Resolvent Estimates for 2D Contact Line Dynamics and Stability Analysis for Active Fluids

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Summary

In the following thesis we consider two different models known from fluid dynamics which are based on Navier-Stokes equations.
The first model is devoted to the so-called 2D contact line dynamics investigating the contact point between fluid and solid phases. Since the fluid and solid phases are moving within time, it is necessary to transform this model to a fixed domain in order to apply known strategies. This leads to a system of Stokes equations subject to transformed free and partial slip boundary conditions which are considered on the sector. Then linear analysis is performed for the resolvent Stokes system leading to the existence of weak solutions. The main result states that the solution triple fulfills corresponding resolvent estimates. Here, we work in the framework of homogeneous Sobolev spaces with $p = 2$. We make use of the fact that in the Hilbert space setting elements from functional analysis, e.g. Lax Milgram's theorem, are available. (In)homogeneous Sobolev spaces in sectors are introduced at the beginning of this thesis complemented by various results which are transferred to the setting of (in)homogeneous spaces in sectors, as e.g. trace theorems, elliptic problems and Korn's inequality.
The second model, that is considered in this thesis, is an active fluid continuum model which describes the motion of self-propelled organisms of high concentration in fluids. This model is based on generalized Navier-Stokes equations having a leading fourth order term which is responsible for global wellposedness. Here, we consider the active fluid continuum model on a bounded domain subject to periodic boundary conditions in Lebesgue spaces with $p = 2$ in $n = 2, 3$. Two stationary states are considered: the disordered isotropic state and the ordered polar state. In this thesis, we focus on the stability analysis of the ordered polar state which indeed forms a manifold. This allows us to apply the generalized principle for normal stability and normal hyperbolicity, respectively. Here, it is essential that we are working on periodic spaces on a bounded domain. Then we can use the Fourier series representation and properties for the spectrum which are necessary to apply the theory. At last the existence of a global attractor for the active fluid continuum model is established. Here, we essentially make use of energy estimates

and perform bootstrapping arguments to obtain a compact absorbing set of arbitrary high regularity. The theory about infinite-dimensional dynamical system yields the existence of such an attractor. Then, several properties of the global attractor are proved, to be precise we show injectivity and finite dimension of the global attractor. At last we even prove the existence of an inertial manifold for $n = 2$ which has even the stronger property of attracting solutions exponentially.

Zusammenfassung

In dieser Arbeit betrachten wir zwei verschiedene Modelle aus dem Bereich der Fluiddynamik. Beide Modelle basieren auf den Navier-Stokes Gleichungen.
Das erste Modell beschreibt die Dynamik von Kontaktlinien in zwei Dimensionen, welche beispielsweise bei der Interaktion von Flüssigkeiten mit Feststoffen und Gas entstehen. Da wir dynamische Modelle betrachten, ist es notwendig, diese in Modelle auf zeitunabhängigen Gebieten zu transformieren um bekannte Methoden zur Lösung von partiellen Differentialgleichungen anzuwenden. Nach der Transformation erhält man ein System von Stokes Gleichungen, welches linear auf einem Sektor gelöst wird. In dieser Arbeit wird das Resolventenproblem untersucht, für welches die Existenz von schwachen Lösungen gezeigt werden kann. Für die Lösung werden Resolventenabschätzungen gezeigt, die das Hauptresultat des Kapitels darstellen. Wir arbeiten in (in)homogenen Sobolevräumen mit $p = 2$, sodass wir Resultate aus der Hilbertraumtheorie verwenden können, wie beispielsweise den Satz von Lax-Milgram. Die (in)homogenen Sobolevräume werden am Anfang dieser Arbeit eingeführt und grundlegende Resultate wie Spursätze, die Lösbarkeit von elliptischen Problemen und die Korn'sche Ungleichung werden gezeigt.
Im zweiten Teil der Arbeit beschäftigen wir uns mit einem Active Fluid Modell, welches die Bewegung von Organismen mit Eigenantrieb in hoher Konzentration in Flüssigkeiten beschreibt. Dieses Modell, welches einen zusätzlichen Term vierter Ordnung besitzt, basiert auf den generalisierten Navier-Stokes Gleichungen. Der Term vierter Ordnung sorgt dafür, dass wir globale Wohlgestelltheit für das System zeigen können. Wir betrachten das Active Fluid Modell auf einem beschränkten Gebiet mit periodischen Randbedingungen in Lebesgueräumen mit $p = 2$ und $n = 2, 3$. Untersucht werden zwei stationäre Zustände, die vorliegen können: Der ungeordnete und der geordnete Zustand. Wir beschränken uns auf die Analyse des geordneten Zustands, der eine Mannigfaltigkeit bildet, sodass wir das generalisierte Prinzip zur normalen Stabilität und normalen Hyperbolizität anwenden können. Die Anwendung von Fourierreihen und Ausnutzung von Eigenschaften des Spektrums aufgrund des beschränkten Gebiets sind hier essentiell. Als letztes zeigen wir die Existenz eines globalen Attraktors für das Active Fluid Modell. Mithilfe von Energieabschätzungen

können wir zeigen, dass kompakte, absorbierende Mengen von beliebig hoher Regularität existieren, welche die Existenz eines globalen Attraktors implizieren. Zusätzlich zeigen wir Injektivität und endliche Dimension des Attraktors. In $n = 2$ können wir außerdem die Existenz einer inertialen Mannigfaltigkeit zeigen, welche Lösungen sogar in exponentieller Geschwindigkeit anzieht.

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Christiane Bui

Contents

Chapter 1

Introduction

Our life is surrounded by mathematics. It is present not only in our daily life when we go grocery shopping and calculate the savings for discounted products or the total amount of our purchases. Especially phenomena in nature can be described by mathematics. How does heat distribute in a room? When water is dropping down to a water surface in a uniform time interval, how do the arising waves on the water surface behave? What will the weather be like in two days? An answer to all these questions can be given when one performs a rigorous analysis of the corresponding mathematical model. This leads to the introduction of so-called partial differential equations. There are many types of PDEs. However, in the following we will consider equations of parabolic type

$$u_t = F(u) \quad (t > 0), \qquad u|_{t=0} = u_0,$$

which describe the dependence of the development of the unknown quantity u on time and space. Of special interest are Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes. The Navier-Stokes equations are widely used to model the physics of many phenomena of scientific and engineering interests, as e.g. weather forecast, the study of ocean currents and modeling of flows of different kinds of fluids in containers. The Navier-Stokes equations for incompressible fluids then read as

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla p + \rho(u \cdot \nabla)u &= \rho f \quad \text{in } (0,T) \times \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } (0,T) \times \Omega, \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{aligned}$$

The equations describe the motion of an incompressible Newtonian fluid with velocity u and pressure p inside some arbitrary domain $\Omega \subseteq \mathbb{R}^n$. The consideration in two and three dimensions is of preferable interest from the physical point of view. The behavior of the fluid is determined by the external body force f given inside the domain Ω and the known initial velocity u_0, which is given at the beginning $t = 0$.

Depending on the viscosity μ and the density ρ we obtain systems that model different phenomena.

Because of the wide application, it is of great interest to study the Navier-Stokes equations and related parabolic systems of partial differential equations in order to have a better understanding of the physics behind natural phenomena. There are different aspects which can be considered when analyzing a system of partial differential equations. In this thesis we focus on two questions:

The first problem is the solvability of the underlying system. Hence, in order to prove so-called wellposedness, it is crucial to choose an appropriate setting (function spaces, domains, regularity of the solution,...). There are different approaches to prove wellposedness. One approach in order to solve parabolic equations is to use the theory of semigroups and maximal regularity which is introduced in [2, 11, 14, 27, 35]. The theory of maximal regularity leads to existence of solutions and corresponding estimates.

The second problem is the long-term behavior of solutions, i.e., how do they behave when time is approaching infinity? Since solutions are normally not explicitly computable, any information about the solution is helpful. Again there are many approaches to study the stability of the system. In this thesis we concentrate on the principle of linearized stability as introduced in [35, 36] and the approach for global attractors from [39, 47].

1.1 2D Contact Line Dynamics

Fluid dynamics appears in many situations in our everyday life without us explicitly noticing the fluid flow. A water drop running down a glass or an ice cube melting in a glass of water are examples for motivating the mathematical analysis of the dynamics of fluids. Inn this thesis we consider the so-called multi-phase model: the interaction of fluid phases with solid phases. Of special interest is the contact line, formed by points where the fluid-fluid interface touches the solid phase. If we consider fluid and solid phases, which are both moving, then the contact line becomes dynamic. The angle between the fluid and the solid phase at the contact line is called the contact angle. There are different points of view how to model such a contact angle problem. The first ansatz follows the idea that the dynamic contact angle is determined by an additional equation, while for the second ansatz one assumes that the contact angle is already fully determined by the appearing dynamic equations for the interface and the fluid. We will mainly focus on the latter ansatz.

In this thesis we consider the contact line dynamics in two dimensions. In the half-space $\mathbb{R}^2_+$ we consider a two-phase model in phases $\Omega(t)$ and $\mathbb{R}^2_+ \backslash \overline{\Omega(t)}$ where we decide to neglect the continuous phase $\mathbb{R}^2_+ \backslash \overline{\Omega(t)}$ for simplicity. In this case the interface is given as $\Gamma(t) := \Gamma_f(t) \cup \Gamma_s(t) \subseteq \mathbb{R}^2_+ \cup \mathcal{C}(t)$, where Γ_f denotes the free boundary, Γ_s the solid boundary and $\mathcal{C}$ the contact point (in two dimensions; in three dimensions one would obtain a contact line).

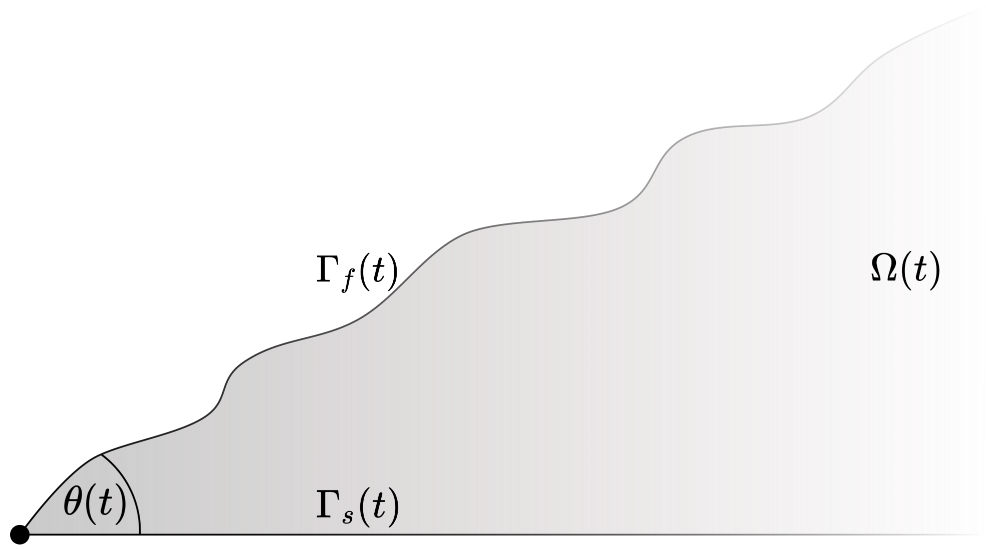

Figure 1: Two-phase model in two dimensions.

The isothermal flow of incompressible fluids is denoted by $(u(t,\cdot), p(t,\cdot)) : \Omega(t) \to \mathbb{R}^3$ for $t \geq 0$. Here, u denotes the velocity field and p the pressure. We assume the fluid to be Newtonian with viscous stress $T(u,p) = 2\mu D(u) - p$ correlating to the rate of deformation $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. At the solid boundary Γ_s we assume impermeability, i.e., $u \cdot n_s = 0$ and a partial slip condition given as:

$$\lambda P_{\Gamma_s} u + P_{\Gamma_s} T(u,p) n_s = 0.$$

Here, n_s denotes the outer normal vector field at Γ_s, $\lambda > 0$ is the constant friction coefficient and $P_{\Gamma_s} = 1 - n_s \otimes n_s$ is the tangential projection. The free boundary Γ_f has constant surface tension $\sigma > 0$ and mean curvature $\kappa = -\operatorname{div} n_f$, where n_f denotes the outer normal vector at Γ_f. The normal interface velocity is given as $V_{n_f} = u \cdot n_f$. At the contact point we assume contact point velocity $V_\mathcal{C} = u \cdot n_\mathcal{C}$, where $n_\mathcal{C}$ denotes the corresponding outer normal vector at the contact point $\mathcal{C}$. Furthermore, the contact angle θ is given through the constitutive equation $\theta = \psi(V_\mathcal{C})$.

Then our full two-phase system reads as

$$
\begin{aligned}
\partial_t u + (u\cdot\nabla)u - \operatorname{div} T(u,p) &= 0 && \text{in } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\Omega(t),\\
\operatorname{div} u &= 0 && \text{in } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\Omega(t),\\
\lambda u^1 + (D(u)n_s)^1 &= 0 && \text{on } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\Gamma_s(t),\\
u^2 &= 0 && \text{on } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\Gamma_s(t),\\
T(u,p)n_f &= \sigma\kappa n_f && \text{on } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\Gamma_f(t),\\
V_{n_f} &= u\cdot n_f && \text{on } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\Gamma_f(t),\\
V_{\mathcal{C}} &= u\cdot n_{\mathcal{C}} && \text{on } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\mathcal{C}(t),\\
\theta &= \psi(V_{\mathcal{C}}) && \text{on } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\mathcal{C}(t),\\
u|_{t=0} &= u_0 && \text{in } \Omega(0),\\
\Gamma_s(0) &= \Gamma_s^0 && \text{in } \Omega(0),\\
\Gamma_f(0) &= \Gamma_f^0 && \text{in } \Omega(0),\\
\mathcal{C}(0) &= \mathcal{C}_0 && \text{in } \Omega(0).
\end{aligned}
\tag{1.1}
$$

It turns out that the conditions at the contact point can be neglected in our setting since the contact point velocity $V_{\mathcal{C}}$ is not defined in this case (this can be observed after transforming (1.1) to a fixed domain). Furthermore, from now on we assume that the contact angle θ is either a given function $\theta = \theta(t)$ or modeled by a constitutive equation $\theta = \psi(V_{\mathcal{C}})$ (if the contact point velocity $V_{\mathcal{C}}$ exists). Transforming (1.1) to a fixed domain $(0,T)\times\Sigma_{\theta_0}$ via a suitable diffeomorphism we end up with the following system of partial differential equations:

$$
\begin{aligned}
\partial_t v - \operatorname{div} T(v,q) &= F_1(v,q,\rho) && \text{in } (0,T)\times\Sigma_{\theta_0},\\
\operatorname{div} v &= F_2(v,\rho) && \text{in } (0,T)\times\Sigma_{\theta_0},\\
\lambda\tau_\Sigma\cdot v + \tau_\Sigma D(v)n_\Sigma &= F_3(v,\rho) && \text{on } (0,T)\times\Gamma_0,\\
n_\Sigma\cdot v &= 0 && \text{on } (0,T)\times\Gamma_0,\\
T(v,q)n_\Sigma + \sigma\tilde{c}(\theta_0)\partial_{y_2}^2\rho n_\Sigma &= F_4(v,\rho) && \text{on } (0,T)\times\Gamma_+,\\
\sin(\theta_0)\partial_t\rho + n_\Sigma\cdot v &= F_5(v,\rho) && \text{on } (0,T)\times\Gamma_+,\\
V_{\mathcal{C}} &= v\cdot n_\Sigma && \text{on } (0,T)\times\{0\},\\
\partial_{x_2}\rho &= \cot(\theta(t)) - \cot(\theta_0) && \text{on } (0,T)\times\{0\},\\
\theta &= \psi(V_{\mathcal{C}}) && \text{on } (0,T)\times\{0\},\\
v|_{t=0} &= v_0 && \text{in } \Sigma_{\theta_0},\\
\rho|_{t=0} &= \rho_0 && \text{on } \Gamma_+,
\end{aligned}
\tag{1.2}
$$

with height function $\rho(t,\cdot) : \Gamma_+ \to \mathbb{R}^3$ and for suitable right-hand sides. Here, $\Sigma_{\theta_0} := \{(x_1,x_2)\in\mathbb{R}^2 : x_1\in(0,\infty),\ 0\le x_2\le\tan(\theta_0)x_1\}$ is the wedge with opening

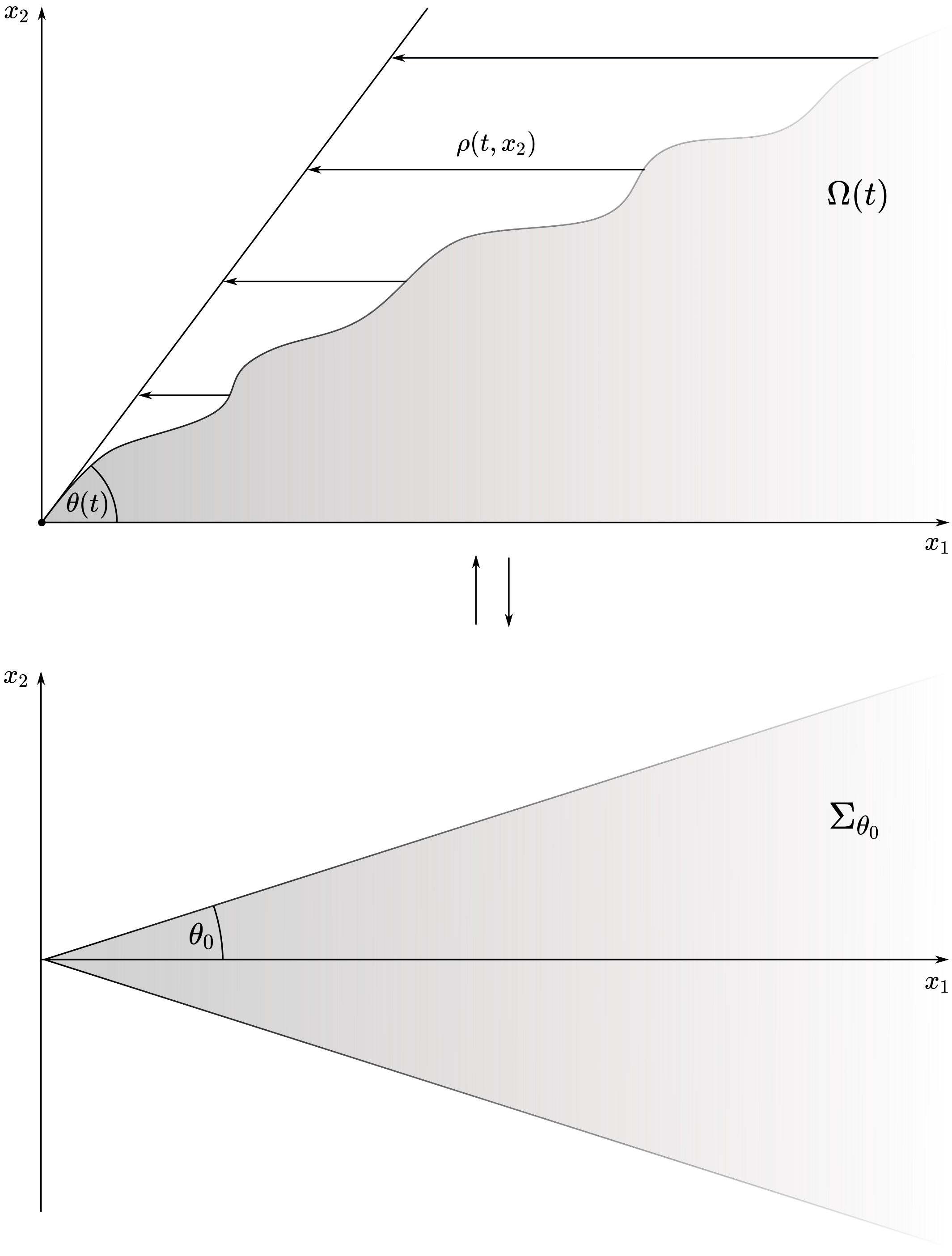

Figure 2: Transformation to a fixed domain.

angle $\theta_0 = \theta(0)$ (the contact angle at $t = 0$) and Γ_0 the lower boundary and Γ_+ the upper boundary of the wedge. By treating (1.2) in the setting of reflection invariant L^2-spaces on the sector $\Sigma_{\theta_0} := \{(x_1, x_2) \in \mathbb{R}^2 \backslash \{0\} : |x_2| < x_1 \tan(\theta_0)\}$ we first observe that V_C and $\partial_{x_2}\rho|_{(0,T)\times\{0\}}$ are not defined in the weak regularity class for $1 < p \leq 2$, hence we can neglect these equations. For $p > 2$ we need to reduce the inhomogeneity which again leads to working in reflection invariant spaces. Furthermore, the partial slip condition on Γ_0 is automatically fulfilled when working in reflection invariant spaces after applying a perturbation argument. Therefore, a full analysis of the system

$$\begin{aligned} \partial_t u - \operatorname{div} T(u,p) &= f_1 && \text{in } (0,T) \times \Sigma_\theta, \\ \operatorname{div} u &= 0 && \text{in } (0,T) \times \Sigma_\theta, \\ T(u,p)n + \sigma c(\theta)\partial_\tau^2 \rho n &= f_4 && \text{on } (0,T) \times \Gamma, \\ \partial_t \rho + \tfrac{1}{\sin(\theta)}(n \cdot u) &= f_5 && \text{on } (0,T) \times \Gamma, \\ u|_{t=0} &= u_0 && \text{in } \Sigma_\theta, \\ \rho|_{t=0} &= \rho_0 && \text{on } \Gamma, \end{aligned}$$

will greatly help our understanding in order to solve the full nonlinear system (1.1) with $\theta = \theta_0$ and $\Gamma := \partial\Sigma_\theta$. A first crucial step is to consider the stationary system

$$\begin{aligned} \lambda u - \operatorname{div} T(u,p) &= f_1 && \text{in } \Sigma_\theta, \\ \operatorname{div} u &= 0 && \text{in } \Sigma_\theta, \\ T(u,p)n + \sigma c(\theta)\partial_\tau^2 \rho n &= f_4 && \text{on } \Gamma, \\ \lambda \rho + \tfrac{1}{\sin(\theta)}(n \cdot u) &= 0 && \text{on } \Gamma \end{aligned} \tag{1.3}$$

for $\lambda \in \Sigma_{\pi/2}$. However, it seems that solving the Stokes system subject to different boundary conditions on wedge type domains is a natural first step in order to solve the linear contact line problem for $\theta \in (0, \pi/2)$. The analysis was addressed by Maier, Köhne, Saal and Westermann in [29, 30, 31, 33].

On the other hand, contact line dynamics have been studied for almost three decades. First pioneering results were derived by Solonnikov in 1995. These results were published in [44], where it was proved that singularities of solutions for the contact line problem vanish and that corresponding solutions have finite Dirichlet integral for fixed contact angle $\theta \in \{0, \pi\}$. Almost two decades later in [55], Wilke proved wellposedness in cylindrical domains for fixed contact angle $\theta = \pi/2$. Both authors observed that these contact angles remove singularities at the contact lines. From the classical Young law it follows that the contact angle is dependent on time if the initial contact angle is not equal to the contact angle at the equilibria. Hence, by considering a fixed contact angle, just an idealized situation is represented. On the other hand, Watanabe proved optimal regularity for $\theta = \pi/2$ in a cylinder in

[53]. Dynamic contact lines were considered by Zhang, Guo and Tice. They proved wellposedness in a 2D vessel in [58] and considered stability analysis in the same setting in [21]. In [16], Fricke, Köhne and Bothe observed that smooth solutions to the dynamic contact line problem are non-physical and that the existence of smooth solutions lead to unstable equilibria. Hence, weak regularity at the contact line needs to be present. By using another approach, namely the interface formation model, Kusaka proved the existence of an axially symmetric solution for the stationary problem in weighted Hölder spaces in [28]. This shows that the study of contact line dynamics is still an interesting challenge in research up to today.

In Chapter 3 we first perform analysis on two-dimensional sectors Σ_θ of opening angle θ. There, we first introduce (in)homogeneous Sobolev spaces since they provide the framework where we want to consider the linearized 2D contact line problem in Chapter 4. Making use of results on the half-space and whole space, we will prove e.g. trace theorems, Korn's inequality and solvability of elliptic problems on sectors. The consideration of the normal Dirichlet trace and Neumann trace is of special interest, which leads to the the multiplication with normal and tangential vector fields at the boundary Γ. Normal and tangential vector fields are given as the sign function in one component. It turns out that multiplication with sgn is not bounded on (in)homogeneous spaces of order $s \geq 1/2$. This leads to the introduction of reflection invariant subspaces in Section 3.2, where the multiplication with sgn is bounded when the correct symmetry is given.

In Chapter 4 we prove wellposedness of the linearized 2D contact line problem (1.3) and corresponding resolvent estimates. We first apply a suitable transformation to (1.1) in order to obtain a system on a fixed domain $(0, T) \times \Sigma_{\theta_0}$ in Section 4.1. Then in Section 4.2 we consider the resolvent problem of (1.3) in the setting of reflection invariant homogeneous Sobolev spaces in $p = 2$. The advantage of working in $p = 2$ lies in the fact that we can derive a corresponding weak formulation of (1.3) to apply the Lax-Milgram theorem to obtain a weak solution u. In this case the pressure p and the height function ρ can be recovered. By making use of the scaling invariance of the sector Σ_θ and the scaling of the norm in homogeneous spaces, it is possible to obtain relevant resolvent estimates for $\lambda \in \Sigma_{\pi/2}$ of large absolute value, i.e., $|\lambda| \geq 1$. Hence, this leads to resolvent estimates for the stationary system (1.3), to be precise we prove the following estimate:

$$
\begin{aligned}
&\|u\|_{\lambda,\, H_0^{-1}(\Sigma_\theta)_R} + |\lambda|^{1/2}\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \sqrt{\sigma}|\lambda|^{1/2}\|\rho\|_{\hat{H}^1(\Gamma)_r} \\
&\qquad + \sigma\|\partial_\tau^2\rho\|_{\lambda,\, \hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} + \|p\|_{\lambda,\, L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r} \\
&\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right).
\end{aligned}
$$

In the context of homogeneous spaces, it turns out that we can expect (u, p, ρ) to have at most the regularity given above since $\hat{H}^{1/2}(\Gamma)$ is the borderline for the non existence of a trace at the singular point $(0, 0)$ (the contact point), whereas for $s > 1/2$ the trace does exist. Furthermore, we observe that $\hat{H}^{1/2}(\Gamma)$ is the borderline where multiplication with normal and tangential vector is still a continuous operator if the right symmetry is given (for $s > 1/2$ the multiplication is not continuous any more), hence it seems that in the weak setting with $p = 2$ we are working in a borderline case.

1.2 Active Fluids

There is a need to study turbulence since it is ubiquitous in nature. Turbulence occurs e.g. in ocean currents and small-scale biological and quantum systems. It is of great interest to study the self-sustained turbulent motion in microbial suspensions. In [54] different experiments and simulations were made in order to model the bacterial dynamics and spontaneous formation of vortex structures of bacteria at high concentration at low Reynolds number [37] adequately. In [54] it is shown that a system of generalized Navier-Stokes equations models the motion and behavior of self-propelled bacteria adequately and this system was then also considered in [12, 13]. Since the bacteria has internal self-propulsion such a model is often referred to as an active fluid continuum model, which is given as

$$\begin{aligned} v_t + \lambda_0 v \cdot \nabla v &= f - \nabla p + \lambda_1 \nabla |v|^2 - (\alpha + \beta |v|^2) v + \Gamma_0 \Delta v - \Gamma_2 \Delta^2 v, \\ \operatorname{div} v &= 0, \\ v|_{t=0} &= v_0. \end{aligned} \tag{1.4}$$

The continuum model is based on two assumptions: At first, the vector field v models the dynamic behavior of the bacterial suspension. Secondly, the bacterial suspension becomes an incompressible active fluid at high concentration.

In (1.4) elements from the Toner-Tu theory [49, 50] and Swift-Hohenberg theory [45] were combined. Corresponding Toner-Tu terms (α - β terms) model flocking which describes the motion of self-propelled organisms and which is responsible for the emergence of turbulence and provide the isotropic equilibrium state. Swift-Hohenberg terms (Γ_2 - Γ_0 terms) were added in order to model the pattern formation and describe the turbulence of the described particles.

Let n be the dimension of the space we are working in. Since we assume that the bacterial suspension becomes incompressible at high concentration, we obtain the well-known divergence condition as known from the Navier-Stokes equations

$$\operatorname{div} v = \nabla \cdot v = 0$$

for the velocity field v. The dynamics of v is governed by generalized Navier-Stokes equations for incompressible fluids

$$(\partial_t v \cdot \nabla)v = -\nabla p - (\alpha + \beta|v|^2)v + \nabla \cdot E. \tag{1.5}$$

Here p denotes the pressure and the rate-of-strain tensor E depends on the velocity field v. The α - β term is called Toner-Tu term and corresponds to a Landau-potential. We demand $\beta \geq 0$ in order to obtain stability, whereas α can be any real number. Stability analysis in Chapter 5 and Chapter 6 show that the relation of α and β is responsible for (in)stability of the system, hence the Toner-Tu term is responsible whether stability or instability occurs.
The symmetric and trace-free rate-of-strain tensor E describes the rate of change of the deformation of the bacterial suspensions. Hence, E depends on v and has the following form (cf. [43]):

$$E_{ij} = \Gamma_0(\partial_i v_j + \partial_j v_i) - \Gamma_2 \Delta(\partial_i v_j + \partial_j v_i) + S q_{ij},$$
$$q_{ij} = v_i v_j - \frac{\delta_{ij}}{n}|v|^2,$$

where δ_{ij} denotes the Kronecker-symbol denoting elements of the unit matrix and S presents an active stress contribution, which depends on the choice of the fluid. For $S = \Gamma_2 = 0$ we obtain the usual rate-of-strain tensor E of a conventional fluid with viscosity Γ_0 as seen e.g. in the usual Navier-Stokes equations. Since we aim to model self-propelled turbulence, negative values for Γ_0 have to be allowed while demanding $\Gamma_2 > 0$ to ensure wellposedness of the system. Defining

$$\lambda_0 = 1 - S, \qquad \lambda_1 = -\frac{S}{n},$$

and inserting everything in (1.5) we finally end up with (1.4). Hence, the Toner-Tu term drives the fluid to a disordered isotropic equilibrium state if

$$v = 0.$$

If $\alpha < 0$ then the Toner-Tu term leads to an ordered global state with characteristic speed

$$|v| = \sqrt{|\alpha|/\beta}.$$

For a further introduction and a more precise derivation of the continuum model we refer to [54], especially the Supporting Appendix.

In order to mathematically justify the observations regarding active turbulence that were made within simulations, the continuum model (1.4) was already considered in various settings. In [57] a full analysis regarding local and global wellposedness in $L^p(\mathbb{R}^n)$ and stability in $L^2(\mathbb{R}^n)$ for the full nonlinear model was proposed. However, the more interesting ordered global state is not covered in this setting, since constants $|v| = \sqrt{|\alpha|/\beta}$ are not contained in $L^2(\mathbb{R}^n)$. In [9] an approach in spaces of Fourier transformed Radon measures $\mathrm{FM}(\mathbb{R}^n)$ was performed with the intention to mathematically justify the ansatz that "waves" of the form $e^{ik\cdot}$ solve (1.4) which makes sense from the physical point of view. However, in this setting the ordered global state is still not covered. To this end, in Chapter 5 and Chapter 6 the continuum model is considered on a bounded domain subject to periodic boundary conditions which seems to fit into the physical setting.

In Chapter 5 and Chapter 6 we work in the periodic $L^2(Q_n)$ setting where Q_n is the n-dimensional cube with side length $L > 0$. Indeed, in this setting the ordered global polar state is contained in $L^2(Q_n)$. Furthermore, in Chapter 5 a proper nonlinear stability analysis can be performed using the generalized principle of normal stability and normal hyperbolicity as known from Prüss, Simonett and Zacher (cf. [35, 36]). In contrast to the setting in $L^2(\mathbb{R}^n)$ and $\mathrm{FM}(\mathbb{R}^n)$, where the corresponding linear operator A to (1.4) has a continuous spectrum, in $L^2(Q_n)$ the operator A has compact resolvent, hence the spectrum only consists of the point spectrum. This fact allows 0 to be an isolated eigenvalue of A such that it is possible to prove that 0 is a semi-simple eigenvalue depending on the choice of the occurring parameters $\Gamma_2, \Gamma_0, \alpha$ and β. Note that the semi-simple eigenvalue assumption is crucial in order to apply the generalized principle. Having proved normal hyperbolicity, we can conclude that the manifold of globally ordered states can be split up in a stable and unstable foliation. The existence of this unstable foliation coincides with the observation of turbulence in [54].
In Chapter 6 the existence of a global attractor can be ensured (in contrast to the results from Chapter 5 the existence is assured for every parameter set $\Gamma_2, \Gamma_0, \alpha, \beta$) by using the approach as known from Robinson and Temam (cf. [39, 47]). This result coincides with the observation from [54] that the simulation of the bacterial suspension reaches some stable final state after a finite time.

Chapter 5 is structured as follows: In Section 5.1 local and global wellposedness of (1.4) is proved for initial values in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ by making use of the H^∞-calculus and maximal L^p-regularity. Applying the generalized principle of linearized (in)stability in Section 5.2 we are able to prove normal stability and normal hyperbolicity for the ordered polar state, respectively. In Chapter 6 we again have to ensure local and global wellposedness of (1.4) at first. In contrast to Section 5.1 we need wellposedness in spaces of lower regularity, i.e., initial values in $L^2_\sigma(Q_n)$. Hence, in Section 6.1 we introduce interpolation-extrapolation scales to transfer the results from Section 5.1. The existence of a global attractor $\mathcal{A}$ is addressed in Section 6.2 by making use of energy methods and in Section 6.3 important properties of this global attractor are proved. At last in Section 6.4 we observe that an inertial manifold $\mathcal{M}$ exists in two dimensions which even has stronger properties: firstly, the global attractor $\mathcal{A}$ has to be contained in $\mathcal{M}$ and secondly, every solution of (1.4) can be approximated by solutions on $\mathcal{M}$ at an exponential rate.

Chapter 2

Preliminaries

Let $n \in \mathbb{N}$ be the dimension. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we denote the j-th component of a vector $x \in \mathbb{K}^n$ by x_j for $j = 1, ..., n$ and the components of a vector field u in $\mathbb{K}^n$ by $u = (u^1, ..., u^n)$. By $x \cdot y = \sum_{j=1}^n x_j y_j = x^T y$ we denote the scalar product of two vectors $x, y \in \mathbb{K}^n$, where x^T is the transpose of x (A^T also denotes the transpose of a matrix A). By $|\cdot|$ we denote the norm in $\mathbb{K}^n$ and $\mathbb{K}^{n \times n}$, respectively.

Let X, Y be Banach spaces. The space $\mathscr{L}(X, Y)$ contains all linear and bounded operators $T : X \to Y$ and is equipped with the usual operator norm $\|\cdot\|_{\mathscr{L}(X,Y)}$. The space $\mathscr{L}_{is}(X, Y)$ is the subspace of $\mathscr{L}(X, Y)$ containing all isomorphisms. If $X = Y$ we write $\mathscr{L}(X)$ and $\mathscr{L}_{is}(X)$. Let $T : D(T) \subseteq X \to X$ be a closed operator. By $N(T)$ we denote the kernel and by $R(T)$ the range of the operator T. We call $\sigma(T)$ the spectrum and $\rho(T)$ the corresponding resolvent set.

The dual space of a Banach space X is denoted by $X' := \mathscr{L}(X, \mathbb{K})$ whereas the dual operator to a linear and bounded operator $T : X \to Y$ is denoted by $T' : Y' \to X'$. For $x \in X$ and a functional $x' \in X'$ we write $\langle x', x \rangle_{X',X}$ for the duality pairing. Then the dual space is endowed with the standard norm

$$\|x'\|_{X'} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} |\langle x', x \rangle_{X',X}| \qquad (x' \in X').$$

If $X = H$ is a Hilbert space we denote by $(\cdot, \cdot)_H$ the corresponding inner product which induces the norm.

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. By $C(\Omega, X)$ we denote the space of all continuous functions $u : \Omega \to X$. The subspace of bounded continuous functions is denoted by $BC(\Omega, X)$ which is a Banach space endowed with the usual $\|\cdot\|_\infty$ norm. By $BUC(\Omega, X)$ we denote the space of uniformly bounded functions. For $k \in \mathbb{N}$ the space $C^k(\Omega, X)$ contains all k-times continuously differentiable functions and we set $C^\infty(\Omega, X) := \cap_{k \in \mathbb{N}} C^k(\Omega, X)$. The space of test functions or infinitely often differentiable functions with compact support is denoted by $C_c^\infty(\Omega, X)$. Furthermore, the space $C_c^\infty(\overline{\Omega}, X)$ contains all restrictions of functions $u \in C_c^\infty(\mathbb{R}^n, X)$ to $\overline{\Omega}$.

The space $C_{c,\sigma}^\infty(\Omega, X)$ is a subspace of $C_c^\infty(\Omega, X)$ consisting of functions which are divergence free additionally. The space $C_{c,\sigma}^\infty(\overline{\Omega}, X)$ is defined accordingly.
The scale of spaces of continuous functions $f : E \to X$ on a Banach space E are defined accordingly.

The X-valued Bochner-Lebesgue spaces for $1 \le p \le \infty$ are denoted by $L^p(\Omega, X)$ endowed with the standard integral norm

$$\|u\|_{L^p(\Omega,X)} = \left(\int_\Omega \|u(x)\|_X^p \, dx\right)^{1/p} \qquad (u \in L^p(\Omega, X))$$

for $1 \le p < \infty$ and $\|u\|_{L^\infty(\Omega,X)} = \operatorname{ess\,sup}_{x\in\Omega}\|u(x)\|_X$ if $p = \infty$. The spaces $W^{k,p}(\Omega, X)$ contain the L^p-functions that are weakly differentiable in the distributional sense equipped with the norm

$$\|u\|_{W^{k,p}(\Omega,X)} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^p(\Omega,X)}^p\right)^{1/p} \qquad (p < \infty)$$

for $k \in \mathbb{N}$ and multi indices $\alpha \in \mathbb{N}_0^n$ with the usual modification if $p = \infty$. Furthermore, we define $W_0^{k,p}(\Omega, X) := \overline{C_c^\infty(\Omega, X)}^{\|\cdot\|_{W^{k,p}}}$. In the Hilbert space setting $p = 2$ and X Hilbert space, we set $H^k(\Omega, X) := W^{k,2}(\Omega, X)$ and $H_0^k(\Omega, X) := W_0^{k,2}(\Omega, X)$ for $k \in \mathbb{N}_0$. By $(\cdot, \cdot)_2$ we denote the standard inner product in $L^2(\Omega, X)$ given as

$$(u, v)_2 = (u, v)_{L^2(\Omega,X)} = \int_\Omega u(x)v(x) \, dx \qquad (u, v \in L^2(\Omega, X)).$$

The Bessel potential spaces of fractional powers with $s \in \mathbb{N}$ are defined via interpolation: $H^{\theta s}(\Omega, X) := [L^2(\Omega, X), H^s(\Omega, X)]_\theta$ and $H_0^{\theta s}(\Omega, X) := [L^2(\Omega, X), H_0^s(\Omega, X)]_\theta$, where $0 < \theta < 1$ and $[\cdot, \cdot]_\theta$ denotes the interpolation functor (cf. [51, Section 1.9]). Bessel potential spaces of negative power $-s < 0$ are defined as $H^{-s}(\Omega, X) := (H_0^s(\Omega, X))'$ and $H_0^{-s}(\Omega, X) := (H^s(\Omega, X))'$, respectively.
The corresponding Bessel potential spaces for $p \neq 2$ are defined accordingly as complex interpolation spaces $W^{\theta s,p}(\Omega, X) := [L^p(\Omega, X), W^{s,p}(\Omega, X)]_\theta$ for $0 < \theta < 1$ and $s \in \mathbb{N}$. Sobolev-Slobodeckij spaces are denoted by $W_p^s(\Omega, X)$ and are defined via real interpolation $W_p^{\theta s}(\Omega, X) := (L^p(\Omega, X), W^{s,p}(\Omega, X))_{\theta,p}$ (see [51, Sections 1.3, 1.4, 1.6] for an introduction to real interpolation).
The space $L_{loc}^1(\Omega, X)$ consists of all functions which are locally integrable. In all cases we drop the space X if $X = \mathbb{K}^n$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ or if X is clear by the context.

2.1 Elements from Functional Analysis

Here, we will list most of the results from functional analysis we will use of in the following chapters of this thesis. However, we will omit the proofs and just give a corresponding reference for the reader's convenience. The first two statements are well-known (see [56, Sections III.6, III.7]). In the context of wellposedness of PDEs those statements are frequently used to ensure the existence of weak solutions in the Hilbert space setting:

2.1.1 Theorem (Riesz' representation theorem). *Let H be a Hilbert space and ℓ a bounded linear functional on H. Then there exists a unique $y \in H$ such that*

$$\ell(x) = (x, y)_H \quad \text{for all } x \in H,$$

and $\|\ell\|_{\mathscr{L}(H,\mathbb{C})} = \|y\|_H$. Conversely, any vector $y \in H$ defines a bounded linear functional ℓ_y on H by

$$\ell_y(x) := (x, y)_H \quad \text{for all } x \in H$$

and $\|\ell_y\|_{\mathscr{L}(H,\mathbb{C})} = \|y\|_H$.

2.1.2 Theorem (Lax and Milgram). *Let H be a separable Hilbert space and the map $a : H \times H \to \mathbb{C}$ a sesquilinear form. We assume that*

(i) there exists a $C > 0$ such that $|a(u, v)| \leq C\|u\|_H\|v\|_H \quad (u, v \in H)$;

(ii) there exists a $\delta > 0$ such that $|a(u, u)| \geq \delta\|u\|_H^2 \quad (u \in H)$.

Then for every linear functional $\ell \in H' = \mathscr{L}(H, \mathbb{C})$ there exists a unique $u_\ell \in H$ such that

$$a(u_\ell, v) = \ell(v) \qquad (v \in H).$$

Next, we add some theorems stating (in)stability of equilibria for nonlinear quasilinear parabolic problems. To be precise, we are quoting the generalized principle of linearized (in)stability for manifolds from [35, 36]. Hence, for the proofs we also refer to [35, 36]. We start with the principle for normal stability as seen in [35, Theorem 5.3.1] and [36, Theorem 2.1]:

2.1.3 Theorem. *Let $1 < p < \infty$ and X_0, X_1 be two Banach spaces where X_1 is densely embedded in X_0. Let $U \subseteq X_\gamma := (X_0, X_1)_{1-1/p,p}$ be open and assume that $(A, F) \in C^1(U, \mathscr{L}(X_1, X_0) \times X_0)$ with*

$$\dot{v}(t) + A(v(t))v(t) = F(v(t)), \qquad t > 0, \qquad v(0) = v_0. \tag{2.1}$$

Suppose $V \in U \cap X_1$ is an equilibrium state of (2.1) and $A(V)$ possesses the property of maximal L^p-regularity. Let

$$A_o u := A(V)u + (DA(V)u)V - DF(V)u$$

for $u \in X_1$ denote the linearization of (2.1) at V. Suppose that V is normally stable, i.e., assume that *(i) near V the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1 of dimension $m \in \mathbb{N}$,*

(ii) the tangent space for $\mathcal{E}$ at V is given by $N(A_o)$,

(iii) 0 is a semi-simple eigenvalue of A_o, i.e., $N(A_o) \oplus R(A_o) = X_0$,

(iv) $\sigma(A_o)\backslash\{0\} \subseteq \{z \in \mathbb{C} : \mathrm{Re}\ z > 0\}$.

Then V is stable in X_γ and there exists $\delta > 0$ such that the unique solution $v(t)$ of (2.1) with initial value $v_0 \in X_\gamma$ satisfying $\|v_0 - V\|_{X_\gamma} < \delta$ converges exponentially to some $V_\infty \in \mathcal{E}$ in X_γ as $t \to \infty$.

Next, we quote the version of the generalized principle corresponding to normally hyperbolic equilibria, cf. [35, Theorem 5.5.1] and [36, Theorem 6.1]:

2.1.4 Theorem. *Let $1 < p < \infty$. Suppose $V \in U \cap X_1$ is an equilibrium of (2.1) and suppose that the functions (A, F) have the same properties as in Theorem 2.1.3. Suppose further that $A(V)$ has the property of maximal L^p-regularity. Let A_o be the linearization of (2.1) at V. Suppose that V is normally hyperbolic, which means that*

(i) near V the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1 of dimension $m \in \mathbb{N}_0$,

(ii) the tangent space for $\mathcal{E}$ at V is given by $N(A_o)$,

(iii) 0 is a semi-simple eigenvalue of A_o, i.e., $N(A_o) \oplus R(A_o) = X_0$,

(iv) $\sigma(A_o) \cap i\mathbb{R} = \{0\}$, $\sigma_u := \sigma(A_o) \cap \mathbb{C}_- = \sigma(A_o) \cap \{z \in \mathbb{C} : \mathrm{Re}\ z < 0\} \neq \emptyset$.

Then V is unstable in X_γ: For each sufficiently small $\rho > 0$ there exists $0 < \delta \leq \rho$ such that the unique solution $v(t)$ of (2.1) with initial value $v_0 \in B_{X_\gamma}(V, \delta)$ either satisfies

- *$dist_{X_\gamma}(v(t_0), \mathcal{E}) > \rho$ for some finite time $t_0 > 0$, or*
- *$v(t)$ exists on $\mathbb{R}_+$ and converges at an exponential rate to some $v_\infty \in \mathcal{E}$ in X_γ as $t \to \infty$.*

2.2 Periodic Sobolev Spaces

In this section we will introduce periodic Sobolev spaces. For a more detailed introduction we refer to [19, Chapter 3]. We fix some $L > 0$ and set $Q_n := [0, L]^n$ such that L is the length of the box Q_n. We set

$$C^k_\pi(Q_n) := \left\{ f \in C^k(Q_n, \mathbb{R}^n) : \partial^\alpha f^m|_{x_j=0} = \partial^\alpha f^m|_{x_j=L}, j, m = 1, ..., n\ \forall\ |\alpha| \leq k \right\},$$

$$C_\pi^\infty(Q_n) := \bigcap_{k=0}^{\infty} C_\pi^k(Q_n),$$

as the spaces of k-times continuously differentiable periodic functions. Then the periodic Sobolev space $L_\pi^2(Q_n, \mathbb{R}^n) := L_\pi^2(Q_n)$ is defined as the completion of $C_\pi^\infty(Q_n)$ w.r.t. the L^2-norm. By [19, Proposition 3.2.1] it follows that the above definition of $L_\pi^2(Q_n, \mathbb{R}^n)$ coincides with the definition of $L^2(Q_n, X)$ with $X = \mathbb{R}^n$ such that we can write $L^2(Q_n, \mathbb{R}^n) = L^2(Q_n) = L_\pi^2(Q_n)$ in this specific case.

When working in periodic Sobolev spaces we can employ the Fourier transform to obtain the Fourier coefficient $\hat{u}(m)$ for $m = (m_1, ..., m_n) \in \mathbb{Z}^n$:

$$\hat{u}(m) := \mathcal{F}(u)(m) := \frac{1}{L^n} \int_{Q_n} u(x) e^{-2\pi i m \cdot x / L} \, dx \qquad (u \in L^2(Q_n)).$$

Also, for a smooth function u the m-th Fourier coefficient of the derivative is given as

$$\widehat{\partial^\alpha u}(m) = \left(\frac{2\pi i}{L}\right)^{|\alpha|} m^\alpha \hat{u}(m) \tag{2.2}$$

for $m \in \mathbb{Z}^n$ and $\alpha \in \mathbb{N}_0^n$, which can be verified by integration by parts. The norm on $L^2(Q_n)$ is induced by the scalar product

$$(f, g)_{2,\pi} := \frac{1}{L^n} \int_{Q_n} u(x) \overline{v(x)} \, dx \qquad (f, g \in L^2(Q_n)).$$

As a consequence we obtain well-known results from the whole space case in the periodic setting:

2.2.1 Theorem. *Let $u, v \in L^2(Q_n)$ be arbitrary.*

(1) The Plancherel theorem holds:

$$\|u\|_{L^2(Q_n)}^2 = \sum_{m \in \mathbb{Z}^n} |\hat{u}(m)|^2.$$

(2) Then Parseval's identity holds:

$$(u, v)_{2,\pi} = \frac{1}{L^n} \int_{Q_n} u(x) \overline{v(x)} \, dx = \sum_{m \in \mathbb{Z}^n} \hat{u}(m) \overline{\hat{v}(m)}.$$

(3) Every function $u \in L^2(Q_n)$ can be represented as the $L^2(Q_n)$-limit of trigonometric polynomials, i.e.,

$$u = \sum_{m \in \mathbb{Z}^n} \hat{u}(m) e^{2\pi i m \cdot / L}.$$

We refer to [19, Proposition 3.2.7] for the proof of Theorem 2.2.1. Next, periodic Sobolev spaces of higher order $k \in \mathbb{N}$ are defined in a natural way as

$$\begin{aligned} H^k_\pi(Q_n) &:= \left\{ u = \sum_{m\in\mathbb{Z}^n} \hat{u}(m) e^{2\pi i m\cdot/L} : \hat{u}(m) = \overline{\hat{u}(-m)}\ \forall\ m \in \mathbb{Z}^n, \|u\|_{\tilde{H}^k_\pi(Q_n)} < \infty \right\} \\ &= \left\{ u \in H^k(Q_n) : \partial^\alpha u^m|_{x_j=0} = \partial^\alpha u^m|_{x_j=L}\ (j, m = 1, ..., n,\ |\alpha| < k) \right\} \\ &= \overline{C^\infty_\pi(Q_n)}^{H^k(Q_n)}, \end{aligned}$$

where the restriction $\hat{u}(m) = \overline{\hat{u}(-m)}$ for $m \in \mathbb{Z}^n$ ensures that u takes real values. In this case the norm is given as

$$\|u\|^2_{\tilde{H}^k_\pi(Q_n)} := \sum_{m\in\mathbb{Z}^n} \left| \left(1 + \left(\frac{2\pi}{L}\right)^k |m|^k\right) \hat{u}(m) \right|^2.$$

We will also make use of homogeneous periodic Sobolev spaces which are defined accordingly:

$$\begin{aligned} \hat{H}^1_\pi(Q_n) &:= \left\{ u \in L^1_{loc}(Q_n) : \nabla u \in L^2(Q_n),\ u|_{x_j=0} = u|_{x_j=L}\ (j = 1, ..., n) \right\} \\ &= \overline{C^\infty_\pi(Q_n)}^{\|\nabla\cdot\|_{L^2}}. \end{aligned}$$

Thanks to Theorem 2.2.1 and formula (2.2) any derivative of $u \in H^k_\pi(Q_n)$ can be represented as the $L^2(Q_n)$-limit

$$\partial^\alpha u = \sum_{m\in\mathbb{Z}^n} \widehat{\partial^\alpha u}(m) e^{2\pi i m\cdot/L} = \sum_{m\in\mathbb{Z}^n} \left(\frac{2\pi i}{L}\right)^{|\alpha|} m^\alpha \hat{u}(m) e^{2\pi i m\cdot/L}$$

for $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le k$. Furthermore, it is straightforward to prove that $\|\cdot\|_{\tilde{H}^k_\pi(Q_n)}$ and $\|\cdot\|_{H^k_\pi(Q_n)}$ are equivalent, where

$$\|u\|^2_{H^k_\pi(Q_n)} := \sum_{|\alpha|\le k} \sum_{m\in\mathbb{Z}^n} \left| \left(\frac{2\pi}{L}\right)^{|\alpha|} m^\alpha \hat{u}(m) \right|^2.$$

Also by (2.2) we now can write the norm in the well-known form:

$$\|u\|^2_{H^k_\pi(Q_n)} = \sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{L^2(Q_n)} = \|u\|^2_{H^k(Q_n)}.$$

Thus, periodic Sobolev spaces of fractional power $s \ge 0$ are defined in the canonical way

$$H^s_\pi(Q_n) := \left\{ u = \sum_{m\in\mathbb{Z}^n} \hat{u}(m) e^{2\pi i m\cdot/L} : \hat{u}(m) = \overline{\hat{u}(-m)}\ \forall\ m \in \mathbb{Z}^n, \|u\|_{H^s_\pi(Q_n)} < \infty \right\},$$

where the norm is modified as

$$\|u\|_{H^s_\pi(Q_n)} := \sum_{m\in\mathbb{Z}^n} \left(1+\left(\frac{2\pi}{L}\right)^2 |m|^2\right)^{s/2} |\hat{u}(m)|^2.$$

It is straightforward to prove that both definitions for periodic Sobolev spaces of higher order coincide for $s \in \mathbb{N}$. We can also obtain a Gagliardo-Nirenberg type estimate just by the fact that $L^2(Q_n)$ is equipped with a scalar product:

2.2.2 Corollary. *Let $k \in \mathbb{N}$ be arbitrary. Then we have*

$$\|\nabla^k u\|^2_{L^2(Q_n)} \leq \|\nabla^{k+1} u\|_{L^2(Q_n)} \|\nabla^{k-1} u\|_{L^2(Q_n)} \qquad (u \in H^{k+1}_\pi(Q_n)).$$

Proof. This is a direct consequence of the Cauchy-Schwarz inequality:

$$\|\nabla u\|^2_{L^2(Q_n)} = -(\Delta u, u)_{2,\pi} \leq \|\Delta u\|_{L^2(Q_n)} \|u\|_{L^2(Q_n)}$$

for $u \in H^2_\pi(Q_n)$. Now for arbitrary $v \in H^{k+1}_\pi(Q_n)$ we insert $u = \nabla^{k-1} v$ in order to obtain the desired estimate. □

Next, we aim to define Fourier multipliers in the setting of periodic Sobolev spaces. Let $m : \mathbb{Z}^n \to \mathbb{C}^{n\times n}$ be a function. We define $T_m : D(T_m) \subseteq L^2(Q_n) \to L^2(Q_n)$ with domain

$$D(T_m) := \left\{ u \in L^2(Q_n) : \|T_m u\|^2_{L^2(Q_n)} = \sum_{k\in\mathbb{Z}^n} |m(k)\hat{u}(k)|^2 < \infty \right\}$$

as the $L^2(Q_n)$-limit

$$T_m u := \sum_{k\in\mathbb{Z}^n} m(k)\hat{u}(k) e^{2\pi i k\cdot/L} \quad (u \in L^2(Q_n)),$$

and the operator T_m is bounded if m is a bounded function with $D(T_m) = L^2(Q_n)$. This is a direct consequence of Theorem 2.2.1. Then we call m a Fourier multiplier.

In order to decompose $L^2(Q_n)$ into a solenoidal subspace $L^2_\sigma(Q_n)$ and a subspace of gradient fields $G_2(Q_n)$ we define the multiplier $\sigma_P : \mathbb{Z}^n \to \mathbb{C}^{n\times n}$ as

$$\sigma_P(m) := I - \frac{mm^T}{|m|^2}$$

for $m \neq 0$ and $\sigma_P(0) = I$, where I denotes the $n \times n$ identity matrix. Then the Helmholtz-Weyl projector on $L^2(Q_n)$ is given as

$$P : L^2(Q_n) \to L^2_\sigma(Q_n), \quad u \mapsto Pu := \sum_{m\in\mathbb{Z}^n} \sigma_P(m)\hat{u}(m) e^{2\pi i m\cdot/L},$$

inducing the desired decomposition

$$L^2(Q_n) = L^2_\sigma(Q_n) \oplus G_2(Q_n),$$

where

$$L^2_\sigma(Q_n) \coloneqq \left\{u \in L^2(Q_n) : \hat{u}(m) = \overline{\hat{u}(-m)},\ m \cdot \hat{u}(m) = 0\ \forall\ m \in \mathbb{Z}^n\right\},$$
$$G_2(Q_n) \coloneqq \left\{u = \nabla g \in L^2(Q_n) : g \in L^1_{loc}(Q_n)\right\},$$

cf. [40, Section 2.1]. We note that $u \in L^2_\sigma(Q_n)$ implies $\operatorname{div} u = 0$. We also observe that P obviously commutes with Bessel potentials and derivatives. As a consequence, P is also a projector on $H^s_\pi(Q_n)$ and $P(H^s_\pi(Q_n)) = H^s_\pi(Q_n) \cap L^2_\sigma(Q_n)$ which yields the decomposition

$$H^s_\pi(Q_n) = (H^s_\pi(Q_n) \cap L^2_\sigma(Q_n)) \oplus (H^s_\pi(Q_n) \cap G_2(Q_n))$$

for fractional Sobolev spaces with $s \geq 0$.

2.3 Global Attractors for Infinite-Dimensional Dynamical Systems

In this section we collect important definitions and theorems from the theory concerning global attractors for infinite-dimensional dynamical systems. For a more detailed introduction to this theory we refer to [39, Chapters 10,13] and [47, Chapters III, VI].

Let H be a Hilbert space. We consider a semidynamical system on the phase space H given by

$$u_t = f(u), \qquad u|_{t=0} = u_0, \tag{2.3}$$

with some nonlinearity f such that for $u_0 \in H$ the system (2.3) has a unique solution $u = u(t; u_0)$ for all positive times, hence we demand global solvability of (2.3). In the context of semidynamical systems, we define the C_0-semigroup of solution operators $S(t) : H \to H$ by $S(t)u_0 \coloneqq u(t; u_0)$. In the following we will consider the semidynamical system $(H, (S(t))_{t\geq 0})$.

2.3.1 Definition. Let $(S(t))_{t\geq 0}$ be a semigroup.

(i) A set $Y \subseteq H$ is called *positively invariant* if $S(t)Y \subseteq Y$ for all $t \geq 0$.

(ii) A set $X \subseteq H$ is called *invariant* if $S(t)X = X$ for all $t \geq 0$.

(iii) The semigroup $(S(t))_{t\geq 0}$ is called *dissipative* if it possesses a compact *absorbing set* $B \subseteq H$, i.e., there exists some compact set $B \subseteq H$ such that for any bounded set $X \subseteq H$ there exists some $t_0(X) \geq 0$ such that

$$S(t)X \subseteq B \qquad \text{for all } t \geq t_0(X).$$

2.3.2 Definition. Let $(S(t))_{t\geq 0}$ be a semigroup. The *global attractor* $\mathcal{A} \subseteq H$ is the maximal compact invariant set such that

$$S(t)\mathcal{A} = \mathcal{A} \qquad \text{for all } t \geq 0$$

and the minimal set that attracts all bounded sets:

$$dist_H(S(t)X, \mathcal{A}) \xrightarrow{t\to\infty} 0,$$

for any bounded set $X \subseteq H$.

The next theorem is crucial in order to prove the existence of an attractor. For the proof we refer to [39, Theorem 10.5]:

2.3.3 Theorem. *Let $(S(t))_{t\geq 0}$ be a semigroup. If $(S(t))_{t\geq 0}$ is dissipative and $B \subseteq H$ is a compact absorbing set then there exists a global attractor*

$$\mathcal{A} = \omega(B) := \bigcap_{t\geq 0} S(t)B.$$

If H is connected then so is $\mathcal{A}$.

In the next results we try to characterize a global attractor more precisely. Again for the proofs we refer to [39, Theorems 10.6, 10.7, 10.10].

2.3.4 Definition. The semigroup $(S(t))_{t\geq 0}$ is injective on a global attractor $\mathcal{A} \subseteq H$ if for any $u_0, v_0 \in \mathcal{A}$ we have

$$S(t)u_0 = S(t)v_0 \in \mathcal{A} \quad \text{for some } t > 0 \qquad \Rightarrow \qquad u_0 = v_0.$$

2.3.5 Theorem. *Let $(S(t))_{t\geq 0}$ be a semigroup which is injective on a global attractor $\mathcal{A} \subseteq H$. Then the following statements hold:*

(i) Every trajectory on $\mathcal{A}$ is defined for all $t \in \mathbb{R}$ and $(\mathcal{A}, (S(t))_{t\in\mathbb{R}})$ is a dynamical system with $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$.

(ii) $\mathcal{A} = \bigcup\{u$ is a complete bounded orbit$\}$, in particular all complete bounded orbits lie in $\mathcal{A}$.

(iii) For every compact invariant set $X \subseteq H$ we have

$$W^u(X) \coloneqq \{u_0 \in H : S(t)u_0 \text{ defined} \; \forall \; t \in \mathbb{R}, \; S(-t)u_0 \xrightarrow{t\to\infty} x \in X\} \subseteq \mathcal{A}.$$

At last we introduce relevant notions in order to prove that an attractor $\mathcal{A} \subseteq H$ has finite fractal and Hausdorff dimension. We refer to [39, Chapter 13] for a precise introduction.

2.3.6 Definition. Let $(S(t))_{t\geq 0}$ be a semigroup. We say that $(S(t))_{t\geq 0}$ is uniformly differentiable on $\mathcal{A}$ if for every $u \in \mathcal{A}$ there exists a linear operator $\Lambda(t,u) : H \to H$ such that for all $t \geq 0$

$$\sup_{u,v\in\mathcal{A};\; 0<\|u-v\|_H\leq\varepsilon} \frac{\|S(t)v - S(t)u - \Lambda(t,u)(v-u)\|_H}{\|v-u\|_H} \xrightarrow{\varepsilon\to 0} 0$$

and

$$\sup_{u\in\mathcal{A}} \|\Lambda(t,u)\|_{\mathscr{L}(H)} < \infty \qquad (t \geq 0).$$

Next, we want to quote the result finite dimension of the global attractor (cf. [39, Theorem 13.16]). In order to introduce the notion of the statement, we consider the semigroup $(S(t))_{t\geq 0}$ and the compact global attractor $\mathcal{A}$ of (2.3). To this end, let $u_0 \in \mathcal{A}$ be arbitrary and $\{\xi_j^0 : j = 1, ...n\} \subseteq H$ where ξ_j^0 are linearly independent. We are in interested in the evolution of $\{\xi_j^0 : j = 1, ...n\}$ near u_0 under the flow of (2.3). Let $\Lambda(t,u)\xi_j^0$ be the solution of the linearized equation about $u(t) \coloneqq S(t)u_0$ with initial value ξ_j^0, to be precise

$$v_t = f'(u)v = L(\cdot, u_0)v \qquad v|_{t=0} = \xi_j^0.$$

with linear operator $L(t;u_0) \coloneqq f'(u(t))$. Then consider the span $\{\Lambda(t,u)\xi_j^0 : j = 1, ..., n\} \subseteq H$ and chose a time-dependent set of orthonormal vectors $\{\varphi^j(t) : j = 1, ..., n\} \subseteq H$ which have the same span. Next, we define the projection $P^{(n)}_{\xi_1^0,...,\xi_n^0}(t)$ to $\{\varphi^j(t) : j = 1, ..., n\}$ and we have

$$P^{(n)}_{\xi_1^0,...,\xi_n^0}(t) = \sum_{i=1}^n \varphi^i(t)(\varphi^i(t), \cdot)_H$$

and

$$\text{Tr}L(t;u_0)P^{(n)}_{\xi_1^0,...,\xi_n^0}(t) = \sum_{i=1}^n (\varphi^{(i)}(t), L(t;u_0)\varphi^{(i)})_H.$$

Hence, the asymptotic growth rate of the n-volume $\{\xi_j^0 : j = 1, ..., n\}$ about the trajectory $u(\cdot) = S(\cdot)u_0$ is given as

$$\lim_{t\to\infty} \exp\left[\frac{1}{t}\int_0^t \text{Tr}L(s;u_0)P^{(n)}_{\xi_1^0,...,\xi_n^0}(s)\,ds\right].$$

We aim to prove that all of the n-volumes decay exponentially for all initial values $u_0 \in \mathcal{A}$ and initial infinitesimal n-volumes $\{\xi_j^0 : j = 1, ..., n\}$. This finally leads to

2.3.7 Theorem. *Let $(S(t))_{t\geq 0}$ be a semigroup. Suppose that $(S(t))_{t\geq 0}$ is uniformly differentiable on $\mathcal{A}$ and that there exists a $t_0 \geq 0$ such that $\Lambda(t, u_0)$ is compact for all $t \geq t_0$. If*

$$\mathcal{TR}_n(\mathcal{A}) := \sup_{u_0 \in \mathcal{A}} \sup_{\substack{\xi_j^0 \in H \\ \|\xi_j^0\|_H = 1, \\ j=1,...,n}} \left\langle \mathrm{Tr} L(t; u_0) P^{(n)}_{\xi_1^0, ..., \xi_n^0}(t) \right\rangle < 0,$$

where $\langle f(t) \rangle = \limsup_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds$ and L and P are defined as above. Then the fractal dimension of $\mathcal{A}$ is finite, to be precise $d_f(\mathcal{A}) \leq n$.

Chapter 3

Analysis on Sectors

In this chapter we perform analysis on sectors which are the natural domain for some PDE systems from fluid dynamics, as e.g. the contact line model from Chapter 4. To this end we will introduce homogeneous and inhomogeneous Sobolev spaces in sectors Σ_θ as well as in smooth sector-like domains Σ_θ^δ in two dimensions. In this section we assume $n = 2$. For a fixed $0 < \theta < \pi/2$ we define the sector

$$\Sigma_\theta := \{x = (x_1, x_2) \in \mathbb{R}^2 \backslash \{0\} : |x_2| < x_1 \tan(\theta)\}$$

and the smooth sector-like domain

$$\Sigma_\theta^\delta := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \backslash \{0\} : \sqrt{x_2^2 + \sin(\theta)^2 \delta} < x_1 \tan(\theta) \right\}, \quad \delta \geq 0.$$

We note that $\Sigma_\theta = \Sigma_\theta^0$. Furthermore, we set $\Gamma := \partial \Sigma_\theta$ and $\Gamma_\delta := \partial \Sigma_\theta^\delta$ for $\delta \geq 0$ as the boundary of the sector. Then normal and tangential vector field at Γ are given as

$$\begin{aligned} n(x) &= n(x_2) = (-\sin(\theta), \operatorname{sgn}(x_2)\cos(\theta)), \\ \tau(x) &= \tau(x_2) = (\operatorname{sgn}(x_2)\cos(\theta), \sin(\theta)), \end{aligned}$$

for $x = (x_1, x_2) \in \Gamma$ (cf. Lemma 3.1.1).

3.1 Sobolev Spaces in Sectors

In this section we introduce (in)homogeneous Sobolev spaces in sectors and prove related results, as e.g. solvability of elliptic problems, density properties and trace theorems. In order to transfer results from the half-space or the whole space to Σ_θ and Σ_θ^δ or its boundary Γ and Γ_δ we need to construct an appropriate transformation. Then it is straightforward to transfer the results. It will be crucial to derive estimates as e.g. for the trace operator uniformly in $\delta \geq 0$ in order to prove results for elliptic problems. We note that then we will especially obtain these estimates for $\delta = 0$ which is the case which we are mostly interested in.

In order to define homogeneous Sobolev spaces in Σ_θ^δ and Γ_δ properly we first parametrize the boundary Γ_δ via the path γ_δ which turns out to be a bi-Lipschitz diffeomorphism:

3.1.1 Lemma. *Let $\delta \geq 0$. Then*

$$\gamma_\delta : \mathbb{R} \to \Gamma_\delta, \quad t \mapsto \begin{pmatrix} \cos(\theta)\sqrt{t^2+\delta} \\ \sin(\theta)t \end{pmatrix}$$

passed through in clockwise direction parametrizes the boundary Γ_δ. Then γ_δ is a bi-Lipschitz diffeomorphism uniformly in $\delta \geq 0$ with

$$\left|\frac{d}{dt}\gamma_\delta(t)\right| = \sqrt{\frac{t^2+\sin(\theta)^2\delta}{t^2+\delta}} \qquad (t \in \mathbb{R},\ \delta \geq 0)$$

and the special case $|d/dt\ \gamma_0(t)| = 1$. Furthermore, outer normal and tangential vector fields on Γ_δ are given as

$$\tilde{\tau}_\delta(t) = \sqrt{\frac{t^2+\delta}{t^2+\sin(\theta)^2\delta}} \begin{pmatrix} \cos(\theta)t/\sqrt{t^2+\delta} \\ \sin(\theta) \end{pmatrix} \qquad (t \in \mathbb{R})$$

and

$$\tilde{n}_\delta(t) = \sqrt{\frac{t^2+\delta}{t^2+\sin(\theta)^2\delta}} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta)t/\sqrt{t^2+\delta} \end{pmatrix} \qquad (t \in \mathbb{R}).$$

Proof. First, we note that $\gamma_\delta(t) \in \Gamma_\delta$ for every $t \in \mathbb{R}$ since $\sqrt{(\gamma_\delta^2)^2 + \sin(\theta)^2\delta} = \gamma_\delta^1 \tan(\theta)$, where $\gamma_\delta = (\gamma_\delta^1, \gamma_\delta^2)^T$. Then γ_δ parametrizes Γ_δ also by the fact that γ_δ is obviously injective. Furthermore, we have

$$\frac{d}{dt}\gamma_\delta(t) = \begin{pmatrix} \cos(\theta)2t\frac{1}{2\sqrt{t^2+\delta}} \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta)\frac{t}{\sqrt{t^2+\delta}} \\ \sin(\theta) \end{pmatrix}$$

for all $t \in \mathbb{R}$ such that

$$\left|\frac{d}{dt}\gamma_\delta(t)\right| = \sqrt{\cos(\theta)^2\frac{t^2}{t^2+\delta} + \sin(\theta)^2} = \sqrt{\frac{t^2+\sin(\theta)^2\delta}{t^2+\delta}}$$

for all $t \in \mathbb{R}$. From this we note that on one hand we have

$$\left|\frac{d}{dt}\gamma_\delta(t)\right| \leq \sqrt{\frac{t^2+\sin(\theta)^2\delta}{t^2+\sin(\theta)^2\delta}} = 1$$

for all $t \in \mathbb{R}$ since $\sin(\theta) \in (0,1)$. On the other hand we also obtain a lower bound:

$$\left|\frac{d}{dt}\gamma_\delta(t)\right| = \sqrt{1 - \frac{\cos(\theta)^2\delta}{t^2+\cos(\theta)^2\delta+\sin(\theta)^2\delta}} \geq \sqrt{1-\cos(\theta)^2} = \sin(\theta),$$

for all $t \in \mathbb{R}$ such that we especially obtain a bound uniformly in $\delta \geq 0$:

$$\sin(\theta) \leq \left\| \frac{d}{dt} \gamma_\delta \right\|_{L^\infty(\mathbb{R})} \leq 1, \tag{3.1}$$

which proves that γ_δ is a bi-Lipschitz diffeomorphism. The tangential vector field follows directly from the derivative of γ_δ since $\tilde{\tau}_\delta = |d/dt\ \gamma_\delta|^{-1}(d/dt\ \gamma_\delta)$. Then it is also straightforward to calculate the outer normal vector field. □

On the other hand we also obtain a bi-Lipschitz diffeomorphism between Σ_θ^δ and the rotated half-space $\mathbb{R}^2_{>0} := \{(\eta, t) \in \mathbb{R}^2 : \eta > 0\}$ which will be used to transfer function spaces on Σ_θ^δ to spaces defined on the half-space.

3.1.2 Lemma. *Let $\delta \geq 0$. We define $\mathbb{R}^2_{>0} := \{(\eta, t) \in \mathbb{R}^2 : \eta > 0\}$. Then*

$$\varphi_\delta : \mathbb{R}^2_{>0} \to \Sigma_\theta^\delta, \quad (\eta, t) \mapsto \begin{pmatrix} \eta + \cos(\theta)\sqrt{t^2 + \delta} \\ \sin(\theta) t \end{pmatrix}$$

is a bi-Lipschitz diffeomorphism uniformly in $\delta \geq 0$, i.e.,

$$\|\nabla \varphi_\delta\|_{L^\infty(\mathbb{R}^2_{>0}, \mathscr{L}(\mathbb{R}^2))}, \|(\nabla \varphi_\delta)^{-1}\|_{L^\infty(\mathbb{R}^2_{>0}, \mathscr{L}(\mathbb{R}^2))} \leq C \qquad (\delta \geq 0)$$

and $\det \varphi_\delta' = \sin(\theta)$ *and* $\det[(\varphi_\delta')^{-1}] = \sin(\theta)^{-1}$.

Proof. Obviously, φ_δ is well-defined. The Jacobi matrix is given as

$$\nabla \varphi_\delta(\eta, t) = \begin{pmatrix} 1 & \frac{\cos(\theta) t}{\sqrt{t^2+\delta}} \\ 0 & \sin(\theta) \end{pmatrix} \quad \text{and} \quad (\nabla \varphi_\delta(\eta, t))^{-1} = \frac{1}{\sin(\theta)} \begin{pmatrix} \sin(\theta) & \frac{-\cos(\theta) t}{\sqrt{t^2+\delta}} \\ 0 & 1 \end{pmatrix}$$

for $(\eta, t) \in \mathbb{R}^2_{>0}$ and we immediately obtain the stated estimates as well as the determinant of both matrices. Hence, φ_δ is a bi-Lipschitz diffeomorphism. □

Finally, we can define homogeneous Sobolev spaces on sectors and sector-like domains, see e.g. [17, 52]. Let $k \in \mathbb{N}$ be arbitrary, then

$$\dot{H}^k(\Sigma_\theta^\delta) := \{u \in L^1_{loc}(\Sigma_\theta^\delta) : \nabla^k u \in L^2(\Sigma_\theta^\delta)\}$$

is a function space which is equipped with the semi-norm $|\cdot|_{\dot{H}^k(\Sigma_\theta^\delta)} = \|\nabla^k \cdot\|_{L^2(\Sigma_\theta^\delta)}$. Now let $\mathcal{P}_k$ be the class of all polynomials of degree $\leq k-1$. Then we set

$$\hat{H}^k(\Sigma_\theta^\delta) := \dot{H}^k(\Sigma_\theta^\delta)/\mathcal{P}_k,$$

such that the homogeneous Sobolev space $\hat{H}^k(\Sigma_\theta^\delta)$ is defined as a factor space. For simplicity we denote elements in $\hat{H}^k(\Sigma_\theta^\delta)$ as u if we refer to their equivalence class

$[u] = u + \mathcal{P}_k$. By [17, Lemma II.6.2] we know that $\hat{H}^k(\Sigma_\theta^\delta)$ is a Hilbert space equipped with the norm

$$\begin{aligned}\|u\|_{\hat{H}^k(\Sigma_\theta^\delta)} &:= \|[u]\|_{\hat{H}^k(\Sigma_\theta^\delta)} = \inf_{p \in \mathcal{P}_k} |u+p|_{\dot{H}^k(\Sigma_\theta^\delta)} = \inf_{p \in \mathcal{P}_k} \|\nabla^k(u+p)\|_{L^2(\Sigma_\theta^\delta)} \\ &= \|\nabla^k u\|_{L^2(\Sigma_\theta^\delta)}\end{aligned}$$

for all $u \in \hat{H}^k(\Sigma_\theta^\delta)$ (in contrast to $\dot{H}^k(\Sigma_\theta^\delta)$ which is only equipped with a semi-norm $|\cdot|_{\dot{H}^k(\Sigma_\theta^\delta)}$). Furthermore, for $k \in \mathbb{N}$ we define

$$\hat{H}_0^k(\Sigma_\theta^\delta) = \overline{C_c^\infty(\Sigma_\theta^\delta)}^{\|\nabla^k \cdot\|_{L^2}}.$$

Homogeneous spaces on the half-space $\mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$, $\mathbb{R}_{>0}^2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ and on the whole space $\mathbb{R}^2$ are defined accordingly. Furthermore, for $k \in \mathbb{N}$ we define

$$\dot{H}_{\mathrm{div}}^k(\Sigma_\theta^\delta) := \left\{u \in \dot{H}^k(\Sigma_\theta^\delta) : \operatorname{div} u = 0\right\}$$

and $\hat{H}_{\mathrm{div}}^k(\Sigma_\theta^\delta) := \dot{H}_{\mathrm{div}}^k(\Sigma_\theta^\delta)/\mathcal{P}_k$ equipped with the same norm as $\hat{H}^k(\Sigma_\theta^\delta)$ since $\hat{H}_{\mathrm{div}}^k(\Sigma_\theta^\delta) \subseteq \hat{H}^k(\Sigma_\theta^\delta)$. In order to define Sobolev spaces of fractional power for $0 < s < 1$ we first observe

3.1.3 Lemma. *Let the bi-Lipschitz diffeomorphism φ_δ from Lemma 3.1.2 be given. Defining push-forward and pull-back through*

$$\Phi_*^\delta u := u \circ \varphi_\delta^{-1} \quad \textit{and} \quad \Phi_\delta^* v := v \circ \varphi_\delta,$$

it holds

$$\begin{aligned}\Phi_*^\delta &\in \mathscr{L}_{is}(\hat{H}^1(\mathbb{R}_{>0}^2), \hat{H}^1(\Sigma_\theta^\delta)) \cap \mathscr{L}_{is}(H^k(\mathbb{R}_{>0}^2), H^k(\Sigma_\theta^\delta)) \\ (\Phi_*^\delta)^{-1} = \Phi_\delta^* &\in \mathscr{L}_{is}(\hat{H}^1(\Sigma_\theta^\delta), \hat{H}^1(\mathbb{R}_{>0}^2)) \cap \mathscr{L}_{is}(H^k(\Sigma_\theta^\delta), H^k(\mathbb{R}_{>0}^2))\end{aligned}$$

for $k = 0, 1$ with norm estimates uniformly in $\delta \geq 0$. In the setting of homogeneous spaces we interpret the composition $[v] \circ \varphi_\delta = [v \circ \varphi_\delta]$ where $[v] \in \hat{H}^1(\Sigma_\theta^\delta)$ with $v \in \dot{H}^1(\Sigma_\theta^\delta)$ by choosing a corresponding representative for $[v]$.

Proof. First we note that the definition in the setting of homogeneous spaces is meaningful. For any constant function c we also have $c \circ \varphi_\delta^{-1} \equiv c$ such that we have $(u + c) \circ \varphi_\delta^{-1} = u \circ \varphi_\delta^{-1} + c$ for any $u \in \dot{H}^1(\mathbb{R}_{>0}^2)$ leading to the same equivalence class for $u \circ \varphi_\delta^{-1}$ by

$$[u] \circ \varphi_\delta = [(u + \mathbb{R}) \circ \varphi_\delta] = [u \circ \varphi_\delta + \mathbb{R}] = [u \circ \varphi_\delta].$$

We can estimate the norm uniformly in $\delta \geq 0$: Let $[v] \in \hat{H}^1(\Sigma_\theta^\delta)$ with $v+c \in \dot{H}^1(\Sigma_\theta^\delta)$ for any constant $c \in \mathbb{R}$. Then

$$\begin{aligned}
\|\nabla \Phi_\delta^*[v]\|_{L^2(\mathbb{R}^2_{>0})}^2 &= \|[\nabla(v+c)\circ\varphi_\delta]\cdot\nabla\varphi_\delta\|_{L^2(\mathbb{R}^2_{>0})}^2 \\
&= \int_{\mathbb{R}^2_{>0}} |\nabla v(\varphi_\delta(\eta,t))\cdot\nabla\varphi_\delta(\eta,t)|^2 \, d(\eta,t) \\
&\leq C\int_{\mathbb{R}^2_{>0}} |\nabla v(\varphi_\delta(\eta,t))|^2 \, d(\eta,t) \\
&= C\int_{\Sigma_\theta^\delta} |\nabla v(x)|^2 |\det[(\varphi_\delta')^{-1}]| \, dx \\
&\leq C\|\nabla v\|_{L^2(\Sigma_\theta^\delta)}^2 = C\|\nabla[v]\|_{L^2(\Sigma_\theta^\delta)}^2
\end{aligned}$$

and we also obtain the converse estimate for Φ_*^δ. Also similarly to (3.3) we obtain the estimate in the L^2-setting where we again put emphasize on the fact that all appearing estimates are uniform in $\delta \geq 0$. Hence the assertion holds. □

3.1.4 Remark. (i) Note that in the context of Lemma 3.1.2 and Lemma 3.1.3 we obtain a bi-Lipschitz transformation of Σ_θ^δ to $\mathbb{R}^2_{>0}$. However, the results that we want to transfer later are formulated on $\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. It is clear that we can transform $\mathbb{R}^2_{>0}$ to $\mathbb{R}^2_+$ via a rotation matrix such that in fact we obtain

$$\begin{aligned}
\Phi_*^\delta &\in \mathscr{L}_{is}(\hat{H}^1(\mathbb{R}^2_+), \hat{H}^1(\Sigma_\theta^\delta)) \cap \mathscr{L}_{is}(H^k(\mathbb{R}^2_+), H^k(\Sigma_\theta^\delta)) \\
(\Phi_*^\delta)^{-1} = \Phi_\delta^* &\in \mathscr{L}_{is}(\hat{H}^1(\Sigma_\theta^\delta), \hat{H}^1(\mathbb{R}^2_+)) \cap \mathscr{L}_{is}(H^k(\Sigma_\theta^\delta), H^k(\mathbb{R}^2_+))
\end{aligned}$$

in Lemma 3.1.3 with norm estimates uniform in $\delta \geq 0$.

(ii) We also observe that due to the fact that φ_δ is not more regular than bi-Lipschitz and not C^1, we cannot transform spaces of higher regularity than $k = 1$ from Σ_θ^δ to the half-space $\mathbb{R}^2_{>0}$ (using this diffeomorphism).

By making use of the push-forward from Lemma 3.1.2 we can now define homogeneous Sobolev spaces of fractional power $0 < s < 1$: We set

$$\hat{H}^s(\Sigma_\theta^\delta) := \Phi_*^\delta \hat{H}^s(\mathbb{R}^2_+) = \Phi_*^\delta \dot{H}^s(\mathbb{R}^2_+)/\mathcal{P}_1,$$

which can also be defined via interpolation as we will observe. Here $(\cdot,\cdot)_{s,p}$ denotes the real interpolation functor for $s \in (0,1)$ and $1 < p < \infty$ (see [51, Chapter 1]). Interpolation of the homogeneous Sobolev spaces $\hat{H}^s(\Sigma_\theta^\delta)$ can be interpreted in the following way: At first we consider interpolation on the whole space $\mathbb{R}^n$ (see [52, Section 5.1]): Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions defined in the usual way (cf. [52, Section 1.2.1]). Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inversion in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. Then we define the following subspace of $\mathcal{S}(\mathbb{R}^n)$ equipped with the same topology

$$\mathcal{Z}(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \partial^\alpha \mathcal{F}\varphi(0) = 0 \ \forall\ \alpha \in \mathbb{N}_0^n\}\,.$$

Then $\mathcal{Z}(\mathbb{R}^n)$ is a locally convex space and $\mathcal{Z}'(\mathbb{R}^n)$ denotes its dual. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ the restriction to $\mathcal{Z}(\mathbb{R}^n)$ yields $f|_{\mathcal{Z}(\mathbb{R}^n)} \in \mathcal{Z}'(\mathbb{R}^n)$ and

$$(f+p)(\varphi) = f(\varphi) \qquad (\varphi \in \mathcal{Z}(\mathbb{R}^n)),$$

if $p \in \mathcal{P}_\infty$ is any polynomial. Here $\mathcal{P}_\infty$ denotes the set of all polynomials of degree $n \in \mathbb{N}$. Conversely, any $f \in \mathcal{Z}'(\mathbb{R}^n)$ can be extended to $\tilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ linearly and continuously, where $\tilde{f}_1 - \tilde{f}_2$ is a polynomial if $\tilde{f}_1, \tilde{f}_2$ are two extensions of f. Hence, we may identify $\mathcal{Z}'(\mathbb{R}^n)$ with the factor space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_\infty$ via a corresponding isomorphism ι. Hence by [52, Section 5.1.3, Definition 2] we can regard $\hat{H}^k(\mathbb{R}^n)$ for $k \in \mathbb{N}$ as a subspace of $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_\infty$ and via the isomorphism

$$\iota : \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_\infty \to \mathcal{Z}'(\mathbb{R}^n) \tag{3.2}$$

we can regard $\hat{H}^k(\mathbb{R}^n)$ for $k \in \mathbb{N}$ as a subspace of $\mathcal{Z}'(\mathbb{R}^n)$. Hence, making use of Lemma 3.1.2, the extension operator $E_1 : \hat{H}^1(\mathbb{R}^2_+) \to \hat{H}^1(\mathbb{R}^2)$ from [10, Proposition 3.19] and the extension by zero E_0 yields

$$\begin{aligned} \hat{H}^1(\Sigma_\theta^\delta) &\xrightarrow{\Phi_\delta^*} \hat{H}^1(\mathbb{R}^2_+) \xrightarrow{E_1} \hat{H}^1(\mathbb{R}^2) \xrightarrow{\iota} \mathcal{Z}'(\mathbb{R}^2) \\ L^2(\Sigma_\theta^\delta) &\xrightarrow{\Phi_\delta^*} L^2(\mathbb{R}^2_+) \xrightarrow{E_0} L^2(\mathbb{R}^2) \to \mathcal{Z}'(\mathbb{R}^2) \end{aligned}$$

by regarding $L^2(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}_\infty \hookrightarrow \mathcal{Z}'(\mathbb{R}^2)$. Thus, $\left\{L^2(\Sigma_\theta^\delta), \hat{H}^1(\Sigma_\theta^\delta)\right\}$ is an interpolation couple. Interpolation of the diffeomorphism Φ_*^δ then yields

$$\Phi_*^\delta \in \mathscr{L}_{is}\left(\hat{H}^s(\mathbb{R}^2_+), (L^2(\Sigma_\theta^\delta), \hat{H}^1(\Sigma_\theta^\delta))_{s,2}\right)$$

by [10, Proposition 3.22]. This explicitly yields the desired characterization of fractional spaces via interpolation $\hat{H}^s(\Sigma_\theta^\delta) \cong (L^2(\Sigma_\theta^\delta), \hat{H}^1(\Sigma_\theta^\delta))_{s,2}$. Homogeneous Sobolev spaces of negative order are defined as dual spaces. For $0 < s \leq 1$ we set $\hat{H}_0^{-s}(\Sigma_\theta^\delta) := (\hat{H}^s(\Sigma_\theta^\delta))'$, endowed with the canonical norm.

Analogously we can now define homogeneous spaces on the boundary Γ_δ: Using the path from Lemma 3.1.1 we can identify $L^2(\Gamma_\delta)$ with $L^2(\mathbb{R})$ (see Lemma 3.1.5

below):

$$\begin{aligned} \|u \circ \gamma_\delta\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |u(\gamma_\delta(t))|^2 \, dt \leq \sin(\theta)^{-2} \int_{\mathbb{R}} |u(\gamma_\delta(t))|^2 |\dot{\gamma_\delta}(t)|^2 \, dt \\ &= \sin(\theta)^{-2} \int_{\Gamma_\delta} |u|^2 \, d\eta = \sin(\theta)^{-2} \|u\|^2_{L^2(\Gamma_\delta)} \\ &\leq \sin(\theta)^{-2} \int_{\mathbb{R}} |u(\gamma_\delta(t))|^2 \, dt = \sin(\theta)^{-2} \|u \circ \gamma_\delta\|^2_{L^2(\mathbb{R})}, \end{aligned} \tag{3.3}$$

where we essentially made use of the uniform estimate of $\dot{\gamma_\delta}$ from (3.1). Thus, by defining $\zeta_\delta^* u := u \circ \gamma_\delta$ the calculation above yields $\zeta_\delta^* \in \mathscr{L}_{is}(L^2(\Gamma_\delta), L^2(\mathbb{R}))$ with $\zeta_*^\delta := (\zeta_\delta^*)^{-1}$. Sobolev spaces on the boundary Γ_δ can be defined as

$$\dot{H}^1(\Gamma_\delta) := \left\{ u : \Gamma_\delta \to \mathbb{R}^n : u \circ \gamma_\delta \in \dot{H}^1(\mathbb{R}) \right\} = \zeta_*^\delta \dot{H}^1(\mathbb{R}).$$

Then $\dot{H}^1(\Gamma_\delta)$ is equipped with the semi-norm $|u|_{\dot{H}^1(\Gamma_\delta)} := \|\partial_{\tau_\delta} u\|_{L^2(\Gamma_\delta)}$. Note that for any constant function c the function $\zeta_*^\delta c$ is still a constant. Hence, the following definition is meaningful:

$$\hat{H}^1(\Gamma_\delta) := \dot{H}^1(\Gamma_\delta)/\mathcal{P}_1 = \zeta_*^\delta \left(\dot{H}^1(\mathbb{R})/\mathcal{P}_1 \right) = \zeta_*^\delta \hat{H}^1(\mathbb{R}) = \zeta_*^\delta \dot{H}^1(\mathbb{R})/\mathcal{P}_1$$

where $\hat{H}^1(\Gamma_\delta)$ again is a Hilbert space by [17, Lemma II.6.2] with the corresponding norm $\|[u]\|_{\hat{H}^1(\Gamma_\delta)} = \|\partial_{\tau_\delta} u\|_{L^2(\Gamma_\delta)}$, where the proof can easily be modified such that it holds in the 1-dimensional case as well. Note that similarly to the case of Σ_θ^δ we will denote elements in $\hat{H}^1(\Gamma_\delta)$ by u although we want to refer to their equivalence class $[u] = u + \mathcal{P}_1$.

Sobolev spaces of fractional power $0 < s < 1$ are again defined as

$$\hat{H}^s(\Gamma_\delta) := \zeta_*^\delta \hat{H}^s(\mathbb{R}) = \zeta_*^\delta \dot{H}^s(\mathbb{R})/\mathcal{P}_1,$$

where $\hat{H}^s(\mathbb{R})$ is defined via interpolation $\hat{H}^s(\mathbb{R}) = (L^2(\mathbb{R}), \hat{H}^1(\mathbb{R}))_{s,2}$ (see [52, Section 5.2.5, Theorem 5.2.3.1(ii), Theorem 2.4.2]. Since $\zeta_\delta^* \in \mathscr{L}_{is}(\hat{H}^1(\Gamma_\delta), \hat{H}^1(\mathbb{R}))$ (elementary calculation as seen in Lemma 3.1.5), we can use the same arguments as for Σ_θ^δ to deduce

$$\hat{H}^1(\Gamma_\delta) \xrightarrow{\zeta_\delta^*} \hat{H}^1(\mathbb{R}) \xrightarrow{\iota} \mathcal{Z}'(\mathbb{R}),$$

$$L^2(\Gamma_\delta) \xrightarrow{\zeta_\delta^*} L^2(\mathbb{R}) \to \mathcal{Z}'(\mathbb{R}),$$

hence $\left\{ L^2(\Gamma_\delta), \hat{H}^1(\Gamma_\delta) \right\}$ is an interpolation couple and we obtain

$$\zeta_\delta^* \in \mathscr{L}_{is}((L^2(\Gamma_\delta), \hat{H}^1(\Gamma_\delta))_{s,2}, \hat{H}^s(\mathbb{R})),$$

which again yields the characterization of spaces of fractional order by interpolation

$$\hat{H}^s(\Gamma_\delta) \cong (L^2(\Gamma_\delta), \hat{H}^1(\Gamma_\delta))_{s,2}.$$

Spaces of negative order are again defined as dual spaces $\hat{H}^{-s}(\Gamma_\delta) := (\hat{H}^s(\Gamma_\delta))'$ for $0 < s \leq 1$. Note that thanks to Lemma 3.1.5 we have $(\hat{H}^s(\Gamma_\delta))' \cong (\hat{H}_0^s(\Gamma_\delta))'$ since $\dot{H}^s(\mathbb{R}) = \dot{H}_0^s(\mathbb{R}) := \overline{C_c^\infty(\mathbb{R})}^{|\cdot|_{H^s}}$.

These observations lead to the desire to identify (in)homogeneous spaces on the boundary Γ_δ with well-studied (in)homogeneous spaces on the whole space $\mathbb{R}$ for the scale $\hat{H}^s(\Gamma_\delta)$ and $H^s(\Gamma_\delta)$ for $s \in [-1,1]$:

3.1.5 Lemma. *Let $s \in [-1,1]$ and the path γ_δ from Lemma 3.1.1 be given. We define*

$$\zeta_*^\delta u := u \circ \gamma_\delta^{-1} \quad \textit{and} \quad \zeta_\delta^* v := v \circ \gamma_\delta,$$

Then

$$\zeta_*^\delta \in \mathscr{L}_{is}(\hat{H}^s(\mathbb{R}), \hat{H}^s(\Gamma_\delta)) \quad \textit{and} \quad (\zeta_*^\delta)^{-1} = \zeta_\delta^* \in \mathscr{L}_{is}(\hat{H}^s(\Gamma_\delta), \hat{H}^s(\mathbb{R}))$$

for $s \in [0,1]$ and

$$|\dot{\gamma}_\delta \circ \gamma_\delta^{-1}|^{-1}\zeta_*^\delta \in \mathscr{L}_{is}(\hat{H}^s(\mathbb{R}), \hat{H}^s(\Gamma_\delta)) \quad \textit{and} \quad |\dot{\gamma}_\delta|\zeta_\delta^* \in \mathscr{L}_{is}(\hat{H}^s(\Gamma_\delta), \hat{H}^s(\mathbb{R}))$$

for $s \in [-1,0]$ where all norm estimates are uniform in $\delta \geq 0$. Furthermore, the statement also holds in the case of inhomogeneous spaces H^s.

Proof. As observed in Lemma 3.1.3 the path γ_δ transforms constants to constants such that the definition is meaningful. In this proof we just consider the proof of the statement in the homogeneous spaces. We first prove the assertion for $s = 0, 1$. First, we define

$$C_{Lip}(\Gamma_\delta) := \zeta_*^\delta C_{Lip}(\mathbb{R}) := \zeta_*^\delta \{u \in C(\mathbb{R}) : |u(x) - u(y)| \leq L|x-y| \ (x,y \in \mathbb{R})\}$$

since γ_δ is a bi-Lipschitz transform by Lemma 3.1.1. Hence for $u \in C_{Lip}(\Gamma_\delta)$ we observe that

$$\frac{d}{dt}u(\gamma_\delta(t)) = \nabla u(\gamma_\delta(t)) \cdot \dot{\gamma_\delta}(t) = (\partial_{\tau_\delta} u)(\gamma_\delta(t))|\dot{\gamma_\delta}(t)| \tag{3.4}$$

for a.e. $t \in \mathbb{R}$ where we used the fact that $\dot{\gamma_\delta} = \tau_\delta|\dot{\gamma_\delta}|$ from Lemma 3.1.1. Hence, for $\zeta_\delta^* u := u \circ \gamma_\delta$ we infer

$$\begin{aligned}
\left\|\frac{d}{dt}(u \circ \gamma_\delta)\right\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left|\frac{d}{dt}u(\gamma_\delta(t))\right|^2 dt = \int_{\mathbb{R}} |(\partial_{\tau_\delta} u)(\gamma_\delta(t))|^2 |\dot{\gamma_\delta}(t)|^2 \, dt \\
&\leq \int_{\mathbb{R}} |(\partial_{\tau_\delta} u)(\gamma_\delta(t))|^2 |\dot{\gamma_\delta}(t)| \, dt = \int_{\Gamma_\delta} |\partial_{\tau_\delta} u|^2 \, d\eta = \|\partial_{\tau_\delta} u\|_{L^2(\Gamma_\delta)}^2 \\
&\leq \sin(\theta)^{-1} \int_{\mathbb{R}} |(\partial_{\tau_\delta} u)(\gamma_\delta(t))|^2 |\dot{\gamma_\delta}(t)|^2 \, dt
\end{aligned}$$

$$= \sin(\theta)^{-1} \left\| \frac{d}{dt}(u \circ \gamma_\delta) \right\|^2_{L^2(\mathbb{R})},$$

where we essentially made use of the uniform estimate of $|\dot{\gamma_\delta}|$ from (3.1). By the density of $C_{Lip}(\Gamma_\delta)$ in $\hat{H}^1(\Gamma_\delta)$ this estimate also holds for all $u \in \hat{H}^1(\Gamma_\delta)$ such that we can define push-forward and pull-back as stated in the proposition. The estimate above combined with (3.3) shows that

$$\zeta_*^\delta \in \mathscr{L}_{is}(\hat{H}^1(\mathbb{R}), \hat{H}^1(\Gamma_\delta)) \cap \mathscr{L}_{is}(H^s(\mathbb{R}), H^s(\Gamma_\delta)),$$
$$(\zeta_*^\delta)^{-1} = \zeta_\delta^* \in \mathscr{L}_{is}(\hat{H}^1(\Gamma_\delta), \hat{H}^1(\mathbb{R})) \cap \mathscr{L}_{is}(H^s(\Gamma_\delta), H^s(\mathbb{R})),$$

for $s = 0, 1$. Interpolation yields the statement in $\hat{H}^s$ and H^s for $s \in [0, 1]$. Again, we put emphasize on the fact that all arising norm estimates are uniform in $\delta \geq 0$. In case of negative s we need to calculate the dual operator $(\zeta_*^\delta)' : \hat{H}^{-1}(\Gamma_\delta) \to \hat{H}^{-1}(\mathbb{R})$: For $\varphi \in C^\infty_{c,m}(\Gamma_\delta), \psi \in C_{Lip}(\mathbb{R})$ (where $C^\infty_{c,m}(\Gamma_\delta)$ is defined in Lemma 3.1.11) we infer

$$\begin{aligned}
\langle (\zeta_*^\delta)'\varphi, \psi \rangle_{\hat{H}^{-1}(\mathbb{R}), \hat{H}^1(\mathbb{R})} &:= \langle \varphi, \zeta_*^\delta \psi \rangle_{\hat{H}^{-1}(\Gamma_\delta), \hat{H}^1(\Gamma_\delta)} = \int_{\Gamma_\delta} \varphi \zeta_*^\delta \psi \, d\eta \\
&= \int_{\Gamma_\delta} \varphi \psi \circ \gamma_\delta^{-1} \, d\eta = \int_{\Gamma_\delta} (\varphi \circ \gamma_\delta) \psi |\dot{\gamma_\delta}| \, d\eta \\
&= \int_{\mathbb{R}} |\dot{\gamma_\delta}| \zeta_\delta^* \varphi \psi \, dt \\
&= \langle |\dot{\gamma_\delta}| \zeta_\delta^* \varphi, \psi \rangle_{\hat{H}^{-1}(\mathbb{R}), \hat{H}^1(\mathbb{R})},
\end{aligned}$$

where we made use of the fact that by Lemma 3.1.11 functions of the form $(f, \cdot)_{2,\Gamma_\delta}$ with $f \in C^\infty_{c,m}(\Gamma_\delta)$ are dense in $(\hat{H}^1(\Gamma_\delta))'$ and by [17, Theorem II.8.1] this density also holds for $\mathbb{R}$. Then the calculation above yields $(\zeta_*^\delta)' = |\dot{\gamma_\delta}| \zeta_\delta^*$. On the other hand, almost the same calculation gives us

$$\begin{aligned}
\langle (\zeta_\delta^*)'\varphi, \psi \rangle_{\hat{H}^{-1}(\Gamma_\delta), \hat{H}^1(\Gamma_\delta)} &:= \langle \varphi, \zeta_\delta^* \psi \rangle_{\hat{H}^{-1}(\mathbb{R}), \hat{H}^1(\mathbb{R})} = \int_{\mathbb{R}} \varphi \zeta_\delta^* \psi \, dx \\
&= \int_{\mathbb{R}} |\dot{\gamma_\delta}| \varphi \zeta_\delta^* \psi |\dot{\gamma_\delta}|^{-1} \, dt \\
&= \int_{\Gamma_\delta} (\varphi \circ \gamma_\delta^{-1}) \psi |\dot{\gamma_\delta} \circ \gamma_\delta^{-1}|^{-1} \, d\eta \\
&= \int_{\Gamma_\delta} \zeta_*^\delta \varphi \psi |\dot{\gamma_\delta} \circ \gamma_\delta^{-1}|^{-1} \, d\eta \\
&= \langle |\dot{\gamma_\delta} \circ \gamma_\delta^{-1}|^{-1} \zeta_*^\delta \varphi, \psi \rangle_{\hat{H}^{-1}(\Gamma_\delta), \hat{H}^1(\Gamma_\delta)}
\end{aligned}$$

again for smooth φ, ψ such that $(\zeta_\delta^*)' = |\dot{\gamma_\delta} \circ \gamma_\delta^{-1}|^{-1} \zeta_*^\delta$. We note that both calculations also hold in L^2 such that the dual operator in the L^2-setting is given in the same way. Thanks to the boundedness of the dual operator we obtain

$$|\dot{\gamma_\delta} \circ \gamma_\delta^{-1}|^{-1} \zeta_*^\delta \in \mathscr{L}_{is}(\hat{H}^{-1}(\mathbb{R}), \hat{H}^{-1}(\Gamma_\delta)) \cap \mathscr{L}_{is}(H^{-s}(\mathbb{R}), H^{-s}(\Gamma_\delta)),$$

$$|\dot{\gamma}_\delta|\zeta_\delta^* \in \mathscr{L}_{is}(\hat{H}^{-1}(\Gamma_\delta), \hat{H}^{-1}(\mathbb{R})) \cap \mathscr{L}_{is}(H^{-s}(\Gamma_\delta), H^{-s}(\mathbb{R}))$$

for $s = 0, 1$ where the operator norms are uniformly in $\delta \geq 0$. Again, interpolation yields the statement for $\hat{H}^s$ and H^s for $s \in [-1, 0]$ with norms uniform in $\delta \geq 0$. □

3.1.6 Remark. It is clear that we can transform $\mathbb{R}$ to $\partial\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 = 0\}$ and $\partial\mathbb{R}^2_{>0} := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \in \mathbb{R}\}$ via a smooth linear mapping (using a transformation matrix). Hence, this yields that from Lemma 3.1.5 we even obtain

$$\zeta_*^\delta \in \mathscr{L}_{is}(\hat{H}^s(\partial\mathbb{R}^2_+), \hat{H}^s(\Gamma_\delta)) \qquad \text{and} \qquad (\zeta_*^\delta)^{-1} = \zeta_\delta^* \in \mathscr{L}_{is}(\hat{H}^s(\Gamma_\delta), \hat{H}^s(\partial\mathbb{R}^2_+)),$$
$$\zeta_*^\delta \in \mathscr{L}_{is}(\hat{H}^s(\partial\mathbb{R}^2_{>0}), \hat{H}^s(\Gamma_\delta)) \qquad \text{and} \qquad (\zeta_*^\delta)^{-1} = \zeta_\delta^* \in \mathscr{L}_{is}(\hat{H}^s(\Gamma_\delta), \hat{H}^s(\partial\mathbb{R}^2_{>0}))$$

with norm estimates uniform in $\delta \geq 0$. In the following we denote every push-forward and pull-back from Γ_δ to $\mathbb{R}, \partial\mathbb{R}^2_+, \partial\mathbb{R}^2_{>0}$ by ζ_*^δ and ζ_δ^*.

3.1.1 Embeddings, Dual Spaces and Related Results

For a precise characterization and well-understanding of (in)homogeneous spaces on sectors, we collect some embeddings and related results in this section. The strategy for the proof of most of the results is the following: By applying the bi-Lipschitz transforms from Lemma 3.1.1 and Lemma 3.1.2 we transfer results known from the half-space setting to sectors. We will mainly focus on results on sectors Σ_θ with boundary Γ but we note that all results hold true on smooth sector-like domains Σ_θ^δ with boundary Γ_δ and corresponding norm estimates are uniformly in $\delta > 0$. We start with some embedding theorems:

3.1.7 Lemma. *Let $s \in (0, 1]$. Then we have*

$$C_c^\infty(\overline{\Sigma_\theta^\delta}) \overset{d}{\hookrightarrow} H^s(\Sigma_\theta^\delta) \overset{d}{\hookrightarrow} \hat{H}^s(\Sigma_\theta^\delta).$$

Here, the embeddings above can be interpreted by considering the isomorphism ι from (3.2).

Proof. From the observation above we know that $\{L^2(\Sigma_\theta^\delta), H^1(\Sigma_\theta^\delta)\}$ as well as $\{L^2(\Sigma_\theta^\delta), \hat{H}^1(\Sigma_\theta^\delta)\}$ are interpolation couples. By [51, Theorem 1.6.2] we obtain for $s \in (0, 1)$:

$$H^1(\Sigma_\theta^\delta) = L^2(\Sigma_\theta^\delta) \cap \hat{H}^1(\Sigma_\theta^\delta) \overset{d}{\hookrightarrow} (L^2(\Sigma_\theta^\delta), \hat{H}^1(\Sigma_\theta^\delta))_{s,2} = \hat{H}^s(\Sigma_\theta^\delta),$$
$$H^1(\Sigma_\theta^\delta) = L^2(\Sigma_\theta^\delta) \cap H^1(\Sigma_\theta^\delta) \overset{d}{\hookrightarrow} (L^2(\Sigma_\theta^\delta), H^1(\Sigma_\theta^\delta))_{s,2} = H^s(\Sigma_\theta^\delta).$$

Furthermore,

$$H^s(\Sigma_\theta^\delta) = (L^2(\Sigma_\theta^\delta), H^1(\Sigma_\theta^\delta))_{s,2} \hookrightarrow (L^2(\Sigma_\theta^\delta), \hat{H}^1(\Sigma_\theta^\delta))_{s,2} = \hat{H}^s(\Sigma_\theta^\delta).$$

Since $C_c^\infty(\overline{\Sigma_\theta^\delta}) \overset{d}{\hookrightarrow} H^1(\Sigma_\theta^\delta)$ by [25, Lemma 12.4] we deduce

$$C_c^\infty(\overline{\Sigma_\theta^\delta}) \overset{d}{\hookrightarrow} H^1(\Sigma_\theta^\delta) \overset{d}{\hookrightarrow} H^s(\Sigma_\theta^\delta) \overset{d}{\hookrightarrow} \hat{H}^s(\Sigma_\theta^\delta).$$

The assertion for $s = 1$ follows from [29, Lemma B1]. □

3.1.8 Lemma. *Let $m \in \mathbb{N}$ and $H^m_{\mathrm{div}}(\Sigma_\theta) := \{u \in H^m(\Sigma_\theta, \mathbb{R}^2) : \operatorname{div} u = 0\}$. Then $H^m_{\mathrm{div}}(\Sigma_\theta)$ is dense in $H^1_{\mathrm{div}}(\Sigma_\theta)$.*

Proof. Let $(\varphi_\eta)_{\eta>0} \subseteq C_c^\infty(\mathbb{R}^2)$ be a mollifier such that $\varphi_\eta \geq 0$ and $\operatorname{supp} \varphi_\eta \subseteq \overline{B_\eta(0)}$ for all $\eta > 0$. We fix $u \in H^1_{\mathrm{div}}(\Sigma_\theta)$ and $\varepsilon > 0$. For $\omega > 0$ we denote by $\Sigma_\theta^\omega := \Sigma_\theta - \omega e_1$ the shifted sector. Next, we define the shifted $u_\omega \in H^1_{\mathrm{div}}(\Sigma_\theta^\omega)$ as

$$u_\omega(x_1, x_2) := u(x_1 + \omega, x_2) \qquad \text{for } x = (x_1, x_2) \in \Sigma_\theta^\omega.$$

By construction we have $u_\omega|_{\Sigma_\theta} \to u$ in $H^1(\Sigma_\theta, \mathbb{R}^2)$ as $\omega \to 0$. We first fix $\omega > 0$ such that

$$\|v|_{\Sigma_\theta} - u\|_{H^1(\Sigma_\theta, \mathbb{R}^2)} < \frac{\varepsilon}{2} \qquad \text{for } v := u_{4\omega}.$$

We choose $\delta > 0$ such that $\Sigma_\theta + \overline{B_\delta(0)} \subseteq \Sigma_\theta^\omega$. By $\chi := \chi_{\Sigma_\theta^{2\omega}} \in L^\infty(\mathbb{R}^2)$ we denote the characteristic function of the shifted sector $\Sigma_\theta^{2\omega}$. For the convolution $\psi := \varphi_\delta * \chi \in C^\infty(\mathbb{R}^2)$ we observe that

(i) $\int_{\mathbb{R}^2} \varphi_\delta(x_1, x_2)\, d(x_1, x_2) = 1$ and $0 \leq \chi \leq 1$ imply $0 \leq \psi \leq 1$;

(ii) $\chi|_{\Sigma_\theta^{2\omega}} \equiv 1$ and $\Sigma_\theta^\omega + \overline{B_\delta(0)} \subseteq \Sigma_\theta^{2\omega}$ imply that $\psi|_{\overline{\Sigma_\theta^\omega}} \equiv 1$;

(iii) $\chi|_{\mathbb{R}^2 \setminus \Sigma_\theta^{2\omega}} \equiv 0$ and $\mathbb{R}^2 \setminus \Sigma_\theta^{3\omega} + \overline{B_\delta(0)} \subseteq \mathbb{R}^2 \setminus \Sigma_\theta^{2\omega}$ imply that $\psi|_{\mathbb{R}^2 \setminus \Sigma_\theta^{3\omega}} \equiv 0$;

(iv) $0 \leq \chi \leq 1$ implies $\|\psi\|_{BC^1(\mathbb{R}^2)} \leq \|\varphi_\delta\|_{W^{1,1}(\mathbb{R}^2)} < \infty$.

By definition

$$w(x_1, x_2) := \begin{cases} \psi(x_1, x_2) v(x_1, x_2), & \text{if } (x_1, x_2) \in \Sigma_\theta^{4\omega}, \\ 0, & \text{otherwise}, \end{cases} \qquad (x = (x_1, x_2) \in \mathbb{R}^2)$$

leads to a well-defined vector field $w \in H^1(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{div} w = 0$ in Σ_θ^ω since $\psi|_{\overline{\Sigma_\theta^\omega}} \equiv 1$ and $\operatorname{div} v = 0$ in Σ_θ^ω.

Then by construction we have $\varphi_\eta * w \in H^m(\mathbb{R}^2, \mathbb{R}^2)$ for all $\eta > 0$ with

$$\varphi_\eta * w \xrightarrow{\eta \to 0} w \qquad \text{in } H^1(\mathbb{R}^2, \mathbb{R}^2).$$

Furthermore, if $0 < \eta < \delta$ then $\Sigma_\theta + \overline{B_\eta(0)} \subseteq \Sigma_\theta^\omega$ such that $\operatorname{div}(\varphi_\eta * w) = 0$ in Σ_θ due to the fact that $\operatorname{div} w = 0$ in Σ_θ^ω. Choosing $0 < \eta < \delta$ such that $\|\varphi_\eta * w - w\|_{H^1(\mathbb{R}^2,\mathbb{R}^2)} < \frac{\varepsilon}{2}$ we also obtain

$$\|\tilde{u} - w|_{\Sigma_\theta}\|_{H^1(\Sigma_\theta,\mathbb{R}^2)} \leq \frac{\varepsilon}{2} \qquad \text{for } \tilde{u} := (\varphi_\eta * w)|_{\Sigma_\theta}.$$

Then we have $\tilde{u} \in H^m(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{div} \tilde{u} = 0$, i.e., $\tilde{u} \in H^m_{\mathrm{div}}(\Sigma_\theta)$ and

$$\|\tilde{u} - u\|_{H^1(\Sigma_\theta,\mathbb{R}^2)} < \varepsilon.$$

□

3.1.9 Lemma. *Let $s \in [0, 1/2]$. Then it holds*

(i) $C_c^\infty(\Gamma_\pm) \overset{d}{\hookrightarrow} H^s(\Gamma_\pm) \overset{d}{\hookrightarrow} \hat{H}^s(\Gamma_\pm)$;

(ii) $C_c^\infty(\Gamma\backslash\{0\}) \overset{d}{\hookrightarrow} H^s(\Gamma) \overset{d}{\hookrightarrow} \hat{H}^s(\Gamma)$.

Proof. (i) We just prove the assertion for $+$. This is a consequence of [51]. Note that by [51, Theorem 2.9.3(d)] the space $C_c^\infty(\mathbb{R}_+)$ is dense in $H^s(\mathbb{R}_+)$ for $s \in [0, 1/2]$. This also holds for $\hat{H}^s(\mathbb{R}_+)$: Applying [51, Theorem 1.6.2] to the interpolation couples $\{L^2(\mathbb{R}_+), \hat{H}^1(\mathbb{R}_+)\}$ and $\{L^2(\mathbb{R}_+), H^1(\mathbb{R}_+)\}$ yields

$$\begin{aligned} H^1(\mathbb{R}_+) &= L^2(\mathbb{R}_+) \cap \hat{H}^1(\mathbb{R}_+) \overset{d}{\hookrightarrow} (L^2(\mathbb{R}_+), \hat{H}^1(\mathbb{R}_+))_{s,2} = \hat{H}^s(\mathbb{R}_+), \\ H^1(\mathbb{R}_+) &= L^2(\mathbb{R}_+) \cap H^1(\mathbb{R}_+) \overset{d}{\hookrightarrow} (L^2(\mathbb{R}_+), H^1(\mathbb{R}_+))_{s,2} = H^s(\mathbb{R}_+). \end{aligned}$$

Since also

$$H^s(\mathbb{R}_+) = (L^2(\mathbb{R}_+), H^1(\mathbb{R}_+))_{s,2} \hookrightarrow (L^2(\mathbb{R}_+), \hat{H}^1(\mathbb{R}_+))_{s,2} = \hat{H}^s(\mathbb{R}_+),$$

we can deduce for $s \in [0, 1/2]$

$$C_c^\infty(\mathbb{R}_+) \overset{d}{\hookrightarrow} H^s(\mathbb{R}_+) \overset{d}{\hookrightarrow} \hat{H}^s(\mathbb{R}_+),$$

and the assertion follows by rotating $\mathbb{R}_+$ to Γ_+.

(ii) We pick $v \in \hat{H}^{1/2}(\Gamma)$. Then we define $v_\pm := v|_{\Gamma_\pm} \in \hat{H}^{1/2}(\Gamma_\pm)$. Now let $(v_\pm^k) \subseteq C_c^\infty(\Gamma_\pm)$ be a sequence such that

$$v_\pm^k \xrightarrow{k\to\infty} v_\pm \qquad \text{in } \hat{H}^{1/2}(\Gamma_\pm).$$

Now we set $v^k(x) := \chi_{\Gamma_+}(x) v_+^k(x_1, x_2) + \chi_{\Gamma_-}(x) v_-^k(x_1, x_2) \in C_c^\infty(\Gamma\backslash\{0\})$ where $x = (x_1, x_2)$. Obviously by definition we know that $(v^k)_k$ is a Cauchy sequence in $\hat{H}^{1/2}(\Gamma)$ such that there exists $\psi \in \hat{H}^{1/2}(\Gamma)$ with $v^k \to \psi$ in $\hat{H}^{1/2}(\Gamma)$. Then it also follows $v_k \to \psi$ in $L^1_{loc}(\Gamma\backslash\{0\})$. On the other hand by construction we have $v^k \to v$ in $\hat{H}^{1/2}(\Gamma_\pm)$, hence $v^k \to v$ in $L^1_{loc}(\Gamma\backslash\{0\})$. Since the limit is unique, we deduce $v = \psi$. The statement for in the inhomogeneous setting follows with the same arguments. □

Next, we characterize dual spaces of homogeneous Sobolev spaces on sectors Σ_θ^δ and the boundary Γ_δ in order to be able to use the representation of the duality pairings as an integral. To be precise we want to prove the density of mean value free infinitely differentiable functions with compact support in those dual spaces. This result is well-known for e.g. the whole space $\mathbb{R}^n$ and half-space $\mathbb{R}^n_+$ where the restriction that the functions have to be mean value free can be dropped. In this case we need this restriction such that the integral is well-defined. The approach follows the ideas as seen in [29, Appendix A].

3.1.10 Lemma. *Let $r > 0$ and $\Omega_1 := \Sigma_\theta^\delta \cap B_r(0)$ and $\Omega_2 := (-r, r)$. The Poincaré inequality holds for $u \in \dot{H}^1_m(\Omega_i) := \{u \in \dot{H}^1(\Omega_i) : \int_{\Omega_i} u\, dx = 0\}$ for $i = 1, 2$. To be precise we have*

$$\|u\|_{L^2(\Omega_i)} \leq C \|\nabla u\|_{L^2(\Omega_i)}.$$

Proof. This is a direct consequence of the Poincaré inequality for mean value free functions in the version of [46, Lemma 10.2(vi)]. Note that by [17, Remark II.6.1] we infer that $H^1(\Omega_i)$ and $\dot{H}^1(\Omega_i)$ are equal algebraically, i.e., $H^1(\Omega_i) = \dot{H}^1(\Omega_i)$ since Ω_i is bounded and Lipschitz. Note that [17, Remark II.6.1] follows from [17, Lemma II.6.1] which holds also in the 1-dimensional case if we modify its proof, hence [17, Remark II.6.1] holds for $\Omega_i,\ i = 1, 2$, simultaneously. Then we can apply [46, Lemma 10.2(vi)]. □

3.1.11 Lemma. *We set $\Omega_1 := \Sigma_\theta^\delta$ and $\Omega_2 := \Gamma_\delta$. We define the set of functionals which are given through a regular distribution, as*

$$\mathcal{S}_i := \left\{ \mathcal{F} \in (\hat{H}^1(\Omega_i))' : \mathcal{F}(u) = (f, u)_{2,i} \text{ for } f \in C^\infty_{c,m}(\Omega_i) \right\},$$

where the space of mean value free infinitely differentiable functions with compact support are defined as

$$C^\infty_{c,m}(\Omega_i) := \left\{ f \in C^\infty_c(\Omega_i) : \int_{\Omega_i} f(x)\, dx = 0 \right\}.$$

Here, the functionals are defined as

$$(f, u)_{2,1} := (f, u)_2 := \int_{\Sigma_\theta^\delta} f(x) u(x)\, dx \qquad (f \in C^\infty_{c,m}(\Sigma_\theta^\delta), u \in \hat{H}^1(\Sigma_\theta^\delta)),$$

$$(f, u)_{2,2} := (f, u)_{2,\Gamma_\delta} := \int_{\Gamma_\delta} f(x) u(x)\, d\eta(x) \qquad (f \in C^\infty_{c,m}(\Gamma_\delta), u \in \hat{H}^1(\Gamma_\delta)).$$

Then $\mathcal{S}_i$ is dense $\hat{H}^1(\Omega_i)'$ for $i = 1, 2$. It also holds that $\mathcal{S}_i$ is dense in $\hat{H}^1_{\mathrm{div}}(\Sigma_\theta^\delta)'$.

Proof. First, we need to prove that $\mathcal{S}_i$ is a subset of $\hat{H}^s(\Omega_i)'$. To this end, let $\mathcal{F} \in \mathcal{S}_i$ be arbitrary with $\mathcal{F}(u) = (f, u)_{2,i}$ for one fixed $f \in C^\infty_{c,m}(\Omega_i)$ and for all $u \in \hat{H}^s(\Omega_i)$. Note that due to the fact that f is mean value free, the integral $(f, u)_{2,i}$ is well-defined for $i = 1, 2$. It is obvious that $\mathcal{F}$ is a linear operator. Now let $r > 0$ such that $\operatorname{supp} f \subseteq B_r(0)$. We set $K_i := \Omega_i \cap B_r(0) \subseteq \mathbb{R}^2$. For $i = 1$ we first obtain by choosing a representative of $u \in \hat{H}^1(\Sigma^\delta_\theta)$ with $\int_{K_1} u(x)\,dx = 0$:

$$\begin{aligned} |(f,u)_{2,1}| &= \left| \int_{K_1} f(x)u(x)\,dx \right| \le \|f\|_{L^2(K_1)} \|u\|_{L^2(K_1)} \\ &\le C\|f\|_{L^2(\Sigma^\delta_\theta)} \|u\|_{\hat{H}^1(K_1)} \le C\|f\|_{L^2(\Sigma^\delta_\theta)} \|u\|_{\hat{H}^1(\Sigma^\delta_\theta)}, \end{aligned}$$

where we applied Lemma 3.1.10. For $i = 2$ we note that the path γ_δ from Lemma 3.1.1 maps bounded sets to bounded sets, in fact $\gamma_\delta^{-1}(K_2) = (-r, r)$. Then we obtain by choosing a representative of $u \in \hat{H}^1(\Gamma_\delta)$ with $\int_{K_2} u\,d\eta = \int_{-r}^{r} u \circ \gamma_\delta |\dot{\gamma}_\delta|\,dt = 0$ with Lemma 3.1.1:

$$\begin{aligned} |(f,u)_{2,2}| &= \left| \int_{K_2} f(x)u(x)\,d\eta(x) \right| \\ &= \left| \int_{-r}^{r} f(\gamma_\delta(t))u(\gamma_\delta(t))\dot{\gamma}_\delta(t)\,dt \right| \\ &\le \|f \circ \gamma_\delta\|_{L^2((-r,r))} \|u \circ \gamma_\delta |\dot{\gamma}_\delta|\|_{L^2((-r,r))}. \end{aligned}$$

Now we note that thanks to $u \in \hat{H}^1(K_2)$ we have $u \circ \gamma_\delta |\dot{\gamma}_\delta| \in \hat{H}^1((-r, r))$. Again by applying the Poincaré inequality from Lemma 3.1.10 we obtain

$$\begin{aligned} |(f,u)_{2,2}| &\le \|f \circ \gamma_\delta\|_{L^2((-r,r))} \|u \circ \gamma_\delta |\dot{\gamma}_\delta|\|_{L^2((-r,r))} \\ &\le C\|f\|_{L^2(K_2)} \|u \circ \gamma_\delta |\dot{\gamma}_\delta|\|_{\hat{H}^1((-r,r))} \\ &\le C\|f\|_{L^2(\Gamma_\delta)} \|u\|_{\hat{H}^1(\Gamma_\delta)}. \end{aligned}$$

Hence, we proved $\mathcal{S}_i \subseteq \hat{H}^1(\Omega_i)'$.

The density follows with a functional analytic argument as in [17, Lemma II.8.1]. Assume that $\mathcal{S}_i$ is not dense in $\hat{H}^1(\Omega_i)'$. Since $\hat{H}^1(\Omega_i)'$ as a dual space is equipped with a norm we can apply Hahn-Banach ([17, Theorem 1.7(b)] to the result that there exists some $Z \in \hat{H}^1(\Omega_i)''$ such that $Z \neq 0$ and

$$Z(\mathcal{F}) = 0 \qquad (\mathcal{F} \in \mathcal{S}_i).$$

Since $\dot{H}^1(\Omega_i)$ is reflexive by [17, Exercise II.6.2] we know that $\hat{H}^1(\Omega_i) = \dot{H}^1(\Omega_i)/\mathbb{R}$ is also reflexive and that $\hat{H}^1(\Omega_i)'' \cong \hat{H}^1(\Omega_i)$ such that the condition simplifies as

$$\mathcal{F}(z) = (f, z)_{2,i} = 0 \qquad (\mathcal{F} \in \mathcal{S} \text{ such that } f \in C^\infty_c(\Omega_i)).$$

Let $\overline{U} \subseteq \Omega_i$ such that U is open and bounded. Let $\tilde{z} \in [z]$ be a representative of $z \in \hat{H}^1(\Omega_i)$ such that

$$\tilde{z}|_U \in L^2_m(U) := \left\{ \tilde{z} \in L^2(U) : \int_U \tilde{z}\, dx = 0 \right\}.$$

(Note that $\tilde{z} \in L^2_{loc}(\Omega_i)$ such that $\tilde{z} \in L^2(U)$). Then we have

$$(\tilde{z}, f)_2 = \int_U \tilde{z} f\, dx = 0 \qquad (f \in C^\infty_{c,m}(U)).$$

Since $L^2_m(U) = \overline{C^\infty_{c,m}(U)}^{L^2}$ we even deduce $(\tilde{z}, f)_2 = 0$ for all $f \in L^2_m(U)$ such that $\tilde{z} = 0$ a.e. in U follows. Hence $[z] = [\text{const}]$ in U, such that $z = 0$ since $U \subseteq \Omega_i$ is an arbitrary open and bounded subset with $\overline{U} \subseteq \Omega_i$.

The statement for the subspace of divergence free functions $\hat{H}^1_{\text{div}}(\Sigma_\theta)$ follows analogously. Then the assertion is proved. □

3.1.12 Remark. In the following we will interpret Lemma 3.1.11 in the following way: Instead of saying that $\mathcal{S}_i$ is dense in $(\hat{H}^1(\Omega_i))'$, we will say that

$$C^\infty_{c,m}(\Omega_i) \overset{d}{\hookrightarrow} \hat{H}^1(\Omega_i)'$$

by using the duality pairing defined in Lemma 3.1.11.

Next, we will consider some results regarding the tangential derivative operator on the boundary Γ by transferring corresponding results from $\mathbb{R}$ by making use of Lemma 3.1.5.

3.1.13 Corollary. *For every $s \in [0, 1]$ the tangential derivative operator satisfies*

$$\partial_\tau \in \mathscr{L}_{is}(\hat{H}^s(\Gamma), \hat{H}^{s-1}(\Gamma)) \cap \mathscr{L}(H^s(\Gamma), H^{s-1}(\Gamma)).$$

Proof. First, we prove the assertion for $s = 0, 1$. Note that for $u \in C_{Lip}(\Gamma_\delta)$ it follows from (3.4) that

$$\partial_\tau = \zeta^0_* \frac{d}{dt} \zeta^*_0.$$

Since d/dt is the derivative operator on the whole space $\mathbb{R}$ we can use the definition of Bessel potential spaces for $s \in \mathbb{R}$ using the Fourier transform $\mathcal{F}$:

$$\dot{H}^s(\mathbb{R}) = \left\{ u \in L^1_{loc}(\mathbb{R}) : \mathcal{F}^{-1} |\xi|^s \mathcal{F} u \in L^2(\mathbb{R}) \right\}.$$

Then it is straightforward to prove that $d/dt : \hat{H}^1(\mathbb{R}) \to L^2(\mathbb{R})$ is an isomorphism. Combining this with Lemma 3.1.5 we immediately deduce

$$\partial_\tau \in \mathscr{L}_{is}(\hat{H}^1(\Gamma), L^2(\Gamma)).$$

Next, we have to compute the dual operator $(\partial_\tau)' : L^2(\Gamma) \to \hat{H}^{-1}(\Gamma)$. For $\varphi \in C^\infty_{c,m}(\Gamma)$ and $\psi \in C_{Lip}(\Gamma)$ we infer

$$\begin{aligned}
\langle (\partial_\tau)'\varphi, \psi\rangle_{\hat{H}^{-1}(\Gamma),\hat{H}^1(\Gamma)} &:= \langle \varphi, \partial_\tau \psi\rangle_{L^2(\Gamma),L^2(\Gamma)} = \int_\Gamma \varphi \partial_\tau \psi \, d\eta \\
&= \int_\Gamma \varphi \zeta^0_* \frac{d}{dt} \zeta^*_0 \psi \, d\eta \\
&= \int_{\mathbb{R}} (\zeta^*_0 \varphi) \frac{d}{dt} \zeta^*_0 \psi \, dt = -\int_{\mathbb{R}} \frac{d}{dt} (\zeta^*_0 \varphi)(\zeta^*_0 \psi)\, dt \\
&= -\int_\Gamma (\partial_\tau \varphi)\psi \, d\eta = \langle -\partial_\tau \varphi, \psi\rangle_{\hat{H}^{-1}(\Gamma),\hat{H}^1(\Gamma)}
\end{aligned}$$

again thanks to Lemma 3.1.11 which yields $(\partial_\tau)' = -\partial_\tau$ and from the boundedness of the dual operator we deduce

$$\partial_\tau \in \mathscr{L}_{is}(L^2(\Gamma), \hat{H}^{-1}(\Gamma)).$$

Interpolation then yields the result for the homogeneous case.

In order to obtain the statement in the inhomogeneous case we only need to replace $d/dt : \hat{H}^1(\mathbb{R}) \to L^2(\mathbb{R})$ with $d/dt : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ which is obviously bounded. □

3.1.14 Corollary. *Let $s \in [0,1]$. Then the shifted tangential derivative operator on Γ is an isomorphism, i.e.,*

$$1 + \partial_\tau \in \mathscr{L}_{is}(H^s(\Gamma), H^{s-1}(\Gamma)).$$

Proof. The proof is essentially the same proof as of Corollary 3.1.13. Then we have $1 \pm \partial_\tau = \zeta^0_* (1 \pm d/dt)\, \zeta^*_0$. Clearly we have $1 \pm d/dt \in \mathscr{L}_{is}(H^1(\mathbb{R}), L^2(\mathbb{R}))$ which yields $1 \pm \partial_\tau \in \mathscr{L}(H^1(\Gamma), L^2(\Gamma))$. Furthermore, we also infer $(1 \pm \partial_\tau)' = 1 \mp \partial_\tau \in \mathscr{L}_{is}(L^2(\Gamma), H^{-1}(\Gamma))$ and interpolation yields the result. □

Weyl Projections

Dealing with divergence free functions almost always leads to introducing corresponding Weyl and Helmholtz projections on the corresponding function spaces. The solenoidal subspaces of $H^1(\Sigma_\theta)^2, \hat{H}^1(\Sigma_\theta)^2$ and $L^2(\Sigma_\theta)^2$ will be denoted by $H^1_{\text{div}}(\Sigma_\theta), \hat{H}^1_{\text{div}}(\Sigma_\theta)$ and $L^2_{\text{div}}(\Sigma_\theta)$. Here, we will only consider the divergence free subspaces which are defined as the range of a Weyl projection. The Weyl projection can be defined by making use of the Dirichlet problem (cf. Lemma 3.1.15). However, another approach to obtain divergence free subspaces is to consider the range of the Helmholtz projection which is defined by using the Neumann problem. Using the Helmholtz projection leads to the definition of the $L^2_\sigma(\Sigma_\theta)$ spaces. In the context of Chapter 4 we are only interested in $L^2_{\text{div}}(\Sigma_\theta)$, hence in the following we will only consider Weyl projections.

Here, we will use standard techniques in order to prove the existence of the Weyl projection and corresponding properties. In fact, we will make use of the Dirichlet problem, cf. Section 3.1.3.

3.1.15 Lemma (Weyl projection on $L^2(\Sigma_\theta)^2$)**.** *We define the Weyl map as*

$$P_W : L^2(\Sigma_\theta)^2 \to L^2(\Sigma_\theta)^2, \qquad P_W\varphi := \varphi - \nabla\Phi,$$

where Φ is the weak solution of the homogeneous Dirichlet problem

$$\Delta\Phi = \operatorname{div}\varphi \quad \text{in } \Sigma_\theta, \quad \Phi = 0 \quad \text{on } \Gamma. \tag{3.5}$$

Then P_W is a projection along $\nabla\hat{H}^1_0(\Sigma_\theta)$ and there exists a direct orthogonal decomposition

$$L^2(\Sigma_\theta)^2 = L^2_{\text{div}}(\Sigma_\theta) \oplus \nabla\hat{H}^1_0(\Sigma_\theta).$$

Proof. At first we observe the following regarding the divergence: Note that the definition of the divergence $\operatorname{div} : L^2(\Sigma_\theta) \to \hat{H}^{-1}(\Sigma_\theta)$ is meaningful by setting

$$\langle \operatorname{div}\varphi, \psi\rangle_{\hat{H}^{-1}(\Sigma_\theta),\hat{H}^1_0(\Sigma_\theta)} := (\varphi, \nabla\psi)_2 \qquad (\psi \in C_c^\infty(\Sigma_\theta)).$$

Integration by parts then yields the consistency of the definition of the divergence with the definition in spaces of positive order with $\|\operatorname{div}\varphi\|_{\hat{H}^{-1}(\Sigma_\theta)} \leq \|\varphi\|_{L^2(\Sigma_\theta)}$. Hence, for $\varphi \in L^2(\Sigma_\theta)^2$ we have $\operatorname{div}\varphi \in \hat{H}^{-1}(\Sigma_\theta)$ and by Lemma 3.1.31 there exists a unique $\Phi \in \hat{H}^1_0(\Sigma_\theta)$ solving (3.5). Then P_W is well-defined. Furthermore, for $\varphi \in L^2(\Sigma_\theta)^2$ we observe

$$P_W^2\varphi = P_W P_W\varphi = P_W(\varphi - \nabla\Phi) = \varphi - \nabla\Phi - \nabla\Psi$$

where Φ solves (3.5) and Ψ solves

$$\Delta\Psi = \operatorname{div}(\varphi - \nabla\Phi) = 0 \quad \text{in } \Sigma_\theta, \quad \Psi = 0 \quad \text{on } \Gamma.$$

Since (3.5) is uniquely solvable we infer $\Psi = 0$ and $P_W^2\varphi = \varphi - \nabla\Phi = P_W\varphi$. It is obvious that P_W is a linear map. Furthermore, P_W is bounded:

$$\begin{aligned}\|P_W\varphi\|_{L^2(\Sigma_\theta)} &= \|\varphi - \nabla\Phi\|_{L^2(\Sigma_\theta)} \leq \|\varphi\|_{L^2(\Sigma_\theta)} + C\|\operatorname{div}\varphi\|_{\hat{H}^{-1}(\Sigma_\theta)} \\ &\leq C\|\varphi\|_{L^2(\Sigma_\theta)}\end{aligned}$$

where we used the estimate from Lemma 3.1.31. Hence, P_W is a projection.
Since $L^2(\Sigma_\theta)^2$ is a Hilbert space it is a well-known fact that there exists a direct orthogonal decomposition

$$L^2(\Sigma_\theta)^2 = N(P_W) \oplus_\perp R(P_W).$$

where $N(P_W)$ and $R(P_W)$ denote the kernel and the range of P_W, respectively. We prove $N(P_W) = \nabla \hat{H}^1_0(\Sigma_\theta)$ and $R(P_W) = L^2_{\text{div}}(\Sigma_\theta)$ and the assertion then follows.
At first we prove $R(P_W) = L^2_{\text{div}}(\Sigma_\theta)$. If $\psi \in R(P_W)$ then there exists some $\varphi \in L^2(\Sigma_\theta)^2$ such that $P_W\varphi = \varphi - \nabla\Phi = \psi$ where Φ solves (3.5). But then we have

$$\operatorname{div}\psi = \operatorname{div}(\varphi - \nabla\Phi) = \operatorname{div}\varphi - \Delta\Phi = 0$$

and $\psi \in L^2_{\text{div}}(\Sigma_\theta)$. If $\varphi \in L^2_{\text{div}}(\Sigma_\theta)$ then we note that $P_W\varphi = \varphi - \nabla\Phi = \varphi$, where Φ solves (3.5) with right-hand side $\operatorname{div}\varphi = 0$. By the uniqueness it follows $\Phi = 0$ and $R(P_W) = L^2_{\text{div}}(\Sigma_\theta)$.
Next, we will show $N(P_W) = \nabla\hat{H}^1_0(\Sigma_\theta)$. Let $\varphi \in N(P_W)$, then we have $P_W\varphi = \varphi - \nabla\Phi = 0$, hence $\varphi = \nabla\Phi$. Since Φ solves (3.5) we know that $\Phi \in \hat{H}^1_0(\Sigma_\theta)$ by Lemma 3.1.31. Then we have $\varphi \in \nabla\hat{H}^1_0(\Sigma_\theta)$.
Now let $\varphi \in \hat{H}^1_0(\Sigma_\theta)$. Then

$$P_W(\nabla\varphi) = \nabla\varphi - \nabla\Phi,$$

where Φ is the solution of (3.5) with right-hand side $\operatorname{div}\nabla\varphi = \Delta\varphi$. Obviously, since $\nabla\varphi \in \nabla\hat{H}^1_0(\Sigma_\theta)$ we know that φ solves (3.5) with the same right-hand side. By the solution's uniqueness we obtain $\Phi = \varphi$ and $P_W(\nabla\varphi) = 0$. Then the assertion follows. □

3.1.16 Corollary (Weyl projection on $H^1(\Sigma_\theta)^2$)**.** *We define the Weyl map as*

$$P_W : H^1(\Sigma_\theta)^2 \to H^1(\Sigma_\theta)^2, \quad P_W\varphi := \varphi - \nabla\Phi$$

where Φ solves (3.5) *strongly. Then P_W is a projection and there exists a direct orthogonal decomposition*

$$H^1(\Sigma_\theta)^2 = H^1_{\text{div}}(\Sigma_\theta) \oplus \nabla(\hat{H}^1_0(\Sigma_\theta) \cap \hat{H}^2(\Sigma_\theta)).$$

Proof. This follows from the consistency of P_W. Due to the fact that in Lemma 3.1.31 the weak and strong solution of (3.5) are consistent, the Weyl projection P_W is also consistent on $L^2(\Sigma_\theta)^2$ and $H^1(\Sigma_\theta)^2$. Then the assertion follows by making use of $H^1(\Sigma_\theta) \overset{d}{\hookrightarrow} L^2(\Sigma_\theta)$ and

$$N(P_W|_{H^1(\Sigma_\theta)^2}) \subseteq N(P_W|_{L^2(\Sigma_\theta)^2}) \quad \text{and} \quad R(P_W|_{H^1(\Sigma_\theta)^2}) \subseteq R(P_W|_{L^2(\Sigma_\theta)^2}).$$

□

As a direct consequence of the consistency of the Weyl projection P_W due to Lemma 3.1.31 we also obtain the Weyl projection on $\hat{H}^1(\Sigma_\theta)^2$. We only replace $\hat{H}^1_0(\Sigma_\theta) \cap \hat{H}^2(\Sigma_\theta)$ by $\hat{H}^2_D(\Sigma_\theta)$ due to regularity reasons.

3.1.17 Corollary (Weyl projection on $\hat{H}^1(\Sigma_\theta)^2$). *We define the Weyl map as*

$$P_W : \hat{H}^1(\Sigma_\theta)^2 \to \hat{H}^1(\Sigma_\theta)^2, \qquad P_W\varphi := \varphi - \nabla\Phi,$$

where Φ is the strong solution of the Dirichlet problem (3.5). Then P_W is a projection and there exists a direct orthogonal decomposition

$$\hat{H}^1(\Sigma_\theta)^2 = \hat{H}^1_{\mathrm{div}}(\Sigma_\theta) \oplus \nabla\hat{H}^2_D(\Sigma_\theta),$$

where $\hat{H}^2_D(\Sigma_\theta) := \{u \in \hat{H}^2(\Sigma_\theta) : u|_\Gamma = 0\}$.

3.1.18 Remark. We put emphasize on the fact that the Weyl projections on $L^2(\Sigma_\theta), H^1(\Sigma_\theta)$ and $\hat{H}^1(\Sigma_\theta)$ from the previous Lemma 3.1.15, Corollary 3.1.16 and Corollary 3.1.17 are consistent by construction and by Lemma 3.1.31.

3.1.2 Trace Theorems

In this section we collect various trace theorems dealing with Dirichlet and Neumann traces. Using the bi-Lipschitz diffeomorphism from Lemma 3.1.2 we are able to transfer results from the half-space $\mathbb{R}^2_+$ to sectors Σ_θ. Note that trace theorems in the framework of inhomogeneous spaces are well-known from [34, Theorem 2] since Σ_θ is a convex domain.

3.1.19 Theorem. *The trace operator*

$$T : \hat{H}^1(\Sigma_\theta) \to \hat{H}^{1/2}(\Gamma)$$

exists and is linear and bounded and satisfies the estimate

$$\|Tu\|_{\hat{H}^{1/2}(\Gamma)} \leq C\|u\|_{\hat{H}^1(\Sigma_\theta)}$$

with a constant $C > 0$ independent of u. Furthermore, T is a retraction: There exists a bounded linear extension operator

$$\tilde{E} : \hat{H}^{1/2}(\Gamma) \to \hat{H}^1(\Sigma_\theta)$$

such that if $\tilde{u} \in \hat{H}^{1/2}(\Gamma)$ then $u := \tilde{E}\tilde{u} \in \hat{H}^1(\Sigma_\theta)$ with $Tu = \tilde{u}$ and

$$\|u\|_{\hat{H}^1(\Sigma_\theta)} \leq C\|\tilde{u}\|_{\hat{H}^{1/2}(\Gamma)}$$

where $C > 0$ is again independent of $\tilde{u}$.
Likewise, in the inhomogeneous case there also exists a trace operator T from $H^1(\Sigma_\theta)$ to $H^{1/2}(\Gamma)$ and a bounded extension operator $\tilde{E} : H^{1/2}(\Gamma) \to H^1(\Sigma_\theta)$ fulfilling corresponding estimates.

Proof. In order to prove the result, we want to transfer the trace operator T from the half-space $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ to our setting. [17, Theorem II.10.2] states the existence of a bounded trace operator $\tilde{T} : \dot{H}^1(\mathbb{R}^2_+) \to \dot{H}^{1/2}(\partial\mathbb{R}^2_+)$ with $\partial\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Hence, by definition of the spaces $\hat{H}^1$ we immediately also infer the boundedness of $\tilde{T} : \hat{H}^1(\mathbb{R}^2_+) \to \hat{H}^{1/2}(\partial\mathbb{R}^2_+)$. [17, Theorem II.10.2] also states the surjectivity of $\tilde{T}$. By functional analytic arguments we immediately infer that $\tilde{T}$ is a retraction since we are in the Hilbert space setting. We denote the coretraction by $\tilde{E} \in \mathscr{L}(\hat{H}^{1/2}(\partial\mathbb{R}^2_+), \hat{H}^1(\mathbb{R}^2_+))$.

Using the same arguments as in Remark 3.1.4 and Remark 3.1.6 we can extend the trace operator to

$$\tilde{T} \in \mathscr{L}(\hat{H}^1(\mathbb{R}^2_{>0}), \hat{H}^{1/2}(\mathbb{R})) \quad \text{and} \quad \tilde{E} \in \mathscr{L}(\hat{H}^{1/2}(\mathbb{R}), \hat{H}^1(\mathbb{R}^2_{>0}))$$

where $Tu := u|_{x_1=0}$.
We now want to transfer this result to our case. Let push-forward and pull-back Φ^0_*, Φ^*_0 and ζ^0_*, ζ^*_0 from Lemma 3.1.3 and Lemma 3.1.5 be given. For $v \in C^\infty_c(\overline{\Sigma^\delta_\theta})$ we observe that

$$(v \circ \varphi_0)(0, t) = v\,(\cos(\theta)|t|, \sin(\theta)t) = (v \circ \gamma_\delta)(t) \qquad (t \in \mathbb{R}).$$

Consequently,

$$(v \circ \varphi_0|_{\eta=0}) \circ \gamma_0^{-1} = v|_\Gamma$$

or equivalently

$$(\zeta^0_* \circ \tilde{T} \circ \Phi^*_0)v = v|_\Gamma.$$

Hence, we can define $T := \zeta^0_* \circ \tilde{T} \circ \Phi^*_0$. Furthermore, we define $E = \Phi^0_* \circ \tilde{E} \circ \zeta^*_0$ and by construction we have $TE = I_{\hat{H}^{1/2}(\Gamma)}$. Then E is also linear and bounded .

For the statement in the inhomogeneous case we replace the trace operator T and the extension operator E by the operators in the inhomogeneous case, cf. [17, Theorem II.4.3] with $\Omega = \mathbb{R}^2_{>0}$. □

3.1.20 Remark. Note that [17, Theorem II.10.2] actually states the existence of a bounded extension operator $E \in \mathscr{L}(\dot{H}^{1/2}(\partial\mathbb{R}^2_+), \dot{H}^1(\mathbb{R}^2_+))$ but the linearity of E is not stated. However, E is actually linear by construction. This can be observed by having a look at [26, Theorem 2.7, Corollary 1]. The proof of [26, Theorem 2.7] is similar to the proof of [26, Theorem 2.6]. If we consider the proof of [26, Theorem 2.6] in the case $r = 1$ then the corresponding extension u is defined as

$$E(\tilde{u})(x_1, ..., x_n) = u(x_1, ..., x_n) = \sum_{k=0}^{r-1} u_k(x_1, ..., x_n) = u_0(x_1, ..., x_n)$$

$$= \frac{1}{x_n^{n-1}} \int_{x_1}^{x_1+x_n} \dots \int_{x_{n-1}}^{x_{n-1}+x_n} \tilde{u}(t_1, \dots, t_{n-1})\, dt_1 \dots dt_{n-1}$$

where we used [26, Formula (2.46),(2.49)]. Then it is obvious that E is a linear operator, hence the trace T on the half-space is a retraction. This also holds in the non Hilbert space setting.

Next, we consider Neumann trace operators and prove the existence of the Neumann trace operator on $\hat{H}^2_D(\Sigma_\theta)$ of functions with vanishing Dirichlet trace. Furthermore, we prove that the Neumann trace operator has dense range. For this purpose, we consider the Dirichlet-Neumann trace pair taking the trace of $f \in \hat{H}^2(\Sigma_\theta)$ simultaneously. The main idea is to consider the trace in a neighborhood at the vortex (the critical point) and separately on $\Gamma_\pm$ as seen in e.g. [20]. Note that the Neumann trace operator on $\hat{H}^2(\Sigma_\theta)$ doesn't have to exist since by interpreting $\partial_n u = \nabla u \cdot n$ for $u \in \hat{H}^2(\Sigma_\theta)$ it is now clear if $\partial_n u \in \hat{H}^{1/2}(\Gamma)$ since multiplication with the normal vector field n is not continuous in general. This is caused by the fact that multiplication with sgn is not bounded in $H^{1/2}(\mathbb{R})$ (cf. [51, Section 2.10.2, Remark 1]). Hence, in Lemma 3.1.21 we will observe that the Dirichlet-Neumann trace pair maps into a rather unnatural space $\mathrm{DN}(\Gamma)$ where functions, which fulfill compatibility conditions, are contained. However, later we will observe that the Neumann trace of functions with vanishing Dirichlet trace, does indeed map into $\hat{H}^{1/2}(\Gamma)$.

3.1.21 Lemma. *The trace pair*

$$(T, T_n) \coloneqq (T, \partial_n) : \hat{H}^2(\Sigma_\theta) \to \mathrm{DN}(\Gamma)$$

is well-defined and continuous with

$$\mathrm{DN}(\Gamma) \coloneqq \left\{ (g_0,\, g_1) \in \widetilde{\mathrm{DN}}(\Gamma)/(\mathcal{P}_1 \times \mathcal{P}_0) : \|(\partial_\tau g_0)\tau + g_1 n\|_{\hat{H}^{1/2}(\Gamma) \times \hat{H}^{1/2}(\Gamma)} < \infty \right\}$$

where

$$\widetilde{\mathrm{DN}}(\Gamma) \coloneqq \left\{ (g_0, g_1) \in L^2_{loc}(\Gamma) \times L^2_{loc}(\Gamma) : \begin{array}{c} \|g_j|_{\Gamma_\pm}\|_{\dot{H}^{3/2-j}(\Gamma_\pm)} < \infty \text{ for } j = 0, 1, \\ g_0|_{\Gamma_+}(0) = g_0|_{\Gamma_-}(0) \end{array} \right\}.$$

Proof. Let $[u] \in \hat{H}^2(\Sigma_\theta)$ be fixed and $u \in \dot{H}^2(\Sigma_\theta)$ be any representative (u will be specified later on). At first we take care of the trace pair on $\Gamma_\pm = \Gamma \cap (\mathbb{R} \times \mathbb{R}_\pm)$. Due to the density there exists a sequence $(u_k)_{k\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^2)$ such that $u_k|_{\Sigma_\theta} \to u$ in $\dot{H}^2(\Sigma_\theta)$ as $k \to \infty$. Now, for every $k \in \mathbb{N}$ the trace pair $(Tu_k, T_n u_k) \in \dot{H}^{3/2}(\Lambda_+) \times \dot{H}^{1/2}(\Lambda_+)$ is well-defined on the line $\Lambda_+ \coloneqq \Gamma_+ \cup (-\Gamma_+) \cup \{0\}$ and depends linearly and continuously

on u_k (w.r.t. the topology on $\dot{H}^2(\mathbb{R}^2)$). Here, $\dot{H}^{3/2}(\Lambda_+)$ is defined as $\dot{H}^{3/2}(\mathbb{R})$ since Λ_+ is a rotation of $\mathbb{R}$.
Clearly, if $v, w \in \dot{H}^2(\mathbb{R}^2)$ such that $v|_{\Sigma_\theta} = w|_{\Sigma_\theta}$, then $(Tv, T_n v) = (Tw, T_n w)$ on Γ_+. Hence, the trace (T, T_n) is only dependent of the function inside Σ_θ. Then for every $k \in \mathbb{N}$ the trace pair $(Tu_k, T_n u_k) \in \dot{H}^{3/2}(\Gamma_+) \times \dot{H}^{1/2}(\Gamma_+)$ is well-defined and depends linearly and continuously on $u_k|_{\Sigma_\theta}$ (w.r.t. the topology on $\dot{H}^2(\Sigma_\theta)$).

This shows that the trace pair $(Tu, T_n u) \in \dot{H}^{3/2}(\Gamma_+) \times \dot{H}^{1/2}(\Gamma_+)$ is well-defined (as the limit of the trace pairs of the u_k) and depends linearly and continuously on u (w.r.t. the topology on $\dot{H}^2(\Sigma_\theta)$). Of course, the same observations are valid for the trace pair $(Tu, T_n u) \in \dot{H}^{3/2}(\Gamma_-) \times \dot{H}^{1/2}(\Gamma_-)$.

From [17, Remark II.6.1] we have $\dot{H}^2(\Sigma_\theta^1) = H^2(\Sigma_\theta^1)$ algebraically for $\Sigma_\theta^1 := \Sigma_\theta \cap B_1(0)$. Due to the Sobolev embedding (cf. [1, Theorem 4.12]) we can choose a continuous representative $u \in \hat{H}^2(\Sigma_\theta \cap B_1(0)) \subseteq BUC(\Sigma_\theta^1)$ which yields $Tu|_{\Gamma_+}(0) = Tu|_{\Gamma_-}(0)$. In summary, the trace pair

$$(T, T_n) : \hat{H}^2(\Sigma_\theta) \to \widetilde{\mathrm{DN}}(\Gamma)/(\mathcal{P}_1 \times \mathcal{P}_0)$$

is well-defined, linear and continuous.

We now detect the compatibility condition at the vortex point of Γ. We extend normal and tangential vector field to Σ_θ by extending constantly. Then $\tau, n \in L^\infty(\Sigma_\theta)$. Moreover, a straightforward calculation shows $\nabla u = (\tau \cdot \nabla u)\tau + (n \cdot \nabla u)n$ a.e. in Σ_θ, which implies that

$$(\partial_\tau u)\tau + (\partial_n u)n = T(\nabla u) \in \hat{H}^{1/2}(\Gamma) \times \hat{H}^{1/2}(\Gamma)$$

by Theorem 3.1.19, where $T : \hat{H}^1(\Sigma_\theta) \to \hat{H}^{1/2}(\Gamma)$ denotes the trace operator. Thus, if $(g_0, g_1) = (Tu, T_n u) \in \widetilde{\mathrm{DN}}(\Gamma)$, then we necessarily have

$$(\partial_\tau g_0)\tau + g_1 n \in \hat{H}^{1/2}(\Gamma) \times \hat{H}^{1/2}(\Gamma).$$

This can be seen as a compatibility condition at the vortex of Γ. Then the assertion follows. $\square$

As a consequence we obtain the existence of the Neumann trace operator (see Corollary 3.1.24). By making use of the density of $C_c^\infty(\Gamma\backslash\{0\})$ in $\hat{H}^{1/2}(\Gamma)$ from Lemma 3.1.9 we can actually prove that the Neumann trace operator has dense range. However, we prove this statement for $v_\pm \in W^{1-1/p,p}(\Gamma_\pm)$ for $1 < p < \infty$ with the restriction of fulfilling the compatibility condition $\lim_{t\to 0} v_\pm(t \cdot \tau_\pm) = 0$ for the trace at the vortex if $p > 2$. This is reasoned by the fact that in the proof of Lemma 3.1.22 we have to apply Hardy's inequality which is not valid in $p = 2$ which is the case we are interested in. However, the results then follows by interpolation.

3.1.22 Lemma. *Let* $1 < p < \infty$. *We set* $\Gamma_+ := \Gamma \cap (\mathbb{R} \times (0,\infty)) = (0,\infty) \cdot \tau_+$ *and* $\Gamma_- := \Gamma \cap (\mathbb{R} \times (-\infty,0)) = (-\infty,0) \cdot \tau_-$, *where*

$$\tau_\pm = (\pm\cos(\theta), \sin(\theta)), \qquad n_\pm = (-\sin(\theta), \pm\cos(\theta)).$$

Let $v_\pm \in W_p^{1-1/p}(\Gamma_\pm)$ *such that* $\lim_{t\to 0} v_\pm(t\cdot\tau_\pm) = 0$ *for the trace at the vortex if* $p > 2$ *and* $vn \in \dot{H}^{1/2}(\Gamma)^2$ *if* $p = 2$. *Then there exists some* $u \in W^{2,p}(\Sigma_\theta)$ *such that*

$$u = 0, \quad \partial_n u = v_\pm \qquad \text{on} \quad \Gamma_\pm.$$

Proof. Step 1. At first we consider the case $p \neq 2$. In this step we follow the ideas of the proof of [29, Proposition 4.1]. First we rotate Σ_θ anti-clockwise with θ such that $\Gamma_- = (-\infty, 0)\cdot\tau_-$ and $\Gamma_+ = (0,\infty)\cdot\tau_+$ where

$$\begin{aligned} \tau_+ &= (\cos(2\theta), \sin(2\theta)), & n_+ &= (-\sin(2\theta), \cos(2\theta)), \\ \tau_- &= -e_1, & n_- &= -e_2. \end{aligned}$$

Note that by rotation we still have $v_\pm \in W^{1-1/p,p}(\Gamma_\pm)$.
Let $\tilde{G} = (0,\infty)^2$ be the wedge with opening angle $\pi/2$. For the boundary we define $\tilde{\Gamma}_+ := \{0\}\times(0,\infty)$ and $\tilde{\Gamma}_- := (0,\infty)\times\{0\}$. Furthermore, we set $\rho := |x| = \sqrt{x_1^2 + x_2^2}$ for $x \in \mathbb{R}^2$. Defining the transformation

$$\Psi : \Sigma_\theta \to \tilde{G}, \quad \Psi(x_1,x_2) = \left(\rho\cos\left(\frac{\pi}{4\theta}\arccos\left(\frac{x_1}{\rho}\right)\right), \rho\sin\left(\frac{\pi}{4\theta}\arccos\left(\frac{x_1}{\rho}\right)\right)\right),$$

it is straightforward to verify that Ψ is well-defined as well as the fact that Ψ is a C^∞-diffeomorphism. Next, we set $\tilde{v}_- := v_-$ and $\tilde{v}_+(t\cdot e_2) := v_+(t\cdot\tau_+)$ for $t > 0$ such that

$$\tilde{v}_\pm \in W_p^{1-1/p}(\tilde{\Gamma}_\pm) \quad \text{with} \quad \lim_{t\to 0}\tilde{v}_+(t\cdot e_2) = \lim_{t\to 0}\tilde{v}_-(t\cdot e_1) = 0 \quad (p > 2) \tag{3.6}$$

Then we are in position to apply [4, Theorem VIII.1.8.5] on the corner $\tilde{G}$ and [4, Theorem VIII.1.8.5] implies the existence of $\tilde{u} \in W^{2,p}(\tilde{G})$ with $\tilde{u} = 0$ on $\tilde{\Gamma}_\pm$ and $\partial_n\tilde{u} = \tilde{v}_\pm$ on $\tilde{\Gamma}_\pm$. Then we set $u := \tilde{u}\circ\Psi$ and by construction u satisfies the desired boundary conditions. Furthermore, we have $u \in W^{2,p}(\Sigma_\theta)$ by considering the following: we note that $\partial_{x_j}\Psi \sim \rho$ and $\partial_{x_j}\partial_{x_k}\Psi \sim \rho^{-1}$ for $\rho \to 0$ and $\rho \to \infty$ and $j,k = 1,2$. Then we have $\partial_{x_j}\Psi^n, \rho\partial_{x_j}\partial_{x_k}\Psi^n \in L^\infty(\Sigma_\theta)$ for $j,k,n = 1,2$. Regarding the derivatives of $u = \tilde{u}\circ\Phi$ we obtain (note that $\det\nabla\Psi \equiv \pi/4\theta$) for $j,k = 1,2$

$$\begin{aligned} \|\partial_{x_k}(\tilde{u}\circ\Psi)\|_{L^p(\Sigma_\theta)} &= \|(\partial_{x_1}\tilde{u}\circ\Psi)\partial_{x_k}\Psi^1 + (\partial_{x_2}\tilde{u}\circ\Psi)\partial_{x_k}\Psi^2\|_{L^p(\Sigma_\theta)} \\ &\le \|\partial_{x_1}\tilde{u}\circ\Psi\|_{L^p(\Sigma_\theta)} + \|\partial_{x_2}\tilde{u}\circ\Psi\|_{L^p(\Sigma_\theta)} \le C\|\nabla\tilde{u}\|_{L^p(\tilde{G})} < \infty \end{aligned}$$

and

$$\begin{aligned}
&\|\partial_{x_j}\partial_{x_k}(\tilde{u}\circ\Psi)\|_{L^p(\Sigma_\theta)} \\
&= \left\| \sum_{m,n=1}^{2} ((\partial_{x_m}\partial_{x_n}\tilde{u})\circ\Psi)\partial_{x_j}\Psi^m\partial_{x_k}\Psi^n + \sum_{n=1}^{2}(\partial_{x_n}\tilde{u}\circ\Psi)\partial_{x_j}\partial_{x_k}\Psi^n \right\|_{L^p(\Sigma_\theta)} \\
&\leq \sum_{m,n=1}^{2} \|(\partial_{x_m}\partial_{x_n}\tilde{u})\circ\Psi\|_{L^p(\Sigma_\theta)} + \sum_{n=1}^{2} \|\rho^{-1}\partial_{x_n}\tilde{u}\circ\Psi\|_{L^p(\Sigma_\theta)} \leq C\|\nabla^2\tilde{u}\|_{L^p(\tilde{G})},
\end{aligned}$$

where we applied Hardy's inequality in the version of [29, Lemma A.2] in the last step. Note that in order to apply Hardy's inequality we need to fulfill the boundary condition $\lim_{t\to 0}\partial_n\tilde{u}(t\cdot e_j) = 0$ for $j = 1,2$ (if $p > 2$) which is fulfilled due to (3.6). Thus, then we have $u = \tilde{u}\circ\Psi \in W^{2,p}(\Sigma_\theta)$. At last we rotate Σ_θ clockwise with θ to obtain our original sector Σ_θ.
This, in fact, proves that the operator T_p extending $v_\pm \in W_p^{1-1/p}(\Gamma_\pm)$ to some $u \in W^{2,p}(\Sigma_\theta)$ with $u = 0$ and $\partial_n u = v_\pm$ on $\Gamma_\pm$ is bounded since the coretraction from [4, Theorem VIII.1.8.5] is bounded.

Step 2. In order to obtain $p = 2$ we apply an interpolation argument. Note that in Step 1, [4, Theorem VIII.1.8.5] yields a coretraction which is universal such that the operators T_p for $p \neq 2$ are consistent. Hence, interpolation of

$$\begin{aligned}
&T_p : D(T_p) := \left\{ v \in W_p^{1-1/p}(\Gamma) : \lim_{t\to 0} v(t\cdot\tau) = 0 \right\} \to W^{2,p}(\Sigma_\theta) && (p > 2), \\
&T_p : D(T_p) := W_p^{1-1/p}(\Gamma) \to W^{2,p}(\Sigma_\theta) && (p < 2)
\end{aligned}$$

yields the assertion for $p = 2$ and

$$D(T_2) = \left\{ v \in H^{1/2}(\Gamma) : vn \in \dot{H}^{1/2}(\Gamma)^2 \right\}.$$

□

3.1.23 Remark. (i) We are only able to obtain the result in Lemma 3.1.22 since by demanding $u = 0$ on $\Gamma_\pm$ as a boundary condition we don't have any other compatibility conditions as in [29, Proposition 4.1] and for every $p > 2$ the operator T_p has the same domain

$$D(T_p) = \left\{ v \in W_p^{1-1/p}(\Gamma) : \lim_{t\to 0} v(t\cdot\tau) = 0 \right\}$$

and $D(T_p) = W_p^{1-1/p}(\Gamma)$ for $p < 2$.

(ii) In the second step of Lemma 3.1.22 we can interpolate

$$(D(T_{3/2}), D(T_3))_{1/2,2} = \{v \in H^{1/2}(\Gamma) : vn \in \dot{H}^{1/2}(\Gamma)^2\},$$

which leads to Lions-Magenes type spaces. These occur since we interpolate spaces with boundary conditions (in our case $D(T_3)$) with spaces with no boundary conditions (in our case $D(T_{3/2})$). For an introduction to the Lions-Magenes spaces we refer to [46, Chapter 33] and [51, Section 2.10].

Summarizing the results from above we obtain the following result on the Neumann trace operator:

3.1.24 Corollary. *Let $\hat{H}^2_D(\Sigma_\theta) := \{u \in \hat{H}^2(\Sigma_\theta) : u|_\Gamma = 0\}$. The Neumann trace operator $T_n : \hat{H}^2_D(\Sigma_\theta) \to \hat{H}^{1/2}(\Gamma)$ is bounded and has dense range.*

Proof. The assertion essentially follows from Lemma 3.1.21 and Lemma 3.1.22. If $u \in \hat{H}^2_D(\Sigma_\theta)$, then $(Tu, T_n u) = (0, g_1) \in \mathrm{DN}(\Gamma)$ such that

$$g_1 n \in \hat{H}^{1/2}(\Gamma) \times \hat{H}^{1/2}(\Gamma).$$

Since $n(x) = (-\sin(\theta), \mathrm{sgn}(x_2)\cos(\theta))$ we infer $-\sin(\theta) g_1 \in \hat{H}^{1/2}(\Gamma)$. Furthermore, by the density of $C_c^\infty(\Gamma \backslash \{0\})$ in $\hat{H}^{1/2}(\Gamma)$ by Lemma 3.1.9 and the fact that functions in $C_c^\infty(\Gamma \backslash \{0\})$ fulfill the assumptions of Lemma 3.1.22, for every $g \in C_c^\infty(\Gamma \backslash \{0\})$ we can find $u \in H^2(\Sigma_\theta)$ with $u|_\Gamma = 0$. Thus, $u \in \hat{H}^2_D(\Sigma_\theta)$ and T_n has dense range. □

Next, we consider the normal trace operator. Note that the normal vector field at Γ is given as $n = (-\sin(\theta), \mathrm{sgn}(x_2)\cos(\theta))$ which shows that by taking the normal trace we have to multiply with sgn. Since multiplication with sgn is not a bounded operator in $H^{1/2}(\mathbb{R})$ (cf. [51, Section 2.10.2, Remark 1]) we cannot expect the normal trace operator $T_0 : H^1(\Sigma_\theta)^2 \to H^{1/2}(\Gamma)$ to be well-defined. In Lemma 3.2.8 we will observe that the normal trace operator on Γ actually exists if the correct symmetry is given. However, the coretraction exists even if we don't assume any symmetry properties. In the following we will construct such a coretraction. The strategy will be as follows: We will divide Σ_θ into a bounded Lipschitz domain (containing the vortex $(0,0)$ of Σ_θ) and a smooth sector-like domain. Then for given $h \in H^{1/2}(\Gamma) \cap \hat{H}^{-1/2}(\Gamma)$ we will construct a function v on the bounded Lipschitz domain fulfilling the boundary condition on the corresponding boundary via a Stokes system. On the unbounded domain we will prove the existence of such a function w fulfilling the boundary conditions by solving a divergence equation. The solvability of the divergence equation is stated at first:

3.1.25 Lemma. *Let $H_\omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > \omega(x_1)\}$ be the bent half-space with $\omega \in C^\infty(\mathbb{R}, \mathbb{R})$, where ω is linear in $\mathbb{R} \backslash B_r(0)$ for some $r > 0$. Then for every $h \in H^{1/2}(\partial H_\omega)^2 \cap \hat{H}^{-1/2}(\partial H_\omega)^2$ there exists a unique $u \in H^1(H_\omega)^2$ solving*

$$\operatorname{div} u = 0 \quad \text{in } H_\omega, \qquad u = h \quad \text{on } \partial H_\omega.$$

and fulfilling the estimate

$$\|u\|_{H^1(H_\omega)} \leq C\|h\|_{H^{1/2}(\partial H_\omega)\cap \hat{H}^{-1/2}(\partial H_\omega)}$$

with $C > 0$ independent of h.

Proof. We prove the assertion by solving the Stokes equation

$$\begin{aligned} (1-\Delta)v - \nabla q &= 0 \quad \text{in } \mathbb{R}^2_+, \\ \operatorname{div} v &= 0 \quad \text{in } \mathbb{R}^2_+, \\ v &= \tilde{h} \quad \text{on } \partial\mathbb{R}^2_+ \end{aligned} \tag{3.7}$$

on the half-space and then transform the solution to H_ω. The solvability and corresponding estimates in the half-space can be based on explicit solution formulas as displayed e.g. in [24, Section 2.6]. Hence, we can reduce the proof to considering a suitable transformation.

We define the transformation

$$\varphi : \overline{H_\omega} \to \overline{\mathbb{R}^2_+}, \qquad (x_1, x_2) \mapsto (x_1, x_2 - \omega(x_1)),$$

which is clearly a C^∞-diffeomorphism due to the fact that $\omega \in C^\infty(\mathbb{R}, \mathbb{R})$ and $\omega' \in BC^\infty(\mathbb{R}, \mathbb{R})$. Note that

$$\varphi'(x) = \begin{pmatrix} 1 & 0 \\ -\omega'(x_1) & 1 \end{pmatrix} \quad \text{and} \quad \det \varphi'(x) = 1 \qquad (x \in \overline{H_\omega}).$$

We set $\Phi(u) := u \circ \varphi^{-1}$ for $u : H_\omega \to \mathbb{R}^2$. By the fact that ω is linear in $\mathbb{R}\backslash B_r(0)$ for some $r > 0$, we deduce that ω' is constant in $\mathbb{R}\backslash\overline{B_r(0)}$ and that $\operatorname{supp}(\omega^{(k)}) \subseteq B_r(0)$ for $k \geq 2$. Then we immediately infer

$$\Phi \in \mathscr{L}_{is}(W^{s,p}(H_\omega)^2, W^{s,p}(\mathbb{R}^2_+)^2)$$

for all $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ which follows as in Lemma 3.1.3. Here, we can transform derivatives of higher order since the derivatives of φ are bounded. Furthermore, we observe that

$$\varphi' \circ \varphi^{-1} = \varphi' = \begin{pmatrix} 1 & 0 \\ -\omega' & 1 \end{pmatrix} \in BC^\infty(\mathbb{R}^2_+, \mathbb{R}^{2\times 2}).$$

Then for $u : H_\omega \to \mathbb{R}^2$ we set $\Psi u := \varphi' \Phi u$. Then the divergence transforms as follows:

$$\begin{aligned} \operatorname{div}_{\mathbb{R}^2_+} \Psi u &= \partial_{x_1}(\Phi u)^1 + \partial_{x_2}[-\omega'(\Phi u)^1 + (\Phi u)^2] \\ &= (\partial_{x_1} u^1) \circ \varphi^{-1} + [(\partial_{x_2} u^1) \circ \varphi^{-1}]\omega' - [(\partial_{x_2} u^1) \circ \varphi^{-1}]\omega' + (\partial_{x_2} u^2) \circ (\varphi^{-1}) \end{aligned}$$

$$= (\operatorname{div}_{H_\omega} u) \circ \varphi^{-1}$$

and $\Psi \in \mathscr{L}_{is}(W^{s,p}(H_\omega)^2, W^{s,p}(\mathbb{R}^2_+)^2)$ for all $s \in \mathbb{R}$ and $1 \le p \le \infty$ as well. Note that obviously $\Psi \in \mathscr{L}_{is}(W^{s,p}(\partial H_\omega)^2, W^{s,p}(\partial\mathbb{R}^2_+)^2) \cap \mathscr{L}_{is}(\hat{H}^k(\partial H_\omega)^2, \hat{H}^k(\partial\mathbb{R}^2_+)^2)$ as well for $k \in [-1, 1]$.

Let $h \in H^{1/2}(\partial H_\omega)^2 \cap \hat{H}^{-1/2}(\partial H_\omega)^2$. Then $\tilde{h} := \Psi h \in H^{1/2}(\partial\mathbb{R}^2_+)^2 \cap \hat{H}^{-1/2}(\partial\mathbb{R}^2_+)^2$. Then there exists $v \in H^1_{\operatorname{div}}(\mathbb{R}^2_+)$ solving (3.7) in $\mathbb{R}^2_+$ with $v = \tilde{h}$ on $\partial\mathbb{R}^2_+$. We set $u := \Psi^{-1} v \in H^1(H_\omega)^2$ and thanks to the observation above we also obtain $\operatorname{div}_{H_\omega} u = (\operatorname{div}_{\mathbb{R}^2_+} v) \circ \varphi = 0$. Furthermore, $u = h$ on ∂H_ω by construction and

$$\|u\|_{H^1(H_\omega)} \le C\|v\|_{H^1(\mathbb{R}^2_+)} \le C\|\tilde{h}\|_{H^{1/2}(\partial\mathbb{R}^2_+)\cap\hat{H}^{-1/2}(\partial\mathbb{R}^2_+)} \le C\|h\|_{H^{1/2}(\partial H_\omega)\cap\hat{H}^{-1/2}(\partial H_\omega)}$$

with $C > 0$ independent of h. □

3.1.26 Remark. We note that the transformation Ψ from Lemma 3.1.25 preserves the normal trace. Hence, using Ψ it is possible to transfer normal and tangential trace from the half-space. This can be observed by

$$n_{\partial H_\omega} \cdot u = (n_{\partial\mathbb{R}^2_+} \cdot v) \circ \varphi = (n_{\partial\mathbb{R}^2_+} \cdot \tilde{h}) \circ \varphi = n_{\partial H_\omega} \cdot h.$$

3.1.27 Lemma. *There exists a linear and bounded operator*

$$R_0 : H^{1/2}(\Gamma) \cap \hat{H}^{-1/2}(\Gamma) \to H^1_{\operatorname{div}}(\Sigma_\theta)$$

such that

$$(R_0 g \cdot n)|_\Gamma = g \quad \text{for all } g \in H^{1/2}(\Gamma) \cap \hat{H}^{-1/2}(\Gamma).$$

Proof. We will prove the assertion in several steps. To this end, we consider the bounded Lipschitz domain $G := \Sigma_\theta \cap B_4(0)$ and the smooth sector-like domain Ω by smoothing out the vortex (cf. Figure 3). We will make use of the Stokes equation in G and the divergence equation in Ω in order to construct functions v, w in G and Ω which fulfill the given boundary conditions, i.e., $u \cdot n = g$ on Γ for $u = v + w \in H^1_{\operatorname{div}}(\Sigma_\theta)$ and given $g \in H^{1/2}(\Gamma) \cap \hat{H}^{-1/2}(\Gamma)$.

Let $g \in C_c^\infty(\Gamma\backslash\{0\})$ be arbitrary but fixed. We choose $\chi \in C_c^\infty(\Gamma)$ such that $\chi \equiv 1$ on $\Gamma \cap \overline{B_{7/3}(0)}$ and $\chi \equiv 0$ on $\Gamma\backslash B_{8/3}(0)$. We note that $\operatorname{supp}(\chi g) \in (\Gamma \cap B_3(0))\backslash\{0\}$ and $\operatorname{supp}((1-\chi)g) \subseteq \Gamma\backslash\overline{B_2(0)}$.

In the following we denote the extension operator, which extends functions by zero to a set $U \subseteq \mathbb{R}^2$, as E_0^U.

Step 1. We construct a linear operator $M : F \to F$ where $F := \{h \in C_c^\infty(\Gamma \cap B_3(0)) : 0 \notin \operatorname{supp}(h)\}$. The operator M will have the following properties: it holds

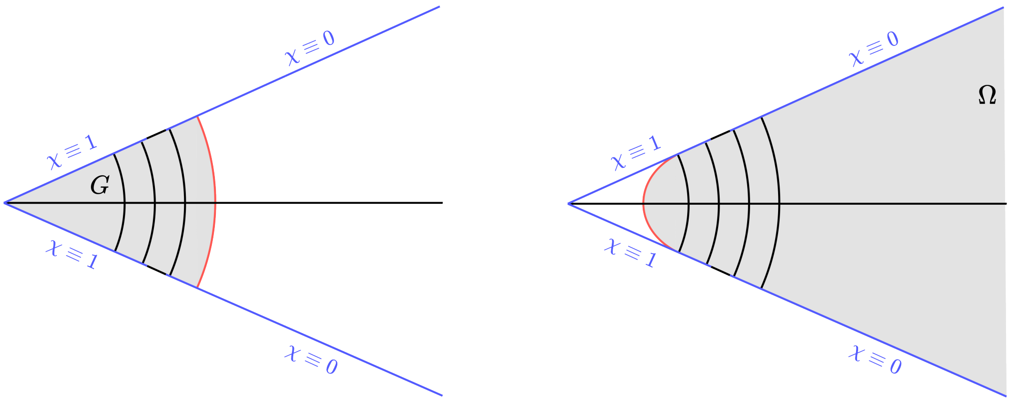

Figure 3: Domains G and Ω

$\operatorname{supp}(Mh) \subseteq \Gamma \cap (B_3(0)\backslash\overline{B_2(0)})$, the extended function $E_0^{\partial G}(h + Mh)$ is mean value free on ∂G and

$$\|Mh\|_{H^{1/2}(\Gamma\cap B_3(0))\cap \hat{H}^{-1/2}(\Gamma\cap B_3(0))} \leq C\|h\|_{H^{1/2}(\Gamma\cap B_3(0))\cap \hat{H}^{-1/2}(\Gamma\cap B_3(0))} \tag{3.8}$$

for $h \in F$. We set

$$Mh(t\tau^{\pm}) := \begin{cases} -3h((3t-6)\tau^{\pm}), & 2 \leq |t| < 3, \\ 0, & 0 \leq |t| < 2, \end{cases}$$

where $t\tau^+ = t(\cos(\theta), \sin(\theta))$ if $t > 0$ and $t\tau^- = t(-\cos(\theta), \sin(\theta))$ if $t < 0$. Then we can calculate

$$\begin{aligned} \int_{\partial G} E_0^{\partial G}(h + Mh)\, d\eta &= \int_{\Gamma\cap B_3(0)} (h + Mh)\, d\eta \\ &= \int_{0<|t|<3} h(t\tau^{\pm})\, dt - 3\int_{2<|t|<3} h((3t-6)\tau^{\pm})\, dt \\ &= 0. \end{aligned}$$

Furthermore, it is straight forward to prove that (3.8) holds for all $h \in F$ with a constant $C > 0$ independent of h by making use of the Slobodeckij seminorm. A density argument then yields the assertion.

Step 2. First we consider the bounded Lipschitz domain G. We set $g_1 := E_0^{\partial G}(\chi g + M(\chi g)) \in H^{1/2}(\partial G) \cap \hat{H}^{-1/2}(\partial G)$. We consider the Stokes equations on G:

$$\begin{aligned} -\Delta\tilde{v} - \nabla\tilde{p} &= 0 && \text{in } G, \\ \operatorname{div}\tilde{v} &= 0 && \text{in } G, \\ \tilde{v} &= g_1 n_{\partial G} && \text{on } \partial G, \end{aligned}$$

where $n_{\partial G}$ denotes the normal vector field at ∂G. Note that since $\chi g \in C_c^\infty((\Gamma \cap B_3(0))\backslash\{0\})$ by construction, multiplication with $n_{\partial G}$ is well-defined (no singularities at the vortices occur) and yields $g_1 n_{\partial G} \in H^{1/2}(\partial G)^2$. By [17, Theorem IV.1.1] there exists a unique weak solution $\tilde{v} \in H^1_{\mathrm{div}}(G)$ to the Stokes system if

$$\int_{\partial G} g_1 n_{\partial G} \cdot n_{\partial G} \, d\eta = \int_{\partial G} E_0^{\partial G}(\chi g + M(\chi g)) \, d\eta = 0.$$

Furthermore, $\tilde{v}$ satisfies the estimate

$$\|\tilde{v}\|_{H^1(G)} \leq C\|g_1 n_{\partial G}\|_{H^{1/2}(\partial G)} \leq C\|g_1\|_{H^{1/2}(\partial G)} \leq C\|g\|_{H^{1/2}(\Gamma)\cap\hat{H}^{-1/2}(\Gamma)}. \tag{3.9}$$

Since $\tilde{v} = g_1 n_{\partial G} = 0$ on $\Sigma_\theta \cap \partial G$ by construction, we infer $v := E_0^{\Sigma_\theta}\tilde{v} \in H^1_{\mathrm{div}}(\Sigma_\theta)$.

Step 3. Now we consider the unbounded smooth sector-like domain Ω. We set $g_2 := E_0^{\partial\Omega}((1-\chi)g - M(\chi g)) \in H^{1/2}(\partial\Omega) \cap \hat{H}^{-1/2}(\partial\Omega)$ and $h := g_2 n_{\partial\Omega} \in H^{1/2}(\partial\Omega)^2 \cap \hat{H}^{-1/2}(\partial\Omega)$ where $n_{\partial\Omega}$ denotes the normal vector field at $\partial\Omega$. Since Ω is smooth, $n_{\partial\Omega}$ is smooth as well (cf. Lemma 3.1.1) such that $g_2 n_{\partial\Omega} \in H^{1/2}(\partial\Omega)^2 \cap \hat{H}^{-1/2}(\partial\Omega)^2$. Due to Lemma 3.1.25 there exists a $\tilde{w} \in H^1_{\mathrm{div}}(\Omega)$ such that $\tilde{w} = g_2 n_{\partial\Omega} = 0$ on $\partial\Omega \cap \overline{B_1(0)}$ by construction. Furthermore, $\tilde{w}$ satisfies $\tilde{w} \cdot n_{\partial\Omega} = g_2$ by construction and

$$\begin{aligned} \|\tilde{w}\|_{H^1(\Omega)} &\leq C\|g_2 n_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)\cap\hat{H}^{-1/2}(\partial\Omega)} \leq C\|g_2\|_{H^{1/2}(\partial\Omega)\cap\hat{H}^{-1/2}(\partial\Omega)} \\ &\leq C\|g\|_{H^{1/2}(\Gamma)\cap\hat{H}^{-1/2}(\Gamma)} \end{aligned} \tag{3.10}$$

since $\mathrm{supp}(g_2) \subseteq \Gamma\backslash\overline{B_2(0)}$. Then we obtain $w := E_0^{\Sigma_\theta}\tilde{w} \in H^1_{\mathrm{div}}(\Sigma_\theta)$.

Step 4. We set $R_0 g := u := v + w \in H^1_{\mathrm{div}}(\Sigma_\theta)$. Furthermore, we have

$$n \cdot u = n \cdot v + n \cdot w = \chi g + M(\chi g) + (1-\chi)g - M(\chi g) = g \qquad \text{on } \Gamma$$

as well as $\|u\|_{H^1(\Sigma_\theta)} \leq C\|g\|_{H^{1/2}(\Gamma)\cap\hat{H}^{-1/2}(\Gamma)}$ by (3.9) and (3.10). Since $C_c^\infty(\Gamma\backslash\{0\})$ is dense in $H^{1/2}(\Gamma) \cap \hat{H}^{-1/2}(\Gamma)$ by Lemma 3.1.9 and Lemma 3.1.11, a density argument yields the assertion. □

At last we prove the existence of the generalized trace by following the ideas from [6]. From the discussion ahead of Lemma 3.1.27 one would expect that the generalized trace should also not exist. Actually, due to the lack of regularity and by construction via the generalized principle of integration, it turns out that the generalized trace $T_n : L^2_{\mathrm{div}}(\Sigma_\theta) \to \hat{H}^{-1/2}(\Gamma)$ does exist.

3.1.28 Lemma (Generalized trace theorem). *Let*

$$T_0 : L^2_{\mathrm{div}}(\Sigma_\theta) \to \hat{H}^{-1/2}(\Gamma)$$

be defined by

$$T_0 v(\psi) = \langle n \cdot v, \psi \rangle_{\hat{H}^{-1/2}(\Gamma), \hat{H}^{1/2}(\Gamma)} := \langle v, \nabla E\psi \rangle_{L^2(\Sigma_\theta), L^2(\Sigma_\theta)},$$

for $\psi \in \hat{H}^{1/2}(\Gamma)$ *where* $E : \hat{H}^{1/2}(\Gamma) \to \hat{H}^1(\Sigma_\theta)$ *is the linear and bounded extension operator to the trace operator* $T : \hat{H}^1(\Sigma_\theta) \to \hat{H}^{1/2}(\Gamma)$*, characterized by the inhomogeneous Dirichlet problem:*

$$\Delta E\psi = 0 \quad \text{in } \Sigma_\theta, \qquad E\psi = \varphi \quad \text{on } \Gamma,$$

see Corollary 3.1.33. Then T_0 *is well-defined (especially independent of the choice of the extension operator* E*) and bounded.*

Proof. The proof follows the proof of [6, Proposition 3.4]. Let $\varphi \in \hat{H}^1(\Sigma_\theta)$ be arbitrary. Then $T\varphi = \varphi|_\Gamma \in \hat{H}^{1/2}(\Gamma)$ exists by Theorem 3.1.19 and we have $E\varphi - \varphi \in \hat{H}^1_0(\Sigma_\theta)$ since $E\varphi \in \hat{H}^1(\Sigma_\theta)$ and by construction we have $(E\varphi - \varphi)|_\Gamma = 0$ (the trace exists). Then also $\nabla(E\varphi - \varphi) = \nabla E\varphi - \nabla\varphi \in \nabla\hat{H}^1_0(\Sigma_\theta)$. From Lemma 3.1.15 we know that the Weyl projection P_W projects along $\nabla\hat{H}^1_0(\Sigma_\theta)$ such that

$$P_W(\nabla E\varphi - \nabla\varphi) = 0 \qquad \Rightarrow \quad P_W(\nabla E\varphi) = P_W(\nabla\varphi).$$

Making use of the fact that $v \in L^2_{\text{div}}(\Sigma_\theta) = R(P_W)$ (note that P_W is a symmetric operator) we obtain

$$\begin{aligned}
&\langle v, \nabla E\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)} \\
&\quad = \langle P_W v, \nabla E\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)} = \langle v, P_W \nabla E\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)} \\
&\quad = \langle v, P_W \nabla\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)} = \langle P_W v, \nabla\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)} \\
&\quad = \langle v, \nabla\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)}
\end{aligned}$$

which gives us the generalized version of integration by parts

$$\langle T_0 v, T\varphi\rangle_{\hat{H}^{-1/2}(\Gamma),\hat{H}^{1/2}(\Gamma)} = \langle v, \nabla\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)}$$

for $\varphi \in \hat{H}^1(\Sigma_\theta)$ and $v \in L^2_{\text{div}}(\Sigma_\theta)$. In particular we have

$$\langle T_0 v, \psi\rangle_{\hat{H}^{-1/2}(\Gamma),\hat{H}^{1/2}(\Gamma)} = \langle v, \nabla\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)}$$

for any $\varphi \in \hat{H}^1(\Sigma_\theta)$ that fulfills $T\varphi = \psi$ such that the definition of T_0 is independent of the extension operator E, hence T_0 is well-defined. By making use of the boundedness of the extension operator E from Theorem 3.1.19 we obtain the boundedness of T_0:

$$\begin{aligned}
\|T_0 v\|_{\hat{H}^{-1/2}(\Gamma)} &= \sup_{\substack{\psi \in \hat{H}^{1/2}(\Gamma) \\ \|\psi\|_{\hat{H}^{1/2}(\Gamma)} = 1}} |\langle T_0 v, \psi\rangle_{\hat{H}^{-1/2}(\Gamma),\hat{H}^{1/2}(\Gamma)}| \\
&= \sup_{\substack{\psi \in \hat{H}^{1/2}(\Gamma) \\ \|\psi\|_{\hat{H}^{1/2}(\Gamma)} = 1}} |\langle v, \nabla E\psi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)}|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\substack{\psi\in\hat{H}^{1/2}(\Gamma)\\ \|\psi\|_{\hat{H}^{1/2}(\Gamma)}=1}} \|E\psi\|_{\hat{H}^1(\Sigma_\theta)}\|v\|_{L^2(\Sigma_\theta)}\\
&\leq C \sup_{\substack{\psi\in\hat{H}^{1/2}(\Gamma)\\ \|\psi\|_{\hat{H}^{1/2}(\Gamma)}=1}} \|\psi\|_{\hat{H}^{1/2}(\Gamma)}\|v\|_{L^2(\Sigma_\theta)}\\
&\leq C\|v\|_{L^2(\Sigma_\theta)}
\end{aligned}
$$

for arbitrary $v \in L^2_{\mathrm{div}}(\Sigma_\theta)$. □

3.1.3 Elliptic Problems

The study of elliptic problems is as usual of great interest. Elliptic problems were already considered in various kinds of settings and domains. However, the closest results we could find for elliptic problems on sectors, were found in [20] where the setting in inhomogeneous spaces and bounded non-smooth domains were considered.

In this section we collect results in the framework of homogeneous spaces. We will mainly focus on the Dirichlet problem in the strong, weak and very weak setting. The strategy to solve the strong problem is as follows: It will be necessary to approximate Σ_θ with smooth sector-like domains Σ_θ^δ where results are known. Then passing to the limit we obtain the results for Σ_θ. We start with a result derived in [20]:

3.1.29 Lemma (Theorem 3.1.1.1 in [20])**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a C^2-domain. Then the equality*

$$
\begin{aligned}
&\int_\Omega |\operatorname{div} v|^2 - \sum_{i,j=1}^{2}\int_\Omega \partial_i v^i \partial_j v^j \, dx\\
&= -2\langle \partial_\tau(n\cdot v), (\tau\cdot v)\rangle_{\hat{H}^{-1/2}(\partial\Omega),\hat{H}^{1/2}(\partial\Omega)}\\
&\quad - \int_{\partial\Omega\setminus\{0\}} [\mathscr{B}(t\cdot v, t\cdot v) + tr\mathscr{B}(n\cdot v)^2]\, d\eta
\end{aligned}
\tag{3.11}
$$

holds for every $v \in C_c^\infty(\overline{\Omega})$. Here, $\mathscr{B}$ denotes the second fundamental quadratic form corresponding to the boundary of the underlying domain ($tr\mathscr{B}$ denotes its trace), see [20, Section 3.1.1].

3.1.30 Remark. We remark that in [20, Theorem 3.1.1.1] Ω is assumed to be bounded. However, following the lines of the proof it is easily checked that the boundedness assumption can be dropped. In fact, the calculations in the proof of [20, Theorem 3.1.1.1] work verbatim for all $v \in C_c^\infty(\overline{\Omega})$ since then the existence of all appearing integrals are given.

The strategy to derive strong solvability of related Dirichlet problems on Σ_θ is now as follows where we adapt the main ideas from [20, Section 3.1]: As already mentioned we approximate Σ_θ by the smoothed out convex sector-like domains Σ_θ^δ. Since Σ_θ^δ is of class C^∞ we will then obtain smooth solutions for the corresponding elliptic problem with estimates in $\hat{H}^2(\Sigma_\theta^\delta)$ that are uniform in $\delta > 0$ thanks to Lemma 3.1.29. Note that in (3.11) the boundary terms $\mathscr{B}$ drop out due to the convexity of Σ_θ^δ. Then passing $\delta \to 0$ then yields $\hat{H}^2$ regularity for the considered elliptic problems on Σ_θ.

We now consider the Dirichlet problem with homogeneous boundary conditions which is formulated as follows:

$$\begin{aligned} -\Delta p &= f \quad \text{in } \Sigma_\theta, \\ p &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.12}$$

3.1.31 Lemma (Strong and weak homogeneous Dirichlet problem). *We assume $f \in \hat{H}^{-1}(\Sigma_\theta)$. Then there exists a unique solution $p \in \hat{H}_0^1(\Sigma_\theta)$ of (3.12) in the weak sense satisfying*

$$\|\nabla p\|_{L^2(\Sigma_\theta)} \leq C\|f\|_{\hat{H}^{-1}(\Sigma_\theta)}$$

with $C > 0$ independent of f and p. If, in addition, $f \in L^2(\Sigma_\theta)$, then $\nabla p \in H^1(\Sigma_\theta)$ and

$$\|\nabla^2 p\|_{L^2(\Sigma_\theta)} \leq C\|f\|_{L^2(\Sigma_\theta)}.$$

Proof. First we note that since $C_{c,m}^\infty(\Sigma_\theta) \overset{d}{\hookrightarrow} \hat{H}^{-1}(\Sigma_\theta)$ by Lemma 3.1.11, we can assume $f \in C_{c,m}^\infty(\Sigma_\theta)$. As already mentioned, at first we consider (3.12) on Σ_θ^δ in its weak formulation

$$\int_{\Sigma_\theta^\delta} \nabla p_\delta \cdot \nabla \varphi \, dx = \int_{\Sigma_\theta^\delta} f_\delta \varphi \, dx \qquad (\varphi \in \hat{H}_0^1(\Sigma_\theta^\delta)), \tag{3.13}$$

where we define $f_\delta := f|_{\Sigma_\theta^\delta}$ via restriction and use the representation of the duality pairing from Lemma 3.1.11. Then the Riesz representation Theorem 2.1.1 yields a unique solution $p \in \hat{H}_0^1(\Sigma_\theta^\delta)$ for (3.13) satisfying

$$\|\nabla p_\delta\|_{L^2(\Sigma_\theta^\delta)} \leq C\|f\|_{\hat{H}^{-1}(\Sigma_\theta)} \qquad (\delta > 0). \tag{3.14}$$

Since Σ_θ^δ is a uniform C^∞ domain, we have $\nabla^2 p_\delta \in L^2(\Sigma_\theta^\delta)$ if $f \in L^2(\Sigma_\theta^\delta)$ additionally, see [20]. Hence, we can apply Lemma 3.1.29 to the result

$$\begin{aligned} \|\nabla^2 p_\delta\|_{L^2(\Sigma_\theta^\delta)} &\leq \|\Delta p_\delta\|_{L^2(\Sigma_\theta^\delta)} + 2\langle \partial_{\tau_\delta}(\partial_{n_\delta} p_\delta), \partial_{\tau_\delta} p_\delta \rangle_{\hat{H}^{-1/2}(\Gamma_\delta), \hat{H}^{1/2}(\Gamma_\delta)} \\ &\leq \|f\|_{L^2(\Sigma_\theta)} \end{aligned} \tag{3.15}$$

uniformly in $\delta > 0$. Here, we took into account the fact that $p_\delta = 0$ implies $\partial_{\tau_\delta} p_\delta = 0$ on Γ_δ and that the $\mathscr{B}$ term drops out due to the convexity of Σ_θ^δ. We note that formula (3.11) still holds for p since $C_c^\infty(\Sigma_\theta^\delta)$ is dense in $\hat{H}_0^1(\Sigma_\theta^\delta)$ by definition.

For $\varphi \in C_c^\infty(\Sigma_\theta^\delta) \hookrightarrow \hat{H}_0^1(\Sigma_\theta)$ we consider the weak formulation (3.13) again. Then by the continuity of the integral we obtain

$$\int_{\Sigma_\theta^\delta} f_\delta \varphi \, dx \xrightarrow{\delta \to 0} \int_{\Sigma_\theta} f \varphi \, dx.$$

Now, we denote by $\tilde{h}$ the extension to Σ_θ by zero for some function $h : \Sigma_\theta^\delta \to \mathbb{R}^n$. Then by (3.14) we obtain the estimate

$$\|\widetilde{\nabla p_\delta}\|_{L^2(\Sigma_\theta)} \leq C \|f\|_{\hat{H}^{-1}(\Sigma_\theta)}$$

uniformly in $\delta > 0$ which yields the boundedness of $(\widetilde{\nabla p_\delta})_{\delta > 0}$ in $L^2(\Sigma_\theta)$. Then $(\widetilde{\nabla p_\delta})_{\delta > 0}$ has a weak limit

$$\widetilde{\nabla p_\delta} \to w \quad \text{weak in } L^2(\Sigma_\theta),$$

i.e.,

$$\int_{\Sigma_\theta^\delta} \nabla p_\delta \varphi \, dx = \int_{\Sigma_\theta} \widetilde{\nabla p_\delta} \varphi \, dx \xrightarrow{\delta \to 0} \int_{\Sigma_\theta} w \varphi \, dx$$

for all $\varphi \in C_c^\infty(\Sigma_\theta)$. For $\varphi \in C_{c,\sigma}^\infty(\Sigma_\theta)$ we can choose $\delta > 0$ so small that $\varphi \in C_{c,\sigma}^\infty(\Sigma_\theta^\delta)$. Then we can calculate

$$\int_{\Sigma_\theta} \widetilde{\nabla p_\delta} \varphi \, dx = \int_{\Sigma_\theta^\delta} \nabla p_\delta \varphi \, dx = \int_{\Sigma_\theta^\delta} p_\delta \operatorname{div} \varphi \, dx = 0$$

where we made use of the fact that $\operatorname{div} \varphi = 0$. Thus, $\langle w, \varphi \rangle_{L^2(\Sigma_\theta), L^2(\Sigma_\theta)} = 0$ for all $\varphi \in C_{c,\sigma}^\infty(\Sigma_\theta)$ and de Rahm's theorem implies $w = \nabla p$ for some $p \in L_{loc}^2(\Sigma_\theta)$. So for $\delta \to 0$ in (3.13) we infer

$$\int_{\Sigma_\theta} \nabla p \cdot \nabla \varphi \, dx = \int_{\Sigma_\theta} f \varphi \, dx \qquad (\varphi \in C_c^\infty(\Sigma_\theta)),$$

which means that $p \in \hat{H}_0^1(\Sigma_\theta)$ is the weak solution of (3.12). Note that $\nabla \hat{H}_0^1(\Sigma_\theta) \ni \widetilde{\nabla p_\delta} \to \nabla p$ in $L^2(\Sigma_\theta)$ such that $p \in \hat{H}_0^1(\Sigma_\theta)$. Also by the Riesz representation theorem it is clear that p is unique.

Finally, for the $\hat{H}^2$-regularity we remark that by (3.15) we obtain

$$\|\widetilde{\nabla^2 p_\delta}\|_{L^2(\Sigma_\theta)} \leq \|f\|_{L^2(\Sigma_\theta)} \qquad (\delta > 0),$$

which on one hand yields th boundedness of $(\widetilde{\nabla^2 p_\delta})_{\delta > 0}$ in $L^2(\Sigma_\theta)$ which on the other hand gives us the weak convergence of $(\widetilde{\nabla^2 p_\delta})_{\delta > 0}$ in $L^2(\Sigma_\theta)$, i.e.,

$$\int_{\Sigma_\theta} \widetilde{\nabla^2 p_\delta} \varphi \, dx \xrightarrow{\delta \to 0} \int_{\Sigma_\theta} W \varphi \, dx$$

for all $\varphi \in C_c^\infty(\Sigma_\theta)$. Then again for $\varphi \in C_c^\infty(\Sigma_\theta)$ we can choose δ so small that $\varphi \in C_c^\infty(\Sigma_\theta^\delta)$ such that

$$\int_{\Sigma_\theta} \varphi \widetilde{\partial_j \partial_k p_\delta}\, dx = \int_{\Sigma_\theta^\delta} \varphi \partial_j \partial_k p_\delta \, dx = -\int_{\Sigma_\theta^\delta} \partial_j \varphi \partial_k p_\delta \, dx = -\int_{\Sigma_\theta} \partial_j \varphi \widetilde{\partial_k p_\delta}\, dx$$
$$\xrightarrow{\delta \to 0} -\int_{\Sigma_\theta} \partial_j \varphi \partial_k p \, dx = \int_{\Sigma_\theta} \varphi \partial_j \partial_k p \, dx$$

for $k, j = 1, 2$ by the weak convergence of $(\widetilde{\nabla p_\delta})_{\delta > 0}$ by the argumentation above. By the uniqueness of the limit we deduce $\nabla^2 p = W \in L^2(\Sigma_\theta)$. Now we have proved that p is the unique strong solution to the homogeneous Dirichlet problem (3.12) satisfying the estimates as claimed for smooth f. Then a density argument yields the assertions for all f. □

3.1.32 Remark. (i) For the reader's convenience we want to compare the results of [20, Theorem 3.1.2.1, Theorem 3.1.2.3] to our above results. In [20] in order to obtain the full $H^2(\Omega)$ regularity, Poincaré's inequality is applied where the constant from Poincaré's inequality depends on the diameter of the bounded domain. Hence, the constant C of the corresponding norm estimates for the solution also depend on the diameter of Ω. However, in the context of [20] it is also possible to obtain a constant C independent of the domain if we only consider $\hat{H}^k(\Omega), k = 1, 2$, regularity. Thus, in homogeneous spaces the approach developed in [20] also works on a class of unbounded convex domains. The only condition we used above is that $C_c^\infty(\Omega)$ is dense in $\hat{H}_0^1(\Omega)$ which holds via definition.

As the results about surjectivity of trace theorems and solvability of elliptic problems are closely related, it is possible to solve the weak Dirichlet problem with inhomogeneous boundary conditions by applying the trace theorem (cf. Theorem 3.1.19):

3.1.33 Corollary (Weak inhomogeneous Dirichlet problem). *For every pair of data $(f, g) \in \hat{H}^{-1}(\Sigma_\theta) \times \hat{H}^{1/2}(\Gamma)$ there exists a unique solution $p \in \hat{H}^1(\Sigma_\theta)$ of*

$$\begin{aligned} -\Delta p &= f \quad \text{in } \Sigma_\theta, \\ p &= g \quad \text{on } \Gamma, \end{aligned} \tag{3.16}$$

in the weak sense satisfying

$$\|\nabla p\|_{L^2(\Sigma_\theta)} \le C \left(\|f\|_{\hat{H}^{-1}(\Sigma_\theta)} + \|g\|_{\hat{H}^{1/2}(\Gamma)} \right)$$

with $C > 0$ independent of f, g and p.

Proof. Let $(f,g) \in \hat{H}^{-1}(\Sigma_\theta) \times \hat{H}^{1/2}(\Gamma)$ be given. Thanks to Theorem 3.1.19 there exists some $\tilde{p} = E(g) \in \hat{H}^1(\Sigma_\theta)$ such that $\tilde{p} = g$ on Γ. Now let $\overline{p} \in \hat{H}^1(\Sigma_\theta)$ be the solution of

$$-\Delta\overline{p} = f + \Delta\tilde{p} \qquad \text{in } \Sigma_\theta, \qquad \overline{p} = 0 \qquad \text{on } \Gamma$$

which exists thanks to Lemma 3.1.31. By the observation regarding divergence from Lemma 3.1.15, we deduce $\Delta\tilde{p} = \operatorname{div}\nabla\tilde{p} \in \hat{H}^{-1}(\Sigma_\theta)$ since $\nabla\tilde{p} \in L^2(\Sigma_\theta)$.
Then $p = \tilde{p} + \overline{p} \in \hat{H}^1(\Sigma_\theta)$ solves (3.16). The solution is also unique: Let p and $\tilde{p}$ be two solutions solving (3.16). It follows that $\overline{p} = p - \tilde{p}$ solves

$$-\Delta\overline{p} = 0 \quad \text{in } \Sigma_\theta, \qquad \overline{p} = 0 \quad \text{on } \Gamma,$$

which is uniquely solvable thanks to Lemma 3.1.31. Hence we infer $\overline{p} = 0$ and $p = \tilde{p}$. The estimate also follows from Theorem 3.1.19 and Lemma 3.1.31. □

At last we solve the very weak Dirichlet problem which we want to prove by a simplified approach for spaces of low regularity as seen in [3, Theorem 1.1]:

3.1.34 Lemma. *Let* $\hat{H}^2_D(\Sigma_\theta) := \{u \in \hat{H}^2(\Sigma_\theta) : u|_\Gamma = 0\}$ *and let* $\hat{H}^{-2}_D(\Sigma_\theta) := (\hat{H}^2_D(\Sigma_\theta))'$ *be the corresponding dual space. For* $h \in \hat{H}^{-1/2}(\Gamma)$ *we define*

$$\ell_h(\varphi) := -\langle h, T_n\varphi\rangle_{\hat{H}^{-1/2}(\Gamma),\hat{H}^{1/2}(\Gamma)} \qquad (\varphi \in \hat{H}^2_D(\Sigma_\theta)),$$

where T_n *denotes the Neumann trace operator from Corollary 3.1.24. Then we have* $\ell_h \in \hat{H}^{-2}_D(\Sigma_\theta)$ *and using this identification we obtain*

$$\hat{H}^{-1/2}(\Gamma) \hookrightarrow \hat{H}^{-2}_D(\Sigma_\theta).$$

Proof. Fix an arbitrary $h \in \hat{H}^{-1/2}(\Gamma)$. Let ℓ_h be defined as above. Then ℓ_h is obviously well-defined since $T_n\varphi \in \hat{H}^{1/2}(\Gamma)$ which follows from Corollary 3.1.24. Obviously, using Corollary 3.1.24 we also see that $\ell_h \in \hat{H}^{-2}_D(\Sigma_\theta)$:

$$|\ell_h(\varphi)| \leq \|h\|_{\hat{H}^{-1/2}(\Gamma)}\|T_n\varphi\|_{\hat{H}^{1/2}(\Gamma)} \leq C\|h\|_{\hat{H}^{-1/2}(\Gamma)}\|\varphi\|_{\hat{H}^2(\Sigma_\theta)}$$

for all $\varphi \in \hat{H}^2_D(\Sigma_\theta)$ which gives us

$$\|\ell_h\|_{\hat{H}^{-2}_D(\Sigma_\theta)} \leq C\|h\|_{\hat{H}^{-1/2}(\Gamma)}.$$

□

The strategy to prove the very weak inhomogeneous Dirichlet problem is given as follows: By considering the solution operator L^{-1} to the strong homogeneous Dirichlet problem from Lemma 3.1.31, we will observe that the dual operator $(L^{-1})'$ actually is the solution operator to the very weak (in)homogeneous Dirichlet problem. By the low regularity that is assumed, we even obtain the solution operator for inhomogeneous boundary conditions.

3.1.35 Lemma (Very weak inhomogeneous Dirichlet problem). *For every data* $g \in \hat{H}^{-1/2}(\Gamma)$ *there exists a unique solution* $p \in L^2(\Sigma_\theta)$ *of*

$$\begin{aligned} -\Delta p &= 0 \quad \text{in } \Sigma_\theta, \\ p &= g \quad \text{on } \Gamma, \end{aligned} \tag{3.17}$$

in the very weak sense satisfying

$$\|p\|_{L^2(\Sigma_\theta)} \leq C\|g\|_{\hat{H}^{-1/2}(\Gamma)}$$

with $C > 0$ *independent of* g *and* p.

Proof. As already mentioned before, it will turn out that the solution operator for the very weak formulation corresponds with the solution operator for the strong formulation of the Dirichlet problem with homogeneous boundary conditions from Lemma 3.1.31. Also by Lemma 3.1.31 we know that for every $f \in L^2(\Sigma_\theta)$ there exists a unique $p \in \hat{H}^2_D(\Sigma_\theta)$, where $\hat{H}^2_D(\Sigma_\theta) = \{p \in \hat{H}^2(\Sigma_\theta) : p|_\Gamma = 0\}$, solving the Dirichlet problem for homogeneous boundary conditions (here $\hat{H}^2_D(\Sigma_\theta)$ is as defined as in Lemma 3.1.34). In fact, that means that the solution operator $L^{-1} : L^2(\Sigma_\theta) \to \hat{H}^2_D(\Sigma_\theta)$ to

$$L : \hat{H}^2_D(\Sigma_\theta) \to L^2(\Sigma_\theta), \quad Lp := \Delta p,$$

exists and that L^{-1} is bounded (by the estimate from Lemma 3.1.31). Considering the dual operator $L' : L^2(\Sigma_\theta) \to \hat{H}^{-2}_D(\Sigma_\theta)$ we infer for $u, \varphi \in L^2(\Sigma_\theta) \cap \hat{H}^2_D(\Sigma_\theta)$:

$$\langle L'u, \varphi\rangle_{\hat{H}^{-2}_D(\Sigma_\theta),\hat{H}^2_D(\Sigma_\theta)} = \langle u, L\varphi\rangle_{L^2(\Sigma_\theta),L^2(\Sigma_\theta)} = (u, \Delta\varphi)_2 = (-\Delta u, \varphi)_2,$$

which shows that L' is a consistent extension of L and $(L')^{-1} = (L^{-1})' : \hat{H}^{-2}_D(\Sigma_\theta) \to L^2(\Sigma_\theta)$ is the corresponding solution operator. Then to every $H \in \hat{H}^{-2}_D(\Sigma_\theta)$ there exists a unique $p \in L^2(\Sigma_\theta)$ such that $L'p = H$ in $\hat{H}^{-2}_D(\Sigma_\theta)$ with the desired estimate. Now thanks to Lemma 3.1.34 we even get a solution for the very weak Dirichlet problem with inhomogeneous boundary conditions (3.17): By Lemma 3.1.34 we can identify every $g \in \hat{H}^{-1/2}(\Gamma)$ with some functional

$$\ell_g = -\langle g, T_n \cdot\rangle_{\hat{H}^{-1/2}(\Gamma),\hat{H}^{1/2}(\Gamma)} = H \in \hat{H}^{-2}_D(\Sigma_\theta).$$

Then there exists a unique $p \in L^2(\Sigma_\theta)$ solving $L'p = H$ which in particular gives us

$$-\langle g, T_n\varphi\rangle_{\hat{H}^{-1/2}(\Gamma),\hat{H}^{1/2}(\Gamma)} = \langle H, \varphi\rangle_{\hat{H}^{-2}_D(\Sigma_\theta),\hat{H}^2_D(\Sigma_\theta)} = (p, \Delta\varphi)_2$$

for every $\varphi \in \hat{H}^2_D(\Sigma_\theta)$ which is indeed the very weak formulation of (3.17) showing that p solves (3.17) in the very weak sense. Also the estimate follows:

$$\|p\|_{L^2(\Sigma_\theta)} = \|(L^{-1})'H\|_{L^2(\Sigma_\theta)} \leq \|(L^{-1})'\|_{\mathscr{L}(\hat{H}^{-2}_D(\Sigma_\theta),L^2(\Sigma_\theta))}\|H\|_{\hat{H}^{-2}_D(\Sigma_\theta)}$$

$$\leq \|L^{-1}\|_{\mathscr{L}(L^2(\Sigma_\theta),\hat{H}^2_D(\Sigma_\theta))}\|g\|_{\hat{H}^{-1/2}(\Gamma)}$$

with $C = \|L^{-1}\|_{\mathscr{L}(L^2(\Sigma_\theta),\hat{H}^2_D(\Sigma_\theta))} > 0$ being the same constant C as in Lemma 3.1.31. □

3.1.4 Korn's Inequality for Convex and Non-Convex Wedges

For $0 < \theta < \pi$ we define the upper wedge of opening angle θ as

$$\Sigma_\theta^+ := \left\{(x_1,\, x_2) \in \mathbb{R}^2 : \operatorname{arccot}\left(\tfrac{x_1}{x_2}\right) < \theta, x_2 > 0\right\}$$

and the lower wedge of opening angle θ as $\Sigma_\theta^- := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1, -x_2) \in \Sigma_\theta^+\}$. Finally, the wedge of opening angle 2θ (sector of opening angle θ) is given as $\Sigma_\theta := (0, \infty) \times \{0\} \cup \Sigma_\theta^+ \cup \Sigma_\theta^-$. Note that $\Sigma_{\pi/2}$ is the right half plane $(0, \infty) \times \mathbb{R}$. Now, the following variant of Korn's inequality is available for the (right) half plane; cf. [7]:

3.1.36 Lemma. *Let $\mathbb{R}^2_{>0} := \Sigma_{\pi/2}$. There exists a constant $C > 0$ such that*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \leq C\left(\|u\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^2)} + \|D(u)\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})}\right)$$

for $u \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)$.

Proof. Following the proof of [7, Lemma IV.7.6] the assertion follows immediately by replacing Nečas' inequality for bounded domains with Nečas' inequality for the half-space [7, Proposition IV.1.5] in the last step. □

In order to transfer Korn's inequality to convex and non-convex wedges, we first prove Korn's inequality on the first and second quadrant:

3.1.37 Corollary. *Let $\mathbb{R}^2_{>0,+} := \Sigma^+_{\pi/2}$. There exists a constant $C > 0$ such that*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} \leq C\left(\|u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^2)} + \|D(u)\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})}\right)$$

for all $u \in H^1(\mathbb{R}^2_{>0,+}, \mathbb{R}^2)$.

Proof. Step 1. For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ we define the extension operators

$$\mathcal{E}_\lambda, \mathcal{E}'_\lambda : L^2(\mathbb{R}^2_{>0,+}) \to L^2(\mathbb{R}^2_{>0})$$

by setting

$$(\mathcal{E}_\lambda f)(x_1, x_2) := \begin{cases} f(x_1, x_2), & \text{if } x_2 > 0, \\ \lambda_1 f(x_1, -x_2) + \lambda_2 f(x_1, -2x_2), & \text{if } x_2 < 0, \end{cases}$$

for $x \in \mathbb{R}^2_{>0}$ as well as

$$(\mathcal{E}'_\lambda f)(x_1,x_2) := \begin{cases} f(x_1,x_2), & \text{if } x_2 > 0, \\ -\lambda_1 f(x_1,-x_2) - 2\lambda_2 f(x_1,-2x_2), & \text{if } x_2 < 0, \end{cases}$$

for $x \in \mathbb{R}^2_{>0}$ for functions $f \in L^2(\mathbb{R}^2_{>0,+})$, respectively. Clearly, both operators are linear and continuous with

$$\|f\|^2_{L^2(\mathbb{R}^2_{>0,+})} \leq \|\mathcal{E}_\lambda f\|^2_{L^2(\mathbb{R}^2_{>0})} \leq (1+|\lambda_1|^2+2|\lambda_2|^2)\|f\|^2_{L^2(\mathbb{R}^2_{>0,+})} =: C_\lambda \|f\|^2_{L^2(\mathbb{R}^2_{>0,+})}$$

and

$$\|f\|^2_{L^2(\mathbb{R}^2_{>0,+})} \leq \|\mathcal{E}'_\lambda f\|^2_{L^2(\mathbb{R}^2_{>0})} \leq (1+|\lambda_1|^2+4|\lambda_2|^2)\|f\|^2_{L^2(\mathbb{R}^2_{>0,+})} =: C'_\lambda \|f\|^2_{L^2(\mathbb{R}^2_{>0,+})}$$

for $f \in L^2(\mathbb{R}^2_{>0,+})$, respectively.

Step 2. Now, let $\mathbb{R}^2_{>0,-} := \Sigma^-_{\pi/2}$ and let $\lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2$ with $\lambda_1+\lambda_2 = 1$. For $f \in H^1(\mathbb{R}^2_{>0,+})$ we then have $\partial_1 \mathcal{E}_\lambda f = \mathcal{E}_\lambda \partial_1 f$ as well as

$$\begin{aligned}
&\int_{\mathbb{R}^2_{>0}} (\partial_2\varphi)\mathcal{E}_\lambda f \, dx \\
&= \int_{\mathbb{R}^2_{>0,+}} (\partial_2\varphi) f \, dx + \int_{\mathbb{R}^2_{>0,-}} (\partial_2\varphi(x_1,x_2))\,(\lambda_1 f(x_1,-x_2)+\lambda_2 f(x_1,-2x_2))\, dx \\
&= \int_0^\infty \varphi(x_1,0)\underbrace{((\lambda_1 f(x_1,0)+\lambda_2 f(x_1,0)) - f(x_1,0))}_{=0}\, dx_1 - \int_{\mathbb{R}^2_{>0,+}} \varphi(\partial_2 f)\, dx \\
&\quad - \int_{\mathbb{R}^2_{>0,-}} \varphi(x_1,x_2)\,(-\lambda_1\partial_2 f(x_1,-x_2) - 2\lambda_2\partial_2 f(x_1,-2x_2))\, dx \\
&= -\int_{\mathbb{R}^2_{>0}} \varphi(\mathcal{E}'_\lambda \partial_2 f)\, dx,
\end{aligned}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^2_{>0})$ which shows that $\partial_2\mathcal{E}_\lambda f = \mathcal{E}'_\lambda\partial_2 f$. Note that the condition $\lambda_1+\lambda_2 = 1$ is necessary to obtain the latter relation. It follows that $\mathcal{E}_\lambda : H^1(\mathbb{R}^2_{>0,+}) \to H^1(\mathbb{R}^2_{>0})$ is well-defined, linear and continuous, provided that we assume $\lambda_1+\lambda_2 = 1$.

Step 3. Now, let $\lambda = (\lambda_1,\lambda_2) := (3,-2)$ and let $\mu = (\mu_1,\mu_2) := (-3,4)$. Then we have $\lambda_1+\lambda_2 = 1$, $\mu_1+\mu_2 = 1$, $-\lambda_1 = \mu_1$ and $-2\lambda_2 = \mu_2$. Note that the latter two relations ensure that $\mathcal{E}'_\lambda = \mathcal{E}_\mu$. We fix $u \in H^1(\mathbb{R}^2_{>0,+},\mathbb{R}^2)$ and define $v \in H^1(\mathbb{R}^2_{>0},\mathbb{R}^2)$ as $v = (v_1,v_2) := (\mathcal{E}_\lambda u_1, \mathcal{E}_\mu u_2)$. Now, we have

$$\|u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^2)} \leq \|v\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^2)} \leq \max\{C_\lambda, C_\mu\}\|u\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^2)}$$

using the estimate from above as well as

$$\nabla v = \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 \\ \partial_1 v_2 & \partial_2 v_2 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_\lambda \partial_1 u_1 & \mathcal{E}_\mu \partial_2 u_1 \\ \mathcal{E}_\mu \partial_1 u_2 & \mathcal{E}'_\mu \partial_2 u_2 \end{pmatrix}$$

and

$$\begin{aligned} D(v) &= \begin{pmatrix} \partial_1 v_1 & \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) \\ \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) & \partial_2 v_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{E}_\lambda \partial_1 u_1 & \mathcal{E}_\mu \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2) \\ \mathcal{E}_\mu \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2) & \mathcal{E}'_\mu \partial_2 u_2 \end{pmatrix}, \end{aligned}$$

which implies that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} \le \|\nabla v\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \le \max\{C_\lambda, C_\mu, C'_\mu\} \|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})}$$

and

$$\begin{aligned} \|D(u)\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} &\le \|D(v)\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \\ &\le \max\{C_\lambda, C_\mu, C'_\mu\} \|D(u)\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})}. \end{aligned}$$

Now, the assertion is a direct consequence of Lemma 3.1.36. □

As a consequence we also obtain

3.1.38 Corollary. *Let $\mathbb{R}^2_{>0,-} := \Sigma^-_{\pi/2}$. There exists a constant $C > 0$ such that*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^{2\times 2})} \le C \left(\|u\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^2)} + \|D(u)\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^{2\times 2})} \right)$$

for $u \in H^1(\mathbb{R}^2_{>0,-}, \mathbb{R}^2)$.

Proof. This result follows with the same arguments as used in the proof of Corollary 3.1.37 from Lemma 3.1.36. Alternatively, this result can be deduced from Corollary 3.1.37 by means of suitable reflections w.r.t. the half axis $(0,\infty) \times \{0\}$. □

For our next result we introduce the following notation: For $\mathcal{M} \in L^\infty(\mathbb{R}^2_{>0}, \mathbb{R}^{2\times 2})$ we define the modified rate of deformation tensor as

$$D_\mathcal{M}(u) := \frac{1}{2}\left((\nabla u)\mathcal{M} + \mathcal{M}^T(\nabla u)^T\right) \qquad (u \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)).$$

Note that $D_\mathcal{M}(u) \in L^2(\mathbb{R}^2_{>0}, \mathbb{R}^{2\times 2})$ for $u \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)$. Also note that for $\mathcal{M} \equiv M \in \mathbb{R}^{2\times 2}$ we have $\nabla(Mu) = M\nabla u$ and $D(Mu) = \frac{1}{2}(M(\nabla u) + (\nabla u)^T M^T)$ for $u \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)$, which yields

$$D(M^{-T}u) = M^{-T}\frac{1}{2}\left((\nabla u)M + M^T(\nabla u)^T\right) M^{-1} = M^{-T} D_\mathcal{M}(u) M^{-1}$$

for $u \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)$ provided that $\det(M) \neq 0$.

3.1.39 Corollary. *Let* $\mathbb{R}^2_{>0} := \Sigma_{\pi/2}$. *Let* $M_\pm \in \mathbb{R}^{2\times 2}$ *with* $\det(M_\pm) \neq 0$. *Let* $\mathcal{M} \in L^\infty(\mathbb{R}^2_{>0}, \mathbb{R}^{2\times 2})$ *such that* $\mathcal{M}|_{\mathbb{R}^2_{>0,\pm}} = M_\pm$ *for* $\mathbb{R}^2_{>0,\pm} := \Sigma^\pm_{\pi/2}$. *Then there exists a constant* $C > 0$ *such that*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \leq C \left(\|u\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^2)} + \|D_{\mathcal{M}}(u)\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \right)$$

for $u \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)$.

Proof. Let $C_M := \max\{|M_+^{-1}|, |M_-^{-1}|, |M_+^T|, |M_+^{-T}|, |M_-^T|, |M_-^{-T}|\} \geq 1$ and let $C > 0$ be the constant in Korn's inequality for $\mathbb{R}^2_{>0,\pm}$ obtained in Corollary 3.1.37 and Corollary 3.1.38. For $u \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)$ we then have

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} &= \|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} + \|\nabla u\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^{2\times 2})} \\
&\leq C_M^2 \left(\|M_+^{-T} \nabla u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} + \|M_-^{-T} \nabla u\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^{2\times 2})} \right) \\
&= C_M^2 \left(\|\nabla (M_+^{-T} u)\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} + \|\nabla (M_-^{-T} u)\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^{2\times 2})} \right) \\
&\leq C C_M^2 \Big(\|M_+^{-T} u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^2)} + \|D(M_+^{-T} u)\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} \\
&\qquad + \|M_-^{-T} u\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^2)} + \|D(M_-^{-T} u)\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^{2\times 2})} \Big) \\
&\leq C C_M^6 \Big(\|u\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^2)} + \|D_{\mathcal{M}}(u)\|^2_{L^2(\mathbb{R}^2_{>0,+},\mathbb{R}^{2\times 2})} \\
&\qquad + \|u\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^2)} + \|D_{\mathcal{M}}(u)\|^2_{L^2(\mathbb{R}^2_{>0,-},\mathbb{R}^{2\times 2})} \Big) \\
&= C C_M^6 \left(\|u\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^2)} + \|D_{\mathcal{M}}(u)\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \right),
\end{aligned}$$

which is the asserted estimate. □

Finally, we are able to transfer a variant of Korn's inequality from the half-space to convex and non-convex wedges:

3.1.40 Corollary. *Let* $0 < \theta < \pi$. *There exists a constant* $C > 0$ *such that*

$$\|\nabla u\|^2_{L^2(\Sigma_\theta,\mathbb{R}^{2\times 2})} \leq C \left(\|u\|^2_{L^2(\Sigma_\theta,\mathbb{R}^2)} + \|D(u)\|^2_{L^2(\Sigma_\theta,\mathbb{R}^{2\times 2})} \right) \qquad (u \in H^1(\Sigma_\theta, \mathbb{R}^2)).$$

Proof. We fix $u \in H^1(\Sigma_\theta, \mathbb{R}^2)$. Using the transformation $\Phi : \mathbb{R}^2_{>0} \to \Sigma_\theta$ given as

$$\Phi(x_1, x_2) := (x_1 + |x_2| \cos(\theta), x_2) \qquad (x \in \Sigma_\theta),$$

we set $v := u \circ \Phi \in H^1(\mathbb{R}^2_{>0}, \mathbb{R}^2)$. The inverse $\Phi^{-1} : \Sigma_\theta \to \mathbb{R}^2_{>0}$ of Φ is given as

$$\Phi^{-1}(x_1, x_2) = (x_1 - |x_2| \cos(\theta), x_2), \qquad (x \in \Sigma_\theta),$$

and we have

$$\nabla\Phi(x) = \begin{pmatrix} 1 & \operatorname{sgn}(x_2)\cos(\theta) \\ 0 & 1 \end{pmatrix} \qquad (x \in \mathbb{R}^2_{>0}),$$

$$\nabla(\Phi^{-1})(x) = \begin{pmatrix} 1 & -\operatorname{sgn}(x_2)\cos(\theta) \\ 0 & 1 \end{pmatrix} \qquad (x \in \mathbb{R}^2_{>0}).$$

Note that the composition with Φ and Φ^{-1} constitute linear isometries from $L^2(\Sigma_\theta)$ onto $L^2(\mathbb{R}^2_{>0})$ and from $L^2(\mathbb{R}^2_{>0})$ onto $L^2(\Sigma_\theta)$, respectively. Now, with $\nabla(\Phi^{-1}) = ((\nabla(\Phi^{-1})) \circ \Phi) \circ \Phi^{-1} =: \mathcal{M} \circ \Phi^{-1}$ the chain rule yields

$$\nabla u = \nabla(v \circ \Phi^{-1}) = ((\nabla v) \circ \Phi^{-1})\nabla(\Phi^{-1}) = ((\nabla v)\mathcal{M}) \circ \Phi^{-1}$$

and, consequently,

$$\begin{aligned} D(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) = \frac{1}{2}\left(((\nabla v)\mathcal{M}) \circ \Phi^{-1} + (\mathcal{M}^T(\nabla v)^T) \circ \Phi^{-1}\right)\right) \\ &= (D_{\mathcal{M}}(v)) \circ \Phi^{-1}. \end{aligned}$$

Moreover, we have

$$\mathcal{M}(x) = \begin{pmatrix} 1 & \mp\cos(\theta) \\ 0 & 1 \end{pmatrix} =: M_\pm \in \mathbb{R}^{2\times 2} \qquad (x \in \mathbb{R}^2_{>0,\pm}),$$

which shows that $\mathcal{M} \in L^\infty(\mathbb{R}^2_{>0}, \mathbb{R}^{2\times 2})$ satisfies the assumptions of Corollary 3.1.39. Therefore, with $C_M := \max\{|M_+|, |M_-|\} > 0$ we obtain

$$\begin{aligned} \|\nabla u\|^2_{L^2(\Sigma_\theta,\mathbb{R}^{2\times 2})} &= \|((\nabla v)\mathcal{M}) \circ \Phi^{-1}\|^2_{L^2(\Sigma_\theta,\mathbb{R}^{2\times 2})} = \|(\nabla v)\mathcal{M}\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \\ &\le C_M^2\|\nabla v\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})} \\ &\le CC_M^2\left(\|v\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^2)} + \|D_{\mathcal{M}}(v)\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})}\right) \\ &= CC_M^2\left(\|u \circ \Phi\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^2)} + \|D(u) \circ \Phi\|^2_{L^2(\mathbb{R}^2_{>0},\mathbb{R}^{2\times 2})}\right) \\ &= CC_M^2\left(\|u\|^2_{L^2(\Sigma_\theta,\mathbb{R}^2)} + \|D(u)\|^2_{L^2(\Sigma_\theta,\mathbb{R}^{2\times 2})}\right), \end{aligned}$$

which is the asserted estimate. □

3.1.5 Scaling of Norms

One advantage of working with homogeneous Sobolev spaces on sectors lies in the fact that the sector Σ_θ is scaling invariant and that the norms in the homogeneous setting have nice scaling properties. Hence, we briefly collect those properties in the following statements. We start with the calculation in Sobolev spaces of positive order which turns out to be the easier case. We will consider the scaling on sectors Σ_θ and on the boundary Γ separately.

3.1.41 Lemma. *Let $\lambda \in \Sigma_{\pi/2}$ and $k = 0, 1$. We define*

$$\begin{aligned} S &: \hat{H}^k(\Sigma_\theta) \to \hat{H}^k(\Sigma_\theta), & (Sf)(x) &:= f\left(\frac{x}{\sqrt{|\lambda|}}\right), \\ S^{-1} &: \hat{H}^k(\Sigma_\theta) \to \hat{H}^k(\Sigma_\theta), & (S^{-1}f)(x) &:= f\left(\sqrt{|\lambda|}x\right). \end{aligned}$$

Then

$$\|Sf\|_{\hat{H}^k(\Sigma_\theta)} = |\lambda|^{-k/2+1/2}\|f\|_{\hat{H}^k(\Sigma_\theta)} \quad \text{and} \quad \|S^{-1}f\|_{\hat{H}^k(\Sigma_\theta)} = |\lambda|^{-1/2+k/2}\|f\|_{\hat{H}^k(\Sigma_\theta)},$$

such that

$$\|S\|_{\mathscr{L}(\hat{H}^k(\Sigma_\theta))} = |\lambda|^{-k/2+1/2} \quad \text{and} \quad \|S^{-1}\|_{\mathscr{L}(\hat{H}^k(\Sigma_\theta))} = |\lambda|^{k/2-1/2}.$$

Proof. It is obvious that S^{-1} is the inverse of S and $SS^{-1} = S^{-1}S = I$ on $\hat{H}^k(\Sigma_\theta)$. Then a straightforward calculation yields

$$\begin{aligned} \|Sf\|_{\hat{H}^k(\Sigma_\theta)} &= \|\nabla^k Sf\|_{L^2(\Sigma_\theta)} \\ &= \left(\int_{\Sigma_\theta} \left|\nabla^k \left(f\left(\frac{\cdot}{\sqrt{|\lambda|}}\right)\right)(x)\right|^2 dx\right)^{1/2} \\ &= |\lambda|^{-k/2}\left(\int_{\Sigma_\theta} \left|\nabla^k f\left(\frac{x}{\sqrt{|\lambda|}}\right)\right|^2 dx\right)^{1/2} \\ &= |\lambda|^{-k/2+1/2}\|\nabla^k f\|_{L^2(\Sigma_\theta)} = |\lambda|^{-k/2+1/2}\|f\|_{\hat{H}^k(\Sigma_\theta)}, \end{aligned}$$

since

$$\begin{aligned} \left\|\nabla^k f\left(\frac{\cdot}{\sqrt{|\lambda|}}\right)\right\|^2_{L^2(\Sigma_\theta)} &= \int_{\Sigma_\theta} \left|\nabla^k f\left(\frac{x}{\sqrt{|\lambda|}}\right)\right|^2 dx \\ &= \int_{\Sigma_\theta} \left|\nabla^k f(y)\right|^2 |\lambda|\, dy \\ &= \|\nabla^k f\|^2_{L^2(\Sigma_\theta)}|\lambda|. \end{aligned}$$

For the inverse we obtain what we have expected:

$$\|S^{-1}f\|_{\hat{H}^k(\Sigma_\theta)} = \left\|\nabla^k S^{-1}f\right\|_{L^2(\Sigma_\theta)} = \left\|\nabla^k\left(f\left(\sqrt{|\lambda|}\cdot\right)\right)\right\|_{L^2(\Sigma_\theta)}$$

$$= |\lambda|^{k/2} \left\| \nabla^k f\left(\sqrt{|\lambda|}\cdot\right) \right\|_{L^2(\Sigma_\theta)}$$
$$= |\lambda|^{-1/2+k/2} \|\nabla^k f\|_{L^2(\Sigma_\theta)} = |\lambda|^{-1/2+k/2} \|f\|_{\hat{H}^k(\Sigma_\theta)}$$

since analogously to the calculation above we have

$$\begin{aligned} \left\| \nabla^k f\left(\sqrt{|\lambda|}\cdot\right) \right\|^2_{L^2(\Sigma_\theta)} &= \int_{\Sigma_\theta} \left| \nabla^k f\left(\sqrt{|\lambda|}x\right) \right|^2 dx \\ &= \int_{\Sigma_\theta} |\nabla^k f(y)|^2 |\lambda|^{-1}\, dy \\ &= \|\nabla^k f\|^2_{L^2(\Sigma_\theta)} |\lambda|^{-1} \end{aligned}$$

using the transform $y = \sqrt{|\lambda|}x$ and $y = x/\sqrt{|\lambda|}$, respectively. This immediately also yields the assertion regarding the operator norm. □

3.1.42 Lemma. *Let* $\lambda \in \Sigma_{\pi/2}$ *and* $k = 0, 1$. *We define* $S, S^{-1} : \hat{H}^k(\Gamma) \to \hat{H}^k(\Gamma)$ *as in Lemma 3.1.41. Then we have*

$$\|Sf\|_{\hat{H}^k(\Gamma)} = |\lambda|^{-k/2+1/4} \|f\|_{\hat{H}^k(\Gamma)} \qquad \text{and} \qquad \|S^{-1}f\|_{\hat{H}^k(\Gamma)} = |\lambda|^{k/2-1/4} \|f\|_{\hat{H}^k(\Gamma)},$$

hence

$$\|S\|_{\mathscr{L}(\hat{H}^k(\Gamma))} = |\lambda|^{-k/2+1/4} \qquad \text{and} \qquad \|S^{-1}\|_{\mathscr{L}(\hat{H}^k(\Gamma))} = |\lambda|^{k/2-1/4}.$$

Proof. Note that by Lemma 3.1.5 we can identify $L^2(\Gamma)$ with $L^2(\mathbb{R})$ on the boundary. Hence, the transformation from Lemma 3.1.41 changes a little bit:

$$\begin{aligned} \|Sf\|_{\hat{H}^k(\Gamma)} = \|\partial_\tau^k Sf\|_{L^2(\Gamma)} &= \left(\int_\Gamma \left| \partial_\tau^k \left(f\left(\frac{\cdot}{\sqrt{|\lambda|}} \right) \right)(x) \right|^2 dx \right)^{1/2} \\ &= |\lambda|^{-k/2} \left(\int_\Gamma \left| \partial_\tau^k f\left(\frac{x}{\sqrt{|\lambda|}} \right) \right|^2 dx \right)^{1/2} \\ &= |\lambda|^{-k/2+1/4} \|\partial_\tau^k f\|_{L^2(\Gamma)} = |\lambda|^{-k/2+1/4} \|f\|_{\hat{H}^k(\Gamma)} \end{aligned}$$

since

$$\begin{aligned} \int_\Gamma \left| \partial_\tau^k f\left(\frac{x}{\sqrt{|\lambda|}} \right) \right|^2 dx &= \int_{\mathbb{R}} \left| \partial^k f\left(\gamma_0\left(\frac{t}{\sqrt{|\lambda|}} \right) \right) \right|^2 \left| \gamma_0'\left(\frac{t}{\sqrt{|\lambda|}} \right) \right| dt \\ &= \int_{\mathbb{R}} |\partial^k f(\gamma_0(t))|^2 |\gamma_0'(t)| \sqrt{|\lambda|}\, dt \\ &= \int_\Gamma |\partial_\tau^k f(x)|^2 \sqrt{|\lambda|}\, dx. \end{aligned}$$

The same holds for S^{-1} (calculation can be done analogously to the Σ_θ case as seen in Lemma 3.1.41):

$$\begin{aligned}\|S^{-1}f\|_{\hat{H}^k(\Gamma)} &= \|\partial_\tau^k S^{-1} f\|_{L^2(\Gamma)} = \left\|\partial_\tau^k \left(f\left(\sqrt{|\lambda|}\cdot\right)\right)\right\|_{L^2(\Gamma)}\\ &= |\lambda|^{k/2}\left\|(\partial_\tau^k f)\left(\sqrt{|\lambda|}\cdot\right)\right\|_{L^2(\Gamma)} = |\lambda|^{k/2-1/4}\|\partial_\tau^k f\|_{L^2(\Gamma)}\\ &= |\lambda|^{k/2-1/4}\|f\|_{\hat{H}^k(\Gamma)}.\end{aligned}$$

Hence, the assertion follows. □

3.1.43 Remark. In the context of Lemma 3.1.42 we will be mainly interested in the norm scaling in $\hat{H}^{1/2}(\Gamma)$. To this end, we apply the Riesz-Thorin interpolation theorem (see e.g. [51]), to $S, S^{-1} : L^2(\Gamma) + \hat{H}^1(\Gamma) \to L^2(\Gamma) + \hat{H}^1(\Gamma)$ (we note that interpolation is meaningful in this case by the observation we made in the beginning of Section 3.1). Then we deduce that $S, S^{-1} : \hat{H}^{1/2}(\Gamma) \to \hat{H}^{1/2}(\Gamma)$ are bounded and

$$\begin{aligned}\|S\|_{\mathscr{L}(\hat{H}^{1/2}(\Gamma))} &\le \|S\|^{1/2}_{\mathscr{L}(L^2(\Gamma))}\|S\|^{1/2}_{\mathscr{L}(\hat{H}^1(\Gamma))} = 1,\\ \|S^{-1}\|_{\mathscr{L}(\hat{H}^{1/2}(\Gamma))} &\le \|S^{-1}\|^{1/2}_{\mathscr{L}(L^2(\Gamma))}\|S^{-1}\|^{1/2}_{\mathscr{L}(\hat{H}^1(\Gamma))} = 1.\end{aligned}$$

Hence, from this we infer for $f \in \hat{H}^{1/2}(\Gamma)$:

$$\|f\|_{\hat{H}^{1/2}(\Gamma)} = \|S^{-1}Sf\|_{\hat{H}^{1/2}(\Gamma)} \le \|Sf\|_{\hat{H}^{1/2}(\Gamma)} \le \|f\|_{\hat{H}^{1/2}(\Gamma)},$$

to be precise this means

$$\|f\|_{\hat{H}^{1/2}(\Gamma)} = \|Sf\|_{\hat{H}^{1/2}(\Gamma)}.$$

Now we need to calculate the norms in Sobolev spaces of negative order. To this end, we again consider the operator S in Σ_θ first for $k = 1$:

3.1.44 Lemma. *Let $\lambda \in \Sigma_\theta$ and*

$$\begin{aligned}\tilde{S}^{-1} : \hat{H}^1(\Sigma_\theta) \to \hat{H}^1(\Sigma_\theta), \quad &\tilde{S}^{-1}f := |\lambda|S^{-1}f,\\ \tilde{S} : \hat{H}^1(\Sigma_\theta) \to \hat{H}^1(\Sigma_\theta), \quad &\tilde{S}f := |\lambda|^{-1}Sf,\end{aligned}$$

where S, S^{-1} are defined as in Lemma 3.1.41. Then for smooth functions $g \in C^\infty_{c,m}(\Sigma_\theta)$ the dual operator $(\tilde{S}^{-1})'$ is given as

$$\begin{aligned}&(\tilde{S}^{-1})' : \hat{H}_0^{-1}(\Sigma_\theta) \to \hat{H}_0^{-1}(\Sigma_\theta),\\ &\langle(\tilde{S}^{-1})'g, \cdot\rangle_{\hat{H}_0^{-1}(\Sigma_\theta),\hat{H}^1(\Sigma_\theta)} = \langle Sg, \cdot\rangle_{\hat{H}_0^{-1}(\Sigma_\theta),\hat{H}^1(\Sigma_\theta)},\end{aligned}$$

with operator norms

$$\|\tilde{S}^{-1}\|_{\mathscr{L}(\hat{H}^1(\Sigma_\theta))} = |\lambda| = \|(\tilde{S}^{-1})'\|_{\mathscr{L}(\hat{H}_0^{-1}(\Sigma_\theta))},$$
$$\|\tilde{S}\|_{\mathscr{L}(\hat{H}^1(\Sigma_\theta))} = |\lambda|^{-1} = \|\tilde{S}'\|_{\mathscr{L}(\hat{H}_0^{-1}(\Sigma_\theta))}.$$

The statement also holds in divergence free spaces, i.e., in $\hat{H}^1_{\mathrm{div}}(\Sigma_\theta)$.

Proof. By Lemma 3.1.41 we immediately deduce

$$\|\tilde{S}^{-1}f\|_{\hat{H}^1(\Sigma_\theta)} = |\lambda|\|f\|_{\hat{H}^1(\Sigma_\theta)} \quad \text{and} \quad \|\tilde{S}f\|_{\hat{H}^1(\Sigma_\theta)} = |\lambda|^{-1}\|f\|_{\hat{H}^1(\Sigma_\theta)}.$$

This yields the assertion regarding the operator norms. Next, we want to calculate the dual operator $(\tilde{S}^{-1})' : \hat{H}_0^{-1}(\Sigma_\theta) \to \hat{H}_0^{-1}(\Sigma_\theta)$:

$$\begin{aligned}
\langle(\tilde{S}^{-1})'g, f\rangle_{\hat{H}_0^{-1}(\Sigma_\theta),\hat{H}^1(\Sigma_\theta)} &= (g, \tilde{S}^{-1}f)_2 = \int_{\Sigma_\theta} g(x)(\tilde{S}^{-1}f)(x)\,dx \\
&= \int_{\Sigma_\theta} g(x) f\left(\sqrt{|\lambda|}x\right)\,dx|\lambda| \\
&= \int_{\Sigma_\theta} g\left(\frac{x}{\sqrt{|\lambda|}}\right) f(x)\,dx \\
&= \langle Sg, f\rangle_{\hat{H}_0^{-1}(\Sigma_\theta),\hat{H}^1(\Sigma_\theta)}
\end{aligned}$$

for every $f \in \hat{H}^1(\Sigma_\theta)$ and $g \in C^\infty_{c,m}(\Sigma_\theta)$. Since the set of functionals of the form $\hat{H}^1(\Sigma_\theta) \ni f \mapsto (g, f)_2$ with $g \in C^\infty_{c,m}(\Sigma_\theta)$ are dense in $(\hat{H}^1(\Sigma_\theta))'$ (see Lemma 3.1.11) we obtain

$$(\tilde{S}^{-1})'g = \lim_{\ell\to\infty} Sg_\ell \quad \text{in } \hat{H}_0^{-1}(\Sigma_\theta)$$

where $(g_\ell)_\ell \subseteq C^\infty_{c,m}(\Sigma_\theta)$ is a sequence where $(g_\ell, \cdot)_2$ is approximating $g \in \hat{H}_0^{-1}(\Sigma_\theta)$. Hence, the assertion holds. □

3.1.45 Lemma. *Let $\lambda \in \Sigma_{\pi/2}$ and $k = 0, 1$. As in Lemma 3.1.44 we set*

$$\tilde{S}^{-1} : \hat{H}^k(\Gamma) \to \hat{H}^k(\Gamma), \qquad (\tilde{S}^{-1}f)(x) = |\lambda|^{1/2}S^{-1}f$$

with S defined as in Lemma 3.1.41. Then the dual operator $(\tilde{S}^{-1})' : \hat{H}^{-k}(\Gamma) \to \hat{H}^{-k}(\Gamma)$ for smooth functions $g \in C^\infty_{c,m}(\Gamma)$ is given as

$$\big\langle(\tilde{S}^{-1})'g, \cdot\big\rangle_{\hat{H}^{-k}(\Gamma),\hat{H}^k(\Gamma)} = \langle Sg, \cdot\rangle_{\hat{H}^{-k}(\Gamma),\hat{H}^k(\Gamma)}$$

with

$$\|\tilde{S}^{-1}\|_{\mathscr{L}(\hat{H}^k(\Gamma))} = |\lambda|^{1/4+k/2} = \|(\tilde{S}^{-1})'\|_{\mathscr{L}(\hat{H}^{-k}(\Gamma))}.$$

Proof. It immediately follows

$$\|\tilde{S}^{-1}f\|_{\hat{H}^k(\Gamma)} = |\lambda|^{1/4+k/2}\|f\|_{\hat{H}^k(\Gamma)}$$

for $k = 0, 1$. Also the dual operator $(\tilde{S}^{-1})' : \hat{H}^{-k}(\Gamma) \to \hat{H}^{-k}(\Gamma)$ is given as

$$\begin{aligned}
\langle(\tilde{S}^{-1})'g, f\rangle_{\hat{H}^{-k}(\Gamma),\hat{H}^k(\Gamma)} &= (g, (\tilde{S}^{-1})f)_{2,\Gamma} = \int_\Gamma g(x)(\tilde{S}^{-1}f)(x)\,dx \\
&= \int_\Gamma g(x) f\left(\sqrt{|\lambda|}x\right)\,dx|\lambda|^{1/2} \\
&= \int_\Gamma g\left(\frac{x}{\sqrt{|\lambda|}}\right) f(x)\,dx \\
&= \langle Sg, f\rangle_{\hat{H}^{-k}(\Gamma),\hat{H}^k(\Gamma)}
\end{aligned}$$

for $f \in H^k(\Gamma)$ where we again identified $L^2(\Gamma)$ with $L^2(\mathbb{R})$ and for $\hat{H}^1(\Gamma)$ we used the representation of the functionals in $\hat{H}^{-1}(\Gamma) = (\hat{H}^1(\Gamma))'$ from Lemma 3.1.11. Using the density argument again we obtain

$$(\tilde{S}^{-1})'g = \lim_{\ell\to\infty} Sg_\ell \quad \text{in } \hat{H}^{-k}(\Gamma)$$

for any sequence $(g_\ell)_\ell \subseteq C^\infty_{c,m}(\Gamma)$ where $(g_\ell, \cdot)_{2,\Gamma}$ is approximating $g \in \hat{H}^{-k}(\Gamma)$. □

3.1.46 Remark. In the context of Lemma 3.1.45 we are interested in the norm scaling in $\hat{H}^{-1/2}(\Gamma)$. To this end, in order to obtain the estimates in the $\hat{H}^{-1/2}(\Gamma)$ norm we again have to apply the interpolation argument: $S^{-1} : L^2(\Gamma)+\hat{H}^1(\Gamma) \to L^2(\Gamma)+\hat{H}^1(\Gamma)$ is bounded. Applying the Riesz-Thorin interpolation theorem (see e.g. [51]) we know that $\tilde{S}^{-1} : \hat{H}^{1/2}(\Gamma) \to \hat{H}^{1/2}(\Gamma)$ is also bounded and its operator norm can be estimated as

$$\|\tilde{S}^{-1}\|_{\mathscr{L}(\hat{H}^{1/2}(\Gamma))} \leq \|\tilde{S}^{-1}\|^{1/2}_{\mathscr{L}(L^2(\Gamma))}\|\tilde{S}^{-1}\|^{1/2}_{\mathscr{L}(\hat{H}^1(\Gamma))} = |\lambda|^{1/8}|\lambda|^{1/8+1/4} = |\lambda|^{1/2}.$$

such that for the dual operator $(\tilde{S}^{-1})' : \hat{H}^{-1/2}(\Gamma) \to \hat{H}^{-1/2}(\Gamma)$ we obtain

$$\|(\tilde{S}^{-1})'\|_{\mathscr{L}(\hat{H}^{-1/2}(\Gamma))} = \|\tilde{S}^{-1}\|_{\mathscr{L}(\hat{H}^{1/2}(\Gamma))} \leq |\lambda|^{1/2}.$$

3.2 Reflection Invariant Sobolev Spaces in Sectors

In this section we introduce subspaces of (in)homogeneous spaces that consist of even and odd functions. This later allows us to consider functions on the wedge $\Sigma^\pm_\theta$ as defined in Section 3.1.4 which is the natural domain to some systems from fluid dynamics, as e.g. the contact line problem from Chapter 4. Another advantage of considering reflection invariant subspaces lies in the fact that given the correct symmetry, multiplication with sgn is a bounded operator which yields that multiplication with the normal vector n is also a bounded operator (see Lemma 3.2.8).

Since many results from Section 3.1 are transferable, we keep most of the proofs short and sometimes also do not formulate the corresponding statements from Section 3.1 in the setting of reflected spaces. However, we put emphasize on the fact that almost all results from Section 3.1 also hold in the setting of reflection invariant spaces.

We start with the definition of a reflection invariant subspace. For a function space E on Σ_θ we set

$$E_{\pm R} := \{u \in E : \pm Ru = u\}$$

in case that functions in E are vector-valued and if functions in E are scalar-valued then we define

$$E_{\pm r} := \{u \in E : \pm ru = u\},$$

where the reflection is defined as $Ru := (ru^1, -ru^2)$ and $rh(x_1, x_2) = h(x_1, -x_2)$ for $u = (u^1, u^2)$ and $h : \Sigma_\theta \to \mathbb{C}$. For matrix-valued functions we set

$$E_{\pm \mathcal{R}} := \{u \in E : \pm \mathcal{R}u = u\},$$

where

$$\mathcal{R}u = \mathcal{R}\begin{pmatrix} u^{1,1} & u^{1,2} \\ u^{2,1} & u^{2,2} \end{pmatrix} = \begin{pmatrix} ru^{1,1} & -ru^{1,2} \\ -ru^{2,1} & ru^{2,2} \end{pmatrix}.$$

At first we collect some basic properties of (in)homogeneous spaces in the framework of reflected spaces. The proof is straightforward and hence is kept short.

3.2.1 Lemma. *The following assertions hold:*

(i) $\pm r \in \mathscr{L}_{is}(\hat{H}^k(\Sigma_\theta)), \|\pm r\|_{\mathscr{L}(\hat{H}^k(\Sigma_\theta))} = 1, (\pm r)^2 = I_{\hat{H}^k(\Sigma_\theta)}, (\pm r)' = \pm r$ *for all* $k \in [0,1]$*. This also holds for* Σ_θ *replaced by* Γ*.*

(ii) $\pm R \in \mathscr{L}_{is}(\hat{H}^k(\Sigma_\theta)^2), \|\pm R\|_{\mathscr{L}(\hat{H}^k(\Sigma_\theta)^2)} = 1, (\pm R)^2 = I_{\hat{H}^k(\Sigma_\theta)^2}, (\pm R)' = \pm R$ *for all* $k \in [0,1]$*. This also holds for* Σ_θ *replaced by* Γ*.*

(iii) For $k \in [0,1]$ *the operator*

$$Q_\pm : \hat{H}^k(\Sigma_\theta)^2 \to \hat{H}^k(\Sigma_\theta)^2, \qquad Q_\pm v := \frac{v \pm Rv}{2}$$

is a bounded projection onto $\hat{H}^k(\Sigma_\theta)_{\pm R}$*. In particular,* $\hat{H}^k(\Sigma_\theta)_{\pm R}$ *is closed in* $\hat{H}^k(\Sigma_\theta)^2$ *and we have*

$$(\hat{H}^k(\Sigma_\theta)_{\pm R})' = (Q_\pm \hat{H}^k(\Sigma_\theta)^2)' = Q_\pm(\hat{H}^k(\Sigma_\theta)^2)' = (\hat{H}^k(\Sigma_\theta))'_{\pm R}.$$

This statement also holds for $\pm R$ *replaced by* $\pm r$ *and* Σ_θ *replaced by* Γ*.*

(iv) For $k \in (0,1]$ *we have* $C_c^\infty(\overline{\Sigma_\theta})_{\pm R} \overset{d}{\hookrightarrow} H^k(\Sigma_\theta)_{\pm R} \overset{d}{\hookrightarrow} \hat{H}^k(\Sigma_\theta)_{\pm R}$.

(v) For $s \in (0,1/2]$ *we have* $C_c^\infty(\Gamma\backslash\{0\})_{\pm R} \overset{d}{\hookrightarrow} H^s(\Gamma)_{\pm R} \overset{d}{\hookrightarrow} \hat{H}^s(\Gamma)_{\pm R}$.

Proof. (i) We prove the statement for $k = 0, 1$: Let $u \in \hat{H}^k(\Sigma_\theta)$ be arbitrary. Then

$$\begin{aligned}
\|ru\|^2_{\hat{H}^k(\Sigma_\theta)} &= \sum_{|\alpha|\le k} \int_{\Sigma_\theta} |\partial^\alpha (ru)(x_1,x_2)|^2 \, dx_1 \, dx_2 \\
&= \sum_{|\alpha|\le k} \int_{\Sigma_\theta} |(r\partial^\alpha u)(x_1,x_2)|^2 \, dx_1 \, dx_2 \\
&= \sum_{|\alpha|\le k} \int_{\Sigma_\theta} |\partial^\alpha u(x_1,-x_2)|^2 \, dx_1 \, dx_2 \\
&= \|u\|^2_{\hat{H}^k(\Sigma_\theta)}
\end{aligned}$$

using the transform $(x_1, x_2) \mapsto (x_1, -x_2)$ and the fact that Σ_θ is scaling invariant. Obviously, r is self-inverse and $r^2 = I_{\hat{H}^k(\Sigma_\theta)}$. The reflection of a distribution $f \in \hat{H}_0^{-k}(\Sigma_\theta)$ is defined as

$$\langle rf, \varphi\rangle_{\hat{H}_0^{-k}(\Sigma_\theta),\hat{H}^k(\Sigma_\theta)} := \langle f, r\varphi\rangle_{\hat{H}_0^{-k}(\Sigma_\theta),\hat{H}^k(\Sigma_\theta)} \qquad (\varphi \in \hat{H}^k(\Sigma_\theta)),$$

which coincides with the definition of the dual operator $r' \in \mathscr{L}(\hat{H}_0^{-k}(\Sigma_\theta))$ such that we obtain $r' = r$. Then the assertion follows for $k = 0, 1$. Interpolation yields the statement for all $k \in [0,1]$.

(ii) The assertion follows from (i) since $R = (r, -r)$.

(iii) Thanks to (ii) it is obvious that $Q_\pm \in \mathscr{L}(\hat{H}^k(\Sigma_\theta)^2)$ for all $k \in [0,1]$. Furthermore, $Q_\pm$ is a projection since

$$Q_\pm^2 v = Q_\pm \frac{v \pm Rv}{2} = \frac{\frac{v\pm Rv}{2} \pm R\frac{v\pm Rv}{2}}{2} = \frac{v \pm Rv}{2} = Q_\pm v.$$

Now we aim to prove $R(Q_\pm) = \hat{H}^k(\Sigma_\theta)_{\pm R}$. Let $v \in \hat{H}^k(\Sigma_\theta)^2$. Then

$$\pm RQ_\pm v = \pm R\frac{v \pm Rv}{2} = \frac{\pm Rv + v}{2} = Q_\pm v,$$

hence $v \in \hat{H}^k(\Sigma_\theta)_{\pm R}$. On the other hand, let $v \in \hat{H}^k(\Sigma_\theta)_{\pm R}$, i.e., $\pm Rv = v$. Then $Q_\pm v = \frac{v\pm Rv}{2} = v$ and $v \in R(Q_\pm)$. Indeed, $\hat{H}^k(\Sigma_\theta)_{\pm R}$ is closed since the range of a projection is always closed. The fact that $Q'_\pm = Q_\pm$ follows as above and as a consequence we obtain $(Q_\pm \hat{H}^k(\Sigma_\theta)^2)' = Q_\pm(\hat{H}^k(\Sigma_\theta)^2)'$.

(iv) First we note that $Q_\pm : C_c^\infty(\overline{\Sigma_\theta})^2 \to C_c^\infty(\overline{\Sigma_\theta})^2$ again with

$$Q_\pm C_c^\infty(\overline{\Sigma_\theta})^2 = C_c^\infty(\overline{\Sigma_\theta})_{\pm R}$$

using the same calculation as in (iii). Then Lemma 3.1.7 combined with (iii) and Lemma 3.2.2(iii) yields the assertion. The proof for (v) follows analogously. □

3.2.2 Lemma. *Let $k \in \mathbb{Z}$ and $1 < p < \infty$. Then the following assertions hold:*

(i) $\pm r \in \mathscr{L}_{is}(W^{k,p}(\Sigma_\theta)), \| \pm r\|_{W^{k,p}(\Sigma_\theta)} = 1, (\pm r)' = \pm r$ *and* $(\pm r)^2 = I_{W^{k,p}(\Sigma_\theta)}$.

(ii) $\pm R \in \mathscr{L}_{is}(W^{k,p}(\Sigma_\theta)^2), \| \pm R\|_{W^{k,p}(\Sigma_\theta)^2} = 1, (\pm R)' = \pm R$ *and* $(\pm R)^2 = I_{W^{k,p}(\Sigma_\theta)^2}$.

(iii) The operator

$$Q_\pm : W^{k,p}(\Sigma_\theta)^2 \to W^{k,p}(\Sigma_\theta)^2, \quad Q_\pm v := \frac{v \pm Rv}{2}$$

is a bounded projection onto $W^{k,p}(\Sigma_\theta)_{\pm R}$ *which is orthogonal for* $p = 2$. *In particular,* $W^{k,p}(\Sigma_\theta)_{\pm R}$ *is closed in* $W^{k,p}(\Sigma_\theta)^2$ *and we have*

$$(W^{k,p}(\Sigma_\theta)_{\pm R})' = (Q_\pm W^{k,p}(\Sigma_\theta)^2)' = Q_\pm (W^{k,p}(\Sigma_\theta)^2)' = (W^{k,p}(\Sigma_\theta))'_{\pm R}.$$

The same assertions remain true, if $W^{k,p}(\Sigma_\theta)$ *is replaced by* $W^{k,p}(\Gamma)$ *and* R *by* r.

(iv) For $k, m \in \mathbb{Z}, k \leq m$, *we have the embeddings* $W^{m,p}(\Sigma_\theta)_{\pm R} \overset{d}{\hookrightarrow} W^{k,p}(\Sigma_\theta)_{\pm R}$ *and* $W^{m,p}(\Gamma)_{\pm R} \overset{d}{\hookrightarrow} W^{k,p}(\Gamma)_{\pm R}$.

(v) For $m \in \mathbb{N}$ *it holds* $H^m_{\mathrm{div}}(\Sigma_\theta)_{\pm R} \overset{d}{\hookrightarrow} H^1_{\mathrm{div}}(\Sigma_\theta)_{\pm R}$.

Proof. (i) - (iv) essentially follow as in Lemma 3.2.1.
(v) By Lemma 3.1.8 we know that the statement holds in the non-reflected case. Then the assertion follows from (iii). □

By the observations we made above we are now able to characterize interpolation of reflected spaces by making use of the bounded projection $Q_\pm$ from Lemma 3.2.1 and Lemma 3.2.2. Hence, interpolation of reflection invariant spaces are then well-defined by applying the standard argument from [51, Section 1.2.4]:

3.2.3 Corollary. *Let* $k \in \mathbb{N}$, $1 \leq p < \infty$ *and* $s \in (0,1)$. *Then*

$$\begin{aligned} \hat{H}^s(\Sigma_\theta)_{\pm R} &= (L^2(\Sigma_\theta)_{\pm R}, \hat{H}^1(\Sigma_\theta)_{\pm R})_{s,2}, \\ W^{sk}_p(\Sigma_\theta)_{\pm R} &= (L^p(\Sigma_\theta)_{\pm R}, W^{k,p}(\Sigma_\theta)_{\pm R})_{s,p}, \\ W^{sk,p}(\Sigma_\theta)_{\pm R} &= [L^p(\Sigma_\theta)_{\pm R}, W^{k,p}(\Sigma_\theta)_{\pm R}]_s, \end{aligned}$$

where $W^k_p(\Sigma_\theta)$ *denotes the Sobolev-Slobodeckij space (see [51, Chapter 4]). The above statements also hold with* $\pm R$ *replaced by* $\pm r$ *and* Σ_θ *replaced by* Γ.

Proof. This follows from the fact that $Q_\pm$ is a bounded projection onto $\hat{H}^1(\Sigma_\theta)_{\pm R}$ and $W^{k,p}(\Sigma_\theta)_{\pm R}$ by Lemma 3.2.1(iii) and Lemma 3.2.2(iii). Hence, $Q_\pm$ is a retraction and the assertion follows from [51, Theorem 1.2.4]. □

3.2.4 Remark. It is straightforward to verify that all results from Section 3.1 can be transferred to the corresponding reflection invariant setting. We will not state all results again in this section but focus on results which are essential within this thesis. However, transferring the results almost always makes use of the projection $Q_\pm$ from Lemma 3.2.1 and Lemma 3.2.2.

3.2.1 Multiplication with the Sign Function

In this section we will prove that multiplication with the sign function sgn is a bounded operator if the correct symmetry is given. Hence, then also multiplication with the normal vector $n = (-\sin(\theta), \operatorname{sgn}\cos(\theta))$ is well-defined. This shows that introducing reflection invariant subspaces is somehow natural in order to perform analysis on sectors.

At first we briefly define the reflection invariant subspace of a function space where functions $f : \mathbb{R} \to \mathbb{R}^n$ are contained. By making use of push-forward ζ_*^0 and pull-back ζ_0^* from Lemma 3.1.5, we can reduce the boundedness of the tangential and normal traces to corresponding estimates on $\mathbb{R}$.

3.2.5 Corollary. *Let $s \in [-1,1]$ and ζ_*^0, ζ_0^* from Lemma 3.1.5 be given. Then*

$$\zeta_*^0 \in \mathscr{L}_{is}(\hat{H}^s(\mathbb{R})_{\pm\tilde{r}}, \hat{H}^s(\Gamma_\delta)_{\pm r}) \quad \text{and} \quad \zeta_0^* \in \mathscr{L}_{is}(\hat{H}^s(\Gamma_\delta)_{\pm r}, \hat{H}^s(\mathbb{R})_{\pm\tilde{r}})$$

for $s \in [-1,1]$. Here, we define for any scalar-valued function space E on $\mathbb{R}$

$$E_{\pm\tilde{r}} := \{u \in E : \pm\tilde{r}u(t) := \pm u(-t) = u(t)\}$$

and any vector-valued function space E on $\mathbb{R}$

$$E_{\pm\tilde{R}} := \{u = (u^1, u^2) \in E : \pm\tilde{R}(u^1, u^2)(t) = \pm(u^1(-t), -u^2(-t)) = u(t)\}.$$

Proof. Let $s \in [-1,1]$. The assertion essentially follows from Lemma 3.1.5. We note that if $f \in \hat{H}^s(\Gamma_\delta)_{\pm r}$ is smooth with $f(x_1, -x_2) = \pm f(x_1, x_2)$ then

$$\zeta_0^* f(-t) = f\left(\cos(\theta)|t|, -\sin(\theta)t\right) = \pm f\left(\cos(\theta)|t|, \sin(\theta)t\right) = \pm\zeta_0^*(f)(t).$$

On the other hand, if $f \in \hat{H}^s(\mathbb{R})_{\pm\tilde{r}}$ with $f(-t) = \pm f(t)$ then

$$\zeta_*^\delta f\left(\cos(\theta)|t|, -\sin(\theta)t\right) = f(-t) = \pm f(t) = \pm\zeta_*^\delta f\left(\cos(\theta)|t|, -\sin(\theta)t\right).$$

And the assertion follows. □

Next, we consider the boundedness of the operations $\tau\cdot$ and $n\cdot$ from $\hat{H}^{\pm 1/2}(\Gamma,\mathbb{R}^2)$ to $\hat{H}^{\pm 1/2}(\Gamma,\mathbb{R})$ where τ and n are tangential and outer normal vector fields given by

$$\begin{aligned} n(x) &= n(x_2) = (-\sin(\theta), \operatorname{sgn}(x_2)\cos(\theta)),\\ \tau(x) &= \tau(x_2) = (\operatorname{sgn}(x_2)\cos(\theta), \sin(\theta)), \end{aligned}$$

for $x = (x_1, x_2) \in \Gamma$ (cf. Lemma 3.1.1). However, as we will observe this is only achievable if the functions have the correct symmetry, since multiplication with sgn is not a bounded operator on $H^{1/2}(\mathbb{R})$ in general, cf. [51, Section 2.10.2, Remark 1]. We make use of Corollary 3.2.5 to transfer results from $\mathbb{R}$ to Γ. We finally prove

3.2.6 Lemma. *For $s \in [0, 1/2]$ the multiplication with the sign function is a bounded operator on $\hat{H}^s(\mathbb{R})_{-\tilde{r}}$ and $\hat{H}^{-s}(\mathbb{R})_{\tilde{r}}$, to be precise:*

$$\operatorname{sgn}\cdot \in \mathscr{L}(\hat{H}^s(\mathbb{R})_{-\tilde{r}}, \hat{H}^s(\mathbb{R})_{\tilde{r}}) \cap \mathscr{L}(\hat{H}^{-s}(\mathbb{R})_{\tilde{r}}, \hat{H}^{-s}(\mathbb{R})_{-\tilde{r}}).$$

The statement also holds in the inhomogeneous setting.

Proof. Obviously, we have $\operatorname{sgn}\cdot \in \mathscr{L}(L^2(\mathbb{R}))$ since sgn is bounded. Let $f \in \hat{H}^{1/2}(\mathbb{R})_{-\tilde{r}}$. Then we calculate by using the Slobodeckij norm:

$$\begin{aligned} \|\operatorname{sgn} f\|^2_{\hat{H}^{1/2}(\mathbb{R})_{\tilde{r}}} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\operatorname{sgn}(x)f(x) - \operatorname{sgn}(y)f(y)|^2}{|x-y|^2}\,dy\,dx \\ &= \int_{-\infty}^0\int_{-\infty}^0 \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dy\,dx + \int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dy\,dx \\ &\quad + 2\int_{-\infty}^0\int_0^\infty \frac{|f(x)+f(y)|^2}{|x-y|^2}\,dy\,dx \\ &\le C\|f\|^2_{\hat{H}^{1/2}(\mathbb{R}_+)} \le C\|f\|^2_{\hat{H}^{1/2}(\mathbb{R})_{-\tilde{r}}}, \end{aligned}$$

where we note that

$$\begin{aligned} 2\int_{-\infty}^0\int_0^\infty \frac{|f(x)+f(y)|^2}{|x-y|^2}\,dy\,dx &= 2\int_0^\infty\int_0^\infty \frac{|f(-z)+f(y)|^2}{|y+z|^2}\,dy\,dz \\ &= 2\int_0^\infty\int_0^\infty \frac{|f(y)-f(z)|^2}{|y+z|^2}\,dy\,dz \\ &\le 2\int_0^\infty\int_0^\infty \frac{|f(y)-f(z)|^2}{|y-z|^2}\,dy\,dz \\ &\le 2\|f\|_{\hat{H}^{1/2}(\mathbb{R}_+)} \end{aligned}$$

using the transform $z = -x$ and making use of the symmetry $f(-z) = -f(z)$ and $|y+z|^2 = (y+z)^2 \ge (y-z)^2 = |y-z|^2$ for $y, z > 0$. Interpolation (cf. Corollary 3.2.3) yields the assertion for $s \in [0, 1/2]$. The assertion for $\hat{H}^{-s}(\mathbb{R})_{\tilde{r}}$ follows by duality since $(\operatorname{sgn}\cdot)' = \operatorname{sgn}\cdot$ in $L^2(\mathbb{R})$ and $(\hat{H}^{1/2}(\mathbb{R})_{\pm\tilde{r}})' = \hat{H}^{-1/2}(\mathbb{R})_{\pm\tilde{r}}$ by Lemma 3.2.1. □

3.2.7 Remark. (i) Since Lemma 3.2.6 holds, we can immediately deduce that $\operatorname{sgn}\cdot : H^{1/2}(\mathbb{R})_{\tilde{r}} \to H^{1/2}(\mathbb{R})_{-\tilde{r}}$ cannot be bounded. Since then by decomposing $H^{1/2}(\mathbb{R}) = H^{1/2}(\mathbb{R})_{\tilde{r}} + H^{1/2}(\mathbb{R})_{-\tilde{r}}$ in even and odd functions would imply that $\operatorname{sgn}\cdot$ would be bounded on $H^{1/2}(\mathbb{R})$ which definitely does not hold by [51, Section 2.10.2, Remark 1].

(ii) Multiplication with sgn is known to be a bounded operator on $H^s(\mathbb{R})$ for $s \in [0, 1/2)$. Hence in Lemma 3.2.6 we can actually drop the symmetry restrictions for $s \in [0, 1/2)$.

3.2.8 Lemma. *For $s \in [0, 1/2]$ we have*

$$(v \mapsto n \cdot v) \in \mathscr{L}(\hat{H}^s(\Gamma)_R, \hat{H}^s(\Gamma)_r),$$
$$(v \mapsto \tau \cdot v) \in \mathscr{L}(\hat{H}^s(\Gamma)_{-R}, \hat{H}^s(\Gamma)_r),$$

and

$$(v \mapsto n \cdot v) \in \mathscr{L}(\hat{H}^{-s}(\Gamma)_{-R}, \hat{H}^{-s}(\Gamma)_{-r}),$$
$$(v \mapsto \tau \cdot v) \in \mathscr{L}(\hat{H}^{-s}(\Gamma)_R, \hat{H}^{-s}(\Gamma)_{-r}).$$

The same assertions hold true for the inhomogeneous counterparts of the spaces.

Proof. This is a direct consequence of Corollary 3.2.5 and Lemma 3.2.6. □

3.2.2 Elliptic Problems

In this section we briefly collect the results from Section 3.1.3 transferred to the framework of reflection invariant spaces. However, the strategy is always as follows: Since we assume the data to have a certain symmetry, it follows by the uniqueness that the solution also has to have a certain symmetry.

3.2.9 Corollary (Strong and weak homogeneous Dirichlet problem)**.** *For every $f \in \hat{H}^{-1}(\Sigma_\theta)_{\pm r}$ there exists a unique solution $p \in \hat{H}^1_0(\Sigma_\theta)_{\pm r}$ of (3.12) in the weak sense satisfying*

$$\|\nabla p\|_{L^2(\Sigma_\theta)_{\pm R}} \leq C\|f\|_{\hat{H}^{-1}(\Sigma_\theta)_{\pm r}}$$

with $C > 0$ independent of f and p. If, in addition, $f \in L^2(\Sigma_\theta)_{\pm r}$, then we have $\nabla p \in H^1(\Sigma_\theta)_{\pm R}$ and

$$\|\nabla^2 p\|_{L^2(\Sigma_\theta)_{\pm\mathcal{R}}} \leq C\|f\|_{L^2(\Sigma_\theta)_{\pm r}}.$$

Proof. Let p be the unique solution of (3.12) from Lemma 3.1.31. We assume $f = rf$. Then it is straightforward to prove that rp solves $\Delta(rp) = r\Delta p = rf = f$. From the uniqueness of the solution we infer $p = rp$. □

The proofs for the weak and very weak (in)homogeneous Dirichlet problem are given accordingly:

3.2.10 Corollary (Weak inhomogeneous Dirichlet problem). *For every pair of data* $(f, g) \in \hat{H}^{-1}(\Sigma_\theta)_{\pm r} \times \hat{H}^{1/2}(\Gamma)_{\pm r}$ *there exists a unique solution* $p \in \hat{H}^1(\Sigma_\theta)_{\pm r}$ *of (3.16) in the weak sense satisfying*

$$\|\nabla p\|_{L^2(\Sigma_\theta)_{\pm R}} \leq C\left(\|f\|_{\hat{H}^{-1}(\Sigma_\theta)_{\pm r}} + \|g\|_{\hat{H}^{1/2}(\Gamma)_{\pm r}}\right)$$

with $C > 0$ *independent of* f, g *and* p.

3.2.11 Corollary (Very weak inhomogeneous Dirichlet problem). *For every data* $g \in \hat{H}^{-1/2}(\Gamma)_{\pm r}$ *there exists a unique solution* $p \in L^2(\Sigma_\theta)_{\pm r}$ *of (3.17) in the very weak sense satisfying*

$$\|p\|_{L^2(\Sigma_\theta)_{\pm r}} \leq C\|g\|_{\hat{H}^{-1/2}(\Gamma)_{\pm r}}$$

with $C > 0$ *independent of* g *and* p.

3.2.3 Trace Theorems

In this section we collect trace theorems from Section 3.1.2 and transfer them to the setting of reflected (in)homogeneous spaces. Here, we will essentially make use of the projection $Q_\pm$ from Lemma 3.2.1 and Lemma 3.2.2 which helps us to construct even and odd functions.

Furthermore, thanks to the symmetry property we are able to prove the existence and surjectivity of the normal trace $T_0 : H^1_{\text{div}}(\Sigma_\theta)_R \to H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$ which we don't obtain in the unreflected setting due to the unboundedness of multiplication with the normal vector field.

3.2.12 Corollary (Trace theorem). *The trace operator*

$$T : \hat{H}^1(\Sigma_\theta)_{\pm R} \to \hat{H}^{1/2}(\Gamma)_{\pm R}$$

exists and is linear and bounded and satisfies

$$\|Tu\|_{\hat{H}^{1/2}(\Gamma)_{\pm R}} \leq C\|u\|_{\hat{H}^1(\Sigma_\theta)_{\pm R}}$$

with a constant $C > 0$ *independent of* u*. Furthermore,* T *is a retraction: There exists a bounded linear extension operator*

$$E : \hat{H}^{1/2}(\Gamma)_{\pm R} \to \hat{H}^1(\Sigma)_{\pm R}$$

such that if $\tilde{u} \in \hat{H}^{1/2}(\Gamma)_{\pm R}$ *then we have* $u := E\tilde{u} \in \hat{H}^1(\Sigma_\theta)_{\pm R}$ *with* $Tu = \tilde{u}$ *and*

$$\|u\|_{\hat{H}^1(\Sigma_\theta)_{\pm R}} \leq C\|\tilde{u}\|_{\hat{H}^{1/2}(\Gamma)_{\pm R}}$$

where $C > 0$ is again independent of $\tilde{u}$.
All statements also hold for $\hat{H}^1(\Sigma_\theta)_{\pm r}$ and $\hat{H}^{1/2}(\Gamma)_{\pm r}$ and the corresponding inhomogeneous counterparts.

Proof. We only prove the case $+R$. The other case $-R$ follows analogously. In view of Theorem 3.1.19 we only need to ensure that if $Ru = u$ in Σ_θ then $RTu = Tu$ on Γ and the other way around.
Let T be the same trace operator from Theorem 3.1.19. First we observe that from Lemma 3.2.1(iv) we obtain $C_c^\infty(\overline{\Sigma_\theta})_R \overset{d}{\hookrightarrow} \hat{H}^1(\Sigma_\theta)_R$. Let $u \in \hat{H}^1(\Sigma_\theta)_R$ be arbitrary. Then due to the density there exists a sequence $(u_k)_k \subseteq C_c^\infty(\overline{\Sigma_\theta})_R$ with $u_k \to u$ in $\hat{H}^1(\Sigma_\theta)_R$. Then obviously $RTu_k = R(u_k|_\Gamma) = u_k|_\Gamma = Tu_k$ since u_k is continuous and even in $\overline{\Sigma_\theta}$. Since T is a continuous operator thanks to Theorem 3.1.19 we infer

$$RTu = RT\left(\lim_{k\to\infty} u_k\right) = R\lim_{k\to\infty} Tu_k = \lim_{k\to\infty} RTu_k = \lim_{k\to\infty} Tu_k = Tu$$

where we also made use of the fact that by Lemma 3.2.1(ii) $\pm R : \hat{H}^{1/2}(\Gamma) \to \hat{H}^{1/2}(\Gamma)$ is bounded.

Now let $\tilde{u} \in \hat{H}^{1/2}(\Gamma)_R$ be arbitrary, i.e., $R\tilde{u} = \tilde{u}$. Then due to Theorem 3.1.19 there exists $v = \tilde{E}\tilde{u} \in \hat{H}^1(\Sigma_\theta)$ with $Tv = \tilde{u}$. Setting $u := 1/2(v + Rv) = Q_+v \in \hat{H}^1(\Sigma_\theta)_R$ (where Q_+ is the projection from Lemma 3.2.1(iii)) we deduce

$$Tu = T\left(\frac{1}{2}(v + Rv)\right) = \frac{1}{2}(Tv + TRv) = \frac{1}{2}(Tv + RTv) = \frac{1}{2}(\tilde{u} + R\tilde{u}) = \tilde{u},$$

where $TRv = RTv$. In fact this holds for continuous $v \in C_c^\infty(\overline{\Sigma_\theta})$ and since the continuous functions $C_c^\infty(\overline{\Sigma_\theta})$ are dense in $\hat{H}^1(\Sigma_\theta)$ and R and T are bounded we obtain this equality for all $v \in \hat{H}^1(\Sigma_\theta)$. Then we can set $E\tilde{u} := u = Q_+\tilde{E}\tilde{u}$ and obviously $E : \hat{H}^{1/2}(\Gamma)_R \to \hat{H}^1(\Sigma_\theta)_R$ is linear and bounded since E inherits the linearity and boundedness of $\tilde{E}$ and Q_+.
Replacing R by r we obtain the exact same statements for $\hat{H}^1(\Sigma_\theta)_r$ and $\hat{H}^{1/2}(\Gamma)_r$ as well as for the inhomogeneous case. □

3.2.13 Corollary. *Let $\hat{H}^2_D(\Sigma_\theta)_{\pm r}$ be defined as in Corollary 3.1.24. Then the Neumann trace operator $T_n : \hat{H}^2_D(\Sigma_\theta)_{\pm r} \to \hat{H}^{1/2}(\Gamma)_{\pm r}$ is bounded and has dense range.*

Proof. By Corollary 3.1.24 the trace operator T_n is well-defined by the same arguments as used in Corollary 3.2.12.

Now let such $v \in C_c^\infty(\Gamma\backslash\{0\})_{\pm r}$ be given. Due to Corollary 3.1.24 there exists some $\tilde{u} \in \hat{H}^2_D(\Sigma_\theta)$ with $T_n\tilde{u} = \partial_n\tilde{u} = v$ on Γ. Using the same arguments as in

Corollary 3.2.12 we deduce that $u = 1/2(\tilde{u} \pm r\tilde{u}) = Q_\pm u \in \hat{H}^2_D(\Sigma_\theta)_{\pm r}$ (where the operator $Q_\pm$ is defined as in Lemma 3.2.1). Then we observe

$$\partial_n u = \frac{1}{2}\partial_n(\tilde{u} \pm r\tilde{u}) = \frac{1}{2}(v \pm rv) = v \quad \text{on } \Gamma,$$

since $v = \pm rv$. In the second step we made use of the fact

$$r\partial_n \tilde{u} = r(n \cdot \nabla T\tilde{u}) = Rn \cdot RT\nabla\tilde{u} = n \cdot T\nabla r\tilde{u} = \partial_n r\tilde{u},$$

where the third equality $RT\nabla\tilde{u} = T\nabla r\tilde{u}$ holds on Γ: Note that the equality holds for smooth $\tilde{u} \in C_c^\infty(\overline{\Sigma_\theta}) \overset{d}{\hookrightarrow} \hat{H}^2(\Sigma_\theta)$ and $R, r : \hat{H}^2_D(\Sigma_\theta) \to \hat{H}^2_D(\Sigma_\theta)$ and the trace operator $T : \hat{H}^1(\Sigma_\theta) \to \hat{H}^{1/2}(\Gamma)$ is bounded due to Theorem 3.1.19 and Lemma 3.2.2(iii). □

3.2.14 Corollary (Generalized trace theorem). *Let*

$$T_0 : L^2_{\text{div}}(\Sigma_\theta)_{\pm R} \to \hat{H}^{-1/2}(\Gamma)_{\pm r}$$

be defined by

$$T_0 v(\psi) = \langle n \cdot v, \psi \rangle_{\hat{H}^{-1/2}(\Gamma)_{\pm r}, \hat{H}^{1/2}(\Gamma)_{\pm r}} := \langle v, \nabla E\psi \rangle_{L^2(\Sigma_\theta)_{\pm R}, L^2(\Sigma_\theta)_{\pm R}},$$

for $\psi \in \hat{H}^{1/2}(\Gamma)_{\pm r}$ *where* $E : \hat{H}^{1/2}(\Gamma)_{\pm r} \to \hat{H}^1(\Sigma_\theta)_{\pm r}$ *is the linear and bounded extension operator to the trace operator* $T : \hat{H}^1(\Sigma_\theta)_{\pm r} \to \hat{H}^{1/2}(\Gamma)_{\pm r}$, *characterized by the inhomogeneous Dirichlet problem:*

$$\Delta E\psi = 0 \quad \text{in } \Sigma_\theta, \qquad E\psi = \varphi \quad \text{on } \Gamma,$$

see Corollary 3.2.10. Then T_0 *is well-defined (especially independent of the choice of the extension operator* E*) and bounded.*

Proof. Note that the Weyl decomposition from Lemma 3.1.15 also holds in the setting of reflection invariant spaces since the weak inhomogeneous Dirichlet problem (cf. Corollary 3.2.10) can be solved in the setting. Then the statement can be proved as in the unreflected setting (Lemma 3.1.28) and all cited results can be replaced by the corresponding results from the reflection invariant setting. □

Since boundedness of sgn · is not given in $H^{1/2}(\mathbb{R})$ due to [51, Section 2.10.2, Remark 1], we cannot expect the normal trace operator T_0 to be bounded in $H^1(\Sigma_\theta)$. However, Lemma 3.1.27 at least states that for any $g \in H^{1/2}(\Gamma) \cap \hat{H}^{-1/2}(\Gamma)_r$ we can find $u \in H^1_{\text{div}}(\Sigma_\theta)$ with $g = u \cdot n$ on Γ. Thanks to Lemma 3.2.8, Corollary 3.2.12 and Corollary 3.2.14, we now immediately obtain existence and boundedness of the trace operator and thanks to Lemma 3.1.27 we can even prove that T_0 is a retraction.

3.2.15 Lemma. *The normal trace operator*

$$T_0 : H^1_{\mathrm{div}}(\Sigma_\theta)_R \to H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$$

is linear, bounded and a retraction. The coretraction is given by $\tilde{R}_0 := Q_+ R_0 : H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r \to H^1_{\mathrm{div}}(\Sigma_\theta)_R$ *where* R_0 *is the linear, bounded operator from Lemma 3.1.27 and* Q_+ *is the projection from Lemma 3.2.2.*

Proof. As already observed above the normal trace operator T_0 is well-defined and bounded by Lemma 3.2.8, Corollary 3.2.12 and Corollary 3.2.14. Note that $T_0 : H^1_{\mathrm{div}}(\Sigma_\theta)_R \to H^{1/2}(\Gamma)_r$ and T_0 from Corollary 3.2.14 are consistent by the formula for integration by parts. We apply the same arguments as in Corollary 3.2.12 and Corollary 3.2.13 to obtain the symmetry properties. Furthermore, we infer that $\tilde{R}_0 := Q_+ R_0 : H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r \to H^1_{\mathrm{div}}(\Sigma_\theta)_R$ is well-defined, linear and bounded. Note that $Q_+ H^1_{\mathrm{div}}(\Sigma_\theta) \subseteq H^1_{\mathrm{div}}(\Sigma_\theta)$. Hence, we have $T_0 \tilde{R}_0 = I_{H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r}$ and the assertion follows. □

3.2.16 Remark. By the construction of the linear operator R_0 in Lemma 3.1.27 we can already deduce that R_0 preserves symmetry properties, i.e., $R_0 = Q_+ R_0$.

3.2.17 Corollary. *The normal trace operator*

$$T_0 : H^1_{\mathrm{div}}(\Sigma_\theta)_R \to \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$$

is linear, bounded and a retraction.

Proof. By Lemma 3.2.15 the normal trace operator $T_0 : H^1_{\mathrm{div}}(\Sigma_\theta)_R \to H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$ is bounded and surjective. We observe that

$$H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r \hookrightarrow \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r \hookrightarrow L^2(\Gamma)_r,$$

since $(\hat{H}^{-1/2}(\Gamma)_r, \hat{H}^{1/2}(\Gamma)_r)_{1/2,2} = L^2(\Gamma)_r$. This shows that the $L^2(\Gamma)_r$ norm can be estimated by the norm in $\hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$. Then we can deduce that

$$H^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r = \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$$

topologically. Then in fact $T_0 : H^1_{\mathrm{div}}(\Sigma_\theta)_R \to \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$. □

Chapter 4

2D Contact Line Dynamics

In this chapter we consider the following set-up of the contact line problem (here in 2D where the contact line is actually a contact point):

$$
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \operatorname{div} T(u,p) &= 0 && \text{in } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \Omega(t), \\
\operatorname{div} u &= 0 && \text{in } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \Omega(t), \\
\lambda u^1 + (D(u) n_s)^1 &= 0 && \text{on } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \Gamma_s(t), \\
u^2 &= 0 && \text{on } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \Gamma_s(t), \\
T(u,p) n_f &= \sigma \kappa n_f && \text{on } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \Gamma_f(t), \\
V_{n_f} &= u \cdot n_f && \text{on } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \Gamma_f(t), \\
V_{\mathcal{C}} &= u \cdot n_{\mathcal{C}} && \text{on } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \mathcal{C}(t), \\
\theta &= \psi(V_{\mathcal{C}}) && \text{on } \textstyle\bigcup_{t \in (0,T)} \{t\} \times \mathcal{C}(t), \\
u|_{t=0} &= u_0 && \text{in } \Omega(0), \\
\Gamma_s(0) &= \Gamma_s^0, && \text{in } \Omega(0), \\
\Gamma_f(0) &= \Gamma_f^0 && \text{in } \Omega(0), \\
\mathcal{C}(0) &= \mathcal{C}_0 && \text{in } \Omega(0).
\end{aligned}
\tag{4.1}
$$

Here, $\Omega(t) \subseteq \mathbb{R}^2$ is a two-dimensional domain at time $t \in (0,T)$ which is moving within the time t. By Γ_s we denote the solid surface whereas by Γ_f we refer to the free upper surface (cf. Figure 1). Furthermore, n_f and n_s denote the normal outer vector at Γ_f and Γ_s, respectively. The surface tension coefficient is given as σ and the mean curvature as κ. The normal velocity of the free surface Γ_f is denoted by V_{n_f}. Again, the stress tensor is written as

$$
T(u,p) := 2\mu D(u) - Ip, \qquad D(u) := \frac{1}{2}(\nabla u + \nabla u^T),
$$

where μ is the viscosity. At the contact point $\mathcal{C}$ the contact point velocity is denoted by $V_{\mathcal{C}}$ and $n_{\mathcal{C}}$ is the corresponding normal vector at the contact point.

Note that the third and fourth equation of (4.1) corresponds to partial slip boundary conditions with slip length λ and the fifth equation is the kinematic condition. The sixth equation describes the normal velocity, whereas the seventh equation addresses the contact point velocity. Note that both equations about the normal velocity and the contact point velocity show that there is no phase transition at the interface and at the contact point. The constitutive equation $\theta = \psi(V_{\mathcal{C}})$ models the contact angle at the contact point, which is the point where the upper free surface Γ_f gets in contact with the solid surface Γ_s.

The chapter is structured as follows: At first we transform (4.1) to a fixed wedge-type domain $(0,T) \times \Sigma_\theta$ by applying a suitable transformation in Section 4.1 leading to the following resolvent Stokes system:

$$\begin{aligned} \lambda u - \operatorname{div} T(u,p) &= f_1 \quad \text{in } \Sigma_\theta, \\ \operatorname{div} u &= f_2 \quad \text{in } \Sigma_\theta, \\ T(u,p)n + \sigma c(\theta)\partial_\tau^2 \rho n &= f_4 \quad \text{on } \partial\Sigma_\theta, \\ \lambda\rho + \tfrac{1}{\sin(\theta)}(n \cdot u) &= f_5 \quad \text{on } \partial\Sigma_\theta. \end{aligned}$$

Then we study the resulting Stokes system in Section 4.2 assuming $f_2 = 0$ and $f_5 = 0$ for simplicity. We prove the existence of a triple (u,p,ρ) which solves the system in the weak sense. Furthermore, the triple (u,p,ρ) fulfills corresponding resolvent estimates.

4.1 Transformation

In this section we want to apply a transformation to (4.1) such that we have a fixed Ω that is not moving in time such that a rigorous analysis is simplified. To this end, we assume the origin to be located at the meeting point of Γ_s^0 and Γ_f^0. At first we note that we can write the solid surface and the free surface as

$$\begin{aligned} \Gamma_s(t) &= \{(y_1,0) : y_1 \in (y^*,\infty)\}, \\ \Gamma_f(t) &= \{(y_1,h(t,y_1)) : y_1 \in (y^*,\infty)\}, \end{aligned}$$

where $h(t,\cdot)$ is the height function. The contact point is obviously $\mathcal{C}(t) = (y^*,0)$ where $h(t,y^*) = 0$, i.e., where the free surface meets the solid surface at $y_2 = 0$. We want to parametrize Γ_f w.r.t. the y_2-axis, then we have

$$\Gamma_f(t) = \left\{(h^{-1}(t,y_2),y_2) : y_2 \in (0,\infty)\right\},$$

if we assume that h is monotone increasing. Then we set $b(t,y_2) := h^{-1}(t,y_2)$. Now we get another parametrization of Γ_s and Γ_f:

$$\Gamma_s(t) = \{(y_1,0) : y_1 \geq b(t,0)\},$$

$$\Gamma_f(t) = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_2 \in (0,\infty), y_1 = b(t, y_2)\right\},$$

and the contact point is now at $\mathcal{C}(t) = (b(t,0),0)$. Note that the initial free surface

$$\Gamma_f^0 = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_2 \in (0,\infty),\ y_1 = b(0, y_2) = h^{-1}(0, y_2)\right\}$$

is given with initial contact angle $\theta_0 \in (0, \pi/2)$ between Γ_f^0 and Γ_s^0. Furthermore, we deduce $\partial_{y_2} h(0,0) = \tan(\theta_0)$ by the fact that θ_0 is the initial contact angle and that $h(0,\cdot)$ parametrizes Γ_f^0. Now we define the wedge for $\theta_0 \in (0, \pi/2)$ as

$$\Sigma_{\theta_0} := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (0,\infty),\ 0 \le x_2 < \tan(\theta_0) x_1\right\},$$

where we note that for simplicity we make the assumption

$$\lim_{y_1 \to \infty} \partial_{y_1} h(0, y_1) = \tan(\theta_0) \tag{4.2}$$

(which means that at the initial free surface Γ_f^0 even for large y_1 we still have the contact angle θ_0). Basically, we then have $h(0, y_1) \sim \tan(\theta_0) y_1$ for large y_1. The transformation from the fixed wedge domain to the free moving domain along the x_1-axis is given as:

$$\begin{aligned}
&\Theta(t) : \Sigma_{\theta_0} \to \Omega(t),\\
&(x_1, x_2) \mapsto (y_1(t), y_2) = \Theta(t, x_1, x_2) := (x_1 - \partial_{x_2} b_0(0) x_2 + b(t, x_2), x_2),
\end{aligned}$$

where

$$\begin{aligned}
\Omega(t) &= \{(x_1, x_2) : x_1 = b(t, y_2) \text{ for a } y_2 \in (0,\infty),\ x_2 \in (0, h(t, x_1))\}\\
&= \{(x_1, x_2) : x_1 \in \mathbb{R},\ x_2 \in (0, h(t, x_1))\}\\
&= \{(x_1, x_2) : x_2 \in (0,\infty),\ x_1 > b(t, x_2)\}
\end{aligned}$$

and $b_0(x_2) := b(0, x_2)$. First, we need to assure that the transformation is well-defined which means showing that for a fixed $(x_1, x_2) \in \Sigma_{\theta_0}$ we have $\Theta(t, x_1, x_2) \in \Omega(t)$. Now let $(x_1, x_2) \in \Sigma_{\theta_0}$. According to the definition of $\Omega(t)$ it is sufficient to ensure that $\tilde{x}_1 > b(t, \tilde{x}_2)$ for $(\tilde{x}_1, \tilde{x}_2) = \Theta(t, x_1, x_2) = (x_1 - \partial_{x_2} b_0(0) x_2 + b(t, x_2), x_2)$. It follows $\tilde{x}_2 = x_2$ such that we need to prove $x_1 - \partial_{x_2} b_0(0) x_2 + b(t, x_2) > b(t, x_2)$, which actually means that we have to assure $x_1 - \partial_{x_2} b_0(0) x_2 > 0$. Since $0 \le x_2 < \tan(\theta_0) x_1$ and $\partial_{x_2} b_0(0) = \cot(\theta_0)$ we deduce

$$x_1 - \partial_{x_2} b_0(0) x_2 = x_1 - \cot(\theta_0) x_2 > x_1 - \cot(\theta_0)\tan(\theta_0) x_1 = 0.$$

(It holds $\partial_{x_2} b(0,0) = (\partial_{x_1} h(0,0))^{-1} = \tan(\theta_0)^{-1} = \cot(\theta_0)$ by the rule for derivation of inverse functions.) Then $\Theta(t)$ is well-defined for every $t \in (0,T)$. The full time-space transformation is presented by

$$\Phi : (0,T) \times \Sigma_{\theta_0} \to \bigcup_{t \in (0,T)} \{t\} \times \Omega(t),$$

$$(t, x_1, x_2) \mapsto (t, y_1(t), y_2) = \Phi(t, x_1, x_2) := (t, \Theta(t, x_1, x_2)).$$

Obviously, then Φ is also well-defined. Moreover, we define

$$\rho(t, x_1, x_2) := \rho(t, x_2) := b(t, x_2) - \partial_{x_2} b_0(0) x_2 \qquad ((t, x_1, x_2) \in [0, T) \times \Sigma_{\theta_0}).$$

Then we also have $\rho(0, x_2) = b(0, x_2) - \partial_{x_2} b_0(0) x_2 = b(0, x_2) - \cot(\theta_0) x_2$, which results in

$$\lim_{x_2 \to \infty} \frac{\rho(0, x_2)}{x_2} = \lim_{x_2 \to \infty} \frac{b(0, x_2) - b(0, 0)}{x_2 - 0} - \cot(\theta_0) = \lim_{x_2 \to \infty} \partial_{x_2} b(0, x_2) - \cot(\theta_0) = 0$$

by (4.2) such that $\rho(0, x_2) \to 0$ as $x_2 \to \infty$. Then we can simplify the definition of the transformation as

$$\begin{aligned} (y_1, y_2) &= \Theta(t, x_1, x_2) = (x_1 + \rho(t, x_2), x_2) : \Sigma_{\theta_0} \to \Omega(t), \\ (x_1, x_2) &= \Theta^{-1}(t, y_1, y_2) = (y_1 - \rho(t, y_2), y_2) : \Omega(t) \to \Sigma_{\theta_0}. \end{aligned}$$

We note that Θ^{-1} is well-defined as well: To this end, let $(y_1, y_2) \in \Omega(t)$. By the definition of $\Omega(t)$ and the property of h and b by being strictly increasing we know that $y_2 < h(t, y_1)$ since $(y_1, h(t, y_1)) \in \Gamma_f(t)$ and on the other hand $y_1 > b(t, y_2)$ since $(b(t, y_2), y_2) \in \Gamma_f(t)$. Then we deduce

$$x_1 = y_1 - \rho(t, y_2) = y_1 - b(t, y_2) + \partial_{y_2} b_0(0) y_2 > \cot(\theta_0) y_2 > 0$$

since $\theta_0 \in (0, \pi/2)$ and $y_2 > 0$ since $(y_1, y_2) \in \Omega(t)$. For the second component x_2 we obtain

$$\begin{aligned} 0 \le x_2 = y_2 &= \tan(\theta_0) \cot(\theta_0) y_2 = \tan(\theta_0) \partial_{y_2} b_0(0) y_2 \\ &\le \tan(\theta_0)(y_1 - b(t, y_2)) + \tan(\theta_0) \partial_{y_2} b_0(0) y_2 \\ &= \tan(\theta_0)(y_1 - \rho(t, y_2)) = \tan(\theta_0) x_1, \end{aligned}$$

which yields $(x_1, x_2) \in \Sigma_{\theta_0}$. Furthermore, it is obvious that Θ and Θ^{-1} are inverse to each other. Then also

$$\begin{aligned} &\Phi : (0, T) \times \Sigma_{\theta_0} \to \bigcup_{t \in (0, T)} \{t\} \times \Omega(t), \\ &(t, x_1, x_2) \mapsto (t, y_1(t), y_2) = \Phi(t, x_1, x_2) := (t, \Theta(t, x_1, x_2)), \end{aligned}$$

and

$$\Phi^{-1} : \bigcup_{t \in (0, T)} \{t\} \times \Omega(t) \to (0, T) \times \Sigma_{\theta_0},$$

$$(t, y_1, y_2) \mapsto (t, x_1(t), x_2) = \Phi^{-1}(t, y_1, y_2) := (t, \Theta^{-1}(t, y_1, y_2))$$

are inverse to each other. We note that it depends on $b(t,\cdot)$ which regularity the transformation has.

Next, we want to apply the transformation to our main system (4.1) in order to get a system on a fixed time-space domain $(0,T) \times \Sigma_{\theta_0}$. To this end, we denote push forward and pull back by

$$\begin{aligned} u = \Phi^* v = v \circ \Phi^{-1} &: \bigcup_{t \in (0,T)} \{t\} \times \Omega(t) \to \mathbb{R}^2, \\ p = \Phi^* q = q \circ \Phi^{-1} &: \bigcup_{t \in (0,T)} \{t\} \times \Omega(t) \to \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} v = \Phi_* u = u \circ \Phi &: (0,T) \times \Sigma_{\theta_0} \to \mathbb{R}^2, \\ q = \Phi_* p = p \circ \Phi &: (0,T) \times \Sigma_{\theta_0} \to \mathbb{R}, \end{aligned}$$

where (u, p) is the solution of the original system (4.1) and (v, q) will be the solution of the transformed system.

Now, we need to transform the (u, p) terms in (4.1) to terms depending on (v, q), since they are defined on a fixed domain. We note that if h and b, respectively, are smooth then the Jacobian (derivation in space dimension) of Θ is given by

$$D\Theta(t, x_1, x_2) = \begin{pmatrix} 1 & \partial_{x_2}\rho(t, x_2) \\ 0 & 1 \end{pmatrix}, \qquad D\Theta^{-1}(t, y_1, y_2) = \begin{pmatrix} 1 & -\partial_{y_2}\rho(t, y_2) \\ 0 & 1 \end{pmatrix}.$$

In order to obtain (4.1) in terms of (v, q) we need to apply Φ_* to (4.1). At first we calculate the transformation of the time derivative $\partial_t u$:

$$\begin{aligned} \Phi_* \partial_t u = \partial_t u \circ \Phi &= (\partial_t u)(t, \Theta(t, x_1, x_2)) \\ &= \partial_t (u(t, \Theta(t, x_1, x_2)) - (\partial_{x_1} u)(t, \Theta(t, x_1, x_2)) \partial_t \rho \\ &= \partial_t v - \partial_{x_1} v \partial_t \rho, \end{aligned}$$

since

$$\begin{aligned} \partial_t (u(t, \Theta(t, x_1, x_2)) &= (\partial_t u)(t, \Theta(t, x_1, x_2)) + (\nabla u)(t, \Theta(t, x_1, x_2))^T (\partial_t \rho, 0) \\ &= (\partial_t u)(t, \Theta(t, x_1, x_2)) + \underbrace{(\partial_{x_1} u)(t, \Theta(t, x_1, x_2))}_{\substack{= (\nabla u)(t, \Theta(t, x_1, x_2)) \partial_{x_1} \Theta(t, x_1, x_2) \\ = \partial_{x_1}(u(t, \Theta(t, x_1, x_2)))}} \partial_t \rho. \end{aligned}$$

In general we can transform derivatives in x_1 as follows for arbitrary functions φ:

$$\Phi_* \partial_{x_1} \varphi = \partial_{x_1} \varphi \circ \Phi = (\partial_{x_1} \varphi)(t, \Theta(t, x_1, x_2))$$

$$
\begin{aligned}
&= (\nabla\varphi))(t,\Theta(t,x_1,x_2))^T \underbrace{(1,0)}_{=\partial_{x_1}\Theta(t,x_1,x_2)} \\
&= \partial_{x_1}(\varphi(t,\Theta(t,x_1,x_2))) = \partial_{x_1}\Phi_*\varphi.
\end{aligned}
$$

It will be more difficult to calculate the first and second order derivatives in x_2 since the transformation in x_2 involves the height function ρ. At first we calculate the first derivative:

$$
\begin{aligned}
\Phi_*\partial_{x_2}u^k = (\partial_{x_2})u^k \circ \Phi = \partial_{x_2}u^k(t,\Theta(t,x_1,x_2)) \\
&= \partial_{x_2}(u^k(t,\Theta(t,x_1,x_2)) - (\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}\rho \\
&= \partial_{x_2}v^k - \partial_{x_1}v^k\partial_{x_2}\rho
\end{aligned}
$$

for $k = 1, 2$, since

$$
\begin{aligned}
\partial_{x_2}(u^k(t,\Theta(t,x_1,x_2)) &= (\nabla u^k)(t,\Theta(t,x_1,x_2))^T \underbrace{(\partial_{x_2}\rho, 1)}_{=\partial_{x_2}\Theta(t,x_1,x_2)} \\
&= (\partial_{x_2}u^k)(t,\Theta(t,x_1,x_2)) + \underbrace{(\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))}_{=\partial_{x_1}(u^k(t,\Theta(t,x_1,x_2)))}\partial_{x_2}\rho.
\end{aligned}
$$

The transformation of the second derivative is given as:

$$
\begin{aligned}
\Phi_*\partial_{x_2}^2 u^k &= (\partial_{x_2}^2 u^k)(t,\Theta(t,x_1,x_2)) \\
&= \partial_{x_2}^2(u^k(t,\Theta(t,x_1,x_2)) - (\partial_{x_1}\partial_{x_2}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}\rho \\
&\quad - (\partial_{x_1}^2 u^k)(t,\Theta(t,x_1,x_2))(\partial_{x_2}\rho)^2 - (\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}^2\rho \\
&= \partial_{x_2}^2 v^k - 2\partial_{x_1}\partial_{x_2}v^k\partial_{x_2}\rho - \partial_{x_1}^2 v^k(\partial_{x_2}\rho)^2 - \partial_{x_1}v^k\partial_{x_2}^2\rho
\end{aligned}
$$

for $k = 1, 2$. In the following we will show how we transformed derivatives in x_2 of second order more precisely: The difficulty lies in the fact that now mixed derivatives in x_1 and x_2 are also involved now. First, we have

$$
\begin{aligned}
\partial_{x_2}^2(u^k(t,\Theta(t,x_1,x_2)) &= \partial_{x_2}(\partial_{x_2}(u^k(t,\Theta(t,x_1,x_2))) \\
&= \partial_{x_2}((\partial_{x_2}u^k)(t,\Theta(t,x_1,x_2))) \\
&\quad + \partial_{x_2}((\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}\rho).
\end{aligned}
\tag{4.3}
$$

For the first part of (4.3) we have using the calculation of $\partial_{x_2}(u^k(t,\Theta(t,x_1,x_2)))$ above:

$$
\begin{aligned}
\partial_{x_2}((\partial_{x_2}u^k)(t,\Theta(t,x_1,x_2))) &= (\partial_{x_2}^2 u^k)(t,\Theta(t,x_1,x_2)) \\
&\quad + (\partial_{x_1}\partial_{x_2}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}\rho
\end{aligned}
$$

and for the second term in (4.3) we calculate:

$$\begin{aligned}
&\partial_{x_2}((\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}\rho)\\
&= \partial_{x_2}(\partial_{x_1}u^k(t,\Theta(t,x_1,x_2)))\partial_{x_2}\rho + (\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}^2\rho\\
&= (\partial_{x_1}^2u^k)(t,\Theta(t,x_1,x_2))(\partial_{x_2}\rho)^2 + (\partial_{x_2}\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}\rho\\
&\quad + (\partial_{x_1}u^k)(t,\Theta(t,x_1,x_2))\partial_{x_2}^2\rho.
\end{aligned}$$

Hence, we obviously obtain with the calculation we made above:

$$\Phi_*\partial_{x_1}\partial_{x_2}u^k = \partial_{x_1}\Phi_*\partial_{x_2}u^k = \partial_{x_1}(\partial_{x_2}v^k - \partial_{x_1}v^k\partial_{x_2}\rho) = \partial_{x_1}\partial_{x_2}v^k - \partial_{x_1}^2v^k\partial_{x_2}\rho.$$

Using the calculations for u from above we are able to transform the pressure p immediately:

$$\Phi_*\nabla p = (\partial_{x_1}q, \partial_{x_2}q - (\partial_{x_1}q)(\partial_{x_2}\rho)) = \nabla q - (0, (\partial_{x_1}q)(\partial_{x_2}\rho)).$$

In order to transform the first two equations of (4.1) we also need to calculate

$$\Phi_* \operatorname{div} u = \partial_{x_1}v^1 + \partial_{x_2}v^2 - \partial_{x_1}v^2\partial_{x_2}\rho = \operatorname{div} v - (\partial_{x_1}v^2)(\partial_{x_2}\rho)$$

for the divergence. Transforming the stress tensor, we first observe that

$$\begin{aligned}
T(u,p) &= 2\mu D(u) - Ip = \mu(\nabla u + \nabla u^T) - Ip\\
&= \mu\left(\begin{pmatrix}\partial_{x_1}u^1 & \partial_{x_2}u^1\\ \partial_{x_1}u^2 & \partial_{x_2}u^2\end{pmatrix} + \begin{pmatrix}\partial_{x_1}u^1 & \partial_{x_1}u^2\\ \partial_{x_2}u^1 & \partial_{x_2}u^2\end{pmatrix}\right) - \begin{pmatrix}p^1 & 0\\ 0 & p^2\end{pmatrix},
\end{aligned}$$

hence the divergence of the stress tensor is given as

$$\begin{aligned}
&\operatorname{div} T(u,p)\\
&= \mu\nabla\cdot\left(\begin{pmatrix}\partial_{x_1}u^1 & \partial_{x_2}u^1\\ \partial_{x_1}u^2 & \partial_{x_2}u^2\end{pmatrix} + \begin{pmatrix}\partial_{x_1}u^1 & \partial_{x_1}u^2\\ \partial_{x_2}u^1 & \partial_{x_2}u^2\end{pmatrix}\right) - \nabla\cdot\begin{pmatrix}p^1 & 0\\ 0 & p^2\end{pmatrix}\\
&= \mu(\partial_{x_1}^2u^1 + \partial_{x_1}\partial_{x_2}u^2, \partial_{x_1}\partial_{x_2}u^1 + \partial_{x_2}^2u^2) + \mu(\partial_{x_1}^2u^1 + \partial_{x_2}^2u^1, \partial_{x_1}^2u^2 + \partial_{x_2}^2u^2) - \nabla p\\
&= \mu(2\partial_{x_1}^2u^1 + \partial_{x_2}^2u^1 + \partial_{x_1}\partial_{x_2}u^2, \partial_{x_1}^2u^2 + 2\partial_{x_2}^2u^2 + \partial_{x_1}\partial_{x_2}u^1) - \nabla p\\
&= \mu\Delta u + \mu\nabla\operatorname{div} u - \nabla p.
\end{aligned}$$

Having this form we can now transform the divergence of the stress tensor more easily by using the transformations that we calculated before:

$$\begin{aligned}
&\Phi_* \operatorname{div} T(u,p)\\
&= \mu(2\Phi_*\partial_{x_1}^2u^1 + \Phi_*\partial_{x_2}^2u^1 + \Phi_*\partial_{x_1}\partial_{x_2}u^2, \Phi_*\partial_{x_1}^2u^2 + 2\Phi_*\partial_{x_2}^2u^2 + \Phi_*\partial_{x_1}\partial_{x_2}u^1)
\end{aligned}$$

$$
\begin{aligned}
&- (\Phi_* \partial_{x_1} p, \Phi_* \partial_{x_2} p) \\
&= \mu(2\partial_{x_1}^2 v^1 + \partial_{x_2}^2 v^1 - 2\partial_{x_1}\partial_{x_2} v^1 \partial_{x_2}\rho + \partial_{x_1}^2 v^1 (\partial_{x_2}\rho)^2 - \partial_{x_1} v^1 \partial_{x_2}^2 \rho \\
&\quad + \partial_{x_1}\partial_{x_2} v^2 - \partial_{x_1}^2 v^2 \partial_{x_2}\rho, \\
&\quad \partial_{x_1}^2 v^2 + 2(\partial_{x_2}^2 v^2 - 2\partial_{x_1}\partial_{x_2} v^2 \partial_{x_2}\rho + \partial_{x_1}^2 v^2 (\partial_{x_2}\rho)^2 - \partial_{x_1} v^2 \partial_{x_2}^2 \rho) \\
&\quad + \partial_{x_1}\partial_{x_2} v^1 - \partial_{x_1}^2 v^1 \partial_{x_2}\rho) \\
&\quad - (\partial_{x_1} q, \partial_{x_2} q - (\partial_{x_1} q)(\partial_{x_2}\rho)) \\
&= \operatorname{div} T(v,q) + \mu(-2\partial_{x_1}\partial_{x_2} v^1 \partial_{x_2}\rho + \partial_{x_1}^2 v^1 (\partial_{x_2}\rho)^2 - \partial_{x_1} v^1 \partial_{x_2}^2\rho - \partial_{x_1}^2 v^2 \partial_{x_2}\rho, \\
&\quad - 4\partial_{x_1}\partial_{x_2} v^2 \partial_{x_2}\rho + 2\partial_{x_1}^2 v^2 (\partial_{x_2}\rho)^2 - 2\partial_{x_1} v^2 \partial_{x_2}^2 \rho - \partial_{x_1}^2 v^1 \partial_{x_2}\rho) \\
&\quad + (0, (\partial_{x_1} q)(\partial_{x_2}\rho)) \\
&= \operatorname{div} T(v,q) + \mu(-2\partial_{x_1}\partial_{x_2} v \partial_{x_2}\rho + \partial_{x_1}^2 v (\partial_{x_2}\rho)^2 - \partial_{x_1} v \partial_{x_2}^2 \rho - \partial_{x_1}^2 (v^2, v^1) \partial_{x_2}\rho \\
&\quad + (0, -2\partial_{x_1}\partial_{x_2} v^2 \partial_{x_2}\rho + \partial_{x_1}^2 v^2 (\partial_{x_2}\rho)^2 - \partial_{x_1} v^2 \partial_{x_2}^2 \rho) \\
&\quad + (0, \partial_{x_1} q)(\partial_{x_2}\rho).
\end{aligned}
$$

The Navier-Stokes nonlinearity is known to be written as

$$
(u \cdot \nabla) u = \sum_{j=1}^{2} (u^j \partial_{x_j}) u = u^1 \partial_{x_1} u + u^2 \partial_{x_2} u,
$$

which leads to

$$
\begin{aligned}
\Phi_*(u \cdot \nabla) u &= \Phi_* u^1 \Phi_* \partial_{x_1} u + \Phi_* u^2 \Phi_* \partial_{x_2} u \\
&= v^1 \partial_{x_1} v + v^2 \partial_{x_2} v - v^2 \partial_{x_1} v (\partial_{x_2}\rho) \\
&= (v \cdot \nabla) v - v^2 \partial_{x_1} v (\partial_{x_2}\rho).
\end{aligned}
$$

The deformation tensor is given as

$$
D(u) = \frac{1}{2}(\nabla u + \nabla u^T) = \frac{1}{2}\left(\begin{pmatrix} \partial_{x_1} u^1 & \partial_{x_2} u^1 \\ \partial_{x_1} u^2 & \partial_{x_2} u^2 \end{pmatrix} + \begin{pmatrix} \partial_{x_1} u^1 & \partial_{x_1} u^2 \\ \partial_{x_2} u^1 & \partial_{x_2} u^2 \end{pmatrix} \right),
$$

such that for the transformation of the deformation tensor we obtain

$$
\begin{aligned}
&\Phi_* D(u) \\
&= \frac{1}{2}\left(\begin{pmatrix} \Phi_*\partial_{x_1} u^1 & \Phi_*\partial_{x_2} u^1 \\ \Phi_*\partial_{x_1} u^2 & \Phi_*\partial_{x_2} u^2 \end{pmatrix} + \begin{pmatrix} \Phi_*\partial_{x_1} u^1 & \Phi_*\partial_{x_1} u^2 \\ \Phi_*\partial_{x_2} u^1 & \Phi_*\partial_{x_2} u^2 \end{pmatrix} \right) \\
&= \frac{1}{2}\left(\begin{pmatrix} \partial_{x_1} v^1 & \partial_{x_2} v^1 - \partial_{x_1} v^1 \partial_{x_2}\rho \\ \partial_{x_1} v^2 & \partial_{x_2} v^2 - \partial_{x_1} v^2 \partial_{x_2}\rho \end{pmatrix} + \begin{pmatrix} \partial_{x_1} v^1 & \partial_{x_1} v^2 \\ \partial_{x_2} v^1 - \partial_{x_1} v^1 \partial_{x_2}\rho & \partial_{x_2} v^2 - \partial_{x_1} v^2 \partial_{x_2}\rho \end{pmatrix} \right)
\end{aligned}
$$

$$= D(v) + \frac{1}{2}\left(\begin{pmatrix} 0 & -\partial_{x_1}v^1\partial_{x_2}\rho \\ 0 & -\partial_{x_1}v^2\partial_{x_2}\rho \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\partial_{x_1}v^1\partial_{x_2}\rho & -\partial_{x_1}v^2\partial_{x_2}\rho \end{pmatrix}\right)$$

$$= D(v) + \frac{1}{2}\left(\nabla v \begin{pmatrix} 0 & -\partial_{x_2}\rho \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\partial_{x_2}\rho & 0 \end{pmatrix} \nabla v^T\right)$$

and using the transformation of the deformation tensor we can obtain the transformation of the stress tensor:

$$\begin{aligned}
\Phi_* T(u,p) &= 2\mu \Phi_* D(u) - \Phi_* Ip \\
&= 2\mu D(v) - Iq + \mu\left(\nabla v \begin{pmatrix} 0 & -\partial_{x_2}\rho \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\partial_{x_2}\rho & 0 \end{pmatrix} \nabla v^T\right) + \begin{pmatrix} 0 \\ (\partial_{x_1} q)(\partial_{x_2}\rho) \end{pmatrix} \\
&= T(v,q) + \mu\left(\nabla v \begin{pmatrix} 0 & -\partial_{x_2}\rho \\ 0 & -0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\partial_{x_2}\rho & 0 \end{pmatrix} \nabla v^T\right) + \begin{pmatrix} 0 \\ (\partial_{x_1} q)(\partial_{x_2}\rho) \end{pmatrix}.
\end{aligned}$$

Hence, the first two equations of (4.1) are transformed.
Now we need to take care of the terms on the boundary. In order to transform the third and fourth equation of (4.1) we denote by τ_Σ and n_Σ the tangential and exterior normal vector at $\partial\Sigma_{\theta_0}$, respectively. We note that at $\Gamma_0 := \Phi_*\Gamma_s$ we have $\tau_\Sigma = (1,0)^T$ and $n_\Sigma = (0,-1)^T$ where

$$\Gamma_0 = \Phi_*\Gamma_s(t) = \{(x_1 + b(t,0), 0) : x_1 + b(t,0) \geq b(t,0)\} = \{(x_1, 0) : x_1 \geq 0\}$$

as desired. Then we transform the third equation (note that τ_s and n_s denote the tangential and outer normal vector at Γ_s, respectively, i.e., $\tau_s = (1,0)$ and $n_s = (0,-1)$ and that the third and fourth equation are defined on Γ_s):

$$\begin{aligned}
&\Phi_*(\lambda u^1 + (D(u)n_s)^1) \\
&= \Phi_*(\lambda \tau_s \cdot u + \tau_s \cdot D(u) n_s) \\
&= \lambda \Phi_*\tau_s \cdot \Phi_* u + \Phi_*\tau_s \cdot \Phi_* D(u) \Phi_* n_s) \\
&= \lambda \tau_\Sigma \cdot v + \tau_\Sigma \cdot \left(D(v) + \frac{1}{2}\left(\nabla v \begin{pmatrix} 0 & -\partial_{x_2}\rho \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\partial_{x_2}\rho & 0 \end{pmatrix} \nabla v^T\right)\right) n_\Sigma \\
&= \lambda \tau_\Sigma \cdot v + \tau_\Sigma D(v) n_\Sigma + \frac{1}{2}\partial_{x_1} v^1 \partial_{x_2}\rho,
\end{aligned}$$

and

$$\Phi_* u^2 = \Phi_*(-n_s \cdot u) = -n_\Sigma \cdot v,$$

with both equations now being defined on Γ_0.
Next, we have to transform the equations defined on the free boundary. To this end, we first need to transform tangential and exterior normal vector. We want to

calculate the exterior normal vector at $(b(t,y_2),y_2) \in \Gamma_f(t)$. Then $\tan(\theta)$ of the angle θ at $(b(t,y_2),y_2)$ is given by

$$\tan(\theta) = \lim_{h\to 0} \frac{y_2+h-y_2}{b(t,y_2+h)-b(t,y_2)} = \lim_{h\to 0} \frac{h}{b(t,y_2+h)-b(t,y_2)} = (\partial_{y_2} b(t,y_2))^{-1}$$

and for $\varphi + \theta = \pi/2$ we have

$$\tan(\varphi) = \tan(\pi/2-\theta) = \frac{\sin(\pi/2-\theta)}{\cos(\pi/2-\theta)} = \frac{\cos(\theta)}{\sin(\theta)} = \tan(\theta)^{-1} = \partial_{y_2} b(t,y_2).$$

Then for the (not normed) exterior normal vector at $(b(t,y_2),y_2)$ with $(\tilde{n}_f)^1 = -1$ we conclude

$$(\tilde{n}_f)^2 = -\tan(\varphi)(n_f)^1 = \tan(\varphi) = \partial_{y_2} b(t,y_2),$$

such that for the (not normed) exterior normal vector we obtain:

$$\tilde{n}_f(t,(b(t,y_2),y_2)) = \tilde{n}_f(t,y_2) = (-1, \partial_{y_2} b(t,y_2)).$$

Hence, the generalized exterior normal vector at $\Gamma_f(t)$ is given as

$$n_f(t,(b(t,y_2),y_2)) = n_f(t,y_2) = \frac{(-1,\partial_{y_2} b(t,y_2))^T}{\sqrt{1+\partial_{y_2}(b(t,y_2))^2}}.$$

Next, we have to transform $\Gamma_f(t)$. To this end, we obtain

$$\begin{aligned}\Gamma_+ := \Phi_* \Gamma_f(t) &= \{(b(t,y_2) + \partial_{y_2} b_0(0) y_2 - b(t,y_2), y_2) \in \mathbb{R}^2 : y_2 \in (0,\infty)\} \\ &= \{(\partial_{y_2} b_0(0) y_2, y_2) \in \mathbb{R}^2 : y_2 \in (0,\infty)\} \\ &= \{(\cot(\theta_0) y_2, y_2) \in \mathbb{R}^2 : y_2 \in (0,\infty)\}.\end{aligned}$$

Doing the exact same calculation as above for $(\cot(\theta_0)y_2, y_2) \in \Gamma_+$, we see that the generalized exterior normal vector at Γ_+ is given as

$$\begin{aligned} n_\Sigma(t,(\cot(\theta_0)y_2,y_2)) = n_\Sigma(t,y_2) &= \frac{(-1,\cot(\theta_0))^T}{\sqrt{1+\cot(\theta_0)^2}} = \frac{(-1,\partial_{y_2} b_0(0))^T}{\sqrt{1+(\partial_{y_2} b_0(0))^2}} \\ &= (-\sin(\theta_0), \cos(\theta_0))^T \end{aligned}$$

since

$$1+\cot(\theta_0)^2 = 1 + \frac{\cos(\theta_0)^2}{\sin(\theta_0)^2} = \frac{\sin(\theta_0)^2+\cos(\theta_0)^2}{\sin(\theta_0)^2} = \frac{1}{\sin(\theta_0)^2}.$$

We note that n_Σ is then independent of t and y_2. For the fifth and sixth equation in (4.1) it is sufficient to see how the exterior normal vector n_Σ at Γ_+ is given. This now yields

$$\Phi_* T(u,p) n_f = (\Phi_* T(u,p)) \Phi_* n_f$$

$$
\begin{aligned}
&= T(v,q)n_f \\
&\quad + \mu\left(\nabla v \begin{pmatrix} 0 & -\partial_{x_2}\rho \\ 0 & -0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\partial_{x_2}\rho & 0 \end{pmatrix} \nabla v^T\right) n_f + \begin{pmatrix} 0 \\ (\partial_{x_1} q)(\partial_{x_2}\rho) \end{pmatrix} n_f \\
&= T(v,q)n_\Sigma + T(v,q)(n_f - n_\Sigma) \\
&\quad + \mu\left(\nabla v \begin{pmatrix} 0 & -\partial_{x_2}\rho \\ 0 & -0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\partial_{x_2}\rho & 0 \end{pmatrix} \nabla v^T\right) n_f + \begin{pmatrix} 0 \\ (\partial_{x_1} q)(\partial_{x_2}\rho) \end{pmatrix} n_f,
\end{aligned}
$$

since n_f is only dependent on x_2 but not on x_1

$$
\begin{aligned}
\Phi_* n_f(t,x_1,x_2) = n_f(t,\Theta(t,x_1,x_2)) &= n_f(t, x_1 - \partial_{x_2} b_0(0)x_2 + b(t,x_2), x_2) \\
&= n_f(t,x_2) = n_f(t,x_1,x_2).
\end{aligned}
$$

For the fifth equation in (4.1) we have to take a look at the mean curvature κ: First we note that similar to our calculations for n_f we can also obtain the tangential vector τ_f. Here, it is even easier: If

$$
(\tilde{\tau}_f(t,(b(t,y_2),y_2)))^1 = (\tilde{\tau}_f(t,y_2))^1 = \partial_{y_2} b(t,y_2)
$$

then we obtain

$$
(\tilde{\tau}_f(t,(b(t,y_2),y_2)))^2 = (\tilde{\tau}_f(t,y_2))^2 = \tan(\theta)(\tilde{\tau}_f(t,y_2))^1 = 1
$$

such that the normed tangential vector is given by

$$
\tau_f(t,y_2) = \frac{(\partial_{y_2} b(t,y_2), 1)^T}{\sqrt{1 + (\partial_{y_2} b(t,y_2))^2}}.
$$

At this point we also calculate the tangential vector τ_Σ at Γ_+ as for the exterior normal vector. Here, for $(\cot(\theta_0)y_2, y_2) \in \Gamma_+$ we obtain (using the exact same arguments as for τ_f):

$$
\begin{aligned}
\tau_\Sigma(t,(\cot(\theta_0)y_2,y_2)) = \frac{(\cot(\theta_0),1)^T}{\sqrt{1+\cot(\theta_0)^2}} &= \tau_\Sigma(t,y_2) = \frac{(\partial_{y_2} b_0(0),1)^T}{\sqrt{1+(\partial_{y_2} b_0(0))^2}} \\
&= (\cos(\theta_0), \sin(\theta_0))^T,
\end{aligned}
$$

where we again made use of the observations above. We need the tangential vector at $\Gamma_f(t)$ to calculate the mean curvature (we omit the arguments (t,y_2) in this calculation):

$$
\kappa = -\operatorname{div}_{\Gamma_f} n_f = -\frac{1}{1+(\partial_{y_2} b)^2} \begin{pmatrix} \partial_{y_2} b \\ 1 \end{pmatrix} \cdot \partial_{y_2} \left(\frac{1}{\sqrt{1+(\partial_{y_2} b)^2}} \begin{pmatrix} -1 \\ \partial_{y_2} b \end{pmatrix} \right)
$$

$$
\begin{aligned}
&= -\frac{1}{1+(\partial_{y_2}b)^2}\begin{pmatrix}\partial_{y_2}b\\ 1\end{pmatrix}\cdot\begin{pmatrix}\frac{(\partial_{y_2}b)\partial_{y_2}^2 b}{(1+(\partial_{y_2}b)^2)^{3/2}}\\ \frac{\partial_{y_2}^2 b}{\sqrt{1+(\partial_{y_2}b)^2}}-\frac{(\partial_{y_2}b)^2\partial_{y_2}^2 b}{(1+(\partial_{y_2}b)^2)^{3/2}}\end{pmatrix}\\
&= -\frac{(\partial_{y_2}b)^2\partial_{y_2}^2 b}{(1+(\partial_{y_2}b)^2)^{5/2}}-\frac{\partial_{y_2}^2 b}{(1+(\partial_{y_2}b)^2)^{3/2}}+\frac{(\partial_{y_2}b)^2\partial_{y_2}^2 b}{(1+(\partial_{y_2}b)^2)^{5/2}}\\
&= -\frac{\partial_{y_2}^2 \rho}{(1+(\partial_{y_2}b)^2)^{3/2}}\\
&= -\frac{\partial_{y_2}^2 \rho}{(1+(\partial_{y_2}b_0(0)^2)^{3/2}}-\frac{\partial_{y_2}^2 \rho}{(1+(\partial_{y_2}b)^2)^{3/2}}+\frac{\partial_{y_2}^2 \rho}{(1+(\partial_{y_2}b_0(0)^2)^{3/2}}\\
&= -\sin(\theta_0)^3\partial_{y_2}^2\rho-\partial_{y_2}^2\rho\left(\frac{1}{(1+(\partial_{y_2}b)^2)^{3/2}}-\sin(\theta_0)^3\right)
\end{aligned}
$$

since $\rho(t,y_2)=b(t,y_2)-\partial_{y_2}b_0(0)y_2$ such that $\partial_{y_2}^2 b(t,y_2)=\partial_{y_2}^2\rho(t,y_2)$ and where we used the calculation for $1+(\partial_{y_2}b_0(0))^2=1+\cot(\theta_0)^2=\sin(\theta_0)^{-2}$ from above. This also shows that κ is only dependent on x_2 but not on x_1. Hence, transforming κ yields

$$
\begin{aligned}
\Phi_*\kappa(t,x_1,x_2)&=\kappa(t,\Theta(t,x_1,x_2))=\kappa(t,x_1-\partial_{y_2}b_0(0)x_2+b(t,x_2),x_2)=\kappa(t,x_2)\\
&=\kappa(t,x_1,x_2).
\end{aligned}
$$

This gives us all terms for the fifth equation of (4.1). For the sixth equation of (4.1) we need to transform the normal velocity V_{n_f}. To this end, we define

$$
\gamma_{y_2}:(0,T)\to\mathbb{R}^2,\quad t\mapsto(b(t,y_2),y_2)
$$

for a fixed $y_2\in(0,\infty)$. Then γ_{y_2} is a C^1-path on $\bigcup_{t\in(0,T)}\{t\}\times\Gamma_f(t)$ since $\gamma_{y_2}(t)\in\Gamma_f(t)$ for each $t\in(0,T)$. Then for the normal velocity we obtain

$$
\begin{aligned}
V_{n_f}(t,y_2)&=\gamma_{y_2}'(t)\cdot n_f(t,y_2)\\
&=\frac{1}{\sqrt{1+(\partial_{y_2}b(t,y_2))^2}}(\partial_t b(t,y_2),0)\begin{pmatrix}-1\\ \partial_{y_2}b(t,y_2)\end{pmatrix}\\
&=-\frac{\partial_t b(t,y_2)}{\sqrt{1+(\partial_{y_2}b(t,y_2))^2}}\\
&=-\frac{\partial_t \rho(t,y_2)}{\sqrt{1+(\partial_{y_2}b(t,y_2))^2}},
\end{aligned}
$$

since $\rho(t,y_2)=b(t,y_2)-\partial_{y_2}b_0(0)y_2$ such that $\partial_t\rho(t,y_2)=\partial_t b(t,y_2)$. Using the same arguments as for the mean curvature κ, we observe that V_{n_f} is only dependent on x_2 but not on x_1 such that

$$
\Phi_*V_{n_f}(t,x_1,x_2)=V_{n_f}(t,x_1,x_2).
$$

Hence the kinematic condition (sixth equation of (4.1)) transforms as

$$\begin{aligned}
V_{n_f} &= \Phi_* V_{n_f} = \Phi_*(n_f \cdot u) = n_f \cdot v \\
&= \frac{1}{\sqrt{1+(\partial_{x_2} b)^2}}(-1, \partial_{x_2} b)\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \\
&= -\frac{v^1}{\sqrt{1+(\partial_{x_2} b)^2}} + \frac{(\partial_{x_2} b) v^2}{\sqrt{1+(\partial_{x_2} b)^2}} \\
&= -\frac{v^1}{\sqrt{1+(\partial_{x_2} b)^2}} + \frac{(\partial_{x_2} \rho) v^2}{\sqrt{1+(\partial_{x_2} b)^2}} + \frac{\cot(\theta_0) v^2}{\sqrt{1+(\partial_{x_2} b)^2}}
\end{aligned}$$

where we note that

$$\partial_{x_2}\rho(t, x_2) = \partial_{x_2} b(t, x_2) - \partial_{x_2} b_0(0) = \partial_{x_2} b(t, x_2) - \cot(\theta_0).$$

Hence, inserting V_{n_f} we arrive at (again omitting the arguments here)

$$\partial_t \rho = v^1 - (\partial_{x_2}\rho) v^2 - \cot(\theta_0) v^2$$

for the sixth equation of (4.1).
It is obvious that we also obtain $(0,0) = \Phi_* \mathcal{C}(t)$ for the contact point such that $\Phi_* n_{\mathcal{C}} = \Phi_*(-1,0) = n_\Sigma = (-1,0)$ in this case. Hence, for the contact point velocity we observe

$$\Phi_* V_{\mathcal{C}} = V_{\mathcal{C}} = u \cdot n_\Sigma = \Phi_*(v \cdot n_{\mathcal{C}}),$$

since $V_{\mathcal{C}}$ is independent of x_1 and x_2. Furthermore, regarding the angle which is determined thanks to the (derivative of the) height function ρ we obtain

$$\partial_{x_2}\rho(t, 0) = \partial_{x_2} b(t, 0) - \cot(\theta_0) = \cot(\theta(t)) - \cot(\theta_0)$$

at the contact point $(0,0)$ now.

Now collecting all terms we obtain the following system after the transformation:

$$\begin{aligned}
\partial_t v - \operatorname{div} T(v,q) &= F_1(v,q,\rho) && \text{in } (0,T)\times\Sigma_{\theta_0}, \\
\operatorname{div} v &= F_2(v,\rho) && \text{in } (0,T)\times\Sigma_{\theta_0}, \\
\lambda\tau_\Sigma \cdot v + \tau_\Sigma D(v) n_\Sigma &= F_3(v,\rho) && \text{on } (0,T)\times\Gamma_0, \\
n_\Sigma \cdot v &= 0 && \text{on } (0,T)\times\Gamma_0, \\
T(v,q)n_\Sigma + \sigma\tilde{c}(\theta_0)\partial_{y_2}^2 \rho n_\Sigma &= F_4(v,\rho) && \text{on } (0,T)\times\Gamma_+, \\
\sin(\theta_0)\partial_t\rho + n_\Sigma \cdot v &= F_5(v,\rho) && \text{on } (0,T)\times\Gamma_+, \\
V_{\mathcal{C}} &= v \cdot n_\Sigma && \text{on } (0,T)\times\{0\}, \\
\partial_{x_2}\rho &= \cot(\theta(t)) - \cot(\theta_0) && \text{on } (0,T)\times\{0\}, \\
\theta &= \psi(V_{\mathcal{C}}) && \text{on } (0,T)\times\{0\}, \\
v|_{t=0} &= v_0 && \text{in } \Sigma_{\theta_0}, \\
\rho|_{t=0} &= \rho_0 && \text{on } \Gamma_+,
\end{aligned} \tag{4.4}$$

with

$$\begin{aligned}
F_1(v,q,\rho) &= (\partial_{x_1} v)\partial_t\rho - (v\cdot\nabla)v + v^2(\partial_{x_1} v)\partial_{x_2}\rho \\
&\quad + \mu(-2\partial_{x_1}\partial_{x_2} v\partial_{x_2}\rho + \partial_{x_1}^2 v(\partial_{x_2}\rho)^2 - \partial_{x_1} v\partial_{x_2}^2\rho + \partial_{x_1}^2(v^2,v^1)\partial_{x_2}\rho \\
&\quad + (-2\partial_{x_1}\partial_{x_2} v^2\partial_{x_2}\rho + \partial_{x_1}^2 v^2(\partial_{x_2}\rho)^2 - \partial_{x_1} v^2\partial_{x_2}^2\rho)(0,1)) \\
&\quad + (\partial_{x_1} q)(\partial_{x_2}\rho)(0,1), \\
F_2(v,\rho) &= (\partial_{x_1} v^2)\partial_{x_2}\rho, \\
F_3(v,\rho) &= -1/2(\partial_{x_1} v^1)\partial_{x_2}\rho, \\
F_4(v,\rho) &= T(v,q)(n_\Sigma - n) - \mu\left(\nabla v\begin{pmatrix}0 & -\partial_{x_2}\rho\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ -\partial_{x_2}\rho & 0\end{pmatrix}\nabla v^T\right)n \\
&\quad - \sigma\partial_{y_2}^2\rho\left(\frac{1}{\sqrt{1+(\partial_{y_2} b)^2}} - \tilde{c}(\theta_0)\right)n \\
&\quad - \sigma\tilde{c}(\theta_0)\partial_{y_2}^2\rho(n-n_\Sigma) - \begin{pmatrix}0\\ (\partial_{x_1} q)(\partial_{x_2}\rho)\end{pmatrix}n, \\
F_5(v,\rho) &= -\sin(\theta_0)(\partial_{x_2}\rho)v^2
\end{aligned}$$

and $\tilde{c}(\theta_0) := \sin(\theta_0)^3 > 0$. Here for simplicity we write $n := n_f$ and $v_0 := \Phi_* u_0$ and $\rho_0 = b_0$ are the given initial conditions. Note, that the contact point velocity $V_\mathcal{C}$ has to be determined as well as the contact point via $\mathcal{C}(t) = (\rho(t,0),0)$, whereas the contact angle θ is given by the constitutive equation $\theta = \psi(V_\mathcal{C})$ or prescribed.

4.2 Resolvent Stokes Equations on Sectors

In this section we provide a full analysis in the weak setting of the linearized version of (4.4) on a sector Σ_θ with angle $0 < \theta < \pi/2$ (see definition below). Note that in contrast to system (4.4) we will consider (4.5) on the reflected wedge which yields the sector. However, in the framework of reflection invariant spaces we obtain the boundary conditions on Γ_0 after restricting the solution to the wedge again and only boundary conditions at Γ_+ have to be imposed: In the framework of reflected spaces (cf. Section 3.2) we demand u^1 to be an even function and u^2 to be an odd function w.r.t. to x_2. Then we especially can ensure for $\Gamma_0 = (0,\infty)\times\{0\}$ with $n_\Sigma = (0,-1)$ and $\tau_\Sigma = (-1,0)$ that

$$\begin{aligned}
(n_\Sigma\cdot u)(t,x_1,0) &= -u^2(t,x_1,0) = u^2(t,x_1,0) = 0, \\
(\tau_\Sigma D(v)n_\Sigma)(t,x_1,0) &= \frac{1}{2}(\partial_{x_2} u^1(t,x_1,0) + \partial_{x_1} u^2(t,x_1,0)) = 0,
\end{aligned}$$

for $t \in (0,T)$ and $x_1 \in (0,\infty)$. Then applying a perturbation argument yields the original boundary conditions. Also we will simplify the fifth equation of (4.4). Since ρ only depends on the y_2 component of the argument we deduce $\partial^k_{\tau_\Sigma}\rho = \sin(\theta_0)^k \partial^k_{y_2}\rho$ for $k = 1,2$:

$$\partial_{\tau_\Sigma}\rho = (\cos(\theta_0), \sin(\theta_0))^T \begin{pmatrix} \partial_{y_1}\rho \\ \partial_{y_2}\rho \end{pmatrix} = \sin(\theta_0)\partial_{y_2}\rho,$$

$$\partial^2_{\tau_\Sigma}\rho = (\cos(\theta_0), \sin(\theta_0))^T \sin(\theta_0) \begin{pmatrix} \partial_{y_2}\partial_{y_1}\rho \\ \partial^2_{y_2}\rho \end{pmatrix} = \sin(\theta_0)^2\partial^2_{y_2}\rho.$$

Also note that in our framework all equations at the contact point vanish since in our desired regularity class

$$\begin{aligned} u &\in H^1((0,T), H_0^{-1}(\Sigma_\theta)) \cap H^{1/2}((0,T), L^2_{\mathrm{div}}(\Sigma_\theta)) \cap L^2((0,T), \hat{H}^1(\Sigma_\theta)), \\ \rho &\in H^{3/2}((0,T), \hat{H}^{-1/2}(\Gamma)) \cap H^{1/2}((0,T), H^1(\Gamma)), \\ \partial_\tau\rho &\in L^2((0,T), \hat{H}^{1/2}(\Gamma)), \end{aligned}$$

the contact point velocity $V_C = -u^2$ and $\partial_{x_2}\rho$ are not defined at the contact point, hence the corresponding equations drop out of (4.4).

This section is structured as follows: We prove the existence of weak solutions for the inhomogeneous stationary system (4.5). Furthermore, we will prove corresponding resolvent estimates. The strategy is as follows: In Section 4.2.1 we first consider the weak formulation of (4.5) in the Hilbert space setting in order to obtain weak solutions of the system with corresponding resolvent estimates for $|\lambda| = 1$. In Section 4.2.2 we will apply a scaling argument to finally obtain resolvent estimates for λ with arbitrary large absolute value.

In the sequel we will consider the following stationary system with data (f_1, f_4) that have the suitable regularity:

$$\begin{aligned} \lambda u - \operatorname{div} T(u,p) &= f_1 \quad \text{in } \Sigma_\theta, \\ \operatorname{div} u &= 0 \quad \text{in } \Sigma_\theta, \\ T(u,p)n + \sigma c(\theta)\partial^2_\tau\rho n &= f_4 \quad \text{on } \Gamma, \\ \lambda\rho + \tfrac{1}{\sin(\theta)}(n\cdot u) &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.5}$$

where $c(\theta) = \sin(\theta) > 0$ for $\theta \in (0,\pi/2)$. In the following $\theta \in (0,\pi/2)$ will be a fixed angle throughout the section and (4.5) will be considered on the sector Σ_θ as introduced in Chapter 3:

$$\Sigma_\theta = \{x = (x_1,x_2) \in \mathbb{C}\backslash\{0\} : |\arg x| < \theta\},$$

where $\Gamma := \partial\Sigma_\theta$ denotes the boundary. First we assume $\lambda \in \Sigma_{\pi/2}$ but later in the section we will assume that $|\lambda|$ is large. For the reader's convenience we recall exterior normal vector field n and tangential vector field τ at Γ which are given in Lemma 3.1.1 for $\delta = 0$:

$$n = n(x_2) = (-\sin(\theta), \operatorname{sgn}(x_2)\cos(\theta)) = \begin{cases} n_+ = (-\sin(\theta), \cos(\theta)), & x_2 > 0, \\ n_- = (-\sin(\theta), -\cos(\theta)), & x_2 < 0, \end{cases}$$

and

$$\tau = \tau(x_2) = (\operatorname{sgn}(x_2)\cos(\theta), \sin(\theta)) = \begin{cases} \tau_+ = (\cos(\theta), \sin(\theta)), & x_2 > 0, \\ \tau_- = (-\cos(\theta), \sin(\theta)), & x_2 < 0. \end{cases}$$

As we were originally interested in solving (4.4) on the wedge, we will consider (4.5) on the reflected wedge (which is a sector) in the framework of homogeneous spaces as introduced in Section 3.2. In the following we will assume for the data

$$f_1 \in \hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R := (\hat{H}^1_{\mathrm{div}}(\Sigma_\theta)_R)' \qquad \text{and} \qquad f_4 \in \hat{H}^{-1/2}(\Gamma)_R.$$

4.2.1 Remark. We set $\hat{H}^{-1}_0(\Sigma_\theta) = (\hat{H}^1(\Sigma_\theta))'$ and define the divergence of a functional in $\hat{H}^{-1}_0(\Sigma_\theta)$ as

$$\operatorname{div} : \hat{H}^{-1}_0(\Sigma_\theta) \to \mathcal{D}'(\Sigma_\theta), \ \operatorname{div} f(\varphi) := \langle f, \nabla\varphi \rangle_{\hat{H}^{-1}_0(\Sigma_\theta), \hat{H}^1(\Sigma_\theta)} \qquad (\varphi \in C_c^\infty(\Sigma_\theta)),$$

where $\mathcal{D}'(\Sigma_\theta)$ denotes the space of distributions on Σ_θ. We note that indeed we have $\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta) \subseteq \{u \in \hat{H}^{-1}_0(\Sigma_\theta) : \operatorname{div} u = 0\}$ by the following observation: Note that the Weyl projection from Corollary 3.1.17 is symmetric, hence we obtain the orthogonal decomposition

$$(\hat{H}^1(\Sigma_\theta)^2)' = (\hat{H}^1_{\mathrm{div}}(\Sigma_\theta))' \oplus (\nabla \hat{H}^2_D(\Sigma_\theta))',$$

with

$$(\hat{H}^1_{\mathrm{div}}(\Sigma_\theta))' \cong (\nabla \hat{H}^2_D(\Sigma_\theta))^\perp,$$

where $M^\perp$ denotes the polar of M. We observe that the polar contains $u \in \hat{H}^{-1}_0(\Sigma_\theta)$ having the property

$$\langle u, \nabla\varphi \rangle_{\hat{H}^{-1}_0(\Sigma_\theta), \hat{H}^1(\Sigma_\theta)} = 0 \qquad (\varphi \in \hat{H}^2_D(\Sigma_\theta)),$$

which especially yields the equality for all $\varphi \in C_c^\infty(\Sigma_\theta)$. Thus, $\operatorname{div} u = 0$ by definition.

4.2.1 Weak Solutions and Resolvent Estimates for $|\lambda| = 1$

In this section we prove existence of a triple (u, p, ρ) solving (4.5) in the weak sense and fulfilling corresponding resolvent estimates. To be precise, we want to prove the following

4.2.2 Proposition. *Let $\sigma > 0$, $\lambda \in \Sigma_{\pi/2}$ with $|\lambda| = 1$. Furthermore, we assume*

$$f_1 \in \hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R \qquad \text{and} \qquad f_4 \in \hat{H}^{-1/2}(\Gamma)_R.$$

Then there exists a unique weak solution $(u, p, \rho) \in H^1_{\mathrm{div}}(\Sigma_\theta)_R \times L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r \times \hat{H}^1(\Gamma)_r$ of (4.5) fulfilling the resolvent estimate

$$\begin{aligned} &\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \sqrt{\sigma}\|\rho\|_{\hat{H}^1(\Gamma)_r} \\ &\qquad + \sigma\|\partial_\tau^2\rho\|_{\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} + \|p\|_{L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r} \\ &\le C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right) \end{aligned} \tag{4.6}$$

with $C > 0$ independent of $\sigma, \lambda, u, p, \rho$ and the data f_1, f_4.

The strategy to prove Proposition 4.2.2 is as follows: At first we prove the existence of the velocity field u by considering the weak formulation of (4.5) and applying the Lax-Milgram theorem. From this we directly obtain resolvent estimates for u. Next, we define the height function ρ by making use of the fourth equation of (4.5). In order to reconstruct the pressure p we need to prove that ρ has higher regularity. Then we can solve the very weak and weak Dirichlet problem to obtain p and corresponding resolvent estimates are also obtained.

At first we need to derive the weak formulation of (4.5). To this end, let $\varphi \in C^\infty_{c,\sigma}(\overline{\Sigma_\theta})$ and data f_1, f_4 be sufficiently smooth. Then we calculate

$$\begin{aligned} &(\lambda u, \varphi)_2 - (\operatorname{div} T(u,p), \varphi)_2 \\ &= (\lambda u, \varphi)_2 + (T(u,p), \nabla\varphi)_2 - (T(u,p)n, \varphi)_{2,\Gamma} \\ &= (\lambda u, \varphi)_2 + 2\mu(D(u), \nabla\varphi)_2 - (p \cdot I, \nabla\varphi)_2 - (T(u,p)n, \varphi)_{2,\Gamma} \\ &= (\lambda u, \varphi)_2 + 2\mu(D(u), D(\varphi))_2 - (p, \operatorname{div}\varphi)_2 - (f_4 - \sigma c(\theta)\partial_\tau^2\rho n, \varphi)_{2,\Gamma} \\ &= (\lambda u, \varphi)_2 + 2\mu(D(u), D(\varphi))_2 - (f_4, \varphi)_{2,\Gamma} + \left(\sigma c(\theta)\partial_\tau^2\rho, n\cdot\varphi\right)_{2,\Gamma} \\ &= (\lambda u, \varphi)_2 + 2\mu(D(u), D(\varphi))_2 - \sigma c(\theta)(\partial_\tau\rho, \partial_\tau(n\cdot\varphi))_{2,\Gamma} - (f_4, \varphi)_{2,\Gamma} \\ &= (\lambda u, \varphi)_2 + 2\mu(D(u), D(\varphi))_2 + \frac{\sigma c(\theta)}{\sin(\theta)\lambda}(\partial_\tau(n\cdot u), \partial_\tau(n\cdot\varphi))_{2,\Gamma} - (f_4, \varphi)_{2,\Gamma} \\ &= (f_1, \varphi)_2, \end{aligned}$$

where we inserted the equations from (4.5). The calculation above then leads to the weak formulation of (4.5) for u given as

$$\begin{aligned}(\lambda u,\varphi)_2 + 2\mu(D(u),D(\varphi))_2 + \frac{\sigma c(\theta)}{\sin(\theta)\lambda}(\partial_\tau(n\cdot u),\partial_\tau(n\cdot\varphi))_{2,\Gamma}\\ = (f_1,\varphi)_2 + (f_4,\varphi)_{2,\Gamma}.\end{aligned}\tag{4.7}$$

In order to apply the Lax-Milgram theorem from Theorem 2.1.2 we have to consider the weak formulation (4.7) in a suitable setting. To this end, we set

$$\mathbb{H}^1 := \left\{u \in H^1(\Sigma_\theta)^2 : \operatorname{div} u = 0,\ n\cdot u|_\Gamma \in H^1(\Gamma)\right\},$$

equipped with the norm $\|u\|_{\mathbb{H}^1} := \left(\|u\|^2_{H^1(\Sigma_\theta)} + \|n\cdot u\|^2_{H^1(\Gamma)}\right)^{1/2}$ which is the natural function space to apply Lax-Milgram. We demand higher regularity for the boundary term $n\cdot u|_\Gamma$ such that the term $(\partial_\tau(n\cdot u),\partial_\tau(n\cdot\varphi))_{2,\Gamma}$ is well-defined. Thus, at first we prove

4.2.3 Lemma. *Let $\theta \in (0,\pi/2)$ and $\lambda \in \Sigma_{\pi/2}$. Then there exists a unique weak solution $u \in \mathbb{H}^1_R$ of the linearized problem (4.7). If $|\lambda| = 1$ then the solution u can be estimated as*

$$\begin{aligned}&\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \sqrt{\sigma}\|\partial_\tau(n\cdot u)\|_{L^2(\Gamma)_{-r}}\\ &\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right)\end{aligned}\tag{4.8}$$

with $C > 0$ independent of λ, σ, u and the data f_1, f_4.

Proof. As already mentioned above we want to apply Theorem 2.1.2. To this end, we define the corresponding form to (4.7) as

$$\begin{aligned}&a_\lambda : \mathbb{H}^1_R \times \mathbb{H}^1_R \to \mathbb{C},\\ &a_\lambda(u,\varphi) = (\lambda u,\varphi)_2 + 2\mu(D(u),D(\varphi))_2 + \frac{\sigma c(\theta)}{\sin(\theta)\lambda}(\partial_\tau(n\cdot u),\partial_\tau(n\cdot\varphi))_{2,\Gamma}.\end{aligned}$$

Furthermore, the functional ℓ on the right-hand side is given as (now assuming that f_1 and f_4 have the assumed regularity)

$$\begin{aligned}&\ell : \mathbb{H}^1_R \to \mathbb{C},\\ &\ell(\varphi) = \langle f_1,\varphi\rangle_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R,\hat{H}^1_{\mathrm{div}}(\Sigma_\theta)_R} + \langle f_4,\varphi\rangle_{\hat{H}^{-1/2}(\Gamma)_R,\hat{H}^{1/2}(\Gamma)_R}\end{aligned}$$

and ℓ is linear (obviously), well-defined and bounded:

$$|\ell(\varphi)| \leq \|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R}\|\varphi\|_{\hat{H}^1(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\|\varphi\|_{\hat{H}^{1/2}(\Gamma)_R}$$

$$\begin{aligned}&\leq \|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R}\|\varphi\|_{\hat{H}^1(\Sigma_\theta)_R} + C\|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\|\varphi\|_{\hat{H}^1(\Sigma_\theta)_R}\\ &\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right)\|\varphi\|_{H^1(\Sigma_\theta)_R},\end{aligned}$$

where we made use of the boundedness of the trace operator $T : \hat{H}^1(\Sigma_\theta)_{\pm R} \to \hat{H}^{1/2}(\Gamma)_{\pm R}$ from Corollary 3.2.12. Hence, $\ell \in (\mathbb{H}^1_R)'$ such that we only need to prove that a_λ is a coercive sesquilinear form in order to apply Theorem 2.1.2.

It is obvious that a_λ is sesquilinear in both arguments. However, in order to prove the coercitivity we need to apply Korn's inequality from Corollary 3.1.40. Then we infer by the fact that $\lambda, \lambda^{-1} \in \Sigma_{\pi/2}$

$$\begin{aligned}&|a_\lambda(u,u)|\\ &= \left|\lambda\|u\|^2_{L^2(\Sigma_\theta)_R} + 2\mu\|D(u)\|^2_{L^2(\Sigma_\theta)_\mathcal{R}} + \frac{\sigma c(\theta)}{\sin(\theta)\lambda}\|\partial_\tau(n\cdot u)\|^2_{L^2(\Gamma)_{-r}}\right|\\ &\geq C\left(|\lambda|\|u\|^2_{L^2(\Sigma_\theta)_R} + 2\mu\|D(u)\|^2_{L^2(\Sigma_\theta)_\mathcal{R}} + \frac{\sigma c(\theta)}{\sin(\theta)|\lambda|}\|\partial_\tau(n\cdot u)\|^2_{L^2(\Gamma)_{-r}}\right)\\ &\geq C(\lambda)\left(\|u\|^2_{L^2(\Sigma_\theta)_R} + \|\nabla u\|^2_{L^2(\Sigma_\theta)_\mathcal{R}} + \|\partial_\tau(n\cdot u)\|^2_{L^2(\Gamma)_{-r}} + \|n\cdot u\|^2_{L^2(\Gamma)_r}\right)\\ &\geq C(\lambda)\|u\|^2_{\mathbb{H}^1_R},\end{aligned}$$

by the boundedness of the trace operator $T : H^1(\Sigma_\theta)_{\pm R} \to H^{1/2}(\Gamma)_{\pm R}$ from Corollary 3.2.12 and $n\cdot : L^2(\Gamma)_R \to L^2(\Gamma)_r$ from Lemma 3.2.8:

$$\begin{aligned}&\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_R} = \|u\|_{H^1(\Sigma_\theta)_R}\\ &\geq C\|u\|_{H^{1/2}(\Gamma)_R} \geq C\|u\|_{L^2(\Gamma)_R} \geq C\|n\cdot u\|_{L^2(\Gamma)_r}.\end{aligned}$$

Hence, thanks to the Lax-Milgram Theorem (cf. Theorem 2.1.2) we can find a unique $u \in \mathbb{H}^1_R$ such that

$$a_\lambda(u,\varphi) = \ell(\varphi) \qquad (\varphi \in \mathbb{H}^1_R).$$

Now setting $\varphi = u$ we immediately obtain the important resolvent estimate for u:

$$\begin{aligned}&\left|\lambda\|u\|^2_{L^2(\Sigma_\theta)_R} + 2\mu\|D(u)\|^2_{L^2(\Sigma_\theta)_\mathcal{R}} + \frac{\sigma c(\theta)}{\sin(\theta)\lambda}\|\partial_\tau(n\cdot u)\|^2_{L^2(\Gamma)_{-r}}\right|\\ &= \left|\langle f_1, u\rangle_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R, \hat{H}^1_{\mathrm{div}}(\Sigma_\theta)_R} + \langle f_4, u\rangle_{\hat{H}^{-1/2}(\Gamma)_R, \hat{H}^{1/2}(\Gamma)_R}\right|,\end{aligned}$$

and since $\lambda \in \Sigma_{\pi/2}$ we immediately deduce

$$\begin{aligned}&|\lambda|\|u\|^2_{L^2(\Sigma_\theta)_R} + 2\mu\|D(u)\|^2_{L^2(\Sigma_\theta)_\mathcal{R}} + \frac{\sigma c(\theta)}{\sin(\theta)|\lambda|}\|\partial_\tau(n\cdot u)\|^2_{L^2(\Gamma)_{-r}}\\ &\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R}\|u\|_{\hat{H}^1(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\|u\|_{\hat{H}^1(\Sigma_\theta)_R}\right).\end{aligned}$$

Thanks to Korn's inequality from Corollary 3.1.40 and $u \in \mathbb{H}^1_R$ we have

$$\|u\|_{\hat{H}^1(\Sigma_\theta)_R} = \|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} \le C \left(\|u\|_{L^2(\Sigma_\theta)_R} + \|D(u)\|_{L^2(\Sigma_\theta)_\mathcal{R}} \right),$$

such that we can absorb all $\|u\|_{\hat{H}^1(\Sigma_\theta)_R}$ terms on the right hand side by applying Young's inequality. At this point we are not able to absorb the $\|u\|_{\hat{H}^1(\Sigma_\theta)_R}$ term as well as the remaining $\|u\|_{L^2(\Sigma_\theta)_R}$ term without leaving some terms containing $|\lambda|$ as a factor on the right-hand side. Hence, we set $|\lambda| = 1$ to obtain

$$\begin{aligned} &\|u\|^2_{L^2(\Sigma_\theta)_R} + \|\nabla u\|^2_{L^2(\Sigma_\theta)_\mathcal{R}} + \sigma \|\partial_\tau (n \cdot u)\|^2_{L^2(\Gamma)_{-r}} \\ &\le C \left(\|f_1\|^2_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|^2_{\hat{H}^{-1/2}(\Gamma)_R} \right), \end{aligned}$$

and (4.8) follows for $|\lambda| = 1$. □

4.2.4 Remark. (i) The reason why we don't immediately get estimates for arbitrary $\lambda \in \Sigma_{\pi/2}$ follows from the proof: By applying Korn's inequality in the form of Corollary 3.1.40 we obtain an extra $\|u\|_{L^2(\Sigma_\theta)_R}$ term on the right hand side which cannot be absorbed without leaving some terms containing λ as a factor. However, if we had the stronger Korn inequality

$$\|\nabla u\|_{L^2(\Sigma_\theta, \mathbb{R}^{2\times 2})} \le C \|D(u)\|_{L^2(\Sigma_\theta, \mathbb{R}^{2\times 2})}, \tag{4.9}$$

then this problem wouldn't occur and we would have obtained estimates for all $\lambda \in \Sigma_{\pi/2}$. To the best knowledge of the author, up to now it is not known whether (4.9) holds or not.

(ii) Without setting $f_5 = 0$ the linear form ℓ would have had another term $-\sigma c(\theta)\lambda^{-1}(\partial_\tau f_5, \partial_\tau(n \cdot \varphi))_{2,\Gamma}$ which is difficult to handle in view of getting the resolvent estimate for u. Even by setting $|\lambda| = 1$, we cannot absorb terms containing σ term fully such that a term containing σ as a factor would be left on right-hand side of the estimate. Hence, the scaling argument from Section 4.2.1 cannot be applied.

Next, we reconstruct the pressure p by solving a corresponding very weak and weak Dirichlet problem. To this end, we need higher regularity for $n \cdot u$. Note that by the last equation in (4.5) we can reconstruct the height function ρ by setting

$$\rho := -\frac{1}{\lambda \sin(\theta)} (n \cdot u) \in H^1(\Gamma)_r, \tag{4.10}$$

where u is the solution from Lemma 4.2.3. As mentioned before we need to prove higher regularity for u to reconstruct the pressure p:

4.2.5 Lemma. *Let $\sigma > 0$ and $\lambda \in \Sigma_{\pi/2}$ with $|\lambda| = 1$ and u be given from in Lemma 4.2.3. Then we have*

$$\sigma \|\partial_\tau^2(n \cdot u)\|_{\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} \leq C \left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \right) \tag{4.11}$$

with $C > 0$ independent of λ, σ, u and the data f_1, f_4.

Proof. We will prove that $\partial_\tau^2(n \cdot u) \in \hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r$ with a corresponding estimate. Let $\varphi \in \mathbb{H}^1_R$. Since u is the solution from Lemma 4.2.3 it fulfills $a_\lambda(u, \varphi) = \ell(\varphi)$ for $\varphi \in \mathbb{H}^1_R$ which yields

$$\begin{aligned} \frac{\sigma c(\theta)}{\sin(\theta)\lambda} \int_\Gamma \partial_\tau(n \cdot u)\partial_\tau(n \cdot \varphi)\, d\eta = -(\lambda u, \varphi)_2 - 2\mu(D(u), D(\varphi))_2 \\ + \langle f_1, \varphi \rangle_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R, \hat{H}^1_{\mathrm{div}}(\Sigma_\theta)_R} \\ + \langle f_4, \varphi \rangle_{\hat{H}^{-1/2}(\Gamma)_R, \hat{H}^{1/2}(\Gamma)_R}. \end{aligned}$$

Estimating this we arrive at

$$\begin{aligned} &\sigma \left| \int_\Gamma \partial_\tau(n \cdot u)\partial_\tau(n \cdot \varphi)\, d\eta \right| \\ &= \frac{\sin(\theta)|\lambda|}{c(\theta)} \Big| -(\lambda u, \varphi)_2 - 2\mu(D(u), D(\varphi))_2 \\ &\qquad + \langle f_1, \varphi \rangle_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R, \hat{H}^1_{\mathrm{div}}(\Sigma_\theta)_R} + \langle f_4, \varphi \rangle_{\hat{H}^{-1/2}(\Gamma)_R, \hat{H}^{1/2}(\Gamma)_R} \Big| \\ &\leq C|\lambda| \big(|\lambda| \|u\|_{L^2(\Sigma_\theta)_R} \|\varphi\|_{L^2(\Sigma_\theta)_R} + \|D(u)\|_{L^2(\Sigma_\theta)_\mathcal{R}} \|D(\varphi)\|_{L^2(\Sigma_\theta)_\mathcal{R}} \\ &\qquad + \|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} \|\varphi\|_{\hat{H}^1(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \|\varphi\|_{\hat{H}^1(\Sigma_\theta)_R} \Big) \\ &\leq C|\lambda| \Big(|\lambda| \|u\|_{L^2(\Sigma_\theta)_R} + \|D(u)\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} \\ &\qquad + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \Big) \cdot \|\varphi\|_{H^1(\Sigma_\theta)_R}. \end{aligned}$$

Since we only have estimate (4.8) for $|\lambda| = 1$ we also have to assume $|\lambda| = 1$ here; then we can make use of (4.8) and apply integration by parts to the left-hand term to obtain

$$\sigma \left| \int_\Gamma \partial_\tau^2(n \cdot u)(n \cdot \varphi)\, d\eta \right| \leq C \left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \right) \|\varphi\|_{H^1(\Sigma_\theta)_R}$$

where we note that $C > 0$ is independent of σ and λ. (Note that $r(\partial_\tau(n \cdot u)) = -(\partial_\tau(n \cdot u))$ by the symmetry.)

By Corollary 3.2.17 the normal trace operator $T_0 : H^1_{\mathrm{div}}(\Sigma_\theta)_R \to \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$ is bounded and a retraction. Then $N(T_0) := \{\varphi \in H^1_{\mathrm{div}}(\Sigma_\theta)_R : n \cdot \varphi|_\Gamma = 0\}$ is closed and we obtain the orthogonal decomposition

$$H^1_{\mathrm{div}}(\Sigma_\theta)_R = N(T_0) \oplus_\perp N(T_0)^\perp.$$

Then there exists a projection $P \in \mathscr{L}(H^1_{\mathrm{div}}(\Sigma_\theta)_R)$ such that $R(P) = N(T_0)^\perp$ and $T_0|_{N(T_0)^\perp} : N(T_0)^\perp \to \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$ is an isomorphism. Then we observe that

$$\begin{aligned}
&\sigma \left| \int_\Gamma \partial_\tau^2 (n \cdot u)(n \cdot \varphi)\, d\eta \right| \\
&\le C \left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \right) \|n \cdot \varphi\|_{\hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r}
\end{aligned}$$

holds for $\varphi \in \mathbb{H}^1_R \cap N(T_0)^\perp$.

Next, we observe that $P(\mathbb{H}^1_R) \overset{d}{\hookrightarrow} P(H^1_{\mathrm{div}}(\Sigma_\theta)_R) = N(T_0)^\perp$: Since $H^2_{\mathrm{div}}(\Sigma_\theta)_R \hookrightarrow \mathbb{H}^1_R \hookrightarrow H^1_{\mathrm{div}}(\Sigma_\theta)_R$ and $H^2_{\mathrm{div}}(\Sigma_\theta)_R \overset{d}{\hookrightarrow} H^1_{\mathrm{div}}(\Sigma_\theta)_R$ by Lemma 3.2.2(iv), we deduce $\mathbb{H}^1_R \overset{d}{\hookrightarrow} H^1_{\mathrm{div}}(\Sigma_\theta)_R$ which yields the desired density statement.

We finally conclude that $T_0 : P(\mathbb{H}^1_R) \to \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$ has dense range. Then we finally deduce

$$\begin{aligned}
&\sigma \|\partial_\tau^2 (n \cdot u)\|_{\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} \\
&= \sup_{\substack{g \in \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r \\ \|g\|_{\hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r} = 1}} \sigma |(\partial_\tau^2 (n \cdot u), g)_\Gamma| \\
&= \sup_{\substack{\varphi \in P(\mathbb{H}^1_R) \\ \|n\cdot\varphi\|_{\hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r} = 1}} \sigma |(\partial_\tau^2 (n \cdot u), n \cdot \varphi)_\Gamma| \\
&\le C \sup_{\substack{\varphi \in P(\mathbb{H}^1_R) \\ \|n\cdot\varphi\|_{\hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r} = 1}} \left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \right) \|n \cdot \varphi\|_{\hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r} \\
&\le C \left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \right),
\end{aligned}$$

where $C > 0$ is independent of λ and σ for $|\lambda| = 1$. We remark that $(\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r)' = \hat{H}^{1/2}(\Gamma)_r \cap \hat{H}^{-1/2}(\Gamma)_r$. Then the assertion follows. □

As a consequence of the above result we obtain the following estimates for the height function ρ if $|\lambda| = 1$:

$$\begin{aligned}
\|\rho\|_{\hat{H}^1(\Gamma)_r} &= \left\| -\frac{1}{\lambda \sin(\theta)} (n \cdot u) \right\|_{\hat{H}^1(\Gamma)_r} \le \frac{C}{|\lambda|} \|\partial_\tau (n \cdot u)\|_{L^2(\Gamma)_{-r}} \\
&\le \frac{C}{\sqrt{\sigma}} \left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} \right)
\end{aligned}$$

and

$$\begin{aligned}\|\partial_\tau^2\rho\|_{\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r} &= \left\|-\frac{1}{\lambda\sin(\theta)}\partial_\tau^2(n\cdot u)\right\|_{\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r}\\ &\leq \frac{C}{|\lambda|}\left\|\partial_\tau^2(n\cdot u)\right\|_{\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r}\\ &\leq \frac{C}{\sigma}\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R}+\|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right).\end{aligned}$$

At last we need to reconstruct the pressure p. Finally, we are able to prove (weak) solvability of the linearized problem (4.5). Hence, we finally give the proof for Proposition 4.2.2:

Proof of Proposition 4.2.2. In Lemma 4.2.3 and by (4.10) we already proved the existence of a unique u and ρ solving (4.5) (in the weak sense). At last we need to reconstruct the pressure p. Since $\partial_\tau^2\rho \in \hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r$ we can choose $\rho_1 \in \hat{H}^{-1/2}(\Gamma)_r$ and $\rho_2 \in \hat{H}^{1/2}(\Gamma)_r$ such that $\partial_\tau^2\rho = \rho_1 + \rho_2$ and

$$\|\partial_\tau^2\rho\|_{\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r} \leq \|\rho_1\|_{\hat{H}^{1/2}(\Gamma)_r} + \|\rho_2\|_{\hat{H}^{1/2}(\Gamma)_r} < \|\partial_\tau^2\rho\|_{\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r} + \varepsilon/\sigma$$

for every $\varepsilon > 0$. In order to construct the pressure we set $p := q\cdot n + \tilde{p}$ where q and $\tilde{p}$ solve the very weak and the weak Dirichlet problem, respectively. In particular, we consider

$$-\Delta q = 0 \quad \text{in } \Sigma_\theta, \quad q = -f_4 + 2\mu D(u)n + \sigma c(\theta)\rho_1 n \quad \text{on } \Gamma \tag{4.12}$$

as a very weak Dirichlet problem and

$$-\Delta\tilde{p} = 0 \quad \text{in } \Sigma_\theta, \quad \tilde{p} = \sigma c(\theta)\rho_2 \quad \text{on } \Gamma \tag{4.13}$$

as a weak Dirichlet problem. Considering (4.12) we obtain this very weak formulation by calculating for $\varphi \in \hat{H}^2_D(\Sigma_\theta)_R$ from Lemma 3.1.34:

$$\begin{aligned}-(q,\Delta\varphi)_2 &= -\langle q, \partial_n\varphi\rangle_{\hat{H}^{-1/2}(\Gamma)_R,\hat{H}^{1/2}(\Gamma)_R}\\ &= -\left\langle -f_4 + 2\mu D(u)n + \sigma c(\theta)\rho_1 n, \partial_n\varphi\right\rangle_{\hat{H}^{-1/2}(\Gamma)_R,\hat{H}^{1/2}(\Gamma)_R}\end{aligned} \tag{4.14}$$

where we took (4.12) into consideration. By Corollary 3.2.11 such a unique solution $q \in L^2(\Sigma_\theta)_R$ exists if

$$-f_4 + 2\mu D(u)n + \sigma c(\theta)\rho_1 n \in \hat{H}^{-1/2}(\Gamma)_R.$$

We note that by $D(u)n = 1/2(\nabla u^T n + (n^T\nabla u^T)^T)$ with $\operatorname{div}\nabla u^T = 0$ it follows $D(u)n \in \hat{H}^{-1/2}(\Gamma)_R$ by Corollary 3.2.14. Furthermore, since we know that the normal trace $T_0 : L^2_{\mathrm{div}}(\Sigma_\theta)_{\pm R} \to \hat{H}^{-1/2}(\Gamma)_{\pm r}$ from Corollary 3.2.14 is bounded, this yields

$$\|D(u)n\|_{\hat{H}^{-1/2}(\Gamma)_R} \leq C\|D(u)\|_{L^2(\Sigma_\theta)_{\mathcal{R}}} < \infty$$

by using (4.8). The last term is also in $\hat{H}^{-1/2}(\Gamma)_R$ which follows from $\rho_1 \in \hat{H}^{-1/2}(\Gamma)_r$ and Lemma 3.2.6. Hence, by Corollary 3.2.11 there exists a unique $q \in L^2(\Sigma_\theta)_R$ which can be estimated as

$$\begin{aligned}\|q\|_{L^2(\Sigma_\theta)_R} &\leq C\,\|-f_4 + 2\mu D(u)n + \sigma c(\theta)\rho_1 n\|_{\hat{H}^{-1/2}(\Gamma)_R} \\ &\leq C\left(\|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R} + 2\mu\|\nabla u\|_{L^2(\Sigma_\theta)_{\mathcal{R}}} + \sigma\|\rho_1\|_{\hat{H}^{-1/2}(\Gamma)_r}\right)\end{aligned}$$

where we used the estimates that we proved before.

Regarding (4.13) we observe that $\sigma c(\theta)\rho_2 \in \hat{H}^{1/2}(\Gamma)_r$. Then by Corollary 3.1.33 there exists a unique $\tilde{p} \in \hat{H}^1(\Sigma_\theta)_r$ such that $\tilde{p}$ solves (4.13) with

$$\|\tilde{p}\|_{\hat{H}^1(\Sigma_\theta)_r} \leq C\sigma\|\rho_2\|_{\hat{H}^{1/2}(\Gamma)_r}.$$

Then summing up we end up by using $\rho_1 + \rho_2 = \partial_\tau^2\rho$

$$q \cdot n + \tilde{p} = -f_4 \cdot n + 2\mu D(u)n \cdot n + \sigma c(\theta)(\rho_1 + \rho_2) = p \qquad \text{on } \Gamma,$$

which shows that p as a solution of the Dirichlet problem is unique and for $\varepsilon > 0$:

$$\begin{aligned}&\|p\|_{L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r} \\ &\leq C\left(\|f_4\|_{\hat{H}^{-1/2}(\Gamma)_r} + 2\mu\|\nabla u\|_{L^2(\Sigma_\theta)_{\mathcal{R}}} + \sigma\|\partial_\tau^2\rho\|_{\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} + \varepsilon\right).\end{aligned}$$

We extend the normal vector $n = (-\sin(\theta), \mathrm{sgn}(x_2)\cos(\theta))$ constantly to the entire sector Σ_θ. Recovering the pressure p by setting $p = q \cdot n + \tilde{p}$ we first note that $p \in L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r$ by Lemma 3.2.8. Furthermore, we also know $\Delta p = 0$ in distributional sense since for $\varphi \in C_c^\infty(\Sigma_\theta \backslash \{x_2 = 0\})_r$ we deduce (since p is even it is sufficient to consider φ even since the integral vanishes for odd φ):

$$\begin{aligned}&\int_{\Sigma_\theta} p(x)\Delta\varphi(x)\,dx \\ &= \int_{\Sigma_\theta} (-\sin(\theta)q^1(x) + \mathrm{sgn}(x_2)\cos(\theta)q^2(x))\Delta\varphi(x)\,dx + \int_{\Sigma_\theta} \tilde{p}(x)\Delta\varphi(x)\,dx \\ &= -\sin(\theta)\int_{\Sigma_\theta} q^1(x)\Delta\varphi(x)\,dx + \cos(\theta)\int_{\Sigma_\theta} \mathrm{sgn}(x_2)q^2(x)\Delta\varphi(x)\,dx \\ &\quad + \int_{\Sigma_\theta} \tilde{p}(x)\Delta\varphi(x)\,dx \\ &= 0.\end{aligned}$$

Note that the first integral vanishes since $\Delta q^1 = 0$ by the weak formulation in (4.14) and the third integral vanishes since $\tilde{p}$ is the solution of (4.13). The weak formulation (4.14) for q^1 first holds for $\varphi \in C_c^\infty(\Sigma_\theta)_r$. However, since q^1 is even, (4.14) also holds for $\varphi \in C_c^\infty(\Sigma_\theta)_{-r}$, hence it holds for all $\varphi \in C_c^\infty(\Sigma_\theta)$. The same arguments can be applied to q^2 since q^2 is odd and (4.14) holds for $\varphi \in C_c^\infty(\Sigma_\theta)_{-r}$. Regarding the latter integral we make use of q_2 odd and φ even such that for $\Sigma_\theta^\pm := \Sigma_\theta \cap \{\pm x_2 > 0\}$ we obtain:

$$\begin{aligned}\int_{\Sigma_\theta} \operatorname{sgn}(x_2) q^2(x) \Delta\varphi(x)\, dx &= \int_{\Sigma_\theta^+} q^2(x)\Delta\varphi(x)\, dx - \int_{\Sigma_\theta^-} q^2(x)\Delta\varphi(x)\, dx \\ &= 2\int_{\Sigma_\theta^+} q^2 \Delta\varphi(x)\, dx \\ &= 2\int_{\Sigma_\theta^+} \Delta q^2(x)\varphi(x)\, dx - 2\int_{\partial\Sigma_\theta^-} \partial_n q^2(x)\varphi(x)\, d\eta \\ &= 0,\end{aligned}$$

since $\{x_2 = 0\} \cap \operatorname{supp} \varphi = \emptyset$. Hence, $\Delta p = 0$ in the distributional sense in $\Sigma_\theta \backslash \{x_2 = 0\}$ and p fulfills the estimate:

$$\begin{aligned}&\|p\|_{L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r} \\ &\leq C\left(\|f_4\|_{\hat{H}^{-1/2}(\Gamma)_r} + 2\mu\|\nabla u\|_{L^2(\Sigma_\theta)_{\mathcal{R}}} + \sigma\|\partial_\tau^2 \rho\|_{\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r}\right) \\ &\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right).\end{aligned}$$

Collecting all terms from Lemma 4.2.3 and from above of this lemma, we arrive at the resolvent estimate in case if $|\lambda| = 1$:

$$\begin{aligned}&\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_{\mathcal{R}}} + \sqrt{\sigma}\|\rho\|_{\hat{H}^1(\Gamma)_r} \\ &\qquad + \sigma\|\partial_\tau^2\rho\|_{\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} + \|p\|_{L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r} \\ &\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right).\end{aligned}$$

Then the assertion is proved. □

4.2.2 Weak Solutions and Resolvent Estimates for large $|\lambda|$

In this section we prove solvability of the weak linearized problem (4.7) for $\lambda \in \Sigma_{\pi/2}$ of large absolute value and corresponding resolvent estimates as in (4.6). To this end, we apply a scaling argument since the sector $\Sigma_{\pi/2}$ is obviously scaling invariant. We note that it is sufficient to have results on solvability of (4.5) for $|\lambda| = 1$ and arbitrary $\sigma > 0$ since we will see that (4.5) is equivalent to

$$\begin{aligned} \tfrac{\lambda}{|\lambda|}\tilde{u} - \mu\Delta\tilde{u} + \nabla\tilde{p} &= \tilde{f}_1 && \text{in } \Sigma_\theta, \\ \operatorname{div}\tilde{u} &= 0 && \text{in } \Sigma_\theta, \\ T(\tilde{u},\tilde{p})n + \tfrac{\sigma c(\theta)}{|\lambda|^{1/2}}\partial_\tau^2\tilde{\rho}n &= \tilde{f}_4 && \text{on } \Gamma, \\ \tfrac{\lambda}{|\lambda|}\tilde{\rho} + \tfrac{1}{\sin(\theta)}(n\cdot\tilde{u}) &= 0 && \text{on } \Gamma. \end{aligned} \tag{4.15}$$

Hence if (u,p,ρ) is a weak solution of (4.5) then $(\tilde{u},\tilde{p},\tilde{\rho})$ is a weak solution of (4.15) where the relation between both solution triples is given as

$$\tilde{u}(x) = |\lambda| u\left(\frac{x}{\sqrt{|\lambda|}}\right), \quad \tilde{p}(x) = \sqrt{|\lambda|}p\left(\frac{x}{\sqrt{|\lambda|}}\right), \quad \tilde{\rho}(x) = |\lambda|^2\rho\left(\frac{x}{\sqrt{|\lambda|}}\right)$$

and left-hand side (at first for smooth f_1, f_4, the definition will be adjusted to later on)

$$\tilde{f}_1 = f_1\left(\frac{x}{\sqrt{|\lambda|}}\right), \quad \tilde{f}_4 = \sqrt{|\lambda|}f_4\left(\frac{x}{\sqrt{|\lambda|}}\right),$$

for all $x \in \Sigma_\theta$ and $x \in \Gamma$, respectively (note that if $x \in \Sigma_\theta$ then $x/\sqrt{|\lambda|} \in \Sigma_\theta$ for all $\lambda \in \Sigma_{\pi/2}$ since Σ_θ is scaling invariant; the same holds for Γ). Equivalence of (4.5) and (4.15) can be observed by a straightforward calculation:

$$\begin{aligned} &\lambda u(x) - \mu\Delta u(x) + \nabla p(x) \\ &\quad = \frac{\lambda}{|\lambda|}\tilde{u}\left(\sqrt{|\lambda|}x\right) - \frac{\mu}{|\lambda|}\Delta\left(\tilde{u}\left(\sqrt{|\lambda|}\cdot\right)\right)(x) + \frac{1}{\sqrt{|\lambda|}}\nabla\left(\tilde{p}\left(\sqrt{|\lambda|}\cdot\right)\right)(x) \\ &\quad = \frac{\lambda}{|\lambda|}\tilde{u}\left(\sqrt{|\lambda|}x\right) - \mu\Delta\tilde{u}\left(\sqrt{|\lambda|}x\right) + \nabla\tilde{p}\left(\sqrt{|\lambda|}x\right) \\ &\quad = \tilde{f}_1\left(\sqrt{|\lambda|}x\right) = f_1(x) \end{aligned}$$

and

$$\operatorname{div} u(x) = \operatorname{div}\left(\frac{1}{|\lambda|}\tilde{u}\left(\sqrt{|\lambda|}\cdot\right)\right)(x) = \frac{1}{\sqrt{|\lambda|}}\operatorname{div}\tilde{u}\left(\sqrt{|\lambda|}x\right) = 0.$$

Furthermore, by the fact that $\tau(x) = \tau\left(\sqrt{|\lambda|}x\right)$ and $n(x) = n\left(\sqrt{|\lambda|}x\right)$ since $\operatorname{sgn}(x_2) = \operatorname{sgn}\left(\sqrt{|\lambda|}x_2\right)$ we infer

$$\begin{aligned} &T(u,p)n + \frac{\sigma}{\sin(\theta)}\partial_\tau^2\rho(x)n \\ &= \left[\mu\left(\nabla\left(\frac{1}{|\lambda|}\tilde{u}\left(\sqrt{|\lambda|}\cdot\right)\right)(x) + \nabla\left(\frac{1}{|\lambda|}\tilde{u}\left(\sqrt{|\lambda|}\cdot\right)\right)^T(x)\right) - \frac{1}{\sqrt{|\lambda|}}\tilde{p}\left(\sqrt{|\lambda|}x\right)\right]n \\ &\quad + \sigma c(\theta)\partial_\tau^2\left(\frac{1}{|\lambda|^2}\tilde{\rho}\left(\sqrt{|\lambda|}\cdot\right)\right)(x)n \end{aligned}$$

$$= \left[\frac{1}{\sqrt{|\lambda|}} \mu \left(\nabla \tilde{u} \left(\sqrt{|\lambda|} x \right) + \nabla \tilde{u} \left(\sqrt{|\lambda|} x \right)^T \right) - \frac{1}{\sqrt{|\lambda|}} \tilde{p} \left(\sqrt{|\lambda|} x \right) n \left(\sqrt{|\lambda|} x \right) \right]$$
$$+ \frac{1}{\sqrt{|\lambda|}} \frac{\sigma c(\theta)}{\sqrt{|\lambda|}} \partial_\tau^2 \tilde{\rho} \left(\sqrt{|\lambda|} x \right)$$
$$= \frac{1}{\sqrt{|\lambda|}} \tilde{f}_4 \left(\sqrt{|\lambda|} x \right) = f_4(x)$$

and at last

$$\lambda \rho(x) + \frac{1}{\sin(\theta)} (n(x) \cdot u(x))$$
$$= \frac{\lambda}{|\lambda|^2} \tilde{\rho} \left(\sqrt{|\lambda|} x \right) + \frac{1}{\sin(x)|\lambda|} \left(n \left(\sqrt{|\lambda|} x \right) \cdot \tilde{u} \left(\sqrt{|\lambda|} x \right) \right)$$
$$= 0.$$

Then (u, p, ρ) is a weak solution to (4.5) if and only if $(\tilde{u}, \tilde{p}, \tilde{\rho})$ is a solution to (4.15). By Proposition 4.2.2 we know that (4.15) is weakly solvable for $\tilde{\sigma} = \sigma / |\lambda|^{1/2}$ and arbitrary $\lambda \in \Sigma_{\pi/2}$. As a consequence, we obtain the generalization of Proposition 4.2.2:

4.2.6 Corollary. *Let $\sigma > 0$ and $\lambda \in \Sigma_{\pi/2}$. Furthermore, we assume*

$$f_1 \in \hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R \quad \textit{and} \quad f_4 \in \hat{H}^{-1/2}(\Gamma)_R.$$

Then there exists a unique weak solution

$$(u, p, \rho) \in H^1_{\mathrm{div}}(\Sigma_\theta)_R \times L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r \times \hat{H}^1(\Gamma)_r$$

of (4.5).

Thus, it was possible to transfer the solvability of (4.5) by using the scaling argument. In the following we will investigate in which sense estimate (4.6) can be transferred to the case if $\lambda \in \Sigma_{\pi/2}$ is of large absolute value. Furthermore, we aim to also have an estimate of u in the corresponding $H_0^{-1}(\Sigma_\theta)_R$ norm. To this end, we first define equivalent norms in $L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r$ and $\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r$:

$$\|q\|_{\lambda, L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r} := \inf \left\{ \begin{array}{c} \|q_0\|_{L^2(\Sigma_\theta)_r} + |\lambda|^{-1/2} \|q_1\|_{\hat{H}^1(\Sigma_\theta)_r} : \\ q = q_0 + q_1, q_0 \in L^2(\Sigma_\theta)_r, q_1 \in \hat{H}^1(\Sigma_\theta)_r \end{array} \right\},$$
$$\|h\|_{\lambda, \hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} := \inf \left\{ \begin{array}{c} \|h_0\|_{\hat{H}^{-1/2}(\Gamma)_r} + |\lambda|^{-1/2} \|h_1\|_{\hat{H}^{1/2}(\Gamma)_r} : \\ h = h_0 + h_1, h_0 \in \hat{H}^{-1/2}(\Gamma)_r, h_1 \in \hat{H}^{1/2}(\Gamma)_r \end{array} \right\}.$$

We can now take advantage of the fact that for the scaled system (4.15) we can use our estimates from (4.5) by setting $\tilde{\sigma} = \sigma / |\lambda|^{1/2}$. Then if $(\tilde{u}, \tilde{p}, \tilde{\rho})$ is the weak solution of (4.15) with right-hand side $(\tilde{f}_1, \tilde{f}_4)$ we obtain the following estimates from (4.6):

$$\begin{aligned} &\|\tilde{u}\|_{L^2(\Sigma_\theta)_R} + \|\nabla\tilde{u}\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \frac{\sqrt{\sigma}}{|\lambda|^{1/4}}\|\tilde{\rho}\|_{\hat{H}^1(\Gamma)_r} \\ &\quad + \frac{\sigma}{|\lambda|^{1/2}}\|\partial_\tau^2\tilde{\rho}\|_{\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r} + \|\tilde{p}\|_{L^2(\Sigma_\theta)_r+\hat{H}^1(\Sigma_\theta)_r} \\ &\leq C\left(\|\tilde{f}_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|\tilde{f}_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right). \end{aligned} \tag{4.16}$$

with $C > 0$ independent of $u, p, \rho, \lambda, \sigma$ and the data f_1, f_4. Next, we make use of the norm scaling that we already considered in Section 3.1.5 to obtain these norms in terms of u, p, ρ, f_1 and f_4. Then inserting all norm calculations from Section 3.1.5 in (4.16) we deduce

$$\begin{aligned} &|\lambda|^{3/2}\|u\|_{L^2(\Sigma_\theta)_R} + |\lambda|\|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \sqrt{\sigma}|\lambda|^{3/2}\|\rho\|_{\hat{H}^1(\Gamma)_r} \\ &\quad + \sigma|\lambda|\|\partial_\tau^2\rho\|_{\lambda,\,\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r} + |\lambda|\|p\|_{\lambda,\,L^2(\Sigma_\theta)_r+\hat{H}^1(\Sigma_\theta)_r} \\ &\leq C\left(|\lambda|\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + |\lambda|\|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right) \end{aligned}$$

which simplifies to

$$\begin{aligned} &|\lambda|^{1/2}\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \sqrt{\sigma}|\lambda|^{1/2}\|\rho\|_{\hat{H}^1(\Gamma)_r} \\ &\quad + \sigma\|\partial_\tau^2\rho\|_{\lambda,\,\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r} + \|p\|_{\lambda,\,L^2(\Sigma_\theta)_r+\hat{H}^1(\Sigma_\theta)_r} \\ &\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right). \end{aligned}$$

As a consequence we are finally able to prove the full resolvent estimate for the Stokes system (4.5) in the weak setting. Here, we want to prove resolvent estimates in $H_0^{-1}(\Sigma_\theta)_R$. Again, we consider an equivalent norm defined as

$$\|v\|_{\lambda,\,H_0^{-1}(\Sigma_\theta)_R} := \sup_{\substack{\varphi\in H^1(\Sigma_\theta)_R,\\ |\lambda|^{-1/2}\|\varphi\|_{L^2(\Sigma_\theta)_R}+|\lambda|^{-1}\|\varphi\|_{\hat{H}^1(\Sigma_\theta)_R}\leq 1}} |\langle v, \varphi\rangle_{H_0^{-1}(\Sigma_\theta)_R, H^1(\Sigma_\theta)_R}|. \tag{4.17}$$

4.2.7 Theorem. *Let $\sigma > 0$, $\lambda \in \Sigma_{\pi/2}$ with $|\lambda| \geq 1$. Furthermore, we assume*

$$f_1 \in \hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R \quad \text{and} \quad f_4 \in \hat{H}^{-1/2}(\Gamma)_R.$$

Then there exists a unique weak solution $(u, p, \rho) \in H^1_{\mathrm{div}}(\Sigma_\theta)_R \times L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r \times \hat{H}^1(\Gamma)_r$ of the Stokes system (4.5) fulfilling the resolvent estimate

$$\begin{aligned} &\|u\|_{\lambda,\,H_0^{-1}(\Sigma_\theta)_R} + |\lambda|^{1/2}\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \sqrt{\sigma}|\lambda|^{1/2}\|\rho\|_{\hat{H}^1(\Gamma)_r} \\ &\quad + \sigma\|\partial_\tau^2\rho\|_{\lambda,\,\hat{H}^{-1/2}(\Gamma)_r+\hat{H}^{1/2}(\Gamma)_r} + \|p\|_{\lambda,\,L^2(\Sigma_\theta)_r+\hat{H}^1(\Sigma_\theta)_r} \\ &\leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right) \end{aligned} \tag{4.18}$$

with $C > 0$ independent of $\lambda, \sigma, u, p, \rho$ and the data f_1, f_4.

Proof. Our last aim is to obtain the resolvent estimate of $\tilde{u}$ in the $H_0^{-1}(\Sigma_\theta)_R$ norm given in (4.17). We again note that (4.5) and (4.15) are equivalent problems, i.e., if (u, p, ρ) solves (4.5) then $(\tilde{u}, \tilde{p}, \tilde{\rho})$ solves (4.15) in the weak sense with $\tilde{\sigma} = \sigma|\lambda|^{-1/2}$ and $\tilde{\lambda} = \lambda/|\lambda|$.

By Proposition 4.2.2 we know that $\tilde{u}$ satisfies the weak formulation (4.7) such that (4.7) holds for $\varphi \in \mathbb{H}_R^1$ with the corresponding right-hand side $(\tilde{f}_1, \tilde{f}_4)$. By taking $\tilde{u}, \varphi \in L^2(\Sigma_\theta)$ into account, we first observe that

$$\left\langle \frac{\lambda}{|\lambda|}\tilde{u}, \varphi \right\rangle_{H_0^{-1}(\Sigma_\theta)_R, H^1(\Sigma_\theta)_R} = \left(\frac{\lambda}{|\lambda|}\tilde{u}, \varphi \right)_2 \qquad (\varphi \in H^1(\Sigma_\theta)_R).$$

We note that in this case we have the decomposition

$$H^1(\Sigma_\theta)_R = H^1_{\mathrm{div}}(\Sigma_\theta)_R \oplus \nabla(\hat{H}^1_0(\Sigma_\theta)_r \cap \hat{H}^2(\Sigma_\theta)_r)$$

from Corollary 3.1.16. Now let $\varphi \in H^1(\Sigma_\theta)_R$ be arbitrary. Thanks to the decomposition there exists $\psi \in H^1_{\mathrm{div}}(\Sigma_\theta)_R$ and $\Phi \in \hat{H}^1_0(\Sigma_\theta)_r \cap \hat{H}^2(\Sigma_\theta)_r$ such that $\varphi = \psi + \nabla\Phi$. Then we have

$$\left(\frac{\lambda}{|\lambda|}\tilde{u}, \varphi \right)_2 = \left(\frac{\lambda}{|\lambda|}\tilde{u}, \psi \right)_2 + \left(\frac{\lambda}{|\lambda|}\tilde{u}, \nabla\Phi \right)_2 = \left(\frac{\lambda}{|\lambda|}\tilde{u}, \psi \right)_2,$$

since

$$\left(\frac{\lambda}{|\lambda|}\tilde{u}, \nabla\Phi \right)_2 = -\left(\frac{\lambda}{|\lambda|}\operatorname{div}\tilde{u}, \Phi \right)_2 + \left(\frac{\lambda}{|\lambda|}\tilde{u}\cdot n, \Phi \right)_\Gamma = 0$$

because of $\operatorname{div}\tilde{u} = 0$ in Σ_θ and $\Phi = 0$ on Γ. Now, we again make use of the fact that $\mathbb{H}_R^1 \overset{d}{\hookrightarrow} H^1_{\mathrm{div}}(\Sigma_\theta)_R$ (as observed in the proof of Lemma 4.2.5). Hence, it is sufficient to obtain an estimate for $\psi \in \mathbb{H}_R^1$. Then from (4.7) we obtain the identity:

$$\begin{aligned}\left(\frac{\lambda}{|\lambda|}\tilde{u}, \psi \right)_2 &= -2\mu(D(\tilde{u}), D(\psi))_2 - \frac{\sigma c(\theta)|\lambda|}{\sin(\theta)\lambda|\lambda|^{1/2}}(\partial_\tau(n\cdot\tilde{u}), \partial_\tau(n\cdot\psi))_\Gamma \\ &\quad + \langle \tilde{f}_1, \psi\rangle_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R, \hat{H}^1_{\mathrm{div}}(\Sigma_\theta)_R} + \langle \tilde{f}_4, \psi\rangle_{\hat{H}^{-1/2}(\Gamma)_R, \hat{H}^{1/2}(\Gamma)_R}\end{aligned}$$

from which we obtain the estimate:

$$\begin{aligned}\left|\left(\frac{\lambda}{|\lambda|}\tilde{u}, \psi \right)_2\right| &\le 2\mu\|D(\tilde{u})\|_{L^2(\Sigma_\theta)_\mathcal{R}}\|D(\psi)\|_{L^2(\Sigma_\theta)_\mathcal{R}} \\ &\quad + \frac{\sigma c(\theta)}{\sin(\theta)|\lambda|^{1/2}}\|\partial_\tau^2(n\cdot\tilde{u})\|_{\hat{H}^{1/2}(\Gamma)_r + \hat{H}^{-1/2}(\Gamma)_r}\|n\cdot\psi\|_{\hat{H}^{-1/2}(\Gamma)_r\cap\hat{H}^{1/2}(\Gamma)_r} \\ &\quad + \|\tilde{f}_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R}\|\psi\|_{\hat{H}^1(\Sigma_\theta)_R} + C\|\tilde{f}_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\|\psi\|_{\hat{H}^{1/2}(\Gamma)_R}.\end{aligned}$$

We can use (4.8) to handle $\|D(\tilde{u})\|_{L^2(\Sigma_\theta)_\mathcal{R}}$. Since $\tilde{\sigma} = \sigma|\lambda|^{-1/2}$ we can use (4.11) to estimate $\|\partial_\tau^2(n \cdot \tilde{u})\|_{\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r}$. Also thanks to Lemma 3.2.15 and the observation that $\hat{H}^{-1/2}(\Gamma)_r + H^{1/2}(\Gamma)_r$ and $\hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r$ coincide topologically (cf. Lemma 4.2.5) we can estimate

$$\|n \cdot \psi\|_{\hat{H}^{-1/2}(\Gamma)_r \cap \hat{H}^{1/2}(\Gamma)_r} \leq C\|T_0\psi\|_{\hat{H}^{-1/2}(\Gamma)_r \cap H^{1/2}(\Gamma)_r} \leq C\|\psi\|_{H^1(\Sigma_\theta)_R}.$$

Hence, we have the following estimate

$$\left|\left(\frac{\lambda}{|\lambda|}\tilde{u}, \psi\right)_2\right| \leq C\left(\|\tilde{f}_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|\tilde{f}_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right)\|\psi\|_{H^1(\Sigma_\theta)_R}.$$

Note that $\varphi = \psi + \nabla\Phi$ such that $\psi = \varphi - \nabla\Phi$ and then

$$\|\psi\|_{H^1(\Sigma_\theta)_R} \leq \|\varphi\|_{H^1(\Sigma_\theta)_R} + \|\nabla\Phi\|_{H^1(\Sigma_\theta)_R} \leq C\|\varphi\|_{H^1(\Sigma_\theta)_R},$$

where the second estimate follows from the fact that Φ is the unique solution of the strong Dirichlet problem (3.5) (cf. Lemma 3.1.31). Since

$$\langle\lambda/|\lambda|\tilde{u}, \varphi\rangle_{H_0^{-1}(\Sigma_\theta)_R, H^1(\Sigma_\theta)_R} = \langle\lambda/|\lambda|\tilde{u}, \psi\rangle_{H_0^{-1}(\Sigma_\theta)_R, H^1(\Sigma_\theta)_R}$$

we finally arrive at

$$\left|\left\langle\frac{\lambda}{|\lambda|}\tilde{u}, \varphi\right\rangle_{H_0^{-1}(\Sigma_\theta)_R, H^1(\Sigma_\theta)_R}\right| \leq C\left(\|\tilde{f}_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|\tilde{f}_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right)\|\varphi\|_{H^1(\Sigma_\theta)_R}$$

and, hence,

$$\left\|\frac{\lambda}{|\lambda|}\tilde{u}\right\|_{H_0^{-1}(\Sigma_\theta)_R} \leq C\left(\|\tilde{f}_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|\tilde{f}_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right).$$

Furthermore, we note that

$$\|\tilde{u}\|_{H_0^{-1}(\Sigma_\theta)_R} \geq C|\lambda|\|u\|_{\lambda, H_0^{-1}(\Sigma_\theta)_R}$$

with $C > 0$ independent of λ by taking the calculations from Section 3.1.5 into consideration. Now inserting the calculations from above for $\tilde{f}_1, \tilde{f}_4$ as in Section 3.1.5 we obtain the estimate we aimed for:

$$\|u\|_{\lambda, H_0^{-1}(\Sigma_\theta)_R} \leq C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right).$$

Altogether we proved the desired resolvent estimate (4.18). □

4.2.8 Remark. Taking Section 4.2 into consideration we observe that the proof of higher regularity for $n \cdot u$ in Lemma 4.2.5 is an essential step for the whole section. However, if we would get a better estimate for $n \cdot u$, i.e., $n \cdot u \in \hat{H}^{-1/2}(\Gamma)_r$, this would also lead to better regularity classes for p and u: Then we would be able to prove that the pressure p is in $L^2(\Sigma_\theta)_r$ and estimates for $|\lambda|\|u\|_{\hat{H}_0^{-1}(\Sigma_\theta)_R}$.

Chapter 5

Stable and Unstable Flow Regimes for Active Fluids

In this chapter we will consider a continuum model which models the motion of self-propelled organisms in fluids, e.g. of bacteria. This model was proposed in [54] and is formulated as

$$\begin{aligned} v_t + \lambda_0 v \cdot \nabla v &= f - \nabla p + \lambda_1 \nabla |v|^2 - (\alpha + \beta |v|^2) v \\ &\qquad + \Gamma_0 \Delta v - \Gamma_2 \Delta^2 v \quad \text{in } (0,T) \times Q_n, \\ \operatorname{div} v &= 0 \qquad \text{in } (0,T) \times Q_n, \\ v|_{t=0} &= v_0 \qquad \text{in } Q_n. \end{aligned} \tag{5.1}$$

Here, we consider the model in the physically relevant dimensions $n = 2, 3$ and $Q_n := [0, L]^n$ denotes the box of length L as in Section 2.2. We will investigate (5.1) in the framework of periodic Sobolev spaces as introduced in Section 2.2. The bacterial velocity field is denoted by v whereas the pressure is denoted by p. Regarding the occurring parameters we will assume $\Gamma_2, \beta > 0$ and $\Gamma_0, \lambda_0, \lambda_1 \in \mathbb{R}$.

This chapter is structured as follows: In the first section we will provide a theorem stating global wellposedness of (5.1). In the second section we will investigate a manifold of stationary solutions which can be proved to be (in)stable depending on the occurring parameters. In Chapter 6 we prove the existence of a global attractor to (5.1) which is even contained in an inertial manifold in the two-dimensional case.

There are two known steady states. The disordered isotropic state is given as

$$(v, p) = (0, p_0),$$

where p_0 is a constant. If $\alpha < 0$ we even obtain a manifold of globally ordered polar states given as

$$(v, p) = (V, p_0),$$

where $V \in B_{\alpha,\beta} := \{x \in \mathbb{R}^n : |x| = \sqrt{-\alpha/\beta}\}$, i.e., V denotes a constant vector with arbitrary orientation and fixed swimming speed $|V| = \sqrt{\alpha/\beta}$ and p_0 is again a constant.

5.0.1 Remark. It is not known whether aside from the disordered and the ordered polar states there are more physically relevant stationary states. Note that in the whole space setting (see [57]) there is a larger manifold of stationary solutions given as

$$v(x) = v_0, \qquad p(x) = p_0 - (\alpha + \beta|v_0|^2)v_0 \cdot x, \qquad x \in \mathbb{R}^n,\ p_0 \in \mathbb{R}.$$

For $v_0 = 0$ and $|v_0| = \sqrt{-\alpha/\beta}$ we then obtain the disordered isotropic and the manifold of globally ordered polar states, respectively. Whereas for arbitrary $v_0 \in \mathbb{R}^n$ the pressure p takes negative values for large $x \in \mathbb{R}^n$ such that this stationary state doesn't make sense from the physical point of view. However, this kind of stationary states are not contained in $L^2_\pi(Q_n)$ since p is not periodic.

5.1 Global Wellposedness in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$

In order to prove global wellposedness we will consider a generalized system of (5.1) that includes the linearization at the corresponding stationary state (disordered and ordered state):

$$\begin{aligned} u_t + \lambda_0[(u+V)\cdot\nabla]u + (M + \beta|u|^2)u & & & \\ -\Gamma_0\Delta u + \Gamma_2\Delta^2 u + \nabla q &= f + N(u) & &\text{in } (0,T)\times Q_n, \\ \operatorname{div} u &= 0 & &\text{in } (0,T)\times Q_n, \\ u|_{t=0} &= v_0 & &\text{in } Q_n. \end{aligned} \tag{5.2}$$

where $q = p - \lambda_1|u|^2$ and $M \in \mathbb{R}^{n\times n}$ is a symmetric matrix and the nonlinearity N of second order is given as $N(u) = \sum_{j,k} a_{jk}u^j u^k$ with $(a_{jk})_{j,k=1}^n \in \mathbb{R}^{n\times n}$. Regarding the occurring parameters we assume

$$\lambda_0, \lambda_1, \Gamma_0, \alpha \in \mathbb{R}, \qquad \Gamma_2, \beta > 0,$$

throughout this and the next sections. Note that from (5.2) we obtain (5.1) linearized about the disordered isotropic state by setting

$$V = 0, \qquad M = \alpha I, \qquad N(u) = 0$$

for $u = v$ where I denotes the $n \times n$ identity matrix and α is a scalar. By setting

$$V \in B_{\alpha,\beta}, \qquad M = 2\beta V V^T, \qquad N(u) = -\beta|u|^2 V - 2\beta(u\cdot V)u$$

we obtain the system corresponding to the ordered polar state for $u = v - V$. In order to prove global wellposedness we first consider the linearization of (5.2) where we already applied the Helmholtz-Weyl projection P from Section 2.2:

$$\begin{aligned} u_t + \lambda_0(V \cdot \nabla)u + PMu - \Gamma_0\Delta u + \Gamma_2\Delta^2 u &= f \quad \text{in } (0,T) \times Q_n, \\ u|_{t=0} &= u_0 \quad \text{in } Q_n. \end{aligned} \tag{5.3}$$

Then we define the operator associated to (5.3) as

$$\begin{aligned} A_{LF}u &:= \lambda_0(V \cdot \nabla)u + PMu - \Gamma_0\Delta u + \Gamma_2\Delta^2 u, \\ D(A_{LF}) &:= H^4_\pi(Q_n) \cap L^2_\sigma(Q_n), \end{aligned} \tag{5.4}$$

and the corresponding Fourier symbol as

$$\sigma_{A_{LF}}(\ell) := \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \lambda_0 \left(\frac{2\pi i}{L}\right)(V \cdot \ell) + \sigma_P(\ell)M$$

for $\ell \in \mathbb{Z}^n$. By considering the leading term

$$A_{SH}u := \Gamma_2\Delta^2 u, \qquad D(A_{SH}) := H^4_\pi(Q_n) \cap L^2_\sigma(Q_n),$$

we observe that A_{SH} is a selfadjoint operator. Furthermore, by making use of the fact that for $\lambda \in \rho(A_{SH})$ the resolvent $(\lambda - A_{SH})^{-1} : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n)$ is compact by the Rellich-Kondrachov theorem [39, Theorem A.4, Corollary A.5], we conclude that A_{SH} has compact resolvent, hence the spectrum $\sigma(A_{SH})$ is discrete and $\sigma(A_{SH}) = \sigma_P(A_{SH})$ where $\sigma_P(A_{SH})$ denotes the point spectrum of A_{SH}. Hence, we can further characterize the (point) spectrum of A_{SH} as

$$\lambda - A_{SH} \text{ is not injective } \Leftrightarrow \lambda = \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 \text{ for some } \ell \in \mathbb{Z}^n,$$

where we made use of the fact that $\sigma_{A_{SH}}(\ell) := \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4$. Hence the spectrum is given as $\sigma(A_{SH}) = \{\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 : \ell \in \mathbb{Z}^n\}$ and as a consequence of the spectral theorem in its functional calculus form (e.g. [38, Theorem VIII.5]), we observe that for some $\omega > 0$ the operator $\omega + A_{SH}$ admits a bounded H^∞-calculus on $L^2_\sigma(Q_n)$ with H^∞-angle $\varphi^\infty_{w+A_{SH}} = 0$. For a proper introduction to the notion of a bounded H^∞-calculus, we refer to [22]. Next, by defining the perturbation as

$$\begin{aligned} Bu &:= \lambda_0(V \cdot \nabla)u + PMu - \Gamma_0\Delta u, \\ D(B) &:= H^2_\pi(Q_n) \cap L^2_\sigma(Q_n), \end{aligned}$$

we immediately observe that B is a perturbation of lower order. Now applying a perturbation theorem for the H^∞-calculus [27, Proposition 13.1] we deduce

5.1.1 Proposition. *There exists an* $\omega > 0$ *such that* $\omega + A_{LF}$ *admits a bounded* H^∞*-calculus on* $L^2_\sigma(Q_n)$ *with* H^∞*-angle* $\varphi^\infty_{\omega + A_{LF}} = 0$.

As an immediate consequence we obtain that A_{LF} enjoys maximal L^p-regularity on intervals $(0, T)$ with $T < \infty$ and $-A_{LF}$ is the generator of an analytic C_0-semigroup $(\exp(-tA_{LF}))_{t \geq 0}$ on $L^2_\sigma(Q_n)$:

5.1.2 Proposition. *Let* $T \in (0, \infty)$. *For* $f \in L^2((0,T), L^2_\sigma(Q_n))$ *and initial value* $u_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n) = (L^2_\sigma(Q_n), H^4_\pi(Q_n) \cap L^2_\sigma(Q_n))_{1/2,2}$ *there exists a unique solution* (u, q) *of* (5.2) *such that*

$$\begin{aligned} &\|u\|_{H^1((0,T),L^2_\sigma(Q_n))} + \|u\|_{L^2((0,T),H^4_\pi(Q_n))} + \|\nabla q\|_{L^2((0,T),L^2(Q_n))} \\ &\leq C(T) \left(\|f\|_{L^2((0,T),L^2_\sigma(Q_n))} + \|u_0\|_{H^2_\pi(Q_n)} \right). \end{aligned}$$

In order to prove local wellposedness we use the common approach by combining the maximal L^p-regularity with the local inverse theorem to construct a solution (u, q). By making use of energy estimates we even obtain global wellposedness by proceeding as in [57, Section 3.2]:

5.1.3 Theorem (Global wellposedness). *Let* $\Gamma_2, \beta > 0$ *and* $\Gamma_0, \alpha, \lambda_0 \in \mathbb{R}$ *and* $T \in (0, \infty)$. *Let the initial value* $u_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ *and an exterior force* $f \in L^2((0,T), L^2_\sigma(Q_n))$ *be given. Then there exists a unique pair* (u, q) *with*

$$\begin{aligned} u &\in H^1((0,T), L^2_\sigma(Q_n)) \cap L^2((0,T), H^4_\pi(Q_n)), \\ \nabla q &\in L^2((0,T), L^2(Q_n)), \end{aligned}$$

solving (5.2) *for periodic boundary conditions.*

5.1.4 Remark. Note that, in contrast to the classical incompressible Navier-Stokes equations, we can prove global wellposedness since the convective term $(u \cdot \nabla)u$ in (5.2) is dominated by the fourth order term Δ^2. In this case, we are able to prove corresponding energy estimates which lead to global strong solvability.

5.2 Stability Analysis for the Ordered Polar State

In this section we perform a full stability analysis for the manifold of ordered polar states, i.e., for $(v, p) = (V, p_0)$ where $V \in B_{\alpha,\beta} = \{x \in \mathbb{R}^n : |x| = \sqrt{-\alpha/\beta}\}$ and p_0 is a constant. We will proceed as follows: At first we will consider linear (in)stability. In fact, we will prove that depending on the relation of the occurring parameters we will obtain stability or instability for the ordered polar state. Hence, those observations are fundamental to prove nonlinear stability and turbulence, respectively. Here, we will apply the generalized principle of linearized stability as provided in [35, 36].

5.2.1 Remark. Note that in this section we are not considering the disordered polar state $(v, p) = (0, p_0)$ where p_0 is a constant. In this case a full stability analysis can also be performed but is straightforward by making use of energy methods. For the full stability analysis for the disordered polar state we refer to [8].

5.2.1 Linear Stability

In this section we consider linear stability for the ordered polar state. To this end, we are making use of properties of the analytic semigroup $(\exp(-tA_{LF}))_{t\geq 0}$ which is generated by the operator $-A_{LF}$, see (5.4). It is straightforward to verify the identity

$$\exp(-tA_{LF})v = \sum_{\ell\in\mathbb{Z}^n} \exp(-t\sigma_{A_LF}(\ell))\hat{v}(\ell)e^{2\pi i\ell\cdot/L}$$

for $v \in L^2_\sigma(Q_n)$. Using this representation we can characterize linear (in)stability by basically examining the Fourier symbol $\sigma_{A_{LF}}$. Next, we set $V \in B_{\alpha,\beta}$, and $M = 2\beta VV^T$ to obtain the operator A_o corresponding to the ordered polar state:

$$A_o u = \Gamma_2\Delta^2 u - \Gamma_0\Delta u + \lambda_0(V\cdot\nabla)u + 2\beta PVV^T u \qquad (u \in H^4_\pi(Q_n)\cap L^2_\sigma(Q_n)). \quad (5.5)$$

Then the Fourier symbol σ_{A_o} is given as

$$\begin{aligned}\sigma_{A_o}(\ell) := \Gamma_2\left(\frac{2\pi}{L}\right)^4|\ell|^4 + \Gamma_0\left(\frac{2\pi}{L}\right)^2|\ell|^2 + \lambda_0\left(\frac{2\pi i}{L}\right)(V\cdot\ell) \\ + 2\beta\sigma_P(\ell)VV^T\sigma_P(\ell).\end{aligned} \quad (5.6)$$

Then we can state the following result on linear (in)stability:

5.2.2 Proposition. *Let* $\Gamma_2 > 0$. *Then the semigroup* $(\exp(-tA_o))_{t\geq 0}$ *corresponding to the ordered polar state is*

(1) stable if $\Gamma_0 \geq 0$;

(2) exponentially unstable if $\Gamma_0 < 0$ *and*

(i) if for $n = 2$ *there exists some* $0 \neq \ell_0 \in \mathbb{Z}^n$ *such that*

$$\Gamma_2\left(\frac{2\pi}{L}\right)^2|\ell_0|^4 + \Gamma_0|\ell_0|^2 < 2\alpha; \quad (5.7)$$

(ii) if for $n = 3$ *there exists some* $0 \neq \ell_0 \in \mathbb{Z}^n$ *such that*

$$\Gamma_2\left(\frac{2\pi}{L}\right)^2|\ell_0|^2 + \Gamma_0 < 0. \quad (5.8)$$

Proof. At first we consider the case if $\Gamma_0 \geq 0$. Then we can infer that the Fourier symbol

$$\text{Re } \sigma_{A_o}(\ell) = \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + 2\beta\sigma_P(\ell)VV^T\sigma_P(\ell) \in \mathbb{R}^{n\times n} \tag{5.9}$$

is positive semi-definite since $\sigma_P(\ell)VV^T\sigma_P(\ell)$ is positive semi-definite and the two remaining terms are positive. Then we can estimate the norm of the semigroup as

$$\begin{aligned}\|\exp(-tA_o)v\|^2_{L^2(Q_n)} &= \sum_{\ell\in\mathbb{Z}^n} |e^{-t\sigma_{A_o}(\ell)}\hat{v}(\ell)|^2 \leq |\hat{v}(0)|^2 + \sum_{\ell\in\mathbb{Z}^n\backslash\{0\}} |e^{-t\sigma_{A_o}(\ell)}\hat{v}(\ell)|^2 \\ &\leq \|v\|^2_{L^2(Q_n)},\end{aligned}$$

where we applied Theorem 2.2.1. Hence, the estimate yields stability for $\Gamma_0 \geq 0$.

Next, we examine instability. For this purpose we assume $\Gamma_0 < 0$. In order to prove exponential instability we have to find some $0 \neq \ell_0 \in \mathbb{Z}^n$ such that the matrix Re $\sigma_{A_o}(\ell_0) \in \mathbb{R}^{n\times n}$ is negative definite or indefinite. Then the growth bound of the semigroup $(\exp(-tA_o))_{t\geq 0}$ is strictly positive and we obtain exponential instability of the semigroup.
To prove the negative definiteness or indefiniteness we have to find some $x \in \mathbb{R}^n\backslash\{0\}$ such that $x^T\text{Re } \sigma_{A_o}(\ell_o)x < 0$, which in fact results in

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell_0|^4|x|^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2|x|^2 + 2\beta x^T\sigma_P(\ell_0)VV^T\sigma_P(\ell_0)x < 0.$$

For $\ell_0 \in \mathbb{Z}^n\backslash\{0\}$ we are able to find some $x \in \mathbb{R}^n\backslash\{0\}$ such that $\ell_0 \perp x$. Then $\sigma_P(\ell_0)x = \left(I - \ell_0\ell_0^T/|\ell_0|^2\right)x = x$ such that we end up with

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell_0|^4|x|^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2|x|^2 + 2\beta|V\cdot x|^2 < 0$$

which we want to prove. Thanks to the fact that $|V|^2 = -\alpha/\beta$ this is equivalent to the condition

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell_0|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2 < -\frac{2\beta|V\cdot x|^2}{|x|^2} \in [2\alpha, 0].$$

Indeed if we assume the existence of some $\ell_0 \in \mathbb{Z}^n\backslash\{0\}$ which fulfills (5.7) we can choose some $x \in \mathbb{R}^n\backslash\{0\}$ with $x \perp \ell_0$ such that $x^T\text{Re } \sigma_{A_o}(\ell_0)x < 0$. This yields the exponential instability for $n = 2, 3$.
For the three dimensional case we can even improve the condition (5.7) a little bit. In three dimensions we have enough degrees of freedom to choose $x \in \mathbb{R}^3\backslash\{0\}$ with $x \perp \ell_0$ and $x \perp V$. Then

$$x^T\text{Re } \sigma_{A_o}(\ell_0)x = \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell_0|^4|x|^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2|x|^2 < 0,$$

if $\ell_0 \in \mathbb{Z}^3 \backslash \{0\}$ fulfills (5.8). Hence, also in this case we obtain exponential instability and the assertion is proved. □

5.2.3 Remark. We want to briefly compare this result to the continuous setting considered in [57, Section 3.1] and [9, Section 3.1]. Note that in the continuous setting we don't have the restricted assumption (5.7) for $n = 2$ and $\ell_0 \in \mathbb{R}^2 \backslash \{0\}$ since in the continuous setting some $\ell_0 \in \mathbb{R}^2 \backslash \{0\}$ such that ℓ_0 is parallel to V can always be found such that we can find some $x \in \mathbb{R}^2 \backslash \{0\}$ with $x \perp V$ and $x \perp \ell_0$. Hence in the continuous case assumption (5.8) for $\ell_0 \in \mathbb{R}^2 \backslash \{0\}$ is sufficient to prove instability.

5.2.2 Nonlinear Stability

In this section we study nonlinear stability of the manifold of ordered polar states. As already mentioned we will apply the generalized principle of linearized stability [35, Theorem 5.3.1] or [36, Theorem 2.1] to prove normal stability. To observe normal hyperbolicity we will use the principle of normally hyperbolic equilibria [35, Theorem 5.5.1] or [36, Theorem 6.1]. For the reader's convenience we formulated the corresponding theorems in Theorem 2.1.3 and Theorem 2.1.4.

In order to apply both principles we formulate our setting in the notation of Theorem 2.1.3 and Theorem 2.1.4. In our case we first neglect the pressure and consider system (5.1) after applying the Helmholtz-Weyl projection

$$\begin{aligned} v_t + \lambda_0 P(v \cdot \nabla) v &= f - (\alpha + \beta P|v|^2) v + \Gamma_0 \Delta v - \Gamma_2 \Delta^2 v && \text{in } (0,T) \times Q_n, \\ v|_{t=0} &= v_0 && \text{in } Q_n. \end{aligned} \tag{5.10}$$

Here, we have $U = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n),\ X_0 = L^2_\sigma(Q_n),\ X_1 = H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for the spaces and $\mathcal{E} = B_{\alpha,\beta}$ for the manifold. Furthermore, we set

$$\begin{aligned} A(v)\tilde{v} &\coloneqq A\tilde{v} \coloneqq \Gamma_2 \Delta^2 \tilde{v} - \Gamma_0 \Delta \tilde{v} + \alpha \tilde{v} \qquad (\tilde{v} \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)), \\ F(v) &\coloneqq -\lambda_0 P(v \cdot \nabla) v - \beta P|v|^2 v. \end{aligned}$$

for $v \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. By the structure of A and F (linear and semilinear, respectively) it is straightforward to see that

$$(A, F) \in C^1(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n), \mathscr{L}(H^4_\pi(Q_n) \cap L^2_\sigma(Q_n), L^2_\sigma(Q_n)) \times L^2_\sigma(Q_n)).$$

Moreover, a quick calculation shows that the operator A_o from (5.5) is indeed the linearized operator of (5.1) at V. By Proposition 5.1.2 we also know that A and A_o enjoy maximal L^p-regularity on $(0,T)$ for $T < \infty$.

In order to apply both principles of normal stability and normal hyperbolicity, we first provide the following

5.2.4 Lemma. *Let $n = 2, 3$ and $V \in B_{\alpha,\beta}$ be arbitrary but fixed. Then near V the set of equilibria $B_{\alpha,\beta}$ is a C^1-manifold in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ of dimension $n-1 \in \mathbb{N}$. The tangent space $T_V B_{\alpha,\beta}$ at V is given as*

$$T_V B_{\alpha,\beta} = \langle V \rangle^T.$$

Proof. It is straightforward to define a C^1-function which maps into $B_{\alpha,\beta}$. If $n = 2$ we can write every given $V \in B_{\alpha,\beta}$ as

$$V = \sqrt{-\frac{\alpha}{\beta}} \begin{pmatrix} \cos(\varphi_V) \\ \sin(\varphi_V) \end{pmatrix}$$

with a unique fixed $\varphi_V \in [0, 2\pi)$. Then we can define a corresponding C^1 map as

$$\Psi_2 : [0, 2\pi) \to H^4_\pi(Q_n) \cap L^2_\sigma(Q_n), \qquad z \mapsto \Psi_2(z) := \sqrt{-\frac{\alpha}{\beta}} \begin{pmatrix} \cos(\varphi_V + z) \\ \sin(\varphi_V + z) \end{pmatrix}.$$

For $n = 3$ analogously we can write every $V \in B_{\alpha,\beta}$ as

$$V = \sqrt{-\frac{\alpha}{\beta}} \begin{pmatrix} \sin(\theta_V)\cos(\varphi_V) \\ \sin(\theta_V)\sin(\varphi_V) \\ \cos(\theta_V) \end{pmatrix},$$

for fixed and unique $\theta_V \in [0, \pi]$ and $\varphi_V \in [0, 2\pi)$. The corresponding C^1 map then can be defined as

$$\Psi_3 : [0, \pi] \times [0, 2\pi) \to H^4_\pi(Q_n) \cap L^2_\sigma(Q_n),$$
$$\begin{pmatrix} y \\ z \end{pmatrix} \mapsto \Psi_3(y, z) := \sqrt{-\frac{\alpha}{\beta}} \begin{pmatrix} \sin(\theta_V + y)\cos(\varphi_V + z) \\ \sin(\theta_V + y)\sin(\varphi_V + z) \\ \cos(\theta_V + y) \end{pmatrix}.$$

Hence $\Psi_2(z) \in B_{\alpha,\beta}$ and $\Psi_3(y, z) \in B_{\alpha,\beta}$, respectively, are constant functions in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for every $z \in [0, 2\pi)$ and $(y, z) \in [0, \pi] \times [0, 2\pi)$, respectively, satisfying $\Psi_2(0) = V$ and $\Psi_3(0, 0) = V$.

Obviously then the corresponding tangent space $T_V B_{\alpha,\beta}$ at V is $n - 1 \in \mathbb{N}$ dimensional and a straightforward calculation shows $T_V B_{\alpha,\beta} = \langle V \rangle^T$. □

5.2.5 Lemma. *Let $V \in B_{\alpha,\beta}$ be arbitrary and A_o be defined as in (5.5). Assume that the occurring parameters are chosen such that*

$$N(A_o) \subseteq \{u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) : u \text{ constant and } u \perp V\}. \tag{5.11}$$

Then the spectrum of A_o is discrete, consists only of the point spectrum and 0 is a semi-simple eigenvalue of A_o, i.e., $L^2_\sigma(Q_n) = N(A_o) \oplus R(A_o)$, where

$$N(A_o) = \{u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) : u \text{ constant and } u \perp V\}.$$

Proof. First we observe that for $\lambda \in \rho(A_o)$

$$(\lambda - A_o)^{-1} : L^2_\sigma(Q_n) \to D(A_o) = H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) \overset{c}{\hookrightarrow} L^2_\sigma(Q_n)$$

is a compact operator by the Rellich-Kondrachov theorem [39, Theorem A.4, Corollary A.5]. Hence, A_o has compact resolvent and A_o has a discrete spectrum which just consists of the point spectrum, i.e., $\sigma(A_o) = \sigma_p(A_o)$, where $\sigma_p(A_o)$ denotes the point spectrum of A_o. Next, we prove that $0 \in \sigma(A_o)$ and

$$N(A_o) \supseteq \{u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) : u \text{ constant and } u \perp V\}.$$

Let $u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ be a constant vector in $\mathbb{R}^n$ and perpendicular to V. Then we immediately observe that

$$A_o u = \Gamma_2 \Delta^2 u - \Gamma_o \Delta u + \lambda_0 (V \cdot \nabla) u + 2\beta P V V^T u = 0$$

by the properties of u. Hence, 0 is an eigenvalue and by assumption (5.11) we even obtain the equality in (5.11).

At last we need to prove that 0 is a semi-simple eigenvalue of A_o, i.e., we will show that the decomposition $N(A_o) \oplus R(A_o) = L^2_\sigma(Q_n)$ holds. In order to prove this we define the following projection

$$S : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \qquad Su := \frac{1}{L^n} \int_{Q_n} S_* u(x)\, dx,$$

where $S_* : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n)$ is given by $S_* u(x) = (I - VV^T/|V|^2) u(x)$, where I again denotes the identity matrix in n dimensions. First we note that if $u \in L^2_\sigma(Q_n)$ then Su is constant and $Su \in L^2_\sigma(Q_n)$. Note that $S_* S_* = S_*$, hence we also obtain $S^2 = S$ such that S is a projection. Then there exists a decomposition $S(L^2_\sigma(Q_n)) \oplus (I - S)(L^2_\sigma(Q_n)) = L^2_\sigma(Q_n)$ and we need to prove that on one hand $S(L^2_\sigma(Q_n)) = N(A_o)$ holds and on the other hand $(I - S)(L^2_\sigma(Q_n)) = R(A_o)$.

First we claim $N(A_o) = S(L^2_\sigma(Q_n))$. To see the inclusion $S(L^2_\sigma(Q_n)) \subseteq N(A_o)$ we assume $u \in S(L^2_\sigma(Q_n))$. Then $u = Su$ is constant and perpendicular to V since

$$V^T u = V^T S u = \frac{1}{L^n} \int_{Q_n} V^T u(x) dx - \frac{1}{L^n} \int_{Q_n} \frac{1}{|V|^2} V^T V V^T u(x)\, dx = 0,$$

hence $u \in N(A_o)$ by the already proved equality in (5.11). Conversely, let $u \in N(A_o)$. Then by the fact that u is constant and perpendicular to V we observe

$$Su = \frac{1}{L^n} \int_{Q_n} u\, dx - \frac{1}{L^n} \int_{Q_n} \frac{1}{|V|^2} V V^T u\, dx = u \left(\frac{1}{L^n} \int_{Q_n} dx \right) = u,$$

such that $u \in S(L^2_\sigma(Q_n))$. Hence, the claim is proved.

Since $L^2_\sigma(Q_n)$ is a Hilbert space and S is a selfadjoint projection it is well-known that $L^2_\sigma(Q_n) = S(L^2_\sigma(Q_n)) \oplus (I - S)(L^2_\sigma(Q_n))$ is an orthogonal decomposition. If we take $u \in D(A_o)$ and show that $A_o u$ is perpendicular to any $w \in N(A_o)$ then we have $R(A_o) \subseteq (I - S)(L^2_\sigma(Q_n))$:

$$\begin{aligned}(A_o u, w)_{2,\pi} &= \Gamma_2(\Delta u, \Delta w)_{2,\pi} + \Gamma_0(\nabla u, \nabla w)_{2,\pi} \\ &\qquad - \lambda_0(u, (V \cdot \nabla)w)_{2,\pi} + 2\beta(V^T u, V^T w)_{2,\pi} \\ &= 0,\end{aligned}$$

because w is constant and perpendicular to V. In fact, by the orthogonal decomposition of $L^2_\sigma(Q_n)$ we just proved $N(A_o) \cap R(A_o) = \{0\}$.
Since A_o has compact resolvent it follows from [14, Corollary 1.19] that the spectral value 0 is a pole of the resolvent. Then by [32, Remark A.2.4] it suffices to show that

$$N(A_o) = N(A_o^2)$$

to prove that 0 is a semi-simple eigenvalue of A_o. It is obvious that $N(A_o) \subseteq N(A_o^2)$. To observe the converse inclusion let $u \in N(A_o^2)$ such that $A_o^2 u = 0$. Then we conclude $A_o u \in N(A_o) \cap R(A_o) = \{0\}$ by our observation above. Hence, $A_o u = 0$ and $u \in N(A_o)$ such that $N(A_o^2) = N(A_o)$. Finally, from [32, Proposition A.2.2, Remark A.2.4] it follows that 0 is a semi-simple eigenvalue. □

At first we will show that for the unstable regime the manifold of ordered polar states is normally hyperbolic. From Theorem 2.1.4 we recall that an equilibrium V is called normally hyperbolic in our setting if

(i) near V the set of equilibria $B_{\alpha,\beta}$ is a C^1-manifold in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ of dimension $n \in \mathbb{N}$;

(ii) the tangent space $T_V B_{\alpha,\beta}$ for $B_{\alpha,\beta}$ at V is isomorphic to $N(A_o)$;

(iii) 0 is a semi-simple eigenvalue of A_o, i.e., $L^2_\sigma(Q_n) = N(A_o) \oplus R(A_o)$;

(iv) $\sigma(A_o) \cap i\mathbb{R} = \{0\}$ and $\sigma_u := \sigma(A_o) \cap \{z \in \mathbb{C} : \operatorname{Re} z < 0\} \neq \emptyset$.

This means instability in the following sense: For each sufficiently small $\rho > 0$ there exists $0 < \delta \leq \rho$ such that the unique solution v of (5.1) with initial value $v_0 \in B_{H^2}(V, \delta) := \{v \in H^2_\pi(Q_n) : \|v - V\|_{H^2_\pi(Q_n)} < \delta\}$ either satisfies

(i) $dist_{H^2}(v(t_0), B_{\alpha,\beta}) := \inf_{V \in B_{\alpha,\beta}} \|v(t_0) - V\|_{H^2_\pi(Q_n)} > \rho$ for a finite time $t_0 > 0$ or

(ii) $(v(t), p(t))$ exists on $\mathbb{R}_+$ and converges at exponential rate to some pair $(V_\infty, p_\infty) \in B_{\alpha,\beta} \times \mathbb{R}$ in $(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)) \times \hat{H}^1_\pi(Q_n)$ as $t \to \infty$.

Finally, we gathered all relevant properties to prove normal hyperbolicity by applying the principle of normally hyperbolic equilibria from Theorem 2.1.4:

5.2.6 Theorem. *Let $\Gamma_2, \beta > 0$ and $\alpha < 0$ and $\lambda_0 \in \mathbb{R}$. The ordered polar state is normally hyperbolic if*

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 \notin [2\alpha, 0], \qquad \ell \in \mathbb{Z}^n \backslash \{0\} \tag{5.12}$$

for $\Gamma_0 < 0$ and if there exists some $\ell_0 \in \mathbb{Z}^n$ such that (5.8) holds. Thus, the ordered polar state is unstable in the sense given above.

Proof. As mentioned before we will first consider the projected system (5.10) and neglect the pressure first. In order to apply Theorem 2.1.4 we need to show that every equilibrium $V \in B_{\alpha,\beta}$ is normally hyperbolic. By Lemma 5.2.4 we already know that the manifold of ordered polar states $B_{\alpha,\beta}$ forms a C^1-manifold of equilibria. In order to obtain the results from Lemma 5.2.5 we only need to prove the inclusion (5.11).

For this purpose let $u \in N(A_o)$ such that $A_o u = 0$. Using the Fourier series representation from Theorem 2.2.1 we obtain

$$\|A_o u\|^2_{L^2(Q_n)} = \sum_{\ell \in \mathbb{Z}^n} |\sigma_{A_o}(\ell)\hat{u}(\ell)|^2 = 0$$

where σ_{A_o} is defined as in (5.6). Then $\sigma_{A_o}(\ell)\hat{u}(\ell) = 0$ for every $\ell \in \mathbb{Z}^n$, hence also $\overline{\hat{u}(\ell)}^T \sigma_{A_o}(\ell)\hat{u}(\ell) = 0$ such that we obtain

$$\begin{aligned} 0 &= \operatorname{Re}\left(\overline{\hat{u}(\ell)}^T \sigma_{A_o}(\ell)\hat{u}(\ell)\right) \\ &= \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 |\hat{u}(\ell)|^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 |\hat{u}(\ell)|^2 + 2\beta \overline{\hat{u}(\ell)}^T \sigma_P(\ell) V V^T \sigma_P(\ell) \hat{u}(\ell) \end{aligned}$$

for all $\ell \in \mathbb{Z}^n$. Note that $\hat{u}(\ell) \in \mathbb{C}^n$ for $\ell \in \mathbb{Z}^n$. We recall that $\sigma_P(\ell)$ is a symmetric matrix for all $\ell \in \mathbb{Z}^n$ and that $\sigma_P(\ell)\hat{u}(\ell) = \hat{u}(\ell)$ since u is divergence free by assumption such that $\ell \cdot \hat{u}(\ell) = 0$ (see definition of $L^2_\sigma(Q_n)$ in Section 2.2). Then we infer

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 |\hat{u}(\ell)|^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 |\hat{u}(\ell)|^2 + 2\beta |V \cdot \hat{u}(\ell)|^2 = 0 \qquad (\ell \in \mathbb{Z}^n).$$

Setting $\ell = 0$ yields $|V \cdot \hat{u}(0)| = 0$ such that $V \perp \hat{u}(0)$. By considering the remaining $\ell \neq 0$ with $\hat{u}(\ell) \neq 0$ we obtain

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 = -\frac{2\beta |V \cdot \hat{u}(\ell)|^2}{|\hat{u}(\ell)|^2} \in [2\alpha, 0]$$

by the fact that $|V|^2 = -\alpha/\beta$. Since we assumed (5.12) this is a contradiction such that $\hat{u}(\ell) = 0$ for $\ell \neq 0$. Altogether we just proved that $u \in N(A_o)$ is constant and perpendicular to V, indeed we just proved (5.11). By Lemma 5.2.5 we know that 0 is a semi-simple eigenvalue of A_o and combined with Lemma 5.2.4 this yields $T_V B_{\alpha,\beta} = N(A_o)$. Hence (i)-(iii) in Theorem 2.1.4 are fulfilled.

Finally, we have to verify (iv) from Theorem 2.1.4. Note that by assumptions (5.8) and (5.12) we deduce that also (5.7) is fulfilled such that the arguments work for both dimensions $n = 2, 3$. Hence by Proposition 5.2.2(2) we infer $\sigma(A_o) \cap \mathbb{C}_- \neq \emptyset$, since the ordered polar state is linearly exponentially unstable in this case.

At last we need to verify $\sigma(A_o) \cap i\mathbb{R} = \{0\}$. Let $\lambda \in \sigma(A_o)$ with Re $\lambda = 0$. Let $u \neq 0$ be the corresponding eigenfunction. Then $(\lambda - A_o)u = 0$ which again results in

$$\|(\lambda - A_o)u\|^2_{L^2(Q_n)} = \sum_{\ell \in \mathbb{Z}^n} |(\lambda - \sigma_{A_o}(\ell))\hat{u}(\ell)|^2 = 0$$

again by Theorem 2.2.1. Then by applying the same arguments as above we obtain that $\overline{\hat{u}(\ell)}^T(\lambda - \sigma_{A_o}(\ell))\hat{u}(\ell) = \lambda|\hat{u}(\ell)|^2 - \overline{\hat{u}(\ell)}^T\sigma_{A_o}(\ell)\hat{u}(\ell) = 0$ for every $\ell \in \mathbb{Z}^n$ which results in

$$\text{Re}\,\left(\overline{\hat{u}(\ell)}^T\sigma_{A_o}(\ell)\hat{u}(\ell)\right) = \text{Re}\,\lambda|\hat{u}(\ell)|^2 = 0 \qquad (\ell \in \mathbb{Z}^n).$$

By applying exactly the same arguments as in the first part of the proof this implies $\hat{u}(\ell) = 0$ for all $\ell \neq 0$ and $\hat{u}(0) \perp V$. Hence $u \in N(A_o)$ such that $\lambda = 0$ since all eigenspaces $N(\lambda - A_o)$ corresponding to the eigenvalues $\lambda \in \sigma(A_o)$ are disjoint. (Note that by Lemma 5.2.5 we know $\sigma(A_o) = \sigma_P(A_o)$.) Finally by Theorem 2.1.4 the assertion for V follows.

At last it remains to prove the convergence of the pressure in case (ii). We assume in this case that $v(t)$ exists on $\mathbb{R}_+$ and $v(t) \to V_\infty$ exponentially in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for some $V_\infty \in B_{\alpha,\beta}$ as $t \to \infty$. We still need to prove the existence of p and the convergence $p(t) \to p_\infty$ in $\hat{H}^1_\pi(Q_n)$ exponentially for some $p_\infty \in \mathbb{R}$ as $t \to \infty$. Note that we can recover the pressure gradient ∇p by applying the projection $(I - P)$ to (5.1). Hence, we then obtain

$$\nabla p = (I - P)\left[-\lambda_0(v \cdot \nabla)v + \lambda_1\nabla|v|^2 - \beta|v|^2 v\right] = (I - P)G(v),$$

with $G(v) := -\lambda_0(v \cdot \nabla)v + \lambda_1\nabla|v|^2 - \beta|v|^2 v$ for our solution $v(t) \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Note that $G \in C^1(H^2_\pi(Q_n), L^2(Q_n))$ by the estimate

$$\begin{aligned}
\|G(w)\|_{L^2(Q_n)} &\leq C\|w\|_{L^4(Q_n)}\|\nabla w\|_{L^4(Q_n)} + C\|w\|^2_{H^1_\pi(Q_n)} + C\|w\|^3_{L^6(Q_n)} \\
&\leq C\|w\|_{H^1_\pi(Q_n)}\|\nabla w\|_{H^1_\pi(Q_n)} + C\|w\|^2_{H^1_\pi(Q_n)} + C\|w\|^3_{H^1_\pi(Q_n)} \\
&\leq C\left(\|w\|^2_{H^2_\pi(Q_n)} + \|w\|^3_{H^2_\pi(Q_n)}\right)
\end{aligned}$$

and the fact that G consists of bi- and trilinear forms which are known to be continuous on $H^2_\pi(Q_n)$. Note that in the estimate we used two Sobolev embeddings [5, Corollary 1.2]

$$H^1_\pi(Q_n) \hookrightarrow L^6(Q_n) \quad \text{and} \quad H^1_\pi(Q_n) \hookrightarrow L^4(Q_n),$$

where we only consider dimensions $n = 2, 3$. The Fréchet derivative of G at $w \in H^2_\pi(Q_n)$ reads as

$$\begin{aligned} DG(w)z = &-\lambda_0(w \cdot \nabla)z - \lambda_0(z \cdot \nabla)w + 2\lambda_1(\nabla w)z + 2\lambda_1 w(\nabla z) \\ &- 2\beta(w \cdot z)w - \beta(w \cdot w)z \end{aligned}$$

for all $z \in H^2_\pi(Q_n)$. Since $v(t) \to V_\infty$ exponentially in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ we infer that the solution v remains in a ball in $H^2_\pi(Q_n)$ for all times that also includes V_∞, to be precise we have $v(t) \in B_{H^2}(V_\infty, R)$ for all $t < \infty$ for some $R > 0$. Hence, we can estimate the Fréchet derivative DG in this ball as:

$$\begin{aligned} &\|DG(\xi)z\|_{L^2(Q_n)} \\ &\quad \le C\left(\|\xi\|_{L^4(Q_n)}\|\nabla z\|_{L^4(Q_n)} + \|z\|_{L^4(Q_n)}\|\nabla\xi\|_{L^4(Q_n)} + \|\xi\|^2_{L^6(Q_n)}\|z\|_{L^6(Q_n)}\right) \\ &\quad \le C\left(\|\xi\|_{H^1_\pi(Q_n)}\|z\|_{H^2_\pi(Q_n)} + \|z\|_{H^1_\pi(Q_n)}\|\xi\|_{H^2_\pi(Q_n)} + \|\xi\|^2_{H^1_\pi(Q_n)}\|z\|_{H^1_\pi(Q_n)}\right) \\ &\quad \le C\|z\|_{H^2_\pi(Q_n)} \end{aligned}$$

such that

$$\|DG(\xi)\|_{\mathscr{L}(H^2_\pi(Q_n), L^2(Q_n))} \le C$$

for all $\xi \in B_{H^2}(V_\infty, R)$. Note that $\|\xi\|_{H^2_\pi(Q_n)} \le \|\xi - V_\infty\|_{H^2_\pi(Q_n)} + \|V_\infty\|_{H^2_\pi(Q_n)} \le C$ for some $C > 0$ independent of ξ in this case. Hence, applying the Taylor expansion for $v(t), V_\infty \in B_{H^2}(V_\infty, R)$ in the convex ball $B_{H^2}(V_\infty, R)$ yields

$$\begin{aligned} \|G(v(t)) - G(V_\infty)\|_{L^2(Q_n)} &= \|DG(\xi)(v(t) - V_\infty)\|_{L^2(Q_n)} \\ &\le \sup_{\xi \in B_{H^2}(V_\infty, R)} \|DG(\xi)\|_{\mathscr{L}(H^2_\pi(Q_n), L^2(Q_n))}\|v(t) - V_\infty\|_{H^2_\pi(Q_n)} \\ &\le C\|v(t) - V_\infty\|_{H^2_\pi(Q_n)}. \end{aligned}$$

Since we assumed $v(t) \to V_\infty$ in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ as $t \to \infty$ at an exponential rate, this inequality shows $G(v(t)) \to G(V_\infty)$ in $L^2(Q_n)$ as $t \to \infty$ also at an exponential rate. Since $I - P : L^2(Q_n) \to L^2(Q_n)$ is a bounded operator, we then even obtain

$$\nabla p(t) = (I - P)G(v(t)) \xrightarrow{t \to \infty} (I - P)G(V_\infty)$$

at an exponential rate. On the other hand, we recall that every $(V_\infty, p_1) \in B_{\alpha,\beta} \times \mathbb{R}$ is a stationary solution of (5.1). Since $V_\infty \in L^2_\sigma(Q_n)$ we deduce $(I-P)G(V_\infty) = \alpha(I-P)V_\infty = 0$ and thus

$$\nabla p(t) \xrightarrow{t\to\infty} 0.$$

Finally, $p(t)$ converges in $\hat{H}^1_\pi(Q_n)$ to some constant $p_\infty \in \mathbb{R}$ at an exponential rate and the proof is complete. □

Next, we will show that for the stable regime the manifold of ordered polar states is normally stable. From Theorem 2.1.3 we recall that an equilibrium V is called normally stable in our setting if (i)-(iii) from the definition of normal hyperbolicity hold and

(iv) $\sigma(A_o)\backslash\{0\} \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Finally, in our last result regarding nonlinear stability, we are able to prove that if $\Gamma_0 \geq 0$ then every stationary solution $(V, p_0) \in B_{\alpha,\beta} \times \mathbb{R}$ is exponentially stable in the following sense by applying the principle of normal stability:

5.2.7 Theorem. *Let $\Gamma_2, \beta > 0, \Gamma_0 \geq 0, \alpha < 0$ and $\lambda_0 \in \mathbb{R}$. Let (V, p_0) with $V \in B_{\alpha,\beta}$ be a stationary state of (5.1). Then (V, p_0) is stable in the space $(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)) \times \hat{H}^1_\pi(Q_n)$ and there exists some $\delta > 0$ such that if (v, p) is a solution to (5.1) with initial data $v_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ and $\|v_0 - V\|_{H^2_\pi(Q_n)} < \delta$ then (v, p) converges to some $(V_\infty, p_\infty) \in B_{\alpha,\beta} \times \mathbb{R}$ exponentially in $(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)) \times \hat{H}^1_\pi(Q_n)$.*

Proof. As in the proof of Theorem 5.2.6 we first consider the projected system (5.10) and neglect the pressure. We proceed as in the proof of Theorem 5.2.6 and prove that every equilibrium $V \in B_{\alpha,\beta}$ is normally stable. Again by Lemma 5.2.4 the first condition (i) is fulfilled. In order to verify (ii) and (iii) we again apply Lemma 5.2.5: Here we need to show that assumption (5.11) holds. Therefore, let $u \in N(A_o)$ be arbitrary. Then $A_o u = 0$ and by testing with u we obtain

$$(\Gamma_2 \Delta^2 u, u)_{2,\pi} - (\Gamma_0 \Delta u, u)_{2,\pi} + (\lambda_0 (V \cdot \nabla) u, u)_{2,\pi} + 2\beta(PVV^T u, u)_{2,\pi} = 0.$$

Then exploiting integration by parts yields

$$\Gamma_2 \|\Delta u\|^2_{L^2(Q_n)} + \Gamma_0 \|\nabla u\|^2_{L^2(Q_n)} + 2\beta \|V \cdot u\|^2_{L^2(Q_n)} = 0$$

by the fact that the λ_0 term is skew-symmetric. By assumption we have $\Gamma_2, \beta > 0$ and $\Gamma_0 \geq 0$ such that we infer $\|\Delta u\|_{L^2(Q_n)} = \|V \cdot u\|_{L^2(Q_n)} = 0$ which yields on one hand that u is constant by the fact that

$$\|\Delta u\|^2_{L^2(Q_n)} = \sum_{\ell \in \mathbb{Z}^n} |\ell|^2 |\hat{u}(\ell)|^2 = 0,$$

hence $\hat{u}(\ell) = 0$ for all $\ell \neq 0$. On the other hand $\|V \cdot u\|_{L^2(Q_n)} = 0$ yields that u is perpendicular to V. Altogether we just verified (5.11). By Lemma 5.2.4 and Lemma 5.2.5 assumptions (ii) and (iii) are now fulfilled.

The fact that $\sigma(A_o) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\ \lambda \geq 0\}$ follows from Proposition 5.2.2(i) since the manifold of ordered polar states is linearly stable. In (5.9) we observed that $\lambda = 0$ is the only possible eigenvalue with Re $\lambda = 0$ since for $\ell \neq 0$ the symbol Re $\sigma_{A_o}(\ell)$ is always positive definite. Hence, (iv) follows and V is normally stable. By Theorem 2.1.3 the assertion follows for V.

Convergence for the pressure p can be obtained completely analogously to the proof of Theorem 5.2.6. □

Chapter 6

Global Attractor for an Active Fluid Continuum Model

In this chapter we consider the active fluid continuum model from the last Chapter 5. In contrast to the result from the last chapter, we prove the existence of a global attractor in two and three dimensions. At last we prove some properties of the global attractor. For the reader's convenience we recall the model. In the following we consider:

$$\begin{aligned} v_t + \Gamma_2\Delta^2 v - \Gamma_0\Delta v + (\alpha+\beta|v|^2)v & & & \\ + \lambda_0(v\cdot\nabla)v - \lambda_1\nabla|v|^2 + \nabla p &= 0 && \text{in } (0,T)\times Q_n, \\ \operatorname{div} v &= 0 && \text{in } (0,T)\times Q_n, \\ v|_{t=0} &= v_0 && \text{in } Q_n, \end{aligned} \tag{6.1}$$

again with subject to periodic boundary conditions on $L^2(Q_n)$ with $Q_n = [0,L]^n$ for $n = 2,3$ where the length $L > 0$ is arbitrary chosen but fixed. In this chapter we assume $\Gamma_2, \beta > 0$ and $\Gamma_0, \alpha, \lambda_0, \lambda_1 \in \mathbb{R}$. In contrast to the last chapter we will not distinguish between the cases $\Gamma_0 \geq 0$ and $\Gamma_0 < 0$.

We will proceed as follows: Since we are working in a different setting as in the last chapter, we first have to make sure that (6.1) is globally wellposed in $L^2_\sigma(Q_n)$. Then we will prove the existence of absorbing sets of arbitrary regularity in order to prove existence of a global attractor which turns out to have finite Hausdorff and fractal dimension. At last we can deduce that the model (6.1) even has an inertial manifold in two dimensions which attracts all solutions at an exponential rate.

6.1 Global Wellposedness in $L^2_\sigma(Q_n)$

In order to investigate the existence of a global attractor as well as its finite dimension we need to prove the existence of a semigroup solving (6.1) for $u_0 \in L^2_\sigma(Q_n)$ in this case. To this end, we first neglect the pressure p and consider the projected system. Then the system where we applied the Helmholtz-Weyl projection from Section 2.2 to (6.1) reads as

$$\begin{aligned} v_t + \Gamma_2 \Delta^2 v - \Gamma_0 \Delta v + (\alpha + P\beta |v|^2) v + \lambda_0 P(v \cdot \nabla) v = 0 \quad &\text{in } (0,T) \times Q_n, \\ v|_{t=0} = v_0 \quad &\text{in } Q_n. \end{aligned} \tag{6.2}$$

In this section we aim to prove the existence of a semigroup

$$S(t) : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \qquad v_0 \mapsto S(t)v_0,$$

where $S(t)v_0$ solves (6.2). Recovering the pressure p by applying $(I-P)$ to (6.1) we also obtain the existence of a pair $(v, \nabla p) \in L^2_\sigma(Q_n) \times L^2(Q_n)$ which solves (6.1).

Note that in contrast to (global) wellposedness in Section 5.1 we need the existence of a semigroup $(S(t))_{t\geq 0}$ for initial values $v_0 \in L^2_\sigma(Q_n)$ and not in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ as in Section 5.1. Hence, applying the theory of interpolation-extrapolation scales we will be able to transfer results from Section 5.1 to the desired setting.
To this end, we set $E_0 := L^2_\sigma(Q_n)$ and

$$A : D(A) \subseteq L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \qquad Au := \Gamma_2 \Delta^2 u,$$

where $D(A) := H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$. By Section 5.1 we know that $\omega + A$ admits a bounded H^∞-calculus for $\omega > 0$ large. In order to define the interpolation-extrapolation scale we choose $\lambda > \omega$ such that $0 \in \rho(\lambda + A)$ and define $\mathbb{A} := \lambda + A$. By [2, Theorem V.1.5.4] we deduce that $[(E_\alpha, \mathbb{A}_\alpha) : \alpha \in [-1, \infty)]$ is a densely injected interpolation-extrapolation scale generated by $(E_0, \mathbb{A})$, where

$$E_\alpha \doteq \begin{cases} \big(D(\mathbb{A}^\alpha), \|\mathbb{A}^\alpha \cdot\|_{L^2(Q_n)}\big) & 0 \leq \alpha < \infty, \\ \big(E_0, \|\mathbb{A}^\alpha \cdot\|_{L^2(Q_n)}\big)^\sim & -1 \leq \alpha < 0, \end{cases}$$

and $(E_0, \|\mathbb{A}^\alpha \cdot\|_{L^2(Q_n)})^\sim$ denotes the completion of E_0 w.r.t. the norm $\|\mathbb{A}^\alpha \cdot\|_{L^2(Q_n)}$. For the E_α-realization of $\mathbb{A}$ we also obtain $\mathbb{A}_\alpha \in \mathscr{L}(E_{\alpha+1}, E_\alpha)$. By [2, Theorem V.1.5.15] we know that the scale $[(E_\alpha, \mathbb{A}_\alpha) : \alpha \in [-1, \infty)]$ consists of Hilbert spaces equipped with the canonical inner product since $\mathbb{A}$ is self-adjoint and positive. Corresponding dual spaces w.r.t. the duality pairing $(\cdot, \cdot)_{2,\pi}$ are characterized as

$$(E_\alpha)' = E_{-\alpha}, \qquad (\mathbb{A}_\alpha)' = \mathbb{A}_{-\alpha} \qquad (-1 \leq \alpha \leq 1),$$

by [2, Theorem V.1.4.12]. At last we note that we have

$$(E_\alpha, E_\beta)_{\theta,2} \doteq [E_\alpha, E_\beta]_\theta \doteq E_{(1-\theta)\alpha + \theta\beta} \qquad (-1 \leq \alpha < \beta < \infty,\ 0 < \theta < 1)$$

by [2, Theorem V.1.5.4] and [51, Chapter 1.18.10, Remark 3]. This also shows why extrapolation of the setting in Section 5.1 leads us to wellposedness with initial values in $L^2_\sigma(Q_n)$: For $\alpha = -1/2$ and $\beta = 1/2$ we then obtain $(E_{-1/2}, E_{1/2})_{1/2,2} = E_0 = L^2_\sigma(Q_n)$, hence we solve (6.2) in $E_{-1/2}$.
In the following we give a short overview of the steps we have to follow in order to prove global wellposedness of (6.2) with initial values $v_0 \in L^2_\sigma(Q_n)$.

Step 1 (H^∞-calculus and maximal L^p-regularity). Let $A_{-1/2}$ denote the realization of the operator A in $E_{-1/2}$, i.e., $A_{-1/2} : D(A_{-1/2}) \subseteq E_{-1/2} \to E_{-1/2}$ and let $B_{-1/2} : D(B_{-1/2}) = E_0 \subseteq E_{-1/2} \to E_{-1/2}$ be the $E_{-1/2}$-realization of the perturbation

$$B : D(B) \subseteq L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \qquad Bu := -\Gamma_0 \Delta u + \alpha,$$

where $D(B) = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. By [23, Theorem 6.5] we deduce that $\mathbb{A}_{-1/2} = A_{-1/2} + \lambda$ admits a bounded H^∞-calculus on $E_{-1/2}$. Applying a standard perturbation result for the H^∞-calculus (e.g. [27, Proposition 13.1]), it is straightforward to conclude that $\mu + \mathbb{A}_{-1/2} + B_{-1/2}$ also admits an H^∞-calculus for $\mu > 0$ large. As a consequence we observe that $A_{-1/2} + B_{-1/2}$ enjoys maximal L^p-regularity on intervals $(0, T)$ with $T < \infty$:

6.1.1 Proposition. *Let $T \in (0, \infty)$. For data $f \in L^2((0,T), E_{-1/2})$ and initial value $v_0 \in L^2_\sigma(Q_n) = (E_{-1/2}, E_{1/2})_{1/2,2}$ there exists a unique solution (v, p) of the linearization of (6.1), i.e.,*

$$\begin{aligned} v_t + \Gamma_2 \Delta^2 v - \Gamma_0 \Delta v + \alpha v + \nabla p &= f \quad \text{in } (0,T) \times Q_n, \\ \operatorname{div} v &= 0 \quad \text{in } (0,T) \times Q_n, \\ v|_{t=0} &= v_0 \quad \text{in } Q_n, \end{aligned}$$

such that the following estimate holds:

$$\begin{aligned} &\|v\|_{H^1((0,T),E_{-1/2})} + \|v\|_{L^2((0,T),E_{1/2})} + \|\nabla p\|_{L^2((0,T),E_{-1/2})} \\ &\leq C(T) \left(\|f\|_{L^2((0,T),E_{-1/2})} + \|v_0\|_{L^2(Q_n)} \right). \end{aligned}$$

Thus, wellposedness of the linearization of (6.1) is ensured and by applying standard techniques (fixed point argument) we can obtain local wellposedness of the full nonlinear problem (6.1).

Step 2 (local wellposedness). At first we define relevant function spaces as

$$\begin{aligned} \mathbb{E}_T &:= H^1((0,T), E_{-1/2}) \cap L^2((0,T), E_{1/2}), \\ \mathbb{F}^1_T &:= L^2((0,T), E_{-1/2}), \\ \mathbb{F}^2 &:= (E_{-1/2}, E_{1/2})_{1/2,2} = E_0 = L^2_\sigma(Q_n), \end{aligned}$$

$$\mathbb{F}_T := \mathbb{F}_T^1 \times \mathbb{F}^2.$$

Making use of the isomorphism which is stated within maximal L^p-regularity $L : \mathbb{E}_T \to \mathbb{F}_T$ with $Lu = (\dot{u} + A_{-1/2} + B_{-1/2}, u|_{t=0})$ it is possible to rephrase (6.2) as

$$F : \mathbb{E}_T \to \mathbb{F}_T, \qquad F(u) := Lu + (H(u), 0),$$

with $H : \mathbb{E}_T \to \mathbb{F}_T^1$ defined as $H(u) := \beta P_{-1/2}|u|^2 u + \lambda_0 P_{-1/2}(u \cdot \nabla)u$. Here, $P_{-1/2}$ is the consistent extension of the Helmholtz-Weyl projection P on $L^2(Q_n)$ from Section 2.2 to $\tilde{E}_{-1/2}$. Here, the interpolation-extrapolation scale $[(\tilde{E}_\alpha, \mathbb{A}_\alpha) : \alpha \in [-1, \infty)]$ is generated by the $(\tilde{E}_0, \mathbb{A}) = (L^2(Q_n), \mathbb{A})$. Note that E_α denotes the scale generated by the projected spaces $L^2_\sigma(Q_n)$ and $\tilde{E}_\alpha$ is generated by $L^2(Q_n)$. It is straightforward to prove $H \in C^1(\mathbb{E}_T, \mathbb{F}_T^1)$: Making use of the embedding

$$\mathbb{E}_T \hookrightarrow L^\infty((0,T), L^2_\sigma(Q_n))$$

by [2, Theorem III.4.10.2], $\operatorname{div} \in \mathscr{L}((\tilde{E}_{-1/4})^{n\times n}, (\tilde{E}_{-1/2})^n)$ and the fact that $\tilde{E}_{1/2} = H^2_\pi(Q_n) \hookrightarrow L^\infty(Q_n)$ and $L^2(Q_n) = \tilde{E}_0 \hookrightarrow \tilde{E}_{-1/4}$ we can estimate the latter term of H as:

$$\begin{aligned}\|P_{-1/2}(u \cdot \nabla)u\|_{\mathbb{F}_T^1} &\le C\| \operatorname{div}(u \otimes u)\|_{L^2((0,T),\tilde{E}_{-1/2})} \le C\||u|^2\|_{L^2((0,T),\tilde{E}_{-1/4})} \\ &\le C\|u\|_{L^\infty((0,T),L^2_\sigma(Q_n))}\|u\|_{L^2((0,T),L^\infty(Q_n))} \le C\|u\|^2_{\mathbb{E}_T}\end{aligned}$$

and the first term as

$$\begin{aligned}\|P_{-1/2}|u|^2 u\|_{\mathbb{F}_T^1} &\le C\||u|^2 u\|_{L^2((0,T),\tilde{E}_{-1/2})} \le C\|u\|^2_{L^\infty((0,T),L^2_\sigma(Q_n))}\|u\|_{L^2((0,T),L^\infty(Q_n))} \\ &\le C\|u\|^3_{\mathbb{E}_T}\end{aligned}$$

by additionally making use of the embedding $L^1(Q_n) \hookrightarrow \tilde{E}_{-1/2}$ which holds thanks to the estimate

$$\left|\langle \varphi, u\rangle_{\tilde{E}_{-1/2},\tilde{E}_{1/2}}\right| = \left|\frac{1}{L^n}\int_{Q_n} \varphi(x)u(x)\,dx\right| \le C\|\varphi\|_{L^1(Q_n)}\|u\|_{\tilde{E}_{1/2}}$$

for $\varphi \in L^1(Q_n)$ and $u \in \tilde{E}_{1/2}$. We also observe that $L + (DH(v), 0) \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T)$ for arbitrary $v \in \mathbb{E}_T$: This can be proved by using the same arguments as in [57, Lemma 3]:

$$\begin{aligned}\|(u \cdot \nabla)v(t)\|_{\tilde{E}_{-1/2}} &\le C\|u \otimes v(t)\|_{\tilde{E}_{-1/4}} \le C\|u\|_{L^\infty(Q_n)}\|v(t)\|_{L^2(Q_n)} \\ &\le C\|u\|_{E_{\alpha/4}}\|v\|_{\mathbb{E}_T},\end{aligned}$$

and

$$
\begin{aligned}
\| |v(t)|^2 u \|_{\tilde{E}_{-1/2}} &\leq C \| |v(t)|^2 u \|_{L^1(Q_n)} \leq C \|u\|_{L^\infty(Q_n)} \|v(t)\|^2_{L^2(Q_n)} \\
&\leq C \|u\|_{E^{\alpha/4}} \|v\|^2_{\mathbb{E}_T}
\end{aligned}
$$

and $E_{\alpha/4} = D(\mathbb{A}^{\alpha/4}) = H^\alpha_\pi(Q_n) \cap L^2_\sigma(Q_n) \hookrightarrow L^\infty(Q_n) \cap L^2_\sigma(Q_n)$ for $\alpha > 3/2$ since $\alpha - n/2 > 0$. At last local wellposedness can be proved by making use of maximal L^p-regularity and applying the local inverse theorem as seen in [57, Theorem 1].

6.1.2 Theorem. *Let* $\Gamma_2, \beta > 0$, $\Gamma_0, \alpha, \lambda_0, \lambda_1 \in \mathbb{R}$. *For every initial value* $v_0 \in L^2_\sigma(Q_n)$ *and data* $f \in L^2((0,T), E_{-1/2})$ *there exists* $0 < T_* < T$ *and a unique solution* (v,p) *of (6.1) such that*

$$
\begin{aligned}
v &\in H^1((0,T_*), E_{-1/2}) \cap L^2((0,T_*), E_{1/2}), \\
\nabla p &\in L^2((0,T_*), E_{-1/2}).
\end{aligned}
$$

Having proved local wellposed leads to the question whether global wellposedness can also be obtained.

Step 3 (global wellposedness). Global wellposedness can be obtained by using energy estimates as in Theorem 5.1.3 and [57, Theorem 2], to be precise we can show

$$
\|v\|_{L^\infty((0,T), L^2_\sigma(Q_n))} \leq C(T) \|v_0\|_{L^2(Q_n)},
$$

which proves that (v,p) from Step 2 exists globally.

6.2 Existence of a Global Attractor

In this section we prove the existence of a global attractor of arbitrary high regularity. We proceed as in [39, Chapter 10] in order to prove the existence. For instance we will show that there exists some compact absorbing set such that we can apply Theorem 2.3.3 in order to obtain the result. In Section 2.3 we collected all relevant definitions and theorems regarding the global attractor theory.

In order to prove the existence of some compact absorbing set in $L^2_\sigma(Q_n)$, it is crucial to prove the existence of an absorbing set in $L^2_\sigma(Q_n)$ in general at first. Using a bootstrapping argument and the Rellich-Kondrachov compact embedding theorem we then obtain the compactness of the corresponding absorbing set.
Let $v_0 \in L^2_\sigma(Q_n)$ be some initial value such that $v(t) = S(t)v_0$ for $t \geq 0$ is the corresponding solution of (6.2). Testing the first equation of (6.2) with v w.r.t. the inner product in $L^2_\sigma(Q_n)$ yields

$$
\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(Q_n)} = -\Gamma_2 \|\Delta v(t)\|^2_{L^2(Q_n)} - \Gamma_0 \|\nabla v(t)\|^2_{L^2(Q_n)}
$$

$$- \alpha \|v(t)\|^2_{L^2(Q_n)} - \frac{\beta}{L^n} \|v(t)\|^4_{L^4(Q_n)},$$

where we note that the λ_0 term vanishes since it is skew-symmetric. Again by making use of the Fourier series representation for periodic functions (Theorem 2.2.1) we end up with

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(Q_n)} \\ &= - \sum_{\ell \in \mathbb{Z}^n} \left(\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \alpha \right) |\widehat{v(t)}(\ell)|^2 - \frac{\beta}{L^n}\|v(t)\|^4_{L^4(Q_n)}. \end{aligned}$$

Using the same arguments as in [8, Corollary 3.5] we observe that there exists a finite set $U \subseteq \mathbb{Z}^n$ such that

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \alpha \leq 0 \quad \Leftrightarrow \quad \ell \in U, \tag{6.3}$$

where U depends on the relation between the occurring parameters $\Gamma_2, \Gamma_0, \alpha$. This is justified by the fact that the paraboloid given in (6.3) can be fully analyzed and describes a paraboloid which is open to the top. Hence, we can also find some $\gamma_1 > 0$ such that

$$-\gamma_1 \leq \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \alpha \leq 0 \qquad (\ell \in U)$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(Q_n)} \\ &\leq - \sum_{\ell \in U} \left(\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \alpha \right) |\widehat{v(t)}(\ell)|^2 - \frac{\beta}{L^n}\|v(t)\|^4_{L^4(Q_n)} \\ &\leq \gamma_1 \|v(t)\|^2_{L^2(Q_n)} - \frac{\beta}{L^n}\|v(t)\|^4_{L^4(Q_n)} \\ &\leq \gamma_1 \|v(t)\|^2_{L^2(Q_n)} - \gamma_2 \|v(t)\|^4_{L^2(Q_n)} \end{aligned}$$

for $\gamma_1, \gamma_2 > 0$, where we applied the Sobolev embedding $L^4(Q_n) \cap L^2_\sigma(Q_n) \hookrightarrow L^2_\sigma(Q_n)$ in the last step such that $\gamma_2 = \beta$. In order to obtain an L^2 bound for $v(t)$ uniformly in $t < \infty$ we need to examine the differential inequality

$$\frac{d}{dt}\varphi(t) \leq 2\gamma_1 \varphi(t) - 2\gamma_2 \varphi(t)^2 \qquad (t > 0). \tag{6.4}$$

By regarding the differential inequality (6.4) as a differential equation, some elementary calculations yield

6.2.1 Remark. Let $t^* \geq 0$ be arbitrary. Then the ordinary differential equation

$$\frac{d}{dt}\psi(t) = 2\gamma_1\psi(t) - 2\gamma_2\psi(t)^2 \quad (t > t^*), \qquad \psi(t^*) = \|v(t^*)\|^2_{L^2(Q_n)}$$

has a maximal unique solution

$$\psi(t) = \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2\psi(t^*)}{\psi(t^*)}\exp(-2\gamma_1(t - t^*)) + \gamma_2} \qquad (t \in (\tilde{t}, \infty)),$$

for $\psi(t^*) \in \mathbb{R}_{\geq 0} \setminus \left\{0, \frac{\gamma_1}{\gamma_2}\right\}$ and

$$\tilde{t} := \begin{cases} -\infty, & \text{if } \frac{\gamma_1}{\gamma_2} > \psi(t^*), \\ -(2\gamma_1)^{-1}\log(\gamma_2\psi(t^*)/(\gamma_2\psi(t^*) - \gamma_1)) + t^*, & \text{if } \frac{\gamma_1}{\gamma_2} < \psi(t^*), \end{cases}$$

and $\psi(t) = \psi(t^*)$ for $\psi(t^*) \in \{0, \frac{\gamma_1}{\gamma_2}\}$ and $t \in \mathbb{R}$. If $\frac{\gamma_1}{\gamma_2} > \psi(t^*)$ then $\psi(t) \nearrow \frac{\gamma_1}{\gamma_2}$ as $t \to \infty$ and if $\frac{\gamma_1}{\gamma_2} < \psi(t^*)$ then $\psi(t) \searrow \frac{\gamma_1}{\gamma_2}$ as $t \to \infty$. Furthermore, we observe that

$$\mathcal{A} := \left\{\frac{\gamma_1}{\gamma_2}, 0\right\}$$

are stationary solutions and that $\mathcal{A}$ attracts all solutions ψ.

We aim to apply a comparison theorem in order to obtain a (uniform) bound for $\|v(t)\|_{L^2(Q_n)}$ for $t \geq t_0$ starting from a certain $t_0 > 0$. Since we set $\varphi(t) = \|v(t)\|^2_{L^2(Q_n)}$ it suffices to find any bound for φ. Hence, we can finally prove

6.2.2 Lemma. *Let v denote the solution of (6.2). Then there exists some $t_0 > 0$ and $R_0 > 0$ both independent of the initial value $v_0 \in L^2_\sigma(Q_n)$ such that*

$$\|v(t)\|^2_{L^2(Q_n)} \leq R_0 \qquad (t \geq t_0).$$

Proof. Using the differential inequality (6.4) we want to find a bound for φ and hence for $\|v(t)\|_{L^2(Q_n)}$. At first we aim to apply the comparison theorem from [48, Theorem 1.3] to $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, f(z) := 2\gamma_1 z - 2\gamma_2 z^2$ and $\psi : [0, \infty) \to \mathbb{R}$, the solution from Remark 6.2.1 and φ, the function from the differential inequality (6.4). Then we have

$$\varphi(0) = \|v_0\|^2_{L^2(Q_n)} = \psi(0), \qquad \varphi'(t) - f(\varphi(t)) \leq 0 = \psi'(t) - f(\psi(t)) \tag{6.5}$$

for $t > 0$. Note that f is locally Lipschitz w.r.t. z and that φ and ψ are weakly differentiable in $(0, \infty)$. Then [48, Theorem 1.3] states

$$\varphi(t) \leq \psi(t) \qquad (t \geq 0).$$

Note that in fact we first just obtain the estimate for $t > 0$. However, by the continuity of ψ and φ we indeed obtain the estimate for $t \geq 0$.
Hence, we are now able to obtain bounds for $\|v(t)\|_{L^2(Q_n)}$ uniformly in the initial data $v_0 \in L^2_\sigma(Q_n)$. If $\|v_0\|^2_{L^2(Q_n)} = 0$ then $\psi \equiv 0$, hence $\|v(t)\|^2_{L^2(Q_n)} = \varphi(t) = 0$ for all $t \geq 0$. If $\|v_0\|^2_{L^2(Q_n)} = \frac{\gamma_1}{\gamma_2}$ then $\psi \equiv \frac{\gamma_1}{\gamma_2}$ and $\|v(t)\|^2_{L^2(Q_n)} = \varphi(t) \leq \frac{\gamma_1}{\gamma_2}$ for all $t \geq 0$. Let $\|v_0\|^2_{L^2(Q_n)} < \frac{\gamma_1}{\gamma_2}$. Then by Remark 6.2.1 we observe that $\psi(t) \nearrow \frac{\gamma_1}{\gamma_2}$ as $t \to \infty$, hence ψ is monotonically increasing such that $\psi(t) \leq \frac{\gamma_1}{\gamma_2}$ for $t \geq 0$. Hence, also in this case we infer $\|v(t)\|^2_{L^2(Q_n)} = \varphi(t) \leq \frac{\gamma_1}{\gamma_2}$ for all $t \geq 0$.
At last we consider the case $\|v_0\|^2_{L^2(Q_n)} > \frac{\gamma_1}{\gamma_2}$. Then integrating the differential inequality (6.4) yields

$$\varphi(t) \leq \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2\varphi(0)}{\varphi(0)} \exp(-\gamma_1 2t) + \gamma_2} \qquad (t \geq 0).$$

Note that we can estimate the denominator on the right-hand side as

$$\begin{aligned}\frac{\gamma_1 - \gamma_2\varphi(0)}{\varphi(0)} \exp(-\gamma_1 2t) + \gamma_2 = \gamma_2 - \frac{\gamma_2\varphi(0) - \gamma_1}{\varphi(0)} \exp(-\gamma_1 2t) \\ = \gamma_2(1 - \exp(-\gamma_1 2t)) + \frac{\gamma_1}{\varphi(0)} \exp(-\gamma_1 2t) \\ \geq \gamma_2(1 - \exp(-\gamma_1 2t))\end{aligned}$$

for all $t \geq 0$. Furthermore, we observe that for every $\varepsilon > 0$ there exists some $t_0 > 0$ such that

$$\frac{\gamma_1}{\gamma_2(1 - \exp(-\gamma_1 2t))} - \frac{\gamma_1}{\gamma_2} = \left| \frac{\gamma_1}{\gamma_2(1 - \exp(-\gamma_1 2t))} - \frac{\gamma_1}{\gamma_2} \right| < \varepsilon \qquad (t \geq t_0)$$

due to the convergence. Hence for fixed $\varepsilon > 0$ and corresponding $t_0 = t_0(\varepsilon) > 0$ we finally end up with

$$\begin{aligned}\varphi(t) &\leq \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2\varphi(0)}{\varphi(0)} \exp(-\gamma_1 2t) + \gamma_2} \leq \frac{\gamma_1}{\gamma_2(1 - \exp(-\gamma_1 2t))} \\ &= \frac{\gamma_1}{\gamma_2(1 - \exp(-\gamma_1 2t))} - \frac{\gamma_1}{\gamma_2} + \frac{\gamma_1}{\gamma_2} < \varepsilon + \frac{\gamma_1}{\gamma_2}\end{aligned}$$

for $t \geq t_0$. Summing up we finally obtain

$$\|v(t)\|^2_{L^2(Q_n)} = \varphi(t) \leq \varepsilon + \frac{\gamma_1}{\gamma_2} := R_0 \qquad (t \geq t_0)$$

independent of $v_0 \in L^2_\sigma(Q_n)$. □

6.2.3 Remark. We can apply [48, Theorem 1.3] in the setting of Lemma 6.2.2 by weakening the assumptions from [48, Theorem 1.3]. In fact in the original formulation differentiability of φ and ψ in the classical sense and (6.5) for $t \geq 0$ is required. Actually, the proof of [48, Theorem 1.3] shows that we can assume weak differentiability of φ and ψ and that (6.5) does not have to hold for $t = 0$ in order to obtain the result.

6.2.4 Corollary. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then the set $\mathcal{B}_0 := B_{L^2}\left(0, \sqrt{R_0}\right)$ is a bounded absorbing set in $L^2_\sigma(Q_n)$.*

Proof. This is an immediate consequence of Lemma 6.2.2. □

Next, we will prove the existence of an absorbing set in $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ in order to apply the Rellich-Kondrachov compact embedding theorem [39, Theorem A.4, Corollary A.5] to obtain a compact absorbing set in $L^2_\sigma(Q_n)$.

6.2.5 Lemma. *Let v denote the solution of (6.2). Then there exists some $t_1 > 0$ and $R_1 > 0$ both independent of the initial value $v_0 \in L^2_\sigma(Q_n)$ such that*

$$\|v(t)\|^2_{H^1_\pi(Q_n)} \leq R_1 \qquad (t \geq t_1).$$

Proof. By Lemma 6.2.2 we already have an L^2 bound for any solution v of (6.2). Hence we only need to find some estimate for ∇v in $L^2(Q_n)$. To this end, we apply energy methods. We test (6.2) with $-\Delta v$ w.r.t. to the inner product in $L^2(Q_n)$. Then we infer

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2_{L^2(Q_n)} + \alpha\|\nabla v(t)\|^2_{L^2(Q_n)} + \Gamma_0\|\Delta v(t)\|^2_{L^2(Q_n)} + \Gamma_2\|\Delta\nabla v(t)\|^2_{L^2(Q_n)}\\ &+ \frac{\beta}{L^n}\int_{Q_n} \nabla\left(|v(t)|^2 v(t)\right)\nabla v(t)\,dx = -\lambda_0((v(t)\cdot\nabla)v(t), \Delta v(t))_{2,\pi}.\end{aligned}$$

As in [57, Theorem 3.2] we observe that the β term is positive and hence can be dropped. Furthermore, in order to estimate the Γ_0 term we apply Corollary 2.2.2 for $k = 2$ and Young's inequality to absorb the $\Delta\nabla v$ term with the Γ_2 term. We then infer

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2_{L^2(Q_n)} + \frac{\Gamma_2}{2}\|\Delta\nabla v(t)\|^2_{L^2(Q_n)}\\ &\qquad\qquad \leq |\lambda_0||((v(t)\cdot\nabla)v(t), \Delta v(t))_{2,\pi}| + C_1\|\nabla v(t)\|^2_{L^2(Q_n)}\end{aligned}$$

with $C_1 > 0$. We can estimate the λ_0 term as

$$\begin{aligned}((v(t)\cdot\nabla)v(t), \Delta v(t))_{2,\pi} &= (\operatorname{div}(v(t)\otimes v(t)), \Delta v(t))_{2,\pi}\\ &= -(v(t)\otimes v(t), \Delta\nabla v(t))_{2,\pi}\\ &\leq \|v(t)\otimes v(t)\|_{L^2(Q_n)}\|\Delta\nabla v(t)\|_{L^2(Q_n)}\end{aligned}$$

$$\leq C(\varepsilon)\|v(t)\|_{L^4(Q_n)}^4 + \varepsilon\|\Delta\nabla v(t)\|_{L^2(Q_n)}^2$$

such that we end up with

$$\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_{L^2(Q_n)}^2 + \frac{\Gamma_2}{4}\|\Delta\nabla v(t)\|_{L^2(Q_n)}^2 \leq C_1\|\nabla v(t)\|_{L^2(Q_n)}^2 + C_2\|v(t)\|_{L^4(Q_n)}^4$$

for some new constants $C_1, C_2 > 0$ independent of v and t. Hence, we also obtain the estimate

$$\frac{d}{dt}\|\nabla v(t)\|_{L^2(Q_n)}^2 \leq C_1\|\nabla v(t)\|_{L^2(Q_n)}^2 + C_2\|v(t)\|_{L^4(Q_n)}^4. \tag{6.6}$$

Next, we have to ensure the integrability of the terms in (6.6) in order to apply the generalized Gronwall lemma from [47, Lemma III.1.1]. To this end, let $r \geq 0$ and $t \geq t_0$ where $t_0 > 0$ is the same t_0 from Lemma 6.2.2. Testing (6.2) with v and integrating in Q_n w.r.t. the space variable and from t to $t+r$ w.r.t. the time variable leads to

$$\begin{aligned}\|v(t+r)\|_{L^2(Q_n)}^2 &+ \Gamma_2\int_t^{t+r}\|\Delta v(s)\|_{L^2(Q_n)}^2\,ds + \Gamma_0\int_t^{t+r}\|\nabla v(s)\|_{L^2(Q_n)}^2\,ds\\ &+ \alpha\int_t^{t+r}\|v(s)\|_{L^2(Q_n)}^2\,ds + \frac{\beta}{L^n}\int_t^{t+r}\|v(s)\|_{L^4(Q_n)}^4\,ds\\ &= \|v(t)\|_{L^2(Q_n)}^2.\end{aligned}$$

Again, we apply Corollary 2.2.2 and Young's inequality in order to absorb the Γ_0 term with the Γ_2 and α term. Hence, we obtain

$$\begin{aligned}\|v(t+r)\|_{L^2(Q_n)}^2 &+ \frac{\Gamma_2}{2}\int_t^{t+r}\|\Delta v(s)\|_{L^2(Q_n)}^2\,ds + \frac{\beta}{L^n}\int_t^{t+r}\|v(s)\|_{L^4(Q_n)}^4\,ds\\ &\leq \|v(t)\|_{L^2(Q_n)}^2 + C\int_t^{t+r}\|v(s)\|_{L^2(Q_n)}^2\,ds,\end{aligned}$$

with some constant $C > 0$. Since $t \geq t_0$ from Lemma 6.2.2 we obtain

$$\|v(t+r)\|_{L^2(Q_n)}^2 + \frac{\Gamma_2}{2}\int_t^{t+r}\|\Delta v(s)\|_{L^2(Q_n)}^2\,ds + \frac{\beta}{L^n}\int_t^{t+r}\|v(s)\|_{L^4(Q_n)}^4\,ds \leq C(r)$$

for $t \geq t_0$ where $C(r) > 0$ is dependent of $r \geq 0$. Then we obtain

$$\int_t^{t+r}\|v(s)\|_{L^4(Q_n)}^4\,ds \leq C(r) \qquad (t \geq t_0)$$

and

$$\int_t^{t+r}\|\nabla v(s)\|_{L^2(Q_n)}^2\,ds \leq \int_t^{t+r}\|\Delta v(s)\|_{L^2(Q_n)}^2\,ds + \int_t^{t+r}\|v(s)\|_{L^2(Q_n)}^2\,ds \leq C(r)$$

for $t \geq t_0$ where we again applied Corollary 2.2.2 for $k = 1$. Hence we can finally apply [47, Lemma III.1.1] to (6.6) in order to obtain

$$\|\nabla v(t)\|_{L^2(Q_n)}^2 \leq C \qquad (t \geq t_1),$$

where we set $t_1 := t_0 + r$ for some fixed $r > 0$ now. Combined with Lemma 6.2.2 this finally yields

$$\|v(t)\|_{H^1_\pi(Q_n)}^2 = \|v(t)\|_{L^2(Q_n)}^2 + \|\nabla v(t)\|_{L^2(Q_n)}^2 \leq R_1 \qquad (t \geq t_1)$$

for some $R_1 \geq R_0 > 0$. □

Hence, as a direct consequence we obtain:

6.2.6 Corollary. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then the set $\mathcal{B}_1 := B_{H^1}(0, \sqrt{R_1})$ is a bounded absorbing set in $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$.*

Then we can finally prove the existence of a global attractor in $L^2_\sigma(Q_n)$:

6.2.7 Proposition. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then there exists a global attractor $\mathcal{A}_0 \subseteq H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ such that*

$$S(t)\mathcal{A}_0 = \mathcal{A}_0 \qquad \text{for all} \quad t \geq 0.$$

Proof. We apply Theorem 2.3.3 to prove the assertion. We need to verify that $(S(t))_{t\geq 0}$ is dissipative, hence we have to ensure the existence of a compact absorbing set $B \subseteq L^2_\sigma(Q_n)$.
From Corollary 6.2.6 we infer that $\mathcal{B}_1$ is a bounded absorbing set in $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Hence by Rellich's compact embedding theorem ([39, Theorem A.4, Corollary A.5]) we obtain that $\mathcal{B}_1$ is relatively compact in $L^2_\sigma(Q_n)$. As a consequence $B := \overline{\mathcal{B}_1}^{L^2}$ is compact in $L^2_\sigma(Q_n)$ and Theorem 2.3.3 yields the existence of a global attractor $\mathcal{A}_0 \subseteq L^2_\sigma(Q_n)$.

We can even prove that the global attractor $\mathcal{A}_0$ has H^1-regularity. Note that $S(t)\mathcal{A}_0 = \mathcal{A}_0$ for all $t \geq 0$ by definition of a global attractor. This especially holds for $t = t_1$ where t_1 is the same as in Lemma 6.2.5. Hence we deduce

$$\mathcal{A}_0 = S(t_1)\mathcal{A}_0 \subseteq \mathcal{B}_1 \subseteq H^1_\pi(Q_n) \cap L^2_\sigma(Q_n),$$

where $\mathcal{B}_1$ is defined as in Corollary 6.2.6. □

Next, we prove that the global attractor in fact has arbitrary high regularity which again can be obtained by considering energy estimates:

6.2.8 Lemma. *Let $k \in \mathbb{N}$ and v denote the solution of (6.2). Then there exists some $t_k > 0$ and $R_k > 0$ both independent of the initial value $v_0 \in L^2_\sigma(Q_n)$ such that*

$$\|v(t)\|^2_{H^k_\pi(Q_n)} \leq R_k \qquad (t \geq t_k).$$

Proof. We will prove the assertion via an induction argument. For $k = 1$ the statement was proved in Corollary 6.2.6. For $k \in \mathbb{N}$ we assume the existence of an absorbing set in $H^{k-1}_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Then there exist $R_j > 0$ and $t_j > 0$ with

$$\|v(t)\|^2_{H^j_\pi(Q_n)} \leq R_j \qquad (t \geq t_j)$$

for $j = 0, ..., k-1$ by assumption. Note that we can assume $t_{k-1} > ... > t_0 > 0$. Now testing (6.2) with $(-1)^k \Delta^k v(t)$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^k v(t)\|^2_{L^2(Q_n)} + \Gamma_2\|\nabla^{k+2}v(t)\|^2_{L^2(Q_n)} + \Gamma_0\|\nabla^{k+1}v(t)\|^2_{L^2(Q_n)} \\
&+ \alpha\|\nabla^k v(t)\|^2_{L^2(Q_n)} + \lambda_0(\nabla^{k-2}(v(t)\cdot\nabla)v(t), \nabla^{k+2}v(t))_{2,\pi} \\
&+ \beta(\nabla^{k-2}|v(t)|^2 v(t), \nabla^{k+2}v(t))_{2,\pi} = 0.
\end{aligned}$$

Again, we aim to absorb the Γ_0 term with the Γ_2 and α terms by applying Corollary 2.2.2 and Young's inequality (as seen in the proof of Corollary 6.2.6). Furthermore, we obtain

$$\begin{aligned}
\|\nabla^{k-2}|v(t)|^2 v(t)\|_{L^2(Q_n)} &\leq \sum_{j,i,\ell=0}^{k-2} \|\nabla^j v(t)\|_{L^6(Q_n)}\|\nabla^i v(t)\|_{L^6(Q_n)}\|\nabla^\ell v(t)\|_{L^6(Q_n)} \\
&\leq C\sum_{j,i,\ell=0}^{k-2} \|\nabla^j v(t)\|_{H^1_\pi(Q_n)}\|\nabla^i v(t)\|_{H^1_\pi(Q_n)}\|\nabla^\ell v(t)\|_{H^1_\pi(Q_n)} \\
&\leq C\sum_{j,i,\ell=0}^{k-2} \|v(t)\|_{H^{j+1}_\pi(Q_n)}\|v(t)\|_{H^{i+1}_\pi(Q_n)}\|v(t)\|_{H^{\ell+1}_\pi(Q_n)} \\
&\leq C\|v(t)\|^3_{H^{k-1}_\pi(Q_n)},
\end{aligned}$$

where we used the Sobolev embedding $H^1_\pi(Q_n) \hookrightarrow L^6(Q_n)$ by [5, Corollary 1.2] since $n = 2, 3$. Hence we can estimate the β term as

$$\begin{aligned}
|(\nabla^{k-2}|v(t)|^2 v(t), \nabla^{k+2}v(t))_{2,\pi}| &\leq \varepsilon\|\nabla^{k+2}v(t)\|^2_{L^2(Q_n)} + C(\varepsilon)\|v(t)\|^6_{H^{k-1}_\pi(Q_n)} \\
&\leq \varepsilon\|\nabla^{k+2}v(t)\|^2_{L^2(Q_n)} + C(\varepsilon)R^3_{k-1}
\end{aligned}$$

for $\varepsilon > 0$ arbitrary small and $C(\varepsilon) > 0$ dependent on ε. Hence we can absorb the $\nabla^{k+2}v(t)$ term with the Γ_2 term. We also observe

$$\|\nabla^{k-2}(v(t)\cdot\nabla)v(t)\|_{L^2(Q_n)} \leq \sum_{j,i=0}^{k-2} \|\nabla^j v(t)\|_{L^4(Q_n)}\|\nabla^{i+1}v(t)\|_{L^4(Q_n)}$$

$$\begin{aligned}
&\leq C \sum_{j,i=0}^{k-2} \|\nabla^j v(t)\|_{H^1_\pi(Q_n)} \|\nabla^{i+1} v(t)\|_{H^1_\pi(Q_n)} \\
&\leq C \sum_{j,i=0}^{k-2} \|v(t)\|_{H^{j+1}_\pi(Q_n)} \|v(t)\|_{H^{i+2}_\pi(Q_n)} \\
&\leq C \|v(t)\|_{H^{k-1}_\pi(Q_n)} \|v(t)\|_{H^k_\pi(Q_n)} \\
&\leq C \|v(t)\|_{H^{k-1}_\pi(Q_n)} \left(\|v(t)\|^2_{L^2(Q_n)} + \|\nabla^k v(t)\|^2_{L^2(Q_n)} \right)^{1/2},
\end{aligned}$$

where we again made use of the Sobolev embedding $H^1_\pi(Q_n) \hookrightarrow L^4(Q_n)$ by [5, Corollary 1.2] and the fact that

$$\|v(t)\|_{H^k_\pi(Q_n)} = \left(\sum_{j=0}^{k} \|\nabla^j v(t)\|^2_{L^2(Q_n)} \right)^{1/2} \leq C \left(\|v(t)\|^2_{L^2(Q_n)} + \|\nabla^k v(t)\|^2_{L^2(Q_n)} \right)^{1/2}$$

by Corollary 2.2.2. Then the λ_0 term can be estimated as

$$\begin{aligned}
&|(\nabla^{k-2}(v(t)\cdot\nabla)v(t), \nabla^{k+2} v(t))_{2,\pi}| \\
&\leq \varepsilon \|\nabla^{k+2} v(t)\|^2_{L^2(Q_n)} \\
&\qquad + C(\varepsilon) \left(\|v(t)\|^2_{H^{k-1}_\pi(Q_n)} \|\nabla^k v(t)\|^2_{L^2(Q_n)} + \|v(t)\|^2_{H^{k-1}_\pi(Q_n)} \|v(t)\|^2_{L^2(Q_n)} \right) \\
&\leq \varepsilon \|\nabla^{k+2} v(t)\|^2_{L^2(Q_n)} + C(\varepsilon) R_{k-1} \|\nabla^k v(t)\|^2_{L^2(Q_n)} + C(\varepsilon) R_0 R_{k-1}.
\end{aligned}$$

Here the remaining terms $\nabla^{k+2} v(t)$ and $\nabla^k v(t)$ can be absorbed by the Γ_2 and α term. Note that all estimates hold for $t \geq t_{k-1}$. Summing up we arrive at

$$\frac{d}{dt} \|\nabla^k v(t)\|^2_{L^2(Q_n)} + \frac{\Gamma_2}{2} \|\nabla^{k+2} v(t)\|^2_{L^2(Q_n)} \leq C_1 \|\nabla^k v(t)\|^2_{L^2(Q_n)} + C_2 \qquad (t \geq t_{k-1})$$

for some constants $C_1, C_2 > 0$ which are independent of t and v. Note that we can do the same calculation for $k-2$ such that we obtain

$$\frac{d}{dt} \|\nabla^{k-2} v(t)\|^2_{L^2(Q_n)} + \frac{\Gamma_2}{2} \|\nabla^{k} v(t)\|^2_{L^2(Q_n)} \leq C_1 \|\nabla^{k-2} v(t)\|^2_{L^2(Q_n)} + C_2 \qquad (t \geq t_{k-1}),$$

which yields for $t \geq t_{k-1}$ and $r \geq 0$ arbitrary

$$\int_t^{t+r} \|\nabla^k v(s)\|^2_{L^2(Q_n)}\, ds \leq C_1 \int_t^{t+r} \|\nabla^{k-2} v(s)\|^2_{L^2(Q_n)}\, ds + C_2 \int_t^{t+r} ds \leq C(r),$$

where we made use of the fact that $\|\nabla^{k-2} v(t)\|^2_{L^2(Q_n)} \leq R_{k-2}$ for $t \geq t_{k-1}$ by the induction assumption. Again by applying the generalized Gronwall lemma (cf. [47, Lemma III.1.1] we end up with

$$\|\nabla^k v(t)\|^2_{L^2(Q_n)} \leq C \qquad (t \geq t_k),$$

for some $t_k \geq 0$ and therefore

$$\|v(t)\|^2_{H^k_\pi(Q_n)} \leq R_k \qquad (t \geq t_k)$$

for some $R_k > 0$ independent of v_0. Hence, the assertion is proved. □

6.2.9 Corollary. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then the set $\mathcal{B}_k := B_{H^k}(0, \sqrt{R_k})$ is a bounded absorbing set in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$.*

6.2.10 Corollary. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then the global attractor $\mathcal{A}_0$ from Proposition 6.2.7 has H^k-regularity for all $k \in \mathbb{N}$, i.e.,*

$$\mathcal{A}_0 \subseteq H^k_\pi(Q_n) \cap L^2_\sigma(Q_n).$$

Proof. Proposition 6.2.7 states that $S(t)\mathcal{A}_0 = \mathcal{A}_0$ for all $t \geq 0$. This especially holds for $t_k > 0$ for all $k \in \mathbb{N}$, hence

$$\mathcal{A}_0 = S(t_k)\mathcal{A}_0 \subseteq \mathcal{B}_k \subseteq H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$$

by Corollary 6.2.9. □

Next, we prove the existence of a global attractor $\mathcal{A}_4 \subseteq H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for the semigroup $(\tilde{S}(t))_{t\geq 0}$ in the context of Section 5.1 and Theorem 5.1.3. Note that there we proved that solutions even have H^4-regularity for initial data $v_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Then even $(H^4_\pi(Q_n) \cap L^2_\sigma(Q_n), \tilde{S}(t))_{t\geq 0}$ is a semidynamical system and we can expect higher regularity for the global attractor $\mathcal{A}_4$. At last we will prove that the attractor $\mathcal{A}_0$ from Proposition 6.2.7 and $\mathcal{A}_4$ actually coincide.

6.2.11 Proposition. *Let $(\tilde{S}(t))_{t\geq 0}$ be the semigroup from Section 5.1. Then there exists a global attractor $\mathcal{A}_4 \subseteq H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ such that*

$$\tilde{S}(t)\mathcal{A}_4 = \mathcal{A}_4 \qquad \textit{for all } t \geq 0.$$

Proof. We apply the same arguments as in Proposition 6.2.7. Then we only have to prove that $(\tilde{S}(t))_{t\geq 0}$ is dissipative by Theorem 2.3.3. Note that already by Corollary 6.2.9 we obtain a bounded absorbing set $\mathcal{B}_5 \subseteq H^5_\pi(Q_n) \cap L^2_\sigma(Q_n)$ where $\overline{\mathcal{B}_5}^{H^4}$ is compact in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ by Rellich's embedding theorem (cf. [39, Theorem A.4, Corollary A.5]). □

6.2.12 Lemma. *Let $(S(t))_{t\geq 0}$ and $(\tilde{S}(t))_{t\geq 0}$ be the semigroups from Section 6.1 and Section 5.1, respectively. Then the corresponding attractors $\mathcal{A}_0$ and $\mathcal{A}_4$ coincide.*

Proof. First we note that $\mathcal{A}_0 \subseteq H^5_\pi(Q_n) \cap L^2_\sigma(Q_n)$ is bounded and invariant, such that by Rellich's compact embedding theorem (cf. [39, Theorem A.4, Corollary A.5]) we infer that $\overline{\mathcal{A}_0}^{H^4}$ is compact and absorbing in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Furthermore, we observe that $\overline{\mathcal{A}_0}^{H^4} = \mathcal{A}_0$ since $\mathcal{A}_0$ is compact (and especially closed) in $L^2_\sigma(Q_n)$ which is equipped with a weaker topology than $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Hence,

$$\mathcal{A}_0 = \overline{\mathcal{A}_0}^{H^4} \subseteq \mathcal{A}_4,$$

since $\mathcal{A}_4$ is the maximal compact, invariant set in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$. On the other hand by making use of the representation of the global attractor from Theorem 2.3.3, we obtain

$$\mathcal{A}_4 = \bigcap_{t\geq 0} S(t)\overline{\mathcal{B}_5}^{H^2} \subseteq \bigcap_{t\geq 0} S(t)\overline{\mathcal{B}_5}^{L^2} = \mathcal{A}_0,$$

since $\overline{\mathcal{B}_5}^{L^2}$ is obviously also a compact, absorbing set in $L^2_\sigma(Q_n)$ and since the global attractor $\mathcal{A}_0$ is unique. Thus, the assertion is proved. □

6.3 Injectivity and Finite Dimension

In this section we try to characterize the global attractor $\mathcal{A}_0$ from Proposition 6.2.7 more precisely. To this end, we will first prove injectivity of the semigroup $(S(t))_{t\geq 0}$ from Section 6.1 on $\mathcal{A}_0$ which will yield some properties of the global attractor, i.e., $(\mathcal{A}_0, S(t))_{t\in\mathbb{R}}$ is a dynamical system (we put emphasize on the fact that the semigroup then exists for all $t \in \mathbb{R}$) and that the global attractor just consists of complete and bounded orbits. At last we will prove that the global attractor $\mathcal{A}_0$ has finite (fractal and Hausdorff) dimension m.

6.3.1 Lemma. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then $(S(t))_{t\geq 0}$ is injective on its global attractor $\mathcal{A}_0 \subseteq L^2_\sigma(Q_n)$. Furthermore, all properties stated in Theorem 2.3.5 hold, i.e.,*

(i) every trajectory on $\mathcal{A}_0$ is defined for all $t \in \mathbb{R}$ and $(\mathcal{A}_0, S(t))_{t\in\mathbb{R}}$ is a dynamical system;

(ii) $\mathcal{A}_0 = \bigcup\{v \text{ is a complete bounded orbit}\}$;

(iii) for every compact invariant set $X \subseteq L^2_\sigma(Q_n)$ the unstable manifold of X

$$W^u(X) := \left\{v_0 \in L^2_\sigma(Q_n) : S(t)v_0 \text{ defined } \forall t \in \mathbb{R},\ S(-t)v_0 \xrightarrow{t\to\infty} x \in X\right\}$$

is contained in the global attractor $\mathcal{A}_0$.

Proof. We follow the ideas of [39, Theorem 12.8] in order to prove that $(S(t))_{t\geq 0}$ is injective on $\mathcal{A}_0$. Hence, we need to show that if $S(T)u_0 = S(T)v_0 \in \mathcal{A}_0$ for some $T > 0$, then $u_0 = v_0$ already follows.
Hence, let $u_0, v_0 \in \mathcal{A}_0$ and let $u = S(\cdot)u_0$ and $v = S(\cdot)v_0$ be the corresponding solutions of (6.2) with $u(T) = S(T)u_0 = S(T)v_0 = v(T)$ for some $T > 0$. We set $w := u - v$ and then w solves

$$w_t + \Gamma_2\Delta^2 w - \Gamma_0\Delta w + \alpha w + P\beta(|u|^2u - |v|^2v) + P\lambda_0((u\cdot\nabla)u - (v\cdot\nabla)v) = 0$$

in $(0,T)\times Q_n$. Using the same energy estimates as in [57, Theorem 2] we obtain $u, v \in L^\infty((0,T), H^2_\pi(Q_n)) \cap L^2((0,T), H^4_\pi(Q_n) \cap L^2_\sigma(Q_n))$ by Corollary 6.2.9 since $u_0, v_0 \in \mathcal{A}_0 \subseteq H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Hence, also $w \in L^\infty((0,T), H^2_\pi(Q_n)) \cap L^2((0,T), H^4_\pi(Q_n) \cap L^2_\sigma(Q_n))$. By defining

$$B : H^2_\pi(Q_n) \cap L^2_\sigma(Q_n) = E_{1/2} \subseteq E_{-1/2} \to E_{-1/2}, \; Bw = \Gamma_2\Delta^2 w - \Gamma_0\Delta w + \alpha w,$$

we observe that B is a bounded and linear operator (see Section 6.1). We aim to apply [39, Lemma 11.9, Theorem 11.10] with $H = L^2_\sigma(Q_n)$ and $V = H^2_\pi(Q_n)\cap L^2_\sigma(Q_n)$. Defining

$$h(t, w(t)) := P\beta(|u(t)|^2u(t) - |v(t)|^2v(t)) + P\lambda_0((u(t)\cdot\nabla)u(t) - (v(t)\cdot\nabla)v(t)),$$

it remains to show $\|h(t,w(t))\|_{L^2(Q_n)} \leq k(t)\|w(t)\|_{H^2_\pi(Q_n)}$ for $k(t) \in L^2((0,T),\mathbb{R})$. We first consider the first difference in h. Applying the Taylor expansion to $G \in C^1(H^1_\pi(Q_n) \cap L^2_\sigma(Q_n), L^2_\sigma(Q_n))$, $G(u) = |u|^2u$ with $DG(\xi)\lambda = 2(\xi\cdot\lambda)\xi + |\xi|^2\lambda$ for $\xi, \lambda \in H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ we arrive at

$$\begin{aligned}
\||u|^2u - |v|^2v\|_{L^2(Q_n)} &= \|G(u) - G(v)\|_{L^2(Q_n)} = \|DG(\xi)(u-v)\|_{L^2(Q_n)} \\
&\leq C \sup_{\xi\in\mathcal{B}_1} \|DG(\xi)\|_{\mathscr{L}(H^1_\pi(Q_n)\cap L^2_\sigma(Q_n), L^2_\sigma(Q_n))}\|u-v\|_{H^1_\pi(Q_n)} \\
&\leq C \sup_{\xi\in\mathcal{B}_1} \sup_{\|\lambda\|_{H^1_\pi(Q_n)}=1} \|2(\xi\cdot\lambda)\xi + |\xi|^2\lambda\|_{L^2(Q_n)}\|u-v\|_{H^1_\pi(Q_n)} \\
&\leq C \sup_{\xi\in\mathcal{B}_1} \sup_{\|\lambda\|_{H^1_\pi(Q_n)}=1} \|\xi\|^2_{H^1_\pi(Q_n)}\|\lambda\|_{H^1_\pi(Q_n)}\|u-v\|_{H^1_\pi(Q_n)} \\
&\leq C\|u-v\|_{H^1_\pi(Q_n)},
\end{aligned}$$

where we applied the Taylor expansion to $u, v \in \mathcal{A}_0 \subseteq \mathcal{B}_1$ by Corollary 6.2.6 and making use of the Sobolev embedding $H^1_\pi(Q_n) \hookrightarrow L^6(Q_n)$ from [5, Corollary 1.2]. The second difference in h is estimated as

$$\|(u\cdot\nabla)u - (v\cdot\nabla)v\|_{L^2(Q_n)}$$

$$\begin{aligned}
&= \|(u\cdot\nabla)(u-v) - ((u-v)\cdot\nabla)v\|_{L^2(Q_n)}\\
&\le \|u\|_{L^4(Q_n)}\|\nabla(u-v)\|_{L^4(Q_n)} + \|u-v\|_{L^4(Q_n)}\|\nabla v\|_{L^4(Q_n)}\\
&\le C\left(\|u\|_{H^1_\pi(Q_n)}\|u-v\|_{H^2_\pi(Q_n)} + \|u-v\|_{H^1_\pi(Q_n)}\|v\|_{H^2_\pi(Q_n)}\right)\\
&\le C\|u-v\|_{H^2_\pi(Q_n)},
\end{aligned}$$

using the same arguments as before. Then making use of the boundedness of the Helmholtz-Weyl projection yields $\|h(t,w(t))\|_{L^2(Q_n)} \le C\|w(t)\|_{H^2_\pi(Q_n)}$ where $C>0$ is independent of t. Hence, all assumptions for [39, Theorem 11.10] hold. Since $S(T)u_0 = S(T)v_0$ for some $T>0$ we have $w(T) = u(T)-v(T) = 0$ and [39, Theorem 11.10] yields $w(t) = u(t)-v(t) = 0$ for all $0\le t\le T$, which especially holds for $t=0$. Hence, $u_0 = v_0$ and $(S(t))_{t\ge 0}$ is injective on $\mathcal{A}_0$. □

Next, we attempt to obtain dimensional bounds for the Hausdorff and fractal dimension for the attractor $\mathcal{A}_0$ following the approach in [39, Chapter 13]. Even though $\mathcal{A}_0 \subseteq L^2_\sigma(Q_n)$ is a subset of an infinitely-dimensional phase space, we are able to prove (fractal) finite-dimension which shows that the dynamics of the whole system (6.2) can be determined by a finite degree of freedom. We aim to apply Theorem 2.3.7. To this end, we first prove uniform differentiability of the semigroup $(S(t))_{t\ge 0}$:

6.3.2 Lemma. *Let $(S(t))_{t\ge0}$ be the semigroup from Section 6.1 and $\mathcal{A}_0$ be the global attractor. For $u_0, v_0 \in \mathcal{A}_0$ we then have*

$$\|S(t)u_0 - S(t)v_0\|^2_{L^2(Q_n)} \le e^{Ct}\|u_0 - v_0\|^2_{L^2(Q_n)} \qquad (t\ge 0).$$

This especially yields uniqueness of solutions with initial value $v_0 \in \mathcal{A}_0$.

Proof. We apply a standard argument which is often used in order to prove uniqueness of solutions. Let $u_0, v_0 \in \mathcal{A}_0$ be initial values and $u = S(\cdot)u_0$ and $v = S(\cdot)v_0$ be corresponding solutions of (6.2). We define the difference $w := u - v$ and $w_0 = u_0 - v_0$. Then w solves

$$\begin{aligned}
w_t + \Gamma_2\Delta^2 w - \Gamma_0\Delta w + \alpha w + \beta P(|u|^2u - |v|^2v) + \lambda_0 P((u\cdot\nabla)u - (v\cdot\nabla)v) = 0,\\
w|_{t=0} = w_0.
\end{aligned}$$

Testing with w w.r.t. the scalar product in $L^2(Q_n)$ then yields the following (we omit the variable t):

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(Q_n)} + \Gamma_2\|\Delta w\|^2_{L^2(Q_n)} + \Gamma_0\|\nabla w\|^2_{L^2(Q_n)} + \alpha\|w\|^2_{L^2(Q_n)} + \beta(|u|^2u, w)_{2,\pi}$$

$$+ \beta(|v|^2 v, w)_{2,\pi} + \lambda_0((u \cdot \nabla)u, w)_{2,\pi} - \lambda_0((v \cdot \nabla)v, w)_{2,\pi} = 0.$$

We aim to apply Gronwall's inequality. To this end, we observe that $((v\cdot\nabla)v, v)_{2,\pi} = ((v\cdot\nabla)u, u)_{2,\pi} = 0$ since u, v are divergence free and $((v\cdot\nabla)v, u)_{2,\pi} = -((v\cdot\nabla)u, v)_{2,\pi}$, hence the λ_0 terms simplify as

$$((u \cdot \nabla)u, w)_{2,\pi} - ((v \cdot \nabla)v, w)_{2,\pi} = -((w \cdot \nabla)u, w)_{2,\pi}.$$

Again by applying the Sobolev embedding $H^1_\pi(Q_n) \hookrightarrow L^4(Q_n)$ from [5, Corollary 1.2] and making use of the fact that $\mathcal{A}_0 \subseteq \mathcal{B}_2$, Corollary 6.2.9 yields for the λ_0 term:

$$|((w \cdot \nabla)u, w)_{2,\pi}| \leq C\|w\|_{H^1_\pi(Q_n)}\|w\|_{L^2(Q_n)}\|u\|_{H^2_\pi(Q_n)} \leq C\|w\|_{H^1_\pi(Q_n)}\|w\|_{L^2(Q_n)},$$

since $u(t) = S(t)u_0 \in \mathcal{A}_0$ for all $t \geq 0$. Concerning the β term we apply the Taylor expansion exactly as in the proof of Lemma 6.3.1. Hence collecting all estimates and making use of Corollary 2.2.2 for all ∇w terms leads us to

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(Q_n)} + \frac{\Gamma_2}{8}\|\Delta w\|^2_{L^2(Q_n)} \leq C\left(\|w\|_{H^1_\pi(Q_n)}\|w\|_{L^2(Q_n)} + \|w\|^2_{L^2(Q_n)}\right)$$

and in the end

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(Q_n)} + \frac{\Gamma_2}{8}\|w\|^2_{H^2_\pi(Q_n)} \leq C\|w\|^2_{L^2(Q_n)} \tag{6.7}$$

with $C > 0$ independent of u_0, v_0 and t. Applying Gronwall's inequality yields the assertion:

$$\|w(t)\|^2_{L^2(Q_n)} \leq e^{Ct}\|w_0\|^2_{L^2(Q_n)} \qquad (t \geq 0),$$

where $C > 0$ is independent of t. □

6.3.3 Lemma. *The semigroup $(S(t))_{t\geq 0}$ from Section 6.1 is uniformly differentiable on the attractor $\mathcal{A}_0 \subseteq L^2_\sigma(Q_n)$ in the sense of Definition 2.3.6. Furthermore, the solution operator $\Lambda(t, v_0)$ of (6.8) is a compact operator for $t > 0$ and $v_0 \in \mathcal{A}_0$.*

Proof. In order to prove the assertion we need to show that for every $v_0 \in \mathcal{A}_0$ there exists a linear operator $\Lambda(t, v_0)$ such that for all $t \geq 0$ we have

$$\sup_{u_0, v_0 \in \mathcal{A}_0, 0 < \|v_0 - u_0\|_{L^2(Q_n)} < \varepsilon} \frac{\|S(t)u_0 - S(t)v_0 - \Lambda(t, v_0)(u_0 - v_0)\|_{L^2(Q_n)}}{\|u_0 - v_0\|_{L^2(Q_n)}} \xrightarrow{\varepsilon \to 0} 0$$

and

$$\sup_{v_0 \in \mathcal{A}_0} \|\Lambda(t, v_0)\|_{\mathscr{L}(L^2_\sigma(Q_n))} < \infty.$$

We claim that $w(t) = \Lambda(t, v_0)w_0$ is given as the solution of the equation (6.2) linearized about the solution v of (6.2) with initial value v_0:

$$\begin{aligned} w_t + \Gamma_2\Delta^2 w - \Gamma_0\Delta w + \alpha w + P\lambda_0((v\cdot\nabla)w - (w\cdot\nabla)v) & \\ +P\beta(2(v\cdot w)v - |v|^2 w) &= 0, \\ w|_{t=0} &= w_0. \end{aligned} \tag{6.8}$$

Hence, by Section 6.1 we observe that (6.8) is wellposed for data $w_0 \in \mathcal{A}_0$ (and $w_0 \in L^2_\sigma(Q_n)$ for the second part of the proof) by applying a perturbation argument once again.

At first we will prove the first assumption on uniform differentiability. To this end, let u, v be solutions of (6.2) to corresponding initial values $u_0, v_0 \in \mathcal{A}_0$. Let w be the solution of (6.8) with initial value $w_0 = u_0 - v_0$. Then we define the error $\theta := u - v - w$ which then fulfills the following equation

$$\begin{aligned} &\theta_t + \Gamma_2\Delta^2\theta - \Gamma_0\Delta\theta + \alpha\theta + P\lambda_0((v\cdot\nabla)\theta + (\theta\cdot\nabla)v + ((v-u)\cdot\nabla)(v-u)) \\ &\quad + P\beta(|u|^2u - |v|^2v - 2(v\cdot w)v - |v|^2w) = 0. \end{aligned} \tag{6.9}$$

Note that we can write the β term as

$$\begin{aligned} &|u|^2u - |v|^2v - 2(v\cdot w)v - |v|^2w \\ &= |u|^2u - |v|^2v - |v|^2(u-v) - 2(v\cdot(u-v))v + 2(v\cdot\theta)v + |v|^2\theta \end{aligned}$$

such that (6.9) can be written as

$$\begin{aligned} &\theta_t + \Gamma_2\Delta^2\theta - \Gamma_0\Delta\theta + \alpha\theta + P\lambda_0((v\cdot\nabla)\theta + (\theta\cdot\nabla)v + ((v-u)\cdot\nabla)(v-u)) \\ &\quad + P\beta(2(v\cdot\theta)v + |v|^2\theta + g(u,v)) = 0, \end{aligned} \tag{6.10}$$

with $g(u, v) = |u|^2u - |v|^2v - |v|^2(u-v) - 2(v\cdot(u-v))v$. We aim to estimate $\|\theta\|_{L^2(Q_n)}$ in order to obtain the desired convergence. To this end, we test (6.10) with θ w.r.t. the scalar product in $L^2(Q_n)$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(Q_n)} + \Gamma_2\|\Delta\theta\|^2_{L^2(Q_n)} + \Gamma_0\|\nabla\theta\|^2_{L^2(Q_n)} + \alpha\|\theta\|^2_{L^2(Q_n)} + \lambda_0((\theta\cdot\nabla)v,\theta)_{2,\pi} & \\ + \lambda_0(((u-v)\cdot\nabla)(u-v),\theta)_{2,\pi} - \beta(g(u,v),\theta)_{2,\pi} & \\ + 2\beta((v\cdot\theta)v,\theta)_{2,\pi} + \beta(|v|^2,|\theta|^2)_{2,\pi} = 0. & \end{aligned}$$

We aim to apply Gronwall's inequality to obtain corresponding estimates for the term $\|\theta\|_{L^2(Q_n)}$. Hence, we consider all terms separately. Using similar estimates and arguments (Sobolev embeddings) as in Lemma 6.2.5 we obtain for the λ_0 terms

$$|((\theta\cdot\nabla)v,\theta)_{2,\pi}| \leq C\|\theta\|_{H^1_\pi(Q_n)}\|\theta\|_{L^2(Q_n)}\|v\|_{H^2_\pi(Q_n)},$$

$$|(((u-v)\cdot\nabla)(u-v),\theta)_{2,\pi}| \le C\|u-v\|_{H^2_\pi(Q_n)}\|u-v\|_{L^2(Q_n}\|\theta\|_{H^1_\pi(Q_n)}.$$

Furthermore, since $\beta > 0$ we deduce that $\beta(|v|^2,|\theta|^2)_{2,\pi} \ge 0$ and that $\beta((v\cdot\theta)v,\theta)_{2,\pi} = \beta\|v\cdot\theta\|^2_{L^2(Q_n)} \ge 0$. Considering the $(g(u,v),\theta)_{2,\pi}$ term we observe that $g(u,v) = G(u)-G(v)-DG(v)(u-v)$ with $G(x) = |x|^2x$, hence considering the Taylor expansion of $G \in C^2(H^1_\pi(Q_n)\cap L^2_\sigma(Q_n), L^2_\sigma(Q_n))$ with $D^2G(x)[y,z] = 2(x\cdot y)z+2(y\cdot z)x+2(x\cdot z)y$ for $x,y,z \in H^1_\pi(Q_n)\cap L^2_\sigma(Q_n)$ yields

$$\begin{aligned}
&\|g(u,v)\|_{L^2(Q_n)}\\
&\le C\sup_{x\in\mathcal{B}_1}\sup_{\|(y,z)\|_{H^1_\pi(Q_n)\times H^1_\pi(Q_n)}=1}\|(x\cdot y)z+(y\cdot z)x+(x\cdot z)y\|_{L^2(Q_n)}\|u-v\|^2_{H^1_\pi(Q_n)}\\
&\le C\sup_{x\in\mathcal{B}_1}\sup_{\|(y,z)\|_{H^1_\pi(Q_n)\times H^1_\pi(Q_n)}=1}\|x\|_{H^1_\pi(Q_n)}\|y\|_{H^1_\pi(Q_n)}\|z\|_{H^1_\pi(Q_n)}\|u-v\|^2_{H^1_\pi(Q_n)}\\
&\le C\|u-v\|^2_{H^1_\pi(Q_n)},
\end{aligned}$$

because $u,v\in\mathcal{A}_0\subseteq\mathcal{B}_1$, since $u_0,v_0\in\mathcal{A}_0$ (cf. Corollary 6.2.6). At last by applying Corollary 2.2.2 to the Γ_0 term, collecting all estimates from above and making use of $u,v\in\mathcal{A}_0\subseteq\mathcal{B}_2$ (Corollary 6.2.9) we arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(Q_n)}+\frac{\Gamma_2}{4}\|\Delta\theta\|^2_{L^2(Q_n)} \le C\Big(&\|\theta\|^2_{H^1_\pi(Q_n)}+\|u-v\|^2_{H^1_\pi(Q_n)}\|\theta\|_{L^2(Q_n)}\\
&+\|u-v\|_{H^2_\pi(Q_n)}\|u-v\|_{L^2(Q_n)}\|\theta\|_{H^1_\pi(Q_n)}\Big).
\end{aligned}$$

Again, by making use of Corollary 2.2.2 (applied to the ∇ terms) and Young's inequality we finally conclude

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(Q_n)}+\frac{\Gamma_2}{8}\|\Delta\theta\|^2_{L^2(Q_n)} \le C\left(\|\theta\|^2_{L^2(Q_n)}+\|u-v\|^2_{H^2_\pi(Q_n)}\|u-v\|^2_{L^2(Q_n)}\right).$$

Finally, applying the Gronwall inequality then yields (note that $\theta(0)=0$):

$$\|\theta(t)\|^2_{L^2(Q_n)} \le C(t)\int_0^t \|u(s)-v(s)\|^2_{H^2_\pi(Q_n)}\|u(s)-v(s)\|^2_{L^2(Q_n)}\,ds.$$

Note that the right-hand side is bounded by Lemma 6.3.2; in the proof of Lemma 6.3.2 we tested (6.7) with $\|u-v\|^2_{L^2(Q_n)}$ w.r.t. the time t to conclude

$$\int_0^t \|u(s)-v(s)\|^2_{H^2_\pi(Q_n)}\|u(s)-v(s)\|^2_{L^2(Q_n)}\,ds \le C(t)\|u_0-v_0\|^4_{L^2(Q_n)},$$

which then finally yields

$$\frac{\|u(t)-v(t)-w(t)\|^2_{L^2(Q_n)}}{\|u_0-v_0\|^2_{L^2(Q_n)}} \le C(t)\|u_0-v_0\|^2_{L^2(Q_n)} \xrightarrow{u_0\,\to\,v_0} 0.$$

At last we prove that $\Lambda(t, v_0)$ is a compact operator for $t > 0$ and $v_0 \in \mathcal{A}_0$ and has an operator norm uniform in v_0. To this end, we will prove that $\Lambda(t, v_0)w_0$ can be bounded in L^2 and H^1_π such that we can apply the Rellich compact embedding theorem. We recall that $w(t) = \Lambda(t, v_0)w_0$, where w is the solution of (6.8). Testing (6.8) with w w.r.t. the L^2 inner product and making use of the same arguments as in the proof before (positivity of the β term, Corollary 2.2.2, Corollary 6.2.6, Sobolev inequality), we obtain

$$\frac{d}{dt}\|w(t)\|^2_{L^2(Q_n)} + \|\Delta w(t)\|^2_{L^2(Q_n)} \leq C\|w(t)\|^2_{L^2(Q_n)}. \tag{6.11}$$

Hence, applying Gronwall's inequality yields

$$\|w(t)\|^2_{L^2(Q_n)} \leq e^{Ct}\|w_0\|^2_{L^2(Q_n)}, \qquad \|w(t)\|^2_{L^2(Q_n)} \leq e^{Ct}\|w(t/2)\|^2_{L^2(Q_n)} \tag{6.12}$$

for $t > 0$ with $C > 0$ independent of v_0, w_0. The first inequality then already yields the L^2 bound for $\Lambda(t, v_0)$ for $t > 0$ and $v_0 \in \mathcal{A}_0$. In order to obtain the H^1_π bound we test (6.8) with $-\Delta w$ and apply Corollary 2.2.2 and Sobolev embeddings to obtain

$$\frac{d}{dt}\|\nabla w(t)\|^2_{L^2(Q_n)} + \|\Delta\nabla w(t)\|^2_{L^2(Q_n)} \leq C\left(\|\nabla w(t)\|_{L^2(Q_n)} + \|w(t)\|_{L^2(Q_n)}\right). \tag{6.13}$$

We aim to derive a bound for ∇w. To this end, we integrate (6.11) from $t/2$ to t to obtain

$$\int_{t/2}^t \|w(s)\|^2_{H^2_\pi(Q_n)}\, ds \leq C\left(\int_{t/2}^t \|w(s)\|^2_{L^2(Q_n)}\, ds + \|w(t/2)\|^2_{L^2(Q_n)}\right),$$

which especially yields by applying (6.12):

$$\int_{t/2}^t \|\nabla w(s)\|^2_{L^2(Q_n)}\, ds \leq C(t)\|w_0\|^2_{L^2(Q_n)}.$$

At last we integrate (6.13) first from s to t with $s \in (t/2, t)$ and then from $t/2$ to t to end up with

$$\begin{aligned}&\frac{t}{2}\|\nabla w(t)\|_{L^2(Q_n)}\\ &\leq C\left(\int_{t/2}^t \|\nabla w(s)\|^2_{L^2(Q_n)}\, ds + \frac{t}{2}\int_{t/2}^t \|\nabla w(s)\|^2_{L^2(Q_n)} + \frac{t}{2}\int_{t/2}^t \|w(s)\|^2_{L^2(Q_n)}\, ds\right),\end{aligned}$$

which finally leads us to

$$\|\nabla w(t)\|^2_{L^2(Q_n)} \leq C(t)\|w_0\|^2_{L^2(Q_n)}.$$

This proves that for any bounded set $\mathcal{M} \subseteq L^2_\sigma(Q_n)$ the range $\Lambda(t, v_0)\mathcal{M} \subseteq H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ is bounded, such that $\Lambda(t, v_0)\mathcal{M} \subseteq L^2_\sigma(Q_n)$ is relatively compact. Thus, $\Lambda(t, v_0) \in \mathscr{L}(L^2_\sigma(Q_n))$ is compact with bounds uniformly in $v_0 \in \mathcal{A}_0$. The proof is now completed. □

6.3.4 Theorem. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then the fractal and Hausdorff dimension of the global attractor $\mathcal{A}_0$ from Proposition 6.2.7 is finite, i.e., there exists $m \in \mathbb{N}$ such that $d_H(\mathcal{A}_0) \leq d_f(\mathcal{A}_0) \leq m$, where d_H and d_f denote the Hausdorff and fractal dimension, respectively.*

Proof. We want to apply Theorem 2.3.7 to obtain the result. By Lemma 6.3.3 we only need to prove the trace condition, i.e.,

$$\mathcal{TR}_m(\mathcal{A}_0) = \sup_{v_0 \in \mathcal{A}_0} \sup_{\substack{\xi_j^0 \in L^2_\sigma(Q_n), \\ \|\xi_j^0\|_{L^2(Q_n)}=1, \\ j=1,\dots,m}} \langle \mathrm{Tr} L(t,v_0) P^{(m)}_{\xi_1^0,\dots,\xi_m^0}(t)\rangle < 0$$

(see Theorem 2.3.7 regarding the notation).

To this end, we fix $m \in \mathbb{N}$ and consider $\{\xi_j^0 : j = 1,\dots,m\} \subseteq L^2_\sigma(Q_n)$ where ξ_j^0 are linearly independent. Let $v_0 \in \mathcal{A}_0$ and v be the corresponding solution of (6.2). Then by $L(t,v_0)$ we denote the linearized operator in (6.8) and by $\Lambda(t,v_0)$ its corresponding solution operator (as defined in Lemma 6.3.3). We consider the linear span

$$\mathcal{M}(t) \coloneqq \left\{\xi_j(t) \coloneqq \Lambda(t,v_0)\xi_j^0 : j = 1,\dots,m\right\} \subseteq L^2_\sigma(Q_n),$$

which is a finite dimensional subspace of $L^2_\sigma(Q_n)$, hence we can find a projection $P^{(m)}_{\xi_1^0,\dots,\xi_m^0}(t)$ onto $\mathcal{M}(t)$ for every $t \geq 0$ such that $P^{(m)}_{\xi_1^0,\dots,\xi_m^0}(t)L^2_\sigma(Q_n) = \mathcal{M}(t)$. However, w.l.o.g. we can choose an orthonormal span $\{\varphi_j(t) : j = 1,\dots,m\} \subseteq H^m_\pi(Q_n) \cap L^2_\sigma(Q_n)$ of $P^{(m)}_{\xi_1^0,\dots,\xi_m^0}(t)L^2_\sigma(Q_n)$ w.r.t. the L^2 norm. Then testing $L(t,v_0)\varphi_j(t)$ with $\varphi_j(t)$ w.r.t. the L^2 inner product yields

$$\begin{aligned}
&(L(t,v_0)\varphi_j(t), \varphi_j(t))_{2,\pi} \\
&= -\Gamma_2\|\Delta\varphi_j(t)\|^2_{L^2(Q_n)} - \Gamma_0\|\nabla\varphi_j(t)\|^2_{L^2(Q_n)} - \alpha\|\varphi_j(t)\|^2_{L^2(Q_n)} \\
&\quad - \lambda_0((\varphi_j(t)\cdot\nabla)v(t), \varphi_j(t))_{2,\pi} - \lambda_0((v(t)\cdot\nabla)\varphi_j(t), \varphi_j(t))_{2,\pi} \\
&\quad - 2\beta((v(t)\cdot\varphi_j(t))v(t), \varphi_j(t))_{2,\pi} - \beta(|v|^2(t)\varphi_j(t), \varphi_j(t))_{2,\pi} \\
&\leq -\Gamma_2\|\Delta\varphi_j(t)\|^2_{L^2(Q_n)} - \Gamma_0\|\nabla\varphi_j(t)\|^2_{L^2(Q_n)} - \alpha\|\varphi_j(t)\|^2_{L^2(Q_n)} \\
&\quad + C\|\varphi_j(t)\|_{H^2_\pi(Q_n)}\|\varphi_j(t)\|_{L^2(Q_n)}\|v(t)\|_{H^2_\pi(Q_n)} \\
&\leq -\frac{\Gamma_2}{4}\|\Delta\varphi_j(t)\|^2_{L^2(Q_n)} + C,
\end{aligned}$$

taking into account Corollary 2.2.2, $\|\varphi_j(t)\|_{L^2(Q_n)} = 1$, $v(t) \in \mathcal{B}_2$ by Corollary 6.2.9 and Sobolev embeddings with some constant $C > 0$ independent of φ_j, t and v_0. Also note that the β terms are positive. Now, summing up all $j = 1,\dots,m$ and making use of the definition of $\langle\cdot\rangle$ from Theorem 2.3.7 yields

$$\langle \mathrm{Tr} L(t,v_0)P^{(m)}(t)\rangle \leq -\sum_{j=1}^m \frac{\Gamma_2}{4}\left\langle \|\Delta\varphi_j(t)\|^2_{L^2(Q_n)}\right\rangle + mC.$$

Next, we want to apply the Sobolev-Lieb-Thierring inequality as seen in [18, Proposition 3.1, Remark 3.2] to estimate the $\|\Delta\varphi_j(t)\|^2_{L^2(Q_n)}$ term. In order to apply [18, Proposition 3.1] we write all appearing terms in the setting of [18, Proposition 3.1]:

$$\rho(x) := \sum_{j=1}^{m} |\varphi_j(t,x)|^2 \qquad (x \in Q_n)$$

which yields for $p = 3/2$ and $n, m = 2$:

$$\begin{aligned}
\left(\int_{Q_n} \rho(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} &= \int_{Q_n} \rho(x)^3\,dx \\
&\le C\left(\sum_{j=1}^{m}\int_{Q_n}\sum_{|\alpha|=2} |\partial^\alpha \varphi_j(t,x)|^2\,dx + \int_{Q_n}\rho(x)\,dx\right) \\
&\le C\sum_{j=1}^{m} \|\varphi_j(t)\|^2_{H^2_\pi(Q_n)} \\
&\le C\sum_{j=1}^{m}\left(\|\varphi_j(t)\|^2_{L^2(Q_n)} + \|\Delta\varphi_j(t)\|^2_{L^2(Q_n)}\right) \\
&\le C\left(m + \sum_{j=1}^{m}\|\Delta\varphi_j(t)\|^2_{L^2(Q_n)}\right),
\end{aligned}$$

where we again made use of the fact that $\{\varphi_j(t)\}_{j=1,\dots,m}$ is an orthonormal system. Hence, by observing

$$\begin{aligned}
m^3 &= \left(\sum_{j=1}^{m}\|\varphi_j(t)\|^2_{L^2(Q_n)}\right)^3 = \left(\frac{1}{L^n}\int_{Q_n}\rho(x)\,dx\right)^3 \le C\int_{Q_n}\rho(x)^3\,dx \\
&\le C\left(m + \sum_{j=1}^{m}\|\Delta\varphi_j(t)\|^2_{L^2(Q_n)}\right)
\end{aligned}$$

for all $t > 0$, we finally obtain

$$\langle \mathrm{Tr} L(t,v_0) P^{(m)}_{\xi_1^0,\dots,\xi_m^0}(t)\rangle \le C(m - m^3) < 0,$$

which holds for a chosen $m \in \mathbb{N}$ large enough. For $n = 3$ we can obtain the same results by applying [18, Proposition 3.1] with $p = 7/4$ such that we get the same estimate with the leading term $m^{7/3}$. Thus, we infer

$$\mathcal{TR}_m(\mathcal{A}_0) = \sup_{v_0\in\mathcal{A}_0} \sup_{\substack{\xi_j^0 \in L^2_\sigma(Q_n), \\ \|\xi_j^0\|_{L^2(Q_n)}=1, \\ j=1,\dots,m}} \langle \mathrm{Tr} L(t,v_0) P^{(m)}_{\xi_1^0,\dots,\xi_m^0}(t)\rangle < 0,$$

and Theorem 2.3.7 yields the assertion. □

6.4 Existence of a 2D Inertial Manifold

Another approach to analyze long-term behavior is the examination of the existence of an inertial manifold such that the underlying system (6.2) reduces to an ordinary differential equation on the inertial manifold - hence stability analysis of (6.2) can be simplified. In the following we will prove the existence of an inertial manifold for (6.2) in $n = 2$ by following the approach in [15] and [41, Chapter 8].

6.4.1 Theorem. *Let $(S(t))_{t\geq 0}$ be the semigroup from Section 6.1. Then there exists an inertial manifold $\mathcal{M}$ for (6.2) having the following properties:*

(i) $\mathcal{M}$ is a finite dimensional, Lipschitz continuous manifold in $H^{3/2}_\pi(Q_2)\cap L^2_\sigma(Q_2)$;

(ii) $\mathcal{M}$ is positively invariant;

(iii) $\mathcal{M}$ is exponentially attracting, i.e., there exists $\eta > 0$ such that for every $v_0 \in L^2_\sigma(Q_2)$ there is some $K = K(v_0) > 0$ such that

$$dist_{L^2}(S(t)v_0, \mathcal{M}) \leq Ke^{-\eta t} \qquad (t \geq 0).$$

Proof. We aim to apply [41, Theorem 81.2] to prove the result. To this end, in the setting of [41] we set $H = L^2_\sigma(Q_2)$ and $A_\omega : D(A_\omega) \subseteq L^2_\sigma(Q_2) \to L^2_\sigma(Q_n)$ with

$$A_\omega v = \Gamma_2\Delta^2 v - \Gamma_0\Delta v + \alpha v + \omega v,$$
$$D(A_\omega) = H^4_\pi(Q_2) \cap L^2_\sigma(Q_2),$$

where $\omega > 0$ is chosen arbitrary large such that A_ω is a linear, positive operator. By Section 5.1 it is known that A_ω has compact resolvent and admits a bounded H^∞-calculus such that by [22, Theorem 6.6.9, Theorem 7.3.1] the family of interpolation spaces $V^{2\alpha} = D(A^\alpha_\omega) = [D(A^\beta_\omega), D(A^\gamma_\omega)]_\theta$ for $(1-\theta)\beta + \theta\gamma = \alpha,\ \theta \in [0,1]$ and $0 \leq \beta < \gamma$ generated by fractional powers of A_ω are defined for $\alpha \geq 0$. Furthermore, we set

$$F_\omega(v) := -P\beta|v|^2 v - P\lambda_0(v\cdot\nabla)v + \omega v$$

such that (6.2) can be rewritten as $v_t + A_\omega v = F_\omega(v)$. For $\beta = 3/8$ and $V^{2\beta} = D(A^\beta_\omega) = [L^2_\sigma(Q_2), H^4_\pi(Q_2) \cap L^2_\sigma(Q_2)]_{3/8}$ we then infer for the nonlinearity that $F_\omega \in C_{\mathrm{Lip,loc}}(H^{3/2}_\pi(Q_2)\cap L^2_\sigma(Q_2), L^2_\sigma(Q_2))$ since the derivative is given as $DF_\omega(v)u = -2P\beta(u\cdot v)v - P\beta|v|^2u - P\lambda_0(u\cdot\nabla)v - P\lambda_0(v\cdot\nabla)u + \omega u$ for $u, v \in H^{3/2}_\pi(Q_2)\cap L^2_\sigma(Q_2)$. Here, $C_{Lip,loc}$ denotes the space of all locally Lipschitz continuous functions. Then we can estimate the occurring terms locally

$$\||v|^2u\|_{L^2(Q_2)} \leq \|v\|^2_{L^6(Q_2)}\|u\|_{L^6(Q_2)} \leq C\|v\|^2_{H^{3/2}_\pi(Q_2)}\|u\|_{H^{3/2}_\pi(Q_2)},$$

$$\|(u \cdot \nabla)v\|_{L^2(Q_2)} \le \|\nabla v\|_{L^3(Q_2)} \|u\|_{L^6(Q_2)} \le C\|v\|_{H^{3/2}_\pi(Q_2)} \|u\|_{H^{3/2}_\pi(Q_2)},$$

by making use of the embedding $H^{3/2}_\pi(Q_2) \hookrightarrow W^{1,3}(Q_2)$. Furthermore, the so-called spectral gap condition

$$\lambda_{k+1} - \lambda_k \ge K(\lambda_{k+1}^\beta + \lambda_k^\beta) \qquad (k \in \mathbb{N})$$

(see [41, Formula 81.16]) has to hold for $K \ge 0$ and ordered eigenvalues $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_{k-1} \le \lambda_k \le ...$ of the bi-Laplacian Δ^2. It is known that the corresponding eigenvalues behave as $\lambda_k \sim k^2 = k^{4/n}$ and [41, Table 8.1] states that the spectral gap condition is fulfilled in two dimensions whenever $0 \le \beta < 1/2$ which is given in our case. Hence [41, Theorem 81.2] yields the assertion. □

6.4.2 Remark. (i) Note that by the third property of the inertial manifold $\mathcal{M}$ in Theorem 6.4.1, we observe that the global attractor $\mathcal{A}_0$ from Proposition 6.2.7 is contained in $\mathcal{M}$. Furthermore, in contrast to the result from Proposition 6.2.7 we obtain exponential attraction of the inertial manifold $\mathcal{M}$ which means that after a rather short time every solution of (6.2) can be approximated by solutions on the inertial manifold $\mathcal{M}$.

(ii) We cannot satisfy the spectral gap condition for $n = 3$ due to the fact that we need $\beta \ge 1/4$ in order to estimate the nonlinearity but we need $\beta < 1/4$ in order to fulfill the spectral gap condition (see [41, Table 8.1]).

Chapter 7

Conclusion

In this thesis we considered two systems of partial differential equations: A 2D contact line model and an active fluid continuum model. Both are based on the Navier-Stokes equations which describe the motion of viscous fluids. As the dynamic 2D contact line model can be transformed to a Stokes system on a fixed sector, we first performed analysis on a sector in Chapter 3. There we introduced (in)homogeneous Sobolev spaces in sectors and we gave results on e.g. trace theorems, Korn's inequality and solvability of elliptic problems. We also introduced reflection invariant subspaces since there multiplication with the sign function sgn is bounded for $s = 1/2$ which is not the case in the setting of (in)homogeneous spaces $\hat{H}^s$ for $s > 1/2$.

For the linearized 2D contact line model, existence of weak solutions for the stationary system was proved resulting in resolvent estimates for the corresponding solution triple. The active fluid continuum model was considered in the periodic setting. At first (in)stability depending on the involved parameters was proved, hence in order to obtain results matching the observations in [54], the existence of a global attractor and characterizations were considered in the second part.

2D Contact Line Dynamics

Chapter 4 is devoted to the analysis of a contact line model in two dimensions which corresponds to the Navier-Stokes equations subject to partial slip conditions at the solid boundary and free slip conditions at the free boundary in a time-dependent domain. At first a suitable transformation was applied in order to obtain a system on a fixed domain $(0, T) \times \Sigma_\theta$ where $\theta \in (0, \pi/2)$ is the initial contact angle at time $t = 0$ and Σ_θ denotes the sector with opening angle θ and Γ its boundary. This leads to the resolvent Stokes system:

$$\begin{aligned} \lambda u - \operatorname{div} T(u,p) &= f_1 \quad \text{in } \Sigma_\theta, \\ \operatorname{div} u &= 0 \quad \text{in } \Sigma_\theta, \\ T(u,p)n + \sigma c(\theta)\partial_\tau^2 \rho n &= f_4 \quad \text{on } \Gamma, \\ \lambda\rho + \tfrac{1}{\sin(\theta)}(n\cdot u) &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{7.1}$$

which is solved in the setting of homogeneous Sobolev spaces with $p = 2$. At first the resolvent Stokes system was analyzed by making use of the fact that we are working in the Hilbert space setting. Here, it was crucial to have a proper introduction to the homogeneous Sobolev spaces $\hat{H}^s$ for $s \in [-1,1]$ on the sector Σ_θ and its boundary Γ which was addressed in Chapter 3. We again put emphasize on the fact that the boundedness of the multiplication with sgn was crucial throughout Chapter 4 which we only obtained in the setting of reflection invariant (in)homogeneous spaces with the correct symmetry. Since literature dealing with homogeneous spaces in sectors Σ_θ is limited, we applied a bi-Lipschitz transformation to transfer as much results as possible from the whole space $\mathbb{R}^n$ and the half-space $\mathbb{R}^2_+$ to the sector Σ_θ. Furthermore, elliptic problems and additional trace results, and Korn's inequality on convex and non-convex wedges were proved. Then solving the resolvent problem (7.1) in the weak setting can be achieved by using Hilbert space theory, leading to resolvent estimates for $|\lambda| = 1$. By making use of the scaling invariance of (7.1) and the scaling of the norm in homogeneous spaces, it was possible to obtain resolvent estimates for λ with large absolute value, leading to the important resolvent estimates

$$\begin{aligned} &\|u\|_{\lambda,\, H_0^{-1}(\Sigma_\theta)_R} + |\lambda|^{1/2}\|u\|_{L^2(\Sigma_\theta)_R} + \|\nabla u\|_{L^2(\Sigma_\theta)_\mathcal{R}} + \sqrt{\sigma}|\lambda|^{1/2}\|\rho\|_{\hat{H}^1(\Gamma)_r} \\ &\qquad + \sigma\|\partial_\tau^2\rho\|_{\lambda,\, \hat{H}^{-1/2}(\Gamma)_r + \hat{H}^{1/2}(\Gamma)_r} + \|p\|_{\lambda,\, L^2(\Sigma_\theta)_r + \hat{H}^1(\Sigma_\theta)_r} \\ &\le C\left(\|f_1\|_{\hat{H}^{-1}_{0,\mathrm{div}}(\Sigma_\theta)_R} + \|f_4\|_{\hat{H}^{-1/2}(\Gamma)_R}\right). \end{aligned}$$

Several observations within the development of the results were made: In the context of homogeneous spaces we can expect at most the regularity stated above for the triple (u,p,ρ) since $\hat{H}^{1/2}(\Gamma)$ (the space for the data on the boundary) is the borderline for the non existence of the trace at the singular point $(0,0)$ and for the boundedness of the multiplication with normal and tangential vector fields in spaces with the correct symmetry.

Further challenges are to develop methods to reduce the divergence of the velocity field and f_5 (second and fourth equation of (7.1)). E.g. it is possible to reduce f_5 by solving a corresponding weak and strong inhomogeneous Neumann problem. As soon as (7.1) can be solved with arbitrary data f_5 it is possible to apply the Laplace transform to the time-dependent system in order to obtain maximal regularity type estimates as seen e.g. in [42]. Solving the linear 2D contact line model automatically leads to the question whether the nonlinear problem can also be solved in this setting. This is left as a future challenge.

Active Fluids

In Chapter 5 we presented a full stability analysis of an active fluid continuum model with results depending on the occurring parameters in two and three dimensions. Here, the model is given as generalized Navier-Stokes equations with a leading fourth order term Δ^2 subject to periodic boundary conditions:

$$\begin{aligned} v_t + \lambda_0 v \cdot \nabla v &= f - \nabla p + \lambda_1 \nabla |v|^2 - (\alpha + \beta |v|^2) v + \Gamma_0 \Delta v - \Gamma_2 \Delta^2 v, \\ \operatorname{div} v &= 0, \\ v|_{t=0} &= v_0. \end{aligned} \tag{7.2}$$

The model was investigated in the periodic L^2-setting in a box Q_n of length $L > 0$ such that it was possible to take advantage of the fact that in bounded domains results like the compactness theorem by Rellich-Kondrachov could be applied. At first we ensured global wellposedness of (7.2) for initial values in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ using the theory of maximal L^p-regularity. Global wellposedness can then be obtained by using energy estimates.
Two physical relevant stationary states occur: the disordered polar state $(0, p_0)$ with $p_0 \in \mathbb{R}$ and the manifold of ordered polar states consisting of (V, p_0) with $p_0 \in \mathbb{R}$ and constant vectors V of length $\sqrt{-\alpha/\beta}$. Here, we focus on the analysis of the manifold of ordered polar states. Every function in $L^2(Q_n)$ can be represented by a Fourier series, hence by making use of this Fourier series expansion it was possible to prove that V is normally stable if $\Gamma_0 \geq 0$, hence applying the generalized principle of linearized stability from [35, 36] yields stability. Depending on the occurring parameters $\Gamma_2, \Gamma_0, \alpha, \beta$ it was possible to show that V is normally hyperbolic, by again making use of the Fourier series expansion. Then again [35, 36] yields the existence of a stable and an unstable foliation, stating especially that the case of instability occurs which matches the observation of turbulence in [54].
However, in [54] it is also observed that the simulation of bacterial suspensions reaches some stable final state after a finite time. This was mathematically justified in Chapter 6 where we proved the existence of a global attractor independent of the relation of the occurring parameters. At first we again ensured global wellposedness for initial values in $L^2(Q_n)$ in contrast to the setting in Chapter 5. Following the

approach from [39, 47] it was possible to show the existence of compact absorbing sets of arbitrary high regularity such that we could deduce the existence of a global attractor, which is a maximal set which attracts all solutions of (7.2). Here, we again made use of the Fourier series expansion, and used a bootstrapping argument to obtain estimates in spaces of higher regularity. At least we tried to characterize the global attractor by showing properties like injectivity and finite fractal and Hausdorff dimension, which leads to the observation that the dynamics of (7.2) can be determined by finite degrees of freedom. At last we proved the existence of an inertial manifold in two dimensions which was possible due to the fourth order term and the nonlinearity of second and third order. Then a so-called spectral gap condition could be fulfilled. The advantage of having an inertial manifold lies in the fact that an inertial manifold has more structure (and is indeed a manifold) which attracts exponentially and where the global attractor is contained.

Hence, the inertial manifold might also contain stationary states that we haven't considered before. Also it is not clear whether we also obtain an inertial manifold in three dimensions. In this case the spectral gap condition is not fulfilled, however, it is still open if the spectral gap condition is mandatory to obtain the existence of an inertial manifold. Furthermore, in [54] it was observed that the stable final state forms a hexagonal grid. Up to today, it is still open if there are any stationary solutions corresponding to this hexagonal grid. Also it could be helpful to have a more precise characterization of the global attractor which could be achieved with numerical simulations.

Contributions

The content of this thesis is based on joint work with other contributors.

Chapter 3, which introduce the mathematical framework used in Chapter 4, and Chapter 4, are based on a joint work of Jürgen Saal, Matthias Köhne and the author of this thesis. The results in Chapter 3 are the result of several working sessions of Jürgen Saal, Matthias Köhne and the author of this thesis. In Chapter 4 the results concerning weak solvability and corresponding resolvent estimates were established by Matthias Köhne, Jürgen Saal and the author of this thesis.

Chapter 5 and Chapter 6 resulted from a joint work with Jürgen Saal, Christian Gesse and the author of this thesis. The results of Chapter 5 were published in [8]. Here, global wellposedness with initial values in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ was established by Jürgen Saal and the author of this thesis. The full nonlinear stability analysis of the ordered polar state, to be precise results regarding normal stability and normal hyperbolicity, were developed by Christian Gesse, Jürgen Saal and the author of this thesis.

In Chapter 6 results regarding global wellposedness with initial values in $L^2_\sigma(Q_n)$, existence and corresponding properties of the attractor were established by Christian Gesse and the author of this thesis in equal parts complemented by many discussions with Jürgen Saal. The existence of an inertial manifold in two dimensions is due to Christian Gesse, Jürgen Saal and the author of this thesis, inspired by a discussion regarding this topic with Edriss Titi.

Bibliography

[1] R. Adams and J. Fournier. *Sobolev Spaces.* Academic Press, 2003.

[2] H. Amann. *Linear and Quasilinear Parabolic Problems.* Birkhäuser, Basel, 1995.

[3] H. Amann. Nonhomogeneous Navier-Stokes equations in spaces of low regularity. In *Papers from the meeting in honor of John G. Heywood on the occasion of his 60th birthday held at Capo Miseno, May 27-30, 2000*, pages 13–31. Department of Mathematics, Seeconda Università di Napoli, 2002.

[4] H. Amann. *Linear and Quasilinear Parabolic Problems. Part II. Function Spaces.* Birkhäuser, Basel, 2019.

[5] A. Bényi and T. Oh. The Sobolev inequality on the torus revisited. *Publ. Math. Debrecen*, 83:359–374, 2013.

[6] D. Bothe, M. Köhne, and J. Prüss. On a Class of Energy Preserving Boundary Conditions for Incompressible Newtonian Flows. *SIAM J. Math. Anal*, 45:3768–3822, 2013.

[7] F. Boyer and P. Fabrie. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models.* Springer, New York, 2013.

[8] C. Bui, C. Gesse, and J. Saal. Stable and unstable flow regimes for active fluids in the periodic setting. *Nonlinear Anal. Real World Appl.*, 69:103707, 2023.

[9] C. Bui, H. Löwen, and J. Saal. Turbulence in active fluids caused by self-propulsion. *Asymptot. Anal.*, 113:195–205, 2019.

[10] R. Danchin, M. Hieber, P. Mucha, and P. Tolksdorf. Free Boundary Problems by Da Prato - Grisvard theory. *arXiv:2011.07918*, 2021.

[11] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Am. Math. Soc.*, 788, 2003.

[12] J. Dunkel, S. Heidenreich, M. Bär, and R. E. Goldstein. Minimal continuum theories of structure formation in dense active fluids. *New Journal of Physics*, 15:045016, 2013.

[13] J. Dunkel, S. Heidenreich, K. Drescher, H. Wensink, M. Bär, and R. Goldstein. Fluid dynamics of bacterial turbulence. *Phys. Rev. Lett.*, 110:228192, 2013.

[14] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Springer, Berlin-Heidelberg, 2000.

[15] C. Foias, G. R. Sell, and E. S. Titi. Exponential Tracking and Approximation of Inertial Manifolds for Dissipative Nonlinear Equations. *Journal of Dynamics and Differential Equations*, 1:199–244, 1989.

[16] M. Fricke, M. Köhne, and D. Bothe. A kinematic evolution equation for the dynamic contact angle and some consequences. *Phys. D*, 394:26–43, 2019.

[17] G. P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems.* Springer, New York, 1994.

[18] J.-M. Ghidaglia, M. Marion, and R. Temam. Generalization of the Sobolev-Lieb-Thirring inequalities and Applications to the Dimension of attractors. *Differential and Integral Equatioins*, 1:1–21, 1988.

[19] L. Grafakos. *Classical Fourier Analysis.* Springer, New York, 2008.

[20] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Society for Industrial and Applied Mathematics, 2011.

[21] Y. Guo and I. Tice. Stability of contact lines in fluids: 2D Stokes flow. *Arch. Ration. Mech. Anal.*, 227:767–854, 2018.

[22] M. Haase. *The Functional Calculus for Sectorial Operators.* Birkhäuser, Basel, 2006.

[23] M. Haase. Operator-valued H^∞-calculus in Inter- and Extrapolation Spaces. *Integral Equations and Operator Theory*, 56:197–228, 2006.

[24] M. Hieber and J. Saal. The Stokes Equation in the l^p-setting: Well Posedness and Regularity Properties. In *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, pages 117–2016. Springer, Cham, 2016.

[25] P. Hobus and J. Saal. Stokes and Navier-Stokes equations subject to partial slip on uniform $C^{2,1}$-domains in L_q-spaces. *J. Differ. Equ.*, 284:374–432, 2021.

[26] L. D. Kudrjavcev. Imbedding Theorem For a Class of Functions Defined on the Entire Space or on a Half Space. Part II. In *Ten Papers on Analysis*, pages 227–260. Amer. Math. Soc. Transl., 1968.

[27] P. C. Kunstmann and L. Weis. *Maximal L_p-regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-functional calculus.* Springer, Berlin-Heidelberg, 2004.

[28] Y. Kusaka. Solvability of a moving contact-line problem with interface formation for an incompressible viscous fluid. *Boundary Value Problems*, 1, 2022.

[29] M. Köhne, J. Saal, and L. Westermann. The Dirichlet Stokes Operator on a 2D Wedge Domain in L^p: Sectoriality and Optimal Regularity. In *Optimal Regularity for the Stokes Equations on a 2D Wedge Domain Subject to Perfect Slip, Dirichlet and Navier Boundary Conditions*, pages 47–78. Shaker Verlag, 2021.

[30] M. Köhne, J. Saal, and L. Westermann. Optimal Regularity of the Stokes Equations on a 2D Wedge Domain Subject to Navier Boundary Conditions. In *Optimal Regularity for the Stokes Equations on a 2D Wedge Domain Subject to Perfect Slip, Dirichlet and Navier Boundary Conditions*, pages 80–102. Shaker Verlag, 2021.

[31] M. Köhne, J. Saal, and L. Westermann. Optimal Sobolev regularity for the Stokes equations on a 2D wedge domain. *Mathematische Annalen*, 379:377–413, 2021.

[32] A. Lunardi. *Analytic Semigroups and optimal Regularity in Parabolic Problems.* Birkhäuser, Basel, 1995.

[33] S. Maier and J. Saal. Stokes and Navier Stokes Equations with Perfect Slip on Wedge Type Domains. *Discrete Contin. Dyn. Syst. - Series S*, 7:1045–1063, 2014.

[34] J. Marschall. The trace of Sobolev-Slobodeckij spaces on Lipschitz domains. *Manuscripta Math*, 58:47–65, 1987.

[35] J. Prüss and G. Simonett. *Moving Interfaces and Quasilinear Parabolic Evolution Equations.* Birkhäuser Cham, 2016.

[36] J. Prüss, G. Simonett, and R. Zacher. On convergence of solutions to equilibria for quasilinear parabolic problems. *Journal of Differential Equations*, 246:3902–3931, 2009.

[37] E. M. Purcell. Life at Low Reynolds Number. *Am. J. Phys.*, 45:3–11, 1977.

[38] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis, second ed.* Academic Press, New York, 1980.

[39] J. C. Robinson. *Infinite-Dimensional Dynamical Systems. An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors.* Cambridge University Press, 2001.

[40] J. C. Robinson, J. L. Rodrigo, and W. Sadowski. *The Three-Dimensional Navier-Stokes Equations.* Cambridge University Press, 2016.

[41] G. R. Sell and Y. You. *Dynamics of Evolutionary Equations.* Springer, New York, 2002.

[42] Y. Shibata. On the $\mathcal{R}$-boundedness of solution operators for the Stokes equations with free boundary condition. *Differ. Integral Equ.*, 27:313–368, 2014.

[43] R. A. Simha and S. Ramaswamy. Hydrodanamic Fluctuations and Instabilities in Ordered Suspensions of Self-Propelled Particles. *Phys. Rev. Lett.*, 89:058101, 2002.

[44] V. A. Solonnikov. On some free boundary problems for the Navier-Stokes equations with moving contact points and lines. *Math. Ann.*, 302:743–772, 1995.

[45] J. B. Swift and P. C. Hohenberg. Hydrodynamic fluctuations at the convective instability. *Phys. Rev. A*, 15:319–328, 1977.

[46] L. Tartar. *An introduction to Sobolev Spaces and Interpolation Spaces.* Springer, Berlin-Heidelberg, 2007.

[47] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Springer, New York, 1998.

[48] G. Teschl. *Ordinary Differential Equations and Dynamical Systems.* American Mathematical Society, 2012.

[49] J. Toner and Y. Tu. Long-range order in a two-dimensional dynamical xy model: How birds fly together. *Phys. Rev. Lett.*, 75:4326–4329, 1995.

[50] J. Toner and Y. Tu. Flocks, herds, and schools: A quantitative theory of flocking. *Phys. Rev. E*, 58:4828–4858, 1998.

[51] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland Publishing Company, 1978.

[52] H. Triebel. *Theory of Function Spaces*. Birkhäuser, Basel, 2010.

[53] K. Watanabe. Local well-posedness of incompressible viscous fluids in bounded cylinders with 90°-contact angle. *Nonlinear Anal. Real World. Appl.*, 65:103489, 2022.

[54] H. H. Wensink, J. Dunkel, S. Heidenreich, K. Drescher, R. E. Goldstein, H. Löwen, and J. M. Yeomans. Meso-scale turbulence in living fluids. *Proc. Natl. Acad. Sci. USA*, 109:14308–14313, 2012.

[55] M. Wilke. Rayleigh-Taylor instability for the two-phase Navier-Stokes equations with surface tension in cylindrical domains. *Habilitations-Schrift Universität Halle, Naturwissenschaftliche Fakultät II*, 2013.

[56] K. Yosida. *Functional Analysis*. Springer, Berlin-Heidelberg, 1974.

[57] F. Zanger, H. Löwen, and J. Saal. Analysis of a Living Fluid Continuum Model. In *Mathematics for Nonlinear Phenomena - Analysis and Computation*, pages 285–303. Springer International Publishing, 2017.

[58] Y. Zheng and I. Tice. Local well posedness of the near-equilibrium contact line problem in 2-dimensional Stokes flow. *SIAM J. Math. Anal.*, 49:899–953, 2017.