Long-term Dynamics for Living Fluids and Heterogeneous Catalysis and an Approach to the Stokes Equations via Duality Scales

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Summary

In this thesis we consider (in)stability and long-term behavior of a living fluids model, stability of a model for the heterogenous catalysis process and as a last topic the use of duality scales on complemented subspaces with regard to partial differential equations. The first model to be considered, a living fluids model, is given as generalized Navier-Stokes equations and describes dense bacterial suspensions at low Reynolds number. In real world experiments as well as in numerical simulations turbulence was observed, which should be respected by the mathematical model. We establish a complete analysis of linear and nonlinear stability and instability in the periodic L^2-setting about the two relevant types of equilibria and find parameter sets corresponding to stability and instability. It has to be noted that one type of equlibria, the ordered polar states, yields a manifold of equilibria. Therefore, the theory of normal stability and normal hyperbolicity is applied to the system. Afterwards, we show that the living fluids model possesses a global attractor of finite dimension and arbitrary high regularity, which characterizes the long-term behavior of the model. To this end, we first show that the equations admit a unique solution with initial values of L^2-regularity and obtain a semigroup from these solutions. Then, theory from infinite dimensional dynamical systems is applied to show the existence of compact absorbing sets and hence the existence of a global attractor. In a last step, the properties of this global attractor are analyzed in more detail to obtain results about regularity and dimensionality.
The second model considered in this thesis stems from chemical engineering and describes the process of heterogeneous catalysis in a cylinder-shaped domain. Since the catalysis considered is heterogeneous, we assume the catalyzer to be on the lateral boundary of the cylinder, which results in a coupled system of equations in the bulk and on the lateral boundary. We show stability and instability for the heterogeneous catalysis model in the L^p-setting dependent on the chemical reaction which is chosen on the lateral boundary. To this end, we apply the principle of linearized stability to isolated equilibria. One example for such equilibria is given as the state of chemical balance.
In the last part of this thesis we consider the concept of duality scales of Banach spaces, which gives a more precise meaning to the powerful concept of duality. Roughly speak-

ing, the existence of a duality scale with two scales of Banach spaces $(E_q)_{q\in I_0}$ and $(F_q)_{q\in I_0}$ and a scale of bilinear continuous duality pairings $\mathfrak{a}_q(\cdot,\cdot)$ (which can e.g. stem from the weak formulation of elliptic problems) yields solubility of a corresponding (elliptic) partial differential equation. We show that under certain assumptions regarding a consistent projection P on these scales, the property of being a duality scale is preserved if we consider the complemented subspaces $(P(E_q))_{q\in I_0}$ and $(P(F_q))_{q\in I_0}$. This is especially useful in order to obtain solutions for the well known Stokes equations. As an example, we consider the Stokes operator with mixed-type boundary conditions on C^3-domains with compact boundary and apply the theory of projected duality scales in order to obtain solutions of regularity $W^{1+\varepsilon,q}$, where $q \in (1,\infty)$ and $0 < \varepsilon < \min\{1/q, 1/q'\}$. Finally, we give some abstract results regarding duality scales on complemented subspaces which exploit the property of compactness (which can, e.g. stem from the compact boundary of the underlying domain).

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Contents

1 Introduction

Dynamically changing processes occur in many phenomena in our world, e.g. in fluid flows, evolution of populations, movements of particles or distribution of heat. Therefore, it is of great interest to develop mathematical models and tools in order to analyze such processes and learn about their behavior. This leads to the class of evolution equations, which is a subclass of the so-called partial differential equations in mathematics.

In general, we observe an evolving quantity u (e.g. a temperature, a density of some substance, a velocity field, etc.), which depends on the time t. Therefore, $u(t)$ describes the value or state of the quantity at time t. In many cases, we assume the time to be positive, i.e. $t \geqslant 0$, and prescribe some initial value u_0, which describes the state of the quantity at the beginning ($t = 0$), which, for example, could be a real world measurement of the quantity. Then, a general evolution equation (or system) formally reads as

$$\dot{u} = F(u) \quad (t > 0), \quad u(0) = u_0, \tag{1.1}$$

where $\dot{u}$ is the time-derivative of u, i.e. the rate of change of the quantity u w.r.t. the time t, and F is some mapping which describes how the quantity evolves. Note that in most cases, F depends on the quantity u itself. Given an evolution equation, one usually has to find spaces X, the solution space where the solution u shall *live in*, and Y, the data space where the initial value u_0 comes from. These spaces characterize the properties of the solution and the initial value, which gives rise to different questions regarding the mathematical treatment. For instance, one can ask for:

- *Local solubility:* Given any initial value $u_0 \in Y$, does there exist a solution $u \in X$ of (1.1) on some time interval $(0, T)$, where $0 < T < \infty$ may depend on u_0?
- *Global solubility:* Given any initial value $u_0 \in Y$, does there exist a solution $u \in X$ of (1.1) on any time interval $(0, T)$, where $T \in (0, \infty]$?
- *Continuous dependence on the data:* If one chooses some initial data u_1 which deviates from u_0 only at a small scale, does the solution v corresponding to u_1 de-

viate from u corresponding to u_0 at a small scale too? In other words, continuous dependence on the data ensure that small perturbations of the initial values only cause small perturbations in the solutions. This property becomes important if one wants to use real world measurements for (numerical) simulations, since these measurements are usually only accurate up to some degree.

- *Uniqueness of the solution:* Can there exist two ore more different solutions in X corresponding to the same initial value u_0?

- *Stability:* Given a steady state (also called equilibrium) of (1.1), i.e. a $u_* \in X$ with $F(u_*) = 0$ such that the system is *at rest* and does not evolve anymore, do solutions which start from u_0 near u_* converge back to a steady state? A stable equilibrium will force the system back into some steady state if it is perturbed, whereas an unstable equilibrium does not have this property and perturbations can cause the system to evolve further away from the steady state.

- *Long-term dynamics:* Does there exist a (small) subset $\mathcal{A} \subseteq X$ of possible solutions which attracts all other solutions of the system as time evolves? If such an (global) attractor $\mathcal{A}$ exists, the solution to *any* initial value $u_0 \in Y$ will evolve towards the attractor as $t \to \infty$. In such a case, the dynamics of the whole system can be reduced to the dynamics on the attractor $\mathcal{A}$, which is in some cases less complex to analyze.

In many examples of typical evolution equations, the equations have a more concrete structure than in (1.1), which leads to the development of mathematical tools exploiting the concrete structure of the equation in order to answer some of these questions. This is the case for so-called linear, semilinear or quasilinear evolution equations.
This thesis aims to contribute to the last two questions in particular, but also presents some mathematical theory which helps to answer the first questions.
First, we analyze the model of a living fluid and find conditions for stability and instability. Afterwards, we consider the long-term dynamics of this model and show that there exists a global attractor of finite dimension which characterizes the long-term behavior of any solution. Then, we turn to another model describing the process of heterogeneous catalysis and establish settings for stability and instability. In the last part of this thesis, we consider the concept of duality scales. This concept will help to show the existence of solutions to certain partial differential equations in a more general way. We will shortly introduce the topics and the results in the following sections.

1.1 Living Fluids

The first model which is analyzed in this thesis is a so-called active or living fluid model. One example for the dynamics of a living fluid is the motion of a dense bacterial suspension at low Reynolds number. In this case, a major driving force is the self-propulsion of the bacteria which causes the motion in the fluid. Observations and simulations of such suspensions show that active turbulence and the formation of vortices are likely to occur, cf. e.g. [59, 57, 13].
Given these results, it is highly desirable to analyze the behavior of active fluids rigorously in a mathematical model. A suitable model using a generalized Navier-Stokes equations was proposed in [59] and also used e.g. in [14, 36]. This model extends the Navier-Stokes equations with Swift-Hohenberg and Toner-Tu terms that respect the behavior of the self-propulsed motion of the bacteria. In this thesis, we use this model and analyze stability and instability as well as the long-term dynamics. We consider the equations

$$\begin{aligned} \dot{v} + \lambda_0 v \cdot \nabla v &= f - \nabla p + \lambda_1 \nabla |v|^2 - (\alpha + \beta |v|^2) v + \Gamma_0 \Delta v - \Gamma_2 \Delta^2 v, \\ \operatorname{div} v &= 0, \\ v(0) &= v_0 \end{aligned} \tag{1.2}$$

in a box $Q_n := [0, L]^n$ with periodic boundary conditions, where $L > 0$ and $n \in \{2, 3\}$. Here, v is the (divergence-free) velocity field of the suspension with n components. The parameter $\lambda_0 \in \mathbb{R}$ describes the advection, where $\lambda_1 \in \mathbb{R}$ describes the active pressure contribution. In addition, we have a viscosity parameter $\Gamma_0 \in \mathbb{R}$ and the Swift-Hohenberg term of fourth order lead by $\Gamma_2 > 0$, which contributes to pattern formation of the bacteria. In order to describe the flocking-like behavior of the bacteria we consider the Toner-Tu term characterized by $\alpha \in \mathbb{R}$ and $\beta > 0$, which determines the manifold of ordered polar equilibrium states if $\alpha < 0$. By p we denote the (scalar) pressure contribution, which only appears as a gradient in the equations.
Note that a rigorous mathematical analysis of this model was first considered in $L^2(\mathbb{R}^n)$ (cf. [61]). There, a stability analysis is performed for the isotropic disordered and the ordered polar state. A major drawback in this analysis is the fact that the manifold of ordered polar states $B_{\alpha,\beta}$, which consists of vectors of constant velocity of a given length, and the wave functions for a classical wave ansatz do not belong to $L^2(\mathbb{R}^n)$.
Therefore, another analysis was carried out in spaces of Fourier transformed Radon measures $\mathrm{FM}(\mathbb{R}^n)$ in [9]. Here, the functions of a wave ansatz $\exp(ik \cdot x + \sigma t)$ are contained in $\mathrm{FM}(\mathbb{R}^n)$. However, in both approaches it seems difficult to capture the instability behavior of the ordered polar states on the manifold $B_{\alpha,\beta}$. It is shown that a

single polar state admits instability. But it is unclear, if the solution converges back to another equilibrium on $B_{\alpha,\beta}$. This behavior near the manifold of equilibria is important to describe the behavior of active turbulence which is observed in active fluids.

Therefore, our approach carries out the analysis in a bounded box $Q_n := [0, L]^n$ in spatial dimensions $n = 2$ and $n = 3$ with periodic boundary conditions. There are several advantages using this setting: on the one hand, all considered equilibrium states belong to $L^2(Q_n)$. On the other hand, the corresponding linearized operators have a discrete point spectrum due to compactness of the resolvent, which makes the analysis of the spectrum easier and allows us to prove that $\lambda = 0$ is a semi-simple eigenvalue. This gives rise to the application of the theory of normally stable and normally hyperbolic equilibria (cf. [39, 40, 38] and Section 2.3), which describes the behavior of solutions near $B_{\alpha,\beta}$ rigorously.

In Chapter 3 we address the issue of stability and instability. Therefore, we first collect the relevant equlibria of (1.2). It turns out that the relevant equilibria consist of the already mentioned manifold of ordered polar states $B_{\alpha,\beta}$, which are fixed vectors of a given length, and the disordered isotropic state, where the velocity field equals zero. Then, we mention basic results of (global) well-posedness and carry out an analysis of (in)stability in the linear setting, which is based on the Fourier series symbols of the corresponding linear operators. We see that stability or instability depends on the parameter set $(\Gamma_2, \Gamma_0, \alpha)$ and - in the case of instability - additionally on the existence of unstable Fourier modes. Finally, we apply the theory of normally hyperbolic equilibria to the manifold of ordered polar states $B_{\alpha,\beta}$ in order to show evidence for turbulence under certain conditions for $\Gamma_0 < 0$. Additionally we use the theory of normally stable equilibria to the nonlinear system in order to show stability for $\Gamma \geqslant 0$ in the phase space $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Nonlinear (in)stability of the disordered isotropic state will also be discussed. With these results we have given a complete analysis of stability and instability of the disordered isotropic and the ordered polar states in the nonlinear setting. The results presented in this regard where first proposed in [8].

While the behavior of solutions near equilibria already gives some insight into the dynamics of the living fluid system, we are also interested in the long-term dynamics of the whole system. Therefore we carry out a complete analysis in this regard in Chapter 4, based on the theory from [41, 55, 47]. First we show the well-posedness of (1.2), projected by the Helmholtz projection, with initial values in $L^2_\sigma(Q_n)$ in order to obtain some semigroup $S(t)$ as a solution operator. In the next step, we see that there exists a global attractor $\mathcal{A}$ for this system, which can be roughly described as a compact set that attracts all solutions. We prove that this attractor is of arbitrary high regularity. In a last step it is shown that $\mathcal{A}$ is finite dimensional in the fractal and the

Hausdorff dimension. These results imply that the long-term dynamics of (4.1) can be reduced to a finite dimensional system. If $n = 2$, then there exists an inertial manifold that attracts all solutions at an exponential rate and contains the global attractor.
To the best of the author's knowledge, such types of rigorous analysis where not carried out to this model in the periodic setting before. However, there exist examples of a formal stability analysis based on the standard wave ansatz e.g. in [59], and, as already noted, an analysis of stability and instability in the $\mathbb{R}^n$ setting in [61] and [9]. Regarding the long-term behavior, a complete analysis of the global attractor was carried out, for instance, for the classical Navier-Stokes equations and the Kuramoto-Sivashinsky equation (cf. [55]), but not for the living fluids model.

1.2 Heterogeneous Catalysis

The process of catalysis is an important tool in the field in chemical engineering. By exploiting the reaction of chemical substances with each other, catalysis allows for an increase in the speed of chemical reactions or for a change of the selectivity in favor of a desired product of a reaction. Thus, it is used in many real-world applications e.g. in the automotive industry or laboratory chemical syntheses. For more information on catalysis in general we refer to [33, 43] and the references therein. For our purpose of obtaining and analyzing a mathematical model, one can roughly differentiate between homogeneous catalysis, where the catalyst itself is in the same phase as the other reactants, and heterogeneous catalysis, where the catalyst is in a different phase and usually given on a solid wall. In the latter case, a high area-to-volume ratio is required, which is fulfilled in porous media. For more information on heterogeneous catalysis we refer to [28, 4, 60]. In this thesis we only consider the case of heterogeneous catalysis and assume that the catalyzer has the shape of a cylinder, i.e. there is some sufficiently smooth, simply connected domain $G \subseteq \mathbb{R}^2$, which is the floor and the lid of the cylinder, and a length $h > 0$ given. We then consider the cylindric domain $\Omega = G \times (0, h)$. We may decompose the smooth part of the boundary of this domain into several parts: the inflow surface $\Gamma_{\text{in}} = A \times \{0\}$, the outflow surface $\Gamma_{\text{out}} = A \times \{h\}$ and the lateral surface $\Sigma = \partial A \times (0, h)$ (cf. Figure 1.1).
The basic idea of the catalysis model is given as follows. We have some velocity field u, which pushes the substrate in a dilute phase through the cylinder by advection. At the inflow surface Γ_{in}, the substrate is transported into the catalyzer. In the bulk phase Ω, the substrate is moved by the advection due to the velocity field u and by diffusion due to diffusive fluxes. The chemical reactants are transported onto the lateral boundary, the *active surface* Σ, by adsorption. On the active surface, the chemical reaction of the catalysis process as well as diffusion take place and the product is then transported

back into the bulk phase by desorption. Finally, the velocity field transports the substrate out of the cylinder through the outflow surface Γ_{out}. Note that adsorption, desorption and diffusion take place on a larger time scale than the chemical reaction. Furthermore, the reaction and the sorption processes take place on the lateral, two dimensional surface of the cylinder, which may help in the process of mathematical analysis.
The mathematical model of the heterogeneous catalysis process we use in this thesis was first considered in [7]. It is derived from continuum mechanics w.r.t. the second law of thermodynamics in the isothermal case and partial mass balances on bulk and surface for molar mass concentrations. Additionally, due to the dilute phase in the bulk, there are diffusive fluxes governed by Fickian diffusion with constant coefficients. On the lateral boundary, there is no dilute phase in general, but for the mathematical model the same type of diffusion with constant coefficients is chosen. This results in a coupled system of diffusion-advection equations in the bulk and reaction-diffusion-sorption equations on the active surface. For a more detailed outline of the modeling we refer to [30, 7] and the references therein.

Let $A \subseteq \mathbb{R}^2$ be a bounded, simply connected C^2-domain and $h > 0$. Furthermore, let $\Omega = A \times (0, h)$ be a finite three-dimensional cylinder and $T > 0$. We consider the following coupled system

$$\begin{aligned}
\partial_t c_i + (u \cdot \nabla) c_i - d_i \Delta c_i &= 0 && \text{in} \quad (0,T) \times \Omega, \\
\partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &= r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) && \text{on} \quad (0,T) \times \Sigma, \\
(u \cdot \nu) c_i - d_i \partial_\nu c_i &= g_i^{\text{in}} && \text{on} \quad (0,T) \times \Gamma_{\text{in}}, \\
-d_i \partial_\nu c_i &= r_i^{\text{sorp}}(c_i, c_i^\Sigma) && \text{on} \quad (0,T) \times \Sigma, \\
-d_i \partial_\nu c_i &= 0 && \text{on} \quad (0,T) \times \Gamma_{\text{out}}, \\
-d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &= 0 && \text{on} \quad (0,T) \times \partial\Sigma, \\
c_i|_{t=0} &= c_{i,0} && \text{in} \quad \Omega, \\
c_i^\Sigma|_{t=0} &= c_{i,0}^\Sigma && \text{on} \quad \Sigma,
\end{aligned} \tag{1.3}$$

where $N \in \mathbb{N}$ denotes the number of involved species, $c := (c_i)_{i=1}^N$ denote the bulk concentrations and $c^\Sigma := (c_i^\Sigma)_{i=1}^N$ denote the surface concentrations of the involved chemical species $(C_i)_{i=1}^N$. A prescribed velocity field u is given in the transport term. The functions $(r_i^{\text{sorp}})_{i=1}^N$ describe the rates of adsorption and desorption on the lateral surface, whereas $(r_i^{\text{ch}})_{i=1}^N$ describe the chemical reaction rates.
Throughout this thesis we restrict our choice for r_i^{sorp} to the linear case

$$r_i^{\text{sorp}}(c_i, c_i^\Sigma) = k_i^{\text{ad}} c_i - k_i^{\text{de}} c_i^\Sigma, \tag{1.4}$$

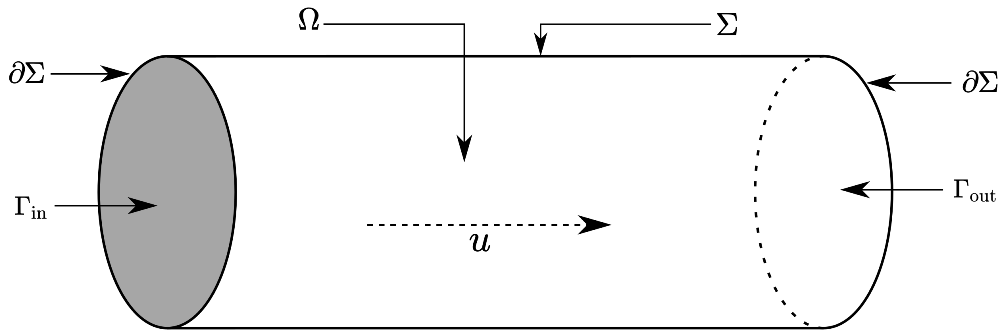

Figure 1.1: An exemplary cylinder for the heterogeneous catalysis model.

where $k_i^{\mathrm{ad}}, k_i^{\mathrm{de}} > 0$. For the choice of r_i^{ch}, we assume that the reaction of N species is given as a reversible reaction

$$\sum_{k=1}^{N} \alpha_k C_k \underset{\kappa_b}{\overset{\kappa_f}{\rightleftarrows}} \sum_{k=1}^{N} \beta_k C_k.$$

Here, $\kappa_f > 0$ denotes the forward reaction rate and $\kappa_b > 0$ the backward reaction rate, while $(\alpha_k)_{k=1}^N \in (\{0\} \cup [1,\infty))^N$ and $(\beta_k)_{k=1}^N \in (\{0\} \cup [1,\infty))^N$, $\alpha, \beta \neq 0$ denote the stoichiometric coefficients. The reaction rate for this reaction is given through

$$r_i^{\mathrm{ch}}(c^\Sigma) := (\alpha_i - \beta_i) \left(\kappa_b \prod_{k=1}^{N} (c_k^\Sigma)^{\beta_k} - \kappa_f \prod_{k=1}^{N} (c_k^\Sigma)^{\alpha_k} \right). \tag{1.5}$$

In Chapter 5, we analyze the stability and instability behavior of the equations (1.3). To this end, we first cite some result on well-posedness of the linearized equations and propose conditions regarding the equilibria and the velocity field u. Then we prove a stability result in the L^p-setting for $p \in [2,\infty) \setminus \{3\}$ and non-negative equilibria (c_*, c_*^Σ) using the principle of linearized stability (cf. Section 2.3). This result yields stability dependent on the first derivative of the chemical reaction rates and the Poincaré constant of the lateral boundary of the cylinder. We provide some examples for isolated positive equilibria and show that under special circumstances stability is to be expected without a smallness condition on the first derivative of the chemical reaction rates. In a last step we present a result on instability.

The equations modeling heterogeneous catalysis processes considered in this thesis were proposed in [7], where a mathematical analysis of linear and nonlinear local well-posedness is carried out. Additionally, global well-posedness is proved taking into consideration a triangular structure for the chemical reactions. There are more recent results on the mathematical modeling of the heterogeneous catalysis process, see e.g. [53] for a detailed approach modeling a coupled system of equations in a suitable thermodynamic framework. In [6], several limit models are derived taking into account the time scale on which the chemical reaction and the sorption occur. Additionally, a three component model problem is analyzed in terms of well-posedness, positivity of solutions, blow-up criteria and a-priori bounds. The approach is extended to more general systems in [5]. Recent results regarding global well-posedness of volume-surface reaction-diffusion systems were presented in [34]. However, no cylindrical structure with inflow and outflow surface is considered in these works and the results do not cover stability or instability of equilibria. In [46], a general theory regarding stable and unstable manifolds is developed for quasilinear problems with nonlinear dynamical boundary conditions. However, the functional analytic setting in this work differs from the one we use in this thesis.

1.3 Duality Scales

In the field of fluid dynamics, the Navier-Stokes equations are a standard model to describe the movement of an incompressible fluid in a given domain $\Omega \subseteq \mathbb{R}^n$, where $n \in \{2,3\}$. For a mathematical analysis of these equations the stationary Stokes equations often play a central role. These are given as

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega, \\ Bu &= g \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\lambda > 0$, u is the n-dimensional velocity field, p is the pressure, B represents the imposed (linear) boundary conditions and f and g is the data in appropriately chosen spaces. Often it is useful to assume homogeneous boundary conditions for a start, i.e. $g = 0$. In this case, we can write the equations as

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega, \end{aligned}$$

where u is in some Banach space X_B where the boundary conditions are fulfilled.
In order to get rid of the divergence condition and the pressure term, one possibility may be to use a Helmholtz-Weyl projection P and obtain the projected equations

$$\lambda u - \Delta u = f \quad \text{in} \quad \Omega$$

on $X_{B,\sigma} := P(X_B)$. In this setting, we want to analyze the mathematical properties of the appearing operator $A_{B,\sigma}u := \Delta_B u$ in $X_{B,\sigma}$ subject to the boundary conditions, where $A_{B,\sigma}$ is called the Stokes operator. The question whether the functional analytic properties of the (often) well-known Laplace operator $A_B u := \Delta_B u$ on X_B can be carried over to the projected Stokes operator $A_{B,\sigma}$ on $X_{B,\sigma}$ arises naturally in this context. Since there are many results on the Stokes operator or the Stokes equations on various types of domains (e.g. [52, 20, 10, 3, 51, 35, 50, 16, 48, 49, 31]; see also the survey [24] and the references in [44]), one could ask if there exists some abstract theory which links the properties of the operators defined on X_B and $X_{B,\sigma} = P(X_B)$, respectively.

One possible approach arises if we think of weak solutions of elliptic problems, which lead to the introduction of duality scales. Consider for instance the weak Neumann problem in

$$W^{1,q}_\nu(\Omega) = \{u \in W^{1,q}(\Omega, \mathbb{R}^3) \,:\, \nu \cdot u|_{\partial\Omega} = 0\}$$

for some sufficiently smooth domain $\Omega \subseteq \mathbb{R}^3$ and $1 < p < \infty$. We have the weak formulation

$$\mathfrak{a}(u,v) := \lambda \int_\Omega uv\,dx + \int_\Omega \nabla u\,\nabla v = \ell(u) \qquad (u \in W^{1,q'}_\nu(\Omega))$$

for $1/q+1/q' = 1$ and $\ell \in (W^{1,q'}_\nu(\Omega))'$. This problem has a unique solution $v \in W^{1,q}_\nu(\Omega)$ if and only if $W^{1,q}_\nu(\Omega)$ is a representation of the dual space of $W^{1,q'}_\nu(\Omega)$ for $1/q+1/q' = 1$ and $q \in (1,\infty)$ w.r.t. $\mathfrak{a}$. Then we call $(W^{1,q'}_\nu(\Omega), W^{1,q}_\nu(\Omega), \mathfrak{a})$ a duality system.

Considering the Stokes equations, it would be helpful to know whether the duality system is preserved if we restrict to projected subspaces. In our example, this would be the application of the Helmholtz projection P. By setting $W^{1,q}_{\nu,\sigma}(\Omega) := P(W^{1,q}_\nu(\Omega))$ this leads to the question, if $(W^{1,q'}_{\nu,\sigma}(\Omega), W^{1,q}_{\nu,\sigma}(\Omega), \mathfrak{a})$ also is a duality system, i.e. if

$$(W^{1,q'}_{\nu,\sigma}(\Omega))' = \{\mathfrak{a}(\cdot,y) \,:\, y \in W^{1,q}_{\nu,\sigma}(\Omega)\} \qquad (q \in (1,\infty), 1/q + 1/q' = 1).$$

In a more general notation, let E, F be two Banach spaces that embed into a linear Haussdorff space $\mathcal{H}$ and let $\mathfrak{a} : E \times F \to \mathbb{C}$ be bilinear and continuous such that

$$E' = \{\mathfrak{a}(\cdot,y) \,:\, y \in F\}, \;\; F' = \{\mathfrak{a}(x,\cdot) \,:\, x \in E\},$$

i.e. such that $(E, F, \mathfrak{a})$ is a duality system. If P is a projection on E and F, then we ask whether the equations

$$P(E)' = \{\mathfrak{a}(\cdot, y) : y \in P(F)\}, \;\; P(F)' = \{\mathfrak{a}(x, \cdot) : x \in P(E)\}$$

hold, such that $P(E)$ and $P(F)$ are representatives of the duals of $P(F)$ and $P(E)$ w.r.t. $\mathfrak{a}$.
Note that if P is a symmetric projection w.r.t. $\mathfrak{a}$ or if $(E, \mathfrak{a})$ is a Hilbert space, the result is true. But if both conditions are not met, it is a priori unclear if the property of duality is preserved for the projected subspaces.

These considerations lead to the introduction of duality systems and duality scales, which where first considered in [44] and are introduced in Chapter 2.
As a first step, we establish a more precise meaning of duality between Banach spaces. In general, for a Banach space E we define its dual by

$$E' := \mathscr{L}(E, \mathbb{K}),$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and E' is equipped with the induced operator norm. However, this does not reveal many details about the structure of E'. Especially if we want to *work* with E', it is more practicable to have it represented by a well known space, e.g. a L^p or a Sobolev space. This often leads to some imprecise notations. One example is the commonly used notation

$$(L^p(\Omega))' = L^{p'}(\Omega), \;\; 1 < p < \infty, \; \frac{1}{p} + \frac{1}{p'} = 1 \tag{1.6}$$

for a domain $\Omega \subseteq \mathbb{R}^n$. Here, the representative of $(L^p(\Omega))'$ is also given as a L^p space, but w.r.t. the duality pairing

$$\langle u, v \rangle_{L^p, L^{p'}} = \int_\Omega uv \, dx \qquad (u \in L^p(\Omega), v \in L^{p'}(\Omega)).$$

There can be infinitely many further representatives of $(L^p(\Omega))'$ w.r.t. different duality pairings, which differ from each other in structure and properties. Consequently, the notation used in (1.6) is not precise and can lead to wrong conclusions about the relationship of Banach spaces to each other.
This motivates a more precise notation of duality by the use of duality systems of the already mentioned form $(E, F, \mathfrak{a})$. This notion will be developed further to duality scales $(E_q, F_q, \mathfrak{a}_q)_{q \in I_0}$. Here, we consider complex interpolation scales of Banach spaces

like $(E_q)_{q\in I_0}$ and $(F_q)_{q\in I_0}$, where I_0 is some appropriately chosen interval $I_0 \subseteq \mathbb{R}$. The scale parameter q can refer to the integrability of the underlying spaces (e.g. if we work in L^q spaces) or to the regularity (e.g. if we work in Sobolev spaces). In the latter case we usually denote the scale parameter by s. In order to work with these duality scales, the property of strong consistency will play an important role.

In Chapter 6 we deal with the central and already motivated question if the property of being a duality scale can be preserved if we restrict to complemented subspaces $E_{q,P}$ and $F_{q,P}$ with some projection P on E_q and F_q. This question was already answered in [44] for a duality scale consisting of a single scale $(E_q)_{q\in I_0}$ of Banach spaces. But it is - to the best of the author's knowledge - open if we consider two different scales of Banach spaces, which leads to new possibilities regarding the application of the results. In the main theorem of this Chapter it is shown that the property of being a duality scale can be preserved if, roughly spoken, $1 \in \rho(P'(1-P))$ is fulfilled on the whole scale, where P' is the dual of P w.r.t. $\mathfrak{a}$ in this case.

Next, we apply this theorem to a stationary Stokes-Neumann problem on C^3-domains with compact boundary and show that we may obtain unique solutions to this problem in $W^{1+\varepsilon,q}_{\nu,\sigma}(\Omega)$, where $q \in (1,\infty)$, $0 < \varepsilon < \min\{1/q, 1/q'\}$ and $1/q + 1/q' = 1$. Note that we only require solubility of the problem in the non-projected spaces $W^{1+\varepsilon,q}_{\nu}(\Omega)$ and use abstract theory in order to obtain solubility in the projected subspaces. A direct approach to achieve similar results without applying the theory of duality scales can be found e.g. in [25]. In our case, we may exploit the compactness of $P'(1-P)$, since it makes the spectrum of the operator invariant for the whole scale. Finally, we generalize the principles used in the application and present theorems for duality scales on complemented subspaces which rely on compactness.

2 Preliminaries

2.1 Notation

Let $\mathbb{N} = \{1, 2, 3, ...\}$ denote the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N})$ be the integers. Moreover, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be the field of the real or complex numbers and $m, n \in \mathbb{N}$. For a vector $x \in \mathbb{K}^n$ we denote by x_j, $j \in \{1, ..., n\}$ the j-th component and for a matrix $A \in \mathbb{K}^{m\times n}$ by a_{ij}, $(i, j) \in \{1, ..., m\} \times \{1, ..., n\}$ the entry in the i-th row and the j-th column. By x^T and A^T we describe the corresponding transposed vector or matrix. For $k = 1, ..., n$, let e_k denote the unit vector in the k-th direction.
For two matrices $A, B \in \mathbb{K}^{n\times n}$ we define

$$A : B := \sum_{i,j=1}^{n} a_{ij} b_{ij}. \tag{2.1}$$

If $D \in \mathbb{K}^{n\times n}$ is a diagonal matrix with diagonal values $d_1, ...d_n \in \mathbb{K}$, we set

$$\operatorname{diag}(d_1, ..., d_n) := \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}.$$

Moreover, given $x, y \in \mathbb{R}^n$ we write

$$x \cdot y := (x, y)_{\mathbb{R}^n} = \sum_{j=1}^{n} x_j y_j$$

for the standard scalar product and

$$|x| := \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2}$$

for the Eucledian norm defined on $\mathbb{R}^n$.
For a value $z \in \mathbb{C}$ we denote by $\operatorname{Re} z$ and $\operatorname{Im} z$ its real and imaginary part, so that $z = \operatorname{Re} z + i \operatorname{Im} z$, where i is the imaginary unit. We write $\arg(z) \in \mathbb{R}$ for the argument,

i.e. the angle of a complex value and

$$\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\} \,:\, |\arg(z)| < \varphi\} \subseteq \mathbb{C}$$

for a sector with opening angle $\varphi \in [0, \pi]$. Moreover, given $z = \operatorname{Re} z + i \operatorname{Im} z \in \mathbb{C}$, we define the complex conjugation as $\overline{z} := \operatorname{Re} z - i \operatorname{Im} z$. Then, the absolute value of z is given as

$$|z| := \sqrt{z \cdot \overline{z}} \qquad (z \in \mathbb{C}),$$

and the eucledian norm of a complex vector as

$$|z| := \left(\sum_{j=1}^{n} |z_j|^2 \right)^{1/2} \qquad (z = (z_1, ..., z_n)^T \in \mathbb{C}^n).$$

For $u, z \in \mathbb{C}^n$ we write

$$u \cdot z = (u, z)_{\mathbb{C}^n} = \sum_{j=1}^{n} u_j \overline{z_j}.$$

for the Hermitian inner product on $\mathbb{C}^n$. Additionally, we set

$$\mathbb{C}_+ := \{z \in \mathbb{C} \,:\, \operatorname{Re} z > 0\}, \;\; \mathbb{C}_- := \{z \in \mathbb{C} \,:\, \operatorname{Re} z < 0\}.$$

Let $\alpha = (a_1, ..., a_n) \in [0, \infty)^n$ and $x = (x_1, ..., x_n) \in \mathbb{K}^n$ be vectors, then

$$x^\alpha := \prod_{k=1}^{N} x_k^{\alpha_k}.$$

For a metric space (X, d), $r > 0$ and $x \in X$ let $\mathbb{B}_X(x, r)$ be the open ball of radius r centered at x and $\overline{\mathbb{B}}_X(x, r)$ the closed ball respectively. For $T \subseteq X$ let T° be the inner, ∂T the boundary, $\overline{T}$ the completion and T^c the complement of T in X. Moreover, we write $\operatorname{dist}_X(x, T)$ for the distance of a point $x \in X$ to a set T measured in X and $\operatorname{dist}_X(M, T)$ for the distance of two sets $M, T \subseteq X$ in X. If $\Omega \subseteq \mathbb{K}^n$ is a (sufficiently smooth) domain, we denote by $\partial\Omega$ its boundary and write $\nu(x) := \nu$ for the outer normal unit vector at $x \in \partial\Omega$.

For a Banach space X we denote its norm by

$$\|x\|_X \qquad (x \in X)$$

and for a Hilbert space H we use

$$(x, y)_H \qquad (x, y \in H)$$

as the scalar product. For Banach spaces X, Y we write $\mathscr{L}(X, Y)$ for the space of bounded, linear operators from X to Y and set $\mathscr{L}(X, X) = \mathscr{L}(X)$. For an (unbounded) operator $A : D(A) \subseteq X \to X$ with domain of definition $D(A)$ let $\sigma(A)$ be the spectrum of A and $\rho(A) = \mathbb{C} \setminus \sigma(A)$ be the resolvent set of A. Sometimes we write $\sigma(A, X)$ and $\rho(A, X)$ respectively if we want to emphasize in which space the operator A is defined. Furthermore, $N(A)$ and $R(A)$ denote kernel and range of A. Additionally, we set $\mathscr{L}_{is}(X, Y)$ for the set of all bounded linear isomorphisms between X and Y, where $X \doteq Y$ denotes that two spaces are isomorphic and such an isomorphism exists. We write $X' = \mathscr{L}(X, \mathbb{K})$ for the (abstract) dual space of X equipped with the induced norm

$$\|\ell\|_{X'} := \sup_{0 \neq x \in X} \frac{|\ell(x)|}{\|x\|_X} = \sup_{x \in E,\, \|x\|_X = 1} |\ell(x)|.$$

The (abstract) duality pairing between X and X' is then denoted by $\langle \cdot, \cdot \rangle_{X,X'}$ such that we can write

$$\ell(x) = \langle x, \ell \rangle_{X,X'} \qquad (x \in X, \ell \in X').$$

For a subspace $M \subseteq X$ we define the annihilator as

$$M^\perp := \{\ell \in X' : \ell(x) = 0 \quad (x \in M)\}.$$

The direct decomposition of X in two complemented subspaces X_1 and X_2 is denoted by $X = X_1 \oplus X_2$. I.e., $X_1 \cap X_2 = \{0\}$ and for every $x \in X$ there exist uniquely determined $x_1 \in X_1$ and $x_2 \in X_2$ such that $x = x_1 + x_2$. By $X \hookrightarrow Y$ we define the injective, continuous embedding of X into Y, while with $X \overset{d}{\hookrightarrow} Y$ we denote an embedding that is injective, continuous and dense. If an injective, continuous embedding is compact, we write $X \overset{c}{\hookrightarrow} Y$. If two Banach spaces X and Y are embedded into a common Hausdorff space $\mathcal{H}$, then we define the Banach spaces $X \cap Y$ with norm

$$\|z\|_{X \cap Y} = \|z\|_X + \|z\|_Y \qquad (z \in X \cap Y)$$

and $X+Y := \{z = x+y \,:\, x \in X,\, y \in Y\}$ with norm

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y \,:\, z = x+y \quad (x \in X,\, y \in Y)\}.$$

For a general functional $T : X \to Y$ and a subset $M \subseteq X$ we write $T|_M$ for the restriction of T onto M.
Let $n \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$ be a domain. For a mapping $f : \Omega \to X$ we denote by $\partial_j f$ for $j = 1, ..., n$ the partial derivative in the j-th direction and by $\partial^\alpha f$ the (distributional) derivative w.r.t. $\alpha \in \mathbb{N}_0^n$. For $f : \Omega \to \mathbb{K}$ we denote by $\nabla f = (\partial_1 f, ..., \partial_n f)^T$ the gradient and by $\Delta f = \nabla \cdot \nabla f$ the Laplacian of f, where $\nabla = (\partial_1, ..., \partial_n)$. For $f = (f_1, ..., f_m) : \Omega \to \mathbb{K}^m$ with $m \in \mathbb{N}$ we write $\nabla f = (\nabla f_1, ..., \nabla f_m)^T$ for the Jacobian matrix. The Laplacian then has the form $\Delta f = (\Delta f_1, ..., \Delta f_m)$. The divergence of $f : \Omega \to \mathbb{K}^n$ is given as $\operatorname{div} f = \nabla \cdot f$.
If $\Omega \subseteq \mathbb{K}$, we use f' for the first derivative. If a mapping u depends (among other variables) on a time variable $t \in \mathbb{R}$, sometimes we will make use of the notation $\dot{u}$ or u_t for the first derivative w.r.t t. If a mapping $\phi : M \to X$ is defined on the (sufficiently smooth) surface M, then $\nabla_M \phi := (\nabla u)|_M - \nu(\nu \cdot (\nabla u)|_M)$ denotes the surface gradient and $\Delta_M \phi := \nabla_M \cdot \nabla_M \phi$ denotes the Laplace-Beltrami operator.
If $\Omega \subseteq \mathbb{R}^n$ is a domain with (sufficiently smooth) boundary $\partial\Omega$ and $f : \Omega \to \mathbb{K}^m$, we denote by $\partial_\nu f$ the outer normal derivative w.r.t. $\partial\Omega$. The projection onto the normal and tangential part of a mapping $f : \Omega \to \mathbb{R}^n$ at the boundary $\partial\Omega$ is given as $\Pi_\nu f := (\nu\nu^T) f$ and $\Pi_\tau f := (I - \nu\nu^T) f$.
Let $\Omega \subseteq \mathbb{R}^n$ be a domain or $\Omega = \mathbb{R}^n$. For a Banach space X and $k \in \mathbb{N}$ we will denote by $C(\Omega, X)$ the space of continuous functions, by $C^k(\Omega, X)$ the space of k-times continuously differentiable functions and by $C^\infty(\Omega, X)$ the space of infinitely often differentiable functions. Moreover, let $C_{\text{Lip}}(\Omega, X)$ be the space of globally Lipschitz continuous functions, whereas $C_{\text{Lip, loc}}(\Omega, X)$ is the space of locally Lipschitz continuous functions. Note that these notations remain valid if $\Omega \subseteq Y$ is an open subset of a Banach space Y. Furthermore, let $C_c^\infty(\Omega, X)$ be the space of infinitely often differentiable functions with compact support. For $X = \mathbb{K}^n$ we will usually drop the second parameter. Then, $\mathcal{D}(\Omega)$ denotes the space of test functions, i.e. $C_c^\infty(\Omega)$ equipped with the standard topology, and $\mathcal{D}'(\Omega)$ the space of distributions.
For $1 \leqslant p \leqslant \infty$ let $L^p(\Omega, X)$ be the Bochner-Lebesgue space equipped with the norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega \|f(x)\|_X^p dx\right)^{1/p}, & 1 < p < \infty, \\ \operatorname{ess\,sup}_{x \in \Omega} \|f(x)\|_X, & p = \infty, \end{cases}$$

and $W^{k,p}(\Omega, X)$ be the Sobolev space of order $k \in \mathbb{N}_0$ equipped with the norm

$$\|f\|_{W^{k,p}(\Omega,X)} = \begin{cases} \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha f\|^p_{L^p(\Omega,X)} \right)^{1/p}, & 1 < p < \infty, \\ \max_{|\alpha| \leqslant k} \|\partial^\alpha f(x)\|_{L^\infty(\Omega,X)}, & p = \infty. \end{cases}$$

Sometimes we will drop the parameters X and Ω if no confusion is possible. For $s \in (0, \infty) \setminus \mathbb{N}$ we denote by

$$W^{s,p}(\Omega, X) := [L^p(\Omega, X), W^{k,p}(\Omega, X)]_{s/k},$$
$$W^s_p(\Omega, X) := (L^p(\Omega, X), W^{k,p}(\Omega, X))_{s/k,p}$$

the Sobolev and Sobolev-Slobodeckii spaces of order s, where $[\cdot, \cdot]$ and $(\cdot, \cdot)$ denote the complex and real interpolation functors and $s < k \in \mathbb{N}$. If $p = 2$ and $s \in [0, \infty)$, we set $H^s(\Omega, X) = W^{s,2}(\Omega, X)$ and remark that $H^s(\Omega, X)$ is a Hilbert space. We utilize some of these spaces on the boundary $\Gamma = \partial\Omega$ of a domain by implicitly taking trace or on a manifold Σ by the implicit use of local charts.

In estimates, we will make use of positive constants, which we usually denote by $C > 0$. If we want to make a distinction between several constants, we sometimes use the notation $C_1, C_2, ...$ or C_X, where X is a set of values on which the constant does depend on.

Throughout this thesis some general results regarding parabolic evolution equations, especially the theory of analytic semigroups, maximal L^p-regularity, the H^∞-calculus and $\mathcal{R}$-bounded multipliers, will implicitly play an important role. Since these results are generally well known and part of many other works, we refer the reader to [12, 38, 27] and the references therein for the definitions and results regarding this topic.

2.2 Sobolev Spaces in the Periodic Setting

In order to analyze the long-time behavior of so-called living fluids, we will assume periodic boundary conditions for the equations. In this setting, Fourier series and Sobolev spaces with periodic boundary conditions will play a central role. In the following section, we give a short introduction into this topic. For more detailed information we refer to [21, 41, 42] and the references therein.
Let $L > 0$, $\ell, n \in \mathbb{N}$ and $Q_n := [0, L]^n$. We define the spaces of smooth functions with periodic boundary conditions as

$$C_\pi^k(Q_n) := \left\{ f \in C^k(Q_n, \mathbb{R}^\ell) : \partial^\alpha f|_{x_j=0} = \partial^\alpha f|_{x_j=L} \ (\forall |\alpha| \leqslant k) \right\},$$
$$C_\pi^\infty(Q_n) := \bigcap_{k=0}^{\infty} C_\pi^k(Q_n).$$

The L^2 space with periodic boundary conditions is defined as

$$L_\pi^2(Q_n, \mathbb{R}^\ell) := \overline{C_\pi^\infty(Q_n, \mathbb{R}^\ell)}^{L^2(Q_n, \mathbb{R}^\ell)}. \tag{2.2}$$

In order to simplify the notation we set $L^2(Q_n) := L_\pi^2(Q_n) := L_\pi^2(Q_n, \mathbb{R}^\ell)$. Indeed, by [21, Proposition 3.2.1] it follows that the definitions of $L^2{}_\pi(Q_n, \mathbb{R}^\ell)$ and $L^2(Q_n, \mathbb{R}^\ell)$ as a standard Lebesgue space are equivalent. In the following we will just write $C_\pi^k(Q_n)$, $C_\pi^\infty(Q_n)$ and $L_\pi^2(Q_n)$ if no confusion is likely.
Working in $L^2(Q_n)$ enables the use of Fourier series in order to represent L^2-functions, which gives several advantages. Let $f \in L^2(Q_n)$. The Fourier coefficient $\widehat{f}(m)$ for $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ is defined as the integral

$$\widehat{f}(m) := \mathcal{F}f(m) := \frac{1}{L^n} \int_{Q_n} f(x) e^{-2\pi i m \cdot x / L} \, dx.$$

Using integration by parts one easily verifies the identity

$$\widehat{\partial^\alpha f}(m) = \left(\frac{2\pi i}{L} \right)^{|\alpha|} m^\alpha \widehat{f}(m) \tag{2.3}$$

if $f \in C_\pi^{|\alpha|}(Q_n)$, $m \in \mathbb{Z}^n$ and $\alpha \in \mathbb{N}_0^n$. Then the scalar product in $L^2(Q_n)$ is defined as

$$(f, g)_{L_\pi^2(Q_n)} := \frac{1}{L^n} \int_{Q_n} f(x) \overline{g(x)} \, dx \qquad (f, g \in L^2(Q_n))$$

and by $\|\cdot\|_{L_\pi^2}$ we denote the induced norm on $L^2(Q_n)$, which differs by a factor of $L^{-n/2}$ from the standard L^2 norm. Here we will keep the notation L_π^2 or $L_\pi^2(Q_n)$ as a subscript to emphasize that we are working in the periodic setting. Some important properties of the Fourier series and $L^2(Q_n)$ functions are listed below ([21, Proposition 3.2.7]).

2.1 Proposition. *Let $f, g \in L^2(Q_n)$ be arbitrary. The following properties hold in $L^2(Q_n)$.*

(i) *Plancherel theorem:*

$$\|f\|^2_{L^2_\pi(Q_n)} = \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)|^2.$$

(ii) *Parseval's identity:*

$$(f,g)_{L^2_\pi} = \frac{1}{L^n}\int_{Q_n} f(x)\overline{g(x)}\,dx = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)}.$$

(iii) *The function f can be represented as the $L^2(Q_n)$-limit of trigonometric polynomials, i.e. as the Fourier series*

$$f = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot/L}.$$

The periodic Sobolev spaces for $k \in \mathbb{N}$ are defined as follows.

$$\begin{aligned} H^k_\pi(Q_n) &:= \Big\{ u = \sum_{m\in\mathbb{Z}^n} \widehat{u}(m)e^{2\pi i m\cdot/L} \,:\, \widehat{u}(m) = \overline{\widehat{u}(-m)}, \|u\|_{\tilde{H}^k_\pi(Q_n)} < \infty \Big\} \\ &= \Big\{ u \in H^k(Q_n) : \partial^\alpha u|_{x_j=0} = \partial^\alpha u|_{x_j=L},\ (|\alpha| < k,\ j = 1,...,n) \Big\} \\ &= \overline{C^\infty_\pi(Q_n)}^{H^k(Q_n)}, \end{aligned}$$

where the norm above is defined as

$$\|u\|^2_{\tilde{H}^k_\pi} := \sum_{m\in\mathbb{Z}^n} \left| \left(1 + \left(\frac{2\pi}{L}\right)^k |m|^k\right) \widehat{u}(m) \right|^2,$$

cf. [41, Chapter 5.10]. We will also make use of the homogeneous periodic Sobolev space

$$\begin{aligned} \widehat{H}^1_\pi(Q_n) &= \overline{C^\infty_\pi(Q_n)}^{\|\nabla\cdot\|_{L^2(Q_n)}} \\ &= \Big\{ u \in L^1_{loc}(Q_n) \,:\, \nabla u \in L^2(Q_n),\, u|_{x_j=0} = u|_{x_j=L}\ (j = 1,...,n) \Big\} \end{aligned}$$

equipped with the semi-norm $\|\nabla\cdot\|_{L^2_\pi(Q_n)}$ (note that the trace in the second characterization is defined in a local sense). If $u \in H^k_\pi(Q_n)$ and $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leqslant k$, then the derivative $\partial^\alpha u$ can be written as the $L^2(Q_n)$-limit

$$\partial^\alpha u = \sum_{k\in\mathbb{Z}^n} \widehat{\partial^\alpha u}(k) e^{2\pi i k\cdot/L} = \sum_{k\in\mathbb{Z}^n} \left(\frac{2\pi i}{L}\right)^{|\alpha|} k^\alpha \hat{u}(k) e^{2\pi i k\cdot/L},$$

where we used the fact that the identity in (2.3) also holds for $u \in H^k_\pi(Q_n)$. It follows that the $\|\cdot\|_{\tilde{H}^k_\pi(Q_n)}$ and the $\|\cdot\|_{H^k_\pi(Q_n)}$ norms are equivalent, where

$$\|u\|^2_{H^k_\pi(Q_n)} := \sum_{|\alpha|\leqslant k} \sum_{m\in\mathbb{Z}^n} \left|\left(\frac{2\pi}{L}\right)^{|\alpha|} m^\alpha \, \hat{u}(m)\right|^2,$$

by the Plancherel theorem. Periodic Sobolev spaces of fractional powers are defined in the canonical way. For $s \geqslant 0$ set

$$H^s_\pi(Q_n) = \left\{ u = \sum_{m\in\mathbb{Z}^n} \hat{u}(m) e^{2\pi i m\cdot/L} : \hat{u}(m) = \overline{\hat{u}(-m)}, \ \|u\|_{H^s_\pi} < \infty \right\},$$

where

$$\|u\|^2_{H^s_\pi} := \sum_{m\in\mathbb{Z}^n} \left(1 + \left(\frac{2\pi}{L}\right)^2 |m|^2\right)^{s/2} |\hat{u}(m)|^2,$$

and it is straightforward to see that for $s \in \mathbb{N}$ these two definitions for Sobolev spaces coincide.

Finally, let $m : \mathbb{Z}^n \to \mathbb{C}^{n\times n}$ be a function. We define $T_m : D(T_m) \subseteq L^2(Q_n) \to L^2(Q_n)$ as the $L^2(Q_n)$-limit

$$T_m f := \sum_{k\in\mathbb{Z}^n} m(k) \hat{f}(k) e^{2\pi i k\cdot/L}$$

for a function $f \in D(T_m)$, where

$$D(T_m) := \left\{ f \in L^2(Q_n) : \|T_m f\|^2_{L^2(Q_n)} = \sum_{k\in\mathbb{Z}^n} |m(k)\hat{f}(k)|^2 < \infty \right\}.$$

Note that T_m is well-defined and a bounded operator by the Plancherel theorem if m is a bounded function. Then m is called a Fourier multiplier on $L^2(Q_n)$.
As an important example we introduce the Helmholtz-Weyl projection on $L^2(Q_n)$. Here

we use the Fourier multiplier

$$\sigma_P : \mathbb{Z}^n \longrightarrow \mathbb{C}^{n\times n}, \;\; m \mapsto \begin{cases} I - \dfrac{mm^T}{|m|^2}, & m \neq 0, \\ I, & m = 0, \end{cases}$$

and set

$$P : L^2(Q_n) \longrightarrow L^2(Q_n), \; u \mapsto Pu := \sum_{m\in\mathbb{Z}^n} \sigma_p(m)\widehat{u}(m)e^{2\pi i m\cdot/L}. \tag{2.4}$$

We obtain the Helmholtz decomposition as $L^2(Q_n) = L^2_\sigma(Q_n) \oplus G_2(Q_n)$ with

$$\begin{aligned} L^2_\sigma(Q_n) &:= \Big\{u \in L^2(Q_n) \,:\, \widehat{u}(m) = \overline{\widehat{u}(-m)},\, m\cdot\widehat{u}(m) = 0 \;\forall m \in \mathbb{Z}^n\Big\} = P(L^2(Q_n)), \\ G_2(Q_n) &:= \Big\{u = \nabla g \in L^2(Q_n) \,:\, g \in L^1_{loc}(Q_n)\Big\} = (I-P)(L^2(Q_n)). \end{aligned}$$

We note that P admits higher regularity for $k \in \mathbb{N}$, i.e. P is a projection on $H^k_\pi(Q_n)$ with $P(H^k_\pi(Q_n)) = H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$. With [58] we immediately obtain the following lemma.

2.2 Lemma. *Let $\theta \in [0,1]$ and $k \in \mathbb{N}$. Then we have*

$$[L^2_\sigma(Q_n), H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)]_\theta = H^{k\theta}_\pi(Q_n) \cap L^2_\sigma(Q_n).$$

Sobolev-Lieb-Thirring inequality

We recall the Sobolev-Lieb-Thirring inequality in its form for periodic Sobolev spaces from [19].

2.3 Proposition (Sobolev-Lieb-Thirring inequality)**.** *Let $m,n \in \mathbb{N}$. For every p satisfying*

$$\max\left\{1, \frac{n}{2m}\right\} < p \leqslant 1 + \frac{n}{2m}$$

there exists constants $C_1, C_2 > 0$ such that for every finite family $\{\varphi_j\}_{j=1}^N \subseteq H^m_\pi(Q_n)$ ($N \in \mathbb{N}$) which is orthonormal in $L^2(Q_n)$ we have

$$\left(\int_{Q_n} \rho(x)^{p/(p-1)}\,dx\right)^{2m(p-1)/n} \leqslant C_1 \sum_{j=1}^N \int_{Q_n} \sum_{|\alpha|=m} |\partial^\alpha \varphi_j(x)|^2\,dx + C_2 \int_{Q_n} \rho(x)\,dx,$$

where

$$\rho(x) = \sum_{j=1}^{N} |\varphi_j(x)|^2.$$

2.4 Remark. Note that the constants C_1 and C_2 only depend on k, m, n and p, where $k \in \mathbb{N}$ stems from $H^m_\pi(Q_n) = H^m_\pi(Q_n, \mathbb{R}^k)$.

2.3 Stability Theory for Quasilinear Parabolic Problems

The principle of linearized stability is an important tool to investigate the behavior of ordinary differential equations near equilibrium points. For quasilinear parabolic problems a similar theory is available, which will be of significant importance for the analysis of equilibria of the living fluids problem (cf. chapter 3) and the heterogeneous catalysis model (cf. chapter 5). Here, we outline the most important points of this theory, which is taken from and described in greater detail in [39, 40, 38, 37].
Let E_0, E_1 be Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$ and $1 < p < \infty$. Let $I_p := (E_0, E_1)_{1-1/p,p}$ and $V \subseteq I_p$ open. We consider the quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)),\ t > 0,\ \ u(0) = u_0, \tag{2.5}$$

where $(A, F) \in C^1(V, \mathscr{L}(E_1, E_0) \times E_0)$ and $u_0 \in V$. We denote by $\mathcal{E} \subseteq V \cap E_1$ the set of equilibria of (2.5) fulfilling

$$u \in \mathcal{E} \ \Leftrightarrow\ u \in V \cap E_1,\ A(u)u = F(u).$$

Especially we have $\dot{u}_* = 0$ for $u_* \in \mathcal{E}$. Next, we define a manifold of equilibria for a quasilinear problem.

2.5 Definition. Let $u_* \in \mathcal{E}$. We say that u_* lies on a m-dimensional manifold of equilibria, if there exists an open $U \subseteq \mathbb{R}^m$ (where $m \in \mathbb{N}_0$) with $0 \in U$ and a mapping $\Psi \in C^1(U, E_1)$ such that

(i) $\Psi(U) \subseteq \mathcal{E}$ and $\Psi(0) = u_*$;

(ii) the rank of $\Psi'(0)$ equals m;

(iii) $A(\Psi(\eta))\Psi(\eta) = F(\Psi(\eta)),\ \eta \in U$.

In the following, we will assume that about a fixed $u_* \in \mathcal{E}$ there exist no other equilibria than those that lie on the manifold itself, i.e. there exists a $\rho = \rho(u_*) > 0$ such that

$\mathcal{E} \cap \mathbb{B}_{E_1}(u_*, \rho) = \Psi(u)$. Moreover, we will impose conditions on the tangent space $T_{u_*}\mathcal{E} = \Psi(0)\mathbb{R}^m$.
In order to analyze the behavior of (2.5) near an equilibrium we will use a first order linearization at the point u_*. To this end we assume that $A(u_*)$ possesses maximal L^p-regularity on E_0. For $u_0 \in V$ and $u_* \in \mathcal{E}$ we set $v := u - u_*$ as the deviation of the solution from the equilibrium and linearize (2.5) as follows.

$$\dot{v}(t) + A_0 v(t) = G(v(t)),\ t > 0,\ \ v(0) = v_0.$$

Here, $v_0 := u_0 - u_*$,

$$A_0 v := A(u_*)v + (A'(u_*)v)u_* - F'(u_*) \qquad (v \in E_1), \tag{2.6}$$

and $G(v) := G_1(v) + G_2(v, v)$, where

$$\begin{aligned} G_1(v) &:= (F(u_* + v) - F(u_*) - F'(u_*)v) - (A(u_* + v) - A(u_*) - A'(u_*)v)u_*, \\ G_2(v, w) &:= -(A(u_* + v) - A(u_*))w \end{aligned}$$

for $w \in E_1$, $v \in V_* := V - u_*$. It follows that $G_1 \in C^1(V_*, E_0)$ and $G_2 \in C^1(V_* \times E_1, E_0)$ with

$$G_1(0) = G_2(0, 0) = 0,\ \ G_1'(0) = G_2'(0, 0) = 0.$$

Setting $\psi(\eta) := \Psi(\eta) - u_*$ we obtain

$$A_0\psi(\eta) = G(\psi(\eta))$$

for all $\eta \in U$, which yields $A_0\psi'(0) = 0$ and $T_{u_*}\mathcal{E} \subseteq N(A_0)$.
With these prerequisites we are able to formulate the theorem for normally stable equilibria, cf. [38, Theorem 5.3.1] and [39].

2.6 Theorem. *Let $1 < p < \infty$, $(A, F) \in C^1(V, \mathscr{L}(E_1, E_0) \times E_0)$ and $u_* \in V \cap E_1$ be an equilibrium of (2.5). Moreover, assume that $A(u_*)$ possesses maximal L^p-regularity. Let A_0 be given as in (2.6). Then u_* is called normally stable, if the following conditions are fulfilled.*

(i) *near u_* the set of equilibria $\mathcal{E}$ is a C^1-manifold in E_1 of dimension $m \in \mathbb{N}$,*

(ii) *the tangent space at u_*, $T_{u_*}\mathcal{E}$, is isomorphic to $N(A_0)$,*

(iii) *the eigenvalue $\lambda = 0$ of A_0 is semi-simple, i.e. $E_0 = R(A_0) \oplus N(A_0)$,*

(iv) $\sigma(A_0) \setminus \{0\} \subseteq \mathbb{C}_+$.

If u_* is normally stable, then there exists $\rho > 0$ such that the unique solution u of (2.5) for $u_0 \in \mathbb{B}_{I_p}(u_*, \rho)$ exists on $\mathbb{R}_+$ and converges to a $u_\infty \in \mathcal{E}$ in I_p for $t \to \infty$ at an exponential rate.

The proof of this theorem exploits the spectral properties of A_0 to obtain a decomposition of A_0 and X_0 into a stable and a center part. The equation is then analyzed on these parts in order to obtain the exponential convergence to an equilibrium $u_\infty \in \mathcal{E}$. We make the following observations.

2.7 Remark. (i) It does not necessarily hold that $u_* = u_\infty$. Normal stability only ensures that every solution u starting near the manifold of equilibria $\mathcal{E}$ converges back onto this manifold.

(ii) In case that $m = 0$, the manifold of equilibria just consists of one point u_*, where u_* is an isolated equilibrium. In this case Theorem 2.6 reduces to the principle of linearized stability and the solution u converges back to $u_\infty = u_*$ at an exponential rate. We can even relax the conditions imposed on A and F to $A : I_p \to \mathscr{L}(E_1, E_0)$ and $F : I_p \to E_0$ being locally Lipschitz continuous. Additionally, A and F then need to meet the following condition: There exist constants $\rho, \varepsilon, L > 0$ such that

$$\begin{aligned} \|A(u_* + v)u_* - A(u_*)u_* - (A'(u_*)v)u_*\|_{E_0} &\leqslant \varepsilon \|v\|_{I_p}, \\ \|F(u_* + v) - F(u_*) - F'(u_*)v\|_{E_0} &\leqslant \varepsilon \|v\|_{I_p}, \\ \|A(u_* + v) - A(u_*)\|_{\mathscr{L}(E_1,E_0)} &\leqslant L\|v\|_{I_p}, \end{aligned}$$

for $\|v\|_{I_p} \leqslant r$. For details we refer to [37].

Another case occurs if we have unstable parts in $\sigma(A_0)$ and there is a *spectral gap* in $\mathbb{C}$ (cf. [38, Theorem 5.4.1]).

2.8 Theorem. *Let $1 < p < \infty$, $(A, F) \in C^1(V, \mathscr{L}(E_1, E_0) \times E_0)$ and $u_* \in V \cap \mathcal{E}$ be an equilibrium of (2.5). Moreover, assume that $A(u_*)$ possesses maximal L^p-regularity. If there exists $\kappa \geqslant 0$ such that*

$$\sigma(-A_0) \cap [\kappa + i\mathbb{R}] = \emptyset, \quad \sigma(-A_0) \cap \{z \in \mathbb{C} \,:\, \mathrm{Re} z > \kappa\} \neq \emptyset,$$

then $u_* \in \mathcal{E}$ is unstable in I_p.

A special form of instability occurs in the case of normally hyperbolic equilibria (cf. [38, Theorem 5.5.1] and [39, Theorem 6.1]), where we have stable and unstable foliations near u_*.

2.9 Theorem. *Let $1 < p < \infty$, $(A, F) \in C^1(V, \mathscr{L}(E_1, E_0) \times E_0)$ and $u_* \in V \cap E_1$ be an equilibrium of (2.5). Moreover, assume that $A(u_*)$ possesses maximal L^p-regularity. Let A_0 be given as in (2.6). Then u_* is called normally hyperbolic, if the following conditions are fulfilled.*

(i) *near u_* the set of equilibria $\mathcal{E}$ is a C^1-manifold in E_1 of dimension $m \in \mathbb{N}$,*

(ii) *the tangent space at u_*, $T_{u_*}\mathcal{E}$, is isomorphic to $N(A_0)$,*

(iii) *the eigenvalue $\lambda = 0$ of A_0 is semi-simple, i.e. $E_0 = R(A_0) \oplus N(A_0)$,*

(iv) *$\sigma(A_0) \cap i\mathbb{R} = \{0\}$ and $\sigma_u := \sigma(A_0) \cap \mathbb{C}_- \neq \emptyset$.*

If u_* is normally hyperbolic, then u_* is unstable in I_p. For each sufficiently small $\rho > 0$ there exists $0 < \delta \leqslant \rho$, such that the unique solution u of (2.5) for $u_0 \in \mathbb{B}_{I_p}(u_*, \delta)$ either satisfies

- $\operatorname{dist}_{I_p}(u(t_0), \mathcal{E}) > \rho$ *for some finite $t_0 > 0$ or*
- *$u(t)$ exists on $\mathbb{R}_+$ and converges to an $u_\infty \in \mathcal{E}$ in I_p for $t \to \infty$ at an exponential rate.*

2.4 Infinite-dimensional Dynamical Systems and Global Attractors

Besides the analysis of the behavior of solutions near equilibria of partial differential equations, another important topic is the long-time behavior of solutions. This leads to the theory of dynamical systems and their attractors. Especially the existence and properties of a so-called global attractor, which can be roughly described as a compact set which attracts all solutions after some time, is of special interest. Since we will use this theory in order to analyze the long-time behavior of the living fluids model, we give a brief introduction. We use the terminology and results from [41], but refer also to [55, 47].

In order to describe time-dependent processes, we want analyze (time-dependent) partial differential equations and therefore work with evolution equations of type

$$\frac{d}{dt}u(t) = F(u(t)), \quad u(0) = u_0, \tag{2.7}$$

where $u(t), u_0 \in X$ for some Banach space X and $F : X \to X$. In case of parabolic partial differential equations a solution u for (2.7) only exists for $t \geqslant 0$. In order to describe such types of solutions, we introduce the notion of semigroups.

2.10 Definition. A familiy of mappings $(S(t))_{t\geqslant 0} \subseteq C(X)$ is called C_0-semigroup, if the following conditions are fulfilled.

(i) $S(0) = I$, where I is the identity operator on X,

(ii) $S(s+t) = S(s)S(t) = S(t)S(s)$ for $s, t \geqslant 0$,

(iii) and $S(t)x$ is continuous in $t \geqslant 0$ for $x \in X$.

If $S(t)$ additionally exists for $t < 0$ fulfilling these properties, we call $(S(t))_{t\in\mathbb{R}} \subseteq C(X)$ a (C_0)-group. Note that $S(t)$ is nonlinear in general. We will just write S for a semigroup if no confusion is likely.

One can use semigroups to write the solution of $u(t; u_0)$ of (2.7) with initial value $u_0 \in X$ in the form $u(t; u_0) = S(t)u_0$. This gives rise to the notation of a semidynamical system, which is given by $(X, (S(t))_{t\geqslant 0})$, where S is sometimes denoted as a semiflow. If the solution exists also for $t < 0$, we call $(X, (S(t))_{t\in\mathbb{R}})$ a dynamical system.

2.11 Remark. Note that especially in the theory of analytic semigroups, the semigroup S is generally assumed to be generated by a linear operator A. In the context of dynamical systems however, the semigroup S mostly corresponds to some nonlinear function F since it can be seen as a (nonlinear) solution operator to an evolution equation.

As outlined at the beginning of the section, we usually want to find a set $M \subseteq X$ which attracts the solutions of the equation. To formalize this notion, we introduce dissipative semigroups as follows.

2.12 Definition. A set $B \subseteq X$ is called absorbing set, if for any bounded set $M \subseteq X$ there exists a finite $t_0(M) \geqslant 0$ such that

$$S(t)M \subseteq B \text{ for all } t \geqslant t_0(M),$$

where $S(t)M$ describes the image of M under $S(t)$. A (semi-)group S is called dissipative if it possesses a compact absorbing set.

2.13 Definition. A set $M \subseteq X$ is called positively invariant under the semigroup S, if

$$S(t)M \subseteq M \qquad (t \geqslant 0).$$

It is called invariant under S, if

$$S(t)M = M \qquad (t \geqslant 0).$$

One may note that in general we want absorbing sets to be compact since we work in infinite dimensional Banach spaces. The intention of dissipativity is that it helps to find a global attractor $\mathcal{A}$, which is defined as follows.

2.14 Definition. The global attractor $\mathcal{A}$ of a semigroup S is given as a set $\mathcal{A} \subseteq X$ which is the maximal compact invariant set under S and the minimal set that attracts all bounded sets $M \subseteq X$, i.e.

$$\operatorname{dist}_X(S(t)M, \mathcal{A}) \xrightarrow{t\to\infty} 0.$$

From the definition it is not clear how to obtain such a set $\mathcal{A}$. We introduce the notion of the ω-limit set, which is given as

$$\omega(M) := \{x \in X \,:\, \exists (t_n)_{n\in\mathbb{N}} \subseteq (0,\infty), t_n \xrightarrow{n\to\infty} \infty,\ (x_n)_{n\in\mathbb{N}} \subseteq M \text{ with } S(t_n)x_n \xrightarrow{n\to\infty} x\}$$

and can also be characterized as

$$\omega(M) = \bigcap_{t\geqslant 0} \overline{\bigcup_{s\geqslant t} S(s)M}.$$

If M is an absorbing set, then the ω-limit set simplifies to

$$\omega(M) = \bigcap_{t\geqslant 0} S(t)M. \tag{2.8}$$

Using the notion of an ω-limit set we are able to describe the existence of a global attractor for a dissipative semigroup as follows.

2.15 Theorem. *Let the semigroup S be dissipative and $B \subseteq X$ be a compact absorbing set. Then there exists a global attractor $\mathcal{A}$ which is given as $\mathcal{A} = \omega(B)$. In addition, $\mathcal{A}$ is connected if X is connected.*

This theorem reduces the task of finding a global attractor to showing that the semigroup possesses a compact absorbing set. In particular, the dynamics of S on the global attractor $\mathcal{A}$ is of interest. If the semigroup yields injectivity on $\mathcal{A}$, then the dynamics on $\mathcal{A}$ is even defined for $t \in \mathbb{R}$ and we obtain a dynamical system on $\mathcal{A}$.

2.16 Theorem. *Let S be a semigroup on X which possesses a global attractor $\mathcal{A}$. If S is injective on $\mathcal{A}$, i.e.*

$$S(t)u_0 = S(t)u_1 \text{ for some } t > 0 \Rightarrow u_0 = u_1 \qquad (u_0, u_1 \in \mathcal{A}), \tag{2.9}$$

then every trajectory of S in $\mathcal{A}$ is defined for $t \in \mathbb{R}$ and we have $S(t)\mathcal{A} = \mathcal{A}$ for $t \in \mathbb{R}$. Moreover, $(\mathcal{A}, (S(t))_{t\in\mathbb{R}})$ is a dynamical system.

In order to prove injectivity of S on $\mathcal{A}$ in a Hilbert space setting, the following Lemma can be helpful.

2.17 Lemma. *Let H and V be Hilbert spaces and V' be the dual of V such that $V \overset{c}{\hookrightarrow} H \doteq H' \hookrightarrow V'$ by the Riesz representation theorem. Suppose that*

$$w \in L^\infty((0,T),V) \cap L^2((0,T),D(A))$$

satisfies

$$\dot{w} + Aw = h(t,w)$$

as an equality in $L^2((0,T),H)$, where $A \in \mathscr{L}(V,V')$ and

$$\|h(t,w(t))\|_H \leqslant k(t)\|w(t)\|_V$$

with $k \in L^2((0,T))$. If $w(t_0) = 0$ for some $t_0 > 0$, then $w(t) = 0$ for all $t \in (0,t_0)$.

Given a global attractor $\mathcal{A}$, studying the long-term dynamics of an evolution equation can now be reduced to studying the dynamics on $\mathcal{A}$, which - depending on the structure of $\mathcal{A}$ - can be a somewhat easier task. Indeed, we will see that in some cases the global attractor admits a finite dimensional structure in opposite to the infinite dimensional phase space X. Consequently, it seems to be useful to analyze the structure of a global attractor in greater detail. One first result is given for injective semigroups on $\mathcal{A}$.

2.18 Definition. Let S be a semigroup on X with global attractor $\mathcal{A}$. A complete orbit is a solution of the (partial) differential equation (2.7) that is defined for all $t \in \mathbb{R}$.

2.19 Theorem. *Let S be a semigroup on X with global attractor $\mathcal{A}$. Then all complete bounded orbits lie in $\mathcal{A}$. If S is injective on $\mathcal{A}$ in the sense of (2.9), then $\mathcal{A}$ is the union of all complete bounded orbits.*

Another fact about the global attractor concerns the so-called unstable manifold of compact invariant sets. The stable and unstable manifolds are defined as follows.

2.20 Definition. Let S be a semigroup on X and $x \in X$ be fixed. The stable manifold of x is given as

$$W^s(x) := \{u_0 \in X \,:\, S(t)u_0 \xrightarrow{t\to\infty} x\}$$

and the unstable manifold of x is given as

$$W^u(x) := \{u_0 \in X \,:\, S(t)u_0 \text{ is defined for } t \in \mathbb{R},\ S(-t)u_0 \xrightarrow{t\to\infty} x\}.$$

If $M \subseteq X$ is an invariant set, then we have

$$W^u(M) := \{u_0 \in X \,:\, S(t)u_0 \text{ is defined for } t \in \mathbb{R},\ \operatorname{dist}_X(S(-t)u_0, M) \xrightarrow{t\to\infty} 0\}.$$

We have the following result.

2.21 Theorem. *Let S be a semigroup on X with global attractor $\mathcal{A}$. If M is a compact invariant set, then $W^u(M) \subseteq \mathcal{A}$.*

Now that we have some results about the global attractor at hand and it is somehow clear that the long-term dynamics of (2.7) is connected to the global attractor, one may ask if the relationship between the latter two can be expressed detailed. In fact, we have the following result.

2.22 Theorem. *Let S be a semigroup on X with global attractor $\mathcal{A}$. Given a solution u of (2.7), there exists a sequence of errors $(\varepsilon_k)_{k\in\mathbb{N}} \subseteq (0,\infty)$ with $\varepsilon_k \to 0$ for $k \to \infty$ and an increasing sequence of times $(t_k)_{k\in\mathbb{N}} \subseteq (0,\infty)$ with*

$$t_{k+1} - t_k \longrightarrow \infty \ \text{ for } \ k \longrightarrow \infty$$

and a sequence of points on the attractor $(v_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ such that

$$\|u(t) - S(t-t_k)v_k\|_X \leqslant \varepsilon_k \ \text{ for all } \ t_k \leqslant t \leqslant t_{k+1}.$$

Moreover, $\|v_{k+1} - S(t_{k+1}-t_k)v_k\|_X$ decreases to zero.

The result shows that each solution of the evolution equation is tracked by the dynamics on the attractor after some time. Roughly speaking, instead of analyzing the long-term behavior of (2.7) we can analyze the dynamics of the global attractor.
In most cases, the global attractor is a smaller subset of X such that we expect a better understanding of the dynamics. However, the question remains in which sense $\mathcal{A}$ is smaller than X given the fact that X is usually infinite dimensional. A finite dimensional attractor would reduce the dynamics to a simpler structure which is easier to analyze. Although $\mathcal{A}$ is not a manifold of finite dimension in general, we are interested in dimensional bounds on $\mathcal{A}$ by using different notions of dimensions. To this end, we introduce the fractal and the Hausdorff dimension, which can be used to bound the dimension of $\mathcal{A}$.

2.23 Definition (Fractal dimension)**.** Let $M \subseteq X$ be relatively compact and $N(M,\varepsilon)$ be the minimum number of balls of radius $\varepsilon > 0$ that are needed to cover M. Then the fractal dimension of M is defined as

$$d_f(M) := \limsup_{\varepsilon \to 0} \frac{\log(N(M,\varepsilon))}{\log(1/\varepsilon)}.$$

Here, the balls that are used to cover a set M have to be of equal radius $\varepsilon > 0$. For the Hausdorff dimension, we require the radius of the balls to be bounded by $\varepsilon > 0$, but allow different radii. This results in the definition of the Hausdorff measure.

2.24 Definition (Hausdorff measure)**.** Let $M \subseteq X$ and $\varepsilon, d > 0$. We set

$$\mu(M,d,\varepsilon) := \inf \left\{ \sum_k r_k^d \,:\, 0 < r_k \leqslant \varepsilon,\ M \subseteq \bigcup_k \mathbb{B}_X(x_k, r_k) \right\}$$

and define the d-dimensional Hausdorff-measure of M as

$$\mathcal{H}^d(M) := \lim_{\varepsilon \to 0} \mu(M,d,\varepsilon).$$

One can see that for a fixed compact set $M \subseteq X$ there may be one $\delta > 0$ such that $\mathcal{H}^d(M) = 0$ for $d < \delta$ and $\mathcal{H}^d(M) = \infty$ for $d > \delta$. This gives rise to the definition of the Hausdorff dimension.

2.25 Definition (Hausdorff dimension)**.** Let $M \subseteq X$ be a compact set. Then the Hausdorff dimension is defined as

$$d_H(M) := \inf_{d > 0} \{ d \,:\, \mathcal{H}^d(M) = 0 \}.$$

A rough comparison between these two definitions of dimensions is given in the following remark. For more properties we refer to [41, Chapter 13].

2.26 Remark. Let $M \subseteq X$ be compact. Then $d_H(M) \leqslant d_f(M)$.

Next we want to find conditions under which the global attractor $\mathcal{A}$ is expected to be finite dimensional in the sense introduced above. We restrict to the case that $X = H$ is a Hilbert space since the application in this thesis restricts to this case, too. The first assumption we have to make is the uniform differentiability of the semigroup S on the attractor.

2.27 Definition. Let S be a semigroup on H with global attractor $\mathcal{A}$. Then S is uniformly differentiable on $\mathcal{A}$, if for every $x \in \mathcal{A}$ there exists a bounded linear operator $\Lambda(t,x) \in \mathscr{L}(H)$ such that for all $t \geqslant 0$ we have

$$\sup_{x,y \in \mathcal{A};\, 0 < \|x-y\|_H \leqslant \varepsilon} \frac{\|S(t)y - S(t)x - \Lambda(t,x)(y-x)\|_H}{\|y-x\|_H} \xrightarrow{\varepsilon \to 0} 0$$

and additionally

$$\sup_{x\in\mathcal{A}}\|\Lambda(t,x)\|_{\mathscr{L}(H)} < \infty$$

is fulfilled for each $t \geqslant 0$.

In addition we need to impose a rather technical condition which arises from the proof of Theorem 2.28 (cf. [41, Chapter 13]). We consider the linearized equation (2.7) in the form

$$\frac{d}{dt}v = F'(S(t)u_0)v(t), \quad v(0) = v_0, \tag{2.10}$$

where we assume F to be differentiable, $v_0 \in H$ and S to be the semigroup corresponding to the original equation (2.7) with initial value $u_0 \in H$. Moreover, we set $L(t,u_0) := F'(S(t)u_0)$ and assume that $\Lambda(t,u)v_0$ is the solution of (2.10).
Now, we fix $m \in \mathbb{N}$ and $\{\xi_j^0 \,:\, j = 1, ...m\} \subseteq H$ where the ξ_j^0 are linearly independent. Then $\xi_j(t) = \Lambda(t,u_0)\xi_j^0$ is the solution of (2.10) with initial value ξ_j^0. We define $P^m_{\xi_1^0,\dots,\xi_m^0}(t)$ as the projection onto the subspace spanned by the vectors $\xi_j(t)$, $j = 1, ..., m$ for $t \geqslant 0$ and set

$$\mathcal{TR}_m(\mathcal{A}) := \sup_{u_0\in\mathcal{A}} \sup_{\substack{\xi_j^0\in H\\ \|\xi_j^0\|_H\leqslant 1\\ j=1,\dots,m}} \Big\langle \operatorname{Tr}\Big(L(t,u_0)P^m_{\xi_1^0,\dots,\xi_m^0}(t)\Big)\Big\rangle, \tag{2.11}$$

where Tr is the trace of the operator and $\langle f(t)\rangle$ is the time-average given by

$$\langle f(t)\rangle = \limsup_{t\to\infty}\frac{1}{t}\int_0^t f(s)ds.$$

With these notions clarified, we can state the theorem for finite dimensional global attractors.

2.28 Theorem. *Suppose that the semigroup S on H with global attractor $\mathcal{A}$ is uniformly differentiable on $\mathcal{A}$ and that there exists a $t_0 > 0$ such that $\Lambda(t,u_0)$ is compact for all $t \geqslant t_0$. Then $d_f(\mathcal{A}) \leqslant m$, if $\mathcal{TR}_m(\mathcal{A}) < 0$.*

With this final result we close the short introduction into infinite dimensional systems and their global attractors and refer the reader to the literature stated at the beginning of this section for further details, especially regarding the proofs of the theorems.

Gronwall inequalities

An important tool to obtain estimates in the setting of dynamical systems is the lemma of Gronwall. We will present the standard and a generalized Gronwall lemma, which are taken from [55, III. Section 1.1.3]. The standard Gronwall lemma reads as.

2.29 Lemma (Standard Gronwall Lemma). *Let g, h, y be three locally integrable functions on (t_0, ∞) for some $t_0 \geqslant 0$ that satisfy the inequality*

$$\dot{y}(t) \leqslant g(t)y(t) + h(t) \qquad (t \geqslant t_0),$$

where $\dot{y} = \frac{dy}{dt}$ is locally integrable. Then we have

$$y(t) \leqslant y(t_0) \exp\Big(\int_{t_0}^t g(\tau)\, d\tau\Big) + \int_{t_0}^t h(s) \exp\Big(\int_s^t g(\tau)\, d\tau\Big)\, ds \qquad (t \geqslant t_0).$$

While this version of the Gronwall lemma is especially useful for bounded values of t, it is of interest to have a version where the bound does not grow exponentially in t. This leads to the generalized Gronwall lemma, which gives a uniform bound for $t \geqslant t_0$.

2.30 Lemma (Generalized Gronwall Lemma). *Let g, h, y be three positive, locally integrable functions on (t_0, ∞) for some $t_0 \geqslant 0$, such that $\dot{y}$ is locally integrable on (t_0, ∞) and*

$$\dot{y}(t) \leqslant g(t)y(t) + h(t) \qquad (t \geqslant t_0)$$

is satisfied. Furthermore, assume that

$$\int_t^{t+r} g(s)\, ds \leqslant C_1, \quad \int_t^{t+r} h(s)\, ds \leqslant C_2, \quad \int_t^{t+r} y(s)\, ds \leqslant C_3 \qquad (t \geqslant t_0),$$

where $r > 0$ and $C_1, C_2, C_3 > 0$ may depend on r but not on t. Then we have

$$y(t+r) \leqslant \Big(\frac{C_3}{r} + C_2\Big) \exp(C_1) \qquad (t \geqslant t_0).$$

2.5 Duality of Banach Spaces and Projections

2.5.1 Duality Scales

Based on [44] we give a short introduction to duality pairings and duality scales. We skip the proofs and refer to the original work for more details. Most of them are rather straight forward and elementary. In the following, let E and F be real or complex valued Banach spaces. We begin with a first definition of a duality pairing.

2.31 Definition (Duality systems). (1) A duality pairing for (E, F) is a bilinear form

$$\mathfrak{a}(\cdot,\cdot) : E \times F \to \mathbb{C}$$

that is continuous in the sense of

$$|\mathfrak{a}(x,y)| \leqslant C\|x\|_E\|y\|_F \quad (x \in E,\ y \in F)$$

for some $C > 0$.

(2) A triple $(E, F, \mathfrak{a})$ is called right duality system, if the mapping

$$r_{\mathfrak{a}} : F \to E',\ y \longmapsto r_{\mathfrak{a}}(y) := \mathfrak{a}(\cdot, y)$$

is bijective. We call F the dual space of E w.r.t. $\mathfrak{a}$ in that case and write $F = E'_{\mathfrak{a}}$.

(3) A triple $(E, F, \mathfrak{a})$ is called left duality system, if the mapping

$$\ell_{\mathfrak{a}} : E \to F',\ x \longmapsto \ell_{\mathfrak{a}}(x) := \mathfrak{a}(x, \cdot)$$

is bijective. We call E the dual space of F w.r.t. $\mathfrak{a}$ in that case and write $E = F'_{\mathfrak{a}}$.

(4) A triple $(E, F, \mathfrak{a})$ is called duality system if it is a left and a right duality system.

This definition gives rise to some facts that we will collect in the following remark.

2.32 Remark. (1) If $(E, F, \mathfrak{a})$ is a right duality system, then we have equivalence of $\|\cdot\|_F$ and the induced norm

$$\|y\|_F^{\mathfrak{a}} := \sup_{0 \neq x \in E} \frac{|\mathfrak{a}(x,y)|}{\|x\|_E}. \tag{2.12}$$

Furthermore, we have

$$E' = \{\mathfrak{a}(\cdot, y) \,:\, y \in F\},$$

which justifies the notation $F = E'_{\mathfrak{a}}$. The corresponding assertions also hold in the case that $(E, F, \mathfrak{a})$ is a left duality system.

(2) Let $(E, F, \mathfrak{a})$ be a right duality system. Then $(E, F, \mathfrak{a})$ is a left duality system if and only if E (and hence F) is reflexive. The corresponding assertion also holds, if $(E, F, \mathfrak{a})$ is a left duality system.

(3) We may introduce operators

$$A^r : F \to E',\ y \longmapsto A^r y := \mathfrak{a}(\cdot, y),$$
$$A^\ell : E \to F',\ x \longmapsto A^\ell x := \mathfrak{a}(x, \cdot),$$

which are continuous since $\mathfrak{a}$ is continuous. We have $A^r \in \mathscr{L}_{is}(F, E')$ if and only if $(E, F, \mathfrak{a})$ is a right duality system and $A^\ell \in \mathscr{L}_{is}(E, F')$ is and only if $(E, F, \mathfrak{a})$ is a left duality system.

After introducing these basic notations, we consider duality pairings on scales of Banach spaces. We will use two types of parameters for these scales in this thesis, i.e.

- Let $q_0 \in (2, \infty]$. Then we set $I_0 := (q_0', q_0)$ where $1/q_0 + 1/q_0' = 1$ and consider scales $(E_q)_{q \in I_0}$ of Banach spaces which are centered at $q = 2$. Typically, this type of scale parameter q describes a *integrability* parameter, e.g., for spaces $E_q = L^q$.
- Let $s_0 > 0$. Then we set $I_0 := (-s_0, s_0)$ and consider scales $(E_s)_{s \in I_0}$ of Banach spaces which are centered at $s = 0$. Typically, this type of scale parameter s describes a *regularity* parameter, e.g. for spaces $E_s = W^{s,q}$ with a fixed $q \in [1, \infty]$.

In the following, we will consider the latter case of a regularity parameter s, but all arguments work with an integrability parameter q after some minor modifications. We will assume that $E_s \hookrightarrow \mathcal{H}$ for all $s \in I_0$ and some fixed Hausdorff space $\mathcal{H}$. The next definition clarifies what we understand under a complex interpolation scale.

2.33 Definition. A scale $(E_s)_{s \in I_0}$ of Banach spaces is called complex interpolation scale, if the following conditions hold.

(i) $E_r \cap E_s \overset{d}{\hookrightarrow} E_q$ for $q \in [r, s]$, $r, s \in I_0$,

(ii) $E_q = [E_r, E_s]_\theta$ for $q, r, s \in I_0$, $\theta \in [0, 1]$ with $q = \theta s + (1 - \theta) r$.

In order to define a scale of duality systems, we must consider consistency of duality pairings on the scales of Banach spaces they are operating on. While consistency ensures that a mapping is well-behaved on a scale of spaces, we will introduce the stronger notion of strong consistency, which is necessary to obtain well-behaved scales of duality pairings.

2.34 Definition ((Strong) consistency)**.** Let $(E_s)_{s \in I_0}$, $(F_s)_{s \in I_0}$ be two complex interpolation scales and $(\mathfrak{a}_s)_{s \in I_0}$ be a scale of duality pairings $\mathfrak{a}_s : E_{-s} \times F_s \to \mathbb{C}$.

(i) Let $(A_s)_{s\in I_0}$ be a scale of mappings $A_s : E_s \to \mathcal{K}$ into some fixed Hausdorff space $\mathcal{K}$. We say that $(A_s)_{s\in I_0}$ is consistent on $(E_s)_{s\in I_0}$, if

$$A_r|_{E_r\cap E_s} = A_s|_{E_r\cap E_s} \quad (r, s \in I_0).$$

(ii) The scale $(\mathfrak{a}_s)_{s\in I_0}$ is called consistent on $(E_s, F_s)_{s\in I_0}$, if $(a_s)_{s\in I_0}$ is consistent on $(E_{-s} \times F_s)_{s\in I_0}$ in the sense introduced in (i).

(iii) The scale $(\mathfrak{a}_s)_{s\in I_0}$ is called strongly right consistent on $(E_s, F_s)_{s\in I_0}$, if for arbitrary $r, s \in I_0$ and $x \in F_r$, $y \in F_s$ we have

$$\mathfrak{a}_r(x', x) = \mathfrak{a}_s(x', y) \;\; (x' \in E_{-r} \cap E_{-s}) \Leftrightarrow x = y \text{ in } \mathcal{H}.$$

The scale $(\mathfrak{a}_s)_{s\in I_0}$ is called strongly left consistent on $(E_s, F_s)_{s\in I_0}$, if for arbitrary $-r, -s \in I_0$ and $x' \in E_{-r}$, $y' \in E_{-s}$ we have

$$\mathfrak{a}_r(x', x) = \mathfrak{a}_s(y', x) \;\; (x \in F_r \cap F_s) \Leftrightarrow x' = y' \text{ in } \mathcal{H}.$$

Here, we assume $E_s \hookrightarrow \mathcal{H}$ and $F_s \hookrightarrow \mathcal{H}$ for all $s \in I_0$ and a Hausdorff space $\mathcal{H}$. The scale $(\mathfrak{a}_s)_{s\in I_0}$ is called strongly consistent on $(E_s, F_s)_{s\in I_0}$, if it is strongly left consistent and strongly right consistent.

2.35 Remark. Let $(E_1, F_1, \mathfrak{a}_1)$ and $(E_2, F_2, \mathfrak{a}_2)$ be duality systems with $E_2 \hookrightarrow E_1$, $F_1 \hookrightarrow F_2$ and $\mathfrak{a}_1$ and $\mathfrak{a}_2$ consistent. Moreover, let $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$\mathfrak{a}_1(x_1, x') = \mathfrak{a}_2(x_2, x') \qquad (x' \in F_1 \cap F_2). \tag{2.13}$$

Then we already have $x_2 \in E_1$ and $x' \in F_1 = F_1 \cap F_2$. It follows by the fact that $(E_1, F_1, \mathfrak{a}_1)$ is a duality system, that $x_1 = x_2$. If in converse $x_1 = x_2$, then (2.13) follows directly, so that we have proved strong left consistency for the two duality pairings. In the same way one can show strong right consistency in this case, such that we obtain strong consistency.

The definition of strong consistency enables us to define a duality scale on two scales of Banach spaces, which is given as follows.

2.36 Definition (Duality scales). We assume that for every $s \in I_0$ a duality pairing $\mathfrak{a}_s(\cdot,\cdot) : E_{-s} \times F_s \to \mathbb{C}$ is given. Then the scale $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ is called duality scale, if

(1) $(E_s)_{s\in I_0}$ and $(F_s)_{s\in I_0}$ are complex interpolation scales;

(2) $(\mathfrak{a}_s)_{s\in I_0}$ is strongly consistent on $(E_s, F_s)_{s\in I_0}$;

(3) $(E_{-s}, F_s, \mathfrak{a}_s)_{s\in I_0}$ is a duality system for every $s \in I_0$.

2.37 Remark. (i) It is clear that every duality scale $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ consists of reflexive Banach spaces due to Remark 2.32 (2)

(ii) If $(E_s)_{s\in I_0}$ and $(F_s)_{s\in I_0}$ are decreasing scales, i.e. $E_r \hookrightarrow E_s$ and $F_r \hookrightarrow F_s$ for $r \geqslant s$, then strong consistency is automatically fulfilled if the duality pairings are consistent (cf. Remark 2.35).

The next result deals with the dual space of sums and intersections of Banach spaces and underlines the importance of strong consistency.

2.38 Lemma. *Let $E, F, \tilde{E}, \tilde{F} \hookrightarrow \mathcal{H}$ be reflexive Banach spaces with $E \cap F \overset{d}{\hookrightarrow} E, F$ and $\tilde{E} \cap \tilde{F} \overset{d}{\hookrightarrow} \tilde{E}, \tilde{F}$. For two right duality systems $(E, \tilde{E}, \mathfrak{a}_E)$ and $(F, \tilde{F}, \mathfrak{a}_F)$ we define the mapping*

$$\mathfrak{b}^{\mathfrak{a}} : (E \cap F) \times (\tilde{E} + \tilde{F}) \to \mathbb{C}, \quad \mathfrak{b}^{\mathfrak{a}}(x, x') = \mathfrak{a}_E(x, x'_E) + \mathfrak{a}_F(x, x'_F)$$

for $x' = x'_E + x'_F$ with $x'_E \in \tilde{E}$ and $x'_F \in \tilde{F}$. Then are equivalent:

(i) *for $x' \in \tilde{E}$ and $y' \in \tilde{F}$ it holds that*

$$\mathfrak{a}_E(\cdot, x')|_{(E\cap F)} = \mathfrak{a}_F(\cdot, y')|_{(E\cap F)} \;\Leftrightarrow x' = y' \text{ in } \mathcal{H}.$$

(ii) *$(E \cap F, \tilde{E} + \tilde{F}, \mathfrak{b}^{\mathfrak{a}})$ is a right duality system.*

(iii) *For*

$$\begin{aligned} \Phi_E &: \tilde{E} \to E', \quad x \mapsto \mathfrak{a}_E(\cdot, x), \\ \Phi_F &: \tilde{F} \to F', \quad x \mapsto \mathfrak{a}_F(\cdot, x), \end{aligned}$$

we have $\Phi_E x = \Phi_F x$ in $E' + F'$ for $x \in \tilde{E} \cap \tilde{F}$ as well as $(\Phi_E)^{-1}\ell = (\Phi_F)^{-1}\ell$ for $\ell \in E' \cap F'$ in $\mathcal{H}$.

2.39 Remark. (1) Note that strong consistency implies consistency in the sense of Definition 2.34 (ii). By the consistency, in turn, the map $\mathfrak{b}^{\mathfrak{a}}$ is well-defined.

(2) By Remark 2.32 (2) we have that $(E \cap F, \tilde{E} + \tilde{F}, \mathfrak{b}^{\mathfrak{a}})$ is even a duality system if condition (i) is fulfilled. The analogous assertions hold if the lemma is formulated for left duality systems.

(3) The consequence of this lemma underlines the importance of the notion of strong consistency. Especially, we obtain that $(E_{-r} \cap E_{-s}, F_r + F_s, \mathfrak{b}^{\mathfrak{a}})$ is a duality system for $r, s \in I_0$, given a duality scale $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$.

2.5.2 Dual Operators

In order to define dual operators w.r.t. a duality pairing we consider two duality systems $(E_1, \tilde{E}_1, \mathfrak{a}_{E_1})$ and $(E_2, \tilde{E}_2, \mathfrak{a}_{E_2})$. Then, for an operator $T \in \mathscr{L}(E_1, E_2)$ we can define the dual operator w.r.t. the duality pairings as follows. For every $x' \in \tilde{E}_2$ we see that

$$(x \mapsto \mathfrak{a}_{E_2}(Tx, x')) \in E_1'$$

such that (by the meaning of a duality system) there exists a unique $y' \in \tilde{E}_1$ with

$$\mathfrak{a}_{E_1}(x, y') = \mathfrak{a}_{E_2}(Tx, x') \quad (x \in E_1).$$

Thus, we can define

$$T'_{\mathfrak{a}_E} : \tilde{E}_2 \to \tilde{E}_1, \;\; x' \mapsto T'_{\mathfrak{a}_E} x' := y'$$

as the dual operator of T w.r.t. $(\mathfrak{a}_{E_1}, \mathfrak{a}_{E_2})$. Now, set $\ell_{x'} = \mathfrak{a}_{E_2}(\cdot, x') \in E_2'$ and let $T' \in \mathscr{L}(E_2', E_2')$ be the standard dual operator. Then we have

$$(T'\ell_{x'})(x) = \ell_{x'}(Tx) = \mathfrak{a}_{E_2}(Tx, x') = \mathfrak{a}_{E_1}(x, T'_{\mathfrak{a}} x') \;\; (x \in E_1).$$

This shows that - as with the definition of the dual space itself - we have to be careful with the definition of the dual operator, since we can have different representations of T' depending on the duality pairing we chose.
This definition of the dual operator gives rise to the question of consistency of dual operators, which is answered by the following lemma.

2.40 Lemma. *For $j = 1, 2$ let $(E_j, \tilde{E}_j, \mathfrak{a}_{E_j})$ and $(F_j, \tilde{F}_j, \mathfrak{a}_{F_j})$ be duality systems satisfying the assumptions of Lemma 2.38. Moreover, let one of the equivalences of Lemma 2.38 be satisfied for $j = 1, 2$. Then for all $T_E \in \mathscr{L}(E_1, E_2)$, $T_F \in \mathscr{L}(F_1, F_2)$ such that $T_E = T_F$ on $E_1 \cap F_1$, we have $T'_{\mathfrak{a}_E} = T'_{\mathfrak{a}_F}$ on $\tilde{E}_2 \cap \tilde{F}_2$ for the corresponding dual operators.*

2.41 Remark. Note that Lemma 2.40 especially implies consistency of dual operators on duality scales.

2.5.3 Projections and Duality

For a great number of partial differential equations, especially concerning fluid dynamics, projections on subspaces, e.g. the Helmholtz projection, are of special interest. Given a duality scale $(E_s, F_s, \mathfrak{a}_s)_{s \in I_0}$ and consistent scales of projections $(P_{s,F})_{s \in I_0}$ on

$(F_s)_{s\in I_0}$ and $(P_{s,E})_{s\in I_0}$ on $(E_s)_{s\in I_0}$, one could ask if $(P_E E_s, P_F F_s, \mathfrak{a}_s)_{s\in I_0}$ is a duality scale, too. Here, we skipped the indices of the different projections due to their consistency. We will treat this question in chapter 6, but give a short overview over the important results regarding projections and duality.
We begin with some basic facts about projections. Let P be a projection on a Banach space E, i.e. $P \in \mathscr{L}(E)$ and $P^2 = P$. We set $E_P := P(E)$. It is well known that E_P is a closed complemented subspace of E, i.e.

$$E = E_P \oplus E_{1-P}.$$

We collect some properties of the dual of P. Therefore, let $(E, F, \mathfrak{a})$ be a duality system and $P \in \mathscr{L}(E)$ be a projection on E. We set $P' := (P)'_{\mathfrak{a}}$ as the dual operator of P w.r.t. $\mathfrak{a}$. Additionally, we define the annihilators of subsets $A \subseteq E$ and $B \subseteq F$ w.r.t $\mathfrak{a}$ as follows.

$$\begin{aligned} A_{\mathfrak{a}}^{\perp} &:= \{x' \in F \,:\, \mathfrak{a}(x, x') = 0 \ (x \in A)\} \\ B_{\mathfrak{a}}^{\perp} &:= \{x \in E \,:\, \mathfrak{a}(x, x') = 0 \ (x' \in B)\} \end{aligned}$$

With these information in mind we state some results on projections and their dual operators.

2.42 Lemma. *Let E, F be Banach spaces, P be a projection on E and $(E, F, \mathfrak{a})$ be a duality system. We write P' for $P'_{\mathfrak{a}}$ since no confusion is possible here.*

(i) *P' is a projection on F with*

$$(E_P)_{\mathfrak{a}}^{\perp} = F_{1-P'} \ \text{ and } \ (E_{1-P})_{\mathfrak{a}}^{\perp} = F_{P'}$$

and

$$(F_{1-P'})_{\mathfrak{a}}^{\perp} = E_P \ \text{ and } \ (F_{P'})_{\mathfrak{a}}^{\perp} = E_{1-P}.$$

w.r.t. $\mathfrak{a}$. Moreover,

$$(E_P, F_{P'}, \mathfrak{a}) \ \text{ and } \ (E_{1-P}, F_{1-P'}, \mathfrak{a})$$

are duality systems.

(ii) *If X is another Banach space and $T \in \mathscr{L}_{is}(E, X)$, then*

$$X = T(E_P) \oplus T(E_{1-P}).$$

(iii) *If $T \in \mathscr{L}(E)$, then $T \in \mathscr{L}_{is}(E_P) \cap \mathscr{L}_{is}(E_{1-P})$ implies $T \in \mathscr{L}_{is}(E)$.*

Now we will consider the situation that scales of projections are given. In the following, we set $(P_{s,E})' = P'_{s,E} \in \mathscr{L}(F_{-s})$ to simplify notation. We obtain the following result.

2.43 Lemma. *Let $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ be a duality scale and $(P_{s,E})_{s\in I_0}$, $(P_{s,F})_{s\in I_0}$ be consistent scales of projections on $(E_s)_{s\in I_0}$ and $(F_s)_{s\in I_0}$ respectively. Then the following assertions hold.*

(i) *$(P'_{-s,E})_{s\in I_0}$ is a consistent scale of projections on $(F_s)_{s\in I_0}$ and $(P'_{-s,F})_{s\in I_0}$ is a consistent scale of projections on $(E_s)_{s\in I_0}$.*

(ii) *For every $r, s \in I_0$ we have*

$$\begin{aligned} E_r \cap E_s &= \left(E_{r,P_{r,E}} \cap E_{s,P_{s,E}}\right) \oplus \left(E_{r,1-P_{r,E}} \cap E_{s,1-P_{s,E}}\right), \\ F_r \cap F_s &= \left(F_{r,P_{r,F}} \cap F_{s,P_{s,F}}\right) \oplus \left(F_{r,1-P_{r,F}} \cap F_{s,1-P_{s,F}}\right). \end{aligned}$$

(iii) *For every $r, s \in I_0$ we have*

$$\begin{aligned} E_r + E_s &= \left(E_{r,P_{r,E}} + E_{s,P_{s,E}}\right) \oplus \left(E_{r,1-P_{r,E}} + E_{s,1-P_{s,E}}\right), \\ F_r + F_s &= \left(F_{r,P_{r,F}} + F_{s,P_{s,F}}\right) \oplus \left(F_{r,1-P_{r,F}} + F_{s,1-P_{s,F}}\right). \end{aligned}$$

Next, we consider a pair of projections $P, Q \in \mathscr{L}(X)$ on a Banach space E. In our application of these results in Chapter 6 we will set $Q = P'$ as the dual projection, but for now we can assume two general projections. First, we state a basic equivalence which is due to [26].

2.44 Lemma. *Let P, Q be projections on E. Then the following assertions are equivalent:*

(i) *$E = E_Q \oplus E_{1-P}$,*

(ii) *$P \in \mathscr{L}_{is}(E_Q, E_P)$,*

(iii) *$1 - Q \in \mathscr{L}_{is}(E_{1-P}, E_{1-Q})$.*

Note that P and Q (and later on P and P') do not commute in general, but the quadratic difference

$$R := (P - Q)^2 = P + Q - PQ - QP \in \mathscr{L}(E)$$

of these two does commute with P, Q as well as with $I-P$ and $I-Q$. For the following lemma, we define some auxiliary operators

$$U := 1 - P - QP, \ \ V := 1 - Q - PQ$$

and obtain $1 - R = UV = VU$. We state another result from [26].

2.45 Lemma. *Let $R, U, V \in \mathscr{L}(E)$ be defined as above. Then the following assertions are equivalent.*

(i) $1 \in \rho(R)$,

(ii) $V, U \in \mathscr{L}_{is}(E)$,

(iii) $E = E_P \oplus E_{1-Q} = E_Q \oplus E_{1-P}$.

If one of these assertions is fulfilled, we have

$$V^{-1} = (1-R)^{-1}U \in \mathscr{L}(E) \ \text{ and } \ U^{-1} = (1-R)^{-1}V \in \mathscr{L}(E).$$

Moreover, the projection subject to the decomposition $E = E_P \oplus E_{1-Q}$ respectively $E = E_Q \oplus E_{1-P}$ is given by

$$\begin{aligned} \mathbb{Q} &:= VQV^{-1} = PQ(1-R)^{-1}, \\ \mathbb{P} &:= UPU^{-1} = QP(1-R)^{-1}. \end{aligned}$$

Here, we have $\mathbb{Q}(E) = E_P$, $(1-\mathbb{Q})(E) = E_{1-Q}$, $\mathbb{P}(E) = E_Q$ and $(1-\mathbb{P})(E) = E_{1-P}$.

Although the value of this result may not be obvious at this moment, it will be a useful tool in order to characterize projected duality scales.

2.5.4 Scales of Compact Operators

By an extension of a corresponding lemma in [44] we show that the spectrum of compact operators on a duality scale is independent of the parameter $s \in I_0$. This invariance of the spectrum is useful for some applications of duality scales.

2.46 Lemma. *Let $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ be a duality scale and $(T_s)_{s\in I_0}$ with $T_s \in \mathscr{L}(E_s)$ for $s \in I_0$ be a consistent scale of compact operators on $(E_s)_{s\in I_0}$. Then the spectrum of T is s-invariant, i.e. $\sigma(T_s, E_s) = \sigma(T_0, E_0)$ for $s \in I_0$.*

Proof. Using Schauder's theorem and Lemma 2.40, we first note that the scale $(T_s')_{s\in I_0}$ of compact dual operators w.r.t. $\mathfrak{a}$ given by $T_{-s}' := (T_s)_{\mathfrak{a}}'$ is consistent. Let $\lambda \in \rho(T_0, E_0)$

and $s \in I_0$. Then we have $\lambda \in \rho(T_0', F_0)$, too. Due to the fact that the intersection of relatively compact sets remains relatively compact, we have that

$$T' : F_0 \cap F_{-s} \to F_0 \cap F_{-s} \quad (s \in I_0)$$

is compact, where we omitted the parameter s of T_{-s}' due to consistency. Additionally, by $F_0 \cap F_{-s} \hookrightarrow F_0$ and $\lambda \in \rho(T_0', F_0)$ we have injectivity of $\lambda - T'$ on $F_0 \cap F_{-s}$. Fredholm's alternative then yields $\lambda - T' \in \mathscr{L}_{is}(F_0 \cap F_{-s})$.
By reflexivity and Lemma 2.38 we have $\lambda - T \in \mathscr{L}_{is}(E_0 + E_s)$. Then, $E_s \subseteq E_0 + E_s$ and applying Fredolm's alternative again gives $\lambda - T \in \mathscr{L}_{is}(E_s)$ and $\lambda \in \rho(T_s, E_s)$ due to compactness, which completes the proof. □

2.47 Remark. Note that an equivalent assertion also holds if we assume $(T_s)_{s\in I_0}$ to be defined on $(F_s)_{s\in I_0}$.

2.5.5 Consistency of Operator Scales

In this last section we cite two results from [44] concerning the consistency of operator scales. In the original reference the assertions are proved for q-scales, but they can easily be transferred to the case of s-scales used here.
The first result shows that consistency of analytic operator families can be easily extended from one point to the whole domain.

2.48 Lemma. *Let $s_0 > 0$, $I_0 = (-s_0, s_0)$, $\emptyset \neq K \subseteq \mathbb{C}$ be a domain and $(E_s)_{s\in I_0}$ be a complex interpolation scale. Let $(T_s(z))_{s\in I_0, z\in K}$ be a family of operators such that $T_s(z) \in \mathscr{L}(E_s)$ for every $s \in I_0$ and $z \in K$. Moreover, let $K \ni z \mapsto T_s(z)$ be analytic for each $s \in I_0$. Then the following assertions are equivalent:*

(i) *The scale $(T_s(z))_{s\in I_0}$ is consistent for one $z \in K$.*

(ii) *The scale $(T_s(z))_{s\in I_0}$ is consistent for all $z \in K$.*

The second result shows that the scale of resolvents of consistent operators is also consistent, which can be useful to prove consistency of operators that can be represented (among others) by the resolvent.

2.49 Lemma. *Let $s_0 > 0$, $I = (-s_0, s_0)$ and $(E_s)_{s\in I_0}$ be a complex interpolation scale. Moreover, let $(T_s)_{s\in I_0}$ be a consistent scale of operators $T_s \in \mathscr{L}(E_s)$. Let $G \subseteq \mathbb{C}$ be an unbounded domain such that $G \subseteq \bigcap_{s\in I} \rho(T_s, E_s)$. Then for every $\lambda \in G$ the scale $((\lambda - T_s)^{-1})_{s\in I_0}$ is consistent on $(E_s)_{s\in I_0}$.*

3 Stable and Unstable Flow Regimes for Living Fluids

In this chapter we analyze the behavior of the active fluid model (1.2) near an equilibrium (steady) state and determine, whether stability or instability is likely to occur. To this end, we first present results regarding the well-posedness of the equations (1.2) and list the (physically) relevant equilibria, i.e., the disordered isotropic and the globally ordered polar states. Afterwards, an analysis of stability and instability for the linearized equations is carried out before we consider stability and instability in the nonlinear setting. Here, the theory from Section 2.5 plays a central role in order to analyze the manifold of globally ordered polar states.

3.1 Equilibria

In order to analyze (in)stability of the living fluids model (1.2), we will consider the following physically relevant equilibria.

- *Disordered isotropic state:* For $\alpha \in \mathbb{R}$, set

$$(v, p) = (0, p_0), \tag{3.1}$$

  where the pressure p_0 is constant.

- *Globally ordered polar state:* For $\alpha < 0$ set

$$(v, p) = (V, p_0), \tag{3.2}$$

  where $V \in B_{\alpha,\beta} := \left\{ x \in \mathbb{R}^n : |x| = \sqrt{-\alpha/\beta} \right\}$ is a constant vector of arbitrary orientation and p_0 is constant.

Especially the latter case will be of interest since we obtain a manifold of equilibria which allows to apply theory from Section 2.3.

We use a generalization of (1.2) in order to consider the two relevant types of equilibria.

$$\begin{aligned} \dot{u} + \lambda_0[(u+V)\cdot\nabla]u + (M+\beta|u|^2)u - \Gamma_0\Delta u + \Gamma_2\Delta^2 u + \nabla q = f + N(u), \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{aligned} \tag{3.3}$$

where $q = p - \lambda_1|v|^2$, $M \in \mathbb{R}^{n\times n}$ symmetric, $N(u) = \sum_{j,k} a_{jk}u_j u_k$ with $(a_{jk})_{j,k=1}^n$ is a nonlinearity of second order and the spatial dimension is $n = 2$ or $n = 3$. Additionally we assume

$$\lambda_0, \lambda_1, \Gamma_0, \alpha \in \mathbb{R}, \quad \Gamma_2, \beta > 0 \tag{3.4}$$

for the parameters in this chapter. In order to recover the original system (1.2) from (3.3), we set $u := v - V$ and

- for the disordered state (3.1) we assume

$$V = 0,\ M = \alpha I,\ N(u) = 0, \tag{3.5}$$

  where $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix.

- for the ordered polar state (3.2) we assume

$$V \in B_{\alpha,\beta},\ M = 2\beta VV^T,\ N(u) = -\beta|u|^2V - 2\beta(u\cdot V)u. \tag{3.6}$$

Note that from now on the equilibrium is always denoted by V and the deviation of the solution v from the equilibrium is denoted by u.

3.2 Linear Stability

In order to analyze stability of the nonlinear equations, it is helpful to analyze the linearized equations in a first step. Following the generalization in (3.3), we consider the following linearized system.

$$\begin{aligned} \dot{u} + \lambda_0(V\cdot\nabla)u + Mu - \Gamma_0\Delta u + \Gamma_2\Delta^2 u + \nabla q &= f \quad \text{in} \quad (0,\infty)\times Q_n, \\ \operatorname{div} u &= 0 \quad \text{in} \quad (0,\infty)\times Q_n, \\ u(0) &= u_0 \quad \text{in} \quad Q_n. \end{aligned} \tag{3.7}$$

Here, we assume periodic boundary conditions

$$\partial^\alpha u|_{x_j=0} = \partial^\alpha u|_{x_j=L} \ \text{ for } |\alpha| < 4, j = 1, ..., n.$$

We define the operator resulting from the linearization as

$$\begin{aligned} A_{LF}u &:= \lambda_0(V\cdot\nabla)u + PMu - \Gamma_0\Delta u + \Gamma_2\Delta^2 u,\\ D(A_{LF}) &:= H^4_\pi(Q_n)\cap L^2_\sigma(Q_n), \end{aligned}$$

where P is the Helmoltz-Weyl projection as in Section 2.2. The corresponding Fourier symbol is given as

$$\sigma_{A_{LF}}(\ell) := \Gamma_2\left(\frac{2\pi}{L}\right)^4|\ell|^4 + \Gamma_0\left(\frac{2\pi}{L}\right)^2|\ell|^2 + \lambda_0\left(\frac{2\pi i}{L}\right)(V\cdot\ell) + \sigma_P(\ell)M \quad (\ell\in\mathbb{Z}^n).$$

First, we get the following result.

3.1 Proposition. *There exists an $\omega>0$ such that $\omega + A_{LF}$ admits a bounded H^∞-calculus on $L^2_\sigma(Q_n)$ with H^∞-angle $\varphi^\infty_{\omega+A_{LF}} = 0$.*

Proof. Using $\Gamma_2>0$ one can immediately see that $A_{SH}u := \Gamma_2\Delta^2 u$ is selfadjoint with $D(A_{SH}) = D(A_{LF})$, hence $\omega + A_{SH}$ is selfadjoint and positive for $\omega>0$ and admits a bounded H^∞-calculus with $\varphi^\infty_{\omega+A_{SH}} = 0$. Perturbation theory for perturbations of lower order then yields the assertion (cf. [27, Proposition 13.1]). □

Consequently we obtain maximal L^p-regularity of A_{LF} on time intervals $(0,T)$ for $0<T<\infty$ and $-A_{LF}$ is generator of an analytic C_0-semigroup on $L^2_\sigma(Q_n)$.
In order to characterize linear (in)stability, we use the representation

$$\exp(-tA_{LF})v = \sum_{\ell\in\mathbb{Z}^n}\exp(-t\sigma_{A_{LF}}(\ell))\widehat{v}(\ell)e^{2\pi i\ell\cdot/L} \quad (v\in L^2_\sigma(Q_n)), \tag{3.8}$$

which is easy to verify.

3.2.1 Disordered Isotropic State

In this case we set $A_d := A_{LF}$ with $V=0$ and $M=\alpha I$. From the definition of P we have that P commutes with M and $PMu = \alpha u$ for $u\in L^2_\sigma(Q_n)$. By using representation (3.8) and

$$\sigma_{A_d}(\ell) := \Gamma_2\left(\frac{2\pi}{L}\right)^4|\ell|^4 + \Gamma_0\left(\frac{2\pi}{L}\right)^2|\ell|^2 + \alpha \quad (\ell\in\mathbb{Z}^n).$$

we can characterize (in)stability of the disordered isotropic state as follows.

3.2 Proposition. *Let $\Gamma_2>0$ and $\Gamma_0,\alpha\in\mathbb{R}$. Then the semigroup $(\exp(-tA_d))_{t\geqslant 0}$ corresponding to the disordered state (3.1) is*

(1) *stable, if* $\sigma_{A_d} \geqslant 0$;

(2) *exponentially stable, if* $\sigma_{A_d} \geqslant \delta > 0$;

(3) *exponentially unstable, if there exists some* $\ell_0 \in \mathbb{Z}^n$ *such that* $\sigma_{A_d}(\ell_0) < 0$.

We can give a more precise characterization based on the involved parameters by substituting $z = |\ell|^2$ in σ_{A_d} and analyzing the intersection points of the resulting parabola

$$p(z) := \Gamma_2 \left(\frac{2\pi}{L}\right)^4 z^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 z + \alpha.$$

This gives the following characterization of stability.

3.3 Lemma. *Let* $\Gamma_2 > 0$. *If* $\Gamma_0 < 0$ *and* $4\alpha > \Gamma_0^2/\Gamma_2$ *or if* $\Gamma_0 \geqslant 0$ *and* $\alpha > 0$, *then the semigroup* $(\exp(-tA_d))_{t\geqslant 0}$ *is exponentially stable. To be precise, the semigroup corresponding to the disordered state (3.1) is*

(1) *stable, if* $\Gamma_0 < 0$ *and* $4\alpha \geqslant \Gamma_0^2/\Gamma_2$ *or if* $\Gamma_0 \geqslant 0$ *and* $\alpha \geqslant 0$;

(2) *exponentially stable, if* $\Gamma_0 < 0$ *and* $4\alpha > \Gamma_0^2/\Gamma_2$ *or if* $\Gamma_0 \geqslant 0$ *and* $\alpha > 0$ *or if* $\Gamma_0 < 0$ *and* $4\alpha = \Gamma_0^2/\Gamma_2$ *with* $|\ell|^2 \neq -\frac{\Gamma_0}{2\Gamma_2}\left(\frac{L}{2\pi}\right)^2$ *for all* $\ell \in \mathbb{Z}^n$.

3.2.2 Ordered Polar State

In this case we set $A_o := A_{LF}$ with $V \in B_{\alpha,\beta}$ and $M = 2\beta VV^T$. Then we have

$$\sigma_{A_o} := \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \lambda_0 \left(\frac{2\pi i}{L}\right)(V \cdot \ell) + 2\beta\sigma_P(\ell)VV^T\sigma_P(\ell) \quad (\ell \in \mathbb{Z}^n).$$

If $\Gamma_0 \geqslant 0$, then

$$\operatorname{Re}\sigma_{A_o} := \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + 2\beta\sigma_P(\ell)VV^T\sigma_P(\ell) \in \mathbb{R}^{n\times n}$$

is positive semi-definite for every $\ell \in \mathbb{Z}^n$ and positive definite for $\ell \neq 0$ due to the fact that $\sigma_P(\ell)VV^T\sigma_P(\ell)$ is positive semi-definite. We can use

$$\|\exp(-tA_o)v\|^2_{L^2(Q_n)} \leqslant |\widehat{v}(0)|^2 + \sum_{\ell\in\mathbb{Z}^n\setminus\{0\}} |e^{-t\sigma_{A_o}(\ell)}|^2|\widehat{v}(\ell)|^2$$

in order to obtain stability. Conversely, if $\Gamma_0 < 0$ and if there exists $0 \neq \ell_0 \in \mathbb{Z}^n$ such that

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2 + \Gamma_0 < 0, \tag{3.9}$$

then for $n = 3$ we can choose $x \in \mathbb{R}^n \setminus \{0\}$ with $x \perp V$ and $x \perp \ell_0$ such that $x^T \mathrm{Re}\, \sigma_{A_o}(\ell_0) x < 0$. In case of $n = 2$, due to

$$\frac{2\beta |V \cdot \widehat{v}(\ell)|^2}{|\widehat{v}(\ell)|^2} \in [0, -2\alpha],$$

we assume the existence of a $0 \neq \ell_0 \in \mathbb{Z}^n$ such that

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell_0|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2 < 2\alpha. \tag{3.10}$$

This yields $x^T \mathrm{Re}\, \sigma_{A_o}(\ell_0) x < 0$ for some $x \in \mathbb{R}^n \setminus \{0\}$ such that $x \perp \ell_0$. Consequently, the matrix $\mathrm{Re}\, \sigma_{A_o}(\ell_0) \in \mathbb{R}^{n \times n}$ is negative semi-definite or indefinite and the growth bound of $(\exp(-tA_o))_{t \geqslant 0}$ strictly positive. These considerations yield the following result.

3.4 Proposition. *Let $\Gamma_2 > 0$. Then the semigroup $(\exp(-tA_o))_{t \geqslant 0}$ corresponding to the ordered polar state is*

(1) *stable, if $\Gamma_0 \geqslant 0$;*

(2) *exponentially unstable, if $\Gamma_0 < 0$ and*

 (a) if there exists some $0 \neq \ell_0 \in \mathbb{Z}^n$ such that (3.10) holds for $n = 2$;

 (b) if there exists some $0 \neq \ell_0 \in \mathbb{Z}^n$ such that (3.9) holds for $n = 3$.

3.5 Remark. (i) In the situation in [61, Section 3.1], where $L^2(\mathbb{R}^n)$ is considered, we have a continuous $\xi \in \mathbb{R}^n$. Thus we can always find a ξ parallel to V, which in general is not possible in the discrete case where $\ell \in \mathbb{Z}^n$. As a consequence, for $n = 2$ we can always find a nontrivial $x \in \mathbb{R}^2$ satisfying $x \perp V$ and $x \perp \xi$ in the continuous case and the more restrictive condition (3.10) does not appear in [61, 9] for $n = 2$.

(ii) Note that for $n = 2$ the condition (3.10) does not impose any restrictions regarding the analysis of nonlinear instability due to condition (3.12) in Theorem 3.11.

3.3 Global Well-posedness

It is possible to obtain local and global well-posedness of (3.3) by analogous arguments as in [61, Section 3.2].

3.6 Theorem (Global well-posedness). *Let $\Gamma_2, \beta > 0$ and $\Gamma_0, \alpha, \lambda_0 \in \mathbb{R}$ and $T \in (0, \infty)$. Let an initial value $u_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ and an exterior force $f \in L^2((0,T), L^2_\sigma(Q_n))$ be given. Then there exists a unique solution (u, q) for (3.3) with periodic boundary conditions satisfying*

$$\begin{aligned} u &\in H^1((0,T), L^2_\sigma(Q_n)) \cap L^2((0,T), H^4_\pi(Q_n)), \\ \nabla q &\in L^2((0,T), L^2(Q_n)). \end{aligned}$$

3.7 Remark. Note that in contrast to the Navier-Stokes equations the leading term $\Gamma_2 \Delta^2 u$ dominates the convective term, yielding global existence by standard energy techniques even for $n = 3$.

3.4 Nonlinear Stability and Turbulence

Now we consider nonlinear (in)stability of the equilibrium states of (1.2). We analyze the disordered isotropic state and the ordered polar state separately as follows.

- *Disordered isotropic state:* We apply energy methods in combination with the lemma of Gronwall (cf. Lemma 2.29) in order to obtain stability results. For instability results, Henry's instability theorem [23, Corollary 5.1.6] is used.
- *Ordered polar state:* Since we have a manifold of equilibria at hand, we consider normal hyperbolicity and normally stability as in Section 2.5.

In the following, we will only give a short outline of the disordered isotropic state and focus on the ordered polar state.

3.4.1 Disordered Isotropic State

First we collect the following property of the nonlinearity in (3.3).

3.8 Lemma. *Let $H(u) := \beta P|u|^2 u + \lambda_0 P(u \cdot \nabla)u - PN(u)$. Then, for $\eta \geqslant 5/4$ we have $H \in C^1(H^\eta_\pi(Q_n) \cap L^2_\sigma(Q_n), L^2_\sigma(Q_n))$ and H can be estimated as follows.*

$$\|H(u)\|_{L^2_\pi(Q_n)} \leqslant C\|u\|^2_{H^\eta_\pi(Q_n)} \quad (\|u\|^2_{H^\eta_\pi(Q_n)} \leqslant 1).$$

Proof. Follows in an analogous way as in [61, Lemma 4]. □

Given a global solution (u, q) to (3.3) we can now consider the isotropic state (3.1) and obtain

$$u_t + \Gamma_2 \Delta^2 u - \Gamma_0 \Delta u + \lambda_0 (u \cdot \nabla) u + (\alpha + \beta |u|^2) u + \nabla q = 0.$$

Testing this equation with u yields the energy equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_\pi(Q_n)} + \Gamma_2\|\Delta u(t)\|^2_{L^2_\pi(Q_n)} + \Gamma_0\|\nabla u(t)\|^2_{L^2_\pi(Q_n)} \\ + \alpha\|u(t)\|_{L^2_\pi(Q_n)} + \beta\tfrac{1}{L^n}\|u(t)\|^4_{L^4(Q_n)} = 0\end{aligned} \tag{3.11}$$

for $t > 0$.

3.9 Theorem. *Let $\Gamma_2, \beta > 0$ and $\Gamma_0, \alpha, \lambda_0 \in \mathbb{R}$. Then the velocity $V \equiv 0$ of the isotropic disordered state (3.1) is nonlinearly*

(1) *stable in $L^2_\sigma(Q_n)$, if $\Gamma_0 \geqslant 0$ and $\alpha \geqslant 0$ or if $\Gamma_0 < 0$ and $4\alpha \geqslant \Gamma_0^2/\Gamma_2$;*

(2) *(globally) exponentially stable in $L^2_\sigma(Q_n)$, if $\Gamma_0 \geqslant 0$ and $\alpha > 0$ or if $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 < 0$ and $4\alpha = \Gamma_0^2/\Gamma_2$ and $|\ell|^2 \neq \frac{-\Gamma_0}{2\Gamma_2}\left(\frac{L}{2\pi}\right)^2$ for all $\ell \in \mathbb{Z}^n$;*

(3) *unstable in $H^{4\gamma}_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for $\gamma \in [5/16, 1)$ if there exists some $\ell_0 \in \mathbb{Z}^n$ such that $\sigma_{A_d}(\ell_0) < 0$.*

Proof. We will only give a short outline of the proof and refer the reader to [8] for the details. Using the energy equality (3.11) and estimates of the Fourier symbol one can obtain the (exponential) stability by an application of the Gronwall lemma (cf. Lemma 2.29) in order to show (1) and (2).
On the other hand, exploiting the H^∞-calculus of $\omega + A_d$ for $\omega > 0$, spectral properties of A_d under the given assumptions as well as the consequences of Lemma 3.8, an application Henry's instability [23, Corollary 5.1.6] yields instability of the disordered isotropic state in $H^{4\gamma}_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for $\gamma \in [5/16, 1)$ and therefore assertion (3). □

3.10 Remark. Note that we neglected stability for the pressure p_0 of the corresponding equilibrium in Theorem 3.9. Since we can prove convergence of u in the stronger H^2_π-norm (e.g. by an application of [38, Theorem 5.3.1] combined with remark [38, Remark 5.3.2(a)]), we may show exponential convergence of the pressure $p(t)$ in $\widehat{H}^1_\pi(Q_n)$ to some constant p_∞ as $t \to \infty$ in the same way as in the proof of Theorem 3.11.

3.4.2 Ordered Polar State

Next we consider $\alpha < 0$ and fix an ordered polar state $V \in B_{\alpha,\beta}$. First we show that under certain restrictions regarding the parameters of (3.3) the equilibrium V is normally hyperbolic.

3.11 Theorem. *Let $\Gamma_2, \beta > 0$, $\alpha < 0$ and $\lambda_0 \in \mathbb{R}$. Let (V, p_0) be an ordered polar steady state with $V \in B_{\alpha,\beta}$. Then V is normally hyperbolic, if*

$$\Gamma_2\left(\frac{2\pi}{L}\right)^4|\ell|^4 + \Gamma_0\left(\frac{2\pi}{L}\right)^2|\ell|^2 \notin [2\alpha, 0],\ \ell \in \mathbb{Z}^n \setminus \{0\} \tag{3.12}$$

for $\Gamma_0 < 0$ *and if there exists some* $\ell_0 \in \mathbb{Z}^n$ *such that (3.9) holds. Thus, for each sufficient small* $\rho > 0$ *there exists* $0 < \delta < \rho$ *such that the unique solution* (v, p) *of (1.2) with initial value* $v_0 \in \mathbb{B}_{H^2(Q_n)}(V, \delta)$ *either satisfies*

(i) $\mathrm{dist}_{H^2(Q_n)}(v(t_0), B_{\alpha,\beta}) > \rho$ *for some finite time* $t_0 > 0$ *or*

(ii) *the solution* $(v(t), p(t))$ *exists on* $\mathbb{R}_+$ *and converges at exponential rate to some* $(V_\infty, p_\infty) \in B_{\alpha,\beta} \times \mathbb{R}$ *in* $(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)) \times \widehat{H}^1_\pi(Q_n)$ *as* $t \to \infty$.

Proof. We apply Theorem 2.9. In the notation of Theorem 2.9 we have $E_0 = L^2_\sigma(Q_n)$, $E_1 = H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ and $V = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. The manifold of equilibria is given as $\mathcal{E} = B_{\alpha,\beta}$ und $u_* = V$ is the equilibrium. The structure of the quasilinear problem is

$$\begin{aligned} A\tilde{v} &:= A(v)\tilde{v} := \Gamma_2 \Delta^2 \tilde{v} - \Gamma_0 \Delta \tilde{v} + \alpha \tilde{v} \qquad (\tilde{v} \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)) \\ F(v) &:= -\lambda_0 P(v \cdot \nabla) v - \beta P |v|^2 v \end{aligned}$$

for $v \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. We first consider the projected version of system (1.2)

$$\dot{v} + Av = F(v), \quad v(0) = v_0 \tag{3.13}$$

and will recover the pressure in the last step. By the structure of A and F (linear and semilinear respectively) it is obvious that

$$(A, F) \in C^1(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n), \mathscr{L}(H^4_\pi(Q_n) \cap L^2_\sigma(Q_n), L^2_\sigma(Q_n)) \times L^2_\sigma(Q_n)).$$

Furthermore, we see that A_o is the linearization of (3.13) at V, where A_o has maximal L^p-regularity on $(0, T)$ for $0 < T < \infty$, cf. Proposition 3.1. We split the proof of the conditions of Theorem 2.9 into several steps.

Step 1: Characterization of $N(A_o)$

Let $u \in N(A_o)$. By

$$(A_o u, A_o u)_{L^2_\pi(Q_n)} = \sum_{\ell \in \mathbb{Z}^n} |\sigma_{A_o}(\ell) \widehat{u}(\ell)|^2 = 0$$

we obtain $\sigma_{A_o}(\ell)\widehat{u}(\ell) = 0$ for every $\ell \in \mathbb{Z}^n$. This yields

$$\begin{aligned} 0 &= \mathrm{Re}\left(\overline{\widehat{u}(\ell)}^T \sigma_{A_o}(\ell) \widehat{u}(\ell) \right) \\ &= \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 |\widehat{u}(\ell)|^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 |\widehat{u}(\ell)|^2 + 2\beta \overline{\widehat{u}(\ell)}^T \sigma_P(\ell) V V^T \sigma_P(\ell) \widehat{u}(\ell) \end{aligned}$$

for all $\ell \in \mathbb{Z}^n$. We exploit that $\sigma_P(\ell)$ is symmetric and $\sigma_P(\ell)\widehat{u}(\ell) = \widehat{u}(\ell)$ to obtain

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 |\widehat{u}(\ell)|^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 |\widehat{u}(\ell)|^2 + 2\beta |V \cdot \widehat{u}(\ell)|^2 = 0.$$

We have $V \perp \widehat{u}(0)$ by setting $\ell = 0$. Moreover, for $\ell \neq 0$ and $\widehat{u}(\ell) \neq 0$ we see that

$$\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 = -\frac{2\beta |V \cdot \widehat{u}(\ell)|^2}{|\widehat{u}(\ell)|^2} \in [2\alpha, 0] \tag{3.14}$$

due to $|V|^2 = -\alpha/\beta$. With (3.12) it follows that $\widehat{u}(\ell) = 0$ in this case. Moreover, any constant $u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ that fulfills $u \perp V$ yields

$$A_o u = \Gamma_2 \Delta^2 u - \Gamma_0 \Delta u + \lambda_0 (V \cdot \nabla) u + 2\beta P V V^T u = 0,$$

such that we have

$$N(A_o) = \{u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) \,:\, u \text{ constant and } u \perp V\}.$$

Obviously, $N(A_o)$ has dimension $n-1$.

Step 2: $B_{\alpha,\beta}$ is a C^1-manifold in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ of dimension $n-1$
We will only show the case $n = 3$ here, since the steps for $n = 2$ are analogous. So let $n = 3$, then $V \in B_{\alpha,\beta}$ can be written as

$$V = \sqrt{-\frac{\alpha}{\beta}} \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

for fixed angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We define a C^1-map as

$$\Psi : [0,\pi] \times [0, 2\pi) \to H^4_\pi(Q_n) \cap L^2_\sigma(Q_n),$$
$$\begin{pmatrix} y \\ z \end{pmatrix} \longmapsto \Psi(y,z) := \begin{pmatrix} \sin(\theta + y)\cos(\theta + z) \\ \sin(\theta + y)\sin(\theta + z) \\ \cos(\theta + y) \end{pmatrix}.$$

Hence, $\Psi(y,z) \in \mathbb{B}_{\alpha,\beta}$ is a constant function in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for every value $(y,z) \in [0,\pi] \times [0, 2\pi)$. Moreover, we have $\Psi(0,0) = V$. The tangent space of $B_{\alpha,\beta}$ is of dimension $m = 2$ and obviously given as

$$T_V B_{\alpha,\beta} = \langle V \rangle^\perp.$$

Combining this with our result from step 1, we have

$$N(A_o) = \langle V \rangle^\perp = T_V B_{\alpha,\beta},$$

so the tangent space of $B_{\alpha,\beta}$ at V is isomorphic to $N(A_o)$.

Step 3: Characterization of $\sigma(A_o)$
We first note that due to $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) \overset{c}{\hookrightarrow} L^2_\sigma(Q_n)$ by the Rellich-Kondrachov theorem (cf. [1] and [41, Theorem A.4, Corollary A.5]), the resolvent

$$(\lambda - A_o)^{-1} : L^2_\sigma(Q_n) \to D(A_o) \overset{c}{\hookrightarrow} L^2_\sigma(Q_n)$$

is compact for $\lambda \in \rho(A_o)$. Therefore $\sigma(A_o)$ is discrete and consists only of the point spectrum, i.e. $\sigma(A_o) = \sigma_p(A_o)$. So it is sufficient to restrict to eigenvalues in order to characterize $\sigma(A_o)$. In step 1 we have already seen that $\lambda = 0$ is an eigenvalue of A_o. By assumption (3.12) condition (3.10) is fulfilled and Proposition 3.4(2) ensures that $\sigma(A_o) \cap \mathbb{C}_- \neq \emptyset$.
In order to show $\sigma(A_o) \cap i\mathbb{R} = \{0\}$ we fix $\lambda \in \sigma(A_o)$ such that $\operatorname{Re}\lambda = 0$. Let $u \neq 0$ be a corresponding eigenfunction, then

$$\operatorname{Re}\left(\overline{\widehat{u}(\ell)}^T \sigma_{A_o}(\ell)\widehat{u}(\ell)\right) = \operatorname{Re}\lambda|\widehat{u}(\ell)|^2 = 0 \qquad (\ell \in \mathbb{Z}^n).$$

By our argumentation in step 1 this implies $\widehat{u}(\ell) = 0$ for all $\ell \in \mathbb{Z}^n \setminus \{0\}$ and $\widehat{u}(0) \perp V$. This yields $u \in N(A_o)$, i.e. $\lambda = 0$.
It remains to show that $\lambda = 0$ is a semi-simple eigenvalue, i.e. $L^2_\sigma(Q_n) = N(A_o) \oplus R(A_o)$. To this end, we define the map

$$S : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \;\; u \mapsto Su := \frac{1}{L^n}\int_{Q_n} S_* u(x)\, dx,$$

where $S_* : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n)$ is the map given by $S_*u(x) = (I - VV^T/|V|^2)u(x)$. Here, I denotes the identity matrix in $\mathbb{R}^{n\times n}$. We first note that if $u \in L^2_\sigma(Q_n)$, then Su is constant and $Su \in L^2_\sigma(Q_n)$.
One can directly prove that S is a projection and that there exists a decomposition $S(L^2_\sigma(Q_n)) \oplus (I - S)(L^2_\sigma(Q_n)) = L^2_\sigma(Q_n)$, where I here denotes the identity operator on $L^2_\sigma(Q_n)$. We proceed in two substeps.

$N(A_o) = S(L^2_\sigma(Q_n))$: First, we show $S(L^2_\sigma(Q_n)) \subseteq N(A_o)$. Let $u \in S(L^2_\sigma(Q_n))$. As already seen, $u = Su$ is constant and we have

$$V^T u = V^T Su = \frac{1}{L^n}\int_{Q_n} V^T u(x)\, dx - \frac{1}{L^n}\int_{Q_n} \frac{1}{|V|^2} V^T V V^T u(x)\, dx$$

$$= \frac{1}{L^n}\int_{Q_n} V^T u(x)\,dx - \frac{1}{L^n}\int_{Q_n} V^T u(x)\,dx = 0,$$

which yields $u \perp V$ and thus $u \in N(A_o)$ by our characterization of $N(A_o)$ in Step 1. To see the inverse inclusion, let $u \in N(A_o)$. We know that u is constant and $u \perp V$. This yields

$$\begin{aligned} Su &= \frac{1}{L^n}\int_{Q_n} u(x)\,dx - \frac{1}{L^n}\int_{Q_n} \frac{1}{|V|^2} VV^T u(x)\,dx \\ &= u\left(\frac{1}{L^n}\int_{Q_n} dx\right) = u, \end{aligned}$$

hence $u \in S(L^2_\sigma(Q_n))$ and the claim is proved.
$R(A_o) = (I-S)(L^2_\sigma(Q_n))$: We note that $L^2_\sigma(Q_n) = S(L^2_\sigma(Q_n)) \oplus (I-S)(L^2_\sigma(Q_n))$ is orthogonal due to the fact that $L^2_\sigma(Q_n)$ is a Hilbert space and S is self-adjoint. Let $u \in D(A_o)$ and $w \in N(A_o)$, then we have

$$\begin{aligned} (A_o u, w)_{L^2_\pi(Q_n)} &= \Gamma_2(\Delta u, \Delta w)_{L^2_\pi(Q_n)} + \Gamma_0(\nabla u, \nabla w)_{L^2_\pi(Q_n)} \\ &\quad - \lambda_0(u, (V\cdot\nabla)w)_{L^2_\pi(Q_n)} + 2\beta(V^T u, V^T w)_{L^2_\pi(Q_n)} = 0 \end{aligned}$$

since w is constant and $w \perp V$. This yields $R(A_o) \perp N(A_0) = S(L^2_\sigma(Q_n))$ and by the orthogonality of the decomposition we obtain $R(A_o) \subseteq (I-S)(L^2_\sigma(Q_n))$.
We note again that A_o has compact resolvent. Following [29, Remark A.2.4] it suffices to show that

$$N(A_o) = N(A_o^2)$$

to prove that $\lambda = 0$ is a semi-simple eigenvalue of A_o. The inclusion $N(A_o) \subseteq N(A_o^2)$ is obvious. To show the inverse inclusion, fix $u \in N(A_o^2)$. Then $A_o^2 u = 0$ such that $A_o u \in N(A_0) \cap R(A_0)$. Due to $N(A_0) \perp R(A_0)$ we have $A_o u = 0$, thus $u \in N(A_o)$. Finally, $N(A_o) = N(A_o^2)$ and $\lambda = 0$ is semi-simple.

We now have proved that the conditions required by Theorem 2.9 are fulfilled, thus V is a normally hyperbolic equilibrium of (3.13). It remains to recover the pressure p. Observe that

$$\nabla p = (I-P)G(v),$$

where P denotes the Helmholtz-Weyl projection and

$$G(v) = [-\lambda_0(v\cdot\nabla)v + \lambda_1\nabla|v|^2 - \beta|v|^2 v].$$

Since V is normally hyperbolic, there are two possibilities:

- $\mathrm{dist}_{H^2(Q_n)}(v(t_0), B_{\alpha,\beta}) > \rho$ for some finite time $t_0 > 0$. Then there is nothing else to show since (v,p) is already unstable.
- $v(t)$ exists on $\mathbb{R}_+$ and converges to some $V_\infty \in B_{\alpha,\beta}$ in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ at an exponential rate as $t \to \infty$. Analogously to Lemma 3.8 we can show that $G \in C^1(H^2_\pi(Q_n), L^2(Q_n))$. Due to the convergence of the solution v it remains in a ball $B \subseteq H^2_\pi(Q_n)$ for $t \geqslant 0$. Note that this ball also includes V_∞. Then we have

  $$\|DG(w)\|_{\mathscr{L}(H^2_\pi(Q_n), L^2_\pi(Q_n))} \leqslant C \qquad (w \in B).$$

  An application of the mean-value theorem yields

  $$\|G(v) - G(V_\infty)\|_{L^2_\pi(Q_n)} \leqslant C\|v - V_\infty\|_{H^2_\pi(Q_n)}.$$

  Thus $(I-P)G(v)$ converges to $(I-P)G(V_\infty)$ in $L^2_\pi(Q_n)$ at an exponential rate. On the other hand, the fact that (V_∞, p_1) is a stationary solution of (1.2) for every $p_1 \in \mathbb{R}$ implies that $(I-P)G(V_\infty) = 0$. It follows that $p(t)$ converges to some constant $p_\infty \in \mathbb{R}$ in $\widehat{H}^1_\pi(Q_n)$ at an exponential rate.

This completes the proof and the assertions for (v,p) follow. □

3.12 Remark. It is easy to verify that, e.g. by setting $L = 2\pi$, $\Gamma_2 = 4$, $\Gamma_0 = -5$ and $\alpha = -1/4$ all conditions of Theorem 3.11 are satisfied, which yields unstable equilibria on $B_{\alpha,\beta}$. Thus, condition (3.12) is meaningful.

3.13 Remark. Note that a normally hyperbolic equilibrium implies the existence of a stable and an unstable foliation near V. In fact, if V is normally hyperbolic, then there exists $r > 0$ and a manifold $\mathcal{M}^s$, called the stable foliation, such that for each $v_0 \in \mathbb{B}_{H^2_\pi(Q_n)}(V, r)$ we have that $v_0 \in \mathcal{M}^s$ if and only if the solution $v(t, v_0)$ exists on $\mathbb{R}_+$ and converges to a $V_\infty \in B_{\alpha,\beta}$ at an exponential rate. Moreover, the projection onto the stable part of A_o is exactly the projection onto the tangent space of $\mathcal{M}^s$ at V (cf. [40, Theorem 3.1]). Analogously, there exists an unstable foliation $\mathcal{M}^u$ ([40, Theorem 4.1]).

In order to complete the analysis of (in)stability of the ordered polar state, let us now turn to the case of normal stability of a given $V \in B_{\alpha,\beta}$.

3.14 Theorem. *Let $\Gamma_2, \beta > 0$, $\Gamma_0 \geqslant 0$, $\alpha < 0$ and $\lambda_0 \in \mathbb{R}$. Let (V, p_0) with $V \in B_{\alpha,\beta}$ be an ordered polar stationary state of (1.2). Then V is normally stable, thus (V, p_0) is stable in the space $(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)) \times H^1_\pi(Q_n)$. There exists a $\delta > 0$ such that* *if (v,p) is a solution to (1.2) with initial data $v_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n) \cap \mathbb{B}_{H^2_\pi(Q_n)}(V,\delta)$, then (v,p) exists globally on $\mathbb{R}_+$ and converges to some $(V_\infty, p_\infty) \in B_{\alpha,\beta} \times \mathbb{R}$ as $t \to \infty$ at an exponential rate in $(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)) \times H^1_\pi(Q_n)$.*

Proof. From the proof of Theorem 3.11 we already know that

- $B_{\alpha,\beta}$ is a C^1-manifold of equilibria of dimension $n-1$ and
- we have $\sigma(A_o) = \sigma_p(A_o)$ and $\lambda = 0$ is a semi-simple eigenvalue of A_o,

since the proof can be carried out in the same way for the parameter set assumed in Theorem 3.14. We will prove the remaining assumptions in two several steps.
Step 1: Characterization of $N(A_o)$
Let $u \in N(A_o)$ such that $A_o u = 0$. Testing this equation with u yields

$$\begin{aligned} 0 &= \Gamma_2(\Delta^2 u, u)_{L^2_\pi(Q_n)} - \Gamma_0(\Delta u, u)_{L^2_\pi(Q_n)} \\ &\quad + \lambda_0((V\cdot\nabla)u, u)_{L^2_\pi(Q_n)} + 2\beta(PVV^T u, u)_{L^2_\pi(Q_n)}. \end{aligned}$$

By taking the real part and applying partial integration we obtain

$$0 = \Gamma_2\|\Delta u\|^2_{L^2_\pi(Q_n)} + \Gamma_0\|\nabla u\|^2_{L^2_\pi(Q_n)} + 2\beta\|V\cdot u\|^2_{L^2_\pi(Q_n)}$$

due to the fact that the λ_0-term is skew-symmetric. Since by assumption $\Gamma_2, \beta > 0$ and $\Gamma_0 \geqslant 0$, we arrive at

$$\|\Delta u\|^2_{L^2_\pi(Q_n)} = \|V\cdot u\|^2_{L^2_\pi(Q_n)} = 0,$$

hence $u \perp V$ and

$$\|\Delta u\|^2_{L^2_\pi(Q_n)} = \sum_{\ell\in\mathbb{Z}^n} |\ell|^2 |\hat{u}(\ell)|^2 = 0,$$

so u is constant. Altogether we have the same characterization for $N(A_o)$ as in the proof of Theorem 3.11, i.e.

$$N(A_o) = \{u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) \,:\, u \text{ constant and } u \perp V\}.$$

Step 2: Characterization of $\sigma(A_o)$
We need to show that

$$\sigma(A_o) \subseteq \mathbb{C}_+ \cup \{0\}.$$

To this end, let $\lambda \in \sigma(A_o) = \sigma_p(A_o)$ and $u \in D(A_o)$ be a nontrivial eigenvector such that $(\lambda - A_o)u = 0$. Testing the equation and applying partial integration yields

$$\begin{aligned} 0 = &\lambda \|u\|^2_{L^2_\pi(Q_n)} - \Gamma_2 \|\Delta u\|^2_{L^2_\pi(Q_n)} - \Gamma_0 \|\nabla u\|^2_{L^2_\pi(Q_n)} \\ &- \lambda_0((V \cdot \nabla)u, u)_{L^2_\pi(Q_n)} - 2\beta \|V \cdot u\|^2_{L^2_\pi(Q_n)}. \end{aligned}$$

By taking the real part the λ_0-term vanishes and we obtain

$$0 = \operatorname{Re} \lambda \|u\|^2_{L^2_\pi(Q_n)} - \Gamma_2 \|\Delta u\|^2_{L^2_\pi(Q_n)} - \Gamma_0 \|\nabla u\|^2_{L^2_\pi(Q_n)} - 2\beta \|V \cdot u\|^2_{L^2_\pi(Q_n)},$$

which gives us

$$\operatorname{Re} \lambda = \Gamma_2 \left(\frac{\|\Delta u\|_{L^2_\pi(Q_n)}}{\|u\|_{L^2_\pi(Q_n)}} \right)^2 + \Gamma_0 \left(\frac{\|\nabla u\|_{L^2_\pi(Q_n)}}{\|u\|_{L^2_\pi(Q_n)}} \right)^2 + 2\beta \left(\frac{\|V \cdot u\|_{L^2_\pi(Q_n)}}{\|u\|_{L^2_\pi(Q_n)}} \right)^2 \geqslant 0.$$

Furthermore, if $\operatorname{Re} \lambda = 0$, then u is constant and $u \perp V$, thus $u \in N(A_o)$ by step 1 and $\lambda = 0$. This yields $\sigma(A_o) \subseteq \mathbb{C}_+ \cup \{0\}$.

By Theorem 2.6 we obtain that V is normally stable for (3.13). We may recover the pressure p and prove the assertion in the same way as in the proof of Theorem 3.11. □

4 A Global Attractor for the Living Fluids Problem

In Chapter 3 we proved parameter settings for stability and instability of concrete equilibria of the living fluid problem (1.2). In this chapter, we analyze the long-term dynamics of the problem in a more general way and show that there exists a global attractor of finite dimension in $n = 2$ and $n = 3$ dimensions. Moreover, in $n = 2$ dimensions it will be possible to show the existence of an inertial manifold which attracts all solutions by an exponential rate. These results reduce the analysis of the dynamics of the living fluids problem into a finite dimensional problem.
The outline of this chapter is as follows. First, we provide a setting and show global well-posedness of the living fluids problem in $L^2_\sigma(Q_n)$ in order to obtain a corresponding semiflow. Then we prove the existence of a global attractor with arbitrary high regularity. Afterwards, we show that this attractor has finite dimension and obtain an inertial manifold in $n = 2$ dimensions.

4.1 Semiflow on $L^2_\sigma(Q_n)$

4.1.1 The Setting

We consider problem (1.2) with periodic boundary conditions, where we apply the Helmholtz projection and obtain

$$\begin{aligned} \partial_t u + \Gamma_2 \Delta^2 u - \Gamma_0 \Delta u + (\alpha + P\beta |u|^2) u + P\lambda_0 (u \cdot \nabla u) &= f, \\ u|_{t=0} &= u_0, \end{aligned} \tag{4.1}$$

on $L^2_\sigma(Q_n)$,where $n = 2, 3$, $L > 0$, $Q_n = [0, L]^n$, $\Gamma_2, \beta > 0$, $\Gamma_0, \alpha, \lambda_0 \in \mathbb{R}$ and P denotes the Helmholtz projection on $L^2(Q_n)$ as in Chapter 3. Additionally, an external force f is given. We already know from Theorem 3.6 that the problem is globally well-posed if we choose $u_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$, but for an application of results from the theory of dynamical systems we aim to have a semiflow

$$S(t) : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \;\; u_0 \mapsto S(t)u_0 = u(t),$$

where $u(t)$ solves (4.1) for $t > 0$. Thus we need to decrease the regularity of the initial data.
In order to prove the existence of a semiflow S we will use interpolation-extrapolation theory from [2, Chapter V], thus we will introduce some new notation: Let $E := L^2_\sigma(Q_n)$ and

$$A : D(A) \subseteq L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \quad u \mapsto Au := \Gamma_2 \Delta^2 u,$$

where $D(A) := E_1 := H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$, be a closed operator. By Theorem 3.1 there exists an $\omega > 0$ such that $\mathbb{A} := \omega + A$ has a bounded H^∞-calculus and $\mathbb{A} \in \mathscr{L}_{is}(E_1, E_0)$. Thus we obtain a densely injected consistent interpolation-extrapolation scale $[(E_\alpha, \mathbb{A}_\alpha) : \alpha \in [-1, \infty)]$, where $\mathbb{A}_0 := \mathbb{A}$, $E_0 := E$ and $\mathbb{A}_\alpha \in \mathscr{L}_{is}(E_{\alpha+1}, E_\alpha)$. Note that we have

$$D(\mathbb{A}_0^\theta) = E_\theta = [E_0, E_1]_\theta = H^{4\theta}_\pi(Q_n) \cap L^2_\sigma(Q_n) \qquad (\theta \in [0,1]).$$

Due to the fact that $\mathbb{A}_0$ restricts or extends consistently to an operator $\mathbb{A}_\alpha$ on E_α, we will sometimes just write $\mathbb{A}$ in the following. Moreover, due to the fact that $\mathbb{A}$ is self-adjoint and positive, $[(E_\alpha, \mathbb{A}_\alpha) : \alpha \in [-1, \infty)]$ is a Hilbert scale. We have $(E_\alpha)' = E_{-\alpha}$ and $(\mathbb{A}_\alpha)' = \mathbb{A}_{-\alpha}$ for $0 \leqslant \alpha \leqslant 1$ w.r.t. the canonical duality pairing induced by the scalar product $(\cdot, \cdot)_{E_0} = (\cdot, \cdot)_{L^2_\pi}$.
Note that

$$(E_\alpha, E_\beta)_{\eta,2} \doteq [E_\alpha, E_\beta]_\eta \doteq E_{(1-\eta)\alpha + \eta\beta} \quad (-1 \leqslant \alpha < \beta < \infty,\ 0 < \eta < 1) \tag{4.2}$$

and $\mathbb{A}^{\alpha-\beta} \in \mathscr{L}_{is}(E_\alpha, E_\beta)$ for $-1 \leqslant \alpha < \beta < \infty$.

4.1.2 H^∞-calculus and Maximal Regularity

In this section we will consider the linear part of (4.1) on $E_{-1/2}$ and prove well-posedness of the linear equations. To this end, let

$$A_{-1/2} : D(A_{-1/2}) = E_{1/2} = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n) \subseteq E_{-1/2} \to E_{-1/2}$$

be the $E_{-1/2}$ realization of A. Moreover, we set

$$B : D(B) := E_{1/2} = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \quad u \mapsto Bu := -\Gamma_0 \Delta u + \alpha u$$

and denote by

$$B_{-1/2} : D(B_{-1/2}) = L^2_\sigma(Q_n) = E_0 \subseteq E_{-1/2} \to E_{-1/2}$$

the $E_{-1/2}$ realization of B. Then we obtain the following result.

4.1 Theorem. *Let $T > 0$, $A_{-1/2}$ and $B_{-1/2}$ be defined as above. Moreover, let the initial value be given as $u_0 \in (E_{-1/2}, E_{1/2})_{1/2,2} = E = L^2_\sigma(Q_n)$ and the exterior force as $f \in L^2((0,T), E_{-1/2})$.*
Then there exists a unique solution

$$u \in H^1((0,T), E_{-1/2}) \cap L^2((0,T), E_{1/2})$$

of the problem

$$\begin{aligned} u_t + (A_{-1/2} + B_{-1/2})u &= f \quad \text{in } (0,T) \times Q_n, \\ u|_{t=0} &= u_0 \quad \text{in } Q_n. \end{aligned}$$

Proof. We note that $\mathbb{A}$ is an injective, sectorial operator on E. Then $\mathbb{A}_{-1/2}$ has an $\mathcal{R}$-bounded H^∞-calculus on $E_{-1/2}$ with $\varphi^\infty_{\mathbb{A}_{-1/2}} = 0$ (cf. [22, Theorem 6.5]), where $\varphi^\infty_{\mathbb{A}_{-1/2}}$ denotes the H^∞-angle.
Using perturbation theory for the H^∞-calculus, e.g. [27, Proposition 13.1], we will show that $\mathbb{A}_{-1/2} + B_{-1/2} + \nu$ possesses a bounded H^∞-calculus for $\nu \geqslant 0$ large enough. First, it is obvious that $D(\mathbb{A}_{-1/2}) \subseteq D(B_{-1/2})$. Let $u \in D(\mathbb{A}_{-1/2})$, then we have

$$\|B_{-1/2}u\|_{E_{-1/2}} = \|B_{-1/2}(\mathbb{A}_{-1/2})^{-1/2}(\mathbb{A}_{-1/2})^{1/2}u\|_{E_{-1/2}} \leqslant C\|(\mathbb{A}_{-1/2})^{1/2}u\|_{E_{-1/2}} \quad (4.3)$$

where we used that $(\mathbb{A}_{-1/2})^{-1/2} \in \mathscr{L}_{is}(E_{-1/2}, D((\mathbb{A}_{-1/2})^{1/2}))$ with domain of definition $D((\mathbb{A}_{-1/2})^{1/2}) = L^2_\sigma(Q_n)$. Moreover, it is easy to see that $B \in \mathscr{L}(E_{1/2}, E)$. Thus by a standard duality argument we obtain $B' = B_{-1/2} \in \mathscr{L}(E, E_{-1/2})$. This yields

$$B_{-1/2}(\mathbb{A}_{-1/2})^{-1/2} \in \mathscr{L}(E_{-1/2}),$$

which justifies estimate (4.3). Hence, by perturbation arguments there exists a $\nu \geqslant 0$ such that $\nu + \mathbb{A}_{-1/2} + B_{-1/2}$ possesses a bounded H^∞-calculus. Therefore, the operator $\nu + \lambda + A_{-1/2} + B_{-1/2}$ has maximal L^p regularity in $E_{-1/2}$ and

$$A_{-1/2} + B_{-1/2} : E_{1/2} \subseteq E_{-1/2} \to E_{-1/2}$$

enjoys maximal L^p-regularity on finite time intervals $(0,T)$ for $T > 0$. This completes the assertion. □

4.1.3 Local and Global Well-Posedness

In this section we want to make use of the maximal L^p-regularity for the linear problem in order to solve the nonlinear problem locally and globally. To this end, we first define the relevant solution and data spaces for $T \in (0, \infty)$.

$$\begin{aligned} \mathbb{E}_T &:= H^1((0,T), E_{-1/2}) \cap L^2((0,T), E_{1/2}), \\ \mathbb{F}_T^1 &:= L^2((0,T), E_{-1/2}), \\ \mathbb{F}^2 &:= (E_{-1/2}, E_{1/2})_{1/2,2} = E_0 = L^2_\sigma(Q_n), \\ \mathbb{F}_T &:= \mathbb{F}_T^1 \times \mathbb{F}^2. \end{aligned}$$

We begin with a definition and some auxiliary lemmas.

4.2 Definition. We set $\tilde{E} := \tilde{E}_0 := L^2(Q_n)$, $\tilde{E}_1 := H^4_\pi(Q_n)$ and define

$$\tilde{E}_\alpha := [\tilde{E}_0, \tilde{E}_1]_\alpha = H^{4\alpha}_\pi(Q_n), \qquad \tilde{E}_{-\alpha} = \tilde{E}_\alpha' \qquad (\alpha \in [0,1]).$$

Moreover, let $P = P_0 \in \mathscr{L}(L^2(Q_n)) = \mathscr{L}(\tilde{E}_0)$ be the Helmholtz projection.

4.3 Lemma. *The Helmholtz projection $P_0 \in \mathscr{L}(\tilde{E}_0)$ extends consistently to a projection $P_{-1/2} \in \mathscr{L}(\tilde{E}_{-1/2})$ with $P_{-1/2}(\tilde{E}_{-1/2}) = E_{-1/2}$.*

Proof. It is known that the Helmholtz projection P_0 on $\tilde{E}$ admits higher regularity on $\tilde{E}_\alpha$ for $\alpha \in [0,1]$ such that it restricts consistently to a projection P_α with $P(\tilde{E}_\alpha) = E_\alpha$ (cf. Lemma 2.2). By duality, we may obtain an operator $P_{-1/2} \in \mathscr{L}(\tilde{E}_{-1/2})$. In fact, due to symmetry of P_0, $P_{-1/2}$ extends P_0 consistently and is also a projection with $P_{-1/2}(\tilde{E}_{-1/2}) = E_{-1/2}$. □

4.4 Lemma. *The nonlinearity*

$$H : \mathbb{E}_T \to \mathbb{F}_T^1, \quad H(u) := \beta P_{-1/2}|u|^2 u + \lambda_0 P_{-1/2}(u \cdot \nabla) u$$

fulfills $H \in C^1(\mathbb{E}_T, \mathbb{F}_T^1)$.

Proof. First we prove that $H : \mathbb{E}_T \to \mathbb{F}_T^1$ is well-defined. From [2, III, Theorem 4.10.2] we know that

$$\mathbb{E}_T \hookrightarrow L^\infty((0,T), I_2(\mathscr{A})) = L^\infty((0,T), L^2_\sigma(Q_n))$$

since $I_2(\mathscr{A}) = \mathbb{F}^2 = L^2_\sigma(Q_n)$. We have

$$\nabla \in \mathscr{L}(H^2_\pi(Q_n)^n, H^1_\pi(Q_n)^{n\times n}) = \mathscr{L}(D(\tilde{\mathbb{A}}^{1/2})^n, D(\tilde{\mathbb{A}}^{1/4})^{n\times n}) = \mathscr{L}((\tilde{E}_{1/2})^n, \tilde{E}_{1/4})^{n\times n}),$$

where we clarify the dimensions w.r.t. n in this case in order to avoid confusion. Since we can identify $((\tilde{E}_{1/4})^{n\times n})'$ with $(\tilde{E}_{-1/4})^{n\times n}$ and $((\tilde{E}_{1/2})^{n})'$ with $(\tilde{E}_{-1/2})^{n}$, we can use duality arguments in order to obtain

$$\operatorname{div} \in \mathscr{L}((\tilde{E}_{1/4})^{n\times n}, (\tilde{E}_{-1/2})^{n}).$$

We first note that

$$\tilde{E}_{1/2} = H^2_\pi(Q_n) \hookrightarrow L^\infty(Q_n) \tag{4.4}$$

and estimate the second nonlinearity as

$$\begin{aligned}
\|P_{-1/2}(u\cdot\nabla)u\|_{\mathbb{F}^1_T} &= \|P_{-1/2}(u\cdot\nabla)u\|_{L^2((0,T),E_{-1/2})} \\
&\leqslant C\|\operatorname{div}(u\otimes u)\|_{L^2((0,T),\tilde{E}_{-1/2})} \\
&\leqslant C\|u\otimes u\|_{L^2((0,T),\tilde{E}_{-1/4})} \\
&\leqslant C\|u\otimes u\|_{L^2((0,T),L^2_\pi(Q_n))} \\
&\leqslant C\|u\|_{L^2((0,T),L^\infty(Q_n))}\|u\|_{L^\infty((0,T),L^2_\pi(Q_n))} \\
&\leqslant C\|u\|_{L^2((0,T),\tilde{E}_{1/2})}\|u\|_{L^\infty((0,T),L^2_\pi(Q_n))} \\
&\leqslant C\|u\|^2_{\mathbb{E}_T}.
\end{aligned}$$

Regarding the first nonlinearity we will again use (4.4): by choosing $f \in L^1(Q_n)$ and $\varphi \in \tilde{E}_{1/2}$ we have

$$|\langle f,\varphi\rangle_{\tilde{E}_{-1/2},\tilde{E}_{1/2}}| \leqslant C\int_{Q_n}|f(x)\varphi(x)|\,dx \leqslant C\|f\|_{L^1(Q_n)}\|\varphi\|_{L^\infty(Q_n)} \leqslant C\|f\|_{L^1(Q_n)}\|\varphi\|_{\tilde{E}_{1/2}},$$

which leads to

$$\|f\|_{\tilde{E}_{-1/2}} = \sup_{0\neq\varphi\in\tilde{E}_{1/2}}\frac{|\langle f,\varphi\rangle_{\tilde{E}_{-1/2},\tilde{E}_{1/2}}|}{\|\varphi\|_{\tilde{E}_{1/2}}} \leqslant C\|f\|_{L^1(Q_n)}.$$

Summarized we have shown that

$$L^1(Q_n) \hookrightarrow \tilde{E}_{-1/2} \tag{4.5}$$

and can estimate as follows.

$$\begin{aligned}
\|P_{-1/2}|u|^2u\|_{\mathbb{F}^1_T} &\leqslant C\||u|^2u\|_{L^2((0,T),\tilde{E}_{-1/2})} \leqslant C\||u|^2u\|_{L^2((0,T),L^1(Q_n))} \\
&\leqslant C\|u\|^2_{L^\infty((0,T),L^2(Q_n))}\|u\|_{L^2((0,T),L^\infty(Q_n))}
\end{aligned}$$

$$\leqslant C\|u\|^2_{L^\infty((0,T),L^2(Q_n))}\|u\|_{L^2((0,T),\tilde{E}_{1/2})} \leqslant \|u\|^3_{\mathbb{E}_T}.$$

Hence, the nonlinearity $H : \mathbb{E}_T \to \mathbb{F}^1_T$ is well defined and the derivative is given as

$$DH(u)v = \beta P_{-1/2}|u|^2 v + 2\beta P_{-1/2}(u\cdot v)u + \lambda_0 P_{-1/2}(v\cdot\nabla)u + \lambda_0 P_{-1/2}(u\cdot\nabla)v$$

for $u, v \in \mathbb{E}_T$. By similar estimates as for H itself we see that $DH : \mathbb{E}_T \to \mathscr{L}(\mathbb{E}_T, \mathbb{F}^1_T)$ is well-defined and that $H \in C^1(\mathbb{E}_T, \mathbb{F}^1_T)$. □

We obtain the following result on local well-posedness.

4.5 Theorem (Local well-posedness)**.** *Let $T > 0$ and $(f, u_0) \in \mathbb{F}_T$. Then there exists a $0 < T' < T$ such that (4.1) possesses a unique solution $u \in \mathbb{E}_{T'}$.*

Proof. We already know that

$$\mathscr{A} : E_{1/2} \subseteq E_{-1/2} \to E_{-1/2}, \quad u \mapsto \mathscr{A}u := A_{-1/2}u + B_{-1/2}u$$

has maximal L^p-regularity in $E_{-1/2}$ due to Theorem 4.1 and that

$$L : \mathbb{E}_T \to \mathbb{F}_T, \quad Lu := (\dot{u} + \mathscr{A}u, u(0))$$

is an isomorphism.
According to Lemma 4.3 and Lemma 4.4 we may write (4.1) as

$$\begin{aligned} u_t + \mathscr{A}u + H(u) &= f, \\ u|_{t=0} &= u_0 \end{aligned}$$

and rephrase the problem as $F(u) = (f, u_0)$, where

$$F : \mathbb{E}_T \to \mathbb{F}_T, \quad F(u) := Lu + (H(u), 0).$$

Next, we pick $v \in \mathbb{E}_T$ arbitrary and show that the perturbed linear solution operator $L+(DH(v), 0)$ is an isomorphism, where we have already proved that $L \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T)$. To this end, we want to show that

$$\|DH(v(t))u\|_{E_{-1/2}} \leqslant \frac{C_v}{\lambda^{1/2-\alpha/4}}\|(\lambda + \mu + \mathscr{A})u\|_{E_{-1/2}}$$

for $u \in E_{-1/2}$, $\lambda > 0$ arbitrary, $\mu > 0$ large, a fixed $v \in \mathbb{E}_T$ and some $\alpha \in (3/2, 2)$. We estimate

$$\|(u\cdot\nabla)v(t)\|_{\tilde{E}_{-1/2}} = \|\operatorname{div}(u\otimes v(t))\|_{\tilde{E}_{-1/2}} \leqslant C\|u\otimes v(t)\|_{\tilde{E}_{-1/4}} \leqslant C\|u\otimes v(t)\|_{L^2_\pi(Q_n)}$$

$$\begin{aligned} &\leqslant C\|u\|_{L^\infty(Q_n)}\|v(t)\|_{L^2_\pi(Q_n)} \leqslant C\|u\|_{E_{\alpha/4}}\|v\|_{L^\infty((0,T),L^2_\pi(Q_n))} \\ &\leqslant C\|u\|_{E_{\alpha/4}}\|v\|_{\mathbb{E}_T}, \end{aligned}$$

where we use $v \in \mathbb{E}_T \hookrightarrow L^\infty((0,T), L^2_\sigma(Q_n))$ and

$$E_{\alpha/4} = H^\alpha_\pi(Q_n) \cap L^2_\sigma(Q_n) \hookrightarrow L^\infty(Q_n) \cap L^2_\sigma(Q_n)$$

for $\alpha > 3/2$.
Next, we choose $\mu > 0$ arbitrary such that $\mu + \mathscr{A}$ admits an H^∞-calculus in $E_{-1/2}$. Then we have

$$(\lambda + \mu + \mathscr{A})^{-1} \in \mathscr{L}(E_{-1/2}, E_{1/2}) \cap \mathscr{L}(E_{-1/2})$$

and

$$\|(\lambda + \mu + \mathscr{A})^{-1}\|_{\mathscr{L}(E_{-1/2},E_{1/2})} < \infty, \quad \|(\lambda + \mu + \mathscr{A})^{-1}\|_{\mathscr{L}(E_{-1/2})} \leqslant \frac{C}{\lambda}$$

for $\lambda > 0$. By a standard interpolation result for bounded linear operators, applied with $\theta = 1/2 + \alpha/4$ we obtain

$$(\lambda + \mu + \mathscr{A})^{-1} \in \mathscr{L}([E_{-1/2}, E_{-1/2}]_\theta, [E_{-1/2}, E_{1/2}]_\theta) = \mathscr{L}(E_{-1/2}, E_{\alpha/4})$$

and

$$\|(\lambda + \mu + \mathscr{A})^{-1}\|_{\mathscr{L}(E_{-1/2},E_{\alpha/4})} \leqslant \frac{C}{\lambda^{1-1/2-\alpha/4}}.$$

Thus, we obtain the estimate

$$\|(u\cdot\nabla)v(t)\|_{\tilde{E}_{-1/2}} \leqslant C\|u\|_{E_{\alpha/4}}\|v\|_{\mathbb{E}_T} \leqslant \frac{C_v}{\lambda^{1/2-\alpha/4}}\|(\lambda + \mu + \mathscr{A})u\|_{E_{-1/2}},$$

where $C_v > 0$ depends on $\|v\|_{\mathbb{E}_T}$. The $(v(t)\cdot\nabla)u$-term can be estimated analogously since $(v(t)\cdot\nabla)u = \operatorname{div}(v(t)\otimes u)$.
For the second nonlinearity we use (4.5) and have

$$\begin{aligned} \||v(t)|^2u\|_{\tilde{E}_{-1/2}} &\leqslant C\||v(t)|^2u\|_{L^1(Q_n)} \leqslant C\|u\|_{L^\infty(Q_n)}\|v(t)\|^2_{L^2_\pi(Q_n)} \leqslant C\|u\|_{E_{\alpha/4}}\|v\|^2_{\mathbb{E}_T} \\ &\leqslant \frac{C_v}{\lambda^{1/2-\alpha/4}}\|(\lambda + \mu + \mathscr{A})u\|_{E_{-1/2}}, \end{aligned}$$

where $C_v > 0$ depends on $\|v\|^2_{\mathbb{E}_T}$. We can estimate the $(v(t)\cdot u)v(t)$-term analogously.

Summing up, we have the estimate

$$\|DH(v(t))u\|_{E_{-1/2}} \leqslant \frac{C_v}{\lambda^{1/2-\alpha/4}} \|(\lambda + \mu + \mathscr{A})u\|_{E_{-1/2}}$$

for $\lambda > 0$ arbitrary, $\mu > 0$ large, $v \in \mathbb{E}_T$ and $u \in E_{1/2}$. By choosing $\mu > 0$ large enough we set $\{\lambda + \mu + \mathscr{A}\}_{t\in[0,T]}$ as a constant family of bounded invertible operators having maximal L^p-regularity. Choosing $\lambda > 0$ large such that

$$\frac{C_v}{\lambda^{1/2-\alpha/4}} < 1,$$

we obtain maximal L^p-regularity of $\lambda + \mu + \mathscr{A} + DH(v)$ in $E_{-1/2}$ by applying [45, Theorem 2.5]. Hence, $\mathscr{A} + DH(v)$ has maximal L^p-regularity in $E_{-1/2}$ as well on time intervals $(0, T)$ for $0 < T < \infty$. Summarized we have

$$L + (DH(v), 0) \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T). \tag{4.6}$$

Finally we want to prove local well-posedness using the maximal L^p-regularity. We already know that

$$L : \mathbb{E}_T \to \mathbb{F}_T, \quad Lu = (\dot{u} + \mathscr{A}u, u(0))$$

is an isomorphism. We set $u^* := L^{-1}(f, u_0)$ as a reference solution for a given pair of data $(f, u_0) \in \mathbb{F}_T$. We want to apply the local inverse theorem to

$$F : \mathbb{E}_T \to \mathbb{F}_T, \quad u \mapsto F(u) = Lu + (H(u), 0).$$

Next, we prove that F fulfills the assumptions of the local inverse theorem. To this end, we note that

$$DF(u^*) = L + (DH(u^*), 0) \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T)$$

and $F \in C^1(\mathbb{E}_T, \mathbb{F}_T)$. Using the local inverse theorem there exist $\varepsilon > 0$ and $\delta > 0$ such that $F : \mathbb{B}_{\mathbb{E}_T}(u^*, \varepsilon) \to \mathbb{B}_{\mathbb{F}_T}(F(u^*), \delta)$ is bijective. Let $0 < T' < T$. We define

$$f_{T'} : (0, T) \to E_{-1/2}, \;\; t \mapsto f_{T'}(t) := \begin{cases} f(t), & t \in (0, T'), \\ f(t) + H(u^*)(t), & t \in [T', T) \end{cases}$$

and see that

$$\|f_{T'} - (f + H(u^*))\|^2_{\mathbb{F}^1_T} = \int_0^T \|f_{T'}(t) - (f(t) + H(u^*)(t))\|^2_{E_{-1/2}} \, dt$$

$$= \int_0^{T'} \|H(u^*)(t)\|_{E_{-1/2}}^2 \, dt \xrightarrow{T' \to 0} 0$$

by applying the dominated convergence theorem. We choose $0 < T' < T$ such that

$$\|f_{T'} - (f + H(u^*))\|_{\mathbb{F}_T^1} < \delta.$$

By noting that $F(u^*) = (f + H(u^*), u_0)$ we see that

$$\|(f_{T'}, u_0) - F(u^*)\|_{\mathbb{F}_T} < \delta$$

and hence $(f_{T'}, u_0) \in \mathbb{B}_{\mathbb{F}_T}(F(u^*), \delta)$. Then there exists a unique $u \in \mathbb{B}_{\mathbb{E}_T}(u^*, \varepsilon)$ such that $F(u) = (f_{T'}, u_0)$ and therefore $Lu + (H(u), 0) = (f, u_0)$ in $(0, T')$, which completes the assertion. □

In order to obtain a semigroup

$$S(t) : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n), \quad u_0 \mapsto S(t)u_0 = u(t)$$

that solves (4.1) for $0 < t < \infty$, it remains to show that a local solution u which is given by Theorem 4.5 is a global solution. By making use of energy methods this will be proved in the following Theorem.

4.6 Theorem (Global well-posedness). *Let $0 < T < \infty$ and $(f, u_0) \in \mathbb{F}_T$. Then there exists a unique solution $u \in \mathbb{E}_T$ to (4.1).*

Proof. Let $(f, u_0) \in \mathbb{F}_T$ and $u \in \mathbb{E}_{T'}$ be the unique local solution from Theorem 4.5 for some $0 < T' < T$. Let $0 < t < T'$. We test the equation (4.1) with u and integrate from 0 to t with the result

$$\begin{aligned} \frac{1}{2}\|u(t)\|_{L^2_\pi}^2 + \Gamma_2 \int_0^t \|\Delta u(s)\|_{L^2_\pi}^2 \, ds - \Gamma_0 \int_0^t (\Delta u(s), u(s))_{L^2_\pi} \, ds + \alpha \int_0^t \|u(s)\|_{L^2_\pi}^2 \, ds \\ + \frac{\beta}{L^n} \int_0^t \|u(s)\|_{L^4}^4 \, ds = \frac{1}{2}\|u_0\|_{L^2_\pi}^2. \end{aligned}$$

By applying the Cauchy-Schwarz and the Young inequality on the Γ_0-term we arrive at

$$\begin{aligned} \frac{1}{2}\|u(t)\|_{L^2_\pi}^2 + \frac{\Gamma_2}{2} \int_0^t \|\Delta u(s)\|_{L^2_\pi}^2 \, ds + \left(\alpha - \frac{|\Gamma_0|^2}{2\Gamma_2}\right) \int_0^t \|u(s)\|_{L^2_\pi}^2 \, ds \\ + \frac{\beta}{L^n} \int_0^t \|u(s)\|_{L^4}^4 \, ds \leqslant \frac{1}{2}\|u_0\|_{L^2_\pi}^2, \end{aligned}$$

which yields

$$\|u(t)\|_{L^2_\pi}^2 \leqslant \|u_0\|_{L^2_\pi}^2 + C\int_0^t \|u(s)\|_{L^2_\pi}^2\, ds,$$

where $C = C(\alpha, \Gamma_0, \Gamma_2) > 0$. We apply the Gronwall lemma (cf. Lemma 2.29) and obtain

$$\|u(t)\|_{L^2_\pi}^2 \leqslant \exp(Ct)\|u_0\|_{L^2_\pi}^2 \leqslant \exp(CT)\|u_0\|_{L^2_\pi}^2$$

for $0 < t < T$. This yields the bound

$$\|u\|_{L^\infty((0,T),L^2_\sigma(Q_n))} \leqslant C\|u_0\|_{L^2_\pi}$$

with some $C = C(T, \alpha, \Gamma_0, \Gamma_2) > 0$, which gives global well-posedness. □

4.2 Global Attractor

In this chapter we will prove the existence of a global attractor $\mathcal{A}$ for the problem (4.1). Additionally, we show that this attractor has H^k-regularity for $k \in \mathbb{N}$. We will construct the attractor in several steps which are laid out in the next sections. Note that from now on we will assume $f = 0$ in (4.1) for simplification. Furthermore, we will omit the domain Q_n in the subscript of norms which are used.

4.2.1 Absorbing Set in $L^2_\sigma(Q_n)$

First we prove the existence of an absorbing set in $L^2_\sigma(Q_n)$ which serves as a starting point for the bootstrapping arguments we will use to obtain higher regularity.

4.7 Lemma. *There exists some $R_0 > 0$ and $t_0 \in (0,\infty)$ such that*

$$\|u(t)\|_{L^2_\pi}^2 \leqslant R_0 \quad (t \geqslant t_0)$$

for any (global) solution u to (4.1). Note that t_0 does not depend on the initial value $u_0 \in L^2_\sigma(Q_n)$. Thus there exists a bounded absorbing set $\mathcal{B}_0 := \mathbb{B}_{L^2_\pi}(0, \sqrt{R_0})$ in $L^2_\sigma(Q_n)$.

Proof. Let u be a (global) solution to (4.1) with initial value $u_0 \in L^2_\sigma(Q_n)$. We test equation (4.1) with u w.r.t. the inner scalar-product $(\cdot,\cdot)_{L^2_\pi}$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2_\pi}^2 = -\Gamma_2\|\Delta u(t)\|_{L^2_\pi}^2 - \Gamma_0\|\nabla u(t)\|_{L^2_\pi}^2 - \alpha\|u(t)\|_{L^2_\pi}^2 - \frac{\beta}{L^n}\|u(t)\|_{L^4}^4, \tag{4.7}$$

where the λ_0-term vanishes due to its anti-symmetry and the pressure term λ_1 vanishes due to the fact that u is divergence free. By using the Fourier expansion we obtain

$$u(t,x) = \sum_{\ell \in \mathbb{Z}^n} u_\ell(t) \exp(2\pi i \ell \cdot x/L),$$

where $(u_\ell(t))_{\ell \in \mathbb{Z}^n} \subseteq \mathbb{R}^n$ denote the Fourier coefficients. Plugging this into (4.7) yields

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2_\pi}^2 = -\sum_{\ell \in \mathbb{Z}^n} \left(\Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \alpha\right) |u_\ell(t)|^2 - \frac{\beta}{L^n}\|u(t)\|_{L^4}^4.$$

By substituting $z := |\ell|^2$ we obtain a parabola

$$p(z) := \Gamma_2 \left(\frac{2\pi}{L}\right)^4 z^2 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 z + \alpha,$$

for which the set $U := \{\ell \in \mathbb{Z}^n : p(|\ell|^2) < 0\}$ is finite. This leads to the estimate

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2_\pi}^2 &\leqslant -\sum_{\ell \in U} p(|\ell|^2)|u_\ell(t)|^2 - \frac{\beta}{L^n}\|u(t)\|_{L^4}^4 \\ &\leqslant \delta \sum_{\ell \in U} |u_\ell(t)|^2 - \frac{\beta}{L^n}\|u(t)\|_{L^4}^4 \\ &\leqslant \delta \|u(t)\|_{L^2_\pi}^2 - \beta\|u(t)\|_{L^2_\pi}^4 \end{aligned}$$

with some $\delta := \delta(\Gamma_2, \Gamma_0, \alpha) > 0$, where we used $L^4(Q_n) \cap L^2_\sigma(Q_n) \hookrightarrow L^2_\sigma(Q_n)$ with

$$\|u(t)\|_{L^2_\pi} \leqslant \frac{|\Omega|^{1/2-1/4}}{L^{n/2}}\|u(t)\|_{L^4} = L^{-n/4}\|u(t)\|_{L^4}$$

in the last step. By setting $\gamma_1 := \delta$, $\gamma_2 := \beta$ and $\varphi(t) := \|u(t)\|_{L^2_\pi}^2$ we arrive at the differential inequality

$$\frac{d}{dt}\varphi(t) \leqslant 2\gamma_1\varphi(t) - 2\gamma_2\varphi(t)^2 \quad (t > 0). \tag{4.8}$$

Thus in order to analyze the long-time behavior of u in $L^2_\pi(Q_n)$ we can analyze the differential inequality (4.8). We will first take a look at (4.8) as a differential equation to get an idea of the long-term behavior: let $t_* \geqslant 0$. We consider the ordinary differential equation

$$\frac{d}{dt}\psi(t) = 2\gamma_1\psi(t) - 2\gamma_2\psi(t)^2 \quad (t > t_*), \quad \psi(t_*) = \|u(t_*)\|_{L^2_\pi}^2. \tag{4.9}$$

By using a separation ansatz and calculating we obtain the following solutions to (4.9).

- If $\psi(t_*) = 0$, then $\psi(t) = 0$ for $t > 0$ is a unique global solution.
- If $\psi(t_*) > 0$, then

$$\psi(t) = \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2\psi(t_*)}{\psi(t_*)}\exp(-2\gamma_1(t-t_*)) + \gamma_2} \tag{4.10}$$

is a unique solution. We have to distinguish two cases: if $\gamma_1 - \gamma_2\psi(t_*) > 0$, then the maximal existence interval is given by $(-\infty, \infty)$. If $\gamma_1 - \gamma_2\psi(t_*) < 0$, then we have the maximal existence interval $(\tilde{t}, \infty)$, where

$$\tilde{t} := -\frac{1}{2\gamma_1}\ln\left(\frac{\gamma_2\psi(t_*)}{\gamma_2\psi(t_*) - \gamma_1}\right) + t_* < t_*.$$

We obtain

$$\psi(t) = \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2\psi(t_*)}{\psi(t_*)}\exp(-2\gamma_1(t-t_*)) + \gamma_2} \overset{t\to\infty}{\nearrow} \frac{\gamma_1}{\gamma_2}$$

if $\gamma_1 - \gamma_2\psi(t_*) > 0$ and

$$\psi(t) = \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2\psi(t_*)}{\psi(t_*)}\exp(-2\gamma_1(t-t_*)) + \gamma_2} \overset{t\to\infty}{\searrow} \frac{\gamma_1}{\gamma_2}$$

if $\gamma_1 - \gamma_2\psi(t_*) < 0$. Finally, for $\gamma_1 - \gamma_2\psi(t_*) = 0$ we have $\psi(t) := \gamma_1/\gamma_2$ as a unique solution to the initial value $\psi(t_*) = \gamma_1/\gamma_2$ for $t \in (-\infty, \infty)$.

We observe that

$$V := \left\{\frac{\gamma_1}{\gamma_2}, 0\right\}$$

is a global attractor for the ordinary differential equation (4.9).

Next, we compare φ from the inequality (4.8) to the solution ψ of (4.9) in order to obtain information about the long-term behavior of φ and therefore of $\|u(t)\|^2_{L^2_\pi}$. To this end, let u be a global solution to (4.1) with initial value $u_0 \in L^2_\sigma(Q_n)$.
Moreover, let $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $g(z) := 2\gamma_1 z - 2\gamma_2 z^2$, $\psi : [t_*, \infty) \to \mathbb{R}$ be the solution of the differential equation (4.9) to the initial value $\psi(0)$ and φ be the function satisfying the differential inequality (4.8). Obviously g is locally Lipschitz continuous and φ, ψ are both almost everywhere differentiable in $(0, \infty)$. We have

$$\varphi(0) = \|u_0\|^2_{L^2_\pi} = \psi(0), \quad \frac{d}{dt}\varphi(t) - g(\varphi(t)) \leqslant 0 = \frac{d}{dt}\psi(t) - g(\psi(t)) \quad (t > 0). \tag{4.11}$$

By applying [56, Theorem 1.3] we obtain the estimate

$$\varphi(t) \leqslant \psi(t) \quad (t \geqslant 0). \tag{4.12}$$

4.8 Remark. Note that [56, Theorem 1.3] originally requires differentiable functions φ, ψ (in the classical sense) and the inequality in (4.11) to hold for $t \geqslant 0$, which we cannot guarantee. However, it is easy to see from the proof that we can also work with functions that are differentiable almost everywhere and that it is sufficient if the inequality only holds for $t > 0$ as in our case. We then obtain inequality (4.12) almost everywhere for $t \geqslant 0$, which yields the inequality for all $t \geqslant 0$ due to continuity.

We distinguish the following cases.

- $\varphi(0) = \psi(0) = 0$: Then we have $\varphi(t) \leqslant \psi(t) = 0$ for $t \geqslant 0$.
- $\varphi(0) = \psi(0) = \frac{\gamma_1}{\gamma_2}$: Then we have $\varphi(t) \leqslant \psi(t) = \frac{\gamma_1}{\gamma_2}$ for $t \geqslant 0$.
- $0 < \varphi(0) = \psi(0) < \frac{\gamma_1}{\gamma_2}$: Then we have

  $$\varphi(t) \leqslant \psi(t) = \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2 \psi(0)}{\psi(0)} \exp(-2\gamma_1 t) + \gamma_2} \leqslant \frac{\gamma_1}{\gamma_2}$$

  for $t \geqslant 0$.
- $\varphi(0) = \psi(0) > \frac{\gamma_1}{\gamma_2}$: Then we have

  $$\varphi(t) \leqslant \psi(t) = \gamma_1 \frac{1}{\frac{\gamma_1 - \gamma_2 \psi(0)}{\psi(0)} \exp(-2\gamma_1 t) + \gamma_2}.$$

  Due to

  $$\begin{aligned} \frac{\gamma_1 - \gamma_2 \psi(0)}{\psi(0)} \exp(-2\gamma_1 t) + \gamma_2 &= \gamma_2(1 - \exp(-2\gamma_1 t)) + \frac{\gamma_1}{\psi(0)} \exp(-2\gamma_1 t) \\ &\geqslant \gamma_2(1 - \exp(-2\gamma_1 t)) \end{aligned}$$

  we have

  $$\varphi(t) \leqslant \frac{\gamma_1}{\gamma_2(1 - \exp(-2\gamma_1 t))} \xrightarrow{t \to \infty} \frac{\gamma_1}{\gamma_2}.$$

  Thus for every $\varepsilon > 0$ there exists some $t_0 = t_0(\varepsilon) > 0$ such that

  $$\frac{\gamma_1}{\gamma_2(1 - \exp(-2\gamma_1 t))} - \frac{\gamma_1}{\gamma_2} = \left| \frac{\gamma_1}{\gamma_2(1 - \exp(-2\gamma_1 t))} - \frac{\gamma_1}{\gamma_2} \right| < \varepsilon.$$

and

$$\varphi(t) \leqslant \frac{\gamma_1}{\gamma_2(1-\exp(-2\gamma_1 t))} - \frac{\gamma_1}{\gamma_2} + \frac{\gamma_1}{\gamma_2} < \varepsilon + \frac{\gamma_1}{\gamma_2} \tag{4.13}$$

for $t \geqslant t_0$ independently of $\varphi(0) = \psi(0)$ (i.e. u_0).

Finally, let $\varepsilon > 0$ be arbitrary and $t_0 := t_0(\varepsilon) > 0$ be chosen as in (4.13). Putting everything together, we have shown that for every $u_0 \in L^2_\sigma(Q_n)$ the corresponding solution u to (4.1) is bounded as

$$\|u(t)\|^2_{L^2_\pi} \leqslant \frac{\gamma_1}{\gamma_2} + \varepsilon =: R_0 \qquad (t \geqslant t_0), \tag{4.14}$$

where t_0 does not depend on u_0. We conclude that $\mathcal{B}_0 := \mathbb{B}_{L^2_\sigma}(0, \sqrt{R_0})$ is a bounded absorbing set in $L^2_\sigma(Q_n)$. □

4.9 Remark. With (4.14) in the proof of Lemma 4.7 we proved a stronger, uniform estimate than it would be required for an absorbing set (cf. Definition 2.12). This will also be the case in the proofs for absorbing sets with higher regularity in Lemma 4.10 and Theorem 4.11.
Moreover, we note that $\mathbb{B}_{L^2_\sigma}(0, \gamma_1/\gamma_2) \subseteq L^2_\sigma(Q_n)$ is forward invariant thanks to (4.12). We have that

$$\|u(t)\|^2_{L^2_\pi} = \varphi(t) \leqslant \psi(t) \xrightarrow{t\to\infty} \frac{\gamma_1}{\gamma_2}$$

for $u_0 \neq 0$, hence $\lim_{t\to\infty} u(t) \in \overline{\mathbb{B}}_{L^2_\sigma}(0, (\gamma_1/\gamma_2)^{1/2})$.

4.2.2 Higher Regularity for Absorbing Sets

In order to show the existence of a global attractor $\mathcal{A}$ for (4.1), we want to make use of compact embeddings of the type $H^k_\pi(Q_n) \overset{c}{\hookrightarrow} H^{k-1}_\pi(Q_n)$ for $k \in \mathbb{N}$. Thus we will show in two steps that we can find absorbing sets for (4.1) of arbitrary high regularity. In the first step, we will show this assertion for H^1_π-regularity.

4.10 Lemma. *There exists some $R_1 > 0$ and $t_1 \geqslant 0$ such that*

$$\|u(t)\|^2_{H^1_\pi} \leqslant R_1 \quad (t \geqslant t_1)$$

for any (global) solution u to (4.1). Note that t_1 does not depend on the initial value u_0. Thus there exists a bounded absorbing set $\mathcal{B}_1 := \mathbb{B}_{H^1_\pi}(0, \sqrt{R_1})$ in $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$.

Proof. Let u be a (global) solution to (4.1) with initial value $u_0 \in L^2_\sigma(Q_n)$. We test (4.1) with $-\Delta u$ w.r.t. the L^2_π scalar product and infer

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2_\pi} + \alpha\|\nabla u(t)\|^2_{L^2_\pi} + \Gamma_0\|\Delta u(t)\|^2_{L^2_\pi} + \Gamma_2\|\Delta\nabla u(t)\|^2_{L^2_\pi}$$
$$+ \frac{\beta}{L^n}\int_{Q_n}(\nabla(|u(t)|^2\, u(t))) : \nabla u(t)\, dx = -\lambda_0((u(t)\cdot\nabla)u(t), \Delta u(t))_{L^2_\pi}.$$

A direct calculation yields

$$\int_{Q_n}(\nabla(|u(t)|^2\, u(t))) : \nabla u(t)\, dx \geqslant 0,$$

see e.g. [61, Theorem 2]. By using the estimate

$$\|\nabla u\|^2_{L^2_\pi} = |(\Delta u, u)_{L^2_\pi}| \leqslant \|\Delta u\|_{L^2_\pi}\|u\|_{L^2_\pi} \tag{4.15}$$

and applying Young's inequality we infer

$$\|\Delta u\|^2_{L^2_\pi} \leqslant \varepsilon\|\Delta\nabla u(t)\|^2_{L^2_\pi} + C\|\nabla u(t)\|^2_{L^2_\pi},$$

where $\varepsilon > 0$ will be chosen later and $C := C(\varepsilon) > 0$. Then we can estimate as

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2_\pi} + \frac{\Gamma_2}{2}\|\Delta\nabla u(t)\|^2_{L^2_\pi} \leqslant C\|\nabla u(t)\|^2_{L^2_\pi} + |\lambda_0||((u(t)\cdot\nabla)u(t), \Delta u(t))_{L^2_\pi}|$$

with $C > 0$. We estimate the λ_0-term as

$$\begin{aligned}|((u(t)\cdot\nabla)u(t), \Delta u(t))_{L^2_\pi}| &= |(\operatorname{div}(u(t)\otimes u(t)), \Delta u(t))_{L^2_\pi}|\\ &\leqslant \|u(t)\otimes u(t)\|_{L^2_\pi}\|\Delta\nabla u(t)\|_{L^2_\pi}\\ &\leqslant \varepsilon\|\Delta\nabla u(t)\|^2_{L^2_\pi} + C\|u(t)\|^4_{L^4}\end{aligned}$$

with $\varepsilon > 0$ arbitrary and $C := C(\varepsilon, Q_n) > 0$. Plugging this into our inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2_\pi} + \frac{\Gamma_2}{4}\|\Delta\nabla u(t)\|^2_{L^2_\pi} \leqslant C_1\|\nabla u(t)\|^2_{L^2_\pi} + C_2\|u(t)\|^4_{L^4} \tag{4.16}$$

with $C_1, C_2 > 0$ independent of u, which leads to

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2_\pi} \leqslant C_1\|\nabla u(t)\|^2_{L^2_\pi} + C_2\|u(t)\|^4_{L^4}. \tag{4.17}$$

We want to apply the generalized Gronwall lemma (cf. Lemma 2.30) to (4.17) in order

to obtain a uniform bound of u in $H^1_\pi(Q_n)$. To this end, we define $f(t) := \|\nabla u(t)\|^2_{L^2_\pi}$, $g(t) := C_1$ and $h(t) := C_2\|u(t)\|^4_{L^4}$. We have to show that f, g and h are uniformly integrable on $[t, t+r]$ for $t \geqslant t'$, where $t' > 0$ has to be chosen later.
Therefore let $R_0 > 0$ and $t_0 > 0$ as in Lemma 4.7 such that $\|u(t)\|^2_{L^2_\pi} \leqslant R_0$ for $t \geqslant t_0$. Moreover, let $r \geqslant 0$ and $t \geqslant t_0$ be arbitrary. By multiplying (4.1) with u and integrating from t to $t+r$ we obtain

$$\frac{1}{2}\|u(t)\|^2_{L^2_\pi} = \frac{1}{2}\|u(t+r)\|^2_{L^2_\pi} + \Gamma_2 \int_t^{t+r} \|\Delta u(s)\|^2_{L^2_\pi}\, ds + \Gamma_0 \int_t^{t+r} \|\nabla u(s)\|^2_{L^2_\pi}\, ds$$
$$+ \alpha \int_t^{t+r} \|u(s)\|^2_{L^2_\pi}\, ds + \frac{\beta}{L^n} \int_t^{t+r} \|u(s)\|^4_{L^4}\, ds.$$

By using (4.15) with Young's inequality on $\|\nabla u(t)\|^2_{L^2_\pi}$ and by exploiting the uniform boundedness of u in $L^2_\sigma(Q_n)$ for $t \geqslant t_0$ according to Lemma 4.7 we arrive at

$$\begin{aligned}\frac{1}{2}\|u(t+r)\|^2_{L^2_\pi} + \frac{\Gamma_2}{2} \int_t^{t+r} \|\Delta u(s)\|^2_{L^2_\pi}\, ds + \frac{\beta}{L^n} \int_t^{t+r} \|u(s)\|^4_{L^4}\, ds \\ \leqslant \frac{1}{2}\|u(t)\|^2_{L^2_\pi} + C_3 \int_t^{t+r} \|u(s)\|^2_{L^2_\pi}\, ds \leqslant C_4\end{aligned} \tag{4.18}$$

for $t \geqslant t_0$, where $C_3 := C_3(\Gamma_2, \Gamma_0, \alpha) > 0$ and $C_4 := C_4(C_3, R_0, r) > 0$. This yields

$$\int_t^{t+r} h(s)\, ds = C_2 \int_t^{t+r} \|u(t)\|^4_{L^4}\, ds \leqslant C_5$$

with $0 < C_5 := C_5(C_4, \beta, Q_n) < \infty$. Obviously we have

$$\int_t^{t+r} g(s)\, ds = \int_t^{t+r} C_1\, ds = C_6$$

with $0 < C_6 := C_6(C_1, r) < \infty$. Finally we obtain

$$\int_t^{t+r} f(s)\, ds = \int_t^{t+r} \|\nabla u\|^2_{L^2_\pi}\, ds \leqslant C_7,$$

where $0 < C_7 := C_7(C_4, \Gamma_2, \Gamma_0, \alpha, R_0, r) < \infty$ and we used (4.15) and (4.18). Applying the generalized Gronwall lemma (cf. Lemma 2.30) and choosing a fixed $r > 0$ we obtain

$$f(t) = \|\nabla u(t)\|^2_{L^2_\pi} \leqslant C_8 \exp(C_6) \qquad (t \geqslant t_0 + r)$$

with $0 < C_8 := C_8(C_5, C_7, r)$. This yields a uniform bound on the H^1-norm as

$$\|u(t)\|^2_{H^1_\pi} = \|u(t)\|^2_{L^2_\pi} + \|\nabla u(t)\|^2_{L^2_\pi} \leqslant R_0 + C_8 \exp(C_6) =: R_1 \qquad (t \geqslant t_1).$$

By defining the ball

$$\mathcal{B}_1 := \{u \in H^1_\pi(Q_n) \cap L^2_\sigma(Q_n) \ : \ \|u(t)\|^2_{H^1_\pi} \leqslant R_1\}$$

we see that

$$\mathrm{dist}_{H^1_\pi}(u(t), \mathcal{B}_1) = 0 \qquad (t \geqslant t_1)$$

and $\mathcal{B}_1$ is a bounded absorbing set for (4.1) in $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ and therefore also in $L^2_\sigma(Q_n)$. □

Using the results for bounded absorbing sets in $L^2_\sigma(Q_n)$ and $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ we can prove the existence of bounded absorbing sets with arbitrary high regularity by induction.

4.11 Theorem. *Let $k \geqslant 0$. There exists some $R_k > 0$ and $t_k \geqslant 0$ such that*

$$\|u(t)\|^2_{H^k_\pi} \leqslant R_k \qquad (t \geqslant t_k)$$

for any (global) solution u to (4.1). Note that t_k does not depend on the initial value u_0. Thus there exists a bounded absorbing set $\mathcal{B}_k := \mathbb{B}_{H^k_\pi}(0, \sqrt{R_k})$ in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$.

Proof. We have shown the assertion for $k = 0$ in Lemma 4.7 and for $k = 1$ in Lemma 4.10. Therefore, let $k \geqslant 2$ and assume that there exist $R_{k-1} \geqslant ... \geqslant R_0 > 0$ and $t_{k-1} \geqslant ... \geqslant t_0 > 0$ such that $\|u(t)\|^2_{H^j_\pi} \leqslant R_j$ for $j \in \{0, ..., k-1\}$ and $t \geqslant t_j$.
Testing (4.1) with $(-1)^k \Delta^k u(t)$ yields

$$\begin{aligned} 0 = \frac{1}{2}\frac{d}{dt}\|\nabla^k u(t)\|^2_{L^2_\pi} + \Gamma_2\|\nabla^{k+2}u(t)\|^2_{L^2_\pi} + \Gamma_0\|\nabla^{k+1}u(t)\|^2_{L^2_\pi} + \alpha\|\nabla^k u(t)\|^2_{L^2_\pi} \\ + (-1)^k\lambda_0(\nabla^{k-2}(u(t)\cdot\nabla)u(t), \nabla^{k+2}u(t))_{L^2_\pi} + (-1)^k\beta(\nabla^{k-2}(|u(t)|^2u(t)), \nabla^{k+2}u(t))_{L^2_\pi}. \end{aligned}$$

By using (4.15) we have

$$\Gamma_0\|\nabla^{k+1}u(t)\|^2_{L^2_\pi} \leqslant \varepsilon\|\nabla^{k+2}u(t)\|^2_{L^2_\pi} + C(\varepsilon)\|\nabla^k u(t)\|^2_{L^2_\pi}$$

with $\varepsilon > 0$ and $C(\varepsilon) > 0$. From now on let $t \geqslant t_{k-1}$. Regarding the β-term we have

$$\begin{aligned} \|\nabla^{k-2}(|u(t)|^2u(t))\|_{L^2_\pi} &\leqslant C\sum_{j,k,\ell=0}^{k-2}\|\nabla^j u(t)\|_{L^6}\|\nabla^i u(t)\|_{L^6}\|\nabla^\ell u(t)\|_{L^6} \\ &\leqslant C\sum_{j,k,\ell=0}^{k-2}\|\nabla^j u(t)\|_{H^1_\pi}\|\nabla^i u(t)\|_{H^1_\pi}\|\nabla^\ell u(t)\|_{H^1_\pi} \end{aligned}$$

$$
\begin{aligned}
&\leqslant C \sum_{j,k,\ell=0}^{k-2} \|u(t)\|_{H_\pi^{j+1}} \|u(t)\|_{H_\pi^{i+1}} \|u(t)\|_{H_\pi^{\ell+1}} \\
&\leqslant C \|u(t)\|_{H_\pi^{k-1}}^3
\end{aligned}
$$

with $C := C(Q_n) > 0$, where we used the Sobolev embedding as $H_\pi^1(Q_n) \hookrightarrow L^6(Q_n)$ (cf. [1] and [41, Theorem A.3]), and therefore

$$
\begin{aligned}
|\beta| |(\nabla^{k-2}(|u(t)|^2 u(t)), \nabla^{k+2} u(t))_{L_\pi^2}| &\leqslant \varepsilon \|\nabla^{k+2} u(t)\|_{L_\pi^2}^2 + C(\varepsilon) \|u(t)\|_{H_\pi^{k-1}}^6 \\
&\leqslant \varepsilon \|\nabla^{k+2} u(t)\|_{L_\pi^2}^2 + C(\varepsilon) R_{k-1}^3.
\end{aligned}
$$

Regarding the λ_0-term we can use similar arguments and obtain

$$
\begin{aligned}
\|\nabla^{k-2}(u(t) \cdot \nabla) u(t)\|_{L_\pi^2} &\leqslant C \sum_{i,j=0}^{k-2} \|\nabla^j u(t)\|_{L^4} \|\nabla^{i+1} u(t)\|_{L^4} \\
&\leqslant C \sum_{i,j=0}^{k-2} \|\nabla^j u(t)\|_{H_\pi^1} \|\nabla^{i+1} u(t)\|_{H_\pi^1} \\
&\leqslant C \sum_{i,j=0}^{k-2} \|u(t)\|_{H_\pi^{j+1}} \|u(t)\|_{H_\pi^{i+2}} \\
&\leqslant C \|u(t)\|_{H_\pi^{k-1}} \|u(t)\|_{H_\pi^k} \\
&\leqslant C \|u(t)\|_{H_\pi^{k-1}} (\|u(t)\|_{L_\pi^2}^2 + \|\nabla^k u(t)\|_{L_\pi^2}^2)^{1/2}
\end{aligned}
$$

with $C := C(Q_n) > 0$ where we used $H_\pi^1(Q_n) \hookrightarrow L^4(Q_n)$ (cf. [1] and [41, Theorem A.3]), and therefore

$$
\begin{aligned}
&|\lambda_0| |(\nabla^{k-2}(u(t) \cdot \nabla) u(t), \nabla^{k+2} u(t))_{L_\pi^2}| \\
&\leqslant \varepsilon \|\nabla^{k+2} u(t)\|_{L_\pi^2}^2 + C(\varepsilon)(\|u(t)\|_{H_\pi^{k-1}}^2 (\|u(t)\|_{L_\pi^2}^2 + \|\nabla^k u(t)\|_{L_\pi^2}^2)) \\
&\leqslant \varepsilon \|\nabla^{k+2} u(t)\|_{L_\pi^2}^2 + C(\varepsilon) R_{k-1} \|\nabla^k u(t)\|_{L_\pi^2}^2 + C(\varepsilon) R_0 R_{k-1}.
\end{aligned}
$$

This yields

$$
\frac{d}{dt} \|\nabla^k u(t)\|_{L_\pi^2}^2 + \frac{\Gamma_2}{2} \|\nabla^{k+2} u(t)\|_{L_\pi^2}^2 \leqslant C(\alpha, \varepsilon, R_{k-1}) \|\nabla^k u(t)\|_{L_\pi^2}^2 + C(\varepsilon, R_0, R_{k-1})
$$

for $t \geqslant t_{k-1}$. By applying the generalized Gronwall lemma (cf. Lemma 2.30) we have

$$
\|\nabla^k u(t)\|_{L_\pi^2}^2 \leqslant C \qquad (t \geqslant t_k)
$$

for some $t_k > t_{k-1}$ and therefore

$$\|u(t)\|^2_{H^k_\pi} \leqslant R_k \qquad (t \geqslant t_k)$$

for a $R_k > 0$ chosen accordingly. We set

$$\mathcal{B}_k := \{u \in H^k_\pi(Q_n) \cap L^2_\sigma(Q_n) \ : \ \|u\|^2_{H^k_\pi} \leqslant R_k\}$$

in order to obtain an absorbing set for (4.1) in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$. □

4.2.3 Existence of a Global Attractor

Since we have proved the existence of absorbing sets of arbitrary high regularity, we can show the existence of global attractors of arbitrary high regularity, too. Moreover, we will see that the attractors of different regularity coincide.

4.12 Remark. If it is known that the semigroup S regularizes such that the solution $u(t) = S(t)u_0$ to some initial value $u_0 \in L^2_\sigma(Q_n)$ is of arbitrary high spatial regularity $u \in H^4_\pi(Q_n) \cap C^\infty(Q_n)$, then it is clear that the global attractor is of arbitrary high regularity due to $S(t)\mathcal{A} = \mathcal{A}$ for $t > 0$. Since this property of the semigroup is not shown in this thesis, we use another approach to prove that the global attractor has arbitrary high regularity.

4.13 Theorem. *Let $k \in \mathbb{N}_0$ be chosen arbitrarily. Then there exists a global attractor $\mathcal{A}_k \subseteq H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for (4.1). Moreover, all attractors of different regularity coincide, i.e. $\mathcal{A}_k = \mathcal{A}_j$ for $j, k \in \mathbb{N}_0$. Consequently, we write $\mathcal{A}$ for the unique global attractor of (4.1).*

Proof. Let $k \in \mathbb{N}_0$. Due to the Rellich embedding theorem (cf. [1] and [41, Corollary A.5]) we know that $H^{k+1}_\pi(Q_n) \cap L^2_\sigma(Q_n) \overset{c}{\hookrightarrow} H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Then the bounded absorbing set $\mathcal{B}_{k+1} \subseteq H^{k+1}_\pi(Q_n) \cap L^2_\sigma(Q_n)$, which exists due to Theorem 4.11, is a relatively compact absorbing set in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Thus, $\overline{\mathcal{B}_{k+1}}^{H^k_\pi}$ is a compact absorbing set. There exists a global attractor $\mathcal{A}_k \subseteq H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$, which is due to Theorem 2.15 and given as $\mathcal{A}_k = \omega(\overline{\mathcal{B}_{k+1}}^{H^k_\pi})$ in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$.
Now let $j, k \in \mathbb{N}_0$ and w.l.o.g. $j < k$. Let $S(t) : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n)$ be the semigroup to (4.1) . Since $\mathcal{A}_j$ is a global attractor, we know that

$$\mathcal{A}_j = S(t_{k+1})\mathcal{A}_j \subseteq \mathcal{B}_{k+1} \subseteq H^{k+1}_\pi(Q_n) \cap L^2_\sigma(Q_n)$$

such that $\mathcal{A}_j$ actually admits H^{k+1}_π-regularity and is a bounded, invariant set in $H^{k+1}_\pi(Q_n) \cap L^2_\sigma(Q_n)$. We obtain that $\overline{\mathcal{A}_j}^{H^k_\pi} \subseteq H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$ is a compact set, using the compact embedding. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_j$ be a sequence such that there exists a $u \in H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$ with $u_n \xrightarrow{n\to\infty} u$ in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Then we have

$$\|u_n - u\|_{H^j_\pi} \leqslant \|u_n - u\|_{H^k_\pi} \xrightarrow{n\to\infty} 0$$

and - since $\mathcal{A}_j$ is closed in $H^j_\pi(Q_n) \cap L^2_\sigma(Q_n)$ - we obtain $u \in \mathcal{A}_j$. Thus $\mathcal{A}_j = \overline{\mathcal{A}_j}^{H^k_\pi}$ and $\mathcal{A}_j$ is a compact invariant set in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Since $\mathcal{A}_k$ is the maximal compact invariant set in that space, we obtain $\mathcal{A}_j \subseteq \mathcal{A}_k$.
In order to retrieve the opposite inclusion, we fix the absorbing set $\mathcal{B}_{k+1}$ and note that $\overline{\mathcal{B}_{k+1}}^{H^k_\pi}$ and $\overline{\mathcal{B}_{k+1}}^{H^j_\pi}$ are compact absorbing sets in $H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)$ and respectively in $H^j_\pi(Q_n) \cap L^2_\sigma(Q_n)$ due to Rellich's compact embedding theorem. Obviously, we have $\overline{\mathcal{B}_{k+1}}^{H^k_\pi} \subseteq \overline{\mathcal{B}_{k+1}}^{H^j_\pi}$. Then (2.8) and Theorem 2.15 yield

$$\mathcal{A}_k = \bigcap_{t\geqslant 0} S(t)\overline{\mathcal{B}_{k+1}}^{H^k_\pi} \subseteq \bigcap_{t\geqslant 0} S(t)\overline{\mathcal{B}_{k+1}}^{H^j_\pi} = \mathcal{A}_j,$$

thus $\mathcal{A}_j = \mathcal{A}_k$ and the assertion is proved. □

4.2.4 Injectivity on the Attractor

Injectivity of the semigroup $S(t)$ on the attractor $\mathcal{A}$ yields some interesting consequences, such that it is worthwhile to investigate it. At first we will prove injectivity in the sense of Theorem 2.16.

4.14 Theorem. *Let $\mathcal{A}$ be the global attractor of (4.1) as in Theorem 4.13. Then the semigroup S is injective on $\mathcal{A}$ in the sense of Theorem 2.16.*

Proof. Let u and v be solutions to (4.1) for initial values $u_0 \in \mathcal{A}$ and $v_0 \in \mathcal{A}$, where $u(t) = S(t)u_0$ and $v(t) = S(t)v_0$ for $t > 0$. We define $w := u - v$ which solves

$$\partial_t w + \Gamma_2\Delta^2 w - \Gamma_0\Delta w + \alpha w + P\beta(|u|^2u - |v|^2v) + P\lambda_0((u\cdot\nabla)u - (v\cdot\nabla)v) = 0. \tag{4.19}$$

We already know that $u, v \in L^2((0,T), H^4_\pi(Q_n) \cap L^2_\sigma(Q_n))$ for $T > 0$, which comes from the arbitrary high regularity of $u_0, v_0 \in \mathcal{A}$ (cf. Theorem 3.6), and that $u, v \in L^\infty((0,T), H^2_\pi(Q_n) \cap L^2_\sigma(Q_n))$. We use Lemma 2.17 with $H = L^2_\sigma(Q_n)$ and $V = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. For this purpose, we note that

$$\mathbb{M} : E_{1/2} = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n) \to E_{-1/2},\quad w \mapsto \Gamma_2\Delta^2 w - \Gamma_0\Delta w + \alpha w$$

is a bounded operator (cf. Section 4.1). Finally, it remains to show the estimate

$\|h(t,w(t))\|_{L^2_\pi} \leqslant K(t)\|w(t)\|_{H^2_\pi}$ with $K \in L^2((0,T))$ and

$$h(t,w(t)) := P\beta(|u(t)|^2u(t) - |v(t)|^2v(t)) + P\lambda_0((u(t)\cdot\nabla)u(t) - (v(t)\cdot\nabla)v(t)).$$

Now, consider $G(u) = |u|^2u$ with Fréchet-derivative $G'(\xi)\lambda = 2(\xi\cdot\lambda)\xi + |\xi|^2\lambda$. We obtain

$$\begin{aligned}\||u|^2u - |v|^2v\|_{L^2_\pi} &= \|G(u)-G(v)\|_{L^2_\pi} \\ &\leqslant C\sup_{\xi\in\mathcal{B}_2}\|G'(\xi)\|_{\mathscr{L}(H^1_\pi,L^2)}\|u-v\|_{H^1_\pi} \\ &\leqslant \sup_{\xi\in\mathcal{B}_2}\sup_{\|\lambda\|_{H^1_\pi}=1}\|2(\xi\cdot\lambda)\xi + |\xi|^2\lambda\|_{L^2_\pi}\|u-v\|_{H^1_\pi} \\ &\leqslant C\|u-v\|_{H^1_\pi},\end{aligned}$$

where $C := C(Q_n,R_2) > 0$ and we used the generalized mean value theorem, Sobolev embeddings as well as $\mathcal{A} \subseteq \overline{\mathcal{B}_2}^{H^1_\pi}$. Moreover, we have

$$\begin{aligned}\|(u\cdot\nabla)u - (v\cdot\nabla)v\|_{L^2_\pi} &= \|(u\cdot\nabla)(u-v) - ((v-u)\cdot\nabla)v\|_{L^2_\pi} \\ &\leqslant C(\|u\|_{L^4}\|\nabla(u-v)\|_{L^4} + \|u-v\|_{L^4}\|\nabla v\|_{L^4}) \\ &\leqslant C(\|u\|_{H^1_\pi}\|u-v\|_{H^2_\pi} + \|v\|_{H^2_\pi}\|u-v\|_{H^1_\pi}) \\ &\leqslant C\|u-v\|_{H^2_\pi},\end{aligned}$$

where $C := C(Q_n,R_2) > 0$ and we used $H^1(Q_n) \hookrightarrow L^4(Q_n)$. This yields the desired estimate on h and we can use Lemma 2.17 to obtain the following:
If $S(t_0)u_0 = S(t_0)v_0$ for some $t_0 > 0$, then $S(t)u_0 = S(t)v_0$ for $0 \leqslant t \leqslant t_0$ and especially $u_0 = v_0$, which means injectivity in the sense of Theorem 2.16. □

4.15 Remark. Note that Theorem 4.14 yields some further consequences regarding the global attractor $\mathcal{A}$.

- $(\mathcal{A}, S(t))_{t\in\mathbb{R}}$ is a dynamical system (cf. Theorem 2.16),
- $\mathcal{A} = \bigcup\{u \text{ is a complete, bounded orbit}\}$ (cf. Theorem 2.19).

As a last step we prove some lemma to estimate w in equation (4.19). We will need this estimate in the next section where we show uniform differentiability of S.

4.16 Lemma. *Let u and v be solutions to (4.1) for initial values $u_0 \in \mathcal{A}$ and $v_0 \in \mathcal{A}$, where $u(t) = S(t)u_0$ and $v(t) = S(t)v_0$ for $t > 0$. Then $w := u - v$ satisfies the estimate*

$$\frac{d}{dt}\|w\|^2_{L^2_\pi} + C_1\|w\|^2_{H^2_\pi} \leqslant C_2\|w\|^2_{L^2_\pi}, \tag{4.20}$$

where $C_1, C_2 > 0$ *denote some constants independent of* u_0 *and* v_0.

Proof. Let $w := u - v$ be given. Then w fulfills equation (4.19). Testing this equation with w yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2_\pi} &= -\Gamma_2\|\Delta w\|^2_{L^2_\pi} - \Gamma_0\|\nabla w\|^2_{L^2_\pi} - \alpha\|w\|^2_{L^2_\pi} - \beta(|u|^2u, w)_{L^2_\pi} + \beta(|v|^2v, w)_{L^2_\pi} \\ &\quad - \lambda_0((u\cdot\nabla)u, w)_{L^2_\pi} + \lambda_0((v\cdot\nabla)v, w)_{L^2_\pi}.\end{aligned}$$

Due to the fact that u and v are divergence free we know that

$$((v\cdot\nabla)v, v)_{L^2_\pi} = ((v\cdot\nabla)u, u)_{L^2_\pi} = 0 \tag{4.21}$$

and

$$((v\cdot\nabla)v, u)_{L^2_\pi} = -((v\cdot\nabla)u, v)_{L^2_\pi}.$$

We can then rewrite the λ_0-terms as

$$\begin{aligned}-((u\cdot\nabla)u, w)_{L^2_\pi} + ((v\cdot\nabla)v, w)_{L^2_\pi} &= -((u\cdot\nabla)u, w)_{L^2_\pi} - ((v\cdot\nabla)u, v)_{L^2_\pi} \\ &= -((w\cdot\nabla)u, w)_{L^2_\pi}.\end{aligned}$$

Using the embedding $H^1_\pi(Q_n) \hookrightarrow L^4(Q_n)$ and $u(t) \in \mathcal{A}$ for $t \geqslant 0$, we obtain the estimate

$$|((w\cdot\nabla)u, w)_{L^2_\pi}| \leqslant C\|w\|_{L^2_\pi}\|w\|_{H^1_\pi}\|u\|_{H^2_\pi} \leqslant C\|w\|_{L^2_\pi}\|w\|_{H^1_\pi}$$

with $C = C(Q_n, R_2) > 0$. We apply (4.15) to that term and to the Γ_0-term and obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2_\pi} \leqslant -\frac{\Gamma_2}{4}\|\Delta w\|^2_{L^2_\pi} + C\|w\|^2_{L^2_\pi} - \beta(|u|^2u, w)_{L^2_\pi} + \beta(|v|^2v, w)_{L^2_\pi}.$$

By the same arguments as in the proof of Theorem 4.14 we have

$$|(|u|^2u - |v|^2v, w)_{L^2_\pi}| \leqslant C\|w\|_{H^1_\pi}\|w\|_{H^2_\pi}$$

with $C = C(\beta, R_1)$ and by an application of (4.15) also

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2_\pi} \leqslant -\frac{\Gamma_2}{8}\|\Delta w\|^2_{L^2_\pi} + C\|w\|^2_{L^2_\pi}, \tag{4.22}$$

which completes the proof. □

4.17 Remark. Note that by an application of the Gronwall lemma (cf. Lemma 2.29) to (4.20) we can also show

$$\|w(t)\|_{L^2_\pi}^2 \leqslant C(t)\|w(0)\|_{L^2_\pi}^2 \tag{4.23}$$

with $C(t) > 0$ monotonically increasing in t.

4.3 Dimensional Bounds for the Global Attractor

In the last section we proved that there exists a global attractor $\mathcal{A}$ for (4.1) that has arbitrary high regularity. In this section we want to show that the long-term dynamics is basically finite dimensional, i.e. the global attractor has finite (Hausdorff or fractal) dimension. To this end we proceed in two steps: At first we show that the semigroup S corresponding to (4.1) is uniformly differentiable in the sense of Definition 2.27. In the second step, we will prove that $\mathcal{TR}_m(\mathcal{A}) < 0$ is fulfilled for some $m \in \mathbb{N}$ in order to apply Theorem 2.28.
Following this scheme, we start with the uniform differentiability.

4.18 Lemma. *The semigroup S corresponding to (4.1) is uniformly differentiable on $\mathcal{A}$ in $L^2_\sigma(Q_n)$ in the sense of Definition 2.27. Moreover, $\Lambda(t, v_0) \in \mathscr{L}(L^2_\sigma(Q_n))$ is compact for $v_0 \in \mathcal{A}$ and $t > 0$.*

Proof. We will proceed in two steps: first, we want to find the operator $\Lambda(t, v_0)$ in order to show uniform differentiability. Then we show compactness of this operator.
Step 1: Uniform differentiability
Let $v_0 \in \mathcal{A}$. By (4.6) and the inverse function theorem it is clear that the derivative $\Lambda(t, v_0)V_0$ is given as the solution of the linearized problem

$$\begin{aligned}\frac{dV}{dt} =& -\Gamma_2\Delta^2 V + \Gamma_0\Delta V - \alpha V - P\lambda_0((v\cdot\nabla)V \\ &+ (V\cdot\nabla)v) - P\beta((2(v\cdot V)v) + |v|^2 V)\end{aligned} \tag{4.24}$$

with $V(0) = V_0 \in L^2_\sigma(Q_n)$, where v is the solution of (4.1) with initial value v_0. This equation is locally and globally well-posed due to (4.6).
Let v_1, v_2 be two solutions of (4.1) with initial values $v_0^1, v_0^2 \in \mathcal{A} \subseteq H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$. Let V be a solution of (4.24) with v_1 and initial value $V(0) = v_0^2 - v_0^1$. The error due to the linearization is given as $\theta := v_2 - v_1 - V$ and fulfills the equation

$$\begin{aligned}\frac{d\theta}{dt} =& -\Gamma_2\Delta^2\theta + \Gamma_0\Delta\theta - \alpha\theta - P\lambda_0((v_1\cdot\nabla)\theta + (\theta\cdot\nabla)v_1 \\ &+ ((v_1 - v_2)\cdot\nabla)(v_1 - v_2)) - P\beta(|v_2|^2 v_2 \\ &- |v_1|^2 v_1 - 2(v_1\cdot V)v_1 - |v_1|^2 V).\end{aligned} \tag{4.25}$$

By calculations on the β-term we obtain

$$\begin{aligned}
&|v_2|^2 v_2 - |v_1|^2 v_1 - 2(v_1 \cdot V)v_1 - |v_1|^2 V \\
=& |v_2|^2 v_2 - |v_1|^2 v_1 - 2(v_1 \cdot (v_2 - v_1 - \theta))v_1 - |v_1|^2 (v_2 - v_1 - \theta) \\
=& |v_2|^2 v_2 - |v_1|^2 v_1 - |v_1|^2 (v_2 - v_1) - 2(v_1 \cdot (v_2 - v_1))v_1 + 2(v_1 \cdot \theta)v_1 + |v_1|^2 \theta \\
=& g(v_1, v_2) + 2(v_1 \cdot \theta)v_1 + |v_1|^2 \theta,
\end{aligned}$$

where

$$g(v_1, v_2) := |v_2|^2 v_2 - |v_1|^2 v_1 - |v_1|^2 (v_2 - v_1) - 2(v_1 \cdot (v_2 - v_1))v_1.$$

We want to apply Taylor's formula in order to estimate g. To this end we write g with $G : H^1_\pi(Q_n) \cap L^2_\sigma(Q_n) \to L^2_\sigma(Q_n)$, $G(x) := |x|^2 x$ as

$$g(v_1, v_2) = G(v_2) - G(v_1) - G'(v_1)(v_2 - v_1),$$

where G' denotes the Fréchet-derivative, which is given as

$$G'(u)v = 2(u \cdot v)u + |u|^2 u \quad (u, v \in H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)).$$

The Gâteau-derivative of G' for $u, v, w \in H^1_\pi(Q_n) \cap L^2_\pi(Q_n)$ is easily calculated as

$$G''(u)[v, w] = 2(u \cdot v)w + 2(w \cdot v)u + 2(u \cdot w)v.$$

We apply a Taylor expansion on G in a ball $\mathscr{B} \subseteq H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ such that $\mathcal{A} \subseteq \mathscr{B}$ and estimate the remainder:

$$\begin{aligned}
\|g(v_1, v_2)\|_{L^2_\pi} &\leqslant C \sup_{0 \leqslant t \leqslant 1} \|G''(v_2 + t(v_1 - v_2))\|_{\mathscr{L}(H^1_\pi \times H^1_\pi, L^2_\pi)} \|v_2 - v_1\|^2_{H^1_\pi} \\
&\leqslant C \sup_{u \in \mathscr{B}} \sup_{\|(v,w)\|_{H^1_\pi \times H^1_\pi} = 1} \|G''(u)[v, w]\|_{L^2_\pi} \|v_2 - v_1\|^2_{H^1_\pi} \\
&\leqslant C \sup_{u \in \mathscr{B}} \sup_{\|(v,w)\|_{H^1_\pi \times H^1_\pi} = 1} \|u\|_{H^1_\pi} \|v\|_{H^1_\pi} \|w\|_{H^1_\pi} \|v_2 - v_1\|^2_{H^1_\pi} \\
&\leqslant C \|v_2 - v_1\|^2_{H^1_\pi}.
\end{aligned}$$

Here, we used $H^1_\pi(Q_n) \hookrightarrow L^6(Q_n)$ and $C := C(Q_n, R_1) > 0$ denotes a constant.

By testing (4.25) with θ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2_\pi}^2 = &-\Gamma_2\|\Delta\theta\|_{L^2_\pi}^2 - \Gamma_0\|\nabla\theta\|_{L^2_\pi}^2 - \alpha\|\theta\|_{L^2_\pi}^2 - \lambda_0((\theta\cdot\nabla)v_1,\theta)_{L^2_\pi}\\ &- \lambda_0(((v_2-v_1)\cdot\nabla)(v_2-v_1),\theta)_{L^2_\pi} - \beta(g(v_1,v_2),\theta)_{L^2_\pi}\\ &- 2\beta((v_1\cdot\theta)v_1,\theta)_{L^2_\pi} - \beta(|v_1|^2,|\theta|^2)_{L^2_\pi}.\end{aligned} \tag{4.26}$$

First, we note that

$$\beta(|v_1|^2,|\theta|^2)_{L^2_\pi} \geqslant 0$$

and

$$\beta((v_1\cdot\theta)v_1,\theta)_{L^2_\pi} = L^{-n}\int_{Q_n}|v_1(x)\cdot\theta(x)|^2\,dx \geqslant 0.$$

Next, we apply the estimates

$$|((\theta\cdot\nabla)v_1,\theta)_{L^2_\pi}| \leqslant C\|\theta\|_{L^4}\|\nabla v_1\|_{L^4}\|\theta\|_{L^2_\pi} \leqslant C\|\theta\|_{H^1_\pi}\|v_1\|_{H^2_\pi}\|\theta\|_{L^2_\pi} \tag{4.27}$$

and

$$\begin{aligned}|(((v_2-v_1)\cdot\nabla)(v_2-v_1),\theta)_{L^2_\pi}| &\leqslant C\|v_2-v_1\|_{L^2_\pi}\|\nabla(v_2-v_1)\|_{L^4}\|\theta\|_{L^4}\\ &\leqslant C\|v_2-v_1\|_{L^2_\pi}\|v_2-v_1\|_{H^2_\pi}\|\theta\|_{H^1_\pi},\end{aligned}$$

where $C := C(Q_n) > 0$ and we used the embedding $H^1_\pi(Q_n)\hookrightarrow L^4(Q_n)$. Furthermore, we have

$$\begin{aligned}|(g(v_1,v_2),\theta)_{L^2_\pi}| &\leqslant \frac{1}{2}\|g(v_2,v_2)\|_{L^2_\pi}^2 + \frac{1}{2}\|\theta\|_{L^2_\pi}^2\\ &\leqslant C\|v_2-v_1\|_{H^1_\pi}^4 + \frac{1}{2}\|\theta\|_{L^2_\pi}^2\\ &\leqslant C(\|v_2-v_1\|_{L^2_\pi}^4 + \|v_2-v_1\|_{H^2_\pi}^2\|v_2-v_1\|_{L^2_\pi}^2 + \|\theta\|_{L^2_\pi}^2),\end{aligned}$$

where $C := C(Q_n, R_1) > 0$ and we used $\|\nabla(v_2-v_1)\|_{L^2_\pi}^4 \leqslant C\|v_2-v_1\|_{H^2_\pi}^2\|v_2-v_1\|_{L^2_\pi}^2$ by (4.15) in the last step. Plugging these estimates into (4.26) and applying (4.15) onto the Γ_0-term yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2_\pi}^2 \leqslant &-\frac{\Gamma_2}{2}\|\Delta\theta\|_{L^2_\pi}^2 + C\Big(\|\theta\|_{L^2_\pi}^2 + \|\theta\|_{H^1_\pi}\|v_1\|_{H^2_\pi}\|\theta\|_{L^2_\pi}\\ &+\|v_2-v_1\|_{L^2_\pi}\|v_2-v_1\|_{H^2_\pi}\|\theta\|_{H^1_\pi} + \|v_2-v_1\|_{L^2_\pi}^4\\ &+\|v_2-v_1\|_{H^2_\pi}^2\|v_2-v_1\|_{L^2_\pi}^2 + \|\theta\|_{L^2_\pi}^2\Big)\end{aligned}$$

with $C := C(\Gamma_2, \Gamma_0, \alpha, \beta, \lambda_0, Q_n, R_1) > 0$. We apply (4.15) to the $\|\theta\|_{H^1_\pi}$-terms and obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2_\pi} \leqslant -\frac{\Gamma_2}{4}\|\Delta\theta\|^2_{L^2_\pi} + C\left(\|\theta\|^2_{L^2_\pi} + \|v_2 - v_1\|^4_{L^2_\pi} + \|v_1 - v_2\|^2_{L^2_\pi}\|v_1 - v_2\|^2_{H^2_\pi}\right),$$

where again $C := C(\Gamma_2, \Gamma_0, \alpha, \beta, \lambda_0, Q_n, R_1) > 0$. By dropping the Γ_2-term and using the embeddings $H^2_\pi(Q_n) \hookrightarrow H^1_\pi(Q_n) \hookrightarrow L^2_\pi(Q_n)$ we finally have

$$\frac{d}{dt}\|\theta\|^2_{L^2_\pi} \leqslant C\left(\|\theta\|^2_{L^2_\pi} + \|v_1 - v_2\|^2_{L^2_\pi}\|v_1 - v_2\|^2_{H^2_\pi}\right)$$

and can apply the standard Gronwall inequality (cf. Lemma 2.29, note that $\theta(0) = 0$) to obtain

$$\|\theta(t)\|^2_{L^2_\pi} \leqslant C(t)\int_0^t \|v_1(s) - v_2(s)\|^2_{L^2_\pi}\|v_1(s) - v_2(s)\|^2_{H^2_\pi}\, ds$$

with $C(t) > 0$. Testing (4.22) with $\|v_2 - v_1\|^2_{L^2_\pi}$ (note that $w = v_2 - v_1$), integrating w.r.t. t and using (4.23) yields

$$\begin{aligned}\int_0^t \|v_1(s) - v_2(s)\|^2_{L^2_\pi}\|v_1(s) - v_2(s)\|^2_{H^2_\pi}\, ds &\leqslant C\left(\|v_2^0 - v_1^0\|^4_{L^2_\pi} + \int_0^t \|v_2(s) - v_1(s)\|^4_{L^2_\pi}\, ds\right)\\ &\leqslant C(t)\|v_2^0 - v_1^0\|^4_{L^2_\pi}\end{aligned}$$

such that we have

$$\|\theta(t)\|_{L^2_\pi} \leqslant C(t)\|v_2^0 - v_1^0\|^2_{L^2_\pi}.$$

For a fixed $t > 0$ we finally obtain

$$\frac{\|v_2(t) - v_1(t) - V(t)\|_{L^2_\pi}}{\|v_2^0 - v_1^0\|_{L^2_\pi}} = \frac{\|\theta(t)\|_{L^2_\pi}}{\|v_2^0 - v_1^0\|_{L^2_\pi}} \leqslant C(t)\|v_2^0 - v_1^0\|_{L^2_\pi} \longrightarrow 0$$

as $v_0^2 \to v_0^1$ in $L^2_\pi(Q_n)$. Hence, we obtain uniform differentiability of S on $\mathcal{A}$ with $V(t) = \Lambda(t, v_0^1)V(0) = \Lambda(t, v_0^1)(v_0^2 - v_0^1)$.

Step 2: Compactness of $\Lambda(t, v_0)$

Next we want to show that $\Lambda(t, v_0)$ is a compact operator for $t > 0$ and $v_0 \in \mathcal{A}$. In order to apply Rellich's embedding theorem we first prove the existence of L^2_π and H^1_π bounds for $\Lambda(t, v_0)V_0$. Let v be a solution of (4.1) for $v_0 \in \mathcal{A}$. Testing (4.24) with its solution V yields

$$\frac{1}{2}\frac{d}{dt}\|V\|^2_{L^2_\pi} = -\Gamma_2\|\Delta V\|^2_{L^2_\pi} - \Gamma_0\|\nabla V\|^2_{L^2_\pi} - \alpha\|V\|^2_{L^2_\pi} - \lambda_0((v\cdot\nabla)V, V)_{L^2_\pi}$$

$$- \lambda_0((V \cdot \nabla)v, V)_{L^2_\pi} - 2\beta((v \cdot V)v, V)_{L^2_\pi} - \beta(|v|^2 V, V)_{L^2_\pi}.$$

With similar arguments as before we deduce

$$\frac{1}{2}\frac{d}{dt}\|V\|^2_{L^2_\pi} \leqslant -\frac{\Gamma_2}{4}\|\Delta V\|^2_{L^2_\pi} + C\|V\|^2_{L^2_\pi} \tag{4.28}$$

with $C = C(\Gamma_2, \Gamma_0, \alpha, \lambda_0, R_2) > 0$. Then we may apply the standard Gronwall lemma (cf. Lemma 2.29) to obtain L^2_π-bounds for $t > 0$:

$$\|V(t)\|^2_{L^2_\pi} \leqslant C(t)\|V(0)\|^2_{L^2_\pi}, \tag{4.29}$$

where $C(t) > 0$ is monotonically increasing in t. This inequality gives us a bound on the operator norm of $\Lambda(t, v_0)$ that does not depend on v_0. To obtain bounds for higher regularity, we test (4.24) with $-\Delta V$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla V\|^2_{L^2_\pi} =& - \Gamma_2\|\Delta\nabla V\|^2_{L^2_\pi} - \Gamma_0\|\Delta V\|^2_{L^2_\pi} - \alpha\|\nabla V\|^2_{L^2_\pi} + \lambda_0((v \cdot \nabla)V, \Delta V)_{L^2_\pi} \\ &+ \lambda_0((V \cdot \nabla)V, \Delta V)_{L^2_\pi} + 2\beta((v \cdot V)v, \Delta V)_{L^2_\pi} + \beta(|v|^2 V, \Delta V)_{L^2_\pi}.\end{aligned}$$

By the usual use of (4.15) and $H^1_\pi(Q_n) \hookrightarrow L^6(Q_n), L^4(Q_n)$ we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\nabla V\|^2_{L^2_\pi} \leqslant -\frac{\Gamma_2}{2}\|\Delta\nabla V\|^2_{L^2_\pi} + C(\|\nabla V\|^2_{L^2_\pi} + \|V\|^2_{L^2_\pi}), \tag{4.30}$$

where $C > 0$ does not depend on t, V and v_0. We integrate (4.28) from $t/2$ to t, neglect the $\|V(t)\|^2_{L^2_\pi}$ on the left-hand side and add $\int_{t/2}^t \|V(s)\|^2_{L^2_\pi}\,ds$ on both sides to obtain

$$\int_{t/2}^t \|V(s)\|^2_{H^2_\pi}\,ds \leqslant C\left(\int_{t/2}^t \|V(s)\|^2_{L^2_\pi}\,ds + \|V(t/2)\|^2_{L^2_\pi}\right).$$

We use this inequality in combination with (4.29), which yields

$$\int_{t/2}^t \|\nabla V(s)\|^2_{L^2_\pi}\,ds \leqslant C\int_{t/2}^t \|V(s)\|^2_{H^2_\pi}\,ds \leqslant C(t)\|V(0)\|^2_{L^2_\pi}. \tag{4.31}$$

Next, dropping the $\|\Delta\nabla V\|^2_{L^2_\pi}$-term in (4.30) and integrating this inequality from s to t with $t/2 \leqslant s \leqslant t$ gives us

$$\begin{aligned}\|\nabla V(t)\|^2_{L^2_\pi} &\leqslant \|\nabla V(s)\|^2_{L^2_\pi} + C\left(\int_s^t \|\nabla V(r)\|^2_{L^2_\pi}\,dr + \int_s^t \|V(r)\|^2_{L^2_\pi}\,dr\right) \\ &\leqslant \|\nabla V(s)\|^2_{L^2_\pi} + C\left(\int_{t/2}^t \|\nabla V(r)\|^2_{L^2_\pi}\,dr + \int_{t/2}^t \|V(r)\|^2_{L^2_\pi}\,dr\right).\end{aligned}$$

By integrating this inequality from $t/2$ to t w.r.t. s we have

$$\frac{t}{2}\|\nabla V(t)\|_{L^2_\pi}^2 \leqslant \int_{t/2}^t \|\nabla V(s)\|_{L^2_\pi}^2\, ds + \frac{t}{2}C\left(\int_{t/2}^t \|\nabla V(r)\|_{L^2_\pi}^2\, dr + \int_{t/2}^t \|V(r)\|_{L^2_\pi}^2\, dr\right)$$

and by applying (4.29) and (4.31) we arrive at

$$\|\nabla V(t)\|_{L^2_\pi}^2 \leqslant C(t)\|V(0)\|_{L^2_\pi}^2.$$

Thus $\Lambda(t, v_0) : L^2_\sigma(Q_n) \to L^2_\sigma(Q_n)$ is bounded as

$$\|\Lambda(t, v_0)V(0)\|_{H^1_\pi}^2 = \|V(t)\|_{H^1_\pi}^2 \leqslant C(t)\|V(0)\|_{L^2_\pi}^2$$

and therefore a compact operator for $t > 0$ and $v_0 \in \mathcal{A}$: Let $\mathcal{M} \subseteq L^2_\sigma(Q_n)$ be a bounded set. For $V(0) \in \mathcal{M}$ we obtain the boundedness of $\Lambda(t, v_0)V(0)$ in $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for $t > 0$, where the bound may depend on t. Hence, the image $\Lambda(t, v_0)\mathcal{M}$ is bounded in $H^1_\pi(Q_n) \cap L^2_\sigma(Q_n)$ and therefore relatively compact in $L^2_\sigma(Q_n)$. □

Next, we prove a bound on $\mathcal{TR}_m(\mathcal{A})$.

4.19 Lemma. *Let $\mathcal{A}$ be the global attractor of (4.1). Then there exists a $m_0 \in \mathbb{N}$ such that we have $\mathcal{TR}_m(\mathcal{A}) < 0$ for $m_0 \leqslant m \in \mathbb{N}$, where $\mathcal{TR}_m$ is defined as in (2.11).*

Proof. Fix $m \in \mathbb{N}$ and choose $\{\xi_j^0 : j = 1, ..., m\} \subseteq L^2_\sigma(Q_n)$ arbitrary but linearly independent. Let v be a solution of (4.1) with initial value $v_0 \in \mathcal{A}$. Moreover, let $L(t, v_0)$ be the linear operator of equation (4.24) (which is equation (4.1) linearized about v) and $\Lambda(t, v_0)$ be the corresponding solution operator as in the proof of Lemma 4.18.
Then $\xi_j(t) := \Lambda(t, v_0)\xi_j^0$ is a solution to the linearized problem with initial value ξ_j^0. Let $P^m_{\xi_1^0,...,\xi_m^0}(t)$ be the projection onto the subspace spanned by $\{\xi_j(t) : j = 1, ..., m\}$. For a fixed $t > 0$ we may obtain an orthonormal base $\{\varphi_j(t) : j = 1, ..., m\}$ of $P^m_{\xi_1^0,...,\xi_m^0}(t)(L^2_\sigma(Q_n))$ w.r.t. the L^2_π scalar product. Due to $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n) \overset{d}{\hookrightarrow} L^2_\sigma(Q_n)$ we may choose $\{\varphi_j(t) : j = 1, ..., m\} \subseteq H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$.
Testing $L(t, v_0)\varphi_j(t)$ with $\varphi_j(t)$ w.r.t the L^2_π-scalar product yields

$$\begin{aligned}(L(t, v_0)\varphi_j, \varphi_j)_{L^2_\pi} =& -\Gamma_2\|\Delta\varphi_j\|_{L^2_\pi}^2 - \Gamma_0\|\nabla\varphi_j\|_{L^2_\pi}^2 - \alpha\|\varphi_j\|_{L^2_\pi}^2 - \lambda_0((\varphi_j \cdot \nabla)v, \varphi_j)_{L^2_\pi}\\ &- \lambda_0((v \cdot \nabla)\varphi_j, \varphi_j)_{L^2_\pi} - 2\beta((v \cdot \varphi_j)v, \varphi_j)_{L^2_\pi} - \beta(|v|^2\varphi_j, \varphi_j)_{L^2_\pi}\\ \leqslant& -\Gamma_2\|\Delta\varphi_j\|_{L^2_\pi}^2 - \Gamma_0\|\nabla\varphi_j\|_{L^2_\pi}^2 - \alpha + C|\lambda_0|\|v\|_{H^2_\pi}\|\varphi_j\|_{H^1_\pi},\end{aligned}$$

where we used (4.21), (4.27), $\|\varphi_j\|_{L^2_\pi} = 1$ and the fact that

$$2\beta((v \cdot \varphi_j)v, \varphi_j)_{L^2_\pi} + \beta(|v|^2\varphi_j, \varphi_j)_{L^2_\pi} \geqslant 0.$$

Note that we omit t in the following and that all appearing constants are independent of m and t. Applying (4.15) and $v \in \mathcal{A}$ then yields

$$(L(t, v_0)\varphi_j, \varphi_j)_{L^2_\pi} \leqslant -\frac{\Gamma_2}{4}\|\Delta\varphi_j\|^2_{L^2_\pi} + K,$$

where $K := K(\Gamma_2, \Gamma_0, \lambda_0, \alpha, R_2, Q_n) > 0$. By summing up for $j = 1, ..., m$ and time averaging we obtain

$$\left\langle \mathrm{Tr}\Big(L(t, v_0)P^m_{\xi^0_1,\dots,\xi^0_m}(t)\Big)\right\rangle \leqslant -\sum_{j=1}^{m} \frac{\Gamma_2}{4}\left\langle \|\Delta\varphi_j\|^2_{L^2_\pi}\right\rangle + mK. \tag{4.32}$$

Next, we want to apply the Sobolev-Lieb-Thirring inequality (cf. Proposition 2.3) in order to estimate $\|\Delta\varphi_j\|^2_{L^2_\pi}$ accordingly. To this end, we define

$$\rho(x) := \sum_{j=1}^{m} |\varphi_j(x)|^2 \quad (x \in Q_n). \tag{4.33}$$

First, let $n = 2$. Then the Sobolev-Lieb-Thirring inequality applied with $p = 3/2$ yields

$$\begin{aligned}
\left(\int_{Q_n} \rho(x)^{p/(p-1)}\, dx\right)^{2\cdot 2\cdot (p-1)/n} &= \int_{Q_n} \rho(x)^3\, dx \\
&\leqslant C\left(\sum_{j=1}^{m} \int_{Q_2} \sum_{|\alpha|=2} |\partial^\alpha \varphi_j(x)|^2\, dx + \int_{Q_2} \rho(x)\, dx\right) \\
&\leqslant C\sum_{j=1}^{m} \|\varphi_j\|^2_{H^2_\pi} \\
&\leqslant C\left(m + \sum_{j=1}^{m} \|\Delta\varphi_j\|^2_{L^2_\pi}\right)
\end{aligned}$$

with $C > 0$ independent of m. Applying the Hölder inequality gives us

$$\begin{aligned}
m^3 = \left(\sum_{j=1}^{m} \|\varphi_j\|^2_{L^2_\pi}\right)^3 &= \left(\frac{1}{L^2}\int_{Q_2} \rho(x)\, dx\right)^3 \\
&\leqslant C\left(\left(\int_{Q_2} 1^{3/2}\, dx\right)^{2/3} \left(\int_{Q_2} \rho(x)^3\right)^{1/3}\right)^3 \\
&\leqslant C\int_{Q_2} \rho(x)^3\, dx
\end{aligned}$$

$$\leqslant C\left(m + \sum_{j=1}^{m} \|\Delta\varphi_j\|_{L^2_\pi}^2\right).$$

Inserting this estimate into (4.32) yields

$$\begin{aligned}\left\langle \operatorname{Tr}\Big(L(t,v_0)P^m_{\xi^0_1,\dots,\xi^0_m}(t)\Big)\right\rangle &\leqslant -\frac{\Gamma_2}{4}\left(\frac{m^3}{C} - m\right) + mK \\ &= -C_1 m^3 + C_2 m,\end{aligned}$$

where $C_1 > 0$ and $C_2 \in \mathbb{R}$ are constants independent of m and t. Then there exists a $m_0 \in \mathbb{N}$ such that for all $m_0 \leqslant m \in \mathbb{N}$ we have

$$\left\langle \operatorname{Tr}\Big(L(t,v_0)P^m_{\xi^0_1,\dots,\xi^0_m}(t)\Big)\right\rangle < 0.$$

A similar result can be achieved for $n = 3$ if we choose $p = 7/4$ in the calculations above, resulting in a leading term $-C_1 m^{7/3}$.
Altogether we have

$$\mathcal{TR}_m(\mathcal{A}) = \sup_{v_0\in\mathcal{A}} \sup_{\substack{\xi^0_j \in L^2_\sigma(Q_n)\\ \|\xi^0_j\|_{L^2_\pi}\leqslant 1\\ j=1,\dots,m}} \left\langle \operatorname{Tr}\Big(L(t,v_0)P^m_{\xi^0_1,\dots,\xi^0_m}(t)\Big)\right\rangle < 0$$

for $m > m_0$. □

Now, an application of Theorem 2.28 gives the following main result of this section. It guarantees that the global attractor $\mathcal{A}$ of (4.1) is of finite dimension.

4.20 Theorem. *Let $\mathcal{A}$ be the global attractor of (4.1). Then there exists a $m \in \mathbb{N}$ such that $d_H(\mathcal{A}) \leqslant m$ and $d_f(\mathcal{A}) \leqslant m$, where d_H and d_f denote the Hausdorff and the fractal dimension.*

Proof. Follows directly from Lemma 4.18 and Lemma 4.19 by an application of Theorem 2.28. □

4.4 Inertial Manifold in 2D

For $n = 2$ dimensions, we can even prove more than the existence of a global, finite dimensional attractor for (4.1). It turns out that there exists a so-called *inertial manifold* $\mathcal{M}$ with the following properties (cf. [47, Chapter 8]).

(i) $\mathcal{M}$ is a finite dimensional, Lipschitz continuous manifold in $H^{3/2}_\pi(Q_2) \cap L^2_\sigma(Q_2)$;

(ii) $\mathcal{M}$ is positively invariant under the semigroup S corresponding to (4.1);

(iii) $\mathcal{M}$ is exponentially attracting, i.e. there exists a $\delta > 0$ such that for every initial value $u_0 \in L^2_\sigma(Q_2)$ there is a $C = C(u_0)$ with

$$\mathrm{dist}_{L^2_\pi}(S(t)u_0, \mathcal{M}) \leqslant C \exp(-\delta t) \quad (t \geqslant 0).$$

It is clear that possessing an inertial manifold is a stronger property than possessing a (finite dimensional) global attractor. In fact, the global attractor lies within the inertial manifold, that is $\mathcal{A} \subseteq \mathcal{M}$. For more information about inertial manifolds and their derivation we refer to [47, Chapter 8] and [15]. We are able to prove the following result.

4.21 Theorem. *There exists an inertial manifold $\mathcal{M} \subseteq H^{3/2}_\pi(Q_2) \cap L^2_\sigma(Q_2)$ for equation (4.1) in $n = 2$ dimensions.*

Proof. We want to apply the theory from [47, Chapter 8], especially Theorem 81.1 and Theorem 81.2. To this end, we define

$$\begin{gathered} A_\omega : D(A_\omega) = H^4_\pi(Q_2) \cap L^2_\sigma(Q_2) \subseteq L^2_\sigma(Q_2) \to L^2_\sigma(Q_2), \\ u \mapsto A_\omega u := \Gamma_2 \Delta^2 u - \Gamma_0 \Delta u + \alpha u + \omega u \end{gathered}$$

and choose $\omega > 0$ such that A_ω is a positive, self-adjoint linear operator with compact resolvent (cf. Chapter 3). Moreover, in the setting of [47, Chapter 8] we choose $\beta = 3/8$. Then clearly $D((A_\omega)^{3/8}) = H^{3/2}_\pi(Q_2) \cap L^2_\sigma(Q_2)$. Moreover, we define

$$F_\omega : H^{3/2}_\pi(Q_2) \cap L^2_\sigma(Q_2) \to L^2_\sigma(Q_2), u \mapsto F_\omega(u) := -P\beta|u|^2 u - P\lambda_0(u \cdot \nabla)u + \omega u$$

such that (4.1) is equivalent to

$$\partial_t u + A_\omega u = F_\omega(u).$$

It is easy to see that

$$F_\omega \in C_{\text{Lip, loc}}(H^{3/2}_\pi(Q_2) \cap L^2_\sigma(Q_2), L^2_\sigma(Q_2))$$

by using

$$\begin{aligned} \||u|^2 v\|_{L^2_\pi} &\leqslant C\|u\|^2_{L^6}\|v\|_{L^6} \leqslant C\|u\|^2_{H^{3/2}_\pi}\|v\|_{H^{3/2}_\pi}, \\ \|(u \cdot \nabla)v\|_{L^2_\pi} &\leqslant C\|u\|_{L^6}\|\nabla v\|_{L^3} \leqslant C\|u\|_{H^{3/2}_\pi}\|v\|_{H^{3/2}_\pi} \end{aligned}$$

and the embedding $H^{3/2}_{\pi}(Q_2) \hookrightarrow W^{1,3}(Q_2)$.
Moreover, it is well known that the ordered eigenvalues $0 < \tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant ...$ of A_ω behave as $\tilde{\lambda}_k \sim Ck^2 = Ck^{4/n}$ (cf. [47, Table 8.1]). Therefore, they fulfill the spectral gap condition in [47, Theorem 81.1]. This completes the assertion. □

4.22 Remark. Note that it is not possible to satisfy the spectral gap condition in [47, Theorem 81.1] for $n = 3$. This is due to the fact that we need $\beta \geqslant 1/4$ in order to estimate the nonlinearity F accordingly, but at the same time we need $\beta < 1/4$ in order to fulfill the gap condition, cf. [47, Table 8.1].

5 Stability for a Class of Heterogeneous Catalysis Models

In this chapter we analyze the model of a heterogeneous catalysis process, which is given as in (1.3), in terms of stability and instability. First, we define the solution and data spaces and give a result concerning maximal L^p-regularity for the linearized equations. Then, we show stability of isolated equilibria using the principle of linearized stability (cf. Remark 2.7(ii)) and give an example for equilibria. In the last section we present a result on instability.

5.1 Maximal Regularity for the Linearized Equations

In order to apply the principle of linearized stability to (1.3), we first need to obtain results on maximal L^p-regularity for a linearized version of (1.3). The results presented here are due to [7] and will only be cited without giving a proof.
Let $T \in (0, \infty)$. First, we need to define the corresponding solution spaces

$$\begin{aligned}
\mathbb{E}_p^{\Omega}(T) &:= W^{1,p}((0,T), L^p(\Omega)) \cap L^p((0,T), W^{2,p}(\Omega)),\\
\mathbb{E}_p^{\Sigma}(T) &:= W^{1,p}((0,T), L^p(\Sigma)) \cap L^p((0,T), W^{2,p}(\Sigma)),
\end{aligned}$$

and the data spaces, given by appropriate trace theorems.

$$\begin{aligned}
\mathbb{F}_p^{\Omega}(T) &:= L^p((0,T) \times \Omega),\\
\mathbb{F}_p^{\Sigma}(T) &:= L^p((0,T) \times \Sigma),\\
\mathbb{G}_p^{\mathrm{in}}(T) &:= W_p^{1/2-1/2p}((0,T), L^p(\Gamma_{\mathrm{in}})) \cap L^p((0,T), W_p^{1-1/p}(\Gamma_{\mathrm{in}})),\\
\mathbb{G}_p^{\Sigma}(T) &:= W_p^{1/2-1/2p}((0,T), L^p(\Sigma)) \cap L^p((0,T), W_p^{1-1/p}(\Sigma)),\\
\mathbb{G}_p^{\mathrm{out}}(T) &:= W_p^{1/2-1/2p}((0,T), L^p(\Gamma_{\mathrm{out}})) \cap L^p((0,T), W_p^{1-1/p}(\Gamma_{\mathrm{out}})),\\
\mathbb{I}_p(\Omega) &:= W_p^{2-2/p}(\Omega),\\
\mathbb{I}_p(\Sigma) &:= W_p^{2-2/p}(\Sigma).
\end{aligned}$$

Now we define the tupel data space for (1.3) without initial data as

$$\mathbb{F}_p^{\Omega,\Sigma}(T) := \mathbb{F}_p^{\Omega}(T) \times \mathbb{F}_p^{\Sigma}(T) \times \mathbb{G}_p^{\text{in}}(T) \times \mathbb{G}_p^{\Sigma}(T) \times \mathbb{G}_p^{\text{out}} \times \{0\}$$

and the corresponding space with initial data as

$$\mathbb{F}_{p,I}^{\Omega,\Sigma}(T) := \mathbb{F}_p^{\Omega,\Sigma}(T) \times \mathbb{I}_p(\Omega) \times \mathbb{I}_p(\Sigma).$$

Moreover, we set

$$\mathbb{E}_p^N := W^{2,p}(\Omega)^N \times W^{2,p}(\Sigma)^N$$

and

$$\mathbb{I}_p^N := \mathbb{I}_p(\Omega)^N \times \mathbb{I}_p(\Sigma)^N.$$

Additionally, we impose the following assumptions regarding the prescribed velocity field u.

- **($\mathbf{A}^{\text{vel}}$)** Let u denote a given velocity field of regularity

  $$u \in \mathbb{U}_p^{\Omega}(T) := W^{1,p}((0,T), L^p(\Omega,\mathbb{R}^3)) \cap L^p((0,T), W^{2,p}(\Omega,\mathbb{R}^3))$$

  fulfilling

  $$u \cdot \nu \leqslant 0 \text{ on } \Gamma_{\text{in}}, \quad u \cdot \nu = 0 \text{ on } \Sigma \;\text{ and }\; u \cdot \nu \geqslant 0 \text{ on } \Gamma_{\text{out}},$$

  and $\nabla \cdot u = 0$ in the distributional sense.

Then we can obtain maximal L^p-regularity for the linearized equations

$$\begin{aligned}
\partial_t c_i + (u\cdot\nabla)c_i - d_i\Delta c_i &= f_i && \text{in} \quad (0,T)\times\Omega,\\
\partial_t c_i^{\Sigma} - d_i^{\Sigma}\Delta_{\Sigma} c_i^{\Sigma} &= f_i^{\Sigma} && \text{on} \quad (0,T)\times\Sigma,\\
(u\cdot\nu)c_i - d_i\partial_\nu c_i &= g_i^{\text{in}} && \text{on} \quad (0,T)\times\Gamma_{\text{in}},\\
-d_i\partial_\nu c_i &= g_i^{\Sigma} && \text{on} \quad (0,T)\times\Sigma,\\
-d_i\partial_\nu c_i &= g_i^{\text{out}} && \text{on} \quad (0,T)\times\Gamma_{\text{out}},\\
-d_i^{\Sigma}\partial_{\nu_\Sigma} c_i^{\Sigma} &= 0 && \text{on} \quad (0,T)\times\partial\Sigma,\\
c_i|_{t=0} &= c_{i,0} && \text{in} \quad \Omega,\\
c_i^{\Sigma}|_{t=0} &= c_{i,0}^{\Sigma} && \text{on} \quad \Sigma.
\end{aligned} \tag{5.1}$$

5.1 Theorem (Bothe, Köhne, Maier and Saal)**.** *Let $T > 0$ be finite and $5/3 < p < \infty$ with $p \neq 3$. Suppose the velocity field u satisfies* (A^{vel})*. Then (1.3) admits a unique* *solution*

$$(c_i, c_i^\Sigma) \in \mathbb{E}_p^\Omega(T) \times \mathbb{E}_p^\Sigma(T),$$

if and only if the data satisfy the regularity condition

$$(f_i, f_i^\Sigma, g_i^{\mathrm{in}}, g_i^\Sigma, g_i^{out}, 0, c_{i,0}, c_{i,0}^\Sigma) \in \mathbb{F}_{p,I}^{\Omega,\Sigma}(T)$$

and in case of $p > 3$ *the compatibility condition*

$$\begin{aligned} c_{i,0}u(0)\cdot\nu - d_i\partial_\nu c_{i,0} &= g_i^{\mathrm{in}}(0) && \text{on } \Gamma_{\mathrm{in}}, \\ -d_i\partial_\nu c_{i,0} &= r_i^{\mathrm{sorp}}(c_{i,0}, c_{i,0}^\Sigma) && \text{on } \Sigma, \\ -d_i\partial_\nu c_{i,0} &= 0 && \text{on } \Gamma_{\mathrm{out}}, \\ -d_i^\Sigma\partial_{\nu_\Sigma} c_{i,0}^\Sigma &= 0 && \text{on } \partial\Sigma. \end{aligned} \tag{5.2}$$

Additionally, the corresponding solution operator ${}_0\mathcal{S}_T$ *w.r.t. zero time trace satisfies*

$$\|{}_0\mathcal{S}_T\|_{\mathscr{L}({}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N,\, {}_0\mathbb{E}_p^\Omega(\tau)^N \times {}_0\mathbb{E}_p^\Sigma(\tau)^N)} \leqslant M \qquad (0 < \tau < T),$$

for a constant $M > 0$ *independent of* τ*, where* ${}_0\mathbb{E}_p^\Omega(\tau)^N$*,* ${}_0\mathbb{E}_p^\Sigma(\tau)^N$ *and* ${}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N$ *denote the corresponding spaces of zero time trace.*

5.2 L^p-stability for Isolated Equilibria

In this section we want to prove a result about stability of equilibria of the catalysis model in the L^p-setting for $p \in [2,\infty)\setminus\{3\}$. First we give a short example of a common type of equilibria which arises from the chemical reaction.

5.2 Remark. One may choose the constant equilibria of chemical balance which are determined by the rates of the chemical reaction. In this case, we have

$$c_{i*} \equiv \psi_i > 0,\ c_{i*}^\Sigma \equiv \xi_i > 0 \qquad (i = 1, ..., N),$$

where

$$\psi_i = \frac{k_i^{\mathrm{de}}}{k_i^{\mathrm{ad}}}\xi_i, \quad \kappa_b\prod_{k=1}^N(\xi_k)^{\beta_k} = \kappa_f\prod_{k=1}^N(\xi_k)^{\alpha_k}$$

such that $r_i^{\mathrm{ch}}(c_*^\Sigma) = 0$. Additionally, we assume that the inflow profile fulfills $g_i^{\mathrm{in}} \leqslant 0$, $g_i^{\mathrm{in}} \neq 0$ on Γ_{in} for every $i = 1, ..., N$. The velocity profile at the inflow is then determined by

$$(u \cdot \nu) = \frac{k_i^{\text{ad}}}{k_i^{\text{de}} \xi_i} g_i^{\text{in}}.$$

This example of equilibria motivates the following assumptions we impose onto all equilibria.

- $(\mathbf{A}_{\mathbf{P}}^{\mathbf{eq}})$ The equilibrium is non-negative, i.e.

$$c_{i*}(x) \geqslant 0 \quad (x \in \Omega, i = 1, .., N), \qquad c_{i*}^{\Sigma}(x) \geqslant 0 \quad (x \in \Sigma, i = 1, .., N).$$

- $(\mathbf{A}_{\mathbf{R}}^{\mathbf{eq}})$ The equilibrium fulfills the following regularity conditions.

$$c_{i*} \in W^{2,p}(\Omega), \;\; c_{i*}^{\Sigma} \in W^{2,p}(\Sigma) \qquad (i = 1, ..., N).$$

- $(\mathbf{A}_{\mathbf{I}}^{\mathbf{eq}})$ The equilibrium is isolated: there exists $\varepsilon > 0$ such that there is no equilibrium in $\mathbb{B}_{\mathbb{E}_p^N}((c_*, c_*^{\Sigma}), \varepsilon) \setminus \{(c_*, c_*^{\Sigma})\}$.

In order to apply the Poincaré inequality we impose an additional assumption regarding the velocity field u.

- $(\mathbf{A}_{\mathbf{in}}^{\mathbf{vel}})$ The velocity field has non-trivial inflow, i.e. $u \cdot \nu \neq 0$ on Γ_{in}.

First we recall the following fact.

5.3 Remark. By [54, Lemma 10.2 (vi)] we have the following: Let $M \subseteq \mathbb{R}^n$ be a nonempty open subset and $1 \leqslant p \leqslant \infty$. Let $V \subseteq W^{1,p}(M)$ be a subspace. If the injection $V \hookrightarrow L^p(M)$ is compact and the constant function $u \equiv 1$ does not belong to V, then there exists a $C > 0$ such that

$$\|u\|_{L^p(M)} \leqslant C \|\nabla u\|_{L^p(M)} \qquad (u \in V)$$

and one says that the Poincaré inequality holds. This assertion also holds if M is replaced by the lateral boundary Σ of a cylindrical domain $\Omega = A \times (0, h)$ with a simply connected C^2-domain $A \subseteq \Omega$.

Then we obtain the following result.

5.4 Theorem. *Let $p \in [2, \infty) \setminus \{3\}$, $T = \infty$ and $g_i^{\text{in}} \in \mathbb{G}_p^{\text{in}}$. Let the sorption rates be given as*

$$r_i^{\text{sorp}}(c_i, c_i^{\Sigma}) := k_i^{\text{ad}} c_i - k_i^{\text{de}} c_i^{\Sigma} \qquad (i = 1, ..., N),$$

and the reaction rates as

$$r_i^{\mathrm{ch}}(c^\Sigma) := (\alpha_i - \beta_i)\left(\kappa_b \left(c^\Sigma\right)^\beta - \kappa_f \left(c^\Sigma\right)^\alpha\right) \qquad (i = 1, ..., N),$$

with $k_i^{\mathrm{ad}}, k_i^{\mathrm{de}} > 0$, $\kappa_b, \kappa_f > 0$ *and* $\alpha, \beta \in (\{0\} \cup [1, \infty))^N$, $\alpha, \beta \neq 0$. *Assume that* (c_*, c_*^Σ) *is an equilibrium of (1.3) satisfying* ($\mathrm{A}_{\mathrm{P}}^{\mathrm{eq}}$), ($\mathrm{A}_{\mathrm{R}}^{\mathrm{eq}}$), ($\mathrm{A}_{\mathrm{I}}^{\mathrm{eq}}$) *and the velocity field* u *satisfies the conditions* ($\mathrm{A}^{\mathrm{vel}}$) *and* ($\mathrm{A}_{\mathrm{in}}^{\mathrm{vel}}$). *Let*

$$\max_\Sigma \left\{|a||b(c_*^\Sigma)|\right\} \leqslant \frac{1}{C_P}, \tag{5.3}$$

where $C_P > 0$ *denotes the Poincaré constant (cf. Remark 5.3) on* Σ *and*

$$\begin{aligned} a_k &:= (\alpha_k - \beta_k), \\ b_k &:= b_k(c_*^\Sigma) := \left(\kappa_b \beta_k \left(c_*^\Sigma\right)^{\beta - e_k} - \kappa_f \alpha_k \left(c_*^\Sigma\right)^{\alpha - e_k}\right). \end{aligned}$$

Then there exists $\rho > 0$ *such that for*

$$(c_0, c_0^\Sigma) \in \mathbb{B}_{\mathbb{I}_p^N}((c_*, c_*^\Sigma), \rho),$$

which in case of $p > 3$ *have to fulfill (5.2), there exists a unique global solution* (c, c^Σ) *satisfying*

$$(c, c^\Sigma) \in W^{1,p}_{loc}(\mathbb{R}_+, L^p(\Omega)^N \times L^p(\Sigma)^N) \cap L^p_{loc}(\mathbb{R}_+, W^{2,p}(\Omega)^N \times W^{2,p}(\Sigma)^N).$$

Moreover, the equilibrium (c_*, c_*^Σ) *is exponentially stable in* $\mathbb{I}_p^N = \mathbb{I}_p(\Omega)^N \times \mathbb{I}_p(\Sigma)^N$.

Proof. In order to obtain the assertions, we want to apply the principle of linearized stability (cf. Theorem 2.6). Since the equilibria are assumed to be isolated points, we want to make use of Remark 2.7 (ii). To shorten the notation throughout the proof, we write e.g. $c = (c_1, ...c_N)^T$ with the corresponding meaning for all other appearing quantities.

Let $(c_*, c_*^\Sigma) \in \mathbb{E}_p^N = W^{2,p}(\Omega)^N \times W^{2,p}(\Sigma)^N$ be an equilibrium fulfilling the assumptions ($\mathrm{A}_{\mathrm{P}}^{\mathrm{eq}}$), ($\mathrm{A}_{\mathrm{R}}^{\mathrm{eq}}$), ($\mathrm{A}_{\mathrm{I}}^{\mathrm{eq}}$). We carry out the proof in three steps.

Step 1: Translation of the system and mapping properties

Let $(\tilde{c}, \tilde{c}^\Sigma)$ be a local solution of (1.3) for initial values $(\tilde{c}_0, \tilde{c}_0^\Sigma)$. We write the system in the following form:

$$\begin{aligned} \partial_t(\tilde{c}, \tilde{c}^\Sigma) + \tilde{A}(\tilde{c}, \tilde{c}^\Sigma) &= \tilde{F}(\tilde{c}, \tilde{c}^\Sigma) && \text{in} \quad (0, T) \times (\Omega \times \Sigma), \\ \tilde{B}(\tilde{c}, \tilde{c}^\Sigma) &= 0 && \text{on} \quad (0, T) \times \Pi, \\ (\tilde{c}, \tilde{c}^\Sigma)|_{t=0} &= (\tilde{c}_0, \tilde{c}_0^\Sigma) && \text{on} \quad \Omega \times \Sigma, \end{aligned} \tag{5.4}$$

where $\Pi := \Gamma_{\text{in}} \times \Sigma \times \Gamma_{\text{out}} \times \partial\Sigma$ and the linear operator $\tilde{A}$ is given as

$$\tilde{A} := \begin{pmatrix} U_\nabla - D_\Delta & 0 \\ -K^{\text{ad}} & -D_{\Delta_\Sigma} + K^{\text{de}} \end{pmatrix} : D(\tilde{A}) \to L^p(\Omega)^N \times L^p(\Sigma)^N,$$
$$D(\tilde{A}) := \mathbb{E}_p^N = W^{2,p}(\Omega)^N \times W^{2,p}(\Sigma)^N.$$

Note that we implicitly take the trace on Σ in the second component of $\tilde{A}(\tilde{c}, \tilde{c}^\Sigma)$. The nonlinearity is denoted by

$$\tilde{F}(\tilde{c}, \tilde{c}^\Sigma) := \begin{pmatrix} 0 \\ r^{\text{ch}}(\tilde{c}^\Sigma) \end{pmatrix},$$

and the inhomogeneous boundary conditions are given as

$$\tilde{B}(\tilde{c}, \tilde{c}^\Sigma) := (U_\nu \tilde{c} - D_\nu \tilde{c} - g^{\text{in}}, -D_\nu \tilde{c} - K^{\text{ad}} \tilde{c} + K^{\text{de}} \tilde{c}^\Sigma, -D_\nu \tilde{c}, -D_{\nu_\Sigma} \tilde{c}^\Sigma)|_\Pi.$$

We also set

$$D_\Delta := \operatorname{diag}(d_1\Delta, ..., d_N\Delta),\ D_{\Delta_\Sigma} := \operatorname{diag}(d_1^\Sigma \Delta_\Sigma, ..., d_N^\Sigma \Delta_\Sigma)$$

and

$$U_\nabla := \operatorname{diag}(u \cdot \nabla, ..., u \cdot \nabla),\ U_\nu := \operatorname{diag}(u \cdot \nu, ..., u \cdot \nu)$$

in N dimensions as well as

$$K^{\text{ad}} := \operatorname{diag}(k_1^{\text{ad}}, ..., k_N^{\text{ad}}),\ K^{\text{de}} := \operatorname{diag}(k_1^{\text{de}}, ..., k_N^{\text{de}}).$$

Additionally, we have

$$D_\nu := \operatorname{diag}(d_1\partial_\nu, ..., d_N\partial_\nu),\ D_{\nu_\Sigma} := \operatorname{diag}(d_1^\Sigma \partial_{\nu_\Sigma}, ..., d_N^\Sigma \partial_{\nu_\Sigma}).$$

We write r^{ch} for the vector of chemical reaction rates $(r_i^{\text{ch}})_{i=1}^N$ and g^{in} for the vector of inflow profiles $(g_i^{\text{in}})_{i=1}^N$.
In order to apply the principle of linearized stability, we decompose the solution $(\tilde{c}, \tilde{c}^\Sigma)$ into the equilibrium part (c_*, c_*^Σ) and the deviation part (c, c^Σ) as follows.

$$(\tilde{c}, \tilde{c}^\Sigma) = (c_*, c_*^\Sigma) + (c, c^\Sigma).$$

By subtracting the system for the equilibrium from the whole system (5.4) we arrive at

$$\begin{aligned} \partial_t(c, c^\Sigma) + A(c, c^\Sigma) &= F(c, c^\Sigma) \quad \text{in} \quad (0,T)\times(\Omega\times\Sigma), \\ (c, c^\Sigma)|_{t=0} &= (c_0, c_0^\Sigma) \quad \text{on} \quad \Omega\times\Sigma, \end{aligned} \tag{5.5}$$

where

$$A := \tilde{A}|_{N(B)},\ D(A) := \{(c, c^\Sigma) \in D(\tilde{A})\ :\ B(c, c^\Sigma) = 0\}$$

with linear boundary conditions

$$B(c, c^\Sigma) := (U_\nu c - D_\nu c, -D_\nu c - K^{\mathrm{ad}} c + K^{\mathrm{de}} c^\Sigma, -D_\nu c, -D_{\nu_\Sigma} c^\Sigma)|_\Pi.$$

Note that we got rid of the dependency on g^{in} in the boundary conditions. Moreover, we have

$$F(c, c^\Sigma) := \tilde{F}(c_* + c, c_*^\Sigma + c^\Sigma) - \tilde{F}(c_*, c_*^\Sigma)$$

and $c_0 := \tilde{c}_0 - c_*$, $c_0^\Sigma := \tilde{c}_0^\Sigma - c_*^\Sigma$. Note that it is equivalent whether we analyze system (5.5) about its equilibrium $(0, 0)$ or system (1.3) about its equilibrium (c_*, c_*^Σ).
Next, we want to assure that the assumptions regarding the nonlinearity in Theorem 2.6 and Remark 2.7 (ii) are fulfilled. For a fixed r^{ch} it is clear that we have a polynomial growth bound of type

$$|r^{\mathrm{ch}}(y)| \leqslant M(1 + |y|^\gamma) \quad (y \in [0, \infty)^N),$$

where $M > 0$ and $\gamma \in [1, \infty)$. Therefore, we may apply [7, Remark 4.1] to obtain

$$r^{\mathrm{ch}} : L^{p\gamma}(\Sigma)^N \to L^p(\Sigma)^N.$$

Since Σ is a manifold of dimension $m = 2$, by $\mathbb{I}_p(\Sigma) \hookrightarrow L^{p\gamma}(\Sigma)$ we obtain

$$F : \mathbb{I}_p^N \to L^p(\Omega)^N \times L^p(\Sigma)^N.$$

Here, we used that $2 - 2/p - 2/p \geqslant -2/\gamma p$ for $p \in [2, \infty)$. Again by [7, Remark 4.1], we can show that F is locally Lipschitz. To this end, let $\rho > 0$. Then there exists $C(\rho, c_*^\Sigma) > 0$ such that

$$\begin{aligned} \|F(c, c^\Sigma) - F(z, z^\Sigma)\|_{L^p(\Omega)^N \times L^p(\Sigma)^N} &\leqslant \|r^{\mathrm{ch}}(c_*^\Sigma + c^\Sigma) - r^{\mathrm{ch}}(z^\Sigma + c_*^\Sigma)\|_{L^p(\Sigma)^N} \\ &\leqslant C(\rho, c_*^\Sigma)\|c^\Sigma - z^\Sigma\|_{L^{p\gamma}(\Sigma)^N} \leqslant C(\rho, c_*^\Sigma)\|(c - z, c^\Sigma - z^\Sigma)\|_{\mathbb{I}_p(\Omega)^N \times \mathbb{I}_p(\Sigma)^N} \end{aligned}$$

for $(c, c^\Sigma), (z, z^\Sigma) \in \overline{\mathbb{B}}_{\mathbb{I}_p^N}((0,0), \rho)$, which yields the assertion.
Now we consider the Fréchet derivative of the nonlinearity F at $(0,0)$. First, we see that

$$\sum_{k=1}^{N} \partial_k r_i^{\mathrm{ch}}(c_*^\Sigma) c_k^\Sigma = \sum_{k=1}^{N} (\alpha_i - \beta_i) \left(\kappa_b \beta_k \left(c_*^\Sigma\right)^{\beta - e_k} - \kappa_f \alpha_k \left(c_*^\Sigma\right)^{\alpha - e_k} \right) c_k^\Sigma.$$

To shorten the notation, we introduce

$$\begin{aligned} a_k &:= (\alpha_k - \beta_k), \\ b_k &:= b_k(c_*^\Sigma) := \left(\kappa_b \beta_k \left(c_*^\Sigma\right)^{\beta - e_k} - \kappa_f \alpha_k \left(c_*^\Sigma\right)^{\alpha - e_k} \right) \end{aligned}$$

for $k = 1, ..., N$, where $\alpha := (\alpha_1, ..., \alpha_N)$ and $\beta := (\beta_1, ..., \beta_N)$. Furthermore, we set $a := (a_1, ..., a_N)$ and $b := b(c_*^\Sigma) := (b_1, ..., b_n)$. Now we can write the derivative of the chemical reaction rates as

$$\tilde{M} := \tilde{M}(c_*^\Sigma) := a \otimes b = \begin{pmatrix} a_1 b_1 & \dots & a_1 b_N \\ \vdots & \ddots & \vdots \\ a_N b_1 & \dots & a_N b_N \end{pmatrix}.$$

One can easily see that, for a and b linearly independent, we have $\dim(N(\tilde{M})) = N - 1$ and $\sigma(\tilde{M}) = \{\lambda_1, ..., \lambda_N\}$ is given by $\lambda_1 = a^T b$ and $\lambda_2 = ... = \lambda_N = 0$. Note that b and therefore λ_1 may depend on $x \in \Sigma$. For a fixed $x \in \Sigma$, the symmetric part $S := \frac{1}{2}(\tilde{M} + \tilde{M}^T)$ of $\tilde{M}$ has the eigenvalues

$$\sigma(S) = \left\{ \tfrac{1}{2} \left(ab \pm |a||b| \right), 0 \right\}$$

if a and b are linearly independent and

$$\sigma(S) = \left\{ \tfrac{1}{2} \left(ab + |a||b| \right), 0 \right\}$$

if a and b are linearly dependent. We write $F'(0,0) = M(c_*^\Sigma)$ for the first Fréchet derivative of F at $(0,0)$ and obtain

$$M(c_*^\Sigma) : L^p(\Omega)^N \times L^p(\Sigma)^N \to L^p(\Omega)^N \times L^p(\Sigma)^N, \quad \begin{pmatrix} c \\ c^\Sigma \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M} \end{pmatrix} \begin{pmatrix} c \\ c^\Sigma \end{pmatrix}$$

for the L^p-realization of the multiplication operator corresponding to $\tilde{M}$. Since we have

$c_*^\Sigma \in W^{2,p}(\Sigma)^N$, $\tilde{M}$ is bounded on Σ and

$$M := M(c_*^\Sigma) \in \mathscr{L}(L^p(\Omega)^N \times L^p(\Sigma)^N).$$

Finally we have to show that A and F fulfill the assumptions in Remark 2.7 (ii). To this end, note that A is a constant linear operator which does not depend on (c, c^Σ), so we only have to take F into consideration: let $r > 0$. By [7, Remark 4.1] and $M \in \mathscr{L}(L^p(\Omega)^N \times L^p(\Sigma)^N)$ we obtain

$$\begin{aligned}
&\|F(c,c^\Sigma) - F(0,0) - M(c_*^\Sigma)(c,c^\Sigma)\|_{L^p(\Omega)^N \times L^p(\Sigma)^N} \\
\leqslant\ &\|r^{\mathrm{ch}}(c_*^\Sigma + c^\Sigma) - r^{\mathrm{ch}}(c_*^\Sigma)\|_{L^p(\Sigma)^N} + \|M(c_*^\Sigma)(c,c^\Sigma)\|_{L^p(\Omega)^N \times L^p(\Sigma)^N} \\
\leqslant\ &C(r,\tilde{M},c_*^\Sigma)\left(\|c^\Sigma\|_{L^{p\gamma}(\Sigma)^N} + \|(c,c^\Sigma)\|_{L^p(\Omega)^N \times L^p(\Sigma)^N}\right) \\
\leqslant\ &C(r,\tilde{M},c_*^\Sigma)\ \|(c,c^\Sigma)\|_{\mathbb{I}_p(\Omega)^N \times \mathbb{I}_p(\Sigma)^N}
\end{aligned}$$

for $(c, c^\Sigma) \in \overline{\mathbb{B}}_{\mathbb{I}_p^N}((0,0), r)$ and a $C(r, \tilde{M}, c_*^\Sigma) > 0$, which completes the necessary estimates.

Step 2: Linearization
By a first-order linearization of (5.5) about $(0,0)$ we obtain the linear system

$$\begin{aligned}
\partial_t(c,c^\Sigma) + A_0(c,c^\Sigma) &= G(c,c^\Sigma) \quad \text{in} \quad (0,T)\times(\Omega\times\Sigma), \\
(c,c^\Sigma)|_{t=0} &= (c_0,c_0^\Sigma) \quad \text{on} \quad \Omega\times\Sigma,
\end{aligned} \tag{5.6}$$

where

$$A_0 := A - M$$

with

$$D(A_0) := \left\{\left(c,c^\Sigma\right) \in W^{2,p}(\Omega)^N \times W^{2,p}(\Sigma)^N \ :\ B(c,c^\Sigma) = 0\right\} = D(A).$$

Note that for $(c, c^\Sigma) \in D(A_0)$ constant we have $(c, c^\Sigma) = 0$ immediately due to $B(c, c^\Sigma) = 0$ and $(\mathrm{A}_{\mathrm{in}}^{\mathrm{vel}})$. Thus, we can use the Poincaré inequality (cf. Remark 5.3). Regarding G we have

$$G(c,c^\Sigma) := F(c,c^\Sigma) - F(0,0) - M(c,c^\Sigma).$$

It remains to note that, due to the fact that A has maximal L^p-regularity (cf. Theorem 5.1), we also obtain maximal L^p-regularity for A_0 by perturbation theory.

Step 3: Characterization of the spectrum
We will use the notation

$$K_\beta^\alpha := \left(K^{\mathrm{ad}}\right)^\alpha \left(K^{\mathrm{de}}\right)^\beta \qquad (\alpha, \beta \in \mathbb{R})$$

in the following and note the special cases

$$K_0^0 = \mathrm{Id},\; K_0^\alpha = \left(K^{\mathrm{ad}}\right)^\alpha,\;\; K_\beta^0 = \left(K^{\mathrm{de}}\right)^\beta.$$

Moreover, we have that K_β^α commutes with D_Δ, D_{Δ_Σ}, U_∇, U_ν, D_ν and D_{ν_Σ}.
Since Ω and Σ are bounded, it is easy to see that A_0 has compact resolvent. Therefore, we have $\sigma(A_0) = \sigma_p(A_0)$, where σ_p denotes the point spectrum. Furthermore, due to compactness the spectrum of A_0 is p-invariant. We will exploit this fact and analyze the L^2-spectrum of A_0 only, since the results are also applicable for A_0 in the L^p-setting for $p \in [2, \infty) \setminus \{3\}$.
Let $(f_\Omega, f_\Sigma) \in D(A_0)$ be an eigenvector corresponding to the eigenvalue $\lambda \in \sigma(A_0)$. We set

$$\begin{pmatrix} c \\ c^\Sigma \end{pmatrix} := \begin{pmatrix} K_{-1}^1 & 0 \\ 0 & K_{-1}^1 \end{pmatrix} \begin{pmatrix} f_\Omega \\ f_\Sigma \end{pmatrix} \in D(A_0).$$

Testing in the L^2-setting yields

$$\begin{aligned}
&\mathrm{Re}\left(\lambda \begin{pmatrix} K_1^{-1} & 0 \\ 0 & K_1^{-1} \end{pmatrix} \begin{pmatrix} c \\ c^\Sigma \end{pmatrix}, \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & K_1^{-1} \end{pmatrix} \begin{pmatrix} c \\ c^\Sigma \end{pmatrix}\right)_{L^2(\Omega)^N \times L^2(\Sigma)^N} \\
&= \mathrm{Re}\left(A_0 \begin{pmatrix} K_1^{-1} & 0 \\ 0 & K_1^{-1} \end{pmatrix} \begin{pmatrix} c \\ c^\Sigma \end{pmatrix}, \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & K_1^{-1} \end{pmatrix} \begin{pmatrix} c \\ c^\Sigma \end{pmatrix}\right)_{L^2(\Omega)^N \times L^2(\Sigma)^N} \\
&= \mathrm{Re}\left(\begin{pmatrix} U_\nabla - D_\Delta & 0 \\ -K_0^1 & -D_{\Delta_\Sigma} + K_1^0 - \tilde{M} \end{pmatrix} \begin{pmatrix} K_1^{-1} c \\ K_1^{-1} c^\Sigma \end{pmatrix}, \begin{pmatrix} c \\ K_1^{-1} c^\Sigma \end{pmatrix}\right)_{L^2(\Omega)^N \times L^2(\Sigma)^N} \\
&= \mathrm{Re}\, F_\Omega + \mathrm{Re}\, F_\Sigma,
\end{aligned}$$

where

$$\begin{aligned}
F_\Omega :=& (K_1^{-1} U_\nabla c, c)_{L^2(\Omega)^N} - (K_1^{-1} D_\Delta c, c)_{L^2(\Omega)^N}, \\
F_\Sigma :=& -(K_2^{-1} c, c^\Sigma)_{L^2(\Sigma)^N} - (K_2^{-2} D_{\Delta_\Sigma} c^\Sigma, c^\Sigma)_{L^2(\Sigma)^N} \\
&+ (K_3^{-2} c^\Sigma, c^\Sigma)_{L^2(\Sigma)^N} - (\tilde{M} K_1^{-1} D_{\Delta_\Sigma} c^\Sigma, K_1^{-1} c^\Sigma)_{L^2(\Sigma)^N}.
\end{aligned}$$

First we look at F_Ω. Here we obtain

$$\operatorname{Re} F_\Omega = \operatorname{Re} \sum_{i=1}^{N} \left(- \left(\left(k_i^{\mathrm{ad}}\right)^{-1} k_i^{\mathrm{de}} d_i \int_\Omega \Delta c_i \overline{c_i} dx \right) + \left(k_i^{\mathrm{ad}}\right)^{-1} k_i^{\mathrm{de}} \int_\Omega (u \cdot \nabla) c_i \overline{c_i} dx \right).$$

Applying Green's formula to the first addend and using the boundary conditions leads to

$$\begin{aligned} \operatorname{Re} \left(d_i \int_\Omega \Delta c_i \overline{c_i} dx \right) &= \operatorname{Re} \left(d_i \int_{\partial\Omega} \partial_\nu c_i \overline{c_i} d\sigma \right) - d_i \int_\Omega |\nabla c_i|^2 dx \\ &= \int_{\Gamma_{\mathrm{in}}} (u \cdot \nu) |c_i|^2 d\sigma - \int_\Sigma k_i^{\mathrm{ad}} |c_i|^2 d\sigma + \operatorname{Re} \left(\int_\Sigma k_i^{\mathrm{de}} c_i^\Sigma \overline{c_i} d\sigma \right) \\ &\quad - d_i \int_\Omega |\nabla c_i|^2 dx. \end{aligned}$$

The second addend can be written as

$$\begin{aligned} \int_\Omega (u \cdot \nabla) c_i \overline{c_i} dx &= \frac{1}{2} \int_{\partial\Omega} (u \cdot \nu) |c_i|^2 d\sigma \\ &= \frac{1}{2} \int_{\Gamma_{\mathrm{in}}} (u \cdot \nu) |c_i|^2 d\sigma + \frac{1}{2} \int_{\Gamma_{\mathrm{out}}} (u \cdot \nu) |c_i|^2 d\sigma \end{aligned}$$

using partial integration, boundary conditions in (5.6) and ($\mathrm{A}^{\mathrm{vel}}$). Combining the results yields

$$\begin{aligned} \operatorname{Re} F_\Omega = &\, \| K_{1/2}^{-1/2} D_\nabla c \|^2_{L^2(\Omega)^{N \times N}} - \tfrac{1}{2} (K_1^{-1} U_\nu c, c)_{L^2(\Gamma_{\mathrm{in}})^N} + \tfrac{1}{2} (K_1^{-1} U_\nu c, c)_{L^2(\Gamma_{\mathrm{out}})^N} \\ &+ \| K_{1/2}^0 c \|^2_{L^2(\Sigma)^N} - \operatorname{Re} (K_2^{-1} c, c^\Sigma)_{L^2(\Sigma)^N}, \end{aligned}$$

where

$$D_\nabla := \operatorname{diag} \left(\sqrt{d_1} \nabla, ..., \sqrt{d_N} \nabla \right), \quad D_{\nabla_\Sigma} := \operatorname{diag} \left(\sqrt{d_1^\Sigma} \nabla_\Sigma, ..., \sqrt{d_N^\Sigma} \nabla_\Sigma \right).$$

Examining F_Σ leads to

$$\begin{aligned} \operatorname{Re} F_\Sigma = &\, \| K_1^{-1} D_{\nabla_\Sigma} c^\Sigma \|^2_{L^2(\Sigma)^{N \times N}} + \| K_{3/2}^{-1} c^\Sigma \|^2_{L^2(\Sigma)^N} \\ &- \operatorname{Re} (K_2^{-1} c, c^\Sigma)_{L^2(\Sigma)^N} - \operatorname{Re} (\tilde{M} K_1^{-1} c^\Sigma, K_1^{-1} c^\Sigma)_{L^2(\Sigma)^N}. \end{aligned}$$

Finally, we obtain

$$\operatorname{Re} \lambda \left(\| K_{1/2}^{-1/2} c \|^2_{L^2(\Omega)^N} + \| K_1^{-1} c^\Sigma \|^2_{L^2(\Sigma)^N} \right) = \operatorname{Re} F_\Omega + \operatorname{Re} F_\Sigma. \tag{5.7}$$

In order to determine the sign of $\operatorname{Re} \lambda$ we first note that the norms are non-negative

and

$$-\tfrac{1}{2}(K_1^{-1}U_\nu c, c)_{L^2(\Gamma_{\mathrm{in}})^N},\ \tfrac{1}{2}(K_1^{-1}U_\nu c, c)_{L^2(\Gamma_{\mathrm{out}})^N} \geqslant 0,$$

due to ($\mathrm{A}^{\mathrm{vel}}$). It remains to find appropriate estimates for the remaining terms. Using the Cauchy-Schwarz and the Young inequality we have

$$\begin{aligned} 2\Big|\mathrm{Re}(K_2^{-1}c, c^\Sigma)_{L^2(\Sigma)^N}\Big| &\leqslant 2\Big|(K_{1/2}^0 c, K_{3/2}^{-1}c^\Sigma)_{L^2(\Sigma)^N}\Big| \\ &\leqslant \|K_{1/2}^0 c\|_{L^2(\Sigma)^N}^2 + \|K_{3/2}^{-1}c^\Sigma\|_{L^2(\Sigma)^N}^2, \end{aligned}$$

which leaves the term which is caused by the chemical reaction to be estimated.

$$\begin{aligned} \Big|\mathrm{Re}\,(\tilde{M}(c_*^\Sigma)K_1^{-1}c^\Sigma, K_1^{-1}c^\Sigma)_{L^2(\Sigma)^N}\Big| &= \Big|\mathrm{Re}\,(S(c_*^\Sigma)K_1^{-1}c^\Sigma, K_1^{-1}c^\Sigma)_{L^2(\Sigma)^N}\Big| \\ &\leqslant \max_\Sigma |S(c_*^\Sigma)|_2 \|K_1^{-1}c^\Sigma\|_{L^2(\Sigma)^N}^2 \\ &\leqslant \max_\Sigma \Big\{\tfrac{1}{2}|a(b(c_*^\Sigma))^T \pm |a||b(c_*^\Sigma)||\Big\} \|K_1^{-1}c^\Sigma\|_{L^2(\Sigma)^N}^2 \\ &\leqslant C_P \max_\Sigma \Big\{|a||b(c_*^\Sigma)|\Big\} \|K_1^{-1}D_{\nabla_\Sigma}c^\Sigma\|_{L^2(\Sigma)^{N\times N}}^2, \end{aligned}$$

where $C_P > 0$ denotes the Poincaré constant on Σ which does not depend on c^Σ. In order to obtain the stability result we want to show that $\mathrm{Re}\,\lambda$ has positive sign, so we impose the condition

$$\max_\Sigma \Big\{|a||b(c_*^\Sigma)|\Big\} \leqslant \frac{1}{C_P}.$$

Now, let $\mathrm{Re}\,\lambda = 0$. From (5.7) we obtain $(c, c^\Sigma) = 0$ such that every eigenvector corresponding to λ is zero. This yields $\lambda \in \rho(A_0)$. Since $\rho(A_0)$ is open, we obtain that for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda = 0$ there exists an $\varepsilon_\lambda > 0$ such that $\mathbb{B}(\lambda, \varepsilon_\lambda) \subseteq \rho(A_0)$. Additionally, we know that A_0 has maximal L^p-regularity, so $A_0 + \mu$ is sectorial with angle $\varphi_{\mu+A_0} < \pi/2$ for some $\mu \geqslant 0$. This gives us

$$\mathbb{C}_\varepsilon := \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < \varepsilon\} \subseteq \rho(A_0)$$

for some $\varepsilon > 0$ and thus $\sigma(A_0) \subseteq \mathbb{C}_+$, which yields the assertion. □

5.5 Remark. If (c_*, c_*^Σ) is an equilibrium of chemical balance (cf. Remark 5.2) and $a = \varphi b$ for some $\varphi \in \mathbb{R}$, i.e. a and b are linearly dependent, then the situation simplifies as follows. The spectrum of the symmetric part S of $\tilde{M}$ consists of the eigenvalues

$$\lambda_1 = \phi|b|^2, \lambda_2 = ... = \lambda_N = 0,$$

such that we obtain stability immediately if $\varphi \leqslant 0$ due to the fact that the corresponding bilinear form is negative semidefinite. Moreover, by

$$c_{i,*}^{\Sigma} b_i = \left(\kappa_b \beta_i \left(c_*^{\Sigma}\right)^{\beta} - \kappa_f \alpha_i \left(c_*^{\Sigma}\right)^{\alpha}\right) = -(\alpha_i - \beta_i)\kappa_f \left(c_*^{\Sigma}\right)^{\alpha} = -\kappa_f \left(c_*^{\Sigma}\right)^{\alpha} a_i$$

we obtain that the condition $\varphi \leqslant 0$ is always fulfilled in this case and therefore the stability result in L^p for $p \in [2,\infty) \setminus \{3\}$ holds without condition (5.3) if a and b are linearly dependent.
Additionally, we note that such equilibria exist if $\alpha \neq \beta$ since we have

$$c_{1,*}^{\Sigma} = \dots = c_{N,*}^{\Sigma} =: \gamma > 0$$

and

$$\kappa_b \prod_{i=1}^{N} \gamma^{\beta_k} - \kappa_f \prod_{i=1}^{N} \gamma^{\alpha_k} = 0 \Leftrightarrow \kappa_b \gamma^{|\beta|} - \kappa_f \gamma^{|\alpha|} = 0 \Leftrightarrow \left(\frac{\kappa_b}{\kappa_f}\right)^{\frac{1}{|\alpha|-|\beta|}} = \gamma.$$

5.3 Instability

Having proved a result on stability of the heterogeneous catalysis model, we may drop condition $(\mathrm{A}_{\mathrm{in}}^{\mathrm{vel}})$ since the Poincaré inequality is not longer needed to prove a corresponding result regarding the instability of equilibria.

5.6 Theorem. *Let $p \in [2,\infty) \setminus \{3\}$, $T = \infty$ and $g_i^{\mathrm{in}} \in \mathbb{G}_p^{\mathrm{in}}$. Let the sorption rates be given as*

$$r_i^{\mathrm{sorp}}(c_i, c_i^{\Sigma}) := k_i^{\mathrm{ad}} c_i - k_i^{\mathrm{de}} c_i^{\Sigma}, \qquad (i = 1, \dots, N)$$

and the reaction rates as

$$r_i^{\mathrm{ch}}(c^{\Sigma}) := (\alpha_i - \beta_i)\left(\kappa_b \left(c^{\Sigma}\right)^{\beta} - \kappa_f \left(c^{\Sigma}\right)^{\alpha}\right) \qquad (i = 1, \dots, N)$$

with $k_i^{\mathrm{ad}}, k_i^{\mathrm{de}} > 0$, $\kappa_b, \kappa_f > 0$ and $\alpha, \beta \in (\{0\} \cup [1,\infty))^N$, $\alpha, \beta \neq 0$. Assume (c_*, c_*^{Σ}) is an equilibrium of (1.3) satisfying $(\mathrm{A}_{\mathrm{P}}^{\mathrm{eq}})$, $(\mathrm{A}_{\mathrm{R}}^{\mathrm{eq}})$, $(\mathrm{A}_{\mathrm{I}}^{\mathrm{eq}})$ and that u satisfies $(\mathrm{A}^{\mathrm{vel}})$. Furthermore, we assume that there exists an eigenvector (c, c^{Σ}) of A_0, such that

$$(b(c_*^{\Sigma})c^{\Sigma}, ac^{\Sigma})_{L^2(\Sigma)^N} > \left|(A(c, c^{\Sigma}), (c, c^{\Sigma}))_{L^2(\Omega)^N \times L^2(\Sigma)^N}\right| \tag{5.8}$$

where A_0, A, a and $b(c_*^{\Sigma})$ are defined as in Theorem 5.4.
Then (c_*, c_*^{Σ}) is unstable in $\mathbb{I}_P^N = \mathbb{I}_p(\Omega)^N \times \mathbb{I}_p(\Sigma)^N$ and there exists a $\rho > 0$ such that

for $\eta > 0$ there is a

$$(c_0, c_0^\Sigma) \in \overline{\mathbb{B}}_{\mathbb{I}_p^N}((c_*, c_*^\Sigma), \eta),$$

in case of $p > 3$ satisfying (5.2), but the solution (c, c^Σ) corresponding to (c_0, c_0^Σ) satisfies

$$\|(c(t_\eta), c^\Sigma(t_\eta)) - (c_*, c_*^\Sigma)\|_{\mathbb{I}_p^N} > \rho$$

for some finite time $t_\eta > 0$.

Proof. Let (c, c^Σ) be an eigenvector of A_0 corresponding to the eigenvalue $\lambda \in \sigma(A_0)$ fulfilling the assumption (5.8). As in the proof of Theorem 5.4 we remark that it is sufficient to characterize the spectrum of A_0 in the L^2-setting to obtain the result for all $p \in [2, \infty) \setminus \{3\}$ due to compactness. By testing $A_0(c, c^\Sigma) = \lambda(c, c^\Sigma)$ with (c, c^Σ) and taking the real part we obtain

$$\operatorname{Re} \lambda(\|c\|^2_{L^2(\Omega)^N} + \|c^\Sigma\|^2_{L^2(\Sigma)^N}) = ((A - M)\,(c, c^\Sigma), (c, c^\Sigma))_{L^2(\Omega)^N \times L^2(\Sigma)^N}$$

and

$$(A\,(c, c^\Sigma), (c, c^\Sigma))_{L^2(\Omega)^N \times L^2(\Sigma)^N} \geqslant 0.$$

Then condition (5.8) yields $\operatorname{Re} \lambda < 0$ and there exists a $\lambda_0 \in \sigma(A_0) \cap \mathbb{C}_-$. As before we exploit that A_0 has compact resolvent, which implies that A_0 has a discrete point spectrum. Moreover, since $\mu + A_0$ is sectorial for a $\mu \geqslant 0$ with angle $\varphi_{\mu+A_0} < \pi/2$, we know that $\sigma(A_0) \cap \mathbb{C}_-$ is compact and therefore we obtain a spectral gap in $\mathbb{C}_-$. This means that there exists a $\delta \in (\operatorname{Re} \lambda_0, 0)$ such that $\sigma(A_0) \cap [\delta + i\mathbb{R}] = \emptyset$. Then Theorem 2.8 yields the result. □

5.7 Remark. (1) Note that due to dropping the condition ($\mathrm{A}_{\mathrm{in}}^{\mathrm{vel}}$) there may be constant functions in $D(A_0)$, which is not the case in the stable setting.

(2) Even with condition ($\mathrm{A}_{\mathrm{in}}^{\mathrm{vel}}$) dropped, there does not exist some constant eigenvector for eigenvalues in $\sigma(A_0) \setminus \{0\}$. Indeed, let $\lambda \in \sigma(A_0) \setminus \{0\}$ and (c, c^Σ) be a corresponding constant eigenvector. Due to $B(c, c^\Sigma) = 0$ we have $K^{\mathrm{ad}} c = K^{\mathrm{de}} c^\Sigma$ and therefore $c, c^\Sigma \neq 0$. Then

$$\lambda(c, c^\Sigma) = A_0(c, c^\Sigma) = (A - M)(c, c^\Sigma) = -(0, \tilde{M} c^\Sigma)$$

yields the contradiction.

(3) In general it is not clear if an eigenvector fulfilling (5.8) exists. Especially the condition $b(c_*^\Sigma)c^\Sigma, ac^\Sigma \neq 0$ has to be fulfilled in such a case. Due to the fact that A_0 is not normal in general, it is not even clear if there exists a basis of $L^2(\Omega)^N \times L^2(\Sigma)^N$ consisting of eigenvectors of A_0.

6 Duality Scales for Partial Differential Equations

In this chapter we analyze the coherence between duality scales and complemented subspaces, projected by some projection P. First, we collect some results regarding the resolvent set of the operator $R = (P - P')^2$ and give a characterization of duality scales on complemented subspaces in Theorem 6.2. Then we apply the developed theory to a Stokes problem on a C^3-domain with compact boundary and see that compactness may simplify the application of Theorem 6.2. Therefore, we develop some abstract results relying on compactness in the last section.

6.1 Projections on Duality Scales

In the following, let $s_0 > 0$ and $I_0 := (-s_0, s_0)$. We assume that $(E_s)_{s\in I_0}$, $(F_s)_{s\in I_0}$ are complex interpolation scales of Banach spaces which are continuously embedded into a common Hausdorff space $\mathcal{H}$ and that $(P_{s,E})_{s\in I_0}$, $(P_{s,F})_{s\in I_0}$ are consistent scales of projections on these interpolation scales. Let $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ be a duality scale. We will use the notation $E_{s,P_{s,E}} := P_{s,E}(E_s)$ for simplicity. Note that in the following $(P_{s,E})'_{\mathfrak{a}} = P'_{s,E} \in \mathscr{L}(F_{-s})$ always denotes the dual operator w.r.t. the pairing $\mathfrak{a}$. Moreover, we will use the following notations according to Lemma 2.45.

$$R_{s,E} := (P_{s,E} - P'_{-s,F})^2, \quad R_{s,F} := (P_{s,F} - P'_{-s,E})^2,$$

as well as

$$\mathbb{Q}_{s,E} := P_{s,E}P'_{-s,F}(1 - R_{s,E})^{-1}, \quad \mathbb{Q}_{s,F} := P_{s,F}P'_{-s,E}(1 - R_{s,F})^{-1},$$
$$\mathbb{P}_{s,E} := P'_{-s,F}P_{s,E}(1 - R_{s,E})^{-1}, \quad \mathbb{P}_{s,F} := P'_{-s,E}P_{s,F}(1 - R_{s,F})^{-1}.$$

First we prove an auxiliary result.

6.1 Lemma. *Let $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ be a duality scale and $(P_{s,E})_{s\in I_0}$, $(P_{s,F})_{s\in I_0}$ be consistent scales of projections. Let $\lambda \in \mathbb{C}$. Consider the following assertions.*

(i) *For every $s \in I_0$ we have $\lambda \in \rho(R_{s,E}, E_s)$.*

(ii) *For every* $s \in I_0$ *we have* $\lambda \in \rho(R_{s,F}, F_s)$.

(iii) *For every* $s \in I_0$ *we have*

$$\lambda \in \rho(P'_{-s,F}(1-P_{s,E}), E_{s,P'_{-s,F}}) \cap \rho((1-P'_{-s,F})P_{s,E}, E_{s,1-P'_{-s,F}}).$$

(iv) *For every* $s \in I_0$ *we have*

$$\lambda \in \rho(P'_{-s,E}(1-P_{s,F}), F_{s,P'_{-s,E}}) \cap \rho((1-P'_{-s,E})P_{s,F}, F_{s,1-P'_{-s,E}}).$$

(v) *For every* $s \in I_0$ *we have*

$$\lambda \in \rho(P'_{-s,F}(1-P_{s,E}), E_s) \cap \rho(P'_{-s,E}(1-P_{s,F}), F_s).$$

Then we have

$$\text{(v)} \Rightarrow \text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)}.$$

Moreover, if $\lambda = 1$, *then all assertions are equivalent.*

Proof. (i) $\Leftrightarrow$ (ii): Follows immediately due to

$$R'_{s,E} = [(P_{s,E} - P'_{-s,F})^2]' = [((P'_{-s,F})' - P'_{s,E})^2] = [P_{-s,F} - P'_{s,E}]^2 = R_{-s,F}$$

and vice versa taking into account reflexivity of E_s and F_s.

(i) $\Rightarrow$ (iii): By restriction of $R_{s,E}$ onto subspaces we obtain

$$R_{s,E}x = (P_{s,E} - P'_{-s,F})^2 x = P'_{-s,F}(1-P_{s,E})x \qquad (x \in E_{s,P'_{-s,F}}) \tag{6.1}$$

$$R_{s,E}x = (P_{s,E} - P'_{-s,F})^2 x = (1-P'_{-s,F})P_{s,E}x \qquad (x \in E_{s,1-P'_{-s,F}}). \tag{6.2}$$

Then, $R_{s,E}(E_{s,P'_{-s,F}}) \subseteq E_{s,P'_{-s,F}}$ and

$$\lambda - R_{s,E} : E_{s,P'_{-s,F}} \longrightarrow E_{s,P'_{-s,F}}$$

is injective with closed range for $s \in I_0$. Due to $R'_{s,E} = R_{-s,F}$ for $s \in I_0$ w.r.t. the duality pairing $(E_{s,P'_{-s,F}}, F_{-s,P_{s,F}}, \mathfrak{a}_{-s})$ and due to $R_{-s,F}(F_{-s,P_{-s,F}}) \subseteq F_{-s,P_{-s,F}}$ we also see that

$$(\lambda - R_{s,E})' = \lambda - R_{-s,F} : F_{-s,P_{-s,F}} \to F_{-s,P_{-s,F}}$$

is injective and by duality arguments that $\lambda - R_{s,E} \in \mathscr{L}_{is}(E_{s,P'_{-s,F}})$. By (6.1) we have that

$$\lambda \in \rho(P'_{-s,F}(1 - P_{s,E}), E_{s,P'_{-s,F}}). \tag{6.3}$$

Using analogous arguments and (6.2) we also obtain $\lambda \in \rho((1-P_{s,E})P'_{-s,F}), E_{s,1-P'_{-s,F}})$.

(ii) ⇒ (iv): Follows in the same way as (i) ⇒ (iii).

(iii) ⇒ (i): Follows from (6.1), (6.2) and Lemma 2.42(iii).

(iv) ⇒ (ii): Follows in the same way as (iii) ⇒ (i).

(v) ⇒ (i): We dualize $\lambda \in \rho(P'_{s,E}(1-P_{-s,F}), F_{-s})$ w.r.t. the duality pairing $(E_s, F_{-s}, \mathfrak{a}_{-s})$ for $s \in I_0$ and obtain

$$\lambda \in \rho((1 - P'_{-s,F})P_{s,E}, E_s).$$

With (6.1) and (6.2) we again have that

$$\begin{aligned} \lambda - R_{s,E} &= \lambda - P'_{-s,F}(1 - P_{s,E}) : E_{s,P'_{-s,F}} \to E_{s,P'_{-s,F}}, \\ \lambda - R_{s,E} &= \lambda - (1 - P'_{-s,F})P_{s,E} : E_{s,1-P'_{-s,F}} \to E_{s,1-P'_{-s,F}} \end{aligned}$$

are injective with closed range. Moreover,

$$(\lambda - R_{s,E})(E_{s,P'_{-s,F}}) \oplus (\lambda - R_{s,E})(E_{s,1-P'_{-s,F}}) \subseteq E_s$$

is closed in E_s. Thus,

$$\lambda - R_{s,E} : E_s = E_{s,P'_{-s,F}} \oplus E_{s,1-P'_{-s,F}} \to (\lambda - R_{s,E})(E_{s,P'_{-s,F}}) \oplus (\lambda - R_{s,E})(E_{s,1-P'_{-s,F}})$$

is isomorphic and therefore $\lambda - R_{s,E} : E_s \to E_s$ is injective with closed range. By applying the same arguments to $\lambda - R_{-s,F}$ and using duality we obtain $\lambda - R_{s,E} \in \mathscr{L}_{is}(E_s)$.

From now on let $\lambda = 1$. We prove
(i) ⇒ (v): We already know that (i) implies $1 \in \rho(P'_{-s,F}(1 - P_{s,E}), E_{s,P'_{-s,F}})$. Due to $1 \in \rho(R_{s,E}, E_s)$ by assumption, an application of Lemma 2.45 yields

$$E_s = E_{s,P_{s,E}} \oplus E_{s,1-P'_{-s,F}} = E_{s,P'_{-s,F}} \oplus E_{s,1-P_{s,E}} \tag{6.4}$$

and an application of Lemma 2.44 results in

$$1 - P'_{-s,F} \in \mathscr{L}is(E_{s,1-P_{s,E}}, E_{s,1-P'_{-s,F}}). \tag{6.5}$$

Let $x \in E_{s,1-P_{s,E}}$, then $(1 - (P'_{-s,F}(1 - P_{s,E}))x = (1 - P'_{-s,F})x$. Thus we have

$$1 - P'_{-s,F}(1 - P_{s,E}) \in \mathscr{L}is(E_{s,P'_{-s,F}} \oplus E_{s,1-P_{s,E}}, E_{s,P'_{-s,F}} \oplus E_{s,1-P'_{-s,F}}) = \mathscr{L}is(E_s)$$

by (6.3), (6.4) and 6.5. The assertion

$$1 - P'_{-s,E}(1 - P_{s,F}) \in \mathscr{L}is(F_s)$$

follows by using (i) $\Leftrightarrow$ (ii) and similar arguments. □

Regarding the central question if $(E_{s,P_{s,E}}, F_{s,P_{s,F}}, \mathfrak{a}_s)_{s\in I_0}$ is a duality scale, we obtain the following main result.

6.2 Theorem. *Let $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ be a duality scale and $(P_{s,E})_{s\in I_0}$, $(P_{s,F})_{s\in I_0}$ be consistent scales of projections. Then the following assertions are equivalent.*

(i) *We have $1 \in \rho(R_{s,E}, E_s)$ for every $s \in I_0$.*

(ii) *We have $1 \in \rho(R_{s,F}, F_s)$ for every $s \in I_0$.*

(iii) *For every $s \in I_0$ we have $E_s = E_{s,P'_{-s,F}} \oplus E_{s,1-P_{s,E}}$ and $F_s = F_{s,P'_{-s,E}} \oplus F_{s,1-P_{s,F}}$ as well as $E_s = E_{s,P_{s,E}} \oplus E_{s,1-P'_{-s,F}}$ and $F_s = F_{s,P_{s,F}} \oplus F_{s,1-P'_{-s,E}}$.*

(iv) *There exist consistent scales of projections $(\mathbb{P}_{s,E})_{s\in I_0}$ and $(\mathbb{P}_{s,F})_{s\in I_0}$ as well as $(\mathbb{Q}_{s,E})_{s\in I_0}$ and $(\mathbb{Q}_{s,F})_{s\in I_0}$ which are symmetric w.r.t. each other, i.e. $\mathbb{P}'_{s,E} = \mathbb{P}_{-s,F}$, $\mathbb{P}'_{s,F} = \mathbb{P}_{-s,E}$, $\mathbb{Q}'_{s,E} = \mathbb{Q}_{-s,F}$ and $\mathbb{Q}'_{s,F} = \mathbb{Q}_{-s,E}$, such that $\mathbb{P}_{s,E}(E_s) = E_{s,P'_{-s,F}}$, $\mathbb{P}_{s,F}(F_s) = F_{s,P'_{-s,E}}$, $\mathbb{Q}_{s,E}(E_s) = E_{s,P_{s,E}}$ and $\mathbb{Q}_{s,F}(F_s) = F_{s,P_{s,F}}$.*

(v) *$(E_{-s,P'_{s,F}}, F_{s,P'_{-s,E}}, \mathfrak{a}_s)$ and $(E_{-s,P_{-s,E}}, F_{s,P_{s,F}}, \mathfrak{a}_s)$ are duality systems for all $s \in I_0$.*

(vi) *$(E_{s,P'_{-s,F}}, F_{s,P'_{-s,E}}, \mathfrak{a}_s)_{s\in I_0}$ and $(E_{s,P_{s,E}}, F_{s,P_{s,F}}, \mathfrak{a}_s)_{s\in I_0}$ are duality scales.*

(vii) *We have*

$$1 \in \rho(P'_{-s,F}(1 - P_{s,E}), E_s) \cap \rho(P'_{-s,E}(1 - P_{s,F}), F_s)$$

for every $s \in I_0$.

Proof. We will prove the assertion in several steps.

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (vii): Follows from Lemma 6.1.

(i) $\Leftrightarrow$ (iii): We can use (i) $\Leftrightarrow$ (ii) and apply Lemma 2.45 to obtain the equivalence by setting $Q := P'_{-s,F}$ or $Q := P'_{-s,E}$ respectively.

Next, we show (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) without consistency of $(\mathbb{P}_{s,E})_{s\in I_0}$, $(\mathbb{P}_{s,F})_{s\in I_0}$, $(\mathbb{Q}_{s,E})_{s\in I_0}$ and $(\mathbb{Q}_{s,F})_{s\in I_0}$.

(iii) $\Rightarrow$ (iv): Lemma 2.45 yields scales of projections $(\mathbb{P}_{s,E})_{s\in I_0}$ on $(E_s)_{s\in I_0}$ and $(\mathbb{P}_{s,F})_{s\in I_0}$ on $(F_s)_{s\in I_0}$. We already have $R'_{s,E} = R_{-s,F}$ for $s \in I_0$. Furthermore, we know that $(1 - R_{s,E})$ commutes with $P_{s,E}$ and $P'_{-s,F}$. This leads to

$$\mathbb{P}'_{s,E} = [P'_{-s,F}P_{s,E}(1 - R_{s,E})^{-1}]' = P'_{s,E}P_{-s,F}(1 - R_{-s,F})^{-1} = \mathbb{P}_{-s,F},$$

thus the symmetry of $\mathbb{P}_{s,E}$. Symmetry of $\mathbb{P}_{s,F}$, $\mathbb{Q}_{s,E}$ and $\mathbb{Q}_{s,F}$ follows in the same manner.

(iv) $\Rightarrow$ (v): The symmetry of $(\mathbb{P}_{s,E})_{s\in I_0}$ and $(\mathbb{P}_{s,F})_{s\in I_0}$ and Lemma 2.42 imply that $(E_{-s,\mathbb{P}_{-s,E}}, F_{s,\mathbb{P}_{s,F}}, \mathfrak{a}_s)$ is a duality system for every $s \in I_0$. Lemma 2.45 implies $E_{-s,\mathbb{P}_{-s,E}} = E_{-s,P'_{s,F}}$ and $F_{s,\mathbb{P}_{s,F}} = F_{s,P'_{-s,E}}$, which yields that $(E_{-s,P'_{s,F}}, F_{s,P'_{-s,E}}, \mathfrak{a}_s)$ is a duality system. The assertion for $(E_{-s,P_{-s,E}}, F_{s,P_{s,F}}, \mathfrak{a}_s)$ follows in an analogous way.

(v) $\Rightarrow$ (iii): We consider the bounded map $P_{s,E} : E_{s,P'_{-s,F}} \to E_{s,P_{s,E}}$ and pick $z \in E_{s,P_{s,E}}$ arbitrary. Then we have $\mathfrak{a}_{-s}(z,\cdot) \in (F_{-s,P'_{s,E}})'$. Since $(E_{s,P'_{-s,F}}, F_{-s,P'_{s,E}}, \mathfrak{a}_{-s})$ is a duality system, there exists a unique $y \in E_{s,P'_{-s,F}}$ such that

$$\mathfrak{a}_{-s}(P_{s,E}y, x) = \mathfrak{a}_{-s}(y, P'_{s,E}x) = \mathfrak{a}_{-s}(y, x) = \mathfrak{a}_{-s}(z, x) \quad (x \in F_{-s,P'_{s,E}}).$$

From Lemma 2.42 we already know that $(E_{s,P_{s,E}}, F_{-s,P'_{s,E}}, \mathfrak{a}_{-s})$ is a duality system. This yields $P_{s,E}y = z$ and $P_{s,E} \in \mathscr{L}is(E_{s,P'_{-s,F}}, E_{s,P_{s,E}})$, thus

$$E_s = E_{s,P'_{-s,F}} \oplus E_{s,1-P_{s,E}}$$

by Lemma 2.44. The same arguments lead to the remaining decompositions in (iii).

It remains to show consistency of $(\mathbb{P}_{s,E})_{s\in I_0}$, $(\mathbb{P}_{s,F})_{s\in I_0}$, $(\mathbb{Q}_{s,E})_{s\in I_0}$ and $(\mathbb{Q}_{s,F})_{s\in I_0}$, which will be postponed to the last step of the proof. Next, we show the equivalence (v) $\Leftrightarrow$ (vi). The conclusion (vi) $\Rightarrow$ (v) is clear, so we only give a proof for the opposite direction.

(v) $\Rightarrow$ (vi): Let (v) and therefore (iii) and (iv) be fulfilled. It remains to show that $(E_{s,P'_{-s,F}})_{s\in I_0}$ and $(F_{s,P'_{-s,E}})_{s\in I_0}$ are complex interpolation scales and that $(\mathfrak{a}_s)_{s\in I_0}$ is strongly consistent. Let $r,s\in I_0$ and $q\in[r,s]$. Due to the fact that $(F_s)_{s\in I_0}$ is a complex interpolation scale, we immediately obtain $F_r\cap F_s \overset{d}{\hookrightarrow} F_q$. We know that $(P'_{-s,E})_{s\in I_0}$ is a consistent scale of projections on $(F_s)_{s\in I_0}$, such that

$$F_{r,P'_{-r,E}}\cap F_{s,P'_{-s,E}} \overset{d}{\hookrightarrow} F_{q,P'_{-q,E}}$$

follows immediately. On the other hand, we have

$$[F_{r,P'_{-r,E}},F_{s,P'_{-s,E}}]_\theta = P'_E[F_r,F_s]_\theta \quad (\theta\in(0,1), r,s\in I_0)$$

by [58, 1.2.4], which yields the complex interpolation scale. Note that the notation P'_E without parameter s is justified due to consistency. It remains to show strong consistency of $(\mathfrak{a}_s)_{s\in I_0}$. To this end, we first show strong right consistency. Pick $r,s\in I_0$ and $x\in F_{r,P'_{-r,E}}$, $y\in F_{s,P'_{-s,F}}$ such that

$$\mathfrak{a}_r(x',x)=\mathfrak{a}_s(x',y) \quad (x'\in E_{-r,P'_{r,F}}\cap E_{-s,P'_{s,F}}). \tag{6.6}$$

Now, let $x'\in E_{-r}\cap E_{-s}$ and remember that by (iv) $(\mathbb{P}_{r,E})_{r\in I_0}$ is a consistent scale of (symmetric) projections on $(E_r)_{r\in I_0}$ with $\mathbb{P}_{r,E}(E_r)=E_{r,P'_{-r,F}}$. We have $\mathbb{P}'_{-r,E}=\mathbb{P}_{r,F}$ and $\mathbb{P}'_{r,F}=\mathbb{P}_{-r,E}$ with $\mathbb{P}_{r,F}(F_r)=F_{r,P'_{-r,E}}$, where $(\mathbb{P}_{r,F})_{r\in I_0}$ is consistent on $(F_r)_{r\in I_0}$. Combining these facts with (6.6) we obtain

$$\mathfrak{a}_r(x',x)=\mathfrak{a}_r(x',\mathbb{P}_{r,F}x)=\mathfrak{a}_r(\mathbb{P}_{-r,E}x',x)=\mathfrak{a}_s(\mathbb{P}_{-s,E}x',y)=\mathfrak{a}_s(x',\mathbb{P}_{s,F}y)=\mathfrak{a}_s(x',y)$$

for $x'\in E_{-r}\cap E_{-s}$. Due to strong consistency of $(\mathfrak{a}_s)_{s\in I_0}$ on $(E_s,F_s)_{s\in I_0}$ this leads to $x=y$ in $\mathcal{H}$, thus to strong right consistency of $(\mathfrak{a}_s)_{s\in I_0}$ on $(E_{s,P'_{-s,F}},F_{s,P'_{-s,E}})_{s\in I_0}$. Next, pick $r,s\in I_0$ and $x\in E_{-r,P'_{r,F}}$, $y\in E_{-s,P'_{s,F}}$ such that

$$\mathfrak{a}_r(x,x')=\mathfrak{a}_s(y,x') \quad (x'\in F_{r,P'_{-r,E}}\cap F_{s,P'_{-s,E}}).$$

By the same arguments as above we obtain

$$\mathfrak{a}_r(x,x')=\mathfrak{a}_r(\mathbb{P}_{-r,E}x,x')=\mathfrak{a}_r(x,\mathbb{P}_{r,F}x')=\mathfrak{a}_s(y,\mathbb{P}_{s,F}x')=\mathfrak{a}_s(\mathbb{P}_{-s,E}y,x')=\mathfrak{a}_s(y,x')$$

for all $x' \in F_r \cap F_s$, thus strong left consistency of $(\mathfrak{a}_s)_{s\in I_0}$ on $(E_{s,P'_{-s,F}}, F_{s,P'_{-s,F}})_{s\in I_0}$. This shows that $(E_{s,P'_{-s,F}}, F_{s,P'_{-s,E}}, \mathfrak{a}_s)_{s\in I_0}$ is a duality scale. The assertion for $(E_{s,P_{s,E}}, F_{s,P_{s,F}}, \mathfrak{a}_s)_{s\in I_0}$ follows analogously.

Finally, we show consistency of $(\mathbb{P}_{s,E})_{s\in I_0}$, $(\mathbb{P}_{s,F})_{s\in I_0}$, $(\mathbb{Q}_{s,E})_{s\in I_0}$ and $(\mathbb{Q}_{s,F})_{s\in I_0}$. We just show consistency of $(\mathbb{P}_{s,F})_{s\in I_0}$. From (iv) we obtain an operator $\mathbb{P}_{0,F} : F_0 \to F_0$ onto $F_{0,P'_{0,E}}$. We fix $s \in I_0$. For $x \in F_0 \cap F_s$ and $x' \in E_0 \cap E_{-s,P'_{s,F}}$ we have

$$\begin{aligned}|\mathfrak{a}_0(x', \mathbb{P}_{0,F}x)| &= |\mathfrak{a}_0(P'_{s,F}x', \mathbb{P}_{0,F}x)| = |\mathfrak{a}_0(\mathbb{P}_{0,E}P'_{0,F}x', x)| \\ &= |\mathfrak{a}_0(P'_{s,F}x', x)| = |\mathfrak{a}_s(x', x)| \leqslant C\|x'\|_{E_{-s}}\|x\|_{F_s},\end{aligned}$$

by consistency of $(\mathfrak{a}_s)_{s\in I_0}$ and $(P'_{-s,F})_{s\in I_0}$ regarding Lemma 2.43 and $E_{0,P'_{0,F}} = E_{0,\mathbb{P}_{0,E}}$. From assertion (v) we know that $E_{-s,P'_{s,F}} = (F_{s,P'_{-s,E}})'_{\mathfrak{a}}$. By $E_{-s} \cap E_0 \overset{d}{\hookrightarrow} E_{-s}$ we have $E_{-s,P'_{s,F}} \cap E_0 \overset{d}{\hookrightarrow} E_{-s,P'_{s,F}}$. This yields

$$\|\mathbb{P}_{0,F}x\|_{F_s} = \|\mathbb{P}_{0,F}x\|_{F_{s,P'_{-s,E}}} = \sup_{0\neq x'\in E_{-s,P'_{s,F}}\cap E_0} \frac{|\mathfrak{a}_0(x', \mathbb{P}_{0,F}x)|}{\|x'\|_{E_{-s}}} \leqslant C\|x\|_{F_s}$$

for $x \in F_0 \cap F_s \overset{d}{\hookrightarrow} F_s$. So for each $s \in I_0$ the operator $\mathbb{P}_{0,F}$ on $F_0 \cap F_s$ extends to an operator $\tilde{\mathbb{P}}_{s,F} \in \mathscr{L}(F_s)$. In the same way we obtain a scale of extended operators $(\tilde{\mathbb{P}}_{s,E})_{s\in I_0}$ with $\tilde{\mathbb{P}}_{s,E}|_{E_s\cap E_0} = \mathbb{P}_{0,E}$ for $s \in I_0$. Next, we show that $(\tilde{\mathbb{P}}_{s,E})_{s\in I_0}$ and $(\tilde{\mathbb{P}}_{s,F})_{s\in I_0}$ are consistent scales of symmetric projections on $(E_s)_{s\in I_0}$ and $(F_s)_{s\in I_0}$. Let $s \in I_0$.

(1) $\tilde{\mathbb{P}}_{s,E}$ *and* $\tilde{\mathbb{P}}_{s,F}$ *are projections:* We start with $\tilde{\mathbb{P}}_{s,E}$. Let $x \in E_s$ and an approximating sequence $(x_k)_{k\in\mathbb{N}} \subseteq E_s \cap E_0$ be given, such that $x_k \to x$ in E_s for $k \to \infty$ by $E_s \cap E_0 \overset{d}{\hookrightarrow} E_s$. Then we have

$$\|\tilde{\mathbb{P}}_{s,E}\tilde{\mathbb{P}}_{s,E}x - \tilde{\mathbb{P}}_{s,E}x\|_{E_s} \leqslant \|\tilde{\mathbb{P}}_{s,E}\tilde{\mathbb{P}}_{s,E}x - \tilde{\mathbb{P}}_{s,E}\tilde{\mathbb{P}}_{s,E}x_k\|_{E_s} + \|\tilde{\mathbb{P}}_{s,E}x_k - \tilde{\mathbb{P}}_{s,E}x\|_{E_s} \overset{k\to\infty}{\longrightarrow} 0$$

due to $\tilde{\mathbb{P}}_{s,E} = \mathbb{P}_{0,E}$ on $E_s \cap E_0$. The assertion for $\tilde{\mathbb{P}}_{s,F}$ follows in the same way.

(2) $(\tilde{\mathbb{P}}_{s,E})_{s\in I_0}$ *and* $(\tilde{\mathbb{P}}_{s,F})_{s\in I_0}$ *are symmetric w.r.t. each other:* Let $x \in F_s$, $x' \in E_{-s}$ for $s \in I_0$ and $(x_k)_{k\in\mathbb{N}} \subseteq F_s \cap F_0$ such that $x_k \to x$ in F_s for $k \to \infty$. Moreover, let $(x'_k)_{k\in\mathbb{N}} \subseteq E_{-s} \cap E_0$ such that $x'_k \to x'$ in E_{-s} for $k \to \infty$. Then we have

$$\begin{aligned}\mathfrak{a}_s(x', \tilde{\mathbb{P}}_{s,F}x) &= \lim_{j,k\to\infty} \mathfrak{a}_s(x'_j, \tilde{\mathbb{P}}_{s,F}x_k) = \lim_{j,k\to\infty} \mathfrak{a}_s(x'_j, \mathbb{P}_{0,F}x_k) \\ &= \lim_{j,k\to\infty} \mathfrak{a}_s(\mathbb{P}_{0,E}x'_j, x_k) = \lim_{j,k\to\infty} \mathfrak{a}_s(\tilde{\mathbb{P}}_{-s,E}x'_j, x_k) = \mathfrak{a}_s(\tilde{\mathbb{P}}_{-s,E}x', x)\end{aligned}$$

by $\mathbb{P}'_{0,F} = \mathbb{P}_{0,E}$ and $\mathbb{P}'_{0,E} = \mathbb{P}_{0,F}$, which yields the symmetry w.r.t each other.

(3) $(\tilde{\mathbb{P}}_{s,E})_{s\in I_0}$ *and* $(\tilde{\mathbb{P}}_{s,F})_{s\in I_0}$ *are consistent:* We only show consistency for $(\tilde{\mathbb{P}}_{s,E})_{s\in I_0}$. Let $r, s \in I_0$ and w.l.o.g. $r < s$. If $r < 0 < s$, then $E_r \cap E_s \hookrightarrow E_0$ such that

$$\tilde{\mathbb{P}}_{r,E}x = \mathbb{P}_{0,E}x = \tilde{\mathbb{P}}_{s,E}x \qquad (x \in E_r \cap E_s).$$

If $r < s < 0$, then we know from Lemma 2.38 that $(E_{-\eta} + E_0, F_\eta \cap F_0, \mathfrak{b}^{\mathfrak{a}}_\eta)$ with $\mathfrak{b}^{\mathfrak{a}}_\eta = \mathfrak{a}_\eta + \mathfrak{a}_0$ is a duality system for each $\eta \in [0, -r]$. Moreover, we know that $(\tilde{\mathbb{P}}_{\eta,F})_{\eta\in[0,-r]}$ is consistent on $(F_\eta \cap F_0)_{\eta\in[0,-r]}$. Let $(\tilde{\mathbb{P}}_{\eta,F})'_{\mathfrak{b}^{\mathfrak{a}}}$ be the dual operator on $E_{-\eta} + E_0$ w.r.t. $\mathfrak{b}^{\mathfrak{a}}_\eta$ for each $\eta \in [0, -r]$.
Since we have $F_{-r} \cap F_0 \hookrightarrow F_{-s} \cap F_0$, we may apply Remark 2.35 and Lemma 2.40 to obtain consistency of $(\tilde{\mathbb{P}}_{-r,F})'_{\mathfrak{b}^{\mathfrak{a}}}$ and $(\tilde{\mathbb{P}}_{-s,F})'_{\mathfrak{b}^{\mathfrak{a}}}$. By exploiting $E_\eta \overset{d}{\hookrightarrow} E_\eta + E_0$ and symmetry of the operator scales it is easy to see that

$$(\tilde{\mathbb{P}}_{-\theta,F})'_{\mathfrak{b}^{\mathfrak{a}}} : E_\theta + E_0 \to E_\theta + E_0$$

is the unique extension of

$$(\tilde{\mathbb{P}}_{-\theta,F})'_{\mathfrak{a}} = \tilde{\mathbb{P}}_{\theta,E} : E_\theta \to E_\theta$$

for $\theta \in \{r, s\}$. Now, let $x \in E_r \cap E_s \hookrightarrow E_r + E_0, E_s + E_0$. Then we have

$$\tilde{\mathbb{P}}_{r,E}x = (\tilde{\mathbb{P}}_{-r,F})'_{\mathfrak{b}^{\mathfrak{a}}}x = (\tilde{\mathbb{P}}_{-s,F})'_{\mathfrak{b}^{\mathfrak{a}}}x = \tilde{\mathbb{P}}_{s,E}x$$

by consistency of $(\tilde{\mathbb{P}}_{-r,F})'_{\mathfrak{b}^{\mathfrak{a}}}$ and $(\tilde{\mathbb{P}}_{-s,F})'_{\mathfrak{b}^{\mathfrak{a}}}$, which yields consistency of $\tilde{\mathbb{P}}_{r,E}$ and $\tilde{\mathbb{P}}_{s,E}$. If $0 < r < s$, the assertion follows in an analogous way.

Finally we have to show that $\tilde{\mathbb{P}}_{s,E}(E_s) = E_{s,P'_{-s,F}}$, $(1 - \tilde{\mathbb{P}}_{s,E})(E_s) = E_{s,1-P_{s,E}}$ as well as $\tilde{\mathbb{P}}_{s,F}(F_s) = F_{s,P'_{-s,E}}$ and $(1 - \tilde{\mathbb{P}}_{s,F})(F_s) = F_{s,1-P_{s,F}}$. We will only prove the first assertion, the other ones follow in a similar manner.

- $\tilde{\mathbb{P}}_{s,E}(E_s) \subseteq E_{s,P'_{-s,F}}$: First, let $x \in \tilde{\mathbb{P}}_{s,E}(E_s) \cap E_0$. It is obvious by definition of $(\tilde{\mathbb{P}}_{s,E})_{s\in I_0}$ that $x = \tilde{\mathbb{P}}_{s,E}x = \mathbb{P}_{0,E}x$. Due to $\mathbb{P}_{0,E}(E_0) = E_{0,P'_{0,F}}$ and consistency of $(P'_{-s,F})_{s\in I_0}$ on $(E_s)_{s\in I_0}$ we also have $x = P'_{0,F}x = P'_{-s,F}x$ for $x \in \tilde{\mathbb{P}}_{s,E}(E_s) \cap E_0$, which yields $\tilde{\mathbb{P}}_{s,E}(E_s) \cap E_0 \subseteq E_{s,P'_{-s,F}} \cap E_0$
  Since $E_s \cap E_0 \overset{d}{\hookrightarrow} E_s$ and by the fact that $\tilde{\mathbb{P}}_{s,E}, P'_{-s,F}$ are bounded projections we have $\tilde{\mathbb{P}}_{s,E}(E_s) \cap E_0 \overset{d}{\hookrightarrow} \tilde{\mathbb{P}}_{s,E}(E_s)$ as well as $E_{s,P'_{-s,F}} \cap E_0 \overset{d}{\hookrightarrow} E_{s,P'_{-s,F}}$. For each $x \in \tilde{\mathbb{P}}_{s,E}(E_s)$ there exists a sequence $(x_k)_{k\in\mathbb{N}} \subseteq \tilde{\mathbb{P}}_{s,E}(E_s) \cap E_0$ such that $\|x_k - x\|_{E_s} \to 0$ for $k \to \infty$. Thus, we have $(x_k)_{k\in\mathbb{N}} \subseteq E_{s,P'_{-s,F}} \cap E_0$ and, since

$E_{s,P'_{-s,F}}$ is closed in E_s, also $x \in E_{s,P'_{-s,F}}$.

- $E_{s,P'_{-s,F}} \subseteq \tilde{\mathbb{P}}_{s,E}(E_s)$: Follows in an analogous way by using consistency of $(P'_{-s,F})_{s\in I_0}$ on $(E_s)_{s\in I_0}$ and the fact that $\tilde{\mathbb{P}}_{s,E}|_{E_s \cap E_0} = \mathbb{P}_{0,E}$.

We obtain $\tilde{\mathbb{P}}_{s,E} = \mathbb{P}_{s,E}$ due to the fact that every decomposition has a unique projection. Thus, $(\mathbb{P}_{s,E})_{s\in I_0}$ and by the same arguments $(\mathbb{P}_{s,F})_{s\in I_0}$, $(\mathbb{Q}_{s,E})_{s\in I_0}$ and $(\mathbb{Q}_{s,F})_{s\in I_0}$ are consistent. □

In addition to this main result we will present special case which will be helpful in order to fulfill the condition $1 \in \rho(R_{s,E}, E)$.

6.3 Lemma. *Let E be a complex Banach space and $\mathfrak{a} : E \times E \to \mathbb{C}$ a duality pairing such that $(E, E, \mathfrak{a})$ is a duality system and E equipped with the scalar product $\mathfrak{a}(\,\cdot\,, \overline{\cdot}\,)$ is a Hilbert space. Moreover, let $P : E \to E$ be a projection with $\overline{Px} = P\overline{x}$ for $x \in E$, P' be its dual w.r.t. $\mathfrak{a}$ and $R = (P - P')^2$. Then we have $1 \in \rho(R, E)$.*

Proof. First we note that $\|\cdot\|_{\mathfrak{a}} := \sqrt{\mathfrak{a}(\,\cdot\,, \overline{\cdot}\,)}$ is a norm on E which is equivalent to $\|\cdot\|_E$. We denote by $E_{\mathfrak{a}}$ the space E equipped with the norm $\|\cdot\|_{\mathfrak{a}}$. By $\overline{Px} = P\overline{x}$ for $x \in E_{\mathfrak{a}}$ we also have $\overline{P'x} = P'\overline{x}$. Let $x \in E_{\mathfrak{a}}$. Then we can write

$$\begin{aligned} \mathfrak{a}(Rx, \overline{x}) &= \mathfrak{a}((P - P')^2 x, \overline{x}) = -\mathfrak{a}((P - P')x, \overline{(P - P')x}) \\ &= -\|(P - P')x\|_{\mathfrak{a}}^2 = \mathfrak{a}(x, \overline{Rx}). \end{aligned}$$

This shows that R is dissipative and symmetric on $E_{\mathfrak{a}}$ and hence there exists an $r > 0$ such that $\sigma(R, E) \subseteq [-r, 0]$. □

6.2 Application to the Stokes Equations

As an application of the theory of duality scales we want to consider the stationary Stokes equation with mixed-type boundary conditions in a C^3-domain $\Omega \subseteq \mathbb{R}^3$ with compact boundary for some $\lambda > 0$.

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f && \text{in} \quad \Omega, \\ \operatorname{div} u &= 0 && \text{in} \quad \Omega, \\ \Pi_\tau \partial_\nu u &= 0 && \text{on} \quad \partial\Omega, \\ \nu \cdot u &= 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{6.7}$$

where we note that $\Pi_\tau \partial_\nu u = \Pi_\tau \nabla u \cdot \nu$ with Π_τ given as the tangential projection onto $\partial\Omega$. By perturbation with the lower order term $\Pi_\tau (\nabla u)^T \cdot \nu$ (cf. [25, Lemma 9.1]), solubility of this problem leads to solubility of the Stokes problem with perfect slip

boundary conditions. We want to show that the use of duality scales yields solutions with regularity up to $W^{1+\varepsilon,q}(\Omega)$ for $0 < \varepsilon < 1/\max\{q, q'\}$, where $q \in (1, \infty)$ and $1/q + 1/q' = 1$.
In order to formulate the problem in terms of duality scales, we introduce the following spaces.

$$\begin{aligned} W^{s,q}_{N,\nu}(\Omega) \;&:= \{u \in W^{s,q}(\Omega, \mathbb{R}^3) \,:\, \Pi_\tau \partial_\nu u|_{\partial\Omega} = 0,\ \nu \cdot u|_{\partial\Omega} = 0\} \quad (s > 1 + 1/q),\\ W^{s,q}_{\nu}(\Omega) \;&:= \{u \in W^{s,q}(\Omega, \mathbb{R}^3) \,:\, \nu \cdot u|_{\partial\Omega} = 0\} \qquad\qquad\qquad\qquad (s > 1/q). \end{aligned}$$

First we define the ν-Laplace operator as follows.

6.4 Definition. Let $\Omega \subseteq \mathbb{R}^3$ be a C^3-domain with compact boundary and $1 < q < \infty$. Then the ν-Laplace operator is defined as

$$\begin{aligned} &A_{N,\nu,q} : D(A_{N,\nu,q}) \subseteq L^q(\Omega) \to L^q(\Omega), \;\; u \mapsto A_{N,\nu,q} u = \Delta u,\\ &D(A_{N,\nu,q}) := W^{2,q}_{N,\nu}(\Omega). \end{aligned}$$

Next, for a fixed $\lambda > 0$, where λ has to be chosen accordingly, we define the operator

$$\mathbb{A}_{0,q} : D(A_{N,\nu,q}) \subseteq L^q(\Omega) \to L^q(\Omega), \;\; u \mapsto (\lambda - A_{N,\nu,q})u,$$

which admits a bounded H^∞-calculus and $0 \in \rho(\mathbb{A}_{0,q})$ (cf. [17, 25]). Now, let $\mathbb{E}_0 := L^{q'}(\Omega)$ and $\mathbb{F}_0 := L^q(\Omega)$ with $1/q + 1/q' = 1$. By [2, Chapter V], the pair $(\mathbb{A}_{0,q}, \mathbb{F}_0)$ generates a densely injected interpolation-extrapolation scale $[(\mathbb{F}_\alpha, \mathbb{A}_{\alpha,q})]_{\alpha\in[-1,\infty)}$ with

$$\mathbb{F}_\alpha := \begin{cases} W^{2\alpha,q}_{N,\nu}(\Omega) & \text{for } \alpha \in (1/2 + 1/2q, \infty),\\ W^{2\alpha,q}_{\nu}(\Omega) & \text{for } \alpha \in (1/2q, 1/2 + 1/2q),\\ W^{2\alpha,q}(\Omega, \mathbb{R}^3) & \text{for } \alpha \in [0, 1/2q),\\ W^{2\alpha,q}_0(\Omega, \mathbb{R}^3) & \text{for } \alpha \in (-1/2 + 1/2q, 0),\\ W^{2\alpha,q}_{\nu}(\Omega) & \text{for } \alpha \in (-1 + 1/2q, -1/2 + 1/2q),\\ W^{2\alpha,q}_{N,\nu}(\Omega) & \text{for } \alpha \in [-1, -1 + 1/2q), \end{cases}$$

where $W^{-s,q}_0(\Omega, \mathbb{R}^3) = (W^{s,q'}(\Omega, \mathbb{R}^3))'$ for $0 < s < 1/q'$, $W^{-s,q}_\nu(\Omega) = (W^{s,q'}_\nu(\Omega))'$ for $1/q' < s < 1 + 1/q'$ and $W^{-s,q}_{N,\nu}(\Omega) = (W^{s,q'}_{N,\nu}(\Omega))'$ for $s > 1 + 1/q'$. Here we left out the critical cases since they do not play a role in our further considerations. Note that $\mathbb{A}_{\alpha,q} \in \mathscr{L}_{is}(\mathbb{F}_{\alpha+1}, \mathbb{F}_\alpha)$ and $(\mathbb{A}_{\alpha,q})^{\alpha-\beta} \in \mathscr{L}_{is}(\mathbb{F}_\alpha, \mathbb{F}_\beta)$ for $-1 \leqslant \alpha < \beta < \infty$ and the scales of operators $(\mathbb{A}_{\alpha,q})_{\alpha\in[-1,\infty)}$ and $((\mathbb{A}_{\alpha,q})^{-1})_{\alpha\in[-1,\infty)}$ are consistent. By the same arguments we obtain a densely injected consistent interpolation-extrapolation scale $[(\mathbb{E}_\alpha, \mathbb{A}_{\alpha,q'})]_{\alpha\in[-1,\infty)}$ if we exchange the roles of q and q'. Due to the consistency we will write $\mathbb{A}_{\alpha,q} = \mathbb{A}_q$ where no confusion is likely. Note that $\mathbb{E}_0 \cong (\mathbb{F}_0)'$ and vice versa w.r.t.

the standard pairing

$$\hat{\mathfrak{a}}(u,v) := \int_\Omega uv\,dx \qquad (u \in \mathbb{E}_0, v \in \mathbb{F}_0)$$

and we have $(\mathbb{A}_{0,q})' = \mathbb{A}_{0,q'}$ w.r.t. this pairing. Moreover, there is a duality system of the form

$$\mathfrak{d}_\alpha^\lambda : \mathbb{E}_{-\alpha} \times \mathbb{F}_\alpha, \quad (u,v) \mapsto \int_\Omega (\mathbb{A}_{q'})^{-\alpha} u \, (\mathbb{A}_q)^\alpha v\,dx \tag{6.8}$$

for $\alpha \in [-1,1]$ such that we can identify $(\mathbb{F}_\alpha)' \cong \mathbb{E}_{-\alpha}$ and $(\mathbb{E}_\alpha)' \cong \mathbb{F}_{-\alpha}$.
Throughout this section we use the following notation corresponding to the obtained interpolation-extrapolation scale.

6.5 Definition. Let $1 < q < \infty$, $1/q + 1/q' = 1$, $s_0 := \min\{1/q, 1/q'\}$ and $I_0 := (-s_0, s_0)$. We set $E_s := W_\nu^{1+s,q'}(\Omega)$, $F_s := W_\nu^{1+s,q}(\Omega)$ and

$$\mathfrak{a}_s^\lambda : E_{-s} \times F_s \to \mathbb{C}, (u,v) \mapsto \int_\Omega (\mathbb{A}_{q'})^{\frac{1-s}{2}} u \, (\mathbb{A}_q)^{\frac{1+s}{2}} v\,dx \tag{6.9}$$

for $s \in I_0$.

6.6 Remark. We note that $E_s = \mathbb{E}_{(1+s)/2}$ and $F_s = \mathbb{F}_{(1+s)/2}$ for $s \in I_0$ as well as $(\mathbb{A}_{q'})^{\frac{1-s}{2}} \in \mathscr{L}_{is}(E_{-s}, \mathbb{E}_0)$ and $(\mathbb{A}_q)^{\frac{1+s}{2}} \in \mathscr{L}_{is}(F_s, \mathbb{F}_0)$.

Moreover, we make use of a symmetric duality pairing for which it is straightforward to show that admits a duality system.

6.7 Lemma. *Let*

$$\mathfrak{b}^\lambda : E_0 \times F_0 \to \mathbb{C}, \quad (u,v) \mapsto \lambda \int_\Omega uv\,dx + \int_\Omega \nabla u : \nabla v\,dx. \tag{6.10}$$

Then $(E_0, F_0, \mathfrak{b}^\lambda)$ *is a duality system.*

Now we are able to show that Definition 6.5 indeed yields a duality scale which is consistent with $\mathfrak{b}^\lambda$.

6.8 Lemma. *The scale* $(E_s, F_s, \mathfrak{a}_s^\lambda)_{s \in I_0}$ *is a duality scale which is consistent with* $\mathfrak{b}^\lambda$.

Proof. By using Remark 6.6 we see that $(E_{-s}, F_s, \mathfrak{a}_s^\lambda)$ is a duality system for $s \in I_0$. Due to the considerations from above it is clear that $(E_s)_{s \in I_0}$ and $(F_s)_{s \in I_0}$ are complex interpolation scales. It remains to show the strong consistency in order to obtain a duality scale.

To this end, we utilize the induced isomorphisms

$$\Phi_{s,E} : E_s \to (F_{-s})', \;\; u \mapsto \mathfrak{a}^\lambda_{-s}(u, \cdot),$$
$$\Phi_{s,F} : F_s \to (E_{-s})', \;\; v \mapsto \mathfrak{a}^\lambda_{s}(\cdot, v).$$

By

$$\mathbb{A}_{q'} : \mathbb{E}_{(1+s)/2} \to \mathbb{E}_{(-1+s)/2} = (F_{-s})', \;\; u \mapsto (\mathbb{A}_{q'}u)(\cdot)$$

we obtain

$$\begin{aligned}(\mathbb{A}_{q'}u)(v) &= \mathfrak{d}^\lambda_{(1-s)/2}(\mathbb{A}_{q'}u, v) = \int_\Omega \mathbb{A}_{q'}(\mathbb{A}_{q'})^{(-1+s)/2}u\,(\mathbb{A}_q)^{(1-s)/2}v\,dx\\ &= \int_\Omega (\mathbb{A}_{q'})^{(1+s)/2}u\,(\mathbb{A}_q)^{(1-s)/2}v\,dx = \mathfrak{a}^\lambda_{-s}(u,v) = (\Phi_{s,E}u)(v)\end{aligned}$$

for all $u \in E_s$ and $v \in F_{-s}$, thus $\Phi_{s,E} = \mathbb{A}_{q'}$. In an analogous way one can show $\Phi_{s,F} = \mathbb{A}_q$. Thus, by consistency of $\mathbb{A}_q$, $(\mathbb{A}_q)^{-1}$ and $\mathbb{A}_{q'}$, $(\mathbb{A}_{q'}^{-1})$ respectively we obtain strong consistency of $\mathfrak{a}^\lambda$ by Lemma 2.38. Finally, $(E_s, F_s, \mathfrak{a}^\lambda_s)_{s\in I_0}$ is a duality scale.
Next, we show consistency of $\mathfrak{a}^\lambda_s$ with $\mathfrak{b}^\lambda$ on $E_0 \times F_s$ for $s \in [0, s_0)$. By choosing $u \in E_0 \overset{d}{\hookrightarrow} E_{-s}$ and $v \in \mathbb{F}_1 \overset{d}{\hookrightarrow} F_s$ and using partial integration we obtain

$$\begin{aligned}\mathfrak{a}^\lambda_s(u,v) &= \int_\Omega (\mathbb{A}_{q'})^{\frac{1-s}{2}}u\,(\mathbb{A}_q)^{\frac{1+s}{2}}v\,dx\\ &= \int_\Omega u\,(\mathbb{A}_q v)\,dx = \int_\Omega u\,(\lambda - \Delta)v\,dx\\ &= \lambda\int_\Omega uv\,dx + \int_\Omega \nabla u : \nabla v\,dx = \mathfrak{b}^\lambda(u,v).\end{aligned}$$

Now, let u be as before and $v \in F_s$ with $(v_k)_{k\in\mathbb{N}} \subseteq \mathbb{F}_1$ such that $v_k \to v$ in F_s for $k \to \infty$. Then

$$\begin{aligned}\mathfrak{a}^\lambda_s(u,v) &= \int_\Omega (\mathbb{A}_{q'})^{\frac{1-s}{2}}u\,(\mathbb{A}_q)^{\frac{1+s}{2}}v\,dx\\ &= \lim_{k\to\infty}\int_\Omega (\mathbb{A}_{q'})^{\frac{1-s}{2}}u\,(\mathbb{A}_q)^{\frac{1+s}{2}}v_k\,dx\\ &= \lim_{k\to\infty}\lambda\int_\Omega uv_k\,dx + \int_\Omega \nabla u : \nabla v_k\,dx\\ &= \lambda\int_\Omega uv\,dx + \int_\Omega \nabla u : \nabla v\,dx = \mathfrak{b}^\lambda(u,v).\end{aligned}$$

Here we use that

$$\begin{aligned}|\mathfrak{a}^\lambda_s(u, v - v_k)| &\leqslant \|(\mathbb{A}_{q'})^{\frac{1-s}{2}}u\|_{\mathbb{E}_0}\|(\mathbb{A}_q)^{\frac{1+s}{2}}(v - v_k)\|_{\mathbb{F}_0}\\ &\leqslant \|(\mathbb{A}_q)^{\frac{1+s}{2}}\|_{\mathscr{L}(F_s,\mathbb{F}_0)}\|v - v_k\|_{F_s}\|(\mathbb{A}_{q'})^{\frac{1-s}{2}}u\|_{\mathbb{E}_0} \xrightarrow{k\to\infty} 0\end{aligned}$$

and

$$\left|\int_\Omega u(v-v_k)\,dx\right|, \left|\int_\Omega \nabla u : \nabla(v-v_k)\,dx\right| \leqslant \|u\|_{E_0}\|v-v_k\|_{F_s} \xrightarrow{k\to\infty} 0.$$

In an analogous manner one can show consistency of $\mathfrak{a}_s^\lambda$ with $\mathfrak{b}^\lambda$ on $E_{-s}\times F_0$ for $s\in(-s_0,0]$. □

As usual in the (Navier-)Stokes setting, we have to deal with the divergence condition $\operatorname{div} u = 0$ and the pressure gradient ∇p. In the following, we want to address this issue with help of the Helmholtz projection P, which maps onto divergence free functions. Finally, we want to show that $(E_{s,P}, F_{s,P}, \mathfrak{a}_s)_{s\in I_0}$ is also a duality scale. To this end, it is necessary to introduce the Helmholtz projection and show its higher regularity in $W^{1+s,q}(\Omega)$ for $s\in I_0$.

6.9 Definition (Helmholtz projection)**.** For $1<q<\infty$ the projection P_q subject to the decomposition

$$L^q(\Omega,\mathbb{R}^3) = L^q_\sigma(\Omega)\oplus G_q(\Omega)$$

with

$$\begin{aligned} L^q_\sigma(\Omega) &:= \overline{\{u\in C_c^\infty(\Omega,\mathbb{R}^3)\,:\,\operatorname{div} u = 0 \text{ in } \Omega\}}^{L^q},\\ G_q(\Omega) &:= \{u\in L^q(\Omega,\mathbb{R}^3)\,:\, u=\nabla p \text{ for some } p\in W^{1,p}_{loc}(\Omega)\}\end{aligned}$$

is called Helmholtz projection.

It is known (cf. [38, Corollary 7.4.4]) that the Helmholtz projection exists on $L^q(\Omega,\mathbb{R}^3)$ for $1<q<\infty$ and C^1-domains Ω with compact boundary and hence also for C^3-domains with compact boundary. Moreover, P_q is consistent w.r.t. q. We show that in our setting P_q admits higher regularity.

6.10 Lemma. *Let P_q be the Helmholtz projection on $L^q(\Omega)$ for $1<q<\infty$. Then there exists a consistent scale of Helmholtz projections $(P_{s,q})_{s\in I_0}$ on $(W^{1+s,q}_\nu(\Omega))_{s\in I_0}$ (and therefore on $(F_s)_{s\in I_0}$), such that $P_q|_{W^{1+s,q}_\nu(\Omega)} = P_q|_{F_s} = P_{s,q}$.*

Proof. Let $1<q<\infty$. First we show that P_q admits $W^{2,q}$-regularity. To this end, for a given $u\in W^{2,q}(\Omega,\mathbb{R}^3)$ we consider the following problem.

$$\begin{aligned} \Delta p &= \operatorname{div} u \quad \text{in} \quad \Omega,\\ \partial_\nu p &= u\cdot\nu \quad \text{on} \quad \partial\Omega.\end{aligned}$$

Then there exists an - up to a constant - unique solution p with $\nabla p \in W^{2,q}(\Omega, \mathbb{R}^3)$ and

$$\|\nabla p\|_{W^{2,q}(\Omega,\mathbb{R}^3)} \leqslant C \left(\|\operatorname{div} u\|_{W^{1,q}(\Omega)} + \|u \cdot \nu\|_{W_q^{2-1/q}(\partial\Omega)} \right),$$

cf. [38, Corollary 7.4.5]. We define

$$P_{2,q} u := u - \nabla p$$

and obtain

$$\begin{aligned}
\|P_{2,q} u\|_{W^{2,q}(\Omega,\mathbb{R}^3)} &= \|u - \nabla p\|_{W^{2,q}(\Omega,\mathbb{R}^3)} \\
&\leqslant \|u\|_{W^{2,q}(\Omega,\mathbb{R}^3)} + \|\nabla p\|_{W^{2,q}(\Omega,\mathbb{R}^3)} \\
&\leqslant \|u\|_{W^{2,q}(\Omega,\mathbb{R}^3)} + C \left(\|\operatorname{div} u\|_{W^{1,q}(\Omega)} + \|u \cdot \nu\|_{W_q^{2-1/q}(\partial\Omega)} \right) \\
&\leqslant C \|u\|_{W^{2,q}(\Omega,\mathbb{R}^3)},
\end{aligned}$$

thus $P_{2,q} \in \mathscr{L}(W^{2,q}(\Omega, \mathbb{R}^3))$. Moreover, it is easy to check that the boundary condition $u \cdot \nu = 0$ on $\partial\Omega$ is left invariant by $P_{2,q}$ and we have $P_{2,q} \in \mathscr{L}(W_\nu^{2,q}(\Omega))$. By construction it is clear that $P_{2,q}$ is a projection and $P_q|_{W_\nu^{2,q}(\Omega)} = P_{2,q}$, c.f. [16, Lemma III.1.2]. We obtain $W_\nu^{2,q}(\Omega) = W_{\nu,\sigma}^{2,q}(\Omega) \oplus G_q^2(\Omega)$ with

$$W_{\nu,\sigma}^{2,q}(\Omega) = W_\nu^{2,q}(\Omega) \cap L_\sigma^q(\Omega), \;\; G_q^2(\Omega) = W_\nu^{2,q}(\Omega) \cap G_q(\Omega).$$

Then, we have consistent projections $P_{t,q} \in \mathscr{L}(W_\nu^{t,q}(\Omega))$ for $t \in (0, 2)$ by interpolation theory (cf. [58, 1.2.3]) and $W_\nu^{t,q}(\Omega) = W_{\nu,\sigma}^{t,q}(\Omega) \oplus G_q^t(\Omega)$, where

$$W_{\nu,\sigma}^{t,q}(\Omega) = W_\nu^{t,q}(\Omega) \cap L_\sigma^q(\Omega), \;\; G_q^t(\Omega) = W_\nu^{t,q}(\Omega) \cap G_q(\Omega).$$

Hence, the Helmholtz projection P_q on $L^q(\Omega, \mathbb{R}^3)$ admits higher regularity and we obtain a scale of consistent projections $(P_{s,q})_{s \in I_0}$ on $(W_\nu^{1+s,q}(\Omega))_{s \in I_0}$. □

6.11 Remark. Due to symmetry of the Helmholtz projection w.r.t. the $L^q - L^{q'}$ pairing we can also extend the scale of projections consistently onto $\mathbb{E}_\alpha$ and $\mathbb{F}_\alpha$ for $\alpha \in (0, -1]$.

In the following results we work with dual projections of $P_{s,q}$ and $P_{s,q'}$ w.r.t. to different pairings, which we define in the following.

6.12 Definition. Let $s \in I_0$, $P_{s,q}$ be the Helmholtz projection on $F_s = W_\nu^{1+s,q}(\Omega)$ and $P_{s,q'}$ be the Helmholtz projection on $E_s = W_\nu^{1+s,q'}(\Omega)$ for $1 < q < \infty$ and $1/q + 1/q' = 1$. Then we set $P'_{s,q'} = (P_{s,q'})'_{\mathfrak{a}^\lambda} \in \mathscr{L}(F_{-s})$ and $P'_{s,q} = (P_{s,q})'_{\mathfrak{a}^\lambda} \in \mathscr{L}(E_{-s})$, i.e. the dual w.r.t. the pairing $\mathfrak{a}^\lambda$. Moreover, for $s = 0$ we denote by $\hat{P}'_{0,q'} = (P_{0,q'})'_{\mathfrak{b}^\lambda} \in \mathscr{L}(F_0)$ and $\hat{P}'_{0,q} = (P_{0,q})'_{\mathfrak{b}^\lambda} \in \mathscr{L}(E_0)$ the dual w.r.t. the pairing $\mathfrak{b}^\lambda$ as in (6.10).

An important ingredient in order to show that the projected scale $(E_{s,P_{s,q'}}, F_{s,P_{s,q}}, \mathfrak{a}_s^\lambda)_{s\in I_0}$ is a duality scale is the fact that the operator $\hat{P}'_{0,q'}(1 - P_{0,q}) : F_0 \to F_0$ is compact. The proof of this assertion is based on [44] and will be carried out in the following.

6.13 Remark. Note that by [25, Lemma 9.1] there exists a $V \in W^{1,\infty}(\partial\Omega, \mathbb{R}^{3\times3})$ such that for all $u \in W^{2,q}(\Omega, \mathbb{R}^3)$ satisfying $\nu \cdot u = 0$ on $\partial\Omega$ we have

$$\Pi_\tau(\nabla u + (\nabla u)^T)\nu = (\nabla u - (\nabla u)^T)\nu + \Pi_\tau V u \qquad \text{on } \partial\Omega.$$

6.14 Lemma. *The operator* $\hat{P}'_{0,q'}(1 - P_{0,q}) : F_0 \to F_0$ *is compact.*

Proof. First, we note that $E_0 = W^{1,q'}_\nu(\Omega)$ and $F_0 = W^{1,q}_\nu(\Omega)$. Let $u \in W^{1,q}_\nu(\Omega)$. Then we have $(1 - P_{0,q})u = \nabla p \in W^{1,q}_\nu(\Omega)$ for some p by definition of the Helmholtz projection. Let $v \in W^{1,q'}_\nu(\Omega)$, then

$$\begin{aligned}\mathfrak{b}^\lambda(v, \hat{P}'_{0,q'}(1 - P_{0,q})u) &= \mathfrak{b}^\lambda(P_{0,q'}v, \nabla p)\\ &= \lambda \int_\Omega P_{0,q'}v \cdot \nabla p \, dx + \int_\Omega \nabla^2 p : \nabla P_{0,q'}v \, dx\\ &= \int_\Omega \nabla^2 p : \nabla P_{0,q'}v \, dx,\end{aligned}$$

where we know by Lemma 6.10 that $P_{0,q'}v \in W^{1,q}_\nu(\Omega)$. It is $\operatorname{div}(\nabla P_{0,q'}v)^T = 0$ such that the normal trace $(\nabla P_{0,q'}v)^T\nu$ is well-defined in $W^{-1/q'}_{q'}(\partial\Omega, \mathbb{R}^3)$. We calculate

$$\operatorname{div}[(\nabla P_{0,q'}v)\nabla p] = \nabla\operatorname{div} P_{0,q'}v \cdot \nabla p + \nabla P_{0,q'}v : \nabla^2 p = \nabla P_{0,q'}v : \nabla^2 p$$

in $\mathcal{D}'(\Omega)$. Applying the generalized Gauß theorem yields

$$\int_\Omega \nabla^2 p : \nabla P_{0,q'}v \, dx = \langle \nabla p, (\nabla P_{0,q'}v)^T\nu \rangle_{W^{1-1/q}_q(\partial\Omega,\mathbb{R}^3), W^{-1/q'}_{q'}(\partial\Omega,\mathbb{R}^3)}.$$

We use the tangential and normal projection on the boundary to the result

$$(\nabla P_{0,q'}v)^T\nu = \Pi_\nu(\nabla P_{0,q'}v)^T\nu + \Pi_\tau(\nabla P_{0,q'}v)^T\nu.$$

This leads to

$$\mathfrak{b}^\lambda(v, \hat{P}'_{0,q'}(1 - P_{0,q})u) = \langle \nabla p, (\nabla P_{0,q'}v)^T\nu \rangle_{W^{1-1/q}_q(\partial\Omega,\mathbb{R}^3), W^{-1/q'}_{q'}(\partial\Omega,\mathbb{R}^3)}$$

$$
\begin{aligned}
&= \langle \Pi_\nu \nabla p, \Pi_\nu (\nabla P_{0,q'} v)^T \nu \rangle_{W_q^{1-1/q}(\partial\Omega,\mathbb{R}^3), W_{q'}^{-1/q'}(\partial\Omega,\mathbb{R}^3)} \\
&\quad + \langle \nabla p, \Pi_\tau (\nabla P_{0,q'} v)^T \nu \rangle_{W_q^{1-1/q}(\partial\Omega,\mathbb{R}^3), W_{q'}^{-1/q'}(\partial\Omega,\mathbb{R}^3)} \\
&= \langle \nu \cdot u, \Pi_\nu (\nabla P_{0,q'} v)^T \nu \rangle_{W_q^{1-1/q}(\partial\Omega,\mathbb{R}^3), W_{q'}^{-1/q'}(\partial\Omega,\mathbb{R}^3)} \\
&\quad + \int_{\partial\Omega} (\nabla p)^T \, \Pi_\tau V P_{0,q'} v \, d\sigma \\
&= \int_{\partial\Omega} (\nabla p)^T \, \Pi_\tau V P_{0,q'} v \, d\sigma
\end{aligned}
$$

for some $V \in L^\infty(\partial\Omega, \mathbb{R}^{3\times 3})$ (cf. Remark 6.13).
Next, set $G := \Omega$ if Ω is bounded. If Ω is exterior, choose a ball $B \subseteq \mathbb{R}^3$ such that $\overline{\mathbb{R}^3 \setminus \Omega} \subseteq B$ and set $G = \Omega \cap B$. For $\alpha \in (1/q, 1]$ the trace operator $\gamma : u \mapsto u|_{\partial\Omega}$ satisfies (cf. [32])

$$
\gamma \in \mathscr{L}(W^{\alpha,q}(G,\mathbb{R}^3), W_q^{\alpha-1/q}(\partial\Omega,\mathbb{R}^3)) \cap \mathscr{L}(W^{1,q'}(\Omega,\mathbb{R}^3), L^{q'}(\partial\Omega,\mathbb{R}^3)).
$$

Thus we can estimate as

$$
\begin{aligned}
|\mathfrak{b}^\lambda(v, \hat{P}'_{0,q'}(1-P_{0,q})u)| &\leqslant C\|\gamma(1-P_{0,q})u\|_{L^q(\partial\Omega,\mathbb{R}^3)} \|\gamma P_{0,q'} v\|_{L^{q'}(\partial\Omega,\mathbb{R}^3)} \\
&\leqslant C\|u\|_{W^{1-\varepsilon,q}(G,\mathbb{R}^3)} \|P_{0,q'} v\|_{W^{1,q'}(\Omega,\mathbb{R}^3)} \\
&\leqslant C\|u\|_{W^{1-\varepsilon,q}(G,\mathbb{R}^3)} \|v\|_{W^{1,q'}(\Omega,\mathbb{R}^3)}
\end{aligned}
$$

for all $v \in W_\nu^{1,q'}(\Omega)$ with some $\varepsilon > 0$ sufficiently small. According to Remark 2.32 (1) this yields

$$
\|\hat{P}'_{0,q'}(1-P_{0,q})u\|_{W^{1,q}(\Omega,\mathbb{R}^3)} \leqslant C\|u\|_{W^{1-\varepsilon,q}(G,\mathbb{R}^3)} \qquad (u \in W_\nu^{1,q}(\Omega)). \tag{6.11}
$$

Let $(u_k)_{k\in\mathbb{N}} \subseteq W_\nu^{1,q}(\Omega)$ be a bounded sequence. Since the standard norms of $W^{1,q}(\Omega,\mathbb{R}^3)$ and $W_\nu^{1,q}(\Omega)$ coincide and the embedding $W^{1,q}(\Omega,\mathbb{R}^3) \hookrightarrow W^{1-\varepsilon,q}(G,\mathbb{R}^3)$ is compact, there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ that converges in $W^{1-\varepsilon,q}(G,\mathbb{R}^3)$ and therefore is a Cauchy sequence in this space. Due to (6.11), $(\hat{P}'_{0,q'}(1-P_{0,q})u_{k_j})_{j\in\mathbb{N}}$ is a Cauchy sequence in $W_\nu^{1,q}(\Omega)$ and converges. Hence, $\hat{P}'_{0,q'}(1-P_{0,q}) : F_0 \to F_0$ is compact. □

6.15 Remark. By the same arguments we may also obtain compactness of the operator $\hat{P}'_{0,q}(1-P_{0,q'}) : E_0 \to E_0$.

The results from above allow us to consider the duality scale of projected subspaces.

6.16 Theorem. *Let* $1 < q < \infty$, $1/q + 1/q' = 1$, $E_s := W_\nu^{1+s,q'}(\Omega)$, $F_s := W_\nu^{1+s,q}(\Omega)$, $s_0 := \min\{1/q, 1/q'\}$ *and* $I_0 := (-s_0, s_0)$. *Then* $(E_{s,P_{s,q'}}, F_{s,P_{s,q}}, \mathfrak{a}_s^\lambda)_{s\in I_0}$ *is a duality scale.*

6.17 Remark. Note that $E_{s,P_{s,q'}} = W_{\nu,\sigma}^{1+s,q'}(\Omega)$ and $F_{s,P_{s,q}} = W_{\nu,\sigma}^{1+s,q}(\Omega)$.

Proof of Theorem 6.16. We split the proof into several steps.

Step 1: Compactness of $P'_{s,q'}(1 - P_{s,q})$ *for* $s \in I_0$.
Let $(P_{s,q})_{s\in I_0}$ and and $(P_{s,q'})_{s\in I_0}$ be the consistent scales of the Helmholtz projections on $(F_s)_{s\in I_0} = (W_\nu^{1+s,q}(\Omega))_{s\in I_0}$ and $(E_s)_{s\in I_0} = (W_\nu^{1+s,q'}(\Omega))_{s\in I_0}$ as in Lemma 6.10. From Lemma 6.14 we already know that $\hat{P}'_{0,q'}(1 - P_{0,q}) : F_0 \to F_0$ is compact. We want to exploit the compactness in order to obtain a projected duality scale. To this end, we first note that

$$\mathfrak{b}^\lambda(u, \hat{P}'_{0,q'}v) = \mathfrak{b}^\lambda(P_{0,q'}u, v) = \mathfrak{a}_0^\lambda(P_{0,q'}u, v) = \mathfrak{a}_0^\lambda(u, P'_{0,q'}v) = \mathfrak{b}^\lambda(u, P'_{0,q'}v)$$

for $u \in E_0$ and $v \in F_0$ due to consistency of $\mathfrak{a}_0^\lambda$ and $\mathfrak{b}^\lambda$. Since $\mathfrak{b}^\lambda$ is a dualitiy pairing between E_0 and F_0, this yields $\hat{P}'_{0,q'} = P'_{0,q'}$ and therefore $\hat{P}'_{0,q'}(1-P_{0,q}) = P'_{0,q'}(1-P_{0,q})$ and compactness of $P'_{0,q'}(1 - P_{0,q})$.
Now, fix $s \in [0, s_0)$ and $s < \varepsilon < s_0$. We know that $F_{\pm s}$ and F_0 are complex interpolation spaces of type $[F_{-\varepsilon}, F_{+\varepsilon}]_\theta$ for certain $\theta \in (0, 1)$. Furthermore, $(P'_{-s,q'}(1 - P_{s,q}))_{s\in I_0}$ is consistent by assumption and Lemma 2.40. Due to compactness of $P'_{0,q'}(1-P_{0,q})$ and the extrapolation result for compactness in [11, Theorem 2.1] we obtain compactness of $P'_{\pm s,q'}(1 - P_{\mp s,q})$ for $s \in [0, s_0)$.

Step 2 (from $W_\nu^{1,2}$ *to* $W_\nu^{1,q}$*):* $1 \in \rho(P'_{0,q'}(1 - P_{0,q}), F_0)$ *for* $1 < q < \infty$.
From Lemma 6.3 we know that $1 \in \rho(R_{0,2}, W_\nu^{1,2}(\Omega))$ and thus, by Lemma 6.1, also $1 \in \rho(P'_{0,2}(1 - P_{0,2}), W_\nu^{1,2}(\Omega))$ if we restrict the scale to the case $s = 0$ and $q = 2$. Moreover, we know that $P'_{0,q'}(1-P_{0,q}) : W_\nu^{1,q}(\Omega) \to W_\nu^{1,q}(\Omega)$ is compact for $1 < q < \infty$. By Lemma 2.46 (use the duality scale $(W_\nu^{1,q'}(\Omega), W_\nu^{1,q}(\Omega), \mathfrak{b}^\lambda)_{q\in I_0}$ with $I_0 = (q'_0, q_0)$ for $q_0 > 2$) we obtain that $1 \in \rho(P'_{0,q'}(1 - P_{0,q}), W_\nu^{1,q}(\Omega))$ for $1 < q < \infty$.

Step 3 (from $W_\nu^{1,q}$ *to* $W_\nu^{1+s,q}$*):* $1 \in \rho(P'_{-s,q'}(1 - P_{s,q}), F_s)$ *for* $s \in I_0$.
Now, fix $1 < q < \infty$ and consider the scale $(E_s, F_s, \mathfrak{a}_s^\lambda)_{s\in I_0}$ again. Due to the compactness of $P'_{-s,q'}(1 - P_{s,q})$ for $s \in I_0$ we may apply Lemma 2.46 again in order to obtain $1 \in \rho(P'_{-s,q'}(1 - P_{s,q}), F_s)$ for $s \in I_0$. By using the same arguments we also obtain $1 \in \rho(P'_{-s,q}(1 - P_{s,q'}), E_s)$ for $s \in I_0$. Then, an application of Theorem 6.2 yields that $(E_{s,P_{s,q'}}, F_{s,P_{s,q}}, \mathfrak{a}_s^\lambda)_{s\in I_0}$ is a duality scale. □

6.18 Theorem. *Let* $1 < q < \infty$*,* $1/q + 1/q' = 1$*,* $s_0 := \min\{1/q, 1/q'\}$ *and* $s \in [0, s_0)$*. Let* $\lambda > 0$ *be chosen accordingly large. Then for every* $f \in W^{-1+s,q}_{\nu,\sigma}(\Omega)$*, where* $W^{-1+s,q}_{\nu,\sigma}(\Omega) := \mathscr{L}(W^{1-s,q'}_{\nu,\sigma}(\Omega), \mathbb{C})$*, the Stokes resolvent problem*

$$\int_\Omega (\mathbb{A}_{q'})^{\frac{1-s}{2}} u \, (\mathbb{A}_q)^{\frac{1+s}{2}} v \, dx = \langle u, f \rangle_{W^{1-s,q'}_{\nu,\sigma}(\Omega), W^{-1+s,q}_{\nu,\sigma}(\Omega)} \quad (u \in W^{1-s,q'}_{\nu,\sigma}(\Omega))$$

possesses a unique solution $v \in W^{1+s,q}_{\nu,\sigma}(\Omega)$ *satisfying*

$$\|v\|_{W^{1+s,q}_{\nu,\sigma}(\Omega)} \leqslant C \|f\|_{W^{-1+s,q}_{\nu,\sigma}(\Omega)}$$

with $C > 0$ *independent of* f*.*

Proof. From Theorem 6.16 we know that $\mathfrak{a}^\lambda_s$ is a duality system between $W^{1-s,q'}_{\nu,\sigma}(\Omega)$ and $W^{1+s,q}_{\nu,\sigma}(\Omega)$. Hence, for $f \in W^{-1+s,q}_{\nu,\sigma}(\Omega)$ there exists a unique $v \in W^{1+s,q}_{\nu,\sigma}(\Omega)$ such that

$$\mathfrak{a}^\lambda_s(\cdot, v) = \langle \cdot, f \rangle_{W^{1-s,q'}_{\nu,\sigma}(\Omega), W^{-1+s,q}_{\nu,\sigma}(\Omega)} \in (W^{1-s,q'}_{\nu,\sigma}(\Omega))', \tag{6.12}$$

which yields the unique solution of the Stokes resolvent problem. Moreover, we have

$$\begin{aligned} \|v\|_{W^{1+s,q}_{\nu,\sigma}(\Omega)} &= \sup_{0 \neq u \in W^{1-s,q'}_{\nu,\sigma}(\Omega)} \frac{|\mathfrak{a}^\lambda_s(u, v)|}{\|u\|_{W^{1-s,q'}_{\nu,\sigma}(\Omega)}} \\ &= \sup_{0 \neq u \in W^{1-s,q'}_{\nu,\sigma}(\Omega)} \frac{\left| \langle u, f \rangle_{W^{1-s,q'}_{\nu,\sigma}(\Omega), W^{-1+s,q}_{\nu,\sigma}(\Omega)} \right|}{\|u\|_{W^{1-s,q'}_{\nu,\sigma}(\Omega)}} \leqslant C \|f\|_{W^{-1+s,q}_{\nu,\sigma}(\Omega)} \end{aligned}$$

with $C > 0$ independent of f. □

6.3 A Criterion for Projected Duality Scales based on Compactness

In the application of Theorem 6.2 to the Stokes operator in the last section, compactness playes a central role. This motivates the assumption that compactness of the operator $R_{s,E}$ or $R_{s,F}$ enables us to formulate another criterion under which a duality scale is preserved when it is restricted to complemented subspaces. In this section we will trade of the condition $1 \in \rho(R)$ for compactness of R and injectivity of $1 - R$ in order to obtain projected duality scales. In fact, we observe that in case of compactness of R, we see that $1 \in \rho(R)$ if $1 - R$ is only injective due to Fredholm's alternative, which will play an important role in Theorem 6.20.

At first we prove a lemma, where we use the following notation: For a projection P on a Banach space E and $x \in E$ let $x_P := Px$ and $x_{1-P} := (1-P)x$ as well as $E_P := P(E)$ and $E_{1-P} := (1-P)(E)$.

6.19 Lemma. *Let E be a Banach space and $P, Q \in \mathscr{L}(E)$ be projections on E. Then the following assertions are equivalent:*

(i) $1 - R : E \to E$ *is injective, where* $R := (P - Q)^2$.

(ii) $E_P \cap E_{1-Q} = E_Q \cap E_{1-P} = \{0\}$.

(iii) $P : E_Q \to E_P$ *and* $Q : E_P \to E_Q$ *are injective.*

(iv) $1 - P : E_{1-Q} \to E_{1-P}$ *and* $1 - Q : E_{1-P} \to E_{1-Q}$ *are injective.*

Proof. (ii) $\Rightarrow$ (iii): Consider $P : E_Q \to E_P$ and $Px = 0$ for a $x \in E_Q$. Then we have $x = (1 - P)x \in E_Q \cap E_{1-P}$, thus $x = 0$ by (ii). The assertion for $Q : E_P \to E_Q$ follows in the same way.

(iii) $\Rightarrow$ (ii): Obviously, we have $Px = 0$ for $x \in E_Q \cap E_{1-P}$ and thus, due to injectivity of $P : E_Q \to E_P$ by (iii), $x = 0$. This yields $E_Q \cap E_{1-P} = \{0\}$. The assertion for the intersection $E_P \cap E_{1-Q}$ follows in an analogous way.

(ii) $\Leftrightarrow$ (iv): Follows in the same way as (iii) $\Leftrightarrow$ (ii) by interchanging the roles of P and Q with $1 - P$ and $1 - Q$.

(i) $\Rightarrow$ (iii): We have

$$(1 - R)x = (1 - (P + Q - PQ - QP))x = PQx \qquad (x \in E_P).$$

Furthermore $1 - R$ is injective on E due to (i), so PQ is injective on $E_P \subseteq E$. This yields injectivity of Q on E_P. The injectivity of P on E_Q follows in the same manner.

(ii) and (iii) and (iv) $\Rightarrow$ (i): Let $x \in E$ with $(1 - R)x = 0$. Due to the fact that $E = E_P \oplus E_{1-P}$, we have uniquely determined $x_P \in E_P$ and $x_{1-P} \in E_{1-P}$ such that $x = x_P + x_{1-P}$. This yields

$$0 = (1 - R)x = (1 - R)x_P + (1 - R)x_{1-P},$$

consequently $(1 - R)x_P = P(1 - R)x = 0$ and $(1 - R)x_{1-P} = (1 - P)(1 - R)x = 0$ since $(1 - R)$ commutes with P. We obtain

$$PQx_P = (1 - R)x_P = 0 \tag{6.13}$$

and

$$(1 - Q + PQ)x_{1-P} = (1 - R)x_{1-P} = 0. \tag{6.14}$$

By (6.13) and (iii) it follows that $x_P = 0$. Moreover, by (6.14) and (ii) we have $(1-Q)x_{1-P} = 0$, which yields $x_{1-P} = 0$ by using (iv). Finally, it is $x = x_P + x_{1-P} = 0$, which completes the proof. □

Using the result from above and Theorem 6.2, we obtain the following result.

6.20 Theorem. *Let $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ be a duality scale and $(P_{s,E})_{s\in I_0}$ and $(P_{s,F})_{s\in I_0}$ be consistent scales of projections. Let $R_{s,E} := (P_{s,E} - P'_{-s,F})^2$ and $R_{s,F} := (P_{s,F} - P'_{-s,E})^2$. If there exist $s_1, s_2 \in I_0$ such that $R_{s_1,E}$ is compact and one of the conditions*

(i) $1 - R_{s_2,E} : E_{s_2} \to E_{s_2}$ *is injective,*

(ii) $E_{P_{s_2,E}} \cap E_{1-P'_{-s_2,F}} = E_{P'_{-s_2,F}} \cap E_{1-P_{s_2,E}} = \{0\}$,

(iii) $P_{s_2,E} : E_{P'_{-s_2,F}} \to E_{P_{s_2,E}}$ *and* $P'_{-s_2,F} : E_{P_{s_2,E}} \to E_{P'_{-s_2,F}}$ *are injective,*

(iv) $1 - P_{s_2,E} : E_{1-P'_{-s_2,F}} \to E_{1-P_{s_2,E}}$ *and* $1 - P'_{-s_2,F} : E_{1-P_{s_2,E}} \to E_{1-P'_{-s_2,F}}$ *are injective,*

is fulfilled, then $(E_{s,P_{s,E}}, F_{s,P_{s,F}}, \mathfrak{a}_s)_{s\in I_0}$ is a duality scale.

Proof. At first we note that due to Lemma 6.19 the conditions (i)-(iv) are equivalent. Thus we will only work with condition (i).
Next, by [11, Theorem 2.1] and compactness of $R_{s_1,E}$ we obtain compactness of $R_{s,E}$ for $s \in I_0$ and, by the duality $(R_{s,E})' = R_{-s,F}$ and vice versa, compactness of $R_{s,F}$ for $s \in I_0$. Moreover, due to Fredholm's alternative we have $1 \in \rho(R_{s_2,E})$. Then Lemma 2.46 yields $1 \in \rho(R_{s,E})$ for $s \in I_0$ and the assertion follows by an application of Theorem 6.2. □

6.21 Remark. (i) Theorem 6.20 also holds if we place the conditions upon $R_{s_1,F}$ and $R_{s_2,F}$ on the scale $(F_s)_{s\in I_0}$.

(ii) Note that in contrast to Theorem 6.2, only injectivity of $1 - R_{s,E}$ is needed, but this comes with the tradeoff that we need compactness. Section 6.2 shows that such a condition is meaningful in some applications.

Finally, by similar arguments we obtain the following lemma.

6.22 Lemma. *Let $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ be a duality scale and $(P_{s,E})_{s\in I_0}$ and $(P_{s,F})_{s\in I_0}$ be consistent scales of projections. Let $R_{s,E} := (P_{s,E} - P'_{-s,F})^2$ and $R_{s,F} := (P_{s,F} - P'_{-s,E})^2$. Assume that there exist $s_1, s_2, s_3 \in I_0$ such that*

(i) $1 - R_{s_1,E} : E_{s_1} \to E_{s_1}$ *is injective,*

(ii) $P'_{-s_2,F}(1 - P_{s_2,E}) : E_{s_2} \to E_{s_2}$ *is compact,*

(iii) $P'_{-s_3,E}(1 - P_{s_3,F}) : F_{s_3} \to F_{s_3}$ *is compact.*

Then $(E_{s,P_{s,E}}, F_{s,P_{s,F}}, \mathfrak{a}_s)_{s \in I_0}$ *is a duality scale.*

Proof. First we note that by an application of [11, Theorem 2.1] we obtain compactness of $(P'_{-s,E}(1 - P_{s,F})_s)_{s \in I_0}$ and $(P'_{-s,F}(1 - P_{s,E})_s)_{s \in I_0}$. Due to Schauder's theorem and reflexivity we have

$$[P'_{s,E}(1 - P_{-s,F})]'_{\mathfrak{a}} = (1 - P'_{-s,F})P_{s,E} \in \mathscr{L}(E_s)$$

compactly for $s \in I_0$. Note that we may obtain the same results with E replaced by F. We have that

$$\begin{aligned} R_{s,E}|_{P'_{-s,F}(E_s)} &= P'_{-s,F}(1 - P_{s,E}), \\ R_{s,E}|_{(1-P'_{-s,F})(E_s)} &= (1 - P'_{-s,F})P_{s,E}, \end{aligned}$$

which yields compactness of $R_{s,E} \in \mathscr{L}(E_s)$ for $s \in I_0$, especially for $s_1 \in I_0$. Then, an application of Fredholm's alternative, Lemma 2.46 and Theorem 6.2 yields the assertion. □

7 Conclusions

In this thesis, we considered three different topics related to the field of partial differential equations: first, the analysis of stability, instability and the long-term behavior of a living fluid model, second the analysis of stability and instability for a class of heterogeneous catalysis models and third the theory of duality scales on complemented subspaces with an application to the Stokes equations on C^3-domains. New results related to all three topics were presented and discussed, leaving space for some future considerations.

Living Fluids

We analyzed generalized Navier-Stokes equations with fourth order terms, which describes the self propulsed motion of living fluids, e.g. bacteria in some liquid fluid. First, we noted that the model (1.2) is globally well-posed in the periodic L^2-setting and listed the physically relevant equilibria, i.e., the disordered isotropic and the ordered polar states. Next we considered linear stability and instability. It turned out that stability and instability can be characterized by the model parameters Γ_2, Γ_0, α and the existence of unstable Fourier modes for instability.
The essential part of Chapter 3 deals with the topic of nonlinear stability and instability. Especially the behavior of solutions about the ordered polar states, which build a manifold of equilibria, is of interest. We showed that - depending on the parameter set and the existence of unstable Fourier modes - the equilibria on this manifold turn out to be normally stable or normally hyperbolic. The latter case is a potential indication for active turbulence, which was previously observed in real world experiments. From this point of view, the theoretical results confirm the assumptions and simulations that where previously made for this model.
In Chapter 4, we performed a complete analysis of the long-term behavior of the living fluids model. We showed that there exists a finite dimensional global attractor of arbitrary high regularity. The long-term dynamics of the living fluids model is therefore determined by a finite dimensional subset of modes. Especially the fact that the global attractor is finite dimensional in terms of the Hausdorff and the fractal dimension in-

dicates, that the long-term dynamics reduce to a simpler structure than one would assume given the infinite dimensional phase space. Utilizing some further results on inertial manifolds, we proved the existence of such a manifold for the model in $n = 2$ dimensions by using a spectral gap condition.
Regarding the living fluids model there are still some aspects left open for future considerations. It is unknown if there exist further physically relevant equilibria for this model, if they build a structure like a manifold and how solutions behave in their surrounding. It is known that there exist more equilibria (cf. [61]), but these are not relevant in a physical context. Given the global attractor, there is not much more known about its concrete structure. Considering the fact that it is finite dimensional (at least in some sense), it may be worth to do some numerical simulations in order to obtain more information regarding the structure. Moreover, it is unclear if the existence of an inertial manifold can be proved for $n = 3$ dimensions without relying on a spectral gap condition.

Heterogeneous Catalysis

In Chapter 5 of this thesis we dealt with stability and instability of a heterogeneous catalysis model in a cylindrical domain. One feature of the model is the coupling of equations in the bulk and nonlinear equations on the lateral surface of the cylinder, modeling the chemical reaction which occurs during the catalysis process.
Based on previous results regarding the maximal regularity of the linearized equations, we showed a stability result in the L^p-setting that indicates that the behavior of solutions near stationary points of the system is determined by the chemical reactions. In our result, stability of equilibria is given dependent on a bound on the first derivative of the chemical reaction rates. As an example, we considered the chemical equilibria in which the chemical reaction itself is at rest.
Based on the analysis of the stability behavior we extracted a result for instability, too. It seems to be difficult to give a concrete example fulfilling these conditions for instability. Several approaches to this end were made, e.g. considering the concrete calculations for several types of chemical reactions and a reduction to the half space setting in order to apply the Fourier transform in two directions. Especially the attempts to find (abstract or concrete) eigenvectors fulfilling the conditions of the instability result suffered from some hindrances, cf. Remark 5.7 for more details. Consequently the topic of a detailed characterization of instability of the heterogeneous catalysis is left open for future considerations.

Duality Scales and Partial Differential Equations

Concerning the functional analytic properties of linear operators on projected subspaces, e.g. the well-known Stokes operator, we addressed scales of Banach spaces, duality pairings and duality scales as well as projections on those scales in Chapter 6. To the best of the author's knowledge, the concept of duality scales for complemented subspaces was first introduced in [44] for one scale of Banach spaces. In the first section of this chapter, we proved conditions under which the property of being a duality scale $(E_s, F_s, \mathfrak{a}_s)_{s\in I_0}$ is preserved if we project the Banach spaces E_s and F_s. One of the equivalent conditions was given as $1 \in \rho(P'(1-P))$, where P' denotes the dual of the projection P w.r.t. $\mathfrak{a}$. Note that in contrast to the results from [44], the scales $(E_s)_{s\in I_0}$ and $(F_s)_{s\in I_0}$ can consist of different Banach spaces.
We used these results in order to show well-posedness of the Neumann-Stokes operator on a C^3-domain Ω with compact boundary in $W^{1+\varepsilon,q}_{\nu,\sigma}(\Omega)$, where $1 < q < \infty$ and $0 \leqslant \varepsilon < \min\{1/q, 1/q'\}$. Based on the functional analytic properties of the Neumann-Laplace operator in $W^{1+\varepsilon,q}_{\nu,\sigma}(\Omega)$, we were able to prove the well-posedness of the projected equations, where the projection P is given by the well-known Helmholtz projection. In the proof of the result, compactness of the operator $P'(1-P)$ played a central role, since it allowed us to generalize properties of the spectrum and the resolvent set of the involved operator to the whole scale.
Regarding duality scales, there are still questions open for future considerations. On one hand, one could ask for more general results that do not require compactness of the involved operators. The requirement of compactness is a rather strong condition and fails e.g., if we modify the boundary conditions of (6.7) to general Neumann boundary conditions, i.e. $\partial_\nu u = 0$ on $\partial\Omega$. To the best of the author's knowledge it is not possible to show compactness of $P'(1-P)$ in this case, such that results in Section 6.3 can not be applied. Furthermore, the theory developed in [44] and this thesis only applies to stationary equations by now. In order to solve non-stationary equations, an extension of this theory seems to be required.

Contributions

The content of this thesis is based on joint work with other contributors.

The Chapters 3 and 4 are based on joint work of Jürgen Saal, Christiane Bui and the author of this thesis. The results of Chapter 3 are published in [8]. The part regarding global well-posedness with initial values in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ and linear and nonlinear (in)stability of the disordered states is due to Christiane Bui and Jürgen Saal. The results regarding linear (in)stability, normal stability and normal hyperbolicity of the ordered polar states were developed by Christiane Bui, Jürgen Saal and the author of this thesis.
The global well-posedness of the living fluids model with initial values in $L^2_\sigma(Q_n)$, the existence, regularity and properties of the global attractor as well as the finite dimensionality in Chapter 4 were developed by Christiane Bui and the author of this thesis in equal parts, complemented with some fruitful discussions of these two with Jürgen Saal. The existence of the inertial manifold in $n = 2$ dimensions is the result of common work of all three contributors, supplemented by a discussion with Edriss Titi.

The content of Chapter 5 is the result of joint work of Matthias Köhne, Jürgen Saal and the author of this thesis, where the author of this thesis contributed substantial parts of the theorems characterizing stability and instability of equilibria. The results are going to be published in [18]. Much effort was put into a more concrete characterization of instability in several working sessions of Matthias Köhne, Jürgen Saal and the author of this thesis.

The results regarding duality scales and the application to a Stokes system are based on joint work by Jürgen Saal and the author of this thesis. The results in Sections 6.1 and 6.3 are substantially contributed by the author of this thesis, based on former work of Jürgen Saal. The results regarding the application to a Stokes system in Section 6.2 are the result of several working sessions of Jürgen Saal and the author of this thesis.

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