Proof. From $gv \equiv_{[G,G]} g \not\equiv_{[G,G]} g^{-1}$ it follows that g and gv are not inverse-conjugate in G. They are not conjugate to one another either, as $gv \neq g[g,w] = g^w$ for all $w \in G$. Thus $\langle g \rangle^G = \langle gv \rangle^G$ shows that G does not have the Magnus property. □

For short we say that (g,v) is a *basic* $\neg(\mathsf{MP})$*-witness pair* for G, viz. a witness pair for G not having the Magnus property, if g, v satisfy the conditions in Lemma 8.2.6. Part (1) of the following lemma is straightforward; compare with Lemma 8.2.2. Part (2) is established by following the proof of Proposition 8.2.4 as per contrapositive.

Lemma 8.2.7. *Let G be a group, and let $N \trianglelefteq G$.*

(1) *If $G = H \ltimes N$ splits over N, then every basic $\neg(\mathsf{MP})$-witness pair for H is also a basic $\neg(\mathsf{MP})$-witness pair for G.*

(2) *Suppose that $N \subseteq [G,G]$ and that $g, v \in G$ are such that their images modulo N form a basic $\neg(\mathsf{MP})$-witness pair $(\bar{g}, \bar{v})$ for G/N. If $\Omega_{gN} = \big\{ \langle gz \rangle^G \mid z \in N \big\}$ satisfies the minimal condition, then $(\bar{g}, \bar{v})$ lifts to a basic $\neg(\mathsf{MP})$-witness pair (g_0, v_0) for G, with $g_0 \equiv_N g$ and $v_0 \equiv_N v$.*

Another useful tool is the *co-centraliser* of an element g in a group G, that is

$$\mathrm{C}^*_G(g) = \langle\, [g,w] \mid w \in G \rangle \leq G;$$

this group is closely related to the normal closure $\langle g \rangle^G$ and thus of interest to us.

Lemma 8.2.8. *Let G be a group, and let $g \in G$. Then*

(i) *$\mathrm{C}^*_G(g) \trianglelefteq G$ and $\langle g \rangle^G = \langle g \rangle \, \mathrm{C}^*_G(g)$; in particular, $\langle g \rangle^G / \, \mathrm{C}^*_G(g)$ is cyclic;*
(ii) *$\mathrm{C}^*_G(g) = [\langle g \rangle^G, G]$, i.e., $\mathrm{C}^*_G(g)$ is the smallest normal subgroup N of G such that $N \subseteq \langle g \rangle^G$ and G acts trivially by conjugation on $\langle g \rangle^G / N$;*
(iii) *$\mathrm{C}^*_G(h) \leq \mathrm{C}^*_G(g)$ for all $h \in \langle g \rangle^G$;*
(iv) *if $h \in G$ with $\langle g \rangle^G = \langle h \rangle^G$ then $\mathrm{C}^*_G(g) = \mathrm{C}^*_G(h)$.*

Proof. We set $X = \{[g,w] \mid w \in G\}$ so that $\mathrm{C}^*_G(g) = \langle X \rangle$ and $\langle g \rangle^G = \langle \{g\} \cup X \rangle$. From the identity $[g,w]^v = [g,v]^{-1}[g,wv]$, for $w, v \in G$, we deduce that $\mathrm{C}^*_G(g) \trianglelefteq G$. This establishes (1) and (2). Claims (3) and (4) are immediate consequences of (2). □

8.3 — Locally nilpotent groups

Clearly, if G is an MP-group then so is its centre $\mathrm{Z}(G)$. We are interested in sufficient conditions so that the factor group $G/\mathrm{Z}(G)$ inherits the Magnus property. First we observe a useful feature of co-centralisers in torsion-free, locally nilpotent groups.

Lemma 8.3.1. *Let G be a torsion-free, locally nilpotent group. Let $g \in G \smallsetminus \mathrm{Z}(G)$. Then the co-centraliser of g satisfies*

$$\langle g \rangle^G \cap \mathrm{Z}(G) \le \mathrm{C}^*_G(g). \tag{8.1}$$

Proof. Choose $v \in G$ such that $[g,v] \neq 1$. For every $z \in \langle g \rangle^G \cap \mathrm{Z}(G)$ there exist finitely many elements $w_1, \ldots, w_n \in G$ such that $z \in \langle g^{w_1}, \ldots, g^{w_n} \rangle$. If the claim holds true for the nilpotent group $H = \langle g, v, w_1, \ldots, w_n \rangle \le G$, we conclude that $z \in \mathrm{C}^*_H(g) \subseteq \mathrm{C}^*_G(g)$. Thus we may assume without loss that G is nilpotent.

Let c denote the nilpotency class of G, and let us fix the position where g makes its appearance within the upper central series: $g \in \mathrm{Z}_{i+1}(G) \smallsetminus \mathrm{Z}_i(G)$ for suitable $i \in \{1, \ldots, c-1\}$. Since G is torsion-free, so is $G/\mathrm{Z}_i(G)$; see [32, Cor. 2.20]. Hence we deduce from $\mathrm{C}^*_G(g) \le [\mathrm{Z}_{i+1}(G), G] \le \mathrm{Z}_i(G)$ that $\langle g \rangle^G = \langle g \rangle \ltimes \mathrm{C}^*_G(g)$ and consequently $\mathrm{C}^*_G(g) = \langle g \rangle^G \cap \mathrm{Z}_i(G)$. This implies $\langle g \rangle^G \cap \mathrm{Z}(G) \le \mathrm{C}^*_G(g)$. $\square$

With this insight we are ready to deal with torsion-free, class-2 nilpotent groups.

Proposition 8.3.2. *Let G be a torsion-free, class-2 nilpotent group. Then G has the Magnus property.*

Proof. Let $g, h \in G$ such that $\langle g \rangle^G = \langle h \rangle^G$. If $g \in \mathrm{Z}(G)$, then $\langle g \rangle = \langle g \rangle^G = \langle h \rangle^G = \langle h \rangle$ and, because G is torsion-free, we conclude that $g \in \{h, h^{-1}\}$.

Now suppose that $g \notin \mathrm{Z}(G)$. Since $G/\mathrm{Z}(G)$ is torsion-free abelian, we deduce from $\langle g \rangle \mathrm{Z}(G) = \langle h \rangle \mathrm{Z}(G)$ that $g \equiv_{\mathrm{Z}(G)} h^{\pm 1}$; replacing g by its inverse if necessary, we may assume without loss that $g \equiv_{\mathrm{Z}(G)} h$, hence $g^{-1}h \in \mathrm{Z}(G) \cap \langle g \rangle^G \le \mathrm{C}^*_G(g)$ by Lemma 8.3.1. This implies $g^{-1}h = \prod_{i=1}^k [g, v_i]^{e_i}$ for suitable $k \in \mathbb{N}_0$, $v_1, \ldots, v_k \in G$ and $e_1, \ldots, e_k \in \{1, -1\}$. Since G has nilpotency class 2, we obtain

$$h = g \prod_{i=1}^k [g, v_i]^{e_i} = g \left[g, \prod_{i=1}^k v_i^{e_i}\right] = g^v \qquad \text{for } v = \prod_{i=1}^k v_i^{e_i}. \qquad \square$$

Example 8.3.3. It would be interesting to complement Proposition 8.3.2 by characterising (or even classifying) finite, class-2 nilpotent groups with the Magnus property. Each such group is necessarily a $\{2,3\}$-group and hence a direct product $G = P \times Q$ of its Sylow-2 and its Sylow-3 subgroup, each of which is again an MP-group.

However, it does not seem easy to give a succinct characterisation of finite, class-2 nilpotent 2- or 3-groups, in terms of canonical subgroups or quotients. A halfway practical criterion for 3-groups is the following: a finite, class-2 nilpotent 3-group G has the Magnus property if and only if (i) $\mathrm{Z}(G)$ and $G/\mathrm{Z}(G)$ are elementary abelian and (ii) for every ($\mathrm{Z}(G)$-coset of an) element g of order 9 there exists (a $\mathrm{Z}(G)$-coset of) an element h such that $[g,h] = g^3$.

With this criterion it is, for instance, easy to see that the class-2 nilpotent group

$$G = \langle t, a, b \mid t^3 = a^9 = b^9 = [a,b] = [a,t]b^3 = [b,t]a^3 = 1\rangle,$$

satisfies $\mathrm{Z}(G) = [G,G] \cong C_3^2$ and $G/\mathrm{Z}(G) \cong C_3^3$, but does not have the Magnus property; for instance, a and a^4 have the same normal closure $\langle a, b^3\rangle$, but are neither conjugate nor inverse-conjugate to one another. Incidentally, examples of such kind illustrate that the condition of torsion-freeness is not redundant in Lemma 8.3.1.

Lemma 8.3.4. *Let G be a group and let $g \in G \smallsetminus \mathrm{Z}(G)$ be such that Eq.* (8.1) *holds. Then every $z \in \langle g\rangle^G \cap \mathrm{Z}(G)$ satisfies $\langle gz\rangle^G = \langle g\rangle^G$.*

Proof. Let $z \in \langle g\rangle^G \cap \mathrm{Z}(G)$. Clearly, $gz \in \langle g\rangle^G$ and it remains to show that $g \in \langle gz\rangle^G$. Since $z \in \langle g\rangle^G \cap \mathrm{Z}(G) \subseteq \mathrm{C}^*_G(g)$, there exist $k \in \mathbb{N}$, $v_1, \ldots, v_k \in G$ and $e_1, \ldots, e_k \in \{1, -1\}$ such that $z = \prod_{i=1}^k [g, v_i]^{e_i}$. Since z is central, we deduce that $z = \prod_{i=1}^k [gz, v_i]^{e_i} \in \mathrm{C}^*_G(gz) \subseteq \langle gz\rangle^G$, and consequently $g \in \langle gz\rangle^G$. ☐

Lemma 8.3.5. *Let G be an* MP*-group, and suppose that Eq.* (8.1) *holds for each $g \in G \smallsetminus \mathrm{Z}(G)$. Then $G/\mathrm{Z}(G)$ is an* MP*-group.*

Proof. Lemma 8.3.4 shows that, for each $g \in G \smallsetminus \mathrm{Z}(G)$, any two distinct elements of $\Omega_{g\,\mathrm{Z}(G)}$ are $\subseteq$-incomparable. Thus Proposition 8.2.4 applies. ☐

From Lemma 8.3.1 and Lemma 8.3.5 we see that within the class of torsion-free, locally nilpotent groups the Magnus property passes from G to $G/\mathrm{Z}(G)$; this is a useful insight for a future characterisation (or even classification) of finitely generated, torsion-free nilpotent groups with the Magnus property.

Corollary 8.3.6. *Let G be a torsion-free, locally nilpotent group. If G is an* MP*-group then so is $G/\mathrm{Z}(G)$.*

For the next result we recall the notion of a basic $\neg$(MP)-witness pair in the wake of Lemma 8.2.6.

Proposition 8.3.7. *Let G be a free class-c nilpotent group of rank at least* 2*, where $c \in \mathbb{N}_{\geq 3}$. Then there exists a basic $\neg$(MP)-witness pair for G.*

Proof. Clearly, any two elements of a free generating set for G generate a subgroup that is free class-c nilpotent of rank 2 and has a normal complement in G. By part (1) of Lemma 8.2.7 we may hence suppose that $G = \langle x, y\rangle$ is freely generated by two elements. Furthermore, by part (2) of Lemma 8.2.7 and induction on c we may suppose that $c = 3$.

In accordance with Witt's formula, the non-trivial sections of the lower central series of G are: $G/\gamma_2(G) = \langle \overline{x}, \overline{y} \rangle \cong C_\infty \times C_\infty$, $\gamma_2(G)/\gamma_3(G) = \langle \overline{[y,x]} \rangle \cong C_\infty$ and

$$\mathrm{Z}(G) = \gamma_3(G) = \langle [y,x,x], [y,x,y] \rangle \cong C_\infty \times C_\infty;$$

compare with [32, Chap. 3].

Put $v = [y,x,y] \in \mathrm{Z}(G)$. Clearly, x and xv have the same normal closure in G, namely $\langle x \rangle^G = \langle x \rangle \gamma_2(G) = \langle xv \rangle^G$. Moreover, x has infinite order modulo $[G,G]$. It remains to prove that $[x,w] \neq v$ for all $w \in G$.

Let $w \in G$. For $[x,w] = v$ it is necessary that $[x,w] \in \mathrm{Z}(G)$ and consequently $w \in \langle x \rangle \gamma_2(G)$. But $w \equiv_{\mathrm{Z}(G)} x^m [y,x]^n$ for $m,n \in \mathbb{Z}$ gives

$$[x,w] = [x,[y,x]^n] = [y,x,x]^{-n} \neq [y,x,y] = v. \qquad \square$$

The following straightforward example illustrates that there are finitely generated, nilpotent MP-groups (with non-trivial 3-torsion) of any prescribed nilpotency class.

Example 8.3.8. For $c \in \mathbb{N}$ the 2-generated group

$$G = \langle t, a \mid a^{3^c} = [a,t]a^{-3} = 1 \rangle \cong C_\infty \ltimes C_{3^c}$$

is nilpotent of class c and an MP-group.

Indeed, let $g,h \in G$ with $\langle g \rangle^G = \langle h \rangle^G$. If $g = h = 1$ then there is nothing to show. Now suppose that g and h are non-trivial. There are unique parameters $l,m \in \mathbb{Z}$ with $0 \leq m < 3^c$ such that $g = t^l a^m$. Put $n = n(g) = 1 + \min\{v_3(l), v_3(m)\} \in \mathbb{N}$, where $v_3(k)$ denotes the 3-adic valuation of an integer k. Lemma 8.2.8 and a routine calculation show that

$$M = \mathrm{C}^*_G(g) = \mathrm{C}^*_G(h) = \langle a^{3^n} \rangle = \{[g,w] \mid w \in G\};$$

further details are given at the end of Section 8.5, where a related group H is considered. It suffices to show that $g \equiv_M h$ or $g \equiv_M h^{-1}$.

If $l = 0$, then $n = 1 + v_3(m)$ and $\langle g \rangle^G = \langle g \rangle = \langle a^{3^{n-1}} \rangle$ gives $\langle g \rangle^G/M = \langle h \rangle^G/M \cong C_3$; it follows that $g \equiv_M h$ or $g \equiv_M h^{-1}$. If $l \neq 0$, then g and h generate the same infinite cyclic subgroup modulo M, and thus $g \equiv_M h$ or $g \equiv_M h^{-1}$.

It would be interesting to construct finitely generated, nilpotent MP-groups of prescribed nilpotency class that are torsion-free. In fact, it is already a challenge to construct explicitly a finitely generated, torsion-free, class-3 nilpotent MP-group, as we do below. In principle, Corollary 8.3.6 suggests that one proceeds by induction on the nilpotency class.

Lemma 8.3.9. *Let G be a group such that $\mathrm{Z}(G)$ and $G/\mathrm{Z}(G)$ are MP-groups. Suppose that for every $g \in G \smallsetminus \mathrm{Z}(G)$, the set $\{[g,w] \mid w \in G\}$ contains $\langle g \rangle^G \cap \mathrm{Z}(G)$. Then G is an MP-group.*

Proof. Suppose that $g, h \in G$ have the same normal closure in G. If $g \in \mathrm{Z}(G)$ then $h \in \mathrm{Z}(G)$; furthermore $\langle g \rangle = \langle h \rangle$ implies $g \in \{h, h^{-1}\}$, because $\mathrm{Z}(G)$ is an MP-group.

Now suppose that $g \notin \mathrm{Z}(G)$. Since $G/\mathrm{Z}(G)$ is an MP-group, there exist $v \in G$ and $e \in \{-1, 1\}$ such that $g^v \equiv_{\mathrm{Z}(G)} h^e$, and hence $g^v z = h^e$ for suitable $z \in \langle g \rangle^G \cap \mathrm{Z}(G)$. Choose $w \in G$ such that $g^w = gz$; then $g^{wv} = (gz)^v = g^v z = h^e$. □

Corollary 8.3.10. *Let G be a torsion-free group such that $G/\mathrm{Z}(G)$ is an MP-group. If $\mathrm{C}^*_G(g) \cap \mathrm{Z}(G) \subseteq \{[g, w] \mid w \in G\}$ for every $g \in G \smallsetminus \mathrm{Z}(G)$, then G is an MP-group.*

Proof. Since G is torsion-free, $\mathrm{Z}(G)$ is an MP-group. Let $g \in G \smallsetminus \mathrm{Z}(G)$. Since $G/\mathrm{Z}(G)$ is torsion-free, we deduce that $\langle g \rangle^G \cap \mathrm{Z}(G) = \mathrm{C}^*_G(G) \cap \mathrm{Z}(G)$. Thus the claim follows from Lemma 8.3.9. □

Example 8.3.11. We use Corollary 8.3.10 to construct a 4-generated, torsion-free, class-3 nilpotent MP-group of Hirsch length 9.

Let $F = \langle \dot{x}, \dot{y}, \dot{z}, \dot{w} \rangle$ be a free class-3 nilpotent group on four generators and consider

$$G = \langle x, y, z, w \rangle = F \,/\, \langle \{[\dot{z}, \dot{y}], [\dot{w}, \dot{z}]\} \cup R \rangle^F,$$

where x, y, z, w denote the images of $\dot{x}, \dot{y}, \dot{z}, \dot{w}$ and

$$R = \left\{ \begin{array}{llll} [\dot{y}, \dot{x}, \dot{x}], & [\dot{y}, \dot{x}, \dot{y}], & [\dot{y}, \dot{x}, \dot{z}][\dot{z}, \dot{x}, \dot{y}]^{-1}, & [\dot{y}, \dot{x}, \dot{w}], \\ [\dot{z}, \dot{x}, \dot{x}], & & [\dot{z}, \dot{x}, \dot{z}], & [\dot{z}, \dot{x}, \dot{w}], \\ [\dot{w}, \dot{x}, \dot{x}][\dot{z}, \dot{x}, \dot{y}]^{-1}, & [\dot{w}, \dot{x}, \dot{y}], & [\dot{w}, \dot{x}, \dot{z}], & [\dot{w}, \dot{x}, \dot{w}], \\ & [\dot{w}, \dot{y}, \dot{y}], & [\dot{w}, \dot{y}, \dot{z}] & [\dot{w}, \dot{y}, \dot{w}][\dot{z}, \dot{x}, \dot{y}]^{-1} \end{array} \right\}.$$

Clearly, G is a 4-generated nilpotent group of class at most 3. The precise structure of G can be determined as follows. The collection process, subject to the initial ordering $\dot{x} < \dot{y} < \dot{z} < \dot{w}$, yields a Hall basis for F consisting of $4 + 6 + 20 = 30$ basic commutators; for instance, see [32, Section 3.1.3]. The relators in R simply tell us to cancel or identify certain basic commutators of degree 3. The additional relators $[\dot{z}, \dot{y}]$ and $[\dot{w}, \dot{z}]$ are basic commutators of degree 2 to be cancelled; they also tell us to cancel the basic commutators $[\dot{z}, \dot{y}, \dot{y}], [\dot{z}, \dot{y}, \dot{z}], [\dot{z}, \dot{y}, \dot{w}], [\dot{w}, \dot{z}, \dot{z}], [\dot{w}, \dot{z}, \dot{w}]$ of degree 3. The relations $[z, y, x] = [w, z, x] = [w, z, y] = 1$, which also come with the relators $[\dot{z}, \dot{y}]$ and $[\dot{w}, \dot{z}]$, are already consequences of the relators in R and the Witt identity. For instance, $[z, y, x] = [z, y, x][y, x, z][z, x, y]^{-1} = [z, y, x][y, x, z][x, z, y] = 1$. In this way we see that G is torsion-free of nilpotency class 3 and admits a poly-C_∞ basis

$$x,\ y,\ z,\ w, \quad [y, x],\ [z, x],\ [w, x],\ [w, y], \quad [y, x, z] = [z, x, y] = [w, x, x] = [w, y, w]$$

such that

$$G/\gamma_2(G) = \langle \overline{x}, \overline{y}, \overline{z}, \overline{w} \rangle \cong C_\infty^4, \quad \gamma_2(G)/\gamma_3(G) = \langle \overline{[y, x]}, \overline{[z, x]}, \overline{[w, x]}, \overline{[w, y]} \rangle \cong C_\infty^4$$

and

$$Z = \mathrm{Z}(G) = \gamma_3(G) = \langle [z,x,y] \rangle \cong C_\infty.$$

In particular, G has Hirsch length 9. The group commutator induces a bi-additive map $\beta\colon \gamma_2(G)/Z \times G/\gamma_2(G) \to Z$ whose values on pairs of basis elements are given by the following commutator table:

$[\cdot,\cdot]$	x	y	z	w
$[y,x]$	$[y,x,x]=1$	$[y,x,y]=1$	$[y,x,z]=[z,x,y]$	$[y,x,w]=1$
$[z,x]$	$[z,x,x]=1$	$[z,x,y]$	$[z,x,z]=1$	$[z,x,w]=1$
$[w,x]$	$[w,x,x]=[z,x,y]$	$[w,x,y]=1$	$[w,x,z]=1$	$[w,x,w]=1$
$[w,y]$	$\underline{[w,y,x]=1}$	$[w,y,y]=1$	$[w,y,z]=1$	$[w,y,w]=[z,x,y]$

.

The underlined entry is the only one that perhaps still requires a short explanation:

$$[w,y,x] = [w,y,x][y,x,w][w,x,y]^{-1} = [w,y,x][y,x,w][x,w,y] = 1,$$

using again relators from R and the Witt identity. The table shows that β is a perfect pairing between $\gamma_2(G)/Z$ and $G/\gamma_2(G)$.

It remains to verify that G is an MP-group. By Proposition 8.3.2, the quotient G/Z has the Magnus property. By Corollary 8.3.10 it suffices to check that $\mathrm{C}^*_G(g) \cap Z \subseteq \{[g,v] \mid v \in G\}$ for all $g \in G \smallsetminus Z$.

If $g \in \gamma_2(G)$ then $g \equiv_Z [y,x]^{m_1}[z,x]^{m_2}[w,x]^{m_3}[w,y]^{m_4}$ for suitable $m_1,\ldots,m_4 \in \mathbb{Z}$; since β is a perfect pairing, this implies that $\mathrm{C}^*_G(g) = \langle [z,x,y]^n \rangle = \{[g,v] \mid v \in G\}$ for $n = \gcd(m_1,m_2,m_3,m_4)$. Now suppose that $g \notin \gamma_2(G)$. Since G is nilpotent of class 3, the commutator identities

$$[g,v]^{-1} = [g,v^{-1}]\underbrace{[g,v,v^{-1}]}_{\in Z} \quad \text{and} \quad [g,v][g,w] = [g,wv]\underbrace{[g,w,v]^{-1}}_{\in Z}, \quad \text{for } v,w \in G,$$

hold. From these we deduce that every $h \in \mathrm{C}^*_G(g) \cap Z$ is of the form $h = h_1h_2$ with

$$h_1 \in \{[g,v] \mid v \in G\} \cap Z \quad \text{and} \quad h_2 \in \langle [g,w_1,w_2] \mid w_1,w_2 \in G \rangle \leq Z.$$

Observe that $g \equiv_{\gamma_2(G)} x^{m_1}y^{m_2}z^{m_3}w^{m_4}$, for suitable $m_1,\ldots,m_4 \in \mathbb{Z}$, and put $n = \gcd(m_1,m_2,m_3,m_4)$. Since β is a perfect pairing, we deduce as previously that

$$\begin{aligned}\langle [g,w_1,w_2] \mid w_1,w_2 \in G \rangle = \{[g,w_1,w_2] \mid w_1,w_2 \in G\} &= \langle [z,x,y]^n \rangle \\ &= \{[g,w] \mid w \in \gamma_2(G)\}.\end{aligned}$$

Writing $h_1 = [g,v]$ and $h_2 = [g,w]$, we obtain $h = h_1h_2 = [g,v][g,w]^v = [g,wv]$.

8.4 — Polynilpotent groups

In this section we prove Theorems 8.1.1 and 8.1.2. To achieve this, we show first that the restricted wreath product $C_\infty \wr C_\infty$ does not have the Magnus property.

Proposition 8.4.1. *There is a basic* $\neg(\mathsf{MP})$*-witness pair for the group* $C_\infty \wr C_\infty$.

Proof. We realise the wreath product as the group $G = \langle t \rangle \ltimes A \cong C_\infty \wr C_\infty$, where $A = \mathbb{Z}[T^{\pm 1}] = \mathbb{Z}[T, T^{-1}]$ is written additively and the action of t on A by conjugation is given by multiplication by T. The commutator of elements $x = t^m a$ and $y = t^n b$, with $m, n \in \mathbb{Z}$ and $a, b \in A$, is

$$[x, y] = [t^m a, t^n b] = -a - b \cdot T^m + a \cdot T^n + b = a \cdot (T^n - 1) - b \cdot (T^m - 1).$$

In particular, $[G, G] = (T-1)\,\mathbb{Z}[T^{\pm 1}] \leq A$ is equal to the ideal of the ring $\mathbb{Z}[T^{\pm 1}]$ generated by $(T-1)$. If $x = t^m a \in G \smallsetminus A$ then $\langle x \rangle^G = \langle x \rangle \ltimes \mathrm{C}^*_G(x)$ and

$$\mathrm{C}^*_G(x) = I_x = a \cdot (T-1)\ \mathbb{Z}[T^{\pm 1}] + (T^m - 1)\ \mathbb{Z}[T^{\pm 1}] \tag{8.2}$$

is the ideal of the ring $\mathbb{Z}[T^{\pm 1}]$ generated by $a \cdot (T-1)$ and $T^m - 1$.

Choose a prime $p \geq 5$ and consider the ring of integers $\mathcal{O} = \mathbb{Z}[\zeta] \cong \mathbb{Z}[T]/\Phi_p\,\mathbb{Z}[T]$ of the p^{th} cyclotomic field, where ζ denotes a primitive p^{th} root of unity and Φ_p the p^{th} cyclotomic polynomial. It is well known that $\mathcal{O}/(\zeta-1)\mathcal{O} \cong \mathbb{F}_p$ is a field with p elements. By the Dirichlet unit theorem, the torsion-free rank of the unit group $\mathcal{O}^\times$ is $(p-1)/2 - 1 \geq 1$. Consider the $(p-1)^{\text{st}}$ power ν of an element of infinite order, for instance, the power of a cyclotomic unit such as $\nu = (\zeta + 1)^{p-1}$, and write $\nu = f(\zeta)$ for a suitable polynomial $f \in 1 + (T-1)\ \mathbb{Z}[T]$. Since ν has infinite order in $\mathcal{O}^\times$, we deduce that

$$f \not\equiv_{T^p - 1} T^n \qquad \text{for all } n \in \mathbb{Z}. \tag{8.3}$$

Furthermore, we observe that ν^{-1} can be written in a similar form: $\nu^{-1} = \bar{f}(\zeta)$ for suitable $\bar{f} \in 1 + (T-1)\ \mathbb{Z}[T]$. The embedding of rings

$$\mathbb{Z}[T^{\pm 1}]/(T^p - 1) \ \hookrightarrow\ \mathbb{Z}[T]/(T-1) \times \mathbb{Z}[T]/\Phi_p\,\mathbb{Z}[T] \ \cong\ \mathbb{Z} \times \mathcal{O}$$

shows that

$$f \cdot \bar{f} \equiv_{T^p - 1} 1. \tag{8.4}$$

Now consider $g = t^p e \in G$, where $e \in A$ denotes the constant polynomial 1, and $v = f - 1 \in (T-1)\,\mathbb{Z}[T] = [G, G] \subseteq A$. Clearly, g has infinite order modulo $A \supseteq [G, G]$. Commutators $[g, w]$, with $w = t^n b \in G$ for $n \in \mathbb{Z}$ and $b \in \mathbb{Z}[T^{\pm 1}]$, are of the form

$$(T^n - 1) - b \cdot (T^p - 1) \equiv_{T^p - 1} T^n - 1;$$

thus Eq. (8.3) shows that $v \notin \{[g,w] \mid w \in G\}$. It remains to prove that $\langle g\rangle^G = \langle gv\rangle^G$, and for this it is enough to show that $v = f - 1 \in (T-1)\,\mathbb{Z}[T^{\pm 1}]$ lies in $I_g \cap I_{gv}$. Indeed, the general description provided in Eq. (8.2) yields directly

$$I_g = (T-1)\,\mathbb{Z}[T^{\pm 1}] + (T^p - 1)\,\mathbb{Z}[T^{\pm 1}] = (T-1)\,\mathbb{Z}[T^{\pm 1}],$$

and Eq. (8.4) implies that

$$I_{gv} = f \cdot (T-1)\,\mathbb{Z}[T^{\pm 1}] + (T^p - 1)\,\mathbb{Z}[T^{\pm 1}] = (T-1)\,\mathbb{Z}[T^{\pm 1}]. \qquad \square$$

Example 8.4.2. It easy to produce explicit examples of elements g and v in the proof of Proposition 8.4.1, for which the claims could be checked directly by computation. For $p = 5$ one may take $\nu = (1+\zeta)^4 = 1 + (\zeta - 1) f_0(\zeta)$ with $f_0 = 3 + 2T - T^2 - 2T^3$ and $\nu^{-1} = 1 + (\zeta - 1)\bar{f}_0(\zeta)$ with $\bar{f}_0 = -T + T^2 - 2T^3$. A routine calculation yields

$$\begin{aligned}&\big(1 + (T-1) f_0(T)\big)\big(1 + (T-1)\bar{f}_0(T)\big)\\ &\quad = 1 + (T-1)(3 + T - 4T^3) + (T-1)^2(-3T + T^2 - 3T^3 - 3T^4 + 4T^6)\\ &\qquad \equiv_{T^5-1} 1 + (T-1)(3 + T - 4T^3) + (T-1)^2(3 + 4T + 4T^2) = 1.\end{aligned}$$

Next we would like to use Proposition 8.4.1 to prove that, for instance, the free metabelian group G of rank 2 does not have the Magnus property. If G had a subgroup $H \cong C_\infty \wr C_\infty$ with a normal complement in G, then Lemma 8.2.2 would immediately yield the desired conclusion. But, in fact, there is no such subgroup H. Assume, for a contradiction, that $G = H \ltimes K$ with $H \cong C_\infty \wr C_\infty$. Then $[G,G] = [H,H] \ltimes [K,G]$ implies $C_\infty \times C_\infty \cong G/[G,G] \cong H/[H,H] \times K/[K,G] \cong C_\infty \times C_\infty \times K/[K,G]$, hence $[K,G] = K$. But G is residually a finite nilpotent group; compare with [75]. Thus there is a finite-index normal subgroup $N \trianglelefteq G$ such that G/N is nilpotent and $K \not\subseteq N$. This gives $1 \neq KN/N \trianglelefteq G/N$ with $[KN/N, G/N] = KN/N$, a contradiction.

In the proof of the next proposition, which deals more generally with free abelian-by-(class-c nilpotent) groups G, we by-pass the obstacle that there is no subgroup $H \cong C_\infty \wr C_\infty$ with normal complement in G.

Proposition 8.4.3. *Let $c \in \mathbb{N}$, and let G be an $\mathfrak{N}_{(c,1)}$-free group of rank 2 viz. a free abelian-by-(class-c nilpotent) group that is freely generated by two elements. Then there exists a basic $\neg(\mathsf{MP})$-witness pair for G.*

Proof. Let $H = \langle \bar{x}, \bar{y}\rangle = G/\gamma_{c+1}(G)$, the free class-$c$ nilpotent group of rank 2, and let $R = \mathbb{Z}H$ denote the associated integral group ring. Let $V = \mathbf{e}R \oplus \mathbf{f}R$ be the free right R-module on two free generators. The Magnus embedding

$$G \hookrightarrow H \ltimes V, \qquad w \mapsto \begin{pmatrix} \bar{w} & 0 \\ \mathbf{v}_w & 1 \end{pmatrix}$$

allows us to realise the relatively free group G as a group of 2×2 matrices with entries $\bar{w} \in H$ and $\mathbf{v}_w \in V$; compare with [156, § 2.1]. In this embedding, the matrices

$$x = \begin{pmatrix} \bar{x} & 0 \\ \mathbf{e} & 1 \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} \bar{y} & 0 \\ \mathbf{f} & 1 \end{pmatrix}.$$

constitute free generators of the relatively free group G.

Let us consider the iterated commutator

$$z = [y, \underbrace{x, \ldots, x}_{c}] \in \gamma_{c+1}(G).$$

We observe that $z \neq 1$; for instance, one can deduce $z \notin \gamma_{c+2}(G)$ from the fact that z is a basic commutator in the Hall collection process, or carry out an explicit calculation as below. It follows that the subgroup $\langle x, z \rangle = \langle x \rangle \ltimes A \leq G$, with $\langle x \rangle \cong C_\infty$ and $A = \langle z^{x^m} \mid m \in \mathbb{Z} \rangle \leq \gamma_{c+1}(G)$ free abelian, is isomorphic to the wreath product $C_\infty \wr C_\infty$: employing a transversal for the subgroup $\langle \bar{x} \rangle \cong C_\infty$ of H, we can regard V as a free $\mathbb{Z}\langle \bar{x} \rangle$-module (of infinite rank) and, accordingly, the non-trivial element z of this module is moved about freely by the action of x which is simply multiplication by the scalar $\bar{x} \in \mathbb{Z}\langle \bar{x} \rangle$.

Following precisely the proof of Proposition 8.4.1, we consider elements

$$g = x^p z \qquad \text{and} \qquad v = z^{f(x)-1},$$

where $p \geq 5$ is a prime and a one-unit of infinite order in the p^{th} cyclotomic field is used to produce a polynomial $f \in 1 + (T-1)\,\mathbb{Z}[T]$ such that g and $h = gv$ are not conjugate in $\langle x, z \rangle \cong C_\infty \wr C_\infty$, but generate the same normal closure in this subgroup and hence in G. Clearly, $g \equiv_{[G,G]} x^p \not\equiv_{[G,G]} 1$ has infinite order in $G/[G,G]$ and $v \in [\langle x, z \rangle, \langle x, z \rangle] \subseteq [G,G]$. It suffices to prove that $v \neq [g, w]$, or equivalently $g^w \neq h$, for all $w \in G$.

As in the proof of Proposition 8.4.1, let $\mathcal{O} = \mathbb{Z}[\zeta] \cong \mathbb{Z}[T]/\Phi_p\,\mathbb{Z}[T]$ denote the ring of integers of the p^{th} cyclotomic field, with ζ a primitive p^{th} root of unity; let $\nu = f(\zeta) \in \mathcal{O}^\times$ denote the one-unit of infinite order. The kernel of the natural projection of rings $\pi \colon R \to \mathcal{O}$ specified by $\bar{x}^\pi = \zeta$ and $\bar{y}^\pi = 1$ is generated, as a two-sided ideal, by the elements $\Phi_p(\bar{x})$ and $\bar{y} - 1$. It induces a π-equivariant projection $\vartheta \colon V \to V/V\ker(\pi) \cong \dot{\mathbf{e}}\mathcal{O} \oplus \dot{\mathbf{f}}\mathcal{O}$, from the free R-module V onto a free $\mathcal{O}$-module of rank 2 such that $(\mathbf{e}r + \mathbf{f}s)^\vartheta = \dot{\mathbf{e}}(r^\pi) + \dot{\mathbf{f}}(s^\pi)$ for all $r, s \in R$.

It is straightforward to work out that

$$x^p = \begin{pmatrix} \bar{x}^p & 0 \\ \mathbf{v}_{x^p} & 1 \end{pmatrix}, \quad \text{where } {\mathbf{v}_{x^p}}^\vartheta = \dot{\mathbf{e}}\,\Phi_p(\zeta) = 0,$$

$$z = \begin{pmatrix} 1 & 0 \\ \mathbf{v}_z & 1 \end{pmatrix}, \quad \text{where } {\mathbf{v}_z}^\vartheta = \dot{\mathbf{f}}(\zeta - 1)^c.$$

From this we continue to see that

$$g = \begin{pmatrix} \bar{x}^p & 0 \\ \mathbf{v}_g & 1 \end{pmatrix}, \quad \text{where } {\mathbf{v}_g}^\vartheta = {\mathbf{v}_z}^\vartheta = \dot{\mathbf{f}}\,(\zeta-1)^c,$$

$$h = \begin{pmatrix} \bar{x}^p & 0 \\ \mathbf{v}_h & 1 \end{pmatrix}, \quad \text{where } {\mathbf{v}_h}^\vartheta = ({\mathbf{v}_z}^\vartheta)\, f(\zeta) = \dot{\mathbf{f}}\,(\zeta-1)^c\nu.$$

Finally, we conjugate g by an arbitrary element $w \in G$ to obtain

$$\begin{pmatrix} \bar{g}^{\bar{w}} & 0 \\ \mathbf{v}_{g^w} & 1 \end{pmatrix} = g^w = \begin{pmatrix} \bar{w} & 0 \\ \mathbf{v}_w & 1 \end{pmatrix}^{-1} \begin{pmatrix} \bar{x}^p & 0 \\ \mathbf{v}_g & 1 \end{pmatrix} \begin{pmatrix} \bar{w} & 0 \\ \mathbf{v}_w & 1 \end{pmatrix} = \begin{pmatrix} \bar{w}^{-1}\bar{x}^p\bar{w} & 0 \\ -\mathbf{v}_w\bar{w}^{-1}\bar{x}^p\bar{w} + \mathbf{v}_g\bar{w} + \mathbf{v}_w & 1 \end{pmatrix}$$

and, for $\bar{w}^\pi = \zeta^m$ with suitable $m \in \{0, 1, \dots, p-1\}$, it follows that

$$(\mathbf{v}_{g^w})^\vartheta = ({\mathbf{v}_w}^\vartheta)\underbrace{(1-\zeta^p)}_{=0} + ({\mathbf{v}_g}^\vartheta)\zeta^m = \dot{\mathbf{f}}\,(\zeta-1)^c\zeta^m.$$

By construction, $\nu \in \mathcal{O}^\times$ has infinite order so that $(\zeta-1)^c\nu \neq (\zeta-1)^c\zeta^m$. Comparison with our computations for h and g^w yields ${\mathbf{v}_h}^\vartheta \neq {\mathbf{v}_{g^w}}^\vartheta$, and hence $h \neq g^w$. □

We require another variant of Propositions 8.3.7 and 8.4.3. First, we record a proposition which is perhaps folklore; we include a proof for completeness.

Proposition 8.4.4. *Let $G = \langle x, y\rangle$ be an $\mathcal{N}_{\mathbf{c}}$-free group of rank 2, where $l \in \mathbb{N}_{\geq 2}$ and $\mathbf{c} \in \mathbb{N}^l$. Let $n \in \mathbb{N}$. Then the centraliser of x^n in G is $\mathrm{C}_G(x^n) = \langle x\rangle$.*

Proof. We write $\mathbf{c} = (c_1, \dots, c_l)$, $\mathbf{c}' = (c_1, \dots, c_{l-1})$ and put

$$K = \gamma_{(c_1+1,\dots,c_{l-1}+1)}(G) \trianglelefteq G$$

so that G/K is an $\mathcal{N}_{\mathbf{c}'}$-free group of rank 2 and K is a free class-c_l nilpotent group.

Step 1. First we argue by induction on l that $\mathrm{C}_G(x^n) = \langle x\rangle\,\mathrm{C}_K(x^n)$. Indeed, it suffices to fill in the base of the induction. Suppose that $l = 2$, and put $c = c_1$ so that $K = \gamma_{c+1}(G)$. We observe that $H = \langle \bar{x}, \bar{y}\rangle = G/\gamma_{c+2}(G)$ is a free class-$(c+1)$-nilpotent group of rank 2. It suffices to show that $\mathrm{C}_H(\bar{x}^n) = \langle \bar{x}\rangle\gamma_{c+1}(H)$.

Clearly, $\mathrm{C}_H(\bar{x}^n) \subseteq \langle \bar{x}\rangle\gamma_2(H)$. Hence it is enough to show that, for each $k \in \{2, \dots, c\}$, the homomorphism of abelian groups

$$\gamma_k(H)/\gamma_{k+1}(H) \to \gamma_{k+1}(H)/\gamma_{k+2}(H),$$
$$w\gamma_{k+1}(H) \mapsto [\bar{x}^n, w]\gamma_{k+2}(H) = [\bar{x}, w]^n\gamma_{k+2}(H)$$

is injective. We may interpret the torsion-free sections $L_k = \gamma_k(H)/\gamma_{k+1}(H)$ as the first few homogeneous components of the free Lie ring $L = \bigoplus_{i=1}^\infty L_i$ on $\tilde{x}, \tilde{y}$, the images of $\bar{x}, \bar{y}$ in L_1. Furthermore, we may think of L as a Lie subring (generated by $\tilde{x}, \tilde{y}$) of the commutation Lie ring on a free associative ring A, where A is freely generated by non-commuting indeterminates $\tilde{x}, \tilde{y}$; compare with [32, Chap. 3]. The free ring A admits a natural $\mathbb{N}_0$-grading $A = \bigoplus_{i=0}^{\infty} A_i$, by means of the total $\{\tilde{x}, \tilde{y}\}$-degree function, which in turn induces the natural $\mathbb{N}$-grading $L = \bigoplus_{i=1}^{\infty} L_i$ of L.

Fix $i \in \mathbb{N}_{\geq 2}$ and let $a \in L_i = L \cap A_i$ be a non-zero homogeneous Lie element of degree i. We are to show that the Lie commutator $[\tilde{x}, a]_{\mathrm{Lie}}$ is non-zero. The monomials in $\tilde{x}, \tilde{y}$ of degree i form a $\mathbb{Z}$-basis for the component A_i; we order them lexicographically

$$\tilde{x}^i < \tilde{x}^{i-1}\tilde{y} < \tilde{x}^{i-2}\tilde{y}\tilde{x} < \tilde{x}^{i-2}\tilde{y}^2 < \ldots < \tilde{y}^{i-1}\tilde{x} < \tilde{y}^i,$$

and proceed similarly with the monomials of degree $i+1$. Suppose that a, expressed as a $\mathbb{Z}$-linear combination of monomials, has leading term $m\, u(\tilde{x}, \tilde{y})$ for $m \in \mathbb{Z} \smallsetminus \{0\}$ and $u(\tilde{x}, \tilde{y})$ the smallest monomial occurring with non-zero coefficient. Since $a \in L$ is a Lie element, we deduce that $u(\tilde{x}, \tilde{y}) \neq \tilde{x}^i$ and hence the Lie commutator $[\tilde{x}, a]_{\mathrm{Lie}} = \tilde{x}a - a\tilde{x} \in L_{i+1} \subseteq A_{i+1}$ is non-zero with leading term $m\, \tilde{x}u(\tilde{x}, \tilde{y})$.

Step 2. It remains to prove that $\mathrm{C}_K(x^n) = 1$. Put $c = c_l$. Let $L = \bigoplus_{k=1}^{c} L_k$ denote the free class-c nilpotent Lie ring associated to K and its lower central series; thus $L_k \cong \gamma_k(K)/\gamma_{k+1}(K)$ for $1 \leq k \leq c$ as a free $\mathbb{Z}$-module, and the Lie commutator of two homogeneous elements is induced by the group commutator, as in Step 1 above. Extension of scalars yields the free class-c nilpotent $\mathbb{Q}$-Lie algebra $\mathcal{L} = \bigoplus_{k=1}^{c} \mathcal{L}_k$, with $\mathcal{L}_k = \mathbb{Q} \otimes_{\mathbb{Z}} L_k$ for each k. Clearly, conjugation by x induces an automorphism ξ of $\mathcal{L}$ which respects the natural grading. It suffices to prove that, for each k, the only element of $\mathcal{L}_k$ fixed by ξ^n is 0.

Put $H = G/K$ and $R = \mathbb{Z}H$. The action of G on L factors through H, and $\bar{x} \in H$ generates an infinite cyclic subgroup. The Magnus embedding for the group $G/[K,K]$ shows that the $\mathbb{Z}\langle\bar{x}\rangle$-module L_1 embeds into a free $\mathbb{Z}\langle\bar{x}\rangle$-module (of infinite rank); compare with the proof of Proposition 8.4.3. Thus, the $\mathbb{Q}\langle\bar{x}\rangle$-module $\mathcal{L}_1$ embeds into a free $\mathbb{Q}\langle\bar{x}\rangle$-module. Observe that $\mathbb{Q}\langle\bar{x}\rangle$ is just the ring of Laurent polynomials over $\mathbb{Q}$ and, in particular, a principal ideal domain. Therefore $\mathcal{L}_1$ is itself a free $\mathbb{Q}\langle\bar{x}\rangle$-module, with $\mathbb{Q}\langle\bar{x}\rangle$-basis $\mathbf{e}_1, \mathbf{e}_2, \ldots$, say. Notice that $\mathcal{L}_1$ admits the $\mathbb{Q}$-basis

$$\mathbf{f}_{i,m} = \mathbf{e}_i \bar{x}^m, \qquad i \in \mathbb{N} \text{ and } m \in \mathbb{Z};$$

these basis elements are at the same time free generators of the free class-c nilpotent $\mathbb{Q}$-Lie algebra $\mathcal{L}$.

Now fix $k \in \{1, \ldots, c\}$ and consider the action of ξ on $\mathcal{L}_k$. We observe that $\mathcal{L}_k$ is the $\mathbb{Q}$-span of the iterated Lie commutators

$$\mathbf{F}_{\underline{i},\underline{m}} = [\mathbf{f}_{i_1,m_1}, \mathbf{f}_{i_2,m_2}, \ldots, \mathbf{f}_{i_k,m_k}]_{\mathrm{Lie}},$$

where $\underline{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ and $\underline{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$. Furthermore, we understand the action of ξ and hence of iterates ξ^r, $r \in \mathbb{N}$, on these Lie commutators:

$$\mathbf{F}_{\underline{i},\underline{m}}\, \xi^r = \mathbf{F}_{\underline{i},\underline{m}+(r,r,\ldots,r)}.$$

Let $\mathbf{v} \in \mathcal{L}_k \smallsetminus \{0\}$, written as a $\mathbb{Q}$-linear combination

$$\mathbf{v} = \sum_{\underline{i}\in\mathbb{N}^k,\underline{m}\in\mathbb{Z}^k} v(\underline{i},\underline{m})\, \mathbf{F}_{\underline{i},\underline{m}},$$

where $v\colon \mathbb{N}^k \times \mathbb{Z}^k \to \mathbb{Q}$ is such that its 'support' $S = \{(\underline{i},\underline{m}) \in \mathbb{N}^k \times \mathbb{Z}^k \mid v(\underline{i},\underline{m}) \neq 0\}$ is finite. Also the 'fine support' in the second coordinate

$$S_{\text{fine}} = \bigcup_{(\underline{i},\underline{m})\in S} \{m_1,\ldots,m_k\} \subseteq \mathbb{Z}$$

is finite. Choose $r \in \mathbb{N}$ sufficiently large so that

$$S_{\text{fine}} \cap \big(S_{\text{fine}} + rn\big) = \varnothing.$$

Let $\mathcal{I} \trianglelefteq \mathcal{L}$ be the Lie ideal generated by the following selection of free generators of $\mathcal{L}$: $\mathbf{f}_{\underline{i},\underline{m}}$, $(\underline{i},\underline{m}) \in \mathbb{N}^k \times (\mathbb{Z}^k \smallsetminus S_{\text{fine}})$. Then $\mathbf{v} \not\equiv_{\mathcal{I}} 0 \equiv_{\mathcal{I}} \mathbf{v}\xi^{rn} = \mathbf{v}(\xi^n)^r$ implies $\mathbf{v} \neq \mathbf{v}\xi^n$. □

Proposition 8.4.5. *Let $c \in \mathbb{N}$, and let G be an $\mathcal{N}_{(c,2)}$-free group of rank 2, viz. a free (class-2 nilpotent)-by-(class-c nilpotent) group that is freely generated by two elements. Then there exists a basic $\neg$(MP)-witness pair for G.*

Proof. The basic idea is to extend the proof of Proposition 8.4.3. For this purpose we put $K = \gamma_{c+1}(G)$, which is a free class-2 nilpotent group of countably infinite rank. In particular, the abelianisation $K^{\text{ab}} = K/[K,K]$ is free abelian, and $[K,K] \cong K^{\text{ab}} \wedge K^{\text{ab}}$ is the exterior square of K^{ab} and also free abelian. Regarding K^{ab} as a free $\mathbb{Z}$-module, written additively, the exterior square is defined as

$$K^{\text{ab}} \wedge K^{\text{ab}} = (K^{\text{ab}} \otimes_{\mathbb{Z}} K^{\text{ab}}) \;/\; \mathbb{Z}\text{-span}\{a\otimes b + b \otimes a \mid a,b \in K^{\text{ab}}\}.$$

Let $G = \langle x, y\rangle$, with free generators x and y. The group commutators

$$z_1 = [y, \underbrace{x,\ldots,x}_{c}] \in \gamma_{c+1}(G) \qquad \text{and} \qquad z_2 = [z_1, y] = [y, \underbrace{x,\ldots,x}_{c}, y] \in \gamma_{c+2}(G)$$

lie in K and their images $\breve{z}_1, \breve{z}_2$ modulo $[K,K]$ yield $\mathbb{Z}$-linear independent generators of K^{ab}; this follows, for instance, from the fact that z_1 and z_2 are basic commutators and form part of a Hall basis for $\gamma_{c+1}(G)/\gamma_{c+2}(G)$ and $\gamma_{c+2}(G)/\gamma_{c+3}(G)$, respectively, where $\gamma_{c+3}(G) \supseteq \gamma_{2c+2}(G) \supseteq [K,K]$. Hence the group commutator

$$z = [z_1, z_2] \in [K,K] \smallsetminus \{1\}$$

is non-trivial; in the additive notation, it corresponds to $\check{z}_1 \wedge \check{z}_2 \in K^{\mathrm{ab}} \wedge K^{\mathrm{ab}} \smallsetminus \{0\}$. The action of G on K^{ab} and on $[K,K] \cong K^{\mathrm{ab}} \wedge K^{\mathrm{ab}}$ factors through $H = G/K$; concretely, $z^w = [z_1, z_2]^w = [z_1^w, z_2^w]$ translates to $(\check{z}_1 \wedge \check{z}_2).\bar{w} = (\check{z}_1.\bar{w}) \wedge (\check{z}_2.\bar{w})$ for $w \in G$ with image $\bar{w} \in H$.

Let $R = \mathbb{Z}H$ denote the integral group ring associated to $H = \langle \bar{x}, \bar{y} \rangle$, and let $V = \mathbf{e}R \oplus \mathbf{f}R$ denote the free right R-module of rank 2. We compose reduction modulo $[K,K]$ with the Magnus embedding for $G/[K,K]$ to obtain a homomorphism

$$\eta \colon G \to G/[K,K] \hookrightarrow H \ltimes V, \qquad w \mapsto \begin{pmatrix} \bar{w} & 0 \\ \mathbf{v}_w & 1 \end{pmatrix};$$

compare with the proof of Proposition 8.4.3. The generators x, y of G are mapped to

$$x\eta = \begin{pmatrix} \bar{x} & 0 \\ \mathbf{e} & 1 \end{pmatrix} \qquad \text{and} \qquad y\eta = \begin{pmatrix} \bar{y} & 0 \\ \mathbf{f} & 1 \end{pmatrix}.$$

The exterior square $V \wedge V$ of the $\mathbb{Z}$-module V is an R-module via the diagonal action. In fact, $V \wedge V$ is a free R-module (of countably infinite rank): if $H_0 \subseteq H \smallsetminus \{1\}$ is a set of representatives for the equivalence classes $\{\bar{w}, \bar{w}^{-1}\} \subseteq H \smallsetminus \{1\}$ of the relation "equal or inverse to one another", then the elements

$$\mathbf{e} \wedge \mathbf{e}\bar{w}, \ \ (\text{for } \bar{w} \in H_0), \quad \mathbf{f} \wedge \mathbf{f}\bar{w} \ \ (\text{for } \bar{w} \in H_0), \quad \mathbf{e} \wedge \mathbf{f}\bar{w} \ \ (\text{for } \bar{w} \in H)$$

constitute an R-basis for $V \wedge V$. Since the R-module K^{ab} embeds into V, the R-module $K^{\mathrm{ab}} \wedge K^{\mathrm{ab}}$ embeds into the free R-module $V \wedge V$. A similar argument as in the proof of Proposition 8.4.3 shows that the subgroup $\langle x, z \rangle = \langle x \rangle \ltimes \langle z^{x^m} \mid m \in \mathbb{Z} \rangle$ is isomorphic to the wreath product $C_\infty \wr C_\infty$. As before, we consider elements

$$g = x^p z \qquad \text{and} \qquad v = z^{f(x)-1},$$

where $f \in 1 + (T-1)\,\mathbb{Z}[T]$ is such that g and gv are not conjugate in $\langle x, z \rangle$, but generate the same normal closure in this subgroup and hence in G. Clearly, $g \equiv_{[G,G]} x^p \not\equiv_{[G,G]} 1$ has infinite order in $G/[G,G]$ and $v \in [K,K] \subseteq [G,G]$. It suffices to prove that $[g,w] \neq v$ for all $w \in G$. Reduction modulo $[K,K]$ shows that for $[g,w] = v$ it would be necessary that $[x^p, w] \equiv_{[K,K]} 1$; hence by Proposition 8.4.4 we only need to prove that $[g,w] \neq v$ for $w \in [K,K]$.

As before let $\mathcal{O} = \mathbb{Z}[\zeta]$ denote the ring of integers of the p^{th} cyclotomic field, with ζ a primitive p^{th} root of unity; by construction, $\nu = f(\zeta) \in \mathcal{O}^\times$ has infinite order. We consider the ring of Laurent polynomials $\mathcal{O}[Y^{\pm 1}] = \mathcal{O}[Y, Y^{-1}]$. The natural projection of rings $\pi \colon R \to \mathcal{O}[Y^{\pm 1}]$ specified by $\bar{x}^\pi = \zeta$ and $\bar{y}^\pi = Y$ induces a π-equivariant projection $\vartheta \colon V \to \dot{\mathbf{e}}\mathcal{O}[Y^{\pm 1}] \oplus \dot{\mathbf{f}}\mathcal{O}[Y^{\pm 1}] = \dot{V}$, from the free R-module V onto a free $\mathcal{O}[Y^{\pm 1}]$-module $\dot{V}$ on $\dot{\mathbf{e}}, \dot{\mathbf{f}}$.

It is straightforward to work out that

$$z_1\eta = \begin{pmatrix} 1 & 0 \\ \mathbf{v}_{z_1} & 1 \end{pmatrix}, \quad \text{where } {\mathbf{v}_{z_1}}^{\vartheta} = \dot{\mathbf{e}}\,(1-Y)(\zeta-1)^{c-1} + \dot{\mathbf{f}}\,(\zeta-1)^{c}.$$
$$z_2\eta = \begin{pmatrix} 1 & 0 \\ \mathbf{v}_{z_2} & 1 \end{pmatrix}, \quad \text{where } {\mathbf{v}_{z_2}}^{\vartheta} = \dot{\mathbf{e}}\,(1-Y)^2(\zeta-1)^{c-1} + \dot{\mathbf{f}}\,(1-Y)(\zeta-1)^{c}.$$

Restriction of scalars turns $\dot{V}$ into a free $\mathcal{O}$-module, with an $\mathcal{O}$-basis consisting of

$$\dot{\mathbf{e}}Y^m \ (m \in \mathbb{Z}), \quad \dot{\mathbf{f}}Y^n \ (n \in \mathbb{Z}).$$

Thus the exterior square $\dot{V} \wedge_{\mathcal{O}} \dot{V}$ over $\mathcal{O}$ is a free $\mathcal{O}$-module, with $\mathcal{O}$-basis

$$\dot{\mathbf{e}}Y^m \wedge \dot{\mathbf{e}}Y^n \text{ (for } m<n), \quad \dot{\mathbf{f}}Y^m \wedge \dot{\mathbf{f}}Y^n \text{ (for } m<n), \quad \dot{\mathbf{e}}Y^m \wedge \dot{\mathbf{f}}Y^n, \quad \text{where } m,n \in \mathbb{Z};$$

below we express elements of $\dot{V} \wedge_{\mathcal{O}} \dot{V}$ with respect to this $\mathcal{O}$-basis, keeping track of the coefficients of the basis element $\dot{\mathbf{e}}Y \wedge \dot{\mathbf{e}}Y^2$.

Next we consider the composition $\psi = \hat{\vartheta}\varrho$ of the π-equivariant morphism

$$\hat{\vartheta}\colon [K,K] \cong K^{\mathrm{ab}} \wedge K^{\mathrm{ab}} \hookrightarrow V \wedge V \to \dot{V} \wedge \dot{V}$$

with the canonical homomorphism of $\mathbb{Z}$-modules $\varrho\colon \dot{V} \wedge \dot{V} \to \dot{V} \wedge_{\mathcal{O}} \dot{V}$. A routine computation yields

$$\begin{aligned} z^{\psi} &= [z_1, z_2]^{\hat{\vartheta}\varrho} = \left({\mathbf{v}_{z_1}}^{\vartheta} \wedge {\mathbf{v}_{z_2}}^{\vartheta}\right)^{\varrho} \\ &= \Big(\Big(\big(\dot{\mathbf{e}}\,(1-Y) + \dot{\mathbf{f}}\,(\zeta-1)\big) \wedge \big(\dot{\mathbf{e}}\,(1-Y)^2 + \dot{\mathbf{f}}\,(Y-1)(\zeta-1)\big)\Big)(\zeta-1)^{2c-2}\Big)^{\varrho} \\ &\qquad = \ldots + (\dot{\mathbf{e}}Y \wedge \dot{\mathbf{e}}Y^2)\,(-(\zeta-1)^{2c-2}) + \ldots, \end{aligned}$$

where on the far right-hand side we only display the $\dot{\mathbf{e}}Y \wedge \dot{\mathbf{e}}Y^2$-term. From this we deduce that

$$v^{\psi} = \left(z^{f(x)-1}\right)^{\psi} = z^{\psi}\,(f(\zeta)-1) = \ldots + (\dot{\mathbf{e}}Y \wedge \dot{\mathbf{e}}Y^2)\,(-(\zeta-1)^{2c-2}(\nu-1)) + \ldots,$$

where we again only record the $\dot{\mathbf{e}}Y \wedge \dot{\mathbf{e}}Y^2$-term to see that v^{ψ} is non-zero.

Finally, we deduce that $[g,w] \neq v$ for all $w \in [K,K]$ from

$$[g,w]^{\psi} = [x^p z, w]^{\psi} = [x^p, w]^{\psi} = (w^{1-x^p})^{\psi} = w^{\psi}\,(1-\zeta^p) = 0 \neq v^{\psi}. \qquad \square$$

Proof of Theorem 8.1.1. The group G is an $\mathcal{N}_{\mathbf{c}}$-free group of rank d, for parameters $d, l \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{N}^l$. If $d = 1$ or $\mathbf{c} \in \{(1),(2)\}$ then G is nilpotent of class at most 2; in this situation Proposition 8.3.2 implies that G has the Magnus property.

Now suppose that $d \geq 2$ and that $\mathbf{c} \notin \{(1),(2)\}$. We are to show that G does not have the Magnus property, and by Corollary 8.2.3 we may suppose that $d = 2$. If $l = 1$ then G is an $\mathcal{N}_{(c)}$-free group with $c = c_1 \geq 3$, and Proposition 8.3.7 shows that G does not have the Magnus property. Likewise, if $l = 2$ and $c_2 \in \{1, 2\}$, then G is an $\mathcal{N}_{(c,1)}$-free or $\mathcal{N}_{(c,2)}$-free group of rank 2 for $c = c_l$, and G does not have the Magnus property by Propositions 8.4.3 and 8.4.5.

Thus, we may suppose that we are in none of these special circumstances. We write $\mathbf{c} = (c_1, \ldots, c_l) \in \mathbb{N}^l$ and distinguish two cases.

Case 1: $c_l \geq 3$. In this case $l \geq 2$ and we put $N = \gamma_{(c_1+1,\ldots,c_{l-1}+1)}(G)$. We note that G/N is an $\mathcal{N}_{\mathbf{c}'}$-free group of rank 2, for $\mathbf{c}' = (c_1, \ldots, c_{l-1})$, while N is a free class-c nilpotent group with $c = c_l \geq 3$ of countably infinite rank.

Case 2: $c_l \in \{1, 2\}$. In this case $l \geq 3$ and we put $N = \gamma_{(c_1+1,\ldots,c_{l-2}+1)}(G)$. We note that G/N is an $\mathcal{N}_{\mathbf{c}'}$-free group of rank 2, for $\mathbf{c}' = (c_1, \ldots, c_{l-2})$, while N is an $\mathcal{N}_{(c,1)}$-free or $\mathcal{N}_{(c,2)}$-free group with $c = c_{l-1}$ of countably infinite rank.

In any case, Propositions 8.3.7, 8.4.3 and 8.4.5 and part (1) of Lemma 8.2.7 provide $g, v \in N$ such that (g, v) is a basic $\neg(\mathsf{MP})$-witness pair for N. Clearly, g and $h = gv$ have the same normal closure $\langle g \rangle^G = \langle h \rangle^G$ in G, and it suffices to prove that g and h are neither conjugate nor inverse-conjugate to one another in G.

For a contradiction, assume that $g^w \in \{h, h^{-1}\}$ for some $w \in G$. We put $H = G/N$ and $R = \mathbb{Z}H$. Observe that $G/[N,N]$ is a free abelian-by-$\mathcal{N}_{\mathbf{c}'}$ group of rank 2. The Magnus embedding for this group yields an embedding of the R-module $N^{\mathrm{ab}} = N/[N,N]$ into a free R-module. Our assumption yields $\mathbf{v}_g \bar{w} \in \{\mathbf{v}_g, -\mathbf{v}_g\}$, hence $\mathbf{v}_g(\bar{w} - 1) = 0$ or $\mathbf{v}_g(\bar{w} + 1) = 0$, where $\mathbf{v}_g$ denotes the image of g in N^{ab}, regarded as a module element, and $\bar{w} \in R$ denotes the image of w in $H \subseteq R$. We observe that $g \notin [N,N]$ implies that $\mathbf{v}_g \neq 0$. The group H is right-orderable and thus the group ring R has no zero-divisors; see [118, Ch. 13, Thm. 1.11], where the result is attributed to Bovdi [25]. This implies $\bar{w} - 1 = 0$ or $\bar{w} + 1 = 0$ in R. From $\bar{w} \in H$ we see that $\bar{w} \neq -1$. Hence $\bar{w} = 1$ and $w \in N$, in contradiction to the initial choice of g and $h = gv$ which precludes that they are conjugate in N. □

Finally, we extend Theorem 8.1.1 to yet another class of relatively free groups. For any $d, l \in \mathbb{N}$ and $\mathbf{c} = (c_1, \ldots, c_l) \in \mathbb{N}^l$, the *free centre-by-$\mathcal{N}_{\mathbf{c}}$ group* of rank d can be constructed as the quotient $F/[\gamma_{(c_1+1,\ldots,c_l+1)}(F), F]$ of an absolutely free group of rank d.

Proposition 8.4.6. *Let G be a free centre-by-$\mathcal{N}_{\mathbf{c}}$ group of rank 2, where $\mathbf{c} \in \mathbb{N}^l$ with $l \in \mathbb{N}_{\geq 2}$. Then there exists a basic $\neg(\mathsf{MP})$-witness pair for the group G.*

Proof. Write $G = \langle x, y \rangle$, with free generators x, y, and $\mathbf{c} = (c_1, \ldots, c_l)$. Let $Z = \gamma_{(c_1+1,\ldots,c_l+1)}(G) \subseteq [G,G]$. Consider $g = x$ and any $v \in Z \smallsetminus \{1\}$. Clearly, g has infinite order modulo $[G,G]$. It remains to show that $\langle g \rangle^G = \langle gv \rangle^G$ and $v \notin \{[g,w] \mid w \in G\}$.

Since $v \in [G,G] \subseteq \langle x \rangle^G$, we find $k \in \mathbb{N}$, $e_1, \ldots, e_k \in \{1, -1\}$ and $w_1, \ldots, w_k \in G$ such that $v = \prod_{i=1}^k (x^{e_i})^{w_i}$. From $x^{\sum_{i=1}^k e_i} \equiv_{[G,G]} v \equiv_{[G,G]} 1$ we conclude that $\sum_{i=1}^k e_i = 0$.

Since v is central in G, this gives

$$v = v^{\sum_{i=1}^k e_i} \prod_{i=1}^k (x^{e_i})^{w_i} = \prod_{i=1}^k \left((xv)^{e_i}\right)^{w_i} \in \langle g \rangle^G \cap \langle gv \rangle^G$$

and hence $\langle g \rangle^G = \langle gv \rangle^G$.

Next assume, for a contradiction, that $[g, w] = v$ for some $w \in G$. Then $[x, w] \equiv_Z 1$, and Proposition 8.4.4 implies $w = x^m z$, for suitable $m \in \mathbb{Z}$ and $z \in Z$. This gives $[g, w] = [x, x^m z] = 1 \neq v$, a contradiction. □

Proof of Theorem 8.1.2. The group G is a free centre-by-$\mathcal{N}_{\mathbf{c}}$ group G of rank d, for parameters $d, l \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{N}^l$. If $d = 1$ or $\mathbf{c} = (1)$ then G is nilpotent of class at most 2; in this situation Proposition 8.3.2 implies that G has the Magnus property.

Now suppose that $d \geq 2$ and that $\mathbf{c} \neq (1)$. We are to show that G does not have the Magnus property, and by Corollary 8.2.3 we may suppose that $d = 2$. If $l = 1$ then G is an $\mathcal{N}_{(c+1)}$-free group with $c = c_1 \geq 2$, and Proposition 8.3.7 shows that G does not have the Magnus property. If $l \geq 2$ then Proposition 8.4.6 shows that G does not have the Magnus property. □

8.5 — Torsion-free nilpotent groups with the Magnus property

In this section we establish Theorem 8.1.3. Let $c \in \mathbb{N}$. In order to construct a torsion-free, nilpotent MP-group of prescribed nilpotency class c we aim to build an ultraproduct $\mathcal{G} = \left(\prod_{p \in P} G_p\right)/{\sim_{\mathfrak{U}}}$ of suitable finite p-groups G_p, where $P = \mathbb{P}_{>2}$ denotes the set of all odd primes, and to appeal to Łoś's theorem.

Being class-c nilpotent, not having q-torsion for a given prime q, and possessing the Magnus property are first-order properties; thus it would suffice to construct a family of finite nilpotent MP-groups G_p, $p \in P$, such that each group has nilpotency class c and such that for any prime q there are only finitely many $p \in P$ with $q \mid |G_p|$. However, finite nilpotent MP-groups are necessarily $\{2, 3\}$-groups. More generally, every group with the Magnus property is *inverse semi-rational*, that is, every pair of elements generating the same subgroup (not necessarily normal) is already a pair of conjugate or inverse-conjugate elements. Finite groups with this property can be characterised using character theory. Chillag and Dolfi [30] establish that all finite soluble inverse semi-rational groups are $\{2, 3, 5, 7, 13\}$-groups. Consequently, no family as described above exists. However, we can salvage our strategy by considering a variant of the Magnus property.

For convenience we use $[k, l]_{\mathbb{Z}=\{m \in \mathbb{Z} \mid k \leq m \leq l\}}$, for $k, l \in \mathbb{Z}$, as a short notation for intervals in $\mathbb{Z}$. Suppose that G_p, $p \in P$, is a family of groups with the following properties: (i)(i)

(i) for each p, the group G_p is a metabelian finite p-group of nilpotency class c;
(ii) there exists a non-decreasing function $f \colon \mathbb{N} \to \mathbb{N}$ such that, for each $p \in P$, the group

G_p satisfies the following uniform, but 'weak' Magnus property:

$$\begin{aligned}
&\forall g, h \in G_p \quad \forall N \in \mathbb{N} \quad \forall k, l \in [0, N]_{\mathbb{Z}} \\
&\qquad \forall e_1, \ldots, e_k \in \{1, -1\} \quad \forall v_1, \ldots, v_k \in G_p \\
&\qquad \forall d_1, \ldots, d_l \in \{1, -1\} \quad \forall w_1, \ldots, w_l \in G_p : \\
&\Big(h = \prod\nolimits_{i=1}^{k} (g^{e_i})^{v_i} \;\wedge\; g = \prod\nolimits_{j=1}^{l} (h^{d_j})^{w_j}\Big) \\
&\quad \Longrightarrow \Big(\exists r, s \in [-f(N), f(N)]_{\mathbb{Z}} \quad \exists v, w \in G_p\colon \quad g = (h^r)^v \wedge h = (g^s)^w\Big).
\end{aligned} \tag{wM_f}$$

We observe that the quantifier over the integer N can be eliminated by passing to a countable collection of sentences in the first-order language of groups; the quantifiers over k, l are purely for convenience and can be eliminated directly.

Let $\mathfrak{U}$ be a non-principal ultrafilter on the index set P. Then, by Łoś's theorem, the ultraproduct

$$\mathcal{G} = \Big(\prod\nolimits_{p \in P} G_p\Big) / \sim_{\mathfrak{U}}$$

is a metabelian, torsion-free, class-c nilpotent group satisfying the uniform 'weak' Magnus property (wM_f); compare with [29, Thm. 4.1.9]. But, since $\mathcal{G}$ is torsion-free nilpotent, the latter implies that $\mathcal{G}$ has the Magnus property. Indeed, suppose that $g, h \in \mathcal{G}$ are such that $\langle g \rangle^{\mathcal{G}} = \langle h \rangle^{\mathcal{G}}$. If $g = 1$ then $h = 1$, and $g = h$ so that g and h are certainly conjugate. Now suppose that $g \neq 1$. Then (wM_f) yields $r, s \in \mathbb{Z}$ and $v, w \in \mathcal{G}$ such that $g = (h^r)^v$ and $h = (g^s)^w$, thus

$$g = (h^r)^v = \Big(\big((g^s)^w\big)^r\Big)^v = (g^{rs})^{vw}.$$

Consider the upper central series $1 = \mathrm{Z}_0(\mathcal{G}) \leq \mathrm{Z}_1(\mathcal{G}) \leq \ldots \leq \mathrm{Z}_c(\mathcal{G}) = \mathcal{G}$ of the nilpotent group $\mathcal{G}$. Since $g \neq 1$, we find $i \in [1, c]_{\mathbb{Z}}$ such that $g \in \mathrm{Z}_i(\mathcal{G}) \smallsetminus \mathrm{Z}_{i-1}(\mathcal{G})$. Since $\mathrm{Z}_i(\mathcal{G})/\mathrm{Z}_{i-1}(\mathcal{G})$ is torsion-free, g generates an infinite cyclic group modulo $\mathrm{Z}_{i-1}(\mathcal{G})$ and the congruence $g \equiv_{\mathrm{Z}_{i-1}(\mathcal{G})} g^{rs}$ implies that $rs = 1$. Thus $r \in \{1, -1\}$, and $g = (h^r)^v$ is conjugate to h or to h^{-1} in $\mathcal{G}$.

Finally, since all relevant properties of $\mathcal{G}$, including the ordinary Magnus property, are expressible in terms of first-order sentences, the Löwenheim–Skolem theorem [29, Cor. 2.1.4] shows that there exists a countable MP-group $\dot{\mathcal{G}}$ which is metabelian, torsion-free and nilpotent of class precisely c. This establishes Theorem 8.1.3.

It remains to construct the family of groups G_p, $p \in P$, with the properties (i) and (ii) described above. We give one rather concrete construction. Fix an odd prime p, and consider

$$G = G_p = \langle t, a \mid [a, t] = a^p,\ t^{p^{c-1}} = a^{p^c} = 1 \rangle. \tag{8.5}$$

Clearly, $G = \langle t \rangle \ltimes \langle a \rangle$ is metacyclic, with $\langle t \rangle \cong C_{p^{c-1}}$ and $\langle a \rangle \cong C_{p^c}$. It is easy to work out the lower central series:

$$\gamma_1(G) = G \qquad \text{and} \qquad \gamma_{i+1}(G) = \langle a^{p^i} \rangle \qquad \text{for } i \in \mathbb{N};$$

in particular, G has nilpotency class c. In order to check the 'weak' Magnus property, we make use of the following lemma, which is heuristically a torsion analogue of Proposition 8.3.2.

Lemma 8.5.1. *Let G be a finite nilpotent group such that $\mathrm{C}^*_G(x) = \{[x, w] \mid w \in G\}$ for every $x \in G$. Then G has the 'weak' Magnus property (wM_f) for $f \colon \mathbb{N} \to \mathbb{N}$, $n \mapsto n$.*

Proof. Let $g, h \in G$. Suppose that $k, l \in \mathbb{N}_0$ and

$$h = \prod\nolimits_{i=1}^{k} (g^{e_i})^{v_i} \qquad \text{and} \qquad g = \prod\nolimits_{j=1}^{l} (h^{d_j})^{w_j} \tag{8.6}$$

for suitable $e_1, \ldots, e_k, d_1, \ldots, d_l \in \{1, -1\}$ and $v_1, \ldots, v_k, w_1, \ldots, w_l \in G$. In particular, this implies that $\langle g \rangle^G = \langle h \rangle^G$. By symmetry, it suffices to show that there exist $w \in G$ and $s \in [-k, k]_{\mathbb{Z}}$ such that $h = (g^s)^w$.

If $g = 1$ then $h = 1$, and no further explanations are necessary. Now suppose that $g \neq 1$, and write $M = \mathrm{C}^*_G(g) = \mathrm{C}^*_G(h) \trianglelefteq G$; see Lemma 8.2.8. From Eq. (8.6) we deduce that $h \equiv_M g^s$, for some $s \in [-k, k]_{\mathbb{Z}}$.

We claim that $\langle g \rangle = \langle g^s \rangle$. For this it is enough to show that $p \nmid s$ for every prime p that divides the order of g. The finite nilpotent group G is the direct product of its Sylow subgroups; let $\bar{x}$ denote the image of $x \in G$ under the canonical projection onto the Sylow p-subgroup of G. Then $\bar{g} \neq 1$ implies that $g \in \mathrm{Z}_i(\overline{G}) \smallsetminus \mathrm{Z}_{i-1}(\overline{G})$, for suitable $i \in \mathbb{N}$, hence $\overline{M} \leq \mathrm{Z}_{i-1}(\overline{G})$ and $\bar{g} \notin \overline{M}$. Consequently, $\langle \bar{g} \rangle \overline{M} = \langle \bar{g} \rangle^{\overline{G}} = \langle \bar{h} \rangle^{\overline{G}} = \langle \bar{g}^s \rangle \overline{M}$ implies $p \nmid s$.

Using our general assumption on cocentralisers in G, we deduce from $\langle g \rangle = \langle g^s \rangle$ that $M = \mathrm{C}^*_G(g^s) = \{[g^s, w] \mid w \in G\}$. Therefore $h \equiv_M g^s$ shows that there exists $w \in G$ such that $h = (g^s)^w$. $\square$

It remains to verify that the condition on cocentralisers in Lemma 8.5.1 applies to the concrete groups $G = G_p$, defined in (8.5). Recall that $p > 2$. Actually, it is convenient to check the required property for the compact p-adic analytic group

$$H = \langle t, a \mid [a, t] = a^p \rangle_{\text{pro-}p} \cong (1 + p\,\mathbb{Z}_p) \ltimes \mathbb{Z}_p,$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers and the multiplicative group of one-units $1 + p\,\mathbb{Z}_p = \overline{\langle 1 + p \rangle} = \{(1+p)^\lambda \mid \lambda \in \mathbb{Z}_p\}$ acts naturally on the additive group $\mathbb{Z}_p$. The group H maps naturally onto G, with kernel $\overline{\langle t^{p^{c-1}}, a^{p^c} \rangle}$, and it is easy to see that the condition on cocentralisers that we are interested in is inherited by factor groups.

Let $h \in H$. For $h = 1$, it is clear that $\{[h, y] \mid y \in H\} = \{1\}$ is closed under multiplication. Now suppose that $h \neq 1$. Then h is of the form $h = t^\lambda a^\mu$ for uniquely determined $\lambda, \mu \in \mathbb{Z}_p$, not both equal to 0. Easy computations show:

$$\{[h, a^\nu] \mid \nu \in \mathbb{Z}_p\} = \{a^{(-(1+p)^\lambda+1)\nu} \mid \nu \in \mathbb{Z}_p\} = \{a^\sigma \mid \sigma \in p^{1+v_p(\lambda)}\,\mathbb{Z}_p\},$$
$$\{[h, t^\nu] \mid \nu \in \mathbb{Z}_p\} = \{a^{(-1+(1+p)^\nu)\mu} \mid \nu \in \mathbb{Z}_p\} = \{a^\sigma \mid \sigma \in p^{1+v_p(\mu)}\,\mathbb{Z}_p\},$$

where $v_p\colon \mathbb{Z}_p \to \mathbb{N}_0 \cup \{\infty\}$ denotes the p-adic valuation map.

Put $m = 1+\min\{v_p(\lambda), v_p(\mu)\}$. As $A_m = \{a^\sigma \mid \sigma \in p^m\,\mathbb{Z}_p\}$ is a closed normal subgroup of H, the well-known commutator identity $[a, bc] = [a, c][a, b]^c$ for arbitrary group elements a, b, c shows that

$$\{[h, y] \mid y \in H\} = A_m \trianglelefteq_{\mathrm{c}} H$$

is indeed closed under multiplication.

Appendix to Part II

To this chapter we append some GAP [52] code that we have used in the preparation of this chapter. The computations are furthermore used in the proods of Theorem 7.3.5 and Theorem 7.3.7 and Theorem 7.6.1.

The following function straitforwardly compute, given a group G, if it has the Magnus property or the strong Magnus property, respectively.

```
hasMagnusProperty := function( G )
  local g,h,U,V;
  for g in G do
    U := NormalClosure(G, Subgroup(G, [g]));
     for h in G do
       V := NormalClosure(G, Subgroup(G, [h]));
       if V = U and not IsConjugate(G, g, h)
        and not IsConjugate(G, g, h^-1) then
          return false;
       fi;
    od;
  od;
  return true;
end;

hasStrongMagnusProperty := function( G )
  local g,h,U,V;
  for g in G do
    U := NormalClosure(G, Subgroup(G, [g]));
     for h in G do
       V := NormalClosure(G, Subgroup(G, [h]));
       if V = U and not IsConjugate(G, g, h) then
          return false;
       fi;
    od;
  od;
  return true;
```

```
end;
```

The next segment of code produces a list of all subgroups of $GL_2(5)$ of orders 24 or 48, and checks if any of these subgroups has the strong Magnus property.

```
i := 0;
flag := "No subgroup has the strong Magnus property.";
G := GL(2,5);
possibleOrders := [24,48];
L := ConjugacyClassesSubgroups(G);
N := [];
for c in L do
  H := Representative(c);
  if Size(H) in possibleOrders then Add(N,H); fi;
od;
for H in N do
  Print(IdGroup(H), "\n");
  if hasStrongMagnusProperty(H) then
    flag := "There is a subgroup with
     the strong Magnus property.";
    break;
  fi;
  i := i+1;
od;
Print(flag);
```

It has the output:

```
[ 24, 5 ]
[ 24, 1 ]
[ 24, 2 ]
[ 48, 33 ]
[ 48, 5 ]
No subgroup has the strong Magnus property.
```

The next piece of code produces a list of all subgroups of $GL_2(13)$ of orders 84, 168, 252, 336, 504 or 672, and checks if any of these subgroups has the Magnus property.

```
i := 0;
flag := "No subgroup has the Magnus property.";
G := GL(2,13);
possibleOrders := [84, 168, 252, 336, 504, 672];
L := ConjugacyClassesSubgroups(G);
N := [];
```

```
for c in L do
 H := Representative(c);
  if Size(H) in possibleOrders then Add(N,H); fi;
od;
for H in N do
  Print(IdGroup(H), "\n");
  if hasMagnusProperty(H) then
    flag := "There is a subgroup with the Magnus property.";
    break;
  fi;
  i := i+1;
od;
Print(flag);
```

It has the output:

```
[ 84, 4 ]
[ 84, 6 ]
[ 84, 12 ]
[ 168, 25 ]
[ 168, 4 ]
[ 168, 6 ]
[ 336, 59 ]
No subgroup has the Magnus property.
```

Finally, the last program produces the entries of the table given in the proof of Theorem 7.6.1.

```
maximalOrder := function(G)
  local g, m;
 m := 1;
  for g in G do
    m := Maximum(m, Order(g));
  od;
  return m;
end;

subdirectProductCheck := function(G,H)
  local S, P, l;
  S := SubdirectProducts(G,H);
  l := [];
  for P in S do
    if maximalOrder(P) > 6 then continue; fi;
```

```
      AddSet(l,IdGroup(P));
   od;
   for P in l do Print(P, "\n"); od;
end;

L := [CyclicGroup(2), CyclicGroup(3),
  CyclicGroup(4), SmallGroup([8,4]),
   SmallGroup([12,1]), SmallGroup([24,3])];

for G in L do for H in L do
   Print(IdGroup(G), ", ", IdGroup(H), ": \n");
   subdirectProductCheck(G,H);
   Print("\n");
od;od;
```

Part III:

Representation zeta functions

Chapter 9

Representation zeta functions of subgroups and split extensions of $\mathrm{SL}_2^1(\mathbb{Z}_p)$

Written in collaboration with Margherita Piccolo.

9.1 — Introduction

A topological group G is called *representation rigid*, or just rigid, if the number of isomorphism classes of n-dimensional continuous irreducible complex representations of G is finite for all $n \in \mathbb{N}$. Given a rigid group G, we write $r_n(G)$ for the number of irreducible representations of dimension n and $R_N(G)$ for the sum $\sum_{i=1}^{N} r_n(G)$. It is a fundamental goal of asymptotic representation theory to understand the behaviour of the resulting integer sequences $(r_n(G))_{n\in\mathbb{N}}$ and $(R_N(G))_{N\in\mathbb{N}}$. A group G has *polynomial representation growth* if these sequences grows polynomially.

Following [76], we encode the sequence $(r_n(G))_{n\in\mathbb{N}}$ as a Dirichlet generating function defined over the complex numbers, given by

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}.$$

This function is called the *representation zeta function of* G. If $R_N(G)$ is unbounded and G has polynomial representation growth, it converges on a right half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha(G)\}$ delimited by the *abscissa of convergence* $\alpha(G)$ (which may be infinite) that coincides with the degree of the growth of $(R_N(G))_{N\in\mathbb{N}}$.

We consider the representation zeta functions of certain compact p-adic analytic groups. These appear as the local factors in the Euler product decomposition of arithmetic groups with the congruence subgroup property [98]; if the ambient groups are semi-simple, the congruence subgroup property also implies polynomial representation growth [103].

A finitely generated profinite group G is rigid if and only if it is FAb, i.e. if every open subgroup H has finite abelianisation $H/[H,H]$, see [21, Prop. 2]. For p-adic analytic groups, being FAb is equivalent to the perfectness of the $\mathbb{Q}_p$-Lie algebra associated to G.

For every odd prime p, Jaikin-Zapirain [89] proved that the representation zeta function of a FAb compact p-adic analytic group is close to being a rational function in p^{-s}, see [89, Theorem 1.1]; in particular the representation zeta functions of FAb p-adic analytic pro-p groups are rational functions in p^{-s}. This result was recently extended to $p = 2$ by Stasinski and Zordan, cf. [142]. So far, there is only a small number of FAb compact p-adic analytic Lie groups for which the representation zeta function has been computed explicitly, see [6, 7, 89, 161]. In particular, the representation zeta functions of $\mathrm{SL}_2(\mathbb{Z}_p)$ and $\mathrm{SL}_3(\mathbb{Z}_p)$ are known, but the higher-rank examples remain mysterious; even the abscissae of convergence of the groups $\mathrm{SL}_n(\mathbb{Z}_p)$ have not been computed, though there exist some bounds, cf. [98].

The main technique for the computation of representation zeta functions of compact p-adic analytic groups can be separated into two steps. First, compute the zeta function of a finitely generated torsion-free potent pro-p subgroup of finite index, using the Kirillov orbit method. The latter allows the construction (in case $p \neq 2$) of an explicit bijection between the isomorphism classes of irreducible representations and orbits of the coadjoint action of the group on its associated $\mathbb{Z}_p$-Lie lattice, for details, see [61, 84]. Using this bijection, one reduces to the computation of the zeta function of the subgroup to the evaluation of a p-adic integral related to the structure of the associated Lie lattice. Then, using relative Clifford theory, one may bridge the (finite index) gap to the full group. Both steps are described in detail in [7].

Even though the situation for the 'linearisable' part is backed-up by strong theoretical results, the actual computation of the involved p-adic integrals is highly complicated. We restrict our attention to the computation of the representation zeta functions of finitely generated torsion-free potent pro-p groups.

Our main result concerns the representation zeta function of the semidirect product of a suitable subgroup of the first principal congruence subgroup $\mathrm{SL}_2^1(\mathbb{Z}_p)$ of $\mathrm{SL}_2(\mathbb{Z}_p)$ acting continuously on an abelian group $V \cong \mathbb{Z}_p^d$ of finite $\mathbb{Z}_p$-rank d. Under certain assumptions, the representation zeta function of the semidirect product $G = \mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes V$ is itself a product of the form

$$\zeta_G(s) = \zeta_{\mathrm{SL}_2^1(\mathbb{Z}_p)}(s) \cdot \zeta^G_{\mathrm{SL}_2^1(\mathbb{Z}_p)}(s-1),$$

see Theorem 9.3.6. In particular, it is a multiple of the representation zeta function of the group $\mathrm{SL}_2^1(\mathbb{Z}_p)$. The other factor is the zeta function associated to the representation of G induced from the trivial representation of $\mathrm{SL}_2^1(\mathbb{Z}_p)$.

Zeta functions associated to representations of p-adic analytic groups were recently introduced and studied by Kionke and Klopsch in [92]; these zeta functions are a generalisation of the representation zeta function related to a group. The specific function appearing as a factor above may be seen as the zeta function associated to the action of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ on the cosets of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ in $\mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes V$, i.e. on V. In view of this interpretation, we hope to extend our theorem to a wider class of groups. We give an example of a potent pro-p group where the corresponding equation fails to hold, showing that this behaviour is not universal. However, it is unclear which pairs of groups and subgroups allow for such a decomposition in general. Since the representation zeta function of $\mathrm{SL}_3(\mathbb{Z}_p)$ (and its principal congruence subgroups) is known, it would be particularly interesting to obtain a better understanding of the situation for semidirect products involving these groups. However, at this stage our methods are particular to $\mathrm{SL}_2^1(\mathbb{Z}_p)$ and its subgroups.

Indeed, to obtain Theorem 9.3.6, we use the following useful description of the representation zeta functions of all potent subgroups of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ of finite index. Let H be a open potent pro-p subgroup of $G = \mathrm{SL}_2^1(\mathbb{Z}_p)$. Then

$$\zeta_H(s) = |G : H| \cdot \zeta_G(s),$$

see Theorem 9.3.1 for a more general approach to such statements.

Using these results, we explicitly calculate the representation zeta functions of four semidirect products of abelian groups with $\mathrm{SL}_2^1(\mathbb{Z}_p)$, see Section 9.4. These are the first representation zeta functions of p-adic analytic groups corresponding to non-semisimple algebras to be computed. Because of the great difficulties that come with the computation of zeta functions of this type, we believe these examples to be a useful outlook towards a deeper understanding of the theory. We record our examples in the following theorem.

Theorem 9.1.1. *Let m and $k \in \mathbb{N}$, and let G_k be the subgroup of $G = G_0 = \mathrm{SL}_2^m(\mathbb{Z}_p)$ generated by $\{h^{p^k}, x, y^{p^k}\}$, where h, x, y are a certain standard generating set for G. Let ν_k be the natural action of G_k on $\mathbb{Z}_p^2$, let $\sigma = \mathrm{Sym}^2(\nu_0)$ be the symmetric square of ν_0 acting on $\mathbb{Z}_p^3$, and let δ be the diagonal action of G_0 on $\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2$. Then*

(i) $\zeta_{G_k \ltimes_{\nu_k} \mathbb{Z}_p^2}(s) = p^{ks+5} Q_1/Q_2$, *where*

$$Q_1 = (1 - p^{-(2+s)})(1 - p^{-s})(1 - p^{1-s} - p^{1-2s} + p^{(k+1)(1-s)-1}),$$
$$Q_2 = (1 - p^{1-s})^3(1 + p^{1-s}),$$

with abscissa $\alpha = 1$.

(ii) $\zeta_{G \ltimes_\sigma \mathbb{Z}_p^3}(s) = p^{6m} Q_1/Q_2$, *where*

$$Q_1 = (1 - p^{-2s})(1 - p^{-(2+s)}),$$
$$Q_2 = (1 - p^{3-2s})(1 - p^{1-s}),$$

with abscissa $\alpha = \frac{3}{2}$.

(iii) $\zeta_{G \ltimes_\delta \mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2}(s) = p^{7m}(1 - p^{-(s+2)}) Q_1/Q_2$, *where*

$$Q_1 = p^{3-2s}(1 - p^{1-s}) - p^{-3(1+s)}(1 + p^{4-3s})(1 - p^{4-4s})(1 - p^{4-3s})(1 + p^{2-2s} - p^{-2s}),$$
$$Q_2 = (1 - p^{5-4s})(1 - p^{3-2s})(1 - p^{1-s}).$$

with abscissa $\alpha = \frac{3}{2}$.

Note that even in the specific case of semi-direct products involving $\mathrm{SL}_2^1(\mathbb{Z}_p)$ higher dimensional examples seem troublesome to compute explicitly.

Since we deal mostly with the group $\mathrm{SL}_2^1(\mathbb{Z}_p)$, the prime 2 leads to various technical difficulties in proofs (and to deviations in results). Thus, we exclude the case $p = 2$, and adopt the standing assumption that p is an odd prime.

9.2 — Preliminaries

9.2.1. Integral formalism for potent pro-p groups. — Denote by $\mathrm{Irr}(G)$ the set of (isomorphism classes of) irreducible *smooth* complex representations $\sigma\colon G \to \mathrm{GL}(V)$ of a group G; a representation of a topological group is smooth if all point stabilisers are open subgroups. All groups considered are profinite, and all representations continuous and smooth. In this situation every representation decomposes as a direct sum of irreducible, and all irreducible representations factor through a finite quotient of G. A representation σ of a profinite group G is called *strongly admissible*, if its decomposition into irreducible subrepresentations,

$$\sigma = \bigoplus_{\varphi \in \mathrm{Irr}(G)} m(\sigma, \varphi)\varphi,$$

is such that there are only finitely many constituents (counted with multiplicity) of every dimension, i.e. that the multipliers $m(\sigma, \varphi) \in \mathbb{N}$ in the formula above fulfil

$$\sum_{\substack{\varphi \in \mathrm{Irr}(G) \\ \dim(\varphi) = d}} m(\sigma, \varphi) \in \mathbb{N}$$

for all $d \in \mathbb{N}$. Given a strongly admissible representation σ of G, one forms the formal Dirichlet series

$$\zeta_\sigma(s) = \sum_{\varphi \in \mathrm{Irr}(G)} m(\sigma, \varphi) \dim(\varphi)^{-s},$$

called the *zeta function associated to* σ. If ρ is the *regular representation* of a compact group G, i.e. the (right) translation action of G on the space of continuous functions $G \to \mathbb{C}$, then every irreducible representation φ of G appears with multiplicity $m(\rho, \varphi) = \dim(\varphi)$. If G is rigid, the regular representation is strongly admissible, and we have

$$\zeta_\rho(s) = \sum_{\varphi \in \mathrm{Irr}(G)} \dim(\varphi)^{1-s} = \zeta_G(s-1).$$

In this way, the zeta functions of strongly admissible representations generalise the representation zeta functions of rigid profinite groups.

More generally, if H is a closed subgroup of G, the multiplication action of G on the H-cosets induces a permutation representation, which is equal to the representation of G

induced from the trivial representation of H; choosing $H = 1$ clearly recovers the previous case. Assume that G is finitely generated. Then, according to [92, Theorem A], the representation $\mathrm{Ind}_H^G(\mathbb{1}_H)$ is strongly admissible if and only if the group G is *FAb relative to H*, i.e. if, for every open subgroup K of G, the quotient $K/(H \cap K)[K, K]$ is finite.

Since we are only concerned with representations of the form $\mathrm{Ind}_H^G(\mathbb{1}_H)$ for $H \leq_c G$, we simplify our notation. We call this function the *relative zeta function of G with respect to H* and denote it by $\zeta_H^G(s)$.

A pro-p group G is called *potent* if $\gamma_{p-1}(G) \subseteq G^p$ (recall our standing assumption that $p \neq 2$), where $\gamma_k(G)$ denotes the k^{th} term of the lower central series of G for $k \in \mathbb{N}$. If G is potent, finitely generated, and torsion-free, we call it *uniformly potent*; since such groups are a straight-forward generalisation of uniformly powerful groups.

Every uniformly potent pro-p group is a saturable group in the sense of Lazard, see [60, 93, 100], hence there exists a finite dimensional $\mathbb{Z}_p$-Lie lattice $\log(G)$ defined on the same set. This Lie lattice is itself *potent*, i.e. it fulfils $\gamma_{p-1}(\log(G)) \subseteq p\log(G)$. We remark that every saturable pro-p group of dimension less than p is potent, cf. [63] for similar phenomena. Using the Lie lattices associated to a uniformly potent pro-p group G and to a closed potent subgroup H, one can use the Kirillov orbit method to describe the constituents of the representation $\mathrm{Ind}_H^G(\mathbb{1}_H)$ by certain orbits under the coadjoint action of G on the Pontryagin dual of $\log(G)$. For a detailed explanation, see [92, Section 4].

To state the p-adic integral that results from this description, we need some further notation. We assume that the Lie sublattice $\mathfrak{h} = \log(H)$ is a direct summand of $\mathfrak{g} = \log(G)$. We choose a $\mathbb{Z}_p$-basis Y of $\mathfrak{h}$ and extend this basis to $X \cup Y$, a $\mathbb{Z}_p$-basis of $\mathfrak{g}$. The *commutator matrix of $\mathfrak{g}$ with respect to $X \cup Y$* is the skew-symmetric $\dim(\mathfrak{g})$-by-$\dim(\mathfrak{g})$ matrix with entries in $\mathfrak{g}$ given by

$$\mathrm{Com}(\mathfrak{g}, X \cup Y) = \big([b, b']\big)_{b, b' \in X \cup Y}.$$

Let $w \colon \mathfrak{g} \to \mathbb{Q}_p$ be a linear functional. Write $w\,\mathrm{Com}(\mathfrak{g}, X \cup Y)$ for the entry-wise application of w. This results in a skew-symmetric matrix with entries in $\mathbb{Q}_p$.

Recall that the determinant of every skew-symmetric n-by-n matrix $T = (t_{i,j})_{1 \leq i,j \leq n}$ with entries in a commutative unital ring R is the square (as a polynomial) of the *pfaffian determinant*, which must necessarily be trivial in case n is odd, and which, for even numbers $n = 2k$, is defined by

$$\mathrm{pf}(T) = \frac{1}{2^k k!} \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{l=1}^{k} t_{(2l-1)\sigma, (2l)\sigma}.$$

A *pfaffian minor* of a skew-symmetric matrix T is the pfaffian determinant of a principal submatrix given by the rows and columns with index in some fixed subset of $\{1, \ldots, n\}$ of even cardinality. We write $\mathrm{Pfaff}(T)$ for the set of all pfaffian minors of a skew-symmetric matrix T. Finally, for a subset $S \subseteq \mathbb{Q}_p$, define

$$||S||_p = \max\{|s|_p \mid s \in S\}.$$

Theorem 9.2.1 (Kionke & Klopsch, [92]). *Let G be a uniformly potent pro-p group and $H \leq G$ be a closed potent subgroup such that G is FAb relative to H. Assume that the Lie lattice $\mathfrak{h}$ associated to H is a direct summand of the Lie–lattice $\mathfrak{g}$ associated to G. Let Y be a basis of $\mathfrak{h}$ and $X \cup Y$ a basis of $\mathfrak{g}$. Let $\mathbb{Q}_p X^*$ be the subspace spanned by X^* in $\mathfrak{g}^*$, with X^* as part of the dual basis $(X \cup Y)^*$ of $X \cup Y$. Then*

$$\zeta_H^G(s) = \int_{\mathbb{Q}_p X^*} ||\operatorname{Pfaff}(w \operatorname{Com}(\mathfrak{g}, X \cup Y))||_p^{-1-s} \mathrm{d}\mu(w),$$

where μ denotes the Haar measure of $\mathbb{Q}_p X^*$, normalised such that the $\mathbb{Z}_p$-span of X^* has measure 1.

Note that the factor in the description in [92] which seems absent here is included in our description of the structure matrix.

Using the identity $\zeta_G(s) = \zeta_1^G(s+1)$ and normalising with respect to the standard (dual) basis, we obtain the corresponding integral formula for the representation zeta function of G.

Corollary 9.2.2. *Let G be a uniformly potent FAb pro-p group of dimension d and let X be a basis of the associated Lie lattice $\mathfrak{g}$. Let X be a basis for $\mathfrak{g}$ and χ the $\mathbb{Q}_p$-isomorphism that maps the standard basis $E = \{e_1, \dots, e_d\}$ of $\mathbb{Q}_p^d$ to X. Then*

$$\zeta_G(s) = \int_{(\mathbb{Q}_p^d)^*} ||\operatorname{Pfaff}((w\chi^{-1}[\chi(e_i), \chi(e_j)])_{i,j})||_p^{-2-s} \mathrm{d}\mu(w).$$

9.2.2. Representation theory of semidirect products. — We make the following standing assumptions for the remainder of this section. Let G be a compact topological group and H a closed subgroup such that G decomposes (continuously) as the semidirect product $G = H \ltimes V$, where V is abelian. We describe the irreducible (continuous) representations of G in terms of those of H, using their classical description by Mackey [106], see [137] for a straight-forward proof. To do so, we introduce some notation.

Given a finite index subgroup Δ of a group Γ and a representation σ of Δ, we denote by $\operatorname{Ind}_\Delta^\Gamma(\sigma)$ the representation of Γ *induced* from σ. This is the representation given by the tensor product of the vector space V supporting σ with $\mathbb{C}[\Gamma]$ over $\mathbb{C}[\Delta]$.

We write $\operatorname{char}(\sigma)$ for the character associated to the representation σ. The character of an induced representation can be expressed as the function

$$\operatorname{char}(\operatorname{Ind}_\Delta^\Gamma(\sigma)) \colon \gamma \mapsto \sum_{\substack{x \in \Gamma/\Delta \\ \gamma^x \in \Delta}} \operatorname{char}(\sigma)(\gamma^x),$$

where Γ/Δ denotes a set of coset representatives of Δ in Γ. Now given a homomorphism $\varphi : \Gamma \to \Omega$ into some group Ω and a representation σ of Ω we define the *inflation* of σ along φ by $\mathrm{Inf}_{\Omega}^{\Gamma,\varphi}(\sigma) = \varphi\sigma$. If the homomorphism is implicit or its choice clear from the context, we will drop it from the superscript.

Lastly, given a subgroup Δ of Γ and a representation σ of Γ, we say that σ is *extendable* if there exists a representation $\mathrm{Ext}_{\Delta}^{\Gamma}(\sigma)$ of Γ such that $\mathrm{Ext}_{\Delta}^{\Gamma}(\sigma)|_{\Delta} = \sigma$. Usually such *extensions* do not exist, however, for a pair (G, V) as described at the beginning of this section, a representation $\sigma \in \mathrm{Irr}(V)$ can be extended to the group $H_\sigma = \mathrm{St}_H(\sigma) \ltimes V$, where H acts on $\mathrm{Irr}(V)$ by $\sigma^h(v) = \sigma(v^{h^{-1}})$ for $v \in V$. Indeed, set

$$\mathrm{Ext}_V^{H_\sigma}(\sigma)(vh) = \mathrm{Ext}_V^{H_\sigma}(\sigma)(v)$$

for all $h \in \mathrm{St}_H(\sigma)$. In fact,

$$\mathrm{Ext}_V^{H_\sigma}(\sigma)(vhv'h') = \sigma(vv'^{h^{-1}}) = \sigma(v)\sigma(v') = \mathrm{Ext}_V^{H_\sigma}(\sigma)(vh)\,\mathrm{Ext}_V^{H_\sigma}(\sigma)(v'h'),$$

so $\mathrm{Ext}_V^{H_\sigma}(\sigma)(vh)$ defines a representation of $H_\sigma = \mathrm{St}_H(\sigma) \ltimes V$.

Proposition 9.2.3. *Let $\{\chi_i \mid i \in I\}$ be a set of representatives of the orbits of the action $\mathrm{Irr}(V) \curvearrowleft H$, and let K_i be the stabiliser of χ_i in H. Assume that all K_i are of finite index in H, and assume that I is countable. Then every irreducible representation of $G = H \ltimes V$ is of the form*

$$\mathrm{Ind}_{K_i \ltimes V}^{G}(\mathrm{Inf}_{K_i}^{K_i \ltimes V}(\tau) \otimes \mathrm{Ext}_V^{K_i \ltimes V}(\chi_i)), \tag{$*$}$$

for some $\tau \in \mathrm{Irr}(K_i)$, and two representations of this form are equivalent only if they are given by the same pair (i, τ).

For compact groups, one may define a generalisation of the usual inner product on the set of irreducible characters for finite groups, by setting

$$\langle \mathrm{char}(\sigma), \mathrm{char}(\tau) \rangle_G = \int_G \mathrm{char}(\sigma)(g)\overline{\mathrm{char}(\tau)(g)}\, \mathrm{d}\mu(g),$$

where μ is the normalised (left-)Haar measure of G. As in the setting of finite groups, it is still true that, given an irreducible component τ of σ, the value of $\langle \mathrm{char}(\sigma), \mathrm{char}(\tau) \rangle_G$ equals the multiplicity of τ appearing in the decomposition of σ. For us, the following equality will be of use.

Proposition 9.2.4. *Let G be the semi-direct product $H \ltimes V$. Let $\{\chi_i \mid i \in I\}$ and K_i be as in Proposition 9.2.3. Then*

$$\langle\, \mathrm{char}(\mathrm{Ind}_{K_i \ltimes V}^{G}(\mathrm{Inf}_{K_i}^{K_i \ltimes V}(\tau) \otimes \mathrm{Ext}_V^{K_i \ltimes V}(\chi_i)))\,,\, \mathrm{char}(\mathrm{Ind}_H^G(\mathbb{1}_H))\,\rangle_G = \langle \mathrm{char}(\tau), \mathrm{char}(\mathbb{1}_{K_i}) \rangle_{K_i}.$$

Proof. Since H is closed, Frobenius reciprocity yields

$$\begin{aligned}\langle\, &\mathrm{char}(\mathrm{Ind}^{G}_{K_i\ltimes V}(\,\mathrm{Inf}^{K_i\ltimes V}_{K_i}(\tau)\otimes \mathrm{Ext}^{K_i\ltimes V}_{V}(\chi_i)))\,,\, \mathrm{char}(\mathrm{Ind}^{G}_{H}(\mathbb{1}_H))\,\rangle_G \\ &= \langle\, \mathrm{char}(\mathrm{Res}^{G}_{H}\,\mathrm{Ind}^{G}_{K_i\ltimes V}(\mathrm{Inf}^{K_i\ltimes V}_{K_i}(\tau)\otimes \mathrm{Ext}^{K_i\ltimes V}_{V}(\chi_i)))\,,\, \mathrm{char}(\mathbb{1}_H)\,\rangle_H \\ &= \int_H \sum_{\substack{x\in G/K_i\ltimes V\\ h^x\in K_i\ltimes V}} \mathrm{char}(\tau)(h^x)\cdot \mathrm{char}(\chi_i)(h^x)\,\mathrm{d}\mu(h).\end{aligned}$$

We may choose the representatives of the cosets of $K_i \ltimes V$ in G to be the representatives of the cosets of K_i in H. Then $h^x \in K_i \ltimes V$ precisely when it is in K_i, and the above integral may be transformed as follows:

$$\begin{aligned}\int_H \sum_{\substack{x\in H/K_i\\ h^x\in K_i}} \mathrm{char}(\tau)(h^x)\cdot \mathrm{char}(\chi)(1)\,\mathrm{d}\mu(h) &= \int_H \mathrm{char}(\mathrm{Ind}^{H}_{K_i}(\tau))(h)\,\mathrm{char}(\mathbb{1}_H)(h)\,\mathrm{d}\mu(h) \\ &= \langle \mathrm{char}(\mathrm{Ind}^{H}_{K_i}(\tau)), \mathrm{char}(\mathbb{1}_H)\rangle_H \\ &= \langle \mathrm{char}(\tau), \mathrm{char}(\mathbb{1}_{K_i})\rangle_{K_i}.\end{aligned}$$

The last equation is again won by applying Frobenius reciprocity. □

9.3 — Zeta functions of subgroups and semidirect products

9.3.1. A condition for subgroups to be thetyspectral. — Let G be a uniformly potent pro-p group. Assume that we understand its representation zeta function – what can we deduce about the representation zeta function of an open potent subgroup H? Are there circumstances where G essentially dictates the representation zeta function of H in terms of its own zeta function? One such case is well-known. Given a uniformly potent pro-p, the subgroup G^{p^k} generated by the $(p^k)^{\mathrm{th}}$ powers of G (which is indeed equal to the set of these powers) gives rise to the same representation zeta function as the full group G does, up to a constant factor. We call two groups with this property *thetyspectral* and two groups with identical representation zeta functions *isospectral*.

Using the Lazard correspondence, one obtains a Lie sublattice $\mathfrak{h} \subseteq \mathfrak{g}$ from a pair of groups $H \leq G$ as above. Since the set underlying the Lie lattices are equal to the sets underlying the groups, the index of $\mathfrak{h}$ in $\mathfrak{g}$ is the index of H in G. This index is a finite number, such that $\mathfrak{h}$ has the same dimension as $\mathfrak{g}$. Thus, since $\mathfrak{g}$ is torsion-free, there exists an isomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ of $\mathbb{Z}_p$-modules, that does (in general) not preserve the Lie bracket. This isomorphism naturally extends to a $\mathbb{Q}_p$-isomorphism between $\mathfrak{g}_{\mathbb{Q}_p} = \mathfrak{g}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $\mathfrak{h}_{\mathbb{Q}_p} = \mathfrak{h}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Of course, $\mathfrak{g}_{\mathbb{Q}_p} = \mathfrak{h}_{\mathbb{Q}_p}$. Note that, for the same reason, $\mathfrak{h}^* = \mathfrak{g}^*$. Heuristically, aside from the Lie bracket, the difference between $\mathfrak{g}$ and $\mathfrak{h}$ is only a change of basis.

If φ is indeed an isomorphism of Lie lattices, we have

$$\mathrm{Pfaff}((w[x_i, x_j])_{x_i,x_j \in X}) = \mathrm{Pfaff}((w\varphi^{-1}[\varphi(x_i), \varphi(x_j)])_{x_i,x_j \in X})$$

and thus, as seen directly from the integral formulation, G and H are isospectral. However, a weaker condition is sufficient to establish thetyspectrality.

Theorem 9.3.1. *Let G be a FAb compact uniformly potent pro-p group and let $H \leq G$ be an open potent subgroup of G. Write $\mathfrak{g}$ and $\mathfrak{h}$ for the associated Lie lattices. Let $\beta\colon \mathfrak{g}\wedge\mathfrak{g} \to \mathfrak{g}$ be the linear map induced by the bracket of $\mathfrak{g}$, and let $\varphi\colon \mathfrak{g} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the $\mathbb{Q}_p$-linear isomorphism fixed above. If there exists a $\mathbb{Q}_p$-linear map $\psi\colon \mathfrak{g} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that $\beta|_{\mathfrak{h}\wedge\mathfrak{h}}(\varphi \wedge \varphi) = \psi\beta$, then G and H are thetyspectral with factor $|\det(\varphi\psi^{-1})|_p$.*

Proof. We use the integral formalism introduced earlier. Let $d = \dim(\mathfrak{g})$, let X be a basis for $\mathfrak{g}$ and let χ be the isomorphism mapping the standard basis $E = \{e_1, \dots, e_d\}$ of $\mathbb{Q}_p^d$ to X. Of course, $\varphi(X) = \varphi\chi(E)$ is a basis for $\mathfrak{h}$, and we may write

$$\begin{aligned}
\zeta_H(s) &= \int_{(\mathbb{Q}_p^d)^*} \|\,\mathrm{Pfaff}((w(\varphi\chi)^{-1}[\varphi\chi(e_i), \varphi\chi(e_j)])_{i,j\in\{1,\dots,d\}})\|_p^{-2-s} \mathrm{d}\mu(w) \\
&= \int_{(\mathbb{Q}_p^d)^*} \|\,\mathrm{Pfaff}(w\chi^{-1}\varphi^{-1}\psi\beta(\chi(e_i) \wedge \chi(e_j)))_{i,j\in\{1,\dots,d\}})\|_p^{-2-s} \mathrm{d}\mu(w) \\
&= |\det((\varphi^{-1}\psi)^*)|_p^{-1} \int_{((\varphi^{-1}\psi)^{\chi})^*(\mathbb{Q}_p^d)^*} \|\,\mathrm{Pfaff}(w\chi^{-1}[\chi(e_i), \chi(e_j)])_{i,j})\|_p^{-2-s} \mathrm{d}\mu(w),
\end{aligned}$$

where the last step is a simple change of variables. Of course, $((\varphi^{-1}\psi)^{\chi})^*(\mathbb{Q}_p^d)^* = (\mathbb{Q}_p^d)^*$ and $\det((\varphi^{-1}\psi)^* = \det(\varphi^{-1}\psi)$. Comparing with the integral describing $\zeta_G(s)$, the theorem is immediate. □

This is especially useful in the case that G is $\mathrm{SL}_2^1(\mathbb{Z}_p)$, since it is three-dimensional and potent. We will heavily rely on the following corollary.

Corollary 9.3.2. *Let H be a open potent pro-p subgroup of $G = \mathrm{SL}_2^1(\mathbb{Z}_p)$. Then*

$$\zeta_H(s) = |G : H| \cdot \zeta_G(s).$$

Proof. The Lie lattice corresponding to G is $\mathfrak{g} = p \cdot \mathfrak{sl}_2(\mathbb{Z}_p)$. Since the bracket is not degenerate, the image of $\mathfrak{g} \wedge \mathfrak{g}$ is of finite index in $\mathfrak{g}$; it is indeed equal to $p\mathfrak{g}$. Crucially, since $\dim(\mathfrak{g}) = 3$, the dimension of the exterior square $\mathfrak{g} \wedge \mathfrak{g}$ is also 3. Thus as a $\mathbb{Q}_p$-linear map, β is invertible. Clearly the $\mathbb{Q}_p$-linear map $\psi\colon \mathfrak{g} \to \mathfrak{h}$ defined by

$$\psi = \beta^{-1}(\varphi \wedge \varphi)\beta|_{\mathfrak{h}\wedge\mathfrak{h}}$$

meets the conditions of Theorem 9.3.1. Its determinant satisfies

$$\det(\psi) = \det(\beta)\det(\varphi \wedge \varphi)\det(\beta^{-1}) = \det(\varphi \wedge \varphi) = \det(\varphi)^{\dim(\mathfrak{g})-1} = \det(\varphi)^2.$$

Since $\det(\varphi) = |\mathfrak{g} : \mathfrak{h}| = |G : H|$ is a power of p, we find $|\det(\varphi^{-1}\psi)|_p^{-1} = \det(\varphi^{-1}\psi)$. Combining both equations, we find that the factor $|\det(\varphi^{-1}\psi)|_p^{-1}$ is equal to $|G : H|$. $\square$

Theorem 9.3.1 allows us to recover some known identities. Given a uniformly potent pro-p group G and an integer $n \in \mathbb{N}$, the representation zeta function of the subgroup of $(p^m)^{\text{th}}$ powers G^{p^m} is equal to $p^{\dim(G)m} \cdot \zeta_G(s)$. A similar behaviour was first described in [89], and a variation of this result was used to prove the main result in [62]. We can derive the statement as follows. Under the logarithm map, G^{p^m} is represented by $p^m \cdot \mathfrak{g}$, i.e. the map φ is induced by scalar multiplication by p^m. Thus

$$\beta(\varphi \wedge \varphi)(v \wedge w) = [p^m v, p^m w] = p^{2m}[v, w] = \varphi^2 \beta(v, w),$$

i.e. we may choose φ^2 as our ψ. Clearly $\det(\psi\varphi^{-1}) = \det(\varphi) = p^{m\dim(G)}$, and we have recovered the result mentioned above.

It is an interesting question which subgroups of $\mathrm{SL}_3^1(\mathbb{Z}_p)$ fulfil the conditions of Theorem 9.3.1, and what factors may appear aside from 1 and the index.

9.3.2. Semidirect products. —

Lemma 9.3.3. *Let G be a uniformly potent pro-p group, and let $\sigma\colon G \to \mathrm{GL}_n^1(\mathbb{Z}_p)$ be a finite-dimensional $\mathbb{Z}_p$ representation of G with image in the first principal congruence subgroup. Then the semidirect product $G \ltimes_\sigma \mathbb{Z}_p^n$ is a uniformly potent pro-p group.*

Proof. The semidirect product is clearly torsion-free and finitely generated. We have to prove that $\gamma_{p-1}(G \ltimes \mathbb{Z}_p^n) \leq (G \ltimes \mathbb{Z}_p^n)^p$. This term of the lower central series is generated by $\gamma_{p-1}(G)$ and $[\mathbb{Z}_p^n, G, \ldots, G]$, since $[\mathbb{Z}_p^n, G] \leq \mathbb{Z}_p^n$, an abelian group. The inclusion $\gamma_{p-1}(G) \leq G^p \leq (G \ltimes \mathbb{Z}_p^n)^p$ follows since G is potent. Let $g \in G$ and $v \in \mathbb{Z}_p^n$. Then

$$[v, g] = v^{-1}v^g = v^{g^\rho - \mathrm{Id}^{n\times n}}.$$

But $g^\sigma \in \mathrm{GL}_n^1(\mathbb{Z}_p)$ implies that $g^\rho - \mathrm{Id}^{n\times n} = ph$ for some $h \in \mathrm{GL}_n(\mathbb{Z}_p)$, hence $[v, g] = pv^h \in p\,\mathbb{Z}_p^n$. The subgroup of p^{th} powers is a normal subgroup, hence $[\mathbb{Z}_p^n, G, \ldots, G] \leq p\,\mathbb{Z}_p^n \leq (G \ltimes \mathbb{Z}_p^n)^p$. $\square$

Lemma 9.3.4. *Let G be uniformly potent pro-p group. Let $H, K \leq G$ be two closed potent subgroups. Then $H \cap K$ is potent.*

Proof. Write $\mathfrak{g}$ for the Lie lattice $\log(G)$. Since H and K are potent, they correspond to potent sublattices $\mathfrak{h}$ and $\mathfrak{k}$. It is enough to prove that the intersection $\mathfrak{h} \cap \mathfrak{k}$ is potent; if it is, it corresponds to a potent subgroup of G, which is equal to $H \cap K$, since the underlying sets of the Lie lattices and groups are the same. By the following calculation, the intersection $\mathfrak{h} \cap \mathfrak{k}$ is potent,

$$\gamma_{p-1}(\mathfrak{h} \cap \mathfrak{k}) \subseteq \gamma_{p-1}(\mathfrak{h}) \cap \gamma_{p-1}(\mathfrak{k}) \subseteq p\mathfrak{h} \cap p\mathfrak{k} = p(\mathfrak{h} \cap \mathfrak{k}). \qquad \square$$

Lemma 9.3.5. *Let G be a finitely generated torsion-free pro-p group, and let $\sigma\colon G \to \mathrm{GL}_n^1(\mathbb{Z}_p)$ be a faithful finite-dimensional $\mathbb{Z}_p$-representation of G, such that $G^\sigma \cap \mathrm{GL}_n^2(\mathbb{Z}_p) \leq (G^\sigma)^p$. Let χ be a (continuous) irreducible representation of $\mathbb{Z}_p^n$. Then $\mathrm{St}_G(\chi)$ (with respect to the action induced by σ) is a open potent subgroup of the uniformly potent group G.*

Proof. Without loss of generality, we identify G and its image under σ. Looking at

$$[G,G] = [G \cap \mathrm{GL}_n^1(\mathbb{Z}_p), G \cap \mathrm{GL}_n^1(\mathbb{Z}_p)] \leq G \cap \mathrm{GL}_n^2(\mathbb{Z}_p) \leq G^p,$$

we see that G is potent (even powerful). Since χ is continuous, it factors over some finite-index subgroup of $\mathbb{Z}_p^n$; hence its kernel contains some subgroup of the form $p^m\,\mathbb{Z}_p^n$ for some $m \in \mathbb{N}$. If $g \in G \cap \mathrm{GL}_n^m(\mathbb{Z}_p)$, then g stabilises χ, since χ cannot detect the action of g on its argument. Hence $G_m = G \cap \mathrm{GL}_n^m(\mathbb{Z}_p) \leq \mathrm{St}_G(\chi)$.

If $m = 0$, the representation is trivial and the statement follows immediately. If $m = 1$, since G is contained in $\mathrm{GL}_n^1(\mathbb{Z}_p)$, the full (potent) group stabilises χ. Thus, assume $m > 1$. Since χ factors over $p^m\,\mathbb{Z}_p^n$, it induces a representation of $\mathbb{Z}_p^n/p^m\,\mathbb{Z}_p^n \cong (\mathbb{Z}/p^m\,\mathbb{Z})^n$; which can be described by a dual vector $x \in ((\mathbb{Z}/p^m\,\mathbb{Z})^n)^*$. Then the stabiliser of χ fits into the exact sequence

$$1 \to G_m \to \mathrm{St}_G(\chi) \to \mathrm{St}_{G/G_m}(x) \to 1.$$

Every orbit of $\mathbb{Z}_p^n$ under the action of the group $\mathrm{GL}_n(\mathbb{Z}_p)$ contains an element of the form $(p^k, 0, \ldots, 0)$ for some $k \in \mathbb{N}$; the Smith normal form. Thus, under conjugation by an appropriate element of $\mathrm{GL}_n(\mathbb{Z}_p)$, we may assume that x is of this form. It is easy to see that its stabiliser is the intersection of G/G_m with the general affine group $\mathrm{GA}_n(\mathbb{Z}/p^m\,\mathbb{Z}) \cong \mathrm{GL}_{n-1}(\mathbb{Z}/p^m\,\mathbb{Z}) \ltimes (\mathbb{Z}/p^m\,\mathbb{Z})^{n-1}$. Lifting to G, we find that every element of G is (non-uniquely) a product of an element of G_m and an element of $G \cap \mathrm{GA}_n(\mathbb{Z}_p)$. Since $\mathrm{GA}_n(\mathbb{Z}_p) \cong \mathrm{GL}_{n-1}(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^{n-1}$, both G_m and $G \cap \mathrm{GA}_n(\mathbb{Z}_p)$ are potent group by Lemma 9.3.3 and Lemma 9.3.4.

By our assumption, the intersection $G_m = G \cap \mathrm{GL}_n^m(\mathbb{Z}_p)$ is contained in the subgroup of p^{th} powers. Thus, we find $[G \cap \mathrm{GA}_n(\mathbb{Z}_p), G \cap \mathrm{GL}_n^m(\mathbb{Z}_p)] \leq G^p$. Since $G \cap \mathrm{GA}_n(\mathbb{Z}_p)$ is potent, we deduce that $\mathrm{St}_G(\chi)$ is potent. □

The assumptions for the last lemma seem a bit technical. It would be interesting to describe all actions such that the point stabilisers are potent. However, Lemma 9.3.5 is strong enough to establish that the point stabilisers of irreducible representations under the action induced by

(i) the natural action of $\mathrm{SL}_n^m(\mathbb{Z}_p)$ or $\mathrm{GL}_n^m(\mathbb{Z}_p)$, for $n, m \in \mathbb{N}$ with $m > 1$,
(ii) the symmetric square of the natural action of $\mathrm{SL}_2^m(\mathbb{Z}_p)$, for $m \in \mathbb{N}$, and
(iii) and direct sums of representations fulfilling the assumptions.

The natural actions clearly fulfil the condition $G^\rho \cap \mathrm{GL}_n^m(\mathbb{Z}_p) \leq G^p$, since the congruence subgroups of $\mathrm{GL}_n(\mathbb{Z}_p)$ coincide with the power subgroups. Also, the third statement is clear. For the second statement, consider the embedding given in Example 9.4.2.

We are now able to prove our main result.

Theorem 9.3.6. *Let H be a potent subgroup of $\mathrm{SL}_2^1(\mathbb{Z}_p)$, and let $\rho\colon H \to \mathrm{GL}(V)$ be a finite-dimensional representation of H adhering to the assumptions of Lemma 9.3.5 and such that the semidirect product $G = H \ltimes_\rho V$ is FAb. Then*

$$\zeta_G(s) = \zeta_H(s) \cdot \zeta_H^G(s-1).$$

Proof. By Proposition 9.2.3, all irreducible representations of G are of the form $(*)$ described there, for some representative χ_i of a coset of the stabiliser of the action of H on $\mathrm{Irr}(V)$, and an irreducible representation τ of the stabiliser K_i of χ_i. The dimension of such a representation is given by the product $|H : K_i| \cdot \dim(\tau)$. Thus

$$r_n(G) = \sum_{\substack{a,b\in\mathbb{N}\\ ab=n}} \sum_{\substack{i\in I\\ |H:K_i|=a}} r_b(K_i).$$

By Lemma 9.3.5 the group K_i is a uniformly potent pro-p group, hence Corollary 9.3.2 implies $r_b(K_i) = |H : K_i| \cdot r_b(H)$.

Now to every $i \in I$ we may associate the $|H : K_i|$-dimensional representation

$$\mathrm{Ind}_{K_i\ltimes V}^G(\mathrm{Ext}_V^{K_i\ltimes V}(\chi_i)).$$

In view of Proposition 9.2.4, these are precisely the $|H : K_i|$-dimensional irreducible constituents of $\mathrm{Ind}_H^G(\mathbb{1}_H)$, hence

$$r_n(G) = \sum_{\substack{a,b\in\mathbb{N}\\ ab=n}} \sum_{\substack{i\in I\\ |H:K_i|=a}} a \cdot r_b(H) = \sum_{\substack{a,b\in\mathbb{N}\\ ab=n}} r_a(G,H) \cdot a \cdot r_b(H),$$

where $r_a(G,H)$ denotes the number of a-dimensional irreducible constituents of $\mathrm{Ind}_H^G(\mathbb{1}_H)$. We see that the numbers $r_n(G)$ are equal to the Dirichlet convolutions of $a \cdot r_a(G,H)$ and $r_b(H)$. The factor a corresponds to a shift in the Dirichlet generating function of the sequence $r_a(G,H)$, i.e. $\sum_{a\in\mathbb{N}} r_a(G,H)\cdot a\cdot a^{-s} = \zeta_{G,H}(s-1)$. Since the generating function of a Dirichlet convolution is the product of the corresponding generating functions, this concludes the proof. □

9.4 — Examples

With Theorem 9.3.6 established, we aim to compute the representation zeta functions of semidirect products with $\mathrm{SL}_2^1(\mathbb{Z}_p)$ in detail. However, mirroring the difficulties of computing the zeta-functions (or even the abscissa of convergence) of p-adic analytic groups of high dimension, the relative zeta-functions associated to high-dimensional modules of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ correspond to p-adic integrals that are cumbersome to compute.

We begin with some generalities. Using the terminology of Theorem 9.2.1, we find that the Lie lattice $\mathfrak{v}$ spanned by X, i.e. such that $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{h}$, corresponds to the abelian subgroup V. Thus, the commutator matrix can be written as block matrix of the form

$$\mathrm{Com}(\mathfrak{g}, X \cup Y) = \begin{pmatrix} 0 & A \\ -A^\intercal & \mathrm{Com}(\mathfrak{h}, Y) \end{pmatrix},$$

where A is the matrix $A = ([x, y])_{x \in X, y \in Y}$. This simplifies the integral described in Corollary 9.2.2. Recall that we integrate over the subspace W spanned by the dual basis of $\mathfrak{v}$, while, since $\mathfrak{h}$ is a sublattice, the entries of $\mathrm{Com}(\mathfrak{h}, Y)$ are in $\mathfrak{h}$. Thus, given $w \in W$, after application to all entries we find a matrix of the form

$$\begin{pmatrix} 0 & wA \\ -wA^\intercal & 0 \end{pmatrix}.$$

We have to compute the pfaffian minors of this matrix. Every principal submatrix is still of the form $\left(\begin{smallmatrix} 0 & B \\ -B^\intercal & 0 \end{smallmatrix}\right)$, for some submatrix B of wA obtained by deleting rows and columns. The determinant (and hence the pfaffian determinant) is 0 if B is not a square matrix; otherwise the determinant of such a matrix is equal to $\det(B)^2$, and its pfaffian determinant equals $\det(B)$ up to a sign. Thus, the set $\mathrm{Pfaff}(w\,\mathrm{Com}(\mathfrak{g}, X \cup Y))$ is equal to the set $\mathrm{Min}(wA)$ of all minors of wA. All in all, we find

$$\zeta_H^G(s) = \int_{\mathfrak{v}^*} \|\mathrm{Min}(wA)\|_p^{-1-s} \mathrm{d}\mu(w).$$

Example 9.4.1. For our first example, we compute the representation zeta function of the group $G = \mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2$. As one would expect, this zeta function is very closely related to the zeta function of $H = \mathrm{SL}_2^1(\mathbb{Z}_p)$, which is given by

$$\zeta_{\mathrm{SL}_2^1(\mathbb{Z}_p)}(s) = p^3 \frac{1 - p^{-s-2}}{1 - p^{-s+1}}.$$

This function can be computed without greater difficulties using Corollary 9.2.2; cf. [89], where the representation zeta function of the ambient group $\mathrm{SL}_2(\mathbb{Z}_p)$ (and indeed, for more general rings R) is computed. Furthermore, the actual p-adic integral that needs to be evaluated is similar to the one related to H. All in all, we shall prove that

$$\zeta_G(s) = p^5 \frac{(1 - p^{-2s})(1 - p^{-s-2})}{(1 - p^{1-s})^2(1 + p^{1-s})}.$$

Knowing the representation zeta function of H, using Theorem 9.3.6, it remains to compute the relative zeta function $\zeta^G_{\mathrm{SL}_2^1(\mathbb{Z}_p)}(s)$. Taking a step back, we indeed compute $\zeta_H^G(s)$ for $G = \mathrm{SL}_n^1(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^n$ and $H = \mathrm{SL}_n^1(\mathbb{Z}_p)$ for all $n > 1$ simultaneously. Note that the group $\mathrm{SL}_n^1(\mathbb{Z}_p) \ltimes p^m\,\mathbb{Z}_p^n$ is isomorphic (and in particular isospectral) to our group. It is useful to consider this group instead.

The Lie lattice corresponding to G is $\mathfrak{g} = p^m\mathfrak{sl}_n(\mathbb{Z}_p) \ltimes p^m\,\mathbb{Z}_p^n$, which has dimension n^2+n-1. Since we may embed G into $\mathrm{SL}^1_{n+1}(\mathbb{Z}_p)$, we can do the same for the Lie lattice, i.e. embed $\mathfrak{g}$ into $p^m\mathfrak{sl}_{n+1}(\mathbb{Z}_p)$ as the matrices of the form

$$\begin{pmatrix} p^m\mathfrak{sl}_n(\mathbb{Z}_p) & p^m\,\mathbb{Z}_p^n \\ 0 & 0 \end{pmatrix}.$$

Writing $e_{i,j}$ for the matrix with entry p^m at position (i,j) and all other entries equal to 0, a basis for $\mathfrak{g}$ is given by $X \cup Y$, where

$$\begin{aligned} &X = \{u_k \mid 1 \le k \le n\}, \quad \text{where} \quad u_k = e_{k,n+1} \quad \text{and} \\ &Y = \{h_l \mid 1 \le l \le n-1\} \cup \{x_{i,j} \mid 1 \le i < j \le n\} \cup \{y_{s,t} \mid 1 \le t < s \le n\}, \\ &\quad \text{where} \quad h_l = (e_{l,l} - e_{l+1,l+1}),\ x_{i,j} = e_{i,j} \quad \text{and} \quad y_{s,t} = e_{s,t}. \end{aligned}$$

Note Y forms a basis for $\mathfrak{h} = p^m\mathfrak{sl}_n(\mathbb{Z}_p)$.

By the considerations above, we may restrict our attention to the partial commutator matrix $A = \ ([x,y])_{x\in X, y\in Y}$. Consider the following identities,

$$[u_k, h_l] = \begin{cases} p^m\,u_k & \text{if } k = l, \\ -p^m\,u_k & \text{if } k = l+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$[u_k, x_{i,j}] = \begin{cases} p^m\,u_j & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases} \qquad [u_k, y_{s,t}] = \begin{cases} p^m\,u_t & \text{if } k = s, \\ 0 & \text{otherwise,} \end{cases}$$

which yield the following description of A, in which the rows correspond to the elements $u_1, \dots, u_n$, in this order, the columns of the matrix A_h correspond to $h_1, \dots, h_{n-1}$, the columns of A_j correspond to $x_{1,j}, \dots, x_{j-1,j}, y_{j+1,j} \dots, y_{n,j}$ for $j \in \{1, \dots, n\}$,

$$A = p^m \begin{pmatrix} A_h & A_1 & \dots & A_n \end{pmatrix}$$

with

$$A_h = \begin{pmatrix} \mathrm{diag}(u_1, \dots, u_{n-1}) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mathrm{diag}(u_2, \dots, u_n) \end{pmatrix}$$

and A_j the $(n \times n)$-matrix with a zero row and a zero column at j^{th} position, such that after the deletion of said row and column the matrix $u_j\,\mathrm{Id}^{(n-1)\times(n-1)}$ remain, for every $j \in \{1, \dots, n\}$.

For comparison, in case $n = 3$ we obtain the matrix

$$p^m \begin{pmatrix} u_1 & & & & u_2 & & u_3 & \\ -u_2 & u_2 & u_1 & & & & & u_3 \\ & -u_3 & & u_1 & & u_2 & & \end{pmatrix},$$

where empty entries are zero.

We claim that all minors of A are of the form $\pm p^{km} \prod_{i=0}^{k-1} u_{j_i}$ for some $j_i \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, n\}$ or equal to 0. Let B be a square submatrix of A. If B has a zero-column, we are done. If there is a column associated to an element $x_{i,j}$ or $y_{s,t}$, this column has at most one non-trivial entry, wherefore developing along this column reduces our task to a smaller matrix. Thus, we may assume that all columns are associated to elements of the form h_l. If there is no zero-column, the columns must correspond to a segment $h_l, h_{l+1}, \ldots, h_k$ and the rows must correspond to $u_l, \ldots, u_k$. The resulting matrix is lower triangular and has determinant $p^{m(k-l)} \prod_{i=l}^{k} u_i$. Thus, all minors are of the desired form. It is not hard to see that all possible products of said form are achievable.

Now we may calculate

$$\zeta_H^G(s) = \int_{\mathbb{Q}_p X^*} ||\operatorname{Min}(wA)||_p^{-1-s} \mathrm{d}\mu(w).$$

First notice that all variables $u_i^* \in X^*$ appear with a factor p^m in the integrant. Thus, by change of variables $u_i^* \mapsto p^m u_i^*$, we find

$$\begin{aligned} \zeta_H^G(s) &= p^{mn} \int_{\mathbb{Q}_p X^*} ||\operatorname{Min}(p^{-m} wA)||_p^{-1-s} \mathrm{d}\mu(w) \\ &= p^{mn} \int_{\mathbb{Q}_p X^*} \left|\left| \left\{ \prod_{k=0}^{l} w(u_{j_k}) \;\middle|\; l, j_k \in \{0, \ldots, n\}, j_k \neq 0 \text{ for all } k \right\} \right|\right|_p^{-1-s} \mathrm{d}\mu(w). \end{aligned}$$

Notice that, since the constant polynomial 1 is a minor of $p^{-m}wA$, if $w \in \mathbb{Z}_p X^*$, the maximal valuation is achieved at $|1|_p = 1$. Hence we write

$$\begin{aligned} \zeta_H^G(s) &= p^{mn} \int_{\mathbb{Z}_p X^*} 1 \,\mathrm{d}\mu(w) + p^{mn} \int_{\mathbb{Q}_p X^* \smallsetminus \mathbb{Z}_p X^*} ||\operatorname{Min}(p^{-m} wA)||_p^{-1-s} \mathrm{d}\mu(w) \\ &= p^{mn} \left(1 + \int_{\mathbb{Q}_p X^* \smallsetminus \mathbb{Z}_p X^*} ||\operatorname{Min}(p^{-m} wA)||_p^{-1-s} \mathrm{d}\mu(w) \right). \end{aligned}$$

We consider the following partition

$$\mathbb{Q}_p X^* \smallsetminus \mathbb{Z}_p X^* = \bigcup_{j=1}^{\infty} (p^{-j}\, \mathbb{Z}_p X^*) \smallsetminus (p^{1-j}\, \mathbb{Z}_p X^*)$$

into sets where the minimum of the valuations of the coordinates under w is constant and equal to $-j$. Thus, on each set the maximal possible value is achieved at a product u_i^n for some $i \in \{1, \dots, n\}$, and it is equal to p^{jn}. Hence we find

$$\begin{aligned}
\zeta_H^G(s) &= p^n \left(1 + \sum_{j=1}^{\infty} \int_{(p^{-j}\,\mathbb{Z}_p\,X^*) \setminus (p^{1-j}\,\mathbb{Z}_p\,X^*)} p^{jn(-1-s)} \right) \\
&= p^{mn} \left(1 + \sum_{j=1}^{\infty} (p^{jn} - p^{(j-1)n}) p^{jn(-1-s)} \right) \\
&= p^{mn} \left(1 + (1 - p^{-n}) \sum_{j=1}^{\infty} p^{-jns} \right) = p^{mn} \frac{(1 - p^{-n(s+1)})}{(1 - p^{-ns})}.
\end{aligned}$$

Consequently, the representation zeta function of $G = \mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2$ is given by the expression stated above, and the abscissa of convergence is $\alpha(G) = \alpha(H) = 1$.

Example 9.4.2. We consider the representation zeta function of the semidirect product $G = \mathrm{SL}_2^m(\mathbb{Z}_p) \ltimes \mathrm{Sym}^2(\mathbb{Z}_p^2)$, where $H = \mathrm{SL}_2^m(\mathbb{Z}_p)$ acts on $\mathbb{Z}_p^3$ by the symmetric square of the natural representation of H. Due to the higher dimension of the module, fewer irreducible representations of the module are conjugate under the action of H. One does suspect a fast representation growth, and indeed, the abscissa of convergence of this zeta function is greater than 1, the abscissa of convergence of H.

The image of H under the associated representation is given by

$$\left\{ \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \right\}.$$

One easily sees that Lemma 9.3.5 applies, and Theorem 9.3.6 reduces the calculation to the relative zeta function $\zeta_H^G(s)$.

Let $\mathfrak{g}$ be the 6-dimensional Lie lattice associated to G. We choose as basis $X \cup Y$, where $X = \{u, v, z\}$ and $Y = \{h, x, y\}$ are bases for $p^m\,\mathbb{Z}_p^3$ and $p^m \cdot \mathfrak{sl}_2(\mathbb{Z}_p)$ as in Example 9.4.1. We embed this Lie lattice into $p^m \cdot \mathfrak{sl}_4(\mathbb{Z}_p)$, and we compute the following partial commutator matrix with columns corresponding to the elements of Y and rows corresponding to elements of X

$$A = p^m \begin{pmatrix} 2u & 2v & 0 \\ 0 & z & u \\ -2z & 0 & 2v \end{pmatrix}.$$

The set of minors of A, up to sign and powers of 2, i.e. units of $\mathbb{Z}_p$, which do not contribute to the integral due to our assumption $p \neq 2$, is the given by

$$\mathrm{Min}(A) = \left\{ 1, p^m a, p^{2m} ab \mid a, b \in X \right\}.$$

The polynomials $p^m a$ are irrelevant, since they do not attain the maximal norm, similarly $p^{2m}ab$ does only contribute if $a = b$. Thus

$$\begin{aligned}\zeta_H^G(s) &= \int_{\mathbb{Q}_p X^*} ||\{1, w(p^{2m}a^2) \mid a \in X\}||_p^{-1-s} \mathrm{d}\mu(w) \\ &= p^{3m} \int_{\mathbb{Q}_p^3} ||\{1, a^2 \mid a \in X\}||_p^{-1-s} \mathrm{d}\mu(u, v, z) \\ &= p^{3m}(1 + \sum_{i \in \mathbb{N}} p^{-2j(1+s)}(p^{3j} - p^{3(j-1)})) \\ &= p^{3m} \frac{1 - p^{-2(1+s)}}{1 - p^{1-2s}}.\end{aligned}$$

Then the zeta function of G is

$$\zeta_G(s) = \zeta_H^G(s-1)\zeta_H(s) = p^{6m} \frac{(1 - p^{-2s})(1 - p^{-(2+s)})}{(1 - p^{3-2s})(1 - p^{1-s})},$$

and then

$$\alpha(G) = \frac{3}{2} > 1 = \alpha(H).$$

Example 9.4.3. Let G be the group $\mathrm{SL}_2^1(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2$, again. The group G acts again naturally on $\mathbb{Z}_p^2$. By Lemma 9.3.5, every stabiliser of an irreducible representation of $\mathbb{Z}_p^2$ under the action of G is potent. We will show that the equivalent of Corollary 9.3.2 is not true for G, and we cannot argue as we did in the proof of Theorem 9.3.6 to compute the representation zeta function of $G \ltimes \mathbb{Z}_p^2$. Indeed, we shall see in Example 9.4.4 that the zeta function cannot be factorised into $\zeta_G(s)$ and $\zeta_G^{G \ltimes \mathbb{Z}_p^2}(s-1)$.

We consider the stabiliser G_k of the irreducible representation induced by the representation $\rho\colon \mathbb{Z}_p^2 / p^k \mathbb{Z}_p^2 \to \mathbb{C}$ given by $v \mapsto (1, 0)v$. Using the same embedding into $\mathrm{SL}_3^1(\mathbb{Z}_p)$ as in the previous example, this stabiliser is the subgroup generated by matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

and by the subgroup $\mathrm{SL}_2^k(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2$. It is easily seen that the Lie lattice $\mathfrak{g}_k = \log(G_k)$ is generated by the elements $\{p^k h, x, p^k y, u, v\}$, where

$$\mathbf{h} = \begin{pmatrix} p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{y} = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}.$$

The group G_k has the structure of a semidirect product. Indeed, $G_k = H_k \ltimes \mathbb{Z}_p^2$, where H_k is the stabiliser of ρ in $\mathrm{SL}_2^1(\mathbb{Z}_p)$. Now by Theorem 9.3.6 and Corollary 9.3.2, we may write

$$\zeta_{G_k}(s) = \zeta_{H_k}(s) \cdot \zeta_{H_k}^{G_k}(s-1) = |G_k : H_k| \cdot \zeta_{\mathrm{SL}_2^1(\mathbb{Z}_p)}(s) \cdot \zeta_{H_k}^{G_k}(s-1).$$

Consider the quotient by $\mathrm{SL}_2^k(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2$. The subgroup $H_k(\mathrm{SL}_2^k(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2)$ is the group of upper uni-triangular matrices with one non-trivial entry at position $(1,2)$. Thus, the index of H_k in $\mathrm{SL}_2^1(\mathbb{Z}_p)$ is p^k. It remains to calculate the relative zeta function.

We compute the matrix $A = ([a,b])_{a\in\{u,v\}, b\in\{p^k h, x, p^k y\}}$,

$$A = \begin{pmatrix} p^{k+1}u & pv & 0 \\ -p^{k+1}v & 0 & p^{k+1}u, \end{pmatrix}$$

and the set of minors

$$\mathrm{Min}(wA) = \left\{1, w(p^{k+1}u), w(pv), w(p^{k+1}v), w(p^{k+1}uv), w(p^{2k+2}u^2), w(p^{k+2}v^2)\right\}.$$

After a change of basis $u \mapsto p^{k+1}u$, $v \mapsto pv$ the integral describing the relative zeta function has the form

$$\zeta_{H_k}^{G_k}(s) = p^{k+2} \int_{\mathbb{Q}_p^2} ||1, u, v, uv, u^2, p^k v^2||_p^{-1-s} \mathrm{d}\mu(u,v).$$

Since $|u|_p \leq |u^2|_p$ whenever $|u^2|_p \geq |1|_p$, the polynomial u is irrelevant. Comparing the valuations of the remaining polynomials, we determine that the maximum is reached by 1 in the area $\mathbb{Z}_p^2$, by u^2 within

$$\bigcup_{m\in\mathbb{N}} p^{-m}\,\mathbb{Z}_p^\times \times p^{-m}\,\mathbb{Z}_p,$$

by uv within

$$\bigcup_{m\in\mathbb{N}} p^{-m}\,\mathbb{Z}_p^\times \times p^{-m-k}\,\mathbb{Z}_p \smallsetminus p^{-m+1}\,\mathbb{Z}_p,$$

by $p^k v^2$ within

$$\bigcup_{m\in\mathbb{N}} p^{-m}\,\mathbb{Z}_p \times p^{-m-k}\,\mathbb{Z}_p^\times,$$

and by v in the area

$$\mathbb{Z}_p \times (p^{-k}\,\mathbb{Z}_p \smallsetminus p\,\mathbb{Z}_p).$$

These areas overlap. We cut up $\mathbb{Q}_p^2$ into pieces on which a fixed polynomial is maximal, see Fig. 9.1 for comparison, such that

$$\begin{aligned}\zeta_{H_k}^{G_k}(s) = \int_{A_1} |1|_p^{-1-s}\mathrm{d}\mu(u,v) + \int_{A_2} |u^2|_p^{-1-s}\mathrm{d}\mu(u,v) + \int_{A_3} |uv|_p^{-1-s}\mathrm{d}\mu(u,v) \\ + \int_{A_4} |p^k v^2|_p^{-1-s}\mathrm{d}\mu(u,v) + \int_{A_5} |v|_p^{-1-s}\mathrm{d}\mu(u,v).\end{aligned}$$

Area A_1, where $|1|_p$ is maximal, is defined as $p\,\mathbb{Z}_p^2$, hence

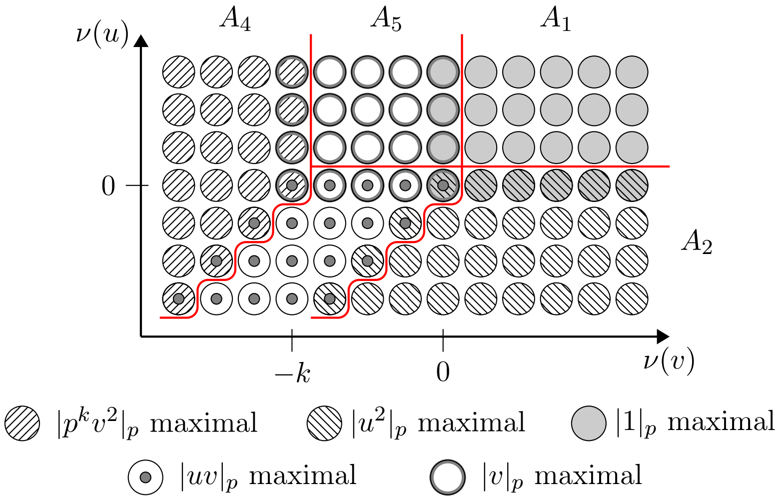

Figure 9.1: A sketch of the partition of $\mathbb{Q}_p^2$ we use for the computation of Example 9.4.3, in case $k = 4$. Every circle represents a subset of fixed valuation, i.e. of the form $p^a\,\mathbb{Z}_p^\times \times p^b\,\mathbb{Z}_p^\times$ of $\mathbb{Q}_p^2$.

$$\int_{A_1} |1|_p^{-1-s} \mathrm{d}\mu(u,v) = p^{-2}.$$

Area A_2 is defined as $\bigcup_{j\in\mathbb{N}} p^{-j}\,\mathbb{Z}_p^\times \times p^{-j+1}\,\mathbb{Z}_p$, such that

$$\begin{aligned}\int_{A_2} |u^2|_p^{-1-s} \mathrm{d}\mu(u,v) &= \sum_{j=0}^{\infty} \int_{p^{-j}\,\mathbb{Z}_p^\times \times p^{-j+1}\,\mathbb{Z}_p} p^{2j(-1-s)} \mathrm{d}\mu(u,v) \\ &= (1-p^{-1})p^{-1} \sum_{j=0}^{\infty} p^{2j+2j(-1-s)} = p^{-1}\frac{(1-p^{-1})}{1-p^{-2s}}.\end{aligned}$$

We put $A_3 = \bigcup_{j\in\mathbb{N}} p^{-j}\,\mathbb{Z}_p^\times \times (p^{-j-k+1}\,\mathbb{Z}_p \smallsetminus p^{-j+1}\,\mathbb{Z}_p)$ and calulate

$$\int_{A_3} |uv|_p^{-1-s} \mathrm{d}\mu(u,v) = \sum_{j=0}^{\infty} \int_{p^{-j}\,\mathbb{Z}_p^\times} |u|_p^{-1-s} \sum_{l=0}^{k-1} \int_{p^{-j-l}\,\mathbb{Z}_p^\times} |v|_p^{-1-s} \mathrm{d}\mu(v)\mathrm{d}\mu(u).$$

Evaluating the inner sum yields

$$\sum_{l=0}^{k-1} \int_{p^{-j-l}\,\mathbb{Z}_p^\times} |v|_p^{-1-s} \mathrm{d}\mu(v) = \sum_{l=0}^{k-1} (1-p^{-1})p^{-(j+l)s} = (1-p^{-1})p^{-js}\frac{1-p^{-ks}}{1-p^{-s}}.$$

Thus, the integral evaluates to the following expression,

$$\begin{aligned}\int_{A_3} |uv|_p^{-1-s} \mathrm{d}\mu(u,v) &= (1-p^{-1})\frac{1-p^{-ks}}{1-p^{-s}}\sum_{j=0}^{\infty} p^{-js}\int_{p^{-j}\,\mathbb{Z}_p^\times} p^{j(-1-s)}\mathrm{d}\mu(u)\\ &= (1-p^{-1})^2\frac{1-p^{-ks}}{(1-p^{-s})(1-p^{-2s})}.\end{aligned}$$

The fourth area is defined by $A_4 = \bigcup_{j\in\mathbb{N}} p^{-j}\,\mathbb{Z}_p \times p^{-j-k}\,\mathbb{Z}_p^\times$, and we compute

$$\begin{aligned}\int_{A_4} |p^k v^2|_p^{-1-s}\mathrm{d}\mu(u,v) &= \sum_{j=0}^{\infty}\int_{p^{-j}\,\mathbb{Z}_p \times p^{-j-k}\,\mathbb{Z}_p^\times} p^{(2(j+k)-k)(-1-s)}\mathrm{d}\mu(u,v)\\ &= (1-p^{-1})\sum_{j=0}^{\infty} p^{-(2j+k)s} = p^{-ks}\frac{1-p^{-1}}{1-p^{-2s}}.\end{aligned}$$

Finally, the last area $A_5 = p\,\mathbb{Z}_p \times (p^{-k+1}\,\mathbb{Z}_p \smallsetminus p\,\mathbb{Z}_p)$ gives rise to

$$\begin{aligned}\int_{A_4} |v|_p^{-1-s}\mathrm{d}\mu(u,v) &= \int_{p\,\mathbb{Z}_p} 1\,\mathrm{d}\mu(u)\sum_{l=0}^{k-1}\int_{p^{-l}\,\mathbb{Z}_p^\times}|v|_p^{-1-s}\mathrm{d}\mu(v)\\ &= p^{-1}(1-p^{-1})\sum_{l=0}^{k-1}p^{-ls} = p^{-1}(1-p^{-1})\frac{1-p^{-ks}}{1-p^{-s}}.\end{aligned}$$

The sum of the five integrals calculated above factorises to

$$\zeta_{H_k}^{G_k}(s) = p^{ks+2}\frac{(1-p^{-(1+s)})(1-p^{-s}-p^{-(1+2s)}+p^{-(k+1)s-1})}{(1-p^{-s})^2(1+p^{-s})},$$

and overall, we have

$$\zeta_{G_k}(s) = p^{ks+5}\frac{(1-p^{-(2+s)})(1-p^{-s})(1-p^{1-s}-p^{1-2s}+p^{(k+1)(1-s)-1})}{(1-p^{1-s})^3(1+p^{1-s})}.$$

Thus, the subgroup G_k of G and G are not thetyspectral. Indeed, the quotient of the respective zeta functions is

$$\frac{\zeta_{G_k}(s)}{\zeta_G(s)} = p^{ks}\frac{(1-p^{1-s}-p^{1-2s}+p^{(k+1)(1-s)-1})}{(1-p^{1-s})(1+p^{-s})}.$$

Example 9.4.4. We now turn our attention to the direct sum $\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2$. Let G be the group $\mathrm{SL}_2^m(\mathbb{Z}_p) \ltimes (\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2)$ and H be $\mathrm{SL}_2^m(\mathbb{Z}_p)$. We have seen that the stabilisers of representation of $\mathbb{Z}_p^2$ in $\mathrm{SL}_2^m(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2$ are not thetyspectral with the full group, thus we are not able to compute the zeta function related to $\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2$ by considering it as the iterated semidirect product $(\mathrm{SL}_2^m(\mathbb{Z}_p) \ltimes \mathbb{Z}_p^2) \ltimes \mathbb{Z}_p^2$, i.e. the relative zeta function of $\mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2$ is not necessarily (and in fact, is not) the square of the relative zeta function associated to the natural action of H on $\mathbb{Z}_p^2$. This is reflected in the set of pfaffian minors associated to the corresponding Lie lattice, that we now compute. Let $\mathfrak{g}$ be the Lie lattice associated to G, with basis $X \cup Y$, where $X = \{u_1, u_2, v_1, v_2\}$, such that $\langle u_1, u_2, v_1, v_2\rangle = \mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2$, and $Y = \{h, x, y\}$.

Analogously to Example 9.4.1, we compute one of the blocks A of the commutator matrix of $\mathfrak{g}$, in which the columns correspond to the elements of Y, and the rows to the elements of X, as

$$A = p^m \begin{pmatrix} u_1 & u_2 & 0 \\ -u_2 & 0 & u_1 \\ v_1 & v_2 & 0 \\ -v_2 & 0 & v_1 \end{pmatrix}.$$

We see that due to the rising dimension, there exist non-trivial rank-3 minors of A, in addition to the rank-2 minors. In fact,

$$\operatorname{Min}(A) = \left\{1, p^{2m}ab, p^{2m}(u_1v_2 - u_2v_1), p^{3m}a(u_1v_2 - u_2v_1) \mid a, b \in X\right\}.$$

As we have seen earlier, when we calculate the maximum of the norms, the polynomials $p^{2m}ab$ are only relevant in case $a = b$, and since $p^{2m}(u_1v_2 - u_2v_1)$ is a factor of another polynomial, we may ignore it as well. Thus

$$\begin{aligned}\zeta_H^G(s) &= \int_{\mathbb{Q}_p X^*} ||\{1, w(p^{2m}a^2), w(p^{3m}a(u_1v_2 - u_2v_1)) \mid a \in X\}||_p^{-1-s} \mathrm{d}\mu(w) \\ &= p^{4m} \int_{\mathbb{Q}_p^4} ||\{1, a^2, a(u_1v_2 - u_2v_1) \mid a \in X\}||_p^{-1-s} \mathrm{d}\mu(u_1, u_2, v_1, v_2).\end{aligned}$$

where we applied the usual change of variables to get rid of the factors p^m. Using the same kind of decomposition as we have used in Example 9.4.1 for $\mathbb{Q}_p^4$, namely

$$\mathbb{Q}_p^4 = \mathbb{Z}_p^4 \cup \bigcup_{j\in\mathbb{N}} p^{-j}(\mathbb{Z}_p^4 \setminus p\,\mathbb{Z}_p^4),$$

we discover that we need to evaluate the integral

$$\begin{aligned}&\int_{p^{-j}(\mathbb{Z}_p^4 \setminus p\,\mathbb{Z}_p^4)} ||\{a^2, a(u_1v_2 - u_2v_1) \mid a \in X\}||_p^{-1-s} \mathrm{d}\mu(u_1, u_2, v_1, v_2) \\ &= p^{4j} \int_{\mathbb{Z}_p^4 \setminus p\,\mathbb{Z}_p^4} ||\{p^{-2j}a^2, p^{-3j}a(u_1v_2 - u_2v_1) \mid a \in X\}||_p^{-1-s} \mathrm{d}\mu(u_1, u_2, v_1, v_2), \\ &= p^{-2j(-1+s)} \int_{\mathbb{Z}_p^4 \setminus p\,\mathbb{Z}_p^4} ||\{a^2, p^{-j}a(u_1v_2 - u_2v_1) \mid a \in X\}||_p^{-1-s} \mathrm{d}\mu(u_1, u_2, v_1, v_2), \quad (\dagger)\end{aligned}$$

for every $j \in \mathbb{N}$. Notice that the integrant is invariant (up to a sign in the determinant-like expression $u_1v_2 - u_2v_1$, which we may ignore due to our standing assumption $p \neq 2$) under the permutations $(u_1\, u_2)(v_1\, v_2)$ and $(u_1\, v_2)(u_2\, v_1)$. This symmetry we shall use to our advantage.

Since one of the variables u_1, u_2, v_1, v_2 must be in $\mathbb{Z}_p^\times$, we look at the four cases separately, excluding the previous cases. Thus, the integral (†), without the factor $p^{-2j(-1+s)}$, equals the sum of integrals

$$\int_{\mathbb{Z}_p^\times}\int_{\mathbb{Z}_p^3} ||1, p^{-j}u_1(u_1v_2 - u_2v_1)||_p^{-1-s} \mathrm{d}\mu(u_2, v_1, v_2)\mathrm{d}\mu(u_1) \qquad (\ddagger_1)$$

$$+ \int_{p\mathbb{Z}_p}\int_{\mathbb{Z}_p^\times}\int_{\mathbb{Z}_p^2} ||1, p^{-j}u_2(u_1v_2 - u_2v_1)||_p^{-1-s} \mathrm{d}\mu(v_1, v_2)\mathrm{d}\mu(u_2)\mathrm{d}\mu(u_1) \qquad (\ddagger_2)$$

$$+ \int_{p\mathbb{Z}_p^2}\int_{\mathbb{Z}_p^\times}\int_{\mathbb{Z}_p} ||1, p^{-j}v_2(u_1v_2 - u_2v_1)||_p^{-1-s} \mathrm{d}\mu(v_1)\mathrm{d}\mu(v_2)\mathrm{d}\mu(u_1, u_2) \qquad (\ddagger_3)$$

$$+ \int_{p\mathbb{Z}_p^3}\int_{\mathbb{Z}_p^\times} ||1, p^{-j}v_1(u_1v_2 - u_2v_1)||_p^{-1-s} \mathrm{d}\mu(v_1)\mathrm{d}\mu(u_1, u_2, v_2). \qquad (\ddagger_4)$$

Permuting the variables such that the (distinguished) minimal variable is always u, and normalising the area of integration, we obtain

$$(\ddagger_1) = (1 - p^{-1}) \int_{\mathbb{Z}_p^3} ||1, p^{-j}(v_2 - u_2v_1)||_p^{-1-s} \mathrm{d}\mu(u_2, v_1, v_2),$$

$$(\ddagger_2) = (1 - p^{-1})p^{-1} \int_{\mathbb{Z}_p^3} ||1, p^{-j}(v_2 - pu_2v_1)||_p^{-1-s} \mathrm{d}\mu(u_2, v_1, v_2),$$

$$(\ddagger_3) = (1 - p^{-1})p^{-2} \int_{\mathbb{Z}_p^3} ||1, p^{1-j}(v_2 - u_2v_1)||_p^{-1-s} \mathrm{d}\mu(u_2, v_1, v_2),$$

$$(\ddagger_4) = (1 - p^{-1})p^{-3} \int_{\mathbb{Z}_p^3} ||1, p^{1-j}(v_2 - pu_2v_1)||_p^{-1-s} \mathrm{d}\mu(u_2, v_1, v_2).$$

Therefore it is enough to compute the following integral for $j \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$,

$$\begin{aligned}\int_{\mathbb{Z}_p^3} &||1,\, p^{-j}(v_2 - p^\varepsilon u_2v_1)||_p^{-1-s} \mathrm{d}\mu(u_2, v_1, v_2) \\ &= \int_{\mathbb{Z}_p^2}\int_{\mathbb{Z}_p} ||1, p^{-j}(v_2 - p^\varepsilon u_2v_1)||_p^{-1-s} \mathrm{d}\mu(v_2)\mathrm{d}\mu(u_2, v_1) \\ &= \int_{\mathbb{Z}_p^2}\int_{\mathbb{Z}_p + p^\varepsilon vx} ||1, p^{-j}v_2||_p^{-1-s} \mathrm{d}\mu(v_2)\mathrm{d}\mu(u_2, v_1).\end{aligned}$$

Since $\mathbb{Z}_p = \mathbb{Z}_p + p^\varepsilon u_2v_1$ for all $u_2, v_1 \in \mathbb{Z}_p$, this equals

$$\int_{\mathbb{Z}_p^2}\int_{\mathbb{Z}_p} ||1, p^{-j}v_2||_p^{-1-s} \mathrm{d}\mu(v_2)\mathrm{d}\mu(u_2, v_1) = \int_{\mathbb{Z}_p} ||1, p^{-j}v_2||_p^{-1-s} \mathrm{d}\mu(v_2) =: I(j).$$

In particular, the value of ε is irrelevant. Reviewing the equations for $(\ddagger_1)$ to $(\ddagger_4)$, we see that

$$\begin{aligned}
\zeta_H^G(s) &= p^{4m}(1 + (1-p^{-1})\sum_{j\in\mathbb{N}} p^{-2j(-1+s)}((1+p^{-1})I(j) + (1+p^{-1})p^{-2}I(j-1))) \\
&= p^{4m}(1 + (1-p^{-2})\sum_{j\in\mathbb{N}} p^{-2j(-1+s)}(I(j) + p^{-2}I(j-1))) \\
&= p^{4m}\left(1 + (1-p^{-2})\Big(p^{-2s} + (1+p^{-2s})\sum_{j\in\mathbb{N}} p^{-2j(-1+s)}I(j)\Big)\right).
\end{aligned}$$

For the last equality we have used the easily obtained equation $I(0) = 1$. More generally, we compute

$$\begin{aligned}
I(j) &= \sum_{k=0}^{j-1}(1-p^{-1})p^{-j(1+s)+ks} + \int_{p^j\,\mathbb{Z}_p} \mathrm{d}\mu(y) \\
&= (1-p^{-1})p^{-j(1+s)}\frac{(1-p^{js})}{(1-p^s)} + p^{-j} \\
&= \frac{-p^{-s}}{(1-p^{-s})}(1-p^{-1})p^{-j(1+s)} + \frac{(1-p^{-(1+s)})}{(1-p^{-s})}p^{-j},
\end{aligned}$$

and consequently

$$\begin{aligned}
\sum_{j\in\mathbb{N}} p^{-2j(-1+s)}I(j) &= \frac{-p^{-s}}{(1-p^{-s})}(1-p^{-1})\sum_{j\in\mathbb{N}} p^{j(1-3s)} + \frac{(1-p^{-(1+s)})}{(1-p^{-s})}\sum_{j\in\mathbb{N}} p^{j(1-2s)} \\
&= \frac{1}{(1-p^{-s})}\left(\frac{-(1-p^{-1})p^{1-4s}}{(1-p^{1-4s})} + \frac{(1-p^{-(1+s)})p^{1-2s}}{(1-p^{1-2s})}\right) \\
&= \frac{p^{1-2s}(1-p^{-s}) - p^{-3s}(1+p^{1-3s})}{(1-p^{1-4s})(1-p^{1-2s})}.
\end{aligned}$$

Combining everything, we find

$$\begin{aligned}
\zeta_H^G(s) &= p^{4m}P_1/P_2 \quad \text{with} \\
P_1 &= (p^{1-2s}(1-p^{-s}) - p^{-3s}(1+p^{1-3s}) \\
&\qquad + (1-p^{-4s})(1-p^{1-3s})(1+p^{-2s}-p^{-2(s+1)})) \\
P_2 &= (1-p^{1-4s})(1-p^{1-2s}),
\end{aligned}$$

and ultimately

$$\begin{aligned}
\zeta_G(s) &= p^{7m}(1-p^{-(s+2)})Q_1/Q_2 \quad \text{with} \\
Q_1 &= p^{3-2s}(1-p^{1-s}) - p^{-3s-3}(1+p^{4-3s}) \\
&\qquad + (1-p^{4-4s})(1-p^{4-3s})(1+p^{2-2s}-p^{-2s}) \\
Q_2 &= (1-p^{5-4s})(1-p^{3-2s})(1-p^{1-s}).
\end{aligned}$$

In particular,

$$\alpha(G) = \frac{3}{2} > 1 = \alpha(\mathrm{SL}_2^m(\mathbb{Z}_p)).$$

It is an interesting question if the growth of the abscissa continues for higher direct sums, i.e. if $\lim_{n\to\infty} \alpha(\mathrm{SL}_2^m(\mathbb{Z}_p) \ltimes \bigoplus_{i=1}^n \mathbb{Z}_p^2) = \infty$.

Bibliography

[1] A. G. Abercrombie. Subgroups and subrings of profinite rings. *Mathematical Proceedings of the Cambridge Philosophical Society*, 116(2):209–222, Sept. 1994. 7

[2] M. Abért. Group laws and free subgroups in topological groups. *Bulletin of the London Mathematical Society*, 37(04):525–534, Aug. 2005. 86, 94

[3] S. V. Aleshin. Finite automata and Burnside's problem for periodic groups. *Mathematical Notes of the Academy of Sciences of the USSR*, 11(3):199–203, Mar. 1972. xiv

[4] T. Alexoudas, B. Klopsch, and A. Thillaisundaram. Maximal subgroups of multi-edge spinal groups. *Groups, Geometry, and Dynamics*, 10(2):619–648, 2016. 59, 62, 100, 105, 113, 117, 134

[5] S. Andreadakis. On the automorphisms of free groups and free nilpotent groups. *Proceedings of the London Mathematical Society. Third Series*, 15:239–268, 1965. 171

[6] N. Avni, B. Klopsch, U. Onn, and C. Voll. Representation zeta functions of some compact p-adic analytic groups. In A. Campillo, G. Cardona, A. Melle-Hernández, W. Veys, and W. Zúñiga-Galindo, editors, *Contemporary Mathematics*, volume 566, pages 295–330. American Mathematical Society, Providence, Rhode Island, 2012. 240

[7] N. Avni, B. Klopsch, U. Onn, and C. Voll. Representation zeta functions of compact p-adic analytic groups and arithmetic groups. *Duke Mathematical Journal*, 162(1), Jan. 2013. xxiii, 240

[8] Y. Barnea and A. Shalev. Hausdorff dimension, pro-p groups, and Kac-Moody algebras. *Transactions of the American Mathematical Society*, 349(12):5073–5091, 1997. 7, 34

[9] L. Bartholdi. *Croissance de Groupes Agissant Sur Des Arbres*. PhD thesis, Université de Genève, 2000. xviii, 58, 59, 62, 70

[10] L. Bartholdi. Endomorphic presentations of branch groups. *Journal of Algebra*, 268(2):419–443, Oct. 2003. 8, 42

[11] L. Bartholdi. Branch rings, thinned rings, tree enveloping rings. *Israel Journal of Mathematics*, 154(1):93–139, Dec. 2006. 7

[12] L. Bartholdi, B. Eick, and R. Hartung. A nilpotent quotient algorithm for certain infinitely presented groups and its applications. *International Journal of Algebra and Computation*, 18(08):1321–1344, Dec. 2008. 8

[13] L. Bartholdi, R. I. Grigorchuk, and Z. Šunik̈. Branch groups. In *Handbook of Algebra*, volume 3, pages 989–1112. Elsevier, 2003. xv, 17, 31, 61, 87, 99, 101, 103, 111, 112

[14] L. Bartholdi, V. A. Kaimanovich, and V. V. Nekrashevych. On amenability of automata groups. *Duke Mathematical Journal*, 154(3), Sept. 2010. 3, 16, 17, 111

[15] L. Bartholdi and V. Nekrashevych. Iterated monodromy groups of quadratic polynomials, I. *Groups, Geometry, and Dynamics*, pages 309–336, 2008. 7

[16] L. Bartholdi and F. Pochon. On growth and torsion of groups. *Groups, Geometry, and Dynamics*, pages 525–539, 2009. 24

[17] L. Bartholdi and S. N. Sidki. The automorphism tower of groups acting on rooted trees. *Transactions of the American Mathematical Society*, 358(1):329–358, Mar. 2005. 113, 114

[18] L. Bartholdi, O. Siegenthaler, and P. Zalesskii. The congruence subgroup problem for branch groups. *Israel Journal of Mathematics*, 187(1):419–450, Jan. 2012. 8

[19] L. Bartholdi and Z. Šunić. On the word and period growth of some groups of tree automorphisms. *Communications in Algebra*, 29(11):4923–4964, Jan. 2001. xv, 5, 61, 62, 84, 98, 103, 134

[20] L. Bartholdi and B. Virág. Amenability via random walks. *Duke Mathematical Journal*, 130(1), Oct. 2005. 3, 16

[21] H. Bass, A. Lubotzky, A. R. Magid, and S. Mozes. The proalgebraic completion of rigid groups. *Geometriae Dedicata*, 95(1):19–58, 2002. xxiii, 239

[22] L. Bieberbach. über die Bewegungsgruppen der Euklidischen Räume. *Mathematische Annalen*, 70(3):297–336, 1911. 193

[23] O. Bogopolski. A surface groups analogue of a theorem of Magnus. In *Geometric Methods in Group Theory*, volume 372 of *Contemp. Math.*, pages 59–69. Amer. Math. Soc., Providence, RI, 2005. xxi, 162, 165

[24] O. Bogopolski and K. Sviridov. A Magnus theorem for some one-relator groups. In *The Zieschang Gedenkschrift*, pages 63–73, Toulouse, France, Apr. 2008. Mathematical Sciences Publishers. 162, 208

[25] A. A. Bovdi. Group rings of torsion-free groups. *Sibirsk. Mat. Ž.*, 1:555–558, 1960. 226

[26] H. Bradford. Quantifying lawlessness in finitely generated groups, Jan. 2022. 83, 84, 94

[27] W. Burnside. On an unsettled question in the theory of discontinuous groups. *The Quarterly Journal of Pure and Applied Mathematics*, 33:230–238, 1902. xiv, 168

[28] L. Carlitz, A. Wilansky, J. Milnor, R. A. Struble, N. Felsinger, J. M. S. Simoes, E. A. Power, R. E. Shafer, and R. E. Maas. Advanced Problems: 5600-5609. *The American Mathematical Monthly*, 75(6):685, June 1968. xiv

[29] C. C. Chang and H. J. Keisler. *Model Theory*. Number v. 73 in Studies in Logic and the Foundations of Mathematics. North-Holland ; Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co, Amsterdam ; New York : New York, NY, USA, 3rd ed edition, 1990. 228

[30] D. Chillag and S. Dolfi. Semi-rational solvable groups. *Journal of Group Theory*, 13(4):535–548, 2010. 178, 179, 227

[31] C. Chou. Elementary amenable groups. *Illinois Journal of Mathematics*, 24(3), Sept. 1980. xvi

[32] A. E. Clement, S. Majewicz, and M. Zyman. *The Theory of Nilpotent Groups*. Springer Science+Business Media, New York, NY, 2017. 213, 215, 216, 222

[33] H. S. M. Coxeter and W. O. J. Moser. *Generators and Relations for Discrete Groups*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. 195

[34] C. W. Curtis and I. Reiner. *Representation Theory of Finite Groups and Associative Algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers (a division of John Wiley & Sons, Inc.), New York-London, 1962. 189

[35] K. Dekimpe. Units in group rings of crystallographic groups. *Fundamenta Mathematicae*, 179(2):169–178, 2003. 195

[36] I. Del Corso and R. Dvornicich. Finite groups of units of finite characteristic rings. *Annali di Matematica Pura ed Applicata. Series IV*, 197(3):661–671, 2018. 184

[37] E. Di Domenico, G. A. Fernández-Alcober, and N. Gavioli. GGS-groups over primary trees: Branch structures. *Monatshefte für Mathematik*, Apr. 2022. 139

[38] E. Di Domenico, G. A. Fernández-Alcober, M. Noce, and A. Thillaisundaram. P-Basilica groups, May 2021. 7

[39] J. D. Dixon, M. Du Sautoy, and D. Segal, editors. *Analytic Pro-p Groups.* Number 61 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge [England] ; New York, 2nd ed. rev. and enl edition, 1999. xxiii

[40] J. D. Dixon and B. Mortimer. *Permutation Groups*, volume 163 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1996. 176

[41] M. P. F. du Sautoy, J. J. McDermott, and G. C. Smith. Zeta functions of crystallographic groups and analytic continuation. *Proceedings of the London Mathematical Society. Third Series*, 79(3):511–534, 1999. 194

[42] M. Edjvet. A Magnus theorem for free products of locally indicable groups. *Glasgow Mathematical Journal*, 31(3):383–387, 1989. 165

[43] J. Fabrykowski and N. Gupta. On groups with sub-exponential growth functions. *The Journal of the Indian Mathematical Society. New Series*, 49(3-4):249–256, 1985. 24, 134

[44] D. R. Farkas. Crystallographic groups and their mathematics. *The Rocky Mountain Journal of Mathematics*, 11(4):511–551, 1981. 195

[45] W. Feit and G. M. Seitz. On finite rational groups and related topics. *Illinois Journal of Mathematics*, 33(1):103–131, 1989. 175

[46] C. Feldkamp. A Magnus theorem for some amalgamated products. *Communications in Algebra*, 47(12):5348–5360, Dec. 2019. xxi, 162, 165, 208

[47] C. Feldkamp. A Magnus extension for locally indicable groups. *Journal of Algebra*, 581:122–172, Sept. 2021. 162, 164

[48] G. Fernández-Alcober and A. Zugadi-Reizabal. GGS-groups: Order of congruence quotients and Hausdorff dimension. *Transactions of the American Mathematical Society*, 366(4):1993–2017, Oct. 2013. xvii, xxi, 34, 39, 40, 117, 134, 135, 138, 142, 145, 150

[49] G. A. Fernández-Alcober, A. Garrido, and J. Uria-Albizuri. On the congruence subgroup property for GGS-groups. *Proceedings of the American Mathematical Society*, 145(8):3311–3322, Jan. 2017. 8, 117, 134, 136

[50] G. A. Fernández-Alcober and A. Thillaisundaram. Congruence quotients of branch path groups. 105

[51] L. Fuchs. *Abelian Groups.* International Series of Monographs on Pure and Applied Mathematics. Pergamon Press, New York-Oxford-London-Paris, 1960. 184

[52] GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.11.1, 2021. xxii, 78, 177, 179, 197, 231

[53] G. Gardam. A counterexample to the unit conjecture for group rings. *Annals of Mathematics. Second Series*, 194(3):967–979, 2021. 183

[54] A. Garrido. On the congruence subgroup problem for branch groups. *Israel Journal of Mathematics*, 216(1):1–13, Oct. 2016. 8

[55] A. Garrido and J. Uria-Albizuri. Pro-C congruence properties for groups of rooted tree automorphisms. *Archiv der Mathematik*, 112(2):123–137, Feb. 2019. 3, 8, 9, 53, 54, 57, 113, 116

[56] P. W. Gawron, V. V. Nekrashevych, and V. I. Sushchansky. Conjugation in tree automorphism groups. *International Journal of Algebra and Computation*, 11(05):529–547, Oct. 2001. 24

[57] D. Gildenhuys. On Pro-p-groups with a Single Defining Relator. *Inventiones mathematicae*, 5:357–366, 1968. 165

[58] E. S. Golod. *On Nil-Algebras and Finitely Approximable p-Groups*, volume 48, pages 103–106. American Mathematical Society, Providence, Rhode Island, 1965. xiv

[59] E. S. Golod and I. R. Šafarevič. *On Class Field Towers*, volume 48, pages 91–102. American Mathematical Society, Providence, Rhode Island, 1965. xiv

[60] J. González-Sánchez. On p-saturable groups. *Journal of Algebra*, 315(2):809–823, Sept. 2007. 243

[61] J. González-Sánchez. Kirillov's Orbit Method for p-Groups and Pro-p Groups. *Communications in Algebra*, 37(12):4476–4488, Nov. 2009. xxiii, 240

[62] J. González-Sánchez, A. Jaikin-Zapirain, and B. Klopsch. The representation zeta function of a FAb compact p-adic Lie group vanishes at -2. *Bulletin of the London Mathematical Society*, 46(2):239–244, Apr. 2014. 248

[63] J. González-Sánchez and B. Klopsch. Analytic pro-p groups of small dimensions. *Journal of Group Theory*, 12(5), Jan. 2009. 243

[64] R. Gow. Groups whose characters are rational-valued. *Journal of Algebra*, 40(1):280–299, 1976. 175

[65] R. Grigorchuk. On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980. xiv, 58, 99, 113

[66] R. Grigorchuk. On the Milnor problem of group growth. *Dokl. Akad. Nauk SSSR*, 271(1):30–33, 1983. xiv, 83, 87

[67] R. Grigorchuk. An example of a finitely presented amenable group not belonging to the class *EG*. *Sbornik: Mathematics*, 189(1):75–95, Feb. 1998. 43

[68] R. I. Grigorchuk. Just infinite branch groups. In *New Horizons in Pro-p Groups*, number 184 in Progr. Math., pages 121–179. Birkhäuser Boston, Boston, MA, 2000. xiv, xviii, 77

[69] R. I. Grigorchuk. Just Infinite Branch Groups. In M. du Sautoy, D. Segal, and A. Shalev, editors, *New Horizons in Pro-p Groups*, pages 121–179. Birkhäuser Boston, Boston, MA, 2000. xvii

[70] R. I. Grigorchuk and S. N. Sidki. The group of automorphisms of a 3-generated 2-group of intermediate growth. *International Journal of Algebra and Computation*, 14(05n06):667–676, Oct. 2004. 114

[71] R. I. Grigorchuk and J. S. Wilson. The uniqueness of the actions of certain branch groups on rooted trees. *Geometriae Dedicata*, 100(1):103–116, 2003. 111, 113, 115, 119, 132

[72] R. I. Grigorchuk and A. Żuk. On a torsion-free weakly branch group defined by a three state automaton. *International Journal of Algebra and Computation*, 12(01n02):223–246, Feb. 2002. xvi, 3

[73] R. I. Grigorchuk and A. Żuk. Spectral properties of a torsion-free weakly branch group defined by a three state automaton. In R. Gilman, V. Shpilrain, and A. G. Myasnikov, editors, *Contemporary Mathematics*, volume 298, pages 57–82. American Mathematical Society, Providence, Rhode Island, 2002. 3, 4, 7, 8, 13, 16, 24, 41, 43, 47, 60, 78

[74] D. Groves and M. Vaughan-Lee. Finite groups of bounded exponent. *Bulletin of the London Mathematical Society*, 35(01):37–40, Jan. 2003. 85

[75] K. W. Gruenberg. Residual Properties of Infinite Soluble Groups. *Proceedings of the London Mathematical Society*, s3-7(1):29–62, 1957. 219

[76] F. J. Grunewald, D. Segal, and G. C. Smith. Subgroups of finite index in nilpotent groups. *Inventiones Mathematicae*, 93(1):185–223, Feb. 1988. xxiii, 239

[77] N. Gupta and S. Sidki. On the Burnside problem for periodic groups. *Mathematische Zeitschrift*, 182(3):385–388, Sept. 1983. xiv, 58, 74, 87, 113, 134

[78] N. Gupta and S. Sidki. Extension of groups by tree automorphisms. In K. I. Appel, J. G. Ratcliffe, and P. E. Schupp, editors, *Contemporary Mathematics*, volume 33, pages 232–246. American Mathematical Society, Providence, Rhode Island, 1984. xviii, 58, 59, 61, 62, 74

[79] M. Hall. Solution to the Burnside Problem for Exponent 6. *Proceedings of the National Academy of Sciences*, 43(8):751–753, Aug. 1957. 168

[80] P. Hegedüs. Structure of solvable rational groups. *Proceedings of the London Mathematical Society. Third Series*, 90(2):439–471, 2005. 175, 176

[81] A. Heller and I. Reiner. Representations of cyclic groups in rings of integers. I. *Annals of Mathematics. Second Series*, 76:73–92, 1962. 188

[82] G. Higman, B. H. Neumann, and H. Neumann. Embedding theorems for groups. *The Journal of the London Mathematical Society*, 24:247–254, 1949. xxi, 165, 209

[83] Graham. Higman. The units of group-rings. *Proceedings of the London Mathematical Society. Second Series*, 46:231–248, 1940. 166, 182

[84] R. Howe. Kirillov theory for compact p -adic groups. *Pacific Journal of Mathematics*, 73(2):365–381, Dec. 1977. xxiii, 240

[85] J. Howie. On pairs of 2-complexes and systems of equations over groups. *Journal für die Reine und Angewandte Mathematik. (Crelle's Journal)*, 324:165–174, 1981. 165

[86] J. Howie. On locally indicable groups. *Mathematische Zeitschrift*, 180(4):445–461, 1982. 165

[87] B. Huppert. *Endliche Gruppen. I.* Die Grundlehren Der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967. 175, 189

[88] A. W. Ingleton. The Rank of Circulant Matrices. *Journal of the London Mathematical Society*, 31(4):445–460, Oct. 1956. 142

[89] A. Jaikin-Zapirain. Zeta function of representations of compact p-adic analytic groups. *Journal of the American Mathematical Society*, 19(1):91–118, Sept. 2005. xxiii, 240, 248, 251

[90] K. Juschenko, V. Nekrashevych, and M. de la Salle. Extensions of amenable groups by recurrent groupoids. *Inventiones mathematicae*, 206(3):837–867, Dec. 2016. 101

[91] O. Kharlampovich and A. Myasnikov. Elementary theory of free non-abelian groups. *Journal of Algebra*, 302(2):451–552, Aug. 2006. 165

[92] S. Kionke and B. Klopsch. Zeta functions associated to admissible representations of compact p-adic Lie groups. *Transactions of the American Mathematical Society*, 372(11):7677–7733, Apr. 2019. xxiv, 240, 243, 244

[93] B. Klopsch. On the Lie theory of p-adic analytic groups. *Mathematische Zeitschrift*, 249(4):713–730, Apr. 2005. 243

[94] B. Klopsch and B. Kuckuck. The Magnus property for direct products. *Archiv der Mathematik*, 107(4):379–388, Oct. 2016. 163, 194, 198, 208

[95] B. Klopsch, L. Mendonça, and J. M. Petschick. Free polynilpotent groups and the Magnus property. *Forum Mathematicum*, 35(2):573–590, Mar. 2023. vi, vii, xiii, xxiv

[96] B. Klopsch and A. Thillaisundaram. Maximal Subgroups and Irreducible Representations of Generalized Multi-Edge Spinal Groups. *Proceedings of the Edinburgh Mathematical Society*, 61(3):673–703, Aug. 2018. 105, 111, 116, 119, 132

[97] L. Kronecker. *Vorlesungen über Zahlentheorie*, volume 1 of *Vorlesungen über allgemeine Arithmetik*. B. G. Teubner, Leipzig, 1901. 142

[98] M. Larsen and A. Lubotzky. Representation growth of linear groups. *Journal of the European Mathematical Society*, pages 351–390, 2008. xxiii, 239, 240

[99] Y. Lavreniuk and V. Nekrashevych. Rigidity of branch groups acting on rooted trees. *Geometriae Dedicata*, 89(1):155–175, 2002. 112, 113

[100] M. Lazard. Groupes analytiques p-adiques. *Publications Mathématiques de l'IHÉS*, (26):5–219, 1965. 243

[101] Y. Leonov. On precisement of estimation of periods' growth for Grigorchuk's 2-groups. 1999. 98

[102] R. Lockhart. *The Theory of Near-Rings*, volume 2295 of *Lecture Notes in Mathematics*. Springer, Cham, 2021. 180

[103] A. Lubotzky and B. Martin. Polynomial representation growth and the congruence subgroup problem. *Israel Journal of Mathematics*, 144(2):293–316, Sept. 2004. 239

[104] R. C. Lyndon and P. E. Schupp. *Combinatorial Group Theory.* Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, Band 89. Springer-Verlag, Berlin-New York, 1977. 170

[105] I. Lysenok. A set of defining relations for the Grigorchuk group. *Mat. Zametki*, 38(4):503–516, 634, 1985. 8

[106] G. W. Mackey. Unitary representations of group extensions. I. *Acta Mathematica*, 99(0):265–311, 1958. 244

[107] W. Magnus. über diskontinuierliche Gruppen mit einer definierenden Relation. (Der Freiheitssatz). *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 163:141–165, 1930. xxi, 162, 207

[108] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial Group Theory.* Dover Publications, Inc., Mineola, NY, second edition, 2004. 162

[109] J. McCool and P. E. Schupp. On one relator groups and *HNN* extensions. *Australian Mathematical Society. Journal. Series A. Pure Mathematics and Statistics*, 16:249–256, 1973. 162

[110] V. Nekrashevych. *Self-Similar Groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, Aug. 2005. xiv, 11, 12, 23

[111] V. Nekrashevych. Palindromic subshifts and simple periodic groups of intermediate growth. *Annals of Mathematics*, 187(3), May 2018. xiv

[112] B. H. Neumann. On the commutativity of addition. *Journal of the London Mathematical Society. Second Series*, 15:203–208, 1940. 180

[113] B. H. Neumann. On characteristic subgroups of free groups. *Mathematische Zeitschrift*, 94:143–151, 1966. 169

[114] P. M. Neumann. Some questions of Edjvet and Pride about infinite groups. *Illinois Journal of Mathematics*, 30(2), June 1986. xiv, 112

[115] M. Noce and A. Thillaisundaram. Hausdorff dimension of the second Grigorchuk group. *International Journal of Algebra and Computation*, 31(06):1037–1047, Sept. 2021. xvii

[116] M. Noce and A. Thillaisundaram. Ramification structures for quotients of the Grigorchuk groups. *Journal of Algebra and Its Applications*, Nov. 2021. 111

[117] D. Osin. Small cancellations over relatively hyperbolic groups and embedding theorems. *Annals of Mathematics. Second Series*, 172(1):1–39, 2010. 165, 209

[118] D. S. Passman. *The Algebraic Structure of Group Rings.* Pure and Applied Mathematics. Wiley, New York, 1977. 226

[119] E. Pervova. Profinite completions of some groups acting on trees. *Journal of Algebra*, 310(2):858–879, Apr. 2007. xv, 62, 110, 111

[120] J. M. Petschick. On conjugacy of GGS-groups. *Journal of Group Theory*, 22(3):347–358, May 2019. xx, 100, 109, 114, 124, 126, 129, 131, 134, 144, 145

[121] J. M. Petschick. The automorphism group of a multi-GGS group, Sept. 2022. xiii

[122] J. M. Petschick. The derived series of GGS-groups, Aug. 2022. vi, vii, xiii

[123] J. M. Petschick. Groups of small period growth, Jan. 2022. xiii

[124] J. M. Petschick. Two periodicity conditions for spinal groups. *Journal of Algebra*, 633:242–269, Nov. 2023. vi, vii, xiii

[125] J. M. Petschick and K. Rajeev. On the Basilica operation. *Groups, Geometry, and Dynamics*, 17(1):331–384, Jan. 2023. vi, vii, xiii, xxiv, 60, 78, 79

[126] J. M. Petschick and A. Thillaisundaram. Conjugacy classes of polyspinal groups, Feb. 2022. vi, vii, xiii, xxiv, 62

[127] R. Pink. Profinite iterated monodromy groups arising from quadratic polynomials, Sept. 2013. 3

[128] W. Plesken. Applications of the theory of orders to crystallographic groups. In *Integral Representations and Applications (Oberwolfach, 1980)*, volume 882 of *Lecture Notes in Math.*, pages 37–92. Springer, Berlin-New York, 1981. 195

[129] A. Prestel. *Einführung in Die Mathematische Logik Und Modelltheorie*, volume 60 of *Vieweg Studium: Aufbaukurs Mathematik [Vieweg Studies: Mathematics Course]*. Friedr. Vieweg & Sohn, Braunschweig, 1986. 164

[130] J. Ritter and S. K. Sehgal. Trivial units in RG. *Mathematical Proceedings of the Royal Irish Academy*, 105A(1):25–39, 2005. 183

[131] D. J. S. Robinson. *A Course in the Theory of Groups*, volume 80 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1996. 52, 167

[132] C. E. Röver. Abstract commensurators of groups acting on rooted trees. *Geometriae Dedicata*, 94(1):45–61, 2002. 113

[133] R. Sandling. Graham Higman's thesis "Units in group rings". In *Integral Representations and Applications (Oberwolfach, 1980)*, volume 882 of *Lecture Notes in Math.*, pages 93–116. Springer, Berlin-New York, 1981. 182

[134] I. N. Sanov. Solution of Burnside's problem for exponent 4. *Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser.*, 10:166–170, 1940. 168, 196

[135] H. Sasse. Basilica-Gruppen und ihre Wirkung auf p-regulären Bäumen. Master's thesis, Heinrich-Heine-Universität Düsseldorf, 2018. 7, 41

[136] Z. Sela. Diophantine geometry over groups. VI. The elementary theory of a free group. *Geometric and Functional Analysis*, 16(3):707–730, 2006. 165

[137] J.-P. Serre. *Linear Representations of Finite Groups*, volume 42 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1977. 244

[138] S. Sidki. On a 2-generated infinite 3-group: Subgroups and automorphisms. *Journal of Algebra*, 110(1):24–55, Oct. 1987. xx, 114, 120, 125, 133

[139] S. Sidki. Automorphisms of one-rooted trees: Growth, circuit structure, and acyclicity. *Journal of Mathematical Sciences*, 100(1):1925–1943, June 2000. 59

[140] S. Sidki and J. S. Wilson. Free subgroups of branch groups. *Archiv der Mathematik*, 80(5):458–463, May 2003. 101

[141] O. Siegenthaler. Hausdorff dimension of some groups acting on the binary tree. *Journal of Group Theory*, 11(4), Jan. 2008. xvii

[142] A. Stasinski and M. Zordan. Rationality of representation zeta functions of compact p-adic analytic groups, July 2020. 240

[143] B. Steinberg and N. Szakács. On the simplicity of Nekrashevych algebras of contracting self-similar groups. *Mathematische Annalen*, July 2022. 101

[144] Z. Šunić. Hausdorff dimension in a family of self-similar groups. *Geometriae Dedicata*, 124(1):213–236, June 2007. xvii, 145

[145] A. Tarski, A. Mostowski, and R. M. Robinson. *Undecidable Theories*. 1953. 165

[146] S. J. Tobin. Groups with exponent four. In *Groups—St. Andrews 1981 (St. Andrews, 1981)*, volume 71 of *London Math. Soc. Lecture Note Ser.*, pages 81–136. Cambridge Univ. Press, Cambridge-New York, 1982. 196

[147] J. Uria-Albizuri. On the concept of fractality for groups of automorphisms of a regular rooted tree. *Reports@SCM*, 2(1):33–44, 2016. 11, 40

[148] M. Vaughan-Lee and E. I. Zel'manov. Bounds in the restricted Burnside problem. *J. Austral. Math. Soc. (Series A)*, 67:261–271, 1999. 85

[149] A. C. Vieira. On the lower central series and the derived series of the Gupta-Sidki 3-group. *Communications in Algebra*, 26(4):1319–1333, Jan. 1998. xxi, 135

[150] C. Voll. Functional equations for zeta functions of groups and rings. *Annals of Mathematics*, 172(2):1181, 2010. xxiii

[151] T. Vovkivsky. Infinite torsion groups arising as generalizations of the second Grigorchuk group. In *Algebra*, pages 357–377, Moscow, 2000. de Gruyter, Berlin. xviii, 59, 134

[152] J. H. C. Whitehead. On equivalent sets of elements in a free group. *Annals of Mathematics. Second Series*, 37(4):782–800, 1936. 169

[153] J. S. Wilson. Groups with every proper quotient finite. *Mathematical Proceedings of the Cambridge Philosophical Society*, 69(3):373–391, May 1971. xiv

[154] J. S. Wilson. On Just Infinite Abstract and Profinite Groups. In M. du Sautoy, D. Segal, and A. Shalev, editors, *New Horizons in Pro-p Groups*, pages 181–203. Birkhäuser Boston, Boston, MA, 2000. xiv

[155] J. S. Wilson. Structure theory for branch groups. In *Geometric and Cohomological Methods in Group Theory. Papers from the London Mathematical Society Symposium on Geometry and Cohomology in Group Theory, Durham, UK, July 2003.*, pages 306–320. Cambridge: Cambridge University Press, 2009. xiv

[156] J. S. Wilson. Free subgroups in groups with few relators. *L'Enseignement Mathématique*, 56(1):173–185, 2010. 220

[157] H. Zassenhaus. über endliche Fastkörper. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 11(1):187–220, 1935. 180

[158] H. Zassenhaus. über einen Algorithmus zur Bestimmung der Raumgruppen. *Commentarii Mathematici Helvetici*, 21:117–141, 1948. 193

[159] E. I. Zel'manov. Solution of the restricted Burnside problem for groups of odd exponent. *Izv. Akad. Nauk SSSR Ser. Mat.*, 54(1):42–59, 221, 1990. xix, 84

[160] E. I. Zel'manov. Solution of the restricted Burnside problem for 2-groups. *Mat. Sb.*, 182(4):568–592, 1991. xix, 84

[161] M. Zordan. *Representation Zeta Functions of Special Linear Groups*. PhD thesis, Universität Bielefeld, 2015. 240

Ich versichere an Eides Statt, dass die Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der "Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf" erstellt worden ist.

Jan Moritz Petschick
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