The Kummer Constructions in Families

Inaugural-Dissertation

Zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

Jakob Studsgaard Bergqvist

aus Hvidovre

Düsseldorf, April 2023

Aus dem Mathematischen Institut
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit der Genehmigung der
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

Referent: Prof. Dr. Stefan Schröer
Koreferent: Prof. Dr. Markus Reineke

Tag der mündlichen Prüfung: 19.06.2023

Summary

The aim of this thesis is to study generalized Kummer constructions in families. This involves many different overlapping areas of study and I introduce them in turn. I provide a thorough introduction to the theory of group schemes, focusing in particular on finite and diagonalizable group schemes. Then, the focus turns to group scheme actions and quotients by these with the existence of quotients being discussed. I explain how an action by a diagonalizable group scheme may be viewed as a grading, and how one may use this to compute quotients. It is also outlined how one may interpret group scheme actions using Lie algebras and how this technique allows for computing a quotient. I briefly outline some general surface theory, focusing on rational double point singularities, and the conditions (S_i) and (R_i) of Serre. To give perspective, the thesis includes a detailed explanation of the classical Kummer construction over fields, and later an outline of how the classical construction behaves in families. Turning the focus to families, I study quotients of families. Special attention is paid to when taking fibers and quotients commute. I discuss simultaneous resolution of singularities, as a preparation for the main result of the thesis. I outline the generalized Kummer constructions in characteristic 2 with α_2 and μ_2 of Schröer and Schröer and Kondo. The thesis then concludes with the new results; I show that the family of singular surfaces $(C \times C)/\mu_2$, with C the cuspidal curve, admits a simultaneous resolution after a finite base change of degree at most 5184, and conclude that the generalized Kummer construction with μ_2 works in families after this base change.

Contents

Introduction

The study of surfaces is a vast area of algebraic geometry. In the study of smooth surfaces, one major landmark is the Enriques-Kodaira classification. The first part was Enriques' classification of projective complex surfaces [24, Chapter VI-VIII] which was followed by Kodaira's classification of the compact complex surfaces [42, 43, 44, 45]. Finally, the classification of algebraic surfaces in positive characteristic was done by Mumford [54] and Bombieri and Mumford [14, 13]. As we are interested in algebraic surfaces, a surface will from now on mean an algebraic surface. In concrete terms, a smooth surface is a smooth, proper, irreducible scheme of dimension 2. The classification separates smooth surfaces X into four classes dependent on their Kodaira dimension, which can be $-\infty$, 0, 1 or 2, and then into further subclasses based on certain invariants.

For example, we have four subclasses of minimal surfaces when the Kodaira dimension is 0. These are determined by their second Betti number b_2. They are the Abelian ($b_2 = 6$), $K3$ ($b_2 = 22$), Enriques ($b_2 = 10$) and Bielliptic ($b_2 = 2$) surfaces respectively. In this thesis, we are interested in the first two of these. One could define these in terms of their Betti numbers. Alternatively one may take the following: Abelian surfaces are those smooth surfaces which are also Abelian varieties. That is, they are the underlying schemes of proper, connected, geometrically reduced, group schemes of dimension 2. They arise naturally from products of elliptic curves or as Jacobians of genus 2 curves. A $K3$ surface is a smooth, proper, geometrically integral scheme of dimension 2 such that $\omega_X = {}_X$ and $\mathrm{H}^1(X, {}_X) = 0$, where ω_X is the dualizing sheaf. Any smooth quartic $X \subseteq \mathbb{P}^3$ is a $K3$ surface. In characteristic not 2, this includes the Fermat quartic $X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$.

This partition of Kodaira dimension 0 surfaces into these four classes leads to questions as to how these classes interact or are related. One example of this is the classical ***Kummer construction***. It gives a process by which one may construct a $K3$ surface from an Abelian surface. It proceeds as follows: An Abelian surface A comes equipped with an involution map ι, which is the inversion morphism of the group scheme structure on A. That this map is self-inverse ensures that we may view this as an action of the group $\{\pm 1\}$ on A. General theory then ensures that in this case one has a quotient scheme $A/\{\pm 1\}$. This scheme is again a surface, however it comes with singularities. Supposing the base field k is of characteristic not 2, there are sixteen singularities which arise from sixteen fixed points of the $\{\pm 1\}$-action on A. As the quotient is again a surface, a minimal resolution of singularities exists [1]. These singularities are as mild as can be, specifically they are rational double points of type A_1 (see Proposition 3.3.3). Furthermore, the minimal resolution of $A/\{\pm 1\}$ is a $K3$ surface. The $K3$ surfaces which arise in this way are called ***Kummer surfaces***. In some sources the unresolved quotient is also called a Kummer surface.

By rational double point we mean the following: A singularity with resolution $\pi \colon Y \to Z$ is ***rational*** if $R^1\pi_{*Y} = 0$ where $R^1\pi_{*Y}$ is the first higher direct image sheaf. The point $z \in Z$ is said to be a ***double point*** if the local ring ${}_{Z,z}$ has multiplicity two. Rational double points are also called rational Gorenstein singularities, Du Val singularities or ADE singularities. They come in various types and are all classified by an associated Dynkin diagram called the dual graph. After passing to a suitable field extension, say an algebraic closure, this diagram describes a singularities' exceptional fiber as either A_r, D_r, E_6, E_7 or E_8, where each node corresponds to a copy of $\mathbb{P}^1$ with self intersection -2 [4, 20, 21, 22] and the edges describe how these components intersect. In the case of the Kummer construction, the 16 singularities are so mild that they are all A_1 singularities. That is, they may be resolved by a single blow-up, with each one having $\mathbb{P}^1$ with self-intersection -2 as exceptional divisor.

In characteristic 2, the classical situation becomes more complicated. Here the construction might fail to yield a $K3$-surface. One may take the quotient all the same, but the singularities can get worse. In a lot of cases, the quotient singularities are still rational, but Shioda and Katsura [65, 38] showed that the quotient obtains an elliptic singularity if and only if A is a so-called supersingular Abelian surface, and that in this case the resolution is not $K3$. It turns out that one instead gets a rational surface [63, Prop. 5.1-3], i.e. a surface birational to $\mathbb{P}^2$.

That this construction can fail in characteristic 2 begs the question if there are analogous constructions in characteristic 2 which do provide $K3$ surfaces, and somehow "fill the gap" for the supersingular case. Such "generalized Kummer constructions" are the overarching subject of this thesis, specifically in families. A construction was presented by Schröer [62], with continued work by Schröer and Kondo [46]. In both of these cases the surface A is replaced by a self-product $C \times C$ of the rational cuspidal curve $C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}]$. This is a natural choice, as the cuspidal curve arises as a degeneration of certain supersingular elliptic curves to characteristic 2, hence $C \times C$ comes from a degeneration of certain supersingular Abelian surfaces. The construction also replaces the group $\mathbb{Z}/2\mathbb{Z}$ by the group scheme of nilpotents of order 2, α_2, in the first article loc. cit. and either α_2 or the group scheme of second roots of unity, μ_2, in the second one. Outside of characteristic 2, the constant group scheme of $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to μ_2. As an action of the abstract group $\mathbb{Z}/2\mathbb{Z}$ may be viewed instead as an action by its associated constant group scheme, it makes sense to view the action in the classical Kummer construction as coming not from $\mathbb{Z}/2\mathbb{Z}$, but instead from μ_2. As such, taking an action by μ_2 when working with a characteristic 2 generalization is a very natural choice. In both cases one obtains only rational double point singularities in the quotient (except for a single degenerate case of the α_2-action). If we assume for simplicity that the base field is algebraically closed, the singularities are (omitting the degenerate case): Both cases have a D_4 singularity [62, Proposition 5.3], [46, Proposition 3.2] coming from the singular point lying on both factors of $C \times C$, what we call the ***quadruple point***. Then there are the fixed point singularities. For α_2 one gets either four D_4 singularities or two D_8 singularities, while the μ_2 case is extremely similar to the classical construction as it has sixteen A_1 singularities.

Now we come to the problem of families which is a central part this thesis. By a family, we mean a flat, proper morphism $\mathcal{X} \to S$ where S is a base scheme and $\mathcal{X}$ is an algebraic space. A family of curves is then a family such that each fiber is a curve. For example, the equation $y^2 = x^3 + 2$ defines a family of curves over $\mathbb{Z}$ with smooth fibers except for the primes 2 and 3 where the fiber is cuspidal. A family of Abelian surfaces is a proper flat morphism of finite presentation $A \to S$ such that A is also a relative group scheme. It can be shown (see Section 4.4) that given a family of Abelian surfaces, one may obtain a quotient family $A/\{\pm 1\}$ where the fibers are the quotients $A_s/\{\pm 1\}$. Furthermore, one may simultaneously resolve the singularities locally on the base, after finite base change. That is to say, the classical Kummer construction works locally in families up to a finite base change.

It is in then natural to ask if the generalized Kummer construction discussed above works in families, and to what extent. Consider a base $S = \operatorname{Spec} R$ of characteristic 2. The α_2-actions of [62] are determined by vector fields $(u^{-2}+r)D_u + (v^{-2}+s)D_v$ where $r, s \in R$ and u, v are the coordinates of each factor in $C \times C \times S$, see Section 2.4. If one looks at the induced α_2 action on each fiber, then it will change, depending on what the residue class of r, s is in a given residue field. So the action moves in the family. It is shown in the article loc.cit. that this generalized construction works in families after possibly a certain inseperable base change. That is to say, the fibers of the quotient family $(C_S \times_S C_S)/\alpha_2$ are the quotients of the fibers, and the family admits a simultaneous resolution of singularities after this mentioned base change.

The objective of this thesis is to investigate whether the generalized Kummer construction with μ_2 presented in [46] works in families. That is, whether one can take the quotient of our family over a general base, obtain a family where the fibers are the quotients arising in the construction over a field, and then get a simultaneous resolution of singularities, to obtain a family, where the fiber are the generalized Kummer surfaces of the fibers in the original family. So fix a base S which is a scheme over some fixed field k of characteristic 2. Let C be the cuspidal curve $\operatorname{Spec}_S S[u^2, u^3] \cup \operatorname{Spec}_S S[u^{-1}]$ over S. We consider the self product $C \times C$ with a diagonal group scheme action determined by a vector field

$$\delta = (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \tau u^{-1} + \lambda_0)D_{u^{-1}} + (\sigma_4 v^{-4} + \sigma_2 v^{-2} + \tau v^{-1} + \sigma_0)D_{v^{-1}},$$

where the $\sigma_i, \lambda_i, \tau$ are all global sections of the base S with λ_4, σ_4 and τ assumed to be units. That τ is a unit ensures that this vector field defines a μ_2 action in every fiber (otherwise τ could be zero in some fiber, which would mean the action degenerated to an α_2 action. The assumption on λ_4, σ_4 is to ensure normality of the fibers [46, Proposition 3.1].

There are two main questions to settle here: First, whether the fibers of the quotient $(C \times C)/\mu_2$ are $(C_s \times_{\operatorname{Spec} \kappa(s)} C_s)/\mu_2$ for $s \in S$, so that the quotient gives us a family of quotients. Secondly, whether this family admits a simultaneous resolution of singularities. We also analyze the singularities over a non-algebraically closed base. It turns out, that since μ_2 is a diagonalizable group scheme, forming quotients commutes with arbitrary base change, so that the first property is immediately satisfied, see Section 1.4 and 2.3. The second question takes a bit more work. This leads us into main result of the thesis, which is Theorem 5.3.1 of Chapter 5:

Theorem. *The quotient family $(C \times C)/\mu_2$ admits a simultaneous resolution of singularities over the base change*

$$S' = S \otimes_{k[\lambda_4,\lambda_2,\lambda_0,\sigma_4,\sigma_2,\sigma_0,\tau]} k\left[\sqrt[3]{\frac{\lambda^2}{\lambda_4^2}}, \sqrt[3]{\frac{\sigma^2}{\sigma_4^2}}, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\right].$$

Here α_i and β_i are the roots of $\lambda_4^2 a^4 + \lambda_2^2 a^2 + \tau^2 a + \lambda_0^2$ and $\sigma_4^2 b^4 + \sigma_2^2 b^2 + \tau^2 b + \sigma_0^2$ respectively, λ and σ are the canonical choice of representatives of λ_4 and σ_4 in k, and $\sqrt[3]{\frac{\lambda^2}{\lambda_4^2}}, \sqrt[3]{\frac{\sigma^2}{\sigma_4^2}}$ are any choice of third roots. This is a finite base change of degree at most $3^2(4!)^2 = 5184$. The only possible prime divisors of the degree of the base change are 2 and 3.

One then obtains the following as an immediate consequence

Corollary. *The generalized Kummer construction with μ_2 works in families after a finite base change of degree at most $3^2(4!)^2 = 5184$. The only possible prime divisors of the degree of the base change are 2 and 3.*

To prove the above theorem we analyze the singularities in two sets. First the one coming from the quadruple point and then the fixed point singularities. For the quadruple point singularity, it turns out one can emulate the analyzis of the quadruple point in [62, Proposition 5.4]. Namely, we will see that the deformation of an infinitesimal neighbourhood of the singularity is isomorphic to a constant deformation, after a base change adding third roots of λ_4^2 and σ_4^2. This implies that this singularity allows for simultaneous resolution. One could expect more trouble from the fixed point singularities. Indeed, for the α_2 construction in [62, Theorem 12.1] these singularities necessitate a large amount of additional techniques. We will see that no such techniques are necessary when the group scheme is μ_2. Instead we settle the question of the fixed point singularities using the same technique as for the quadruple point, by studying deformations. The base change necessary, is to ensure that the polynomial equations defining these fixed point singularities split, so that we do not have a differing number of singularities across different fibers. Using these we will arrive at the above result.

The Structure of the Thesis

The study of these generalized Kummer constructions involve quite a lot of notions, and equivalences. For example, to compute the quotients by μ_2 actions in Chapter 5 we interpret them using Lie algebras, but to argue that this quotient is a family of quotients, we use that a μ_2 action can be interpreted locally as giving a $\mathbb{Z}/2\mathbb{Z}$-grading.

I have attempted to make the thesis more or less self-contained in that most results taken as reference are stated in the text, with the reference given in the proof. This is done so that the reader will have a consistent language, terminology and notation throughout the references. Furthermore, to set the theory presented in the text into perspective, each section contains multiple examples of varying levels of complexity. If one wants to simply read and understand the main result of the thesis, one should skim Sections 1.1-1.4, focusing on anything involving μ_2, and then read Chapter 2, Section 3.1, Sections 4.1-4.2 and Chapter 5.

As the generalized Kummer constructions involve quotients by group scheme actions, Chapter 1 and 2 of the thesis are devoted to studying group schemes and their actions respectively. As the notions presented in these two chapters are so central, they take up nearly half of the thesis. The Sections 1.1 and 1.2 introduce the basic notions of group schemes as well as provide multiple examples. In Section 1.3 and 1.4 we study specific types of group schemes, namely those that are finite and diagonalizable. The group scheme μ_2 falls into both of these classes, and so these sections illuminate this group scheme from different angles. The final section of this chapter, 1.5, provides some more theoretical examples of group schemes, namely the Abelian varieties and the automorphism group scheme of a proper scheme.

In the next chapter, Section 2.1 introduces group scheme actions with some basic properties, while Section 2.2 gets into the meat of a for us central notion: A group scheme quotient. This section deals with existence questions and properties of the quotient. Then, in Section 2.3 we study how actions by diagonalizable group schemes may be reinterpreted in terms of gradings on rings. We see how this allows one to easily give examples of actions and provides a great tool for computing invariant rings and so also quotients. As a curiosity, this section also outlines how the classical Proj-construction of algebraic geometry may be interpreted as a quotient. Finally, Section 2.4 closes out the chapter with an introduction to the Lie algebra of a group functor and we study how one can view a group scheme action as a morphism of Lie algebras.

Chapter 3 is a collection of various topics associated with algebraic surfaces. Section 3.1 provides a basic introduction to the theory of rational double points and their associated Dynkin diagrams. The second section, 3.2, introduces the conditions (R_i) and (S_i) of Serre. These become relevant later, in Section 4.1, when base change properties of quotients are discussed. The final section of this chapter outlines in detail the classical Kummer construction in characteristic not 2.

In Chapter 4, we study families parametrized over a general base. Section 4.1 deals with quotients of families and the fibers of these. In Section 4.2 we study simultaneous resolutions and discuss the challenges that occur. This section also outlines the method used to study simultaneous resolutions in Chapter 5. In Section 4.4 the classical Kummer construction is studied in families.

Chapter 5 contains the main results of the thesis. If one is willing to accept a large amount of black-boxes (or is completely familiar with the prerequisites) it can be read as a self-contained piece. The first section, 5.1, outlines in brief the ideas of the work of Schröer [62], and Kondo and Schröer [46] that form the basis for the work done in this thesis. Section 5.2 forms the bulk of the chapter, and is new content. Section 5.3 then summarizes and collects the results of the previous chapter into the main results. The final section provides a forward looking perspective and outlines two further questions one could investigate.

Acknowledgements

I will briefly mention some of the many people who have in some way, shape, or form helped this thesis come to be:
First and foremost, I would like to thank my advisor Stefan Schröer and co-advisor Markus Reineke for guiding me through this whole process, as well as my colleagues Ivo Kroon, Jan Hennig, Thor Wittich, and Thuong Dang for discussion, mental sparring, and being the occasional rubber duck. A special thanks here to Ivo for our long discussions on both math and life in general, and to Jan for his willingness to double check arguments with me.

A great personal thanks to Ivo Kroon, Jan Hennig, and Josefine Alberte Sjöström for proofreading and comments during the forming of this thesis. Special thanks to Jan, for his careful proofreading of Chapter 5 and its many calculations.

I must thank certain people who are or were members of the department at the University of Copenhagen: Henrik Holm, for disucssing Cohen-Macaulay rings with me during my visit to Copenhagen this past December. Lars Halle, for first introducing me to algebraic geometry and advising my further studies in scheme theory. Fabien Pazuki, for advising me on my master's thesis. Ian Kiming, for advising me on my bachelor's thesis. I am especially grateful towards Fabien and Ian for their warm, honest, seemingly unending support in my life post-graduating from my masters.

Thank you to my cousin Jeppe, my aunt Lone and, of course, my dear mother Anne. Finally, I am forever indebted to my friends abroad who helped keep me sane through the last almost four years: Helene, Josefine, Johanna, Mathias, Mikkel, and Rinoa. Your contributions are less tangible, but no less important than those mentioned above.

While writing this thesis I have been funded by the Deutsche Forshungsgemeinschaft with a PhD position in the research training group *GRK2240: Algebro-Geometric Methods in Algebra, Arithmetic and Topology*.

Chapter 1

Group Schemes

In this chapter we introduce the basic theory of group schemes with examples, before focusing on finite and diagonalizable group schemes. For a general reference on group schemes, see [17] or the original French text [16].

We denote by S a base scheme. Any and all notions presented are considered relative to this base. For example, by "a morphism", we mean "a morphism *over* S" and "a scheme" is an S-scheme. We usually supress the base scheme, but once in a while we make it explicit for emphasis, as is the case with the first definition presented.

1.1 Basic Definition and Examples

Definition 1.1.1. A ***group scheme*** over S is a scheme G over S together with S-morphisms

$$m\colon G \times_S G \longrightarrow G, \qquad \iota\colon G \longrightarrow G, \qquad e\colon S \longrightarrow G,$$

such that for each S-scheme T, these morphisms endow $G(T)$ with the structure of a group. The morphisms m, i, and e are called the ***multiplication***, ***inverse*** and ***identity***, respectively.

Note that the 'neutral element' e is in fact a section. There are numerous equivalent ways of defining group schemes (over S), the shortest of which is to say it is "a group object in the category of S-schemes". One could also omit the morphisms m, ι and e, and simply require that a group scheme is a scheme such that its associated functor of points h_G is a functor of groups. Another equivalent definition, is that the morphisms m, i, and e satisfy the standard commutative diagrams of associativity, identity and inverse:

$$\begin{tikzcd} G \times G \times G \arrow[r, "(\mathrm{id},m)"] \arrow[d, "(m{,}\mathrm{id})"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \text{(associativity)}$$

$$\begin{tikzcd} G \times G \arrow[dr, "m"] & S \times G \arrow[l, "(e{,}\mathrm{id})"'] \arrow[d, "\mathrm{pr}_2"] \\ G \times S \arrow[u, "(\mathrm{id}{,}e)"] \arrow[r, "\mathrm{pr}_1"'] & G \end{tikzcd} \qquad \text{(the identity element)}$$

$$
\begin{tikzcd}
G \arrow[r, "(\iota,\mathrm{id})"] \arrow[d, "(\mathrm{id},\iota)"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\qquad \text{(two sided inverse)}
$$

Here (id, ι) is a shorthand for $G \xrightarrow{\Delta} G \times G \xrightarrow{\mathrm{id}\times\iota} G \times G$, where Δ is the diagonal. Note the particular case where the base is the spectrum of a field k. In this case the identity is determined by an honest closed rational point $e \in G(k)$. With the definition comes also an obvious notion of a subgroup scheme: A subscheme H in G is a subgroup scheme if $H(X)$ is functorially a subgroup of $G(X)$ for any scheme X.

Remark 1.1.2. An important tool when working with group schemes, is the fact that any sheaf on the site (Sch/S) is completely determined by the corresponding sheaf on the site (AffSch/S). In particular, this holds for any scheme. Whether one chooses the Zariski, étale, fppf, or so forth sites, is of no consequence [9, Exposé III Theorem 4.1]. In more "down-to-earth" terms, this means that any scheme is completely determined by its functor of points on affine schemes over S. In particular, it is enough to define G, m, i and e for all affine schemes over S. Furthermore, given a scheme X, to show that X may be endowed with the structure of a group scheme, it is enough to show that $X(T)$ may be given a functorial group structure for each affine scheme $T = \operatorname{Spec} R$ over S. In particular, given a functor of groups on (Sch/S), it is enough to show that the induced functor on (AffSch/S) is representable. Indeed, if a functor is representable at the level of affine schemes, then it is automatically a sheaf, and so extends to all schemes. This last bit is incredibly powerful, as we will see.

Remark 1.1.3 (Hopf Algebras)**.** Dual to the notion of group schemes, there is a corresponding notion in the realm of commutative algebra: The ***Hopf algebras***. A Hopf algebra is an R-algebra A equipped with morphisms $m^{\#}\colon A \to A \otimes_R A$, $e^{\#}\colon R \to A$ and $\iota^{\#}\colon A \to A$ called ***comultiplication***, ***counit*** and ***antipode*** satisfying commutative diagrams

$$
\begin{tikzcd}
A \otimes A \otimes A & & A \otimes A \arrow[ll, "(\mathrm{id},m^{\#})"'] \\
A \otimes A \arrow[u, "(m^{\#},\mathrm{id})"] & & A \arrow[ll, "m^{\#}"] \arrow[u, "m^{\#}"']
\end{tikzcd}
\qquad \text{(coassociativity)}
$$

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "(e^{\#},\mathrm{id})"] \arrow[d, "(\mathrm{id},e^{\#})"'] & & R \otimes A \\
A \otimes R & & A \arrow[ll] \arrow[u] \arrow[ull, "m^{\#}"']
\end{tikzcd}
\qquad \text{(coidentity)}
$$

$$
\begin{tikzcd}
A & A \otimes A \arrow[l] & A \otimes A \arrow[l, "(\iota^{\#},\mathrm{id})"'] \\
A \otimes A \arrow[u] & R \arrow[ul] & \\
A \otimes A \arrow[u, "(\mathrm{id},\iota^{\#})"] & & A \arrow[ll, "m^{\#}"] \arrow[ul, "e^{\#}"'] \arrow[uu, "m^{\#}"']
\end{tikzcd}
\qquad \text{(antipode)}
$$

where in the antipode diagram the homomorphism $R \to A$ is the structure homomorphism and the homomorphisms $A \otimes A \to A$ is the multiplication. We note that these diagrams are just the dual of those following Definition 1.1.1. That is, the notions of Hopf algebras and affine group schemes are anti-equivalent. With this anti-equivalence in hand, we will often, as an abuse of terminology, also refer to the homorphisms defining the Hopf algebra structure as multiplication, identity and inverse, as they are equivalent to these for the Group scheme structure.

This anti-equivalence can be generalized further. A ***sheaf of Hopf*** ${}_S$***-algebras*** is an ${}_S$-algebra $\mathcal{A}$, such that for each open U in S, the algebra $\Gamma(U, \mathcal{A})$ has the structure of a Hopf algebra compatible with the restriction maps of $\mathcal{A}$.

Proposition 1.1.4. *The category of group schemes affine over S is anti-equivalent to the category of quasi-coherent sheaves of Hopf* ${}_S$*-algebras.*

Proof: Recall that we have a one-to-one correspondence between schemes X affine over S and quasi-coherent sheaves of ${}_S$-algebras: If $f\colon X \to S$ is the structure morphism, we denote by $\mathcal{A}(X) = f_{*X}$ and we have $X = \mathrm{Spec}_S\ \mathcal{A}(X)$ where Spec_S denotes the relative spectrum. So what we need to prove is that X is canonically a group scheme if f_{*X} is a sheaf of Hopf algebras and vice versa. Suppose f_{*X} is a sheaf of Hopf algebras. Given an affine open covering $\{U_i\}$ of S, X is covered by $V_i = \mathrm{Spec}\,\Gamma(U_i, \mathcal{A}(X))$. As $\Gamma(U_i, \mathcal{A}(X))$ is a Hopf algebra, each of these spectra are affine group schemes (over U_i). The fact that the Hopf algebra structures are compatible with restriction maps, implies that the morphisms determining the group scheme structures of $\Gamma(U_i, \mathcal{A}(X))$ glue to global morphisms determining a group scheme structure of X. For the sake of brevity, we give the details of how one gets the section of X, and then leave the rest to the reader. Now, each V_i is a group scheme over U_i, meaning it comes with a section $e_i\colon U_i \to V_i$. As the Hopf algebra structures are compatible on overlaps, we have $e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}$, which implies that these glue to a global morphism $e\colon S \to X$.

Suppose now X is an affine group scheme over S. We must show that the group scheme structure of X induces a group scheme structure on each $V_i = \mathrm{Spec}\,\Gamma(U_i, \mathcal{A}(X))$. As f is affine, $f^{-1}(U_i) = V_i$. Furthermore, the structure morphism of $X \times X$ is $f \times f$ and

$$(f \times f)^{-1}(U_i) = V_i \times V_i.$$

This implies that the multiplication $X \times X \to X$, which is a morphism over S, restricts to a morphism $m_i\colon V_i \times V_i \to V_i$. Similarly, we obtain $e_i\colon U_i \to V_i$ and $\iota_i\colon V_i \to V_i$. That these satisfy the desired diagrams then follows from being restrictions of the corresponding morphisms of X.

□

Before we get further into the general theory, let us consider a few examples, to get a feel for what group schemes are. As we will see in these examples, the preceding remark is particularly useful when defining concrete group schemes, as it is often easier to work with a functor concretely on affine schemes than on general ones. Indeed, it all boils down to defining a functor of groups on (AffSch/S) and then proving that the functor is representable i.e. is isomorphic to $h_X = \mathrm{Hom}(-, X)$ for some scheme X. The next three examples also show how ubiquitous group schemes are. Indeed, these three well known functors from (Ring) to (Ab) extend to (Sch) and are in fact representable. We present all of these over $\mathbb{Z}$ to ease notation slightly, but they have obvious analogues over any scheme S. These are obtained either by base change (as we will see in Proposition 1.2.6, the base change of any group scheme yields a group scheme over the new base), or may be constructed correctly in a manner analogous to what we present below. However, if constructing directly over a general base S, one must replace $\mathbb{Z}$ by ${}_S$, the ring R by a quasi-coherent sheaf $\mathcal{F}$, and the spectrum by the relative spectrum.

Example 1.1.5 (The multiplicative group scheme)**.** Consider the functor from (Sch) to (Ab) which to a scheme T associates the abelian group $\Gamma(T, {}^{\times}_{T})$ of units of global sections. We claim that it is representable by the scheme $\operatorname{Spec}\mathbb{Z}[x, x^{-1}]$, and so defines a group scheme. This scheme is called the ***multiplicative group scheme***, and we denote it by $\mathbb{G}_m$. In particular, for an affine scheme $\mathbb{G}_m(R) = R^{\times}$, which explains its name. By base change, this defines a group scheme over any scheme S, which is also denoted $\mathbb{G}_m$ when no confusion is possible.

As noted in Remark 1.1.2, we need only consider affine schemes. So let R be a commutative ring. We must show we have a natural isomorphism

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[x, x^{-1}], R) \cong \Gamma(\operatorname{Spec} R, {}^{\times}_{\operatorname{Spec} R}) = R^{\times}.$$

Now, to give a homomorphism $\mathbb{Z}[x, x^{-1}] \to R$ amounts to giving an element $u \in R$ as the image of x, which must be a unit since x is. Conversely, any such element defines a homomorphism. This defines an isomorphism. To see that it is natural is routine, so our functor is representable.

Let us concretely describe the multiplication map m on $\operatorname{Spec}\mathbb{Z}[x, x^{-1}]$. Denote by φ_u the homomorphism mapping x to $u \in R^{\times}$. Then the group operation on $\operatorname{Hom}(\mathbb{Z}[x, x^{-1}], R)$ is simply

$$\varphi_u \cdot \varphi_{u'} = \varphi_{uu'}.$$

Identifying $\operatorname{Hom}(\mathbb{Z}[x, x^{-1}], R) \times \operatorname{Hom}(\mathbb{Z}[x, x^{-1}], R)$ with $\operatorname{Hom}(\mathbb{Z}[x, x^{-1}] \otimes \mathbb{Z}[x, x^{-1}], R)$, the group structure should coincide with precomposing with $m^{\#}$, i.e. we need an equality

$$\varphi_r \cdot \varphi_{r'} = (\varphi_r \otimes \varphi_{r'}) \circ m^{\#},$$

as homomorphisms $\mathbb{Z}[x, x^{-1}] \to R$. From this description we see that the multiplication must be given by

$$m^{\#} \colon \mathbb{Z}[x, x^{-1}] \longrightarrow \mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}] \qquad x \longmapsto x \otimes x.$$

Similarly, the inverse and identity are

$$\begin{array}{rcl} \iota^{\#} \colon \mathbb{Z}[x, x^{-1}] & \longrightarrow & \mathbb{Z}[x, x^{-1}] \\ x & \longmapsto & x^{-1}, \end{array} \quad \text{and} \quad \begin{array}{rcl} e^{\#} \colon \mathbb{Z}[x, x^{-1}] & \longrightarrow & \mathbb{Z} \\ x & \longmapsto & 1. \end{array}$$

Note that $\mathbb{Z}[x, x^{-1}] = \mathbb{Z}[\mathbb{Z}]$. In Section 1.4 we will see that $\mathbb{G}_m$ is just a special example of the larger class of so-called diagonalizable group schemes which all arise in similar fashion from spectra of group rings.

Example 1.1.6 (The additive group scheme)**.** We now instead look at the functor from Sch to Ab which to T associates the abelian group $(\Gamma(T, {}_{T}), +)$. We claim that this functor is represented by the scheme $\mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[x]$. The resulting group scheme is called the ***additive group scheme*** and is denoted $\mathbb{G}_a$ to distinguish it from the affine line, which is 'just' a scheme with no additional structure. As before, this defines a group scheme over any S via base-change.

The argument is like before. We must show $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[x], R) \cong (R, +)$ in a natural way. But to give a homomorphism $\mathbb{Z}[x] \to R$ is equivalent to giving an element $r \in R$ as the image of x. This is natural and so the functor described is representable by $\operatorname{Spec}\mathbb{Z}[x]$.

As before, we obtain explicit expressions for the multiplication, inverse and identity:

$$m^{\#} \colon \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x], \quad x \longmapsto x \otimes 1 + 1 \otimes x$$

$$\begin{array}{rrcl} \iota^{\#} \colon & \mathbb{Z}[x] & \longrightarrow & \mathbb{Z}[x] \\ & x & \longmapsto & -x \end{array} \quad \text{and} \quad \begin{array}{rrcl} e^{\#} \colon & \mathbb{Z}[x] & \longrightarrow & \mathbb{Z} \\ & x & \longmapsto & 0 \end{array}$$

Example 1.1.7 (The general linear group scheme)**.** Now consider for $n \geq 1$ the functor GL_n from (Sch) to (Ab) which to T associates the group $\mathrm{GL}_n(\Gamma(T, {}_T))$. This is in fact also representable, and so defines a group scheme.

Like the previous two examples, this group scheme is affine. To give the concrete ring, let $d = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n T_{i\sigma(i)}$ i.e. d is the determinant polynomial. We then consider the polynomial ring over $\mathbb{Z}$, localized at d, $\mathbb{Z}[T_{ij}, 1/d]_{1 \leq i,j \leq n}$. We claim this represents GL_n. As before, we regard the set

$$\mathrm{Hom}\left(\mathbb{Z}\left[T_{ij}, \frac{1}{d}\right]_{1 \leq i,j \leq n}, R\right).$$

Then note that giving such a homomorphism is equivalent to giving an element $r_{ij} \in R$ for each T_{ij}, such that the image of d is a unit. Now, the image of d is, by definition,

$$\det \begin{pmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \dots & r_{nn} \end{pmatrix}.$$

So, to give such a homomorphism, is equivalent to giving an invertible matrix with entries in R i.e. an element of $\mathrm{GL}_n(R)$. Hence GL_n is representable by the scheme $\mathrm{Spec}\,\mathbb{Z}[T_{ij}, 1/d]_{1 \leq i,j \leq n}$. We call it the ***general linear group scheme***. Note that the underlying scheme is actually a distinguished open subset of $\mathbb{A}^{n^2}$, namely the set $D(d)$. As before, we give concrete descriptions of the multiplication, inverse and identity morphisms:

$$\begin{array}{rrcl} m^{\#}\colon & \mathbb{Z}[T_{ij}, 1/d] & \longrightarrow & \mathbb{Z}[T_{ij}, 1/d] \otimes_{\mathbb{Z}} \mathbb{Z}[T_{ij}, 1/d] \\ & T_{ij} & \longmapsto & \sum_{l=1}^n (T_{il} \otimes T_{lj}). \end{array}$$

$$\begin{array}{rrcl} \iota^{\#}\colon & \mathbb{Z}[T_{ij}, 1/d] & \longrightarrow & \mathbb{Z}[T_{ij}, 1/d] \\ & T_{ij} & \longmapsto & \dfrac{(-1)^{i+j}}{d} \displaystyle\sum_{\substack{\sigma \in S_n \\ \sigma(i)=i}} \left(\mathrm{sgn}(\sigma) \prod_{\substack{s=1 \\ s \neq j}}^{n} T_{s\sigma(s)} \right) \\ e^{\#}\colon & \mathbb{Z}[T_{ij}, 1/d] & \longrightarrow & \mathbb{Z} \\ & T_{ij} & \longmapsto & \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \end{array}$$

These formulas are more involved than those of the previous examples, and may at first look seem random. But interpreting the first two as the multiplication and inverse of GL_n, we see that the multiplication corresponds to mapping the (i, j)'th entry of a matrix to the (i, j)'th entry of a product. Similarly, the inverse may be viewed as mapping the (i, j)'th entry to the (i, j)'th entry of the inverse matrix via Cramer's rule. Note the special case $n = 1$ where $\mathrm{GL}_1 = \mathbb{G}_m$.

Example 1.1.8 (Subgroup schemes of GL_n)**.** Let us introduce two subgroup schemes of GL_n. The definitions are analogous to those in the preceding examples, so we will be brief. First we have the *special linear group scheme* SL_n which associates to a ring R the special linear group of that ring $\mathrm{SL}_n(R)$. The underlying scheme of this is $\mathrm{Spec}\,\mathbb{Z}[T_{ij}]_{1 \leq i \leq n}/(d-1)$, where d is the determinant polynomial, which is a closed subscheme of both $\mathbb{A}^{n^2}$ and $\mathrm{Spec}\,\mathbb{Z}[T_{ij}, 1/d]_{1 \leq i \leq n}$. As such, SL_n defines a closed subgroup scheme in GL_n.

Secondly, we have the diagonal group scheme D_n which to a ring R associates the invertible diagonal matrices over R. Its underlying scheme is the closed subscheme in $D(d) = \mathrm{Spec}\,\mathbb{Z}[T_{ij}, 1/d]_{1 \leq i,j \leq n}$ defined by the ideal $(T_{ij})_{i \neq j}$. This gives a closed subgroup scheme in GL_n.

Having studied these examples, let us turn our attention to a very specific, but also very important, class of group schemes. Given a group M and a scheme S we may construct a scheme H_S by setting

$$M_S = \coprod_{\sigma \in M} S.$$

Take notice of the particular case where S is the spectrum of a field. In this case M_S becomes a discrete space consisting of closed points, one for each $\sigma \in M$. We will return to this case later. This scheme inherits the structure of a group scheme over S from the group structure of M in the following way: Explicitely the inverse maps the σ component to the σ^{-1} component, and the identity maps S to the component corresponding to 1_M. The multiplication is defined on each component of $M_S \times_S M_S \to M_S$ by mapping the 'component' $S \times_S S \cong S$ corresponding to (σ, σ') identically to the $\sigma\sigma'$ component of M_S. Working over $\mathbb{Z}$, this way of associating a group scheme to a group in fact embeds the category (Grp) as a full subcategory of the category of group schemes.

Definition 1.1.9. The group scheme M_S is called the ***constant group scheme*** over S associated to H.

Proposition 1.1.10. *Given a group M and a scheme T, the T-rational points of the constant group scheme H_S consists of all locally constant functions $f\colon T \to M$.*

Proof: By definition, a T-rational point of M_S is just a morphism $T \to M_S$ or equivalently a T-morphism $f\colon T \to M_S \times_S T = M_T$. As a space H_T consists of a copy of T_σ for each element $\sigma \in M$. Each such copy is open in H_T so by continuity $U_\sigma = f^{-1}(T_\sigma)$ is open in T. These U_σ form a disjoint open cover of T and it follows that defining $g(t) = \sigma$ if $t \in U_\sigma$ gives a locally constant function $g\colon T \to M$. Conversely, given such a g, it defines a morphism of schemes $f\colon T \to M_T$ by simply mapping each set U on which g is constant isomorphically to its copy inside the T_σ part of M_T. □

In particular, if $T = \operatorname{Spec} k$ for some field k we have $M_S(k) = M_{\operatorname{Spec} k}(k) = M$. To make the multiplication of a constant group scheme more explicit, let us consider the following example.

Example 1.1.11. Suppose the base is $S = \operatorname{Spec} \mathbb{Z}$ and take the finite group $M = \mathbb{Z}/2\mathbb{Z}$. To distinguish M from the copies of $\mathbb{Z}$, denote the elements of M by σ, τ, with σ being the neutral element. Then $M_S = \operatorname{Spec}(\mathbb{Z}_\sigma \times \mathbb{Z}_\tau)$ and

$$M_S \times_S M_S = \operatorname{Spec}((\mathbb{Z}_\sigma \times \mathbb{Z}_\tau) \otimes_{\mathbb{Z}} (\mathbb{Z}_\sigma \times \mathbb{Z}_\tau)) \cong \operatorname{Spec}(\mathbb{Z}_{\sigma\sigma} \times \mathbb{Z}_{\tau\sigma} \times \mathbb{Z}_{\sigma\tau} \times \mathbb{Z}_{\tau\tau}),$$

where, as a matter of notation, we write $\mathbb{Z}_g$ for the component corresponding to $g \in G$. As $\sigma = \sigma\sigma = \tau\tau$ and $\tau = \tau\sigma = \sigma\tau$, it follows that the multiplication should correspond to

$$\begin{array}{rcl} \mathbb{Z}_\sigma \times \mathbb{Z}_\tau & \longrightarrow & \mathbb{Z}_{\sigma\sigma} \times \mathbb{Z}_{\tau\sigma} \times \mathbb{Z}_{\sigma\tau} \times \mathbb{Z}_{\tau\tau} \\ (n, m) & \longmapsto & (n, m, m, n). \end{array}$$

Passing through the natural isomorphism

$$\begin{array}{rcl} \mathbb{Z}_{\sigma\sigma} \times \mathbb{Z}_{\tau\sigma} \times \mathbb{Z}_{\sigma\tau} \times \mathbb{Z}_{\tau\tau} & \longrightarrow & (\mathbb{Z}_\sigma \times \mathbb{Z}_\tau) \otimes_{\mathbb{Z}} (\mathbb{Z}_\sigma \times \mathbb{Z}_\tau) \\ (\alpha, \beta, \gamma, \delta) & \longmapsto & (\alpha, \beta) \otimes (1, 0) + (\gamma, \delta) \otimes (0, 1), \end{array}$$

we arrive at the multiplication map

$$\begin{array}{rcl} \mathbb{Z}_\sigma \times \mathbb{Z}_\tau & \longrightarrow & (\mathbb{Z}_\sigma \times \mathbb{Z}_\tau) \otimes_{\mathbb{Z}} (\mathbb{Z}_\sigma \times \mathbb{Z}_\tau) \\ (n, m) & \longmapsto & (n, m) \otimes (1, 0) + (m, n) \otimes (0, 1). \end{array}$$

The inverse map is simply interchanging factors

$$\iota^{\#}\colon \mathbb{Z}_\sigma \times \mathbb{Z}_\tau \longrightarrow \mathbb{Z}_\sigma \times \mathbb{Z}_\tau, \quad (n,m) \longmapsto (m,n),$$

while the unit is projecting from the first factor

$$e^{\#}\colon \mathbb{Z}_\sigma \times \mathbb{Z}_\tau \to \mathbb{Z}, \quad (n,m) \longmapsto n.$$

The above observations generalize, and so in general, if the base is affine $S = \operatorname{Spec} R$, and M is cyclic of order n, the multiplication map of the constant group scheme M_S is given explicitely as $(r_1, r_2, \ldots, r_n) \mapsto \sum_{i=0}^{n-1} \alpha^{-i}(r_1, r_2, \ldots, r_n) \otimes \alpha^i(1,0,0,\ldots,0)$ where α is the permutation $(123\cdots n)$. The unit is projection and inverse is interchanging of factors according to the group law of M.

As a final observation on constant group schemes, we note that given two finite groups M and N we have $(M \times N)_S = M_S \times N_S$. In particular, if M is finitely generated abelian $M = \mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_m\mathbb{Z}$ we obtain a corresponding decomposition of the associated constant group scheme

$$M_S = \mathbb{Z}_S^r \times \mathbb{Z}/n_1\mathbb{Z}_S \times \cdots \times \mathbb{Z}/n_m\mathbb{Z}_S.$$

Proposition 1.1.12. *Every group scheme G over a field k is separated.*

Proof: A rational point $\operatorname{Spec} k \to G$ is a closed immersion, and the diagonal map $\Delta\colon G \to G \times G$ may be obtained as the base change of the identity section $e\colon \operatorname{Spec} k \to G$ by the structure morphism $G \to \operatorname{Spec} k$. $\square$

1.2 Morphisms and More Examples

Having now a number of examples fresh in mind, we establish a bit more formalism. First, we should define what a morphism of group schemes is. In simple terms, it is just a morphism of schemes which is compatible with the multiplication maps. For the sake of formalism, we put this in a strict definition:

Definition 1.2.1. Let G and H be group schemes. A ***morphism of group schemes*** $f\colon G \to H$ is a morphism of schemes $f\colon G \to H$ such that for every scheme T, the induced map $f\colon G(T) \to H(T)$, obtained by post-composition, is a homomorphism of groups. Equivalently, the following diagram is commutative

$$\begin{tikzcd} G \times G \arrow[r, "f \times f"] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_H"] \\ G \arrow[r, "f"] & H. \end{tikzcd}$$

The final bit of this definition makes it easy to define morphisms of certain group schemes. Indeed, by the Yoneda lemma, to define a morphism of schemes, it is enough to define a morphism between their functors of points. As such, to define a morphism of group schemes G and H, we need only give a functorial homomorphism $G(T) \to H(T)$.

Suppose we have a morphism of group schemes $f\colon G \to H$. Then there is an obvious functor associated to this, namely that which to each scheme T associates the kernel of $f_T\colon G(T) \to H(T)$. This is actually representable by a scheme.

Definition 1.2.2. Let G and H be group schemes over a scheme S, and let $f\colon G \to H$ be a morphisms of group schemes. The **kernel** of f, denoted $\ker(f)$ or $\ker f$, is the scheme defined as the pullback of G along the identity $e_H\colon S \to H$

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d] & G \arrow[d] \\
S \arrow[r, "e_H"] & H.
\end{tikzcd}$$

Proposition 1.2.3. *Let G and H be group schemes over a scheme S, and let $f\colon G \to H$ be a morphism of group schemes. Then $\ker(f)(T) = \ker(G(T) \to H(T))$ for any S-scheme T. In particular, $\ker(f)$ has a canonical structure as a group scheme which is a subgroup scheme in G. If H is separated, $\ker(f)$ is closed in G.*

Proof: By definition $\ker(f)$ is the fiber over the identity, so in particular $\ker(f)(T)$ consists of those T valued points of G which map to the identity in $H(T)$. This gives the description of the T-points. For the last point, suppose H is separated. Then the section $e_H\colon S \to H$ is a closed embedded because the composition $S \to H \to S$ is the identity, hence a closed embedding, and $H \to S$ is separated. As $\ker(f) \to G$ is by definition the base change of $e_H\colon S \to H$, it follows that it must be a closed embedding in this case.

□

The fact that any kernel of group schemes with separated target is closed in particular implies that kernels of affine group schemes are again affine. This is the case for the following two examples which give two important group schemes that arise as kernels. The first one is especially important for us as it is the main group scheme considered in Chapter 5.

Example 1.2.4 (n'th roots of unity)**.** Consider the group scheme $\mathbb{G}_m$ and fix an $n \in \mathbb{Z}$. For each scheme T, we define a homomorphism $\mathbb{G}_m(T) \to \mathbb{G}_m(T)$ by mapping $g \in \mathbb{G}_m(T) = \Gamma(T, {}^{\times}_{T})$ to g^n. This is obviously functorial in T, and so defines a morphism of group schemes $\mathbb{G}_m \to \mathbb{G}_m$. As a shorthand, we simply write $\mathbb{G}_m \xrightarrow{g \mapsto g^n} \mathbb{G}_m$. Now, by Proposition 1.2.3 the kernel of this homomorphism is a group scheme, and we denote it by μ_n. The definition of μ_n as the kernel of $g \mapsto g^n$ directly implies that

$$\mu_n(T) = \{f \in \Gamma(T, {}^{\times}_{T}) \mid f^n = 1\}.$$

Hence, we call μ_n the ***scheme of n'th roots of unity***. It is a closed subgroup scheme of $\mathbb{G}_m$. In Proposition 1.3.2 we will see that its underlying scheme is $\mathbb{Z}[x]/(x^n - 1)$. As it is a subgroup scheme of $\mathbb{G}_m$, its multiplication, inverse and neutral element maps are simply the same as those given for $\mathbb{G}_m$ in Example 1.1.5.

The group scheme μ_n is absolutely central to the contents of this thesis. Indeed, we are primarily interested in quotients by actions of this scheme. As we will see later, μ_p is non-canonically isomorphic to $(\mathbb{Z}/p\mathbb{Z})_S$ whenever S is not of characteristic p and has primitive p'th roots of unity. We will return to studying μ_n, when we study finite group schemes in general.

Example 1.2.5 (q'th order nilpotent elements)**.** Suppose our base scheme S has characteristic $p > 0$, and fix a prime power $q = p^n$. Consider then the functor which to an S-scheme T associates

$$\alpha_q(T) = \{f \in \Gamma(T, {}_{T}) \mid f^q = 0\}.$$

As the characteristic of S is p, we have $(f+g)^p = f^q + g^q$ and so this set actually comes with a group structure by addition. This is clearly functorial, giving us a functor of groups. In fact it is a group scheme by Proposition 1.2.3, as it may be realized as the kernel of $\mathbb{G}_a \xrightarrow{g \mapsto g^q} \mathbb{G}_a$. This α_q is called the ***scheme of q'th nilpotent elements*** and is thus a closed subgroup scheme of $\mathbb{G}_a$. Note that it only exists as a group scheme in positive characteristic. For $S = \operatorname{Spec} R$, we will see in Proposition 1.3.3 the underlying scheme of α_p is $R[x]/(x^p)$. The multiplication, inverse and identity are then given on the level of Hopf algebras by extending those of $R[x]$ to the quotient, see Example 1.1.6.

These two group schemes are closely linked. In fact, we will see in Section 1.3 that their underlying schemes are isomorphic and so they are distinguished only by their group scheme structures. While the description of the kernel in Proposition 1.2.3 tells us it has the structure of a group scheme, the fact that it is defined as the pull-back of a group scheme, actually gives us that fact for free, as the following result shows.

Proposition 1.2.6. *Let G be a group scheme and T a scheme. Then the base change $G_T = G \times T$ is a group scheme over T. Furthermore, for any T-scheme Z, viewed as an S-scheme via $Z \to T \to S$, the natural isomorphism $G_T(Z) \cong G(Z)$ of sets is also an isomorphism of groups.*

Proof: The morphisms m_T, ι_T and e_T are all obtained by base-changing m, ι and e respectively along $T \to S$. For example, m_T is obtained via the universal property described by the following diagram

$$\begin{tikzcd}
G_T \times_T G_T \cong G \times_S G \times_S T \arrow[r, "\mathrm{pr}_1 \times \mathrm{pr}_1"] \arrow[dr, dashed, "\exists! m_T"] \arrow[ddr, bend right, "\mathrm{pr}_3"'] & G \times G \arrow[dr, "m"] & \\
& G_T \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & G \arrow[d] \\
& T \arrow[r] & S
\end{tikzcd}$$

That m_T, ι_T and e_T define a group scheme structure on G_T is simply the fact that base change respects composition, i.e. the diagrams of Definition 1.1.1 remain commutative after base-change.

For the second part, we note that, in general, the natural isomorphism $G_T(Z) \to G(Z)$ maps a T-morphism $f\colon Z \to G_T$ to the S-morphism $pr_1 \circ f$. Now, the commutativity in the top parallelogram of the previous diagram, i.e.

$$\begin{tikzcd}
G_T \times_T G_T \arrow[r, "\mathrm{pr}_1 \times \mathrm{pr}_1"] \arrow[d, "m_T"'] & G \times G \arrow[d, "m"] \\
G_T \arrow[r, "\mathrm{pr}_1"] & G
\end{tikzcd}$$

is exactly the requirement that $G_T(Z) \to G(Z)$ is a homomorphism.

$\square$

In chapter 5 we will be working in positive characteristic. So let us discuss some of the unique aspects of working with group schemes in characteristic $p > 0$, starting with the Frobenius.

Remark 1.2.7 (The absolute and relative Frobenius)**.** Suppose now that our base is a field k of characteristic $p > 0$. Then each k-algebra R has a **Frobenius homomorphism** $F_R\colon R \to R$ defined by $F(r) = r^p$. Note that this is not necessarily a k-homomorphisms, but it is a homomorphisms over $\mathbb{F}_p$. However, denote by ${}_F R$ the k-algebra with the underlying ring R, but structure homomorphism $k \xrightarrow{F_k} k \to R$. Then the Frobenius defines a homomorphism of k-algebras $F_R\colon {}_F R \to R$. In the case of schemes, any X over k comes equipped with the so-called **absolute Frobenius** morphism $F_X\colon X \to X$, which we again note is not necessarily a k-morphism, but is an $\mathbb{F}_p$-morphism. It is the identity on topological spaces, while as a map of sheaves ${}_X \to {}_X$, it maps $a \mapsto a^p$. Given an $\mathbb{F}_p$-morphism $f\colon X \to Y$ of schemes over k, the absolute Frobenius fits in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y. \end{tikzcd}$$

Viewing $\operatorname{Spec} k$ as a scheme over $\operatorname{Spec} k$ via the absolute Frobenius, we obtain a k-scheme $X^{(p)}$ as the base change of X to $\operatorname{Spec} k$ along the absolute Frobenius

$$\begin{tikzcd} X^{(p)} = X \times_k \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, "F_{\operatorname{Spec} k}"] & \operatorname{Spec} k. \end{tikzcd}$$

We then view $X^{(p)}$ as a scheme over k via the second projection. The universal property of the fibered product then induces a k-morphism called the **relative Frobenius**, which we denote $F_{X/k}$, via

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists! F_{X/\operatorname{Spec} k}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & \operatorname{Spec} k \arrow[r, "F_{\operatorname{Spec} k}"'] & \operatorname{Spec} k. \end{tikzcd}$$

Note that we could actually have replace $\operatorname{Spec} k$ by any base S of characteristic p. This gives a notion of a relative Frobenius $F_{X/S}$ over any such S. Note also that when viewed as maps of topological spaces, the relative Frobenius and the projection $X^{(p)} \to X$ are homeomorphisms. As such, the Frobenius morphisms are, in a certain sense, purely algebraic in nature. Finally, we highlight the special case where we consider a group scheme G over S. Then Proposition 1.2.6 tells us that $G^{(p)}$ is again a group scheme. In fact, the relative Frobenius is a group scheme morphism as we know prove.

Proposition 1.2.8. *Let S be a base of characteristic $p > 0$ and let G be a group scheme over S. Then the relative Frobenius $F_{G/S}\colon G \to G^{(p)}$ is a morphism of group schemes.*

Proof: Recall that given a k-algebra R, we obtain a homomorphism of k-algebras $F_R\colon {}_F R \to R$. In particular, we obtain a homomorphism of groups

$$G(R) \xrightarrow{G({}_F R)} G({}_F R)$$

We then make two claims:

1. We have a natural isomorphism of groups $\varphi\colon G^{(p)}(R) \longrightarrow G({}_F R)$ given by $f \mapsto \mathrm{pr}_1 \circ f$;

2. We have a commutative diagram of maps of sets

$$\begin{tikzcd} G(R) \arrow[r, "G(F_R)"] \arrow[dr, "F_{G/k}(R)"'] & G({}_F R) \\ & G^{(p)}(R) \arrow[u, "\varphi", "\wr"'] \end{tikzcd}$$

Suppose we have proven these claims. Then $F_{G/k}(R)$ will be a group homomorphism since $F_{G/k}(R)) = \varphi^{-1} \circ G(F_R)$ and each of these are group homomorphisms.

Claim 1: This is a special case of the second part of Proposition 1.2.6. Indeed, ${}_F R$ is just another way of writing R viewed as a k-algebra via $k \xrightarrow{F_k} k \to R$.

Claim 2: The statement to be proven is essentially as follows: Given a morphism $f\colon \operatorname{Spec} R \to G$ we obtain an equality of morphism $\operatorname{Spec} {}_F R \to G$

$$\mathrm{pr}_1 \circ F_{G/k} \circ f = f \circ F_R$$

where the projection is on $G^{(p)}$. To see this, we recall that the absolute Frobenius fits in the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "F_R"] \arrow[d, "f"'] & \operatorname{Spec} R \arrow[d, "f"] \\ G \arrow[r, "F_G"'] & G \end{tikzcd}$$

that is, $f \circ F_R = F_G \circ f$. But by the universal property of $G^{(p)}$ we have $\mathrm{pr}_1 \circ F_{G/k} = F_G$, so we get

$$\mathrm{pr}_1 \circ F_{G/k} \circ f = F_G \circ f = f \circ F_R,$$

as desired. □

The group schemes arising as kernels of iterations of the relative Frobenius are aptly named ***Frobenius kernels***. As we have just seen, both μ_p and α_p are examples of these. Such schemes also have a different name:

Definition 1.2.9. Suppose S is of characteristic $p > 0$. A group scheme G over S is said to be ***of height*** $\leq r$ for $r \geq 0$ if the r'th iterated relative Frobenius map $F^r_{G/S}$ is trivial on G. In particular G is ***of height*** ≤ 1 if it is a kernel for $F_{G/S}$.

In this terminology μ_p and α_p are group schemes of height ≤ 1. Before moving on to finite group schemes, we finish out the section with two results and an example which highlight the difference between group schemes in 0 and positive characteristic. From [59, Premiere Partie, V, Corollaire 3.9] we obtain the following:

Proposition 1.2.10. *Suppose $S = \operatorname{Spec} k$ is of characteristic 0. Then every group scheme over S is geometrically reduced.*

The examples μ_p and α_p are non-reduced in characteristic $p > 0$ and so this result is false in positive characteristic. One could hope that at least the reduced subscheme was a group scheme which one could then work with. However, this is not always the case as the following example (which is supposedly due to Raynaud) shows.

Example 1.2.11. Fix a base field k and consider the additive group $\mathbb{G}_a$. Take the group scheme morphism $\varphi\colon \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ which is defined on points by $(x, y) \mapsto x^p + ty^p$ for some $t \in k^\times$. The kernel N of this is again a group scheme by Proposition 1.2.3. On points $N(R)$ consists of all pairs $(x, y) \in \mathbb{G}_a(R) \times \mathbb{G}_a(R)$ satisfying $x^p + ty^p = 0$. It is affine with underlying scheme $k[x, y]/(x^p + ty^p)$ so it is irreducible. Now, we have to cases, based on whether or not t is a p'th power in k. The simple case, is when $t \in (k^\times)^p$ with $t'^p = t$. Then

$$k[x, y]/(x^p + ty^p) \cong k[x, y]/\left((x + t'y)^p\right) \cong k[x] \otimes_k k[x + t'y]/\left((x + t'y)^p\right).$$

One sees this is compatible with the group operations so that $N \cong \mathbb{G}_a \times \alpha_p$.

Now suppose $t \notin (k^\times)^p$. If k is non-perfect it is always possible to make such a choice. Then N is reduced but consider the group scheme

$$G = N \oplus N = \operatorname{Spec}(k[x, y, u, v]) / (x^p + ty^p, u^p + tv^p) .$$

This is non-reduced as the ideal contains the element

$$v^p (x^p + ty^p) + y^p (u^p + tv^p) = (xv + yu)^p,$$

but not the element $xv + yu$. Now suppose, to derive a contradiction, that G_{red} is a subgroup scheme and consider the quotient $G/(G_{\text{red}})$. We take for granted that such a quotient exists. See [19, Exposé VI_B, Remarque 9.3]. Note that $H = G/(G_{\text{red}})$ is not trivial but H_{red} is, so $H \neq H_{\text{red}}$. Now, since N is commutative, giving any morphism $f\colon G = N \oplus N \to H$ is equivalent to giving a pair of morphisms $f_1, f_2\colon N \to H$. Since N is reduced, these must factor over H_{red}. But then f must also factor over H_{red}. In particular, the quotient map factors implying $H = H_{\text{red}}$ which is a contradiction.

Note how the above example heavily depends on the existence of an element which is not a p'th power. In other words it rests on k not being perfect. This is no coincidence, as over a perfect field, the reduced subscheme is in fact a subgroup scheme.

Proposition 1.2.12. *Let G be a group scheme over a perfect feld. Then G_{red} has a structure of a group scheme induced from that of G.*

Proof: The inclusion $G_{\text{red}} \to G$ gives a morphism

$$G_{\text{red}} \times G_{\text{red}} \longrightarrow G \times G \xrightarrow{m} G.$$

As $G_{\text{red}} \times G_{\text{red}}$ is reduced by [69, Tag 020I and 035Z] this morphism factors over G_{red} giving the multiplication morphism. The rest is similar.

$\square$

1.3 Finite Group Schemes

We now turn our attention to those group schemes which are finite as schemes. Recall that a morphism $f\colon X \to Y$ is ***finite*** if the codomain Y has an open affine cover $\{U_i\}_{i \in I}$ such that each $V_i = f^{-1}(U_i)$ is affine, and $\Gamma(U_i, {}_Y)$ is a finite algebra over $\Gamma(V_i, {}_X)$. One should note that this is property is local on the base. A scheme is finite if its structure morphism is finite. Furthermore, it is useful to keep in mind that a morphism is finite if and only if it is proper and each fiber is a finite discrete set, see [32, Corollaire 18.12.4].

Finite morphisms are in particular affine, so it is immediate from the definition that, a finite scheme over an affine base is again affine. Further, over a field k the finite schemes are spectrums of finite k-algebras. It follows that such a scheme must be the spectrum of a finitely generated Artin ring R i.e. $R = \prod_{i=1}^n R_i$ where the R_i are local Artin rings. As a topological space such a $\operatorname{Spec} R$ is a discrete space consisting of n points corresponding to the factors R_i.

As we will discuss later, quotients by group scheme actions are not necessarily well behaved, and might not even exist. However, if the group scheme is finite, we will see the situation is somewhat simpler. As such, it is desirable to have a collection of finite group schemes. We will give examples in this section, and Section 1.4 will provide even more examples. As non-examples one can take the additive and multiplicative group scheme. Indeed (over S) the underlying scheme of these are $\operatorname{Spec}_S \mathcal{O}_S[x]$ and $\operatorname{Spec}_S \mathcal{O}_S[x, x^{-1}]$ respectively and neither of these are finite.

The first class of examples is provided by the following result:

Proposition 1.3.1. *If G is a finite group, then the constant group scheme G_S is finite over S. In particular, if $S = \operatorname{Spec} k$ for some field k, then G_k is the spectrum of an Artinian ring.*

Proof: The problem is local on the base, so we may assume $S = \operatorname{Spec} R$ to be affine. Let $n = |G|$. In this case $G_S = \bigsqcup_{i=1}^n \operatorname{Spec} R = \operatorname{Spec}\left(\bigoplus_{i=1}^n R\right)$ so G is the spectrum of a finite R-algebra. □

Proposition 1.3.2. *For any $n \in \mathbb{N}$ the underlying scheme of μ_n is $\operatorname{Spec}_S(\mathcal{O}_S[x]/(x^n - 1))$. In particular, the scheme μ_n is finite.*

Proof: As the base change of a finite morphism is finite, and using that $\mu_{n,S} = \mu_{n,\mathbb{Z}} \times S$, it is enough to show that $\mu_{n,\mathbb{Z}}$ is finite. We prove that $\operatorname{Spec}(\mathbb{Z}[x]/(x^n - 1))$ represents the functor μ_n. As the functor of points is completely determined by what it does on affine schemes, we need only prove this in the affine case. So let R be a ring. Giving a homomorphism $\varphi \in \operatorname{Hom}(\mathbb{Z}[x]/(x^n - 1), R)$ is equivalent to giving an element $r \in R$ such that $r^n = 1$, the isomorphism being $\varphi \mapsto \varphi(x)$. This is clearly canonical, and so μ_n is finite over $\mathbb{Z}$. □

An alternative approach to the above direct proof would be the following: In the next section, we will explore the theory of diagonalizable group schemes, which is a class of commutative group schemes constructed from abelian groups. It will turn out that such a group scheme is finite if and only if the associated abelian group is finite, and that μ_n is in fact the diagonal group scheme of $\mathbb{Z}/n\mathbb{Z}$, hence a fortiori it is finite.

Proposition 1.3.3. *Let S be a base of characteristic $p > 0$. The underlying scheme of α_p is $\operatorname{Spec}_S \mathcal{O}_S[x]/(x^p)$. In particular, the scheme α_p is finite.*

Proof: As in Proposition 1.3.2, we prove that $k[x]/(x^p)$ represents the functor α_p on affine schemes over some field k of characteristic p. So let R be a k-algebra. Giving a homomorphism $\varphi \in \operatorname{Hom}(k[x]/(x^p), R)$ is equivalent to giving an element $r \in R$ such that $r^p = 0$, explicitely $\varphi \mapsto \varphi(x)$. As in the previous proposition, this is canonical. □

Remark 1.3.4 (Comparing α_p and μ_p)**.** The schemes $\operatorname{Spec} \mathbb{Z}[x]/(x^p - 1)$ and $\operatorname{Spec} \mathbb{Z}[x]/(x^p)$ both make sense in characteristic zero, and we may even consider μ_n in this case. However, the Frobenius maps only exist in positive characteristic, hence the group scheme α_p only exists in characteristic $p > 0$. Now, in characteristic p we have $x^p - 1 = (x - 1)^p$, so we obtain an explicit isomorphism of schemes $\operatorname{Spec} \mathbb{Z}[x]/(x^p - 1) \to \operatorname{Spec} \mathbb{Z}[x]/(x^p)$ given on rings by $x \mapsto x - 1$. However, this isomorphism is not an isomorphism of group schemes. We will discuss this distinction of μ_p and α_p more in Section 2.4.

As noted finite group schemes are desirable when one wants to form quotients by group scheme actions. However, there is a more special class of group schemes which is even better for this, namely those which are "infinitesimal". In Theorem 2.2.8 we will see that taking a quotient by an infinitesimal group scheme is always possble. We now give the definition and show that μ_p and α_p are examples.

Definition 1.3.5. A group scheme G is ***infinitesimal*** if the structure morphism is finite locally free $G \to S$ and the unit section $e\colon S \to G$ induces a homeomorphism on the underlying topological spaces $e\colon |S| \to |G|$.

Proposition 1.3.6. *Suppose S is of characteristic p. The group schemes μ_p and α_p are infinitesimal.*

Proof: The definition is local so we may assume the base is affine, $S = \operatorname{Spec} R$. We argue in the case of μ_p as the case of α_p is symmetric. We check the defining properties finite locally free and that the distinguished section defines a homeomorphism. The underlying scheme of μ_p is $R[x]/(x^p - 1) = R[x]/(x-1)^p$. As a module $R[x]/(x-1)^p = R\cdot 1 \oplus R\cdot(x-1) \oplus \cdots \oplus R\cdot(x-1)^{p-1}$, which is both finite and free. For the second bit, the prime ideals of $R[x]/(x-1)^p$ are all those of $R[x]$ containing $(x-1)$, as any prime ideal containing $(x-1)^p$ must also contain its radical $\sqrt{(x-1)^p} = (x-1)$. Concretely, the section $e\colon R[x]/(x-1)^p \to R$ maps a polynomial $a_0 + a_1(x-1) + \cdots + a_{p-1}(x-1)^{p-1}$ to a_0. Thus, for any prime ideal $\mathfrak{p} \subset R$ we have

$$e^{-1}(\mathfrak{p}) = \mathfrak{p} \oplus R(x-1) \oplus \cdots \oplus R(x-1)^{p-1} = (\mathfrak{p}, (x-1)^p).$$

Thus e is bijective. The above respects inclusions of ideals, hence is gives homeomorphism on the level of spectra. $\square$

More generally, any finite group scheme of finite height is infinitesimal, by essentially the same argument.

1.4 Diagonalizable Group Schemes

For us, the most important class of group schemes will be those which are *diagonalizable* (a term we will define shortly). As we will see in Proposition 2.3.2, actions by these group schemes may be interpreted in terms of gradings by certain groups, which it turns out will make for a powerful tool when treating quotients. In our case, this is incredibly useful as the scheme μ_n turns out to be diagonalizable.

Suppose we are given an abelian group M. Aside from the constant group scheme, there is a canonical way in which to associate to M a group scheme affine over the base S, which we now outline. To M, there is an associated ${}_S$-algebra, namely ${}_S[M]$ which is the sheaf of group rings over ${}_S$. Concretely, for an open set $U \subset S$, we have $\Gamma(U, {}_S[M]) = \Gamma(U, {}_S)[M]$. This actually comes equipped with the structure of a Hopf ${}_S$-algebra given by

$$\begin{array}{rcl}
m^{\#}\colon & {}_S[M] \longrightarrow {}_S[M] \otimes {}_S[M], & m \longmapsto m \otimes m \\
\iota^{\#}\colon & {}_S[M] \longrightarrow {}_S[M], & m \longmapsto -m \\
e^{\#}\colon & {}_S[M] \longrightarrow {}_S, & m \longmapsto 1
\end{array}$$

As such, the relative spectrum $\operatorname{Spec}_S {}_S[M]$ may in a canonical way be equipped with the structure of a group scheme over S.

Definition 1.4.1. The group scheme with underlying scheme $\operatorname{Spec}_S {}_S[M]$ is denoted $D_S(M)$, or simply $D(M)$ if the base is clear, and is called the ***diagonalizable group scheme of*** M over S.

We note that the group operation on the functor of points is incredibly simple. Indeed, for $S = \operatorname{Spec} R$, some R algebra A and $f, g \in D(M)(A) = \operatorname{Hom}_{R\text{-alg}}(R[M], A) = \operatorname{Hom}_{\text{Grp}}(M, A^\times)$ the product is is the morphism defined by mapping m to $f(m)g(m)$. We note that this is not simply pointwise multiplication, as pointwise multiplication would not be R-linear. The reason for the name "diagonalizable" is that the affine diagonalizable group schemes can be embedded as a subgroup scheme in the diagonal matrices. Given generators $\{m_i\}_{i\in I}$ of M, one maps an element f to the matrix with the images of m_i under f along the diagonal.

Example 1.4.2. The multiplication, inverse and unit of the diagonalizable group schemes should remind the reader of Example 1.1.5 and Example 1.2.4. This is no coincidence. In fact, the homomorphisms are defined in the same way because those previous examples are special cases of this one. Indeed, both $\mathbb{G}_m$ and μ_m are diagonalizable group schemes with $\mathbb{G}_m = \operatorname{Spec} R[x, x^{-1}] \cong \operatorname{Spec} R[\mathbb{Z}]$ and $\mu_n = \operatorname{Spec} R[\mathbb{Z}/n\mathbb{Z}]$ when the base is $S = \operatorname{Spec} R$.

The following proposition is taken from [18, Exposé VIII Proposition 2.1] and provides a nice list of some basic properties of diagonalizable group schemes.

Proposition 1.4.3. *Let M be an abelian group and $G = D(M)$.*

(i) G is faithfully flat and affine over S.

(ii) M is finitely generated if and only if G is of finite type over S if and only if G is of finite presentation over S.

(iii) M is finite if and only if G is finite over S if and only if G is of finite type over S and G is annihilated by some integer $n > 0$. In this case the degree of G over S is $|M|$.

(iv) M is a torsion group if and only if G is integral over S.

(v) M is trivial if and only if G is the unit group over S

(vi) M is finitely generated and the order of its torsion subgroup is prime to the characteristic of the residue fields of S if and only if G is smooth over S.

Proof: (i) By definition $G = \operatorname{Spec}_S {}_S[M]$. All relative spectrums are affine by construction, so this in particular holds for G. Now, as a module ${}_S[M]$ is free, so it is in particular faithfully flat.

(ii) If M is finitely generated with generators $m_1, \ldots, m_n$, then ${}_S[M]$ is the polynomial sheaf ${}_S[m_1, \ldots, m_n]$ so G is of finite type. Among the generators of M there are only finitely many relations (as $\mathbb{Z}$ is a Noetherian ring), so G is actually of finite presentation. For the converse, suppose G is of finite type. Locally, any element $m \in M$ may be viewed as an element of a ring $R[M]$ of finite type over R. Picking generators $x_1, \ldots, x_n$ any m is then some finite polynomial in the x_i. For each x_i, it may be expressed as a finite R-linear combination of elements of M. Taking such a finite subset of M for each x_i, we obtain a finite set of generators of M.

(iii) That M is finite if and only if G is finite is, mutatis mutandis, the same argument as in (ii). For the last bit we note that the requirement that G is annihilated by n is equivalent to $m^n = 1_M$ for any $m \in M$. As G being of finite type is equivalent to M being finitely generated by (ii), this means that combining the two assumptions is equivalent to M being finitely generated with every element of finite order. This is equivalent to M being finite.

(iv) Being integral is a local property so we assume $S = \operatorname{Spec} R$. Then G being integral over S means by definition tohat each element of $R[M]$ satisfies some monic polynomial equation over R. In particular, this means each $m \in M$ viewed as an element of $R[M]$ satisfies some

$$m^n + r_{n-1}m^{n-1} + \cdots + r_1 m + r_0 = 0$$

As $R[M]$ is a free R-module, this means that each $r_i = 0$, so $m^n = 0$.

Finally, (v) is immediate from the definitions and (vi) is proven in detail in [18, Exposé VIII Proposition 2.1 e)]. □

We obtain the following which is a generalization of the structure theorem of finitely generated Abelian groups.

Proposition 1.4.4. *Let $M = \mathbb{Z}^n \times (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$ be a finitely generated group. Then $D(M) \cong \mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_r}$. In particular, any diagonalizable group scheme of finite type is expressible as a product of $\mathbb{G}_m$ and μ_{n_i}.*

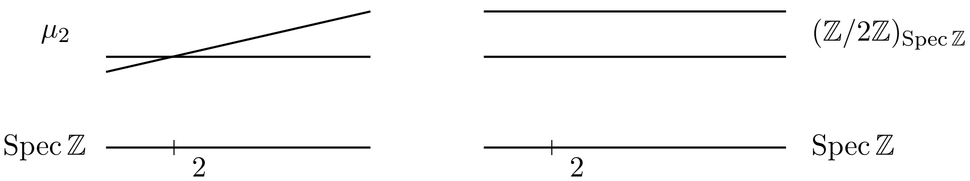
Proof: This follows from the fact that for any groups N and N' we have $R[N \times N'] \cong R[N] \otimes_R R[N']$. The second part follows from Proposition 1.4.3. $\square$

Both diagonalizable group schemes and constant group schemes are built from groups, so it makes sense to wonder if one can compare them in any way. Let us consider an example.

Example 1.4.5 (Comparing μ_2 and $(\mathbb{Z}/2\mathbb{Z})_S$)**.** Suppose we are over an affine base $S = \operatorname{Spec} R$ and that 2 is invertible in R i.e. no residue field of R is of characteristic 2. The underlying scheme of $(\mathbb{Z}/2\mathbb{Z})_S$ is simply $\operatorname{Spec}(R \times R)$, and we have a morphism

$$\operatorname{Spec}(R \times R) \to \operatorname{Spec} R[x]/\left(x^2 - 1\right)$$

determined on rings by $x \mapsto (1, -1)$ which is an isomorphism by the Chinese Remainder Theorem. One can check this respects the explicit morphisms defining the group scheme structures. However, if 2 is not invertible in R, then the two ideals $(x+1)$ and $(x-1)$ are not comaximal and so $R[x]/\left(x^2-1\right) \to R[x]/(x-1) \times R[x]/(x+1) \cong R \times R$ is not surjective. In terms of geometry, 2 not being invertible means that some residue field is of characteristic 2, and so over this point, the group scheme μ_2 collapses to a single non-reduced "thick" point, whereas the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_R$ remains a scheme of two points. This is already the case over the base $\mathbb{Z}$ and the following illustration describes in pictures the difference between the group schemes μ_2 and $(\mathbb{Z}/2\mathbb{Z})_S$:



The previous example generalises, however, one needs to ensure that the polynomial $x^n - 1$ actually splits. Recall that a scheme Y is said to be a twisted form of another scheme X, if there is some base change $S' \to S$ such that there exists an isomorphism $X \times_S S' \cong Y \times_S S'$.

Proposition 1.4.6. *Let $n \geq 1$ be an integer, suppose $S = \operatorname{Spec} R$ is affine and that n is invertible in R. If R contains a primitive n'th root of unity, then μ_n and the constant group scheme $(\mathbb{Z}/n\mathbb{Z})_S$ are isomorphic. The isomorphism is non-canonical except when $n = 2$. In particular, μ_n and $(\mathbb{Z}/n\mathbb{Z})_S$ are twisted forms of each other over any base on which n is invertible.*

Proof: Let ζ_n denote the primitive n'th root of unity. The explicit isomorphism is given by the Chinese Remainder Theorem

$$R[x]/\left(x^n - 1\right) \longrightarrow R[x]/(x-1) \times R[x]/(x-\zeta) \times \cdots R[x]/(x-\zeta^{n-1}) \cong \underbrace{R \times \cdots \times R}_{n-\text{times}}.$$

That the isomorphism is possibly non-canonical follows from the fact that we have to make choice in taking a primitive n'th root of unity, except when $n = 2$. $\square$

Proposition 1.4.7. *If M is finite, the order of M is invertible on the base, and the base contains a primitive $|M|$'th root of unity, then $D(M)$ is isomorphic to the constant group scheme M_S. The isomorphism is usually non-canonical.*

Proof: For any groups N, N' we have $R[N \times N'] \cong R[N] \otimes_R R[N']$. Using the structure theorem of finitely generated Abelian groups, we may express M as a product of finite cyclic groups. Combining these two facts with Proposition 1.4.6 gives the result.

□

More generally, one can prove that the diagonalizable group scheme of M is dual to constant group scheme and vice versa in the sense that $\operatorname{Hom}(M_S, \mathbb{G}_m) = D(M)$, $M_S = \operatorname{Hom}(D(M), \mathbb{G}_m)$, where the hom set is as group schemes over S. This is an equivalent way of defining diagonalizable group schemes, see [18, Exposé VIII, Theorem 1.2] and the general exposé loc. cit. for more details.

1.5 Further Examples

As the title suggest we will spend this last section looking at some more theoretical examples of group schemes. The first we will discuss is the class of Abelian varieties.

1.5.1 Abelian Varieties

In this section, we fix a base field k.

Definition 1.5.1. An **Abelian variety** is a proper, connected, geometrically reduced group scheme over k.

Note that for a proper, connected group scheme, being geometrically reduced is equivalent to smooth, so an Abelian variety is in particular smooth. Any connected group scheme is automatically irreducible [69, Tag 0B7Q], so it follows that any Abelian variety is geometrically irreducible. Before looking at the one-dimensional case, the ubiquitous elliptic curves, we show why these group schemes are called Abelian.

Proposition 1.5.2. *An Abelian variety A is commutative as a group scheme.*

Proof: The main idea is to argue that the inverse $\iota\colon A \to A$ is a morphism of group schemes. One can prove [55, p. 41 Corollary 1] that given abelian varieties A', A'' any morphism of schemes $f\colon A' \to A''$ may be expressed as $f = t_a \circ h$ where h is a morphism of group schemes and t_a is the right translation by a fixed $a \in A''$ defined by $t_a(b) = m(b, a)$ for any $b \in A''$. In particular, the morphism $\iota\colon a \mapsto a^{-1}$ may be expressed in this fashion. As this does not involve any translation, it must be a morphism of group schemes, i.e. it defines a homomorphism of groups on points. It follows that on points $a^{-1}b^{-1} = (ab)^{-1}$. But by its nature $(ab)^{-1} = b^{-1}a^{-1}$. See also [55, p. 41 Corollary 2] □

Now, let us study the 1-dimensional case. A ***Weierstrass equation*** over k is an equation in variables x and y with coefficients $a_1, a_2, a_3, a_4, a_6 \in k$ of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Such an equation is inhomogeneous, but has a so-called homogenization

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

This homogenization then defines a closed subscheme C of $\mathbb{P}^2 = \operatorname{Proj} k[X, Y, Z]$. As is a common convention, we write the variables of the homogenization using capital letters. Except for the point $(0 : 1 : 0)$, this curve is completely contained in the $D_+(Z)$ chart of $\mathbb{P}^2_R$ and the affine equation above defines this curve completely. Formally, one has $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$. As the inhomogeneous equation is enough, one usually writes these equations in this simpler form. Note that such a curve has genus $g = \frac{(3-1)(3-2)}{2} = 1$ by the genus degree formula:

Lemma 1.5.3. *Let C be a closed dimension 1 subscheme of $\mathbb{P}^2$ defined by a homogeneous equation is of degree d. Then C has arithmetic genus*

$$p_a(C) = \frac{(d-1)(d-2)}{2}.$$

Proof: By definition the arithmetic genus is $p_a(C) = 1 - \chi({}_C)$. First, we note that by Grothendieck vanishing theorem the cohomology of a proper scheme vanishes in degrees above dimensions, i.e. $\mathrm{H}^n(C, {}_C) = 0$ for $n \geq 2$. Thus, the only contribution to $\chi({}_C)$ comes from degree 0 and 1. Consider then the standard short exact sequence

$$0 \longrightarrow {}_{\mathbb{P}^2}(-d) \longrightarrow {}_{\mathbb{P}^2} \longrightarrow {}_C \longrightarrow 0.$$

This gives us a long exact sequence in cohomology

$$\begin{array}{l} 0 \longrightarrow \mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}(-d)) \longrightarrow \mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}) \longrightarrow \mathrm{H}^0(C, {}_C) \\ \longrightarrow \mathrm{H}^1(\mathbb{P}^2, {}_{\mathbb{P}^2}(-d)) \longrightarrow \mathrm{H}^1(\mathbb{P}^2, {}_{\mathbb{P}^2}) \longrightarrow \mathrm{H}^1(C, {}_C) \\ \longrightarrow \mathrm{H}^2(\mathbb{P}^2, {}_{\mathbb{P}^2}(-d)) \longrightarrow \mathrm{H}^2(\mathbb{P}^2, {}_{\mathbb{P}^2}) \longrightarrow 0 \longrightarrow \cdots \end{array}$$

Let us analyze the terms of this sequence. First, we use that the intermediate cohomology of any ${}_{\mathbb{P}^2}(n)$ disappears (see [35, III.5.1]), so

$$\mathrm{H}^1(\mathbb{P}^2, {}_{\mathbb{P}^2}(-d)) = 0 = \mathrm{H}^1(\mathbb{P}^2, {}_{\mathbb{P}^2})$$

Next, recall that the space of global sections of ${}_{\mathbb{P}^2}(n)$ is the vector space generate by degree n monomials in 3 variables. This directly gives us $\mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}(-d)) = 0$ and $\mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}) = k$. Putting this together with the previous implies that

$$\mathrm{H}^0(C, {}_C) \cong \mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}) = k.$$

In particular we get an equality $p_a(C) = h^1({}_C)$. We remind that the canonical sheaf on $\mathbb{P}^n$ is ${}_{\mathbb{P}^n}(-n-1)$. It then follows by Serre duality

$$\begin{aligned} \mathrm{H}^2(\mathbb{P}^2, {}_{\mathbb{P}^2}) &= \mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}(-3))^\vee = 0 \\ \mathrm{H}^2((\mathbb{P}^2, {}_{\mathbb{P}^2}(-d)) &= \mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}(-3+d))^\vee \end{aligned}$$

We thus have $\mathrm{H}^1(C, {}_C) \cong \mathrm{H}^2((\mathbb{P}^2, {}_{\mathbb{P}^2}(-d)) \cong \mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}(-3+d))$. If $d = 1, 2$, then $-3+d < 0$, so $\mathrm{H}^0(\mathbb{P}^2, {}_{\mathbb{P}^2}(-3+d)) = 0$. In this case $p_a(C) = 0$, which satisfies the formula. Now, there are $\binom{d+n-1}{n-1}$ monomials of degree d in n variables, so if $d \geq 3$ we have

$$h^0({}_{\mathbb{P}^2}(-3+d)) = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2},$$

which proves the formula. $\square$

One associates to Weiestrass equations a few quantities, one of which is the ***discriminant*** defined as

$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$$

where the b_i are quantities defined from the coefficients a_j by

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, \\ b_4 &= 2a_4 + a_1a_3, \\ b_6 &= a_3^2 + 4a_6, \\ b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \end{aligned}$$

It then holds that C is smooth if and only if $\Delta \neq 0$ [67, Proposition III.1.4(i)]. If $\operatorname{char} k \neq 2$, one can simplify the equation to

$$y^2 = 4x^3 + b_2 + 2b_4x + b_6$$

And if one further assumes $\operatorname{char} k \neq 3$ then one gets the short Weierstrass form

$$y^2 = x^3 + Ax + B,$$

where $A = -27(b_2^2 - 24b_4)$ and $B = -54(-b_2^3 + 36b_2b_4 - 216b_6)$.

Example 1.5.4. Consider the equation $y^2 = x^3 + 5x + 8$. This has determinant $\Delta = -35648 = -2^6 \cdot 557$, hence defines a smooth curve if and only the base field has characteristic $\neq 2, 557$.

We take the following non-standard definition.

Definition 1.5.5. An ***elliptic curve*** is an Abelian variety of dimension 1.

The more standard definition is that an elliptic curve is a smooth, connected, projective, genus one curves with a distinguished rational point. Furthermore, any smooth curve defined by a Weierstrass equation is elliptic as we now argue. The converse is also true, though we will not prove this.

Proposition 1.5.6. *A smooth curve defined by a Weierstrass equation is elliptic.*

Proof: A curve defined by a Weierstrass equation is automaticall projective (hence proper) and irreducible. If it is smooth it is also geometrically reduced. In Section 4.3.1 we will see more generally that any scheme defined by a Weierstrass equation over a general base is in a canonical way a group scheme. □

Example 1.5.7. The curve $y^2 = x^3 + 5x + 8$ in Example 1.5.4 is elliptic over base fields not of characteristic 2 or 557.

For curves given by Weierstrass equations, it is possible to give very concrete descriptions of the group law on rational points, both in terms of geometry by using lines intersecting the curve and by explicit formula. For more on this see [67, Chapter III]. We will not go into more detail as elliptic curves are a gigantic subject in their own right and there are multiple long texts on the subject, such as [66], [67] and [68]. But having this one-dimensional case in mind, we mention the following easy fact, which allows for many examples.

Proposition 1.5.8. *A product of abelian varieties is again an abelian variety.*

Proof: Products of proper, connected and smooth schemes remain so. Furthermore a product of group schemes naturally inherits the structure of a group scheme. □

Example 1.5.9. Using Proposition 1.5.8 one may construct Abelian varieties of arbitrary dimension. For example, the scheme of dimension 3

$$\operatorname{Proj} k[X,Y,Z]/(y^2 - x^3 + 2) \times \operatorname{Proj} k[X,Y,Z]/(y^2 - x^3 + 3) \times \operatorname{Proj} k[X,Y,Z]/(y^2 - x^3 + 13)$$

is an Abelian variety if the base field is of characteristic not $2, 3$ or 13. The scheme

$$\operatorname{Proj} k[X,Y,Z]/(y^2 - x^3 + 1)^{\times 4},$$

is an Abelian variety in all characteristics.

For further reading about Abelian varieties, there are many books on the subject, such as [55].

1.5.2 The Automorphism Group Scheme

The final example we will consider in this section is the automorphism group scheme. We will apply the same tactic as in many of our earliest examples. That is, we will consider a specific functor of groups, and then argue that it is representable by a scheme.

Definition 1.5.10. Let X be a scheme over S. The functor $\mathrm{Aut}_{X/S}$ is defined as the contravariant functor of groups which to an S-scheme T associates the group of T-automorphisms $\mathrm{Aut}_T(X \times_S T)$. To a morphism $T \to T'$, the associated morphism $\mathrm{Aut}_{T'}(X \times_S T') \to \mathrm{Aut}_T(X \times_S T)$ is obtained by pull-back.

Remark 1.5.11. Before considering the automorphism group scheme, we first make an observation on the nature of morphisms of schemes. Consider a morphism $f\colon X \to Y$ of schemes over some base S. Then this morphism comes with its associated graph morphism $\Gamma_f\colon X \to X \times_S Y$. Now, assume Y is separated. This ensures that Γ_f is a closed immersion, and so the image of X under Γ_f yields a closed subscheme of $X \times_S Y$, which we also denote Γ_f. Note that this identification is actually one to one i.e. if $\Gamma_f = \Gamma_g$ then $f = g$. Indeed, we have

$$f = \mathrm{pr}_2 \circ \Gamma_f = \mathrm{pr}_2 \circ \Gamma_g = g,$$

where $\mathrm{pr}_i\colon X \times_S Y \to Y$ is the projection. Furthermore, if $Z \subseteq X \times_S Y$ is a closed subscheme, then $Z = \Gamma_f$ for some f, if and only if $\mathrm{pr}_1|_Z\colon X \to X$ is an isomorphism. The only if part is just a restatement that the morphism Γ_f is a closed immersion. Conversely, if $\mathrm{pr}_1|_Z$ is an isomorphism then it has an inverse $(\mathrm{pr}_1|_Z)^{-1}\colon X \to Z$ so we obtain a morphism $f = \mathrm{pr}_2 \circ (\mathrm{pr}_1|_Z)^{-1}\colon X \to Y$ for which $Z = \Gamma_f$.
Finally, it holds that f is an isomorphism if and only if $\mathrm{pr}_2|_{\Gamma_f}\colon \Gamma_f \to Y$ is an isomorphism. This follows from the fact that $\mathrm{pr}_2|_{\Gamma_f} \circ \Gamma_f = f$, and Γ_f is an isomorphism onto its image.
In conclusion we obtain a 1-1 identification of isomorphisms $X \to Y$ (where Y is separated) with closed subschemes of $X \times_S Y$ which are isomorphic to X and Y via the projections. In particular, automorphisms of a separated scheme X correspond to a certain class of closed subschemes of the fiber product $X \times_S X$.

Proposition 1.5.12. *Let X be a projective scheme over a field k. The functor $\mathrm{Aut}_{X/k}$ which to a k-scheme T associated the group $\mathrm{Aut}(X_T)$ of automorphisms of $X_T = X \times_k T$ over T, is representable by a scheme. This scheme is denoted $\mathrm{Aut}_{X/k}$ and is called the **automorphism group scheme**.*

Proof: As noted in Remark 1.5.11 we may identify the automorphisms $f\colon X_T \to X_T$ with their graph $\Gamma_f \subseteq X_T \times_T X_T$. As proven by Grothendieck in [29, IV: Theorem 3.2 and p. 17] there exists a scheme $\mathrm{Hilb}_{X\times_k X/k}$ parametrizing closed subschemes of $X \times_k X$ in the sense that $\mathrm{Hilb}_{X\times_k X/k}(T)$ corresponds one-to-one to closed subschemes of $X_T \times_T X_T$. Now, as outlined in the discussion of Remark 1.5.11 the automorphisms of $\mathrm{Aut}(X_T)$ correspond to closed subschemes of $X_T \times_T X_T$ such that both projections restrict to an isomorphism onto X_T. Thus, the automorphisms $\mathrm{Aut}(X_T)$ correspond to a specific class of T-rational points of the Hilbert scheme $\mathrm{Hilb}_{X\times_k X/k}$. As explained in the proof of [26, Theorem 5.23] (which our statement is a special case of) the condition that a closed subscheme represents a morphism from X_T to X_T for some T is an open condition. As such, the set of closed subschemes representing automorphisms, may be viewed as the intersection of the two open subschemes representing morphisms from X to X and to X from X (these two are distinct, as direction matters in this case). Thus, the automorphisms of X constitute an open subset of $\mathrm{Hilb}_{X\times_k X/k}$, hence the set of automorphisms inherits a unique induced scheme structure.

□

In fact, one can prove more generally, that the automorphism functor is representable if X is simply proper. This is a main result of [51, Theorem 3.7]

Proposition 1.5.13. *If X is a proper scheme over a field k then $\mathrm{Aut}_{X/k}$ is representable by a scheme which is locally of finite type.*

Chapter 2

Group Schemes in Action

As usual, we fix a base scheme S. Any notion presented is relative to this base.

In this chapter we will study group scheme actions and quotients. This is a central component of the classical and generalized Kummer constructions studied in later chapters, and we will spent quite some time studying how one can interpret actions in varying fashions. As in Chapter 1 we refer to [16, 17] for further reading.

2.1 Definitions and Basic Results

To fix notation, let G be a group scheme.

Definition 2.1.1. An ***action*** of G on a scheme X is a morphism $\sigma\colon G \times X \to X$ such that $\sigma_T\colon G(T) \times X(T) \to X(T)$ defines a group action for each scheme T.

Equivalently, an action of G on X is a morphism $\sigma\colon G \times X \to X$ such that the following diagrams are commutative

$$\begin{tikzcd} G \times G \times X \arrow[r, "\sigma \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_X \times \sigma"'] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X. \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[rrd, "\mathrm{id}_X"'] & S \times X \arrow[r, "e \times \mathrm{id}_X"] & G \times X \arrow[d, "\sigma"] \\ & & X \end{tikzcd}$$

Or, in written equalities $\sigma \circ (\mathrm{id}_G \times \sigma) = \sigma \circ (m \times \mathrm{id}_X)$ and $\sigma \circ (e \times \mathrm{id}_X) = \mathrm{id}_X$. These equalities are just generalisations of the standard axioms of group actions on sets, namely compatibility and identity.

In this chapter we will be juggling actions of both group schemes and groups on X. These notions are of course connected but there are subtleties to keep in mind. Indeed, an action of a group M on X means that each $m \in M$ determines an automorphism of the scheme X, whereas an action of a group scheme G means that each T-rational point of G determines an automorphism of the set $X(T)$. Note that given an action of a group on a scheme, we do get an induced action on each set of T-rational points. Indeed, a T-rational point is a morphism $T \to X$, and so post-composition with the automorphism φ_M gives another T-rational point. This obviously satisfies the axioms of a group action on a set. In fact, we can say more:

Remark 2.1.2. Suppose M is a group acting on a scheme X. Then each element $m \in M$ determines an automorphism $\sigma_m\colon X \to X$. Using this, we obtain a morphism $M_S \times X \to X$, by mapping the m component of $M_S \times X = \coprod_{m \in M} X$ to X via σ_m. Thus we obtain an action of the constant group scheme M_S on X.

Example 2.1.3. Take as a base $S = \operatorname{Spec} R$, and consider as schemes $G = \mathbb{G}_m$ and $X = \mathbb{A}^n$. For any R-algebra A we then have a classical action of $\mathbb{G}_m(A) = A^\times$ on $\mathbb{A}^n(A)$ by scaling.

As we wish to study actions by group schemes, an particularly quotients by these, it becomes useful to have different ways of defining actions. This will lead us to different tools to compute quotients. We are especially interested in finite group schemes. Such group schemes are in particular affine, and actions by such schemes have nice properties. As in Proposition 1.1.4 we let $\mathcal{A}$ denote the functor from the category of schemes affine over S to the category of quasi-coherent ${}_S$-algebras, given by sending a scheme to the pushforward of its structure sheaf.

Lemma 2.1.4. *Suppose G is affine over S. The category of schemes affine over S with G-action is equivalent to the category of ${}_S$-algebra with a $\mathcal{A}(G)$-comodule structure over ${}_S$.*

Proof: To fix notation, let X be a scheme affine over S. An action of G on X is just a morphism $\mu\colon G \times X \to X$ compatible with the group scheme structure on G. Now, the category of affine schemes over S is equivalent to the category of quasi-coherent ${}_S$-algebras, so to give such a μ, is nothing but a morphism of the corresponding sheaves,

$$\mu\colon \mathcal{A}(X) \longrightarrow \mathcal{A}(X) \otimes \mathcal{A}(G),$$

where $\mathcal{A}(G)$ is a Hopf algebra by Proposition 1.1.4 One can then transfer the requirements for μ to be a group scheme action into the category of ${}_S$-algebras, and we see that it corresponds to commutativity of the following diagrams which are dual to those given following Definiton 2.1.1:

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathcal{A}(X) \otimes \mathcal{A}(G) \arrow[d, "\mathrm{id}\otimes\Delta"] \\
\mathcal{A}(X) \otimes \mathcal{A}(G) \arrow[r, "\mu\otimes\mathrm{id}"'] & \mathcal{A}(X) \otimes \mathcal{A}(G) \otimes \mathcal{A}(G),
\end{tikzcd} \tag{CM1}$$

and

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[r, "\mu"] \arrow[rrr, bend right, "\mathrm{id}"'] & \mathcal{A}(X) \otimes \mathcal{A}(G) \arrow[r, "\mathrm{id}\otimes e"] & \mathcal{A}(X) \otimes {}_S \arrow[r, "\sim"] & \mathcal{A}(X),
\end{tikzcd} \tag{CM2}$$

where Δ denotes the comultiplication of the Hopf algebra $\mathcal{A}(G)$ and $e\colon \mathcal{A}(G) \to {}_S$ is the counit. These diagrams are exactly the definition of a comodule structure.

$\square$

This leads us to the following definition:

Definition 2.1.5. Let $\mathcal{F}$ be an ${}_S$-module and suppose G is affine over S. A ***G-action on*** $\mathcal{F}$ is an $\mathcal{A}(G)$-comodule structure of $\mathcal{F}$ over ${}_S$.

Note that by Lemma 2.1.4 this is completely compatible with the definition given earlier.

Example 2.1.6. As an example, take the algebra $A[x_1, \ldots, x_n]$ and the group scheme $\mathbb{G}_m$. We saw in Example 1.1.5 that the underlying scheme of $\mathbb{G}_m$ is $\operatorname{Spec} A[y, y^{-1}]$, and that the multiplication and identity of the Hopf algebra structure are given by $\Delta\colon y \mapsto y \otimes y$ and $e\colon y \mapsto 1$. So to give a comodule structure of $A[x_1, \ldots, x_n]$ over $A[y, y^{-1}]$, we just need to give a homomorphism $A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n] \otimes A[y, y^{-1}]$ that is compatible with these. As an example, take

$$\mu\colon A[x_1, \ldots, x_n] \longrightarrow A[x_1, \ldots, x_n] \otimes A[y, y^{-1}], \quad x_i^n \mapsto x_i^n \otimes y^n.$$

One checks that

$$(\mathrm{id} \otimes \Delta)(\mu(x_i^n)) = x_i^n \otimes y^n \otimes y^n = (\mu \otimes \mathrm{id})(\mu(x_i^n))$$

while $(\mathrm{id} \otimes e)(\mu(x_i^n)) = x_i^n \otimes 1$ as desired. Note that this is just the same action as in Example 2.1.3. One can construct other actions by simply replacing y by y^m, which corresponds to scaling by an m'th power.

Let us now turn to the notion of free actions. As in the classical case of group actions on sets, these are particularly nice actions.

Definition 2.1.7. Suppose we have an action $\sigma\colon G \times X \to X$. The action is said to be ***free*** if the morphism

$$\sigma \times \mathrm{id}_X\colon G \times X \to X \times X, \qquad (g, x) \mapsto (\sigma(g, x), x)$$

is a monomorphism.

One should note that the above definition is different (and less restrictive) than the one taken in [56, Definition 0.8]. In some sense, the definition given here is more natural, as it simply means that for each scheme T the action of $G(T)$ on $G(X)$ is free. Related to this notion is that of a fixed point.

Definition 2.1.8. Suppose we have an action $\sigma\colon G \times X \to X$. A ***schematic fixed point*** is a point $x \in X$ such that $\sigma(g, x) = x$ and such that the induced homomorphism of residue fields $\sigma\colon \kappa(x) \to \kappa(x)$ is the identity.

The first requirement, $\sigma(g, x) = x$, means that the point x is a topological fixed point. The extra requirement on the induced homomorphism is not an empty assumption at all. There really is a difference between topological and schematic fixed points. To illustrate this difference, let us study an explicit example.

Example 2.1.9. Consider a field K and a finite Galois extension L/K. We then have an action of $G = \mathrm{Gal}(L/K)$ on L hence also on $\operatorname{Spec} L$. The topological space $|\operatorname{Spec} L|$ consists of only a single point, which is then invariably a topological fixed point for the action of G. However, the point $\operatorname{Spec} L$, which by its nature carries an algebraic structure, is not a schematic fixed point, as the action of G does not fix the field L.

We also have the following example, which further shows how easily one can happen upon actions that are non-free.

Example 2.1.10. Given a group scheme G it comes equipped with the involution morphism $\iota\colon G \to G$. By its nature this morphism satisfies $\iota \circ \iota = \mathrm{id}$ and so we have a canonical action of $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ on the group scheme G. Furthermore, the multiplication map $G \times G \to G$ defines for each $n \in \mathbb{Z}$ a self map $n_G\colon G \to G$ called multiplication by n. Concretely, a point in G is mapped via the diagonal map to the n-times self-product of G and then back to G via successive multiplication of two factors at a time. This of course determines an action of $\mathbb{Z}$ on G. Furthermore, the fixed scheme of the $\mathbb{Z}/2\mathbb{Z}$ action is exactly the kernel of the map 2_G. For an abelian variety A of dimension g over an algebraically closed field the kernel of n_A consists of n^{2g} points if the characteristic of the base field does not divide n [55, p. 60-61]. Over a non-algebraically closed field, the kernel simply has length n^{2g}. In particular, if A is an abelian surface the fixed scheme of the $\{\pm 1\}$-action consists of exactly $2^4 = 16$ points after passing to an algebraic closure of the base. For the case where the characteristic of k divides n, see the reference loc. cit. or [25, Definition 1.8].

2.2 Quotients

Our main results deal with singularities on quotient families. In this subsection we will have a look at how one constructs quotients in algebraic geometry. We will primarily consider finite group schemes with the only non-finite one being $\mathbb{G}_m$. The concept of quotients by group actions in algebraic geometry is wide ranging topic in its own right, called *Geometric invariant Theory* (GIT). There are many books and articles dedicated to the subject such as Mumford's originating text [53] and its later editions [56]. If one dives into the details of GIT there are many considerations to be made and terms such as "good categorical quotient" crop up. We will simply not bother with these types of considerations, as the situations we consider do not need them. We are content to take the following as out notion of quotient.

Definition 2.2.1. Given a group scheme G acting on a scheme X, a quotient of X by G is a surjective G-invariant morphism $\pi\colon X \to X/G$ to a scheme over which G-invariant morphisms factor i.e. for all G-invariant morphisms $\varphi\colon X \to Y$ there is a unique morphism $\psi\colon X/G \to Y$ such that $\varphi = \psi \circ \pi$. If G is finite, π must be integral.

Of course, the universal property means that any such quotient, if it exists, is unique up to unique isomorphism. The big question is whether such a morphism actually exists. As the next few existence results show, there is no great universal "yes" or "no" answer. It is more of a case-by-case study, depending on the specific group scheme or type thereoff, the type of action, the base or all of these. However, a point to make is that having the group scheme be finite is usually a good starting point. Having the action be free is also desirable, but this is not the situation in the problems we consider later. In any case, we will see that our quotients are locally given by spectra of rings of invariants.

Definition 2.2.2. Let G be a group scheme affine over the base acting on an affine scheme $\operatorname{Spec} A$ via $\sigma\colon G \times \operatorname{Spec} A \to \operatorname{Spec} A$. The ring $A^G = \{a \in A \mid \sigma(a) = 1 \otimes a\}$ is called the ***ring of invariants***, where σ also denotes $\sigma\colon A \to \mathcal{A}(G) \otimes A$. If G acts acts on a scheme $\operatorname{Spec}_S \mathcal{F}$ affine over the base , then $\mathcal{F}^G$ is called the ***sheaf of invariants***.

Note that the above definition is equivalent to letting A^G be the kernel of

$$\sigma - \mathrm{id} \otimes 1\colon A \longrightarrow \mathcal{A}(G) \otimes A.$$

Remark 2.2.3. Take $S = \operatorname{Spec} R$. There is of course also the well known notion of invariant rings under group actions. If M is a group acting on a ring A the M-invariants are

$$A^M = \{a \in A |\ ma = a\ \ \forall m \in M\}.$$

If M is a group, then the two notions of invariants coincide, meaning that if M acts on some R-algebra A, then $A^M = A^{M_S}$. Indeed, the action of M on A gives an action of M on $\operatorname{Spec} A$ which in turn determines an action of M_S on $\operatorname{Spec} A$ as in Remark 2.1.2. Denote this action

$$\sigma\colon M_S \times \operatorname{Spec} A \longrightarrow \operatorname{Spec} A.$$

This corresponds to a homomorphism of R-algebras

$$\sigma\colon A \to (R \times R \times \cdots \times R) \otimes_R A \cong A^{|M|}.$$

Analyzing the morphism in detail, one arrives at the fact that the morphism is

$$\sigma\colon A \longrightarrow A^{|M|} \quad a \longmapsto (1_M a, m_2 a, \ldots, m_{|M|} a).$$

With this interpretation, the M_S invariant elements of A should be those such that $\sigma(a) = (a, a, \ldots, a)$. But the element $(1_M a, m_2 a, \ldots, m_{|M|} a)$ of $A^{|M|}$ corresponds to

$$(1, 0, 0, \ldots,) \otimes a + (0, 1, 0, 0, \ldots, 0) \otimes m_2 a + \cdots + (0, 0, \ldots, 1) \otimes m_{|M|} a,$$

in $(R \times R \times \cdots \times R) \otimes_R A \cong A^{|M|}$. So the criterion $(1_M a, m_2 a, \ldots, m_{|M|} a) = (a, a, \ldots, a)$ translates to

$$\sigma(a) = (1, 1, \ldots, 1) \otimes a = 1 \otimes a.$$

Replacing $S = \operatorname{Spec} R$ by a general base and A by a coherent ${}_S$-algebra $\mathcal{F}$ in the above one gets the same interpretation for the more general case. As such, there is no chance of confusion when talking about invariant rings or sheaves as all notions coincide.

We take the following for granted:

Proposition 2.2.4. *Let M be a finite group acting on a scheme X. Then the quotient $\pi \colon X \to X/M$ exists and the natural homomorphism ${}_{X/M} \to \pi_* {}^G_X$ is an isomorphism if and only if each orbit of the action is contained in an open affine.*

Proof: See [33, Exposé V, Proposition 1.8]. □

As we just saw in Remark 2.2.3 that rings of invariants coincide whether we consider a group or its constant group scheme, this at least gives a criterion for existence of quotients by actions of constant group schemes.

The next result considers quotients over base fields. As we are, in the end, interested in families i.e. schemes over bases with more than one point, this will not be sufficient for our purposes when we reach the main results. However, it is good for computing examples, and it gives existence for the classical kummer construction studied in Section 3.3.

Theorem 2.2.5. *Suppose $S = \operatorname{Spec} k$ for a field k. Let X be a scheme of finite type and G a finite group scheme acting on X. If the orbit of any point is contained in an affine open subset of X, then the quotient $\pi \colon X \to X/G$ exists and the natural homomorphism ${}_{X/G} \to \pi_*({}_X)^G$ is an isomorphism. In particular, if $X = \operatorname{Spec} A$ then $X/G = \operatorname{Spec} A^G$. If the action of G is free, and $\dim \Gamma(G, {}_G) = n$, then π is flat of degree n.*

Proof: This is part of [55, Theorem 1, p.104-105]. The assumption of finite type is not present in the statement loc. cit. but is a standing assumption in the text. The result loc. cit. is only given over an algebraically closed ground field, but the assumption is not necessary for the proof. □

The above reduces computing quotients to simply computing rings of invariants

Example 2.2.6. Consider the action of $\langle -1 \rangle$ on $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[T]$ defined by $[-1] \colon T \mapsto -T$. Then $\mathbb{C}[T]^{\langle -1 \rangle} = \mathbb{C}[T^2]$ so $\mathbb{A}^1/\langle -1 \rangle = \operatorname{Spec} \mathbb{C}[T^2]$. Indeed, each polynomial $f(T) \in \mathbb{C}[T]$ may be written uniquely as

$$f(T) = g(T^2) + h(T^2)T \qquad g(T), h(T) \in \mathbb{C}[T],$$

i.e. it may be split into terms of even and odd degree. Then one simply notes that $[-1]f(T) = f([-1]T) = f(T)$ if and only if $h(T^2) = 0$.

It is important to know that taking quotients rarely commute with other operations such as products and more specifically base change. Later, in Section 4.1 we will focus more on this question, and see examples that not even taking fibers commute with quotients. For now we are content with the following example that products do not behave as one might hope.

Example 2.2.7. Let k be a base field of characteristic not 2. Take as in Example 2.2.6 the action of $G = \langle -1 \rangle$ on $\operatorname{Spec} k[T]$ by $[-1] \colon T \mapsto -T$. Then $\mathbb{A}^1/G \times \mathbb{A}^1/G = \operatorname{Spec} k[T_1^2, T_2^2]$. But, if we consider the induced action on $\mathbb{A}^1 \times \mathbb{A}^1$, then this is determined by

$$[-1] \colon k[T_1] \otimes k[T_2] = k[T_1, T_2] \longrightarrow k[T_1, T_2], \quad T_i \mapsto -T_i.$$

Using a similar argument to before, one finds that $k[T_1, T_2]^G = k[T_1^2, T_2^2, T_1T_2]$. But $k[T_1^2, T_2^2] \ncong k[T_1^2, T_2^2, T_1T_2]$ so

$$\mathbb{A}^1/G \times \mathbb{A}^1/G \ncong (\mathbb{A}^1 \times \mathbb{A}^1)/G.$$

The existence result we will apply for the main results is the following:

Theorem 2.2.8. *If G is a finite infinitesimal group scheme acting on a scheme X, then the quotient X/G exists. The quotient morphism is finite and surjective. If X is affine the quotient is $X/G = \operatorname{Spec} A^G$.*

Proof: This follows from [16, III, §6, 6.1 Corollaire]. The corollaire loc. cit. is a special case of [16, III,§2, Theorem 3.2] for which the precise description of the quotient in the affine case is described in the proof. The result quoted does not mention finiteness of the quotient morphism, but this is obtained from the description of the quotient as the spectrum of the G-invariant ring.

Now, let $\sigma\colon G \times X \to X$ denote the action. According to the first result quoted, we should argue that the projection $p_2\colon G \times X \to X$ is finite locally free and that the set theoretic orbits $[x] = \{y \in X \mid \exists g \in G \text{ s.t. } \sigma(g, x) = y\}$ are all contained in open affines. For the first part, we simply remark that any infinitesimal group scheme is by definition finite locally free, and that this property is preserved by base change. It follows that p_2, which is the base change of the structure morphism $G \to S$ by $X \to S$, is necessarily finite locally free. For the condition on orbits, note that since G is infinitesimal, the identity section $e\colon S \to G$ is a homeomorphism on topological spaces, in particular it is surjective. It follows that on the level of points, any point $g \in G$ is the target of some $s \in S$ and so $m(g, x) = m(e(s), x) = \mathrm{id}_X(x) = x$ for any $x \in X$. Thus these set theoretical orbits consist of just one point each.

□

Note the following fact which is implicit in the proof above: A quotient by an infinitesimal group scheme does not change the underlying topological space. That is, if G is infinitesimal acting on X, then X/G is homeomorphic to X. We now prove a few general lemmas on properties of quotients. In these, we have a standing assumption that the quotient $X/G = \operatorname{Spec} R^G$ actually exists.

Lemma 2.2.9. *Let X be a scheme acted on by a finite group scheme G.*

(i) If the base S is locally Noetherian and X is of finite type, then X/G is of finite type.

(ii) If X is separated then X/G is separated.

(iii) If X is proper then X/G is proper.

Proof: (i): For the quasi compactness suppose $V \subset S$ is an open affine, and let $f\colon X \to S$ and $g\colon X/G \to S$ be the structure morphisms. Then $f^{-1}(V) = \pi^{-1}(g^{-1}(V)$ is quasi-compact. As π is surjective, $\pi(\pi^{-1}(g^{-1}(V)) = g^{-1}(V)$, which is then quasi-compact since π is continuous. For the locally of finite type, we give essentially the same proof as in [55, p.63 Theorem]. We may assume everything is affine $X = \operatorname{Spec} A$ and that $S = \operatorname{Spec} R$ is Noetherian. As A is of finite type, it has some finite generating set $x_1, \ldots, x_n$. As $A^G \to A$ is integral each generator x_i satisfies some monic polynomial equation. Let $B \subset A^G$ be the finite type subalgebra generated by the coefficients of these equations. As A is integral over B, and of finite type over R, hence B, it is finite over B. As B is of finite type over the Noetherian ring R, it is Noetherian. So any B-submodule of A is again finite over B. Thus A^G is finite over B hence of finite type over R.

(ii): We must show the image of X under the diagonal $\Delta_{X/G}\colon X/G \to X/G \times X/G$ is closed. Letting $\pi\colon X \to X/G$ be the quotient map, we have a commutative square

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "\pi"'] & X \times X \arrow[d, "\pi\times\pi"] \\ X/G \arrow[r, "\Delta_{X/G}"] & X/G \times X/G \end{tikzcd}$$

Now, since X is separated, $\Delta_X(X)$ is closed in $X \times X$. But π is finite, hence so is $\pi \times \pi$ and in particular this last map is closed. So $\Delta_{X/G}(X/G) = \pi \times \pi(\Delta_X(X))$ is closed in $X/G \times X/G$.

(iii): As X is separated and of finite type, so is X/G. By its nature, the quotient map $X \to X/G$ is surjective. But then X/G is proper by [48, Proposition 3.3.16 (f)]. □

We will see later, in Example 2.3.17, that the finiteness assumption in parts (ii) and (iii) above is not superfluous so that one can obtain non-separated quotients when the group scheme is not finite. Concerning (i), a famous example by Nagata [57] shows that the ring of invariants of a finite type ring is not necessarily of finite type. However, there are cases of quotients by non-finite group schemes where the quotient of a finite type scheme remains of finite type. For example [18, Exposé VII corollaire 5.8] shows that finite type/presentation is preserved if the group scheme is diagonalizable and acts freely.

The following lemma is immensely useful as it tells us that forming quotient may be done after flat base change. This is useful especially if one is working over a field, as it means one can, without trouble, pass to a field extenstion, for example an algebraic closure.

Lemma 2.2.10. *Taking G invariants commutes with flat base change. In particular, forming quotients commutes with flat base change.*

Proof: By definition $\mathcal{F}^G = \{f \in \mathcal{F} | \mu(f) = f \otimes 1\}$ where $\mu\colon \mathcal{F} \to \mathcal{F} \otimes \mathcal{A}(G)$ is the homomorphism corresponding to the G-action. This may be realized as the kernel of

$$\mathcal{F} \to \mathcal{F} \otimes \mathcal{A}(G), \quad f \mapsto \mu(f) - f \otimes 1.$$

But forming kernels commutes with flat base change, and so we are done.

□

The previous lemma already becomes useful in the following specific case. Suppose R is an integral domain with a G-action. Then there is an induced action on Frac(R) given by acting on numerator and denominator. Now, since Frac(R) is flat over R the lemma immediately gives the equality Frac$(R)^G$ = Frac(R^G).

Lemma 2.2.11. *If X is normal so is X/G.*

Proof: The questions is local, so we consider a local integral domain A. The statement to show is that A^G is integrally closed in its field of fractions. Let F = Frac(A). As noted, Lemma 2.2.10 implies Frac$(A^G) = F^G$. Then let $a \in F^G$ be an element integral over A^G. As $F^G \subset F$ we have $a \in F$. But a is integral over A^G, hence also over A, so A being normal implies $a \in A$. Finally, note that $a \in F^G$ means a is G-invariant, hence $a \in A^G$. So A^G is integrally closed in F^G as desired.

□

Lemma 2.2.12. *If X is irreducible, respectively reduced, then X/G is irreducible, respectively reduced. In particular, if X is integral then X/G is integral.*

Proof: For the irreducibility, we note that the quotient morphism $X \to X/G$ is surjective, hence X/G is the continuous image of an irreducible space, hence irreducible. For the reducedness, we may argue locally. The G-invariant subring A^G of a ring A is exactly that: a subring. Hence A reduced implies A^G is reduced. □

Lemma 2.2.13. *If X is of dimension n and G is finite then X/G is of dimension n.*

Proof: If G is finite, then $X \to X/G$ is finite and surjective hence it preserves dimension, so $\dim(X) = \dim(X/G)$. □

As already alluded to, just the existence of a quotient of a given action is already a big question on its own. As such it is desirable to have certain conditions under which one can easily say the quotients exists. Now as noted, the condition required is often that finite collections of points are contained in open affines. This is the AF ("affine finie") property. This is covered in [61, Appendix B]

Definition 2.2.14. A scheme is said to be ***AF*** if every finite set of points is contained in an open affine subscheme.

Proposition 2.2.15. *If $S = \operatorname{Spec} k$ is a field, X has the AF property and G is finite, then X/G exists as a scheme. In particular this holds if X is projective.*

Proof: As X is AF, every finite collection of points in A is contained in an open affine. In particular, any orbit is contained in an open affine, which implies the quotient exists by Theorem 2.2.5 since G is finite. For the final bit note that projective schemes are AF by [61, Appendix B].

□

2.3 Actions and Gradings

As usual, we fix a base scheme S and work relative to this base. Furthermore, M will denote an Abelian group. Our first goal of this section is to prove that actions by $D(M)$ in a certain sense correspond to M-gradings. Let us illuminate the idea by means of an example.

Example 2.3.1. Consider a ring A and suppose it has a $\mathbb{Z}$-grading $A = \bigoplus A_d$. Then we get an induced action of the group $A^\times$ on A by setting $\alpha * a = \alpha^d a$ for $\alpha \in A^\times$, $a \in A_d$ and distributing over addition. Another way of regarding this, is that the $\mathbb{Z}$-grading induces a $\mathbb{G}_m$ action on $\operatorname{Spec} A$. Indeed, the grading of A induces a grading on $A \otimes B$ for any algebra B in a functorial manner, and so an action of $B^\times$ on this tensor product.
Another perspective is the following: Recall that an action of $\mathbb{G}_m$ on X is nothing but a morphism of schemes $\mathbb{G}_m \times X \to X$ satisfying the equalities listed after Definition 2.1.1. Such a morphism is equivalently a homomorphism of rings $A \to \mathbb{Z}[x, x^{-1}] \otimes A$ such that certain diagrams commute, so to give the desired action it is enough to give such a homomorphism. So we simply use the grading on A and define

$$A \longrightarrow \mathbb{Z}[x, x^{-1}] \otimes A, \quad \sum_d a_d \longmapsto \sum_d (x^d \otimes a_d).$$

One could then use the explicit description of the co-multiplication on $\mathbb{Z}[x, x^{-1}]$ found in Example 1.1.5 to check that this satisfies the necessary equalities.

This way of viewing an action of a group scheme as a comodule structure is the viewpoint we will take. So consider $G = D(M)$ over S. Furthermore, take a scheme X affine over the base. Then we recall from Lemma 2.1.4 that an action of G on X is equivalent to giving a comodule structure $\mu\colon \mathcal{A}(X) \to \mathcal{A}(X) \otimes \mathcal{A}(G)$. Note that since $\mathcal{A}(G) = {}_S[M] = \prod_{m \in M} {}_S m$ we may view this as

$$\mu\colon \mathcal{A} \longrightarrow \prod_{m \in M} \mathcal{A} \otimes m_S.$$

So we may view such a morphism as a collection $(\mu_m)_{m \in M}$ of ${}_S$-module endomorphisms of $\mathcal{A}$ indexed by M. Then we can write $\mu = \sum_{m \in M} \mu_m \otimes m$. Note the similarity with the map $A \to \mathbb{Z}[x, x^{-1}] \otimes A$ in the example above, which is a special example of this. Recall also from Definition 2.1.5 that more generally an action of G on any ${}_S$-module $\mathcal{F}$ is just an $\mathcal{A}(G)$-comodule structure on $\mathcal{F}$.

Proposition 2.3.2. *An action of a diagonalizable group scheme $D(M)$ on a quasi-coherent ${}_S$-module $\mathcal{F}$ is equivalent to a grading of $\mathcal{F}$ by M.*

Proof: As noted just before the proof, an action of $G = D(M)$ determines a morphism $\mu\colon \mathcal{F} \to \mathcal{F} \otimes \mathcal{A}(G)$ or equivalently a collection $(\mu_m)_{m \in M}$ of ${}_S$-module endomorphisms of $\mathcal{F}$ indexed by M. For this to determine an action, μ should fit in the commutative diagrams (CM1) and (CM2) of Lemma 2.1.4. Let us express these axiom diagrams in terms of the μ_m using $\mu = \sum_{m \in M} \mu_m \otimes m$. We have

$$\mu \circ \mu = \mu \left(\sum_{m \in M} \mu_m \otimes m \right) = \sum_{m' \in M} \sum_{m \in M} \mu_m \otimes m \otimes m'.$$

Similarly,

$$\Delta \circ \mu = \Delta \left(\sum_{m \in M} \mu_m \otimes m \right) = \sum_{m \in M} \mu_m \otimes m \otimes m.$$

Now, we see that the two expressions agree, i.e. μ satisfies (CM1), if and only if

$$\mu_{m'} \circ \mu_m = \begin{cases} 0 & \text{if } m \neq m' \\ \mu_m & \text{if } m = m'. \end{cases}$$

In more compact notation

$$\mu_{m'} \circ \mu_m = \delta_{mm'} \mu_m \tag{$*$}$$

where $\delta_{mm'}$ is the Kronecker delta. For (CM2), let φ denote the canonical isomorphism $\mathcal{F} \otimes {}_S \to \mathcal{F}$. Then

$$\varphi \circ (\mathrm{id} \otimes e) \circ \mu = \varphi \circ (\mathrm{id} \otimes \varepsilon) \left(\sum_{m \in M} \mu_m \otimes m \right) = \varphi \left(\sum_{m \in M} \mu_m \otimes 1 \right) = \sum_{m \in M} \mu_m$$

So (CM2) is equivalent to requiring

$$\sum_{m \in M} \mu_m = \mathrm{id}. \tag{$*'$}$$

Now, we claim that having a μ satisfying $(*)$ and $(*')$ are equivalent to giving a grading of $\mathcal{F}$ by M. For the first direction, suppose we are given μ satisfying $(*)$ and $(*')$. Set $\mathcal{F}_m = \mathrm{im}\,(\mu_m)$, i.e. $\mathcal{F}$ is the sheafification of the presheaf associating the image of $\mu_m|_U$ to any open set U of S. We claim $\mathcal{F} = \oplus_{m \in M} \mathcal{F}_m$. Note that $\mathcal{F}_m \hookrightarrow \mathcal{F}$ for all m. So we have a morphism

$$\psi\colon \bigoplus_{m \in M} \mathcal{F}_m \longrightarrow \mathcal{F}, \quad (f_m)_{m \in M} \longmapsto \sum_{m \in M} f_m.$$

Note that the sum $\sum f_m$ does make sense as only finitely many f_m are non-zero. Similarly, we have the canonical map $\varphi\colon \mathcal{F} \to \bigoplus_{m\in M} \mathcal{F}_m$ which locally is given by $f \mapsto (\mu_m(f))_{m\in M}$. These two morphisms are inverses. Indeed, locally

$$\psi(\varphi(f)) = \psi((\mu_m(f))_{m\in M}) = \sum_{m\in M} \mu_m(f) \overset{(*')}{=} f.$$

Similarly, locally

$$\varphi(\psi((f_m)_{m\in M})) = \varphi\left(\sum_{m\in M} f_m\right) = \left(\mu_{m'}\left(\sum_{m\in M} f_m\right)\right)_{m'\in M} = \left(\sum_{m\in M} \mu_{m'}(f_m)\right)_{m'\in M}$$

Since $\mathcal{F}_m = \operatorname{im}\mu_m$, and we are working locally, each $f_m = \mu_m(g_m)$ for some g_m of $\mathcal{F}$. So

$$\begin{aligned}\left(\sum_{m\in M} \mu_{m'}(f_m)\right)_{m'\in M} &= \left(\sum_{m\in M} \mu_{m'}(\mu_m(g_m))\right)_{m'\in M}\\ &\overset{(*)}{=} \left(\sum_{m\in M} \delta_{mm'}(\mu_m(g_m))\right)_{m'\in M}\\ &=(\mu_{m'}(g'_m))_{m'\in M}\\ &=(f_{m'})_{m'\in M}.\end{aligned}$$

Thus φ and ψ are canonical inverses, proving $\mathcal{F} = \bigoplus_{m\in M} \mathcal{F}_m$. To construct μ from the decomposition, one defines for $f = \sum f_m$ in $\bigoplus \mathcal{F}_m$ the morphisms $\mu_m(f) = f_m$. That this satisfies $(*)$ and $(*')$ is immediate.

□

Corollary 2.3.3. *An action of $D(M)$ on a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$ is equivalent to an algebra grading of $\mathcal{A}$ by M.*

Proof: It is enough to argue that the grading of Proposition 2.3.2 as an $\mathcal{O}_S$-module respects the algebra structure of $\mathcal{A}$. Note that we now assume μ to be an algebra homorphism, though the μ_m remain only module homomorphisms. Suppose α and β are sections of $\mathcal{A}$ of degree $m, n \in M$ respectively. Then we get, locally,

$$\mu(\alpha\beta) = \mu(\alpha)\mu(\beta) = \mu_m(\alpha)\mu_n(\beta) = (\alpha \otimes m)(\beta \otimes n) = (\alpha\beta) \otimes mn.$$

We here used $\mu(\alpha) = \mu_m(\alpha)$, which is a consequence of $(*)$ of the previous proof. But this computation exactly shows $\alpha\beta$ is of degree mn. □

We emphasize the previous corollary in a specific case:

Corollary 2.3.4. *An action of $D(M)$ on an affine scheme* $\operatorname{Spec} A$ *is equivalent to a grading of A by M.*

This allows us to give multiple examples of group scheme actions in terms of gradings. For example, we can immediately construct actions by $\mathbb{G}_m$ and μ_n simply by giving gradings by $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$. This immediately gives a whole host of actions on polynomial rings and quotients of these.

Example 2.3.5. Consider the ring $R[x]$. The usual grading by $\mathbb{Z}$ via degree of polynomials corresponds via Lemma 2.3.4 to an action of $D(\mathbb{Z}) = \mathbb{G}_m = \operatorname{Spec} R[x, x^{-1}]$. The concrete map is

$$\mu\colon R[x] \longrightarrow R[x] \otimes R[u, u^{-1}], \quad x^n \longmapsto x^n \otimes u^n,$$

which corresponds to $\mu\colon \mathbb{A}^1 \otimes \mathbb{G}_m \to \mathbb{A}^1$. Let us study this on A rational points, for some R-algebra A. Fix $\varphi \in \mathbb{G}_m(A) = \mathrm{Hom}(A[u, u^{-1}], A) = A^\times$ determined by $\varphi(u) = a \in A^\times$. This gives an endomorphism of $R[x]$ given by

$$R[x] \xrightarrow{\mu} R[x] \otimes R[u, u^{-1}] \xrightarrow{\mathrm{id}\otimes\varphi} R[x], \quad x \mapsto ax,$$

Thus, given concrete $\alpha \in \mathbb{A}^1(A) = A$ i.e. a homomorphism $\psi\colon A[x] \to A$ determined by $\psi(x) = \alpha$, we obtain a new homomorphism via $(\psi \circ \mathrm{id}) \otimes (\phi \circ \mu)$ which simply maps x to $a\alpha$. Thus the usual grading of $R[x]$ in terms of polynomial degree is the grading corresponding to the scalar action of $\mathbb{G}_m$ on $\mathbb{A}^1$. This of course generalizes, and we see that the polynomial degree grading of $R[x_1, \dots, x_n]$ induces an action by $\mathbb{G}_m$ which is exactly the multiplication by scalars. Note also that the fixed scheme of this action is exactly the origin and so there is an induced action on $\mathbb{A}^n \setminus \{0\}$ which is free.

Example 2.3.6. Let $R = k$ be a field and consider the ring $k[x, y]/(y^2 - x^3 - a_2x^2 - a_4x - a_6)$. Let us construct a $\mathbb{Z}/2\mathbb{Z} = \langle -1 \rangle$-grading i.e an action of $D(\mathbb{Z}/2\mathbb{Z}) = \mu_2$-action. The k-algebra is generated by the elements x and y, so let us fix these as homogeneous. Whether y has degree 1 or -1, y^2 must have degree 1, hence so must $x^3 - a_2x^2 - a_4x - a_6$. This forces x to be homogeneous of degree 1. Thus there are only two possible choices of grading. This also works if we take a general Weierstrass equation $y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6)$. Now, suppose $\mathrm{char}\, k \neq 2$. By Proposition 1.4.6 we have a canonical isomorphism of group schemes $(\mathbb{Z}/2\mathbb{Z})_k = D(\mathbb{Z}/2\mathbb{Z})$. Since any automorphism of order two would determine an action by the group $\mathbb{Z}/2\mathbb{Z}$, hence also by the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_k$, this implies that any curve defined by a Weierstrass equation (in particular elliptic curves) can have only a single automorphism of order two fixing the distinguished point if $\mathrm{char}\, k \neq 2$.

When studying isolated singularities, it is often useful to pass to a formal neighbourhood. We will need the following lemma specifically for quotient singularities. As we will see later, it will give that the structure of an action is preserved in fixed points.

Lemma 2.3.7. *Suppose $(A_i)_{i\in I}$ is a direct system of R-algebras graded by an abelian group M such that the maps $A_i \to A_j$ are morphisms of graded algebras. Then there is an induced grading on $A = \varinjlim A_i$.*

Proof: Recall that a grading of A_i by M is nothing but a morphism $A_i \to A_i \otimes R[M]$ satisfying certain axioms. By composition, these give maps $A_i \to A \otimes R[M]$ which by the universal property of A induce a unique $A \to A \otimes R[M]$ which by construction must satisfy the necessary axioms. □

The following lemma is short, but incredibly important, as it tells us exactly how one can compute a quotient from an action given in terms of a grading.

Lemma 2.3.8. *Suppose $G = D(M)$ acts on $\mathrm{Spec}\, R$. Then $R^G = R_e$ where $e \in M$ is the neutral element and R_e is the part of R corresponding to e in the M-grading $R = \bigoplus_{m\in M} R_m$.*

Proof: By definition, R_e is the image of μ_e, i.e. it contains exactly those $r \in R$ such that $\mu(r) = r \otimes e$. This is exactly the definition of R^G.

□

The above naturally leads into the following result on existence of quotients by actions of diagonalizable group schemes.

Theorem 2.3.9. *Let M be an abelian group, $G = D(M)$ and $X = \mathrm{Spec}_S\, \mathcal{F}$ a scheme affine over S defined by the sheaf of algebras $\mathcal{F}$. Suppose G acts freely on X. Then the quotient X/G exists and $X/G = \mathrm{Spec}_S\, \mathcal{F}^G = \mathrm{Spec}_S\, \mathcal{F}_e$. If X is of finite type, respectively of finite presentation, then so is X/G.*

Proof: See [18, Exposé VIII, Théorème 5.1] and [18, Exposé VIII, Corollaire 5.8].

□

This shows that understanding gradings can help us compute quotients. As an aside, we will now study how the Proj construction as a quotient of a scheme by $\mathbb{G}_m$. First, let us consider an example. I personally find this example incredibly fascinating, as it puts the "strange" proj construction into a broader context of projective space being defined as a quotient.

Example 2.3.10. In classical topology, one constructs the projective space as the quotient space of lines through the origin. In algebraic geometry however, one classically obtains the projective n-space over an $\mathbb{N}$-graded ring R as $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$. In light of Theorem 2.3.9 we may now see that the Proj construction is simply a different way of interpreting this quotient construction: Consider $\mathbb{A}^2 \setminus \{0\}$ with the action of $\mathbb{G}_m$ as in Example 2.3.5. Then $\mathbb{A}^2 \setminus \{0\}$ has two $\mathbb{G}_m$-invariant open sets covering it, namely $\operatorname{Spec} k[x,y]_x$ and $\operatorname{Spec} k[x,y]_y$ (removing the second and first axes respectively). Said in another way, the grading by degree induces a grading on $k[x,y]_x$ and $k[x,y]_y$. The degree 0 part, i.e. the $\mathbb{G}_m$-invariant subring, of each of these is exactly $k[x,y]_{(x)}$ and $k[x,y]_{(y)}$. Thus the quotient is exactly

$$(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m \cong \mathbb{P}^1.$$

Thus, we have now recovered the Proj construction of $\mathbb{P}^1$ as a quotient similar to the classical topological situation. Of course, one easily obtains $(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m \cong \mathbb{P}^n$ in a similar manner.

The above example works much more generally. We will not directly need this fact later, but it is an interesting observation and we include the formal statement with a proof for the sake of completion. Before we do so, we first make the following elementary observation, which describes when a grading might "collapse" in a quotient:

Lemma 2.3.11. *Let B be graded by an abelian group M and $I \subset B$ an ideal. Then there is an induced grading on B/I by M if and only if I is a homogeneous ideal i.e. generated by homogeneous elements.*

Proof: By an induced grading we mean that we have a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\mu"] \arrow[d] & B \otimes R[M] \arrow[d] \\
B/I \arrow[r, "\mu_I"] & B/I \otimes R[M].
\end{tikzcd}$$

where the bottom arrow is $\mu_I([f]) = \sum [\mu_m(f)] \otimes m$. Of course, this simply means that this μ_I is a well-defined homomorphism. This means that the quotient $B \to B/I$ should preserve degree except when something is mapped to zero.

In the case that I is homogeneous, the grading of B/I is explicitely given as

$$(B/I)_m = B_m/(I \cap B_m).$$

This is covered in most books on commutative algebra such as [10]. The converse is rarely covered in detail, so we give the full argument here: Suppose I is not generated by homogeneous elements. Then there is some $f_1 + \cdots + f_n \in I$ with all $f_i \notin I$ where each f_i is of distinct degree $\deg f_i = m_i$. Now, obviously the residue class $[f_1 + \cdots + f_n] = 0$, so we must have $\mu_I([f_1 + \cdots + f_n]) = 0$. But we also have

$$\mu_I([f_1 + \cdots + f_n]) = \sum_{i=1}^{n} [\mu_{m_i}(f_i)] \otimes m_i = \sum_{i=1}^{n} [f_i] \otimes m_i.$$

Where the last equality is because $f_i \in B_{m_i}$. Now, we claim that this cannot be zero. Indeed, $R[M]$ is free as an R-module, implying $B/I \otimes R[M] = \bigoplus_M B/I$ as an R-module. But this implies, since the m_i are distinct and $f_i \notin I$, that $\sum_{i=1}^n [f_i] \otimes m_i \neq 0$. Thus, μ_I is not well-defined in this case.

$\square$

Proposition 2.3.12. *Suppose B is an $\mathbb{N}$-graded ring. Let $B_+ = \bigoplus_{n>0} B_n$, the irrelevant ideal. Then $V(B_+)$ is the fixed locus of the action by $G = \mathbb{G}_m = D(\mathbb{Z})$ induced by the grading of B, and $(\operatorname{Spec} B \setminus V(B_+))/G = \operatorname{Proj} B$.*

Proof: First, let us show that $Z = V(B_+)$ is exactly the fixed point locus. For this, we show that it is G-invariant, and in the process we will get for free that the action is trivial. Now, saying that Z is G-invariant is the same as saying that the G-action on $\operatorname{Spec} B$ induces one on Z. Equivalently, the grading on B should induce a grading on B/B_+. This simply means B_+ should be a homogeneous ideal, which it is. Furthermore, $B/B_+ = B_0$, so the induced grading is the trivial one. This corresponds to the trivial action, so Z is G-invariant with trivial action i.e. Z is fixed by G. To see that it contains every fixed point, suppose $\mathfrak{p} \in \operatorname{Spec} B$ is fixed by the action. Then $B/\mathfrak{p}$ has an induced grading which is trivial. This implies $\bar{b} = 0$ for any $b \in B_+$. That is, $B_+ \subset \mathfrak{p}$ or equivalently $\mathfrak{p} \in Z$.

For the final part, we begin by noting that the induced action of G on $\operatorname{Spec} B \setminus V(B_+)$ is free, so the quotient exists as a scheme by Theorem 2.3.9. Next, note that $\operatorname{Spec} B$ is covered by $\operatorname{Spec} B_f$ for homogeneous elements f, as B is generated by homogeneous elements. Now, these $\operatorname{Spec} B_f$ are G-invariant because f is homogeneous, so B_f has an induced grading given by setting

$$\deg(f^{-1}) = -\deg(f).$$

Furthermore, $V(B_+) \subset V(f)$ if $\deg f \neq 0$. So $\operatorname{Spec} B_f$ is disjoint from $V(B_+)$ in this case and thus $\operatorname{Spec} B \setminus V(B_+)$ is covered by the affine schemes $\operatorname{Spec} B_f$ with $\deg f \neq 0$. Thus the quotient $(\operatorname{Spec} B \setminus V(B_+))/\mathbb{G}_m$ is glued together from $\operatorname{Spec} B_f^{\mathbb{G}_m}$. Here $B_f^{\mathbb{G}_m}$ is exactly the degree 0 part of B_f, which is also known as the homogeneous localization $B_{(f)}$. But we know $\operatorname{Proj} B$ is exactly glued together from these.

$\square$

Corollary 2.3.13. *Suppose $\mathcal{A}$ is an $\mathbb{N}$-graded quasi-coherent algebra. Let $\mathcal{A}_+ = \oplus_{n\neq 0}\mathcal{A}_n$, the sheaf of irrelevant ideals. Then $V(\mathcal{A}_+)$ is the fixed locus of the action by $\mathbb{G}_m = D(\mathbb{Z})$ on $\operatorname{Spec}_S \mathcal{A}$ induced by the grading of $\mathcal{A}$. Furthermore, $(\operatorname{Spec}_S \mathcal{A} \setminus V(\mathcal{A}_+))/G = \operatorname{Proj}_S \mathcal{A}$.*

Proof: The scheme $\operatorname{Spec}_S \mathcal{A}$ is covered by affine schemes $\Gamma(U, \mathcal{A})$ for U affine. Each of these $\Gamma(U, \mathcal{A})$ is in a functorial way a graded algebra, hence they constitute G-invariant open affines of $\Gamma(U, \mathcal{A})$. As all the questions here are completely local, we are then immediately reduced to the affine case which is Proposition 2.3.12. $\square$

The fixed locus statement in the above can be generalised to any grading by a commutative monoid with no inverses which embeds in an abelian group. However, for general gradings the situation can be more subtle. If the ring B has a non-trivial grading by a finite group or is graded by a group M such that for some m both B_m and B_{-m} are non-empty, then the irrelevant ideal will not actually be an ideal. Indeed, in this case one has for $b \in B_m$ and $b' \in B_{-m}$ that $bb' \in B_e$.

Example 2.3.14. As an explicit example of the above phenomenon, take $k[x, x^{-1}]$ with degree grading. Then $xx^{-1} = 1$ is not in the ideal generated by non-zero degrees. For an even "worse" example, take $R = k[x]/(x^6)$. As notation let $\mathbb{Z}/3\mathbb{Z} = \{1, \omega_3, \omega_3^2\}$ and define a grading on R such that $\deg x = \omega_3$ and $\deg x^2 = \omega_3^2$. Then x^3 has trivial degree, and the set of polynomials of positive degrees is not an ideal.

As the above example alludes to, if the grading monoid has inverses, say if a ring has a grading by $\mathbb{Z}$ with both non-trivial positive and negative parts, then one needs to be careful, and instead consider the ideal *generated* by non-zero parts.

Proposition 2.3.15. *Let $\mathcal{A}$ be an M-graded quasi-coherent algebra and I the sheaf of ideals $I = (\oplus_{n\neq 0}\mathcal{A}_n)$. Then $V(I)$ is the locus of fixed points for the action of $D(M)$ on $\operatorname{Spec}_S \mathcal{A}$ corresponding to the M-grading.*

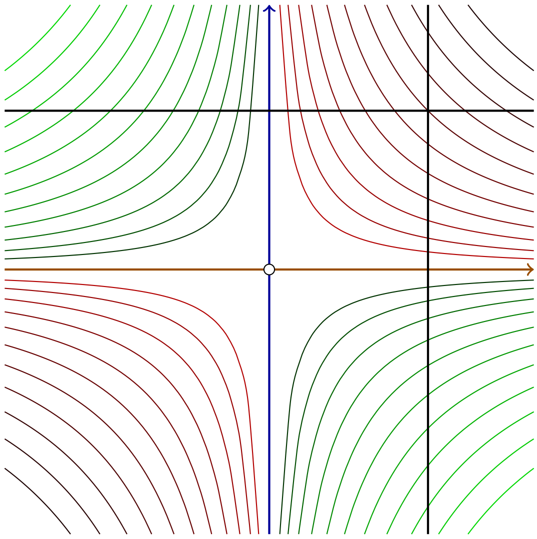
Proof: Proceeds as Proposition 2.3.12. □

Example 2.3.16. Consider the rational cuspidal curve $\operatorname{Spec} R[u^2, u^3] \cup \operatorname{Spec} R[u^{-1}]$ and let us compute the fixed points of a concrete μ_2-action. Such an action corresponds to a grading by $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$. Set the degree of u to be 1. Formally, the element u is in neither ring, but both embed into $R[u, u^{-1}]$. So we may give a grading on this ring, making sure that it induces one on both $R[u^2, u^3]$ and $R[u^{-1}]$. This will ensure that the grading i.e. action is compatible on overlaps, and that the two affine charts are μ_2-invariant. With the degree $\deg u = -1$, we get $\deg u^2 = 1$ and $\deg u^3 = \deg u^{-1} = -1$. It follows that the fixed points on the $R[u^2, u^3]$ chart are given by the ideal (u^3) and the fixed points on the $R[u^{-1}]$ chart are given by the ideal (u^{-1}). Thus each chart has a single fixed point. Note that these points are distinct.

Example 2.3.17 (The affine line with two origins as a quotient)**.** For ease, take a field as the base, and consider $\mathbb{A}^2$ with the $\mathbb{G}_m$ action given by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$. This corresponds to a true $\mathbb{Z}$-grading of $A = k[x, y]$ where x is homogenous of degree 1, and y is homogeneous of degree -1. Thus $A_d = kx^d \oplus kx^{d+1}y \oplus \cdots$ for $d \geq 0$ and $A_d = ky^d \oplus ky^{d+1}x \oplus \cdots$ for $d \leq 0$. Looking topologically, the fixed locus should be the origin, which we confirm algebraically by noting that $I = \left(\bigoplus_{d\neq 0} A_d\right) = (x, y)$. It follows that the action is free on $\mathbb{A}^2 \setminus \{0\}$. Furthermore, the action is invariant on each of the affine charts $U_1 = \operatorname{Spec} k[x, y]_x$ and $U_2 = \operatorname{Spec} k[x, y]_y$ which are the affine plane with each of the axes removed. So we may use these to compute the quotient $(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m$ using these charts. As the G-invariant part is the degree 0 part we have

$$k[x, y]_x^G = k[xy] \quad \text{and} \quad k[x, y]_y^G = k[xy]$$

At first glance one might be tempted into thinking that since these rings are the same, the quotient is just an $\mathbb{A}^1$, but one must be careful here, keeping in mind the gluing datum. The charts U_1 and U_2 are glued by along $U_1 \cap U_2 = \operatorname{Spec} k[x, y]_{xy}$, so $\pi(U_1)$ and $\pi(U_2)$ are glued along $\operatorname{Spec} k[x, y]_{xy}^G = k[xy]_{xy}$. Algebraically speaking, every prime ideal of $k[xy]$ has a corresponding prime ideal in $k[xy]_{xy}$ except for the ideal (xy). Geometrically, this means that the lines $\pi(U_1)$ and $\pi(U_2)$ are identified at every point, except for the origin. Thus the resulting quotient scheme $(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m$ is in fact the affine line with two origins.



The above drawing illustrates the orbits of rational points with an orbit consisting of all lines of the same color. Two of the orbits are the coordinate axes, while the rest are the orbits of diagonal points. Two black lines have been drawn to show how one can parametrize all points but one in two distinct ways, illustrating how the two coordinate axes yield the double origin.

2.4 Group Scheme Actions and Lie Algebras

In this section we will take a look at Lie algebras in algebraic geometry. Specifically, we will give a brief overview how one may reinterpret actions by group schemes in terms of Lie algebras and furthermore how this can sometimes be used to compute quotients. This method of computing quotients will be the one we apply in chapter 5. In fact, we will work with something slightly more general than group schemes. Instead, we will talk about functors of groups, which will implicitely mean functors from the category of R-algebras to the category of groups where R is a fixed base ring. The reason for this level of generality is that with this setting we do not have to worry about representability of the functor $\mathrm{Aut}_{X/R}$. Recall that for a scheme X over a ring R, the automorphism functor $\mathrm{Aut}_{X/R}$ is the functor of groups which to an R-algebra A associates the automorphism group $\mathrm{Aut}_{X/R}(A) = \mathrm{Aut}_A(X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, A)$. We will be brief and refer to [16, II,§4 and §7] for further reading and details. Let us now see how one may express group scheme actions in terms of this functor of groups.

Remark 2.4.1 (Group scheme actions in terms of $\mathrm{Aut}_{X/R}$)**.** Now suppose we have a group scheme G acting on X. The statement that G acts on X, means that for each R-scheme T each element of the group $G(T)$ determines an automorphism of $X(T)$. As usual it is enough to consider affine schemes over $\mathrm{Spec}\, R$. That is, for each R-algebra A we have in a functorial way a homomorphism of groups $G(A) \to \mathrm{Aut}(X(A)) = \mathrm{Aut}(X_{\mathrm{Spec}\, A}(A))$. By functoriality, this is a morphism of functors of groups $G \to \mathrm{Aut}_{X/R}$.

In the following we will write $R[\varepsilon]$ for the ring of dual numbers $R[x]/(x^2)$ over R.

Definition 2.4.2. Let G be a functor from the category of R-algebras to the category of groups. Let $p\colon R[\varepsilon] \to R$ be the quotient homomorphism. We define the ***Lie algebra*** of G, denoted $\mathrm{Lie}(G)$, to be the kernel $\mathrm{Lie}(G) = \ker(G(R[\varepsilon]) \xrightarrow{G(p)} G(R))$.

We remark that this definition is of course functorial in R. As such, one also gets a Lie algebra functor. We will not get into this in more detail, but will simply refer to [16, II, §4]. Next we compute a few examples:

Example 2.4.3 (The Lie algebra of a diagonalizable group scheme)**.** Consider an abelian group M and its diagonalizable group scheme $G = D(M)$. Recall from Section 1.4 that for any R-algebra A we have $D(M)(A) = \mathrm{Hom}_{\mathrm{Grp}}(M, A^\times)$. We have $R[\varepsilon]^\times = \{a + \varepsilon b \mid a \in R^\times, b \in R\}$ since $a + \varepsilon b$ has inverse $(a^{-1} + \varepsilon a^{-2}(-b))$. Thus, by definition, $\mathrm{Lie}(D(M))$ consists of those morphisms $\mathrm{Hom}_{\mathrm{Grp}}(M, R[\varepsilon]^\times)$ which map to the morphism $M \to R^\times$ defined by being identically 1. It follows that mapping $\mathrm{Hom}_{\mathrm{Grp}}(M, R) \to D(M)(R(\varepsilon))$ via $\varphi \mapsto 1 + \varepsilon\varphi$ identifies $\mathrm{Hom}_{\mathrm{Grp}}(M, R)$ with $\mathrm{Lie}(D(M))$. In particular we have

$$\mathrm{Lie}(\mathbb{G}_m) = \mathrm{Lie}(D(\mathbb{Z})) = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}, R) = R.$$

Example 2.4.4 (The Lie algebra of μ_n)**.** The example above gives one description of the Lie algebra of $\mu_n = D(\mathbb{Z}/n\mathbb{Z})$ as

$$\mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}/n\mathbb{Z}, R) = \{r \in R \mid nr = 0\}.$$

It is a good exercise to compute the Lie algebra of this directly from the definitions. Recall from Example 1.2.4 that $\mu_n(A) = \{a \in A^\times \mid a^n = 1\}$. So

$$\mu_n(R[\varepsilon]) = \{a + \varepsilon b \in R[\varepsilon] \mid a \in R^\times,\ b \in R,\ (a + \varepsilon b)^n = 1\}.$$

As $\varepsilon^2 = 0$ we get

$$(a + \varepsilon b)^n = a^n + \sum_{i=1}^{n} \binom{n}{i} a^{n-i}(\varepsilon b)^i = a^n + n\varepsilon a^{n-1} b.$$

The condition $(a + \varepsilon b)^n = 1$ then is equivalent to $nb = 0$. Thus, the Lie algebra of μ_n i.e. the kernel of $\mu_n(R[\varepsilon]) \to \mu_n(R)$, is

$$\mathrm{Lie}(\mu_n) = \{1 + \varepsilon b \in R[\varepsilon] \mid nb = 0\} = \{b \in R \mid nb = 0\}.$$

Note the special case where R is of positive characteristic p and $n = p$. In this case the above simplifies as $pb = 0$ is always satisfied. So in this case we have

$$\mathrm{Lie}(\mu_p) = \{1 + \varepsilon b \in R[\varepsilon]\} \cong \varepsilon R \cong R.$$

Example 2.4.5 (The Lie algebra of α_p)**.** We now consider a base R of characteristic $p > 0$. Recall from Example 1.2.5 that $\alpha_p(A) = \{a \in A | a^p = 0\}$. For $R[\varepsilon]$ these are all $a + \varepsilon b$ such that

$$0 = (a + \varepsilon b)^p = a^p + \varepsilon^p b^p = a^p.$$

As the kernel are all those $a + \varepsilon b$ with $a = 0$ we get

$$\mathrm{Lie}(\alpha_p) = \varepsilon R.$$

Remember that our goal is to reinterpret certain group actions in terms of Lie algebras. With that perspective the previous two examples are a bit demoralizing, as they show μ_p and α_p have Lie algebras which are isomorphic, not just as sets but as groups! So clearly the Lie algebra as we have defined it is not enough on its own to distinguish these two group schemes. As in differential geometry, the Lie algebra comes with an additional structure: The Lie bracket. One could hope this would be the distinguishing feature, but unfortunately this is not enough. Indeed, they are both 1-dimensional, and so the Lie bracket will be trivial. There is however an additional structure distinguishing them, namely the so-called p-map. We will return briefly to this point in a bit, but first we focus on the Lie algebra of the functor $\mathrm{Aut}_{X/R}$.

Proposition 2.4.6. *Let X be a scheme over a ring R. There is an isomorphism between* $\mathrm{Lie}(\mathrm{Aut}_{X/R})$ *and* $\mathrm{Der}_R({}_X, {}_X)$.

Proof: This is [16, II, §4, 2.4 Proposition]. We give the morphism in each direction and otherwise refer the reader to the reference given. Suppose we are given a derivation $d\colon {}_Y \to {}_Y$, then we obtain an automorphism of ${}_{X_{R[\varepsilon]}} = {}_X \oplus \varepsilon_X$ by

$$d \mapsto (\varphi\colon {}_X \oplus \varepsilon_X \to {}_X \oplus \varepsilon_X,\;\; a + \varepsilon b \mapsto a + \varepsilon(b + da))\,.$$

Conversely, suppose we are given $\psi \in \mathrm{Lie}\,\mathrm{Aut}_{X/R}$. This is then an automorphism of $X_{R[\varepsilon]}$ which becomes the identity when pulled back to R. Suppose we have a global section f of ${}_X$. We may view this a global section of ${}_{X_{R[\varepsilon}} = {}_X \oplus \varepsilon_X$ by $f = f + \varepsilon 0$. Then we can apply ψ to this and obtain some $\psi(f) = a + \varepsilon b$. We then set $D_\psi(f) = b$. □

As $\mathrm{Der}_R({}_X, {}_X) = \mathrm{Hom}_X(\Omega^1_{X/R}, {}_X) = \mathrm{H}^0(X, \Theta_{X/R})$,m where $\Theta_{X/R}$ is the dual sheaf of $\Omega_{X/R}$, this result gives us hope that we can interpret certain group scheme actions in terms of $\mathrm{H}^0(X, \Theta_{X/R})$.

Now, we can not avoid mentioning so-called ***restricted Lie algebras*** or ***p-Lie algebras***. We take most of this content as a complete blackbox, referring to [16, II,§7] for the details. The results here are presented merely for the sake of internal reference and to give a broad overview of the ideas. In broad terms, a restricted Lie algebra is a Lie algebra L equipped with a map $L \to L$ usually denoted $x \mapsto x^{[p]}$ satisfying certain axioms with respect to scalar multiplication and the adjoint operator which may also be interpreted in terms of the Lie bracket. A vector $x \in L$ is said to be ***p-closed*** if $x \neq 0$ and $x^{[p]} = \lambda x$ for some $\lambda \in k$, i.e the p-map fixes the linear span of x which is thus a restricted sub-Lie algebra. See [16, II, §7, 3.3] for precise definitions. See also [62, section 1] [46, Section 1] and [64, Section 1] for further perspectives. The reason we are interested in restricted Lie algebras at all are the following results:

Proposition 2.4.7. *Let R be a ring of characteristic $p > 0$. For any group scheme G over R the Lie algebra* $\mathrm{Lie}(G)$ *is in a canonical way a restricted Lie algebra. Furthermore, if X is a scheme over R and $G \to \mathrm{Aut}_{X/R}$ a homomorphism of functors of groups, then the induced* $\mathrm{Lie}(G) \to \mathrm{Der}(X) = \mathrm{H}^0(X, \Theta_{X/R})$ *is a homomorphism of restricted Lie algebras.*

Proof: This is [16, II, §7, 3.4 Proposition]. □

Theorem 2.4.8. *Let R be a ring of characteristic $p >$, X a scheme over R and G be a group scheme over R of height ≤ 1. For each homomorphism of restricted Lie algebras $\psi\colon$* $\mathrm{Lie}(G) \to \mathrm{Der}({}_X) = \mathrm{H}^0(X, \Theta_{X/R})$ *there is a morphism $\varphi : G \to \mathrm{Aut}_{X/R}$, i.e. an action of G on X.*

Proof: See [16, II, §7, 3.10]. □

Note that the p-map of $\mathrm{Der}({}_X)$ is p-fold composition. What this theorem tells us, is that to give an action of a group scheme of height ≤ 1, it is enough to give such a morphism. For α_p and μ_p this simplifies further. It is shown in [16, II, §7, 2.2] that the p-map of α_p is $x^{[p]} = 0$ and that for μ_p we have $\mathrm{Lie}(\mu_p) = \mathrm{Hom}_{\mathrm{grp}}(\mathbb{Z}/p\mathbb{Z}, R)$ and the p-map is p-fold compositiion.

Corollary 2.4.9. *Let R be a ring of characteristic $p > 0$ and let X be a scheme over R.*

(i) Actions of μ_p on X correspond one-to-one to p-closed sections $\delta \in \mathrm{H}^0(X, \Theta_{X/R})$ satisfying $\delta^{[p]} = \delta$;

(ii) Actions of α_p on X correspond one-to-one to p-closed sections $\delta \in \mathrm{H}^0(X, \Theta_{X/R})$ satisfying $\delta^{[p]} = 0$.

Proof: We know that $\mathrm{Lie}(\alpha_p)$ is the free module εR with vanishing p-map and $\mathrm{Lie}(\mu_p)$ is the free module εR with p-map corresponding to p-fold composition. Let G denote either α_p or μ_p. As both Lie algebras are free modules of rank 1, it follows that to define a morphism $\mathrm{Lie}(G) \to \mathrm{H}^0(X, \Theta_{X/R})$ (indeed into any other Lie algebra) it suffices to specify a single section of $\delta \in \mathrm{H}^0(X, \Theta_{X/R})$ as the target of the generating element of $\mathrm{Lie}(G)$. For this morphism to be a morphism of restricted Lie algebras, this δ must satisfy the identities of the p-map of G. For α_p this gives $\delta^{[p]} = 0$. For μ_p the p-map is trivial on the generating vector, giving the desired. That the correspondence is one-to-one is contained in [16, II, §7, 3.12 Corollary]. $\square$

The following proposition is the reason behind this whole section. It tells us that we may compute certain quotients of group scheme actions by computing kernels of derivatons. This is the method employed in Chapter 5.

Proposition 2.4.10. *Let R be a ring of characteristic $p > 0$ and $X = \operatorname{Spec} A$ an affine scheme over R. Let $\delta \in \mathrm{Der}(A) = \mathrm{H}^0(X, \Theta_{X/R})$ determine a $G = \alpha_p$ or $G = \mu_p$ action. Then $X/G =$ $\operatorname{Spec} \ker \delta$.*

Proof: The quotient exists by Proposition 2.2.8. The rest is [16, III,§2,6.4]. $\square$

Example 2.4.11. Let R be a ring of characteristic $p = 2$ and consider the rational cuspidal curve $C = \operatorname{Spec} R[u^2, u^3] \cup \operatorname{Spec} R[u^{-1}]$. It is outlined in the proof of [62, Proposition 3.2] that the Lie algebra $\mathrm{H}^0(X, \Theta_{X/R})$ is 4 dimensional with generators $u^{-2}D_u$, D_u, uD_u and u^2D_u. It also follows from the computations of [62, Proposition 3.1] that each vector of $\mathrm{H}^0(X, \Theta_{X/R})$ is p-closed. It follows that any vector field

$$\delta = (\lambda_0 u^2 + \tau u + \lambda_2\lambda_4 u^{-2})D_u = (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \tau u^{-1} + \lambda_0)D_{u^{-1}}$$

determines an action of either α_2 or μ_2. It is shown in the Proposition 3.2 loc. cit. that the p-map is trivial precisely on the span of u^2D_u, D_u and $u^{-2}D_u$. Thus, δ determines an action of α_2 if and only if $\tau = 0$ and an action of μ_2 otherwise. Now, the quotient C/G, with G either α_2 or μ_2, exists by Proposition 2.2.8. As the vector field descends to a derivation on both $R[u^2, u^3]$ and $R[u^{-1}]$, it follows that the charts determined by these rings are G-invariant. So we may use Proposition 2.4.10 to compute the quotient.

So consider the derivation $\delta\colon R[u^{-1}] \to R[u^{-1}]$ determined by $\delta = (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \tau u^{-1} + \lambda_0)D_{u^{-1}}$. Then

$$\delta(u^{-n}) = \lambda_4 n u^{-n-3} + \lambda_2 n u^{-n-1} + \tau n u^n + \lambda_0 u^{-n+1}.$$

This implies that $\ker \delta = R[u^{-2}]$. A similar computation shows that the kernel on the $R[u^2, u^3]$ chart is $R[u^2]$. It follows that $C/G = \mathbb{P}^1$.

Chapter 3

Surface Theory

3.1 Rational Double Points

Fix a base field k. We assume, for now, that it is algebraically closed. To further fix terminology, a ***surface*** will mean a irreducible scheme of finite type and dimension 2. In this section we will take a look at a specific type of singularity occuring on surfaces. A big part of chapter 5 is analyzing exactly which singularity types arise in the generalized Kummer construction, so we give here a brief introduction to the subject. The book [15], specifically Chapter 3 and 4, are good sources for further reading.

Note that while resolution of singularities is in general an open question in arbitrary characteristic (and we specifically care about characteristic 2 in the end), resolutions of surface singularities exist also in positive characteristic by [1] and [2].

Definition 3.1.1. A normal surface singularity is a pair (Y, y) such that Y is a normal surface and y is a closed point. A ***resolution*** of (Y, y) is a proper morphism $\pi\colon X \to Y$ such that X is regular in a neighbourhood of $\pi^{-1}(y)$ and $\pi|_{X\setminus\pi^{-1}(y)}$ is an isomorphism. In particular, such a resolution is birational. The singularity is said to be ***rational*** if $R^1\pi_{*X} = 0$.

We usually simply write Y with the closed point y being understood. In the setting of proper surfaces, rationality may be reinterpreted in the following way:

Proposition 3.1.2. *Let Y be a proper, normal surface and X a proper, smooth surface. Suppose we have a morphism $\pi\colon X \to Y$ such that there is a closed point $y \in Y$ such that π is an isomorphism outside of $\pi^{-1}(y)$ and $\pi^{-1}(y)$ is a curve, e.g. Y has a singularity at the point y and π is the resolution. Then $R^1\pi_{*X} = 0$ if and only if $\chi({}_X) = \chi({}_Y)$.*

Proof: We have $\pi_{*X} = {}_Y$ by [69, Tag 0AY8]. Since the fibers are all of dimension at most one $R^q\pi_{*X} = 0$ for $q \geq 2$. The Leray-Serre spectral sequence $E_2^{pq} = \mathrm{H}^p(Y, R^q\pi_{*X}) \Rightarrow \mathrm{H}^{p+q}(X, {}_X)$ gives us an exact sequence

$$0 \to \mathrm{H}^1(Y, {}_Y) \to \mathrm{H}^1(X, {}_X) \to \mathrm{H}^0(Y, R^1\pi_{*X}) \to \mathrm{H}^2(Y, {}_Y) \to \mathrm{H}^2(X, {}_X) \to 0.$$

By additivity of the Euler characteristic this implies that $\chi({}_Y) - \chi({}_X) = \dim \mathrm{H}^0(Y, R^1\pi_{*X})$. By definition, the global sections of $R^1\pi_{*X}$ are exactly its stalk at y so $\dim \mathrm{H}^0(Y, R^1\pi_{*X}) = \dim(R^1\pi_{*X})_y$ which is zero if and only if $R^1\pi_{*X} = 0$ since as noted the sheaf is supported only at y. $\square$

See also [15, Lemma 3.8] for further equivalent statements.

Example 3.1.3. Consider $Y = \operatorname{Spec}(k[x,y,z]/(xy-z^2))_{(x,y,z)}$ and let $\pi\colon X \to Y$ be the blowing up in the point p corresponding to (x,y,z). One can show that X is smooth, that π is a minimal resolution and that $\pi^{-1}(p) = \mathbb{P}^1$. The theorem of formal functions [34, III, Theorem 11.1] tells us that

$$(R^1\pi_{*X})^\wedge_p = \varprojlim_n \mathrm{H}^1(X_n, ({}_X)_n)$$

where $X_n = X \times_Y \operatorname{Spec}{}_{Y,y}/\mathfrak{m}_y^n$ is the thickened fiber and $({}_X)_n$ is the pullback of ${}_X$ to X_n, which is just ${}_{X_n}$. In our case, since ${}_{Y,y}/\mathfrak{m}_y = k$, we may write

$$X_n = X \times_Y \operatorname{Spec}{}_{Y,y}/\mathfrak{m}_y \times_k \operatorname{Spec}{}_{Y,y}/\mathfrak{m}_y^n = \mathbb{P}^1 \times_k \operatorname{Spec}{}_{Y,y}/\mathfrak{m}_y^n.$$

Then

$$\mathrm{H}^1(\mathbb{P}^1 \times_k \operatorname{Spec}{}_{Y,y}/\mathfrak{m}_y^n, {}_{X_n}) = \mathrm{H}^1(\mathbb{P}^1, {}_X) \otimes {}_{Y,y}/\mathfrak{m}_y^n = 0.$$

This implies $(R^1\pi_{*X})_p = 0$. As π is an isomorphism outside the exceptional divisor we have $R^1\pi_{*X} = 0$.

We briefly recall the notion of multiplicity and Gorenstein. Consider a ring A and an A-module M. The ***length*** of M, denoted $\operatorname{length}(M)$, is the length of the longest sequence of submodules $M_0 \subset \cdots \subset M_n$ for which all the quotients M_i/M_{i-1} are simple. If there is no upper bound, M is of infinite length. Otherwise it is of finite length n. Suppose now A is finite dimensional and local with maximal ideal $\mathfrak{m}$. For sufficiently large m the length $\operatorname{length}(A/\mathfrak{m}^{m+1})$ is a polynomial function in m of degree $d = \dim A$. The ***multiplicity*** of A is then the leading coefficient of this polynomial times $d!$. Now set $m = \dim A$. The local ring A is said to be ***Gorenstein*** if $\operatorname{Ext}^n(A/\mathfrak{m}, A) = A/\mathfrak{m}$ and $\operatorname{Ext}^i(A/\mathfrak{m}, A) = 0$ for $i \neq m$. If A is of dimension 2 and normal being Gorenstein is equivalent to $\omega_{\operatorname{Spec} A}$ being trivial on $\operatorname{Spec} A \setminus \{\mathfrak{m}\}$, see [15, Corollary 3.13].

Definition 3.1.4. A rational singularity Y is a ***rational double point*** if it satisfies the following equivalent statements

(i) ${}_{Y,y}$ has multiplicity 2;

(ii) ${}_{Y,y}$ is a Gorenstein local ring.

For the equivalence of these statements, see [15, Corolary 4.19]. In my native tongue we have a saying which roughly translates to "A child held dear goes by many names" and this is certainly the case with rational double points. Among other names, they are also known as rational Gorenstein singularities, ADE-singularities, Du Val singularities and Kleinian singularities. Let us consider an example:

Example 3.1.5. Take the rational singularity defined by $A = (k[x,y,z]/(xy-z^2))_{(x,y,z)}$. We have equalities

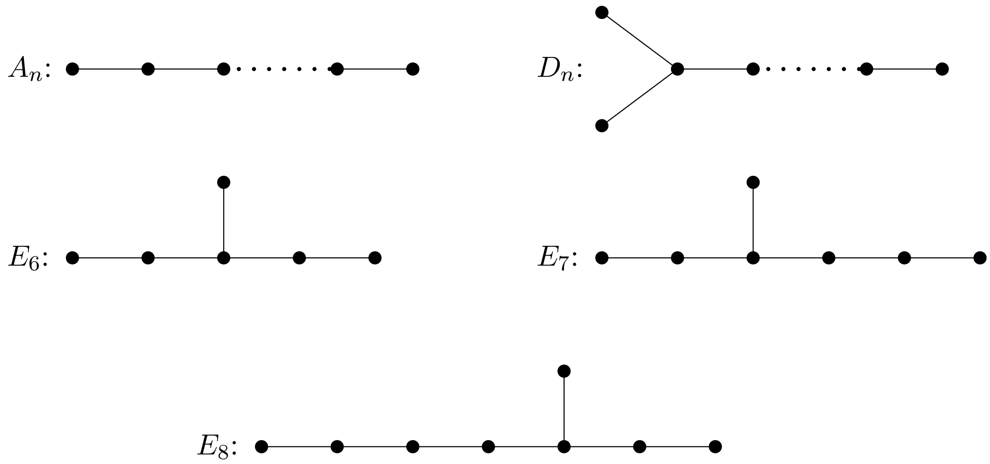
$$\operatorname{length}(A/\mathfrak{m}^{m+1}) = \sum_{i=0}^{m} \operatorname{length}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \sum_{i=0}^{m} \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

Now, for $i = 0$ we have $\dim_k A/\mathfrak{m} = 1$ and for $i \geq 1$ we have $\dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1} = \binom{2+i}{2} - \binom{i}{2} = 1+2i$. Thus

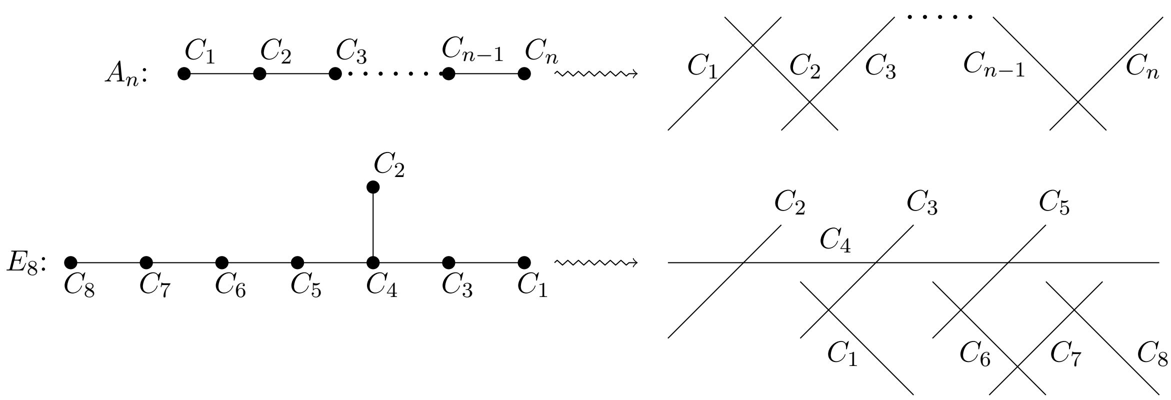
$$\sum_{i=0}^{m} \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1} = 1 + \sum_{i=1}^{n}(1+2i) = 1 + \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i = 1 + n + n\cdot(n-1) = 1 + n^2.$$

As this polynomial is of degree 2 with leading coefficient 1, it follows that the multiplicity of A is 2.

Remark 3.1.6 (Exceptional fibers and Dynkin diagrams)**.** Suppose we have a curve C with irreducible components $C_1, \ldots, C_n$. Then one can simply draw a graph of nodes and edges where one views each node as a curve, and each edge as describing an intersection. For exceptional divisors of rational double points, we obtain simply laced Dynkin diagrams. Given such an exceptional divisor, the Dynkin diagram describing its configuration of irreducible curve is called its ***dual graph***. The possible diagrams we will see are the simply laced Dynkin diagrams, i.e. those with no multiple edges. These come in the following five types, where n denotes the number of nodes:



Here each node corresponds to a copy of $\mathbb{P}^1$ with self-intersection -2 and the edges indicate how these curves intersect. For example the Dynkin diagrams of type A_n and E_8 translate to



Rational double points were studied in three papers by Du Val [20], [21], and [22], explaining one of their many names. He showed that resolving a rational double point on a projective surface yields an exceptional divisor divisor with dual graph A_n, D_n, E_6, E_7 or E_8. Later, Artin [3, Theorem 2.7] proved the following. See also [15, Theorem 3.15].

Theorem 3.1.7. *Let X be a proper, normal surface and $E \subset X$ a curve with irreducible components $E_1, \ldots, E_n$. Then the following are equivalent*

(i) There exists a morphism $\pi\colon X \to Y$ such that Y is normal, $\pi(E) = y$ is a Gorenstein point on Y, $\pi\colon X \setminus E \to Y \setminus \{y\}$ is an isomorphism, and $\pi^*(\omega_Y) \cong \omega_X$;

(ii) The intersection matrix $(E_i \cdot E_j)_{i,j}$ is negative definite, the E_i are nonsingular rational curves, and $E_i^2 = -2$ for all i.

Moreover, if either of these holds $\chi(_X) = \chi(_Y)$.

In [3] is mentioned the following consequence. See also [15, Theorem 3.32].

Corollary 3.1.8. *A proper, normal surface singularity is a rational double point if and only if it is of type A_n, D_n, E_6, E_7, or E_8.*

This equivalence explains why rational double points are also called ADE-singularities. In [4] Artin gives a way to compute multiplicity of points in using the so-called fundamental cycle of their exceptional divisors. Using this, one may separately show that any exceptional divisor with dual graph A_n, D_n, E_6, E_7 or E_8 comes from a point of multiplicity 2.

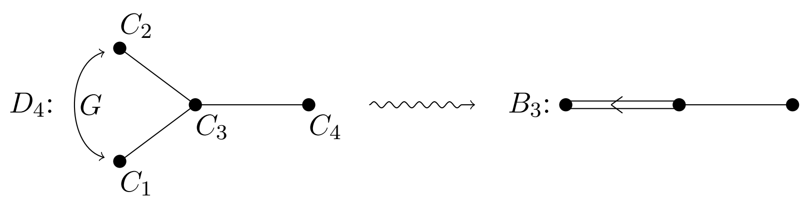
Remark 3.1.9 (Infinitesimal study)**.** An important tool in analyzing singularity types is that of formal neighbourhoods. Suppose $Y = \operatorname{Spec} A$ is a surface singularity. Then $\widehat{Y} = \operatorname{Spec} \widehat{A}$ is called a ***formal neighbourhood*** of the singular point $\mathfrak{m}$, where $\widehat{A}$ denotes the formal completion with respect to the maximal ideal $\mathfrak{m}$. The result [15, Lemma 4.2] tells us that resolving the singularity $\widehat{Y}$ yields the same exceptional divisor as when resolving Y. As such, it is sufficient to determine singularity types after passing to an infinitesimal neighbourhood. This allows for computations in power series rings $k[|x, y, z|]$ rather than just polynomial $k[x, y, z]$. This is useful, as simplifying polynomial equations is easier in such rings. We will see concrete examples of such simplifications in the proofs of Proposition 5.2.5 and Proposition 5.2.8. It also allows for neat lists of simplified equations determining the various singularities. These are called the **normal forms**. Tables can be found in [27, Tables II.1 and II.2]. Some computations and a less neatly arranged list may be found in [8]. For example, our earlier example $xy - z^2$ is the normal form of an A_1 singularity.

Now, what if the base field k is not algebraically closed? The definition rational double point still makes sense, but different configurations may show up in the exceptional divisor.

Remark 3.1.10 (Computing singularity types after base change)**.** Consider a separable polynomial $f \in k[x, y, z]$ defining a rational surface singularity Y. Suppose L/k is a Galois extension. Base changing Y to $Y \times \operatorname{Spec} L$ gives a singularity over L. It turns out we can analyze the singularity Y using $Y \times \operatorname{Spec} L$ and what we know about quotients. We have an action of the Galois group $G = \operatorname{Gal}(L/k)$ on $Y \times \operatorname{Spec} L$ where the quotient exists with

$$(Y \times \operatorname{Spec} L)/G = Y \times \operatorname{Spec} k = Y.$$

Now let $X \to Y$ be a minimal resolution with exceptional divisor E. By [69, Tag 085S] taking a blow-up commutes with flat base change, so the resolution of $Y \times \operatorname{Spec} L$ is simply $X \times \operatorname{Spec} L$ with exceptional divisor $E \times \operatorname{Spec} L$. Furthermore, $X = (X \times \operatorname{Spec} L)/G$ and $E = (E \times \operatorname{Spec} L)/G$. This is useful, as it allows us to study singularities by first passing to a base change and then looking at the Galois action on the irreducible components of the exceptional divisor. The upshot is that the dual graph may no longer be simply laced. For example, given $E \times \operatorname{Spec} L$ with irreducible components $C_1, \ldots, C_4$ in a D_4 configuration the Galois action may identify C_1 and C_2, which makes the quotient E of B_3 type



We will see a very concrete example of this principle in Proposition 5.2.4.

3.2 The Conditions (S_i) and (R_i)

When studying quotients of families it is desirable to have certain base change properties. Ideally, the fibres of the quotient would be the quotient of the fibres. This is true for quotients by finite diagonal group schemes, see Theorem 4.1.5. More generally though, it does not hold, and it takes some work to settle whether this is the case or not. It turns out, see Theorem 4.1.7, that the S_2 condition is useful in certain cases.

We first outline the (S_i)- and (R_i)-conditions following [31, 5.7 and 5.8]. In the following we will need the notion of depth which we quickly recall. For more details, we refer to [30, **0** 16.4] or [34, §3]. Let us fix notation. Let A be a Noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. Recall that a sequence of elements $f_1, \dots, f_n \in A$ is said to be ***M-regular*** or ***regular*** for M if a_i is a non-zero divisor on $M/(f_1, \dots, f_{i-1})M$ i.e. $f_i m = 0$ implies $m = 0$ in $M/(f_1, \dots, f_{i-1})M$. The I-depth of M, denoted $\operatorname{depth}_I M$, is then the largest integer n possible for any M-regular sequence. We refer to the texts loc. cit. for the arguments that this is well-defined i.e. does not depend on the choice of M-regular sequence. If A is local with maximal ideal $\mathfrak{m}$ we simply refer to the $\mathfrak{m}$-depth as the depth. It is a fact [34, Corollary 3.6] that $\operatorname{depth}_I M = \inf_{\mathfrak{p} \in V(I)} \operatorname{depth} M_{\mathfrak{p}}$. Thus, one generalizes depth to schemes in the following way: Suppose X is locally Noetherian, $Y \subset X$ a closed subset and $\mathcal{F}$ a coherent sheaf on X. Then the Y-depth of $\mathcal{F}$ is defined as $\operatorname{depth}_Y \mathcal{F} = \inf_{x \in Y}(\operatorname{depth} \mathcal{F}_x)$.

Definition 3.2.1. A Noetherian ring A is said to satisfy condition (S_i) for an integer i if for all $\mathfrak{p} \in \operatorname{Spec} A$ we have

$$(S_i)\colon\ \operatorname{depth}(A_{\mathfrak{p}}) \geq \inf\{i, \dim A_{\mathfrak{p}}\}.$$

A locally Noetherian scheme X is said to satisfy (S_i) if $_{X,x}$ satisfies (S_i) for all $x \in X$.

There is a corresponding definition for finitely generated modules. We simply give the general definition for sheaves, of which the module definition is a special case. Recall that the dimension of a finitely generated module M is defined as $\dim(A/\mathrm{Ann} M)$. Obviously the dimension of a ring coincides with its dimension as a module over itself.

Definition 3.2.2. Let X be locally Noetherian scheme and a coherent sheaf $\mathcal{F}$ on a locally Noetherian scheme X. We say that $\mathcal{F}$ satisfies (S_i) if for all $x \in X$

$$(S_i)\colon\ \operatorname{depth}(\mathcal{F}_x) \geq \inf\{i, \dim \mathcal{F}_x\}.$$

The other type of conditions, the so-called (R_i)-conditions, are defined as follows:

Definition 3.2.3. A locally Noetherian scheme X is said to satisfy (R_i) or to be ***regular in codimension*** $\leq i$ if

$$(R_i)\colon\ {}_{X,x} \text{ is regular whenever } \dim {}_{X,x} \leq i.$$

Equivalently, the set of non-regular points is of codimension $> i$.

One can apply these conditions in "S_2-type argument". Essentially, an "S_2-type argument" proceeds by arguing that a certain property can be checked outside of a codimension 2 subset. In the case of surfaces, we will see that this usually amounts to arguing that something can be checked outside a dimension 0 closed subset that consists of the singular points. More details will follow in the next section.

It turns out that normal surfaces are in fact determined by satisfying S_2 and R_1. This is a result due to Serre. From [31, Proposition 5.8.5 and Theorem 5.8.6] we have:

Proposition 3.2.4 (Serre's Normality Criterion)**.** *Let X be a locally Noetherian scheme. Then:*

(i) X is reduced if and only if it satisfies properties (S_1) and (R_0);

(ii) X is normal if and only if it satisfies properties (S_2) and (R_1).

We mention as a side remark that this characterization of normality makes it immediate why normal curves are regular.

3.2.1 Cohen–Macaulay Schemes

Definition 3.2.5. A locally Noetherian scheme X is said to be Cohen–Macaulay if it satisfies (S_n) for all n. A coherent sheaf on X is said to be Cohen–Macaulay if it satisfies (S_n) for all n.

This of course also gives us a notion of Cohen–Macaulay rings and modules. The rings are often defined as rings which satisfy $\operatorname{depth} R_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$. As we always have $\operatorname{depth} R_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}}$, the two notions are easily recognized to be equivalent. As our focus is on the use of S_2-conditions on surfaces especially, we take the above as our definition.

Lemma 3.2.6. *Let be an invertible sheaf. If X satisfies (S_i) then so does .*

Proof: The property is local on X and the stalks of coincide with $_{X,x}$. □

For the sake of easier referencing we give the full statement of the following cohomological interpretation of depth which we will need in a few different instances. The cohomology groups are the local cohomology groups which we take as a blackboxed tool. For the theory of these see [34]. Note also that Theorem 3.2.7 gives a characterization of Cohen–Macaulay schemes as those which satisfy $\mathrm{H}^n_x(X, {}_X) = 0$ for all $n < \dim X$ and all $x \in X$.

Theorem 3.2.7 (The Cohomological Interpretation of Depth)**.** *Let X be a locally Noetherian scheme, $Z \subset X$ a closed subset, and $\mathcal{F}$ a coherent sheaf on X. Then the following are equivalent for all $n \in \mathbb{Z}$*

1. $\mathrm{H}^i_Z(\mathcal{F}) = 0$ *for all $i < n$;*
2. $\operatorname{depth}_Z \mathcal{F} \geq n$.

Proof: This is [34, Theorem 3.8]. Note that Hartshorne here uses the dated terminology of prescheme which is just a scheme in modern language. □

Proposition 3.2.8. *A coherent sheaf $\mathcal{F}$ is S_2 if and only if for each closed subset $Z \subset X$ of codimension ≥ 2, the canonical map $\mathcal{F} \to i_* i^* \mathcal{F}$ is an isomorphism, where $i\colon X \setminus Z \to X$ is the canonical embedding.*

Proof: Both S_2 and being an isomorphism of sheaves are local properties (since the morphism exists globally), so we may assume X to be affine and Noetherian. As X is Noetherian and $\mathcal{F}$ is coherent, $\mathcal{F}$ is the sheaf associated to $\mathrm{H}^0(X, \mathcal{F})$ and similarly for $i_*\mathcal{F}$ and $\mathrm{H}^0(X \setminus Z, \mathcal{F})$. So the global isomorphism of sheaves is equivalent to an induced isomorphism of these global sections. Now, by [34, Corollary 1.9] there is an exact sequence linking local cohomology of sheaves with their global cohomology

$$0 \longrightarrow \mathrm{H}^0_Z(X, \mathcal{F}) \longrightarrow \mathrm{H}^0(X, \mathcal{F}) \longrightarrow \mathrm{H}^0(X \setminus Z, i^*\mathcal{F}) \longrightarrow \mathrm{H}^1_Z(X, \mathcal{F}) \longrightarrow 0.$$

Here the last term is zero, since X is affine so $\mathrm{H}^1(X, {}_X) = 0$ by Serre vanishing. Thus $\mathrm{H}^0(X, \mathcal{F}) \cong \mathrm{H}^0(X \setminus Z, i^*\mathcal{F})$ if and only if

$$\mathrm{H}^0_Z(X, \mathcal{F}) = \mathrm{H}^1_Z(X, \mathcal{F}) = 0.$$

By Theorem 3.2.7 this is equivalent to $\operatorname{depth}_Z \mathcal{F} \geq 2$. If we pick x such that Z is the closure of x, this translates into $\operatorname{depth} \mathcal{F}_x \geq 2$. □

Lemma 3.2.9. *Let $f\colon X \to Y$ be a morphism of locally Noetherian schemes and $\mathcal{F}$ an S_2 sheaf on X. Suppose $f_*\mathcal{F}$ is coherent. If there is a a closed codimension at least 2 subscheme $Z \subset X$ such that $f(Z) \subset Y$ is closed of codimension at least 2, then $f_*\mathcal{F}$ is S_2 on Y.*

Proof: Being S_2 is a local condition, so we may assume Y is affine. As notation, let $U = X \setminus Z$ and $V = Y \setminus f(Z)$. Then, by Proposition 3.2.8, it is enough to show $\mathrm{H}^0(Y, f_*\mathcal{F}) = \mathrm{H}^0(V, f_*\mathcal{F}|_V)$. Now, by definition of the pushforward $\mathrm{H}^0(Y, f_*\mathcal{F}) = \mathrm{H}^0(X, \mathcal{F})$. Then we use that $\mathcal{F}$ is an S_2 sheaf, which implies that $\mathrm{H}^0(X, \mathcal{F}) = \mathrm{H}^0(U, \mathcal{F})$. Since $V = Y \setminus f(Z)$, it follows that $f|_U$ maps to V, so

$$\mathrm{H}^0(U, \mathcal{F}) = \mathrm{H}^0(V, f_*\mathcal{F}).$$

Putting all these equalities together gives the desired. □

An immediate situation where the preceding lemmas are applicable is that of normal surfaces. Indeed, by Proposition 3.2.4 these are Cohen–Macaulay.

3.3 The Classical Kummer Construction

In this section we will outline the Kummer construction in the classical setting. This construction gives a canonical way of associating a $K3$ surface to a given Abelian surface.

So let us set the scene. In this section, fix a base field k of characteristic not 2. For simplicity we assume for now that the base field k is algebraically closed. Let A be an abelian surface and consider the sign involution $i\colon A \to A$. This morphism satisfies $i^2 = \mathrm{id}$ and so we recall from Example 2.1.10 how it determines an action on A of the group scheme $(\mathbb{Z}/2\mathbb{Z})_k$ which is isomorphic to μ_2 since $\mathrm{char}\, k \neq 2$. In the following, we generally view it simply as an action of the group $\{\pm 1\}$ for simplicity. Also recall that this action has 16 fixed points, up to base change, which are also called 2-torsion points. In the following, we will give arguments that the quotient $Y = A/\{\pm 1\}$ exists as a scheme, has singularities which we will classify, and that the resolution $X \to Y$ is a $K3$-surface. We fix notation for the quotient map $\pi\colon A \to Y$ and the resolution $f\colon X \to Y$.

Proposition 3.3.1. *The quotient Y exists as a scheme.*

Proof: This can be seen from Proposition 2.2.15 as any abelian variety is projective. □

Thus, we now have a quotient Y to study as a scheme. We begin by listing some properties:

Proposition 3.3.2. *The surface Y is a proper, normal surface.*

Proof: We noted earlier in Lemma 2.2.9 that the quotient of a proper scheme is again proper. We also saw in Lemma 2.2.11 that the quotient of a normal scheme is again normal.

□

As recalled earlier, A has 16 fixed points. As we now prove, Y comes with singularities corresponding to these fixed points. Luckily, however, the singularities are as mild as can be.

Proposition 3.3.3. *Y has 16 isolated singularities corresponding one-to-one to the fixed points of A. Each of these is a rational double point of type A_1. Y is smooth outside of these sixteen points.*

Proof: We first prove that Y is smooth away from these sixteen notable points. As notation, let $a_1 = e, a_2, \ldots, a_{16}$ denote the sixteen fixed points, where e is the distinguished rational point of A. Further, let $y_1, \ldots, y_{16}$ denote the correspondings points on Y. To see that these are the only possible singular points, we note that the action of G is free when restricted to the open set

$U = A \setminus \{a_0, \dots, a_{16}\}$. Indeed, an action by a group of order two is free if and only if it has no fixed points. Thus Theorem 2.2.5 implies that $\pi|_U : U \to V = Y \setminus \{y_1, \dots, y_{16}\}$ is flat. In particular, the fact that U is smooth implies that so is V [50, 21.D Theorem 51].

Now let us analyze the situation at the fixed points. The situation is the same at all of them, so we simply treat one $a \in A$. We note that this point being fixed by the $\{\pm 1\}$-action means we have an induced action on $\operatorname{Spec} {}_{A,a}$. Indeed, the stalk is given as ${}_{A,a} = \varinjlim_{U \subset A} \Gamma(U, {}_A)$. Taking open subsets $U \cap i(U)$ gives a refinement, and taking the limit over these opens makes each U G-invariant hence the $\Gamma(U, {}_A)$ have compatible $\mathbb{Z}/2\mathbb{Z}$-gradings. It follows that so does ${}_{A,a}$ by Lemma 2.3.7. Now pick generators u, v of $\mathfrak{m}_a$. Then we have

$$\hat{}_{A,a} = k[|u, v|].$$

This comes with an induced grading, and after a linear change of basis in u and v, we may assume u and v are homogeneous. In this case both u and v must be of degree -1. Indeed, if this was not the case, the ideal of fixed points in $\hat{}_{A,a}$ would be of height ≤ 1 meaning that a would have some formal neighbourhood on which G-acted trivially. But if both are of degree -1, we get that the G-invariants, i.e the degree 1 part, must be

$$k[|u, v|]^G = k[|u^2, uv, v^2|] = k[|x, y, z|]/(z^2 - xy).$$

This we recognize as the normal form of an A_1-singularity, see Remark 3.1.9 $\square$

Recall that a $K3$ surface is a smooth, proper, geometrically integral scheme of dimension 2 such that $\omega_X = {}_X$ and $\mathrm{H}^1(X, {}_X) = 0$, where ω_X is the canonical sheaf of X. By the Enriques-Kodaira classification [42, 43, 44, 45, 54, 14, 13] these are one of four types of smooth proper surfaces with numerically trivial ω_X. The other three are the Abelian, Enriques and bielliptic surfaces. They may be classified by their second Betti number $b_2 = h^2({}_X)$ as

	Abelian	$K3$	Enriques	Bielliptic
b_2	6	22	10	2

Proposition 3.3.4. *The minimal resolution $f: X \to Y$ is a $K3$ surface.*

Proof: Our tactic of proof is to appeal to the Enriques-Kodaira classification. First we will argue that ω_X is numerically trivial, which will limit the possibilites for what type of smooth surface X can be. Note that since Y has only rational double points, it follows that $\omega_X = f^*\omega_Y$ by Theorem 3.1.7. Thus, it is enough to show that ω_Y is numerically trivial. For this, we must by definition show that $(\omega_Y \cdot C) = 0$ for any curve $C \subset Y$.

First, we make an observation on $\pi^*\omega_Y$. As in the proof of the previous proposition we use that the morphism

$$\pi|_U : U = A \setminus \{a_1, \dots, a_{16}\} \to V = Y \setminus \{y_1, \dots, y_{16}\}$$

is flat by Theorem 2.2.5. In fact, it is also finite by the same result, so $\pi|_U$ is even étale by [32, Corollaire 17.6.2 b)]. This implies $(\pi|_U)^*\omega_Y = \omega_A|_U$. As Y is smooth outside of the fixed points, $\omega_Y|_U$ is locally free. Furthermore, each singularity of Y is a rational double point, in particular Gorenstein, so ω_Y is also locally free at these points as well, hence ω_Y is locally free. Then $\pi^*\omega_Y$ is locally free, as is ω_A because A is smooth. Next, as A is normal, any invertible sheaf on A is S_2 by Proposition 3.2.4 and Lemma 3.2.6. It then follows by Lemma 3.2.8 that $\pi^*\omega_Y = \omega_A$ as they are isomorphic outside of the set $\{a_1, \dots, a_{16}\}$ which has codimension 2.

Let us compute $(\omega_Y \cdot C)$. As Y is projective by [15, Theorem 3.9] we may write ${}_Y(C)$ as a difference of very ample sheaves. Thus it is enough to show $(\omega_Y \cdot C) = 0$ for very ample curves. Moving the curve, we can assume C does not intersect any singularity of Y, and so C is smooth. Then, because π is finite and as noted $\pi^*\omega_Y = \omega_A$

$$2(\omega_Y \cdot C) = \deg(\pi)(\omega_Y \cdot C) = (\pi^*\omega_Y \cdot \pi^{-1}(C)) = (\omega_A \cdot \pi^{-1}(C)).$$

Now, since A is an abelian variety $\omega_A \cong {}_A$ and so we have $\omega_A \cdot C' = 0$ for any curve C' on A. So$(\omega_Y \cdot C) = 0$. Thus ω_Y and hence also ω_X are numerically trivial.

Since ω_X is numerically trivial and X is smooth, the Enriques-Kodaira classification of surfaces tells us that X is either Abelian, $K3$, Enriques or bielliptic, depending on whether $b_2 = \dim_k \mathrm{H}^2(X, {}_X)$ is 6, 22, 10 or 2 respectively. We will argue it cannot be less than 16, hence must be 22 and so X is $K3$. The surface A has exactly sixteen fixed points giving singularities on Y. Thus X has sixteen disjoint distinguished curves $E_1, \dots, E_{16}$ on arising as the exceptional divisors of the singularities on Y. As these singularities are all A_1, i.e. rational double points, these curves are -2-curves. We claim that they give linearly independent classes of $\mathrm{Num}(X)$. This is straightforward. Indeed, as the E_i are disjoint -2-curves

$$(E_i \cdot E_j) = \begin{cases} 0 & \text{if } j \neq i \\ -2 & \text{if } j = i. \end{cases}$$

Then suppose we have an equality $\sum_{i=1}^{16} a_i E_i = 0$ in $\mathrm{Num}(X)$. Then using the intersection numbers just listed gives

$$0 = (E_j \cdot \sum_{i=1}^{16} a_i E_i) = -2a_j.$$

So $a_i = 0$ for all i i.e. the E_i are linearly independent in $\mathrm{Num}(X)$. Thus the Picard number of X defined as $\rho = \operatorname{rank} \mathrm{Num}(X)$ is at least 16. But the Igusa-Severi inequality [37] or [12, Exposé XVIII Proposition 5.2] says that $\rho \leq b_2$. So $16 \leq b_2$, hence $b_2 = 22$ and X is $K3$.

□

Definition 3.3.5. The $K3$ surface X is called the ***Kummer surface*** of A and is denoted $\mathrm{Kum}(A)$.

Remark 3.3.6. The surface $\mathrm{Kum}(A)$ defined above may be defined in a different but equivalent fashion as outlined in [15, 10.5] and [36, Example 1.3 (iii)]. We give a brief description: first, one defines a scheme $g \colon \widetilde{A} \to A$ as the blowing up of A in the closed subscheme of fixed points $\{a_1, \dots, a_{16}\}$. As these points are fixed by the involution i, it follows by the universal property of the blowing up, that there exists a unique morphism $\widetilde{i} \colon \widetilde{A} \to \widetilde{A}$ such that $i \circ g = g \circ \widetilde{i}$. This $\widetilde{i}$ satisfies $\widetilde{i}^2 = \mathrm{id}_{\widetilde{A}}$ and so induces an action of $\{\pm 1\}$ on $\widetilde{A}$. By the same arguments as above, the quotient $\widetilde{A}/\{\pm 1\}$ exists. By the universal property of the quotient, there is a map $\widetilde{A}/\{\pm 1\} \to A/\{\pm 1\}$ which by the universal property of the blowing-up gives a comparison map $\widetilde{A}/\{\pm 1\} \to \mathrm{Kum}(A)$. One can then show that this map is in fact an isomorphism:

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{Kum}(A) = \widetilde{A}/\{\pm 1\} \arrow[r] & A/\{\pm 1\}. \end{tikzcd}$$

Remark 3.3.7 (Non-algebraically close base fields)**.** Most of the arguments made in the above remain valid over non-algebraically closed ground fields. The only point where one has to modify the argument slightly, is in computing b_2. Indeed, if k not algebraically closed, we can not say that the fixed scheme on A has 16 points. However, the invariant b_2 does not depend on the base field, so we may pass to an algebraic closure, to ensure we have the 16 curves, and proceed by the same argument. As such, the classical Kummer construction remains perfectly valid over arbitrary ground fields.

Remark 3.3.8 (The characteristic 2 case)**.** In the preceding, we made the assumption that the base field k should be of characteristic not 2. This is not a superflous assumption. Indeed, in characteristic 2 the construction might fail to yield a $K3$-surface. One may take the quotient all the same, but the singularities can get worse. Indeed, the quotient has an elliptic singularity if and only if A is a so-called supersingular Abelian surface [65, 38] and in this case the resolution is not $K3$. However, the singularities arising in this case are well understood [63, Prop. 5.1-3]. In fact, the resulting surface has Kodaira dimension $-\infty$ and is rational i.e. birational to $\mathbb{P}^2$. The possible failure of this construction in characteristic 2 naturally offers the question whether there is another, possibly better, generalization to characteristic 2. This is in some sense the framing question of Chapter 5 of this thesis.

Chapter 4

Schemes in Families

As usual we fix a base scheme S. In this section, we will study families over S. That is to say, collections of objects parametrized by S. In the end, we are interested in families of surfaces.

Definition 4.0.1. A ***family*** over a scheme S is a proper, flat morphism $\mathcal{X} \to S$ where $\mathcal{X}$ is an algebraic space.

In the outset of any concrete construction we do, we will only consider families that are schematic. However, to talk about simultaneous resolution of singularities in Section 4.2 it is necessary that we allow algebraic spaces. We will not go into much detail on this subject, but we give the definition and indicate how they are generalization of schemes.

By the Yoneda lemma any scheme may be viewed as a functor from the category (Sch/S) to the category of sets. More precisely it determines a sheaf on the étale site (Sch/S), see Remark 1.1.2. With this viewpoing, schemes are simply a certain class of functors, namely those sheaves on the étale site (Sch/S) which are representable. As such, the natural way to try and generalize schemes, is to look at larger classes of sheaves on this site. An ***algebraic space*** $\mathcal{X}$ is a sheaf $\mathcal{X}$ on the étale site (Sch/S) such that

(i) the diagonal morphism $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by schemes, i.e. for any scheme Y and morphism $Y \to \mathcal{X} \times_S \mathcal{X}$ the sheaf obtained by the fiber product $Y \times_{\mathcal{X}\times_S\mathcal{X}} \mathcal{X}$ is representable by a scheme.

(ii) there is a scheme U and a surjective étale morphism $U \to X$.

A scheme X is trivially an algebraic space as $Y \times_{X\times X} X$ is always a scheme and $X \xrightarrow{\mathrm{id}} X$ is an étale surjection. As Olsson [58] puts it, one can think of an algebraic space as "a geometric object obtained by gluing together schemes using the étale topology rather than the Zariski topology". For more see Artins originating papers [5, 6], or the books of Knutson [41] and Olsson [58].

4.1 Quotients of Families

Consider now a base S consisting of more than one point, e.g. a DVR. Suppose further, that we have a scheme X over S, with an action of a group scheme G over S. Suppose a quotient X/G exists, a natural question to then ask is whether the formation of quotients commutes with taking fibers? That is, do we have a canonical isomorphism $X_s/G_s \cong (X/G)_s$? This property is desirable for multiple reasons. Firstly, we would be able to describe the quotient family, by computing quotients over fields. Secondly, the action might be simpler on certain fibers, allowing for easier computations. Say for example one of the fibers is an elliptic curve, and the action turns out to be the involution action on this fiber. Then we know the quotient must be $\mathbb{P}^1$. This can also be fruitful, if one knows the singularities of these quotients in the fibers.

Unfortunately, such an isomorphism does not hold in general, as we will see shortly. From the universal property of the quotient, we at least obtain a comparison map, which would have to be the desired isomorphism (by the universality)

$$X_s/G_s \longrightarrow (X/G)_s,$$

but there is nothing guaranteeing that this is an isomorphism in general. We will explain in more detail how to construct this morphism in Remark 4.1.4. If the action of G is free, then this is an isomorphism, but this is not a necessary condition. Indeed, let us now consider two examples. The first will show that even for non-free actions, taking quotients might commute with fibers, and the second will show that this can fail for non-free actions.

Example 4.1.1. Let $R = \mathbb{C}[|u|]$ and note that this is a DVR. We take as our base $S = \operatorname{Spec} R$ and as usual we denote the closed point by σ and the open point by η. Let $A = \mathbb{C}[|t|]$ and consider $X = \operatorname{Spec} A$ as an S-scheme via $\mathbb{C}[|u|] \to \mathbb{C}[|t|],\ u \mapsto t^2$. There is then an action of $G = \mathbb{Z}/2\mathbb{Z}$ on A over S via $t \mapsto -t$. This action is non-free. Indeed, for the point $x = (t)$ of X the induced morphism on $\kappa(x) = \mathbb{C}$ is just the identity. As the base is $\mathbb{C}$, we may apply $(\mathbb{Z}/2\mathbb{Z})_{\mathbb{C}} \cong \mu_2$. The action gives a grading for which $R = A^G$. So $X/G = S$, hence the fiber of the quotient X/G at any $s \in S$ is just the spectrum of the residue field at the point, i.e.

$$(X/G)_\sigma = \kappa(\sigma) = \operatorname{Spec} \mathbb{C} \quad \text{and} \quad (X/G)_\sigma = \kappa(\eta) = \operatorname{Spec} \mathbb{C}((u))$$

Consider now the fiber X_σ which is the spectrum of

$$\mathbb{C}[|t|] \otimes_{\mathbb{C}[|u|]} \mathbb{C}[|u|]/u = \mathbb{C}[|t|]/t^2 = \mathbb{C}[t]/t^2.$$

The induced action is still given by $t \mapsto -t$. Here the ring of invariants becomes $(\mathbb{C}[t]/t^2)^G = \mathbb{C}$, so

$$X_\sigma/G_\sigma = \operatorname{Spec} \mathbb{C}.$$

Note how in the preceding example the base being characteric 0 allowed us to use $(\mathbb{Z}/2\mathbb{Z})_{\mathbb{C}} \cong \mu_2$. That fibers and quotients commute is then less surprising, as μ_2 is a very well behaved group scheme and indeed we will see in Theorem 4.1.5 that this particular group scheme satisfies this base change property. In the following example, we take instead a base of mixed characteristic and again a $\mathbb{Z}/2\mathbb{Z}$-action. A key fact here is that the base has a point of characteristic 2, so that $(\mathbb{Z}/2\mathbb{Z})_k \ncong \mu_2$. As we will see, things then start to break down.

Example 4.1.2. Now instead consider $R = \mathbb{Z}_2 = \mathbb{Z}[|u|]/(u-2)$ as the base. This ring has field of fraction $\mathbb{Q}_2$. Our idea is to take a degree 2 field extension $L/\mathbb{Q}_2$ such that the integral closure A of R is ramified over $\sigma = 2\mathbb{Z}_2$ as a scheme over S, i.e. $\sigma A = \mathfrak{p}^2$ for some prime ideal $\mathfrak{p}$. Concretely, we take $L = \mathbb{Q}_2(\sqrt{2})$. Then the integral closure of R is $A = \mathbb{Z}_2[\sqrt{2}]$, and $\sigma A = (\sqrt{2}A)^2$. Now set $G = \operatorname{Gal}(L/\mathbb{Q}_2) = \mathbb{Z}/2\mathbb{Z}$. We obtain an induced action of G on A, hence also on $X = \operatorname{Spec} A$. Concretely, the action of -1 (which is the only non-trivial element) is determined by $\sqrt{2} \mapsto -\sqrt{2}$. As $\mathbb{Q}_2$ is the fixed field of G, and $\mathbb{Q}_2 \cap A = R$, we must have $A^G = R$. Thus $X/G = S$, so $(X/G)_\sigma = \operatorname{Spec} \kappa(\sigma)$.

Consider then the fiber over σ. We have $\kappa(\sigma) = \mathbb{F}_2$, and so X_σ is the spectrum of the tensor product $\mathbb{Z}_2[\sqrt{2}] \otimes_{\mathbb{Z}_2} \mathbb{Z}_2/(2) = \mathbb{F}_2[t]/t^2$. As the induced -1 action on $\mathbb{F}_2[t]/t^2$ maps generator to generator (as an $\mathbb{F}_2$-algebra), it must be determined by $\sqrt{2} \mapsto \lambda\sqrt{2}$ for some $\lambda \in \mathbb{F}_2^\times$. But $\mathbb{F}_2$ has a single unit element, so $\lambda = 1$, hence the induced action on X_σ is trivial. But then $X_\sigma/G_\sigma = X_\sigma \neq S$, so the quotient of the fiber differs from the fiber of the quotient.

Now, we will construct and study this comparison map. First we note that the morphism $X_s \to X \to X/G$ is G-equivariant

Proposition 4.1.3. *Let $s \in S$, $f\colon X_s \to X$ be the canonical morphism and $\pi\colon X \to X/G$. The composition $\pi \circ f\colon X_s \to X/G$ is G-equivariant.*

Proof: We must show that the following diagram is commutative

$$\begin{tikzcd}
G \times X_s \arrow[r, "\mu_f"] \arrow[d, "\rho_2"'] & X_s \arrow[d, "\pi\circ f"] \\
X_s \arrow[r, "\pi\circ f"] & X/G,
\end{tikzcd}$$

where ρ_2 is the second projection. For this, we essentially slot this into a larger diagram of three squares known to be commutative.

$$\begin{tikzcd}
G \times X_s \arrow[r, "\mu_f"] \arrow[d, "\mathrm{id}\times f"'] & X_s \arrow[d, "f"] \\
G \times X \arrow[r, "\mu"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times X \arrow[r, "\mu"] \arrow[d, "\rho_2"'] & X \arrow[d, "\pi"] \\
X \arrow[r, "\pi"'] & X/G
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G \times X_s \arrow[dl, "\rho_2"'] \arrow[d, "\mathrm{id}\times f"] \\
A_s \arrow[dr, "f"'] & G \times X \arrow[d, "\rho_2"] \\
 & X.
\end{tikzcd}$$

The individual commutativity of these diagrams is easy. The first is because μ_f is obtained by base change, the second is because π is G-equivariant by construction (it is universal with this property), and the last is just a general fact about fibered products. Slotting these three diagrams together we get a commutative diagram

$$\begin{tikzcd}
 & G \times X_s \arrow[r, "\mu_f"] \arrow[dl, "\rho_2"'] \arrow[d, "\mathrm{id}\times f"] & X_s \arrow[d, "f"] \\
X_s \arrow[dr, "f"'] & G \times X \arrow[r, "\mu"] \arrow[d, "\rho_2"] & X \arrow[d, "\pi"] \\
 & X \arrow[r, "\pi"] & X/G,
\end{tikzcd}$$

which implies the desired by following the two outer paths from $G \times X_s$ to X/G.

□

With this proposition in hand, we can construct the comparison map.

Remark 4.1.4 (The comparison map)**.** We want a canonical comparison morphism $\varphi\colon X_s/G \to (X/G)_s$. As $\pi \circ f\colon X_s \to X/G$ is G-equivariant by Proposition 4.1.3, the universal property of the quotient X_s/G yields the existence of a unique morphism $\psi\colon X_s/G \to X/G$. Secondly, the universal property of the fibered product $(X/G)_s = X/G \times \operatorname{Spec} \kappa(s)$ gives the unique morphism

$$\varphi\colon X_s/G \longrightarrow (X/G)_s.$$

Alternatively, one could use base change to get $\pi \times \mathrm{id}_{\kappa(s)}\colon X_s \to (X/G)_s$ which is G-equivariant since it is the base change of one such map. Then this induces the desired map. We remark that in this way $\pi \times \mathrm{id}_{\kappa(s)} = \varphi \circ \pi_s$ (we will use this to show surjectivity in Proposition 4.1.6).

We want to study this φ in detail in order to give two results on when taking fibers commutes with quotients. The first, Theorem 4.1.5 is very direct, and simply says things go well when the action is by a finite diagonalizable group schemes, e.g. such as μ_n. The second one, Theorem 4.1.7, is more technical in nature.

Theorem 4.1.5. *Let M be a finite abelian group and suppose $G = D(M)$ acts on a scheme X such that the quotient X/G exists e.g. $G = \mu_2$ in characteristic 2. Then the comparison map $\varphi\colon X/G \times_S T \to X \times_S T/G$ is an isomrphism for any S-scheme T.*

Proof: This property is local, so we may assume $X = \operatorname{Spec} A$, $T = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$ affine, hence also $G = \operatorname{Spec} R[M]$ is affine. By Proposition 2.3.2 the action of G on X is equivalent to a grading of A by M, where the G-invariant subring is the summand corresponding to the neutral element e_M,

$$A = \bigoplus_{m \in M} A_m.$$

We compute $X \times T/G$ and $(X \times T/G)_s$ explicitely. Concretely $X \times T$ is the spectrum of $A \otimes_R B$. The action on $X \times T$ is the induced one by base-change. As the group M is finite, the tensor $\otimes B$ distributes over the decomposition of A and so the action on $X \times T$ is expressible in terms of an induced grading

$$A \otimes B = \bigoplus_{m \in M} (A_m \otimes B)$$

where the degree m summand is $A_m \otimes B$. The quotient $X \times T/G$ is then the spectrum of the degree e_M part, which is $A_{e_M} \otimes B$. Now, we look at $(X/G) \times T$. Concretely, $X/G = \operatorname{Spec} A_{e_M}$, and so $(X/G) \times T = \operatorname{Spec} A_{e_M} \otimes B$. As the comparison map is the one induced from the universal property of the tensor product, it must be an isomorphism in this case, in fact the identity. $\square$

Denoting by g the closed embedding $g\colon (X/G)_s \to X/G$, we have by constructions $g \circ \varphi = \psi$. We will use this in the following.

Proposition 4.1.6. *If G is finite over the base and X is proper, then the comparison morphism $\varphi\colon X_s/G \to (X/G)_s$ is finite and surjective .*

Proof: Recall the commutativite diagram illustrating our setup

$$\begin{tikzcd}
X_s \arrow[rr, "f"] \arrow[d, "\pi_s"'] & & X \arrow[d, "\pi"] \\
X_s/G \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & X/G. \\
& (X/G)_s \arrow[ur, "g"'] &
\end{tikzcd}$$

As $\operatorname{Spec} \kappa(s) \to S$ is a monomorphism it is separated, and it follows that the base change

$$g\colon (X/G)_s \longrightarrow X/G,$$

is also separated. Now if the composition $\psi = g \circ \varphi$ is finite and g is separated, then φ is finite [69, Tag 035D]. So to show φ finite it suffices to show ψ is finite. Now, Lemma 2.2.9 tells us that X/G and X_s/G are proper. Since X/G is then also separated, it follows that ψ is proper [48, Proposition 3.3.16]. Thus we are done if we show ψ is also affine.

So suppose $V \subseteq X/G$ is affine, and consider $\psi^{-1}(V)$. By the construction of the quotient, affine open sets on X_s/G correspond exactly to G-invariant affine open sets on X_s. So it is enough to show that $\pi_s^{-1}\psi^{-1}(V)$ is affine and G-invariant. Now, by commutativity of the diagram above,

$$\pi_s^{-1}\psi^{-1}(V) = f^{-1}\pi^{-1}(V).$$

As π is the quotient map it follows that $U = \pi^{-1}(V)$ is both affine and G-invariant. But this implies $f^{-1}(U)$ is also affine and G-invariant. Indeed, it is affine because $f^{-1}(U) = U \times \operatorname{Spec} \kappa(s)$. It is also G-invariant, because $\mu_f \colon G \times X_s \to X_s$ is obtained by base change via f, so $\mu|_{G\times U}$ mapping to U implies that μ_f maps $G \times f^{-1}(U)$ to $f^{-1}(U)$.

To see that φ is surjective we simply note the following: The quotient π is surjective, hence so is the base change $\pi \times \mathrm{id}_{\kappa(s)}$. But as noted in Remark 4.1.4 $\pi \times \mathrm{id}_{\kappa(s)} = \varphi \circ \pi_s$, so φ is also surjective.

□

Theorem 4.1.7. *Suppose X is proper, X_s/G and $(X/G)_s$ are both S_2 and that G is finite. Let $Z \subset X$ be the fixed scheme and suppose further Z_s and $(Z/G)_s$ each have codimension at least 2 in X_s and $(X/G)_s$. Finally, assume that the action of G is free outside of Z. Then the comparison morphism $\varphi\colon X_s/G \to (X/G)_s$ is an isomorphism.*

Proof: In Proposition 4.1.6 we showed that φ is finite. In particular it is affine, so X_s/G is completely determined by the push forward $\mathcal{F} = \varphi_{*X_s/G}$. Thus, to show that φ is an isomorphism, it is enough to show that $\mathcal{F}$ is isomorphic to $_{(X/G)_s}$. We will argue that $\mathcal{F}$ is an S_2-sheaf, and that the two are isomorphic outside a set of codimension at least 2. This will give the desired isomorphism by Proposition 3.2.8.

First let us prove that $\mathcal{F}$ is S_2. For this we will apply Lemma 3.2.9. By assumption, $Z_s = Z \cap X_s$ is of codimension 2 in X_s. As X is proper, both X_s and $(X/G)_s$ are of finite type over $\operatorname{Spec} \kappa(s)$. But a finite and surjective morphism of schemes of finite type over a field preserves codimension, so $\pi \times \mathrm{id}_{\kappa(s)}(Z_s) = \varphi(Z_s/G)$ is of codimension 2 in $(U/G)_s$. Finite morphisms are closed, so this set is closed.

Now, let $U = X \setminus Z$. By assumption, the induced action on U is free, hence the comparison morphism φ is an isomorphism on the restriction $\varphi\colon U_s/G_s \to (U/G)_s$. This implies that $\mathcal{F}$ and $_{(X/G)_s}$ are isomorphic on $(U/G)_s$. Now, by definition $\varphi(Z_s/G)$ is the complement of $(U/G)_s$ in $(X/G)_s$. But as just noted this is closed of codimension 2.

□

4.2 Resolving Singularities in Families

In this section we will discuss the notion of simultaneous resolutions. As we will see, this is a somewhat delicate matter. We suppose S is Noetherian.

Definition 4.2.1. Let $f\colon X \to S$ be a family of surfaces. A ***simultaneous resolution*** of f is a family of smooth surfaces $\mathfrak{X} \to S$ together with a proper morphism $\mathfrak{X} \to X$ such that fiberwise $\mathfrak{X}_s \to X_s$ is a resolution of singularities.

When speaking of a resolution of a family of singularities we always mean a simultaneous one. In [7] Artin showed that the resolution functor which to T associates the set resolutions of the pullback $X \times T$ is representable by an algebraic space, provided the singular locus is finite over S. Unfortunately, this does not mean a resolution exists. Indeed, the functor could be trivial and simply output the empty set. Something can be said though. As alluded to in Section 3.1 rational double points are a particularly nice type of singularity, and indeed [7, Corollary 1.3] says that one can at least locally find a simultaneous resolution of singularities. The main morale is this: Simultaneous resolutions are difficult.

What then are we then to do? First let us study some examples.

Example 4.2.2 (A constant family)**.** Consider a singular surface Y over an algebraically closed base field k. Then $Y \times S$ is a constant family of singular surfaces over S. Now, consider the minimal resolution $\pi\colon X \to Y$ and Z the closed subset of Y such that $X = \mathrm{Bl}_Z(Y)$. Then we obtain a simultaneous resolution of $Y \times S$ by simply blowing up in $Z \times S$ and we get that $\pi_S\colon X \times S \to Y \times S$ is this simultaneous resolution.

Example 4.2.3 (Example of necessary base change)**.** Consider the singularity defined by $z^2 - xy$ i.e. an A_1 singularity, over some field k. We then construct a family from the singularity by adding the indeterminate t to the polynomial $z^2 - xy + t$. This extends our base to $k[t]$. Taking a fiber, then corresponds to setting t equal to some fixed value in k. Computing partial derivatives we get the same Jacobian matrix for each fiber

$$\begin{pmatrix} -y & -x & 2z \end{pmatrix}.$$

A fiber is then singular at a point if this has rank 0. From this, we see that in characteristic not 2, this defines a family of surfaces which are all smooth except for the one above the point (t) which has an A_1 singularity at the point $(0, 0, 0)$. This singularity does not arise in any other fiber as this coordinate set does not satisfy the equation $z^2 - xy + t = 0$ for non-zero t. Now, in characteristic 2, we instead have a family of surfaces which are all singular. Then it is natural to ask if this has a simultaneous resolution of singularity over $k[t]$. The answer is no, as the structure of the singularity is different. Indeed, the singular locus is given by $k[z, t]/(z^2 - t)$. This has non-reduced fiber for t a square, such as $t = 0$, but reduced otherwise. Now, if one passes to the base change $k[\sqrt{t}]$, then the structure of the singularity does not change, and the family allows a simultaneous resolution of singularities.

This previous example illustrates how after a base change we obtain a closed subscheme which we may use as the center of a blowing-up to resolve singularities. This process is the way we will deal with simultaneous resolutions in chapter 5.

Remark 4.2.4 (Formal neighbourhoods in families)**.** In Remark 3.1.9 we saw how one could study infinitesimal neighbourhoods in order to understand a singularity. Given a family of singular surfaces, one can instead study how a formal neighbourhood of the singularity deforms in the family. If one can show that a deformation is constant in the family, it means that one has a center which one may use for blowing-up to resolve the singularities simultaneously. In concrete cases, one then tries to figure out which base change is necessary for such a deformation to be constant, just as in Example 4.2.3. We will see concrete examples of this principle in Proposition 5.2.8.

4.3 Families of Abelian Varietes

Recall from Section 1.5.1 that an abelian variety over a field is a group scheme which is smooth, connected and proper. We will now study a generalization of this notion.

Definition 4.3.1. A ***family of Abelian varieties*** is a proper, flat, morphism of finite presentation $f\colon A \to S$, together with a structure of a relative group scheme on A, such that each fiber of A is an Abelian variety.

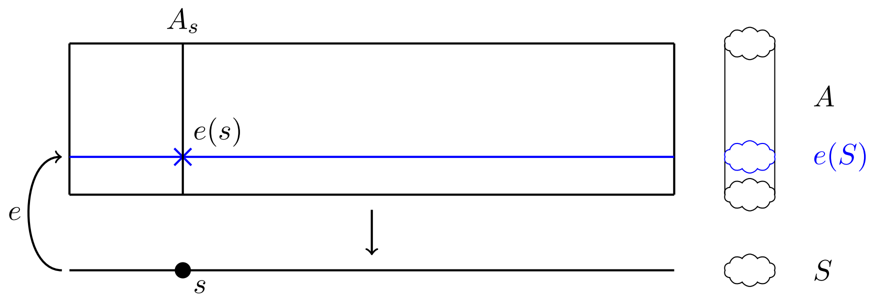
We note that it suffices to check whether the geometric fibers are abelian varieties, that is:

Proposition 4.3.2. *A proper flat relative group scheme A of finite presentation is a family of Abelian varieties if and only if each fiber over a geometric point is an Abelian variety.*

Proof: Indeed, to see this, one compares the fibers $A \otimes \kappa(s)$ and $A \otimes \kappa(s)^{\mathrm{alg}}$. As A is proper, each fiber is automatically proper, so we need to argue $A \otimes \kappa(s)$ is smooth and connected, if and only if $A \otimes \kappa(s)^{\mathrm{alg}}$ is. First, smooth is a geometric property, i.e. holds if and only if it holds over any base change. Furthermore, each fiber is locally Noetherian as it is of finite presentation over a field. Now, it is a fact that a locally Noetherian smooth scheme X is connected if and only if it is geometrically irreducible (in particular geometrically connected). In particular, this is true for smooth schemes of finite type over fields. It is difficult to find this in literature, so we include a proof here for completion's sake:
As X is regular, its local rings are regular, in particular they are integral domains. Since X is locally Noetherian a point lies on only finitely many irreducible components. Putting these two together we get that any point of X lies on a single irreducible component. But X is connected, so there can only be one and X is irreducible. As X is smooth, this argument holds after any base change, so X is geometrically irreducible. □

We note further that the structure morphism of a family of abelian varieties is smooth. Indeed, by [32, Theorem 17.5.1], $f\colon A \to S$ is smooth if and only if it is flat with smooth fibers.

To illustrate a family, we supply the following picture where, for ease, we illustrate S as connected.



As A is a relative group scheme over S, it comes equipped with an identity section $e\colon S \to A$, where the image $e(s)$ of a point $s \in S$ determines the identity section of the Abelian variety arising as the fiber A_s.
Furthermore, one should take note that the group scheme structure on the fibers is intrinsic to A. That is, there is a distinct difference between our definition, and proper flat schemes of finite presentation where each fiber may be endowed with the structure of an abelian variety. Such a family would instead be a called a family of para-abelian varieties, see for example [47] or [11]. The difference is that in a family of abelian varieties, the group structure also "moves in the family", so to say.

This difference is quite important. Indeed, if in the definition above, one replaces the group scheme A by an algebraic space endowed with a relative group law, then a result of Raynaud [25, Theorem 1.9] says that the algebraic space is automatically schematic. So taking A to be a scheme is not at all the restriction as it may seem at first glance. However, this is not the case for families of para-Abelian varieties [60, Chapter XIII, Section 3.2]. In fact, one can show that a family of para-abelian varieties, is a family of abelian varities if and only if it has a rational point, [47, Proposition 4.3].This is similar to the case of para-abelian varieties, which need only have a rational point to be abelian.

Finally, one should note that while Abelian varieties over fields are always projective, this is not true for families over more general bases. Indeed, this fails already over $\mathbb{C}[\varepsilon]$ [60, Remarque XII 4.2]. This creates challenges when forming quotients by finite group schemes. Indeed, projective schemes always satisfy the AF property which makes forming quotients by finite group schemes possible. However, something can be said if the base is affine. Recall from Definition 2.2.14 that a scheme has the AF property if any finite collection of points is contained in an open affine.

Lemma 4.3.3. *If for any finite set of points $a_1, \ldots, a_n \in A$ there is an open affine subset $V \subset S$ such that $a_1, \ldots, a_n$, then there is an open affine $U \subset A$ containing $a_1, \ldots, a_n$. In particular, A has the AF property if S is affine.*

Proof: This is part (b) in the theorem of Raynaud in [25, Theorem 1.9]. □

Before giving a way to construct a whole class of examples, we note the two following properties. The first one is immediate

Proposition 4.3.4. *A product of families of Abelian varieties is again a family of Abelian varieties.*

In Section 4.3.1 we will see that a family defined by a Weierstrass equation is a family of Abelian varieties, provided no fibers are singular. In fact, it gives a family of elliptic curves. This gives a way of constructing families of Abelian varieties with fibers of arbitrary constant dimension. Indeed, we can construct a family of Abelian surfaces, by simply taking the product of two families of elliptic curves.

Proposition 4.3.5. *Let $A \to S$ be a family of Abelian varieties. The function $s \mapsto \dim(A_s)$ is locally constant on S. In particular, all fibers have the same dimension if the base is connected.*

Proof: See [69, Tag 0D4J]. □

4.3.1 An Example: Schemes Defined by Weierstrass Equations

Recall from Section 1.5.1 that a ***Weierstrass equation*** over a field k is an equation in variables x and y with coefficients $a_1, a_2, a_3, a_4, a_6 \in k$ of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Now, there is nothing that prevents us from just taking a general ring R instead of k. The homogenization defines a closed subscheme C of $\mathbb{P}^2 = \operatorname{Proj} R[X, Y, Z]$ all the same, and we can define all the quantities we saw, such as the discriminant Δ. We claim the following:

Theorem 4.3.6. *A scheme E given by a Weierstrass equation over a ring R with $\Delta \in R^\times$ may be endowed with a canonical structure of a group scheme. This makes E a family of Abelian varieties.*

As $\Delta \in R^\times$ is generally a very easy requirement to fulfill, the important part here is of course that this gives us a large class of families of Abelian varieties. As usual, we denote the base by $S = \operatorname{Spec} R$. First, we make the quick observation that $\Delta \in R^\times$ is actually a necessary condition.

Proposition 4.3.7. *If $\Delta \notin R^\times$, then E has a non-smooth fiber. In particular, E has a non-elliptic fiber.*

Proof: If $\Delta \notin R^\times$, then there is some prime ideal $\mathfrak{p} \subset R$ containing Δ. Then the residue class of Δ in $\kappa(\mathfrak{p})$ is 0. But then the fiber $E_\mathfrak{p}$ is non-smooth. □

The real works lies in showing that E may be given the structure of a group scheme, which we show in Theorem 4.3.10. In concrete terms, we will show that $E(T)$ may be identified with the group $\text{Pic}^0(E_T/T)$ in a functorial way. We note that E needs a distinguished section but will see in the theorem loc. cit. that there is a canonical choice. What could in theory go wrong? Well, each fiber might be a group scheme, but the structure might not be well-behaved when passing between fibers. Suppose we have divisors running across the family, then the equations necessary for the addition of points could conceivably change in uncontrollable ways between fibers. However, this can actually not happen, as we will see.
With the assumption that $\Delta \in R^\times$, we proceed by checking the defining properties of a family of Abelian varieties one-by-one.

Proposition 4.3.8. *A scheme E given by a Weierstrass equation over a ring R is proper, flat and of finite presentation over S.*

Proof: First, note that E is projective, hence proper. Furthermore, E is defined by a single equation, hence also of finite presentation. For flatness, we remark that the equation defines a projective scheme E_0 over the Noetherian integral domain $R_0 = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. Then

$$E = E_0 \otimes_{R_0} R.$$

As flatness is preserved by base change, it is sufficient to argue that E_0 is flat. So replace E by E_0 and R by R_0. Then we may apply [35, Theorem III.9.9] which states that in our case E is flat if and only if the polynomials $\chi({}_{E_s}(t))$ in t do not depend on $s \in S$. So we calculate $\chi({}_{E_s}(t))$. Now, each fiber E_s is a degree 3 hypersurface in $\mathbb{P}^2 = \mathbb{P}^2_{\kappa(s)}$ and so ${}_{E_s}$ fits in the short exact sequence

$$0 \longrightarrow {}_{\mathbb{P}^2}(-3) \longrightarrow {}_{\mathbb{P}^2} \longrightarrow {}_{E_s} \longrightarrow 0$$

Tensoring by the locally free sheaf ${}_{\mathbb{P}^2}(t)$ we get another short exact sequence

$$0 \longrightarrow {}_{\mathbb{P}^2}(t-3) \longrightarrow {}_{\mathbb{P}^2}(t) \longrightarrow {}_{E_s}(t) \longrightarrow 0$$

Using the additivity of the Euler characteristic, it then follows that

$$\chi({}_{E_s}(t)) = \chi({}_{\mathbb{P}^2}(t)) - \chi({}_{\mathbb{P}^2}(t-3)) = \binom{2+t}{2} + \binom{-t-3+2}{2} - \binom{2+t-3}{2} - \binom{-t+2}{2}.$$

But this clearly does not depend on s, so by the Theorem loc. cit., E is flat. □

Note that we actually used very few of our assumptions when showing flatness in the preceding proof. Indeed, we used simply the integrality of the base, and the fact that each fiber was a hypersurface of constant degree in $\mathbb{P}^n$. Note also that what we have just shown implies that if $\Delta \in R^\times$, i.e. all fibers of E are smooth, then E is smooth [69, Tag 01V8].

It remains only to check that E may be equipped with a group scheme structure which makes each fiber E_s into an Abelian variety. One may prove, more generally, that a proper scheme of finite presentation and of relative dimension 1 with a distinguished section and geometrically connected smooth fibers each of genus one is in a canonical way a group scheme, see [39, Theorem 2.1.2]. Note that E has smooth fibers and in the coming proof we will show it has a section, so that our case is actually a special case of the one loc. cit.

For this proof, we need to recall the notion of degree of an invertible sheaf as well as discuss certain sets associated with the Picard group. Recall that the degree of an invertible sheaf on a proper scheme C of dimension 1 over a field is $\deg() = \chi() - \chi({}_C)$. For X proper with fibers of dimension 1, we let Pic$^n(X)$ denote the set of sheaves which are fiber-by-fiber of degree n, i.e $\deg(|_{X_s}) = n$ for all s. Now, [69, Tag 0AYV] implies that fiberwise $\deg(\otimes ') = \deg() + \deg(')$ and $\deg(^{-1}) = -\deg()$. Thus, Pic$^0(E)$ in fact becomes a group when equipped with the tensor product as operation. Note that in our case, $\chi({}_{E_s}) = 0$ for any s, as each fiber is an elliptic curve, so here we have $\deg() = \chi()$ fiberwise. In general, the degree measures certain things, among them the existence of global sections, and we will need the following lemma

Lemma 4.3.9. *Let C be an integral proper scheme of dimension 1 over a field. Suppose $\in$ Pic(C) and $\deg() < 0$. Then $h^0() = 0$.*

Proof: We prove the converse, i.e. if there is a global section, then $\deg() \geq 0$. So suppose there is a non-zero $s \in \mathrm{H}^0(C,)$. This global section of defines a map ${}_C \to$ defined concretely as multiplication by s. Since C is integral, the sheaf ${}_C$ consists of integral domains, i.e. has no zero-divisors. But locally, the map is a homomorphism of free modules of rank 1, hence must be injective, as the restriction of s is a non-torsion element. Then we get a short exact sequence

$$0 \longrightarrow {}_C \longrightarrow \quad \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ is the cokernel. In fact, by [39, 1.1.3], $\mathcal{F}$ is the sheaf ${}_D$ of some cartier divisor on C. By the additivity of the Euler characteristic

$$\deg \ = \chi() - \chi({}_C) = \chi({}_D).$$

But D is zero-dimensional, so by Grothendieck vanishing $\chi({}_D) = h^0({}_D) \geq 0$. □

Now, consider the Picard group Pic(S) of the base. We have a morphism Pic$(S) \to$ Pic(E_S) defined by pullback $\mathcal{N} \mapsto f^*\mathcal{N}$. Clearly, this respects the tensor product, i.e. the group structure on the Picard group, and any base change, so we in fact get a morphism of sheaves of groups on the site (AffSch/S). Furthermore, since $\mathcal{N}$ is invertible, we get

$$f^*\mathcal{N}|_{E_s} = f^{-1}\mathcal{N}_s \otimes_{f^{-1}{}_{S,s}} {}_{E_s} = f^{-1}{}_{S,s} \otimes_{f^{-1}{}_{S,s}} {}_{E_s} = {}_{E_s}$$

where by ${}_{S,s}$ we here, as an abuse of notation, mean the sheaf which on E_s is constantly the local ring ${}_{S,s}$. Thus, any invertible sheaf on S gives an invertible sheaf on E which is fiber-by-fiber of degree 0. Thus the morphism Pic$(S) \to$ Pic(E) factors through Pic$^0(E)$. It follows, since the degree is fiberwise additive on E, that if we denote the image of Pic(S) in Pic(E) by G, then Pic$^n(E) \cdot G \subset$ Pic$^n(E)$. Thus we can define Pic$^n(E/S)$ as the image of the set Pic$^n(E)$ in the quotient of groups Pic$(E)/G$. Note that this is apriori only a set, except for Pic$^0(E/S)$ which is a group. Again, this respects any base change.

Proposition 4.3.10. *A smooth scheme E given by a Weierstrass equation over a ring R has a section, denoted e. Furthermore, assuming that $\Delta \in R^\times$, there is a unique structure of a group scheme on E such that e is the zero section.*

Proof: For this, we first need a distinguished section, so let us argue that $E(R)$ is non-empty. In fact, there is even a canonical choice of such a section. Considering the ring $R[X, Y, Z]$, the homogeneous ideal (X, Z) contains the polynomial $X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 - Y^2Z - a_1XYZ - a_3YZ^2$, and so in particular, we obtain an inclusion of closed subschemes in $\mathbb{P}^2$ as

$$V_+(X, Z) \subset E.$$

But $V_+(X, Z) = \operatorname{Proj} R[X, Y, Z]/(X, Z) = \mathbb{P}^0 = \operatorname{Spec} R$, so this inclusion gives a section of E. We will denote this canonical section by e.

Now we come to the most involved part, namely the group scheme structure itself. In our case we follow, in general terms, the same overall approach as [39, Theorem 2.1.2] but deviate slightly by using the existence of a Weierstrass equation to ease certain arguments. We also do not need to reduce to the case of an affine base, as our base is already $\operatorname{Spec} R$.

We must show that $E(T)$ may, in a functorial way, be endowed with a group structure for any R-scheme T. In fact, we may take $T = \operatorname{Spec} A$ to be affine, as noted in Remark 1.1.2. By abuse of notation, we also denote the base change e_T by e. The main idea, is to construct an isomorphism $E(T) \to \text{Pic}^0(E_T/T)$ of sets, and use the group structure on the codomain to induce one on the domain.

Now, consider the set $\text{Pic}^1(E_T/T)$ which by the discussion preceding this proof consists of isomorphism classes of invertible sheaves on E_T which are fiber-by-fiber of degree one, modulo the equivalence relation $\sim \ \otimes f_T^*\mathcal{N}$ for any invertible sheaf $\mathcal{N}$ on T. By [39, Lemma 1.2.7], giving a section of E_T is equivalent to giving an effective Cartier divisor on E_T of degree 1. As such, each section $\sigma\colon T \to E$ has an associated ideal sheaf $_{E_T}(-\sigma)$ which is fiber-by-fiber of degree 1. For now, assume we know that the map

$$E(T) \longrightarrow \text{Pic}^1(E_T/T), \qquad \sigma \longmapsto (\sigma)$$

is a bijection. Then consider the abelian group $\text{Pic}^0(E_T/T)$ consisting of isomorphism classes of invertible sheaves on E_T which are fiberwise of degree zero, modulo the subgroup consisting of those which arise as pullbacks $f_T^*\mathcal{N}$ of invertible sheaves $\mathcal{N}$ on T. As degree is additive when multiplying sheaves on curves of genus one, and $(-e)$ is of degree -1, we then have an obvious bijection of sets

$$\text{Pic}^1(E_T/T) \longrightarrow \text{Pic}^0(E_T/T), \qquad \longmapsto \otimes (-e).$$

Composing these two bijections, we obtain a bijection $E(T) \to \text{Pic}^0(E_T/T)$ defined by $\sigma \mapsto (\sigma) \otimes (-e)$ and so the abelian group law on $\text{Pic}^0(E_T/T)$, which is functorial in T, induces the same on $E(T)$. Note in particular that the distinguished section e gives the neutral element, as $(e) \otimes (-e) = {}_T$.

It remains to show that the map $E(T) \to \text{Pic}^1(E_T/T)$ is actually a bijection. As we could simply replace E by E_T and $\operatorname{Spec} R$ by T, it suffices to treat the case $T = \operatorname{Spec} R$. In fact, we restrict further. As in the proof of Proposition 4.3.8, we note that E may be obtained as the base change of the scheme E_0 defined by the same equation over $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}]$. If the scheme E_0 may be made a group scheme, then it follows that so may E. Indeed, as a functor, E is simply the restriction of E_0 to a smaller category. So we may replace E by E_0 and R by the Noetherian integral domain $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}]$.

Let us prove surjectivity first. So take on E which is fiberwise of degree one. We should construct a section mapping to the class of , or equivalently an effective cartier divisor D of degree one on E such that $= (D)$ in $\text{Pic}^1(E/S)$. Recall that a Cartier divisor may be viewed as an invertible ideal sheaf of . To find one such, we first note that the canonical sheaf of any elliptic curve is trivial. Furthermore, fiberise $\deg({}^{-1}) = -1$, so ${}^{-1}$ has fiberwise no global sections by Lemma 4.3.9. Combining this with Serre duality, we get

$$h^1\left(E_s, |_{E_s}\right) = h^0\left(E_s, {}^{-1}|_{E_s} \otimes \omega_{E_s}\right) = h^0\left(E_s, {}^{-1}|_{E_s}\right) = 0.$$

This implies $h^0(E_s, |_{E_s}) = \chi(|_{E_s})$ and so, since the Euler characteristic and degree coincide on E_s, we have

$$h^0(E_s, |_{E_s}) = \deg(|_{E_s}) = 1.$$

As the base R is integral, we may now apply the criterion of Grauert [35, Corollary III.12.9] which then implies f_* is locally free. Furthermore, this result says the formation of f_* commutes with base change, i.e. $\mathrm{H}^0(E_s, |_{E_s}) \cong \ \otimes \kappa(s)$. Thus, the rank of remains constant and so by our dimension computations must be of rank 1.

Recall then there is a canonical map $f^* f_* \to$. One obtains this in the following manner: As functors, f^* and f_* form an adjoint pair with f^* being left adjoint and f_* being right adjoint. Thus the identity $f_* \to f_*$ determines a map $f^* f_* \to$. When $f^{-1}(f(U))$ is a open for U open affine in E, the map has a concrete description as the multiplication map

$$\Gamma(f^{-1}(f(U)),) \otimes_{\Gamma(f(U),S)} \Gamma(f^{-1}(f(U)), _E) \to \Gamma(f^{-1}(f(U)),)$$

We claim that this canonical map is injective in our case. As R is an integral domain, E itself is integral, so the map being injective, is equivalent to it being non-zero, since is invertible. But on the fibers over S, the map becomes

$$\mathrm{H}^0(E_s, |_{E_s}) \otimes_{\kappa(s)} {}_{E_s} \longrightarrow |_{E_s}$$

As shown, $\mathrm{H}^0(E_s, |_{E_s})$ is a $\kappa(s)$-vector space of dimension 1, and so the above map is completely determined by picking a basis σ of $\mathrm{H}^0(E_s, |_{E_s})$ as $\sigma \otimes 1 \mapsto \sigma$. Hence, the map is non-zero because σ is. So $f^* f_* \to$ is injective. Tensoring by ${}^{-1}$ is exact, and so we get an injective map $f^* f_* \otimes {}^{-1} \to$ so that $f^* f_* \otimes {}^{-1} = (-D)$ for an effective Cartier divisor D. It is necessarily of degree 1, because is. Then D satisfies $= (-D)$ because $(-D)^{-1} = \ \otimes (f^* f_*)^{-1} = \ \otimes f^*(f_*)^{-1}$ which equals in $\mathrm{Pic}^1(E/S)$.

For the injectivity, suppose we are given two effective Cartier divisors of degree one, D_1, D_2, on E, such that $(D_1) = (D_2)$ in $\mathrm{Pic}^1(E/S)$ i.e. $(D_1) \cong (D_2) \otimes f^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on S. We claim it is enough to show that they agree on the generic fiber. Indeed, let η denote the generic point of the base. The D_i arise as sections, so $(D_i)_\eta$ is non-empty, and $\overline{(D_i)_\eta} \subset D_i$. In fact we must have $\overline{(D_i)_\eta} = D_i$ as D_i is reduced. But then it is sufficient to show that $(D_1)_\eta = (D_2)_\eta$.

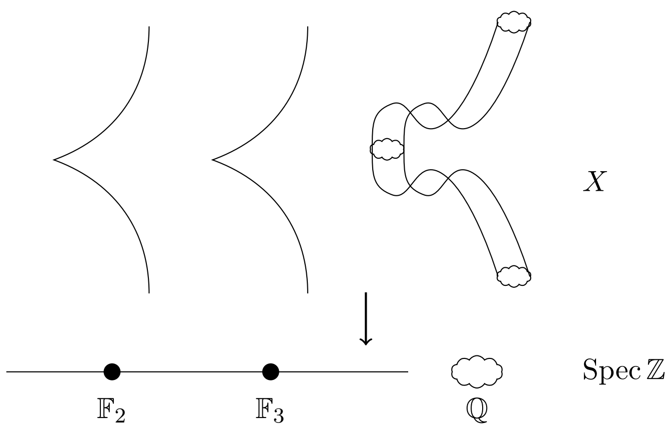
So we may restrict to the case $S = \operatorname{Spec} k$. Then the isomorphism reduces to $(D_1)_s \cong (D_2)_s$, as over a field $\mathcal{N}$ is globally free, so $f^*\mathcal{N} = {}_{E_s}$. Thus the two divisors D_1 and D_2 are linearly equivalent i.e. differ by a principal divisor. But each is effective of degree one, so $D_1 - D_2$ is, as a Weil divisor, the difference of two points. Such a divisor can only be principal if it is trivial, as a rational function on an elliptic curve cannot have only a single root and a single pole. Thus $D_1 = D_2$, and the map is injective as well.

□

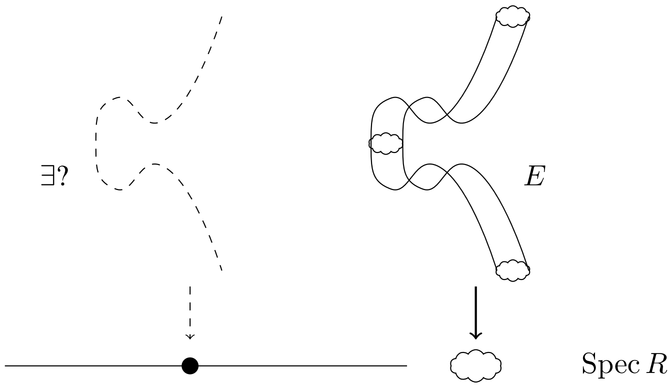
Weierstrass equations over a smooth curve define so called Weierstrass fibrations. These are a special case of elliptic surfaces (i.e. surfaces with almost all fibers elliptic curves). It can be shown that any minimal elliptic surface has an associated Weierstrass fibration (called its Weierstrass model) and that any Weierstrass fibration has a so-called Weierstrass data which realises the fibration in terms of a Weierstrass equation (where the coefficients are now sections of a sheaf). The only possible singular fibers in an elliptic surface, are cuspidal and nodal curves with rational double point singularities. For more, see the excellent notes of Miranda [52]. Now, let us consider some examples.

Example 4.3.11. Consider the equation $y^2 = x^3 + 2$. The equation certainly defines a scheme X of relative dimension 1 over $\mathbb{Z}$. However, the discriminant is $\Delta = -2^6 3^3 \neq \pm 1$ and so is not a unit in $\mathbb{Z}$, hence X has a non-smooth fiber (actually two), by Proposition 4.3.7. The two so-called degenerate fibers are at the primes 2 and 3. Indeed, over $\kappa((2)) = \mathbb{F}_2$, the equation becomes $y^2 = x^3$ which is a cuspidal curve. Over $\kappa((3)) = \mathbb{F}_3$, the equation is still expressed as $y^2 = x^3 + 2$. But

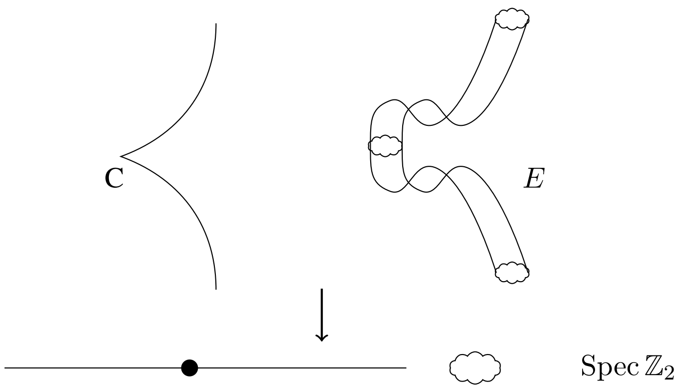
$2 = -1 \pmod 3$, and so the equation is actually $y^2 = (x-1)^3$, which is again a cuspidal curve. In some sense, this can be viewed as starting with the curve $y^2 = x^3 + 2$ over $\mathbb{Q}$ (the generic fiber in this case), and asking if it extends to a family over the ring $\mathbb{Z}$ which has $\mathbb{Q}$ as fraction field, i.e. base at the generic point.



Example 4.3.12. Now that we have this example in mind, a natural question is the following: Suppose we have a DVR R with fraction field F and residue field k. Given an elliptic curve E over F, does it extend to a family of elliptic curves over R?



Even more concretely, the question is what the closed fiber looks like when extending a Weierstrass equation from F to R. From our arguments above, we see that this can fail only when the closed fiber becomes non-smooth, i.e. for us to obtain a family of elliptic curves, we need $\Delta \in R^\times$. As an example take the curve $y^2 = x^3 + 2$ now defined over $\mathbb{Q}_2$. Before we calculated $\Delta = -2^6 3^3$ and as before this is a unit in $\mathbb{Q}_2$, since it is non-zero. The equation certainly defines a scheme X of relative dimension 1 over the DVR $\mathbb{Z}_2$. However, 2 is a divisor of Δ and so Δ is not a unit in $\mathbb{Z}_2$, hence X has a non-smooth fiber, by Proposition 4.3.7. This is very easy to see if one looks directly at the closed fiber. Indeed, the residue field of $\mathbb{Z}_2$ is $\mathbb{F}_2$, so similar to the previous example the closed fiber of X is the singular curve C given by the equation $y^2 = x^3$.



This equation behaves similarly over $\mathbb{Z}_3$, but over $\mathbb{Z}_p$, $p \neq 2, 3$ the discriminant is a unit, and so in these cases we get a family of elliptic curves.

The above examples are closely linked with the ***Tate algorithm***. This algorithm computes the closed fiber of the Néron model of an elliptic curve over a field explicitely from its Wierstrass equation, see [70].

4.4 The Classical Kummer Construction in Families

To finish out the chapter, we study the classical Kummer construction in families. Recall from Section 3.3 that the classical Kummer construction over a field not of characteristic 2 associates a $K3$ surface $\mathrm{Kum}(A)$ to an Abelian surface A by as the minimal resolution of singularites of the quotient A/i. In this section, we will outline that this construction works well in families, and what exactly is meant by this terminology.

So to fix notation, let A be a family of Abelian surfaces over a base scheme S. We assume that 2 is invertible on the base, i.e. no residue field of S has characteristic 2. Further assume that S is Noetherian. The scheme A comes with the structure of a group scheme and so has an inverse morphism $\iota\colon A \to A$. As usual, this gives an action of $G = \{\pm 1\}$ on A. What we would like is a simultaneous resolution of singularities on $\mathfrak{X} \to A/\{\pm 1\}$, with the property that the fibers $\mathfrak{X}_s$ are all precisely the $K3$ surfaces $\mathrm{Kum}(A_s)$. But first of all, we need to study the quotient $A/\{\pm 1\}$ and argue that it even exists.

Lemma 4.4.1. *$A/\{\pm 1\}$ exists as a scheme.*

Proof: By Lemma 4.3.3 A has the AF property, so the quotient exists by Proposition 2.2.4.

□

Next, we consider A_s equipped with the induced G_s action over $\kappa(s)$ or equivalently G action over S. As A is a family of abelian varieties, this is by definition just the involution on the abelian variety A_s. So A_s/G_s is a quotient we understand well, since we studied it in Section 3.3. We would like to compare the fibers of A/G with the quotients of fibers A_s. Specifically, we would like the comparison map $A_s/G_s \to (A/G)_s$ to be an isomorphism (it will be in this case). Recall that this hope of an isomorphism is a non-trivial case. Indeed, we have seen in Example 4.1.2 that this is not always the case. To obtain this map we first make the following observation.

Proposition 4.4.2. *The comparison morphism $A_s/G \to (A/G)_s$ is an isomorphism.*

Proof: As 2 is invertible on the base Proposition 1.4.6 tells us that $(\mathbb{Z}/2\mathbb{Z})_S \cong \mu_2$, and so the action corresponds to a μ_2 action. By Theorem 4.1.5 taking quotients with respect to such actions commutes with arbitrary base change. □

Thus, we know that the quotient family A/G is a family of projective surfaces with rational double point singularities. Furthermore, each fiber has at most sixteen singularities.

Proposition 4.4.3. *Let $f\colon A/G \to S$ be the structure morphism. Each point $s \in S$ has a neighbourhood U such that $f^{-1}(U)$ admits a simultaneous resolution of singularity after a base change. In particular, the Kummer construction works in families up to a base change.*

Proof: As each fiber has only rational double point singularities, this is [7, Corollary 1.3]. □

Chapter 5

Generalized Kummer Constructions and Main Results

5.1 Generalized Kummer Constructions in Characteristic 2

Throughout the rest of the text, I fix a base S of characteristic 2 which is a scheme over a field k of characteristic 2. In this section I will outline the general ideas of [62] and [46] that inspired my own work. This new work is presented in the next few sections.

Suppose for now the base is a field k which is perfect. As outlined in Remark 3.3.8, the classical Kummer construction fails in characteristic 2 and the natural question is then whether there is a more natural generalization to characteristic 2. One such generalization is presented in [62]. This construction is very much in the spirit of the classical one. The idea is the following: The problems of the classical construction arise in the case of supersingular abelian surfaces, i.e. those A which are in fact products $A = E \times E$ of supersingular elliptic curves. The insight in [62] was to replace the curve E by the rational cuspidal curve $C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}]$. The choice of the cuspidal curve is a natural one, as it arises as the degeneration of elliptic curves. Thus one instead considers the non-normal surface $C \times C = (\operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}]) \times (\operatorname{Spec} k[v^2, v^3] \cup \operatorname{Spec} k[v^{-1}])$. Furthermore, the $\mathbb{Z}/2\mathbb{Z}$-action, is replaced by a diagonal action of α_2 determined by a vector field

$$\delta = (u^{-2} + r)D_u + (v^{-2} + s)D_v = (u^{-4} + ru^{-2})D_{u^{-1}} + (v^{-4} + sv^{-2})D_{v^{-1}},$$

as outlined in Section 2.4. To avoid a degenerate case we assume r, s are not both simultaneously zero. The quotient $(C \times C)/\alpha_2$, which exists by Theorem 2.2.8, becomes a normal surface with rational double point singularities. However, the singularities arising from the fixed locus are not the A_1 singularities we saw in Section 3.3. Instead, one gets D_4 or D_8 singularities, depending on the quantities r and s. Moreover, the quotient $(C \times C)/\alpha_2$ has an additional singularity, coming from the singular point $u^2 = u^3 = v^2 = v^3 = 0$ in $C \times C$ which is called the ***quadruple point***. This singularity turns out to be either a D_4 or B_3 singularity depending on whether or not the base field contains a primitive third root of unity. The results of [62] tell us that the minimal resolution of $(C \times C)/\alpha_2$ becomes a $K3$-surface. The close link with the classical construction then makes it reasonable to call this a Kummer surface of $C \times C$, denoted $\operatorname{Km}(C \times C)$.

Now, over a more general base, it is also shown in [62] that forming the quotient $(C \times C)/\alpha_2$ commutes with base change. The result [62, Theorem 12.1] is then that a simultaneous resolution exists for this family of singularities after a purely inseparable base change. In other words, this generalized Kummer construction works in families after this base change.

The work in [46] builds on these principles. The work in the article loc. cit. is over algebraically closed ground fields. The surface considered is the same product $C \times C$, but the actions considered here are determined by vector fields

$$\begin{aligned}\delta &= (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \tau u^{-1} + \lambda_0)D_{u^{-1}} + (\sigma_4 v^{-4} + \sigma_2 v^{-2} + \tau v^{-1} + \sigma_0)D_{v^{-1}} \\ &= (\lambda_0 u^2 + \tau u + \lambda_2 + \lambda_4 u^{-2})D_u + (\sigma_0 v^2 + \tau v + \sigma_2 + \sigma_4 v^{-2})D_v.\end{aligned}$$

Such vector fields determine a diagonal action by a group scheme G which is μ_2 if $\tau \neq 0$ and α_2 otherwise, see Section 2.4. The vector field of [62] is then a special case of this with $\lambda_4 = \sigma_4 = 1$, $\lambda_2 = r$, $\sigma_2 = s$ and $\lambda_0 = \sigma_0 = \tau = 0$. Under the assumption that λ_4 and σ_4 are both non-zero, the quotient surface $(C \times C)/G$ is normal. By [46, Proposition 3.2] it always has a rational double point singularity coming from the quadruple point, which is a D_4 singularity. Furthermore, if $G = \mu_2$ i.e. $\tau \neq 0$, the singularities from the fixed points are sixteen A_1 singularities, which one should note is strikingly similar to the classical case. If $G = \alpha_2$ one gets D_4 or D_8 singularities, as long as one avoids the degenerate case where $\lambda_2 = \sigma_2 = 0$. Finally, the minimal resolution of $(C \times C)/G$ is a $K3$ surface if and only if the singularities are rational double points.

In the following sections, I will study this construction with μ_2-actions in families. I will show that the singularities allow simultaneous resolution after finite separable base change and give concrete descriptions of the base change necessary for the resolution of singularities. In summary, showing that the generalized Kummer construction with μ_2 works in families after a base change.

5.2 Generalized Kummer Constructions in Families

Our goal is to generalize much of the work in [46] from algebraically closed base fields to families over a general base. For this we emulate the work in [62].

Our setup is the following: We take some base S of characteristic 2 and let C be the rational cuspidal curve over S given by $\mathrm{Spec}_S\, {}_S[u^{-2}, u^{-3}] \cup {}_S[u^{-1}]$. We study the product family of surfaces $C \times C$ where we use u as variable in the left factor and v in the right factor. We have a diagonal action on $C \times C$ given by the vector field

$$\begin{aligned}\delta &= (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \tau u^{-1} + \lambda_0)D_{u^{-1}} + (\sigma_4 v^{-4} + \sigma_2 v^{-2} + \tau v^{-1} + \sigma_0)D_{v^{-1}} \\ &= (\lambda_0 u^2 + \tau u + \lambda_2 + \lambda_4 u^{-2})D_u + (\sigma_0 v^2 + \tau v + \sigma_2 + \sigma_4 v^{-2})D_v.\end{aligned}$$

Where the σ_i, λ_i and τ are global sections of ${}_S$. Having an action defined by such a vector field, means that the action moves in the family. Concretely, a coefficient could dissappear in one fiber, while being non-zero in another. In our case, we want a μ_2 action. As outlined in [46, section 1 and 2] over a field this amounts to assuming $\tau \neq 0$. Over a general base, this translates to τ being non-zero everywhere i.e. being a unit. By [46, Proposition 3.1] the surface $(C \times C)/\mu_2$ is normal if and only if λ_4 and σ_4 are non-zero. Again, over a general base, this translates to these two coefficients being units. Thus we make the following standing assumption:

> We assume that the coefficients τ, λ_4 and σ_4 are units of $\Gamma(S, {}_S)$.

The quotient $(C \times C)/\mu_2$ exists by Theorem 2.2.8 and by Theorem 4.1.5 its formation commutes with arbitrary base change. It is covered by four relatively affine charts

$$\begin{array}{cc}\mathrm{Spec}_S\, {}_S[u^2, u^3] \times \mathrm{Spec}_S\, {}_S[v^2, v^3], & \mathrm{Spec}_S\, {}_S[u^2, u^3] \times \mathrm{Spec}_S\, {}_S[v^{-1}], \\ \mathrm{Spec}_S\, {}_S[u^{-1}] \times \mathrm{Spec}_S\, {}_S[v^2, v^3], & \mathrm{Spec}_S\, {}_S[u^{-1}] \times \mathrm{Spec}_S\, {}_S[v^{-1}].\end{array}$$

As the formation commutes with base change $((C \times C)/\mu_2)_s = (C \times C)_s/\mu_2$ for any $s \in S$, so by the work of [46] we know that the singularities of the fibers lie only on the charts of ${}_S[u^2, u^3, v^2, v^3]$ and ${}_S[u^{-1}, v^{-1}]$. We will tackle each of these charts in turn. First though we make the following observation: The base S lives over some field k of characteristic 2 (if nothing else, $\mathbb{F}_2$), and we note that if the coefficients of δ are all in k, and $\lambda_4, \sigma_4, \tau \neq 0$, then the action does not change across the family, and so the quotient family $(C \times C)/\mu_2$ is actually a constant family of singular surfaces, hence simultaneous resolution is possible. The same is true if one assumes $\tau = 0$, but then the quotient is by α_2. These are the trivial cases.

5.2.1 The Quadruple Point

For simplicity, assume $S = \operatorname{Spec} R$ is affine. If one objects to this, one may in the following simply replace R by the sheaf ${}_S$, all spectrums by relative spectrums and elements of R by global sections of ${}_S$.

Remark 5.2.1 (The Universal Situation)**.** As $\mu_2 = D(\mathbb{Z}/2\mathbb{Z})$, it follows by Theorem 4.1.5 that the formation of the quotient $(C \times C \times S)/\mu_2$ commutes with arbitrary base change. In particular, we can reduce the analysis from a complicated base to a more simple one, provided the action also exists over this simpler base. That is, for some $S \to S'$ we want

$$(C \times C \times S)/\mu_2 = (C \times_k C \times_k S')/\mu_2 \times_{S'} S.$$

Essentially we are speaking of a deformation or what we will call a ***universal situation***. We have the following very concrete S' in mind: Suppose our action is determined by the vector field

$$\begin{aligned}\delta &= (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \tau u^{-1} + \lambda_0) D_{u^{-1}} + (\sigma_4 v^{-4} + \sigma_2 v^{-2} + \tau v^{-1} + \sigma_0) D_{v^{-1}} \\ &= (\lambda_0 u^2 + \tau u + \lambda_2 + \lambda_4 u^{-2}) D_u + (\sigma_0 v^2 + \tau v + \sigma_2 + \sigma_4 v^{-2}) D_v.\end{aligned}$$

Everything lives over some base field k of characteristic 2 (if nothing else, $\mathbb{F}_2$), and we have a k-morphism $S \to \operatorname{Spec} k[x_1, \ldots, x_7]$ determined by the k-homomorphism $k[x_1, \ldots, x_7] \longrightarrow \Gamma({}_S, S)$ defined by

$$x_1 \mapsto \lambda_0, \;\; x_2 \mapsto \lambda_2, \;\; x_3 \mapsto \lambda_4, \;\; x_4 \mapsto \sigma_0, \;\; x_5 \mapsto \sigma_2, \;\; x_6 \mapsto \sigma_4, \;\; x_7 \mapsto \tau.$$

Then we may realize the action determined by δ over S by base change of the vector field over $k[x_1, \ldots, x_7]$ given by

$$(x_1 u^2 + x_7 u + x_2 + x_3 u^{-2}) D_u + (x_4 v^2 + x_7 v + x_5 + x_6 v^{-2}) D_v.$$

The upshot is that we can compute the quotient $(C \times C \times S)/\mu_2$ by instead computing the quotient $(C \times C \otimes k[x_1, \ldots, x_7])/\mu_2$ and then taking the base change along $S \to \operatorname{Spec} k[x_1, \ldots, x_7]$. By a slight abuse of notation, we will denote the x_i of $k[x_1, \ldots, x_7]$ by λ_0, λ_2, λ_4, σ_0, σ_2, σ_4 and τ when treating the universal situation.

We will analyze the singularity of the quadruple point. The general analysis for the case $\lambda_4 = \sigma_4 = 1$ and $\tau = \lambda_0 = \sigma_0 = 0$ is treated in [62, Section 5] and it is shown that the quadruple point is a rational double point of either type D_4 or B_3, depending on whether or not the base field contains a third root of unity or not. The case for more general vector fields is the work of Kondo and Schröer [46] which shows that the quadruple point gives a D_4 singularity over an algebraically closed base. As already mentioned in [46] this is similar to the situation in [62, Section 5]. However, the computations are slightly more complicated, there are slight variations in the arguments, and we wish to understand the situation over general base fields which is not covered in [46]. Thus we do the analysis in full here while emulating the proofs of the section 5 loc. cit. It turns out that τ, λ_0 and σ_0 have no influence on the singularity type. However, in Proposition 5.2.4 I show that if $\lambda_4 \neq \sigma_4$ then a G_2 singularity may show up. We consider the chart of $R[u^2, u^3, v^2, v^3]$ so our G-action is given by

$$\delta = (\lambda_0 u^2 + \tau u + \lambda_2 + \lambda_4 u^{-2})D_u + (\sigma_0 v^2 + \tau v + \sigma_2 + \sigma_4 v^{-2})D_v.$$

Proposition 5.2.2. *Let $A = R[u^2, u^3, v^2, v^3]$. The G-invariant subring of* $\operatorname{Spec} A$ *under the action determined by δ is $A^G = R[u^2, v^2, (\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4)u^3 + (\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4)v^3 + \tau u^3 v^3]$.*

Proof: As outlined Remark 5.2.1 it is enough to treat the universal situation where

$$R = k[\lambda_0, \lambda_2, \lambda_4, \sigma_0, \sigma_2, \sigma_4, \tau].$$

We determine A^G by computing the kernel of δ. As $D_u(u^2) = 2u = 0$ and by symmetry $D_v(v^2) = 0$ so at least $R[u^2, v^2] \subset A^G$. Note that since R is a polynomial ring over a field the ring $R[u^2, v^2]$ is a UFD and so A is a free module over $R[u^2, v^2]$ with basis given by 1, u^3, v^3 and u^3v^3. As the entire span of 1 is in the kernel, we need only consider when expressions of the form $\alpha u^3 + \beta v^3 + \gamma u^3 v^3$ are in the kernel for $\alpha, \beta, \gamma \in R[u^2, v^2]$. We find that

$$\begin{aligned}
\delta(\alpha u^3 + \beta v^3 + \gamma u^3 v^3) =& \alpha(\lambda_0 u^4 + \tau u^3 + \lambda_2 u^2 + \lambda_4) + \beta(\sigma_0 v^4 + \tau v^3 + \sigma_2 v^2 + \sigma_4) \\
&+ \gamma((\lambda_0 u^4 + \tau u^3 + \lambda_2 u^2 + \lambda_4)v^3 + (\sigma_0 v^4 + \tau v^3 + \sigma_2 v^2 + \sigma_4)u^3) \\
=& \alpha(\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4) + \beta(\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4) \\
&+ (\gamma(\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4) + \beta\tau)v^3 + (\gamma(\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4) + \alpha\tau)u^3.
\end{aligned}$$

As $R[u^2, u^3, v^2, v^3]$ is free over $R[u^2, v^2]$, the above expression is zero if and only if each coefficient is. That is, we obtain three equations

$$\begin{aligned}
\alpha(\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4) + \beta(\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4) &= 0 \\
\gamma(\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4) + \beta\tau &= 0 \\
\gamma(\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4) + \alpha\tau &= 0.
\end{aligned}$$

The second equation

$$\gamma(\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4) = \beta\tau$$

implies, since $R[u^2, v^2]$ is a UFD, that β is a multiple of $\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4$ and γ is a multiple of τ. Similarly, the third equation

$$\gamma(\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4) = \alpha\tau$$

yields that α must be a multiple of $\sigma_0 v^4 + \tau v^3 + \sigma_2 v^2 + \sigma_4$ (and also that γ is a multiple of τ, but we know that already). The first equation in conjunction with the others finally implies that all other factors of α, β and γ. must be common. Thus we obtain the desired.

□

Having computed this invariant ring, we want to exhibit it as a quotient of a polynomial ring. This will give us a defining equation with which we can analyze the singularity.

Proposition 5.2.3. *We have*

$$A^G = R[a, b, c]/(c^2 + \sigma_0^2 a^3 b^4 + \sigma_2^2 a^3 b^2 + \sigma_4^2 a^3 + \lambda_0^2 a^4 b^3 + \lambda_2^2 a^2 b^3 + \lambda_4^2 b^3 + \tau^2 a^3 b^3)$$

Proof: For ease of notation, let $f(a,b,c) = c^2 + \sigma_0^2 a^3 b^4 + \sigma_2 a^3 b^2 + \sigma_4 a^3 + \lambda_0 a^4 b^3 + \lambda_2 a^2 b^3 + \lambda_4 b^3 + \tau a^3 b^3$ and recall from Proposition 5.2.2 that

$$A^G = R[u^2, v^2, (\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4)u^3 + (\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4)v^3 + \tau u^3 v^3].$$

We have naive candidates for an isomorphism and its inverse namely

$$\begin{array}{rcl} \varphi\colon R[a,b,c]/(f) & \longrightarrow & A^G \\ a & \longmapsto & u^2 \\ b & \longmapsto & v^2 \\ c & \longmapsto & (\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4)u^3 + (\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4)v^3 + \tau u^3 v^3. \end{array}$$

and the assignment in the other direction which we denote ψ. The only real question here is whether or not φ and ψ are well-defined. We have an obvious homomorphism $\overline{\varphi}$ from $R[a,b,c]$ to A^G defined in the same way as φ. Then we simply need to show that this factors over $R[a,b,c]/(f)$ i.e. (f) is in the kernel. We compute

$$\begin{aligned} \overline{\varphi}(c)^2 &= \sigma_0^2 v^8 u^6 + \sigma_2^2 v^4 u^6 + \sigma_4^2 u^6 + \lambda_0^2 u^8 v^6 + \lambda_2^2 u^4 v^6 + \lambda_4^2 v^6 + \tau^2 u^6 v^6 \\ &= \overline{\varphi}(\sigma_0^2 a^3 b^4 + \sigma_2^2 a^3 b^2 + \sigma_4^2 a^3 + \lambda_0^2 a^4 b^3 + \lambda_2^2 a^2 b^3 + \lambda_4^2 b^3 + \tau^2 a^3 b^3), \end{aligned}$$

which implies that (f) is in the kernel of $\overline{\varphi}$, hence φ is well-defined. That ψ is well-defined follows by a symmetric calculation: Let $g = (\sigma_0 v^4 + \sigma_2 v^2 + \sigma_4)u^3 + (\lambda_0 u^4 + \lambda_2 u^2 + \lambda_4)v^3 + \tau u^3 v^3$. Any polynomial $\sum_{i,j,l} \alpha_{i,j,l}(u^2)^i(v^2)^j g^l$ may be expressed uniquely as $\sum_{i,j} \beta_{i,j}(u^2)^i(v^2)^l g + \sum_{i',j'} \gamma_{i',j'}(u^2)^{i'}(v^2)^{j'}$. It may be written in this way using the relaton

$$g^2 = \sigma_0^2 v^8 u^6 + \sigma_2^2 v^4 u^6 + \sigma_4^2 u^6 + \lambda_0^2 u^8 v^6 + \lambda_2^2 u^4 v^6 + \lambda_4^2 v^6 + \tau^2 u^6 v^6,$$

and the fact that it must be unique comes from the fact that $R[u^2, u^3, v^2, v^3]$ is a free module over $R[u^2, v^2]$. Since this method of rewriting only requires use of the relation above, it follows that ψ will be well-defined as long as it respects this relation. But this is a simple computation.

□

With this equation in hand, we can now analyse the singularity completely. The following result should be seen as an analogue of [62, Proposition 5.3], the proof of which we emulate.

Proposition 5.2.4. *Suppose $R = k$. The singularity of the quadruple point is a rational double point. Let $f(b) = b^3 - \frac{\sigma_4^2}{\lambda_4^2}$.*

(i) If k contains no roots of f then the singularity is of type G_2;

(ii) If k contains only a single root of f then the singularity is of type B_3;

(iii) If k contains all roots of f then the singularity is of type D_4.

Proof: We compute the blowing-up

$$p\colon Z \longrightarrow A^G = R[a,b,c]/(c^2 + \sigma_0^2 a^3 b^4 + \sigma_2^2 a^3 b^2 + \sigma_4^2 a^3 + \lambda_0^2 a^4 b^3 + \lambda_2^2 a^2 b^3 + \lambda_4^2 b^3 + \tau^2 a^3 b^3)$$

in the ideal (a,b,c). The blowing up Z is the relative proj of the powers of (a,b,c) and is covered by two affine open charts $D_+(a)$ and $D_+(b)$. Setting $b' = \frac{b}{a}$ and $c' = \frac{c}{a}$, the first chart is the spectrum of

$$k[a,b',c']/(c'^2 + \sigma_0^2 a^5 b'^4 + \sigma_2^2 a^3 b'^2 + \sigma_4^2 a + \lambda_0^2 a^5 b'^3 + \lambda_2^2 a^3 b'^3 + \lambda_4^2 a b'^3 + \tau^2 a^4 b'^3).$$

The exceptional divisor of this is determined by the ideal

$$(a,b,c)k[a,b',c'] = \left(a, \frac{b}{a}\cdot a, \frac{c}{a}\cdot a\right) = (a).$$

Let us compute the singularities on this. The Jacobian of $D_+(a)$ yields

$$\begin{pmatrix}\sigma_4^2 + \lambda_4^2 b'^3 + a^2(\sigma_0^2 a^2 b'^4 + \sigma_2^2 b'^2 + \lambda_0^2 a^2 b'^3 + \lambda_2^2 b'^3) & a(\lambda_0^2 a^4 + \lambda_2^2 a^2 + \lambda_4^2 + \tau a^3)b'^2 & 0\end{pmatrix}$$

We see that this vanishes on the exceptional divisor if and only if $\lambda_4^2 b'^3 = \sigma_4^2$ or equivalently $b'^3 = \frac{\sigma_4^2}{\lambda_4^2}$. As such, the singular points on the exceptional divisor are defined by the ideal $(a, b'^3 - \frac{\sigma_4^2}{\lambda_4^2})$. Now we have a bit of case work based on the splitting of the polynomial $b'^3 - \frac{\sigma_4^2}{\lambda_4^2}$. First suppose k contains all roots of $b'^3 - \frac{\sigma_4^2}{\lambda_4^2}$ and let α denote one such root. Note that since $\lambda_4, \sigma_4 \neq 0$ this also holds for α. We then have a splitting

$$b'^3 - \frac{\sigma_4^2}{\lambda_4^2} = (b'-\alpha)(b'-\zeta_3\alpha)(b'-\zeta_3^2\alpha),$$

where ζ_3 is a primitive third root of unity. Then the non-singular locus on the exceptional divisor consists of three points given by the ideals $(a, b' - \zeta_3^i\sqrt[3]{\alpha})$ for $i = 0,1,2$. We compute the blowing up in one of these, as the other two are symmetric. It will turn out that a single blow-up resolves the singularity replacing it by a $\mathbb{P}^1$. First, we make a linear change of base by mapping $b' \mapsto b' + \alpha$. We substitute this in the defining equation, and keeping in mind $\lambda_4^2\alpha^3 = \sigma_4^2$ i.e. the sum of these is zero, one finds that

$$\begin{aligned} c'^2 =& \lambda_4^2 a(b'^3 + \alpha b'^2 + \alpha^2 b') + a^3(\lambda_2^2 b'^3 + (\sigma_2^2 + \lambda_2^2\alpha)b'^2 + \lambda_2^2\alpha^2 b' + \sigma_2^2\alpha^2 + \lambda_2^2\alpha^3) \qquad (*)\\ &+ \tau^2 a^4(b'^3 + \alpha b'^2 + \alpha^2 b' + \alpha^3) + a^5(\sigma_0^2 b'^4 + \lambda_0^2 b'^3 + \lambda_0^2\alpha b'^2 + \lambda_0^2\alpha^2 b' + \sigma_0^2\alpha^4 + \lambda_0^2\alpha^3).\end{aligned}$$

The blowing up is covered by two new affine charts $D_+(a)$ and $D_+(b')$, where the first is a slight abuse of notation. We wish to show the exceptional divisor is a copy of $\mathbb{P}^1$, so we study it on each of these two charts. The $D_+(a)$ chart replaces b' by $b'' = \frac{b'}{a}$ and c' by $\frac{c'}{a}$. So the equation $(*)$ yields

$$\begin{aligned} c''^2 =& \lambda_4^2\alpha^2 b'' + a(\lambda_4^2\alpha b''^2 + \sigma_2^2\alpha^2 + \lambda_2^2\alpha^3) + a^2(\lambda_4^2 b''^3 + \lambda_2^2\alpha^2 b' + \tau^2\alpha^3) \qquad (**)\\ &+ a^3((\sigma_2^2 + \lambda_2^2\alpha)b''^2 + \tau^2\alpha^2 b'' + \sigma_0^2\alpha^4 + \lambda_0^2\alpha^3) + a^4(\lambda_2 b''^3 + \tau^2\alpha b''^2 + \lambda_0^2\alpha^2 b'')\\ &+ a^5(\tau^2 b''^3 + \lambda_0^2\alpha b''^2) + \lambda_0^2 a^6 b''^3 + \sigma_0^2 a^7 b''^4.\end{aligned}$$

The exceptional divisor is defined by the ideal of (a) in $k[a, \frac{b'}{a}, \frac{c'}{a}]$. Modding out by a in $(**)$ gives the much simpler equation

$$c''^2 = \lambda_4^2\alpha^2 b''.$$

As noted $\lambda_4, \alpha \neq 0$ hence the product of their squares is invertible and we get $\frac{1}{\lambda_4\alpha}c'' = \sqrt{b''}$. Summarizing, it follows that on the $D_+(a)$ chart the exceptional divisor is given by

$$k[a,b'',c'']/(**)/(a) = k[b'',c'']/(c''^2 + \lambda_4^2\alpha^2 b'') = k\left[\sqrt{b''}\right] = k\left[\sqrt{\frac{b'}{a}}\right].$$

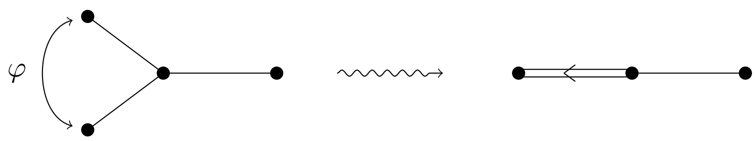
The computation on the $D_+(b')$ are essentially the same, so we give only brief details. One replaces a by $a' = \frac{a}{b'}$ and c by $c''' = \frac{c'}{b'}$ and obtains the following defining equation from $(*)$ as before

$$\begin{aligned} c''^2 =& \lambda_4^2\alpha^2 a' + b'(\lambda_4^2\alpha a' + (\sigma_2^2\alpha^2 + \lambda_2^2\alpha^3)a'^3) + b'^2(\lambda_4^2\alpha a' + \lambda_2^2\alpha^2 a'^3 + \tau^2\alpha^3 a'^5)\\ &+ b'^3((\sigma_2^2 + \lambda_2^2\alpha)a'^3 + \alpha^2 a'^4 + \lambda_0^2\alpha^3 a'^5) + b'^4(\lambda_2^2 a'^3 + \tau^2\alpha a'^4 + \lambda_0^2\alpha^2 a'^5)\\ &+ b'^5(\tau^2 a'^4 + \lambda_0^2\alpha a'^5) + \lambda_0^2 b'^6 a'^5 + \sigma_0^2 b'^7 a'^5.\end{aligned}$$

Using this one finds that the exceptional divisor, which is defined by the ideal (b') is given on the $D_+(b')$ chart as the spectrum of $k[\sqrt{\frac{\alpha}{b'}}]$. It follows that the exceptional divisor is a $\mathbb{P}^1$ as desired. The multiplicity of the singularity may be read from the defining equation as 2, and together with the configuration of curves we see it is a rational double point. Thus the singularity is a D_4 singularity in the case where $b'^3 - \frac{\sigma_4^2}{\lambda_4^2}$ splits completely. We will use this first case, as well as the method outlined in Remark 3.1.10 to easily handle the other two cases. Consider first the case where k contains only a single root of f denoted α as before. Then we have a splitting

$$(b'^3 - \alpha) = (b' - \alpha)(b'^2 + \alpha b' + \alpha^2).$$

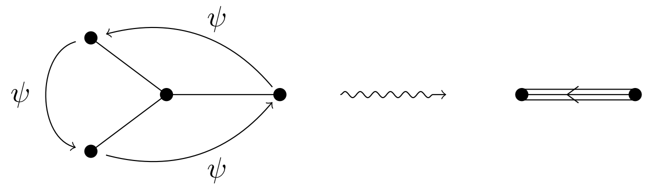
Passing to the splitting field of $(b'^2 + \alpha b' + \alpha^2)$ brings us to the situation of the first case. This splitting field is the degree two extension $k(\zeta_3)$. The Galois group $\mathrm{Gal}(k(\zeta_3)/k)$ is cyclic of order two generated by the k-automorphism φ defined by $\zeta_3 \mapsto \zeta_3^2$. As such, the Galois action permutes the two singularities arising from the splitting of $b'^2 + \alpha \cdot b' + \alpha^2$. Thus the action also permutes the corresponding two components in the resolution:



It follows that in this second case, the singularity is a B_3 singularity.
For the final case we suppose that k contains no roots of f. The splitting field of f is then instead the degree 6 extension $k(\alpha, \zeta_3)$. The Galois group of this is S_3 generated by the two homomorphisms defined by

$$\varphi\colon \begin{cases} \alpha \mapsto \alpha \\ \zeta_3 \mapsto \zeta_3^2 \end{cases} \qquad \psi\colon \begin{cases} \alpha \mapsto \zeta_3\alpha \\ \zeta_3 \mapsto \zeta_3 \end{cases}$$

Similar to before, we see how this action permutes the roots and so identifies the different components in the exceptional divisor. The following diagram illustrates this identification.



Thus the singularity is a G_2 singularity in this final case.

□

We remark the we never used $\tau \neq 0$ in the preceding. Thus this analysis is also valid for α_2 actions determined by our general vector fields. Also note that the first case does not occur if $\lambda_4 = \sigma_4$ as the polynomial $b^3 - 1$ always has the trivial root 1. This explains why the G_2 singularity does not show up in [62, Proposition 5.3] as in this case $\lambda_4 = \sigma_4 = 1$.

We wish to understand this singularity in families. Indeed, our hope is that it gives an honest divisor which may be used as the center of a blowing up. The challenge is that the structure of the singularity may be different depending on the fiber as the action moves in the family. Indeed, as we saw in Example 4.2.3 a singular locus could be reduced in one fiber and non-reduced in another.

Our base is now again a ring R. Suppose $\mathfrak{m} \subset R$ is a maximal ideal such that $R/\mathfrak{m} = k$ i.e. $\mathfrak{m}$ corresponds to a k-rational point of $\operatorname{Spec} R$. To study the singularity in the fiber over $\mathfrak{m}$, we may pass to an infinitesimal neighbourhood of the singularity i.e. we complete with respect to the corresponding ideal. We then wish to show, that as we deform this neighbourhood, the deformation is constant. Similar to the earlier proofs, we can emulate a proof strategy, specifically that of [62, Proposition 5.4].

The following proposition deals with the completed tensor product. We briefly recall the definition in our specific case. Suppose we have a ring A with ideal I and A-algebras B and C with ideals J_B and J_C respectively, such that $IB \subset J_B$ and $IC \subset J_C$. Equipping A, B and C with the topologies defined by these ideals makes B and C into topological algebras over the topological ring A.

$$B\hat{\otimes}_A C = \varprojlim B \otimes_A C/(J_B^n \otimes C + B \otimes J_C^m) = \varprojlim B/J_B^n \otimes C/J_C^m.$$

For more on the completed tensor product see [28, Chapter 0, 7.7] and [31, 7.5]. Note that since k embeds in the k-algebra R, there is a canonical choice of representative of each residue class in $R/\mathfrak{m} = k$.

Proposition 5.2.5. *Let $\mathfrak{m} \in \operatorname{Spec} R$ be a k-rational point,*

$$B = R[a,b,c]/(c^2 + \sigma_0^2 a^3 b^4 + \sigma_2^2 a^3 b^2 + \sigma_4^2 a^3 + \lambda_0^2 a^4 b^3 + \lambda_2^2 a^2 b^3 + \lambda_4^2 b^3 + \tau^2 a^3 b^3)$$

and $B_0 = B \otimes_R R/\mathfrak{m}$. Consider then the ideals $(a,b,c)B$ and $(a,b,c)B_0$. As a deformation of $\widehat{B_0}$ the deformation $\widehat{B}$ is isomorphic to the constant deformation $\widehat{B_0}\hat{\otimes}_{R/\mathfrak{m}} R$ if $\frac{\lambda^2}{\lambda_4^2}$ and $\frac{\sigma^2}{\sigma_4^2}$ have third roots in R, where λ and σ are the canonical representatives of the residue classes of λ_4 and σ_4 in $R/\mathfrak{m} = k$.

Proof: Concretely we have

$$\widehat{B} = R[|a,b,c|]/(c^2 + \sigma_0^2 a^3 b^4 + \sigma_2^2 a^3 b^2 + \sigma_4^2 a^3 + \lambda_0^2 a^4 b^3 + \lambda_2^2 a^2 b^3 + \lambda_4^2 b^3 + \tau^2 a^3 b^3),$$

and

$$\widehat{B_0}\hat{\otimes}_{R/\mathfrak{m}} R = R[|a,b,c|]/(c^2 + [\sigma_0^2] a^3 b^4 + [\sigma_2^2] a^3 b^2 + [\sigma_4^2] a^3 + [\lambda_0^2] a^4 b^3 + [\lambda_2^2] a^2 b^3 + [\lambda_4^2] b^3 + [\tau^2] a^3 b^3),$$

where $[-]$ denotes the residue class in $R/\mathfrak{m} = k$ but we view each class as an element of R via the k-algebra structure embedding k in R. To ease our notation later on, let λ and σ denote the canonical representatives for the residue classes of λ_4 and σ_4 respectively. Note that since we have assumed $\sigma_4, \lambda_4 \in R^\times$ these cannot be in the maximal ideal $\mathfrak{m}$, and so λ and σ will never be 0.

First, we wish to reduce to a simpler case by showing that the equations can be modified. We prove that $\widehat{B} \cong R[|a,b,c|]/(c^2 + \sigma_4^2 a^3 + \lambda_4^2 b^3)$. To do this, we must show there is an R-algebra automorphism of $R[|a,b,c|]$ mapping $c^2 + \sigma_0^2 a^3 b^4 + \sigma_2^2 a^3 b^2 + \sigma_4^2 a^3 + \lambda_0^2 a^4 b^3 + \lambda_2^2 a^2 b^3 + \lambda_4^2 b^3 + \tau^2 a^3 b^3$ to the simpler $c^2 + \sigma_4^2 a^3 + \lambda_4^2 b^3$. This reduction is essentially the content of the proof of [62, Proposition 5.4], but we emulate it here so that the interested reader will not have to translate notation and because our equation is slightly more complicated. To ease notation, set $f = c^2 + \sigma_4^2 a^3 + \lambda_4^2 b^3$ and take any power series $g \in (a^3 b^2, a^2 b^3)$. Note specifically, that the bit of the equation we want to 'delete' is in this ideal. We then show that we can inductively define an automorphism φ such that $\varphi(f + g) = f$. We do this by removing terms of g one-by-one while making sure to also delete any extraneous terms along the way.

So let $g = \sum_{i,j \geq 2,\, i+j \geq 5} \gamma_{ij} a^i b^j$. From this, take the non-zero monomials with minimal total degree $i+j$ and let $\gamma_{mn} a^m b^n$ be the unique one with minimal degree m in a. Note that by assumption $n, m \geq 2$. Let us first suppose $m \geq 3$ and leave the other case for last. We have a concrete automorphism φ_{mn} of $R[|a,b,c|]$ defined by $a \mapsto a + \gamma_{mn} a^{m-2} b^n$ while fixing b and c. Note that for any $l \geq 1$ we have

$$\varphi_{mn}(a^{2l}) = a^{2l} + \sum_{s=1}^{l} \gamma_{mn}^{2s} \binom{l}{s} a^{2l+2(m-3)s} b^{2ns}$$

$$\varphi_{mn}(a^{2l+1}) = a^{2l+1} + \gamma_{mn} a^{m+2(l-1)} b^n + \sum_{s=1}^{l} \binom{l}{s} \gamma_{mn}^{2s} a^{2l+2(m-3)s+1} b^{2ns}$$
$$+ \sum_{s=1}^{l} \binom{l}{s} \gamma_{mn}^{2s+1} a^{2l+2(m-3)s+m-2} b^{n(2s+1)}.$$

We want to ensure φ_{mn} adds only terms of increasing degree to $f+g$. For this, there are a few things to note here. First, that φ_{mn} maps any monomial of degree ≥ 2 in a to itself plus something in (a^2b^3, a^3b^2). Secondly, as $m \geq 3, n \geq 2$ we have for even exponents $i = 2l$

$$2l + 2(m-3)s + 2n + j > 2l + j.$$

While for odd exponents $i = 2l+1$

$$\begin{aligned} m + 2(l-1) + n + j &> 2l + 1 + j, \\ 2l + 2(m-3)s + 1 + 2ns + j &> 2l + 1 + j, \\ 2l + 2(m-3)s + m - 2 + n(2s+1) + j &> 2l + 1 + j. \end{aligned}$$

So any monomial a^ib^j is mapped to itself plus additional summands which are all of total degree $> i + j$. Note that in particular

$$\varphi_{mn}(a^3) = a^3 + \gamma_{mn} a^m b^n + \gamma_{mn}^2 a^{2m-3} b^{2n} + \gamma_{mn}^3 a^{3m-6} b^{3n}.$$

Where we remark that the second term is the least term in g and the last two terms have a total degree strictly greater than the total degree $m+n$ of $\gamma_{mn} a^m b^n$. Thus φ_{mn} maps f to f plus $\gamma_{mn} a^m b^n$ plus some terms of higher total degree. It follows from the above that $\varphi_{mn}(f+g) = f + g'$ where $g' \in (a^2b^3, a^3b^2)$ with all terms of total degree $> m+n$ or total degree $= m+n$ but degree $> m$ in a. This argument also solves the case $m = 2$, as we can simply repeat it with a and b interchanged. As in each step we only add terms of higher degree, we may proceed by defining our desired automorphism φ inductively using the above procedure. Then φ becomes defined by an inductive assignment

$$a \longmapsto a + \sum \gamma ij a^{i-2} b^j, \quad b \longmapsto b + \sum \eta_{ij} a^i b^{j-2} \quad \text{where } \gamma_{ij}, \eta_{ij} \in R.$$

This then shows that $\widehat{B} \cong R[|a,b,c|]/(c^2 + \sigma_4^2 a^3 + \lambda_4^2 b^3)$ and by symmetric argument $\widehat{B_0} \hat{\otimes}_{R/\mathfrak{m}} R \cong R[|a,b,c|]/(c^2 + \sigma^2 a^3 + \lambda^2 b^3)$.

Now we are reduced to showing that there is an automorphism of $R[|a,b,c|]$ which maps the polynomial $c^2 + \sigma_4^2 a^3 + \lambda_4^2 b^3$ to $c^2 + \sigma^2 a^3 + \lambda^2 b^3$. This is where we need our additional assumption. Of course, if $\sigma_4 = \sigma$ and $\lambda_4 = \lambda$ i.e. $\sigma_4, \lambda_4 \in k$ then the equations are on the nose equal and the problem is trivial. However, if they are not necessarily equal, suppose $\frac{\lambda^2}{\lambda_4^2}$ and $\frac{\sigma^2}{\sigma_4^2}$ have third roots $\sqrt[3]{\frac{\lambda^2}{\lambda_4^2}}$ and $\sqrt[3]{\frac{\sigma^2}{\sigma_4^2}}$ in R. As λ_4, σ_4 are both units, it follows that the inverses $\frac{\lambda_4^2}{\lambda^2}$ and $\frac{\sigma_4^2}{\sigma^2}$ also have third roots, which are the inverses of the third roots just described. Then we obtain an automorphism of $R[|a,b,c|]$ defined by

$$a \longmapsto \sqrt[3]{\frac{\sigma^2}{\sigma_4^2}} a, \quad b \longmapsto \sqrt[3]{\frac{\lambda^2}{\lambda_4^2}} b, \quad c \longmapsto c,$$

which is the desired.

□

It might be possible that no base-change is necessary. If one looks at the locus of non-smoothness for the family of formal neighbourhoods $R[|a,b,c|]/(c^2+\sigma_4^2a^3+\lambda_4^2b^3)$, one sees that it is defined by the ideal

$$(\sigma_4^2a^2,\lambda_4^2b^2,c^2+\sigma_4^2a^3+\lambda_4^2b^3)=(\sigma_4^2a^2,\lambda_4^2b^2,c^2)=(a^2,b^2,c^2),$$

where in the last equality we used that λ_4 and σ_4 are units. The reduced locus of non-smoothness is then given by the ideal (a,b,c). This closed subscheme has as its fiber over $s\in S$ the reduced singularity of the fiber $(\operatorname{Spec} B)_s$. Thus (a,b,c) determines a closed subscheme which may be used as the center of a blowing up, which fiberwise gives the first blowing up in a resolution of singularity. However, this single blowing up is not enough to resolve the singularities, and one would then have to study the locus of non-smoothness on this new scheme obtained by blow-up.

5.2.2 The Fixed Point Singularities

We consider now the chart of $A=R[u^{-1},v^{-1}]$. The $G=\mu_2$ action is then given by the derivation

$$\delta=(\lambda_4u^{-4}+\lambda_2u^{-2}+\lambda_0)D_{u^{-1}}+(\sigma_4v^{-4}+\sigma_2v^{-2}+\sigma_0)D_{v^{-1}}+\tau(u^{-1}D_{u^{-1}}+v^{-1}D_{v^{-1}}),$$

where we assume $\lambda_4,\sigma_4,\tau\in R^\times$. The quotient chart corresponding to $\operatorname{Spec} A$ was computed in [46] as:

Proposition 5.2.6. $A^G=R[a,b,c]/(c^2+\tau^2ab+(\sigma_4^2b^4+\sigma_2^2b^2+\sigma_0^2)a+(\lambda_4^2a^4+\lambda_2^2a^2+\lambda_0^2)b)$.

Proof: In [46, Proposition 3.2] this is computed for the case $R=k$ an algebraically closed field, but the computations work for any k-algebra and are completely analogous to Proposition 5.2.2 and Proposition 5.2.3. □

It is also mentioned in [46, Proposition 3.2] that over an algebraically closed field this chart has sixteen A_1 singularities. When we study the deformation in Theorem 5.2.8 we will obtain this for free, as we will see the singularities may be exhibited by the normal form of an A_1 singularity. Now, the above expression for the quotient chart tells us that singularities on this chart are determined by the vanishing of the Jacobian matrix

$$\begin{pmatrix}\tau^2b+\sigma_4^2b^4+\sigma_2^2b^2+\sigma_0^2 & \tau^2a+\lambda_4^2a^4+\lambda_2^2a^2+\lambda_0^2 & 0\end{pmatrix}.$$

So the singular locus is determined by the ideal $(\sigma_4^2b^4+\sigma_2^2b^2+\tau^2b+\sigma_0^2,\lambda_4^2a^4+\lambda_2^2a^2+\tau^2a+\lambda_0^2)$. We wish to derive a result similar to Proposition 5.2.5 to determine when these singularities lie in the family in a uniform manner. Recall that a polynomial is separable in positive characteristic if and only if its formal derivative is non-zero. As we have assumed τ is a unit, we see that each of the two polynomials defining the singular locus are separable. This means that each polynomial has 4 distinct roots, giving a total of sixteen points after base changing so that the polynomials split. Before continuing with the analysis of these we make a brief analysis of how one actually solves such polynomials in characteristic 2. This analysis will be useful if one in the future wishes to compute examples.

Solving Depressed Quartic Equations in Characteristic 2

The solution of the quartic equation is classical, going back to the 16th hundred italian school of mathematics with the work of Ferrari, who settled the quartic dependent on the solution of the cubic, and Cardano, who solved the cubic, in the *Ars Magna* [40]. However, the classical formulas are not directly transferable to characteristic 2 (or 3 for that matter) due to the fractions involved, see [23, p. 630-635]. As such one has to carefully go through the proofs and transfer them to characteristic 2. As I do not know of any source that has these formulas readily available, I have decided to include the process of deriving the quartic solution here. In our case we are interested in factorizing the polynomial $\lambda_4^2 a^4 + \lambda_2^2 a^2 + \tau^2 a \lambda_0^2$ as well as the similar polynomal in b. As we have assumed λ_4 is a unit in R, we may scale by λ_4^{-2} to obtain a monic polynomial $a^4 + \frac{\lambda_2^2}{\lambda_4^2} a^2 + \frac{\tau^2}{\lambda_4^2} a + \frac{\lambda_0^2}{\lambda_4^2}$. This is a depressed quartic so we want to solve such polynomials. We first set

$$p = \frac{\lambda_2^2}{\lambda_4^2}, \quad q = \frac{\tau^2}{\lambda_4^2}, \quad r = \frac{\lambda_0^2}{\lambda_4^2}.$$

Note that as $\tau, \lambda_4 \in R^\times$ we have $q \in R^\times$ and so the polynomial is separable, and remains so when reduced modulo any maximal ideal. In the following we will need the concept of 2-roots: For non-zero γ we call a solution to the Artin-Schrier polynomial $x^2 + x + \gamma$ a 2-***root*** of γ. Given one 2-root, the other may be obtained by adding 1, so we simply denote one root by $R(\gamma)$ and the other $R(\gamma) + 1$. Recall, or confirm, that an equation $ax^2 + bx + c = 0$ in characteristic 2 with a, b units has root $\frac{b}{a} R(\frac{ac}{b^2})$. Further recall that a polynomial of degree n is said to be ***depressed*** if it has term of degree $n - 1$. Polynomial equations involving depressed polynomials are in general easier to solve, and there are classical methods of putting a given polynomial into depressed form. We now treat the solution to a general depressed quartic $x^4 + px^2 + qx + r$ where q is a unit.

Proposition 5.2.7. *A depressed quartic $x^4 + px^2 + qx + r$ with q a unit splits as*

$$\begin{aligned} x^4 + px^2 + qx + r = & \left(x - mR \left(\frac{(m^2+p) R\left(\frac{r}{(m^2+p)^2}\right)}{m^2} \right) \right) \\ & \cdot \left(x - mR \left(\frac{(m^2+p) R\left(\frac{r}{(m^2+p)^2}\right)}{m^2} + 1 \right) \right) \\ & \cdot \left(x - mR \left(\frac{(m^2+p) \left(R\left(\frac{r}{(m^2+p)^2}\right) + 1 \right)}{m^2} \right) \right) \\ & \cdot \left(x - mR \left(\frac{(m^2+p) \left(R\left(\frac{r}{(m^2+p)^2}\right) + 1 \right)}{m^2} + 1 \right) \right), \end{aligned}$$

where m may be taken as

$$m = \sqrt[3]{qR\left(\frac{p^3+q^2}{q^2}\right) + \zeta_3 q} + \sqrt[3]{qR\left(\frac{p^3+q^2}{q^2}\right) + \zeta_3^2 q},$$

or some other root of $m^3 + pm + q = 0$.

Proof: The assumption that q is a unit implies that the polynomial is separable. So we have a splitting

$$x^4 + px^2 + qx + r = (x^2 + mx + n)(x^2 + sx + t).$$

The coefficients m, n, s, t are subject to the relations

$$m + s = 0, \quad n + t + ms = p, \quad mt + ns = q, \quad nt = r.$$

The first of these simply implies $s = m$, so the second and third simplify to

$$n + t + m^2 = p, \qquad m(n+t) = q$$

We can use these to determine m in terms of p, q and r. Indeed, let us first observe that since $q \in R^\times$ and $m(n+t) = q$ implies m is a factor of q, we get that m and $n+t$ are units. So we may isolate to get $n + t = \frac{q}{m}$. Adding this to the equation $n + t + m^2 = p$ results in

$$m^2 = 2n + 2t + m^2 = p + \frac{q}{m},$$

from which we obtain

$$m^3 + pm + q = 0.$$

This is a depressed cubic, and so we need to solve such an equation in characteristic 2. Luckily, this is done in [49]. In this paper Mann shows that a depressed cubic in characteristic 2 has Lagrange resolvents $\sqrt[3]{\theta + \zeta_3 q}$ and $\sqrt[3]{\theta + \zeta_3^2 q}$ where θ is any root of $x^2 + qx + p^3 + q^2$ and ζ_3 is a primitive third root of unity. One then obtains the solutions to $m^3 + pm + q = 0$ in what Mann calls "the usual manner", described in [71, p. 179] as

$$\sqrt[3]{\theta + \zeta_3 q} + \sqrt[3]{\theta + \zeta_3^2 q}, \quad \zeta_3 \sqrt[3]{\theta + \zeta_3 q} + \zeta_3^2 \sqrt[3]{\theta + \zeta_3^2 q} \quad \text{and} \quad \zeta_3^2 \sqrt[3]{\theta + \zeta_3 q} + \zeta_3 \sqrt[3]{\theta + \zeta_3^2 q}.$$

So we are to solve the quadratic equation $x^2 + qx + p^3 + q^2$. The roots of this are given by

$$qR\left(\frac{p^3 + q^2}{q^2}\right) \text{ and } q\left(R\left(\frac{p^3 + q^2}{q^2}\right) + 1\right).$$

Thus, we obtain an expression for m as

$$m = \sqrt[3]{qR\left(\frac{p^3 + q^2}{q^2}\right) + \zeta_3 q} + \sqrt[3]{qR\left(\frac{p^3 + q^2}{q^2}\right) + \zeta_3^2 q}.$$

Next, if we multiply $n + t + m^2 = p$ by n and use $nt = r$ we obtain

$$n^2 + (m^2 + p)n + r = 0.$$

In the same way, multiplying by t instead of n gives $t^2 + (m^2 + p)t + r = 0$. Thus t and r are both roots in the polynomial $x^2 + (m^2 + p)x + r = 0$. But since as noted $n + t$ must be a unit, it follows that they are distinct roots. Thus we get

$$n = (m^2 + p)R\left(\tfrac{r}{(m^2+p)^2}\right) \qquad t = (m^2 + p)\left(R\left(\tfrac{r}{(m^2+p)^2}\right) + 1\right).$$

Note here that we have used $m^2 + p = n + t$ to infer that the first is in fact a unit so the fractions above make sense. Thus, we get a factorization

$$\begin{aligned} a^4 + pa^2 + qa + r &= (a^2 + ma + n)(a^2 + ma + t) \\ &= \left(a - mR\left(\frac{(m^2+p)R\left(\frac{r}{(m^2+p)^2}\right)}{m^2}\right)\right) \\ &\quad \cdot \left(a - mR\left(\frac{(m^2+p)R\left(\frac{r}{(m^2+p)^2}\right)}{m^2} + 1\right)\right) \\ &\quad \cdot \left(a - mR\left(\frac{(m^2+p)\left(R\left(\frac{r}{(m^2+p)^2}\right)+1\right)}{m^2}\right)\right) \\ &\quad \cdot \left(a - mR\left(\frac{(m^2+p)\left(R\left(\frac{r}{(m^2+p)^2}\right)+1\right)}{m^2} + 1\right)\right) \end{aligned}$$

□

In our case, we have explicit descriptions of p, q and r so that $\frac{p^3+q^2}{q^2} = \frac{\lambda_2^6}{\lambda_4^2 \tau^4} + 1$.

5.2.3 Deformation of the Fixed Point Singularities

We wish to understand the fixed point singularities in the family. To do so, we wish to pass to a formal neighbourhood. So first, let us move the singularities to the origin. Consider the polynomials $a^4 + p_1a^2 + q_1a + r_1$ and $b^4 + p_2b^2 + q_2b + r_2$, where

$$p_1 = \tfrac{\lambda_2^2}{\lambda_4^2}, \qquad q_1 = \tfrac{\tau^2}{\lambda_4^2}, \qquad r_1 = \tfrac{\lambda_0^2}{\lambda_4^2}, \qquad p_2 = \tfrac{\sigma_2^2}{\sigma_4^2}, \qquad q_2 = \tfrac{\tau^2}{\sigma_4^2}, \qquad r_2 = \tfrac{\sigma_0^2}{\sigma_4^2}.$$

As noted, τ being a unit implies that these polynomials are separable. Suppose all the roots lie in R (if not, we adjoin them) and let α and β denote some root of $a^4 + p_1a^2 + q_1a + r_1$ and $b^4 + p_2b^2 + q_2b + r_2$ respectively. Using the substitution $a \mapsto a + \alpha$ and $b \mapsto b + \beta$, the polynomial $c^2 + \tau^2ab + (\sigma_4^2b^4 + \sigma_2b^2 + \sigma_0^2)a + (\lambda_4^2a^4 + \lambda_2^2a^2 + \lambda_0^2)b$ becomes

$$\begin{aligned}
&c^2 + \tau^2(a+\alpha)(b+\beta) \\
&+ \left(\sigma_4^2(b+\beta)^4 + \sigma_2^2(b+\beta)^2 + \sigma_0^2\right)(a+\alpha) \\
&+ \left(\lambda_4^2(a+\alpha)^4 + \lambda_2^2(a+\alpha)^2 + \lambda_0^2\right)(b+\beta) = c^2 + \tau^2ab + \left(\sigma_4^2b^4 + \sigma_2^2b^2\right)a + \left(\lambda_4^2a^4 + \lambda_2^2a^2\right)b \\
&\qquad + \lambda_4^2\beta a^4 + \lambda_2^2\beta a^2 + \left(\sigma_4^2\beta^4 + \sigma_2^2\beta^2 + \tau^2\beta + \sigma_0^2\right)a \\
&\qquad + \sigma_4^2\alpha b^4 + \sigma_2^4\alpha b^2 + \left(\lambda_4^2\alpha^4 + \lambda_2^2\alpha^2 + \tau^2\alpha + \lambda_0^2\right)b \\
&\qquad + \left(\lambda_4^2\alpha^4 + \lambda_2^2\alpha^2 + \tau^2\alpha + \lambda_0^2\right)\beta + \left(\sigma_4^2\beta^4 + \sigma_2^2\beta^2 + \sigma_0^2\right)\alpha \\
&\quad = c^2 + \tau^2ab + \left(\sigma_4^2b^4 + \sigma_2^2b^2\right)a + \left(\lambda_4^2a^4 + \lambda_2^2a^2\right)b + \lambda_4^2\beta a^4 \\
&\qquad + \lambda_2^2\beta a^2 + \sigma_4^2\alpha b^4 + \sigma_2^2\alpha b^2 + \tau^2\beta\alpha,
\end{aligned}$$

where we have used that α and β are roots in $\lambda_4^2a^4 + \lambda_2^2a^2 + \tau^2a + \lambda_0^2$ and $\sigma_4^2b^2 + \sigma_2^2b^2 + \tau^2b + \sigma_0^2$ respectively. We then obtain a result analogous to Proposition 5.2.5:

Proposition 5.2.8. *Let* $\mathfrak{m} \in \operatorname{Spec} R$ *be a* k*-rational point and*

$$h = c^2 + \tau^2ab + (\sigma_4^2b^4 + \sigma_2^2b^2)a + (\lambda_4^2a^4 + \lambda_2^2a^2)b + \lambda_4^2\beta a^4 + \lambda_2^2\beta a^2 + \sigma_4^2\alpha b^4 + \sigma_2^2\alpha b^2 + \tau^2\alpha\beta.$$

Further, let $C = R[a,b,c]/(h)$*, and* $C_0 = C \otimes_R R/\mathfrak{m}$*. Consider then the ideals* $(a,b,c)C$ *and* $(a,b,c)C_0$*. As a deformation of* $\widehat{C_0}$ *the deformation* $\widehat{C}$ *is isomorphic to the constant deformation* $\widehat{C_0}\hat{\otimes}_{R/\mathfrak{m}}R$ *which is simply* $R[|a,b,c|]/(c^2+ab)$*.*

Proof: We employ the same method of proof as in Proposition 5.2.5. Like earlier

$$\widehat{C} = R[|a,b,c|]/(h).$$

We will show there is an R-algebra automorphism of $R[|a,b,c|]$ mapping the defining equation to $c^2 + ab$. This implies the desired. We construct this automorphism in steps. First, we note that α and β are in fact squares. Indeed, $\sigma_4^2\alpha^4 + \sigma_2^2\alpha^2 + \tau^2\alpha + \sigma_0^2 = 0$ which implies that

$$\alpha = \frac{1}{\tau^2}\left(\sigma_4^2\alpha^4 + \sigma_2^2\alpha^2 + \sigma_0^2\right),$$

since τ is a unit. As the right hand side is a square, so is α, and by symmetry β. Now, as a matter of notation let α' and β' denote the elements such that $\alpha'^2 = \alpha$ and $\beta'^2 = \beta$ respectively. Then we may define an automorphism φ by fixing a and b, and mapping

$$c \mapsto c + \lambda_4\beta'a^2 + \lambda_2\beta'a + \sigma_4\alpha'b^2 + \sigma_2\alpha'b + \tau\beta'\alpha'.$$

Then

$$\varphi(h) = c^2 + \tau^2ab + (\sigma_4^2b^4 + \sigma_2^2b^2)a + (\lambda_4^2a^4 + \lambda_2^2a^2)b.$$

Next consider the automorphism ψ on $R[|a,b,c|]$ defined by $a \mapsto a + \frac{\sigma_2^2}{\tau^2}ab$. Then

$$\psi(\tau^2 ab) = \tau^2 ab + \sigma_2^2 ab^2,$$

while

$$\psi((\sigma_4^2 b^4 + \sigma_2^2 b^2)a) = (\sigma_4^2 b^4 + \sigma_2^2 b^2)a + \left(\frac{\sigma_4^2\sigma_2^2}{\tau^2} b^5 + \frac{\sigma_2^4}{\tau^2} b^3\right) a.$$

As this only adds new terms of degree 1 in a and total degree > 3, we may continue via induction as in Proposition 5.2.5. Thus we obtain an automorphism ψ' of $R[|a,b,c|]$ defined via induction such that

$$\psi'\left(c^2 + \tau^2 ab + \left(\sigma_4^2 b^4 + \sigma_2^2 b^2\right) a + \left(\lambda_4^2 a^4 + \lambda_2^2 a^2\right) b\right) = c^2 + \tau^2 ab + \left(\lambda_4^2 a^4 + \lambda_2^2 a^2\right) b.$$

Applying the same principle, beginning with the substitution $b \mapsto b + \frac{\lambda_2^2}{\tau^2}ab$, we obtain an inductively defined automorphism ψ'' such that

$$\psi''\left(c^2 + \tau^2 ab + \left(\lambda_4^2 a^4 + \lambda_2^2 a^2\right) b\right) = c^2 + \tau^2 ab.$$

Now we finally apply the automorphism ψ''' defined by $a \mapsto \frac{1}{\tau}a$, $b \mapsto \frac{1}{\tau}b$, $c \mapsto c$ to obtain

$$\psi'''(c^2 + \tau^2 ab) = c^2 + ab.$$

Composing all of these automorphisms now gives the desired.

$\square$

Note that this proof gives for free that the singularities we are treating are all A_1 singularities, as the defining equation is exactly the normal form of this singularity type.

5.3 Concluding Results

The result of the preceding sections lead us to the following results (which are essentially the same result reformulated). First, we combine the results of Proposition 5.2.5 and Proposition 5.2.8 into a statement on the simultaneous resolution of the quotient family $(C \times C)/\mu_2$:

Theorem 5.3.1. *The quotient family $(C \times C)/\mu_2$ admits a simultaneous resolution of singularities over the base change*

$$S' = S \otimes_{k[\lambda_4,\lambda_2,\lambda_0,\sigma_4,\sigma_2,\sigma_0,\tau]} k\left[\sqrt[3]{\frac{\lambda^2}{\lambda_4^2}}, \sqrt[3]{\frac{\sigma^2}{\sigma_4^2}}, \alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4\right].$$

Here α_i and β_i are the roots of $\lambda_4^2 a^4 + \lambda_2^2 a^2 + \tau^2 a + \lambda_0^2$ and $\sigma_4^2 b^4 + \sigma_2^2 b^2 + \tau^2 b + \sigma_0^2$ respectively, λ and σ are the canonical choice of representatives of λ_4 and σ_4 in k, and $\sqrt[3]{\frac{\lambda^2}{\lambda_4^2}}, \sqrt[3]{\frac{\sigma^2}{\sigma_4^2}}$ are any choice of third roots. This is a finite base change of degree at most $3^2(4!)^2 = 5184$. The only possible prime divisors of the degree of the base change are 2 and 3.

Proof: Keep in mind Remark 4.2.4. By this remark, having the deformation of a singularity be constant, means that the singularity may be simultaneously resolved. Now, as outlined in the previous section $(C \times C)/\mu_2$ has fiberwise the fixed point singularities and the singularity coming from the quadruple point. Adjoining the roots of $\lambda_4^2 a^4 + \lambda_2^2 a^2 + \tau^2 a + \lambda_0^2$ and $\sigma_4^2 b^4 + \sigma_2^2 b^2 + \tau^2 b + \sigma_0^2$ allows for the linear substitutions $a \mapsto a + \alpha_i$ and $b \mapsto b + \beta_i$. The result of Proposition 5.2.8 then tells us that each of the fixed point singularities deforms in a constant manner, hence can be simultaneously resolved after the base change which adds these roots.

In the same manner, Proposition 5.2.5 shows that the singularities coming from the quadruple point may be simultaneously resolved after adjoining $\sqrt[3]{\frac{\lambda^2}{\lambda_4^2}}$ and $\sqrt[3]{\frac{\sigma^2}{\sigma_4^2}}$.

□

As an immediate consequence we obtain:

Corollary 5.3.2. *The generalized Kummer construction with μ_2 works in families after a finite base change of degree at most $3^2(4!)^2 = 5184$. The only possible prime divisors of the degree of the base change are 2 and 3.*

5.4 Questions remaining

There are a few immediate questions left related to these generalized constructions that might be of interest to anyone who finds the subject relevant to pursue.

- There is a further case in pure characteristic 2. In the vector field δ we assumed τ to be a unit. This amounts to forcing the action defined by δ to stay a μ_2-action in the whole family. But what if one lifted this restriction? Then the group scheme acting varies in the family. The two main things to solve are then the base change property, i.e. taking fibers and quotients commute, and the simultaneous resolution of singularities. The first part I suspect one can solve by emulating the proof of [62, Proposition4.2]. For the second bit, at least the computations of 5.2.4 remain valid with no assumption on τ, hence also works for α_2-actions. So at least the D_4 singularity should admit simultaneous resolution after the finite base change described. But the other singularities are more tricky as their types vary in the family. I suspect one would need at least the techniques of [62, Theorem 12.1]. At least one could say that the construction would work locally, as each point will have a neighbourhood in which τ is either a unit or identically zero, which restricts to the μ_2 and α_2 cases separately.

- Then there is the case of mixed characteristic. In Section 5.2, we saw how the involution action on a family of Abelian surfaces is just an action by μ_2. As such, one could take a concrete family over a base, say $\mathbb{Z}$, such that the family is Abelian outside of the prime 2, but degenerates to a product of cuspidal curves over 2. Such an example could be the self product of the Weierstrass fibration $y^2 - x^3 - 2$ over $\mathbb{Z}_2$ or $y^2 - x^3 - 2x$ over $\mathbb{Z}$. As the action is always given by μ_2, the quotient exists and we know its fibers. It is then interesting to see if this quotient family admits a simultaneous resolution. I have done some rough preliminary computations with the first equation, and the A_1 singularities seem to work out as in Proposition 5.2.8. The possible problem here seems to be the D_4 singularity as it only appears in one fiber.

References

[1] Abhyankar, S. Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$. *Ann. of Math. (2) 63* (1956), 491–526.

[2] Abhyankar, S. Corrections to “Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$”. *Ann. of Math. (2) 78* (1963), 202–203.

[3] Artin, M. Some numerical criteria for contractability of curves on algebraic surfaces. *Amer. J. Math. 84* (1962), 485–496.

[4] Artin, M. On isolated rational singularities of surfaces. *Amer. J. Math. 88* (1966), 129–136.

[5] Artin, M. The implicit function theorem in algebraic geometry. In *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*. Oxford Univ. Press, London, 1969, pp. 13–34.

[6] Artin, M. *Algebraic spaces*, vol. 3 of *Yale Mathematical Monographs*. Yale University Press, New Haven, Conn.-London, 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969.

[7] Artin, M. Algebraic construction of Brieskorn’s resolutions. *J. Algebra 29* (1974), 330–348.

[8] Artin, M. Coverings of the rational double points in characteristic p. In *Complex analysis and algebraic geometry*. Iwanami Shoten, Tokyo, 1977, pp. 11–22.

[9] Artin, M., Grothendieck, A., and Verdier, J. L., Eds. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, vol. 269 of *Lect. Notes Math.* Springer, Cham, 1972.

[10] Atiyah, M. F., and Macdonald, I. G. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[11] Bergqvist, J., Dang, T., and Schröer, S. Sign involutions on para-abelian varieties. *arXiv e-prints* (Jan. 2023), arXiv:2301.03314.

[12] Berthelot, P., Grothendieck, A., and Illusie, L., Eds. *Séminaire de géométrie algébrique du Bois Marie 1966/67, SGA 6: Théorie des intersections et théorème de Riemann-Roch*, vol. 225 of *Lect. Notes Math.* Springer, Cham, 1971.

[13] Bombieri, E., and Mumford, D. Enriques’ classification of surfaces in char. p. III. *Invent. Math. 35* (1976), 197–232.

[14] Bombieri, E., and Mumford, D. Enriques’ classification of surfaces in char. p. II. In *Complex analysis and algebraic geometry*. Iwanami Shoten, Tokyo, 1977, pp. 23–42.

[15] BĂDESCU, L. *Algebraic surfaces*. Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[16] DEMAZURE, M., AND GABRIEL, P. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice ıt Corps de classes local par Michiel Hazewinkel.

[17] DEMAZURE, M., AND GABRIEL, P. *Introduction to algebraic geometry and algebraic groups*, vol. 39 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam-New York, 1980. Translated from the French by J. Bell.

[18] DEMAZURE, M., AND GROTHENDIECK, A., Eds. *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 : Schémas en groupes (SGA 3). Tome II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, vol. 152 of *Lect. Notes Math.* Berlin, New York, Springer-Verlag, 1970.

[19] DEMAZURE, M., AND GROTHENDIECK, A., Eds. *Séminaire de géométrie algébrique du Bois Marie 1962-64. Schémas en groupes (SGA 3). Tome I: Propriétés générales des schémas en groupes.*, new annotated edition of the 1970 original published by Springer ed., vol. 7 of *Doc. Math. (SMF)*. Paris: Société Mathématique de France, 2011.

[20] DU VAL, P. On isolated singularities of surfaces which do not affect the conditions of adjunction. I. *Proc. Camb. Philos. Soc. 30* (1934), 453–459.

[21] DU VAL, P. On isolated singularities of surfaces which do not affect the conditions of adjunction. II. *Proc. Camb. Philos. Soc. 30* (1934), 460–465.

[22] DU VAL, P. On isolated singularities of surfaces which do not affect the conditions of adjunction. III. *Proc. Camb. Philos. Soc. 30* (1934), 483–491.

[23] DUMMIT, D. S., AND FOOTE, R. M. *Abstract algebra*, third ed. John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[24] ENRIQUES, F. *Le Superficie Algebriche*. Nicola Zanichelli, Bologna, 1949.

[25] FALTINGS, G., AND CHAI, C.-L. *Degeneration of abelian varieties*, vol. 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[26] FANTECHI, B., GÖTTSCHE, L., ILLUSIE, L., KLEIMAN, S. L., NITSURE, N., AND VISTOLI, A. *Fundamental algebraic geometry : Grothendieck's FGA explained.* Mathematical surveys and monographs ; 123. American Mathematical Society, 201 Charles Street, Providence, R.I, 2005.

[27] GREUEL, G.-M., AND KRÖNING, H. Simple singularities in positive characteristic. *Math. Z. 203*, 2 (1990), 339–354.

[28] GROTHENDIECK, A. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, 4 (1960), 228.

[29] GROTHENDIECK, A. *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*. Secrétariat mathématique, Paris, 1962.

[30] GROTHENDIECK, A. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, 20 (1964), 259.

[31] GROTHENDIECK, A. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, 24 (1965), 231.

[32] GROTHENDIECK, A. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, 32 (1967), 361.

[33] GROTHENDIECK, A., Ed. *Séminaire de géométrie algébrique du Bois Marie 1960-61. Revêtements étales et groupe fondamental (SGA 1). Augmenté de deux exposés de M. Raynaud.*, édition recomposée et annotée du original publié en 1971 par Springer ed., vol. 3 of *Doc. Math. (SMF)*. Paris: Société Mathématique de France, 2003.

[34] HARTSHORNE, R. *Local cohomology*. Lecture Notes in Mathematics, No. 41. Springer-Verlag, Berlin-New York, 1967. A seminar given by A. Grothendieck, Harvard University, Fall, 1961.

[35] HARTSHORNE, R. *Algebraic Geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[36] HUYBRECHTS, D. *Lectures on K3 surfaces*, vol. 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[37] IGUSA, J.-I. Betti and Picard numbers of abstract algebraic surfaces. *Proc. Nat. Acad. Sci. U.S.A. 46* (1960), 724–726.

[38] KATSURA, T. On Kummer surfaces in characteristic 2. In *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)* (1978), Kinokuniya Book Store, Tokyo, pp. 525–542.

[39] KATZ, N. M., AND MAZUR, B. *Arithmetic moduli of elliptic curves*, vol. 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[40] KATZ, V. J. *A history of mathematics. An introduction*, 2nd ed. ed. Reading, MA: Addison-Wesley, 1998.

[41] KNUTSON, D. *Algebraic spaces*. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971.

[42] KODAIRA, K. On the structure of compact complex analytic surfaces. I. *Amer. J. Math. 86* (1964), 751–798.

[43] KODAIRA, K. On the structure of compact complex analytic surfaces. II. *Amer. J. Math. 88* (1966), 682–721.

[44] KODAIRA, K. On the structure of compact complex analytic surfaces. III. *Amer. J. Math. 90* (1968), 55–83.

[45] KODAIRA, K. On the structure of complex analytic surfaces. IV. *Amer. J. Math. 90* (1968), 1048–1066.

[46] KONDŌ, S., AND SCHRÖER, S. Kummer surfaces associated with group schemes. *Manuscripta Math. 166*, 3-4 (2021), 323–342.

[47] LAURENT, B., AND SCHRÖER, S. Para-abelian varieties and Albanese maps. *arXiv e-prints* (Jan. 2021), arXiv:2101.10829.

[48] LIU, Q. *Algebraic geometry and arithmetic curves. Transl. by Reinie Erné*, vol. 6 of *Oxf. Grad. Texts Math.* Oxford: Oxford University Press, 2006.

[49] MANN, H. B. On the casus irreducibilis. *Amer. Math. Monthly 71* (1964), 289–290.

[50] MATSUMURA, H. *Commutative algebra*, second ed., vol. 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

[51] MATSUMURA, H., AND OORT, F. Representability of group functors, and automorphisms of algebraic schemes. *Invent. Math. 4* (1967), 1–25.

[52] MIRANDA, R. *The Basic Theory of Elliptic Surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.

[53] MUMFORD, D. *Geometric invariant theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), Band 34. Springer-Verlag, Berlin-New York, 1965.

[54] MUMFORD, D. Enriques' classification of surfaces in $\mathrm{char}\ p$. I. In *Global Analysis (Papers in Honor of K. Kodaira)*. Univ. Tokyo Press, Tokyo, 1969, pp. 325–339.

[55] MUMFORD, D. *Abelian varieties*, vol. 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[56] MUMFORD, D., FOGARTY, J., AND KIRWAN, F. *Geometric invariant theory*, third ed., vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, 1994.

[57] NAGATA, M. On the 14-th problem of Hilbert. *Amer. J. Math. 81* (1959), 766–772.

[58] OLSSON, M. *Algebraic spaces and stacks*, vol. 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[59] PERRIN, D. *Schémas en groupes quasi-compacts sur un corps et groupes henséliens*. Publication Mathématique d'Orsay, Nos. 165–75.46. Université Paris XI, U.E.R. Mathématique, Orsay, 1975.

[60] RAYNAUD, M. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics, Vol. 119. Springer-Verlag, Berlin-New York, 1970.

[61] RYDH, D. Existence and properties of geometric quotients. *J. Algebraic Geom. 22*, 4 (2013), 629–669.

[62] SCHRÖER, S. Kummer surfaces for the self-product of the cuspidal rational curve. *J. Algebraic Geom. 16*, 2 (2007), 305–346.

[63] SCHRÖER, S. The Hilbert scheme of points for supersingular abelian surfaces. *Ark. Mat. 47*, 1 (2009), 143–181.

[64] SCHRÖER, S., AND TZIOLAS, N. The structure of Frobenius kernels for automorphism group schemes. *arXiv e-prints* (May 2021), arXiv:2105.07860.

[65] SHIODA, T. Kummer surfaces in characteristic 2. *Proc. Japan Acad. 50* (1974), 718–722.

[66] SILVERMAN, J. H. *Advanced topics in the arithmetic of elliptic curves*, 2nd ed. ed. Graduate texts in mathematics. Springer-Verlag, 1994.

[67] SILVERMAN, J. H. *The Arithmetic of Elliptic Curves*, 2nd ed. ed. Graduate Texts in Mathematics, 106. Springer New York, New York, NY, 2009.

[68] SILVERMAN, J. H., AND TATE, J. T. *Rational Points on Elliptic Curves*, second ed. Undergraduate Texts in Mathematics. Springer, Cham, 2015.

[69] STACKS PROJECT AUTHORS, T. *Stacks Project*. `http://stacks.math.columbia.edu`, 2018.

[70] TATE, J. *Algorithm for determining the type of a singular fiber in an elliptic pencil*. In *Modular Functions of One Variable IV* (Berlin, Heidelberg, 1975), B. J. Birch and W. Kuyk, Eds., Springer Berlin Heidelberg, pp. 33–52.

[71] VAN DER WAERDEN, B. L. *Modern Algebra. Vol. I.* Frederick Ungar Publishing Co., New York, N. Y., 1949. Translated from the second revised German edition by Fred Blum, With revisions and additions by the author.