Representations of GGS-groups and generalisations of the Basilica group

Inaugural-Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

Karthika Rajeev

aus Ponkunnam, Indien

Düsseldorf, Februar 2022

aus dem Institut für Mathematik
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit Genehmigung der
Mathematisch-Naturwissenschaftlichen Fakultät der
Heinrich-Heine-Universität Düsseldorf

Berichterstatter:

1. Prof. Dr. Benjamin Klopsch
2. Prof. Dr. Gustavo A. Fernández-Alcober

Tag der mündlichen Prüfung: 28 März 2022

This dissertation is dedicated to the loving memory of my grandparents and Professor I. B. S. Passi

Acknowledgements

I would like to express my sincere gratitude to my supervisor Benjamin Klopsch for his unparalleled support and guidance throughout my PhD. His influence has been invaluable on many occasions. I also thank him for introducing me to the world of asymptotic group theory.

I am grateful to my collaborators, Moritz Petschick and Anitha Thillaisundaram, for teaching me so many things about groups acting on rooted trees. I owe a special thanks to both of them for all the mathematical discussions we had, especially during the first pandemic phase. I am grateful to Gustavo A. Fernández-Alcober for being an examiner for this dissertation and for his helpful feedback. I wish to extend my special thanks to Anitha Thillaisundaram and Paula M. Lins de Araujo for inspiring me and being available as mentors. I am also thankful to Alejandra Garrido for her valuable input.

I wish to express my thanks to Hanna Sasse for providing access to her Master thesis on Basilica groups, and to Benjamin Klopsch and Claas Röver for letting me access their unpublished notes on representations of Gupta–Sidki 3-group, which provided a general strategy for the computations presented in this thesis.

I am thankful to Dony, Iker, Margherita, Martina and Moritz for proofreading parts of this thesis. A big thanks to Djurre, Iker, Margherita, Martina, Moritz, Saba, Zeynep and all my colleagues for their support and constant motivation and for making me feel at home while being far away from home. I will always cherish their friendships.

My sincere thanks to Heinrich-Heine-Universität Düsseldorf for the financial support and working environment. I acknowledge the financial support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — 380258175. The research for this thesis was partially conducted in the framework of the DFG-funded research training group "GRK 2240: Algebro-Geometric Methods in Algebra, Arithmetic and Topology".

My appreciation goes to the Junior Scientist and International Researcher Center (JUNO) at HHU for helping me with all bureaucratic work for my stay in Germany.

I am incredibly grateful to Ebi, Aarathy, Hari, Anju, Ashish, Dony, Amal, Megha, Musthu, Vidhya, Sri, Sentil, Muneer, Sony, Manu, Sethu, Aswathy and my other beloved friends. They kept me going through challenging times. I cannot thank them enough for their love and encouragement.

Finally, I express my profound gratitude to my parents, Malu, Kunjunni, Appunni, and other family members, for their unwavering support throughout my years of study. This accomplishment would not have been possible without them.

Abstract

This dissertation is a study of two remarkable classes of groups that admit faithful actions on infinite regular rooted trees and exhibit strong self-similarity features. The groups that we consider are Grigorchuk–Gupta–Sidki groups (GGS-groups) and generalisations of the so-called Basilica group. This thesis is written in the form of a cumulative dissertation consisting of two self-contained parts; each comprises two projects.

The first part contains an investigation of the emerging field of representation zeta functions of groups acting on rooted trees. The representation zeta function of a group G is the Dirichlet generating function that encodes the number of finite-dimensional irreducible complex representations of G. Using representation zeta function as a tool, we prove that a large class of GGS-groups, for instance, the Gupta–Sidki groups, have polynomial representation growth, and provide a bound for the degree of polynomial growth. Furthermore, we carry out explicit computations to describe the representation zeta function of the Gupta–Sidki 3-group. The functional equation which we obtain agrees with the one provided by Bartholdi based on undocumented computer calculations.

The second part of the thesis comprises two articles on generalisations of the Basilica group:

(1) With Jan Moritz Petschick: *On the Basilica operation*, Groups, Geometry, and Dynamics, to appear, available at arXiv:2103.05452;

(2) With Anitha Thillaisundaram: *Maximal subgroups of generalised Basilica groups*, available at arXiv:2103.05452.

Both articles are incorporated into the thesis as self-contained chapters. The first article is supplemented by a detailed proof (for Theorem 6.8) which is not included in the arXiv and accepted versions.

Inspired by the Basilica group, together with Petschick, we introduce a general construction, called the *Basilica operation*, that produces an infinite family of *Basilica groups* from a given group of automorphisms of a rooted tree. We investigate which properties of groups of automorphisms of rooted trees are preserved under the Basilica operation. For groups that display strong self-similarity features, we develop new techniques for computing their Hausdorff dimension, which is generally difficult to calculate. Furthermore, we investigate an analogue of the classical congruence subgroup problem, which is studied in the context of arithmetic groups. In the second article, we study maximal subgroups of certain Basilica groups, and prove that they are of finite index in the corresponding Basilica groups.

Contents

Notation

$\mathbb{N}/\mathbb{R}/\mathbb{C}$	the set of natural/real/complex numbers
$\mathbb{N}_0$	the set of whole numbers
$\mathbb{Z}$	the set of integers
$\mathbb{F}_p$	the field of p elements
$[i,j]$	the interval in $\mathbb{Z}$, i.e., $[i,j] = \{i, i+1, \ldots, j-1, j\}$, for $i, j \in \mathbb{Z}$
$\mathrm{Mat}_n(K)$	the set of all $n \times n$ matrices over a field K
$\mathrm{GL}_n(K)$	the general linear group of degree n over a field K
x^y	$y^{-1}xy$
$[x,y]$	$x^{-1}y^{-1}xy$
C_m	the cyclic group of order m
$\langle S \rangle$	the group generated by a set S
$\langle S \rangle^G$	the normal subgroup generated by a subset $S \subseteq G$ in a group G
G'	commutator subgroup of a group G
$\gamma_n(G)$	n-th term in the lower central series of a group G
$[G:H]$	the index of a subgroup H in a group G
X^n	the sets of words of length n over an alphabet X
X^*	the free monoid generated by an alphabet X
$\mathrm{Sym}(X)$	the symmetric group on a set X
$(0\ 1\ \cdots\ m-1)$	the cyclic permutation mapping 0 to 1, 1 to 2, ..., $m-1$ to 0
T	the m-regular infinite rooted tree, for $m \geqslant 2$
$\mathrm{Aut}\,T$	the automorphism group of the rooted tree T
$\mathrm{st}_G(v)$	the stabiliser of a vertex $v \in T$ in a group $G \leqslant \mathrm{Aut}(T)$
$\mathrm{St}_n(G)$	the n-th level stabiliser in G
$\mathrm{rist}_G(v)$	the rigid vertex stabiliser of the vertex $v \in T$ in a group $G \leqslant \mathrm{Aut}(T)$
$\mathrm{Rist}_G(n)$	the n-th level rigid stabiliser in G
$g\vert_v$	the section of an element $g \in G$ at the vertex $v \in T$

$g\|^v$	the local action of $g \in G$ at the vertex $v \in T$
$G \wr H$	the permutational wreath product of a group G by a group H
ψ	the homomorphism $\psi : G \longrightarrow G \wr \mathrm{Sym}(X)$, $g \mapsto g\|^\epsilon(g\|_0, \ldots, g\|_{m-1})$, where $g\|^\epsilon$ is the local action of g at the root ϵ of T
ψ_n	the induced homomorphism $\psi_n : \mathrm{St}_G(n) \longrightarrow G \times \overset{m^n}{\cdots} \times G$
$\| \cdot \|$	word metric
$r_n(G)$	number of n-dimensional irreducible complex representations of a group G
$R_N(G)$	$\sum_{n=1}^{N} r_n(G)$
$\zeta_G(s)$	the representation zeta function of G, for a complex variable $s \in \mathbb{C}$
$\alpha(G)$	the abscissa of convergence of $\zeta_G(s)$
O	big O-Notation
$\mathrm{Irr}(G)$	the set of irreducible characters of G
$\chi\|_H$	the restriction of a character $\chi \in \mathrm{Irr}(G)$ to a subgroup H
χ^G	the induction of a character $\chi \in \mathrm{Irr}(H)$ to a super group G
$I_G(\chi)$	the inertia group of a character $\chi \in \mathrm{Irr}(H)$ of a subgroup H in a group G
$\mathrm{Bas}_s(\cdot)$	the Basilica operation, for $s \in \mathbb{N}$

Chapter 1

Introduction and general overview

This dissertation contains an investigation of groups acting on infinite regular rooted trees. The groups acting on rooted trees have initially drawn a great deal of attention because they exhibit prominent features and solve several long-standing problems in group theory. Over the last 40 years, the theory of groups acting on rooted trees has been developed substantially and has become an integral part of group theory, with connections to other areas of mathematics such as cryptography and dynamics; see [18, 54, 55, 76].

The most famous example of a group acting on a rooted tree is arguably the (first) Grigorchuk group. It was introduced by Grigorchuk [51] in 1980 as a simple yet elegant example of a finitely generated infinite torsion group. The Grigorchuk group was originally defined as the group of Lebesgue measure-preserving transformations of the set $[0,1]\backslash\{\frac{k}{2^m} \mid k, m \in \mathbb{Z}\}$, where $[0,1]$ is the unit interval in $\mathbb{R}$. Soon, the group was realised as the group of automorphisms of the binary rooted tree. Henceforth, attempts have been made to produce more examples of groups with similar properties that admit faithful actions on rooted trees. For instance, Gupta and Sidki [63] came up with a family of finitely generated infinite p-groups, for each odd prime p.

The Grigorchuk group and the Gupta–Sidki groups are explicit solutions to the General Burnside Problem that asks about the existence of groups of such a kind, a question posted by Burnside in 1902. It was partially proved by Burnside, extended by Schur, and further generalised by Kaplansky that such groups do not exist in the realm of classical matrix groups; cf. § 9 in [71]. That is, every finitely generated torsion subgroup of $\mathrm{GL}_n(K)$, where $n \in \mathbb{N}$ and K is an arbitrary field, is finite. However, the first examples of finitely generated infinite torsion groups was provided by Golod [48] in 1964, based on his work with Shafarevich [49].

The Grigorchuk group also played a significant role in the theory of word growth of groups. Let G be a finitely generated group and let S be a symmetric finite generating set of G (i.e., S is closed under taking inverses). For $n \in \mathbb{N}_0$, let $s_{(G,S)}(n)$ denote the number of distinct elements of G that can be minimally represented by words in S of length less than or equal to n. The non-decreasing function $s_{(G,S)} : \mathbb{N}_0 \longrightarrow \mathbb{N}$ is called a *word growth function* of G. We say that a finitely generated group G has *polynomial word growth* if there exist constants c, $d > 0$ such that $s_{(G,S)}(n) \leqslant c\,n^d$ for all $n \in \mathbb{N}$. A group G has *exponential word growth* if there exist constants $\alpha > 1$ and $c > 0$ such that $s_{(G,S)}(n) \geqslant c\,\alpha^n$ for all $n \in \mathbb{N}$. The word growth function of a group depends on the generating set we choose. However, it can be checked that the *growth type* of the growth function does not depend on the choice of a generating set. It is easy to find examples of groups with polynomial or exponential word growth. For instance, finitely generated free abelian groups have polynomial word growth, and on the other hand, finitely generated non-abelian free groups display exponential word growth. According to the celebrated theorem of Gromov [60], a finitely generated group has polynomial word growth if and only if it is virtually nilpotent.

The Grigorchuk group is the first group shown to have *intermediate word growth* (i.e., neither polynomial nor exponential) [52]; it thus answered a long-standing question of Milnor. In contrast, the word growth of the Gupta–Sidki groups is still not known to be intermediate or exponential. Until recently, the constructions of all known examples of groups of intermediate word growth were inspired by that of Grigorchuk's. However, in [78], Nekrashevych constructed an infinite family of simple groups of intermediate word growth, producing the first examples of such a kind. The groups introduced in [78] are obtained via homeomorphisms of a Cantor set. [1]

In the course of time, various generalisations of early constructions to wider families of groups of automorphisms of rooted trees have been defined and studied. This dissertation focuses on two major classes of generalisations known as *branch groups* and *automaton groups* (groups defined by *automata*).

The concept of branch groups was introduced by Grigorchuk in 1997. From a geometrical point of view, branch groups are groups acting transitively on each level of a rooted tree and having subnormal subgroups similar to the corresponding structure in the full automorphism group of the rooted tree; cf. [54]. The initial examples of groups acting on rooted trees, such as the Grigorchuk group and the Gupta–Sidki groups, are branch groups; cf. [18]. Branch groups naturally arise in the description of just infinite groups. We recall that a group is just

[1] More recently, the study of groups of homeomorphisms of a Cantor set has become an active area of research. They are the main source of examples of totally disconnected locally compact topologically simple groups. Although they are a very interesting class of groups, we will not discuss them in this thesis.

infinite if it is infinite and all of its proper quotients are finite. Within the profinite category, one may think of just infinite groups as generalisations of simple groups. Pioneering work of Wilson [107] provided a basic structure theory for just infinite groups. Based on this Grigorchuk proved that just infinite groups admit a trichotomy in which branch groups occur as one of three cases; cf. [54].

Automaton groups are defined by modelling the self-similarity of rooted trees. The first example of an automaton group was constructed by Aleshin [4]. The Aleshin group is a two-generated infinite torsion group acting on the binary rooted tree and is commensurable to the Grigorchuk group (we say that two groups are commensurable if they are isomorphic up to finite index). The action of an automaton group on a rooted tree can be best described by a finite-state machine, called an *automaton*, whose states correspond to automorphisms; see Section 2.1.2 for an explicit definition. The automaton groups often come with a rich geometry. For instance, the graphs of the action of some automaton groups on each level of a rooted tree (*Schreier graphs*) are of interest; cf. [16, 57]. In certain cases, the finite Schreier graphs converge to some fractal space [32]. The theory of automata and automaton groups has evolved considerably over the last couple of decades. We refer the interested reader to [76] for a survey on the topic.

The groups that we study in this dissertation lie in the intersection of automaton groups and a more general class of groups including all branch groups, called *weakly branch groups*, which are obtained by relaxing some of the algebraic properties of branch groups; see Section 2.3 for more on weakly branch groups. We investigate the properties of two distinct classes of groups, namely Grigorchuk–Gupta–Sidki groups (abbreviated as GGS-groups) and generalisations of the so-called Basilica group, to be discussed shortly, which was introduced by Grigorchuk and Żuk in [58] and [59].

The dissertation is written in the form of a cumulative thesis consisting of two self-contained parts; each comprises two projects. Part I is about the representations of GGS-groups, while Part II studies generalisations of the Basilica group. The content of Part II is available online on the public depository arXiv in the form of two articles; [92] and [94]. The first article is written in collaboration with Jan Moritz Petschick (fellow PhD student at Heinrich-Heine-Universtität Düsseldorf) that has been accepted to the journal "Groups, Geometry and Dynamics" for publication. The second article is the first part of a work in progress with Anitha Thillaisundaram at Lund University and has been submitted for publication in a mathematical journal.

Part I and Part II are preceded by a comprehensive preliminary section (Chapter 2), where we develop the language for groups acting on rooted trees. To facilitate the subsequent discussion, we give a short survey on the Basilica group and the GGS-groups in Chapter 2. Properties of the Basilica group are presented with historical notes that provide a context for the discussion in Part II. Here we give a summary of results from both parts without details. We refer the readers to the respective sections for a formal introduction to the subjects, where we also analyse our results in a historical and a broader mathematical framework. The references for Chapter 1, Chapter 2, Part I and Part II are collected at the end.

Part I is dedicated to the study of the asymptotic distribution of irreducible complex representations of GGS-groups. For a group G, let $r_G(n)$ denote the number of (equivalence classes of) n-dimensional irreducible complex representations. We are interested in groups G such that $r_G(n)$ is finite for all $\mathbb{N}$. We encode the arithmetic sequence $r_G(n)$ in a Dirichlet generating function, known as the *representation zeta function*, given by

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G)\, n^{-s} \quad (s \in \mathbb{C}),$$

and try to link its arithmetic and analytic properties to the algebraic properties of the group G. Part I begins with Chapter 3 that provides a gentle introduction to the theory of representation zeta functions, followed by Chapter 4, where we review key results from the representation theory of finite groups. Our main results on representations of GGS-groups appear in Chapter 5 and Chapter 6.

The GGS-groups are generalisations of the (second) Grigorchuk group and the Gupta–Sidki p-groups, for odd primes p. To each non-zero vector $\mathbf{e} \in \mathbb{F}_p^{p-1}$, one can associate a GGS-group generated by two automorphisms of the p-regular rooted tree; a formal definition can be found in Section 2.4.2. If the defining vector $\mathbf{e}$ of a GGS-group G is non-constant, then G is a branch group [37]. Moreover, there exists a subgroup H of finite-index in G such that H geometrically contains subgroups isomorphic to $H \times \overset{p^n}{\cdots} \times H$ for all $n \in \mathbb{N}$. In particular, if the defining vector $\mathbf{e}$ is also *non-symmetric* (see Definition 2.4.20), by taking $H = G'$, the commutator subgroup of G, one gets the described subgroup structure.

In Chapter 5, we prove that, for a branch GGS-group G, the number $r_n(G)$ is finite for all n. Using the representation zeta function $\zeta_G(s)$, we estimate the growth type of the arithmetic function $N \mapsto R_N(G) = \sum_{n=1}^{N} r_n(G)$. We prove that $R_N(G)$ is polynomially bounded in N. The degree of polynomial growth is given by the *abscissa of convergence* $\alpha(G)$ of $\zeta_G(s)$; see Chapter 3 for details. We set C to be the number of irreducible representations of the commutator subgroup $H = G'$ of G that are invariant under conjugation by G. If the number C is finite, we observe that the numbers $r_n(G)$ are bounded above by a function of n involving the *generalised Catalan numbers*; see Definition 5.3.5. In this case, using the generating function for the generalised Catalan numbers, we provide a bound for $\alpha(G)$, and hence for the degree of representation growth. The key tools are Clifford theory and the in-built self-similarity of G.

Theorem A. *Let G be a* GGS*-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ and let $H = G'$ be the commutator subgroup of G. If the number C of G-invariant (equivalent classes of) irreducible representations of H is finite then the abscissa of convergence $\alpha(G)$ of the representation zeta function $\zeta_G(s)$ satisfies the inequalities*

$$p - 2 \leqslant \alpha(G) \leqslant (p-1)\frac{\log 2}{\log p} + p(p-1)\frac{\log C}{\log p} + (p-1)^2 + (p-1) - 1. \tag{1.1}$$

In particular, G has polynomial representation growth.

We investigate the cases in which the number C is finite. These computations depend on our understanding of the subgroup structure of G, which happens to be determined by the defining vector $\mathbf{e}$ of the GGS-group G. It turns out to be that C is finite, in fact $C \leqslant p$, if the defining vector $\mathbf{e}$ satisfies a polynomial equation in its entries. In this situation, replacing C with p in (1.1), we get that $\alpha(G)$ is bounded above by $O(p^2)$.

Theorem B. *Let G be a* GGS*-group defined by a non-symmetric defining vector $\boldsymbol{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$. We define*

$$\boldsymbol{e}'' = \begin{cases} (e_3 - 2e_2 + e_1, \ldots, e_{i+2} - 2e_{i+1} + e_i, \ldots, e_{p-1} - 2e_{p-2} + e_{p-3}) \in \mathbb{F}_p^{p-3}, & \textit{if } p > 3, \\ \textit{empty tuple}, & \textit{if } p = 3. \end{cases}$$

Assume that the vector $\mathbf{e}''$ is either $(*)$ symmetric, or $(**)$ non-symmetric and the sum

$$\omega(\boldsymbol{e}) = (p-2)(e_1 - e_{p-1}) + (p-4)(e_2 - e_{p-2}) + \cdots + 3(e_{\frac{p-3}{2}} - e_{\frac{p+3}{2}}) + (e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}})$$

is non-zero modulo p. Then the abscissa of convergence $\alpha(G)$ of the representation zeta function $\zeta_G(s)$ of G satisfies the following inequalities

$$p - 2 \leqslant \alpha(G) \leqslant (p-1)\frac{\log 2}{\log p} + 2p^2 - 2p + 1.$$

The definition of (non-)symmetric vectors can be found in Definition 2.4.20. For convention, we take the empty tuple to be symmetric. A large class of GGS-groups satisfies the conditions $(*)$ or $(**)$ in the assertion of Theorem A. For instance, if $p = 3$, then the vector $\mathbf{e}''$ is the empty tuple and it is symmetric by definition. Therefore, every GGS-group defined by a non-symmetric vector $\mathbf{e} \in \mathbb{F}_3^2$ satisfies the condition $(*)$. The condition $(**)$ is valid for all Gupta–Sidki p-groups, for $p \geqslant 5$. The extent to which our results generalise to GGS-groups heavily rely on our good understanding of the algebraic structure of the groups, such as determining the first p terms of the lower central series; see the discussion at the end of Chapter 5.

Using detailed character theory, in Chapter 6, we explicitly compute a recursive description of the representation zeta function of the Gupta-Sidki 3-group G_3 in terms of partial representation zeta functions of its commutator subgroup. The description of the representation zeta function which we obtain agrees with the one provided by Bartholdi in [14] based on undocumented computer calculations.

Theorem C. *Let G_3 be the Gupta–Sidki 3-group. The representation zeta function $\zeta_{G_3}(s)$ of G_3 satisfies the 'functional equation'*

$$\zeta_{G_3}(s) = 9 + 2 \cdot 3^{-s} + 3^{-s}\,\alpha(s) + 2 \cdot 3^{-s}\,\beta(s) + 3^{-s}\,\tau(s) + \frac{1}{9}\,3^{-2s}\,\xi(s),$$

where $\alpha(s)$, $\beta(s)$, $\tau(s)$ and $\xi(s)$ are partial representation zeta functions of the commutator subgroup of G_3, which are defined in Section 6.5.

We refer the reader to Section 6.5 for an explicit formulation of our description of the zeta function $\zeta_{G_3}(s)$. Currently, our computation is limited to this particular case, because it is based on the fact that Clifford theory can be effectively carried out only for branch groups with relatively small *branching quotient*; see Definition 2.3.2. However, we believe that our approach can be used to obtain, in future work, similar results for the Fabrykowski–Gupta group [35], which is the only example of a branch GGS-group acting on the ternary tree that is non-isomorphic to G_3.

Part II is a collection of two research articles [92] and [94] on generalisations of the Basilica group, to be discussed below, incorporated as Chapter 8 and Chapter 9 of the dissertation. We now present selected results from Chapter 8 and Chapter 9. One can find an in-depth discussion indicating the relevance and scope of our main results in the introductory sections of Chapter 8 and Chapter 9. In the short technical Chapter 7, one may find a brief account of the authors' individual contributions.

The Basilica group $\mathcal{B}$ is a two-generated weakly branch, but not branch, group acting on the binary rooted tree, which was introduced in [58] and [59]. It is the first known example of an amenable [24] but not sub-exponentially amenable group [59]. Further, it occurs as the iterated monodromy group of the complex polynomial $z^2 - 1$; see [76, Section 6.12.1]. (For a definition of iterated monodromy group, see Section 2.4.1.2.) Moreover, the Julia set of $z^2 - 1$, known as the *Basilica fractal*, which is the set of accumulations points of the backward iterations of an arbitrary point in the complex plane under $z^2 - 1$, apparently resembles the basilica of San Marcos in Venice, and hence the name. It is shown in [77] that the Basilica fractal can be reconstructed from the Basilica group $\mathcal{B}$. Additionally, the Basilica fractal can be approximated by a sequence of finite Schreier graphs obtained by the action of the Basilica group on each level of the binary rooted tree; cf. [32].

Inspired by the Basilica group $\mathcal{B}$, in [92], we introduced a general construction which produces a family of *Basilica groups* $\mathrm{Bas}_s(G)$, $s \in \mathbb{N}$, from a given group G of automorphisms of a rooted tree. There is a natural bijection between the vertices of the binary rooted tree and the set of all finite words over the alphabet $\{0, 1\}$. The generators of the Basilica group $\mathcal{B}$ can be best described by a three-state automaton given by Figure 1.1. For alphabets $x, y \in \{0, 1\}$ and states $p, q \in \{a, b\}$, we interpret the directed arrow labelled by $x : y$ from the state p to the state q as follows: upon reading the symbol x the state p gives the output y and it enters to the state q. Here, id is the short-hand notation for identity state. The states a and b induce automorphisms of the binary rooted tree.[2] We point out the similarities between these two generators and the single automorphism generating the dyadic odometer $\mathcal{O}_2$; see Figure 1.1. The automorphism b can be interpreted as a delayed version of c that enters the intermediate state a before referring to itself. Modelling this 'delaying effect', we define the *Basilica operation* $\mathrm{Bas}_s(\cdot)$, $s \in \mathbb{N}$; for any group G of automorphisms of a rooted tree, it yields the *s-th Basilica group* $\mathrm{Bas}_s(G)$, by adding $s - 1$ intermediate states to every element of G. For the dyadic odometer $\mathcal{O}_2$, one has $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$.

Figure 1.1: Automaton generating the Basilica group and the dyadic odometer

We investigate which properties of a group G of automorphisms of a rooted tree are preserved under the Basilica operation. It turns out to be that the properties related to the group action of G on a rooted tree (such as self-similarity, fractalness, being weakly branch, contraction, etc.) are inherited by the higher Basilica groups $\mathrm{Bas}_s(G)$. In contrast, word growth type is not preserved under the Basilica operation; see Section 8.3.5.

[2] In [59], the automaton for the Basilica group is provided with the roles of the alphabets 0 and 1 are swapped, and the group acts on the binary tree from the right. Both conventions yield isomorphic groups. To be in consistent with the rest of the thesis and with [92], we employ left actions.

Theorem D. *Let G be a group of automorphisms of a regular rooted tree. Let P be a property from the list below. Then, if G has P, the s-th Basilica group* $\mathrm{Bas}_s(G)$ *of G has P for all $s \in \mathbb{N}$.*

1. *spherically transitive*
2. *self-similar*
3. *(strongly) fractal*
4. *contracting*
5. *weakly branch*
6. *generated by finite-state bounded automorphisms*

In the first part of the article [92], we study the Basilica construction quite generally, and, for a fixed $s \in \mathbb{N}$, we examine the s-th Basilica groups of generalisations of Grigorchuk groups and Gupta–Sidki groups. It can be easily verified that the set of vertices of level n of a rooted tree T is invariant under the action of an automorphism of T, for all $n \in \mathbb{N}$. Therefore, the *n-th level stabiliser*, which is the kernel of the induced action of G on the set of vertices of level n of T, is the natural object to consider when we study automophisms of T. For a group G of automorphisms of a rooted tree T which displays strong self-similarity features, we prove that the level stabilisers in $\mathrm{Bas}_s(G)$ can be obtained from the level stabilisers in G. In Theorem E below, the maps β_i are the algebraic analogues of the added intermediate steps in the definition of the Basilica operation; see Definition 8.2.2. The subgroup K_{s-1} is a normal (possibly trivial) subgroup of G measuring the failure of G to be s-split; being s-split is a notion introduced in Definition 8.4.1 to make sure that the Basilica group $\mathrm{Bas}_s(G)$ closely resembles the original Basilica group $\mathcal{B}$.

Theorem E. *Let G be a self-similar and very strongly fractal group of automorphisms of a regular rooted tree. Assume that G is weakly regular branch over K_{s-1}. Let $n \in \mathbb{N}_0$. Write $n = sq + r$ with $q \geqslant 0$ and $0 \leqslant r \leqslant s-1$. Then, for all $s > 1$, the n-th level stabiliser of* $\mathrm{Bas}_s(G)$ *is given by*

$$\mathrm{St}_{\mathrm{Bas}_s(G)}(n) = \langle \beta_i(\mathrm{St}_G(q+1)), \beta_j(\mathrm{St}_G(q)) \mid 0 \leqslant i < r \leqslant j < s \rangle^{\mathrm{Bas}_s(G)}.$$

Using the description of level stabilisers in Theorem E, we develop new techniques for computing the Hausdorff dimension of the Basilica group $\mathrm{Bas}_s(G)$ from that of G. The Hausdorff dimension of G measures how dense its closure is in an appropriate subgroup of $\mathrm{Aut}\, T$, and is generally difficult to calculate; cf. Section 8.4.2. It is generally analogous to the Hausdorff dimension usually defined over $\mathbb{R}$ as a measure of fractalness; see Section 2.4.1.6 for a formal definition.

The second half of the article specialises on *generalised Basilica groups* $\mathrm{Bas}_s(\mathcal{O}_m^d)$, for d, m, $s \in \mathbb{N}$ with $m, s \geqslant 2$, which are Basilica groups obtained from a direct product of d

copies of a generalisation $\mathcal{O}_m$ of the dyadic odometer $\mathcal{O}_2$. We closely study the structural properties of the generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_m^d)$ and prove that they resemble the original Basilica group $\mathcal{B}$, hence justifying the nomenclature; see Theorem 8.1.6. Moreover, we explicitly compute the Hausdorff dimension of $\mathrm{Bas}_s(\mathcal{O}_m^d)$, which turns out to be independent of the parameter d.

Theorem F. *For all d, m, $s \in \mathbb{N}$ with m, $s \geqslant 2$*

$$\dim_{\mathrm{H}}(\mathrm{Bas}_s(\mathcal{O}_m^d)) = \frac{m(m^{s-1}-1)}{m^s-1}.$$

Furthermore, we investigate an analogue of the classical Congruence Subgroup Problem, which originates from the study of arithmetic lattices in semisimple locally compact groups; see Section 2.4.1.7. Providing an explicit recursive presentation for the generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_m^d)$ allows us to describe the level stabilisers in $\mathrm{Bas}_s(\mathcal{O}_m^d)$ using Theorem E. This enables us to prove a key structural result stating that these groups have a weaker version of the Congruence Subgroup Property in the context of tree actions.

Theorem G. *For all d, $s \in \mathbb{N}$ with $s > 2$, and all primes p, the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_p^d)$ has the p-Congruence Subgroup Property.*

The recursive presentation of the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ obtained in [92] is not finite. However, one can obtain a finite recursive presentation for $\mathrm{Bas}_s(\mathcal{O}_m^d)$, which we stated in [92, Theorem 6.8] without a proof. This proof is included in Chapter 8; see Theorem 8.6.8.

It is worth to point out that, in [92], we observed that if a group law is satisfied by a group G but not by $\mathrm{Bas}_s(G)$, for some $s \in \mathbb{N}$, then $\mathrm{Bas}_s(G)$ is a weakly branch group and it is regular branch over the corresponding verbal subgroup. This enables one to construct a weakly regular branch group over a prescribed verbal subgroup. Therefore, the class of Basilica groups promises to give solutions to problems arising in the theory of groups acting on rooted trees.

In [94], we investigate the maximal subgroups of generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_m)$ for $m, s \geqslant 2$. The groups that we examine are s-generated weakly branch, but not branch, groups. One of the motivations to study the maximal subgroups of (weakly) branch groups is related to a conjecture of Kaplansky; details can be found in Section 2.4.1.8. We point out that, it is the first time that maximal subgroups of a weakly branch, but not branch, group G have been considered for a group G with more than two generators. We prove that all maximal subgroups of the desired generalised Basilica group are of finite index.

Theorem H. *Let m and s be positive integers such that $m, s \geqslant 2$. Then the generalised Basilica group* $\mathrm{Bas}_s(\mathcal{O}_m)$ *does not admit a maximal subgroup of infinite index.*

Since we are considering generalised Basilica groups for an arbitrary $s \geqslant 2$, the final stages of our proof differ from previously seen results. One can also look at the generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_m^d)$, for $d > 1$. We need new insights to tackle this problem as these groups do not follow the usual length decreasing properties. So far, none of the weakly branch, but not branch, groups, whose maximal subgroups have been studied, admit a maximal subgroup of infinite index. Therefore, article [94] is the first step towards either (a) proving that all weakly branch, but not branch, groups have only maximal subgroups of finite index, or (b) classifying the weakly branch, but not branch, groups with maximal subgroups of infinite index.

Chapter 2

Preliminaries

We deliberately dedicate this chapter to establishing the language of groups acting on rooted trees. Section 2.1 focuses on tree automorphisms, where we describe three different ways of expressing a tree automorphism, namely in terms of *portrait, automaton* and *wreath recursion*. In Section 2.2, we discuss the notion of self-similarity. Section 2.3 is a review of branch and weakly branch groups. The final section, Section 2.4, is a survey on GGS-groups and the Basilica group; they are the core ingredients of our studies in Part I and Part II. In Section 2.4.1, we give an account of various properties of the Basilica group. During the process, we recall necessary definitions and provide historical context to different notions related to groups acting on rooted trees, including L-presentation, Congruence Subgroup Property and Hausdorff dimension. Finally, in Section 2.4.2, we summarise known results about GGS-groups and develop new structural results essential for the later discussions.

2.1 Rooted trees and their automorphisms

Let $m \geqslant 2$ be an integer. The *m-regular rooted tree* T is an infinite tree with a distinguish vertex, known as the *root*, of valency m and every other vertex has valency $m+1$. We label the vertices of the rooted tree T by the elements of the free monoid X^* generated by the alphabet $X = \{0, 1, \ldots, m-1\}$ in the following way: the root is labelled by the empty word, denoted by ϵ, and the vertices that are at distance n from the root are labelled lexicographically from left to right with words of length n. In the sequel, we do not differentiate between X^* and vertices of T. The vertices of the set X^n are called *n-th level vertices* and they constitute the *n-th layer* of the rooted tree T. An *automorphism* g of T is a graph automorphism; g has to preserve the root and to keep the adjacency of vertices. As a consequence, the levels X^n of T are invariant under the action of g. The set of all automorphisms of T forms a group and is denoted by $\operatorname{Aut} T$.

Let G be a subgroup of $\operatorname{Aut} T$ and let $v \in X^n$ be a vertex of level n. The *vertex stabiliser* $\mathrm{st}_G(v)$ is the subgroup of G given by

$$\mathrm{st}_G(v) = \{g \in G \mid g(v) = v\},$$

and the *n-th level stabiliser* $\mathrm{St}_G(n)$ is

$$\mathrm{St}_G(n) = \bigcap_{v \in X^n} \mathrm{st}_G(v).$$

The subgroup $\mathrm{St}_G(n)$ is precisely the kernel of the induced action of G on X^n, and hence it has finite index in G. Furthermore, the intersection of all level stabilisers is trivial, which makes the group G residually finite. Taking the set of all level stabilisers as an open neighbourhood system for the identity gives a topology on G. With respect to this topology, which is metrisable, the topological group $\operatorname{Aut} T$ is complete. Indeed, the group $\operatorname{Aut} T$ is profinite:

$$\operatorname{Aut} T \cong \varprojlim_{n \in \mathbb{N}} \operatorname{Aut} T / \mathrm{St}_{\operatorname{Aut} T}(n).$$

Let v be a vertex of the rooted tree T and let T_v denote the subtree rooted at v. The subtree T_v can be identified with the original tree by sending every vertex $vw \in vX^*$ of T_v to the vertex $w \in X^*$ of T. Let $g \in \operatorname{Aut}(T)$. Then g induces an isomorphism between the subtrees T_v and $T_{g(v)}$. Since both of the subtrees T_v and $T_{g(v)}$ are identical to the original tree T, we obtain an automorphism $g|_v : T \to T$, known as the *section of g at v*, which is uniquely determined by the equation

$$g(vw) = g(v)g|_v(w). \tag{2.1}$$

For all g, g_1, $g_2 \in \operatorname{Aut} T$ and v, v_1, $v_2 \in X^*$, it holds that

$$g|_{v_1 v_2} = g|_{v_1}|_{v_2}, \tag{2.2}$$

$$(g_1 \cdot g_2)|_v = g_1|_{g_2(v)} \cdot g_2|_v. \tag{2.3}$$

Now, we shall describe three different ways of expressing automorphisms of rooted trees, namely in terms of *portrait, automaton* and *wreath recursion.*

2.1.1 Portrait of a tree automorphism

For every $g \in \operatorname{Aut} T$ the *portrait* of g is the labelled tree consisting of the tree T in which every vertex v is labelled with an element $g|^v$ of $\mathrm{Sym}(X)$, where $g|^v$ is the action of g on the set of immediate descendants of the vertex v, which is called the *local action of g at v*. An automorphism g is uniquely determined by its portrait: for all $x_1 \cdots x_n \in X^n$,

$$g(x_1\, x_2\, x_3 \cdots x_n) = g(x_1)\, g|^{x_1}(x_2)\, g|^{x_1\, x_2}(x_3)\, \cdots\, g|^{x_1 \cdots x_{n-1}}(x_n).$$

2.1.2 Automata

An *automaton* (A, X, τ) over an alphabet X is given by a *set of states* A and a *transition map* $\tau : A \times X \to X \times A$. For $p \in A$ and $x \in X$, suppose that $\tau(p, x) = (y, q)$, for some $y \in X$ and $q \in A$. The above equality is interpreted as the following: upon reading the input letter x the state p gives the output y and it enters to the state q. We write $y = p(x)$ and $p = q|_x$. If the set of states of A is finite then (A, X, τ) is said to be a *finite-state* automaton. Similarly, we define the automaton (A, X^n, τ_n), for every $n \in \mathbb{N}_0$, in which the input and output are words of length n and the transition map τ_n is given by the equations below.

$$p|_\epsilon = p, \qquad p|_{x\,v} = p|_x|_v, \tag{2.4}$$

$$p(\epsilon) = \epsilon, \qquad p(x\,v) = p(x)\,p|_x(v), \tag{2.5}$$

for $x \in X$ and $v \in X^{n-1}$, where ϵ denotes the empty word. Therefore, the structure of (A, X^n, τ_n) is uniquely determined from that of (A, X, τ).

Now, assume that the set X is finite with cardinality $m \geqslant 2$. We may further assume, without loss of generality, that $X = \{0, 1, \ldots, m-1\}$. Let T be the m-regular rooted tree whose vertices are in bijection with the set X^* of all finite words over X. Consider an automaton (A, X, τ) over the alphabet X. Every state $p \in A$ defines a transformation on X^* which is determined by (2.4) and (2.5). Notice that, for any $v \in X^*$ and $k \in \mathbb{N}$, the first k letters of the word $p(v)$ depends only on the first k letters of the word v. Therefore, the transformation defined by p is an endomorphism of the rooted tree X^*, which in general need not be an automorphism.

Now, let (A, X, τ) and (B, X, ι) be two automata over the alphabet X. Then their *product* $(A \times B, X, \tau \cdot \iota)$ is an automaton, whose set of states is the direct product of A and B. Let $x \in X$, $p_1 \in A$ and $p_2 \in B$. For convenience, we denote the elements of the form (p_1, p_2) from the set $A \times B$ by $(p_1 p_2)$. Then the transition map $\tau \cdot \iota$ of the automaton $(A \times B, X, \tau \cdot \iota)$ is given by the following rules.

$$(p_1 p_2)(x) = p_1(p_2(x)),$$

$$(p_1 p_2)|_x = p_1|_{p_2(x)} p_2|_x.$$

Furthermore, we say that an automaton (A, X, τ) is *invertible* if every $p \in A$ defines an invertible transformation of X^* (or, equivalently, of X). The inverse of (A, X, τ) is given by the automaton (A^{-1}, X, τ^{-1}), whose set of states A^{-1} is in one-to-one correspondence with A, and, for every $p^{-1} \in A^{-1}$ and $x \in X$, the equality $\tau^{-1}(p^{-1}, x) = (y, q^{-1})$ holds if and only if $\tau(p, y) = (x, q)$, for some $y \in X$ and $q \in A$.

Definition 2.1.1. Let (A, X, τ) be an invertible automaton. The *group generated by the automaton* (A, X, τ) is the group $\langle A \rangle$ generated by all transformations of X^* defined by the set of states A of (A, X, τ).

To every automaton (A, X, τ), we can associate a directed graph, known as the *Moore diagram*. The vertices of the Moore diagram representing (A, X, τ) are identified with the set of states A. Two states p, q are connected by a directed edge starting from p if and only if there exist x, $y \in X$ such that $\tau(p, x) = (y, q)$, and the edge is labelled by $x : y$. For convenience, we do not draw the edges of type $\tau(p, x) = (x, p)$. For example, consider Moore diagram Figure 2.1 of the two-state automaton (A, X, τ), where $X = \{0, 1\}$ and $A = \{\mathrm{id}, c\}$. The transition map τ is given by the following rules.

$$\mathrm{id}(0) = 0, \qquad \mathrm{id}(1) = 1, \qquad \mathrm{id}|_0 = \mathrm{id}, \qquad \mathrm{id}|_1 = \mathrm{id},$$
$$c(0) = 1, \qquad c(1) = 0, \qquad c|_0 = c, \qquad c|_1 = \mathrm{id}\,.$$

The automaton (A, X, τ) is invertible and its inverse is given by the Moore diagram Figure 2.1. Observe that the state id induces the identity transformation on X^*, while the action of the element c^{-1} on X^* is equivalent to adding 1 to the dyadic integers from the left. The group $\langle A \rangle$ generated by the automaton (A, X, τ) is called the *dyadic odometer*, denoted by $\mathcal{O}_2$, and is isomorphic to the group of integers. Hence, the group $\mathcal{O}_2$ provides an embedding of the group of integers into the automorphism group of the binary rooted tree.

Figure 2.1: Automaton of generating the dyadic odometer and its inverse automaton.

Without difficulty, one can see that the trivial group, the cyclic group of order two, the Klein four-group, the infinite dihedral group, and the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is the *permutational wreath product* of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$ (cf. see Section 2.1.3), are also generated by two-state automata. In fact, these are the only groups up to isomorphism generated by two-state automata; cf. [57].

2.1.3 Wreath recursion

Let H be a finite group acting on a finite set X from the left and let G be an arbitrary group. Denote by G^X the direct product of $|X|$-many copies of G. If we fix an indexing

$\{x_0, \ldots, x_{m-1}\}$ of the set X, then every element $g \in G^X$ can be written as $(g_0, \ldots, g_{m-1})$, where $g_i \in G$ for each $i \in \{0, 1, \ldots, m-1\}$, and H also acts on the set $\{0, 1, \ldots, m-1\}$ from the left so that $h(x_i) = x_{h(i)}$. The action of H on X induces a right action of H on G^X given by: for every $h \in H$ and $(g_0, \ldots, g_{m-1}) = g \in G$

$$(g_0, \ldots, g_{m-1}) \cdot h = (g_{h(0)}, \ldots, g_{h(m-1)}). \tag{2.6}$$

Thus we can define the semi-direct product $G^X \rtimes H$, where the action of H on G^X is given by (2.6). The semi-direct product $G^X \rtimes H$ is called the *permutational wreath product*, and is denoted by $G \wr_X H$. For any given pair of elements $h\,(g_0, \ldots, g_{m-1})$ and $h'\,(g'_0, \ldots, g'_{m-1})$ in $G \wr_X H$, the multiplication is given by

$$h'\,(g'_0, \ldots, g'_{m-1})\,h\,(g_0, \ldots, g_{m-1}) = h'\,h\,(g'_{h(0)}\,g_0, \ldots, g'_{h(m-1)}\,g_{m-1}). \tag{2.7}$$

If there is no confusion, then we drop the index X from $G \wr_X H$.

Let T be the m-regular rooted tree and let $\operatorname{Aut} T$ be the group of automorphisms of T. Let $g \in \operatorname{Aut} T$. We define the following map

$$\psi : \operatorname{Aut} T \longrightarrow \operatorname{Aut} T \wr \operatorname{Sym}(X), \tag{2.8}$$

by

$$\psi(g) = g|^{\epsilon}\,(g|_0, \ldots, g|_{m-1}), \tag{2.9}$$

where $g|^{\epsilon}$ is the induced action of g on the set X (or equivalently, the local action of g at the root ϵ), and $g|_x$ is the section of g at the vertex x, for $x \in X$. Clearly, ψ is a bijection. It is easy to verify using (2.7) that ψ is a homomorphism. Therefore, the group $\operatorname{Aut} T$ admits the following decomposition

$$\operatorname{Aut} T \cong \operatorname{Aut} T \wr \operatorname{Sym}(X), \tag{2.10}$$

and every element $g \in \operatorname{Aut} T$ can be uniquely written as its image $\psi(g)$. The recursive expression (2.9) is called the *wreath recursion* of g, which provides a convenient way to write down an automorphism. For example, set $X = \{0, 1\}$ and $\sigma \in \operatorname{Sym}(X)$ as the transposition $(0\ 1)$. The wreath recursion

$$c = \sigma(c, \mathrm{id}) \tag{2.11}$$

defines an automorphism of the binary rooted tree and its action on X^* is same as that of the automorphism induced by the state c of the automaton generating the dyadic odometer given by Figure 2.1.

2.2 Self-similarity

For every $v \in T$ and $g \in \operatorname{Aut} T$, we recall that T_v is the subtree rooted at v and $g|_v$ is the section of g at v defined by the equation (2.1). Let G be a subgroup of $\operatorname{Aut} T$. Unlike in $\operatorname{Aut} T$, the set

$$G|_v = \{g|_v \mid g \in G\}$$

is not necessarily a subset of G. The group G is said to be *self-similar* if $G|_v \subseteq G$ for every $v \in T$.

Let T be the m-regular rooted tree and let $G \leqslant \operatorname{Aut} T$ be a self-similar group. Notice that the restriction of the map ψ in (2.8) to G

$$\psi : G \longrightarrow G \wr \operatorname{Sym}(X)$$

embeds G into the wreath product $G \wr \operatorname{Sym}(X)$. (By abuse of notation, we use the same symbol to denote the restriction of ψ to the group G.) Hence, we can regard G as a subgroup of $G \wr \operatorname{Sym}(X)$. Set $A = \{g|_v \mid g \in G,\ v \in T\}$ and $X = \{0, 1, \dots, m-1\}$. It is easy to see that A coincides with G. We define an automaton (A, X, τ_ψ) whose output and the transition functions are determined by the map ψ. The transformations of X^* defined by the states of the automaton (A, X, τ_ψ) determines the action of G on T. We say an automorphism $g \in \operatorname{Aut} T$ is *finite-state* if the set $\{g|_v \mid v \in T\}$ is finite. Suppose that $G \leqslant \operatorname{Aut} T$ is a finitely generated self-similar group, and every element of G is finite-state. In that case, G can be generated by a finite-state automaton obtained by taking the disjoint union of automata defining the generators of G. Conversely, it can be verified that every finite-state automaton generates a finitely generated self-similar group such that its elements are finite-state.

Let G be self-similar and $g \in \operatorname{St}_G(1)$. As g stabilises the vertices of level one, the local action $g|^\epsilon$ of g at the root ϵ is trivial, and hence the element g is uniquely determined by the sections of g at the vertices of level one. Therefore, the wreath recursion of g is given by

$$\psi(g) = (g|_0, \dots, g|_{m-1}),$$

and the induced homomorphism

$$\psi_1 : \operatorname{St}_G(1) \longrightarrow G \times \overset{m}{\cdots} \times G$$

is an embedding. Due to self-similarity of G, the homomorphism ψ_1 extends to all $n \in \mathbb{N}$ in a natural way such that

$$\psi_n : \operatorname{St}_G(n) \longrightarrow G \times \overset{m^n}{\cdots} \times G$$

is injective for all $n \in \mathbb{N}$. If $G = \operatorname{Aut} T$ then the map ψ_n is in fact an isomorphism and $\operatorname{St}_{\operatorname{Aut} T}(n) \cong \operatorname{Aut} T \times \overset{m^n}{\cdots} \times \operatorname{Aut} T$ for every $n \in \mathbb{N}$. For convenience, we often identify an element $g \in \operatorname{St}_G(n)$ with its image $\psi_n(g)$.

For every self-similar group G, taking the section of an element $g \in \operatorname{st}_G(v)$ at a vertex $v \in T$ induces a homomorphism from $\operatorname{st}_G(v)$ to G given by

$$\varphi_v : \operatorname{st}_G(v) \longrightarrow G,\ g \mapsto g|_v.$$

If φ_v is an epimorphism, i.e., $\varphi_v(\operatorname{st}_G(v)) = G$, for all vertices $v \in T$, then the group G is called *fractal*. Clearly, $\operatorname{Aut} T$ is fractal.

2.3 Branch and weakly branch groups

Let T be the m-regular rooted tree and let G be a subgroup of $\operatorname{Aut} T$. Let $v \in X^n$, for some $n \in \mathbb{N}$. The *rigid vertex stabiliser* $\operatorname{rist}_G(v)$ of v is the subgroup of G consisting of elements which fix every vertex outside the subtree rooted at v, i.e.,

$$\operatorname{rist}_G(v) = \{g \in G \mid \forall w \in T \backslash T_v : g(w) = w\}.$$

The group generated by all rigid vertex stabilisers of vertices of level n is called the *n-th rigid level stabiliser*, and is denoted by $\operatorname{Rist}_G(n)$.

Let $v \in X^n$ and $w \in X^\ell$, where $\ell \geqslant n$. It is easy to see that, if v is a prefix of w, i.e, $w = v\,\tilde{w}$, for some $\tilde{w} \in X^{\ell-n}$, then $\operatorname{rist}_G(w) \leqslant \operatorname{rist}_G(v)$. If otherwise v is not a prefix of w, then $\operatorname{rist}_G(v) \cap \operatorname{rist}_G(w) = 1$, and hence the subgroups $\operatorname{rist}_G(v)$ and $\operatorname{rist}_G(w)$ commute. Furthermore, for every $g \in G$, $\operatorname{rist}_G(v)^{g^{-1}} = \operatorname{rist}_G(g(v))$. If G acts transitively on each level of the rooted tree T then the rigid vertex stabilisers are conjugate in G. Therefore, for every $n \in \mathbb{N}$, we get the following equality

$$\operatorname{Rist}_G(n) = \langle \operatorname{rist}_G(v) \mid v \in X^n \rangle = \prod_{v \in X^n} \operatorname{rist}_G(v).$$

Clearly, we have $\operatorname{Rist}_G(n) \leqslant \operatorname{St}_G(n)$. Unlike the stabiliser, the rigid stabiliser may have infinite index in G (may even be trivial; for example in the case of the dyadic odometer $\mathcal{O}_2$ defined by the automaton in Figure 2.1). If $G = \operatorname{Aut} T$ and $v \in T$, then the following holds

$$\operatorname{rist}_{\operatorname{Aut} T}(v) \cong \operatorname{Aut}(T_v) \cong \operatorname{Aut} T,$$

and hence $\operatorname{Rist}_{\operatorname{Aut} T}(n) \cong \operatorname{Aut} T \times \overset{m^n}{\cdots} \times \operatorname{Aut} T$. Since $\operatorname{Rist}_G(n) \leqslant \operatorname{St}_G(n)$, when $G = \operatorname{Aut} T$, we have the equality $\operatorname{Rist}_G(n) = \operatorname{St}_G(n)$.

Similar to $\operatorname{Aut} T$, if G acts transitively on each level of the rooted tree T then G is said to be *spherically transitive*. For a spherically transitive group G, the rigid level stabilisers $\operatorname{Rist}_G(n)$ are either trivial for almost all n or infinite for all n.

Definition 2.3.1. Let $G \leqslant \operatorname{Aut} T$ be a spherically transitive group of automorphisms of the rooted tree T. The group G is *weakly branch* if all $\operatorname{Rist}_G(n)$ are infinite. We say G is a *branch group* if additionally the subgroups $\operatorname{Rist}_G(n)$ have finite index in G.

Notice that not every weakly branch group is branch; for example the Basilica group is weakly branch, but not branch, group; cf. Corollary 2.4.11.

Definition 2.3.2. Let $G \leqslant \operatorname{Aut} T$ be a spherically transitive group of automorphisms of the rooted tree T. We say the group G is *weakly regular branch* if G is self-similar and contains a non-trivial subgroup $H \leqslant G$ such that $H \geqslant \psi^{-1}(H \times \overset{m}{\cdots} \times H)$. The group G is *regular branch* H if such a subgroup H is also of finite index in G. We further say that G is *(weakly) regular branch over* the subgroup H. If G is regular branch over H, then the quotient $H/\psi^{-1}(H \times \overset{m}{\cdots} \times H)$ is a finite group, and is called the *branching quotient* of G.

Every group G that is (weakly) regular branch over a subgroup H is (weakly) branch. Indeed, the subgroup $\psi^{-1}(H \times \overset{m}{\cdots} \times H)$ is contained in $\operatorname{Rist}_G(1)$. One gets by induction that $\psi^{-1}(H \times \overset{m^n}{\cdots} \times H) \leqslant \operatorname{Rist}_G(n)$ for all $n \in \mathbb{N}$. Now, if H is of finite index in G then $\psi^{-1}(H \times \overset{m}{\cdots} \times H)$ has finite index in $\operatorname{St}_G(1)$. Therefore, $\operatorname{Rist}_G(1)$ has finite index in $\operatorname{St}_G(1)$, and hence in G. Again by induction, one can see that $\operatorname{Rist}_G(n)$ has finite index in G for all $n \in \mathbb{N}$. In particular, if G is branch then the groups G and $G \times \overset{m}{\cdots} \times G$ are commensurable as subgroups of $\operatorname{Aut} T$. We recall that two subgroups K_1 and K_2 of a group K are *commensurable* if the intersection $K_1 \cap K_2$ is of finite index in both K_1 and K_2.

Now, we record a fundamental lemma for weakly branch groups, which is crucial for the discussions later. The statement and the proof of the following lemma can be extracted from the proof of [54, Theorem 4], where it is proven for branch groups.

Lemma 2.3.3. *Let T be the m-regular rooted tree and $G \leqslant \operatorname{Aut} T$ be weakly branch. For every non-trivial normal subgroup N of G, there exists $n \in \mathbb{N}$ such that* $\operatorname{Rist}_G(n)' \leqslant N$.

Proof. Since N is non-trivial, there exist $g \in N$ and $v \in T$ such that $g(v) \neq v$. Let $x, y \in \operatorname{rist}_G(v)$. Notice that $y^{g^{-1}} \in \operatorname{rist}_G(g(v))$ and it commutes with x. Since N is normal, N contains

$$[x, [g^{-1}, y]] = [x, (y^{g^{-1}})^{-1} y] = [x, y][x, (y^{g^{-1}})^{-1}]^y = [x, y],$$

that is $[x, y] \in N$, implying that $\mathrm{rist}_G(v)' \leqslant N$. Since G is spherically transitive, the rigid vertex stabilisers are conjugate and it follows that $\mathrm{Rist}_G(n)' \leqslant N$, where $n = |v|$. □

If G is branch, as a corollary of Lemma 2.3.3, one gets that every proper quotient of G is virtually abelian. On the other hand, G itself is not virtually abelian. Indeed, if G is virtually abelian, it follows from the proof of [56, Lemma 2] that $\mathrm{Rist}_G(n)$ is abelian for some n. Since $\mathrm{Rist}_G(n)$ is non-trivial, there exists $g \in \mathrm{Rist}_G(n)$ such that $g(v) \neq v$ for some $v \in T$ with $|v| > n$. This implies that $\mathrm{rist}_G(v) = \mathrm{rist}_G(v)^{g^{-1}} = \mathrm{rist}_G(g(v))$, and in particular $\mathrm{rist}_G(v) \cap \mathrm{rist}_G(g(v)) \neq 1$. Thus $v = g(v)$, and we get a contradiction. Therefore, every branch group is *just non-(virtually abelian)*. We say a group G is just non-P if every proper quotient of G has the property P but G itself does not have the property P. It turns out to be that being just non-(virtually abelian) is one of the characteristic properties of branch groups. Indeed, in [108], Wilson provided a purely group-theoretical characterisation of branch groups. Moreover, there exists a lattice of subnormal subgroups of G, called *structure graph*, on which G acts faithfully as a branch group. Therefore, the algebraic properties of a branch group are independent of its action on a given rooted tree. The construction of structure graphs is greatly dependent on the fact that all proper quotients of branch groups are virtually abelian. Unfortunately, the definition of structure graph does not extend to weakly branch groups, since not all weakly branch groups are just non-(virtually abelian); for example the Basilica group is weakly branch but not just non-(virtually abelian); cf. Corollary 2.4.11. However, it is known that certain algebraic properties of weakly branch groups are independent of their weakly branch actions [45].

2.4 Subgroups of automorphisms of rooted trees

The objective of this section is to set-up a framework for the discussion in Part I and Part II of the dissertation. Let T be the m-regular rooted tree whose set of vertices are in bijection with the set of all words over the alphabet $X = \{0, 1, \dots, m-1\}$. Let σ be an m-cycle in $\mathrm{Sym}(X)$. We define

$$\Gamma = \{g \in \mathrm{Aut}\, T \mid \text{labels in the portrait of } g \text{ are elements of } \langle\sigma\rangle\}.$$

Then Γ a subgroup of $\mathrm{Aut}\, T$ and is isomorphic to

$$\Gamma \cong \varprojlim_{n\in\mathbb{N}} \mathrm{C}_m \wr \overset{n}{\cdots} \wr \mathrm{C}_m.$$

If $m = p$, a prime, then Γ is a Sylow pro-p subgroup of $\mathrm{Aut}\, T$. The groups that we study in this dissertation are abstract subgroups of Γ. Here we give a short survey on the

Basilica group and GGS-groups. Section 2.4.1 contains a review of the Basilica group. We collect various results on the Basilica group and alongside we develop the theory of groups acting on rooted trees. Most of the results from Section 2.4.1 are generalised in Part II. In Section 2.4.2, we list the main features of GGS-groups. Also, we prove new results that are essential for the study in Part I.

2.4.1 The Basilica group

In the sequel, we fix $m = 2$ and $X = \{0, 1\}$. Let T be the binary rooted tree whose vertices are labelled by the elements of the free monoid X^*. Recall that the Basilica group $\mathcal{B}$ is a 2-generated group of automorphisms of the binary rooted tree and is generated by the automaton given by Figure 1.1. For simplicity, we use the notation 1 instead of id to denote the identity element of a group. The standard generators a and b of the group $\mathcal{B}$ can be expressed recursively as follows

$$a = (b, 1), \quad \text{and} \quad b = \sigma(a, 1), \tag{2.12}$$

where σ is the permutation $(0\ 1) \in \mathrm{Sym}(X)$. In [59], the generators a and b are defined with the identity element on the left and σ on the right. Both conventions yield isomorphic groups.

The structural properties of the Basilica group $\mathcal{B}$ were first investigated in [59]. In Theorem 2.4.1 below, we list the important properties of $\mathcal{B}$ that were proved in [59].

Theorem 2.4.1. *Let $\mathcal{B} = \langle a, b\rangle$ be the Basilica group. The following assertions hold.*

1. *$\mathcal{B}$ is self-similar and fractal;*
2. *$\mathcal{B}$ is weakly regular branch over the commutator subgroup $\mathcal{B}'$;*
3. *$\mathcal{B}/\mathcal{B}' \cong \mathbb{Z} \times \mathbb{Z}$;*
4. *$\mathcal{B}$ is torsion-free;*
5. *the semi-group generated by the elements a and b is free, implying that $\mathcal{B}$ has exponential word growth.*

The Basilica group $\mathcal{B}$ is very different from the other famous examples of groups acting on rooted trees, such as the Grigorchuk group and the Gupta–Sidki groups. The properties of $\mathcal{B}$ are of independent interest. In the following, we discuss various notions about $\mathcal{B}$ that are investigated, and use this discussion to provide a historical context for the study of groups acting on rooted trees.

2.4.1.1 Amenability

Although the concept of amenability first appeared in a paper of Banach, the notion of amenable groups was introduced by von Neumann in connection with the Banach–Tarski paradox [79].

Definition 2.4.2. A discrete group G is *amenable* if there is a measure μ from the power set of G to the unit interval $[0, 1]$ such that

(i) μ is a probability measure, in particular $\mu(G) = 1$,

(ii) μ is finitely additive: for every collection $\{A_1, \ldots, A_n\}$ of finitely many disjoint subsets of G, one has $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$,

(iii) μ is left-invariant: for every subset $A \subseteq G$ and every element g of G, the equality $\mu(gA) = \mu(A)$ holds.

Let $\mathcal{AG}$ be the class of all amenable groups. The class $\mathcal{AG}$ contains finite groups and abelian groups, and it is closed under taking subgroups, quotients, extensions and direct limits; cf. [79]. Following Day [31], we denote by $\mathcal{EG}$ the class of *elementary amenable* groups, which is the smallest class of groups containing all finite groups and abelian groups and is closed under taking subgroups, quotients, extensions and direct limits. From the above it is clear that $\mathcal{EG} \subseteq \mathcal{AG}$. The question of whether the class $\mathcal{AG}$ coincides with the class $\mathcal{EG}$ remained open for a long time. In 1980, Chou [30] came up with a characterisation of groups in the class $\mathcal{EG}$. He proved that every group in $\mathcal{EG}$ has either polynomial or exponential word growth. Later, in 1984, Grigorchuk [52] constructed a family of infinite torsion groups of intermediate word growth that contains the Grigorchuk group. It is known that groups of intermediate word growth are amenable, and hence Grigorchuk's family of groups belong to the class $\mathcal{AG}$ but not to the class $\mathcal{EG}$. In fact, the inclusion $\mathcal{EG} \subset \mathcal{AG}$ is proper even if we restrict it to the class of finitely presented groups. Indeed, the Grigorchuk group can be embedded as an amenable group into a finitely presented group [53], even though the Grigorchuk group itself is not finitely presented (however, it admits a recursive presentation as given in [74]). The existence of such an embedding follows from Higman's well-known embedding theorem.

The most prominent example of a non-amenable group is the free group of rank 2 [79]. Since the class $\mathcal{AG}$ is closed under taking subgroups, this asserts that amenable groups do not contain non-abelian free subgroups. Let $\mathcal{NF}$ denote the class of groups which do not contain non-abelian free groups. The problem of the existence of a non-amenable group in the class $\mathcal{NF}$ is known as the von Neumann problem, although the first written evidence of the problem is attributed to Day in [31]. The Thompson group F (introduced by Richard Thompson in 1965) was considered as a potential candidate for a long time, as it is infinite, finitely presented group with no non-abelian free subgroups; see [29] for an introductory survey on Thompson's groups. The amenability of the group F is still an open problem. The first examples of non-amenable groups in the class $\mathcal{NF}$ were constructed by Ol'shanskii [81]. He used the combinatorial characterisation of amenable groups provided by Grigorchuk [50] to prove the existence of non-amenable groups (both torsion-free [80] and torsion [82]) with all essential subgroups cyclic. The non-amenable groups of Ol'shanskii's are not finitely presented. An example of a finitely presented group in the class $\mathcal{NF}$ but not in the class $\mathcal{AG}$ was constructed in [83].

Notice that examples of amenable but not elementary amenable groups constructed in [52] and [53] are of intermediate word growth. Let $\mathcal{SG}$ denotes the class of sub-exponentially amenable groups, i.e., the smallest class of groups of sub-exponential word growth (either polynomial or intermediate) which is closed under taking subgroups, quotients, extensions and direct limits. It is natural to ask whether the classes $\mathcal{AG}$ and $\mathcal{SG}$ coincide; cf. [53]. In [59], Grigorchuk and Żuk proved that the Basilica group $\mathcal{B}$ is not contained in the class $\mathcal{SG}$ but it belongs to the class $\mathcal{NF}$. Later, Bartholdi and Virág proved amenability of $\mathcal{B}$ [24], which makes the Basilica group $\mathcal{B}$ the first known example of an amenable but not sub-exponentially amenable group.

Theorem 2.4.3 ([24, Theorem 1] & [59, Proposition 13]). *The Basilica group $\mathcal{B}$ is amenable but not sub-exponentially amenable.*

Later, Bartholdi, Kaimanovich and Nekrashevych proved that all groups generated from bounded finite-state automorphisms are amenable [20], which includes $\mathcal{B}$; see Section 8.3.2 for details.

2.4.1.2 Iterated monodromy groups

The concept of iterated monodromy group was introduced by Nekrashevych [77] and is used to establish connections between dynamical systems and algebra. Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be an m-fold covering map of a topological space $\mathcal{M}$ by its open subset $\mathcal{M}_1$. Let t be an arbitrary point in $\mathcal{M}$ and let $X_1 = \{t_0, \ldots, t_{m-1}\}$ be the set of m preimages of t under f. Every loop ι based at t lifts to m paths each starting at t_i for some $t_i \in X_1$. Then ι induces a permutation on the set X_1 by sending an element t_i to the end point of the lift of ι starting at t_i. This induced action is called the *monodromy action* of the loop ι and the *monodromy group* is defined to be the subgroup of $\mathrm{Sym}(X_1)$ consisting of the monodromy actions of elements of the fundamental group of $\mathcal{M}$ at t. Now, let f^n be the n-th iterate of f. Let X^* denote the disjoint union of the sets $f^{-n}(t)$ of preimages of t under f^n. Then the set X^* is naturally identified with the m-regular rooted tree and the fundamental group $\pi_1(\mathcal{M})$ acts on X^* via automorphisms of the rooted tree. The action does not depend, up to a conjugacy, on the choice of t; see [77, Proposition 3.2]. This action is called the *iterated monodromy action*, and it may not be faithful in general. The *iterated monodromy group* of f is defined to be the quotient of $\pi_1(\mathcal{M})$ by the kernel of the iterated monodromy action.

It is shown in [77] that the iterated monodromy group of z^2 is $\mathbb{Z}$ and that of $z^2 - 2$ is the infinite dihedral group. Furthermore, the Basilica group is identified as the iterated monodromy group of $z^2 - 1$.

Theorem 2.4.4 ([77, Section 5.2.2])**.** *The iterated monodromy group of the complex polynomial $z^2 - 1$ is the Basilica group $\mathcal{B}$.*

Moreover, one can reconstruct the Julia set of f from its iterated monodromy group, if f is expanding; see [77, Definition 4.5] for the definition of an expanding map.

2.4.1.3 Decision problems

In 1911, Dehn introduced three fundamental algorithmic problems for finitely presented groups: the *word problem*, the *conjugacy problem*, and the *isomorphism problem*. The word problem for a group asks for an algorithm which determines whether two given words in the generators of the group determine the same group element (or equivalently, whether a given word represents the identity of the group). A group has solvable conjugacy problem if there is an algorithm that decides whether two given words represent conjugate elements of the group. Finally, the isomorphism problem is the algorithmic problem of determining whether two given group presentations present isomorphic groups. It is now known by the results of Novikov, Boone, Adjan, and Rabin that all these problems are undecidable in the class of all finitely presented groups. Therefore, there is considerable interest in determining classes of groups with solvable decision problems. The word problem is solvable for many important classes of groups, even outside the realm of finitely presented groups.

In [76, Proposition 2.13.8], Nekrashevych proved an efficient algorithm that solves the word problem for self-similar groups with a suitable 'length reduction property'. A group with such a property is said to be *contracting*. There are several different definitions of contracting groups in the literature. We adopt the one from [76].

Definition 2.4.5 ([76, Lemma 2.11.12])**.** Let G be a subgroup of the automorphism group of a rooted tree T. The group G is said to be contracting if there exist constants $\lambda < 1$ and C, $L \in \mathbb{N}$ such that for every $g \in G$ and for every vertex $v \in T$ of level $n > L$ the following inequality holds

$$|g|_v| < \lambda\,|g| + C, \tag{2.13}$$

where $g|_v$ is the section of g at v and $|\cdot|$ is the usual length function with respect to a finite generating set of G.

Theorem 2.4.6 ([59, Proposition 15])**.** *Let $\mathcal{B}$ be the Basilica group. For every $g \in \mathcal{B}$ and $v \in T$ with $|v| \geqslant 2$, the following inequality holds*

$$|g|_v| < \frac{2}{3}\,|g| + 1,$$

where $|\cdot|$ is the length function with respect to the generating set $\{a, b\}$ of $\mathcal{B}$. In particular, the group $\mathcal{B}$ is contracting with parameters $\lambda = \frac{2}{3}$, $C = 1$ and $L = 1$.

Therefore, by [76, Proposition 2.13.8], the word problem is solvable for the Basilica group. Moreover, it is proved in [58, Theorem 1.1] that the Basilica group has the solvable conjugacy problem.

2.4.1.4 Endomorphic presentation

Definition 2.4.7 ([11, Definition 1.2])**.** An L-presentation (or an endomorphic presentation) is an expression of the form

$$L = \langle Y \mid Q \mid \Phi \mid R \rangle, \tag{2.14}$$

where Y is an alphabet, $Q, R \subset F_Y$ are sets of reduced words in the free group F_Y on Y and Φ is a set of endomorphisms of F_Y. The expression L gives rise to a group G_L defined as

$$G_L = F_Y / \langle Q \cup \langle \Phi \rangle (R) \rangle^{F_Y},$$

where $\langle \Phi \rangle (R)$ denotes the union of the images of R under every endomorphism in the monoid $\langle \Phi \rangle$ generated from Φ. An L-presentation is finite if Y, Q, Φ, R are finite.

It is proved in [11] that every finitely generated, contracting, regular branch group is not finitely presentable, however each such group admits an L-presentation. Unfortunately, this is not applicable to the Basilica group $\mathcal{B}$ as it is not branch by Corollary 2.4.11 below.

Theorem 2.4.8 ([59, Proposition 9]). *The Basilica group $\mathcal{B}$ admits an L-presentation of the form $\langle Y \mid Q \mid \Phi \mid R \rangle$ with $Y = \{a, b\}$, $Q = \varnothing$, $R = \{[a, a^{b^{2\ell+1}}] \mid \ell \in \mathbb{N}_0\}$ and $\Phi = \{\phi\}$, where*

$$\phi : \begin{cases} a & \mapsto b^2, \\ b & \mapsto a. \end{cases} \tag{2.15}$$

The above presentation is not finite since the set R is infinite. However, one can make the above L-presentation of $\mathcal{B}$ finite by introducing another endomorphism θ given by

$$\theta : \begin{cases} a & \mapsto a\, a^{b^2}, \\ b & \mapsto b. \end{cases} \tag{2.16}$$

Theorem 2.4.9 ([59, Proposition 11]). *The Basilica group $\mathcal{B}$ admits the endomorphic presentation $\langle Y \mid Q \mid \Phi \mid R \rangle$ with $Y = \{a, b\}$, $Q = \varnothing$, $R = \{[a, a^b]\}$ and $\Phi = \{\phi, \theta\}$, where ϕ and θ are given by (2.15) and (2.16), respectively.*

The L-presentation of the Basilica group $\mathcal{B}$ is helpful to study the subgroup structure of $\mathcal{B}$, which we discuss in the following section.

2.4.1.5 Quotients and lower central series

Here we consider some of the interesting quotients of the Basilica group $\mathcal{B}$. We start with an easy consequence of Theorem 2.4.8, which is first proved in [46] without using the presentation. (See [39, Proposition 8.3.7] for an alternative prove using the presentation).

Theorem 2.4.10. *Let $\mathcal{B}$ be the Basilica group and let*

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x,\, y,\, z \in \mathbb{Z} \right\}$$

be the discrete Heisenberg group. Then $\mathcal{B}/\gamma_3(\mathcal{B})$ is isomorphic to $H_3(\mathbb{Z})$. Furthermore, $\mathcal{B}'/\gamma_3(\mathcal{B})$ is isomorphic to the infinite cyclic group.

As a consequence Theorem 2.4.10 and Lemma 2.3.3, we get that $\mathcal{B}$ is not branch.

Corollary 2.4.11. *The Basilica group $\mathcal{B}$ is not just non-(virtually abelian). In particular, $\mathcal{B}$ is not branch.*

The parts (i), (ii) and (iii) of Theorem 2.4.12 below are proved in both [59] and [46], but (iii) with a mistake. Here we give a proof of (iii). The part (iv) of Theorem 2.4.12 is new, while part (v) is proved in [15] using the computer algebra system GAP. From (v), it is easy to see that $\gamma_i(\mathcal{B})/\gamma_{i+1}(\mathcal{B})$ is finite for all $i \geqslant 3$. In fact, in [15], the terms $\gamma_i(\mathcal{B})/\gamma_{i+1}(\mathcal{B})$ are computed up to isomorphism type for $1 \leqslant i \leqslant 48$ and a conjectural description is given for all $i > 48$.

Theorem 2.4.12. *Let $\mathcal{B} = \langle a, b\rangle$ be the Basilica group on the standard generators. The following assertions hold.*

(i) $\mathcal{B}' = \psi^{-1}(\mathcal{B}' \times \mathcal{B}') \rtimes \langle [a,b]\rangle$, *and* $\mathcal{B}'/\psi^{-1}(\mathcal{B}' \times \mathcal{B}') \cong \mathbb{Z}$,

(ii) $\mathcal{B}'' = \psi^{-1}(\gamma_3(\mathcal{B}) \times \gamma_3(\mathcal{B}))$,

(iii) $\gamma_3(\mathcal{B}) = \mathcal{B}'' \rtimes \langle [[a,b],b], \psi^{-1}(([a,b],[b,a]))\rangle$, *and* $\gamma_3(\mathcal{B})/\mathcal{B}'' \cong \mathbb{Z} \times \mathbb{Z}$,

(iv) $\gamma_4(\mathcal{B}) = \mathcal{B}'' \rtimes \langle [[[a,b],b],b], \psi^{-1}(([a,b],[b,a])^2)\rangle$, *and* $\gamma_4(\mathcal{B})/\mathcal{B}'' \cong \mathbb{Z} \times \mathbb{Z}$,

(v) $\gamma_3(\mathcal{B})/\gamma_4(\mathcal{B}) \cong C_4$.

Proof of $(iii), (iv) \,\&\, (v)$. (iii) We use the fact that $\gamma_3(\mathcal{B})$ is normally generated from the elements $[[a,b^{-1}],a]$ and $[[a,b^{-1}],b^{-1}]$. Observe first that

$$[a,b^{-1}] = (b^{-1},1)\sigma(a,1)(b,1)(a^{-1},1)\sigma = (b^{-1},b^{a^{-1}}).$$

We have

$$\begin{aligned}
[[a,b^{-1}],a] &= [(b^{-1},b^{a^{-1}}),(b,1)] = 1,\\
[[a,b^{-1}],b^{-1}] &= (b^{-1},b^{a^{-1}})^{-1}\sigma(a,1)(b^{-1},b^{a^{-1}})(a^{-1},1)\sigma = (bb^{a^{-1}},b^{-2a^{-1}}).
\end{aligned}$$

Therefore, $\gamma_3(\mathcal{B}) = \langle [[a,b^{-1}],b^{-1}]\rangle^{\mathcal{B}}$. Consider the element $[[a,b],b^2] \in \gamma_3(\mathcal{B})$. Using Theorem 2.4.12(ii), we get

$$[[a,b],b^2] = [(b^{-1},b),(a,a)] = ([b^{-1},a],[b,a]) = ([a,b]^{b^{-1}},[b,a]) \equiv_{\mathcal{B}''} ([a,b],[b,a]).$$

Set $x = [[a,b^{-1}],b^{-1}]$ and $y = [[a,b],b^2]$. We claim that the quotient group $\gamma_3(\mathcal{B})/\mathcal{B}''$ is generated by the images of the elements x and y. In order to prove the claim, it suffices to show that the group $\langle \mathcal{B}''\,x, \mathcal{B}''\,y\rangle$ is normal in $\mathcal{B}/\mathcal{B}''$. Notice first that the element $\mathcal{B}''\,y$ is central in the quotient group $\mathrm{St}_{\mathcal{B}}(1)/\mathcal{B}''$. We get $y^{a^{\pm 1}} = ([a,b]^{b^{\pm 1}},[b,a]) \equiv_{\mathcal{B}''} y$, and

$$y^b \equiv_{\mathcal{B}''} ([b,a]^a,[a,b]) \equiv_{\mathcal{B}''} y^{-1} \quad \text{and} \quad y^{b^{-1}} \equiv_{\mathcal{B}''} ([b,a],[a,b]^a) \equiv_{\mathcal{B}''} y^{-1}.$$

Furthermore,

$$\begin{aligned}
x^a &= (bb^{a^{-1}}, b^{-2a^{-1}})^{(b,1)} = ((b^2[b,a^{-1}])^b, b^{-2a^{-1}}) \equiv_{\mathcal{B}''} (b^2[b,a^{-1}], b^{-2a^{-1}})\\
&= (bb^{a^{-1}}, b^{-2a^{-1}}) = x,\\
x^{a^{-1}} &= (bb^{a^{-1}}, b^{-2a^{-1}})^{(b^{-1},1)} = ((b^2[b,a^{-1}])^{b^{-1}}, b^{-2a^{-1}}) \equiv_{\mathcal{B}''} (b^2[b,a^{-1}], b^{-2a^{-1}})\\
&= (bb^{a^{-1}}, b^{-2a^{-1}}) = x,\\
x^b &= ((b^{-2a^{-1}})^a, bb^{a^{-1}}) = ((b^{-a^{-1}})^a b^{-1}, (aa^{-1})baba^{-1})\\
&= (b^{-a^{-1}}[b^{-a^{-1}}, a]b^{-1}, aba^{-1}[a^{-1}, b]aba^{-1}) \equiv_{\mathcal{B}''} (b^{-a^{-1}}b^{-1}[b^{-1},a], ab^2a^{-1}[a^{-1},b])\\
&\equiv_{\mathcal{B}''} x^{-1}y\\
x^{b^{-1}} &= (b^{-2a^{-1}}, (bb^{a^{-1}})^{a^{-1}}) = (b^{-2a^{-1}}(bb^{-1}), ab(bb^{-1})b^{a^{-1}}a^{-1})\\
&= (b^{-a^{-1}}[a^{-1},b]b^{-1}, ab^2[b,a^{-1}]a^{-1}) \equiv_{\mathcal{B}''} x^{-1}y^{-1}.
\end{aligned}$$

Therefore, the claim follows: $\gamma_3(\mathcal{B}) = \mathcal{B}''\langle x, y\rangle$.

Next we prove that $\gamma_3(\mathcal{B})/\mathcal{B}'' \cong \mathbb{Z}\times\mathbb{Z}$. Observe first that the quotient group $\gamma_3(\mathcal{B})/\mathcal{B}''$ is abelian as $[\gamma_3(\mathcal{B}), \gamma_3(\mathcal{B})] \leqslant [\mathcal{B}', \mathcal{B}'] = \mathcal{B}''$. It remains to show that, the elements $\mathcal{B}''x$ and $\mathcal{B}''y$ have infinite order and there are no relations between powers of $\mathcal{B}''x$ and powers of $\mathcal{B}''y$. Since $y = ([a,b],[b,a])$, it immediately follows from Theorem 2.4.10 that $\mathcal{B}''y$ has infinite order. Now, assume to the contrary that $\mathcal{B}''x$ has finite order. There exists $n \in \mathbb{Z}$ such that $x^n \equiv_{\mathcal{B}''} 1$. We have

$$x^n \equiv_{\mathcal{B}''} (b^2[a,b], b^{-2}[a,b]^{-2})^n \equiv_{\mathcal{B}''} (b^{2n}[a,b]^n, b^{-2n}[a,b]^{-2n}) = 1.$$

In particular, $b^{-2n}[a,b]^{-2n} = 1$ in $\mathcal{B}/\gamma_3(\mathcal{B})$. This is a contradiction to Theorem 2.4.10. Hence $\mathcal{B}''x$ has infinite order. Assume again to the contrary that, there exists $p, q \in \mathbb{Z}$ such that $x^p y^q \equiv_{\mathcal{B}''} 1$. We obtain

$$\begin{aligned}
x^p y^q &\equiv_{\mathcal{B}''} (b^2[a,b], b^{-2}[a,b]^{-2})^p([a,b],[b,a])^q\\
&\equiv_{\mathcal{B}''} (b^{2p}[a,b]^{p+q}, b^{-2p}[b,a]^{2p+q}) = 1.
\end{aligned}$$

Comparing the coordinates gives,

$$b^{2p}[a,b]^{p+q} \equiv_{\gamma_3(\mathcal{B})} 1 \quad \text{and} \quad b^{-2p}[b,a]^{2p+q} \equiv_{\gamma_3(\mathcal{B})} 1,$$

which is again a contradiction to Theorem 2.4.10, unless $p = q = 0$. Hence, $\gamma_3(\mathcal{B})/\mathcal{B}'' \cong \mathbb{Z}\times\mathbb{Z}$.
(iv) Observe first that $\mathcal{B}'' \leqslant \gamma_4(\mathcal{B})$. We use the fact that $\gamma_4(\mathcal{B})$ is normally generated from the elements $[[[a,b^{-1}],b^{-1}],a]$ and $[[[a,b^{-1}],b^{-1}],b^{-1}]$. From the computation in the proof of Theorem 2.4.12(iii) above, we have

$$[[[a,b^{-1}],b^{-1}],a] \equiv_{\mathcal{B}''} 1, \quad \text{and} \quad [[[a,b^{-1}],b^{-1}],b^{-1}] \equiv_{\mathcal{B}''} x^{-2}y^{-1}.$$

Hence $\gamma_4(\mathcal{B}) = \langle [[[a,b^{-1}],b^{-1}],b^{-1}] \rangle^{\mathcal{B}}$. Set $z = [[[a,b^{-1}],b^{-1}],b^{-1}]$. We claim that the quotient group $\gamma_4(\mathcal{B})/\mathcal{B}''$ is generated by the images of the elements z and $[b^{-1},y] \equiv_{\mathcal{B}''} y^2$. It is enough to prove that the subgroup $\langle \mathcal{B}''\, z, \mathcal{B}''\, y^2 \rangle$ is normal in $\mathcal{B}/\mathcal{B}''$. Recall that the element $\mathcal{B}''\, y$ and hence also $\mathcal{B}''\, y^2$ are central in $\mathrm{St}_{\mathcal{B}}(1)/\mathcal{B}''$. Again, it follows from the proof of Theorem 2.4.12(iii) that

$$(y^2)^{a^{\pm 1}} = y^2, \quad (y^2)^{b^{\pm 1}} \equiv_{\mathcal{B}''} y^{-2}, \quad \text{and} \quad z^{a^{\pm 1}} = z.$$

Furthermore, we get

$$\begin{aligned} z^b &\equiv_{\mathcal{B}''} (x^{-2}y^{-1})^b = x^{-2b}y^{-b} \equiv_{\mathcal{B}''} (x^{-1}y)^{-2}y = z^{-1}y^{-2}, \\ z^{b^{-1}} &\equiv_{\mathcal{B}''} (x^{-2}y^{-1})^{b^{-1}} = x^{-2b^{-1}}y^{-b^{-1}} \equiv_{\mathcal{B}''} (x^{-1}y^{-1})^{-2}y \equiv_{\mathcal{B}''} z^{-1}y^2. \end{aligned}$$

From the above calculations, it follows that the quotient group $\gamma_4(\mathcal{B})/\mathcal{B}''$ is generated by the elements $\mathcal{B}''\, z$ and $\mathcal{B}''y^2$. Clearly, the quotient group $\gamma_4(\mathcal{B})/\mathcal{B}''$ is abelian. To prove the result, it suffices to show that the elements $\mathcal{B}''\, z$ and $\mathcal{B}''y^2$ have infinite order and there are no relations between powers of $\mathcal{B}''\, z$ and powers of $\mathcal{B}''\, y^2$. It follows from Theorem 2.4.12(iii) that $\mathcal{B}''\, y^2$ has infinite order. Furthermore,

$$\langle \mathcal{B}''\, z, \mathcal{B}''\, y^2 \rangle = \langle \mathcal{B}''\, x^{-2}y^{-1}, \mathcal{B}''\, y^2 \rangle \leqslant \langle \mathcal{B}''\, x, \mathcal{B}''\, y \rangle \cong \mathbb{Z} \times \mathbb{Z},$$

and hence we get that $\mathcal{B}''\, z$ is also of infinite order. Now assume that there exist $k, \ell \in \mathbb{Z}$ such that $z^k y^{2\ell} \equiv_{\mathcal{B}''} 1$. We have

$$x^{-2k}y^{-k+2\ell} = (x^{-2}y^{-1})^k y^{2\ell} \equiv_{\mathcal{B}''} z^k y^{2\ell} = 1,$$

and hence we must have $k = 0 = l$ by Theorem 2.4.12(iii) above. Therefore, $\gamma_4(\mathcal{B})/\mathcal{B}'' = \langle z\,\mathcal{B}'', y^2\,\mathcal{B}'' \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

(v) The result follows from Theorem 2.4.12(iii) & (iv). Since $z \equiv_{\mathcal{B}''} x^{-2}y^{-1}$, we get

$$\gamma_3(\mathcal{B})/\gamma_4(\mathcal{B}) = \langle \gamma_4(\mathcal{B})\, x \rangle \cong C_4. \qquad \square$$

Results on quotients of the Basilica group $\mathcal{B}$ lead to the following theorem.

Theorem 2.4.13. *Let $\mathcal{B}$ be the Basilica group. Then $\mathcal{B}$ is*

(i) [59, Proposition 6] *just non-(virtually solvable);*

(ii) [39, Proposition 8.3.6] *just non-(virtually nilpotent).*

2.4.1.6 Hausdorff dimension

The notion of Hausdorff dimension was introduced by Hausdorff and developed by Besicovitch. Although the Hausdorff dimension was initially defined as the measure of fractalness of sets over $\mathbb{R}$, it can be defined over any metric space, and hence it becomes an integral part of fractal geometry; cf. [36]. Pioneered by the work of Abercrombie [1] and of Barnea and Shalev [9], the concept of Hausdorff dimension opened up a rich and interesting field of research in the context of profinite groups.

Let G be a countably based profinite group, i.e., G admits a countable descending chain $\mathcal{F}$

$$\mathcal{F} : G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots \geqslant G_n \geqslant \cdots$$

of open normal subgroups such that $\bigcap_{n=0}^{\infty} G_n = 1$. Such a chain $\mathcal{F}$ is called a *filtration series* of G. The set $\{G_n \mid n \in \mathbb{N}_0\}$ forms a basis of the neighbourhoods of the identity in G. By defining

$$\mathcal{L} = \{G_n\, x \mid x \in G, n \in \mathbb{N}_0\},$$

we obtain an open base of G. Furthermore, the filtration series $\mathcal{F}$ of G induces a translation-invariant metric $d_{\mathcal{F}}$ on G given by

$$d_{\mathcal{F}}(x, y) = \begin{cases} [G : G_n]^{-1} & \text{if } x^{-1}y \in G_n \backslash G_{n+1}, \\ 0 & \text{if } x = y. \end{cases}$$

Let Y be a subset of G. Let $\rho \in \mathbb{R}_{\geqslant 0}$ and let $\mathcal{C}$ be a cover of Y. We say that $\mathcal{C}$ is a ρ-covering of Y if $\operatorname{diam}(S) \leqslant \rho$ for all $S \in \mathcal{C}$, where the diameter of S is defined with respect to the metric $d_{\mathcal{F}}$. For each pair $\delta, \rho \in \mathbb{R}_{\geqslant 0}$, we define

$$\mathcal{H}^{\delta}_{\rho}(Y) = \inf \left\{ \sum_{S \in \mathcal{C}} \operatorname{diam}(S)^{\delta} \mid \mathcal{C} \text{ is a } \rho\text{-covering of } Y \text{ such that } \mathcal{C} \subseteq \mathcal{L} \right\} \tag{2.17}$$

and write

$$\mathcal{H}^{\delta}(Y) = \lim_{\rho \to 0} \mathcal{H}^{\delta}_{\rho}(Y). \tag{2.18}$$

Since $\mathcal{H}^{\delta}_{\rho_1}(Y) \geqslant \mathcal{H}^{\delta}_{\rho_2}(Y)$ whenever $\rho_1 \leqslant \rho_2$, the above limit exists. It is proved in [36] that, there exists $\Delta(Y) \in \mathbb{R}_{\geqslant 0}$ such that

$$\mathcal{H}^{\delta}(Y) = \begin{cases} \infty & \text{for } \delta < \Delta(Y) \\ 0 & \text{for } \delta > \Delta(Y). \end{cases} \tag{2.19}$$

The Hausdorff dimension of Y with respect to the filtration series $\mathcal{F}$, denoted by $\dim_{\mathrm{H}}{}^{\mathcal{F}}(Y)$, is defined to be the number $\Delta(Y)$.

Now, let (X, d) be a metric space and let $Y \subseteq X$. For every $\rho > 0$ define $N_\rho(Y)$ to be the minimal number of sets of diameter at most ρ needed to cover Y. The lower box dimension of the set Y is defined to be

$$\underline{\dim}_B(Y) = \liminf_{\rho \to 0} \frac{\log N_\rho(Y)}{-\log \rho}. \tag{2.20}$$

Let H be a closed subgroup of the profinite group G equipped with the metric $d_{\mathcal{F}}$. By setting $\rho = [G : G_n]^{-1}$, we obtain $N_\rho(H) = [HG_n : G_n] = [H : H \cap G_n]$, yielding that

$$\underline{\dim}_B(H) = \liminf_{n \to \infty} \frac{\log[HG_n : G_n]}{\log[G : G_n]}. \tag{2.21}$$

Based on the work of Abercrombie, Barnea and Shalev prove the following theorem.

Theorem 2.4.14 ([9, Theorem 2.4]). *Let G be a profinite group with a filtration series $\mathcal{F} = \{G_n \mid n \geqslant 0\}$ and let H be a closed subgroup of G. Then the Hausdorff dimension of H with respect to the filtration $\mathcal{F}$ is given by*

$$\dim_{\mathrm{H}}{}^{\mathcal{F}}(H) = \underline{\dim}_B(H) = \liminf_{n \to \infty} \frac{\log[HG_n : G_n]}{\log[G : G_n]} = \liminf_{n \to \infty} \frac{\log[H : H \cap G_n]}{\log[G : G_n]}. \tag{2.22}$$

Now, recall from the beginning of Section 2.4 that Γ is the subgroup of $\operatorname{Aut} T$ isomorphic to $\varprojlim_{n \in \mathbb{N}} \mathrm{C}_m \wr \overset{n}{\cdots} \wr \mathrm{C}_m$. The set of level stabilisers $\{\mathrm{St}_\Gamma(n) \mid n \geqslant 0\}$ of Γ naturally forms a filtration series $\mathcal{F}$ of Γ. For any subgroup $G \leqslant \Gamma$, we define the Hausdorff dimension of G as the Hausdorff dimension of the closure of G in Γ with respect to the filtration series $\mathcal{F}$; it is given by

$$\dim_{\mathrm{H}} G = \dim_{\mathrm{H}}{}^{\mathcal{F}} \overline{G} = \liminf_{n \to \infty} \frac{\log_m |G/\mathrm{St}_G(n)|}{\log_m |\Gamma/\mathrm{St}_\Gamma(n)|} = (m-1) \liminf_{n \to \infty} \frac{\log_m |G/\mathrm{St}_G(n)|}{m^n},$$

where the last equality follows from $\log_m |\Gamma/\mathrm{St}_\Gamma(n)| = \log_m |\mathrm{C}_m \wr \overset{n}{\cdots} \wr \mathrm{C}_m| = \frac{m^n - 1}{m-1}$.

The Hausdorff dimensions of various (weakly) branch subgroups of Γ have been computed; for instance, see [38, 54, 99, 101]. It is proved in [12] and also in [101] that the Hausdorff dimension of a self-similar branch group is always a rational number. However, there are groups acting on rooted trees with irrational Hausdorff dimension. In fact, there exist topologically finitely generated groups of automorphisms of the binary rooted tree with arbitrary Hausdorff dimension in the interval $[0, 1]$; cf. [3]. For explicit examples see [98]. The Hausdorff dimension of the Basilica group $\mathcal{B}$ has also been computed.

Theorem 2.4.15 ([12, Example 2.4.6]). *The Hausdorff dimension of $\mathcal{B}$ is $\frac{2}{3}$.*

2.4.1.7 Congruence Subgroup Property

The congruence subgroup problem for groups acting on rooted trees is a generalisation of the classical congruence subgroup problem defined and studied for arithmetic groups such as $\mathrm{SL}_n(\mathbb{Z})$ for $n \geqslant 2$. The kernel of the canonical epimorphisms

$$\pi_k : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/k\mathbb{Z}),$$

for $k \in \mathbb{N}$, are subgroups of finite index in $\mathrm{SL}_n(\mathbb{Z})$. A subgroup H of $\mathrm{SL}_n(\mathbb{Z})$ containing $\ker(\pi_k)$ for some $k \in \mathbb{N}$ is called a *congruence subgroup*. The classical congruence subgroup problem asks the following question: is every finite index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ a congruence subgroup? Towards the end of the 19th century, Fricke and Klein discovered finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that are not congruence subgroups based on their work on automorphic functions. Later, Bass–Lazard–Serre [25] and independently Mennicke [75] answered the question positively for all $n > 2$; see [102] for a survey on the topic, which treats both cases $n = 2$ and $n > 2$.

Let G be a subgroup of the group of automorphisms $\mathrm{Aut}\, T$ of a rooted tree T. Recall that the level stabiliser $\mathrm{St}_G(n)$ is the kernel of the induced action of G on the n-th level T. In the context of groups acting on rooted trees, the congruence subgroup problem asks whether every subgroup of finite index in G contains some level stabiliser in G. We can reformulate the congruence subgroup problem in terms of profinite completions. By taking the set $\{\mathrm{St}_G(n) \mid n \in \mathbb{N}\}$ as the fundamental system of neighbourhoods of the identity in G, we get a topology on G, called the *congruence topology*. Let $\overline{G}$ be the completion of G with respect to this topology. Then $\overline{G}$ is a profinite group and is called the *congruence completion* of G. Since $\bigcap_{n\in\mathbb{N}} \mathrm{St}_G(n) = 1$, the group G embeds in $\overline{G}$. On the other hand, as G is residually finite, G embeds in its profinite completion $\widehat{G}$, and there is a canonical epimorphism from $\widehat{G}$ onto $\overline{G}$. Then the congruence subgroup problem is equivalent to asking whether the map from $\widehat{G}$ onto $\overline{G}$ is injective. If the two completions coincide then we say G has the *Congruence Subgroup Property* (abbreviated as CSP).

The congruence subgroup problem (or property) for branch groups has been comprehensively studied over the years and it is known that the famous examples of branch groups have the congruence subgroup property, for instance see [22, 37, 45]. It is shown that having CSP is independent of the (weakly) branch action of a (weakly) branch group on a rooted tree [43]. The first known example of branch group without CSP was constructed in [88]. The congruence subgroup problem for groups acting on rooted trees is systematically studied in [22], in which the authors described a general method for computing the kernel of the map $\widehat{G} \longrightarrow \overline{G}$, for a branch group G. Coming back to the Basilica group $\mathcal{B}$, it is easy to see that $\mathcal{B}$ does not have the CSP. Indeed, the quotients of $\mathcal{B}$ by the level stabilisers are finite 2-groups, and on the other hand $\mathcal{B}/\mathcal{B}' \cong \mathbb{Z} \times \mathbb{Z}$ (Theorem 2.4.1). However, $\mathcal{B}$ has a weaker version of the CSP introduced by Garrido and Uria-Albizuri in [46]. Let T be the p-regular rooted tree, for a prime p, and let Γ be a Sylow pro-p subgroup of $\operatorname{Aut} T$. Let $G \leqslant \Gamma$ be a weakly branch group and let $\mathcal{C}$ be the class of all finite p-groups. Observe that, for every $n \in \mathbb{N}$, $G/\operatorname{St}_G(n) \in \mathcal{C}$. The group G has the p-CSP if every subgroup N of G with $G/N \in \mathcal{C}$ contains some level stabiliser in G. In other words, the G has the p-CSP if the pro-p completion $\widehat{G}_p$ of G is isomorphic to the congruence completion $\overline{G}$ of G, where $\widehat{G}_p$ is given by

$$\widehat{G}_p = \varprojlim_{G/N \in \mathcal{C}} G/N.$$

By taking $\mathcal{C}$ to be a pseudo variety of finite groups, in [46], one can find a more general version of CSP, namely $\mathcal{C}$-CSP. Using a similar argument as in [43], one gets that having $\mathcal{C}$-CSP is independent of the weakly branch action of the group, see [46]. In the same article, the authors provided a sufficient condition for weakly branch groups to have the $\mathcal{C}$-CSP. By taking $\mathcal{C}$ as the class of all finite 2-groups, one gets the following result.

Theorem 2.4.16 ([46, Section 4.2]). *The Basilica group $\mathcal{B}$ has the 2-CSP property.*

2.4.1.8 Maximal subgroups

The study of maximal subgroups of branch groups was initiated by Pervova in [86] and [87] by proving that the Grigorchuk group and torsion GGS-groups do not contain maximal subgroups of infinite index. One of the early motivations of this investigation is related to a conjecture of Kaplansky. Let G be a finitely generated group and K be a field of characteristic $p > 0$. Let $\mathcal{J}(K[G])$ be the Jacobson radical and let $\mathcal{A}(K[G])$ be the augmentation ideal of the group algebra $K[G]$. Then $\mathcal{A}(K[G])$ is a maximal right ideal of $K[G]$, and hence it contains $\mathcal{J}(K[G])$. Then Kaplansky conjectured that $\mathcal{J}(K[G]) = \mathcal{A}(K[G])$ if and only if G is a finite p-group; see [67]. In [84], Passman proved that if $\mathcal{J}(K[G]) = \mathcal{A}(K[G])$ then G is a p-group, and moreover, every maximal subgroup of G is normal of index p. Therefore, the class of Burnside groups (finitely generated infinite p-groups) provide potential counter-examples to Kaplansky's conjecture. However, it is shown that the Gupta–Sidki 3-group does not satisfy the equality $\mathcal{J}(K[G]) = \mathcal{A}(K[G])$; cf. [97].

Motivated from Pervova's result, one can ask the following natural question: do all finitely generated branch groups behave in the same way? This was answered negatively by Bondarenko [27] by providing a non-explicit example of a finitely generated branch group that admits a maximal subgroup of infinite index. Thenceforth, attempts have been made to characterise finitely generated branch groups with (or without) maximal subgroups of infinite index and to see how far one can generalise the results and techniques of Pervova. It is now known that the torsion elements in the family of generalisations of Grigorchuk and Gupta–Sidki groups have maximal subgroups only of finite index [5]. On the other hand, non-torsion siblings of Grigorchuk groups acting on the binary rooted tree have maximal subgroups of infinite index [41]. One might suspect that the periodicity of the groups plays a role in the characterisation. However, recent studies by Francoeur and Thillaisundaram [42] show that all non-torsion GGS-groups have maximal subgroups only of finite index, adding more complexity to the characterisation.

The study of maximal subgroups of infinite index extends to the class of weakly branch groups by Francoeur. In [40], he developed new techniques to study the maximal subgroups of weakly branch, but not branch, groups and proved that the classical Basilica group does not contain maximal subgroups of infinite index.

Theorem 2.4.17 ([40, Theorem 4.28])**.** *Every maximal subgroup of the Basilica group $\mathcal{B}$ is of finite index.*

2.4.2 GGS-groups

Grigorchuk–Gupta–Sidki groups (abbreviated as GGS-groups) are generalisations of the (second) Grigorchuk group and the Gupta–Sidki groups. Let T be the m-regular rooted tree and let $\mathbf{e} = (e_1, \dots, e_{m-1}) \in (\mathbb{Z}/m\mathbb{Z})^{m-1}$ be a non-zero vector. To each vector $\mathbf{e}$, we associate a GGS-group $G \leqslant \operatorname{Aut} T$ as follows: $G = \langle a, t\rangle$, where a is the m-cycle $(1\ m\ m{-}1\ \cdots\ 2)$ which interchanges cyclically the m subtrees rooted at the first level of T, while t stabilises the first layer, but acts on the m subtrees rooted at first level recursively as $\psi(t) = (a^{e_1}, \dots, a^{e_{m-1}}, t)$. [1] By setting $m = p$, an odd prime, and $\mathbf{e} = (1, -1, 0, \dots, 0) \in \mathbb{F}_p^{p-1}$, we obtain the Gupta–Sidki p-group. Likewise, the second Grigorchuk group is obtained by setting $m = 4$ and $\mathbf{e} = (1, 0, 1) \in (\mathbb{Z}/4\mathbb{Z})^3$. See Figure 2.2 for an illustration of the action of the generator $t = (a, a^{-1}, t)$ of the Gupta–Sidki 3-group on the ternary rooted tree.

Notice that, for $m = 2$, there is only one non-zero vector $\mathbf{e} = (1)$, and the corresponding GGS-group is an embedding of the infinite dihedral group into the automorphism group of the binary rooted tree. For $m = 3$, there exist 3 non-isomorphic GGS-group associated to

[1] In the literature, the standard generator a is defined by the cycle $(1\ 2\ \cdots\ m)$, and the group G acts on the set of vertices of T from the right. Since we use left actions, we replace a with its inverse $(1\ m\ m{-}1\ \cdots\ 2)$ to be in consistent with the rest of the dissertation.

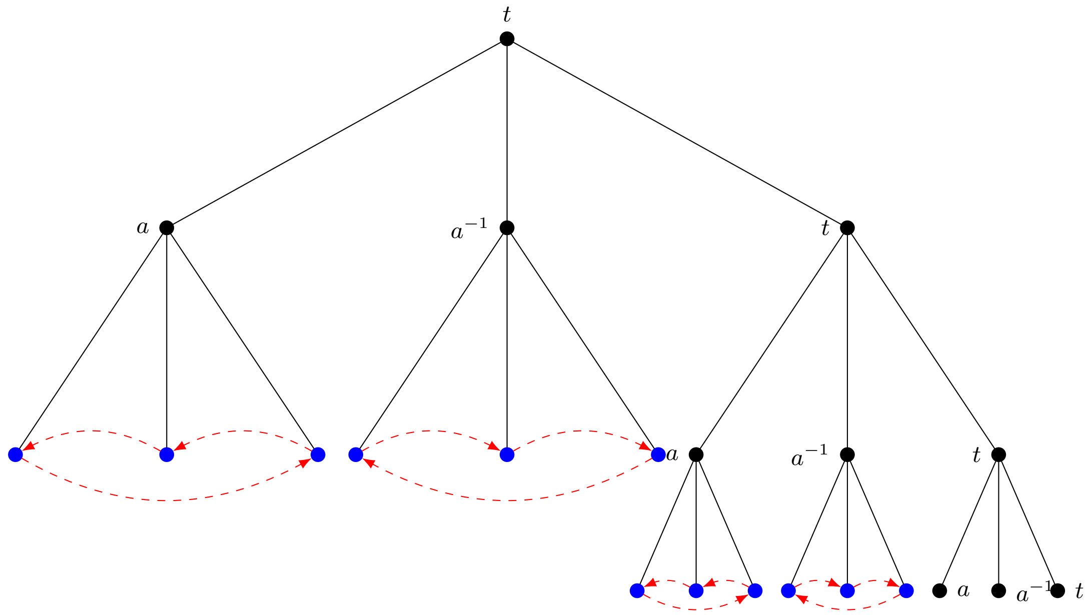

Figure 2.2: Action of the element t of the Gupta–Sidki 3-group on the ternary rooted tree

the vectors $(1,0)$ (Fabrykowski–Gupta group [35]), $(1,1)$ (Bartholdi–Grigorchuk group [17]) and $(1,2)$ (Gupta-Sidki 3-group [63]). If the defining vector $\mathbf{e}$ belongs to $\mathbb{F}_p^{p-1}$ then every multiple of $\mathbf{e}$ defines the same GGS-group. Therefore, there is only one GGS-group with a constant defining vector. In [89], Petschick obtained a sufficient and necessary condition for two GGS-groups acting on the p-regular rooted tree to be isomorphic.

Various properties of GGS-groups acting on p-regular rooted trees have been comprehensively studied over the last couple of years; for instance, see [37, 38, 105]. Here we collect some key results.

The initial examples of GGS-groups, the second Grigorchuk group and the Gupta–Sidki p-groups, are finitely generated infinite torsion groups. On the other hand, the Fabrykowski–Gupta group is not a torsion group; cf. [35]. The condition for a GGS-group acting on the p-regular rooted tree to be torsion is given by the following theorem.

For all $i, j \in \mathbb{Z}$, we use the notation $[i, j]$ to denote the interval in $\mathbb{Z}$.

Theorem 2.4.18 ([105, Theorem 1]). *Let $\mathbf{e} = (e_1, \dots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a non-zero vector and let G be the GGS-group defined by $\mathbf{e}$. Then G is torsion if and only if*

$$\sum_{i \in [1,p-1]} e_i \equiv 0 \pmod p.$$

This result does not apply to the second Grigorchuk group as it is acting on the 4-regular rooted tree. Nonetheless, in the same paper [105], Vovkivsky generalised Theorem 2.4.18 to the subclass of GGS-groups acting on p^n-regular rooted trees, for $n \in \mathbb{N}$. In [10], Bartholdi observed that Vovkivsky's proof also covers the case of composite numbers.

In the sequel, we fix p to be an odd prime.

Theorem 2.4.19 ([38, Theorem 2.1 & Corollary 2.5])**.** *Let* $\boldsymbol{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ *be a non-zero vector and let* $G = \langle a, t\rangle$ *be the* GGS*-group defined by* $\boldsymbol{e}$*. Then the following assertions hold.*

(i) $\mathrm{St}_G(1) = \langle t\rangle^G = \langle t, t^a, \ldots, t^{a^{p-1}}\rangle$ *and* $G = \mathrm{St}_G(1) \rtimes \langle a\rangle$*;*

(ii) $\mathrm{St}_G(2) \leqslant G' \leqslant \mathrm{St}_G(1)$*;*

(iii) $G/G' = \langle G'\, a\rangle \times \langle G'\, t\rangle \cong C_p \times C_p$*. Furthermore,* $G'/\gamma_3(G) = \langle \gamma_3(G)\,[a, t]\rangle \cong C_p$*;*

(iv) $\mathrm{St}_G(2) \leqslant \gamma_3(G)$*.*

The following definition is due to Petschick [90].

Definition 2.4.20. Let $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a non-zero vector. Define

$$\mathbf{e}' = (e'_2, \ldots, e'_{p-1}) = (e_2 - e_1, \ldots, e_{p-1} - e_{p-2}) \in \mathbb{F}_p^{p-2},$$

$$\mathbf{e}'' = \begin{cases} (e''_3, \ldots, e''_{p-1}) = (e'_3 - e'_2, \ldots, e'_{p-1} - e'_{p-2}) \in \mathbb{F}_p^{p-3}, & \text{if } p > 3, \\ \text{empty tuple}, & \text{if } p = 3. \end{cases}$$

We say the vectors $\mathbf{e}, \mathbf{e}'$ and $\mathbf{e}''$ are *symmetric* if $e_i = e_{p-i}$ for all $i \in [1, p-1]$, $e'_i = e'_{p+1-i}$ for all $i \in [2, p-1]$ and $e''_i = e''_{p+2-i}$ for all $i \in [3, p-1]$, respectively. For convention, we take the empty tuple to be symmetric.

Assuming that the defining vector $\mathbf{e}$ is non-symmetric, we get striking properties of the corresponding GGS-group.

Theorem 2.4.21. *Let* $\boldsymbol{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ *be a non-zero vector and let* G *be the* GGS*-group defined by* $\boldsymbol{e}$*. The following assertions hold.*

(i) [38, Lemmas 3.2] *If the defining vector* $\boldsymbol{e}$ *is non-constant then* G *is regular branch over the subgroup* $\gamma_3(G)$*. Moreover,*

$$\psi(\gamma_3(\mathrm{St}_G(1))) = \gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G).$$

(ii) [38, Lemma 3.4 & Theorem 2.14] *If the defining vector* $\boldsymbol{e}$ *is also non-symmetric then* G *is regular branch over its commutator subgroup. Moreover,*

$$\psi(\mathrm{St}_G(1)') = G' \times \overset{p}{\cdots} \times G',$$

and $[G : \mathrm{St}_G(1)'] = p^{p+1}$*.*

(iii) [38, Lemma 4.2] & [37, Theorem 3.7] *If the defining vector* $\mathbf{e}$ *is constant then* G *is weakly regular branch group over* K'*, where* $K = \langle ta^{-1} \rangle$*, but* G *is not a branch group.*

Theorem 2.4.22 ([37, Theorem 2.7])**.** *Let* $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ *be a non-constant vector and let* G *be the* GGS*-group defined by* $\mathbf{e}$*. Then* G *has the congruence subgroup property and is just-infinite. As a consequence, every proper quotient of* G *is a finite* p*-group.*

In light of Theorem 2.4.22, we now state results concerning quotient groups of GGS-groups which will be used in Chapter 5. For convenience, we do not distinguish notationally between the elements of a GGS-group and those of its quotients.

Theorem 2.4.23 ([90, Proposition 3.3])**.** *Let* $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ *be a non-constant vector and let* G *be the* GGS*-group defined by* $\mathbf{e}$*. Then*

$$\log_p [G : G''] = p + 1 + \varepsilon(\mathbf{e}') + \delta(\mathbf{e}'') - \delta(\mathbf{e}),$$

where

$$\delta(\mathbf{d}) = \begin{cases} 1 & \text{if } \mathbf{d} \text{ is symmetric,} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon(\mathbf{d}) = \begin{cases} 1 & \text{if } \mathbf{d} \text{ is constant,} \\ 0 & \text{otherwise,} \end{cases}$$

for $\mathbf{d} \in \{\mathbf{e}, \mathbf{e}', \mathbf{e}''\}$.

Notation 2.4.24. Let $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a non-symmetric vector and let G be the GGS-group defined by $\mathbf{e}$. By Theorem 2.4.22(ii), G is regular branch over the commutator subgroup of G. We denote the commutator subgroup G' of G by H and write $H_1 = H \times \overset{p}{\cdots} \times H = \psi(\mathrm{St}_G(1)')$. Since ψ is a monomorphism, we identify H_1 with the subgroup $\mathrm{St}_G(1)'$ of G: clearly, $H_1 \trianglelefteq H \trianglelefteq G$. Write $x = [a, t]$ and $x_i = x^{a^i}$ for the conjugates of x by powers of a, for $i \in \mathbb{Z}$. Notice that $x_i = x_j$ if and only if $i \equiv j \pmod p$. Further observe that

$$\begin{aligned}
\psi(x_0) &= (t^{-1}a^{e_1}, a^{e'_2}, a^{e'_3}, \ldots, a^{e'_{p-1}}, a^{-e_{p-1}}t), \\
\psi(x_1) &= (a^{-e_{p-1}}t, t^{-1}a^{e_1}, a^{e'_2}, a^{e'_3}, \ldots, a^{e'_{p-1}}), \\
&\vdots \\
\psi(x_{p-2}) &= (a^{e'_3}, \ldots, a^{e'_{p-1}}, a^{-e_{p-1}}t, t^{-1}a^{e_1}, a^{e'_2}), \\
\psi(x_{p-1}) &= (a^{e'_2}, a^{e'_3}, \ldots, a^{e'_{p-1}}, a^{-e_{p-1}}t, t^{-1}a^{e_1}).
\end{aligned}$$

Theorem 2.4.25 below is an easy consequence of Theorem 2.4.21, Theorem 2.4.22 and Theorem 2.4.23. For convenience, here we give a proof.

Theorem 2.4.25. *Let $\boldsymbol{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a non-symmetric vector and let G be the* GGS*-group defined by* $\boldsymbol{e}$. *Let H, H_1 and x_i be as defined in Notation 2.4.24. Then*

(i) $H/H_1 = \langle H_1\, x_i \mid [0, p-2]\rangle \cong C_p \times \overset{p-1}{\cdots} \times C_p$;

(ii) $H_1/[H_1, G] = \langle [H_1, G]\, y\rangle \cong C_p$, *where* $y = \psi^{-1}(x, 1, \ldots, 1)$;

(iii) $[H : H'] = p - 1 + \varepsilon(\boldsymbol{e}') + \delta(\boldsymbol{e}'')$. *Furthermore,*

$$H' = \begin{cases} H_1 & \text{if } \boldsymbol{e}'' \text{ is non-symmetric,} \\ [H_1, G] & \text{if } \boldsymbol{e}'' \text{ is symmetric and } \boldsymbol{e}' \text{ is non-constant,} \\ L & \text{if } \boldsymbol{e}'' \text{ is symmetric and } \boldsymbol{e}' \text{ is constant,} \end{cases}$$

where L is a subgroup of index p in $[H_1, G]$. In particular, $H/H' \cong C_p^{p-1+\varepsilon(\boldsymbol{e}')+\delta(\boldsymbol{e}'')}$.

Proof. (i) Recall that $H_1 = \mathrm{St}_G(1)'$. Set $\overline{G} = G/H_1$. We use the notation $\overline{(\cdot)}$ to denote the images of elements and subgroups of G under the canonical epimorphism $G \to \overline{G}$. Then $\overline{G} = \overline{\mathrm{St}_G(1)} \rtimes \langle \overline{a}\rangle \cong C_p \wr C_p$, which can be seen as follows. From Theorem 2.4.19(i) we see that $\overline{G}$ splits as a semi-direct product; i.e., $\overline{G} = \overline{\mathrm{St}_G(1)} \rtimes \langle \overline{a}\rangle$. Next we analyse the normal subgroup $\overline{\mathrm{St}_G(1)}$. Since, $\mathrm{St}_G(1)$ is generated by the elements of the form t^{a^i}, for $i \in [0, p-1]$, and each of these element has order p, we get that

$$\overline{\mathrm{St}_G(1)} = \left\langle \overline{t}, \overline{t^a}, \ldots, \overline{t^{a^{p-1}}} \right\rangle \cong C_p \times \overset{p}{\cdots} \times C_p.$$

Now, since $[\mathrm{St}_G(1) : G'] = p$, we obtain that

$$H/H_1 = \overline{H} \cong C_p \times \overset{p-1}{\cdots} \times C_p.$$

In fact, we can also identify a minimal generating set for H/H_1 as follows. The quotient group $\overline{H}$ is generated by the images of the elements $x_i = [a, t^{a^i}]$, for $i \in [0, p-1]$. Now observe from the first layer section decomposition of x_i (Notation 2.4.24) that

$$x_{p-1} x_{p-2} \cdots x_0 = 1.$$

Therefore, we conclude that

$$H/H_1 = \langle H_1\, x_i | i \in [0, p-2]\rangle.$$

(ii) Observe first that $[H, G] \times \overset{p}{\cdots} \times [H, G] \leqslant [H_1, G]$. Set $y_k = (1, \overset{k}{\ldots}, 1, x, 1, \overset{p-k-1}{\ldots}, 1)$ for $k \in [0, p-1]$. It is straightforward from Theorem 2.4.19(iii) that the set $\{y_k \mid k \in [0, p-1]\}$ generates the quotient group $H_1/[H_1, G]$. But $y_k = y_0^{a^k} \equiv_{[H_1,G]} y_0$. By setting $y = \psi^{-1}(y_0)$, again it follows from Theorem 2.4.19(iii) that $H_1/[H_1, G] = \langle y_0 \rangle \cong C_p$.

(iii) The result follows from the proof of Theorem 2.4.23. We split the proof into two cases.

Case 1: $\mathbf{e}''$ is non-symmetric. In particular, $\mathbf{e}''$ is a non-zero vector and hence $\mathbf{e}'$ has to be non-constant. If otherwise $\mathbf{e}'$ is constant, then $\mathbf{e}''$ is the zero vector in $\mathbb{F}_p^{p-1}$. Therefore, by Theorem 2.4.23, we get $[H : H'] = p^{p-1} = [H : H_1]$, and hence the result follows as $H' \leqslant H_1$.

Case 2: $\mathbf{e}''$ is symmetric. We prove that $H' \leqslant [H_1, G]$. It suffices to show that $H/[H_1, G]$ is abelian. Again we use the fact that the commutator subgroup H of G is normally generated from the element $x = [a, t]$. We first claim that the quotient group $H/[H_1, G]$ is generated from the set $\{x_0, \dots, x_{p-2}, y\}$. Even though, it follows immediately from Theorem 2.4.25(i) and Theorem 2.4.25(ii), here we give a direct proof, in order to record the following calculations which are useful later. Observe that the set $\{x_0, \dots, x_{p-2}, y\}$ is invariant under conjugation by a modulo $[H_1, G]$. Since $[H, G] \times \overset{p}{\cdots} \times [H, G] \leqslant [H_1, G]$, we get

$$\begin{aligned} x_0^t &= ((t^{-1}a^{e_1})^{a^{e_1}}, (a^{e'_2})^{a^{e_2}}, \dots, (a^{e'_{p-1}})^{a^{e_{p-1}}}, (a^{-e_{p-1}}t)^t) \\ &\equiv_{[H_1,G]} (t^{-1}a^{e_1}[a,t]^{e_1}, a^{e'_2}, \dots, a^{e'_{p-1}}, a^{-e_{p-1}}t[a,t]^{-e_{p-1}}) \\ &\equiv_{[H_1,G]} (t^{-1}a^{e_1}[a,t]^{e_1-e_{p-1}}, a^{e'_2}, \dots, a^{e'_{p-1}}, a^{-e_{p-1}}t) = x_0\, y^{e_1-e_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} x_1^t &= ((a^{-e_{p-1}}t)^{a^{e_1}}, (t^{-1}a^{e_1})^{a^{e_2}}, (a^{e'_2})^{a^{e_3}}, \dots, (a^{e'_{p-2}})^{a^{e_{p-1}}}, (a^{e'_{p-1}})^t) \\ &\equiv_{[H_1,G]} (a^{-e_{p-1}}t[a,t]^{-e_1}, t^{-1}a^{e_1}[a,t]^{e_2}, a^{e'_2}, \dots, a^{e'_{p-2}}, a^{e'_{p-1}}[a,t]^{e'_{p-1}}) \\ &\equiv_{[H_1,G]} (a^{-e_{p-1}}t[a,t]^{e_2-e_1+e'_{p-1}}, t^{-1}a^{e_1}, a^{e'_2}, \dots, a^{e'_{p-2}}, a^{e'_{p-1}}) = x_0\, y^{e'_2+e'_{p-1}}. \end{aligned}$$

For $i \in [2, p-2]$, we have

$$\begin{aligned} x_i^t &= (a^{e'_{p-i+1}}, \dots, a^{e'_{p-1}}, a^{-e_{p-1}}t, t^{-1}a^{e_1}, a^{e'_2}, \dots, a^{e'_{p-i}})^{(a^{e_1},\dots,a^{e_{p-1}},t)} \\ &= (a^{e'_{p-i+1}}, \dots, a^{e'_{p-1}}, a^{-e_{p-1}}t[a^{-e_{p-1}}t, a^{e_i}], t^{-1}a^{e_1}[t^{-1}a^{e_1}, a^{e_{i+1}}], a^{e'_2}, \dots, a^{e'_{p-i}}[a^{e'_{p-i}}, t]) \\ &\equiv_{[H_1,G]} x_i y^{-e_i+e_{i+1}+e'_{p-i}} = x_i y^{e'_{i+1}+e'_{p-i}}. \end{aligned}$$

Finally, we get

$$\begin{aligned} x_{p-1}^t &= (a^{e'_2}, \dots, a^{e'_{p-1}}, a^{e_{p-1}}t, t^{-1}a^{e_1})^{(a^{e_1},\dots,a^{e_{p-1}},t)} \\ &= (a^{e'_2}, \dots, a^{e'_{p-1}}, a^{e_{p-1}}t[a^{e_{p-1}}t, a^{e_{p-1}}], t^{-1}a^{e_1}[t^{-1}a^{e_1}, t]) \\ &\equiv_{[H_1,G]} x_{p-1}y^{e_1-e_{p-1}}. \end{aligned}$$

Since $y \in H_1$ and $[y, t] \in [H_1, G]$, it holds that, for all $j \in \mathbb{Z}$ and $i \in [0, p-1]$, we have $x_i^{t^j} \equiv_{[H_1,G]} x_i\, y^{\omega}$, for some $\omega \in \mathbb{Z}$. Since, the element $[y, a] \in [H_1, G]$, we conclude that the quotient group $H/[H_1, G]$ is generated from the set $\{x_0, \dots, x_{p-2}, y\}$.

Now we prove that $H/[H_1, G]$ is abelian. Notice that, y is a central element of $H/[H_1, G]$. It suffices to prove that, for all $i, j \in [0, p-2]$, the elements $[x_i, x_j]$ are trivial in the quotient group $H/[H_1, G]$. Since $[x_i, x_j] = [x^{a_i}, x^{a_j}] = [x, x^{a^{j-i}}]^{a^i}$ for all $i, j \in [0, p-2]$, we consider the element $[x, x_i]$. Now let $p \geqslant 5$. For $i \in [2, p-2]$, the first layer section decomposition of the element $[x_0, x_i]$ is given by

$$[x_0, x_i]|_k = \begin{cases} [t^{-1}a^{e_1}, a^{e'_{p-(i-1)}}] & \text{if } k = 0, \\ [a^{e_i{}'}, a^{-e_{p-1}}t] & \text{if } k = i-1, \\ [a^{e'_{i+1}}, t^{-1}a^{e_1}] & \text{if } k = i, \\ [a^{-e_{p-1}}t, a^{e'_{p-i}}] & \text{if } k = p-1, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} [x_0, x_i] &\equiv_{[H_1,G]} ([t^{-1}a^{e_1}, a^{e'_{p-(i-1)}}][a^{e_i{}'}, a^{-e_{p-1}}t][a^{e'_{i+1}}, t^{-1}a^{e_1}][a^{-e_{p-1}}t, a^{e'_{p-i}}], 1 \ldots, 1) \\ &\equiv_{[H_1,G]} ([t^{-1}, a^{e'_{p-(i-1)}}][a^{e_i{}'}, t][a^{e'_{i+1}}, t^{-1}][t, a^{e'_{p-i}}], 1 \ldots, 1) \\ &\equiv_{[H_1,G]} ([a, t]^{e'_{p-(i-1)}}[a, t]^{e_i{}'}[a, t]^{-e'_{i+1}}[a, t]^{-e'_{p-i}}, 1 \ldots, 1) \\ &= ([a, t]^{e''_{p-(i-1)} - e''_{i+1}}, 1 \ldots, 1) \\ &\equiv_{[H_1,G]} y^{e''_{p-(i-1)} - e''_{i+1}}. \end{aligned}$$

For $p \geqslant 3$ and for $i = 1$, we have

$$\begin{aligned} [x_0, x_1] &= ([t^{-1}a^{e_1}, a^{-e_{p-1}}t], [a^{e'_2}, t^{-1}a^{e_1}], 1, \ldots, 1, [a^{-e_{p-1}}t, a^{e'_{p-1}}]) \\ &\equiv_{[H_1,G]} ([t^{-1}, a^{-e_{p-1}}][a^{e_1}, t][a^{e'_2}, t^{-1}][t, a^{e'_{p-1}}], 1, \ldots, 1,) \\ &\equiv_{[H_1,G]} ([a, t]^{-e_{p-1}}[a, t]^{e_1}[a, t]^{-e'_2}[a, t]^{-e'_{p-1}}, 1, \ldots, 1,) \\ &= ([a, t]^{e_1 - e_{p-1} - e'_{p-1} - e'_2}, 1, \ldots, 1,) \\ &\equiv_{[H_1,G]} y^{2(e_1 - e_{p-1}) + e_{p-2} - e_2}. \end{aligned}$$

Now since $\mathbf{e}''$ is symmetric, for $p \geqslant 5$, we have $e''_{p-i+1} = e''_{i+1}$ (see Definition 2.4.20), and

$$2(e_1 - e_{p-1}) + (e_{p-2} - e_2) = 0$$

by [90, Lemma 2.4]. For $p = 3$,

$$2(e_1 - e_{p-1}) + (e_{p-2} - e_2) = 3(e_1 - e_2) \equiv 0 \pmod 3.$$

Therefore, $[x, x_i]$ is trivial in the quotient group $H/[H_1, G]$, for all $i \in [1, p-2]$, and hence $H/[H_1, G]$ is abelian. From Theorem 2.4.25(i) and Theorem 2.4.25(ii) and we obtain that

$$H/[H_1, G] = \langle \{x_0, \ldots, x_{p-2}, y\} \rangle \cong C_p^p.$$

In particular $H' \leqslant [H_1, G]$.

Assume that $\mathbf{e}'$ is non-constant. Then by Theorem 2.4.23 we obtain $[H : H'] = p^p = [H : [H_1, G]]$, implying that $H' = [H_1, G]$. If $\mathbf{e}'$ is constant then $[H : H'] = p^{p+1}$ and hence $[[H_1, G] : H'] = p$. □

Remark 2.4.26. Let $\mathbf{e} \in \mathbb{F}_p^{p-1}$ be a non-constant defining vector and let G be the GGS-group defined by $\mathbf{e}$. By Theorem 2.4.21, G is regular branch. If $\mathbf{e}$ is also non-symmetric then G is regular branch over its commutator subgroup $H = G'$ and the branching quotient $H/\psi^{-1}(H \times \overset{p}{\cdots} \times H)$ is elementary abelian (Theorem 2.4.25(i)). If the defining vector $\mathbf{e}$ is non-constant and symmetric then G is regular branch over $\gamma_3(G)$. However, the branching quotient $\gamma_3(G)/\psi^{-1}(\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G))$ is not abelian. Indeed, $\gamma_3(G)$ is normally generated from the elements $[[a,b],a]$ and $[[a,b],b]$. For example, the elements $[[a,b],a]$ and $[[a,b],a]^a$ do not commute modulo $\gamma_3(G) \times \cdots \times \gamma_3(G)$. The fact that the branching quotient being elementary abelian is vital for the computations in Chapter 5 and Chapter 6. Therefore, in Chapter 5 and Chapter 6, we consider GGS-groups that are defined by non-symmetric defining vectors.

Let $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a non-symmetric vector and let G be the GGS-group defined by $\mathbf{e}$. In the rest of this section, we assume that the vector $\mathbf{e}''$ is also non-symmetric. Then, by Theorem 2.4.25, $H' = G'' = H_1$. For convenience, we record the following structural diagram.

$$
\begin{array}{c}
G \\
\Big| \cong C_p \\
\mathrm{St}_G(1) \\
\Big| \cong C_p \\
G' = H \\
\Big| \cong C_p^{p-1} \\
\mathrm{St}_G(1)' = H_1 = H \times \overset{p}{\cdots} \times H \\
\Big| \cong C_p \\
[H_1, G]
\end{array}
$$

Figure 2.3: Structural diagram for the GGS-group defined by a non-symmetric defining vector $\mathbf{e}$ such that $\mathbf{e}''$ is also non-symmetric.

Now we prove two new lemmas that are crucial for the computations in Chapter 5. Before stating the results, we recall the definition of a *circulant matrix*. For any vector

$v = (v_1, \dots, v_n)$, the circulant matrix generated by v is the matrix of size $n \times n$ whose first row is v, and every other row is obtained from the previous one by applying a shift of length one to the right.

Lemma 2.4.27. *Let* $\mathbf{e} = (e_1, \dots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ *be a non-symmetric vector and let* G *be the* GGS*-group defined by* $\mathbf{e}$*. Let* H, H_1 *and* x_i *be as defined in Notation 2.4.24. Suppose that* $\mathbf{e}''$ *is non-symmetric. Define*

$$f : H/H_1 \times H/H_1 \to H_1/[H_1, G]$$

by $f(H_1\, g, H_1\, h) = [H_1, G]\,[g, h]$*. Then* f *is a skew-symmetric bilinear form. With respect to the generating set* $\{x_0, \dots, x_{p-1}\}$ *of* H/H_1*,* f *can be expressed as a* $p \times p$ *circulant matrix* $\mathcal{T} \in \mathrm{Mat}_p(\mathbb{F}_p)$ *generated by the vector*

$$(0, \ell_{1,2}, s_2, s_3, \dots, s_{\frac{p-1}{2}}, -s_{\frac{p-1}{2}}, \dots, -s_3, -s_2, -\ell_{1,2}), \tag{2.23}$$

where $\ell_{1,2} = 2(e_1 - e_{p-1}) + e_{p-2} - e_2$ *and* $s_i = e''_{p-(i-1)} - e''_{i+1}$*, for* $i \in [2, \frac{p-1}{2}]$*. Moreover, there exists at least one* $j \in [2, \frac{p-1}{2}]$ *such that* $s_j \neq 0$*, and in particular,* f *is non-zero.*

Proof. Observe first that, since $\mathbf{e}''$ is non-symmetric, we have $H_1 = H'$ from Theorem 2.4.25. We identify the quotient group $H_1/[H_1, G]$ with the finite filed $\mathbb{F}_p$ and the quotient group H/H_1 with the vector space of dimension $p-1$ over $\mathbb{F}_p$. Therefore, the map f is well-defined. Since f is the commutator map, it is not difficult to see that f is a skew-symmetric bilinear form.

With respect to the generating set $\{x_0, \dots, x_{p-1}\}$ of the quotient group H/H_1, we shall express the f as the $p \times p$ matrix

$$\mathcal{T} = (\ell_{i,j}),$$

where $\ell_{i,j}$ is defined by the equality $f(x_{i-1}, x_{j-1}) = [x_{i-1}, x_{j-1}] \equiv_{[H_1,G]} y^{\ell_{i,j}}$. Since the quotient $H_1/[H_1, G]$ is generated by the element y, the entries $\ell_{i,j}$ are well defined. Since f is a skew-symmetric bilinear form, it is enough to compute $[x_i, x_j]$ for $i < j$. Assume that $i, j \in [0, p-2]$ and $i < j$. We have

$$[x_i, x_j] = [x_0^{a^i}, x_0^{a^j}] \equiv_{[H_1,G]} [x_0, x_0^{a^{j-i}}] = [x_0, x_{j-i}].$$

Therefore, $\mathcal{T}$ can be determined by the values of $\ell_{1,j}$ for all $j \in [1, p-1]$. Clearly, $\ell_{1,1} = 0$. It follows from the proof of Theorem 2.4.25 that

$$\ell_{1,2} = 2(e_1 - e_{p-1}) + e_{p-2} - e_2 \qquad \ell_{1,i+1} = e''_{p-(i-1)} - e''_{i+1},$$

for $i \in [2, p-2]$. Now, for $i \in [2, p-2]$, set $s_i = e''_{p-(i-1)} - e''_{i+1}$. Notice that $s_{p-i} = -s_i$. Since $\mathbf{e}''$ is non-symmetric, there exists $i \in [2, p-2]$ such that $e''_{p-(i-1)} \neq e''_{i+1}$. Consequently, $\ell_{1,i+1} \neq 0$, and hence f is non-zero. For $i = p-1$, we have

$$\begin{aligned}
[x_0, x_{p-1}] &= ([t^{-1}a^{e_1}, a^{e'_2}], 1, \ldots, 1, [a^{e'_{p-1}}, a^{-e_{p-1}}t], [a^{-e_{p-1}}t, t^{-1}a^{e_1}]) \\
&= ([t^{-1}, a^{e'_2}][a^{e'_{p-1}}, t][a^{-e_{p-1}}, t^{-1}][t, a^{e_1}], 1, \ldots, 1,) \\
&= ([a,t]^{e'_2}[a,t]^{e'_{p-1}}[a,t]^{e_{p-1}}[a,t]^{-e_1}, 1, \ldots, 1,) \\
&= ([a,t]^{-e_1+e_{p-1}+e'_{p-1}-e'_2}, 1, \ldots, 1,) \\
&= y^{-2(e_1-e_{p-1})+e_2-e_{p-2}} = y^{-\ell_{1,2}},
\end{aligned}$$

hence $\ell_{1,p} = -\ell_{1,2}$. This completes the proof. □

Observation 2.4.28. Let f be as defined in Lemma 2.4.27 and let $\mathcal{T}$ be the $p \times p$ circulant matrix corresponding to f given by the vector in (2.23). Since f is a skew-symmetric bilinear form, we can find a matrix $M \in \mathrm{GL}_p(\mathbb{F}_p)$ such that

$$M \cdot \mathcal{T} \cdot M^T = \begin{pmatrix} I_1 & & & & & & \\ & I_2 & & & & & \\ & & \ddots & & & & \\ & & & I_r & & & \\ & & & & \ddots & & \\ & & & & & I_{\frac{p-1}{2}} & \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix},$$

where M^T is the transpose of M, and there exists $r \in [1, \frac{p-1}{2}]$ such that $I_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for all $i \leqslant r$, and I_j is the 2×2 zero matrix for all $r < j \leqslant \frac{p-1}{2}$. With the help of the software MAGMA, we have computed the matrix $\mathcal{T}$ and M for Gupta–Sidki p-groups for $p \in \{5, 7, 11, 13\}$. In this situation, $\mathcal{T}$ is of rank $p-3$ and observed that the $(p-2)$-th row of the matrix M is of the form

$$(1, 3, 6, \ldots, \frac{(k+1)(k+2)}{2}, \frac{(p-2)(p-1)}{2}, 0, 0).$$

Moreover, the element

$$x_0 x_1^3 x_2^6 \cdots x_k^{\frac{(k+1)(k+2)}{2}} \cdots x_{p-3}^{\frac{(p-2)(p-1)}{2}}$$

is central in the quotient group $H/[H_1, G]$. We generalise this observation in the following result, which turns out to be one of the main ingredients for the computations in Chapter 5, because on central elements irreducible characters of a group take non-zero values; cf. Lemma 5.3.10.

Lemma 2.4.29. *Let* $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ *be a non-symmetric vector and let* G *be the* GGS*-group defined by* $\mathbf{e}$*. Assume that* $\mathbf{e}''$ *is also non-symmetric. Then the element*

$$z = x_0 x_1^3 x_2^6 \cdots x_k^{\frac{(k+1)(k+2)}{2}} \cdots x_{p-3}^{\frac{(p-2)(p-1)}{2}} x_{p-2}^{\frac{(p-1)p}{2}} x_{p-1}^{\frac{p(p+1)}{2}},$$

is a non-trivial central in the quotient group $H/[H_1, G]$*.*

Proof. We first show that z is non-trivial in the quotient group $H/[H_1, G]$. Observe from the first layer section decomposition of the elements $x_0, \ldots, x_{p-1}$ (Notation 2.4.24) that

$$z \equiv_{[H_1,G]} (ta^{r_1}, ta^{r_2}, \ldots, ta^{r_{p-1}})y^r,$$

for some $r, r_1, \ldots, r_{p-1} \in [0, p-1]$. Thus the element z is non-trivial in the quotient group $H/[H_1, G]$, since $(ta^{r_1}, ta^{r_2}, \ldots, ta^{r_{p-1}})y^r \notin [H_1, G]$; cf. Theorem 2.4.25. Now we prove that the element z is central. From Lemma 2.4.27, the bilinear form f admits the circulant matrix generated by

$$v = (0, \ell_{1,2}, s_2, s_3, \ldots, s_{\frac{p-1}{2}}, -s_{\frac{p-1}{2}}, \ldots, -s_3, -s_2, -\ell_{1,2}),$$

where $\ell_{1,2} = 2(e_1 - e_{p-1}) + e_{p-2} - e_2$ and $s_i = e''_{p-(i-1)} - e''_{i+1}$ for $i \in [2, \frac{p-1}{2}]$. Since $\mathbf{e}''$ is non-symmetric, there is some $i \in [2, p-2]$ such that s_i is non-zero. Observe that

$$\begin{aligned} s_i &= e''_{p-(i-1)} - e''_{i+1} = e'_{p-(i-1)} - e'_{p-i} - e'_{i+1} + e'_i \\ &= e_{p-(i-1)} - e_{p-i} - e_{p-i} + e_{p-i-1} - e_{i+1} + e_i + e_i - e_{i-1} \\ &= e_{p-(i-1)} - e_{i-1} + 2(e_i - e_{p-i}) + e_{p-i-1} - e_{i+1}. \end{aligned}$$

We set $c_i = e_{p-i} - e_i$ for $i \in [1, p-1]$. Therefore, we shall write $\ell_{1,2} = -2c_1 + c_2$. For $i \in [2, \frac{p-3}{2}]$, we get $s_i = c_{i-1} - 2c_i + c_{i+1}$, and

$$s_{\frac{p-1}{2}} = c_{\frac{p-3}{2}} - 2c_{\frac{p-1}{2}} + c_{\frac{p+1}{2}} = c_{\frac{p-3}{2}} - 3c_{\frac{p-1}{2}}.$$

Now, to see that the element z is central in the quotient group $H/[H_1, G]$, it suffices to see that the element

$$\begin{aligned} [x_i, z] &= [x_i, x_0 x_1^3 \cdots x_k^{\frac{(k+1)(k+2)}{2}} \cdots x_{p-1}^{\frac{p(p+1)}{2}}] \\ &\equiv_{[H_1,G]} [x_i, x_0][x_i, x_1]^3 \cdots [x_i, x_k]^{\frac{(k+1)(k+2)}{2}} \cdots [x_i, x_{p-1}]^{\frac{p(p+1)}{2}} \\ &= y^{\ell_{i+1,1}} y^{3\ell_{i+1,2}} \cdots y^{\frac{(k+1)(k+2)}{2}\ell_{i+1,k+1}} \cdots y^{\frac{p(p+1)}{2}\ell_{i+1,p}} \\ &= y^{\ell_{i+1,1} + 3\ell_{i+1,2} + \cdots + \frac{(k+1)(k+2)}{2}\ell_{i+1,k+1} + \cdots + \frac{p(p+1)}{2}\ell_{i+1,p}} \end{aligned}$$

is trivial in $H/[H_1, G]$ for all $i \in [0, p-1]$, where $\ell_{i+1,j}$ denotes the $(i+1, j)$-th entry of the circulant matrix $\mathcal{T}$ generated by v. Let r_i denote the i-th row of the matrix $\mathcal{T}$, for

$i \in [1,p]$, and let w be the element $(1, 3, \ldots, \frac{(k+1)(k+2)}{2}, \ldots, \frac{p(p+1)}{2})$. Therefore, it is enough prove that the coordinate sum of the product of r_i and w is zero for all $i \in [1,p]$. We use the notation $(a_0, \ldots, a_{p-1}) \cdot (b_0, \cdots, b_{p-1})$ to denote the operation which yields the element $a_0 b_0 + \cdots + a_{p-1} b_{p-1}$.

Let $i \in [1,p]$. The coefficient of c_1 in the coordinate sum of the product $r_i \cdot w$ is given by

$$-2\frac{(i+1)(i+2)}{2} + \frac{(i+2)(i+3)}{2} + 2\frac{(i-1)i}{2} - \frac{(i-2)(i-1)}{2} = 0.$$

The coefficient of c_j for $j \in [2, \frac{p-3}{2}]$ in the coordinate sum of the product $r_i \cdot w$ is given by

$$\begin{aligned}\frac{(i+j-1)(i+j)}{2} - 2\frac{(i+j)(i+j+1)}{2} + \frac{(i+j+1)(i+j+2)}{2}& \\ - \frac{(i-j+1)(i-j+2)}{2} + 2\frac{(i-j)(i-j+1)}{2} - \frac{(i-j-1)(i-j)}{2} &= 0.\end{aligned}$$

Finally, the coefficient of $c_{\frac{p-1}{2}}$ in the coordinate sum of the product $r_i \cdot w$ is given by

$$\begin{aligned}\frac{(i+\frac{p-3}{2})(i+\frac{p-1}{2})}{2} - 3\frac{(i+\frac{p-1}{2})(i+\frac{p+1}{2})}{2} - \frac{(i-\frac{p-3}{2})(i-\frac{p-5}{2})}{2} + 3\frac{(i-\frac{p-1}{2})(i-\frac{p-3}{2})}{2}& \\ \equiv 0 \bmod p.&\end{aligned}$$

Therefore $r_i \cdot w = 0$ for all $i \in [1,p]$ and hence we conclude that the element z is central in the quotient group $H/[H_1, G]$. □

Part I

Representations of GGS-groups

Chapter 3

Introduction

Let G be a group and let $r_n(G)$ be the number of (equivalence classes of) n-dimensional irreducible complex representations of G. The group G is said to be *representation rigid* if $r_n(G)$ is finite for all $n \in \mathbb{N}$. Clearly, every finite group is representation rigid. Examples of groups that are not representation rigid are easy to be found. For example, the infinite cyclic group is not representation rigid as it has infinitely many 1-dimensional complex representations. In the sequel, we suppose that the group G is infinite and representation rigid. One of the fundamental problems in asymptotic representation theory is to understand the growth of the function $N \mapsto R_N(G) = \sum_{n=1}^{N} r_n(G)$, where $R_N(G)$ is the number of irreducible representations of G of dimension at most N. We say that the group G has *polynomial representation growth* (abbreviated as PRG) if $R_N(G)$ is polynomially bounded in N, i.e., if $R_N(G) = O(N^\alpha)$ for some $\alpha \in \mathbb{R}_{\geqslant 0}$. To study the representation growth of a PRG group G, following pioneering work of Grunewald, Segal and Smith [61], one introduces the Dirichlet generating function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s} \quad (s \in \mathbb{C}),$$

called the *representation zeta function* of G. From the general theory of Dirichlet generating functions, it is known that the region of convergence of $\zeta_G(s)$ is always a right-half plane of the plane of complex numbers, possibly empty. The *abscissa of convergence*, denoted by $\alpha(G)$, of $\zeta_G(s)$ is the infimum of all $\alpha \in \mathbb{R}$ such that $\zeta_G(s)$ converges and defines a holomorphic function on a right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$. It can be easily verified that G has PRG if and only if $\alpha(G)$ is finite. In fact, if $R_N(G)$ is unbounded then $\alpha(G)$ is given by the formula

$$\alpha(G) = \limsup_{N \to \infty} \frac{\log R_N(G)}{\log N}.$$

Hence, $\alpha(G)$ is minimal with the property that $R_N(G) = O(N^{\alpha(G)+\epsilon})$ for every $\epsilon > 0$, whence it gives the polynomial degree of representation growth, and it bounds the right-half plane of convergence. In favourable circumstances the function $\zeta_G(s)$ may extend meromorphically to a multi-valued analytic function on a larger domain.

The study of zeta functions of groups originated from the study of subgroup growth. Let G be a finitely generated group and let $a_n(G)$ denote the number of subgroups of index n in G. A landmark result in the theory of subgroup growth is the characterisation of groups of polynomial subgroup growth [72]. In the context of representation growth, the analogous question of characterisation, in its full generality, is still open. Nevertheless, remarkable results are obtained in the classes of arithmetic groups, their profinite completions and related compact Lie groups over non-archimedean local fields.

Arithmetic groups naturally arise as lattices, i.e., as discrete subgroups of finite covolume, in locally compact groups, such as $\mathrm{SL}_n(\mathbb{Z}) \subseteq \mathrm{SL}_n(\mathbb{R})$. Let Γ be an arithmetic irreducible lattice in a semisimple locally compact group G of characteristic zero. In [73], Lubotzky and Martin proved that Γ has PRG if and only if Γ has the congruence subgroup property.

In [26, Proposition 2], it is shown that the profinite completion $\widehat{G}$ of a finitely generated discrete group G is representation rigid if and only if G is FAb, i.e., the abelianisation of every finite index subgroup of G is finite. Since every representation of a profinite group factors through a finite quotient, the result follows from an application of Jordan's classical theorem about finite subgroups of linear groups in characteristic zero.

Theorem 3.0.1 (Jordan's theorem)**.** *There exists a function $j : \mathbb{N} \to \mathbb{N}$ such that each finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ has an abelian normal subgroup of index at most $j(n)$.*

In particular, a finitely generated profinite group G is representation rigid if and only if it is FAb. Using techniques from geometric representation theory and model theory, Jaikin-Zapirain [66] established rationality results for representation rigid compact p-adic Lie groups. Key examples of FAb compact p-adic Lie groups are special linear groups $\mathrm{SL}_n(\mathcal{O})$ and their principal congruence subgroups $\mathrm{SL}_n^m(\mathcal{O})$, where $\mathcal{O}$ is a compact discrete valuation ring of characteristic zero with residue field of characteristic p. The study of representation growth of compact p-adic Lie groups is interesting in its own right: it uses tools from various areas of mathematics. One can find interesting results on the representation zeta functions of these groups, including functional equations and explicit formulas, summarised in a series of articles, including [6, 7], by Avni, Klopsch, Onn and Voll.

In this dissertation we consider another important class of groups; the class of self-similar branch groups. The study of representation zeta functions of self-similar branch groups was initiated by Bartholdi and de la Harpe [19]. Let T be the m-regular rooted tree and let $G \leqslant \operatorname{Aut} T$ be a regular branch group over a subgroup H. Recall from Definition 2.3.2 that G is self-similar and it acts transitively on each level of T. Furthermore, G contains a subgroup H of finite index such that $H \geqslant \psi^{-1}(H \times \overset{m}{\cdots} \times H)$. If G is representation rigid (which is the case for many interesting regular branch groups, see Corollary 3.0.3), one may define and study the representation zeta function $\zeta_G(s)$, for $s \in \mathbb{C}$. Since H is a subgroup of finite index in G, as a corollary of [73, Lemma 2.2] one gets that $\alpha(G) = \alpha(H)$. This allows us to study the representation zeta function of H to understand the representation growth of G. Set $H_1 = \psi^{-1}(H \times \overset{m}{\cdots} \times H)$. Every irreducible representation ρ of H can be restricted to a representation of H_1, say $\rho|_{H_1}$. By Clifford's theorem, $\rho|_{H_1}$ can be written as a sum of irreducible representations of H_1. The representations of H_1 are in one-to-one correspondence with products of irreducible representations of H. Moreover, the representations of H_1 induce to representations of H. This process of restriction and induction of representations serves as an essential tool to get a recursive estimate on the number of irreducible characters of H, hence that of G as it is a finite extension of H; see Chapter 4 for a review on Clifford theory.

Let H be a finite group acting transitively on a finite set X with cardinality $m \geqslant 2$. Set $W(H,0) = \{1\}$, $W(H,1) = H$ and, for every $n \in \mathbb{N}$, set

$$W(H, n+1) = W(H,n) \wr_X H \cong H \wr_{X^n} W(H,n).$$

We define the iterated wreath product $W(H)$ of H as the profinite group

$$W(H) = \varprojlim_{k \in \mathbb{N}} W(H,k).$$

Notice that $W(H) \cong W(H) \wr_X H$. Hence $W(H)$ is regular branch over $W(H)$. If H is perfect, i.e., $[H,H] = H$, then it is shown in [19] that $W(H)$ is representation rigid. Furthermore, the abscissa of convergence of the representation zeta function $\zeta_{W(H)}(s)$ of $W(H)$ is positive and finite, and $\zeta_{W(H)}(s)$ satisfies a functional equation involving shifts $\zeta_{W(H)}(es)$ for $e \in \{1, \dots, m\}$. Also, in [19], the authors carried out numerical computation for $H = \mathrm{Alt}(5)$ and $H = \mathrm{PGL}(3,2)$, and obtained approximated values of the abscissa of convergence of the corresponding representation zeta functions.

In [14], Bartholdi generalised the results of [19] to all representation rigid regular branch groups. Akin to the profinite setting, it is claimed that a finitely generated group G that is regular branch over a subgroup H is representation rigid if and only if the abelianisation of H is finite. For a regular branch group G the latter condition is equivalent to the fact that G is FAb; see Theorem 5.1.1. It is easy to see that if G is representation rigid then G is FAb, and in particular, the abelianisation of H is finite. If otherwise, suppose that K is a subgroup of finite index in G with infinite abelianisation. Then K admits infinitely many 1-dimensional representations, and hence G has infinitely many representations of degree at most $[G : K]$, violating the fact that G is representation rigid. However, there is a gap in the proof of the converse statement ([14, Proposition 5.5]), which claims to prove that the kernel of every irreducible representation contains $\psi^{-1}(H \times \overset{m^d}{\cdots} \times H)$, for some $d \in \mathbb{N}$. This can be fixed by a result of Abért [2, Corollary 7] that implies that weakly branch groups are *not linear* over any field. We say that a group is *linear* over a field K if it can be embedded into $\mathrm{GL}_n(K)$ for some n. We prove the following result in Chapter 5.

Theorem 3.0.2. *Let $G \leqslant \operatorname{Aut} T$ be a regular branch group over a subgroup H. Assume that the abelianisation of H is finite. Then G is just infinite and, in particular, every finite-dimensional representation of G factors through a finite quotient.*

For a finitely generated group G that satisfies the assertion of Theorem 3.0.2, one can see that G is representation rigid by an application of Jordan's theorem, as in the proof of [26, Proposition 2]. For convenience, we record the result as the following corollary and its proof can be found in Section 5.1.

Corollary 3.0.3. *Let $G \leqslant \operatorname{Aut}(T)$ be a regular branch group over a subgroup H. If G is representation rigid then the abelianisation of H is finite. The converse holds if G is also finitely generated.*

For any representation rigid group G that is regular branch over a subgroup H, it is proved in [14] that the abscissa of convergence of $\zeta_G(s)$ is positive and finite. Indeed, it is shown that there exist constants $A \in \mathbb{N}$ and $t > 1$ (large enough), both depending on G, such that, for every $n \in \mathbb{N}$,

$$r_n(H) \leqslant A(n/\sigma_0(n))^t,$$

where $\sigma_0(n)$ is the number of divisors of n. By a similar computation as in [19, Proposition 12], one gets a rough upper bound for the abscissa of convergence $\alpha(G)$ of the representation zeta function of G as

$$\alpha(G) = \alpha(H) \leqslant t + 1.$$

Moreover, it is proved that $\zeta_G(s)$ is a linear combination of solutions of a system of functional equations; cf. [14, Theorem A]. This result applies, in particular, to the Grigorchuk group and to the Gupta-Sidki 3-group. The representation zeta functions of these groups are studied in Section 2.1 and Section 2.2 of [14]. Using the computer algebra system GAP, Bartholdi provided the first few terms of the representation zeta function of the Grigorchuk group and the Gupta–Sidki 3-group. Furthermore, he produced a recursive functional equation for the representation zeta function of the Gupta-Sidki 3-group. With the help of computer calculation he reported that the abscissa of convergence of the representation zeta function of the Grigorchuk group and the Gupta–Sidki 3-group are approximately 3.293330470 and 4.250099133, respectively, without providing any error intervals.

In this dissertation, using the representation zeta function as a tool, we study the representation growth of finite-dimensional irreducible complex representations of GGS-groups. Various properties of the GGS-groups have been investigated: periodicity [105], Hausdorff dimension [38], branching, congruence subgroup property [37], etc. Surprisingly little is known about the finite-dimensional representations of these groups.

The boundary representations of GGS-groups have already been investigated in [68]. Let G be a subgroup of the automorphism group $\mathrm{Aut}\, T$ of a regular rooted tree T whose set of vertices are in bijection with the set of all words over an alphabet X. The action of G on the rooted tree T induces an action of G on the boundary ∂T of T, where ∂T is the set of all infinite paths starting at some fixed vertex of T and is homeomorphic to the Cantor set with respect to the natural topology. The action of G on ∂T gives rise to representations of G on spaces of functions on the boundary. The study of boundary representation of groups acting on rooted trees attracts reasonable attention over the last couple of years; for instance see [16, 34, 68]. In [68], Kionke introduced a new notion of local 2-transitivity. A spherically transitive action of G on T is called locally 2-transitive, if for all distinct vertices $u, v \in X^n$ the intersection of the stabilisers $\mathrm{st}_G(u) \cap \mathrm{st}_G(v)$ acts transitively on the set $\{ux \mid x \in X\} \times \{vx \mid x \in X\}$. Under the assumption that G is locally 2-transitive, Kionke provided an explicit decomposition of the boundary representations into irreducible constituents. Furthermore, he established a sufficient and necessary condition for a GGS-group acting on a p^n-regular rooted tree, for an odd prime p and $n \in \mathbb{N}$, to be locally 2-transitive. As a corollary, one gets that if G is a GGS-group acting on the p-regular rooted tree then G is locally 2-transitive.

Our results on representations of GGS-groups are summarised into two chapters; Chapter 5 and Chapter 6. In Chapter 5, we obtain a bound for the abscissa of convergence of the representation zeta function of the GGS-groups. Chapter 6 is devoted to an explicit computation of a recursive formula for the representation zeta function of the Gupta–Sidki 3-group. The resulting functional equation is consistent with the one reported in [14] based on computer calculations.

Let G be the GGS-group defined by a non-constant defining vector $\mathbf{e} \in \mathbb{F}_p^{p-1}$. Then by Theorem 2.4.21, G is regular branch over a subgroup H, and by Theorem 2.4.22, G has the congruence subgroup property and is just-infinite. Without loss of generality, we take $H = \gamma_3(G)$, or if the defining vector $\mathbf{e}$ is also non-symmetric, we take $H = G'$. Since G is just-infinite and the commutator subgroup H' of H is normal in G, the subgroup H' is of finite index in G. Therefore, the abelianisation of H is finite, and from Corollary 3.0.3 we get the following result.

Corollary 3.0.4. *Let G be a GGS-group defined by a non-constant defining vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Then G is representation rigid.*

In [85], Passman and Temple considered the finite-dimensional representations of the Gupta–Sidki p-group G_p, for an odd prime p, over an algebraically closed field K. If $\operatorname{char} K \neq p$ then they obtained a lower bound for the number of irreducible representations of any finite degree n; cf. [85, Theorem 1.3]. In our setting, i.e., $K = \mathbb{C}$, this translates to the fact that

$$\alpha(G_p) \geqslant p - 2. \tag{3.1}$$

They also proved that G_p admits infinitely many representations if $\operatorname{char} K = p$. Using the character theory of finite groups, in the unpublished manuscript [69], Klopsch and Röver obtained partial results that enable us to produce an upper bound for $\alpha(G_p)$. In Chapter 5, we generalise the results from [85] and [69] to GGS-groups. Here we point out that, the results on GGS-groups heavily rely on our good understanding of the algebraic structure of the groups, especially their branching quotients, which in turn depend on the defining vectors. We restrict our attention to the subclass of GGS-groups defined by non-symmetric defining vectors; cf. Remark 2.4.26. In this situation, by Theorem 2.4.21 and Theorem 2.4.25, the corresponding GGS-group G is regular branch over the commutator subgroup $H = G'$ and the branching quotient $H/\psi^{-1}(H \times \overset{p}{\cdots} \times H)$ is elementary abelian. Define C to be the number (possibly infinite) of irreducible representations of the commutator subgroup H of G that are invariant under the action induced by conjugation of G. If the number C is finite, we prove that the coefficients of $\zeta_G(s)$ are bounded above by a function of n involving the *generalised Catalan numbers*; see Definition 5.3.5. In this case, using the generating function for the generalised Catalan numbers, we provide a bound for $\alpha(G)$.

Theorem 3.0.5. *Let G be a GGS-group defined by a non-symmetric defining vector and let $H = G'$ be the commutator subgroup of G. If the number C of G-invariant irreducible* *representations of H is finite then the abscissa of convergence $\alpha(G)$ of the representation zeta function $\zeta_G(s)$ satisfies the inequality*

$$p-2 \leqslant \alpha(G) \leqslant (p-1)\frac{\log 2}{\log p} + p(p-1)\frac{\log C}{\log p} + (p-1)^2 + (p-1) - 1. \tag{3.2}$$

In particular, G has polynomial representation growth.

We investigate the cases in which the number C is finite. It turns out to be that C is finite, in fact $C \leqslant p$, if the defining vector $\mathbf{e}$ satisfies a polynomial equation in its entries. In this situation, replacing C with p in (3.2), we get that $\alpha(G)$ is bounded above by $O(p^2)$.

Theorem 3.0.6. *Let G be a* GGS*-group defined by a non-symmetric defining vector $\boldsymbol{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$. We define*

$$\boldsymbol{e}'' = \begin{cases} (e_3 - 2e_2 + e_1, \ldots, e_{i+2} - 2e_{i+1} + e_i, \ldots, e_{p-1} - 2e_{p-2} + e_{p-3}) \in \mathbb{F}_p^{p-3}, & \text{if } p > 3, \\ \text{empty tuple}, & \text{if } p = 3. \end{cases}$$

Assume that the vector $\boldsymbol{e}''$ is either $(*)$ symmetric, or $(**)$ non-symmetric and the sum

$$\omega(\boldsymbol{e}) = (p-2)(e_1 - e_{p-1}) + (p-4)(e_2 - e_{p-2}) + \cdots + 3(e_{\frac{p-3}{2}} - e_{\frac{p+3}{2}}) + (e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}})$$

is non-zero modulo p. Then

$$p-2 \leqslant \alpha(G) \leqslant (p-1)\frac{\log 2}{\log p} + 2p^2 - 2p + 1. \tag{3.3}$$

For $p = 3$, there are only two non-isomorphic GGS-groups defined by non-symmetric vectors, namely the Fabrykowski–Gupta group defined by $\mathbf{e} = (1, 0)$, and the Gupta–Sidki 3-group defined by $\mathbf{e} = (1, 2)$. In both cases, the vector $\mathbf{e}''$ is the empty tuple, and hence it is symmetric by definition. If $p = 5$ and the defining vector $\mathbf{e}''$ is non-symmetric, we shall prove that the sum $\omega(\mathbf{e})$ has to be non-zero modulo 5; see Lemma 5.3.13. However, there exist GGS-groups that do not satisfy the condition $(**)$. For example, fix $p = 7$. Consider the defining vector $\mathbf{e} = (1, 1, 2, 3, 0, 0)$. Notice that $\mathbf{e}$ and $\mathbf{e}''$ are non-symmetric, but the sum $\omega(\mathbf{e})$ is zero modulo p. We shall take a closer look at the special case where $\omega(\mathbf{e}) \equiv 0 \pmod 7$, and give an alternative proof for the conclusion of Theorem 9.1.1; see Lemma 5.3.16. Therefore, for $p \in \{3, 5, 7\}$, we obtain the following theorem in its full generality.

Theorem 3.0.7. *Let $p \in \{3, 5, 7\}$ and let G be a* GGS*-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Then*

$$p-2 \leqslant \alpha(G) \leqslant (p-1)\frac{\log 2}{\log p} + 2p^2 - 2p + 1.$$

Furthermore, Theorem 9.1.1 can be applied to the infinite family of GGS-groups defined by the defining vectors of type $\mathbf{e} = (1, 2, \ldots, p-1) \in \mathbb{F}_p^{p-1}$, since the vector $\mathbf{e}''$ is symmetric. Now, consider the defining vector $\mathbf{e} = (1, -1, 0, \ldots, 0) \in \mathbb{F}_p^{p-1}$ of the Gupta–Sidki p-group for $p \geqslant 5$. It is easy to see that the defining vector $\mathbf{e}''$ is non-symmetric and the sum $\omega(\mathbf{e})$ is non-zero modulo p. Hence, we record the following result.

Corollary 3.0.8. *Let G be the Gupta–Sidki p-group. The abscissa of convergence $\alpha(G)$ of the representation zeta function $\zeta_G(s)$ satisfies the inequalities* (3.3).

The proofs of the special cases $p = 5$ and $p = 7$ suggest that Theorem 9.1.1 can be generalised to all GGS-groups defined by non-symmetric defining vectors. At the end of Section 5.3 of Chapter 5, we provide partial results that help to generalise Theorem 9.1.1. The general approach requires an understanding of the lower central series (or at least terms up to $\gamma_p(G)$) of the given GGS-group G. So far, the best known result in this direction is the work of Vieira on the Gupta–Sidki 3-group G_3 [104], who proved that the rank of the quotient group $\gamma_i(G_3)/\gamma_{i+1}(G_3)$ is bounded by two, for $i \in \{1, \ldots, 9\}$. Using a nilpotent quotient algorithm, a descriptive bound for $\gamma_i(G_3)/\gamma_{i+1}(G_3)$, for $i \geqslant 2$, is obtained in [15]. With a better insight on lower central series, one would be able to generalise Theorem 9.1.1 to all GGS-groups defined by non-symmetric defining vectors.

If we allow the defining vector $\mathbf{e}$ to be symmetric, two possible cases can occur; either $\mathbf{e}$ is symmetric and non-constant or $\mathbf{e}$ is constant. In the first case, the corresponding GGS-group G is regular branch over the subgroup $\gamma_3(G)$. In this situation, the branching quotient is not abelian anymore. In the latter case, the corresponding GGS-group is merely weakly branch. One might need a different approach to study the representation zeta function of GGS-groups corresponding to symmetric defining vectors. This motivates us to ask the following question.

Question 3.0.9. How far can we generalise results obtained on the representation growth of GGS-groups? How do these results connect to distinctive structural properties of the groups?

In Chapter 6, we explicitly compute the representation zeta function of the Gupta–Sidki 3-group G_3. The group G_3 is regular branch over the commutator subgroup G_3' and its branching quotient $G_3'/\psi^{-1}(G_3' \times G_3' \times G_3')$ is isomorphic to $C_3 \times C_3$. Because of its relatively small branching quotient, we can carry out precise computations to get recursive estimates on the number of irreducible representations of G_3'. Using these estimates, we first obtain a recursive formula for the representation zeta function of G_3'. From that one can easily compute the representation zeta function of G_3. The detailed method of computation is described in Section 6.2. Our calculations are based on partial results obtained in [69] which provided the general strategy.

Theorem 3.0.10. *Let G_3 be the Gupta–Sidki 3-group. The representation zeta function $\zeta_H(s)$ of the commutator subgroup $H = G_3'$ satisfies the functional equation*

$$\zeta_H(s) = 3 + \alpha(s) + 2\beta(s) + \tau(s) + \xi(s),$$

where $\alpha(s)$, $\beta(s)$, $\tau(s)$ and $\xi(s)$ are partial representation zeta functions of H, which are defined in Section 6.5. We get

$$\zeta_{G_3}(s) = 9 + 2 \cdot 3^{-s} + 3^{-s}\,\alpha(s) + 2 \cdot 3^{-s}\beta(s) + 3^{-s}\,\tau(s) + \frac{1}{9}\,3^{-2s}\,\xi(s). \qquad (3.4)$$

An explicit formulation of (3.4) can be found in Section 6.5. We shall show in Section 6.4.2 that the functional equation (3.4) is in agreement with the one provided in [14] based on undocumented computer assisted calculation. In Appendix 10, we give a MAGMA code that produces the first 500 terms of the representation zeta function of G_3, that coincides with all the first 11 terms provided in [14]. Furthermore, the MAGMA code computes a conjectural estimate of the abscissa of convergence based on a truncated representation zeta function of the commutator subgroup G_3' with 500 terms and the value rounded down to the second decimal is 4.25. However, because of the complex recursive nature of the zeta function it is not clear how to obtain a precise value for $\alpha(G_3)$ from (3.4).

Question 3.0.11. Can we find the precise abscissae of convergence of the representation zeta functions of the GGS-groups? Are they rational, algebraic or transcendental? How do they relate to the algebraic properties of the groups?

We emphasise that our computation is limited to G_3. In general, i.e., if G is a GGS-group defined by a non-symmetric defining vector, the branching quotient is not $C_p \times C_p$ of rank 2, but rather $C_p \times \cdots \times C_p$ of rank $p-1$. We need new insights to conduct effective Clifford theory in this increasingly complex setting. As a next step, one can consider the Fabrykowski–Gupta group G defined by the vector $(1, 0)$. It is regular branch over its commutator subgroup and its branching quotient is isomorphic to $C_3 \times C_3$. As pointed out earlier, Theorem 9.1.1 applies to G, and $\alpha(G) \in [1, 12.261895]$.

Before proving the main results, in Chapter 4, we review necessary definitions and results from character theory of finite groups, including Clifford's Theorem. The results from Chapter 4 are essential tools to study the representations of GGS-groups, especially in the computation of the representation zeta function of the Gupta–Sidki 3-group. Theorem 3.0.2, Corollary 3.0.3, Theorem 9.1.1 and Theorem 3.0.7 are proved in Chapter 5. While Chapter 6 is entirely dedicated to prove Theorem 3.0.10.

Chapter 4

Preliminaries from the character theory of finite groups

Here we set up notations, and recall definitions and results from the character theory of finite groups that are vital for the discussion in Chapter 5 and Chapter 6; for details see [65, Chapters 5, 6 and 11]. The results from this chapter will be used several times in the subsequent chapters.

Let A be an arbitrary finite group. A *class function* f of A is a function from A to a field K such that f is constant on the conjugacy classes of A. In this dissertation, we take K to be the field of complex numbers $\mathbb{C}$. The set of all class functions of A forms a vector space over $\mathbb{C}$. For any given pair χ_1, χ_2 of class functions of A, one can define the inner-product

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|A|} \sum_{g \in A} \chi_1(g)\overline{\chi_2(g)},$$

where $\overline{\chi_2(g)}$ is the complex conjugate of $\chi_2(g)$. The *induction* and *restriction* are two operations that are defined between the set of class functions of a group and a given subgroup. Let A be a finite group and B be a subgroup of A. If φ is a class function of A then the restriction of φ to B is a class function of B and is denoted by $\varphi|_B$. Now, let ϑ be a class function of B. The induced class function ϑ^A of A is given by

$$\vartheta^A(g) = \frac{1}{|B|} \sum_{x \in A} \vartheta^\circ(xgx^{-1}),$$

where $\vartheta^\circ(h) = \vartheta(h)$ if $h \in B$, and otherwise, $\vartheta^\circ(h) = 0$. For any given pair φ and ϑ, where φ is a class function of A and ϑ is a class function of B, it is easy to see that

$$\langle \vartheta, \varphi|_B \rangle = \langle \vartheta^A, \varphi \rangle. \tag{4.1}$$

The above equality is known as the *Frobenius reciprocity*.

Let $\rho : A \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a representation of A. The *character* $\chi : A \longrightarrow \mathbb{C}$ afforded by ρ is the class function given by $\chi(g) = \mathrm{tr}(\rho(g))$, for every $g \in A$. Observe that $\chi(1) = n$ is the dimension of ρ, and is called the *degree of the character* χ. The *kernel of the character* χ is given by the set

$$\ker(\chi) = \{g \in A \mid \chi(g) = n\}.$$

Note that a character is not necessarily a homomorphism. However, if the degree of χ is one, then it is a homomorphism, and in that case χ is said to be *linear*. A character χ is said to be *irreducible* if the corresponding representation ρ is irreducible. The set of all irreducible characters of a group A, denoted by $\mathrm{Irr}(A)$, forms an orthogonal basis for the set of all class functions of A. Every character χ of A can be written as sum of irreducible characters $\chi_1, \ldots, \chi_\ell \in \mathrm{Irr}(A)$, for some $\ell \in \mathbb{N}$, as following

$$\chi = m_1\chi_1 + \cdots + m_\ell\chi_\ell,$$

where m_i is the multiplicity of χ_i in χ and is given by the inner-product $\langle \chi, \chi_i \rangle$ for all $i \in \{1, \ldots, \ell\}$. Here the decomposition of χ is unique up to a permutation of its components. We say that χ_i is an *irreducible constituent* of χ.

Let A and B be two finite groups and let $G = A \times B$. Let $\varphi \in \mathrm{Irr}(A)$ and $\vartheta \in \mathrm{Irr}(B)$. We use the notation $\varphi \otimes \vartheta$ to denote the *product of the characters* φ and ϑ, and is given by: for every $g \in A$ and $h \in B$, we have

$$(\varphi \otimes \vartheta)(g, h) = \varphi(g)\vartheta(h).$$

It can be easily verified that $\varphi \otimes \vartheta$ is an irreducible character of G. Moreover, every irreducible character of G can be written uniquely as a product an element of $\mathrm{Irr}(A)$ and an element of $\mathrm{Irr}(B)$.

Theorem 4.0.1 ([65, Theorem 4.21])**.** *Let A and B be two finite groups and let $G = A \times B$. Then*

$$\mathrm{Irr}(G) = \{\varphi \otimes \vartheta \mid \varphi \in \mathrm{Irr}(A), \vartheta \in \mathrm{Irr}(B)\} = \mathrm{Irr}(A) \times \mathrm{Irr}(B).$$

Remark 4.0.2. Let A and B be two finite groups and let $G = A \times B$. Let $\varphi_1, \varphi_2 \in \mathrm{Irr}(A)$ and $\vartheta_1, \vartheta_2 \in \mathrm{Irr}(B)$. It follows from Theorem 4.0.1 that $\varphi_1 \otimes \vartheta_1 = \varphi_2 \otimes \vartheta_2$ if and only if $\varphi_1 = \varphi_2$ and $\vartheta_1 = \vartheta_2$.

Let A be a finite group and let B be a subgroup of A. Assume further that $\varphi \in \mathrm{Irr}(A)$ and $\vartheta \in \mathrm{Irr}(B)$. Then the function ϑ^A and $\varphi|_B$ are again characters of A, resp. B, with $\vartheta^A(1) = [A : B]\,\vartheta(1)$ and $\varphi|_B(1) = \varphi(1)$. It is easy to check that the operations induction and restriction are transitive.

Lemma 4.0.3. *Let B and C be subgroups of A such that $C \leqslant B$. Let $\varphi \in \operatorname{Irr}(A)$ and $\vartheta \in \operatorname{Irr}(C)$. Then $(\vartheta^B)^A = \vartheta^A$, and $(\varphi|_B)|_C = \varphi|_C$.*

Now, suppose that B is a normal subgroup of A. If $\vartheta \in \operatorname{Irr}(B)$ and $g \in A$, then the map $\vartheta^g : B \longrightarrow \mathbb{C}$ given by $\vartheta^g(h) = \vartheta(ghg^{-1})$, for every $h \in B$, is again an irreducible character of B. This gives an action of the group A on the set $\operatorname{Irr}(B)$ by conjugation. The stabiliser of $\vartheta \in \operatorname{Irr}(B)$ under the action of A is the subgroup given by

$$I_A(\vartheta) = \{g \in A \mid \vartheta^g = \vartheta\},$$

and is called the *inertia group* of ϑ in A. Notice that $B \leqslant I_A(\vartheta)$. The induction and restriction of characters from or to the normal subgroup B help to relate the characters of A to the characters of B. One of the fundamental results that guarantees this process is Clifford's theorem, introduced by Clifford in 1937.

Theorem 4.0.4 (Clifford's theorem [65, Theorem 6.4])**.** *Let A be a group (possibly infinite), let B be a normal subgroup of A, and let $\varphi \in \operatorname{Irr}(A)$. Let ϑ be an irreducible component of $\varphi|_B$ and suppose that $\vartheta = \vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are the distinct conjugates of ϑ in A. Then*

$$\varphi|_B = \langle \varphi|_B, \vartheta \rangle \sum_{r=1}^{n} \vartheta_r.$$

For a given character $\vartheta \in \operatorname{Irr}(B)$, Clifford's theorem enables us to construct all irreducible characters $\varphi \in \operatorname{Irr}(A)$ such that $\langle \varphi|_B, \vartheta \rangle \neq 0$.

It is worth to point out that the induced character ϑ^A of $\vartheta \in \operatorname{Irr}(B)$ from the normal subgroup B to A is not necessarily irreducible. However, under certain conditions ϑ^A becomes an irreducible character of A.

Theorem 4.0.5 ([65, Theorem 6.11])**.** *Let B be a normal subgroup of A and let $\vartheta \in \operatorname{Irr}(B)$ with $C = I_A(\vartheta)$. Let*

$$\mathcal{A} = \{\varphi \in \operatorname{Irr}(A) \mid \langle \varphi|_B, \vartheta \rangle \neq 0\}, \qquad \mathcal{C} = \{\eta \in \operatorname{Irr}(C) \mid \langle \eta|_B, \vartheta \rangle \neq 0\}.$$

Then the following assertions hold.

(i) If $\eta \in \mathcal{C}$ then η^A is irreducible, and the map $\eta \mapsto \eta^A$ is a bijection from $\mathcal{C}$ onto $\mathcal{A}$,

(ii) If $\eta^A = \varphi$ for $\eta \in \mathcal{C}$, then $\langle \varphi|_B, \vartheta \rangle = \langle \eta|_B, \vartheta \rangle$.

We say a character $\varphi \in \operatorname{Irr}(A)$ is an *extension* of a character $\vartheta \in \operatorname{Irr}(B)$ if $\varphi|_B = \vartheta$, and we say that ϑ is *extendable*. In this case, the character ϑ is A-invariant, i.e., $I_A(\vartheta) = A$, and $\varphi(1) = \vartheta(1)$. If the character ϑ is extendable and if we identify the set $\operatorname{Irr}(A/B)$ with a subset of $\operatorname{Irr}(A)$ then the induced character ϑ^A can be uniquely described as follows.

Theorem 4.0.6 (Gallagher's Theorem [65, Corollary 6.17])**.** *Let B be a normal subgroup of A and $\varphi \in \mathrm{Irr}(A)$ such that $\varphi|_B = \vartheta \in \mathrm{Irr}(B)$. Then the characters $\varphi\lambda$ for $\lambda \in \mathrm{Irr}(A/B)$ are irreducible, distinct for distinct λ and are all of the irreducible constituents of ϑ^A. Furthermore, each $\varphi\lambda$ occurs in the decomposition of ϑ^A with multiplicity one.*

The theorem below is a special case, under which a character is extendable. This theorem is crucial for the discussion in Chapter 6, where the factor groups considered are mostly cyclic.

Theorem 4.0.7 ([65, Theorem 11.22])**.** *Let B be a normal subgroup of A and A/B be cyclic and $\vartheta \in \mathrm{Irr}(B)$ be such that $I_A(\vartheta) = A$. Then ϑ is extendable to A.*

We now record the following two lemmas for the discussion in Chapter 6.

Lemma 4.0.8. *Let A be a finite group and let B be a proper normal subgroup of A. Let $\vartheta \in \mathrm{Irr}(B)$ with $I_A(\vartheta) = A$. If ϑ extends to an irreducible character φ of A then $\varphi|_{A\setminus B} \neq 0$.*

Proof. Suppose that $\varphi|_{A\setminus B} = 0$. We have,

$$1 = \langle \varphi, \varphi \rangle = \frac{1}{|A|} \sum_{g \in A} \varphi(g)\overline{\varphi(g)} = \frac{1}{|A|} \left(\sum_{h \in B} \varphi(h)\overline{\varphi(h)} + \underbrace{\sum_{g \in A\setminus B} \varphi(g)\overline{\varphi(g)}}_{=0} \right)$$
$$= \frac{1}{|A|} \sum_{h \in B} \vartheta(h)\overline{\vartheta(h)} = \frac{|B|}{|A|},$$

where the last equality follows since $1 = \langle \vartheta, \vartheta \rangle = \frac{1}{|B|} \sum_{h \in B} \vartheta(h)\overline{\vartheta(h)}$. This is a contradiction, as B is a proper subgroup of A, completing the proof. □

Lemma 4.0.9. *Let B and C be normal subgroups of A such that $C \leqslant B$. Let $\varphi \in \mathrm{Irr}(B)$ and let $\eta \in \mathrm{Irr}(C)$ be an irreducible component of $\varphi|_C$. Assume that $\eta = \eta_1, \ldots, \eta_n$ are the distinct conjugates of η in B. If $g \in I_A(\varphi)$ then $\eta^g \in \{\eta_1, \ldots, \eta_n\}$. The converse is also true, if $I_B(\eta) = C$.*

Proof. Thanks to Clifford's theorem we have,

$$\varphi|_C = \langle \varphi|_C, \eta \rangle \sum_{r=1}^{n} \eta_r.$$

Assume that $g \in I_A(\varphi)$. Then by definition $\varphi = \varphi^g$. Therefore $\varphi|_C = \varphi^g|_C = (\varphi|_C)^g$, where the last equality follows because C is normal in A. This further implies that $\eta^g \in \{\eta_1, \ldots, \eta_n\}$.

Conversely, assume that $\eta^g \in \{\eta_1, \ldots, \eta_n\}$. Then

$$\varphi^g|_C = (\varphi|_C)^g = \langle \varphi|_C, \eta \rangle \sum_{r=1}^{n} \eta_r^g = \langle \varphi|_C, \eta \rangle \sum_{r=1}^{n} \eta_{\sigma(r)} = \varphi|_C,$$

where $\sigma \in \mathrm{Sym}(n)$ is a permutation that reflects how conjugation by g permute the B-orbit of η. If we further assume that $I_B(\eta) = C$, then η^B is irreducible and $\eta^B \simeq \varphi$, since

$$0 \neq \langle \varphi|_C, \eta \rangle = \langle \varphi, \eta^B \rangle.$$

Therefore,

$$1 = \langle \varphi, \eta^B \rangle = \langle \varphi|_C, \eta \rangle = \langle \varphi^g|_C, \eta \rangle = \langle \varphi^g, \eta^B \rangle = \langle \varphi^g, \varphi \rangle,$$

which implies $\varphi^g = \varphi$. $\square$

Chapter 5

Representation growth of GGS-groups

The objective of this chapter is to obtain a bound for the abscissa of convergence of the representation zeta function of a GGS-group G defined by a non-symmetric defining vector $\mathbf{e} \in \mathbb{F}_p^{p-1}$. We begin with Section 5.1, where we prove the rigidity result Theorem 3.0.2 for finitely generated branch groups. Thanks to Theorem 3.0.2, every finite-dimensional representation of G factors through a finite quotient. Moreover, it follows from Theorem 2.4.22 that every proper quotient of G is a finite p-group. Therefore, we shall first obtain bounds for the number of irreducible representations of finite p-groups. The results are summarised in Section 5.2. Recall from the statement of Theorem 3.0.5 that C is defined as

$$C = |\{\varphi \in \mathrm{Irr}([G,G]) \mid I_G(\varphi) = G\}|.$$

If C is finite, then the bounds on finite p-groups enable us to prove Theorem 3.0.5 in Section 5.3 using generalised Catalan numbers (see Definition 5.3.5). We prove Theorem 9.1.1 in several steps. The groups that satisfy condition $(*)$ and condition $(**)$ of Theorem 9.1.1 will be treated separately. The proof of Theorem 9.1.1 for a GGS-group G that satisfies $(*)$ is given by Corollary 5.3.9, while that of for $(**)$ is summarised in Corollary 5.3.11. To conclude the discussion, we give some partial results that might help one to generalise Theorem 9.1.1 to all GGS-groups defined by non-symmetric defining vectors. Along the line, we will also prove Theorem 3.0.7.

5.1 Representations of self-similar branch groups

We first record the following theorem, which shows that for a group G that is regular branch over a subgroup H, being FAb is equivalent to the fact that $H/[H,H]$ is finite.

Theorem 5.1.1. *Let G be a regular branch group over a subgroup H. Then G is FAb if and only if the abelianisation of H is finite.*

Proof. If G is FAb then, since H is of finite index in G, the abelianisation of H is finite. To prove the converse, assume that the abelianisation of H is finite. Let K be a subgroup of finite index in G. Then K has only finitely many conjugates in G. Set L as the core of K in G given by taking the intersection of all conjugates on K in G. Notice that L is a normal subgroup of finite index in G. Then the commutator subgroup L' of L is a non-trivial normal subgroup of G. If otherwise $L' = 1$, then G is virtually abelian, which is a contradiction to the fact that G is branch; see the discussion at the end of Section 2.3. Therefore, by Lemma 2.3.3, L' contains the subgroup $\mathrm{Rist}_G(n)'$, for some $n \in \mathbb{N}$. Since G is regular branch over H, we get $\psi^{-1}(H' \times \overset{m^n}{\cdots} \times H') \leqslant \mathrm{Rist}_G(n)' \leqslant L'$, and hence L' has finite index in G. Therefore, K' has finite index in G, and we conclude that abelianisation of K is finite. □

Now, we prove Theorem 3.0.2 and Corollary 3.0.3. The results follow immediately using the fact that weakly branch groups are not linear over any field; see [2, Corollary 7].

Proof of Theorem 3.0.2. Let N be a non-trivial normal subgroup of G. By Lemma 2.3.3, there exists some $d \in \mathbb{N}$ such that $\psi^{-1}(H' \times \overset{m^d}{\cdots} \times H')$ is contained in N. Since the abelianisation of H is finite $\psi^{-1}(H' \times \overset{m^d}{\cdots} \times H')$ has finite index in G, and hence, the quotient G/N is finite. Thus G is just infinite.

Now, let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation of dimension n. Since G is not linear, the kernel $\ker(\rho)$ of ρ is non-trivial normal subgroup of G. Hence $\ker(\rho)$ has finite index in G, as G is just infinite. Therefore, every representation of G factors through a finite quotient. □

Proof of Corollary 3.0.3. Let G be regular branch over a subgroup H. Suppose that G is representation rigid, and assume to the contrary that abelianisation of H is infinite. Then H admits infinitely many 1-dimensional representations, and hence G has infinitely many representations of degree at most $[G : H]$, violating the fact that G is representation rigid.

Now, to prove the converse assume that G is finitely generated and the abelianisation of H is finite. By Theorem 3.0.2 every finite dimensional representation of G factors through a finite quotient. The rest of the proof follows as in the proof of [26, Proposition 2] using Jordan's theorem; cf. Theorem 3.0.1. □

5.2 Upper polynomial bound for finite p-groups

Let p be an odd prime. Here we obtain bounds for the number of irreducible representation of finite p-groups. Since there is a one-to-one correspondence between the equivalence classes of irreducible representations and irreducible characters of finite groups, it suffices to consider the set of irreducible characters. For the time being assume that G is a finite p-group and N is a normal subgroup of G. For $\vartheta \in \mathrm{Irr}(N)$, set

$$\mathrm{Irr}(G, \vartheta) = \{\varphi \in \mathrm{Irr}(G) \mid \langle \varphi|_N, \vartheta \rangle_N \neq 0\}.$$

For $S \subseteq \mathrm{Irr}(N)$, we write $\mathrm{Irr}(G, S) = \bigcup_{\vartheta \in S} \mathrm{Irr}(G, \vartheta)$.

Lemma 5.2.1. *Let G be a finite p-group and let N be a normal subgroup of G. Let $\vartheta \in \mathrm{Irr}(N)$ and put*

$$\mathrm{Irr}^+(G, \vartheta) = \{\varphi \in \mathrm{Irr}(G, \vartheta) \mid \varphi(1) > \vartheta(1)\}.$$

Then the following assertions hold:

(i) $\varphi(1) \geqslant p\,\vartheta(1)$ *for every* $\varphi \in \mathrm{Irr}^+(G, \vartheta)$*;*

(ii) $|\mathrm{Irr}^+(G, \vartheta)| \leqslant p^{-1}[G : N]$.

Proof. (i) Since $\varphi \in \mathrm{Irr}^+(G, \vartheta)$, it holds that $\varphi(1) > \vartheta(1)$. Since G is a finite p-group and $\varphi(1)$ divides the order of G and likewise $\vartheta(1)$ divides the order of N, both $\varphi(1)$ and $\vartheta(1)$ are p-powers. It follows that $\varphi(1) \geqslant p\,\vartheta(1)$.
(ii) The proof proceeds by induction on $[G : N]$. If $[G : N] = 1$, then $\mathrm{Irr}(G, \vartheta) = \{\vartheta\}$. The set $\mathrm{Irr}^+(G, \vartheta)$ is empty and $|\mathrm{Irr}^+(G, \vartheta)| = 0 < p^{-1}[G : N]$. Assume that $[G : N] \geqslant p$. We split the proof into two cases based on the inertia group $I_G(\vartheta)$ of ϑ in G.

Case 1: $I_G(\vartheta) = N$. Then $\vartheta^G \in \mathrm{Irr}(G)$ and $\mathrm{Irr}(G, \vartheta) = \{\vartheta^G\} = \mathrm{Irr}^+(G, \vartheta)$. Hence $|\mathrm{Irr}^+(G, \vartheta)| = 1 \leqslant p^{-1}[G : N]$.

Case 2: $N < I_G(\vartheta) \leqslant G$. Consider the quotient group $I_G(\vartheta)/N$, which is a non-trivial finite p-group, and hence has a non-trivial center. Therefore, there exists a central element Nx in $I_G(\vartheta)/N$ such that $\langle Nx \rangle \cong N\langle x \rangle/N \cong C_p$. By setting $Z = N\langle x \rangle$, we get that $N \trianglelefteq Z \trianglelefteq I_G(\vartheta)$ and $Z/N \cong C_p$. Then ϑ extends to irreducible characters of Z. Indeed, by Theorem 4.0.6 and Theorem 4.0.7, ϑ admits exactly p distinct extensions, namely $\psi_1, \ldots, \psi_p$. Furthermore, $\mathrm{Irr}(Z, \vartheta) = \{\psi_1, \ldots, \psi_p\}$. It is easy to see that $\bigcup_{i=1}^{p} \mathrm{Irr}^+(G, \psi_i) \subseteq \mathrm{Irr}^+(G, \vartheta)$. On the other hand, if $\varphi \in \mathrm{Irr}^+(G, \vartheta)$ then the restriction of φ to Z is a sum of irreducible characters and at least one of these lies in $\mathrm{Irr}(Z, \vartheta)$. Therefore, there exists $k \in [1, p]$ such that ψ_k is an irreducible constitute of $\varphi|_Z$, and since $\varphi(1) > \vartheta(1) = \psi_k(1)$, we get that $\varphi \in \mathrm{Irr}^+(Z, \psi_k)$. Hence we obtain the following equality:

$$\mathrm{Irr}^+(G, \vartheta) = \bigcup_{i=1}^{p} \mathrm{Irr}^+(G, \psi_i).$$

If $I_G(\vartheta) = G$ then Z is a normal subgroup of G, and, by induction, we get

$$|\,\mathrm{Irr}^+(G, \vartheta)| \leqslant \sum_{k=1}^{p} |\,\mathrm{Irr}^+(G, \psi_i)| \leqslant p(p^{-1}[G : Z]) = p^{-1}[G : N].$$

Now, suppose that $I_G(\vartheta)$ is a proper subgroup of G, say $H = I_G(\vartheta)$. Then ϑ does not extend to G, and hence $|\,\mathrm{Irr}^+(G, \vartheta)| = |\,\mathrm{Irr}(G, \vartheta)|$. Using Theorem 4.0.5, we get

$$|\,\mathrm{Irr}^+(G, \vartheta)| = |\,\mathrm{Irr}(H, \vartheta)| = [H : N] \leqslant p^{-1}[G : N],$$

where the last but one equality follows because $\vartheta^H(1) = [H : N]\vartheta(1)$. This completes the proof. □

Notation 5.2.2. Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be two Dirichlet generating functions, where s is a formal variable, later a complex variable, when convergence on some right half-plane is guaranteed. We write

$$f(s) \preceq g(s)$$

if $\sum_{n=1}^{N} a_n \leqslant \sum_{n=1}^{N} b_n$ for all $N \in \mathbb{N}$. Observe that if $g(s)$ converges for some $s \in \mathbb{R}$ and $f(s) \preceq g(s)$ then $f(s) \leqslant g(s)$.

Corollary 5.2.3. *Let G be a finite p-group and N be a normal subgroup of G such that $[G : N] \geqslant p^2$. Let*

$$\Lambda = \{\vartheta \in \mathrm{Irr}(N) \mid \mathrm{Irr}(G, \vartheta) = \mathrm{Irr}^+(G, \vartheta)\}.$$

Then

$$\sum_{\varphi \in \mathrm{Irr}(G,\Lambda)} \varphi(1)^{-s} \preceq p^{-2-s}[G : N] \sum_{\vartheta \in \Lambda} \vartheta(1)^{-s}. \tag{5.1}$$

Remark 5.2.4. The right-hand side of the inequality (5.1) is a Dirichlet generating function, i.e., the corresponding coefficients are non-negative integers. Indeed, in all cases p^2 divides $[G : N]$.

Proof of Corollary 5.2.3. We set $\Lambda_1 = \{\vartheta \in \Lambda \mid I_G(\vartheta) < G\}$ and $\Lambda_2 = \{\vartheta \in \Lambda \mid I_G(\vartheta) = G\}$. Thus Λ is a disjoint union of Λ_1 and Λ_2. Since the equality $I_G(\vartheta^g) = I_G(\vartheta)^g$ holds for every $g \in G$, the sets Λ_1 and Λ_2 are closed under conjugation by G. This partitions $\mathrm{Irr}(G, \Lambda)$ into a disjoint union of $\mathrm{Irr}(G, \Lambda_1)$ and $\mathrm{Irr}(G, \Lambda_2)$, because for any $\varphi \in \mathrm{Irr}(G)$ the irreducible constituents of $\varphi|_N$ are conjugate in G so their inertia groups are also conjugate. We split the proof into two cases.

Case 1: Suppose that $\vartheta \in \Lambda_1$. Then ϑ has at least p distinct conjugates in G. We count the number of irreducible characters $\varphi \in \mathrm{Irr}(G, \Lambda_1)$ such that $\varphi(1) \leqslant p^n$ for every $n \geqslant 1$. Using Lemma 5.2.1 we get the following inequality:

$$\sum_{\substack{\varphi \in \mathrm{Irr}(G,\Lambda_1) \\ \varphi(1) \leqslant p^n}} 1 \leqslant p^{-1} \sum_{\substack{\vartheta \in \Lambda_1 \\ \vartheta(1) \leqslant p^{n-1}}} \sum_{\varphi \in \mathrm{Irr}^+(G,\vartheta)} 1 \leqslant p^{-2}[G:N] \sum_{\substack{\vartheta \in \Lambda_1 \\ \vartheta(1) \leqslant p^{n-1}}} 1.$$

The above equality holds for all $n \geqslant 1$, from Notation 5.2.2 we obtain that

$$\sum_{\varphi \in \mathrm{Irr}(G,\Lambda_1)} \varphi(1)^{-s} \preceq p^{-2-s}[G:N] \sum_{\vartheta \in \Lambda_1} \vartheta(1)^{-s}.$$

Case 2: Suppose that $\vartheta \in \Lambda_2$. The proof follows by induction on $[G:N]$. Assume that $[G:N] = p^2$. As in the proof of Lemma 5.2.1(ii) there exists a normal subgroup Z of G that contains N and such that $Z/N \cong C_p$. Then, by Theorem 4.0.6 and Theorem 4.0.7, ϑ extends to irreducible characters of Z, namely $\psi_1, \ldots, \psi_p$. Notice that $I_G(\psi_i) = Z$ for all $i \in [1,p]$, since $|G:Z| = p$ and $\vartheta \in \Lambda_2 \leqslant \Lambda$. Therefore $\psi_i^G \in \mathrm{Irr}(G)$ for all $i \in [1,p]$. Moreover, since ϑ is G-invariant, it follows that $\psi_1^G = \cdots = \psi_p^G$. This gives a bijection between the sets $\{\vartheta \in \Lambda_2 \mid \vartheta(1) \leqslant p^{n-1}\}$ and $\{\varphi \in \mathrm{Irr}(G,\Lambda_2) \mid \varphi(1) \leqslant p^n\}$ for all $n \in \mathbb{N}$, and hence we get the following inequality

$$\sum_{\varphi \in \mathrm{Irr}(G,\Lambda_2)} \varphi(1)^{-s} \preceq p^{-s} \sum_{\vartheta \in \Lambda_2} \vartheta(1)^{-s} = p^{-2-s}[G:N] \sum_{\vartheta \in \Lambda_2} \vartheta(1)^{-s}.$$

Now, assume that $[G:N] > p^2$. Choose a normal subgroup Z of G such that Z contains N and $Z/N \cong C_p$. Set $\Omega = \mathrm{Irr}(Z, \Lambda_2) = \{\chi \in \mathrm{Irr}(Z) \mid \chi|_N = \vartheta \in \Lambda_2\}$. Observe that for every $n \in \mathbb{N}_0$,

$$\sum_{\substack{\chi \in \Omega \\ \chi(1) \leqslant p^n}} 1 \leqslant p \sum_{\substack{\vartheta \in \Lambda_2 \\ \vartheta(1) \leqslant p^n}} 1.$$

Hence by induction we obtain that

$$\sum_{\substack{\varphi \in \mathrm{Irr}(G,\Lambda_2) \\ \varphi(1) \leqslant p^n}} 1 = \sum_{\substack{\varphi \in \mathrm{Irr}(G,\Omega) \\ \varphi(1) \leqslant p^n}} 1 \leqslant p^{-2}[G:Z] \sum_{\substack{\chi \in \Omega \\ \chi(1) \leqslant p^{n-1}}} 1 \leqslant p^{-2}[G:N] \sum_{\substack{\vartheta \in \Lambda_2 \\ \vartheta(1) \leqslant p^{n-1}}} 1,$$

implying that

$$\sum_{\varphi \in \mathrm{Irr}(G,\Lambda_2)} \varphi(1)^{-s} \preceq p^{-2-s}[G:N] \sum_{\vartheta \in \Lambda_2} \vartheta(1)^{-s}.$$

The result follows from combining the estimates in Case 1 and Case 2. □

5.3 Upper polynomial bound for GGS-groups

Here we prove Theorem 9.1.1 and Theorem 3.0.7. In the sequel, we fix a non-symmetric vector $\mathbf{e} = (e_0, \dots, e_{p-1}) \in \mathbb{F}_p^{p-1}$, and G denotes the GGS-group defined by $\mathbf{e}$. By Theorem 2.4.21, G is a regular branch group over the commutator subgroup $H = G'$. From here onwards, we identify the subgroup $\psi^{-1}(H \times \overset{p}{\cdots} \times H)$ of H with the subgroup $H_1 = H \times \overset{p}{\cdots} \times H$ of $G \times \overset{p}{\cdots} \times G$. Recall from Corollary 3.0.4 that G is representation rigid. Due to results of Lubotzky and Martin [73, Lemma 2.2 & Corollary 2.3], a discrete group has polynomial representation growth (PRG) if and only if every subgroup of finite index has PRG, which in particularly applies to subgroups of finite index in G. We begin with Theorem 5.3.2, which is an immediate corollary of [85, Lemma 1.2]. For convenience, here we state the first part of [85, Lemma 1.2], which is relevant for our context. In our setting, [85, Lemma 1.2] can be reformulated as the following.

Lemma 5.3.1 ([85, Lemma 1.2])**.** *Let G be a finitely generated representation rigid group and let H be a normal subgroup of index k in G. Suppose that H is isomorphic to the direct product of q copies of G, for some non-negative integer $q \geqslant 2$. Let $R_G(n)$ denote the number of irreducible complex representations of G of dimension less that or equal to $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,*

$$R_G(kn^q) \geqslant R_G(n)^q/k.$$

In particular, if $R_G(1) \geqslant k > 1$, then the inequality $R_G(n) \geqslant kn^{q-2}$ is satisfied for infinitely many n and hence

$$\alpha(G) \geqslant q - 2.$$

Theorem 5.3.2. *Let G be a GGS-group defined by a non-symmetric defining vector. We have*

$$\alpha(G) \geqslant p - 2. \tag{5.2}$$

Proof. Let H be a subgroup of finite index in G. It follows directly from [85, Lemma 1.1] or as a consequence of [73, Lemma 2.2] that $\alpha(G) = \alpha(H)$. Now, take H as the commutator subgroup of G and $H_1 = H \times \overset{p}{\cdots} \times H$. From Theorem 2.4.21(ii) and Theorem 2.4.25(i), we obtain that H contains H_1 as a subgroup of index p^{p-1} and H/H_1 is abelian. In particular, $r_1(H) \geqslant p^{p-1}$, where $r_1(H)$ is the number of irreducible 1-dimensional complex representations of H. It follows from [85, Lemma 1.2] that $\alpha(H) \geqslant p - 2$. This completes the proof. ∎

Remark 5.3.3. Let G be a GGS-group defined by a non-symmetric defining vector and let H be a group that is commensurable to G, i.e., there exist subgroups of finite index $G_1 \leqslant G$ and $H_1 \leqslant H$ such that H_1 is isomorphic to G_1. Then $\alpha(H) = \alpha(G) \geqslant p - 2$.

Let G be a GGS-group that is regular branch over a subgroup H. We remark that the proof of [85, Lemma 1.2] uses an inductive argument based on the fact that the number of linear characters of the subgroup H is greater than or equal to the size of the branching quotient $H/\psi^{-1}(H \times \overset{p}{\cdots} \times H)$. Therefore, the proof does not work if the defining vector is non-constant and symmetric. In that case, the corresponding GGS-group G is regular branch over $\gamma_3(G)$. However, the branching quotient $\gamma_3(G)/\psi^{-1}(\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G))$ is not abelian; cf. Remark 2.4.26. Therefore, the number of linear characters of $\gamma_3(G)$ is less than the index $[\gamma_3(G) : \psi^{-1}(\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G))]$.

Form here onwards, let G be the GGS-group defined by a non-symmetric defining vector $\mathbf{e} \in \mathbb{F}_p^{p-1}$ and let H denote its commutator subgroup. To provide an upper bound for $\alpha(G)$, in light of Remark 5.3.3, it suffices to provide an upper bound for $\alpha(H)$. We consider the representation zeta function of H,

$$\zeta_H(s) = \sum_{n=1}^{\infty} r_n(H) n^{-s} = \sum_{\varphi \in \mathrm{Irr}(H)} \varphi(1)^{-s}, \tag{5.3}$$

where $\mathrm{Irr}(H)$ is the set of all irreducible characters of H as defined in Chapter 4. Since every proper quotient of G is a finite p-group (Theorem 2.4.22), by applying the results from Section 5.2 to the group H and its subgroups, here we obtain an upper bound for $\alpha(H)$, and hence for $\alpha(G)$.

We first prove the following theorem with a restriction on the number, say C, of G-invariant irreducible characters of H. It turns out to be that, for a GGS-group, which satisfies either $(*)$ or $(**)$ of Theorem 9.1.1, this number is less than or equal to p (see Corollary 5.3.9 and Corollary 5.3.11). Moreover, the computations in Chapter 6 show that $C = 3$ for the Gupta–Sidki 3-group.

Theorem 5.3.4. *Let G be a GGS-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $H = G'$ be the commutator subgroup of G and $H_1 = H \times \overset{p}{\cdots} \times H \leqslant H$. If*

$$C = |\{\varphi \in \mathrm{Irr}(H) \mid I_G(\varphi) = G\}| < \infty,$$

then the representation zeta function $\zeta_H(s)$ of H satisfies the inequality

$$\zeta_H(s) \preceq C^p [H : H_1] + p^{-2-s} [H : H_1] \zeta_H(s)^p. \tag{5.4}$$

Proof. Define

$$\Lambda = \{\vartheta \in \mathrm{Irr}(H_1) \mid \mathrm{Irr}(H,\vartheta) = \mathrm{Irr}^+(H,\vartheta)\} \qquad \text{and} \qquad \Lambda^{\complement} = \mathrm{Irr}(H_1)\backslash\Lambda.$$

Observe that the set Λ is closed under conjugation by H and it defines a partition of the set $\mathrm{Irr}(H,\Lambda)$ as $\mathrm{Irr}(H) = \mathrm{Irr}(H,\Lambda) \sqcup \mathrm{Irr}(H,\Lambda^{\complement})$. We split the proof into two cases.

Case 1: Let $\varphi \in \mathrm{Irr}(H,\Lambda^{\complement})$ and let $\vartheta \in \mathrm{Irr}(H_1)$ be an irreducible constituent of $\varphi|_{H_1}$. Clearly, $\vartheta \in \Lambda^{\complement}$. By definition of Λ it holds that $\mathrm{Irr}^+(H,\vartheta) \subsetneq \mathrm{Irr}(H,\vartheta)$, implying that there exists some $\chi \in \mathrm{Irr}(H,\vartheta)$ such that $\chi(1) = \vartheta(1)$. Equivalently, χ is an extension of ϑ which further implies that ϑ is H-invariant. Since $\vartheta \in \mathrm{Irr}(H_1)$, by Theorem 4.0.1, we write $\vartheta = \vartheta_0 \otimes \cdots \otimes \vartheta_{p-1}$, for $\vartheta_i \in \mathrm{Irr}(H)$ and $i \in [0,p-1]$. Furthermore, from Theorem 2.4.25(i) we have $H/H_1 = \langle H_1 x_i \mid i \in [0,p-2]\rangle$. Therefore, $\vartheta^{x_i} = \vartheta$ for all $i \in [0,p-2]$. Since the defining vector is non-symmetric by a straightforward calculation using the first layer section decomposition of x_i (Notation 2.4.24), we obtain that ϑ_i is G-invariant for every $i \in [0,p-1]$. From the assumption, the cardinality of the set of G-invariant irreducible characters of H is finite and is equal to C. There are at most C^p irreducible characters of H_1 of the form ϑ such that $\vartheta \in \Lambda^{\complement}$. In particular, the cardinalities of the sets $\Lambda^{\complement}$ and $\mathrm{Irr}(H,\Lambda^{\complement})$ are finite. If every irreducible character from the set $\Lambda^{\complement}$ extends to irreducible characters of H, we get at most $C^p[H:H_1]$ elements in $\mathrm{Irr}(H,\Lambda^{\complement})$. In general, the number $C^p[H:H_1]$ bounds the cardinality of the set $\mathrm{Irr}(H,\Lambda^{\complement})$, and we get

$$|\,\mathrm{Irr}(H,\Lambda^{\complement})| \leqslant C^p[H:H_1].$$

Case 2: We count the number of $\varphi \in \mathrm{Irr}(H,\Lambda)$ such that $\varphi(1) \leqslant p^n$, for $n \in \mathbb{N}$. For every $\varphi \in \mathrm{Irr}(H,\Lambda)$ with $\varphi(1) \leqslant p^n$, recall from Theorem 3.0.2 that φ factors through a finite quotient of H, we obtain that $\ker(\varphi)$ has finite index in H. We set L_φ as the normal core of $\ker(\varphi)$ in G. Then L_φ is a normal subgroup of finite index in G. Define

$$K_n = H_1 \cap \left(\bigcap_{\substack{\varphi\in\mathrm{Irr}(H,\Lambda) \\ \varphi(1)\leqslant p^n}} L_\varphi \right).$$

Notice that K_n is a non-trivial normal subgroup of finite index in G. Hence, for every $n \in \mathbb{N}$, the quotient group H/K_n is a finite p-group (Theorem 2.4.22), and every $\varphi \in \mathrm{Irr}(H,\Lambda)$ of degree $\varphi(1) \leqslant p^n$ factors through the quotient group H/K_n. If $\vartheta \in \mathrm{Irr}(H_1)$ is an irreducible constituent of $\varphi \in \mathrm{Irr}(H,\Lambda)$ of degree $\varphi(1) \leqslant p^n$ then $K_n \leqslant \ker(\vartheta)$. We identify the character ϑ with an irreducible character of H_1/K_n and the character φ with an irreducible character of H/K_n. Define

$$\Lambda_{H/K_n} = \{\vartheta \in \mathrm{Irr}(H_1/K_n) \mid \mathrm{Irr}(H/K_n, \Lambda_{H/K_n}) = \mathrm{Irr}^+(H/K_n, \Lambda_{H/K_n})\}.$$

By replacing G with H/K_n and N with H_1/K_n in Corollary 5.2.3, we obtain

$$\sum_{\substack{\varphi\in\mathrm{Irr}(H,\Lambda)\\ \varphi(1)\leqslant p^n}} 1 = \sum_{\substack{\varphi\in\mathrm{Irr}(H/K_n,\Lambda_{H/K_n})\\ \varphi(1)\leqslant p^n}} 1 \leqslant p^{-2}[H:H_1] \sum_{\substack{\vartheta\in\Lambda_{H/K_n}\\ \vartheta(1)\leqslant p^{n-1}}} 1 = p^{-2}[H:H_1] \sum_{\substack{\vartheta\in\Lambda\\ \vartheta(1)\leqslant p^{n-1}}} 1$$
$$\leqslant p^{-2}[H:H_1] \sum_{\substack{\vartheta\in\mathrm{Irr}(H_1)\\ \vartheta(1)\leqslant p^{n-1}}} 1,$$

where the two inequalities follow because

$$|\{\vartheta\in\Lambda_{H/K_n} \mid \vartheta(1)\leqslant p^{n-1}\}| = |\{\vartheta\in\Lambda \mid \vartheta(1)\leqslant p^{n-1}\}| \leqslant |\{\vartheta\in\mathrm{Irr}(H_1) \mid \vartheta(1)\leqslant p^{n-1}\}|.$$

This implies

$$\sum_{\varphi\in\mathrm{Irr}(H,\Lambda)} \varphi(1)^{-s} \preceq p^{-2-s}[H:H_1] \sum_{\vartheta\in\mathrm{Irr}(H_1)} \vartheta(1)^{-s} = p^{-2-s}[H:H_1]\zeta_{H_1}(s)$$
$$= p^{-2-s}[H:H_1]\zeta_H(s)^p,$$

where the last equality follows as $H_1 = H \times \overset{p}{\cdots} \times H$.

From Case 1 and Case 2, it follows that

$$\zeta_H(s) \preceq \sum_{\varphi\in\mathrm{Irr}(H,\Lambda^{\complement})} \varphi(1)^{-s} + \sum_{\varphi\in\mathrm{Irr}(H,\Lambda)} \varphi(1)^{-s} \preceq C^p[H:H_1] + p^{-2-s}[H:H_1]\zeta_H(s)^p. \qquad \square$$

Now, from the inequality stated in Theorem 5.3.4, we compute an upper bound for the abscissa of convergence $\alpha(H)$ for the representation zeta function of H using *generalised Catalan numbers.*

Definition 5.3.5. For every $n \in \mathbb{N}_0$, the n-th *Catalan number* $c_2(n)$ is the number of ways to parenthesise a string of $n+1$ symbols such that each multiplication is binary. For instance, the expression $(\diamond(\diamond\diamond))(\diamond\diamond)$ is allowed as it uses only binary multiplications, but the expression $(\diamond\diamond\diamond)(\diamond\diamond)$ is invalid because the expression $(\diamond\diamond\diamond)$ represents a product of three symbols. Here we compute the Catalan numbers $c_2(n)$ for $0 \leqslant n \leqslant 3$.

$n=0$	$n=1$	$n=2$	$n=3$
$\diamond$	$\diamond\diamond$	$\diamond(\diamond\diamond)$	$((\diamond\diamond)\diamond)\diamond$
		$(\diamond\diamond)\diamond$	$(\diamond(\diamond\diamond))\diamond$
			$\diamond((\diamond\diamond)\diamond)$
			$\diamond(\diamond(\diamond\diamond))$
			$(\diamond\diamond)(\diamond\diamond)$
$c_2(0)=1$	$c_2(1)=1$	$c_2(2)=2$	$c_2(3)=5$

The Catalan numbers $c_2(n)$ were first described by Euler and named after the mathematician Catalan. The numbers $c_2(n)$ occur as solutions to different counting problems. In [100], one can find 66 different interpretation of the Catalan numbers. Using the description above, we can find a recursive formula for the n-th Catalan number $c_2(n)$; cf. [106]. We set $c_2(0) = 1$. For $n \geqslant 1$, let ω denote a string of length $n+1$. We can write $\omega = \omega_1\,\omega_2$ such that ω_1 is a string of length ℓ, for some $\ell \in [1, n]$, and ω_2 is a string of length $n+1-\ell$. Then there are $c_2(\ell - 1)$ ways to parenthesise a string of ℓ symbols such that each multiplication is binary, and $c_2(n-\ell)$ ways to parenthesise a string of $n+1-\ell$ symbols such that each multiplication is binary. Therefore, we get

$$c_2(n) = \sum_{\ell=1}^{n} c_2(\ell-1)c_2(n-\ell), \qquad (n \geqslant 1). \tag{5.5}$$

Let $F_2(x) = \sum_{\ell=0}^{\infty} c_2(\ell)x^\ell$ be the generating function for the Catalan numbers $c_2(n)$. We follow the convention that $c_2(-\ell) = 0$ for all $\ell \in \mathbb{N}$. Observe that the right-hand side of the equation (5.5) is the n-th coefficient of the product of the series $xF_2(x) = \sum_{\ell=0}^{\infty} c_2(\ell-1)x^\ell$ and the series $F_2(x)$. Since the constant term of the series $xF_2(x)$ is zero, we get

$$F_2(x) - 1 = \sum_{n=1}^{\infty} c_2(n) = \sum_{n=1}^{\infty}\left(\sum_{\ell=1}^{n} c_2(\ell-1)c_2(n-\ell)\right)x^\ell = x\,F_2(x)\,F_2(x) = x\,F_2(x)^2,$$

and hence the generating function $F_2(x)$ satisfies the functional equation

$$F_2(x) = 1 + x\,F_2(x)^2. \tag{5.6}$$

By solving the above functional equation (5.6), one gets that

$$c_2(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Now let p be an odd prime. For every $n \in \mathbb{N}_0$, the n-th *generalised Catalan number* $c_p(n)$ counts the number of ways to parenthesise a string of $n+1$ symbols such that each multiplication is p-ary. Let $F_p(x)$ be the generating function for the generalised Catalan numbers $c_p(n)$. It is shown in [64] that the generating function $F_p(x)$ satisfies the functional equation

$$F_p(x) = 1 + x\,F_p(x)^p.$$

As a corollary, one gets that

$$c_p(n) = \frac{1}{(p-1)n+1}\binom{pn}{n}$$

for all $n \geqslant 0$. Using Stirling's formula, we can approximate $c_p(n)$ as

$$c_p(n) \approx \left(\frac{p^p}{(p-1)^{p-1}}\right)^n \sqrt{\frac{p}{2\pi(p-1)^3}}\,n^{-\frac{3}{2}}, \tag{5.7}$$

where the sign $\approx$ means that the ratio of the two quantities tends to 1 as n tends to infinity.

Lemma 5.3.6. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *be two generating functions with* a_n, $b_n \in \mathbb{N}_0$. *Let* B, $M \geqslant 1$ *be constants. Suppose that the generating function* $g(x)$ *satisfies the functional equation*

$$g(x) = M + B\, x\, g(x)^p. \tag{5.8}$$

Then

$$g(x) = \sum_{n=0}^{\infty} c_p(n)\, M^{n(p-1)+1}\, B^n\, x^n,$$

where $c_p(n)$ *is the* n*-th generalised Catalan number, for* $n \in \mathbb{N}_0$*, defined in Definition 5.3.5. If the the generating function* $f(x)$ *satisfies the inequality*

$$f(x) \preceq M + B\, x\, f(x)^p, \tag{5.9}$$

then $a_n \leqslant b_n$ *for every* $n \geqslant 0$*, and in particular* $f(x) \preceq g(x)$.

Proof. Notice that

$$\sum_{n=0}^{\infty} b_n x^n = g(x) = M + B\, x\, g(x)^p = M + B\, x \sum_{n=0}^{\infty} \left(\sum_{\substack{0 \leqslant r_1, \ldots, r_p \leqslant n \\ r_1 + \cdots + r_p = n}} b_{r_1} \cdots b_{r_p} \right) x^n.$$

We get $b_0 = M$ and for $n \geqslant 1$

$$b_n = B \sum_{\substack{0 \leqslant r_1, \ldots, r_p \leqslant n-1 \\ r_1 + \cdots + r_p = n-1}} b_{r_1} \cdots b_{r_p}.$$

We shall prove by induction that b_n is a multiple of $M\,(B\,M^{p-1})^n$ for every $n \geqslant 0$. Assume that b_n is a multiple of $M\,(B\,M^{p-1})^n$ for every $n \leqslant N$ for some $N \geqslant 0$. Consider

$$b_{N+1} = B \sum_{\substack{0 \leqslant r_1, \ldots, r_p \leqslant N \\ r_1 + \cdots + r_p = N}} b_{r_1} \cdots b_{r_p}.$$

It is easy to see that each summand in the above expression is a multiple of $M\,(B\,M^{p-1})^{N+1}$ and so is b_{N+1}. Hence we conclude by induction that every b_n is a multiple of $M\,(B\,M^{p-1})^n$ for $n \geqslant 0$. Therefore, we can write

$$g(x) = M\,\tau(y),$$

for some generating function $\tau(y)$ with $y = B\,M^{p-1}\,x$. Now, by substituting $g(x)$ as $M\,\tau(y)$, the equation (5.8) becomes

$$\tau(y) = 1 + y\,\tau(y)^p, \tag{5.10}$$

which is the functional equation for the the generalised Catalan numbers. Therefore,

$$\tau(y) = \sum_{n=0}^{\infty} c_p(n)\, y^n,$$

and hence

$$g(x) = M\,\tau(y) = M\sum_{n=0}^{\infty} c_p(n)\,y^n = \sum_{n=0}^{\infty} c_p(n)\,M^{n(p-1)+1}\,B^n\,x^n.$$

Now we prove the second part of the result. Observe from (5.13) that $a_0 \leqslant M = b_0$. The proof proceeds by induction on n. Assume that $a_n \leqslant b_n$ for every $n \leqslant N$ for some $N \geqslant 0$. From (5.8), we get

$$\sum_{n=0}^{N+1} b_n = M + B\sum_{n=0}^{N}\left(\sum_{\substack{0\leqslant r_1,\dots,r_p\leqslant n\\ r_1+\cdots+r_p=n}} b_{r_1}\cdots b_{r_p}\right),$$

hence

$$b_{N+1} = M + B\sum_{n=0}^{N}\left(\sum_{\substack{0\leqslant r_1,\dots,r_p\leqslant n\\ r_1+\cdots+r_p=n\\ (r_1,\dots,r_p)\neq(n,0,\dots,0)}} b_{r_1}\cdots b_{r_p}\right) + \sum_{n=0}^{N} b_n\left(b_0^{p-1}B-1\right),$$

Similarly, from (5.9), we get

$$a_{N+1} \leqslant M + B\sum_{n=0}^{N}\left(\sum_{\substack{0\leqslant r_1,\dots,r_p\leqslant n\\ r_1+\cdots+r_p=n\\ (r_1,\dots,r_p)\neq(n,0,\dots,0)}} a_{r_1}\cdots a_{r_p}\right) + \sum_{n=0}^{N} a_n\left(a_0^{p-1}B-1\right),$$

Therefore, from induction hypothesis we have $a_{N+1} \leqslant b_{N+1}$, and by induction we conclude that $a_N \leqslant b_N$ for all $N \geqslant 0$. In particular, $\sum_{n=0}^{N} a_n \leqslant \sum_{n=0}^{N} b_n$ for all $N \geqslant 0$. This completes the proof. □

Corollary 5.3.7. *Let G be a* GGS*-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ and let $H = G'$ be the commutator subgroup of G. If the number C of G-invariant irreducible characters of H is finite then the abscissa of convergence $\alpha(G)$ of the representation zeta function $\zeta_G(s)$ satisfies the inequalities*

$$p-2 \leqslant \alpha(G) \leqslant (p-1)\frac{\log 2}{\log p} + p(p-1)\frac{\log C}{\log p} + (p-1)^2 + (p-1) - 1. \tag{5.11}$$

In particular, G has polynomial representation growth.

Proof. Thanks to Remark 5.3.3, we have $\alpha(G) = \alpha(H) \geqslant p-2$. Using (5.4), we shall compute an upper bound for $\alpha(H)$. We define

$$x = p^{-s}, \qquad M = C^p[H:H_1] = p^{p-1}C^p, \quad \text{and} \quad B = p^{-2}[H:H_1] = p^{p-3},$$

in particular, if $p = 3$ then $B = 1$. Observe that (5.4) can be restated as

$$\eta(x) \preceq M + B\,x\,\eta(x)^p,$$

where $\eta(x) = \eta(p^{-s}) = \zeta_H(s)$. Now, suppose that $\xi(x)$ is a generating function which satisfies the following functional equation

$$\xi(x) = M + B\,x\,\xi(x)^p. \tag{5.12}$$

Then, by Lemma 5.3.6(ii), we get $\zeta_H(s) \preceq \xi(p^{-s})$. If $\xi(p^{-s})$ converges for some $s \in \mathbb{C}$ and if $\widetilde{\alpha}$ denotes the abscissa of convergence of $\xi(p^{-s})$, then $\zeta_H(s)$ also converges and $\alpha(H) \leqslant \widetilde{\alpha}$, yielding an upper bound for $\alpha(H)$. Thus, it is enough to find an upper bound for $\widetilde{\alpha}$. Again from Lemma 5.3.6, we have

$$\xi(x) = \sum_{n=0}^{\infty} c_p(n)\,M^{n(p-1)+1}\,B^n\,x^n,$$

where $c_p(n)$ is the n-th generalised Catalan number. Because of (5.7), there exist a constant $\kappa_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$c_p(n) \leqslant \kappa_0 \Big(\frac{p^p}{(p-1)^{p-1}}\Big)^n \sqrt{\frac{p}{2\pi(p-1)^3}}\,n^{-\frac{3}{2}},$$

for all $n \geqslant n_0$. Hence there exists a constant $\kappa > 0$ such that, for all $n \geqslant 0$,

$$c_p(n) \leqslant \kappa\, 2^{n(p-1)} p^n. \tag{5.13}$$

Now, we compute an upper bound for the abscissa of convergence $\widetilde{\alpha}$ of $\xi(p^{-s})$ using the inequality (5.13). Define $R_{p^N} = \sum_{n=0}^{N} c_p(n)\,M^{n(p-1)+1}\,B^n$ for $N \in \mathbb{N}_0$. Then for all $N \geqslant 0$, we obtain

$$\begin{aligned}
R_{p^N} &= \sum_{n=0}^{N} c_p(n)\,M^{n(p-1)+1}\,B^n = \sum_{n=0}^{N} c_p(n)\,(p^{p-1}C^p)^{n(p-1)+1}\,(p^{p-3})^n \\
&= \sum_{n=0}^{N} c_p(n) C^{np(p-1)+p} p^{n(p^2-p-2)+p-1} \leqslant \kappa\, C^p p^{p-1} \sum_{n=0}^{N} \big(2^{p-1} C^{p(p-1)} p^{p^2-p-1}\big)^n \\
&= \kappa\, C^p p^{p-1} \frac{\big(2^{p-1} C^{p(p-1)} p^{p^2-p-1}\big)^{N+1} - 1}{2^{p-1} C^{p(p-1)} p^{p^2-p-1} - 1} \leqslant \kappa\, C^p p^{p-1} \frac{\big(2^{p-1} C^{p(p-1)} p^{p^2-p-1}\big)^{N+1}}{2^{p-1} C^{p(p-1)} p^{p^2-p-1} - 1}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\widetilde{\alpha} &= \limsup_{N\to\infty} \log R_{p^N} / \log p^N \\
&\leqslant \limsup_{N\to\infty} \frac{\log(\kappa\, C^p p^{p-1})}{\log p^N} \\
&\quad + \limsup_{N\to\infty} \frac{(N+1)\log\big(2^{p-1} C^{p(p-1)} p^{p^2-p-1}\big) - \log(2^{p-1} C^{p(p-1)} p^{p^2-p-1} - 1)}{N \log p} \\
&= \frac{\log\big(2^{p-1} C^{p(p-1)} p^{p^2-p-1}\big)}{\log p} \\
&= (p-1)\frac{\log 2}{\log p} + p(p-1)\frac{\log C}{\log p} + p^2 - p - 1.
\end{aligned}$$

Therefore, we conclude that

$$\alpha(G) = \alpha(H) \leqslant \tilde{\alpha} \leqslant (p-1)\frac{\log 2}{\log p} + p(p-1)\frac{\log C}{\log p} + p^2 - p - 1. \qquad \square$$

Now, we prove that for a GGS-group, which satisfies either condition $(*)$ or $(**)$ of Theorem 9.1.1, the number C is less than or equal to p.

Lemma 5.3.8. *Let G be a GGS-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ and let $H = G'$ be the commutator subgroup of G. Let $\varphi \in \mathrm{Irr}(H)$ be such that $I_G(\varphi) = G$ and let $\vartheta = \vartheta_0 \otimes \cdots \otimes \vartheta_{p-1}$ be an irreducible constituent of $\varphi|_{H_1}$, where $\vartheta_i \in \mathrm{Irr}(H)$ for all $i \in [0, p-1]$. Then $I_G(\vartheta_i) = G$ for all $i \in [0, p-1]$. Moreover, $\vartheta_0 = \cdots = \vartheta_{p-1}$ and $I_G(\vartheta) = G$.*

Proof. The restriction $\varphi|_{H_1}$ is a sum of conjugates of ϑ under H. Since $\varphi^{at} = \varphi$, it holds that $\vartheta^{at} = \vartheta^h$ for some $h \in H$. Notice that ϑ is of the form $\vartheta_0 \otimes \cdots \otimes \vartheta_{p-1}$ for $\vartheta_i \in \mathrm{Irr}(H)$ and $i \in [0, p-1]$ as indicated in the statement of the lemma. As $H = H_1 \langle x_0, \ldots, x_{p-2} \rangle$ (Theorem 2.4.25(i)), we can write $h = w(x_0, \ldots, x_{p-2}) \bmod H_1$ for some word w in $\{x_0, \ldots, x_{p-2}\}$. Since ϑ is H_1 invariant, by letting $w(x_0, \ldots, x_{p-2}) = (w_0, \ldots, w_{p-1})$, we obtain

$$\vartheta_{p-1}^{a^{e_1}} \otimes \vartheta_0^{a^{e_2}} \otimes \cdots \otimes \vartheta_{p-3}^{a^{e_{p-1}}} \otimes \vartheta_{p-2}^{t} = (\vartheta_0 \otimes \cdots \otimes \vartheta_{p-1})^{at} = \vartheta^{at} = \vartheta^h = \vartheta_0^{w_0} \otimes \cdots \otimes \vartheta_{p-1}^{w_{p-1}}.$$

This further implies that

$$\vartheta_0 = \vartheta_{p-1}^{a^{e_1} w_0^{-1}} = \vartheta_{p-2}^{t w_{p-1}^{-1} a^{e_1} w_0^{-1}} = \cdots = \vartheta_0^{a^{e_2} w_1^{-1} a^{e_3} w_2^{-1} \cdots a^{e_{p-1}} w_{p-2}^{-1} t w_{p-1}^{-1} a^{e_1} w_0^{-1}}. \tag{5.14}$$

Since the product of the components of the first layer decomposition of each $x_i = [a,t]^{a^i}$ is trivial modulo H (cf. Notation 2.4.24), we get

$$a^{e_2} w_1^{-1} a^{e_3} w_2^{-1} \cdots a^{e_{p-1}} w_{p-2}^{-1} t w_{p-1}^{-1} a^{e_1} w_0^{-1} \equiv_H a^{\ell} t,$$

where $\ell = \sum_{i=1}^{p-1} e_i$. Since ϑ_0 is H-invariant, we get $H \langle a^{\ell} t \rangle \leqslant I_G(\vartheta_0)$. Notice that all ϑ_i are conjugate to each other by elements of G. Therefore, $I_G(\vartheta_0) = \cdots = I_G(\vartheta_{p-1})$, and moreover $I_G(\vartheta_0) \in \{H \langle a^{\ell} t \rangle, G\}$. We claim that $I_G(\vartheta_0) = G$. Suppose that $I_G(\vartheta_0) \neq G$, and hence $I_G(\vartheta_i) \neq G$. Thus $I_G(\vartheta_i) = H \langle a^{\ell} t \rangle$ for all $i \in [0, p-1]$. Since $H_1 \trianglelefteq G \times \overset{p}{\cdots} \times G$ and $\psi^{-1}(G \times \overset{p}{\cdots} \times G) = \mathrm{St}_G(1) = H\langle t \rangle$, we have

$$I_H(\vartheta) = \psi^{-1}(I_G(\vartheta_0) \times \cdots \times I_G(\vartheta_{p-1})) \cap H,$$

and

$$I_{H\langle t \rangle}(\vartheta) = \psi^{-1}(I_G(\vartheta_0) \times \cdots \times I_G(\vartheta_{p-1})) \cap H\langle t \rangle.$$

Since φ is G-invariant, we must have

$$[H : I_H(\vartheta)] = [H\langle t\rangle : I_{H\langle t\rangle}(\vartheta)].$$

Since H is a proper subgroup of $H\langle t\rangle$, this implies that $I_H(\vartheta)$ is a proper subgroup of $I_{H\langle t\rangle}(\vartheta)$, or in other words

$$I_G(\vartheta_0) \times \cdots \times I_G(\vartheta_{p-1}) \cap \psi(H) \lneqq I_G(\vartheta_0) \times \cdots \times I_G(\vartheta_{p-1}) \cap \psi(H\langle t\rangle).$$

Thus, there exist $\varepsilon \in [1, p-1]$ and $h \in H$ such that

$$\psi(ht^\varepsilon) \in I_G(\vartheta_0) \times \cdots \times I_G(\vartheta_{p-1}) \cap \psi(H\langle t\rangle).$$

Since $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ is non-zero, there exists $i \in [1, p-1]$ such that $e_i \neq 0$. This, in particular, implies that $G = H\langle a^\ell t, a^{e_i}\rangle \leqslant I_G(\vartheta_{p-1})$, and hence $I_G(\vartheta_i) = G$ for all $i \in [0, p-1]$, which is a contradiction. Therefore, we conclude that $I_G(\vartheta_i) = G$ for all $i \in [0, p-1]$. Furthermore, it follows from (5.14) that $\vartheta_0 = \cdots = \vartheta_{p-1}$. In particular, $I_G(\vartheta) = G$. □

The following corollary gives a proof for Theorem 9.1.1 when the vector $\mathbf{e}''$ is symmetric, i.e., the condition $(*)$ is satisfied.

Corollary 5.3.9. *Let $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ be a non-symmetric vector such that $\mathbf{e}''$ is symmetric. Let G be the* GGS*-group defined by $\mathbf{e}$ and let $H = G'$ be the commutator subgroup of G. If $\varphi \in \mathrm{Irr}(H)$ such that $I_G(\varphi) = G$ then φ is linear. Moreover, $C \leqslant p$, where C is defined as in Theorem 5.3.4, and thus*

$$p - 2 \leqslant \alpha(G) = \alpha(H) \leqslant (p-1)\frac{\log 2}{\log p} + 2p^2 - 2p + 1. \tag{5.15}$$

Proof. Let φ, ϑ and $\vartheta_0, \ldots, \vartheta_{p-1}$ be as defined in Lemma 5.3.8 above. Then $\vartheta_0 = \cdots = \vartheta_{p-1}$ and ϑ are G-invariant. By induction on the dimension of φ we may assume, without loss of generality, that ϑ_0 is linear. Therefore, $\vartheta = \vartheta_0 \otimes \cdots \otimes \vartheta_{p-1}$ is a linear character of H_1, and $[H_1, G] \leqslant \ker(\vartheta)$. Now, since $\mathbf{e}''$ is symmetric, thanks to Theorem 2.4.25(iv), we have $H' \leqslant [H_1, G]$. Thus, ϑ extends to H and hence φ is of the form $\widehat{\vartheta}\lambda$, where $\widehat{\vartheta}$ is an extension of ϑ and $\lambda \in \mathrm{Irr}(H/H_1)$. As $\widehat{\vartheta}, \lambda$ are linear so is φ. Since φ is G-invariant, we get that $[H, G] \leqslant \ker(\varphi)$. Therefore, φ is an extension of the trivial character of $[H, G]$, and hence we conclude that $C \leqslant p$, as $[H : [H, G]] = p$. By substituting p for C in (5.11), we get the desired bounds for the abscissa of convergence $\alpha(G)$. □

The proof of Theorem 9.1.1 when the alternative condition $(**)$ is satisfied is given by Corollary 5.3.11. To simplify the proof of Corollary 5.3.11, we record first the following lemma. In the remaining part of this chapter, for convenience, we do not distinguish notationally between the elements of G and $G/[H_1, G]$.

Lemma 5.3.10. *Let $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ be a non-symmetric vector such that $\mathbf{e}''$ is non-symmetric. Let G be the GGS-group defined by $\mathbf{e}$ and let $H = G'$ be the commutator subgroup of G. Let $\varphi \in \mathrm{Irr}(H)$ be such that $I_G(\varphi) = G$. If there exists a central element z in the quotient group $H/[H_1, G]$ such that $z^t \neq z$, then φ is linear.*

Proof. Let $\varphi \in \mathrm{Irr}(H)$ be such that $I_G(\varphi) = G$. Let $\vartheta \in \mathrm{Irr}(H_1)$ be an irreducible constituent of $\varphi|_{H_1}$. Notice that ϑ is of the form $\vartheta_0 \otimes \cdots \otimes \vartheta_{p-1}$ for $\vartheta_i \in \mathrm{Irr}(H)$ and $i \in [0, p-1]$. From Lemma 5.3.8, we see that $\vartheta_0 = \cdots = \vartheta_{p-1}$ and $I_G(\vartheta_0) = G$, implying that ϑ is G invariant. By induction on the dimension of φ, we may assume, without loss of generality, that ϑ_0 is linear. Therefore, the character ϑ is linear and hence the subgroup $[H_1, G]$ is contained in the kernel of ϑ. Now, since ϑ is an irreducible constituent of $\varphi|_{H_1}$ and ϑ is G-invariant, we have $\varphi|_{H_1} = \ell\,\vartheta$, for some $\ell \in \mathbb{N}$, and furthermore

$$[H_1, G] \leqslant \ker(\vartheta) = \ker(\varphi|_{H_1}) = \ker(\varphi) \cap H_1 \leqslant \ker(\varphi).$$

Hence, we identify the characters φ and ϑ with irreducible characters of $H/[H_1, G]$ and $H_1/[H_1, G]$, respectively.

Now, suppose that there exists a central element $z \in H/[H_1, G]$ such that $z^t \neq z$. It follows from the proof of Theorem 2.4.25 that the element z can be expressed as a product of finitely many elements from the generating set $\{x_0, \ldots, x_{p-2}, y\}$ of $H/[H_1, G]$. Thus

$$z^t = z[z, t] = z\, y^{\omega},$$

for some $\omega \in [1, p-1]$. Since, the element z is central in $H/[H_1, G]$, we have $\varphi(z) \neq 0$. Moreover, since φ is $G/[H_1, G]$-invariant, we have $\varphi^{t^{-1}}(z) = \varphi(z)$. On the other hand,

$$\varphi^{t^{-1}}(z) = \varphi(z^t) = \varphi(z\, y^{\omega}).$$

Therefore,

$$\varphi(z) = \varphi^{t^{-1}}(z) = \varphi(z\, y^{\omega}) = \varphi(z)\frac{\varphi(y^{\omega})}{\varphi(1)}, \tag{5.16}$$

where the last equality follows because both z and y are central in $H/[H_1, G]$. From (5.16), we get that $\varphi(y^{\omega}) = \varphi(1)$, implying that $y^{\omega} \in \ker(\varphi)$. Since y^{ω} is a generator of the cyclic group $H_1/[H_1, G] \cong C_p$, it follows that $y \in \ker(\varphi)$. In particular, ϑ is the trivial character of $H_1/[H_1, G]$ and $(H/[H_1, G])' = H_1/[H_1, G] \leqslant \ker(\varphi)$. Hence we conclude that φ is linear. $\square$

Corollary 5.3.11. *Let $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ be a non-symmetric vector such that $\mathbf{e}''$ is also non-symmetric. Let G be the* GGS*-group defined by $\boldsymbol{e}$ and let H be the commutator subgroup of G. Assume further that the element*

$$\omega(\boldsymbol{e}) = (p-2)(e_1 - e_{p-1}) + (p-4)(e_2 - e_{p-2}) + \cdots + 3(e_{\frac{p-3}{2}} - e_{\frac{p+3}{2}}) + (e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}})$$

is non-zero modulo p. We have

$$p - 2 \leqslant \alpha(G) = \alpha(H) \leqslant (p-1)\frac{\log 2}{\log p} + 2p^2 - 2p + 1. \tag{5.17}$$

Proof. We will prove that there exists a central element z in $H/[H_1, G]$ such that $z^t \neq z$. Then we get from Lemma 5.3.10 that every G-invariant irreducible character of H is linear. By a similar argument as in the proof of Corollary 5.3.9 we get that (5.17) holds.

Now, set

$$z = x_0 x_1^3 \cdots x_i^{\frac{(i+1)(i+2)}{2}} \cdots x_{p-1}^{\frac{p(p+1)}{2}}.$$

By Lemma 2.4.29 the element z is central in $H/[H_1, G]$. From the conjugation relations in the proof of Theorem 2.4.25 we get

$$\begin{aligned} z^t &= x_0 y^{e_1 - e_{p-1}} (x_1 y^{e'_2 + e'_{p-1}})^3 \cdots (x_i y^{e'_{i+1} + e'_{p-i}})^{\frac{(i+1)(i+2)}{2}} \cdots (x_{p-1} y^{e_1 - e_{p-1}})^{\frac{p(p+1)}{2}} \\ &= z y^{e_1 - e_{p-1} + 3(e'_2 + e'_{p-1}) + \cdots + \frac{(i+1)(i+2)}{2}(e'_{i+1} + e'_{p-i}) + \cdots + \frac{p(p+1)}{2}(e_1 - e_{p-1})}. \end{aligned}$$

Let $\omega(\mathbf{e})$ be the exponent sum of y. Then

$$\begin{aligned} \omega(\mathbf{e}) =& (1 + \frac{p(p+1)}{2})(e_1 - e_{p-1}) + (3 + \frac{(p-1)p}{2})(e'_2 + e'_{p-1}) + \cdots + \\ & (\frac{(p-1)(p+1)}{8} + \frac{(p+3)(p+5)}{8})(e'_{\frac{p-1}{2}} + e'_{\frac{p+3}{2}}) + \frac{(p+1)(p+3)}{4} e'_{\frac{p+1}{2}} \\ =& \frac{p^2 + p + 2}{2}(e_1 - e_{p-1}) + \frac{p^2 - p + 6}{2}(e'_2 + e'_{p-1}) + \cdots + \\ & \frac{p^2 + 4p + 7}{4}(e'_{\frac{p-1}{2}} + e'_{\frac{p+3}{2}}) + \frac{p^2 + 4p + 3}{4} e'_{\frac{p+1}{2}}. \end{aligned}$$

Therefore, we can write $\omega(\mathbf{e})$ as

$$\begin{aligned} \omega(\mathbf{e}) =& -2(e_1 - e_{p-1}) - 4(e_2 - e_{p-2}) + \cdots + 3(e_{\frac{p-3}{2}} - e_{\frac{p+3}{2}}) + (e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}}) \\ & \equiv (p-2)(e_1 - e_{p-1}) + (p-4)(e_2 - e_{p-2}) + \cdots + 3(e_{\frac{p-3}{2}} - e_{\frac{p+3}{2}}) + (e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}}), \end{aligned}$$

where the equivalence is taken modulo p. By assumption $\omega(\mathbf{e})$ is non-zero modulo p, and hence $z^t \neq z$. This completes the proof. $\square$

Corollary 5.3.12. *Let G be the Gupta–Sidki p-group defined by the defining vector $\boldsymbol{e} = (1, -1, 0, \ldots, 0) \in \mathbb{F}_p^{p-1}$ and let H be the commutator subgroup of G. Then $\omega(\boldsymbol{e})$ is non-zero modulo p. Furthermore, if $\varphi \in \mathrm{Irr}(H)$ such that $I_G(\varphi) = G$ then φ is linear, and hence the bounds in (5.17) for $\alpha(G) = \alpha(H)$ hold.*

Proof. Observe that $\mathbf{e}''$ is non-symmetric and that

$$\omega(\mathbf{e}) = (p-2)(e_1 - e_{p-1}) + (p-4)(e_2 - e_{p-2}) + \cdots + 3(e_{\frac{p-3}{2}} - e_{\frac{p+3}{2}}) + (e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}}) = 2.$$

Therefore, the result follows immediately from Corollary 5.3.11. □

Now, we shall present some results and ideas to generalise Theorem 9.1.1 to all GGS-groups defined by non-symmetric defining vectors. Also, we will prove Theorem 3.0.7. First we record that, for $p = 5$, the condition that $\omega(\mathbf{e})$ not equal to zero modulo p is automatically satisfied.

Lemma 5.3.13. *Let $\boldsymbol{e} = (e_1, e_2, e_3, e_4) \in \mathbb{F}_5^4$ be a defining vector such that $\boldsymbol{e}''$ is non-symmetric. Then $\omega(\boldsymbol{e}) = 3(e_1 - e_4) + e_2 - e_3$ is non-zero modulo p.*

Proof. Assume to the contrary that $\omega(\mathbf{e}) \equiv 0 \pmod 5$. Then $e_2 - e_3 \equiv 2(e_1 - e_4) \pmod 5$. Consider the vector $\mathbf{e}'' = (e_3 - 2e_2 + e_1, e_4 - 2e_3 + e_2)$. We get

$$\begin{aligned} e_4 - 2e_3 + e_2 - (e_3 - 2e_2 + e_1) &= (e_4 - e_1) + 3(e_2 - e_3) \equiv (e_4 - e_1) + 6(e_1 - e_4) \\ &\equiv 0 \pmod 5. \end{aligned}$$

This contradicts the fact that $\mathbf{e}''$ is non-symmetric. □

As an immediate corollary, we obtain the following result.

Corollary 5.3.14. *Let G be a GGS-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_5^4$. Then the following inequalities are satisfied.*

$$3 \leqslant \alpha(G) \leqslant 42.7227062.$$

Let $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a non-symmetric defining vector such that $\mathbf{e}''$ is also non-symmetric. Let G be the GGS-group defined by $\mathbf{e}$. To obtain an upper bound for $\alpha(G)$, by Lemma 5.3.9, it suffices to prove the existence of a central element z in $H/[H_1, G]$ such that $z^t \neq z$. To be able to do so, one needs a better understanding of the lower central series of G, or at least terms up to $\gamma_p(G)$. We provide an outline of the approach. Let H be the commutator subgroup of G. We recall from Theorem 2.4.25 that $H = H_1\langle x_0, \ldots, x_{p-2}\rangle$, where $H_1 = \mathrm{St}_G(1)' = H \times \overset{p}{\cdots} \times H$, and $H_1 = H'$, since $\mathbf{e}''$ is non-symmetric.

Lemma 5.3.15. *For every odd number $\ell \in [2, p]$, the element*

$$x_i^{-\binom{\ell-2}{0}} x_{i+1}^{\binom{\ell-2}{1}} x_{i+2}^{-\binom{\ell-2}{2}} \cdots x_{i+\ell-3}^{-\binom{\ell-2}{\ell-3}} x_{i+\ell-2}^{\binom{\ell-2}{\ell-2}} \in \gamma_\ell(G)\, H_1,$$

and, for every even number $\ell \in [2, p]$, the element

$$x_i^{\binom{\ell-2}{0}} x_{i+1}^{-\binom{\ell-2}{1}} x_{i+2}^{\binom{\ell-2}{2}} \cdots x_{i+\ell-3}^{-\binom{\ell-2}{\ell-3}} x_{i+\ell-2}^{\binom{\ell-2}{\ell-2}} \in \gamma_\ell(G)\, H_1,$$

where $i \in [0, p - \ell]$.

Proof. We set $\overline{G} = G/H_1$, and use the notation $\overline{(\cdot)}$ to denote the images of elements and subgroups of G under the canonical epimorphism $G \to \overline{G}$. Recall from the proof of Theorem 2.4.25(i) that $\overline{G} = \overline{\mathrm{St}_G(1)} \rtimes \langle \overline{a} \rangle \cong C_p \wr C_p$, $\overline{\mathrm{St}_G(1)} = \left\langle \overline{t}, \overline{t^a}, \ldots, \overline{t^{a^{p-1}}} \right\rangle \cong C_p^p$ and $\overline{H} = \langle \overline{x_0}, \ldots, \overline{x_{p-2}} \rangle \cong C_p^{p-1}$. We identify the quotient group $\overline{H}$ with the vector space of dimension $p-1$ over $\mathbb{F}_p$, and the elements x_i with the vectors $c_i = (0, \overset{i-2}{\ldots}, 0, 1, 0, \ldots, 0)$ in C_p^{p-1}. The action of the element $\overline{a}$ on each c_i is given by $c_i^a = c_{i+1}$, where the subscripts are taken modulo p. Hence the quotient group $\overline{\gamma_3(G)}$ is generated by the set of vectors

$$\{(-1, 1, 0, \ldots, 0), (0, -1, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, -1, 1)\},$$

and the quotient group $\overline{\gamma_4(G)}$ is generated by the set of vectors

$$\{(1, -2, 1, 0, \ldots, 0), (0, 1, -2, 1, 0, \ldots, 0), \ldots, (0, \ldots, 1, -2, 1)\}.$$

Now, observe that the coordinates of the above mentioned vectors are the entries of second and third rows of the Pascal's triangle with alternating signs. By iterating the above process, one sees that, for every $\ell \in [2, p]$, the quotient group $\overline{\gamma_\ell(G)}$ is generated by the vector

$$\left(-\binom{\ell-2}{0}, \binom{\ell-2}{1}, -\binom{\ell-2}{2}, \ldots, \binom{\ell-2}{\ell-3}, -\binom{\ell-2}{\ell-2}\right),$$

and its cyclic shifts if ℓ is odd, and by the vector

$$\left(\binom{\ell-2}{0}, -\binom{\ell-2}{1}, \binom{\ell-2}{2}, \ldots, -\binom{\ell-2}{\ell-3}, \binom{\ell-2}{\ell-2}\right),$$

and its cyclic shifts if ℓ is even. □

Now, we recall from Lemma 2.4.29 that the element z is defined as

$$z = x_0 x_1^3 \cdots x_i^{\frac{(i+1)(i+2)}{2}} \cdots x_{p-1}^{\frac{p(p+1)}{2}},$$

where the exponents are taken modulo p. Observe that

$$z = x_0^{\binom{p-3}{0}} x_1^{-\binom{p-3}{1}} x_2^{\binom{p-3}{2}} \cdots x_{p-4}^{-\binom{p-3}{p-4}} x_{p-3}^{\binom{p-3}{p-3}} \in \gamma_{p-1}(G)\, H_1.$$

We have seen that the element z is central in $H/[H_1, G]$, and $z^t \neq z$ if and only if $\omega(\mathbf{e})$ is non-zero modulo p. Now, assume that $\omega(\mathbf{e}) \equiv 0 \pmod{p}$. In this case, we get $z^t = z$. We shall take a closer look at the case $p = 7$.

Lemma 5.3.16. *Let G be a* GGS*-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Assume that $\mathbf{e}''$ is non-symmetric and $\omega(\boldsymbol{e}) \equiv 0 \pmod p$. Then there exists an element $z_1 \in \gamma_4(G)\, H_1$ such that z_1 is central in $H/[H_1, G]$, and $z_1^t \neq z_1$.*

Proof. Set $z_1 = x_0 x_1^{-2} x_2$. It follows from Lemma 2.4.27 that

$$[x_i, z_1] \equiv_{[H_1,G]} y^{k_i},$$

where k_i is the coordinate sum of the product of the $(i+1)$th row of the matrix $\mathcal{T}$ with the element $(1, -2, 1, 0, 0, 0, 0)$. We get

$$\begin{aligned}
k_1 &= -2\ell_{1,2} + s_2 = -2(2(e_1 - e_6) + e_5 - e_2) + e_6 - e_1 + 2(e_2 - e_5) + e_4 - e_3 \\
&= 2(e_1 - e_6) + 4(e_2 - e_5) + 6(e_3 - e_4) \equiv \omega \bmod 7 = 0. \\
k_2 &= -\ell_{1,2} + \ell_{1,2} = 0. \\
k_3 &= -s_2 + 2\ell_{1,2} = 0. \\
k_4 &= -s_3 + 2s_2 - \ell_{1,2} = -(e_5 - e_2 + 3(e_3 - e_4)) + 2(e_6 - e_1 + 2(e_2 - e_5) + e_4 - e_3) \\
&\quad - (2(e_1 - e_6) + e_5 - e_2) = 3(e_1 - e_6) + 6(e_2 - e_5) + 2(e_3 - e_4) \equiv \omega \bmod 7 = 0. \\
k_5 &= 3s_3 - s_2 = 3(e_5 - e_2 + 3(e_3 - e_4)) - (e_6 - e_1 + 2(e_2 - e_5) + e_4 - e_3) \\
&= e_1 - e_6 + 2(e_2 - e_5) + 3(e_3 - e_4) \equiv \omega \bmod 7 = 0. \\
k_6 &= s_2 - 3s_3 = 0.
\end{aligned}$$

Hence, we conclude z_1 is central in $H/[H_1, G]$. Furthermore,

$$z_1^t \equiv_{[H_1,G]} z_1 y^{e_1 - e_6 - 2(e_6 - e_1 + e_2 - e_5) + e_5 - e_2 + e_3 - e_4} = z_1 y^{3(e_1 - e_6) + 4(e_2 - e_5) + e_3 - e_4}.$$

We claim that $3(e_1 - e_6) + 4(e_2 - e_5) + e_3 - e_4 \neq 0 \pmod 7$. Assume to the contrary that $3(e_1 - e_6) + 4(e_2 - e_5) + e_3 - e_4 = 0 \pmod 7$. Since $\omega(\mathbf{e})$ is also equal to zero modulo 7, from an easy computation one gets $e_1 - e_6 = 5(e_3 - e_5)$ and $e_2 - e_5 = 3(e_3 - e_4)$. Now, set $d = e_3 - e_4$. Then $e_1 = 5d + e_6$, $e_2 = 3d + e_5$ and $e_3 = d + e_4$. Therefore,

$$\begin{aligned}
\mathbf{e} &= (5d + e_6, 3d + e_5, d + e_4, e_4, e_5, e_6) \\
\mathbf{e}' &= (-2d + e_5 - e_6, -2d + e_4 - e_5, -d, e_5 - e_4, e_6 - e_5) \\
\mathbf{e}'' &= (e_4 - 2e_5 + e_6, e_5 - e_4, +d, e_5 - e_4 + d, e_4 - 2e_5 + e_6),
\end{aligned}$$

which implies that $\mathbf{e}''$ is symmetric, hence a contradiction. Therefore, $z_1^t \neq z_1$. □

Therefore, Theorem 9.1.1 holds for $p = 7$ in full generality, and we record the following result.

Corollary 5.3.17. *Let G be a* GGS*-group defined by a non-symmetric defining vector $\boldsymbol{e} \in \mathbb{F}_7^6$. Then the following inequalities are satisfied.*

$$5 \leqslant \alpha(G) \leqslant 87.1372431.$$

Now, Theorem 3.0.7 follows from Corollary 5.3.17 and Theorem 5.3.17. The proof of Lemma 5.3.16 suggests that, if we set

$$z_1 = x_0^{\binom{p-5}{0}} \, x_1^{-\binom{p-5}{1}} \, x_2^{\binom{p-5}{2}} \cdots x_{p-6}^{-\binom{p-5}{p-6}} \, x_{p-5}^{\binom{p-5}{p-5}} \in \gamma_{p-3}(G) \, H_1,$$

then z_1 would be a potential candidate for higher primes. Then one could iterate the process, and make sure that such elements exist for all choices of the defining vector $\mathbf{e}$.

Chapter 6

Representation zeta function of the Gupta–Sidki 3-group

In this chapter, we explicitly compute a recursive formula for the representation zeta function of the Gupta–Sidki 3-group, and hence we give a proof for Theorem 3.0.10. Moreover, we will show that the formula presented in Section 6.5 is in agreement with the one obtained in [14, Section 2.2] by means of computer calculations.

For every odd prime p, the Gupta–Sidki p-group G_p is a GGS-group defined by the vector $\mathbf{e} = (1, -1, 0, \ldots, 0) \in \mathbb{F}_p^{p-1}$. Since the defining vector $\mathbf{e}$ is non-symmetric, recall from Theorem 2.4.21 that G_p is regular branch over the commutator subgroup. We emphasise that the detailed computation presented here is currently limited to the Gupta–Sidki 3-group G_3 because of its relatively small branching quotient isomorphic to $C_3 \times C_3$. However, we begin with Section 6.1, where we review some structural properties of the Gupta–Sidki p-groups. These results help us to have a better understanding about the branching quotient. In Section 6.2, we explain the strategy of computing the representation zeta function of G_3. The crucial computations are carried out in Section 6.3, where we analyse the inertia groups of the irreducible representations of the subgroup G_3' and $\psi^{-1}(G_3' \times G_3' \times G_3')$. The calculations in Section 6.3 give recursive estimates on the number of irreducible characters of G_3'. Using this, we first compute the representation zeta function of G_3' in Section 6.4. Finally, the functional equation summarised in Theorem 3.0.10 is obtained in Section 6.5.

6.1 Gupta–Sidki p-groups

Throughout this section let p denote an odd prime.

Proposition 6.1.1 ([44], Proposition 2.4). *Let G_p be the Gupta–Sidki p-group and let*

$g \in G_p$. *Then* $g \in \mathrm{St}_{G_p}(1)$ *if and only if there exist* $i_0, \dots, i_{p-1} \in [0, p-1]$ *and* $h_0, \dots, h_{p-1} \in G_p'$ *such that*

$$\psi(g) = (h_0 a^{-i_{p-1}+i_0} t^{i_1}, h_1 a^{-i_0+i_1} t^{i_2}, \dots, h_{p-1} a^{-i_{p-2}+i_{p-1}} t^{i_0}). \tag{6.1}$$

Proof. If $g \in G_p$ is of the form (6.1), then it is clear that $g \in \mathrm{St}_{G_p}(1)$. Let $g \in \mathrm{St}_{G_p}(1)$. Then by Theorem 2.4.19(i), $g \equiv t_0^{i_0} t_1^{i_1} \cdots t_{p-1}^{i_{p-1}} \mod \mathrm{St}_{G_p}(1)'$ for some $i_0, \dots, i_{p-1} \in [0, p-1]$, where $t_i = t^{a^i}$ for all $i \in [0, p-1]$. Therefore,

$$\begin{aligned}\psi(g) &= (k_0 a^{i_0} t^{i_1} a^{-i_{p-1}}, k_1 a^{-i_0+i_1} t^{i_2}, \dots, k_{p-1} t^{i_0} a^{-i_{p-2}+i_{p-1}}) \\ &= (h_0 a^{-i_{p-1}+i_0} t^{i_1}, h_1 a^{-i_0+i_1} t^{i_2}, \dots, h_{p-1} a^{-i_{p-2}+i_{p-1}} t^{i_0}),\end{aligned}$$

for some $h_i, k_i \in G_p'$, where $i \in [0, p-1]$, completing the proof. ☐

Proposition 6.1.2 (cf. [44], Lemma 2.5)**.** *Let* G_p *be the Gupta–Sidki* p*-group. The element* $\psi^{-1}(t, \dots, t)$ *lies in* G_p'*. Moreover,* $\mathrm{St}_{G_p}(2)/\mathrm{St}_{G_p}(1)' = \langle \mathrm{St}_{G_p}(1)' \psi^{-1}(t, \dots, t) \rangle \cong C_p$*. In particular, for* $p = 3$*,* $\gamma_3(G_p) = \mathrm{St}_{G_p}(2)$*.*

Proof. A straightforward computation using the section decomposition of x_i (Notation 2.4.24) yields that

$$\psi^{-1}(t, \dots, t) \equiv_{G_p'} x_0 x_1^2 \cdots x_{p-2}^{p-1}. \tag{6.2}$$

Hence $\psi^{-1}(t, \dots, t) \in G_p'$. Now observe that, $\mathrm{St}_{G_p}(1)' \leqslant \mathrm{St}_{G_p}(2)$. Let $g = (g_0, \dots, g_1) \in \mathrm{St}_{G_p}(2)$. In particular, $g \in \mathrm{St}_{G_p}(1)$ and hence g is of the form (6.1) such that the exponent sum of a in each co-ordinate is zero. This implies that $i_0 = i_1 = \cdots = i_{p-1} = i$ for some $i \in [0, p-1]$ and, since $\psi(\mathrm{St}_{G_p}(1)') = G_p' \times \overset{p}{\cdots} \times G_p'$ (Theorem 2.4.19(ii)), we get

$$\psi(g) = (h_0 t^i, h_1 t^i \dots, h_{p-1} t^i) \equiv (t, \dots, t)^i \mod \psi(\mathrm{St}_{G_p}(1)'). \tag{6.3}$$

Furthermore, (6.3) implies that $\mathrm{St}_{G_p}(2)/\mathrm{St}_{G_p}(1)' = \langle \mathrm{St}_{G_p}(1)' \psi^{-1}(t, \dots, t) \rangle \cong C_p$.

Now, suppose that $p = 3$. From Theorem 2.4.19 and Theorem 2.4.21, it holds that $[G_p' : \mathrm{St}_{G_p}(1)'] = 3^2$ and hence $[G_p' : \mathrm{St}_{G_p}(2)] = 3$. From Theorem 2.4.19(iii), we have $\mathrm{St}_{G_p}(2) \leqslant \gamma_3(G_p)$ and $[G_p' : \gamma_3(G_p)] = 3$, resulting that $\gamma_3(G_p) = \mathrm{St}_{G_p}(2)$. ☐

For convenience of the later computations, we state the following immediate corollary of the last part of the proof of Proposition 6.1.2.

Corollary 6.1.3. *Let* G *be the Gupta–Sidki* 3*-group. Let* $H = G'$ *be the commutator subgroup and let and* $H_1 = \psi^{-1}(H \times H \times H)$*. Then* $H = H_1 \langle x, \bar{t} \rangle$*, where* $x = [a, t]$ *and* $\bar{t} = \psi^{-1}(t, t, t)$*.*

6.2 Method of computing the representation zeta function

Before presenting the calculations, we give an outline of the strategy. Thereby, we record the following structural lemma that is vital to the computation of the recursive representation zeta function of G_3, since it illustrates possible subgroups between G_3 and $\psi^{-1}(G_3' \times G_3' \times G_3')$ from (resp. to) which characters induce (resp. restrict).

Lemma 6.2.1. *Let G be the Gupta–Sidki 3-group and let $H = G'$ be the commutator subgroup of G. Let H_1 denote the subgroup $\psi^{-1}(H \times H \times H)$ of G. Then the Hasse diagrams for the sets of subgroups that are sandwiched between G and H, and that are sandwiched between H and H_1 are given by Figure 6.1.*

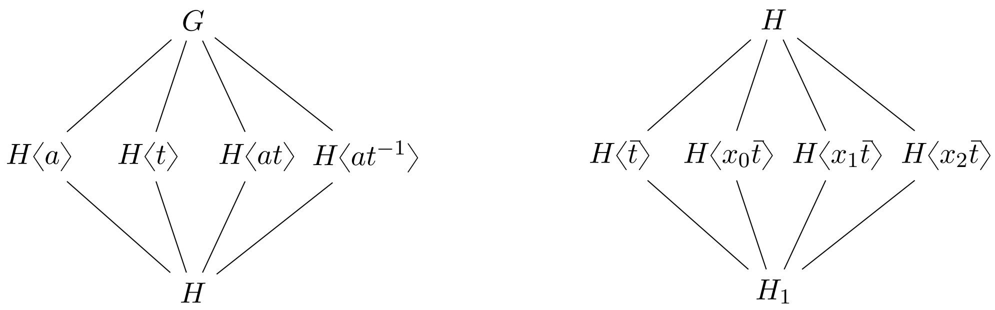

Figure 6.1: Hasse diagrams

Proof. From Theorem 2.4.19, it follows that $G/H = \langle H\, a\rangle \times \langle H\, t\rangle \cong C_3 \times C_3$. Similarly, from Theorem 2.4.25, we get that $H/H_1 = \langle H_1\, x_0\rangle \times \langle H_1\, x_1\rangle \cong C_3 \times C_3$, where

$$\psi(x_0) = [a,t] = (t^{-1}a, a^{-2}, at), \qquad \text{and} \qquad \psi(x_1) = x^a = (at, t^{-1}a, a^{-2}),$$

cf. Notation 2.4.24. The subgroups lying between G and H are given by the set

$$\mathcal{G} = \{G, H\langle a\rangle, H\langle t\rangle, H\langle at\rangle, H\langle at^{-1}\rangle, H\},$$

and the subgroups lying between H and H_1 are given by the set

$$\mathcal{H} = \{H, H_1\langle x_0\rangle, H_1\langle x_1\rangle, H_1\langle x_0x_1\rangle, H_1\langle x_0x_1^{-1}\rangle, H_1\}.$$

Now, since $\bar{t} \in H$ (Corollary 6.1.3), observe that

$$\begin{aligned}
\psi(x_0) &= (t^{-1}a, a, at) \equiv_{H_1} (at, t^{-1}a, a)(t,t,t) = \psi(x_1\bar{t}),\\
\psi(x_1) &= (at, t^{-1}a, a) \equiv_{H_1} (a, at, t^{-1}a)(t,t,t) = \psi(x_2\bar{t}),\\
\psi(x_0x_1) &\equiv_{H_1} (a, at, t^{-1}a)^{-1} \equiv_{H_1} ((t^{-1}a, a, at)(t,t,t))^{-1} = \psi((x_0\bar{t})^{-1}),\\
\psi(x_0x_1^{-1}) &\equiv_{H_1} (t,t,t) = \psi(\bar{t}).
\end{aligned}$$

Therefore, $\mathcal{H} = \{H, H_1\langle\bar{t}\rangle, H_1\langle x_0\bar{t}\rangle, H_1\langle x_1\bar{t}\rangle, H_1\langle x_2\bar{t}\rangle, H_1\}$, yielding the Figure 6.1. $\square$

Since G is regular branch over the commutator subgroup H, we identify the subgroup H_1 with the subgroup $H \times H \times H$. As stated in Theorem 4.0.1, every irreducible character $\rho \in \mathrm{Irr}(H_1)$ can be uniquely expressed as

$$\rho = \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2,$$

for some $\vartheta_i \in \mathrm{Irr}(H)$, where $i \in [0, 2]$. Recall from Chapter 4 that the notation $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ denotes the product of characters $\vartheta_0, \vartheta_1, \vartheta_2$, and is given by

$$\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2(h_0, h_1, h_2) = \vartheta_0(h_0)\vartheta_1(h_1)\vartheta_2(h_2),$$

for every $(h_0, h_1, h_2) \in H \times H \times H$. Conversely, for every choice of $\vartheta_i \in \mathrm{Irr}(H)$, the product $\vartheta_0 \otimes \vartheta_2 \otimes \vartheta_3$ is an element of $\mathrm{Irr}(H_1)$. Therefore, we identify the following sets

$$\mathrm{Irr}(H_1) = \mathrm{Irr}(H) \times \mathrm{Irr}(H) \times \mathrm{Irr}(H),$$

where $\mathrm{Irr}(H)\times\mathrm{Irr}(H)\times\mathrm{Irr}(H) = \{\vartheta_0\otimes\vartheta_1\otimes\vartheta_2 \mid \vartheta_i \in \mathrm{Irr}(H), i \in [0, 2]\}$ as defined in Chapter 4. For every $\rho \in \mathrm{Irr}(H_1)$ (resp. $\varphi \in \mathrm{Irr}(H)$), the inertia group $I_H(\rho)$ in H (resp. $I_G(\varphi)$ in G) belongs to the set $\mathcal{H}$ (resp. $\mathcal{G}$), and hence there are six different possibilities. In Section 6.3, we do a case-by-case study on the inertia groups. In each case, we obtain a sufficient and necessary condition for a character to have a given inertia group. We split the process into two steps.

Step 1: For every $S \in \mathcal{H}$, we provide a sufficient and necessary condition for a character $\rho \in \mathrm{Irr}(H_1)$ to satisfy $I_H(\rho) = S$, in terms of the inertia group $I_G(\vartheta_i)$ of ϑ_i in G.

Now, suppose $\varphi \in \mathrm{Irr}(H)$ and ρ is an irreducible constituent of $\varphi|_{H_1}$ such that $I_H(\rho) = S \in \mathcal{H}$. The inertia group $I_G(\varphi)$ of φ in G is an element of the set $\mathcal{G}$.

Step 2: Using Step 1 and the results discussed in Chapter 4, we give a sufficient and necessary condition for $I_G(\varphi) = T$ for every $T \in \mathcal{G}$, in terms of the the characters ϑ_i and their inertia groups in G.

Using the information from Section 6.3, in Section 6.4, we obtain a recursive procedure for calculating the representation zeta function of H. From each of the cases in Step 2, we count the irreducible characters of H that are obtained either by extension or by induction from a given irreducible character ρ of H_1. In this way, we obtain all irreducible characters of H. For every $T \in \mathcal{G}$, denote by r_d^T the cardinality of the set

$$r_d^T = |\{\varphi \in \mathrm{Irr}(H) \mid \varphi(1) = 3^d, I_G(\varphi) = T\}|, \tag{6.4}$$

and define the partial representation zeta function $\zeta^T(H, s)$ of H associated with T as the

Dirichlet generating function give by

$$\zeta^T(H,s) = \sum_{d=0}^{\infty} r_d^T\, 3^{-ds}, \tag{6.5}$$

for $s \in \mathbb{C}$. By taking the sum of $\zeta^T(H,s)$ over all six $T \in \mathcal{H}$ we obtain the representation zeta function of H given by

$$\zeta(H,s) = \sum_{T\in\mathcal{G}} \zeta^T(H,s). \tag{6.6}$$

Finally in Section 6.5, again by an application of results from Chapter 4, we compute the representation zeta function of G as a recursive function in terms of the representation zeta function of H.

From the computation of partial representation zeta functions of H, we observe that

$$\zeta^G(H,s) = 3,$$

(see Lemma 6.4.1). That is there are exactly three irreducible characters of H that are G-invariant and all of them are linear. Therefore, we get that the cardinality C of G-invariant irreducible characters of H (defined in Theorem 5.3.4) is equal to three.

6.3 Inertia groups

This section comprises of Step 1 (Section 6.3.1) and Step 2 (Section 6.3.2). Here we carefully and elaborately develop the theory that enables us to compute the partial representation zeta functions described in Section 6.2. In the following, we fix $G = G_3$ as the Gupta–Sidki 3-group and $H = G'$ as the commutator subgroup of G. We identify $H_1 = H \times H \times H$ with the subgroup $\psi^{-1}(H \times H \times H)$ of G and the element $\bar{t} = (t,t,t)$ with the element $\psi^{-1}(t,t,t)$. Further, we fix

$$\psi(x_0) = (t^{-1}a, a^{-2}, at), \qquad \psi(x_1) = (at, t^{-1}a, a^{-2}), \qquad \psi(x_2) = (a^{-2}, at, t^{-1}a).$$

We recall from Lemma 6.2.1 the following collection of subgroups:

$$\mathcal{G} = \{G, H\langle a\rangle, H\langle t\rangle, H\langle at\rangle, H\langle at^{-1}\rangle, H\},$$

and

$$\mathcal{H} = \{H, H_1\langle \bar{t}\rangle, H_1\langle x_0\bar{t}\rangle, H_1\langle x_1\bar{t}\rangle, H_1\langle x_2\bar{t}\rangle, H_1\}.$$

6.3.1 Inertia groups: Step 1

The objective of this section is to obtain fundamental results that provide sufficient and necessary conditions for a character $\rho = \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \operatorname{Irr}(H_1)$ to have $I_G(\rho) = S \in \mathcal{H}$

in terms of the inertia groups $I_G(\vartheta_i) \in \mathcal{G}$ for $i \in [0,2]$. We set $K = H\langle t\rangle = \mathrm{St}_G(1)$. For convenience, we identify an element $g \in K$ with its image under the projection map ψ. Also, we adopt the convention that the subscripts of the irreducible characters ϑ_i are taken modulo 3.

Lemma 6.3.1. *Let $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathrm{Irr}(H)$. The following assertion holds*

$$I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = \psi^{-1}(I_G(\vartheta_0) \times I_G(\vartheta_1) \times I_G(\vartheta_2)) \cap H.$$

Proof. Since ψ is a monomorphism from the first level stabiliser $K = \mathrm{St}_G(1)$ to the direct product $G \times G \times G$, it holds that $\psi^{-1}(I_G(\vartheta_0) \times I_G(\vartheta_1) \times I_G(\vartheta_2)) \leqslant K$ and

$$\psi^{-1}(I_G(\vartheta_0) \times I_G(\vartheta_1) \times I_G(\vartheta_2)) \cap H \leqslant I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2).$$

To see the reverse inclusion, consider $g = (g_0, g_1, g_2) \in I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$ and $h = (h_0, h_1, h_2) \in H_1$, where $g_i \in G, h_i \in H$ and $i \in [0,2]$. We have

$$\begin{aligned} \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2(h) = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^g(h) &= (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)(ghg^{-1}) \\ &= \vartheta_0(g_0h_0g_0^{-1})\vartheta_1(g_1h_1g_1^{-1})\vartheta_2(g_2h_2g_2^{-1}) \\ &= \vartheta_0^{g_0}(h_0)\vartheta_1^{g_1}(h_1)\vartheta_2^{g_2}(h_2) = \vartheta_0^{g_0} \otimes \vartheta_1^{g_1} \otimes \vartheta_2^{g_2}(h). \end{aligned}$$

Thus $g_i \in I_G(\vartheta_i)$ for all $i \in [0,2]$; cf. Remark 4.0.2. Hence $g \in \psi^{-1}(I_G(\vartheta_0) \times I_G(\vartheta_1) \times I_G(\vartheta_2)) \cap H$. □

Lemma 6.3.2. *Let $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathrm{Irr}(H)$. Then $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$ if and only if $I_G(\vartheta_i) = G$ for all $i \in [0,2]$.*

Proof. If $I_G(\vartheta_i) = G$ for all $i \in [0,2]$ then it follows from Lemma 6.3.1 that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. To see the converse, assume that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. We get,

$$\begin{aligned} \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 &= (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x_0} = \vartheta_0^{t^{-1}a} \otimes \vartheta_1^{a} \otimes \vartheta_2^{at}, \\ \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 &= (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x_1} = \vartheta_0^{at} \otimes \vartheta_1^{t^{-1}a} \otimes \vartheta_2^{a}, \\ \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 &= (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x_2} = \vartheta_0^{a} \otimes \vartheta_1^{at} \otimes \vartheta_2^{t^{-1}a}, \end{aligned}$$

implying that $I_G(\vartheta_0) = I_G(\vartheta_1) = I_G(\vartheta_2) = G$; cf. Remark 4.0.2. □

Lemma 6.3.3. *Let $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathrm{Irr}(H)$. Then $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t}\rangle$ if and only if $H\langle t\rangle \leqslant I_G(\vartheta_i)$ for all $i \in [0,2]$, and there exists $j \in [0,2]$ such that $I_G(\vartheta_j) \neq G$.*

Proof. Suppose that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t} \rangle$. It follows from Lemma 6.3.2 that there exists $j \in [0,2]$ such that $I_G(\vartheta_j) \neq G$. Let $h = (h_0, h_1, h_2) \in H_1$ and let $g\bar{t}^\varepsilon = (g_0 t^\varepsilon, g_1 t^\varepsilon, g_2 t^\varepsilon) \in H_1\langle \bar{t} \rangle$, where $g_i, h_i \in H$ and $i, \varepsilon \in [0,2]$. We get

$$\begin{aligned}\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2(h) &= (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{g\bar{t}^\varepsilon}(h) = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)(g\bar{t}^\varepsilon h (g\bar{t}^\varepsilon)^{-1}) \\ &= \vartheta_0(g_0 t^\varepsilon h_0 (g_0 t^\varepsilon)^{-1}) \vartheta_1(g_1 t^\varepsilon h_1 (g_1 t^\varepsilon)^{-1}) \vartheta_2(g_2 t^\varepsilon h_2 (g_2 t^\varepsilon)^{-1}) \\ &= \vartheta_0^{g_0 t^\varepsilon}(h_0) \vartheta_1^{g_1 t^\varepsilon}(h_1) \vartheta_2^{g_2 t^\varepsilon}(h_2) = \vartheta_0^{g_0 t^\varepsilon} \otimes \vartheta_1^{g_1 t^\varepsilon} \otimes \vartheta_2^{g_2 t^\varepsilon}(h).\end{aligned}$$

Hence $H\langle t \rangle \leqslant I_G(\vartheta_i)$ for all $i \in [0,2]$.

Now, suppose that $H\langle t \rangle \leqslant I_G(\vartheta_i)$ for all $i \in [0,2]$ and suppose further that there exists $j \in [0,2]$ such that $I_G(\vartheta_j) \neq G$. From Lemma 6.3.1, we have

$$H_1\langle \bar{t} \rangle \leqslant I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) < H,$$

where the strict inequality follows from Lemma 6.3.2. Therefore, $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t} \rangle$, cf. Figure 6.1. ☐

Lemma 6.3.4. *Let $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathrm{Irr}(H)$ and let $j \in [0,2]$. Then $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_j \bar{t} \rangle$ if and only if there exists $k \in [0,2]$ such that $I_G(\vartheta_k) \neq G$ and the following inclusions hold:*

$$H\langle a \rangle \leqslant I_G(\vartheta_j), \qquad H\langle at \rangle \leqslant I_G(\vartheta_{j+1}), \qquad H\langle at^{-1} \rangle \leqslant I_G(\vartheta_{j+2}).$$

Proof. Observe first from the proof of Lemma 6.2.1 that

$$x_0\bar{t} \equiv_{H_1} (a, at, at^{-1}), \qquad x_1\bar{t} \equiv_{H_1} (at^{-1}, a, at), \qquad x_2\bar{t} \equiv_{H_1} (at, at^{-1}, a).$$

We prove the result for the case $j = 0$. The other cases follow in a similar manner. Suppose that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_0\bar{t} \rangle$. It is immediate from Lemma 6.3.2 that there exists $k \in [0,2]$ such that $I_G(\vartheta_k) \neq G$. Furthermore,

$$\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x_0\bar{t}} = \vartheta_0^a \otimes \vartheta_1^{at} \otimes \vartheta_2^{at^{-1}},$$

and hence $H\langle a \rangle \leqslant I_G(\vartheta_0)$, $H\langle at \rangle \leqslant I_G(\vartheta_1)$ and $H\langle at^{-1} \rangle \leqslant I_G(\vartheta_2)$.

Now to see the converse, assume that the given statement is true. Then from Lemma 6.3.2 we have

$$H_1\langle x_0\bar{t} \rangle \leqslant I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) < H,$$

whence $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_0\bar{t} \rangle$, cf. Figure 6.1. ☐

Lemma 6.3.5. *Let $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathrm{Irr}(H)$. Then $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$ if and only if there exists $i \in [0,2]$ such that $\langle t \rangle \nleqslant I_G(\vartheta_i)$ and the following assertion holds:*

$$\neg(\exists j \in [0,2] : H\langle a \rangle \leqslant I_G(\vartheta_j) \quad \wedge \quad H\langle at \rangle \leqslant I_G(\vartheta_{j+1}) \quad \wedge \quad H\langle at^{-1} \rangle \leqslant I_G(\vartheta_{j+2})).$$

Proof. Since for every $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ the inertia group $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$ lies in $\mathcal{H}$, there are only 6 possibilities for $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$. Therefore $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$ if and only $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) \notin \mathcal{H}\backslash\{H_1\}$. By combining Lemma 6.3.2, Lemma 6.3.3 and Lemma 6.3.4 we obtain Lemma 6.3.5. □

Now, we record two lemmas that are helpful for the discussion of G-invariant irreducible characters of H in Section 6.3.2. Let $\vartheta \in \mathrm{Irr}(H)$ be such that $I_G(\vartheta) = G$. Thanks to Corollary 5.3.12, ϑ is linear. In the following, we identify ϑ with an irreducible character of $H/[H, G]$.

Lemma 6.3.6. *Let $\vartheta \in \mathrm{Irr}(H)$ be such that $I_G(\vartheta) = G$. Then ϑ extends to irreducible characters of K. Let $\Theta \in \mathrm{Irr}(K)$ be an extension of ϑ. Then $\Theta^a = \Theta\lambda$ for some $\lambda \in \mathrm{Irr}(K/H)$ and, for each $\varepsilon \in [0, 2]$, $\lambda(t^\varepsilon) = \vartheta([a, t])^\varepsilon$. (Here we identify the quotient group K/H with the cyclic group $\langle t\rangle \cong C_3$). Furthermore, if ϑ extends towards G then $\vartheta = 1_H$, where 1_H is the trivial character of H.*

Proof. Since ϑ is G-invariant and $|K : H| = 3$ is prime, ϑ extends towards K and let Θ be an extension. Since Θ is an extension of ϑ, Θ is linear, and hence Θ is a homomorphism from K to $\mathbb{C}^\times$. Then $\Theta(\ell) \neq 0$ for all $\ell \in K$. We identify Θ with an irreducible character of $K/[H, G]$. Let $\ell = ht^\varepsilon \in K$, where $h \in H$ and $\varepsilon \in [0, 2]\backslash\{0\}$. Then

$$\Theta^a(ht^\varepsilon) = \Theta((ht^\varepsilon)^{a^{-1}}) = \Theta((ht^\varepsilon)[ht^\varepsilon, a^{-1}]) = \Theta(ht^\varepsilon)\vartheta([ht^\varepsilon, a^{-1}]), \tag{6.7}$$

where the last equality follows because Θ is linear. On the other hand, since ϑ is G-invariant, $\Theta^a|_H = (\Theta|_H)^a = \vartheta$. Hence, there exists $\lambda \in \mathrm{Irr}(K/H)$ such that $\Theta^a = \Theta\lambda$. Therefore,

$$\Theta^a(ht^\varepsilon) = (\Theta\lambda)(ht^\varepsilon) = \Theta(ht^\varepsilon)\lambda(t^\varepsilon). \tag{6.8}$$

By comparing (6.7) and (6.8), and using the fact that $[H, G] \leqslant \ker(\vartheta)$, we get

$$\begin{aligned}\lambda(t^\varepsilon) &= \vartheta([ht^\varepsilon, a^{-1}]) = \vartheta([a, ht^\varepsilon]^{a^{-1}}) = \vartheta^a([a, ht^\epsilon]) = \vartheta([a, ht^\varepsilon]) \\ &= \vartheta([a, t^\varepsilon][a, h]^{t^\varepsilon}) = \vartheta([a, t^\varepsilon])\vartheta([a, h]^{t^\varepsilon}) = \vartheta([a, t^\varepsilon]) = \vartheta([a, t])^\varepsilon.\end{aligned}$$

To see the last claim, assume that ϑ admits an extension towards G. Then ϑ is trivial on $[G, G] = H$ whence $\vartheta = 1_H$. □

Lemma 6.3.7. *Let $\vartheta \in \mathrm{Irr}(H)$ be such that $I_G(\vartheta) = G$. Then ϑ extends to irreducible characters of $H\langle at^\delta\rangle$ for every $\delta \in [0, 2]$. Let $\Theta \in \mathrm{Irr}(H\langle at^\delta\rangle)$ be an extension of ϑ. Then $\Theta^t = \Theta\lambda$ for some $\lambda \in \mathrm{Irr}(H\langle at^\delta\rangle/H)$ and, for each $\varepsilon \in [0, 2]$, $\lambda((at^\delta)^\varepsilon) = \vartheta([a, t])^{-\varepsilon}$, (here we identify the quotient group $H\langle at^\delta\rangle/H$ with the cyclic group $\langle at^\delta\rangle \cong C_3$).*

Proof. Let Θ be an extension of ϑ towards $H\langle at^{\delta}\rangle$. Then Θ is linear, and we identify Θ with an irreducible character of $H\langle at^{\delta}\rangle/[H, G]$. Since Θ is linear, $\Theta(\ell) \neq 0$ for all $\ell \in H\langle at^{\delta}\rangle$. Let $\ell = h(at^{\delta})^{\varepsilon} \in H\langle at^{\delta}\rangle$, where $h \in H$ and $\varepsilon \in [0, 2]\backslash\{0\}$. We get

$$\Theta^{t}(h(at^{\delta})^{\varepsilon}) = \Theta(h(at^{\delta})^{\varepsilon})^{t^{-1}}) = \Theta((h(at^{\delta})^{\varepsilon})[h(at^{\delta})^{\varepsilon}, t^{-1}]) = \Theta(h(at^{\delta})^{\varepsilon})\vartheta([h(at^{\delta})^{\varepsilon}, t^{-1}]), \tag{6.9}$$

where the last equality follows because Θ is linear. Now, since ϑ is invariant under the action of G, $\Theta^{t}|_{H} = (\Theta|_{H})^{t} = \vartheta$. Hence, there exists $\lambda \in \mathrm{Irr}(H\langle at^{\delta}\rangle/H)$ such that $\Theta^{t} = \Theta\lambda$. Therefore,

$$\Theta^{t}(h(at^{\delta})^{\varepsilon}) = \Theta\lambda(h(at^{\delta})^{\varepsilon}) = \Theta(h(at^{\delta})^{\varepsilon})\lambda((at^{\delta})^{\varepsilon}). \tag{6.10}$$

By comparing (6.9) and (6.10), and using the fact that $[H, G] \leqslant \ker(\vartheta)$, we get

$$\begin{aligned}\lambda((at^{\delta})^{\varepsilon}) &= \vartheta([h(at^{\delta})^{\varepsilon}, t^{-1}]) = \vartheta([t, h(at^{\delta})^{\varepsilon}]^{t^{-1}}) = \vartheta^{t}([t, h(at^{\delta})^{\varepsilon}]) = \vartheta([t, h(at^{\delta})^{\varepsilon}])\\ &= \vartheta([t, (at^{\delta})^{\varepsilon}][t, h]^{(at^{\delta})^{\varepsilon}}) = \vartheta([t, (at^{\delta})^{\varepsilon}])\vartheta([t, h]^{(at^{\delta})^{\varepsilon}}) = \vartheta([t, (at^{\delta})^{\varepsilon}])\\ &= \vartheta([t, at^{\delta}])^{\varepsilon} = \vartheta([t, t^{\delta}][t, a])^{\varepsilon} = \vartheta([a, t])^{-\varepsilon}.\end{aligned}$$

□

6.3.2 Inertia groups: Step 2

Let $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ for some $\vartheta_i \in \mathrm{Irr}(H)$ and $i \in [0, 2]$, and let $S \in \mathcal{H}$. In Section 6.3.1 we observed the explicit conditions under which $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = S$. Let $\varphi \in \mathrm{Irr}(H)$ be such that $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is an irreducible constituent of $\varphi|_{H_1}$. For every $T \in \mathcal{G}$, here we provide sufficient and necessary conditions for φ to satisfy the equality $I_G(\varphi) = T$. We split the calculation into four cases.

(i) Case 1: $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t}\rangle$.

(ii) Case 2: $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_j\bar{t}\rangle$ for $j \in [0, 2]$.

(iii) Case 3: $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$.

(iv) Case 4: $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$.

Notice from Lemma 6.3.2 that Case 4 occurs if and only if the characters ϑ_i are G-invariant for all $i \in [0, 2]$. Moreover, by Corollary 5.3.12 the characters ϑ_i are then linear and the numbers of such characters are finite. Therefore, there are only finitely many characters which satisfy Case 4. This enables us to explicitly count the number of characters $\varphi \in \mathrm{Irr}(H)$ such that the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$, with inertia group $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$, occurs as an irreducible constituent of $\varphi|_{H_1}$. Further, we prove that these characters contribute to the constant terms of the partial representation zeta functions of H.

6.3.2.1 Case 1: $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t} \rangle$.

We begin by the following observation.

Observation 6.3.8. By Lemma 6.3.3, it follows that $K = H\langle t \rangle \leqslant I_G(\vartheta_i)$ for all $i \in [0, 2]$. Then each ϑ_i extends to an irreducible character of K for all $i \in [0, 2]$. If Θ_i denotes an extension of ϑ_i to K then $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ is an extension of the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ from H_1 to $K \times K \times K$. Since $H_1\langle \bar{t} \rangle = \psi^{-1}(K \times K \times K) \cap H$, we identify $H_1\langle \bar{t} \rangle$ with a subgroup of $K \times K \times K$. Denote by η the restriction of $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ to $H_1\langle \bar{t} \rangle$. Then η is an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ from H_1 to $H_1\langle \bar{t} \rangle$ (Theorem 4.0.3). Since $H_1\langle \bar{t} \rangle < H$, the character η does not extend towards H. Denote by φ the character of H induced from η. Then φ is irreducible (Theorem 4.0.5) and by Clifford's theorem the restriction of φ to $H_1\langle \bar{t} \rangle$ has the following form

$$\varphi|_{H_1\langle \bar{t} \rangle} = \eta + \eta^x + \eta^{x^{-1}}, \tag{6.11}$$

since $H = H_1\langle x, \bar{t} \rangle$ (Corollary 6.1.3) and $H/H_1\langle \bar{t} \rangle \cong C_3$ (cf. Proposition 6.1.2 and Theorem 2.4.19(iii)). Furthermore,

$$\varphi(1) = [H : H_1\langle \bar{t} \rangle] \cdot \eta(1) = 3 \cdot (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)(1) = 3\vartheta_0(1)\vartheta_1(1)\vartheta_2(1).$$

Proposition 6.3.9. *Let $\vartheta_i \in \mathrm{Irr}(H)$, for $i \in [0, 2]$, such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t} \rangle$. Let Θ_i, η and φ be defined as in Observation 6.3.8 above. Then $a \in I_G(\varphi)$ if and only if there exist $\vartheta \in \mathrm{Irr}(H)$ and $\varepsilon \in [0, 2]$ such that $I_G(\vartheta) = K$ and $\vartheta_0 = \vartheta$, $\vartheta_1 = \vartheta^{a^\varepsilon}$ and $\vartheta_2 = \vartheta^{a^{-\varepsilon}}$.*

Proof. From Proposition 6.1.2, observe first that, $H_1\langle \bar{t} \rangle = \mathrm{St}_G(2)$ and set $K_1 = \mathrm{St}_G(2)$. Thanks to Lemma 4.0.9 and (6.11), we get that $a \in I_G(\varphi)$ if and only if $\eta^a \in \{\eta, \eta^x, \eta^{x^{-1}}\}$. If $\eta^a = \eta$ then

$$\begin{aligned}(\Theta_2 \otimes \Theta_0 \otimes \Theta_1)|_{K_1} &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)^a|_{K_1} = ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^a = \eta^a = \eta \\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1}.\end{aligned}$$

Suppose that $\eta^a = \eta^x$. Since $H\langle t \rangle = K \leqslant I_G(\Theta_i)$ for all $i \in [0, 2]$, we get

$$\begin{aligned}(\Theta_2 \otimes \Theta_0 \otimes \Theta_1)|_{K_1} &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)^a|_{K_1} = ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^a = \eta^a = \eta^x \\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)^x|_{K_1} = (\Theta_0^{t^{-1}a} \otimes \Theta_1^a \otimes \Theta_2^{at})|_{K_1} = (\Theta_0^a \otimes \Theta_1^a \otimes \Theta_2^a)|_{K_1}.\end{aligned}$$

If $\eta^a = \eta^{x^{-1}}$, then by a similar argument as above we obtain

$$(\Theta_2 \otimes \Theta_0 \otimes \Theta_1)|_{K_1} = (\Theta_0^{a^{-1}} \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^{a^{-1}})|_{K_1}.$$

By denoting $\vartheta = \vartheta_0 = \Theta_0|_H$, in each of the three cases, we find that $\vartheta_1 = \vartheta^{a^\varepsilon}$ and $\vartheta_2 = \vartheta^{a^{-\varepsilon}}$ for some $\varepsilon \in [0,2]$. Since $I_G(\vartheta^{a^{\pm1}}) = I_G(\vartheta)^{a^{\pm1}}$, by Lemma 6.3.3, for all $i \in [0,2]$, we get that $I_G(\vartheta_i) = I_G(\vartheta) = K$.

Now we prove the reverse implication. Assume that the given statement is true. Let $\Theta \in \mathrm{Irr}(K)$ be an extension of ϑ from H to K. Notice that $\Theta^{a^{\pm1}}$ is an extension of $\vartheta^{a^{\pm1}}$, since $\Theta^{a^{\pm1}}|_H = (\Theta|_H)^{a^{\pm1}} = \vartheta^{a^{\pm1}}$. Set $\eta = (\Theta \otimes \Theta^{a^\varepsilon} \otimes \Theta^{a^{-\varepsilon}})|_{K_1}$. Then η an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Since $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t}\rangle = K_1$, we have $I_H(\eta) = H_1\langle \bar{t}\rangle$. Now consider,

$$\eta^a = ((\Theta \otimes \Theta^{a^\varepsilon} \otimes \Theta^{a^{-\varepsilon}})|_{K_1})^a = (\Theta \otimes \Theta^{a^\varepsilon} \otimes \Theta^{a^{-\varepsilon}})^a|_{K_1} = (\Theta^{a^{-\varepsilon}} \otimes \Theta \otimes \Theta^{a^\varepsilon})|_{K_1}.$$

If $\varepsilon = 0$ then $\eta^a = \eta$, if $\varepsilon = 1$ then $\eta^a = \eta^{x^{-1}}$ and if $\varepsilon = -1$ then $\eta^a = \eta^x$. Therefore $\eta^a \in \{\eta, \eta^x, \eta^{x^{-1}}\}$ and the result follows from Lemma 4.0.9. □

Proposition 6.3.10. *Let $\vartheta_i \in \mathrm{Irr}(H)$, for $i \in [0,2]$, such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t}\rangle$. Let Θ_i, η and φ be defined as in Observation 6.3.8 above. Let $\varepsilon \in \{1,-1\}$. Then $at^\varepsilon \in I_G(\varphi)$ if and only if there exist $\vartheta \in \mathrm{Irr}(H)$ and $j \in [0,2]$ such that $I_G(\vartheta) = K$ and $\vartheta_j = \vartheta$ and $\vartheta_{j+1} = \vartheta_{j+2} = \vartheta^{a^{-\varepsilon}}$.*

Proof. By Lemma 4.0.9 and (6.11), we have $at^\varepsilon \in I_G(\varphi)$ if and only if $\eta^{at^\varepsilon} \in \{\eta, \eta^x, \eta^{x^{-1}}\}$. Suppose that $at^\varepsilon \in I_G(\varphi)$. Since $K \leqslant I_G(\Theta_i)$ for all $i \in [0,2]$, we get

$$(\Theta_2^{a^\varepsilon} \otimes \Theta_0^{a^{-\varepsilon}} \otimes \Theta_1)|_{K_1} = \eta^{at^\varepsilon} = \begin{cases} (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1} & \text{if } \eta^{at^\varepsilon} = \eta, \\ (\Theta_0^a \otimes \Theta_1^a \otimes \Theta_2^a)|_{K_1} & \text{if } \eta^{at^\varepsilon} = \eta^x, \\ (\Theta_0^{a^{-1}} \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^{a^{-1}})|_{K_1}, & \text{if } \eta^{at^\varepsilon} = \eta^{x^{-1}}. \end{cases}$$

We consider the case when $\varepsilon = 1$. In view of the three possibilities described above, this implies that there exists $\vartheta \in \mathrm{Irr}(H)$ such that

$$\vartheta_0 = \vartheta, \vartheta_1 = \vartheta_2 = \vartheta^{a^{-1}}, \quad \text{or} \quad \vartheta_1 = \vartheta, \vartheta_0 = \vartheta_2 = \vartheta^{a^{-1}}, \quad \text{or} \quad \vartheta_2 = \vartheta, \vartheta_0 = \vartheta_1 = \vartheta^{a^{-1}}.$$

Since $I_G(\vartheta^{a^{\pm1}}) = I_G(\vartheta)^{a^{\pm1}}$, Lemma 6.3.3 implies that $I_G(\vartheta_i) = I_G(\vartheta) = K$ for all $i \in [0,2]$. Analogously, we obtain the result for $\varepsilon = -1$.

To prove the reverse implication, again we consider the case when $\varepsilon = 1$. The case when $\varepsilon = -1$ follows in a same manner. Suppose that there exist $\vartheta \in \mathrm{Irr}(H)$ with $I_G(\vartheta) = K$ and $j \in [0,2]$ such that $\vartheta_j = \vartheta$ and $\vartheta_{j+1} = \vartheta_{j+2} = \vartheta^{a^{-1}}$. Then, ϑ extends to irreducible characters of K. Let $\Theta \in \mathrm{Irr}(K)$ be an extension of ϑ. For the case $j = 0$, set $\eta = (\Theta \otimes \Theta^{a^{-1}} \otimes \Theta^{a^{-1}})|_{K_1}$, where $K_1 = H_1\langle \bar{t}\rangle$. Since $\Theta^{a^{-1}}|_H = (\Theta|_H)^{a^{-1}} = \vartheta^{a^{-1}}$, the character

η is indeed an extension of $\vartheta \otimes \vartheta^{a^{-1}} \otimes \vartheta^{a^{-1}}$. Then

$$\begin{aligned}\eta^{at} &= ((\Theta \otimes \Theta^{a^{-1}} \otimes \Theta^{a^{-1}})|_{K_1})^{at} = (\Theta \otimes \Theta^{a^{-1}} \otimes \Theta^{a^{-1}})^{at}|_{K_1} = (\Theta^{a^{-1}} \otimes \Theta \otimes \Theta^{a^{-1}})^{t}|_{K_1} \\ &= (\Theta \otimes \Theta^{a^{-1}} \otimes \Theta^{a^{-1}})|_{K_1} = \eta.\end{aligned}$$

Similarly, for $j = 1$, we get $\eta^{at} = \eta^{x}$ and for $j = 2$, we have $\eta^{at} = \eta^{x^{-1}}$. Therefore, by Lemma 4.0.9, we obtain that $at \in I_G(\varphi)$, where φ is the character induced from η. $\square$

Proposition 6.3.11. *Let $\vartheta_i \in \mathrm{Irr}(H)$, for $i \in [0, 2]$, such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t}\rangle$. Let Θ_i, η and φ be defined as in Observation 6.3.8 above. Then $t \in I_G(\varphi)$ if and only if there exist $\vartheta \in \mathrm{Irr}(H)$ with $I_G(\vartheta) = K$ and $j \in [0, 2]$ such that $\vartheta_j = \vartheta$, and $\vartheta_{j+1} = \vartheta_{j+2}$ with $I_G(\vartheta_{j+1}) = G$.*

Proof. Suppose that $t \in I_G(\varphi)$. Then, by Lemma 4.0.9 and (6.11), $\eta^t \in \{\eta, \eta^x, \eta^{x^{-1}}\}$, or equivalently, $\eta \in \{\eta^{t^{-1}}, \eta^{xt^{-1}}, \eta^{x^{-1}t^{-1}}\}$. By a similar computation as in Proposition 6.3.9 we get $t \in I_G(\varphi)$ if and only if one of the following cases occur:

$$(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1} = \begin{cases} (\Theta_0^{a^{-1}} \otimes \Theta_1^{a} \otimes \Theta_2)|_{K_1} \text{ if } \eta = \eta^{t^{-1}}, \text{ or,} \\ (\Theta_0 \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^{a})|_{K_1} \text{ if } \eta = \eta^{xt^{-1}}, \text{ or,} \\ (\Theta_0^{a} \otimes \Theta_1 \otimes \Theta_2^{a^{-1}})|_{K_1} \text{ if } \eta = \eta^{x^{-1}t^{-1}}, \end{cases}$$

where $K_1 = H_1\langle \bar{t}\rangle$. It is immediate that there exists $j \in [0, 2]$ such that $I_G(\vartheta_{j+1}) = I_G(\vartheta_{j+2}) = G$, and therefore, by Lemma 6.3.3, $I_G(\vartheta_j) = K$. Thanks to Corollary 5.3.12, it follows that ϑ_{j+1} and ϑ_{j+2} are linear. We identify the characters ϑ_{j+1}, ϑ_{j+2} with irreducible characters of $H/[H, G]$ and the characters Θ_{j+1}, Θ_{j+2} with irreducible characters of $K/[H, G]$. In view of Lemma 4.0.8, choose $\ell = (h_0, h_1, h_2)\bar{t}^{\varepsilon}$, where $h_i \in H$, $i \in [0, 2]$ and $\varepsilon \in [0, 2]\backslash\{0\}$, such that $(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) \neq 0$. We obtain that if $\eta = \eta^{t^{-1}}$ then $I_G(\vartheta_0) = I_G(\vartheta_1) = G$ and, by Lemma 6.3.6, the following equality holds:

$$\begin{aligned}(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) = \eta(\ell) = \eta^{t^{-1}}(\ell) = (\Theta_0^{a^{-1}} \otimes \Theta_1^{a} \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\lambda_0(t^{\varepsilon})\lambda_1(t^{\varepsilon}) \\ = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a, t])^{-\varepsilon}\vartheta_1([a, t])^{\varepsilon}\end{aligned}$$

where $\lambda_0, \lambda_1 \in \mathrm{Irr}(K/H)$. Therefore, $\vartheta_0 = \vartheta_1$. By following a similar computation, we get that if $\eta = \eta^{xt^{-1}}$ then $\vartheta_1 = \vartheta_2$ with $I_G(\vartheta_1) = G$, and if $\eta = \eta^{x^{-1}t^{-1}}$ then $\vartheta_0 = \vartheta_2$ with $I_G(\vartheta_0) = G$. The converse follows by reversing the arguments above. $\square$

6.3.2.2 Case 2: $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_j\bar{t}\rangle$ for $j \in [0,2]$.

Observation 6.3.12. We fix an element $j \in [0,2]$. It follows from Lemma 6.3.4 that, $H\langle at^{i-j}\rangle \leqslant I_G(\vartheta_i)$ for all $i \in [0,2]$. Set

$$M_j = H\langle at^{-j}\rangle \times H\langle at^{1-j}\rangle \times H\langle at^{2-j}\rangle.$$

Let Θ_i denote an extension of ϑ_i to $H\langle at^{i-j}\rangle$. Then $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ is an extension of the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ from H_1 to M_j. Since $H_1\langle x_j\bar{t}\rangle = \psi^{-1}(M_j) \cap H$, we identify $H_1\langle x_j\bar{t}\rangle$ with a subgroup of M_j. Let η be the restriction of $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ to $H_1\langle x_j\bar{t}\rangle$. Then η is an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ from H_1 to $H_1\langle x_j\bar{t}\rangle$ (Theorem 4.0.3). Notice that the character η does not extend towards H. Denote by $\varphi \in \mathrm{Irr}(H)$ the character of H induced from η. Observe further that $H_1\langle x_j\bar{t}\rangle$ is normal in H. Moreover, $H = H_1\langle x_j\bar{t}, \bar{t}\rangle$ and $H/H_1\langle x_j\bar{t}\rangle \cong C_3$ (cf. Corollary 6.1.3). By Clifford's theorem the restriction of φ to $H_1\langle x_j\bar{t}\rangle$ has the following form

$$\varphi|_{H_1\langle x_j\bar{t}\rangle} = \eta + \eta^{\bar{t}} + \eta^{\bar{t}^{-1}}. \tag{6.12}$$

Furthermore,

$$\varphi(1) = [H : H_1\langle x_j\bar{t}\rangle] \cdot \eta(1) = 3 \cdot (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)(1) = 3\vartheta_0(1)\vartheta_1(1)\vartheta_2(1).$$

Proposition 6.3.13. *Let $j \in [0,2]$ and let $\vartheta_i \in \mathrm{Irr}(H)$, for $i \in [0,2]$, such that the inertia group $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$ of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is $H_1\langle x_j\bar{t}\rangle$. Let φ be defined as in Observation 6.3.12. Then none of the elements a, at, at^{-1} belongs to $I_G(\varphi)$.*

Proof. Let $\varepsilon \in [0,2]$. By Theorem 4.0.3 and (6.12), we obtain

$$\varphi|_{H_1} = (\varphi|_{H_1\langle x_j\bar{t}\rangle})|_{H_1} = (\eta \oplus \eta^{\bar{t}} \oplus \eta^{\bar{t}^{-1}})|_{H_1} = \sum_{\delta \in [0,2]} \vartheta_0^{t^\delta} \otimes \vartheta_1^{t^\delta} \otimes \vartheta_2^{t^\delta},$$

where η is defined as in Observation 6.3.12. Now assume that $at^\varepsilon \in I_G(\varphi)$, i.e., $\varphi^{at^\varepsilon} = \varphi$, for some $\varepsilon \in [0,2]$. This implies

$$\sum_{\delta \in [0,2]} \vartheta_0^{t^\delta} \otimes \vartheta_1^{t^\delta} \otimes \vartheta_2^{t^\delta} = \varphi|_{H_1} = \varphi^{at^\varepsilon}|_{H_1} = (\varphi|_{H_1})^{at^\varepsilon} = \sum_{\delta \in [0,2]} \vartheta_2^{a^\varepsilon t^\delta} \otimes \vartheta_0^{a^{-\varepsilon} t^\delta} \otimes \vartheta_1^{t^{\varepsilon+\delta}}.$$

An easy calculation yields that $\vartheta_1 = \vartheta_0^{a^{-\varepsilon}} = \vartheta_2$. This implies that $I_G(\vartheta_i) = G$ for all $i \in [0,2]$, since $\langle I_G(\vartheta_{j_1}) \cup I_G(\vartheta_{j_2})\rangle = G$ for $j_1 \neq j_2$, where $j_1, j_2 \in [0,2]$, which is a contradiction to Lemma 6.3.4. □

Proposition 6.3.14. *Let $j \in [0,2]$ and let $\vartheta_i \in \mathrm{Irr}(H)$, for $i \in [0,2]$, such that the inertia group $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$ of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is $H_1\langle x_j\bar{t}\rangle$. Let Θ_i, η and φ be defined as in Observation 6.3.12. Then $t \in I_G(\varphi)$ if and only if one of the following occurs:*

1. $I_G(\vartheta_j) = H\langle a\rangle$ *and* $I_G(\vartheta_{j+1}) = I_G(\vartheta_{j+2}) = G$ *with* $\vartheta_{j+2} = \vartheta_{j+1}^{-1}$*;*

2. $\vartheta_j = 1_H$*, hence* $I_G(\vartheta_j) = G$*, and moreover*

(i) $I_G(\vartheta_{j+1}) = H\langle at\rangle$ *and* $I_G(\vartheta_{j+2}) = G$*, or,*

(ii) $I_G(\vartheta_{j+1}) = G$ *and* $I_G(\vartheta_{j+2}) = H\langle at^{-1}\rangle$*, or,*

(iii) $I_G(\vartheta_{j+1}) = H\langle at\rangle$ *and* $I_G(\vartheta_{j+2}) = H\langle at^{-1}\rangle$*.*

Proof. We shall prove the statement for $j = 0$, the other cases follow in the same way. Assume that $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_0\bar{t}\rangle$. By Lemma 4.0.9 and (6.12), the element $t \in I_G(\varphi)$ if and only if $\eta^t \in \{\eta, \eta^{\bar{t}}, \eta^{\bar{t}^{-1}}\}$. Since $H\langle a\rangle \leqslant I_G(\Theta_0)$, $H\langle at\rangle \leqslant I_G(\Theta_1)$ and $H\langle at^{-1}\rangle \leqslant I_G(\Theta_2)$ (cf. Observation 6.3.12), we get that $t \in I_G(\varphi)$ if and only if one of the following cases occur:

$$(\Theta_0 \otimes \Theta_1^t \otimes \Theta_2^t)|_{H_1\langle x_0\bar{t}\rangle} = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)|_{H_1\langle x_0\bar{t}\rangle} = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)^t|_{H_1\langle x_0\bar{t}\rangle} = \eta^t$$
$$= \begin{cases} (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{H_1\langle x_0\bar{t}\rangle} & \text{if } \eta^t = \eta, \text{ or,} \\ (\Theta_0^t \otimes \Theta_1^t \otimes \Theta_2^t)|_{H_1\langle x_0\bar{t}\rangle} & \text{if } \eta^t = \eta^{\bar{t}}, \text{ or,} \\ (\Theta_0^{t^{-1}} \otimes \Theta_1^{t^{-1}} \otimes \Theta_2^{t^{-1}})|_{H_1\langle x_0\bar{t}\rangle} & \text{if } \eta^t = \eta^{\bar{t}^{-1}}. \end{cases}$$

We split the proof into three cases.

Case 1: Suppose that $\eta^t = \eta$. It is then straightforward that $I_G(\vartheta_1) = G = I_G(\vartheta_2)$ and hence, by Lemma 6.3.4, $I_G(\vartheta_0) = H\langle a\rangle$. Thanks to Corollary 5.3.12, the characters ϑ_1 and ϑ_2 are linear. We identify the characters ϑ_1, ϑ_2 with irreducible characters of $H/[H,G]$ and the characters Θ_1, Θ_2 with the irreducible characters of $H\langle at\rangle/[H,G]$ and $H\langle at^{-1}\rangle/[H,G]$, respectively. By Lemma 4.0.8, there exists $\ell = (h_0a^{\varepsilon}, h_1(at)^{\varepsilon}, h_2(at^{-1})^{\varepsilon})$ for some $h_i \in H$, $i \in [0,2]$ and $\varepsilon \in [0,2]\backslash\{0\}$ such that $(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) = \eta(\ell) \neq 0$. From Lemma 6.3.7, we obtain

$$\begin{aligned}(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) = \eta(\ell) = \eta^t(\ell) &= (\Theta_0 \otimes \Theta_1^t \otimes \Theta_2^t)(\ell) \\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\lambda_1((at)^{\varepsilon})\lambda_2(at^{-1})^{\varepsilon}) \\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_1([a,t])^{-\varepsilon}\vartheta_2([a,t])^{-\varepsilon}\end{aligned}$$

where $\lambda_1 \in \mathrm{Irr}(H\langle at\rangle/H)$ and $\lambda_2 \in \mathrm{Irr}(H\langle at^{-1}\rangle/H)$. Since $[a,t]$ generates H modulo $[H,G]$, we obtain $\vartheta_1 = \vartheta_2^{-1}$.

Case 2: Suppose that $\eta^t = \eta^{\bar{t}}$. We have $(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{H_1\langle x_0\bar{t}\rangle} = (\Theta_0^t \otimes \Theta_1 \otimes \Theta_2)|_{H_1\langle x_0\bar{t}\rangle}$, implying that $I_G(\vartheta_0) = I_G(\Theta_0) = G$, so that ϑ_0 is linear and extends all the way to G. Hence we conclude that $\vartheta_0 = 1_H$. Since $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H\langle x_0\bar{t}\rangle$, the inertia group of ϑ_1 and ϑ_2 can take any of the following values:

(i) $I_G(\vartheta_1) = H\langle at\rangle$ and $I_G(\vartheta_2) = G$, or,

(ii) $I_G(\vartheta_1) = G$ and $I_G(\vartheta_2) = H\langle at^{-1}\rangle$, or,

(iii) $I_G(\vartheta_1) = H\langle at\rangle$ and $I_G(\vartheta_2) = H\langle at^{-1}\rangle$,

and all of the above cases are legitimate by Lemma 6.3.4.

Case 3: Suppose that $\eta^t = \eta^{\bar{t}^{-1}}$. We have

$$(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{H_1\langle x_0\bar{t}\rangle} = (\Theta_0^{t^{-1}} \otimes \Theta_1^t \otimes \Theta_2^t)|_{H_1\langle x_0\bar{t}\rangle}.$$

This implies $I_G(\vartheta_i) = I_G(\Theta_i) = G$ for all $i \in [0,2]$. This is a contradiction to Lemma 6.3.4, hence this case does not occur.

By reversing the above arguments we get the converse, completing the proof. □

6.3.2.3 Case 3: $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$.

Observation 6.3.15. Denote by φ the character of H induced from $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Then by Theorem 4.0.5, φ is irreducible, and hence by Clifford's theorem

$$\varphi|_{H_1} = \sum_{i,j\in[0,2]} (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x^i\bar{t}^j},$$

since $H = H_1\langle x, \bar{t}\rangle$ (Corollary 6.1.3). Furthermore,

$$\varphi(1) = [H : H_1] \cdot (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)(1) = 3^2\vartheta_0(1)\vartheta_1(1)\vartheta_2(1).$$

Also observe from Lemma 4.0.9 that, for each $g \in G$, we have $g \in I_G(\varphi)$ if and only if $(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^g \in \{(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x^i\bar{t}^j} \mid i,j \in [0,2]\}$, where

$$x^i\bar{t}^j = (t^{-1}a, a, at)^i(t,t,t)^j \equiv_{H_1} (a^it^{-i+j}, a^it^j, a^it^{i+j}). \tag{6.13}$$

Proposition 6.3.16. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ such that $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$. Let φ be defined as in Observation 6.3.15 above. Then none of the elements at, at^{-1} belongs to the inertia group $I_G(\varphi)$.*

Proof. Suppose to the contrary that $at \in I_G(\varphi)$. By (6.13), there exist $i,j \in [0,2]$ such that

$$\vartheta_2^a \otimes \vartheta_0^{a^{-1}} \otimes \vartheta_1^t = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{at} = \vartheta_0^{a^it^{-i+j}} \otimes \vartheta_1^{a^it^j} \otimes \vartheta_2^{a^it^{i+j}}.$$

An easy calculation shows that $\vartheta_0 = \vartheta_0^t$, $\vartheta_1 = \vartheta_0^{a^{-1-i}}$ and $\vartheta_2 = \vartheta_0^{a^{-1+i}}$, whence $H\langle t\rangle \leqslant I_G(\vartheta_i)$ for all $i \in [0,2]$. This is a contradiction to Lemma 6.3.5. Analogously, one sees that $at^{-1} \notin I_G(\varphi)$. □

Proposition 6.3.17. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ such that $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$. Let φ be defined as in Observation 6.3.15 above. Then $a \in I_G(\varphi)$ if and only if there exist $\vartheta \in \mathrm{Irr}(H)$ and $i, j \in [0, 2]$ such that $I_G(\vartheta) \in \{H, H\langle a\rangle, H\langle at\rangle, H\langle at^{-1}\rangle\}$ and $\vartheta_0 = \vartheta$, $\vartheta_1 = \vartheta^{a^{-i}t^{-j}}$ and $\vartheta_2 = \vartheta^{a^i t^{-i+j}}$.*

Proof. By Lemma 4.0.9 and (6.13), the element $a \in I_G(\varphi)$ if and only if there exist $i, j \in [0, 2]$ such that

$$\vartheta_2 \otimes \vartheta_0 \otimes \vartheta_1 = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^a = \vartheta_0^{a^i t^{-i+j}} \otimes \vartheta_1^{a^i t^j} \otimes \vartheta_2^{a^i t^{i+j}}. \tag{6.14}$$

Now set $\vartheta = \vartheta_0$. Then (6.14) holds if and only if $\vartheta_1 = \vartheta^{a^{-i}t^{-j}}$ and $\vartheta_2 = \vartheta^{a^i t^{-i+j}}$, implying that $I_G(\vartheta_i) = I_G(\vartheta)$ for all $i \in [0, 2]$. Therefore, by Lemma 6.3.5, $I_G(\vartheta) \notin \{G, H\langle t\rangle\}$. Therefore, by Lemma 6.3.5, $I_G(\vartheta)$ can take any values from the set $\{H, H\langle a\rangle, H\langle at\rangle, H\langle at^{-1}\rangle\}$.

The following table indicates the possible choices of ϑ_1 and ϑ_2 depending on the values of i and j: in each box the first and second entries represent ϑ_1 and ϑ_2, respectively.

$i \backslash j$	0	1	2
0	ϑ ϑ	$\vartheta^{t^{-1}}$ ϑ^{t}	ϑ^{t} $\vartheta^{t^{-1}}$
1	$\vartheta^{a^{-1}}$ $\vartheta^{at^{-1}}$	$\vartheta^{a^{-1}t^{-1}}$ ϑ^{a}	$\vartheta^{a^{-1}t}$ ϑ^{at}
2	ϑ^{a} $\vartheta^{a^{-1}t}$	$\vartheta^{at^{-1}}$ $\vartheta^{a^{-1}t^{-1}}$	ϑ^{at} $\vartheta^{a^{-1}}$

Table 6.1: Values of ϑ_1 and ϑ_2.

Now, to prove the converse, suppose that $\vartheta \in \mathrm{Irr}(H)$ with $I_G(\vartheta) \in \{H, H\langle a\rangle, H\langle at\rangle, H\langle at^{-1}\rangle\}$. Set $\vartheta_0 = \vartheta$, and, for a fixed $(i, j) \in [0, 2] \times [0, 2]$, set ϑ_1 and ϑ_2 as the first and second entries from the ith row and jth column of Table 6.1. An easy computation yields that

$$(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^a = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x^i \bar{t}^j},$$

which completes the proof. □

Proposition 6.3.18. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ such that $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$. Let φ be defined as in Observation 6.3.15 above. Then $t \in I_G(\varphi)$ if and only if there exist ϑ_0, ϑ_1, $\vartheta_2 \in$*

$\mathrm{Irr}(H)$ *and* $i, j \in [0, 2]$ *such that* $\vartheta_0^{a^{-1+i}t^{-i+j}} = \vartheta_0$, $\vartheta_1^{a^{1+i}t^{j}} = \vartheta_1$ *and* $\vartheta_2^{a^{i}t^{-1+i+j}} = \vartheta_2$ *with the inertia groups given by:*

1. *if* $(i, j) \in \{(0,0), (1,0), (2,2)\}$

    (i) $I_G(\vartheta_i) = H\langle a\rangle$, $I_G(\vartheta_{i+1}) = H\langle a\rangle$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$, *or,*

    (ii) $I_G(\vartheta_i) = H\langle a\rangle$, $I_G(\vartheta_{i+1}) = H\langle a\rangle$, $I_G(\vartheta_{i+2}) = G$, *or,*

    (iii) $I_G(\vartheta_i) = H\langle a\rangle$, $I_G(\vartheta_{i+1}) = G$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$, *or,*

    (iv) $I_G(\vartheta_i) = G$, $I_G(\vartheta_{i+1}) = H\langle a\rangle$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$.

2. *if* $(i, j) \in \{(0,2), (1,2), (2,1)\}$

    (i) $I_G(\vartheta_i) = H\langle at\rangle$, $I_G(\vartheta_{i+1}) = H\langle at^{-1}\rangle$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$, *or,*

    (ii) $I_G(\vartheta_i) = H\langle at\rangle$, $I_G(\vartheta_{i+1}) = G$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$, *or,*

    (iii) $I_G(\vartheta_i) = G$, $I_G(\vartheta_{i+1}) = H\langle at^{-1}\rangle$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$.

3. *if* $(i, j) \in \{(0,1), (1,1), (2,0)\}$

    (i) $I_G(\vartheta_i) = H\langle at^{-1}\rangle$, $I_G(\vartheta_{i+1}) = H\langle at\rangle$, $I_G(\vartheta_{i+2}) \in \{H, H\langle a\rangle, H\langle at\rangle, H\langle at^{-1}\rangle, H\langle t\rangle, G\}$, *or,*

    (ii) $I_G(\vartheta_i) = G$, $I_G(\vartheta_{i+1}) = H\langle at\rangle$, $I_G(\vartheta_{i+2}) \in \{H, H\langle a\rangle, H\langle at\rangle, H\langle t\rangle\}$, *or,*

    (iii) $I_G(\vartheta_i) = H\langle at^{-1}\rangle$, $I_G(\vartheta_{i+1}) = G$, $I_G(\vartheta_{i+2}) \in \{H, H\langle a\rangle, H\langle at^{-1}\rangle, H\langle t\rangle\}$, *or,*

    (iv) $I_G(\vartheta_i) = G$, $I_G(\vartheta_{i+1}) = G, I_G(\vartheta_{i+2}) = H$.

Proof. By Lemma 4.0.9 and (6.13), the element t belongs to $I_G(\varphi)$ if and only if there exist $i, j \in [0, 2]$ such that

$$\vartheta_0^a \otimes \vartheta_1^{a^{-1}} \otimes \vartheta_2^t = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^t = \vartheta_0^{a^{i}t^{-i+j}} \otimes \vartheta_1^{a^{i}t^{j}} \otimes \vartheta_2^{a^{i}t^{i+j}}. \tag{6.15}$$

The equality (6.15) holds if and only if $\vartheta_0^{a^{-1+i}t^{-i+j}} = \vartheta_0$, $\vartheta_1^{a^{1+i}t^{j}} = \vartheta_1$ and $\vartheta_2^{a^{i}t^{-1+i+j}} = \vartheta_2$. Observe that the Table 6.2 encodes the information about ϑ_0, ϑ_1 and ϑ_2 based on the values of i and j.

Now, suppose that $(i, j) \in \{(0,0), (1,0), (2,2)\}$. Then, observe that $H\langle a\rangle \leqslant I_G(\vartheta_i)$, $H\langle a\rangle \leqslant I_G(\vartheta_{i+1})$ and $H\langle t\rangle \leqslant I_G(\vartheta_{i+2})$. Therefore, the permitted cases of inertia groups $I_G(\vartheta_i)$ such that $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$ are the following;

(i) $I_G(\vartheta_i) = H\langle a\rangle$, $I_G(\vartheta_{i+1}) = H\langle a\rangle$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$, or,

(ii) $I_G(\vartheta_i) = H\langle a\rangle$, $I_G(\vartheta_{i+1}) = H\langle a\rangle$, $I_G(\vartheta_{i+2}) = G$, or,

$i \backslash j$	0	1	2
0	$\vartheta_0 = \vartheta_0^{a^{-1}}$ $\vartheta_1 = \vartheta_1^{a}$ $\vartheta_2 = \vartheta_2^{t^{-1}}$	$\vartheta_0 = \vartheta_0^{a^{-1}t}$ $\vartheta_1 = \vartheta_1^{at}$ $\vartheta_2 = \vartheta_2$	$\vartheta_0 = \vartheta_0^{a^{-1}t^{-1}}$ $\vartheta_1 = \vartheta_1^{at^{-1}}$ $\vartheta_2 = \vartheta_2^{t}$
1	$\vartheta_0 = \vartheta_0^{t^{-1}}$ $\vartheta_1 = \vartheta_1^{a^{-1}}$ $\vartheta_2 = \vartheta_2^{a}$	$\vartheta_0 = \vartheta_0$ $\vartheta_1 = \vartheta_1^{a^{-1}t}$ $\vartheta_2 = \vartheta_2^{at}$	$\vartheta_0 = \vartheta_0^{t}$ $\vartheta_1 = \vartheta_1^{a^{-1}t^{-1}}$ $\vartheta_2 = \vartheta_2^{at^{-1}}$
2	$\vartheta_0 = \vartheta_0^{at}$ $\vartheta_1 = \vartheta_1$ $\vartheta_2 = \vartheta_2^{a^{-1}t}$	$\vartheta_0 = \vartheta_0^{at^{-1}}$ $\vartheta_1 = \vartheta_1^{t}$ $\vartheta_2 = \vartheta_2^{a^{-1}t^{-1}}$	$\vartheta_0 = \vartheta_0^{a}$ $\vartheta_1 = \vartheta_1^{t^{-1}}$ $\vartheta_2 = \vartheta_2^{a^{-1}}$

Table 6.2: Values of ϑ_0, ϑ_1, ϑ_2.

(iii) $I_G(\vartheta_i) = H\langle a\rangle$, $I_G(\vartheta_{i+1}) = G$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$, or,

(iv) $I_G(\vartheta_i) = G$, $I_G(\vartheta_{i+1}) = H\langle a\rangle$, $I_G(\vartheta_{i+2}) = H\langle t\rangle$.

Observe from Lemma 6.3.5 that all of the above cases are legitimate and that this is the list of all possible cases. Analogously, from Lemma 6.3.5 we obtain the possible cases of inertia groups for the case $(i, j) \in \{(0, 2), (1, 2), (2, 1)\}$ and $(i, j) \in \{(0, 1), (1, 1), (2, 0)\}$.

Now, fix $(i, j) \in [0, 2] \times [0, 2]$. Set ϑ_0, ϑ_1 and ϑ_2 as given in the ith row and jth column of Table 6.2. Further, choose the inertia groups of ϑ_0, ϑ_1 and ϑ_2 from the corresponding list. Then it is easy to see that

$$(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^t = (\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)^{x^i \bar{t}^j},$$

which completes the proof. □

6.3.2.4 Case 4: $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$.

Observation 6.3.19. By Lemma 6.3.2 it follows that $I_G(\vartheta_i) = G$ for all $i \in [0, 2]$. Thanks to Corollary 5.3.12, the characters ϑ_i are linear for all $i \in [0, 2]$. We consider the following four cases;

(i) The characters ϑ_i extend to irreducible characters of G for all $i \in [0, 2]$.

(ii) There exist $i, j \in [0,2]$ with $i \neq j$ such that ϑ_i and ϑ_j extend to irreducible characters of G, while ϑ_k does not extend for $k \in [0,2]\backslash\{i,j\}$.

(iii) There exists exactly one $i \in [0,2]$ such that ϑ_i extends to irreducible characters of G.

(iv) None of the characters ϑ_i extend to irreducible characters of G.

Observe that in each of the above cases the characters ϑ_i extend to irreducible characters of $K = H\langle t\rangle$. Let $\Theta_i \in \mathrm{Irr}(K)$ be an extension of ϑ_i. Observe that Θ_i is linear and hence a homomorphism from K to $\mathbb{C}^\times$. We set $K_1 = H_1\langle \bar{t}\rangle$ and $\eta = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1}$. Then η is an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ to K_1; cf. Observation 6.3.8. In the sequel, we identify the characters ϑ_i with irreducible characters of $H/[H,G]$ and Θ_i with irreducible characters of $K/[H,G]$.

Proposition 6.3.20. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that ϑ_i extends to irreducible characters of G for all $i \in [0,2]$. Then the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ extends to irreducible characters of H. Furthermore, if $\varphi \in \mathrm{Irr}(H)$ is an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ then $I_G(\varphi) = G$ if and only if $\varphi|_{K_1} = 1_{K_1}$, otherwise $I_G(\varphi) = K$.*

Proof. Observe first from Lemma 6.3.6 that $\vartheta_i = 1_H$ for all $i \in [0,2]$. Therefore, the character

$$\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 = 1_H \otimes 1_H \otimes 1_H = 1_{H_1},$$

and it admits an extension towards H as $[H,H] \leqslant H_1 = \ker(1_{H_1})$; cf. Theorem 2.4.25. Let $\varphi \in \mathrm{Irr}(H)$ be an extension of 1_{H_1}. We identify φ with an irreducible character of H/H_1. Since $H = H_1\langle x, \bar{t}\rangle$ (Corollary 6.1.3), the character φ is G-invariant if and only if $\varphi^g(x) = \varphi(x)$ and $\varphi^g(\bar{t}) = \varphi(\bar{t})$ for $g \in \{a,t\}$. Observe that $\bar{t}^a = \bar{t}$, $\bar{t}^t \equiv_{H_1} \bar{t}$, $x^t \equiv_{H_1} x$ and

$$x^{a^{-1}} = x[x,a^{-1}] = x(t^{-1}a,a,at)^{-1}(t^{-1}a,a,at)^{a^{-1}} = x(a^{-1}t,a^{-1},t^{-1}a^{-1})(a,at,t^{-1}a) \equiv_{H_1} x\bar{t}.$$

It is then immediate that $\varphi^t(x) = \varphi(x)$ and $\varphi^t(\bar{t}) = \varphi(\bar{t})$, whence $t \in I_G(\varphi)$. Also, $a \in I_G(\varphi)$ if and only if

$$\varphi(x) = \varphi^a(x) = \varphi(x^{a^{-1}}) = \varphi(x\bar{t}) = \varphi(x)\varphi(\bar{t}) = \varphi(x)\varphi|_{K_1}(\bar{t}),$$

where the last but one equality follows because φ is linear. Hence, $I_G(\varphi) = G$ if and only if $\varphi|_{K_1} = 1_{K_1}$. □

Corollary 6.3.21. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that, for all $i \in [0,2]$, the character ϑ_i extends to irreducible characters of G. Then the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is the trivial character of H_1 and it extends to irreducible characters of H. In* *this way, we get 3 linear characters of H that are G-invariant and 6 linear characters that are K-invariant.*

Proposition 6.3.22. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that there exist $i, j \in [0, 2]$ with $i \neq j$ such that ϑ_i and ϑ_j extend to irreducible characters of G and ϑ_k does not admit an extension towards G for $k \in [0, 2]\backslash\{i, j\}$. Then the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ admits an extension towards K_1. Let η be an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ towards K_1. Then $I_H(\eta) = K_1$ and η does not extend to irreducible characters of H. Denote by $\varphi \in \mathrm{Irr}(H)$ the character induced from η. Then $I_G(\varphi) = K$ and $\varphi|_{H_1} = 3(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$.*

Proof. Observe from Lemma 6.3.6 that $\vartheta_i = \vartheta_j = 1_H$. Since $I_G(\vartheta_{i'}) = G$ for $i' \in [0, 2]$, the characters $\vartheta_{i'}$ extend to irreducible characters of K. Set $\Theta_{i'}$ and η as defined in Observation 6.3.19. Since ϑ_i and ϑ_j extend to irreducible characters of G and ϑ_k does not admit an extension towards G, we get $I_G(\Theta_i) = G = I_G(\Theta_j)$ and $I_G(\Theta_k) = K$. Therefore, from Lemma 6.3.6, we get that $\Theta_k^a = \Theta_k\lambda$, where $1 \neq \lambda \in \mathrm{Irr}(K/H)$. This implies

$$\eta^x = ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^x = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)^x|_{K_1} = (\Theta_0^a \otimes \Theta_1^a \otimes \Theta_2^a)|_{K_1} \neq \eta.$$

Therefore $I_H(\eta) = K_1$ and η does not admit an extension towards H. Denote by $\varphi \in \mathrm{Irr}(H)$ the character induced from of η. Then

$$\varphi(1) = \eta^H(1) = [H : K_1] \cdot \eta(1) = 3(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)(1) = 3\vartheta_0(1)\vartheta_1(1)\vartheta_2(1) = 3,$$

where the last equality follows because ϑ_0, ϑ_1 and ϑ_2 are linear by Corollary 5.3.12. We claim that $t \in I_G(\varphi)$. Then $a \notin I_G(\varphi)$, since φ is non-linear; cf. Corollary 5.3.12. This proves the result. It remains to prove that $t \in I_G(\varphi)$. By Lemma 4.0.9, this happens if and only if $\eta^t \in \{\eta, \eta^x, \eta^{x^{-1}}\}$. We split the proof into three cases.

Case 1: Let $\{i, j\} = \{0, 1\}$. Then $I_G(\Theta_0) = G = I_G(\Theta_1)$ and $I_G(\Theta_2) = K$. We get

$$\eta^t = ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^t = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)|_{K_1} = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1} = \eta.$$

Case 2: Let $\{i, j\} = \{0, 2\}$. Then $I_G(\Theta_0) = G = I_G(\Theta_2)$ and $I_G(\Theta_1) = K$.

$$\begin{aligned}\eta^t &= ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^t = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)|_{K_1} = (\Theta_0 \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)|_{K_1} \\ &= (\Theta_0^{a^{-1}} \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^{a^{-1}})|_{K_1} = ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^{x^{-1}} = \eta^{x^{-1}}.\end{aligned}$$

Case 3: Let $\{i, j\} = \{1, 2\}$. Then $I_G(\Theta_1) = G = I_G(\Theta_2)$ and $I_G(\Theta_0) = K$.

$$\begin{aligned}\eta^t &= ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^t = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)|_{K_1} = (\Theta_0^a \otimes \Theta_1 \otimes \Theta_2)|_{K_1} \\ &= (\Theta_0^a \otimes \Theta_1^a \otimes \Theta_2^a)|_{K_1} = ((\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1})^x = \eta^x.\end{aligned}$$

Finally,

$$\varphi|_{H_1} = (\varphi|_{K_1})|_{H_1} = \eta|_{H_1} + \eta^x|_{H_1} + \eta^{x^{-1}}|_{H_1} = 3(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2). \qquad \square$$

Corollary 6.3.23. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that there exist $i, j \in [0, 2]$ with $i \neq j$ such that ϑ_i and ϑ_j extend to irreducible characters of G and ϑ_k does not admit an extension towards G for $k \in [0, 2]\backslash\{i, j\}$. Then $\vartheta_i = \vartheta_j = 1_H$ and ϑ_k is a non-trivial character. In this way, every choice of $\{i, j\}$ and ϑ_k yields a unique $\varphi \in \mathrm{Irr}(H)$ of degree 3, such that $\langle \varphi|_{H_1}, \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \rangle_{H_1} \neq 0$ with $I_G(\varphi) = K$. In total, we get 6 irreducible characters of H of the described form.*

Proof. For a fixed pair $\{i, j\}$, we have $\vartheta_i = \vartheta_j = 1_H$. Furthermore, ϑ_k is a non-trivial linear character. If otherwise ϑ_k is linear, by Lemma 6.3.6, ϑ_k extends towards G. Since $H/[H, G] \cong C_3$ (Theorem 2.4.19(iii)), there are two possible choices for ϑ_k and each of these choices is legitimate. Therefore, for a fixed pair $\{i, j\}$ we get 2 irreducible characters φ of the described form. Now, since there are three different ways to fix an unordered pair $\{i, j\}$, in total, we obtain 6 irreducible characters of H of the described form. As $\varphi|_{H_1} = 3(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$ determines $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ uniquely, there are no overlaps. $\square$

Lemma 6.3.24. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that there exists exactly one $i \in [0, 2]$ such that ϑ_i extends to irreducible characters of G. Then $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ admits an extension towards K_1. Let $\eta \in \mathrm{Irr}(K_1)$ be an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Then η is H invariant if and only if $\vartheta_j \neq \vartheta_{j'}$ for $j, j' \in [0, 2]\backslash\{i\}$ with $j \neq j'$.*

Proof. Since $I_G(\vartheta_{i'}) = G$ for all $i' \in [0, 2]$, the characters $\vartheta_{i'}$ admit extensions to irreducible characters of K. Set $\Theta_{i'}$ and η as defined in Observation 6.3.19. From Lemma 6.3.6, we get $\vartheta_i = 1_H$ and $I_G(\Theta_i) = G$. Write $[0, 2]\backslash\{i\} = \{j, j'\}$. Then $I_G(\Theta_j) = K = I_G(\Theta_{j'})$. Therefore, again from Lemma 6.3.6 it follows that there exist non-trivial λ_j, $\lambda_{j'} \in \mathrm{Irr}(K/H)\backslash\{1\}$ such that $\Theta_j^a = \Theta_j \lambda_j$ and $\Theta_{j'}^a = \Theta_{j'} \lambda_{j'}$. Since $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ and hence $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ are linear, $\Theta_0 \otimes \Theta_1 \otimes \Theta_2(\ell) \neq 0$ for all $\ell \in K_1$. Let $\ell = (h_0 t^\varepsilon, h_1 t^\varepsilon, h_2 t^\varepsilon) \in K_1$, where $h_0, h_1, h_2 \in H$ and $\varepsilon \in [0, 2]\backslash\{0\}$. We consider the case when $j = 1$ and $j' = 2$; the other cases are dealt with similarly. Then,

$$\begin{aligned}(\eta^x - \eta)(\ell) &= (\Theta_0^a \otimes \Theta_1^a \otimes \Theta_2^a - \Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \lambda_1 \otimes \Theta_2 \lambda_2 - \Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) \\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)(\lambda_1(t^\varepsilon)\lambda_2(t^\varepsilon) - 1).\end{aligned}$$

Hence η is H invariant if and only if $\lambda_1 \lambda_2 = 1$. From Lemma 6.3.6, we obtain

$$\lambda_1(t^\varepsilon)\lambda_2(t^\varepsilon) = \vartheta_1([a, t])^\varepsilon \vartheta_2([a, t])^\varepsilon.$$

Since $\vartheta_1, \vartheta_2 \in \mathrm{Irr}(H/[H, G])\backslash\{1_H\}$, the equality $\lambda_1 \lambda_2 = 1$ holds if and only if $\vartheta_1 \neq \vartheta_2$. $\square$

Proposition 6.3.25. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \operatorname{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that there exists exactly one $i \in [0,2]$ such that ϑ_i extends towards G and $\vartheta_j \neq \vartheta_{j'}$ for j, $j' \in [0,2]\backslash\{i\}$ with $j \neq j'$. Then the character $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2$ extends to irreducible characters of H. Let $\varphi \in \operatorname{Irr}(H)$ be an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Then $I_G(\varphi) = H$.*

Proof. Notice first from Lemma 6.3.6 that $\vartheta_i = 1_H$. Set $\Theta_{i'}$ and η as in Observation 6.3.19, where $i' \in [0,2]$. It follows directly from Lemma 6.3.24 that η is H invariant and η extends to irreducible characters of H. We prove that the set $\{a, t, at, at^{-1}\}$ does not intersect the inertia group $I_G(\varphi)$. This implies $I_G(\varphi) = H$; cf. Figure 6.1.

Assume to the contrary that $at^{\varepsilon} \in I_G(\varphi)$ for some $\varepsilon \in [0,2]$. Since $I_G(\vartheta_{i'}) = G$ for all $i' \in [0,2]$, we have

$$(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)|_{K_1} = \eta = \eta^{at^{\varepsilon}} = (\Theta_2^{a^{\varepsilon}} \otimes \Theta_0^{a^{-\varepsilon}} \otimes \Theta_1^{t^{\varepsilon}})|_{K_1} = (\Theta_2^{a^{\varepsilon}} \otimes \Theta_0^{a^{-\varepsilon}} \otimes \Theta_1)|_{K_1},$$

implying that $\vartheta_0 = \vartheta_1 = \vartheta_2$, which is a contradiction. Thus $at^{\varepsilon} \notin I_G(\varphi)$.

It remains to show that $t \notin I_G(\varphi)$. Assume to the contrary that $\varphi^t = \varphi$. Since φ is an extension of η, we get

$$\eta = \varphi|_{K_1} = \varphi^t|_{K_1} = (\varphi|_{K_1})^t = \eta^t,$$

which implies that $\eta^t = \eta$. Therefore, it suffices to show that $\eta^t \neq \eta$. Since $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2$ and hence $\Theta_0\otimes\Theta_1\otimes\Theta_2$ are linear, $\Theta_0\otimes\Theta_1\otimes\Theta_2(\ell) \neq 0$ for all $\ell \in K_1$. Let $\ell = (h_0t^{\varepsilon}, h_1t^{\varepsilon}, h_2t^{\varepsilon}) \in K_1$, where $h_0, h_1, h_2 \in H$ and $\varepsilon \in [0,2]\backslash\{0\}$. We split the proof into three cases.

Case 1: Let $i = 0$. Then $I_G(\Theta_0) = G$ and $I_G(\Theta_1) = K = I_G(\Theta_2)$. Thanks to Lemma 6.3.6, we obtain

$$\eta^t(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)(\ell) = (\Theta_0 \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_1([a,t])^{-\varepsilon}.$$

Then $\eta^t = \eta$ if and only if $\vartheta_1([a,t])^{-\varepsilon} = 1$, which implies $\vartheta_1 = 1_H$ and ϑ_1 extends towards G. This is a contradiction to the choice of ϑ_1.

Case 2: Let $i = 1$. Then $I_G(\Theta_1) = G$ and $I_G(\Theta_0) = K = I_G(\Theta_2)$. We get

$$\eta^t(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)(\ell) = (\Theta_0^a \otimes \Theta_1 \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{\varepsilon}.$$

Then $\eta^t = \eta$ if and only if $\vartheta_0([a,t])^{\varepsilon} = 1$, which implies $\vartheta_1 = 1_H$ and ϑ_1 extends towards G. This is a contradiction to the choice of ϑ_0.

Case 3: Let $i = 2$. Then $I_G(\Theta_2) = G$ and $I_G(\Theta_0) = K = I_G(\Theta_1)$. Thus

$$\begin{aligned}\eta^t(\ell) &= (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)(\ell)\\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{\varepsilon}\vartheta_1([a,t])^{-\varepsilon},\end{aligned}$$

which is not equal to η, since $\vartheta_0([a,t]) \neq \vartheta_1([a,t])$.

From above three cases we conclude that $t \notin I_G(\varphi)$. Therefore, $I_G(\varphi) = H$. □

Corollary 6.3.26. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that there exists exactly one $i \in [0,2]$ such that ϑ_i extends towards G and $\vartheta_j \neq \vartheta_{j'}$ for j, $j' \in [0,2]\backslash\{i\}$ with $j \neq j'$. Every choice of $i \in [0,2]$, and of a distinct pair of non-trivial G-invariant characters $\vartheta_j, \vartheta_{j'}$ yields 3^2 linear characters of H with inertia group H. In total, we get 54 linear characters of H with inertia group H in this way.*

Proof. For every choice of $i \in [0,2]$ and of a pair of distinct non-trivial linear characters ϑ_j, $\vartheta_{j'}$ as described above, the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ extends to irreducible characters of H. So we get 3^2 linear characters of H, since $[H : H_1] = 3^2$; cf. Figure 6.1. Now, notice that $\vartheta_i = 1_H$. Since $\vartheta_j \neq \vartheta_{j'}$, there are exactly two possible choices selecting such a pair. Since i can be chosen in three different ways, we get a total of 54 linear characters φ of H with inertia group H. As $\varphi|_{H_1} = \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ there can be no overlap. □

Proposition 6.3.27. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that there exists exactly one $i \in [0,2]$ such that ϑ_i extends towards G and $\vartheta_j = \vartheta_{j'}$ for j, $j' \in [0,2]\backslash\{i\}$. Then the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ extends to irreducible characters of K_1. If $\eta \in \mathrm{Irr}(K_1)$ denotes an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ then $I_H(\eta) = K_1$. Let φ be the irreducible character of H induced from η. Then $I_G(\varphi) = K$.*

Proof. It is clear from Lemma 6.3.24, that η is not H invariant and η does not admit an extension towards H. Denote by $\varphi \in \mathrm{Irr}(H)$ the character induced from η. Then

$$\varphi(1) = \eta^H(1) = [H : K_1] \cdot \eta(1) = 3(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)(1) = 3\vartheta_0(1)\vartheta_1(1)\vartheta_2(1) = 3,$$

where the last equality follows because ϑ_0, ϑ_1 and ϑ_2 are linear. We claim that $t \in I_G(\varphi)$. Then $I_G(\varphi) \in \{K, G\}$. Since φ is not linear, $I_G(\varphi) \neq G$ by Corollary 5.3.12. Therefore, $I_G(\varphi) = K$.

Now, we shall prove that $t \in I_G(\varphi)$. From Lemma 4.0.9, $t \in I_G(\varphi)$ if and only if $\eta^t \in \{\eta, \eta^x, \eta^{x^{-1}}\}$. Set $\Theta_{i'}$ as defined in Observation 6.3.19 for all $i' \in [0,2]$. Notice that $I_G(\Theta_i) = G$ and $\vartheta_i = 1_H$, and $I_G(\Theta_j) = K = I_G(\Theta_{j'})$. Observe further that, since $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is linear, $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ is linear, and hence $\Theta_0 \otimes \Theta_1 \otimes \Theta_2(\ell) \neq 0$ for all $\ell \in K_1$. Let $\ell = (h_0 t^\varepsilon, h_1 t^\varepsilon, h_2 t^\varepsilon) \in K_1$, where $h_0, h_1, h_2 \in H$ and $\varepsilon \in [0,2]\backslash\{0\}$. We split the proof into three cases.

Case 1: ϑ_0 extends towards G and $\vartheta_1 = \vartheta_2$. By Lemma 6.3.6, we obtain

$$\eta^t(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)(\ell) = (\Theta_0 \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_1([a,t])^{-\varepsilon}$$
$$= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_1([a,t])^{\varepsilon}\vartheta_2([a,t])^{\varepsilon} = (\Theta_0 \otimes \Theta_1^a \otimes \Theta_2^a)(\ell) = \eta^x(\ell).$$

Case 2: ϑ_1 extends towards G and $\vartheta_0 = \vartheta_2$:

$$\eta^t(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)(\ell) = (\Theta_0^a \otimes \Theta_1 \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{\varepsilon}$$
$$= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{-\varepsilon}\vartheta_2([a,t])^{-\varepsilon} = (\Theta_0^{a^{-1}} \otimes \Theta_1 \otimes \Theta_2^{a^{-1}})(\ell) = \eta^{x^{-1}}(\ell).$$

Case 3: ϑ_2 extends towards G and $\vartheta_0 = \vartheta_1$:

$$\eta^t(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{\varepsilon}\vartheta_1([a,t])^{-\varepsilon}$$
$$= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) = \eta(\ell).$$

Hence, we conclude that $I_G(\varphi) = K$. □

Corollary 6.3.28. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that there exists exactly one $i \in [0,2]$ such that ϑ_i extends towards G and $\vartheta_j = \vartheta_{j'}$ for j, $j' \in [0,2]\backslash\{i\}$. In this way, every $i \in [0,2]$, and every choice of $\vartheta_j = \vartheta_{j'}$ yields a unique irreducible character $\varphi \in \mathrm{Irr}(H)$ such that $\langle \varphi|_{H_1}, \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2\rangle \neq 0$, and φ satisfies $\varphi(1) = 3$ and $I_G(\varphi) = K$. In total, we get 6 irreducible characters φ of H of such a form.*

Proof. First observe from Proposition 6.3.27 that for any fixed $i \in [0,2]$ with $\vartheta_i = 1_H$ and non-trivial G-invariant character $\vartheta_{j'} = \vartheta_j$, where $\{j, j'\} = [0,2]\backslash\{i\}$, we obtain a unique character $\varphi \in \mathrm{Irr}(H)$ of the described form. We identify ϑ_j with a non-trivial character of $H/[H,G]$. There are two different choices for ϑ_j and three different choices for i. Hence we get a total of 6 irreducible characters of H of the described form. As $\varphi|_{H_1} = 3(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$ determines $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ uniquely, there are no overlaps. □

Lemma 6.3.29. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that none of the characters ϑ_i extends towards G for $i \in [0,2]$. The character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ extends to irreducible characters of K_1. Let $\eta \in \mathrm{Irr}(K_1)$ be an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Then η is H-invariant if and only if $\vartheta_0 = \vartheta_1 = \vartheta_2$.*

Proof. Let Θ_i and η be defined as in Observation 6.3.19. Since $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ and hence $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ are linear, $\Theta_0 \otimes \Theta_1 \otimes \Theta_2(\ell) \neq 0$ for all $\ell \in K_1$. Let $\ell = (h_0 t^{\varepsilon}, h_1 t^{\varepsilon}, h_2 t^{\varepsilon}) \in K_1$, where $h_0, h_1, h_2 \in H$ and $\varepsilon \in [0,2]\backslash\{0\}$. From Lemma 6.3.6, we get:

$$\eta^x(\ell) = (\Theta_0^a \otimes \Theta_1^a \otimes \Theta_2^a)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{\varepsilon}\vartheta_1([a,t])^{\varepsilon}\vartheta_2([a,t])^{\varepsilon}.$$

Therefore η is H-invariant if and only if

$$\vartheta_0([a,t])^{\varepsilon}\vartheta_1([a,t])^{\varepsilon}\vartheta_2([a,t])^{\varepsilon} = 1. \tag{6.16}$$

Since there are only two possible choices for each $\vartheta_i \in \mathrm{Irr}(H/[H,G])\backslash\{1_H\}$, the equality (6.16) holds if and only if $\vartheta_0 = \vartheta_1 = \vartheta_2$. □

Proposition 6.3.30. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that none of the characters ϑ_i extends towards G for $i \in [0,2]$. Assume further that $\vartheta_0 = \vartheta_1 = \vartheta_2$. Then the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ extends to irreducible characters of H. Let $\varphi \in \mathrm{Irr}(H)$ be an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Then $I_G(\varphi) = H\langle at^{\varepsilon}\rangle$ if and only if $\varphi|_{K_1}(\bar{t}) = \vartheta_0([a,t])^{-\varepsilon}$ for $\varepsilon \in [0,2]$.*

Proof. Set Θ_i and η as defined in Observation 6.3.19. It is straightforward from Lemma 6.3.29 that η extends to irreducible characters of H. Let φ be an extension of η. Recall that φ is linear, and hence $\varphi(\ell) \neq 0$ for all $\ell \in H$. First we prove that $t \notin I_G(\varphi)$. Indeed, since $\vartheta_0 = \vartheta_1 = \vartheta_2$ and $[H,G] = K_1$ (Proposition 6.1.2), we get:

$$\begin{aligned}
\varphi^t(x) &= \varphi(x^{t^{-1}}) = \varphi(x[x,t^{-1}]) = \varphi(x)\eta([x,t^{-1}]) = \varphi(x)(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)([x,t^{-1}]) \\
&= \varphi(x)\Theta_0([t^{-1}a,a^{-1}])\Theta_1([a,a])\Theta_2([at,t^{-1}]) = \varphi(x)\Theta_0([t^{-1},a^{-1}])\Theta_2([a,t^{-1}]) \\
&= \varphi(x)\vartheta_0([a,t])^{-1}\vartheta_2([a,t])^{-1} \neq \varphi(x).
\end{aligned}$$

Now, let $\varepsilon \in [0,2]$. Note that the character η is linear and recall that $\vartheta_0 = \vartheta_1 = \vartheta_2 \in \mathrm{Irr}(H/[H,G])\backslash\{1_H\}$, thus we obtain

$$\begin{aligned}
\varphi^{at^{\varepsilon}}(x) &= \varphi(x^{t^{-\varepsilon}a^{-1}}) = \varphi(x[x,t^{-\varepsilon}a^{-1}]) = \varphi(x)\eta([x,t^{-\varepsilon}a^{-1}]) = \varphi(x)\eta([x,a^{-1}])\eta([x,t^{-\varepsilon}]) \\
&= \varphi(x)(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(t[t,a],t,t[t^{-1},a])(\Theta_0 \otimes \Theta_1 \otimes \Theta_2)([t^{-1}a,a^{-\varepsilon}],1,[at,t^{-\varepsilon}]) \\
&= \varphi(x)\eta(\bar{t})\vartheta_0([t,a])\vartheta_2([t^{-1},a])\vartheta_0([t^{-1}a,a^{-\varepsilon}])\vartheta_2([at,t^{-\varepsilon}]) \\
&= \varphi(x)\eta(\bar{t})\vartheta_0([a,t])^{-1}\vartheta_2([a,t])\vartheta_0([a,t])^{-\varepsilon}\vartheta_2([a,t])^{-\varepsilon} \\
&= \varphi(x)\eta(\bar{t})\vartheta_0([a,t])^{-2\varepsilon} = \varphi(x)\eta(\bar{t})\vartheta_0([a,t])^{\varepsilon}.
\end{aligned}$$

Therefore φ is at^{ε} invariant if any only if $\vartheta_0([a,t])^{-\varepsilon} = \eta(\bar{t})$. □

Corollary 6.3.31. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that none of the ϑ_i extends towards G for $i \in [0,2]$. Assume further that $\vartheta_0 = \vartheta_1 = \vartheta_2 =: \vartheta$. In this way, for a given ϑ and for every $\varepsilon \in [0,2]$, we get 3 linear characters φ of H, extending $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$, of inertia group $H\langle at^{\varepsilon}\rangle$. In total, for every $\varepsilon \in [0,2]$, we get 6 linear characters of H of inertia group $H\langle at^{\epsilon}\rangle$.*

Proof. Set $\vartheta := \vartheta_0 = \vartheta_1 = \vartheta_2$. Then there are two possible choices for a non-trivial character $\vartheta \in \mathrm{Irr}(H/[H,G])$. Fix a value of ϑ. Then $\vartheta \otimes \vartheta \otimes \vartheta$ extends to irreducible characters of K_1. Let $\eta \in \mathrm{Irr}(K_1)$ be an extension of $\vartheta \otimes \vartheta \otimes \vartheta$. Then the extensions of $\vartheta \otimes \vartheta \otimes \vartheta$ to K_1 are precisely of the form η, $\eta\lambda$, $\eta\lambda^{-1}$ for some $\lambda \in \mathrm{Irr}(K_1/H_1)\backslash\{1_{K_1}\}$. Every character of type $\eta\lambda^\delta$, $\delta \in [0,2]$, extends to irreducible characters of H and gives rise to 3 linear characters with inertia group $H\langle at^\varepsilon\rangle$, where the value of ε is uniquely determined by the value of $\eta\lambda^\delta(\bar{t})$. Since there are two possible choices for the value of ϑ and both of the values are legitimate, for every $\varepsilon \in [0,2]$, we get 6 linear characters of H of inertia group $H\langle at^\varepsilon\rangle$. There is no overlap, as $\varphi|_{H_1} = \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. □

Proposition 6.3.32. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that none of the ϑ_i extends towards G for $i \in [0,2]$. Suppose further that there exist $i, j \in [0,2]$ such that $\vartheta_i \neq \vartheta_j$. The character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ extends to irreducible characters of K_1. Let $\eta \in \mathrm{Irr}(K_1)$ be an extension of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Then η does not extend further to G. Denote by $\varphi \in \mathrm{Irr}(H)$ the character induced from η. Then $I_G(\varphi) = K$.*

Proof. It follows from Lemma 6.3.29 that η is not H-invariant and hence η does not extend further. Denote by $\varphi \in \mathrm{Irr}(H)$ the character induced from η. Then φ is a character of degree 3. We claim that $t \in I_G(\varphi)$. Then $at^\varepsilon \notin I_G(\varphi)$ for any $\varepsilon \in [0,2]$. If otherwise, suppose that $at^\varepsilon \in I_G(\varphi)$, then $I_G(\varphi) = G$, and this is a contradiction to Corollary 5.3.12, since φ is non-linear. Then $I_G(\varphi) = K$.

It remains to show that $t \in I_G(\varphi)$. By Lemma 4.0.9, this happens if and only if $\eta^t \in \{\eta, \eta^x, \eta^{x^{-1}}\}$. Now notice that, since $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathrm{Irr}(H/[H,G])\backslash\{1_H\}$ and $\vartheta_i \neq \vartheta_j$ for $i, j \in [0,2]$, we must have that $\vartheta_k = \vartheta_i$ or $\vartheta_k = \vartheta_j$ for $k \in [0,2]\backslash\{i,j\}$. Set Θ'_i defined as in Observation 6.3.19, where $i' \in [0,2]$. Observe that, since $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is linear, $\Theta_0 \otimes \Theta_1 \otimes \Theta_2$ is linear, and hence $\Theta_0 \otimes \Theta_1 \otimes \Theta_2(\ell) \neq 0$ for all $\ell \in K_1$. Let $\ell = (h_0t^\varepsilon, h_1t^\varepsilon, h_2t^\varepsilon) \in K_1$, where $h_0, h_1, h_2 \in H$ and $\varepsilon \in [0,2]\backslash\{0\}$. We split the proof into three cases.

Case 1: $\vartheta_0 = \vartheta_1 \neq \vartheta_2$. Using Lemma 6.3.6, we obtain

$$\begin{aligned}\eta^t(\ell) &= (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^t)(\ell) = (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)(\ell)\\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^\varepsilon\vartheta_1([a,t])^{-\varepsilon} = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell) = \eta(\ell).\end{aligned}$$

Case 2: $\vartheta_0 = \vartheta_2 \neq \vartheta_1$. Using Lemma 6.3.6, we obtain

$$\begin{aligned}\eta^t(\ell) &= (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^\varepsilon\vartheta_1([a,t])^{-\varepsilon}\\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{-\varepsilon}\vartheta_1([a,t])^{-\varepsilon}\vartheta_2([a,t])^{-\varepsilon}\\ &= (\Theta_0^{a^{-1}} \otimes \Theta_1^{a^{-1}} \otimes \Theta_2^{a^{-1}})(\ell) = \eta^{x^{-1}}(\ell).\end{aligned}$$

Case 3: $\vartheta_1 = \vartheta_2 \neq \vartheta_0$. Using Lemma 6.3.6, we obtain

$$\begin{aligned}\eta^t(\ell) &= (\Theta_0^a \otimes \Theta_1^{a^{-1}} \otimes \Theta_2)(\ell) = (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{\varepsilon}\vartheta_1([a,t])^{-\varepsilon} \\ &= (\Theta_0 \otimes \Theta_1 \otimes \Theta_2)(\ell)\vartheta_0([a,t])^{\varepsilon}\vartheta_1([a,t])^{\varepsilon}\vartheta_2([a,t])^{\varepsilon} = (\Theta_0^a \otimes \Theta_1^a \otimes \Theta_2^a)(\ell) \\ &= \eta^x(\ell).\end{aligned}$$

Hence $I_G(\varphi) = K$. □

Corollary 6.3.33. *Let ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ be such that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Suppose that none of the characters ϑ_i extends towards G for $i \in [0,2]$. Suppose further that there exist $i, j \in [0,2]$ such that $\vartheta_i \neq \vartheta_j$. In this way, each unordered pair $\{i,j\} \subseteq [0,2]$ yields an irreducible character φ of H such that $\langle \varphi|_{H_1}, \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \rangle \neq 0$ with $\varphi(1) = 3$ and $I_G(\varphi) = K$. We get 6 characters of this form.*

Proof. It follows from Proposition 6.3.32 that for a fixed unordered pair $\{i,j\} \subseteq [0,2]$, we get exactly one irreducible character φ of degree three with $I_G(\varphi) = K$. Now, there are three different possible ways to select a pair $\{i,j\}$. For a given pair $\{i,j\}$, there are two choices for ϑ_i. The values of ϑ_j and ϑ_k, for $k \in [0,2]\backslash\{i,j\}$, are uniquely determined from that of ϑ_i. Therefore, we get 6 K-invariant characters of degree 3. There are no overlap, as $\varphi|_{H_1} = 3(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$. □

6.4 Computing the representation zeta function of H

In Section 6.3, we have studied the irreducible characters $\varphi \in \mathrm{Irr}(H)$ which are obtained by extension or induction from an irreducible character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ with a prescribed inertia group $S \in \mathcal{H}$, and computed the sufficient and necessary conditions to have $I_G(\varphi) = T$ for a given $T \in \mathcal{G}$, where $\mathcal{H}$ and $\mathcal{G}$ are defined as in the beginning of Section 6.3.

Here we compute a recursive formula for the representation zeta function of H. Let $\varphi \in \mathrm{Irr}(H)$ and let $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ be an irreducible constituent of $\varphi|_{H_1}$, where $\vartheta_i \in \mathrm{Irr}(H)$ for $i \in [0,2]$. Observe that, by Clifford's theorem, the character $\varphi|_{H_1}$ is a sum of H-conjugates of $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$. Since $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) \in \mathcal{H}$, and the elements of $\mathcal{H}$ are normal in H, the inertia groups of the irreducible constituents of $\varphi|_{H_1}$ are the same and equal to $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2)$. We define

$$a_S^T(d) := \left| \left\{ \varphi \in \mathrm{Irr}(H) \;\middle|\; \begin{array}{l} \varphi(1) = 3^d,\ I_G(\varphi) = T,\ I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = S, \\ \text{where } \vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \text{ is an irreducible constituent of } \varphi|_{H_1} \end{array} \right\} \right|,$$

where $d \in \mathbb{N}_0$, $S \in \mathcal{H}$ and $T \in \mathcal{G}$. For a given $S \in \mathcal{H}$ and $T \in \mathcal{G}$, we define the partial representation zeta function of H as

$$\zeta_S^T(H,s) = \sum_{d=0}^{\infty} a_S^T(d) 3^{-ds}. \tag{6.17}$$

From (6.5) we get

$$\zeta^T(H,s) = \sum_{S \in \mathcal{H}} \zeta_S^T(H,s). \tag{6.18}$$

Further summing $\zeta^T(H,s)$ over all $T \in \mathcal{G}$ gives the representation zeta function of H;

$$\zeta(H,s) = \sum_{T \in \mathcal{G}} \zeta^T(H,s). \tag{6.19}$$

We compute the partial representation zeta functions $\zeta_S^T(H,s)$ in Section 6.4.1. From (6.18) and (6.19), we obtain a recursive formula for the representation zeta function of H in Section 6.4.2.

6.4.1 Computing partial representation zeta functions

For a given $S \in \mathcal{H}$ and $T \in \mathcal{G}$, here we compute the partial representation zeta function $\zeta_S^T(H,s)$. In alignment with the discussion in Section 6.3.2, we divide the computation into four steps depending on the value of S. Whenever there is no reason for confusion, we drop (H,s) from the expression $\zeta_S^T(H,s)$.

We begin with computing $\zeta_H^G(H,s)$, and it turns out be equal to $\zeta^G(H,s)$. We recall that C (defined in Theorem 5.3.4) is the number of G-invariant irreducible characters of H. Therefore, by Lemma 6.4.1 below, we see that $C = 3$.

Lemma 6.4.1. *The equalities $\zeta^G(H,s) = \zeta_H^G(H,s) = 3$ hold.*

Proof. Thanks to Corollary 5.3.12, the G-invariant irreducible characters of H are linear. Notice that the characters of H obtained from characters of type Case 1, Case 2 and Case 3 in Section 6.3.2 are non-linear. Therefore, the linear characters of H must be coming from characters of type Case 4. It is evident from the computations in Section 6.3.2.4 that the only contribution towards the G-invariant irreducible linear characters of H is from Proposition 6.3.20. Therefore, from (6.18) we obtain that

$$\zeta^G(H,s) = \sum_{S \in \mathcal{H}} \zeta_S^G(H,s) = \zeta_H^G(H,s) = 3.$$

□

Case 1: $S = H_1\langle \bar{t}\rangle =: K_1$.

Let $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ with $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = K_1$. Let $\varphi \in \mathrm{Irr}(H)$ be such that $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is an irreducible constituent of $\varphi|_{H_1}$. From Observation 6.3.8, we have

$$\varphi(1) = 3\vartheta_0(1)\vartheta_1(1)\vartheta_2(1).$$

Computing $\zeta^{H\langle a\rangle}_{H_1\langle \bar{t}\rangle}(H, s)$

By Proposition 6.3.9, for every $\varepsilon \in [0, 2]$ and every $\vartheta \in \mathrm{Irr}(H)$ with $I_G(\vartheta) = H\langle t\rangle$, the irreducible character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 = \vartheta \otimes \vartheta^{a^{\varepsilon}} \otimes \vartheta^{a^{-\varepsilon}}$ of H_1 yields three irreducible characters $\varphi \in \mathrm{Irr}(H)$ such that $I_G(\varphi) = H\langle a\rangle$ and $\varphi(1) = 3\vartheta(1)^3$. Indeed, there are three ways to extend $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ to $H_1\langle \bar{t}\rangle$, and each of such extended character induces to an irreducible character φ of H, resulting three distinct irreducible characters of H. Furthermore, this is the necessary condition to yield characters φ of the desired form. However, the restriction $\varphi|_{H_1}$ of each φ to H_1 is of the form

$$\varphi|_{H_1} = \sum_{\delta \in [0,2]} \vartheta^{a^{\delta}} \otimes \vartheta^{a^{\delta+\varepsilon}} \otimes \vartheta^{a^{\delta-\varepsilon}}.$$

Thus three different choices of ϑ yields the same φ. Hence, on average, each choice of $\varepsilon \in [0, 2]$ and $\vartheta \in \mathrm{Irr}(H)$ yields one $\varphi \in \mathrm{Irr}(H)$. For a fixed $\varepsilon \in [0, 2]$, the corresponding partial representation zeta function of H is given by

$$3^{-s}\zeta^{H\langle t\rangle}(H, 3s),$$

and, since there are three choices for ε, we get

$$\zeta^{H\langle a\rangle}_{H_1\langle \bar{t}\rangle}(H, s) = 3^{1-s}\zeta^{H\langle t\rangle}(H, 3s). \tag{6.20}$$

Computing $\zeta^{H\langle at\rangle}_{H_1\langle \bar{t}\rangle}(H, s)$

By Proposition 6.3.10, for each $i \in [0, 2]$ and $\vartheta \in \mathrm{Irr}(H)$ with $I_G(\vartheta) = H\langle t\rangle$, the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H)$ defined by $\vartheta_i = \vartheta$, $\vartheta_{i+1} = \vartheta_{i+2} = \vartheta^{a^{-1}}$ yields three irreducible characters $\varphi \in \mathrm{Irr}(H)$ of degree $\varphi(1) = 3\vartheta(1)^3$ and inertia group $I_G(\varphi) = H\langle at\rangle$. Furthermore, this is the necessary condition to obtain characters φ of the desired form. Similarly, as above, on average, each choice of $i \in [0, 2]$ and $\vartheta \in \mathrm{Irr}(H)$ yields one $\varphi \in \mathrm{Irr}(H)$. For a fixed $i \in [0, 2]$, the corresponding partial recursive representation zeta of H function is given by

$$3^{-s}\zeta^{H\langle t\rangle}(H, 3s),$$

and, since there are three different ways to fix an element $i \in [0, 2]$, we obtain

$$\zeta^{H\langle at\rangle}_{H_1\langle \bar{t}\rangle}(H, s) = 3^{1-s}\zeta^{H\langle t\rangle}(H, 3s). \tag{6.21}$$

Computing $\zeta^{H\langle at^{-1}\rangle}_{H_1\langle \bar{t}\rangle}(H,s)$

By replacing ϑ_{i+1} and ϑ_{i+2} with ϑ^a in the above computation, from Proposition 6.3.10, we obtain the partial representation zeta function $\zeta^{H\langle at^{-1}\rangle}_{H_1\langle \bar{t}\rangle}(H,s)$ of H as

$$\zeta^{H\langle at^{-1}\rangle}_{H_1\langle \bar{t}\rangle}(H,s) = 3^{1-s}\zeta^{H\langle t\rangle}(H,3s). \tag{6.22}$$

Computing $\zeta^{H\langle t\rangle}_{H_1\langle \bar{t}\rangle}(H,s)$

By Propostion 6.3.11, for each $i \in [0,2]$ and $\vartheta \in \mathrm{Irr}(H)$ of inertia group $I_G(\vartheta) = H\langle t\rangle$, the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ of the form $\vartheta_i = \vartheta$ and $\vartheta_{i+1} = \vartheta_{i+2}$ with $I_G(\vartheta_{i+1}) = G$ yields three irreducible characters $\varphi \in \mathrm{Irr}(H)$ such that $I_G(\varphi) = H\langle t\rangle$. Thanks to Corollary 5.3.12, the character $\vartheta_{i+1} = \vartheta_{i+2}$ is linear, and hence $\varphi(1) = 3\vartheta(1)$. Again, this is the necessary condition to obtain characters φ of the desired form, and on average each choice of $i \in [0,2]$ and $\vartheta \in \mathrm{Irr}(H)$ yields one $\varphi \in \mathrm{Irr}(H)$. We identify the character $\vartheta_{i+1} = \vartheta_{i+2}$ with an irreducible character of $H/[H,G]$. For a fixed $i \in [0,2]$ and for a given choice of $\vartheta_{i+1} = \vartheta_{i+2} \in \mathrm{Irr}(H/[H,G])$ the corresponding partial representation zeta function is given by

$$3^{-s}\zeta^{H\langle t\rangle}(H,s).$$

As there are three choices for ϑ_{i+1} and three choices for $i \in [0,2]$, we get

$$\zeta^{H\langle t\rangle}_{H_1\langle \bar{t}\rangle}(H,s) = 3^{2-s}\zeta^{H\langle t\rangle}(H,s). \tag{6.23}$$

Computing $\zeta^{H}_{H_1\langle \bar{t}\rangle}(H,s)$

Notice from Lemma 6.3.3 that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle \bar{t}\rangle$ if and only if $I_G(\vartheta_i) \in \{H\langle t\rangle, G\}$ for all $i \in [0,2]$ given that there exists $j \in [0,2]$ such that $I_G(\vartheta_j) \neq G$. Also observe from Lemma 6.4.1 that $\zeta^{G}_{H_1\langle \bar{t}\rangle}(H,s) = 0$. Thus the partial representation zeta function $\zeta^{H}_{H_1\langle \bar{t}\rangle}(H,s)$ is given by

$$\begin{aligned}\zeta^{H}_{H_1\langle \bar{t}\rangle}(H,s) &= 3^{-s}\left(\left(\zeta^{H\langle t\rangle} + \zeta^{G}\right)^3 - (\zeta^G)^3\right) - \zeta^{H\langle a\rangle}_{H_1\langle \bar{t}\rangle} - \zeta^{H\langle at\rangle}_{H_1\langle \bar{t}\rangle} - \zeta^{H\langle at^{-1}\rangle}_{H_1\langle \bar{t}\rangle} - \zeta^{H\langle t\rangle}_{H_1\langle \bar{t}\rangle} \\ &= 3^{-s}\Big(\zeta^{H\langle t\rangle}(H,s)^3 + 9\zeta^{H\langle t\rangle}(H,s)^2 + 18\zeta^{H\langle t\rangle}(H,s) - 9\zeta^{H\langle t\rangle}(H,3s)\Big).\end{aligned} \tag{6.24}$$

Case 2: $S = H_1\langle x_j\bar{t}\rangle$ for $j \in [0,2]$.

Let $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ of inertia group $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_j\bar{t}\rangle$ for $j \in [0,2]$. Let $\varphi \in \mathrm{Irr}(H)$ be such that $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is an irreducible constituent of $\varphi|_{H_1}$. From Observation 6.3.12, we have

$$\varphi(1) = 3\vartheta_0(1)\vartheta_1(1)\vartheta_2(1).$$

We shall compute the partial representation zeta function $\zeta^{T}_{H_1\langle x_j\bar{t}\rangle}(H,s)$, where $T \in \mathcal{G}$, for a fixed $j \in [0,2]$. However, since the computation is similar for all j, we obtain the same recursive zeta function $\zeta^{T}_{H_1\langle x_j\bar{t}\rangle}(H,s)$ for all j.

Computing $\zeta^{H\langle t\rangle}_{H_1\langle x_j\bar{t}\rangle}(H,s)$ for $j \in [0,2]$

By Proposition 6.3.14, for every $j \in [0,2]$ the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ of the form

1. $\vartheta_j \in \mathrm{Irr}(H)$ with $I_G(\vartheta_j) = H\langle a\rangle$ and $\vartheta_{j+2} = \vartheta_{j+1}^{-1}$ with $I_G(\vartheta_{j+1}) = I_G(\vartheta_{j+2}) = G$, or,
2. $\vartheta_j = 1_H$ with
   - (i) $I_G(\vartheta_{j+1}) = H\langle at\rangle$ and $I_G(\vartheta_{j+2}) = G$, or,
   - (ii) $I_G(\vartheta_{j+1}) = G$ and $I_G(\vartheta_{j+2}) = H\langle at^{-1}\rangle$, or,
   - (iii) $I_G(\vartheta_{j+1}) = H\langle at\rangle$ and $I_G(\vartheta_{j+2}) = H\langle at^{-1}\rangle$,

yields three irreducible characters $\varphi \in \mathrm{Irr}(H)$ of degree $\varphi(1) = 3\vartheta_j(1)\vartheta_{j+1}(1)\vartheta_{j+2}(1)$ with $I_G(\varphi) = H\langle t\rangle$. Moreover, this is a necessary condition for obtaining characters φ of the desired from. Again, $\varphi|_{H_1}$ is a sum of three distinct constituent and we undo the overcounting: on average, each character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$, where ϑ_0, ϑ_1 and ϑ_2 are as described above, yields one $\varphi \in \mathrm{Irr}(H)$. Hence, using Lemma 6.4.1 we get

$$\begin{aligned}\zeta^{H\langle t\rangle}_{H_1\langle x_j\bar{t}\rangle} &= 3^{-s}\left(\zeta^{H\langle a\rangle}\zeta^{G} + \zeta^{H\langle at\rangle}\zeta^{G} + \zeta^{G}\zeta^{H\langle at^{-1}\rangle} + \zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\right)\\ &= 3^{1-s}\left(\zeta^{H\langle a\rangle} + \zeta^{H\langle at\rangle} + \zeta^{H\langle at^{-1}\rangle}\right) + 3^{-s}\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}. \end{aligned} \tag{6.25}$$

Computing $\zeta^{H}_{H_1\langle x_j\bar{t}\rangle}(H,s)$ for $j \in [0,2]$

From Lemma 6.3.4, we get that $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_j\bar{t}\rangle$ if and only if $I_G(\vartheta_j) \in \{H\langle a\rangle, G\}$, $I_G(\vartheta_{j+1}) \in \{H\langle at\rangle, G\}$ and $I_G(\vartheta_{j+2}) \in \{H\langle at^{-1}\rangle, G\}$ given that $I_G(\vartheta_i) \neq G$ for some $i \in [0,2]$. Furthermore, from Proposition 6.3.13, the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ with $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1\langle x_j\bar{t}\rangle$ does not contribute to the partial representation zeta functions $\zeta^{H\langle at^{\varepsilon}\rangle}(H,s)$ for any $\varepsilon \in [0,2]$. Also, it follows from Lemma 6.4.1, that $\zeta^{G}_{H_1\langle x_j\bar{t}\rangle}(H,s) = 0$.

Therefore,

$$\begin{aligned}
\zeta^{H}_{H_1\langle x_j\bar{t}\rangle} &= 3^{-s}\left(\left(\zeta^{H\langle a\rangle}+\zeta^{G}\right)\left(\zeta^{H\langle at\rangle}+\zeta^{G}\right)\left(\zeta^{H\langle at^{-1}\rangle}+\zeta^{G}\right)-(\zeta^{G})^3\right)\\
&\quad -3^{1-s}\left(\zeta^{H\langle a\rangle}+\zeta^{H\langle at\rangle}+\zeta^{H\langle at^{-1}\rangle}\right)-3^{-s}\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\\
&= 3^{-s}\Big(\zeta^{H\langle a\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}+3\zeta^{H\langle a\rangle}\zeta^{H\langle at\rangle}+3\zeta^{H\langle a\rangle}\zeta^{H\langle at^{-1}\rangle}+3\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\\
&\quad +9\zeta^{H\langle a\rangle}+9\zeta^{H\langle at\rangle}+9\zeta^{H\langle at^{-1}\rangle}\Big)-3^{1-s}\left(\zeta^{H\langle a\rangle}+\zeta^{H\langle at\rangle}+\zeta^{H\langle at^{-1}\rangle}\right)\\
&\quad -3^{-s}\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\\
&= 3^{-s}\Big(\zeta^{H\langle a\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}+3\zeta^{H\langle a\rangle}\zeta^{H\langle at\rangle}+3\zeta^{H\langle a\rangle}\zeta^{H\langle at^{-1}\rangle}+2\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\\
&\quad +6\zeta^{H\langle a\rangle}+6\zeta^{H\langle at\rangle}+6\zeta^{H\langle at^{-1}\rangle}\Big). \qquad (6.26)
\end{aligned}$$

Case 3: $S = H_1$.

Let $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2 \in \mathrm{Irr}(H_1)$ be such that $I_H(\vartheta_0\otimes\vartheta_1\otimes\vartheta_2) = H_1$. Let $\varphi \in \mathrm{Irr}(H)$ be such that $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2$ is an irreducible constituent of $\varphi|_{H_1}$. From Observation 6.3.15, we get that φ is the character induced from $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2$ and

$$\varphi(1) = 9\vartheta_0(1)\vartheta_1(1)\vartheta_2(1).$$

Computing $\zeta^{H\langle a\rangle}_{H_1}(H,s)$

By Proposition 6.3.17, for each pair $(i,j) \in [0,2]\times[0,2]$ and for every $\vartheta \in \mathrm{Irr}(H)$ with $I_G(\vartheta) \in \{H, H\langle a\rangle, H\langle at\rangle, H\langle at^{-1}\rangle\}$, the character $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2$ of the form given by $\vartheta_0 = \vartheta$, $\vartheta_1 = \vartheta^{a^{-i}t^{-j}}$ and $\vartheta_2 = \vartheta^{a^it^{-i+j}}$ yields an irreducible character $\varphi \in H$ of degree $\varphi(1) = 9\vartheta(1)^3$ with inertia group $I_G(\varphi) = H\langle a\rangle$. Furthermore, this is the necessary condition to obtain a character φ of the desired form. Also, notice that every conjugate of $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2$ in H gives rise to the same irreducible character φ of H. Hence, for a fixed pair (i,j), the corresponding partial recursive representation zeta function is given by

$$3^{-2-2s}\left(\zeta^{H}(H,3s)+\zeta^{H\langle a\rangle}(H,3s)+\zeta^{H\langle at\rangle}(H,3s)+\zeta^{H\langle at^{-1}\rangle}(H,3s)\right).$$

Since there are 9 choices for the pairs (i,j), we have

$$\zeta^{H\langle a\rangle}_{H_1}(H,s) = 3^{-2s}\left(\zeta^{H}(H,3s)+\zeta^{H\langle a\rangle}(H,3s)+\zeta^{H\langle at\rangle}(H,3s)+\zeta^{H\langle at^{-1}\rangle}(H,3s)\right). \quad (6.27)$$

Computing $\zeta^{H\langle t\rangle}_{H_1}(H,s)$

By Proposition 6.3.18, for every pair $(i,j) \in [0,2]\times[0,2]$ and for characters ϑ_0, ϑ_1, $\vartheta_2 \in \mathrm{Irr}(H)$ satisfying the equalities

$$\vartheta_0^{a^{-1+i}t^{-i+j}} = \vartheta_0, \quad \vartheta_1^{a^{1+i}t^{j}} = \vartheta_1, \quad \text{and} \quad \vartheta_2^{a^{i}t^{-1+i+j}} = \vartheta_2, \qquad (6.28)$$

with prescribed inertia group as given in Proposition 6.3.18, the character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ yields an irreducible character $\varphi \in H$ of degree $\varphi(1) = 9\vartheta_0(1)\vartheta_1(1)\vartheta_2(1)$ and inertia group $I_G(\varphi) = H\langle t\rangle$. Furthermore, this is the necessary condition to obtain such φ.

Set $D_0 = \{(0,0),(1,0),(2,2)\}$, $D_1 = \{(0,2),(1,2),(2,1)\}$ and $D_2 = \{(0,1),(1,1),(2,0)\}$. For every $k \in [0,2]$, fix an element $(i,j) \in D_k$. We define $\zeta_{(i,j)}^{H\langle t\rangle}(H,s)$ as the partial representation zeta function $\zeta_{H_1}^{H\langle t\rangle}(H,s)$ with the additional condition that the characters ϑ_0, ϑ_1 and ϑ_2 satisfy (6.28). Observe from Table 6.2 that, for every $(i,j) \in D_k$, the inertia groups of the characters ϑ_0, ϑ_1 and ϑ_2 are symmetric. Therefore, one gets the same function $\zeta_{(i,j)}^{H\langle t\rangle}(H,s)$ for all $(i,j) \in D_k$. Therefore, the partial representation zeta function $\zeta_{H_1}^{H\langle t\rangle}(H,s)$ is given by

$$\zeta_{H_1}^{H\langle t\rangle}(H,s) = 3\left(\zeta_{(0,0)}^{H\langle t\rangle}(H,s) + \zeta_{(0,2)}^{H\langle t\rangle}(H,s) + \zeta_{(0,1)}^{H\langle t\rangle}(H,s)\right). \tag{6.29}$$

Now, notice that every conjugate of a character $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ in H gives rise to the same irreducible character φ of H. Since $[H : H_1] = 9$, $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ has 9 distinct conjugates. Therefore, we divide the partial representation zeta function by 9 to compensate for overcounting. Using Lemma 6.4.1 we obtain $\zeta_{H_1}^{H\langle t\rangle}(H,s)$ in three steps:

$$\begin{aligned}
\zeta_{(0,0)}^{H\langle t\rangle}(H,s) &= 3^{-2-2s}\left((\zeta^{H\langle a\rangle})^2\zeta^{H\langle t\rangle} + (\zeta^{H\langle a\rangle})^2\zeta^{G} + \zeta^{H\langle a\rangle}\zeta^{G}\zeta^{H\langle t\rangle} + \zeta^{G}\zeta^{H\langle a\rangle}\zeta^{H\langle t\rangle}\right)\\
&= 3^{-2-2s}\left((\zeta^{H\langle a\rangle})^2\zeta^{H\langle t\rangle} + 3(\zeta^{H\langle a\rangle})^2 + 6\zeta^{H\langle a\rangle}\zeta^{H\langle t\rangle}\right),\\
\zeta_{(0,2)}^{H\langle t\rangle}(H,s) &= 3^{-2-2s}\left(\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle} + \zeta^{H\langle at\rangle}\zeta^{G}\zeta^{H\langle t\rangle} + \zeta^{G}\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle}\right)\\
&= 3^{-2-2s}\left(\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle}\right),
\end{aligned}$$

$$\begin{aligned}
\zeta_{(0,1)}^{H\langle t\rangle}(H,s) = 3^{-2-2s}\Big(&\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{G} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle a\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle at\rangle}\\
&+ \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H}\\
&+ \zeta^{G}\zeta^{H\langle at\rangle}\zeta^{H\langle a\rangle} + \zeta^{G}\zeta^{H\langle at\rangle}\zeta^{H\langle at\rangle} + \zeta^{G}\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + \zeta^{G}\zeta^{H\langle at\rangle}\zeta^{H}\\
&+ \zeta^{H\langle at^{-1}\rangle}\zeta^{G}\zeta^{H\langle a\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{G}\zeta^{H\langle at^{-1}\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{G}\zeta^{H\langle t\rangle}\\
&+ \zeta^{H\langle at^{-1}\rangle}\zeta^{G}\zeta^{H} + \zeta^{G}\zeta^{G}\zeta^{H}\Big)\\
= 3^{-2-2s}\Big(&3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle a\rangle} + \zeta^{H\langle at^{-1}\rangle}(\zeta^{H\langle at\rangle})^2\\
&+ (\zeta^{H\langle at^{-1}\rangle})^2\zeta^{H\langle at\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H}\\
&+ 3\zeta^{H\langle at\rangle}\zeta^{H\langle a\rangle} + 3(\zeta^{H\langle at\rangle})^2 + 3\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at\rangle}\zeta^{H}\\
&+ 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle a\rangle} + 3(\zeta^{H\langle at^{-1}\rangle})^2 + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H} + 9\zeta^{H}\Big).
\end{aligned}$$

Then $\zeta_{H_1}^{H\langle t\rangle}(H,s)$ is given by (6.29). Summing $\zeta_{(0,0)}^{H\langle t\rangle}(H,s)$, $\zeta_{(0,2)}^{H\langle t\rangle}(H,s)$ and $\zeta_{(0,1)}^{H\langle t\rangle}(H,s)$, we obtain

$$\begin{aligned}\zeta_{H_1}^{H\langle t\rangle}(H,s) = 3^{-1-2s}\Big(& (\zeta^{H\langle a\rangle})^2\zeta^{H\langle t\rangle} + 3(\zeta^{H\langle a\rangle})^2 + 6\zeta^{H\langle a\rangle}\zeta^{H\langle t\rangle} \\ & + \zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle} \\ & + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle a\rangle} + \zeta^{H\langle at^{-1}\rangle}(\zeta^{H\langle at\rangle})^2 \\ & + (\zeta^{H\langle at^{-1}\rangle})^2\zeta^{H\langle at\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + \zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle at\rangle}\zeta^{H} \\ & + 3\zeta^{H\langle at\rangle}\zeta^{H\langle a\rangle} + 3(\zeta^{H\langle at\rangle})^2 + 3\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at\rangle}\zeta^{H} \\ & + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle a\rangle} + 3(\zeta^{H\langle at^{-1}\rangle})^2 + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at^{-1}\rangle}\zeta^{H} + 9\zeta^{H}\Big).\end{aligned} \tag{6.30}$$

Computing $\zeta_{H_1}^{H}(H,s)$

From Lemma 6.3.5, we get $I_H(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H_1$ if and only if there exists $i \in [0,2]$ such that $\langle t\rangle \nleqslant I_G(\vartheta_i)$ and the following assertion holds:

$$\neg(\exists j \in [0,2] : H\langle a\rangle \leqslant I_G(\vartheta_j) \quad \wedge \quad H\langle at\rangle \leqslant I_G(\vartheta_{j+1}) \quad \wedge \quad H\langle at^{-1}\rangle \leqslant I_G(\vartheta_{j+2})).$$

Furthermore, since every conjugate of $\vartheta_0\otimes\vartheta_1\otimes\vartheta_2 \in \mathrm{Irr}(H)$ gives rise to the same irreducible character φ of H of degree $\varphi(1) = 9\vartheta_0(1)\vartheta_1(1)\vartheta_2(1)$, we have

$$\begin{aligned}\zeta_{H_1}^{H}(H,s) = 3^{-2-2s}\Big(& (\sum_{T\in\mathcal{G}}\zeta^{T})^3 - 3(\zeta^{H\langle a\rangle}+\zeta^{G})(\zeta^{H\langle at\rangle}+\zeta^{G})(\zeta^{H\langle at^{-1}\rangle}+\zeta^{G}) \\ & - (\zeta^{H\langle t\rangle}+\zeta^{G})^3 + 3(\zeta^{G})^3\Big) - \sum_{T\in\mathcal{G}\setminus\{H\}}\zeta_{H_1}^{T}.\end{aligned}$$

It follows from Proposition 6.3.16 that, we $\zeta_{H_1}^{H\langle at\rangle}(H,s) = 0$ and $\zeta_{H_1}^{H\langle at^{-1}\rangle}(H,s) = 0$. Also, it is clear from Lemma 6.4.1 that $\zeta_{H_1}^{G}(H,s) = 0$. Therefore,

$$\begin{aligned}\zeta_{H_1}^{H}(H,s) = 3^{-2-2s}\Big(& (\sum_{T\in\mathcal{G}}\zeta^{T})^3 - 3(\zeta^{H\langle a\rangle}+\zeta^{G})(\zeta^{H\langle at\rangle}+\zeta^{G})(\zeta^{H\langle at^{-1}\rangle}+\zeta^{G}) \\ & - (\zeta^{H\langle t\rangle}+\zeta^{G})^3 + 3(\zeta^{G})^3\Big) - \zeta_{H_1}^{H\langle a\rangle} - \zeta_{H_1}^{H\langle t\rangle}.\end{aligned}$$

Now from the computation of $\zeta_{H_1}^{H\langle a\rangle}(H,s)$ and $\zeta_{H_1}^{H\langle t\rangle}(H,s)$ it follows that

$$\begin{aligned}\zeta_{H_1}^{H}(H,s) = 3^{-2-2s}\Big(&(\zeta^H)^3 + 3(\zeta^H)^2\zeta^{H\langle a\rangle} + 3(\zeta^H)^2\zeta^{H\langle at\rangle} + 3(\zeta^H)^2\zeta^{H\langle at^{-1}\rangle}\\
&+ 3(\zeta^H)^2\zeta^{H\langle t\rangle} + 9(\zeta^H)^2 + 3\zeta^H(\zeta^{H\langle a\rangle})^2 + 6\zeta^H\zeta^{H\langle a\rangle}\zeta^{H\langle at\rangle} + 6\zeta^H\zeta^{H\langle a\rangle}\zeta^{H\langle at^{-1}\rangle}\\
&+ 6\zeta^H\zeta^{H\langle a\rangle}\zeta^{H\langle t\rangle} + 18\zeta^H\zeta^{H\langle a\rangle} + 3\zeta^H(\zeta^{H\langle at\rangle})^2 + 3\zeta^H\zeta^{H\langle at\rangle}\zeta^{H\langle at^{-1}\rangle}\\
&+ 6\zeta^H\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + 9\zeta^H\zeta^{H\langle at\rangle} + 3\zeta^H(\zeta^{H\langle at^{-1}\rangle})^2 + 6\zeta^H\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle}\\
&+ 9\zeta^H\zeta^{H\langle at^{-1}\rangle} + 3\zeta^H(\zeta^{H\langle t\rangle})^2 + 18\zeta^H\zeta^{H\langle t\rangle} + (\zeta^{H\langle a\rangle})^3 + 3(\zeta^{H\langle a\rangle})^2\zeta^{H\langle at\rangle}\\
&+ 3(\zeta^{H\langle a\rangle})^2\zeta^{H\langle at^{-1}\rangle} + 3\zeta^{H\langle a\rangle}(\zeta^{H\langle at\rangle})^2 + 6\zeta^{H\langle a\rangle}\zeta^{H\langle at\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle a\rangle}(\zeta^{H\langle at^{-1}\rangle})^2\\
&+ 6\zeta^{H\langle a\rangle}\zeta^{H\langle at^{-1}\rangle}\zeta^{H\langle t\rangle} + 3\zeta^{H\langle a\rangle}(\zeta^{H\langle t\rangle})^2 + (\zeta^{H\langle at\rangle})^3 + 3(\zeta^{H\langle at\rangle})^2\zeta^{H\langle t\rangle}\\
&+ 3\zeta^{H\langle at\rangle}(\zeta^{H\langle t\rangle})^2 + (\zeta^{H\langle at^{-1}\rangle})^3 + 3(\zeta^{H\langle at^{-1}\rangle})^2\zeta^{H\langle t\rangle} + 3\zeta^{H\langle at^{-1}\rangle}(\zeta^{H\langle t\rangle}))^2\Big)\\
&- 3^{-2s}\Big(\zeta^H(H,3s) + \zeta^{H\langle a\rangle}(H,3s) + \zeta^{H\langle at\rangle}(H,3s) + \zeta^{H\langle at^{-1}\rangle}(H,3s)\Big). \quad (6.31)\end{aligned}$$

Case 4: $S = H$**.**

Let $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2 \in \mathrm{Irr}(H_1)$ with $I_G(\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2) = H$. Let $\varphi \in \mathrm{Irr}(H)$ be such that $\vartheta_0 \otimes \vartheta_1 \otimes \vartheta_2$ is an irreducible constituent of $\varphi|_{H_1}$. From Corollary 6.3.21, Corollary 6.3.23, Corollary 6.3.26, Corollary 6.3.28, Corollary 6.3.31 and Corollary 6.3.33 we obtain

$$\zeta_H^G(H,s) = 3, \quad (6.32)$$

$$\zeta_H^{H\langle a\rangle}(H,s) = 6, \quad (6.33)$$

$$\zeta_H^{H\langle at\rangle}(H,s) = 6, \quad (6.34)$$

$$\zeta_H^{H\langle at^{-1}\rangle}(H,s) = 6, \quad (6.35)$$

$$\zeta_H^{H\langle t\rangle}(H,s) = 6 + 18 \cdot 3^{-s}, \quad (6.36)$$

$$\zeta_H^H(H,s) = 54. \quad (6.37)$$

6.4.2 Computing the representation zeta function of H

Here we compute the representation zeta function $\zeta(H,s)$ of H by combining the partial recursive representation zeta functions from Section 6.4.1. From (6.18) and (6.19) we have

$$\zeta(H,s) = \sum_{T\in\mathcal{G}} \zeta^T(H,s) = \sum_{T\in\mathcal{G}} \sum_{S\in\mathcal{H}} \zeta_S^T(H,s),$$

where $\mathcal{G}$ and $\mathcal{H}$ are defined as in the beginning of Section 6.3. We obtain $\zeta(H,s)$ in six steps, where each step corresponds to the computation of $\zeta^T(H,s)$ for $T \in \mathcal{G}$. Thanks to Lemma 6.4.1, we get

$$\zeta^G(H,s) = \sum_{S\in\mathcal{H}} \zeta_S^G(H,s) = \zeta_H^G(H,s) = 3. \quad (6.38)$$

Now, from (6.20), (6.27) and (6.33), we obtain

$$\begin{aligned}\zeta^{H\langle a\rangle}(H,s) &= \sum_{S\in\mathcal{H}} \zeta_S^{H\langle a\rangle}(H,s) = \zeta_H^{H\langle a\rangle} + \zeta_{H_1\langle\bar{t}\rangle}^{H\langle a\rangle} + \zeta_{H_1}^{H\langle a\rangle} \\ &= 6 + 3^{1-s}\zeta^{H\langle t\rangle}(H,3s) + 3^{-2s}\Big(\zeta^{H}(H,3s) + \zeta^{H\langle a\rangle}(H,3s) + \zeta^{H\langle at\rangle}(H,3s) \\ &\quad + \zeta^{H\langle at^{-1}\rangle}(H,3s)\Big). \end{aligned} \tag{6.39}$$

It follows from (6.21) and (6.34) that

$$\zeta^{H\langle at\rangle}(H,s) = \sum_{S\in\mathcal{H}} \zeta_S^{H\langle at\rangle}(H,s) = \zeta_H^{H\langle at\rangle} + \zeta_{H_1\langle\bar{t}\rangle}^{H\langle at\rangle} = 6 + 3^{1-s}\zeta^{H\langle t\rangle}(H,3s). \tag{6.40}$$

Summing (6.22) and (6.35) we get

$$\zeta^{H\langle at^{-1}\rangle}(H,s) = \sum_{S\in\mathcal{H}} \zeta_S^{H\langle at^{-1}\rangle}(H,s) = \zeta_H^{H\langle at^{-1}\rangle} + \zeta_{H_1\langle\bar{t}\rangle}^{H\langle at^{-1}\rangle} = 6 + 3^{1-s}\zeta^{H\langle t\rangle}(H,3s). \tag{6.41}$$

Now observe that $\zeta^{H\langle at\rangle}(H,s) = \zeta^{H\langle at^{-1}\rangle}(H,s)$. We set

$$\alpha(s) = \zeta^{H\langle a\rangle}(H,s), \quad \text{and} \quad \beta(s) = \zeta^{H\langle at\rangle}(H,s) = \zeta^{H\langle at^{-1}\rangle}(H,s). \tag{6.42}$$

Further, we define

$$\tau(s) = \zeta^{H\langle t\rangle}(H,s), \quad \text{and} \quad \xi(s) = \zeta^{H}(H,s). \tag{6.43}$$

For convenience, we write simply write f instead of $f(s)$, for $f \in \{\alpha,\beta,\tau,\xi\}$. Now, from (6.23), (6.25), (6.30) and (6.36), we have

$$\begin{aligned}\tau(s) &= \sum_{S\in\mathcal{H}} \zeta_S^{H\langle t\rangle}(H,s) = \zeta_H^{H\langle t\rangle} + \zeta_{H_1\langle\bar{t}\rangle}^{H\langle t\rangle} + \zeta_{H_1\langle x_0\bar{t}\rangle}^{H\langle t\rangle} + \zeta_{H_1\langle x_1\bar{t}\rangle}^{H\langle t\rangle} + \zeta_{H_1\langle x_2\bar{t}\rangle}^{H\langle t\rangle} + \zeta_{H_1}^{H\langle t\rangle} \\ &= 6 + 18\cdot 3^{-s} + 3^{1-s}\left(3\tau + 3\alpha + 6\beta + \beta^2\right) + 3^{-1-2s}\left(2\beta^3 + 9\beta^2 + \alpha\beta^2 + 6\alpha\beta + 3\alpha^2\right) \\ &\quad + 3^{-1-2s}\xi\left(\beta^2 + 6\beta + 9\right) + 3^{-1-2s}\tau\left(2\beta^2 + 12\beta + 6\alpha + \alpha^2\right). \end{aligned} \tag{6.44}$$

Finally, it follows from (6.24), (6.26), (6.31) and (6.37) that

$$\begin{aligned}\xi(s) &= \sum_{S\in\mathcal{H}} \zeta_S^{H}(H,s) = \zeta_H^{H} + \zeta_{H_1\langle\bar{t}\rangle}^{H} + \zeta_{H_1\langle x_0\bar{t}\rangle}^{H} + \zeta_{H_1\langle x_1\bar{t}\rangle}^{H} + \zeta_{H_1\langle x_2\bar{t}\rangle}^{H} + \zeta_{H_1}^{H} \\ &= 54 + 3^{-s}\left(\tau^3 + 9\tau^2 + 18\tau - 9\tau(3s)\right) + 3^{1-s}\left(\alpha\beta^2 + 6\alpha\beta + 2\beta^2 + 6\alpha + 12\beta\right) \\ &\quad + 3^{-2-2s}\Big(\xi^3 + \xi^2\left(3\alpha + 6\beta + 3\tau + 9\right) \\ &\quad + \xi\left(3\alpha^2 + 12\alpha\beta + 6\alpha\tau + 18\alpha + 9\beta^2 + 12\beta\tau + 18\beta + 3\tau^2 + 18\tau\right) \\ &\quad + \alpha^3 + 6\alpha^2\beta + 6\alpha\beta^2 + 12\alpha\beta\tau + 3\alpha\tau^2 + 2\beta^3 + 6\beta^2\tau + 6\beta\tau^2\Big) \\ &\quad - 3^{-2s}\big(\xi(3s) + \alpha(3s) + 2\beta(3s)\big). \end{aligned} \tag{6.45}$$

Now, by adding (6.38), (6.39), (6.40), (6.41), (6.44) and (6.45), we get $\zeta(H,s)$:

$$\zeta(H,s) = 3 + \alpha(s) + 2\beta(s) + \tau(s) + \xi(s), \tag{6.46}$$

where the following recursions hold:

$$\begin{aligned}
\alpha(s) &= 6 + 3^{1-s}\tau(3s) + 3^{-2s}\alpha(3s) + 2\cdot 3^{-2s}\beta(3s) + 3^{-2s}\xi(3s),\\
\beta(s) &= 6 + 3^{1-s}\tau(3s),\\
\tau(s) &= 6 + 18\cdot 3^{-s} + 3^{1-s}\left(3\tau + 3\alpha + 6\beta + \beta^2\right) + 3^{-1-2s}\left(2\beta^3 + 9\beta^2 + \alpha\beta^2 + 6\alpha\beta + 3\alpha^2\right)\\
&\quad + 3^{-1-2s}\xi\left(\beta^2 + 6\beta + 9\right) + 3^{-1-2s}\tau\left(2\beta^2 + 12\beta + 6\alpha + \alpha^2\right),\\
\xi(s) &= 54 + 3^{-s}\left(\tau^3 + 9\tau^2 + 18\tau - 9\tau(3s)\right) + 3^{1-s}\left(\alpha\beta^2 + 6\alpha\beta + 2\beta^2 + 6\alpha + 12\beta\right)\\
&\quad + 3^{-2-2s}\Big(\xi^3 + \xi^2\left(3\alpha + 6\beta + 3\tau + 9\right)\\
&\quad + \xi\left(3\alpha^2 + 12\alpha\beta + 6\alpha\tau + 18\alpha + 9\beta^2 + 12\beta\tau + 18\beta + 3\tau^2 + 18\tau\right)\\
&\quad + \alpha^3 + 6\alpha^2\beta + 6\alpha\beta^2 + 12\alpha\beta\tau + 3\alpha\tau^2 + 2\beta^3 + 6\beta^2\tau + 6\beta\tau^2\Big)\\
&\quad - 3^{-2s}\big(\xi(3s) + \alpha(3s) + 2\beta(3s)\big).
\end{aligned}$$

6.5 Computing the representation zeta function of G

In this section we compute the representation zeta function $\zeta(G,s)$ of G using the recursive representation zeta function (6.46) of H from Section 6.4.2. We recall that $\mathcal{G}$ is the set of subgroups that lie between G and H, and is given by

$$\mathcal{G} = \{G, H\langle t\rangle, H\langle a\rangle, H\langle at\rangle, H\langle at^{-1}\rangle, H\};$$

cf. Figure 6.1. We begin with the following observations.

Lemma 6.5.1. *Let $\varphi \in \mathrm{Irr}(H)$ be such that $I_G(\varphi) = G$. Then φ extends to a linear character of G if and only if $\varphi = 1_H$. Otherwise, φ gives rise to an irreducible character of G of degree 3 which restricts to 3φ on H.*

Proof. Thanks to Corollary 5.3.12, we conclude that φ is linear. It is easy to see that, φ extends to G if and only if $H = [G,G] \leqslant \ker(\varphi)$, i.e., $\varphi = 1_H$. This proves the first assertion of the result.

Now, suppose that $\varphi \neq 1_H$. Then φ does not admit an extension towards G. In fact, φ extends to irreducible characters of an intermediate subgroup $L \in \mathcal{G}\backslash\{G,H\}$. Let $\psi \in \mathrm{Irr}(L)$ be an extension of φ. Then $I_G(\psi) = L$ and ψ induces to G and gives rise to the irreducible character $\psi^G \in \mathrm{Irr}(G)$ of degree $\psi^G(1) = [G:L]\,\psi(1) = 3$. Furthermore, $\psi^G|_H = 3\varphi$. □

Lemma 6.5.2. *Let $\varphi \in \mathrm{Irr}(H)$ be such that $I_G(\varphi) = L$, where $L \in \mathcal{G}\backslash\{G, H\}$. Then φ gives rise to three irreducible characters of G, each of degree $3\,\varphi(1)$, which restricts to a sum of three distinct G-conjugates of φ on H.*

Proof. Clearly φ extends in three ways to an irreducible character of L. Let $\psi \in \mathrm{Irr}(L)$ be an extension of φ. Then $I_G(\psi) = L$ and ψ induces to the irreducible character $\psi^G \in \mathrm{Irr}(G)$ of degree $\psi^G(1) = [G : L]\,\psi(1) = 3\,\varphi(1)$. Furthermore, $\psi^G|_H = \varphi_0 + \varphi_1 + \varphi_3$, where $\varphi_1, \varphi_2, \varphi_3$ are the distinct conjugates of φ in G. □

Lemma 6.5.3. *Let $\varphi \in \mathrm{Irr}(G)$ with $I_G(\varphi) = H$. Then φ gives rises to an irreducible character of degree $9\,\varphi(1)$ of G, and each of which restricts to a sum of nine distinct G-conjugates of φ on H.*

Proof. Since $I_G(\varphi) = H$, the character φ induces to the irreducible character $\varphi^G \in \mathrm{Irr}(G)$ of degree $\varphi^G(1) = [G : H]\,\varphi(1) = 9\,\varphi(1)$. Moreover, $\varphi^G|_H = \sum_{i,j\in[0,2]} \varphi^{a^i t^j}$. □

We recall that from (6.42) and (6.43) the notation

$$\alpha(s) = \zeta^{H\langle a\rangle}(H, s), \qquad \beta(s) = \zeta^{H\langle at\rangle}(H, s) = \zeta^{H\langle at^{-1}\rangle}(H, s),$$
$$\tau(s) = \zeta^{H\langle t\rangle}(H, s), \qquad \xi(s) = \zeta^{H}(H, s).$$

By setting $q = 3^{-s}$ and rearranging the terms of $\alpha(s)$, $\beta(s)$, $\tau(s)$ and $\xi(s)$, we get

$$\alpha(s) = 6 + 3q\tau(3s) + q^2\alpha(3s) + 2q^2\beta(3s) + q^2\xi(3s), \tag{6.47}$$

$$\beta(s) = 6 + 3q\tau(3s), \tag{6.48}$$

$$\begin{aligned}\tau(s) = {} & 6 + 18q + 9q\tau + 9q\alpha + 18q\beta + 3q\beta^2 + \frac{2}{3}q^2\beta^3 + 3q^2\beta^2 + \frac{1}{3}q^2\alpha\beta^2 + 2q^2\alpha\beta + q^2\alpha^2 \\ & + \frac{1}{3}q^2\xi\beta^2 + 2q^2\xi\beta + 3q^2\xi + \frac{2}{3}q^2\tau\beta^2 + 4q^2\tau\beta + 2q^2\tau\alpha + \frac{1}{3}q^2\tau\alpha^2,\end{aligned} \tag{6.49}$$

$$\begin{aligned}\xi(s) = {} & 54 + q\tau^3 + 9q\tau^2 + 18q\tau + 3q\alpha\beta^2 + 18q\alpha\beta + 6q\beta^2 + 18q\alpha + 36q\beta + \frac{1}{9}q^2\xi^3 + \frac{1}{3}q^2\xi^2\alpha \\ & + \frac{2}{3}q^2\xi^2\beta + \frac{1}{3}q^2\xi^2\tau + q^2\xi^2 + \frac{1}{3}q^2\xi\alpha^2 + \frac{4}{3}q^2\xi\alpha\beta + \frac{2}{3}q^2\xi\alpha\tau + 2q^2\xi\alpha + q^2\xi\beta^2 \\ & + \frac{4}{3}q^2\xi\beta\tau + 2q^2\xi\beta + \frac{1}{3}q^2\xi\tau^2 + 2q^2\xi\tau + \frac{1}{9}q^2\alpha^3 + \frac{2}{3}q^2\alpha^2\beta + \frac{2}{3}q^2\alpha\beta^2 + \frac{4}{3}q^2\alpha\beta\tau \\ & + \frac{1}{3}q^2\alpha\tau^2 + \frac{2}{9}q^2\beta^3 + \frac{2}{3}q^2\beta^2\tau + \frac{2}{3}q^2\beta\tau^2 - 9q\tau(3s) - q^2\xi(3s) - q^2\alpha(3s) - 2q^2\beta(3s).\end{aligned} \tag{6.50}$$

Now, writing $\zeta_1(s) = \xi(s)$, $\zeta_2(s) = \tau(s)$, $\zeta_3(s) = \alpha(s)$ and $\zeta_4(s) = \beta(s)$, one can easily verify that, for $i \in \{1, 2, 3, 4\}$, the recursive formula for $\zeta_i(s)$, provided in [14, Section 2.2], is precisely that of the corresponding $f(s)$, for $f \in \{\xi, \tau, \alpha, \beta\}$. Therefore, in Theorem 6.5.4 below, we summarise a proof for the recursive representation zeta function of G stated in

[14, Section 2.2], which was obtained by computer assisted calculations not recorded in detail.

Theorem 6.5.4. *The representation zeta function $\zeta(G,s)$ is given by the equation*

$$\begin{aligned}\zeta(G,s) = 9 + 2\cdot 3^{-s} + 3^{-s}\left(\zeta^{H\langle a\rangle}(H,s) + \zeta^{H\langle at\rangle}(H,s) + \zeta^{H\langle at^{-1}\rangle}(H,s) + \zeta^{H\langle t\rangle}(H,s)\right) \\ + 3^{-2-2s}\zeta^{H}(H,s), \qquad (6.51)\end{aligned}$$

and admits the the following form with $q = 3^{-s}$

$$\begin{aligned}\zeta(G,s) = {} & 9 + 2q + q\alpha + 2q\beta + q\tau + 6q^2 + \frac{1}{9}q^3\tau^3 + q^3\tau^2 + 2q^3\tau + \frac{1}{3}q^3\alpha\beta^2 + 2q^3\alpha\beta + \frac{2}{3}q^3\beta^2 \\ & + 2q^3\alpha + 4q^3\beta + \frac{1}{81}q^4\xi^3 + \frac{1}{27}q^4\xi^2\alpha + \frac{2}{27}q^4\xi^2\beta + \frac{1}{27}q^4\xi^2\tau + \frac{1}{9}q^4\xi^2 + \frac{1}{27}q^4\xi\alpha^2 \\ & + \frac{4}{27}q^4\xi\alpha\beta + \frac{2}{27}q^4\xi\alpha\tau + \frac{2}{9}q^4\xi\alpha + \frac{1}{9}q^4\xi\beta^2 + \frac{4}{27}q^4\xi\beta\tau + \frac{2}{9}q^4\xi\beta + \frac{1}{27}q^4\xi\tau^2 \\ & + \frac{2}{9}q^4\xi\tau + \frac{1}{81}q^4\alpha^3 + \frac{2}{27}q^4\alpha^2\beta + \frac{2}{27}q^4\alpha\beta^2 + \frac{4}{27}q^4\alpha\beta\tau + \frac{1}{27}q^4\alpha\tau^2 + \frac{2}{81}q^4\beta^3 \\ & + \frac{2}{27}q^4\beta^2\tau + \frac{2}{27}q^4\beta\tau^2 - q^3\tau(3s) - \frac{1}{9}q^4\xi(3s) - \frac{1}{9}q^4\alpha(3s) - \frac{2}{9}q^4\beta(3s), \qquad (6.52)\end{aligned}$$

where α, β, τ, ξ satisfy the recursive relations specified at the end of Section 6.4.

Proof. The first two summands are coming from the G-invariant irreducible characters of H. We recall from (6.38) that $\zeta^G(H,s) = 3$. It is immediate from Lemma 6.5.1 that only the trivial character of H extends to G, yielding 9 linear characters. The remaining two characters yield two irreducible characters, each of degree 3. By Lemma 6.5.2, every character $\varphi \in \mathrm{Irr}(H)$ with $I_G(\varphi) \in \mathcal{G}\backslash\{G,H\}$, on average, gives rise to one irreducible character of G of degree $3\,\varphi(1)$. This gives the terms with coefficient 3^{-s} in (6.51). Now, suppose that $\varphi^G \in \mathrm{Irr}(G)$ is the character induced from a character $\varphi \in \mathrm{Irr}(H)$ of inertia group $I_G(\varphi) = H$. Since $G/H = \langle a\,H\rangle \times \langle t\,H\rangle \cong C_3 \times C_3$ (Theorem 2.4.19(iii)), by Clifford's theorem we get

$$\varphi^G|_H = \sum_{i,j\in[0,2]} \varphi^{a^i t^j}.$$

Furthermore, each of the conjugates $\varphi^{a^i t^j}$ of φ in G gives rises to the same irreducible character φ^G. Therefore, we divide the partial zeta function $\zeta^H(H,s)$ by 9 to compensate for the overcounting. Considering this fact, we get the last summand in (6.51) from Lemma 6.5.3. Again by writing $q = 3^{-s}$, from (6.47), (6.48), (6.49) and (6.50) we get the described form (6.52). □

Part II

Generalisations of the Basilica group

Chapter 7

Overview

This part comprises the following two articles:

1. With Jan Moritz Petschick: *On the Basilica operation*, Groups Geometry, and Dynamics, to appear, [92];

2. With Anitha Thillaisundaram: *Maximal subgroups of generalised Basilica groups*, available at `arXiv:2103.05452[math.GR]`, [94].

As indicated in Chapter 1, the first article introduces the Basilica operation that associates to any group G of tree automorphisms a family of Basilica groups, $\mathrm{Bas}_s(G)$, for $s \in \mathbb{N}$. In the second article, we study the maximal subgroups of Basilica groups obtained from generalisations of the dyadic odometer. We incorporate the articles as Chapter 8 and Chapter 9. Both chapters are self-contained with references collected at the end. The numbering of the sections and results from the articles [92] and [94] are modified in order to be consistent with the rest of the dissertation. Section or result 'A' in [92] (resp. [94]) will be numbered as 8.A (resp. 9.A). In Chapter 8, we give a proof of [92, Theorem 6.8], which was not provided in [92] because of its technicality. Here we indicate the individual contribution of authors to the articles [92] and [94].

Authors' contribution statement

I declare that the research and the process of writing for the articles [92] and [94] were shared equally among myself and my collaborators. One may find below a detailed account of contributions.

The collaboration between myself and Petschick was kicked off by a basic idea of myself to compute the level stabilisers in certain special cases of the generalised Basilica groups. The research for Section 4 of [92] was conducted by developing this idea. The discussions were mainly carried out in presence by mutual exchange of ideas between myself and Petschick. We contributed equally to the research and formalisation of Theorem 1.4, Theorem 1.5, Theorem 1.6, and Theorem 1.7 of [92]. The general cases of Basilica groups were treated by Petschick, while I focused on the study of generalised Basilica groups. I contributed less than a half to the investigation of Section 2 and Section 3, especially to the specific write-up of Theorem 1.1, Theorem 1.2 and Theorem 1.3 and their proofs. The research for Section 6, Section 7 and Section 8 was mostly carried out by myself, and I have contributed more than a half to the formalisation of Theorem 1.8, Theorem 1.9 and Theorem 1.10 and their proofs. Section 5 is an application of results from Section 2, Section 3 and 4 of [92], to which both of us have equally contributed. Also, the task of writing the introduction was shared equally.

The topic of investigation in [94] was suggested by my collaborator Thillaisundaram. We communicated via emails and using online platforms. The proof of the main result Theorem 1.1 of [94] resulted from several joint attempts some of which remained unsuccessful but gave inspiration for renewed efforts. The given proof is based on an idea of myself, which was inspired by an observation of Thillaisundaram. We contributed equally to the research and the process of writing up Theorem 1.1 and its proof.

Chapter 8

On the Basilica operation

8.1 Introduction

Groups acting on rooted trees play an important role in various areas of group theory, for example in the study of groups of intermediate growth, just infinite groups and groups related to the Burnside problem. Over the years, many groups of automorphisms of rooted trees have been defined and studied. Often they can be regarded as generalisations of early constructions to wider families of groups with similar properties.

In this paper, we consider an operation on the subgroups of the automorphism group $\mathrm{Aut}\, T$ of a rooted tree T with degree $m \geqslant 2$. It is inspired by the *Basilica group* $\mathcal{B}$, a group acting on the binary rooted tree, which was introduced by Grigorchuk and Żuk in [58] and [59]. The Basilica group $\mathcal{B}$ is a particularly interesting example in its own right: it is a self-similar torsion-free weakly branch group, just-(non-soluble) and of exponential word growth. It was the first group known to be not sub-exponentially amenable [59], but amenable [20, 24]. Furthermore, it is the iterated monodromy group of $z^2 - 1$ [76, 93], and it has the 2-congruence subgroup property [46].

The Basilica group $\mathcal{B}$ is usually defined as the group generated by two automorphisms

$$a = (b, \mathrm{id}) \text{ and } b = (0\ 1)(a, \mathrm{id}),$$

acting on the binary rooted tree (in [59] the elements are defined with id on the left, which is merely notational). We point out the similarities between these two generators and the single automorphism generating the *dyadic odometer*. The latter provides an embedding of the infinite cyclic group into the automorphism group $\mathrm{Aut}\, T$ of the binary rooted tree T, given by

$$c = (0\ 1)(c, \mathrm{id}).$$

We can regard b as a delayed version of c, that takes an intermediate step acting as a, before returning to itself. Considering the automata defining the generators of both groups (cf. Figure 8.2), the relationship is even more apparent. We obtain the automaton defining b from the automaton defining c by replacing every edge that does not point to the state of the trivial element with an edge pointing to a new state, which in turn points to the old state upon reading 0 and to the state of the trivial element upon reading any other letter. See Figure 8.1 for an illustration of this replacement rule.

Figure 8.1: Replacement rule for edges.

The same can be done for any automorphism of T and any number s of intermediate states. For any group of automorphisms G, this operation yields a new group of tree automorphisms defined by the automaton with s intermediate steps, which we call $\mathrm{Bas}_s(G)$, the s-th Basilica group of G. A precise, algebraic definition that does not refer to automata will be given in Definition 8.2.3. Figure 8.2 depicts for example the automaton defining $\mathrm{Bas}_8(\mathcal{O}_2)$, while Figure 8.3 depicts the automaton defining the generators of the Gupta–Sidki 3-group $\ddot{\Gamma}$ and the corresponding automaton obtained by the operation Bas_2.

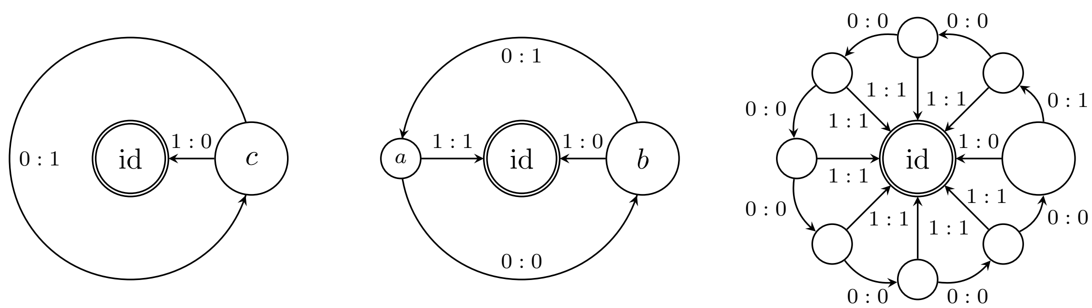

Figure 8.2: Automata for the dyadic odometer $\mathcal{O}_2$, the Basilica group $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$, and $\mathrm{Bas}_8(\mathcal{O}_2)$.

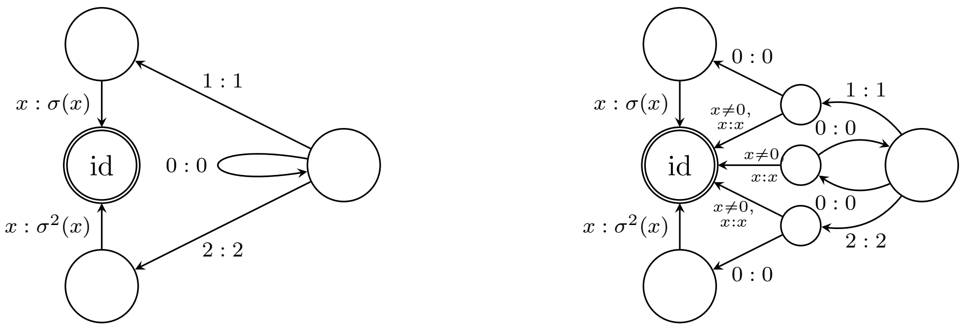

Figure 8.3: Automata for the Gupta–Sidki 3-group $\ddot{\Gamma}$ and $\mathrm{Bas}_2(\ddot{\Gamma})$, where σ is a cyclic permutation.

We prove that many of the desirable properties of the original Basilica group $\mathcal{B}$ are a consequence of the fact that the binary odometer $\mathcal{O}_2$ has those properties and that the properties are preserved under the Basilica operation. We summarise results of this kind for the general Basilica operation in the following theorem.

Theorem 8.1.1. *Let G be a group of automorphisms of a regular rooted tree. Let P be a property from the list below. Then, if G has P, the s-th Basilica group $\mathrm{Bas}_s(G)$ of G has P for all $s \in \mathbb{N}_+$.*

1. *spherically transitive*
2. *self-similar*
3. *(strongly) fractal*
4. *contracting*
5. *weakly branch*
6. *generated by finite-state bounded automorphisms*

As a consequence we derive conditions for $\mathrm{Bas}_s(G)$ to have solvable word problem and to be amenable. Furthermore, we provide a condition for $\mathrm{Bas}_s(G)$ to be a weakly regular branch group given that G satisfies a group law. This enables us to construct a weakly regular branch group over a prescribed verbal subgroup.

The class of spinal groups, defined in [23], is another important class of groups acting on T; it contains the Grigorchuk group and all GGS-groups, see Definition 8.3.7. It is not true that the Basilica operation preserves being spinal, however groups obtained from spinal groups act as spinal groups on another tree $\delta_s T$, obtained by deleting layers from T.

Theorem 8.1.2. *Let G be a spinal group (resp. a GGS-group) acting on T. Then $\mathrm{Bas}_s G$ is a spinal (resp. a GGS-group) acting on $\delta_s T$ for all $s \in \mathbb{N}_+$.*

In contrast to Theorem 8.1.1, the exponential word growth of the original Basilica group $\mathcal{B}$ is not a general feature of groups obtained by the Basilica operation. In fact, the situation appears to be chaotic, for which we provide some examples, see Proposition 8.3.17 and Proposition 8.3.18.

Next we turn our attention to a class of groups G whose Basilica groups $\mathrm{Bas}_s(G)$ more closely resemble the original Basilica group. For this, we introduce the concept of the group G being *s-split* (see Definition 8.4.1). An s-split group decomposes by definition as a semi-direct product, algebraically modelling the property that the image of a delayed automorphism can be detected by observing the layers on which it has trivial labels. We prove that all abelian groups acting locally regular are s-split for all $s \in \mathbb{N}_+$, and that conversely, all s-split groups acting spherically transitive are abelian. Furthermore we obtain the following.

Theorem 8.1.3. *Let $s > 1$ and let G be an s-split self-similar group of automorphisms of a regular rooted tree acting spherically transitively. If G is torsion-free, then $\mathrm{Bas}_s(G)$ is torsion-free. Furthermore $\mathrm{Bas}_s(G)^{\mathrm{ab}} \cong G^s$.*

The $(s-1)$*-th splitting kernel* K_{s-1} is a normal subgroup of G measuring the failure of G to be s-split. A rigorous definition is found in Definition 8.4.1. If G is weakly regular branch over K_{s-1} (allowing K_{s-1} to be trivial, hence including s-split groups), we obtain a strong structural description of the layer stabilisers of $\mathrm{Bas}_s(G)$. The maps β_i are the algebraic analogues of the various added steps delaying an automorphism, defined in Definition 8.2.2.

Theorem 8.1.4. *Let G be a self-similar and very strongly fractal group of automorphisms of a regular rooted tree. Assume that G is weakly regular branch over K_{s-1}. Let $n \in \mathbb{N}_0$. Write $n = sq + r$ with $q \geqslant 0$ and $0 \leqslant r \leqslant s-1$. Then for all $s > 1$*

$$\mathrm{St}_{\mathrm{Bas}_s(G)}(n) = \langle \beta_i(\mathrm{St}_G(q+1)), \beta_j(\mathrm{St}_G(q)) \mid 0 \leqslant i < r \leqslant j < s \rangle^{\mathrm{Bas}_s(G)}.$$

This description allows us to provide an exact relationship between the Hausdorff dimension of a group G fulfilling the conditions of Theorem 8.1.4 and its Basilica groups $\mathrm{Bas}_s(G)$. The precise description makes use of the *series of obstructions* of G, a tailor-made technical construction, see Subsection 8.4.2 for details. Observing this series, we prove that the Hausdorff dimension of $\mathrm{Bas}_s(G)$ is bounded below by the Hausdorff dimension of G for all $s > 1$.

Corollary 8.1.5. *Let $G \leqslant \mathrm{Aut}\, T$ be very strongly fractal, self-similar, weakly regular branch over K_{s-1}, with $\dim_{\mathrm{H}} G < 1$. Then for all $s > 1$*

$$\dim_{\mathrm{H}} G < \dim_{\mathrm{H}} \mathrm{Bas}_s(G).$$

Here we define the Hausdorff dimension of $G \leqslant \Gamma$ as the Hausdorff dimension of its closure in Γ, where Γ is the subgroup of all automorphisms acting locally by a power of a fixed m-cycle. This subgroup is isomorphic to

$$\Gamma \cong \varprojlim_{n \in \mathbb{N}_+} \mathrm{C}_m \wr \overset{n}{\cdots} \wr \mathrm{C}_m.$$

If $m = p$, a prime, then Γ is a Sylow pro-p subgroup of $\operatorname{Aut} T$. The notion of Hausdorff dimension in the profinite setting as above was initially studied by Abercrombie [1] and subsequently by Barnea and Shalev [9]. It is analogous to the Hausdorff dimension defined as usual over $\mathbb{R}$.

In the second half of this paper we study the class of *generalised Basilica groups* $\mathrm{Bas}_s(\mathcal{O}_m^d)$, for d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$, defined by applying Bas_s to the free abelian group of rank d with a self-similar action derived from the m-adic odometer. We remark that the above generalisation of the original Basilica group $\mathcal{B}$ is different from the one given in [21], but it includes the class of p-Basilica groups, where p is a prime, studied recently in [33]. For every odd prime p, we obtain the p-Basilica group by setting $d = 1, m = p$ and $s = 2$ in $\mathrm{Bas}_s(\mathcal{O}_m^d)$. Our construction also includes special cases, $d = 1$ and $m = s = p$, studied by Hanna Sasse in her master's thesis supervised by Benjamin Klopsch. We record the properties of the generalised Basilica groups in the following theorem.

Theorem 8.1.6. *Let d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$. Let $B = \mathrm{Bas}_s(\mathcal{O}_m^d)$ be the generalised Basilica group. The following assertions hold:*

(i) B acts spherically transitively on the corresponding m-regular rooted tree,

(ii) B is self-similar and strongly fractal,

(iii) B is contracting, and has solvable word problem,

(iv) The group $\mathcal{O}_m^d$ is s-split, and $B^{\mathrm{ab}} \cong \mathbb{Z}^{ds}$,

(v) B is torsion-free,

(vi) B is weakly regular branch over its commutator subgroup,

(vii) B has exponential word growth.

Theorem 8.1.6(i) to Theorem 8.1.6(vi) are obtained by direct application of Theorem 8.1.1 and Theorem 8.1.3. The proof of Theorem 8.1.6(vii) is analogous to that of the original Basilica group $\mathcal{B}$ and can easily be generalised from [59, Proposition 4]. Nevertheless, one can prove Theorem 8.1.6 directly by considering the action of the group on the corresponding rooted tree, see [96].

We explicitly compute the Hausdorff dimension of $\mathrm{Bas}_s(\mathcal{O}_m^d)$, which turns out to be independent of the rank d of the free abelian group $\mathcal{O}_m^d$:

Theorem 8.1.7. *For all d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$*

$$\dim_{\mathrm{H}}(\mathrm{Bas}_s(\mathcal{O}_m^d)) = \frac{m(m^{s-1}-1)}{m^s-1}.$$

The above equality agrees with the formula of the Hausdorff dimension of p-Basilica groups given by [33], and also with the Hausdorff dimension of the original Basilica group $\mathcal{B}$ given in [12].

Theorem 8.1.8. *Let d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$. The generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ admits an L-presentation*

$$L = \langle Y \mid Q \mid \Phi \mid R \rangle$$

where the data Y, Q, R and Φ are specified in Section 8.6.

The concrete L-presentation requires unwieldy notation, whence it is not given here. It is analogous to the L-presentation of the original Basilica group $\mathcal{B}$ [59]. The name L-presentation stands as a tribute to Igor Lysionok who obtained such a presentation for the Grigorchuk group in [74]. It is now known that, every finitely generated, contracting, regular branch group admits a finite L-presentation but it is not finitely presentable (cf. [11]). Unfortunately, this result is not applicable to generalised Basilica groups as they are merely weakly branch. Also, the L-presentation of the generalised Basilica group is not finite as the set of relations is infinite. Nonetheless, akin to [59, Proposition 11], we can introduce a set of endomorphisms of the free group on the set of generators of the generalised Basilica group and obtain a finite L-presentation, see Definition 8.6.1, as defined in [11].

Using the concrete L-presentation of a generalised Basilica group, we obtain the following structural result.

Theorem 8.1.9. *Let d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$ and let B be the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$. We have:*

(i) For $s = 2$, the quotient group $\gamma_2(B)/\gamma_3(B) \cong \mathbb{Z}^{d^2}$,

(ii) For $s > 2$, the quotient group $\gamma_2(B)/\gamma_3(B) \cong \mathrm{C}_m^{ds-2} \times \mathrm{C}_{m^2}$.

This implies that the quotients $\gamma_i(B)/\gamma_{i+1}(B)$ of consecutive terms of the lower central series of a generalised Basilica group for $s > 2$ are finite for all $i \geqslant 2$, whereas a similar behaviour happens for the original Basilica group $\mathcal{B}$ from $i \geqslant 3$, see [15] for details.

For a group G of automorphisms of an m-regular rooted tree, we say that G has the congruence subgroup property (CSP) if every subgroup of finite index in G contains some layer stabiliser in G. The congruence subgroup property of branch groups has been studied comprehensively over the years, see [22], [45], [37]. The generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ does not have the CSP as its abelianisation is isomorphic to $\mathbb{Z}^{ds}$ (Theorem 8.1.6). However, the quotients of $\mathrm{Bas}_s(\mathcal{O}_m^d)$ by the layer stabilisers are isomorphic to subgroups of $C_m \wr \overset{n}{\cdots} \wr C_m$, for suitable $n \in \mathbb{N}_0$. If $m = p$, a prime, then these quotients are, in particular, finite p-groups. The class of all finite p-groups is a well-behaved class, i.e., it is closed under taking subgroups, quotients, extensions and direct limits. In light of this, we prove that $\mathrm{Bas}_s(\mathcal{O}_p^d)$ has the p-congruence subgroup property (p-CSP), a weaker version of CSP introduced by Garrido and Uria-Albizuri in [46]. The group G has the p-CSP if every subgroup of index a power of p in G contains some layer stabiliser in G. In [46] one finds a sufficient condition for a weakly branch group to have the p-CSP and it is also proved that the original Basilica group $\mathcal{B}$ has the 2-CSP. This argument is generalised by Fernandez-Alcober, Di Domenico, Noce and Thillaisundaram to see that the p-Basilica groups have the p-CSP. We further generalise these results.

Theorem 8.1.10. *For all $d, s \in \mathbb{N}_+$ with $s > 2$, and all primes p, the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_p^d)$ has the p-congruence subgroup property.*

Even though we follow the same strategy as in [46], the arguments differ significantly because of Theorem 8.1.9. Here we make use of Theorem 8.1.4 to obtain a normal generating set for the layer stabilisers of the generalised Basilica groups (Theorem 8.5.1). We remark that the result of Fernandez-Alcober, Di Domenico, Noce and Thillaisundaram on p-Basilica groups can be generalised to all $d \geqslant 2$ with additional work.

The organisation of the paper is as follows: In Section 8.2, we introduce the basic theory of groups acting on rooted trees and give the formal definition of the Basilica operation, together with important examples. The proofs of Theorem 8.1.1 and Theorem 8.1.2 are given in Section 8.3. Theorem 8.1.3 and related results for s-split groups are contained in Section 8.4, as well as the proofs of Theorem 8.1.4 and Theorem 8.1.7. Section 8.6 contains the proof of Theorem 8.1.8, while Section 8.7 and Section 8.8 contain the proofs of Theorem 8.1.9 and Theorem 8.1.10.

8.2 Preliminaries and Main Definitions

For any two integers i, j, let $[i, j]$ denote the interval in $\mathbb{Z}$. From here on, $T_m = T$ denotes the m-regular rooted tree for an arbitrary but fixed integer $m > 1$. The vertices of T are identified with the elements of the free monoid X^* on $X = [0, m-1]$ by labeling the vertices from left-to-right. We denote the empty word by ϵ. For $n \in \mathbb{N}_0$, the *n-th layer of T* is the set X^n of vertices represented by words of length n.

Every (graph) automorphism of T fixes ϵ and moreover maps the n-th layer to itself for all $n \in \mathbb{N}_0$. The action of the full group of automorphisms $\operatorname{Aut} T$ on each layer is transitive. A subgroup of $\operatorname{Aut} T$ with this property is called *spherically transitive.* The stabiliser of a word u under the action of a group G of automorphisms of T is denoted by $\operatorname{st}_G(u)$ and the intersection of all stabilisers of words of length n is called the *n-th layer stabiliser*, denoted $\operatorname{St}_G(n)$.

Let $a \in \operatorname{Aut} T$ and let u, v be words. Since layers are invariant under a, the equation

$$a(uv) = a(u)a|_u(v)$$

defines a unique automorphism $a|_u$ of T called the *section of a at u.* This automorphism can be thought of as the automorphism induced by a by identifying the subtrees of T rooted at the vertices u and $a(u)$ with the tree T. If G is a group of automorphisms, $G|_u$ will denote the set of all sections of group elements at u. The restriction of the action of the section $a|_u$ to $X^1 = X$ is called the *label of a at u* and it will be written as $a|^u$.

The following holds for all words u, v and all automorphisms a, b:

$$(a|_u)|_v = a|_{uv},$$
$$(ab)|_u = a|_{b(u)}b|_u.$$

The analogous identities hold for the labels $a|^u$, so the action of a on any word $x_0 \dots x_{n-1}$ of length n is given by

$$a(x_0 \dots x_{n-1}) = a|^\epsilon(x_0)a|_{x_0}(x_1 \dots x_{n-1}) = a|^\epsilon(x_0)a|^{x_0}(x_1) \dots a|^{x_0 \dots x_{n-2}}(x_{n-1}).$$

Hence every automorphism a is completely described by the label map $X^* \to \operatorname{Sym}(X)$, $u \mapsto a|^u$, called the *portrait of a.*

For $n \in \mathbb{N}_0$, the isomorphim

$$\psi_n : \operatorname{St}(n) \to (\operatorname{Aut} T)^{m^n}, \ g \mapsto (g|_x)_{x \in X^n},$$

is called the *n-th layer section decomposition.* We will shorten the notation of big tuples arising for example in this way by writing g^{*k} for a sequence of k identical entries g in a tuple, implicitly ordering the vertices lexicographically.

We can uniquely describe an automorphism $g \in \operatorname{Aut} T$ by its label at ϵ and the first layer section decomposition of $(g|^\epsilon)^{-1}g$, i.e. by

$$g = g|^\epsilon \ (g|_x)_{x \in X}.$$

Let $H \leqslant \mathrm{Sym}(X)$ be any subgroup of the symmetric group on X. Then denote by $\Gamma(H)$ the subgroup of $\mathrm{Aut}\, T$ defined as

$$\Gamma(H) = \langle a \in \mathrm{Aut}\, T \mid \forall u \in T, a|^u \in H \rangle.$$

If H is a Sylow-p subgroup of $\mathrm{Sym}(X)$, then $\Gamma(H)$ is a Sylow-pro-p subgroup of $\mathrm{Aut}\, T$. We further fix $\sigma = (0\ 1\ \ldots\ m-1) \in \mathrm{Sym}(X)$ and write Γ for $\Gamma(\langle \sigma \rangle)$.

A group $G \leqslant \mathrm{Aut}\, T$ is called *self-similar* if it is closed under taking sections at every vertex, i.e. if $G|_v \subseteq G$ for all $v \in T$. Self-similar groups correspond to certain automata modelling the behaviour of the section map: there is a state for every element $g \in G$, and an arrow $g \to g|_x$ labelled $x : g(x)$ for every $x \in X$ (for details see [76]).

We follow [103] in the terminology for the first three of the following self-referential properties, and add a fourth one: A group $G \leqslant \mathrm{Aut}\, T$ acting spherically transitively is called

1. *fractal* if $\mathrm{st}_G(u)|_u = G$ for all $u \in T$.
2. *strongly fractal* if $\mathrm{St}_G(1)|_x = G$ for all $x \in X$.
3. *super strongly fractal* if $\mathrm{St}_G(n)|_u = G$ for all $n \in \mathbb{N}_0$ and $u \in X^n$.
4. *very strongly fractal* if $\mathrm{St}_G(n+1)|_x = \mathrm{St}_G(n)$ for all $n \in \mathbb{N}_0$ and $x \in X$.

Notice that for every group H acting regularly on X and $G \leqslant \Gamma(H)$ the properties (1) and (2) coincide. The following lemma will be of great use.

Lemma 8.2.1. *Let $G \leqslant \mathrm{Aut}\, T$ be fractal and self-similar, and let $x, y \in X$. For every $g \in G$ there exists an element $\tilde{g} \in G$ such that $\tilde{g}(x) = y$ and $\tilde{g}|_x = g$. Furthermore, if $H \leqslant G$ is any subgroup of G such that $H \times \{\mathrm{id}\} \times \cdots \times \{\mathrm{id}\} \leqslant \psi_1(K)$ for some normal subgroup $K \trianglelefteq G$, then $(H^G)^m \leqslant \psi_1(K)$.*

Proof. Since G is fractal, it is spherically transitive and in particular it is transitive on the first layer of T. Hence there exists some element $h \in G$ mapping x to y. Also because G is fractal and $h|_x \in G$ by self-similarity, there is some element $k \in \mathrm{st}_G(x)$ such that $k|_x = (h|_x)^{-1} g$. Now $\tilde{g} = hk$ fulfills both $\tilde{g}(x) = y$ and $\tilde{g}|_x = h|_x k|_x = g$.

Assume further that $H \leqslant G$ and $H \times \{\mathrm{id}\} \times \cdots \times \{\mathrm{id}\} \leqslant \psi_1(K)$ for $K \trianglelefteq G$. Let $g \in G$. Choose an element $\tilde{g} \in G$ such that $\tilde{g}(x) = 0$ and $\tilde{g}|_x = g$. Then for every $h \in H$

$$(\mathrm{id}^{*x}, h^g, \mathrm{id}^{*(m-x-1)}) = \psi_1((\tilde{g})^{-1} \psi_1^{-1}(h, \mathrm{id}, \ldots, \mathrm{id}) \tilde{g}) \in \psi_1((\tilde{g})^{-1} K \tilde{g}) = \psi_1(K). \qquad \square$$

From this point on, we fix a positive integer s.

Definition 8.2.2. There is a set of s interdependent monomorphims $\beta_i^s : \operatorname{Aut} T \to \operatorname{Aut} T$ defined by

$$\beta_i^s(g) = (\beta_{i-1}^s(g), \mathrm{id}, \dots, \mathrm{id}) \qquad\qquad \text{for } i \in [1, s-1],$$
$$\beta_0^s(g) = g|^{\epsilon}(\beta_{s-1}^s(g|_0), \dots, \beta_{s-1}^s(g|_{m-1})).$$

We adopt the convention that the subscript for these maps is taken modulo s, whence $\beta_i^s(g)|_x \in \beta_{i-1}^s(\operatorname{Aut} T)$ for all $i \in [0, s-1]$ and $g \in \operatorname{Aut} T$. Whenever there is no reason for confusion, we drop the superscript s.

Definition 8.2.3. Let $G \leqslant \operatorname{Aut} T$. The *$s$-th Basilica group of G* is defined as

$$\operatorname{Bas}_s(G) = \langle \beta_i^s(g) \mid g \in G, i \in [0, s-1] \rangle.$$

Clearly, for $s = 1$ the homomorphism β_0^1 is the identity map and $\operatorname{Bas}_1(G) = G$. In the case of a self-similar group G, the s-th Basilica group of G can be equivalently defined as the self-similar closure of the group $\beta_0^s(G)$, i.e. the smallest self-similar group containing $\beta_0^s(G)$. If G is finitely generated by $g_1, \dots, g_r$, then $\operatorname{Bas}_s(G)$ is generated by $\beta_i^s(g_j)$ with $i \in [0, s-1]$ and $j \in [1, r]$.

The operation Bas_s is multiplicative in s, i.e. for $s, t \in \mathbb{N}_+$ and $G \leqslant \operatorname{Aut} T$ we have $\operatorname{Bas}_s \operatorname{Bas}_t(G) = \operatorname{Bas}_{st}(G)$. This is a consequence of

$$\beta_i^s(\beta_j^t(g)) = \beta_{i+sj}^{st}(g),$$

which is an easy consequence of Definition 8.2.2.

We now describe the monomorphisms β_i^s for $i \in [0, s-1]$ in terms of their portraits. We define a map $\omega_i : T \to T$. For every $k \in \mathbb{N}_0$ and every vertex $u \in X^k$, write $u = x_0 \dots x_{k-1} \in X^k$, and define

$$\omega_i(u) := 0^i \prod_{j=0}^{k-2} \left(x_j 0^{s-1}\right) x_{k-1}.$$

Writing $\omega_i(T)$ for the subgraph of T induced by the image of ω_i, with edges inherited from paths in T, we again obtain an m-regular rooted tree.

Lemma 8.2.4. *Let $g \in \operatorname{Aut} T$ and $i \in [0, s-1]$. Then the portrait of $\beta_i^s(g)$ is given by*

$$\beta_i^s(g)|^u = \begin{cases} g|^v, & \text{if } u = \omega_i(v), \\ \mathrm{id}, & \text{if } u \notin \omega_i(T). \end{cases}$$

In particular $\operatorname{Bas}_s(G) \leqslant \Gamma(H)$, *if* $G \leqslant \Gamma(H)$ *for some* $H \leqslant \operatorname{Sym}(X)$.

Proof. First suppose that $u = \omega_i(v)$ for $v = x_0 \dots x_{k-1}$. From Definition 8.2.2 follows

$$\beta_i^s(g)|^{\omega_i(x_0 \dots x_{k-1})} = \beta_0^s(g)|^{\omega_0(x_0 \dots x_{k-1})} = \beta_{s-1}^s(g|_{x_0})|^{\omega_{s-1}(x_1 \dots x_{k-1})},$$

and iteration establishes $\beta_i^s(g)|^u = g|^v$. Now, if $u = u_0 \dots u_{k-1} \notin \omega_i(T)$, there is some minimal number $n \not\equiv_s i$ such that $u_n \neq 0$. Thus $u = \omega_i(v)0^t u_n \dots u_{k-1}$ for $n \equiv_s t < i$ and some vertex v, hence

$$\beta_i^s(g)|^u = \beta_i^s(g|_v)|^{0^t u_n \dots u_k} = \beta_{i-t}^s(g|_v)|^{u_n \dots u_k} = \mathrm{id}\,. \qquad \square$$

It is interesting to compare the effect of the Basilica operation with another method of deriving new self-similar groups from given ones described by Nekrashevych.

Proposition 8.2.5 ([76, Proposition 2.3.9])**.** *Let $G \leqslant \operatorname{Aut} T$ be a group and let d be a positive integer. There is a set of d injective endomorphisms of $\operatorname{Aut} T$ given by*

$$\pi_0(g) := g|^\epsilon\ (\pi_{d-1}(g|_x))_{x \in X},$$
$$\pi_i(g) := (\pi_{i-1}(g))_{x \in X} \text{ for } i \in [1, d-1].$$

The group $D_d(G) := \langle \pi_i(G) \mid i \in [0, d-1] \rangle$ is isomorphic to the direct product G^d.

We combine both constructions to define a class of groups very closely resembling the original Basilica group $\mathcal{B}$.

Definition 8.2.6. Let d, m, $s \in \mathbb{N}_+$ with $m \geqslant 2$. The *m-adic odometer* $\mathcal{O}_m$ is the infinite cyclic group generated by

$$a = \sigma(a, \mathrm{id}, \dots, \mathrm{id}),$$

where σ is the m-cycle $(m-1\ \ m-2\ \ \dots\ \ 1\ \ 0)$. Write $\mathcal{O}_m^d$ for $D_d(\mathcal{O}_m)$, the d-fold direct product of $\mathcal{O}_m$ embedded into $\operatorname{Aut} T$ by the construction described in Proposition 8.2.5. We call the group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ the *generalised Basilica group*.

Clearly, $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$ is the original Basilica group introduced by Grigorchuk and Żuk in [59].

For illustration we depict explicitly the automaton defining the self-similar action of the dyadic odometer $\mathcal{O}_2$, the automaton defining the action of $D_8(\mathcal{O}_2)$ described above and the automaton defining $\mathrm{Bas}_8(\mathcal{O}_2)$ in Figure 8.4.

We shall prove in the following (cf. Section 8.6, Section 8.7, Section 8.8) that generalised Basilica groups resemble the original Basilica group in many ways, justifying the terminology.

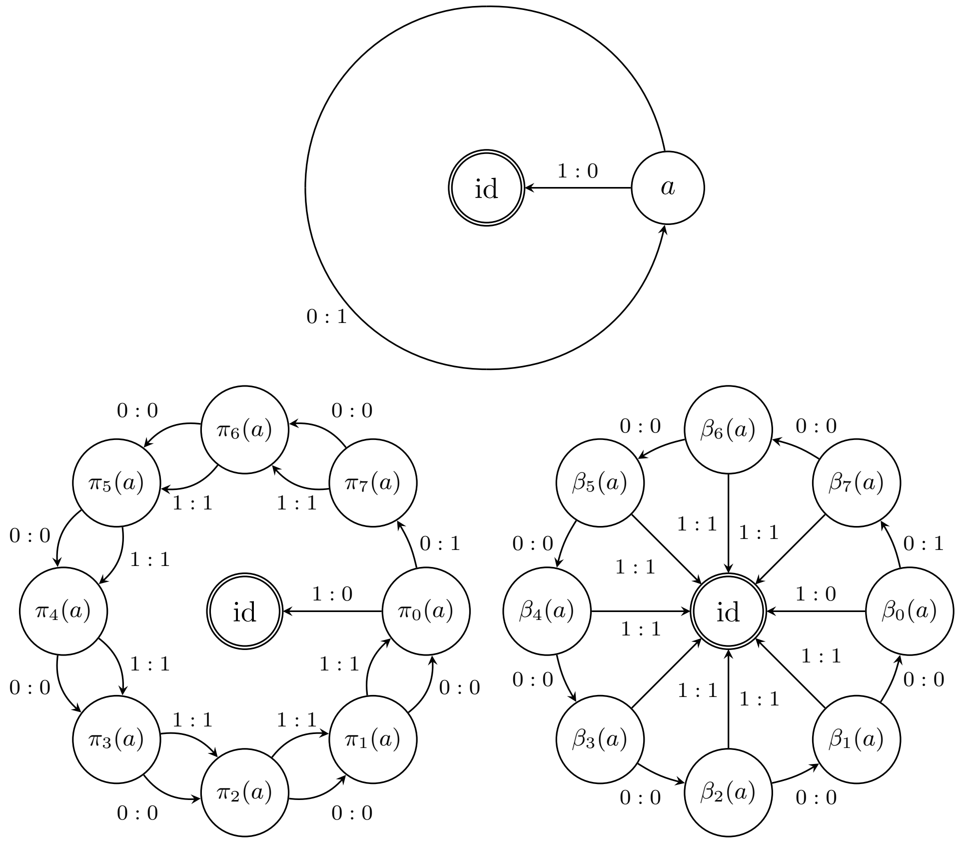

Figure 8.4: The automata defining the generators of $\mathcal{O}_2$, $D_8(\mathcal{O}_2)$ and $\mathrm{Bas}_8(\mathcal{O}_2)$.

Proposition 8.2.7. *Let* $\mathrm{Aut}_{\mathrm{fin}}(T)$ *be the group of all* finitary automorphisms*, i.e. the group generated by all automorphisms* $g_{\tau,v}$ *for* $v \in T$*,* $\tau \in \mathrm{Sym}(X)$ *that have label* τ *at* v *and trivial label everywhere else. For any* $s \in \mathbb{N}_+$

$$\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T)) = \mathrm{Aut}_{\mathrm{fin}}(T).$$

On the other hand $\mathrm{Bas}_s(\mathrm{Aut}\, T)$ *is not of finite index in* $\mathrm{Aut}\, T$ *for all* $s > 1$*.*

Proof. Define for every $n \in \mathbb{N}_0$ a map $\mu_n : \mathrm{Aut}\, T \to \mathbb{N}_0$ by

$$\mu_n(g) = |\{u \in X^n \mid g|_u \neq \mathrm{id}\}|.$$

Lemma 8.2.4 shows that $g_{\tau,v} = \beta_i(g_{\tau,\omega_i^{-1}(v)}) \in \mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T))$ for every $v \in \bigcup_{i=0}^{s-1} \omega_i(T)$. Conjugation with suitable elements produces all other generators, hence $\mathrm{Aut}_{\mathrm{fin}}(T)$ is contained in $\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T))$. On the other hand, $\sum_{n \in \mathbb{N}_0} \mu_n(g) < \infty$ for any $g \in \mathrm{Aut}_{\mathrm{fin}}(T)$, implying that the same holds for all generators (and hence, all elements) of $\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T))$. Thus, $\mathrm{Bas}_s(\mathrm{Aut}_{\mathrm{fin}}(T)) = \mathrm{Aut}_{\mathrm{fin}}(T)$.

For any $g \in \mathrm{Aut}\, T$ we have $\mu_n(g) \leqslant |X^n| = m^n$. But for all generators $\beta_i(g)$ of $\mathrm{Bas}_s(\mathrm{Aut}\, T)$ the stronger inequality $\mu_{sn+i}(\beta_i(g)) \leqslant m^n$ holds, since $\beta_i(g)$ has trivial label at all vertices outside of $\omega_i(T)$. Let $g \in \mathrm{Aut}\, T$ and $q(g) \in \mathbb{Q}_+$ be the infimum of all numbers r such that

$$\limsup_{n\to\infty} \frac{\mu_{sn}(g)}{m^{(1+r)n}} = \infty.$$

Then g cannot be in $\mathrm{Bas}_s(\mathrm{Aut}\, T)$, since the inequality $\mu_n(ab) \leqslant \mu_n(a)+\mu_n(b)$ for $a, b \in \mathrm{Aut}\, T$ implies that it cannot be a finite product of the generators of $\mathrm{Bas}_s(\mathrm{Aut}\, T)$. By the same reason, all elements with different $q(g)$ are in different cosets. Since $q(\mathrm{Aut}\, T) = (0, s-1) \cap \mathbb{Q}$, the second statement follows. □

Question 8.2.8. In view of Proposition 8.2.7 and the original Basilica group $\mathcal{B}$ it seems plausible that the operation Bas_s makes (in some vague sense) big groups smaller and small groups bigger. Let $H \leqslant \mathrm{Sym}(X)$ be a transitive subgroup. Set $\Gamma_{\mathrm{fin}}(H) = \mathrm{Aut}_{\mathrm{fin}}(T) \cap \Gamma(H)$. Replacing $\mathrm{Aut}_{\mathrm{fin}}(T)$ with $\Gamma_{\mathrm{fin}}(H)$ in the proof of Proposition 8.2.7 we obtain the equality $\mathrm{Bas}_s(\Gamma_{\mathrm{fin}}(H)) = \Gamma_{\mathrm{fin}}(H)$.

Is there a group G not of the form $\Gamma_{\mathrm{fin}}(H)$ such that $\mathrm{Bas}_s(G) = G$?

8.3 Properties inherited by Basilica groups

We recall our standing assumptions: m and s are positive integers with $m \neq 1$, $X = [0, m-1]$, and T the m-regular rooted tree. The subscript of the maps β_i^s is taken modulo s, and we will drop the superscript s from now on.

8.3.1 Self-similarity and fractalness

Lemma 8.3.1. *Let $G \leqslant \mathrm{Aut}\, T$ act spherically transitively on T. Then $\mathrm{Bas}_s(G)$ acts spherically transitively on T.*

Proof. It is enough to prove that for any number $n = qs + r \in \mathbb{N}_+$ with $r \in [0, s-1]$ and $q \geqslant 0$, and $y \in X$ there is an element $b \in \mathrm{Bas}_s(G)$ such that $b(0^n 0) = 0^n y$. Let $g \in G$ be such that $g(0^q 0) = 0^q y$ and observe that $\beta_r(g)$ stabilises 0^n. By Lemma 8.2.4 it follows

$$\beta_r(g)(0^n 0) = 0^n \beta_0(g|_{0^q})(0) = 0^n y.$$ □

Lemma 8.3.2. *Let $G \leqslant \mathrm{Aut}\, T$ be self-similar. Then $\mathrm{Bas}_s(G) \leqslant \mathrm{Aut}\, T$ is self-similar.*

Proof. We check that $\beta_i(g)|_v$ is a member of $\mathrm{Bas}_s(G)$ for all $v \in T$. This holds by Definition 8.2.2 for words v of length 1, and follows from $g|_x|_y = g|_{xy}$ by induction for words of any length. □

Lemma 8.3.3. *Let $G \leqslant \mathrm{Aut}\, T$ be self-similar, and fractal (resp. strongly fractal). Then* *(i) The group $B = \mathrm{Bas}_s(G) \leqslant \mathrm{Aut}\, T$ is fractal (resp. strongly fractal).*

(ii) For all $b \in B$ there is an element $c \in \mathrm{st}_B(0)$ (resp. $c \in \mathrm{St}_B(1)$) such that $c|_0 = b$ and $c|_x \in \beta_{s-1}(G)$ for all $x \in [1, m-1]$.

Proof. Lemma 8.3.1 shows that B acts spherically transitively, and by Lemma 8.3.2 the group B is self-similar. First suppose that G is fractal. Since the statement (ii) implies the statement (i), it is enough to prove (ii).

Observe that

$$H = \{g \in \mathrm{st}_B(0) \mid g|_x \in \beta_{s-1}(G) \text{ for all } x \in [1, m-1]\}$$

is a subgroup since $h(x) \neq 0$ and $(gh)|_x = g|_{h(x)} h|_x \in \beta_{s-1}(G)$ for all $g, h \in H, x \in [1, m-1]$. Thus it is enough to show that $\beta_i(G) \leqslant H|_0$ for all $i \in [0, s-1]$.

It is easy to see that $\beta_i(G) \leqslant H$ for $i \neq 0$, hence since $\beta_i(G)|_0 = \beta_{i-1}(G)$ we have $\beta_i(G) \leqslant H|_0$ for $i \neq s-1$. But also $\beta_0(\mathrm{st}_G(0)) \leqslant H$. Note that, since G is fractal, we have $\mathrm{st}_G(0)|_0 = G$. Hence $\beta_{s-1}(G) \leqslant \beta_0(\mathrm{st}_G(0))|_0 \leqslant H|_0$.

If G is strongly fractal, we may replace H by its intersection with $\mathrm{St}_B(1)$ and $\mathrm{st}_G(0)$ by $\mathrm{St}_G(1)$ to obtain a proof for the analogous statement. □

Lemma 8.3.1, Lemma 8.3.3 and Lemma 8.3.2 yield proofs for the statements (1), (2) and (3) of Theorem 8.1.1.

8.3.2 Amenability

The original Basilica group $\mathcal{B}$ was the first example of an amenable, but not subexponentially amenable group. This had been conjectured already in [59], where non-subexponentially amenability of $\mathcal{B}$ was proven. Amenability was proven by Bartholdi and Virág in [24]. Later, Bartholdi, Kaimanovich and Nekrashevych proved that all groups generated from bounded finite-state automorphisms are amenable [20], which includes $\mathcal{B}$. We recall the relevant definitions and then apply the result of Bartholdi, Kaimanovich and Nekrashevych to a wider class of groups produced by the Basilica operation.

Definition 8.3.4. An automorphism $f \in \mathrm{Aut}\, T$ is called

1. *finite-state* if the set $\{f|_u \mid u \in T\}$ is finite, and
2. *bounded* if the sequence $\mu_n(f) := |\{u \in X^n \mid f|_u \neq \mathrm{id}\}|$ is bounded.

Proposition 8.3.5. *Let $G \leqslant \mathrm{Aut}\, T$ be generated from finite-state bounded automorphisms. Then $\mathrm{Bas}_s(G)$ is also generated from finite-state bounded automorphisms.*

Proof. It is enough to prove that for every finite-state bounded $f \in \operatorname{Aut} T$ and $i \in [0, s-1]$ the element $\beta_i(f)$ is again finite-state and bounded. Notice that all sections of f are of the form $\beta_j(f|_u)$ for some $u \in T$, hence there are only finitely many candidates and $\beta_i(f)$ is finite-state. Moreover, by Definition 8.2.2 $\mu_n(\beta_i(f)) = \mu_{\lfloor \frac{n-i}{s} \rfloor}(f)$, bounding $\mu_n(\beta_i(f))$. $\square$

This proves statement (6) of Theorem 8.1.1, and we use [20] to conclude:

Corollary 8.3.6. *Let $G \leqslant \operatorname{Aut} T$ be generated by finite-state bounded automorphisms. Then* $\operatorname{Bas}_s(G)$ *is amenable.*

8.3.3 Spinal Groups

A well-known class of subgroups of $\operatorname{Aut} T$ containing most known branch groups is the class of *spinal groups*, containing both the first and the second Grigorchuk group, and all GGS-*groups*. We use, with modifications for GGS-groups, the definition given in [18].

Definition 8.3.7 (cf. [18, Definition 2.1])**.** Let $R \leqslant \operatorname{Sym}(X)$, let D be a finite group and let

$$\omega = (\omega_{i,j})_{i \in \mathbb{N}_+, j \in [1, m-1]}$$

be a family of homomorphisms $\omega_{i,j} : D \to \operatorname{Sym}(X)$. Identify R with $\{r(\mathrm{id}, \ldots, \mathrm{id}) \mid r \in R\} \leqslant \operatorname{Aut} T$ and identify each $d \in D$ with the automorphism of T given by

$$d|^w := \begin{cases} \omega_{i,j}(d) & \text{if } w = 0^{i-1}j \text{ for } i \in \mathbb{N}_+, j \in [1, m-1], \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Suppose that the following holds:

1. The group R and all groups $\langle \omega_{n,j}(D) \mid j \in [1, m-1] \rangle$, for $n \in \mathbb{N}_+$, act transitively on X.

2. For all $n \in \mathbb{N}_+$,
$$\bigcap_{i=n}^{\infty} \bigcap_{j=1}^{m-1} \ker \omega_{i,j} = 1.$$

Then $\langle R, D \rangle \leqslant \operatorname{Aut} T$ is called the *spinal group acting on T with defining triple* (R, D, ω). The spinal group with defining triple (R, D, ω) is called a GGS-*group acting on T* if $\omega_{n,j} = \omega_{k,j}$ for all $n, k \in \mathbb{N}_+$ and $j \in [1, m-1]$.

We now describe the Basilica groups of spinal groups. For this, we record the following lemma.

Lemma 8.3.8. *Let $i, j \in [0, s-1]$ with $i \neq j$. Denote by $\mathrm{st}(\overline{0})$ the stabiliser of the infinite ray $\overline{0} := \{0^i \mid i \in \mathbb{N}_0\}$ in $\mathrm{Aut}\, T$ (a so-called parabolic subgroup). Then $[\beta_i(\mathrm{st}(\overline{0})), \beta_j(\mathrm{st}(\overline{0}))] = 1$.*

Proof. We prove that for all $g_0, g_1 \in \mathrm{st}(\overline{0})$ the images $b_0 = \beta_i(g_0)$ and $b_1 = \beta_j(g_1)$ commute, using the fact that $\mathrm{st}(\overline{0})|_0 = \mathrm{st}(\overline{0})$. Assume without loss of generality that either $j > i > 0$ or $i = 0$. In the first case both b_0 and b_1 stabilise the i-th layer, we can consider

$$\psi_i([b_0, b_1]) = ([b_0|_{0^i}, b_1|_{0^i}], \mathrm{id}^{*(m^i-1)}) = ([\beta_0(g_0), \beta_{j-i}(g_1)], \mathrm{id}^{*(m^i-1)}),$$

and thus reduce to the second case. Suppose now that $i = 0$. Since the only non-trivial first layer section of b_1 is at the vertex 0 and by assumption b_0 fixes this vertex,

$$\psi_1([b_0, b_1]) = ([b_0|_0, b_1|_0], \mathrm{id}^{*(m-1)}).$$

Since $b_0|_0, b_1|_0 \in \mathrm{st}(\overline{0})$, we conclude by infinite descent that $[b_0, b_1]$ fixes all vertices outside the ray $\overline{0}$, thus acts trivially on the entire tree T. □

The elements $d \in D$ of a spinal group defined by (R, D, ω) can be characterised by the fact that they stabilise the infinite ray (or "spine") $\overline{0}$ and $d|^x \neq \mathrm{id}$ implies that x has distance precisely 1 from $\overline{0}$. Therefore it is easy to see that a Basilica group $B = \mathrm{Bas}_s(G)$ of a spinal group G acting on T cannot act as a spinal group on T, as the elements $\beta_i^s(d)$ have non-trivial labels at vertices of distance s from the ray $\overline{0}$. However, the group B acts as a spinal group on a tree obtained from T by deletion of layers.

Motivated from Examples 8.3.10 and 8.3.11 below, we introduce the following notations.

There is an injection $\iota_s : (X^s)^* \to X^*$ given by

$$(x_{0,0} \cdots x_{0,s-1}) \cdots (x_{n-1,0} \cdots x_{n-1,s-1}) \mapsto x_{0,0} \cdots x_{n-1,s-1},$$

whose image is the union $\bigcup_{n \in \mathbb{N}_0} X^{sn}$. The restriction map induces an injection

$$\iota_s^* : \mathrm{Aut}(X^*) \to \mathrm{Aut}((X^s)^*),$$

and clearly the image $\iota_s^*(\mathrm{Aut}\, T)$ is

$$\Gamma(\mathrm{Sym}(X) \wr \cdots \wr \mathrm{Sym}(X)) \leqslant \mathrm{Aut}((X^s)^*),$$

where the permutational wreath product is iterated s times. Recall that $\Gamma(H)$ for a permutation group G denotes the subgroup of $\mathrm{Aut}\, T$ with every local action a member of H. Define for $i \in [0, s-1]$

$$\tau_i : \mathrm{Sym}(X) \to \mathrm{Sym}(X) \wr \cdots \wr \mathrm{Sym}(X)$$
$$\rho \mapsto \iota_s^*(g_{\rho,0^i})|^\epsilon,$$

where $g_{\rho,0^i}$ is the automorphism with $g|^{0^i} = \rho$ and $g|^x = \mathrm{id}$ everywhere else. It is easy to see that for every transitive permutation group $H \leqslant \mathrm{Sym}(X)$ the group $\langle \tau_k(H) \mid k \in [0, s-1] \rangle$ is isomorphic to the s-fold iterated permutational wreath product $H \wr \cdots \wr H$.

Now given a family of homomorphisms $(\omega_{i,j} : D \to \mathrm{Sym}(X))_{i \in \mathbb{N}_+, j \in X \setminus \{0\}}$ we define a new family $\tilde{\omega} = (\tilde{\omega}_{i,j} : D^s \to \mathrm{Sym}(X^s))_{i \in \mathbb{N}_+, j \in X^s \setminus \{0^s\}}$ by

$$\tilde{\omega}_{n,j} = \begin{cases} \tau_i \circ \omega_{n,x} \circ \pi_i, & \text{if } j = 0^i x 0^{s-i-1} \text{ for some } x \in [1, m-1] \text{ and } i \in [0, s-1], \\ d \mapsto \mathrm{id}, d \in D^s, & \text{otherwise}, \end{cases}$$

where $\pi_i : D^s \to D$ denotes the projection to the $(i+1)$-th factor.

Proposition 8.3.9. *Let G be the spinal group on T with defining triple (R, D, ω). Then $\iota_s^*(\mathrm{Bas}_s(G))$ is the spinal group on $(X^s)^*$ with defining triple $(R \wr \cdots \wr R, D^s, \tilde{\omega})$, by the action of $\mathrm{Bas}_s(G)$ on the m^s-regular tree $\delta_s T$ defined by the deletion of layers.*

If furthermore G is a GGS*-group on T, $\iota_s^*(\mathrm{Bas}_s(G))$ is a* GGS*-group on $(X^s)^*$.*

Proof. First consider the elements of the form $\beta_k(a)$, for $a \in R$, $k \in [0, s-1]$. On $(X^s)^*$ this element acts as $\tau_k(a)$. Since R is transitive, the images of R generate $R \wr \cdots \wr R$, and the first entry of the defining triple is described.

We deal in a similar way with the sections $\beta_i(d|_{0^k y})$ of a directed element for every $d \in D, i \in [0, s-1], k \in \mathbb{N}_0, y \in X \setminus \{0\}$. To obtain the first section decomposition of the action of $\beta_i(d|_{0^k})$ on $\delta_s T$ (which stabilises the first layer) we have to take sections of $\beta_i(d|_{0^k})$ at words $x = x_0 \dots x_{s-1}$ of length s in T. Now by Lemma 8.2.4,

$$\beta_i(d|_{0^k})|_x = \begin{cases} \beta_i(d|_{0^{k+1}}) & \text{if } x = 0^s, \\ \beta_i(\omega_{k+1,x_i}(d)) = \tau_i \omega_{k+1,x_i}(d) & \text{if } x = 0^i x_i 0^{s-i-1}, x_i \neq 0, \\ \mathrm{id} & \text{otherwise}. \end{cases}$$

By Lemma 8.3.8 all pairs $\beta_i(d_1), \beta_j(d_2)$ with $d_1, d_2 \in D$, $i, j \in [0, s-1]$ and $i \neq j$ commute. We identify $\beta_i(D)$ with the $(i+1)$-th direct factor of D^s. Thus $\mathrm{Bas}_s(G)$ is generated by $R \wr \cdots \wr R$ and $\langle \beta_i(D) \mid i \in [0, s-1] \rangle \cong D^s$, where $(\mathrm{id}, \dots, \mathrm{id}, d_i, \mathrm{id}, \dots, \mathrm{id}) \in D^s$ acts on $\delta_s T$ by

$$(\mathrm{id}, \dots, \mathrm{id}, d_i, \mathrm{id}, \dots, \mathrm{id})|_{0^{ks} x} = \beta_i(d|_{0^k})|_x,$$

thus, the elements of D^s are defined by the family $\tilde{\omega}$ of homomorphisms.

It remains to establish the two defining properties of spinal groups. Property (1) holds by the observation that

$$\langle \tilde{\omega}_{i,j}(D^s) \mid j \in [1, m^s - 1] \rangle$$

acts as $\langle \tau_k(\omega_{i,j}(D)) \mid j \in [1, m-1], k \in [0, s-1]\rangle$, hence $\langle \tilde{\omega}_{i,j}(D^s) \mid j \in [1, m^s - 1]\rangle$ acts as the s-fold wreath product of $\langle \omega_{i,j}(D) \mid j \in [1, m-1]\rangle$, in particular, transitively on the first layer of $\delta_s T$.

For (2) consider

$$\ker \tilde{\omega}_{n,j} = \begin{cases} \ker(\omega_{n,x} \circ \pi_i), & \text{if } j = 0^i x 0^{s-i-1}, \text{ for some } x \in [1, m-1], i \in [0, s-1] \\ D^s, & \text{else,} \end{cases}$$

hence

$$\bigcap_{j \in X^s \setminus \{0^s\}} \ker \tilde{\omega}_{n,j} = \left(\bigcap_{j \in X \setminus \{0\}} \ker \omega_{n,j} \right) \times \cdots \times \left(\bigcap_{j \in X \setminus \{0\}} \ker \omega_{n,j} \right).$$

Therefore we see that since (2) holds for G, (2) holds for $\mathrm{Bas}_s(G)$.

The statement regarding GGS-groups follows directly from the description of the defining triple of $\mathrm{Bas}_s(G)$. □

Proposition 8.3.9 yields Theorem 8.1.2.

Example 8.3.10. One of the eponymous examples of a GGS-group is the family of the Gupta–Sidki p-groups acting on the p-regular tree. In the language of spinal groups they are defined by the triple

$$(\langle\sigma\rangle, \langle\sigma\rangle, (\sigma \mapsto \sigma, \sigma \mapsto \sigma^{-1}, \sigma \mapsto \mathrm{id}, \dots, \sigma \mapsto \mathrm{id})_{i \in \mathbb{N}_+}),$$

or in usual notation by the generators $a = \sigma(\mathrm{id}, \dots, \mathrm{id}), b = (b, a, a^{-1}, \mathrm{id}, \dots, \mathrm{id})$. We can describe the generators of the second Basilica group of the Gupta–Sidki 3-group $\ddot{\Gamma}$ by

$$\beta_0^2(a) = \sigma(\mathrm{id}, \mathrm{id}, \mathrm{id}) = a \qquad \beta_0^2(b) = (\beta_1^2(b), \beta_1^2(a), \beta_1^2(a^{-1})),$$
$$\beta_1^2(a) = (a, \mathrm{id}, \mathrm{id}) \qquad \beta_1^2(b) = (\beta_0^2(b), \mathrm{id}, \mathrm{id}).$$

The automaton describing these generators is given explicitly in Figure 8.3. By ordering X^2 reverse lexicographically, the action of the generators on $(X^2)^*$ is

$$\beta_0^2(a) = (00\ 10\ 20)(01\ 11\ 21)(02\ 12\ 22)$$
$$\beta_0^2(b) = (\beta_0^2(b), \beta_0^2(a), \beta_0^2(a)^{-1}, \mathrm{id}, \dots, \mathrm{id})$$
$$\beta_1^2(a) = (00\ 01\ 02)$$
$$\beta_1^2(b) = (\beta_1^2(b), \mathrm{id}, \mathrm{id}, \beta_1^2(a), \mathrm{id}, \mathrm{id}, \beta_1^2(a)^{-1}, \mathrm{id}, \mathrm{id}).$$

Example 8.3.11. The first Grigorchuk group $\mathcal{G}$ is the spinal group acting on the binary tree defined by $\mathrm{C}_2, \mathrm{C}_2^2$ and the sequence $\omega_{i,1}$ of (the three) monomorphisms $\mathrm{C}_2 \to \mathrm{C}_2^2$, where

$\omega_{i,1} = \omega_{j,1}$ holds if and only if $i \equiv_3 j$. Writing a for the non-trivial rooted element and b, c, d for the non-trivial directed elements, one has the descriptions

$$a = (0\ 1)(\mathrm{id}, \mathrm{id}), \quad b = (c, a), \quad c = (d, a), \quad d = bc = (b, \mathrm{id}).$$

By Proposition 8.3.9 $\mathrm{Bas}_2(\mathcal{G})$ is a spinal group on the 4-regular tree $(X^2)^*$, generated by the elements

$$\begin{aligned}
\alpha &:= \beta_0^2(a) &&= (0\ 2)(1\ 3), & \mathrm{A} &:= \beta_1^2(a) &&= (0\ 1),\\
\beta &:= \beta_0^2(b) &&= (\kappa, \alpha, \mathrm{id}, \mathrm{id}), & \mathrm{B} &:= \beta_1^2(b) &&= (\mathrm{K}, \mathrm{id}, \mathrm{A}, \mathrm{id}),\\
\kappa &:= \beta_0^2(c) &&= (\delta, \alpha, \mathrm{id}, \mathrm{id}), & \mathrm{K} &:= \beta_1^2(c) &&= (\Delta, \mathrm{id}, \mathrm{A}, \mathrm{id}),\\
\delta &:= \beta\kappa, && & \Delta &:= \mathrm{BK},
\end{aligned}$$

where we identify $[0, 3]$ with X^2 by the reverse lexicographic ordering.

8.3.4 Contracting groups

For this subsection we fix a self-similar group $G \leqslant \mathrm{Aut}\, T$ and some generating set S of G, which yields a natural generating set $\bigcup_{i\in[0,s-1]} \beta_i(S)$ for $B := \mathrm{Bas}_s(G)$.

The group $G \leqslant \mathrm{Aut}\, T$ is said to be *contracting*, if there exists a finite set $\mathcal{N} \subset G$ (called a *nucleus* of G) such that for all $g \in G$ there is an integer $k(g)$ such that $g|_v \in \mathcal{N}$ for all $v \in T$ with $|v| > k(g)$, where $|\cdot|$ denotes the word norm.

In this section we prove that a contracting group G has contracting Basilica groups $B = \mathrm{Bas}_s(G)$, considering the natural generating set for B. For this we define yet another length function, the *syllable length*, denoted by $\mathrm{syl}(b)$, of an element $b \in B$ as the word length w.r.t. the infinite generating set $\bigcup_{i\in[0,s-1]} \beta_i(G)$, i.e. as

$$\mathrm{syl}(b) := \min\{\ell \in \mathbb{N}_0 \mid b = \prod_{j=0}^{\ell-1} \beta_{i_j}(g_j), \text{ with suitable } i_j \in [0, s-1], g_j \in G\},$$

where $\prod_{j=0}^{\ell-1} \beta_{i_j}(g_j)$ is a word representing b in B with respect to the generating set $\{\beta_i(g) \mid i \in [0, s-1], g \in G\}$. Consequently, we will call a non-trivial element of the given generating set a *syllable* and the corresponding index i its *type*. Since for every non-trivial element $b \in \beta_i(G)$ there is some $u \in X^{ns+i}$ for some $n \in \mathbb{N}_0$ such that $b|^u \neq \mathrm{id}$, while there is no $u \in T \backslash \bigcup_{n\in\mathbb{N}_0} X^{ns+i}$ such that $b|^u \neq \mathrm{id}$, the type of a syllable is unique. Since all sections of a syllable are either trivial or a syllable itself, the syllable length of a section of b is at most $\mathrm{syl}(b)$.

We further define for every $g \in \mathrm{Aut}\, T$,

$$\mathfrak{r}(g) := \begin{cases} \min\{n \in \mathbb{N}_0 \mid g|^{0^n}(0) \neq 0\} & \text{if } g \text{ does not stabilise } \overline{0} = \{0^n \mid n \in \mathbb{N}_0\},\\ \infty & \text{otherwise.} \end{cases}$$

Lemma 8.3.12. *Let $r \in \mathbb{N}_0$. Define*

$$D_r := \{\beta_{a_1}(h_1)\beta_{a_2}(h_2)\beta_{a_3}(h_3) \mid h_1, h_2, h_3 \in G\backslash\{1\},$$
$$a_1, a_2, a_3 \in [0, s-1] \text{ pairwise distinct},$$
$$\mathfrak{r}(\beta_{a_2}(h_2)) = r\}.$$

Then $\mathrm{syl}(c|_u) < 3$ *for* $c \in D_r$ *and all* u *with* $|u| > r$.

Proof. Let $c = \beta_{a_1}(h_1)\beta_{a_2}(h_2)\beta_{a_3}(h_3) \in D_r$, where $a_1, a_2, a_3, h_1, h_2, h_3$ satisfy the conditions stated above. We use induction on r. First consider the case $r = 0$. From $\beta_{a_2}(h_2)(0) \neq 0$ we deduce that $a_2 = 0$. Calculate, for $x \in [0, m-1]$,

$$c|_x = \begin{cases} \beta_{s-1}(h_2|_0)\beta_{a_3-1}(h_3) & \text{if } x = 0, \\ \beta_{a_1-1}(h_1)\beta_{s-1}(h_2|_x) & \text{if } x = h_2^{-1}(0), \\ \beta_{s-1}(h_2|_x) & \text{otherwise.} \end{cases}$$

This shows that $c|_x$ and, by recursion, $c|_u$ for all u with $|u| \geqslant 1$ have syllable length at most 2. Now we assume that $r > 0$. We may reduce to the case that $0 \in \{a_1, a_2, a_3\}$. If $0 \notin \{a_1, a_2, a_3\}$, $c|_0 \in D_{r-1}$ and $c|_x = \mathrm{id}$ for all $x \in X, x \neq 0$. Therefore, by induction $\mathrm{syl}(c|_x|_u) < 3$ for $x \in X$ and $|u| > r-1$, hence $\mathrm{syl}(x|_u) < 3$ for all $|u| > r$.

If $a_3 = 0$, respectively $a_1 = 0$, we have

$$c|_x = \begin{cases} \beta_{a_1-1}(h_1)\beta_{a_2-1}(h_2)\beta_{s-1}(h_3|_x) \in D_{r-1} & \text{if } x = h_3^{-1}(0), \\ \beta_{s-1}(h_3|_x) & \text{otherwise,} \end{cases}$$

$$\text{respectively } c|_x = \begin{cases} \beta_{s-1}(h_1|_0)\beta_{a_2-1}(h_2)\beta_{a_3-1}(h_3) \in D_{r-1} & \text{if } x = 0, \\ \beta_{s-1}(h_1|_x) & \text{otherwise.} \end{cases}$$

In both cases all but one section have length < 3 and the remaining section is contained in D_{r-1}, hence by induction $\mathrm{syl}(c|_{xu}) < 3$ for all $x \in X, |u| > r-1$.

The case $a_2 = 0$ remains. Now $r > 0$ implies $h_2^{-1}(0) = 0$ and we have $\mathfrak{r}(\beta_{s-1}(h_2|_0)) = r-1$. Thus

$$c|_x = \begin{cases} \beta_{a_1-1}(h_1)\beta_{s-1}(h_2|_0)\beta_{a_3-1}(h_3) \in D_{r-1} & \text{if } x = 0, \\ \beta_{s-1}(h_2|_x) & \text{otherwise.} \end{cases}$$

Hence we conclude that $\mathrm{syl}(c|_{xu}) < 3$ for all u with $|u| \geqslant 1$ by induction as before. □

Lemma 8.3.13. *For every element* $b \in B$ *with* $\mathrm{syl}(b) > s+1$ *there is a number* $r \in \mathbb{N}_0$ *such that for all sections* $b|_u$ *with* $|u| > r$,

$$\mathrm{syl}(b|_u) < \mathrm{syl}(b).$$

Proof. Let $b \in B$ be an element with $\operatorname{syl}(b) > s+1$. If b is minimally represented by a word w, it suffices to prove that there is a subword of w representing an element which has a reduction of the syllable length upon taking sections.

Since $\operatorname{syl}(b) > s+1$ there must be at least one syllable type appearing twice, and there is a subword of w that can be written in the form

$$\beta_i(\tilde{g}_1)b_0\beta_i(\tilde{g}_2)b_1 \text{ or } b_1\beta_i(\tilde{g}_1)b_0\beta_i(\tilde{g}_2),$$

where b_0, b_1 are non-trivial and contain neither two syllables of the same type nor a syllable of type i. Passing to the inverse if necessary we restrict to the first case.

Under the assumption of w being minimal it is impossible that both b_0 and $\beta_i(\tilde{g}_2)$ fix the infinite ray $\overline{0}$, since if they did, they would commute by Lemma 8.3.8, and consequently it would be possible to reduce the number of syllables.

Thus there are syllables in $b_0\beta_i(\tilde{g}_2)$ that do not stabilise the ray $\overline{0}$. Among these we choose k such that $r := \mathfrak{r}(\beta_{j_k}(g_k))$ is minimal.

Apply Lemma 8.3.12 to the subword $\beta_{j_{k-1}}(g_{k-1})\beta_{j_k}(g_k)\beta_{j_{k+1}}(g_{k+1})$ of $\beta_i(\tilde{g}_1)b_0\beta_i(\tilde{g}_2)b_1$ consisting only of the syllable $\beta_{j_k}(g_k)$ and its direct neighbours, and obtain for all $u \in T, |u| > r$

$$\operatorname{syl}(b|_u) < \operatorname{syl}(b). \qquad \square$$

Although interesting in its own right we use Lemma 8.3.13 solely to prove the following proposition.

Proposition 8.3.14. *Let $G \leqslant \operatorname{Aut} T$ be contracting. Then $B = \operatorname{Bas}_s(G)$ is contracting.*

Proof. Let $\mathcal{N}(G)$ be a nucleus of G. Define

$$\mathcal{N}(B) := \left\{ \prod_{i=0}^{\ell} \beta_{j_i}(g_i) \mid \ell \leqslant s+1, j_i \in [0, s-1], g_i \in \mathcal{N}(G) \right\}.$$

Since $\mathcal{N}(G)$ is a finite set, $\mathcal{N}(B)$ is finite as well. We will prove that it is a nucleus of B. Let $b \in B$. If $\operatorname{syl}(b) > s+1$, by Lemma 8.3.13 there is a layer, from which onwards all sections of b have syllable length $s+1$ or smaller.

Hence we can assume, that $\operatorname{syl}(b) \leqslant s+1$. Write $b = \prod_{i=0}^{\operatorname{syl}(b)-1} \beta_{j_i}(g_i)$. Since G is contracting, for every g_i there is a number $k(g_i)$ such that $g_i|_u \in \mathcal{N}(G)$ for all $|u| \geqslant k(g_i)$. Set $K := \max\{k(g_i) \mid i \in [0, \operatorname{syl}(b)-1]\}$, and observe that for u with $|u| \geqslant sK$ the section $b|_u$ is a product of at most $\operatorname{syl}(b) \leqslant s+1$ syllables of the form $\beta_i(g)$ with $g \in \mathcal{N}(G)$. Thus $b|_u$ is in $\mathcal{N}(B)$ and B is contracting. $\square$

Proposition 8.3.14 proves statement (4) of Theorem 8.1.1.

As a consequence, the word problem for Basilica groups of self-similar and contracting groups is solvable, since it is solvable for self-similar and contracting groups [76, Proposition 2.13.8].

Corollary 8.3.15. *Let G be self-similar and contracting. Then $\mathrm{Bas}_s(G)$ has solvable word problem.*

Question 8.3.16. Let $G \leqslant \mathrm{Aut}\, T$ be contracting. The fact that $\mathrm{Bas}_s(G)$ is contracting implies the existence of constants $\lambda < 1, L, C \in \mathbb{R}_+$ such that for every $g \in G$, $u \in X^n$ with $n > L$ it holds

$$|g|_u| < \lambda |g| + C.$$

In [59] one set of constants is given for the original Basilica group $\mathcal{B}$, namely $\lambda = \frac{2}{3}$ and $L = C = 1$.

Is there a general formula for the above constants valid for all contracting groups and their Basilica groups, yielding $\lambda = \frac{2}{3}$ for $\mathcal{B}$?

8.3.5 Word growth

We now provide some examples of the possible growth types of Basilica groups. It is known that the original Basilica group $\mathcal{B}$ has exponential word growth, cf. [59, Proposition 4]. The same proof as the one given there also shows that $\mathrm{Bas}_2(\mathcal{O}_m)$ is of exponential growth for all $m \geqslant 2$. This, however, is not a general phenomenon.

Proposition 8.3.17. *Let $a = (0\ 1)(a, \mathrm{id})$ be the generator of the dyadic odometer acting on the binary rooted tree. Then $\mathrm{Bas}_s(\langle\langle(\mathrm{id}, a)\rangle\rangle)$ is a free abelian group of rank s, and is of polynomial growth in particular.*

Proof. The element (id, a) stabilises the ray $\overline{0}$, thus by Lemma 8.3.8 we have

$$[\beta_i(\langle\langle(\mathrm{id}, a)\rangle\rangle), \beta_j(\langle\langle(\mathrm{id}, a)\rangle\rangle)] = \mathrm{id}$$

for distinct $i, j \in [0, s-1]$. Also $\beta_i(\langle\langle(\mathrm{id}, a)\rangle\rangle) \cong \mathbb{Z}$ for all $i \in [0, s-1]$. □

As another example, we prove that there is a group of intermediate word growth such that its second Basilica group has exponential word growth.

Proposition 8.3.18. *Let $G = \langle a = (1\ 2\ 3), b = (a, 1, b)\rangle$ be the Fabrykowski–Gupta group [35] acting on the ternary rooted tree, which is of intermediate growth according to [13]. Then there exists an element $f \in \mathrm{Aut}\, T$ such that the group $\mathrm{Bas}_2(G^f)$ is of exponential growth.*

Proof. The Fabrykowski-Gupta group is a GGS-group. In contrast to the Gupta–Sidki 3-group it is not periodic: an example for an element of infinite order is ab, for which the relation

$$(ab)^3 = (ab, ba, ba)$$

holds. In view of the decomposition it is clear that ab acts spherically transitively on T and thus by a result of Gawron, Nekrashevych and Sushchansky [47] it is $\operatorname{Aut} T$-conjugate to the 3-adic odometer group. Let $f \in \operatorname{Aut} T$ be an element such that $(ab)^f = (1\ 2\ 3)((ab)^f, 1, 1)$. Then the subgroup generated by $\beta_0((ab)^f)$ and $\beta_1((ab)^f)$ in $\operatorname{Bas}_2(G^f)$ is isomorphic to the generalised Basilica group $\operatorname{Bas}_2(\mathcal{O}_3)$, which is of exponential growth by following the proof of [59, Proposition 4] (which is the same result for $\mathcal{B}$) replacing the 2-cycle with a 3-cycle corresponding to $a|^\epsilon$. □

The same idea can be used to obtain the following proposition.

Proposition 8.3.19. *Let $G \leqslant \operatorname{Aut} T$ be a group containing an element acting spherically transitively on T. Then there is an $\operatorname{Aut} T$-conjugate G^f of G such that $\operatorname{Bas}_s(G^f)$ has exponential word growth.*

8.3.6 Weakly Branch Groups

For every vertex $v \in T$ the *rigid vertex stabiliser of v in G* is the subgroup of all elements that fix all vertices outside the subtree rooted at v. For every $n \in \mathbb{N}_0$ the *n-th rigid layer stabiliser* $\operatorname{Rist}_G(n)$ is the normal subgroup generated by all rigid vertex stabilisers of n-th layer vertices. A group $G \leqslant \operatorname{Aut} T$ is called a *weakly branch group*, if G acts spherically transitively and all rigid layer stabilisers $\operatorname{Rist}_G(n)$ are non-trivial. If there is a subgroup $H \leqslant G$ such that $\psi_1(\operatorname{St}_H(1)) \geqslant H \times \cdots \times H$, the group G is said to be *weakly regular branch over H*. Clearly, a group that is weakly regular branch group over a non-trivial subgroup is a weakly branch group.

From Lemma 8.2.4, it follows that elements of the rigid layer stabilisers of G translate to elements of rigid layer stabilisers of $\operatorname{Bas}_s(G)$.

Lemma 8.3.20. *Let $n = qs + r \in \mathbb{N}_0$, with $r \in [0, s-1]$ and $q \geqslant 0$. Let $B = \operatorname{Bas}_s(G)$ for $G \leqslant \operatorname{Aut} T$. Then $\operatorname{Rist}_B(n)$ contains $\beta_i(\operatorname{Rist}_G(q+1))$ and $\beta_j(\operatorname{Rist}_G(q))$ for $0 \leqslant i < r$ and for $r \leqslant j < s$.*

We immediately obtain the following proposition.

Proposition 8.3.21. *Let $G \leqslant \operatorname{Aut} T$ be a weakly branch group. Then $B := \operatorname{Bas}_s(G)$ is again weakly branch.*

This proves the statement (5) of Theorem 8.1.1.

The group $\mathrm{Bas}_s(G)$ can be weakly branch even when G is not weakly branch. We recall that for any group G and an abstract word ω on k letters, the set of ω-elements and the verbal subgroup associated to ω are

$$G_\omega := \{\omega(h_0, \dots, h_{k-1}) \mid h_0, \dots, h_{k-1} \in G\} \text{ and } \omega(G) := \langle G_\omega \rangle \text{ respectively.}$$

Proposition 8.3.22. *Let $G \leqslant \operatorname{Aut} T$ be a self-similar strongly fractal group and let $B := \mathrm{Bas}_s(G)$. Let ω be a law in G, i.e. a word ω such that $\omega(G) = 1$, but let ω not be a law in B. Then B is weakly regular branch over $\omega(B)$.*

Proof. Let $b = \omega(b_0, \dots, b_{k-1}) \neq \mathrm{id}$ with $b_i \in B$ for $i \in [0, k-1]$. By Lemma 8.3.3 there are elements $c_i \in \mathrm{St}_B(1)$ such that $c_i|_0 = b_i$ and $c_i|_x \in \beta_{s-1}(G)$ for all $x \in X \backslash \{0\}$.

For every $x \in X$, let $d_x \in B$ be an element such that $d_x|_x = \mathrm{id}$ and $d_x(x) = 0$ (cf. Lemma 8.2.1). Then $c_i^{d_x}$ stabilises the first layer and has sections $c_i^{d_x}|_x = b_i$ and $c_i^{d_x}|_y = (c_i|_{d_{x(y)}})^{d_x|_y} \in \beta_{s-1}(G)^{d_x|_y}$ for $y \neq x$.

Since $c_i^{d_x}$ stabilises the first layer, the section maps are homomorphisms and

$$\omega(c_0^{d_x}, \dots, c_{k-1}^{d_x})|_y = \omega(c_0^{d_x}|_y, \dots, c_{k-1}^{d_x}|_y) = \begin{cases} b, & \text{if } y = x \\ \mathrm{id} & \text{else,} \end{cases}$$

because in the second case we are evaluating ω in a group isomorphic to G. This shows that $B_\omega \times \dots \times B_\omega$ is geometrically contained in B_ω, and thus the same holds for the verbal subgroups that are generated by these sets. ☐

We point out that, if ω is a law in B, then B cannot be weakly branch as it satisfies an identity. Proposition 8.3.22 allows to obtain examples of groups that are weakly branch over some prescribed verbal subgroup. We provide an easy example:

Example 8.3.23. The group $D := \langle \sigma, b \rangle$, with $\sigma = (0\ 1)$ and $b = (b, \sigma)$, acting on the binary tree is isomorphic to the infinite dihedral group (hence metabelian). It is self-similar and strongly fractal. Considering

$$[[\beta_1(\sigma), \beta_0(\sigma)], [\beta_0(\sigma), \beta_0(\sigma b)]] = ([\beta_0(\sigma), \beta_1(\sigma b)], [\beta_0(\sigma), \beta_1(b^{-1}\sigma)]) \neq \mathrm{id},$$

we see that the second Basilica $\mathrm{Bas}_2(D)$ is not metabelian, and thus it is weakly branch over the second derived subgroup of $\mathrm{Bas}_2(D)$.

8.4 Split groups, Layer Stabilisers and Hausdorff dimension

The subgroup $\beta_i(G) \leqslant \mathrm{Bas}_s(G)$, for $i \in [0, s-1]$, has the property that its elements have non-trivial portrait only at vertices at levels $n \equiv_s i$ for $n \in \mathbb{N}_0$.

We consider an algebraic analogue of this property that will be used to determine the structure of the stabilisers of $\mathrm{Bas}_s(G)$.

Definition 8.4.1. Let $G \leqslant \mathrm{Aut}\, T$ and $B := \mathrm{Bas}_s(G)$. Define:

$$S_i := \langle \beta_j(G) \mid j \neq i \rangle \leqslant B \text{ and } N_i := (S_i)^B \trianglelefteq B.$$

We write $\phi_i : B \to B/N_i$ for the canonical epimorphism with kernel N_i. The quotient B/N_i is isomorphic to the quotient of G by the normal subgroup $K_i := \beta_i^{-1}(\beta_i(G) \cap N_i)$. We call K_i the *i-th splitting kernel* of G. The group G is called *s-split* if its s-th Basilica group B is a split extension of N_i by $\beta_i(G)$ for all $i \in [0, s-1]$, or equivalently if all splitting kernels of G are trivial.

Proposition 8.4.2. *Let $G \leqslant \mathrm{Aut}\, T$ be a group that does not stabilise the vertex* 0*. Then $\beta_i([G,G]) \leqslant N_i$ for $i \in [1, s-1]$. In particular, an s-split group (for $s > 1$) is abelian.*

Proof. Let $g, h \in G$, $k \in G \setminus \mathrm{st}(0)$ and let $i \in [1, s-1]$. Write $\gamma = \beta_{i-1}(g), \eta = \beta_{i-1}(h), \overline{\gamma} = \beta_i(g), \overline{\eta} = \beta_i(h)$ and $\kappa = \beta_0(k)$. Then

$$\begin{aligned}
\kappa^{-1}(\kappa)^{\overline{\gamma}^{-1}}(\kappa^{-1})^{\overline{\gamma}^{-1}\overline{\eta}}(\kappa)^{\overline{\eta}}|_x &= \kappa^{-1}|_{\kappa(x)}\overline{\gamma}|_{\kappa(x)}\kappa|_x(\overline{\gamma}^{-1}\overline{\eta}^{-1}\overline{\gamma})|_x\kappa^{-1}|_{\kappa(x)}\overline{\gamma}^{-1}|_{\kappa(x)}\kappa|_x\overline{\eta}|_x \\
&= \kappa|_x^{-1}\overline{\gamma}|_{\kappa(x)}\kappa|_x(\overline{\gamma}^{-1}\overline{\eta}^{-1}\overline{\gamma})|_x\kappa|_x^{-1}\overline{\gamma}|_{\kappa(x)}^{-1}\kappa|_x\overline{\eta}|_x \\
&= \begin{cases} [\gamma, \eta] & \text{if } x = 0, \\ \mathrm{id} & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus $\kappa^{-1}(\kappa)^{\overline{\gamma}^{-1}}(\kappa^{-1})^{\overline{\gamma}^{-1}\overline{\eta}}(\kappa)^{\overline{\eta}} = ([\gamma, \eta], \mathrm{id}, \ldots, \mathrm{id}) = [\overline{\gamma}, \overline{\eta}]$ is an element of $N_i \cap \beta_i(G)$. □

We remark that $[G,G] \leqslant K_0$ does not necessarily hold. For example, consider a group G such that $[G,G] \nleqslant \mathrm{St}_G(1)$. Since $N_0 \leqslant \mathrm{St}_{\mathrm{Bas}_s(G)}(1)$, the zero-th splitting kernel can not contain $[G,G]$.

Definition 8.4.3. We call a subgroup H of a group G *non-absorbing in G* if for all $h_0, \ldots, h_{m-1} \in H$ such that $\psi_1^{-1}(h_0, \ldots, h_{m-1}) \in G$, implies $\psi_1^{-1}(h_0, \ldots, h_{m-1}) \in H$. If G is weakly branch over H, then H is non-absorbing in G.

Proposition 8.4.4. *Let $G \leqslant \mathrm{Aut}\, T$ be self-similar and such that $G|^{\epsilon}$ acts regularly on X. Assume that $[G,G]$ is non-absorbing in G. Then for $i \in [1, s-1]$ we have $K_i = [G,G]$, and $K_0 \leqslant [G,G]$. In particular, if G is abelian, it is s-split for all $s \in \mathbb{N}_+$.*

Proof. The inclusion $[G, G] \leqslant K_i$ for $i \in [1, s-1]$ is proven in Proposition 8.4.2. Thus we prove $K_i \leqslant [G, G]$ for $i \in [0, s-1]$.

Set $B := \mathrm{Bas}_s(G)$ and define $\mathcal{N} := \bigcup_{i=0}^{s-1}(\beta_i(G) \cap N_i)$. We employ the decomposition in syllables, cf. Subsection 8.3.4. For every $b \in \mathcal{N}$ there is an index $i \in [0, s-1]$ such that b can be written both as an element of the image of some β_i and a word in N_i, i.e.

$$b = \beta_i(g_0) = \prod_{j=1}^{\ell(b)} (h_j)^{\beta_i(g_j)} \tag{$*$}$$

for suitable $\ell(b) \in \mathbb{N}_0$, $g_j \in G$ and $h_j \in S_i$. The minimal possible value of $\ell(b)$ is called the *restricted syllable length*, and from here onwards we use the symbol ℓ for this invariant. Write $\mathcal{C} = \bigcup_{i=0}^{s-1} \beta_i([G, G])$ (notice that this a union of subsets with pairwise trivial intersection), and define

$$\mathcal{M} := \{b \in \mathcal{N} \backslash \mathcal{C} \mid \ell(b) \leqslant \ell(c) \text{ for all } c \in \mathcal{N} \backslash \mathcal{C}\},$$

the set of all non-commutator elements with minimal restricted syllable length.

We shall prove that for every $b \in \mathcal{M}$ there exists a first level vertex $x_i \in X$ such that:

1. $b|_{x_i} \in \mathcal{M}$ and
2. $b|_x = \mathrm{id}$ for all $x \in X \backslash \{x_i\}$.

Furthermore we prove that

3. $b \in \mathrm{St}_B(1)$, i.e. $\mathcal{M} \leqslant \mathrm{St}_B(1)$.

Every subset $\mathcal{M} \subseteq \mathrm{Aut}\, T$ with these properties is empty. Indeed, if $b \in \mathcal{M}$, there is some vertex $u \in T$ such that $b|^u \neq \mathrm{id}$, since b is not trivial. But by properties (1) and (2) $b|_u$ is either trivial or a member of $\mathcal{M}$, hence by property (3) stabilises the first layer, a contradiction.

But if $\mathcal{M}$ is empty, $\mathcal{N}$ is contained in $\mathcal{C}$, hence all splitting kernels are subgroups of $[G, G]$, finishing the proof.

Assume that there is some $b \in \mathcal{M}$. We fix the decomposition and the type given by ($*$), but write ℓ for $\ell(b)$ to shorten the notation.

We first observe that $\ell \neq 1$. If $\ell = 1$, we have $\beta_i(g_0) = h_1^{\beta_i(g_1)}$, consequently $h_1 \in \beta_i(G) \cap S_i$. But $h_1|^u = \mathrm{id}$ for all u with $|u| \equiv_s i$, while $\beta_i(G)|^u = \{\mathrm{id}\}$ for $u \notin \omega_i(T)$ by Lemma 8.2.4. Thus $h_1 = \mathrm{id} = b \notin \mathcal{M}$, which is a contradiction.

We split the proof of statements (1) to (3) into two cases: $i = 0$ and $i \neq 0$.

Case $i = 0$: Since $N_0 \leqslant \mathrm{St}_B(1)$, statement (3) is fulfilled. We have $S_0|_0 = S_{s-1}$ and $S_0|_x = \{\mathrm{id}\}$ for $x \in X\backslash\{0\}$. Also $\beta_0(G)|_x \leqslant \beta_{s-1}(G|_x)$ for $x \in X$, hence $N_0|_x \leqslant N_{s-1}$. Thus all sections $b|_x$ are members of $\beta_{s-1}(G) \cap N_{s-1} \subseteq \mathcal{N}$.

The first layer sections of b are given by

$$b|_x = \beta_{s-1}(g_0|_x) = \prod_{j \in L_x} (h_j|_0)^{\beta_{s-1}(g_j|_x)}, \qquad \text{for } x \in X,$$

where $L_x = \{j \mid 1 \leqslant j \leqslant \ell \text{ and } g_j(x) = 0\}$. The sum $\sum_{x \in X} |L_x|$ equals ℓ. By the minimality of ℓ, either all sections of b are contained in $\beta_{s-1}([G,G])$, or there is some $x_i \in X$ such that $\ell(b|_{x_i}) = |L_{x_i}| = \ell$. In the first case, since $[G,G]$ is non-absorbing in G, this implies $b \in \beta_0([G,G])$, a contradiction. In the second case, $L_x = \varnothing$ for $x \neq x_i$, i.e. $b|_x = \mathrm{id}$ for $x \neq x_i$. This proves statement (2). Furthermore, if $b|_{x_i} \notin \mathcal{M}$, it is contained in $\beta_{s-1}([G,G])$. Since $[G,G]$ is non-absorbing over G, this implies $b \in \beta_0([G,G])$. Thus $b|_{x_i} \in \mathcal{M}$, and statement (1) is true.

Case $i \neq 0$: Recall that $b|_x = \beta_i(g_0)|_x = \mathrm{id}$ for $x \neq 0$. This is statement (2) with $x_i = 0$.

We consider the first layer sections of b. For $x \in X$ and $1 \leqslant j \leqslant \ell$,

$$h_j^{\beta_i(g_j)}|_x = \begin{cases} (h_j|_x)^{\beta_{i-1}(g_j)} & \text{if } x = 0 \text{ and } h_j \in \mathrm{st}_B(0), \\ h_j|_x \beta_{i-1}(g_j) & \text{if } x = 0 \text{ and } h_j \notin \mathrm{st}_B(0), \\ \beta_{i-1}(g_j^{-1}) h_j|_x & \text{if } h_j \notin \mathrm{st}_B(0) \text{ and } x = h_j^{-1}(0), \\ h_j|_x & \text{otherwise.} \end{cases} \tag{†}$$

Since $G|^\epsilon$ acts regularly, $\mathrm{st}_B(0) = \mathrm{St}_B(1)$. We divide the long product in ($*$) into segments that stabilise the first layer: Let $x \in X$, and consider the subsequence $(j_x^{(k)})_{k \in [1,t_x]}$ of $[1,\ell]$ consisting of all indices $j_x^{(k)}$ such that $(\prod_{j=j_x^{(k)}}^{\ell} h_j)(x) = 0$. Clearly $\sum_{x \in X} t_x = \ell$.

Set $j_x^{(0)} = 1$ and $j_x^{(t_x+1)} = \ell + 1$. Then $\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} h_j \in \mathrm{St}_B(1)$ for all $k \in [1, t_x]$, and one may write

$$b = \prod_{k=0}^{t_x} \prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} (h_j)^{\beta_i(g_j)}. \tag{‡}$$

We now make another case distinction.

Subcase $t_x = \ell$ *for some* $x \in X\backslash\{0\}$*:* We will prove that this case can not occur. The equation $t_x = \ell$ implies $h_\ell(x) = 0$ and $h_j \in \mathrm{St}_B(1)$ for all $j \in [1, \ell-1]$. We may assume $g_\ell = \mathrm{id}$, by passing to a conjugate if necessary. Looking at the second and fourth case of

(†), we obtain

$$\beta_{i-1}(g_0) = b|_0 = \prod_{j=1}^{\ell-1}(h_j|_{h_\ell(0)}) \cdot h_\ell|_0 \in N_{i-1}.$$

Thus $\beta_{i-1}(g_0)$ is an element of $\mathcal{N}$ of restricted syllable length at most 1, hence trivial. Consequently g_0 and b are trivial, a contradiction.

Subcase $t_0 = \ell$: This implies $h_j \in \mathrm{St}_B(1)$ for all $j \in [1, \ell]$, and statement (3) holds. By the first case of (†)

$$b|_0 = \prod_{j=1}^{\ell(b)} h_j|_0{}^{\beta_{i-1}(g_j)} \in N_{i-1} \cap \beta_{i-1}(G),$$

which is of restricted syllable length at most ℓ. As we previously argued in the case $i = 0$, we have $b|_0 \notin \beta_{i-1}([G, G])$ and consequently statement (1) holds, since otherwise $b \in \beta_i([G, G])$ because $[G, G]$ is non-absorbing over G.

Subcase $t_x < \ell$ for all $x \in X$: We shall prove that this case can not occur. Combining (‡) with (†) for $x \in X$ we calculate

$$b|_x = \prod_{k=0}^{t_x-1}\left(\left(\Pi_{j=j_x^{(k)}}^{j_x^{(k+1)}-1}(h_j)^{\beta_i(g_j)}\right)|_0\right)\left(\Pi_{j=j_x^{(t_x)}}^{\ell(b)}(h_j)^{\beta_i(g_j)}\right)|_x$$

and for $k \in [1, t_x - 1]$

$$\begin{aligned}\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1}(h_j)^{\beta_i(g_j)}|_0 &= \beta_{i-1}(g_{j_x^{(k)}}^{-1})(\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} h_j|_{\prod_{i=j+1}^{j_x^{(k+1)}-1} h_i(0)})\beta_{i-1}(g_{j_x^{(k+1)}-1}) \\ &= \beta_{i-1}(g_{j_x^{(k)}}^{-1} g_{j_x^{(k+1)}-1})(\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1} h_j|_{\prod_{i=j+1}^{j_x^{(k+1)}-1} h_i(0)})^{\beta_{i-1}(g_{j_x^{(k+1)}-1})}.\end{aligned}$$

Consequently, every segment $\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1}(h_j)^{\beta_i(g_j)}$ of b contributes at most one syllable of N_{i-1} and a member of $\beta_{i-1}(G)$ to $b|_x$. We obtain

$$b|_x \equiv_{N_{i-1}} \begin{cases} \beta_{i-1}\left(g_1^{-1}\prod_{k=1}^{t_x}\left(g_{j_x^{(k)}-1}g_{j_x^{(k)}}^{-1}\right)g_\ell\right) & \text{if } x = 0, \\ \beta_{i-1}\left(\prod_{k=1}^{t_x}\left(g_{j_x^{(k)}-1}g_{j_x^{(k)}}^{-1}\right)\right) & \text{otherwise.}\end{cases}$$

Write $b|_x = \beta_{i-1}(f_x)n_x$ with $n_x \in N_i$ and f_x equal to the corresponding product in G in the last equation. Since the subsequences form a partition, every $\beta_{i-1}(g_{j_x^{(k)}})$ and its inverse appear in precisely one section of b, and we have

$$\prod_{x\in X} b|_x \equiv_{N_{i-1}} \prod_{x\in X}\beta_{i-1}(f_x) \equiv_{\beta_{i-1}([G,G])} \prod_{j=1}^{\ell}\beta_{i-1}(g_j g_j^{-1}) = 1.$$

Now we look at n_x. Since every segment $\prod_{j=j_x^{(k)}}^{j_x^{(k+1)}-1}(h_j)^{\beta_i(g_j)}$ contributes at most one syllable, and $h_j \notin \mathrm{St}_B(1)$ for some $j \in [1,\ell]$, we have $\ell(n_x) \leqslant t_x < \ell$. Also $\beta_{i-1}(f_x)n_x = b|_x = \mathrm{id}$ for $x \neq 0$, hence $n_x = \beta_{i-1}(f_x^{-1}) \in \mathcal{N}$. By minimality, $f_x \in [G,G]$. Then also $f_0 \equiv_{[G,G]} \prod_{x\in X} f_x \equiv_{[G,G]} \mathrm{id}$, and $\beta_{i-1}(f_0^{-1}g_0) = \beta_{i-1}(f_0^{-1})b|_0 = n_0 \in \mathcal{N}$. Again, by minimality, $f_0^{-1}g_0 \in [G,G]$, thus $g_0 \in [G,G]$, a contradiction.

This completes the proof. □

Example 8.4.5. Let $\mathrm{Bas}_s(\mathcal{O}_m^d)$ be a generalised Basilica group (cf. Definition 8.2.6). Since $\mathcal{O}_m^d$ is free abelian and self-similar, and $\mathcal{O}_m^d|^\epsilon$ is cyclic of order m, by Proposition 8.4.4, the group $\mathcal{O}_m^d$ is s-split.

Question 8.4.6. Motivated by the small gap between Proposition 8.4.4 and Proposition 8.4.2 we ask:

Is every abelian group $G \leqslant \mathrm{Aut}\, T$ acting spherically transitive s-split for all $s > 1$?

Corollary 8.4.7. *Let $G \leqslant \mathrm{Aut}\, T$ be a self-similar s-split group. Then the abelianisation $\mathrm{Bas}_s(G)$ is*

$$\mathrm{Bas}_s(G)^{\mathrm{ab}} \cong G^s.$$

Proof. Consider the normal subgroup $H := \langle[\beta_i(G), \beta_j(G)] \mid i,j \in [0, s-1], i \neq j\rangle^{\mathrm{Bas}_s(G)}$ and observe that $H \leqslant N_i$ for all $i \in [0, s-1]$. We obtain an epimorphism $G^s \to \mathrm{Bas}_s(G)/H$, mapping the i-th component of G^s to $\beta_i(G)(H)$, for $i \in [0, s-1]$. This map is also injective. Let $\prod_{i\in[0,s-1]} \beta_i(g_i) \equiv_H \prod_{i\in[0,s-1]} \beta_i(h_i)$ for some $g_i, h_i \in G$. Then for all $x \in X$

$$\beta_x(g_x h_x^{-1}) \equiv_H \prod_{i\in[0,s-1]\setminus\{x\}} \beta_i(g_i^{-1}h_i) \in N_x$$

and $\beta_x(g_x h_x^{-1}) \in N_x$. Since G is s-split, this implies $g_x = h_x$. Thus $\mathrm{Bas}_s(G)/H \cong G^s$. But from Proposition 8.4.2 G is abelian and consequently $H = [\mathrm{Bas}_s(G), \mathrm{Bas}_s(G)]$. □

Proposition 8.4.8. *Let $G \leqslant \mathrm{Aut}\, T$ be a torsion-free self-similar group such that the quotient G/K with $K = \beta_0^{-1}(\beta_0(G) \cap N_0)$ is again torsion-free. Then $\mathrm{Bas}_s(G)$ is torsion-free.*

Proof. Let $b \in \mathrm{Bas}_s(G)$ be a torsion element. Since G/K is torsion-free, we obtain $b \in \ker \phi_0 = N_0 \leqslant \mathrm{St}_{\mathrm{Bas}_s(G)}(1)$. Thus the first layer sections of b are again torsion elements of $\mathrm{Bas}_s(G)$, because $\mathrm{Bas}_s(G)$ is self-similar by Lemma 8.3.2. Hence an iteration of the argument yields $b = \mathrm{id}$. □

Question 8.4.9. On the other end of the spectrum, the group $\mathrm{Bas}_2(\mathcal{G})$ (cf. Example 8.3.11) is periodic as is $\mathcal{G}$, which can be proven analogous to [18, Theorem 6.1], and the second Basilica groups of the periodic Gupta-Sidki-p-groups (cf. Example 8.3.10) are periodic by [91]. Motivated by this observation we ask:

Is there a periodic group $G \leqslant \operatorname{Aut} T$ acting spherically transitive such that $\operatorname{Bas}_s(G)$ is not periodic for some $s \in \mathbb{N}_+$?

Proposition 8.4.8 and Corollary 8.4.7 prove Theorem 8.1.3.

8.4.1 Layer Stabilisers

For an s-split group $G \leqslant \operatorname{Aut} T$ the s-th Basilica decomposes as $\operatorname{Bas}_s(G) = N_i \rtimes \beta_i(G)$. Recall from Definition 8.4.1 that ϕ_i denotes the map to $\operatorname{Bas}_s(G)/N_i$, identified with the quotient G/K_i, such that $\phi_i(n\beta_i(g)) = gK_i$ for all $g \in G, n \in N_i$.

Lemma 8.4.10. *Let $G \leqslant \operatorname{Aut} T$ be a strongly fractal group and let $B = \operatorname{Bas}_s(G)$. Let $b_0, \dots, b_{m-1} \in B$. Then $\psi_1^{-1}(b_0, \dots, b_{m-1})$ is an element of $\operatorname{St}_B(1)$ if and only if there is an element $g \in \operatorname{St}_G(1)$ such that for all $x \in X$*

$$\phi_{s-1}(b_x) = g|_x K_{s-1}.$$

Proof. If there is some element $g \in \operatorname{St}_G(1)$ of the required form, clearly

$$\beta_0(g) \equiv_{\psi_1^{-1}(N_{s-1}^m)} (b_0, \dots, b_{m-1}).$$

Now we claim that $\psi_1(N_0) \geqslant N_{s-1}^m$. Let

$$b = \prod_{j=0}^{\ell-1} h_j^{\beta_{s-1}(g_j)} \in N_{s-1},$$

with $h_j \in S_{s-1}$. Then there are elements $\hat{h}_j = (h_j, \mathrm{id}, \dots, \mathrm{id}) \in S_0$ by the definition of S_{s-1}. Furthermore, since G is strongly fractal, there are elements $\hat{g}_j \in \operatorname{St}_G(1)$ such that $\beta_0(\hat{g}_j)|_0 = \beta_{s-1}(g_j)$, yielding

$$\prod_{j=0}^{\ell-1} \hat{h}_j^{\beta_0(\hat{g}_j)} = (b, \mathrm{id}, \dots, \mathrm{id}).$$

Since G acts spherically transitively, the claim follows by Lemma 8.2.1. Thus there is an element in $N_0\beta_0(g) \leqslant \operatorname{St}_B(1)$ with sections $(b_0, \dots, b_{m-1})$.

Let now $b = \psi_1^{-1}(b_0, \dots, b_{m-1}) \in \operatorname{St}_B(1)$. Then b decomposes as a product $n\beta_0(g)$ with $n \in N_0$ and $g \in \operatorname{St}_G(1)$. This implies, for any $x \in X$,

$$\phi_{s-1}(b_x) = \phi_{s-1}((n\beta_0(g))|_x) = \phi_{s-1}(\beta_{s-1}(g|_x)) = g|_x K_{s-1}. \qquad \square$$

Lemma 8.4.11. *Let G be fractal and self-similar and let $B = \operatorname{Bas}_s(G)$. Let $n \in \mathbb{N}_0$.*

(i) $\psi_1(\beta_i(\mathrm{St}_G(n))^B) = (\beta_{i-1}(\mathrm{St}_G(n))^B)^m$ *for all* $i \neq 0$.

Assuming further that G is very strongly fractal,

(ii) $\psi_1([\beta_0(\mathrm{St}_G(n+1)), N_0]^B) = ([\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m$.

Proof. (i) The inclusion $\psi_1(\beta_i(\mathrm{St}_G(n))^B) \leqslant (\beta_{i-1}(\mathrm{St}_G(n))^B)^m$ is obvious. We prove the other direction. Let $g \in \mathrm{St}_G(n)$ and $b \in B$. Since B is fractal by Lemma 8.3.3, there is an element $c \in \mathrm{st}_B(0)$ such that $c|_0 = b$. Now

$$(\beta_i(g))^c = (\beta_{i-1}(g), \mathrm{id}, \ldots, \mathrm{id})^c = (\beta_{i-1}(g)^b, \mathrm{id}, \ldots, \mathrm{id}),$$

yielding statement (i), by Lemma 8.2.1.

(ii) The inclusion $\psi_1([\beta_0(\mathrm{St}_G(n+1)), N_0]^B) \leqslant ([\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m$ follows directly from $N_0|_x \leqslant N_{s-1}$ and $\beta_0(\mathrm{St}_G(n+1))|_x \leqslant \beta_{s-1}(\mathrm{St}_G(n))$, where $x \in X$. Thanks to Lemma 8.2.1, for the other inclusion it is enough to prove that $([\beta_{s-1}(g), k], \mathrm{id}, \ldots, \mathrm{id})$ is contained in $\psi_1([\beta_0(\mathrm{St}_G(n+1)), N_0]^B)$ for all $g \in \mathrm{St}_G(n)$ and $k \in N_{s-1}$. Let

$$k = \prod_{j=0}^{\ell} (\beta_{i_j}(k_j))^{\beta_{s-1}(k_j')} \in N_{s-1}.$$

Since G is strong fractal there are elements $t_j \in \mathrm{St}_G(1)$ such that $t_j|_0 = k_j'$. Furthermore, since G is very strongly fractal there is an element $h \in \mathrm{St}_G(n+1)$ such that $h|_0 = g$. Then

$$[\beta_0(h), \prod_{j=0}^{\ell} (\beta_{i_j+1}(k_j))^{\beta_0(t_j)}] \in [\beta_0(\mathrm{St}_G(n+1)), N_0]^B$$

and

$$\begin{aligned} [\beta_0(h), \prod_{j=0}^{\ell} (\beta_{i_j+1}(k_j))^{\beta_0(t_j)}]|_x &= [(\beta_0(h))|_x, \prod_{j=0}^{\ell} ((\beta_{i_j+1}(k_j))|_x)^{(\beta_0(t_j))|_x}] \\ &= \begin{cases} [\beta_{s-1}(g), k] & \text{if } x = 0, \\ [\beta_{s-1}(h|_x), \prod_{j=0}^{\ell} \mathrm{id}^{\beta_{s-1}(t_j|_x)}] = \mathrm{id} & \text{otherwise.} \end{cases} \end{aligned}$$

□

Proof of Theorem 8.1.4. Let $B = \mathrm{Bas}_s(G)$. For any $n \in \mathbb{N}_0$, write $n = sq + r$ with $q \geqslant 0$ and $r \in [0, s-1]$. We have to prove

$$\mathrm{St}_B(n) = \langle \beta_i(\mathrm{St}_G(q+1)), \beta_j(\mathrm{St}_G(q)) \mid 0 \leqslant i < r \leqslant j < s \rangle^B.$$

For convenience, we will denote the right-hand side of this equation by H_n. It is clear that $H_n \leqslant \mathrm{St}_B(n)$ for all $n \in \mathbb{N}_0$. It remains to establish the other inclusion. For $n = 0$ the statement is clearly true, so we proceed by induction and assume that the statement is true for some fixed $n = sq + r$ with $q \geqslant 0$ and $r \in [0, s-1]$. Define

$$J := \langle \beta_i(\mathrm{St}_G(q+1)), \beta_j(\mathrm{St}_G(q)), [\beta_{s-1}(\mathrm{St}_G(q)), N_{s-1}]^B \mid 0 \leqslant i \leqslant r-1 < j < s-1 \rangle^B,$$

and observe that by Lemma 8.4.11 we find $J^m \leqslant \psi_1(H_{n+1})$, which yields

$$(\mathrm{St}_B(n))^m/\psi_1(H_{n+1}) = (\beta_{s-1}(\mathrm{St}_G(q)))^m \psi_1(H_{n+1})/\psi_1(H_{n+1}).$$

Hence for every $g \in \mathrm{St}_B(n+1)$, there are elements $t_0, \ldots, t_{m-1} \in \mathrm{St}_G(q)$ such that

$$\psi_1(g) \equiv_{\psi_1(H_{n+1})} (\beta_{s-1}(t_0), \ldots, \beta_{s-1}(t_{m-1})).$$

Since $\phi_{s-1}\beta_{s-1}(t_x) = t_x K_{S-1}$ for all $x \in X$, $g \in \mathrm{St}_B(1)$ and $H_{n+1} \leqslant \mathrm{St}_B(1)$, by Lemma 8.4.10 there are elements $k_0, \ldots, k_{m-1} \in K_{s-1}$ and $h \in \mathrm{St}_G(1)$ such that

$$\psi_1^{-1}(h|_0 k_0, \ldots, h|_{m-1} k_{m-1}) = \psi_1^{-1}(t_0, \ldots, t_{m-1}).$$

Define $\widetilde{h} = h\psi_1^{-1}(k_0, \ldots, k_{m-1})$. Now G is weakly regular branch over K_{s-1}, hence $\psi_1^{-1}(K_{s-1}^m) \leqslant \mathrm{St}_{K_{s-1}}(1)$, and consequently $\widetilde{h} \in \mathrm{St}_G(1)$. But $\widetilde{h}|_x = t_x \in \mathrm{St}_G(q)$ for $x \in X$, whence $\widetilde{h} \in \mathrm{St}_G(q+1)$ and

$$(\beta_{s-1}(t_0), \ldots, \beta_{s-1}(t_{m-1})) = \psi_1(\beta_0(\widetilde{h})) \in \psi_1(\beta_0(\mathrm{St}_G(q+1))) \leqslant \psi_1(H_{n+1}),$$

implying $g \in H_{n+1}$. This completes the proof. □

8.4.2 Hausdorff Dimension

We remind the reader that Γ is the subgroup of $\mathrm{Aut}\, T$ consisting of all automorphisms whose labels are elements of $\langle\sigma\rangle$, with σ being a fixed m-cycle in $\mathrm{Sym}(X)$.

Definition 8.4.12. Let $G \leqslant \Gamma$. The *Hausdorff dimension of G relative to Γ* is defined by

$$\dim_{\mathrm{H}} G := \liminf_{n\to\infty} \frac{\log_m |G/\mathrm{St}_G(n)|}{\log_m |\Gamma/\mathrm{St}_\Gamma(n)|} = (m-1) \liminf_{n\to\infty} \frac{\log_m |G/\mathrm{St}_G(n)|}{m^n}.$$

This relates to the usual definition of Hausdorff dimension over arbitrary spaces by taking the closure, i.e. using this definition, the group G has the same Hausdorff dimension as its closure $\overline{G}$ in Γ, cf. [9]. We drop the base m in $\log_m$ from now on. Denote the quotient $\mathrm{St}_G(n)/\mathrm{St}_G(n+1)$ by $L_G(n)$. The integer series (for $n > 0$) obtained by

$$o_G(n) := \log(|L_G(n-1)|^m) - \log|L_G(n)|$$

is called the *series of obstructions of G*. We set $o_G(0) = -1$ for convenience.

The series of obstructions of a group G determines its Hausdorff dimension, precisely how we will see in Lemma 8.4.13. Nevertheless, one might wonder why it is necessary to define this seemingly impractial invariant. We will demonstrate in Proposition 8.4.16 that it is (to some degree) preserved under $G \mapsto \mathrm{Bas}_s(G)$. Furthermore, many well-studied subgroups of Γ have a well-behaved series of obstructions. For example, it is easy to see that Γ itself has

$$\begin{aligned} o_\Gamma(n) &= \log |\wr^n \mathrm{C}_m / \wr^{n-1} \mathrm{C}_m|^m - \log |\wr^{n+1} \mathrm{C}_m / \wr^n \mathrm{C}_m| \\ &= m \log m^{m^n} - \log m^{m^{n+1}} = 0, \end{aligned}$$

for $n \in \mathbb{N}_+$, where $\wr^n A$ is the n-times iterated wreath product of A, with the convention that $\wr^0 A$ is the trivial group. On the other hand, since the layer stabiliser of $\mathcal{O}_m^d$ are the subgroups generated by $\langle \pi_0(a)^{m^{k+1}}, \dots, \pi_{l-1}(a)^{m^{k+1}}, \pi_l(a)^{m^k}, \dots, \pi_{d-1}(a)^{m^k} \rangle$, the quotients $L_{\mathcal{O}_m^d}(n)$ are all cyclic of order m, and

$$o_{\mathcal{O}_m^d}(n) = m - 1.$$

A Gupta–Sidki p-group G has precisely two terms unequal to 0, a consequence of $\mathrm{St}_G(n) = \mathrm{St}_G(n-1)^p$ for $n \geqslant 3$, cf. [38]. Similarly, the series of obstructions of the Grigorchuk group has only one non-zero term.

Lemma 8.4.13. *Let $G \leqslant \Gamma$ act spherically transitive. Then*

$$\dim_{\mathrm{H}} G = 1 - \limsup_{n \to \infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_G(i).$$

Proof. By definition $\log |L_G(0)| = 1$ and $\log |L_G(n)| = m \log |L_G(n-1)| - o_G(n)$ for $n \geqslant 1$. An inductive argument yields

$$\log |G/\mathrm{St}_G(n+1)| = \log |G/\mathrm{St}_G(n)| - \sum_{k=0}^{n} m^{n-k} o_G(k) \quad = - \sum_{k=0}^{n} \frac{m^{k+1} - 1}{m - 1} o_G(n-k).$$

This gives

$$\begin{aligned} \liminf_{n \to \infty} \frac{(m-1)}{m^{n+1}} \log \left| \frac{G}{\mathrm{St}_G(n+1)} \right| &= - \limsup_{n \to \infty} \sum_{i=0}^{n} (m^{i-n} - m^{-(n+1)}) o_G(n-i) \\ &= 1 - \limsup_{n \to \infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_G(i). \end{aligned}$$ □

Lemma 8.4.14. *Let $G \leqslant \Gamma$ be self-similar. Then for all $n > 0$*

$$o_G(n) = \log[\mathrm{St}_G(n-1)^m : \psi_1(\mathrm{St}_G(n))] - \log[\mathrm{St}_G(n)^m : \psi_1(\mathrm{St}_G(n+1))].$$

Proof. We have, for $n > 0$,

$$\begin{aligned} \left| \frac{\mathrm{St}_G(n-1)^m}{\psi_1(\mathrm{St}_G(n))} \right| &= \frac{|\mathrm{St}_G(n-1)^m / \psi_1(\mathrm{St}_G(n+1))|}{|L_G(n)|} \\ &= \frac{|L_G(n-1)|^m}{|L_G(n)|} \left| \frac{\mathrm{St}_G(n)^m}{\psi_1(\mathrm{St}_G(n+1))} \right|, \end{aligned}$$

hence

$$o_G(n) = \log[\mathrm{St}_G(n-1)^m : \psi_1(\mathrm{St}_G(n))] - \log[\mathrm{St}_G(n)^m : \psi_1(\mathrm{St}_G(n+1))].$$ □

Lemma 8.4.15. *Let G be very strongly fractal, self-similar and weakly regular branch over the splitting kernel K_{s-1}. Then for all $\ell, n \in \mathbb{N}_+$*

$$\begin{aligned}&\psi_1(\beta_0(\mathrm{St}_G(\ell+1)) \cap [\beta_0(\mathrm{St}_G(n+1)), N_0]^B)\\ &= (\beta_{s-1}(\mathrm{St}_G(\ell)) \cap [\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m.\end{aligned}$$

Proof. The left-hand set is clearly contained in the right-hand set. We prove the other inclusion. Let $(b_0, \dots, b_{m-1}) \in (\beta_{s-1}(\mathrm{St}_G(\ell)) \cap [\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B)^m$. By Lemma 8.4.11(ii) there exists $b \in [\beta_0(\mathrm{St}_G(n+1)), N_0]^B \leqslant \mathrm{St}_B(1)$ such that $\psi_1(b) = (b_0, \dots, b_{m-1})$. It remains to prove that $b \in \beta_0(\mathrm{St}_G(\ell+1))$.

Since the set $\beta_{s-1}(\mathrm{St}_G(\ell)) \cap [\beta_{s-1}(\mathrm{St}_G(n)), N_{s-1}]^B$ is contained in $\beta_{s-1}(\mathrm{St}_{K_{s-1}}(1))$ and since G weakly regular branch over K_{s-1}, there is an element $g \in K_{s-1}$ such that

$$\psi_1(g) = (\beta_{s-1}^{-1}(b_0), \dots, \beta_{s-1}^{-1}(b_{m-1})) \in \mathrm{St}_G(\ell)^m.$$

Consequently, $\psi_1(\beta_0(g)) = (b_0, \dots, b_{m-1}) = \psi_1(b)$, and $b = \beta_0(g)$ is a member of $\psi_1(\beta_0(\mathrm{St}_G(\ell+1)) \cap [\beta_0(\mathrm{St}_G(n+1)), N_0]^B)$. □

Proposition 8.4.16. *Let $G \leqslant \Gamma$ be very strongly fractal, self-similar and weakly regular branch over the splitting kernel K_{s-1}. Then the series of obstructions for $B = \mathrm{Bas}_s(G)$ fulfills*

$$o_B(n) = \begin{cases} 0 & \text{if } n \not\equiv_s 0, \\ o_G(\frac{n}{s}) & \text{otherwise.} \end{cases}$$

Proof. Consider first the case $n \equiv_s k \neq 0$. By Theorem 8.1.4 the quotient $L_B(n)$ is normally generated in B by images of elements of $\beta_k(\mathrm{St}_G(\lfloor n/s \rfloor))$. Similarly the images of $\beta_{k-1}(\mathrm{St}_G(\lfloor n/s \rfloor))$ are the normal generators of $L_B(n-1)$. Thus Lemma 8.4.11(i) shows that $o_B(n) = 0$.

Now consider the case $n = qs$. To shorten the notation, we abbreviate

$$\begin{aligned} R_q &:= \beta_0(\mathrm{St}_G(q)) \text{ for } q \in \mathbb{N}_0 \text{ and} \\ T_q &:= \beta_{s-1}(\mathrm{St}_G(q)) \text{ for } q \in \mathbb{N}_0\,. \end{aligned}$$

Define the normal subgroups

$$\begin{aligned} U &= \langle \mathrm{St}_B(n+1) \cup [R_q, N_0]^B \rangle \trianglelefteq B \quad \text{and} \\ V &= \langle \mathrm{St}_B(n) \cup [T_{q-1}, N_{s-1}]^B \rangle \trianglelefteq B. \end{aligned}$$

Using Theorem 8.1.4, we see that U and V, respectively, are normally generated by the sets

$$R_{q+1} \cup \bigcup_{i=1}^{s-1} (\beta_i(\mathrm{St}_G(q))) \cup [R_q, N_0] \quad \text{and} \quad T_q \cup \bigcup_{i=0}^{s-2} (\beta_i(\mathrm{St}_G(q))) \cup [T_{q-1}, N_{s-1}].$$

Let $g \in \mathrm{St}_G(q+1)$ and $b \in B$. We write $b = \beta_0(g_b)n_b$ for $g_b \in G$ and $n_b \in N_0$. Then

$$\beta_0(g)^b = \beta_0(g^{g_b})^{n_b} = \beta_0(g^{g_b})[\beta_0(g^{g_b}), n_b] \in R_{q+1}[R_{q+1}, N_0].$$

Consequently, we drop the conjugates of R_{q+1} in our generating set for U, and write

$$U = \langle R_{q+1} \cup \bigcup_{i=1}^{s-1} \left(\beta_i(\mathrm{St}_G(q))^B\right) \cup [R_q, N_0]^B \rangle.$$

Similarly, the subgroup V is generated by

$$T_q \cup \bigcup_{i=0}^{s-2} \left(\beta_i(\mathrm{St}_G(q))^B\right) \cup [T_{q-1}, N_{s-1}]^B.$$

Using Theorem 8.1.4, it is now easy to see that

$$\mathrm{St}_B(n)/U \cong R_q/(R_q \cap U).$$

Since $\beta_i(\mathrm{St}_G(q)) \leqslant \mathrm{St}_B(n+1)$ for $i \neq 0$, we see that the intersection

$$\langle \beta_1(\mathrm{St}_G(q)) \cup \cdots \cup \beta_{s-2}(\mathrm{St}_G(q)) \cup T_q \rangle^B \cap R_q \leqslant R_{q+1}$$

is contained in R_{q+1}. We conclude

$$R_q \cap U = R_q \cap R_{q+1}[R_q, N_0]^B.$$

Now

$$R_q \cap R_{q+1}[R_q, N_0]^B = R_{q+1}(R_q \cap [R_q, N_0]^B)$$

and

$$[R_q \cap R_{q+1}[R_q, N_0]^B : R_{q+1}] = [R_q \cap [R_q, N_0]^B : R_{q+1} \cap [R_q, N_0]^B].$$

Consequently, the order of $\mathrm{St}_B(n)/U$ equals

$$|L_G(q)| \cdot [R_q \cap [R_q, N_0]^B : R_{q+1} \cap [R_q, N_0]^B]^{-1}.$$

A similar computation shows that the order of $\mathrm{St}_B(n-1)/V$ is

$$|L_G(q-1)| \cdot [T_{q-1} \cap [T_{q-1}, N_{s-1}]^B : T_q \cap [T_{q-1}, N_{s-1}]^B]^{-1}.$$

We now apply Lemma 8.4.15 in the cases $\ell = n = q-1$ and $\ell = n+1 = q$, i.e. we have

$$\psi_1(R_q \cap [R_q, N_0]^B) = (T_{q-1} \cap [T_{q-1}, N_{s-1}]^B)^m \text{ and}$$
$$\psi_1(R_{q+1} \cap [R_q, N_0]^B) = (T_q \cap [T_{q-1}, N_{s-1}]^B)^m.$$

We see that the second factor in the formula for the order of $\mathrm{St}_B(n)/U$ is the m-th power of the corresponding factor for $\mathrm{St}_B(n-1)/V$, and obtain

$$\frac{|\mathrm{St}_B(n-1)/V|^m}{|\mathrm{St}_B(n)/U|} = \frac{|L_G(q-1)|^m}{|L_G(q)|} = m^{o_G(q)}.$$

Now we compare V^m and $\psi_1(U)$. By Lemma 8.4.11(i) and (ii), $\psi_1(U)$ is generated by

$$\psi_1(R_{q+1}) \cup \bigcup_{i=0}^{s-2} (((\beta_i(\mathrm{St}_G(q)))^B)^m) \cup ([T_{q-1}, N_{s-1}]^B)^m.$$

We define yet another subgroup

$$W = \langle \bigcup_{i=0}^{s-2} (((\beta_i(\mathrm{St}_G(q)))^B)^m) \cup ([T_{q-1}, N_{s-1}]^B)^m \rangle \leqslant \psi_1(U) \leqslant B^m.$$

Evidently $W \trianglelefteq B^m$, $W \leqslant N_{s-1}^m$, and $W \trianglelefteq \psi_1(U) \leqslant V^m$. We have

$$\psi_1(U)/W \cong \psi_1(R_{q+1})/(\psi_1(R_{q+1}) \cap W) \text{ and}$$
$$V^m/W \cong T_q^m/(T_q^m \cap W).$$

The two divisors are equal: Clearly $\psi_1(R_{q+1}) \cap W$ is contained in $T_q^m \cap W$. Let

$$(\beta_{s-1}(g_0), \dots, \beta_{s-1}(g_{m-1})) \in T_q^m \cap W \leqslant (T_q \cap N_{s-1})^m.$$

Since $T_q \cap N_{s-1} \leqslant \beta_{s-1}(K_{s-1})$, the elements $g_0, \dots, g_{m-1}$ are members of $K_{s-1} \cap \mathrm{St}_G(q)$. Now since G is weakly regular branch over K_{s-1}, there is an element $k \in K_{s-1} \cap \mathrm{St}_G(q+1)$ such that $\psi_1(k) = (g_0, \dots, g_{m-1})$, and consequently $\beta_0(k) \in R_{q+1}$ fulfills

$$\psi_1(\beta_0(k)) = (\beta_{s-1}(g_0), \dots, \beta_{s-1}(g_{m-1})) \in \psi_1(R_{q+1}) \cap W.$$

We compute

$$\begin{aligned}
[V^m : \psi_1(U)] &= [V^m/W : \psi_1(U)/W] \\
&= [T_q^m : \psi_1(R_{q+1})] \\
&= [(\beta_{s-1} \times \dots \times \beta_{s-1})(\mathrm{St}_G(q)^m) : (\beta_{s-1} \times \dots \times \beta_{s-1})(\psi_1(\mathrm{St}_G(q+1)))] \\
&= [\mathrm{St}_G(q)^m : \psi_1(\mathrm{St}_G(q+1))].
\end{aligned}$$

This implies

$$\begin{aligned}[\mathrm{St}_B(n-1)^m : \psi_1(\mathrm{St}_B(n))] &= [\mathrm{St}_B(n-1)^m/\psi_1(U) : \psi_1(\mathrm{St}_B(n))/\psi_1(U)] \\ &= \frac{[\mathrm{St}_B(n-1)^m : V^m][V^m : \psi_1(U)]}{[\psi_1(\mathrm{St}_B(n)) : \psi_1(U)]} \\ &= \frac{|L_G(q-1)|^m}{|L_G(q)|} \cdot [\mathrm{St}_G(q)^m : \psi_1(\mathrm{St}_G(q+1))].\end{aligned}$$

Since $o_B(k) = 0$ for $k \not\equiv_s 0$, by Lemma 8.4.14,

$$\log[\mathrm{St}_B(n)^m : \psi_1(\mathrm{St}_B(n+1))] = \log[\mathrm{St}_B(n+s-1)^m : \psi_1(\mathrm{St}_B(n+s)],$$

hence

$$\begin{aligned}o_B(n) =&\log[\mathrm{St}_B(n-1)^m : \psi_1(\mathrm{St}_B(n))] - \log[\mathrm{St}_B(n+s-1)^m : \psi_1(\mathrm{St}_B(n+s))] \\ =&o_G(q) + \log\left|\frac{\mathrm{St}_G(q)^m}{\psi_1(\mathrm{St}_G(q+1))}\right| - o_G(q+1) - \log\left|\frac{\mathrm{St}_G(q+1)^m}{\psi_1(\mathrm{St}_G(q+2))}\right| \\ =&o_G(q) - o_G(q+1) + o_G(q+1) \\ =&o_G(q).\end{aligned}$$

□

Proof of Corollary 8.1.5. By Lemma 8.4.13 and Proposition 8.4.16

$$\begin{aligned}\dim_{\mathrm{H}} G &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_G(i) \quad \text{and} \\ \dim_{\mathrm{H}} \mathrm{Bas}_s(G) &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_{\mathrm{Bas}_s(G)}(i) \\ &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-si} - m^{-(sn+1)}) o_G(i).\end{aligned}$$

We prove $m^{-i} - m^{-(n+1)} > m^{-si} - m^{-(sn+1)}$, equivalently $m^{sn+1-i} + 1 > m^{s(n-i)+1} + m^{(s-1)n}$. This is a consequence of $sn+1-i-(s(n-i)+1) = (s-1)i \geqslant 1$ and $sn+1-i-(s-1)n = n-i+1 \geqslant 1$, with equality precisely when $i = 1, s = 2$, resp. $n = i$. Therefore at least one of the differences is greater than 1, and the limit of $\sum_{i=1}^{n}(m^{-si} - m^{-(sn+1)}) o_G(i)$ is strictly greater than the limit of $\sum_{i=1}^{n}(m^{-i} - m^{-(n+1)}) o_G(i)$. The statement follows. □

Example 8.4.17. Let $G \leqslant \mathrm{Aut}(T_p)$, p a prime, be a GGS-group defined by the triple $(\mathsf{C}_p, \mathsf{C}_p, \omega)$, cf. Definition 8.3.7, where C_p denotes the cyclic group of order p acting regularly on X. To be a GGS-group means $\omega_i = \omega_j$ for $i, j \in \mathbb{N}_0$, thus we write ω for ω_1. This is a $(p-1)$-tuple of endomorphisms of C_p. Every such endomorphism is a power map, hence we may identify ω with an element $(e_1, \ldots, e_{p-1})$ of $\mathbb{F}_p^{p-1}$. Assume that

$$e_1 + \cdots + e_{p-1} \equiv_p 0 \qquad (\star)$$

and that there is some $i \in [1, p-1]$

$$e_i \neq e_{p-i}. \qquad (\diamond)$$

In [38] the order of the congruence quotients $G/\operatorname{St}_G(n)$ is explicitly calculated in terms of the rank t of the circulant matrix associated to the vector $(0, e_1, \ldots, e_{p-1})$, i.e. the matrix with rows being all cyclic permutations of the given vector. Under our assumptions ($\star$) and ($\diamond$), for all $n \in \mathbb{N}_+$

$$\log_p(G/\operatorname{St}_G(n+1)) = tp^{n-1} + 1,$$

and $\log_p(G/\operatorname{St}_G(1)) = 1$. Additionaly, ($\star$) is equivalent to $t < p$. By Lemma 8.4.14, for $n > 2$,

$$\begin{aligned} o_G(n) &= p \cdot \log_p(|L_G(n-1)|) - \log_p(|L_G(n)|) \\ &= p \cdot \log_p \frac{p^{t \cdot p^{n-2}+1}}{p^{t \cdot p^{n-3}+1}} - \log \frac{p^{t \cdot p^{n-1}+1}}{p^{t \cdot p^{n-2}+1}} = 0 \end{aligned}$$

and

$$\begin{aligned} o_G(2) &= p \cdot \log \frac{p^{t+1}}{p} - \log \frac{p^{t \cdot p+1}}{p^{t+1}} = tp - t(p-1) = t \quad \text{and} \\ o_G(1) &= p \cdot \log p - \log \frac{p^{t+1}}{p} = p - t. \end{aligned}$$

Consequently, $\dim_{\mathrm{H}} G = t(p-1)/p^2$ (cf. [38] for a more general formula).

We aim to apply Proposition 8.4.16. Condition ($\diamond$) is equivalent to G being weakly regular branch (in fact, regular branch) over $[G, G]$, by [38, Lemma 3.4]. More precisely, we have

$$\psi_1([\operatorname{St}_G(1), \operatorname{St}_G(1)]) = [G, G]^p.$$

By Proposition 8.4.4 this implies that $K_{s-1} = [G, G]$. We now prove that G is very strongly fractal. It is easy to see that $\operatorname{St}_G(1)|_x = G$ for all $x \in X$, and by [38, Lemma 3.3] $\psi_1(\operatorname{St}_G(n)) = \operatorname{St}_G(n-1)^p$ for all $n \geqslant 3$. Thus it remains to check if $\operatorname{St}_G(2)|_x = \operatorname{St}_G(1)$ for all $x \in X$. By the fact that $[\operatorname{St}_G(1), \operatorname{St}_G(1)]|_x = [G, G]$ for all $x \in X$ and $[\operatorname{St}_G(2) : [\operatorname{St}_G(1), \operatorname{St}_G(1)]] = p^{p-t} \geqslant p$ (see again [38]), we see that $\operatorname{St}_G(2)$ contains an element g such that $\psi_1(g) \in \operatorname{St}_G(1)^p \backslash [G, G]^p$. Hence at least for one $x \in X$

$$\operatorname{St}_G(1) \geqslant \operatorname{St}_G(2)|_x > [G, G].$$

But since $[\operatorname{St}_G(1) : [G, G]] = p$ by [38, Theorem 2.1], this implies $\operatorname{St}_G(2)|_x = \operatorname{St}_G(1)$, and since G is spherically transitive, this holds for all $x \in X$, and G is very strongly fractal. We remark that by [103, Proposition 5.1] the condition ($\star$) alone implies that G is super strongly fractal, but our argument additionally needs ($\diamond$), since otherwise $[[G,G]^p : \psi_1([\mathrm{St}_G(1), \mathrm{St}_G(1)])] = p$ (cf. [38, Lemma 3.5]).

Now we may apply Proposition 8.4.16 to calculate the Hausdorff dimension of $\mathrm{Bas}_s(G)$:

$$o_{\mathrm{Bas}_s(G)}(s) = p - t \quad \text{and} \quad o_{\mathrm{Bas}_s(G)}(2s) = t$$

and $o_{\mathrm{Bas}_s(G)}(n) = 0$ for all other values $n \in \mathbb{N}_+$, hence

$$\begin{aligned} \dim_{\mathrm{H}} \mathrm{Bas}_s(G) &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{p^i} - \frac{1}{p^{n+1}} \right) o_{\mathrm{Bas}_s(G)}(i) \\ &= 1 - \limsup_{n\to\infty} \left(\frac{p-t}{p^s} + \frac{t}{p^{2s}} - \frac{p-t+t}{p^{n+1}} \right) \\ &= 1 - \left(\frac{p-t}{p^s} + \frac{t}{p^{2s}} \right) = \frac{p^{s-1}-1}{p^{s-1}} + \frac{t(p^s-1)}{p^{2s}}. \end{aligned}$$

8.5 The generalised Basilica groups

Let d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$. In the subsequent sections 8.5, 8.6, 8.7 and 8.8 we study the generalised Basilica groups, $\mathrm{Bas}_s(\mathcal{O}_m^d)$, where $\mathcal{O}_m^d = D_d(\mathcal{O}_m) = \langle \pi_i(a) \mid i \in [0, d-1] \rangle$ (cf. Proposition 8.2.5 and Definition 8.2.6). For convenience, we use the following notation for the generators of $\mathrm{Bas}_s(\mathcal{O}_m^d)$: let $i \in [0, d-1]$ and $j \in [0, s-1]$, and

$$\begin{aligned} a_{i,j} &:= \beta_j(\pi_i(a)) &&= (a_{i,j-1}, \mathrm{id}, \ldots, \mathrm{id}), && \text{for } j \neq 0 \\ a_{i,0} &:= \beta_0(\pi_i(a)) &&= (a_{i-1,s-1}, \ldots, a_{i-1,s-1}), && \text{for } i \neq 0 \\ a_{0,0} &:= \beta_0(\pi_0(a)) &&= \sigma(a_{d-1,s-1}, \mathrm{id}, \ldots, \mathrm{id}), \end{aligned}$$

where σ is the m-cycle $(0\ 1\ \ldots\ m-1)$. For any fixed j, the elements $a_{i,j}$ commute and are of infinite order.

Now we prove Theorem 8.1.6, which is obtained as corollaries of results from Section 8.3 and Section 8.4.

Proof of Theorem 8.1.6. The statements (i) and (ii) follow directly from Lemma 8.3.1, Lemma 8.3.2 and Lemma 8.3.3. Proposition 8.3.5 together with Corollary 8.3.6 imply the statement (ii). The statement (iii) is a consequence of Proposition 8.3.14 and Corollary 8.3.15. Thanks to Proposition 8.4.4, the group $\mathcal{O}_m^d$ is s-split. Therefore the statements (iv), (v) and (vi) follow from Corollary 8.4.7, Proposition 8.4.8 and Proposition 8.3.22. The proof of (vii) can easily be generalised from [59, Proposition 4]. For the special case $\mathrm{Bas}_p(\mathcal{O}_p)$, where p is a prime, see [96]. □

We use Theorem 8.1.4 to provide a normal generating set for the layer stabilisers of the generalised Basilica groups. This description of layer stabilisers is crucial in proving the p-congruence subgroup property of the generalised Basilica groups (see Section 8.8).

Theorem 8.5.1. *Let $n \in \mathbb{N}_0$. Write $n = sq + r$ with $r \in [0, s-1]$ and $q = dk + l \geqslant 0$ with $l \in [0, d-1]$. Then the n-th layer stabiliser of $B = \mathrm{Bas}_s(\mathcal{O}_m^d)$ is given by*

$$\mathrm{St}_B(n) = \langle a_{i,j}^{m^{k+1}}, a_{i',j'}^{m^k} \mid 0 \leqslant is + j \leqslant ls + r - 1 < i's + j' \leqslant ds - 1 \rangle^B.$$

Proof. Let a be the generator of the m-adic odometer $\mathcal{O}_m$. Set $G = D_d(\mathcal{O}_m) \cong \mathbb{Z}^d$. For every $i \in [0, d-1]$, denote by $a_i = \pi_i(a)$ the generators of G. Since powers of the elements $a_0, \dots, a_{d-1}$ act on vertices of disjoint levels of the m-regular rooted tree T and they commute with each other, we have

$$\mathrm{St}_G(q) = \langle a_0^{m^{k+1}}, \dots, a_{l-1}^{m^{k+1}}, a_l^{m^k}, \dots, a_{d-1}^{m^k} \rangle.$$

Now observe that for every vertex $x \in X$, $i \in [0, d]$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} a_i^{m^k}|_x &= a_{i-1}^{m^k} \\ a_0^{m^k}|_x &= a_{d-1}^{m^{k-1}}. \end{aligned}$$

Therefore $\mathrm{St}_G(q)|_x = \mathrm{St}_G(q-1)$ and hence G is very strongly fractal. A straightforward calculation using Theorem 8.1.4 yields the result. □

Using the description of the layer stabilisers of G, we obtain Theorem 8.1.7 as a direct application of Lemma 8.4.13 and Proposition 8.4.16.

Proof of Theorem 8.1.7. The series of obstructions of $G = \mathcal{O}_m^d$ is constant $m-1$ for all $n \in \mathbb{N}_+$, signifying Hausdorff-dimension 0 (cf. Lemma 8.4.13). We have seen in the proof of Theorem 8.5.1 that $\mathrm{Bas}_s(G)$ is very strongly fractal. Therefore, by Proposition 8.4.16 one has $o_{\mathrm{Bas}_s(G)}(qs) = m-1$ for all $q \in \mathbb{N}_+$ and $o_{\mathrm{Bas}_s(G)}(n) = 0$ for all other levels.

According to Lemma 8.4.13 it is

$$\begin{aligned} \dim_{\mathrm{H}} \mathrm{Bas}_s(G) &= 1 - \limsup_{n\to\infty} \sum_{i=1}^{n} (m^{-i} - m^{-(n+1)}) o_{\mathrm{Bas}_s(G)}(i) \\ &= 1 - (m-1) \limsup_{n\to\infty} \left(m^{-s} \frac{1 - m^{-s\lfloor n/s \rfloor}}{1 - m^{-s}} - \lfloor n/s \rfloor m^{-(n+1)} \right) \\ &= 1 - (m-1) \frac{m^{-s}}{1 - m^{-s}} \\ &= \frac{m^s - m}{m^s - 1}. \end{aligned}$$

In particular, the Hausdorff dimension is independent of d. □

8.6 An L-presentation for the generalised Basilica group

Let d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$. In this section we will provide a concrete L-presentation for the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$, hence proving Theorem 8.1.8. We will later use this presentation to prove that all generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_p^d)$ with p a prime have the p-congruence subgroup property.

Definition 8.6.1. [11, Definition 1.2] An L-presentation (or an endomorphic presentation) is an expression of the form

$$L = \langle Y \mid Q \mid \Phi \mid R \rangle,$$

where Y is an alphabet, $Q, R \subset F_Y$ are sets of reduced words in the free group F_Y on Y and Φ is a set of endomorphisms of F_Y. The expression L gives rise to a group G_L defined as

$$G_L = F_Y / \langle Q \cup \langle \Phi \rangle (R) \rangle^{F_Y},$$

where $\langle \Phi \rangle (R)$ denotes the union of the images of R under every endomorphism in the monoid $\langle \Phi \rangle$ generated from Φ. An L-presentation is finite if Y, Q, Φ, R are finite.

We now set out to prove Theorem 8.1.8. To do this, we follow the strategy from [59] which is motivated from [53]: let

$$Y = \{a_{i,j} \mid i \in [0, d-1], j \in [0, s-1]\}. \tag{8.1}$$

For convenience, we do not distinguish notationally between the generators of $\mathrm{Bas}_s(\mathcal{O}_m^d)$ and the free generators for the presentation. Observe that for a fixed j the generators $a_{i,j}$ and $a_{i',j}$ of $\mathrm{Bas}_s(\mathcal{O}_m^d)$ commute for all $i, i' \in [0, d-1]$. Write

$$Q = \{[a_{i,j}, a_{i',j}] \mid i, i' \in [0, d-1],\ j \in [0, s-1]\} \subseteq F_Y \tag{8.2}$$

and denote by F the quotient of F_Y by the normal closure of Q in F_Y. We identify F with a free product of free abelian groups

$$F = \mathop{\ast}_{j \in [0, s-1]} \langle a_{i,j} \mid i \in [0, d-1] \rangle \cong \mathbb{Z}^d \ast \cdots \ast \mathbb{Z}^d .$$

The group $\mathrm{Bas}_s(\mathcal{O}_m^d)$ is a quotient of F. Let $\mathrm{proj} : F \to \mathrm{Bas}_s(\mathcal{O}_m^d)$ be the canonical epimorphism. Now observe that the subgroup

$$\Delta = \langle a_{i,j}^{a_{0,0}^k},\ a_{0,0}^m \mid (i, j) \in [0, d-1] \times [0, s-1] \backslash \{(0, 0)\},\ k \in [0, m-1] \rangle, \tag{8.3}$$

is normal of index m in F and it is the full preimage of $\mathrm{St}_{\mathrm{Bas}_s(\mathcal{O}_m^d)}(1)$ under the epimorphism proj (cf. Theorem 8.5.1). We define a homomorphism $\Psi : \Delta \to F^m$ modelling the process of taking sections as follows:

$$\begin{aligned}
\Psi(a_{0,0}^m) &= (a_{d-1,s-1}, \dots, a_{d-1,s-1}) && =: z_0, \\
\Psi(a_{i,0}^{a_{0,0}^k}) = \Psi(a_{i,0}) &= (a_{i-1,s-1}, \dots, a_{i-1,s-1}) && =: z_i && \text{for } i \neq 0, \\
\Psi(a_{i,j}^{a_{0,0}^k}) &= (\mathrm{id}^{*k}, a_{i,j-1}, \mathrm{id}^{*(m-k-1)}) && =: x_{i,j,k} && \text{for } j \neq 0, \\
\Psi(a_{i,j}^{a_{0,0}^{-k}}) &= (\mathrm{id}^{*(m-k)}, a_{i,j-1}^{a_{d-1,s-1}^{-1}}, \mathrm{id}^{*(k-1)}),
\end{aligned}$$

where the ranges of i, j and k are as in (8.3). Clearly, $\ker(\Psi) \leqslant \ker(\mathrm{proj})$. Define

$$\alpha(v,k) = a_{0,0}^{mv_0+k} a_{1,0}^{v_1} \cdots a_{d-1,0}^{v_{d-1}} \text{ for } v = (v_0, \dots, v_{d-1}) \in \mathbb{Z}^d \text{ and } k \in [0, m-1], \tag{8.4}$$

$$R = \{[a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \mid i, i' \in [0, d-1],\ j, j' \in [1, s-1],\ k \in [1, m-1],\ v \in \mathbb{Z}^d\}, \tag{8.5}$$

where by abuse of notation we interpret $\alpha(v,k)$ and $r \in R$ both as elements of F_Y and their images in F. We will prove in Proposition 8.6.3 that the kernel of Ψ is normally generated from the image of R in F, implying that the set R belongs to the set of defining relators of $\mathrm{Bas}_s(\mathcal{O}_m^d)$. By definition of the elements $a_{i,j}$, we may obtain the elements of the set R as vertex sections. To incorporate these elements to the set of defining relators we introduce the following endomorphism of F_Y defined as

$$\Phi : \begin{cases} a_{i,j} & \mapsto a_{i,j+1} \text{ for } j \neq s-1, \\ a_{i,s-1} & \mapsto a_{i+1,0} \text{ for } i \neq d-1, \\ a_{d-1,s-1} & \mapsto a_{0,0}^m, \end{cases} \tag{8.6}$$

where $i \in [0, d-1]$ and $j \in [0, s-1]$.

Theorem 8.6.2. *The generalised Basilica group admits the L-presentation*

$$L = \langle Y \mid Q \mid \Phi \mid R \rangle$$

where Y, Q, R and Φ are given by (8.1), (8.2), (8.5) and (8.6).

Observe that for any $g \in Q$ and $r \in \mathbb{N}_0$, it holds that $\Phi^r(g) \in \langle Q^{F_Y} \rangle$. Considering the presentation defining F we may assume that Φ is an endomorphism of F and that R is a subset of F. To prove Theorem 8.6.2, it is enough to show that $\ker(\Psi) = \langle R^F \rangle$ and $\ker(\mathrm{proj}) = \bigcup_{r \in \mathbb{N}_0} \Phi^r(R)$. We will obtain the first part from Proposition 8.6.3 and the latter from Lemma 8.6.5 to Lemma 8.6.7.

Proposition 8.6.3. *Let $\tilde{\Delta}$ be the image of Δ under Ψ. Let z^v be the product $z_0^{v_0}\cdots z_{d-1}^{v_{d-1}}$ for every $v=(v_0,\ldots,v_{d-1})\in\mathbb{Z}^d$. Then $\tilde{\Delta}$ admits the presentation*

$$\langle\, \mathcal{S} \mid \mathcal{R}\,\rangle$$

where $\mathcal{S}=\{x_{i,j,k},\ z_i \mid i\in[0,d-1],\ j\in[1,s-1],\ k\in[0,m-1]\}$ and

$$\mathcal{R}=\left\langle \begin{array}{c} [x_{i,j,k},x_{i',j,k}],\ [x_{i,j,k},x_{i',j',k'}^{z^v}], \\ [z_i,z_{i'}] \end{array} \middle| \begin{array}{l} i,i'\in[0,d-1],\ j,j'\in[1,s-1], \\ k,k'\in[0,m-1] \text{ with } k\neq k',\ v\in\mathbb{Z}^d \end{array} \right\rangle.$$

As a consequence, we obtain that

$$\ker(\Psi)=\langle\{[a_{i,j},a_{i',j'}^{\alpha(v,k)}] \mid i,i'\in[0,d-1],\ j,j'\in[1,s-1],\ k\in[1,m-1],\ v\in\mathbb{Z}^d\}\rangle^F,$$

where $\alpha(v,k)$ is given by (8.4).

Proof. Let $A=\langle a_{i,j}\mid i\in[0,d-1],\ j\in[0,s-2]\rangle^F$ and $Z=\langle z_0,\ldots,z_{d-1}\rangle\cong\mathbb{Z}^d$ be subgroups of F and $\tilde{\Delta}$ respectively. Notice that $\tilde{\Delta}$ is a sub-direct product of m copies of F and the elements $x_{i,j,k}$ and $x_{i',j',k'}$ commute if $k\neq k'$ or if $k=k'$ and $j=j'$. It follows from the definition of Ψ that

$$A^m=\left\langle\ x_{i,j,k}\ \middle|\ i\in[0,d-1],\ j\in[1,s-1],\ k\in[0,m-1]\ \right\rangle^{\tilde{\Delta}}\leqslant\tilde{\Delta}.$$

Hence $\tilde{\Delta}=A^mZ$, yielding $\tilde{\Delta}=A^m\rtimes Z$. Now, since F is a free product of free abelian groups, the group A is freely generated from the elements of the form

$$a_{i,j}^{a_{d-1,s-1}^{v_0}a_{0,s-1}^{v_1}\cdots a_{d-2,s-1}^{v_{d-1}}},$$

where $v_i\in\mathbb{Z}$, $i\in[0,d-1]$ and $j\in[0,s-2]$. Therefore, the group A^m is generated from the elements

$$x_{i,j,k}^{z^v}=(\mathrm{id}^{*k},a_{i,j-1}^{a_{d-1,s-1}^{v_0}a_{0,s-1}^{v_1}\cdots a_{d-2,s-1}^{v_{d-1}}},\mathrm{id}^{*(m-k-1)}),$$

where $i\in[0,s-1]$, $j\in[1,s-1]$, $k\in[0,m-1]$ and

$$z^v=z_0^{v_0}\cdots z_{d-1}^{v_{d-1}}=(a_{d-1,s-1}^{v_0}a_{0,s-1}^{v_1}\cdots a_{d-2,s-1}^{v_{d-1}},\ldots,a_{d-1,s-1}^{v_0}a_{0,s-1}^{v_1}\cdots a_{d-2,s-1}^{v_{d-1}}),$$

with $v_i\in\mathbb{Z}$. We obtain a presentation of A^m as

$$\left\langle\ x_{i,j,k}^{z^v}\ \middle|\ \begin{array}{l} [x_{i,j,k},x_{i',j,k}]=[x_{i,j,k}^{z^v},x_{i',j',k'}^{z^{v'}}]=\mathrm{id},\ i,i'\in[0,d-1],\ j,j'\in[1,s-1], \\ k,k'\in[0,m-1] \text{ with } k\neq k',\ v,v'\in\mathbb{Z}^d \end{array}\right\rangle.$$

Hence $\tilde{\Delta}$, being a semi-direct product, admits the presentation $\langle\mathcal{S}\mid\mathcal{R}\rangle$, since conjugating an element $x_{i,j,k}$ by z_i does not yield a new relation. Therefore, the kernel of Ψ is normally generated from the preimage of the set of defining relators for $\tilde{\Delta}$. Notice that the preimages of the elements $[z_i, z_{i'}]$ and $[x_{i,j,k}, x_{i',j,k}]$ are trivial in Δ. Hence,

$$\ker(\Psi) = \left\langle \, [a_{i,j}^{\alpha(v,k)}, a_{i',j'}^{\alpha(v',k')}] \;\middle|\; \begin{array}{l} i, i' \in [0, d-1],\ j, j' \in [1, s-1], \\ k, k' \in [0, m-1] \text{ with } k \neq k',\ v, v' \in \mathbb{Z}^d \end{array} \right\rangle^{\Delta}.$$

Indeed, $\ker(\Psi)$ is normal in F. Given $v \in \mathbb{Z}^d$ and $k \in [0, m-1]$, define

$$\begin{aligned} \underline{v} &= (\lfloor (mv_0 + k + 1)/m \rfloor, v_1, \ldots, v_{d-1}) \in \mathbb{Z}^d \quad \text{and} \\ \underline{k} &= k + 1 \pmod{m} \in [0, m-1]. \end{aligned}$$

Then

$$\begin{aligned} \alpha(v,k) a_{0,0} &= a_{0,0}^{mv_0+k+1} a_{1,0}^{v_1} \cdots a_{d-1,0}^{v_{d-1}} = \alpha(\underline{v}, \underline{k}) \\ \alpha(v',k') a_{0,0} &= a_{0,0}^{mv_0'+k'+1} a_{1,0}^{v_1'} \cdots a_{d-1,0}^{v_{d-1}'} = \alpha(\underline{v}', \underline{k}') \end{aligned}$$

implies

$$[a_{i,j}^{\alpha(v,k)}, a_{i',j'}^{\alpha(v',k')}]^{a_{0,0}} = [a_{i,j}^{\alpha(v,k)a_{0,0}}, a_{i',j'}^{\alpha(v',k')a_{0,0}}] = [a_{i,j}^{\alpha(\underline{v},\underline{k})}, a_{i',j'}^{\alpha(\underline{v}',\underline{k}')}] \in \ker(\Psi).$$

A similar calculation shows $[a_{i,j}^{\alpha(v,k)}, a_{i',j'}^{\alpha(v',k')}]^{a_{0,0}^{-1}} \in \ker(\Psi)$. We get

$$\ker(\Psi) = \left\langle \, [a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \;\middle|\; i, i' \in [0, d-1],\ j, j' \in [1, s-1],\ k \in [1, m-1],\ v \in \mathbb{Z}^d \right\rangle^{F}. \quad \square$$

Notation 8.6.4. Let $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$, $k \in [1, m-1]$, $v \in \mathbb{Z}^d$ and $n \in \mathbb{N}_0$. Define

$$\begin{aligned} \Omega_0 &:= \ker(\Psi), & \Omega_n &:= \Psi^{-1}(\Omega_{n-1}^m) \text{ for } n \geqslant 1, \\ \tau_{v,k}(i,j,i',j') &:= [a_{i,j}, a_{i',j'}^{\alpha(v,k)}], & X_n &:= \langle \Phi^r(\tau_{v,k}(i,j,i',j')) \mid r \in [0,n] \rangle^F, \end{aligned}$$

where $\alpha(v,k)$ is given by (8.4). Denote further by Ω the kernel of the epimorphism proj : $F \to \mathrm{Bas}_s(\mathcal{O}_m^d)$. We will prove $\Omega_n = X_n$ and $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$, proving Theorem 8.6.2.

Lemma 8.6.5. *For $w \in F'$ the identity $\Psi(\Phi(w)^{a_{0,0}^k}) = (\mathrm{id}^{*k}, w, \mathrm{id}^{*(m-k-1)})$ holds for every $k \in [0, m-1]$.*

Proof. Observe from the definition of Φ that

$$\Phi(F) = \langle a_{i,j}, a_{0,0}^m \mid (i,j) \in [0, d-1] \times [0, s-1] \backslash \{(0,0)\} \rangle \leqslant \Delta.$$

Then by direct calculation using the definition of the homomorphism Ψ and Φ we get the desired identity. $\square$

Lemma 8.6.6. *The equality $\Omega_n = X_n$ holds for all $n \in \mathbb{N}_0$.*

Proof. It follows from Proposition 8.6.3 that $\Omega_0 = \ker(\Psi) = X_0$. The proof proceeds by induction on n. Since $\Phi(F) \leqslant \Delta$, for every $r \in \mathbb{N}_0$, we have $\Phi^r(\Delta) \leqslant \Delta$. Hence $X_n \leqslant \Delta$ for all $n \in \mathbb{N}_0$. Assume for some $n \geqslant 1$ that $\Omega_{n-1} = X_{n-1}$. We will prove that

$$\Psi(X_n) = \Omega_{n-1}^m = \Psi(\Omega_n).$$

Let $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$, $k \in [1, m-1]$, $r \in [1, n]$ and $v \in \mathbb{Z}^d$. For every $\Phi^r(\tau_{v,k}(i, j, i', j')) \in X_n$ and for every $\ell \in [0, m-1]$, since $\Phi^{r-1}(\tau_{v,k}(i, j, i', j')) \in F'$, we obtain from Lemma 8.6.5 that

$$\Psi((\Phi^r(\tau_{v,k}(i, j, i', j')))^{a_{0,0}^{\ell}}) = (\mathrm{id}^{*\ell}, \Phi^{r-1}(\tau_{v,k}(i, j, i', j')), \mathrm{id}^{*(m-\ell-1)}).$$

Since $\tilde{\Delta}$ is a sub-direct product of m copies of F and X_{n-1} is normally generated from the elements of the form $\Phi^{r-1}(\tau_{v,k}(i, j, i', j'))$, we obtain that $\Psi(X_n) = \Omega_{n-1}^m = \Psi(\Omega_n)$.

But since $\ker(\Psi|_{\Omega_n}) = \ker(\Psi) \cap \Omega_n = \Omega_0 = X_0 = \ker(\Psi) \cap X_n = \ker(\Psi|_{X_n})$, we get $\Omega_n = X_n$, and the result follows by induction. □

Lemma 8.6.7. *We have $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$.*

Proof. Write B for $\mathrm{Bas}_s(\mathcal{O}_m^d)$ and recall that $\mathrm{proj} : F \to B$ is the canonical epimorphism. Notice that $\mathrm{St}_B(1)$ is a quotient of Δ and further $\Omega_0 = \ker(\Psi) \leqslant \ker(\mathrm{proj}) = \Omega$. Proceeding by induction on n, we will prove that $\bigcup_{n=0}^{\infty} \Omega_n \leqslant \Omega$. Assume that $\Omega_{n-1} \leqslant \Omega$ for some $n \geqslant 1$. Let $w \in \Omega_n$ and let w_k be the k-th component of $\Psi(w)$. Then $w_k \in \Omega_{n-1}$ for all $k \in [0, m-1]$. Then the first layer sections of $\mathrm{proj}(w) \in \mathrm{St}_B(1)$ act trivially on the subtrees hanging from the vertices of level one of the m-regular rooted tree. Hence $\mathrm{proj}(w)$ acts trivially and $\mathrm{proj}(w) = \mathrm{id}$ in B. It follows by induction that $\Omega_n \leqslant \Omega$ for all $n \in \mathbb{N}_0$. Since $\Omega_{n-1} \leqslant \Omega_n$ for all $n \in \mathbb{N}_+$, we obtain $\bigcup_{n=0}^{\infty} \Omega_n \leqslant \Omega$.

Now, to see the converse choose an arbitrary element $w \in F$ such that $\mathrm{proj}(w) = \mathrm{id}$ in B. Then by Theorem 8.5.1 $\mathrm{proj}(w) \in \mathrm{St}_B(1)$ and hence $w \in \Delta$. Denote by w_k the k-th component of $\Psi(w)$. Then $\mathrm{proj}(w) = \mathrm{id}$ if and only if $\mathrm{proj}(w_k) = \mathrm{id}$ for all $k \in [0, m-1]$, implying that $w_k \in \Delta$ for all $k \in [0, m-1]$. Now repeat this process of taking sections by replacing w with w_k. This process is equivalent to the algorithm solving the word problem for B, cf. [59, Proposition 5]. Thanks to Corollary 8.3.15, the word problem for B is solvable and hence this process terminates in a finite number of steps. This implies the existence of an element $n \in \mathbb{N}_0$ such that $w \in \Omega_n$, completing the proof. □

To conclude this section, we point out that akin to [59, Proposition 11], one can introduce a set of d endomorphisms, each corresponding to a generator $a_{i,0}$, and obtain a finite L-presentation for $\mathrm{Bas}_s(\mathcal{O}_m^d)$.

Theorem 8.6.8. *The group* $\mathrm{Bas}_s(\mathcal{O}_m^d)$ *admits the following* L*-presentation:*

$$\left\langle \begin{array}{c|c|c|c} a_{i,j} & [a_{i,j}, a_{i',j}] & & [a_{i,j}, a_{i',j'}^{\alpha(v,k)}], i, i' \in [0, d-1], \\ i \in [0, d-1] & i, i' \in [0, d-1] & \Phi, \Theta_0, \dots, \Theta_{d-1} & j, j' \in [1, s-1], \; k \in [1, m-1] \\ j \in [0, s-1] & j \in [0, s-1] & & v \in \{0\} \times \{0,1\}^{d-1} \end{array} \right\rangle$$

where $\alpha(v,k)$ *and* Φ *are given by* (8.4) *and* (8.6)*, respectively, and* $\Theta_{i'}$ *are endomorphisms of the free group on the set of generators defined as*

$$\Theta_{i'} : \begin{cases} a_{i,j} \mapsto a_{i,j} a_{i,j}^{a_{i',0}} \text{ for } j \neq 0, i' \neq 0, \\ a_{i,j} \mapsto a_{i,j} a_{i,j}^{a_{0,0}^m} \text{ for } j \neq 0, i' = 0, \\ a_{i,0} \mapsto a_{i,0}. \end{cases}$$

Proof of Theorem 8.6.8 is based on Lemma 8.6.10 below. Before stating the lemma, we set up necessary notations. We define the following sets

$$\Xi = \{\Theta_0, \dots, \Theta_{d-1}\}, \tag{8.7}$$

$$\mathfrak{R} = \{[a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \mid i, i' \in [0, d-1], \; j, j' \in [1, s-1], \; k \in [1, m-1], \; v \in \{0\} \times \{0,1\}^{d-1}\}, \tag{8.8}$$

where $\alpha(v,k)$ is defined as in (8.4), and prove that the generalised Basilica group is given by the finite L-presentation

$$\mathcal{L} = \langle Y \mid Q \mid \Phi \cup \Xi \mid \mathfrak{R} \rangle. \tag{8.9}$$

The idea of the proof is the following: the set R can be obtained from the set $\mathfrak{R}$ by the suitable application of elements from the free monoid Ξ^*. We set up the following notation.

Notation 8.6.9. Let $n \in \mathbb{N}_0$. Set $\Xi_n = \{\Theta_0^{\ell_0} \cdots \Theta_{d-1}^{\ell_{d-1}} \mid \ell_0, \dots, \ell_{d-1} \in [0, n]\}$. We define

$$Y_n = \left\{ \tau_{v,k}(i, j, i', j') \;\middle|\; \begin{array}{l} i, i' \in [0, d-1], \; j, j' \in [1, s-1], \; k \in [1, m-1], \\ v \in [0, n] \times [0, n+1]^{d-1} \end{array} \right\},$$

where $\tau_{v,k}(i, j, i', j')$ is defined as in Notation 8.6.4. Further, we denote

$$U_n = \langle Y_n \rangle^F, \qquad V_n = \langle \xi(Y_0) \mid \xi \in \Xi_n \rangle^F.$$

We shall prove that $U_n = V_n$, which proves Theorem 8.6.8.

Lemma 8.6.10. *The equality* $U_n = V_n$ *holds for all* $n \in \mathbb{N}_0$*.*

Proof. The proof follows by induction on n. For $n = 0$, the equality is true by definition. Assume that $U_n \leqslant V_n$ for some $n \geqslant 1$. We assign a lexicographical ordering on the set $[0, n] \times [0, n+1]^{d-1}$. Let $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$, $k \in [1, m-1]$ and $\xi \in \Xi_{n+1}$ be non-trivial. Then ξ is of the form $\Theta_0^{\ell_0} \cdots \Theta_{d-1}^{\ell_{d-1}}$ for some $\ell_0, \ldots, \ell_{d-1} \in [0, n]$. Set $\ell = (\ell_0, \ldots, \ell_{d-1})$. We have

$$\xi(a_{i,j}) = a_{i,j}\, x\, a_{i,j}^{\alpha(\ell,0)},$$

where x is product of elements of the form $a_{i,j}^{\alpha(\ell',0)}$ such that ℓ' is a non-trivial element of $[0, n]^d$ and $\ell' < \ell$. Observe that, in the quotient group V_{n+1}/U_n (which is well-defined as $U_n \leqslant V_n \leqslant V_{n+1}$), the elements of the form $a_{i,j}^{\alpha(v,k)}$ and $a_{i',j'}^{\alpha(v',k')}$ commute given that

$$(|\lfloor (mv_0 + k - mv_0' - k')/m \rfloor|, |v_1 - v_1'|, \ldots, |v_{d-1} - v_{d-1}'|) \in [0, n] \times [0, n+1]^{d-1},$$

where v_ι and v_ι' are the ι-th coordinate of v and v', respectively. Let $\tau_{\beta,k}(i, j, i', j') \in Y_0$, where $\beta \in \{0\} \times \{0, 1\}^{d-1}$. Then $\xi(\tau_{\beta,k}(i, j, i', j')) \in V_{n+1}$. We get

$$\xi(\tau_{\beta,k}(i, j, i', j')) = \xi([a_{i,j}, a_{i',j'}^{\alpha(\beta,k)}]) = [a_{i,j}\, x\, a_{i,j}^{\alpha(\ell,0)}, a_{i',j'}^{\alpha(\beta,k)}\, y\, a_{i',j'}^{\alpha(\ell+\beta,k)}],$$

where x and y are the product of elements of the form $a_{i,j}^{\alpha(v,0)}$ and $a_{i',j'}^{\alpha(\beta+v',k)}$, respectively, such that v, v' are non-trivial elements of $[0, n]^d$ and $v, v' < \ell$. Then

$$\xi(\tau_{\beta,k}(i, j, i', j')) \equiv_{U_n} [a_{i,j}, a_{i',j'}^{\alpha(\ell+\beta,k)}] = \tau_{\ell+\beta,k}(i, j, i', j').$$

This implies $\tau_{v,k}(i, j, i', j') \in V_{n+1}$ for all $v \in [0, n+1] \times [0, n+2]^{d-1}$, whence $U_{n+1} \leqslant V_{n+1}$. A similar computation gives that $V_{n+1} \leqslant U_{n+1}$. This completes the proof. □

Proof of Theorem 8.6.8. It is immediate from Lemma 8.6.10 that $\Xi^*(\mathfrak{R})$ coincides the set

$$R^+ = \{[a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \mid i, i' \in [0, d-1],\ j, j' \in [1, s-1],\ k \in [1, m-1],\ v \in \mathbb{N}_0^d\}.$$

Furthermore, we get $\langle R^+ \rangle^{F_Y} = \langle R \rangle^{F_Y}$, where F_Y is the free group on the set Y. Hence we conclude that the generalised Basilica group admits the finite L-presentation (8.9). □

8.7 Structural properties of the generalised Basilica groups

Let d, m, $s \in \mathbb{N}_+$ with m, $s \geqslant 2$. Here we prove some structural properties of the generalised Basilica groups $\mathrm{Bas}_2(\mathcal{O}_m^d)$. These result reflect a significant structural dissimilarity between $\mathrm{Bas}_2(\mathcal{O}_m^d)$ and $\mathrm{Bas}_s(\mathcal{O}_m^d)$ for $s > 2$. This structural dissimilarity plays a vital role when we consider the p-congruence subgroup property of the generalised Basilica groups, see Figure 8.5, which is treated in Section 8.8.

For convenience, we omit the subscript from ψ_1 and identify an element $g \in \mathrm{St}_B(1)$ with its image under the map ψ_1.

Proposition 8.7.1. *Let B be the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$. Then $\psi^{-1}((B')^m)$ is a subgroup of B' and*

$$B'/\psi^{-1}((B')^m) = \Big\langle\, c_{i,j,k}\, \psi^{-1}((B')^m) \;\Big|\; i \in [0, d-1], j \in [1, s-1],\, k \in [1, m-1] \,\Big\rangle$$
$$\cong \mathbb{Z}^{d(m-1)(s-1)},$$

where $c_{i,j,k} = [a_{i,j}, a_{0,0}^k]$. In particular, it holds that $\psi^{-1}((B')^m) \geqslant B''$.

Proof. Notice that $B' = \langle [a_{i,j}, a_{i',j'}] \mid i, i' \in [0, d-1],\, j, j' \in [0, s-1] \rangle^B$. For $i, i' \in [0, d-1]$ and $j, j' \in [1, s-1]$, we have $[a_{i,j}, a_{i',j}] = \mathrm{id}$ and for $j \neq j'$

$$[a_{i,j}, a_{i',j'}] = ([a_{i,j-1}, a_{i',j'-1}], \mathrm{id}^{*(m-1)})$$
$$[a_{i,j}, a_{i',0}] = ([a_{i,j-1}, a_{i'-1,s-1}], \mathrm{id}^{*(m-1)}) \text{ for } i' \neq 0,$$
$$[a_{i,j}, a_{0,0}^m] = ([a_{i,j-1}, a_{d-1,s-1}], \mathrm{id}^{*(m-1)}).$$

Therefore, we obtain

$$\langle [a_{i,j}, a_{i',j'}] \mid i, i' \in [0, d-1],\, j, j' \in [0, s-1] \rangle \times \{\mathrm{id}\} \times \cdots \times \{\mathrm{id}\} \leqslant \psi(B'),$$

yielding that $(B')^m \leqslant \psi(B')$ by Lemma 8.2.1.

Now, recall our definition $c_{i,j,k} = [a_{i,j}, a_{0,0}^k]$ and

$$C = \langle c_{i,j,k} \mid i \in [0, d-1],\, j \in [1, s-1],\, k \in [1, m-1] \rangle.$$

We claim that $B'/\psi^{-1}((B')^m) = \overline{C}$, where $\overline{C}$ denotes the image of C in the quotient group. For convenience, we will write the equivalence $\equiv_{\psi^{-1}((B')^m)}$ without the subscript. Observe that, for $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$ and $k \in [1, m-1]$,

$$[a_{i,j}, a_{i',j'}] \equiv \mathrm{id}, \qquad [a_{i,j}, a_{i',0}] \equiv \mathrm{id} \;\text{ for } i' \neq 0, \qquad [a_{i,j}, a_{0,0}] = c_{i,j,1},$$

and

$$c_{i,j,k} = [a_{i,j}, a_{0,0}^k] \equiv (a_{i,j-1}^{-1}, \mathrm{id}^{*(k-1)}, a_{i,j-1}, \mathrm{id}^{*(m-k-1)}).$$

Therefore, to prove the claim, it suffices to show that $\overline{C}$ is normal in $B/\psi^{-1}((B')^m)$. Let $i, i' \in [0, d-1]$, $j, j' \in [1, s-1]$ and $k \in [1, m-1]$. An easy calculation yields

$$c_{i,j,k}^{a_{i',j'}^{\pm 1}} \equiv c_{i,j,k} \qquad \text{and} \qquad c_{i,j,k}^{a_{i',0}^{\pm 1}} \equiv c_{i,j,k} \text{ for } i' \neq 0.$$

Furthermore,

$$\begin{aligned}
c_{i,j,k}^{a_{0,0}} &\equiv (\mathrm{id}, a_{i,j-1}^{-1}, \mathrm{id}^{*(k-1)}, a_{i,j-1}, \mathrm{id}^{*(m-k-2)}) && \equiv c_{i,j,1}^{-1} c_{i,j,k+1} && \text{if } k \neq m-1,\\
c_{i,j,k}^{a_{0,0}} &\equiv (a_{i,j-1}, a_{i,j-1}^{-1}, \mathrm{id}^{*(m-2)}) && \equiv c_{i,j,1}^{-1} && \text{if } k = m-1,\\
c_{i,j,k}^{a_{0,0}^{-1}} &\equiv (\mathrm{id}^{*(k-1)}, a_{i,j-1}, \mathrm{id}^{*(m-k-1)}, a_{i,j-1}^{-1}) && \equiv \begin{cases} c_{i,j,m-1}^{-1} c_{i,j,k-1} & \text{if } k \neq 1,\\ c_{i,j,m-1}^{-1} & \text{if } k = 1, \end{cases}
\end{aligned}$$

implying that $B'/\psi^{-1}((B')^m) = \overline{C}$. Observe that, for a fixed $i \in [0, d-1]$ and $j \in [1, s-1]$,

$$\mathbb{Z}^{m-1} \cong \{(a_{i,j-1}^{x_1}, \dots, a_{i,j-1}^{x_m}) \mid x_r \in \mathbb{Z}, \sum_{r=1}^{m} x_r = 0\} = \langle \overline{c}_{i,j,k} \mid k \in [1, m-1] \rangle \leqslant \overline{C}.$$

Since $B/B' \cong \mathbb{Z}^{ds}$ (Theorem 8.1.6(iv)), this yields

$$B'/\psi^{-1}((B')^m) = \overline{C} = \prod_{(i,j)\in[0,d-1]\times[1,s-1]} \langle \overline{c}_{i,j,k} \mid k \in [1, m-1] \rangle \cong \mathbb{Z}^{d(m-1)(s-1)}.$$

□

Now we prove Theorem 8.1.9. In addition, we provide a generating set for the quotient group $\gamma_2(\mathrm{Bas}_s(\mathcal{O}_m^d))/\gamma_3(\mathrm{Bas}_s(\mathcal{O}_m^d))$.

Theorem 8.7.2. *Let B be the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$. We have:*

(i) For $s = 2$, $B'/\gamma_3(B) = \langle [a_{i,0}, a_{i',1}]\, \gamma_3(B) \mid i, i' \in [0, d-1] \rangle \cong \mathbb{Z}^{d^2}$.

(ii) For $s > 2$, the quotient group $B'/\gamma_3(B) \cong \mathrm{C}_m^{ds-2} \times \mathrm{C}_{m^2}$. Moreover, it is generated from the set

$$\{[a_{i,j}, a_{0,0}]\, \gamma_3(B), [a_{0,1}, a_{i',0}]\, \gamma_3(B) \mid i \in [0, d-1],\ i' \in [1, d-1],\ j \in [1, s-1]\}.$$

Proof. (i) We use Theorem 8.6.2 to obtain a presentation for $B/\gamma_3(B)$. Take Y, Q, Φ and R as given in Theorem 8.6.2 and set $Q' = Q \cup \gamma_3(F_Y)$, where F_Y is the free group on Y. If $s = 2$, the set R becomes

$$R = \{[a_{i,1}, a_{i',1}^{\alpha(v,k)}] \mid i, i' \in [0, d-1],\ k \in [1, m-1],\ v \in \mathbb{Z}^d\}$$

and for every $[a_{i,1}, a_{i',1}^{\alpha(v,k)}] \in R$,

$$[a_{i,1}, a_{i',1}^{\alpha(v,k)}] \equiv_{\gamma_3(F_Y)} [a_{i,1}, a_{i',1}] \in \langle Q' \rangle^{F_Y},$$

where $\alpha(v, k)$ is given by (8.4). Since $\langle Q' \rangle$ is invariant under Φ, the presentation $\langle\ Y \mid Q'\ \rangle$ defines the group $B/\gamma_3(B)$, yielding that

$$B'/\gamma_3(B) = \langle [a_{i,0}, a_{i',1}] \mid i, i' \in [0, d-1] \rangle \cong \mathbb{Z}^{d^2}.$$

(ii) Consider again Y, Q, Φ and R as given in Theorem 8.6.2 and $Q' = Q \cup \gamma_3(F_Y)$. First observe that the element

$$[a_{i,j}, a_{i',j'}^{\alpha(v,k)}] \equiv_{\gamma_3(F_Y)} [a_{i,j}, a_{i',j'}]$$

belongs to $\langle Q' \rangle^{F_Y}$ if and only if $j = j'$. Setting

$$S = \{[a_{i,j}, a_{i',j'}] \mid i, i' \in [0, d-1],\ j, j' \in [1, s-1] \text{ with } j \neq j'\} \subseteq F_Y,$$

we notice that the group $B/\gamma_3(B)$ admits the L-presentation $\langle\, Y \mid Q' \mid \Phi \mid S \,\rangle$. Now, define

$$T = \left\{ \begin{array}{c|c} [a_{i,j}, a_{i',0}], [a_{i'',1}, a_{i',0}], & i \in [0, d-1], \\ [a_{i,j}, a_{0,0}]^m, [a_{i',1}, a_{0,0}]^m, [a_{0,1}, a_{i',0}]^m, & i', i'' \in [1, d-1], \\ [a_{0,1}, a_{0,0}]^{m^2} & j \in [2, s-1] \end{array} \right\}$$

and $N = Q' \cup S \cup T$ as subsets of F_Y. We claim that $\Phi^r(S) \subseteq N^{F_Y}$ for all $r \in \mathbb{N}_0$, and hence the presentation $\langle\, Y \mid N \,\rangle$ defines the group $B/\gamma_3(B)$. Therefore, the commutator subgroup of $B/\gamma_3(B)$ is generated from the set

$$\left\{ \begin{array}{c|c} [a_{i,j}, a_{0,0}], [a_{i',1}, a_{0,0}], & i \in [0, d-1],\ i' \in [1, d-1], \\ [a_{0,1}, a_{i',0}], [a_{0,1}, a_{0,0}] & j \in [2, s-1] \end{array} \right\},$$

yielding that:

$$B'/\gamma_3(B) \cong \mathrm{C}_m^{d(s-2)} \times \mathrm{C}_m^{d-1} \times \mathrm{C}_m^{d-1} \times \mathrm{C}_{m^2} = \mathrm{C}_m^{ds-2} \times \mathrm{C}_{m^2}.$$

Now, let $i, i' \in [0, d-1]$. Observe first that, for $j, j' \in [1, s-2]$,

$$\Phi([a_{i,j}, a_{i',j'}]) = [a_{i,j+1}, a_{i',j'+1}] \in S.$$

To prove the claim, it is enough to consider the elements of the form $\Phi^r([a_{i,j}, a_{i',j'}])$ with either j or j', but not both, equal to $s-1$. Without loss of generality suppose that $1 \leqslant j \leqslant s-2$ and $j' = s-1$. Since $\gamma_3(F_Y) \leqslant N^{F_Y}$, we work modulo $\gamma_3(F_Y)$. We have

$$\Phi([a_{i,j}, a_{i',s-1}]) \equiv \begin{cases} [a_{i,j+1}, a_{i'+1,0}]^m & \text{if } i' = d-1 \\ [a_{i,j+1}, a_{i'+1,0}] & \text{otherwise.} \end{cases}$$

For convenience, the images of $\Phi^2([a_{i,j}, a_{i',s-1}])$ and $\Phi^3([a_{i,j}, a_{i',s-1}])$ are given in the tabular form, see Table 8.1 and Table 8.2.

		$j \neq s-2$	$j = s-2$
$i' \neq d-1$	$i \neq d-1$	$[a_{i,j+2}, a_{i'+1,1}]$	$[a_{i+1,0}, a_{i'+1,1}]$
	$i = d-1$		$[a_{0,0}, a_{i'+1,1}]^m$
$i' = d-1$	$i \neq d-1$	$[a_{i,j+2}, a_{0,1}]^m$	$[a_{i+1,0}, a_{0,1}]^m$
	$i = d-1$		$[a_{0,0}, a_{0,1}]^{m^2}$

Table 8.1: Images of $\Phi^2([a_{i,j}, a_{i',s-1}])$.

<table>
<tr><td></td><td></td><td>$j \notin \{s-3, s-2\}$</td><td>$j = s-2$</td><td>$j = s-3$</td></tr>
<tr><td rowspan="2">$i' \neq d-1$</td><td>$i \neq d-1$</td><td rowspan="2">$[a_{i,j+3}, a_{i'+1,2}]$</td><td>$[a_{i+1,1}, a_{i'+1,2}]$</td><td>$[a_{i+1,0}, a_{i'+1,2}]$</td></tr>
<tr><td>$i = d-1$</td><td>$[a_{0,1}, a_{i'+1,2}]^m$</td><td>$[a_{0,0}, a_{i'+1,2}]^m$</td></tr>
<tr><td rowspan="2">$i' = d-1$</td><td>$i \neq d-1$</td><td rowspan="2">$[a_{i,j+3}, a_{0,2}]^m$</td><td>$[a_{i+1,1}, a_{0,2}]^m$</td><td>$[a_{i+1,0}, a_{0,2}]^m$</td></tr>
<tr><td>$i = d-1$</td><td>$[a_{0,1}, a_{0,2}]^{m^2}$</td><td>$[a_{0,0}, a_{0,2}]^{m^2}$</td></tr>
</table>

Table 8.2: Images of $\Phi^3([a_{i,j}, a_{i',s-1}])$.

Observe that the element $\Phi^r([a_{i,j}, a_{i',s-1}]) \in N^{F_Y}$ for $r \in [1,3]$. By iterating the process we see that $\Phi^r([a_{i,j}, a_{i',j'}]) \in N^{F_Y}$, for all $r \in \mathbb{N}_0$ and $[a_{i,j}, a_{i',j'}] \in S$. □

Lemma 8.7.3. *Let B be the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m^d)$. The following assertions hold:*

(i) For $s = 2$, $B'' = \psi^{-1}(\gamma_3(B)^m)$.

(ii) For $s > 2$, $B'' \gneq \psi^{-1}(\gamma_3(B)^m)$.

Proof. We first prove that $\gamma_3(B)^m \leqslant \psi(B'')$ for all $s \geqslant 2$. From Lemma 8.2.1, since

$$\gamma_3(B) = \langle [[a_{i_1,j_1}, a_{i_2,j_2}], a_{i_3,j_3}] \mid i_1, i_2, i_3 \in [0, d-1],\ j_1, j_2, j_3 \in [0, s-1] \rangle^B,$$

and B is self-similar and fractal (Theorem 8.1.6(ii)), it is enough to prove that the set

$$\{([[a_{i_1,j_1}, a_{i_2,j_2}], a_{i_3,j_3}], \mathrm{id}^{*(m-1)}) \mid i_1, i_2, i_3 \in [0, d-1],\ j_1, j_2, j_3 \in [0, s-1]\} \qquad (*)$$

is contained in $\psi(B'')$. Let $i_1, i_2, i_3 \in [0, d-1]$ and $j_1, j_2, j_3 \in [0, s-1]$. We split the proof into four cases.

Case 1: $j_1 = j_2 = j_3 = s-1$. Clearly, $[[a_{i_1,s-1}, a_{i_2,s-1}], a_{i_3,s-1}] = \mathrm{id}$.

Case 2: $j_3 \neq s-1$. In light of Proposition 8.7.1, the elements $([a_{i_1,j_1}, a_{i_2,j_2}], \mathrm{id}^{*(m-1)})$ and $(a_{i_3,j_3}, a_{i_3,j_3}^{-1}, \mathrm{id}^{*(m-2)}) = [a_{i_3,j_3+1}, a_{0,0}]^{-1}$ belong to $\psi(B')$, implying that

$$([[a_{i_1,j_1}, a_{i_2,j_2}], a_{i_3,j_3}], \mathrm{id}^{*(m-1)}) \in \psi(B'').$$

Now, observe from Proposition 8.7.1 that $\psi(B'') \geqslant (B'')^m$. Therefore, if there exist $g = (g_0, \ldots, g_{m-1}), h = (h_0, \ldots, h_{m-1}) \in B$ such that $g_i \equiv_{B''} h_i$ for all $i \in [0, m-1]$ then $g \equiv_{\psi(B'')} h$.

Case 3: $j_3 = s-1$, $j_1 \neq s-1$ and $j_2 \neq s-1$. Now, from the Hall–Witt identity (see [95, p. 123]), we can easily derive that

$$[[y,x],z][[z,y],x][[x,z],y] \equiv_{B''} [[y,x],z^y][[z,y],x^z][[x,z],y^x] = \mathrm{id},$$

for all $x,y,z \in B$. Setting $x = a_{i_1,j_1}$, $y = a_{i_2,j_2}$ and $z = a_{i_3,j_3}$, we get that the element

$$([[y,x],z],\mathrm{id}^{*(m-1)})^{-1} \equiv_{\psi(B'')} ([[z,y],x][[x,z],y],\mathrm{id}^{*(m-1)})$$

belongs to $\psi(B'')$, as the right-hand side product belongs to $\psi(B'')$ by Case 2.

Case 4: $j_3 = s-1 = j_1$, $j_2 \neq s-1$ or $j_3 = s-1 = j_2$, $j_1 \neq s-1$. Notice that

$$[[a_{i_1,j_1}, a_{i_2,s-1}], a_{i_3,s-1}] \equiv_{B''} [[a_{i_2,s-1}, a_{i_1,j_1}], a_{i_3,s-1}]^{-1},$$

thus, it is enough to consider the first case. We claim that, for every $j \in [0, s-1]$, it holds $[[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}] \equiv_{B''} \mathrm{id}$. Then by taking the j_2-th projection of the element $[[a_{i_1,s-1}, a_{i_2,j_2}], a_{i_3,s-1}]$ we obtain,

$$\begin{aligned}\psi_{j_2}([[a_{i_1,s-1}, a_{i_2,j_2}], a_{i_3,s-1}]) &= ([[a_{i_1,(s-1-j_2)}, a_{i_2,0}], a_{i_3,(s-1-j_2)}], \mathrm{id}^{*(m^{j_2}-1)})\\ &\equiv_{\psi_{j_2}(B'')} \mathrm{id},\end{aligned}$$

implying $[[a_{i_1,s-1}, a_{i_2,j_2}], a_{i_3,s-1}] \equiv_{B''} \mathrm{id}$, and hence $(*)$ follows.

If $i_2 = 0$ or $j = 0$, it is then immediate that $[[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}] = \mathrm{id}$. Assume that $i_2 \neq 0$ and $j \neq 0$. From the presentation of B given in Theorem 8.6.2, we have

$$[[a_{i_1,j}, \alpha(v,k)], a_{i_3,j}] = [a_{i_1,j}^{-1} a_{i_1,j}^{\alpha(v,k)}, a_{i_3,j}] = [a_{i_1,j}^{-1}, a_{i_3,j}]^{a_{i_1,j}^{\alpha(v,k)}} [a_{i_1,j}^{\alpha(v,k)}, a_{i_3,j}] = \mathrm{id},$$

where $\alpha(v,k)$ is given by (8.4). Now, by setting $v = (0^{*(i_2-1)}, 1, 0^{*(m-i_2-1)})$ and $k = 1$, we get $\alpha(v,k) = a_{0,0}a_{i_2,0}$ and consequently

$$\begin{aligned}\mathrm{id} = [[a_{i_1,j}, a_{0,0}a_{i_2,0}], a_{i_3,j}] &= [[a_{i_1,j}, a_{i_2,0}][a_{i_1,j}, a_{0,0}]^{a_{i_2,0}}, a_{i_3,j}]\\ &\equiv_{B''} [[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}][[a_{i_1,j}, a_{0,0}]^{a_{i_2,0}}, a_{i_3,j}] \equiv_{B''} [[a_{i_1,j}, a_{i_2,0}], a_{i_3,j}].\end{aligned}$$

Next we prove (i). Assume that $s = 2$ and notice that it suffices to prove that $B'/\psi^{-1}(\gamma_3(B)^m)$ is abelian. We use the fact that the commutator subgroup can be described by $B' = \langle [a_{i_1,1}, a_{i_2,0}] \mid i_1, i_2 \in [0, d-1] \rangle^B$ as $s = 2$.

Looking at the section decomposition of these generators,

$$\begin{aligned}&[a_{i_1,1}, a_{i_2,0}] = (\ [a_{i_1,0}, a_{i_2-1,1}], \mathrm{id}^{*(m-1)}) \quad \text{for } i_2 \neq 0, \text{ and}\\ &[a_{i_1,1}, a_{0,0}] = (a_{i_1,0}^{-1}, a_{i_1,0}, \mathrm{id}^{*(m-2)}),\end{aligned}$$

we immediately see that they commute modulo $\gamma_3(B)^m$. Thus, $B'/\psi^{-1}(\gamma_3(B)^m)$ is abelian.

(ii) The inclusion $\psi^{-1}(\gamma_3(B)^m) \leqslant B''$ has been already proven above. We prove that $\psi^{-1}(\gamma_3(B)^m)$ is a proper subgroup of B'', by showing that $B'/\psi^{-1}(\gamma_3(B)^m)$ is non-abelian. Suppose to the contrary $B'/\psi^{-1}(\gamma_3(B)^m)$ is abelian. Then, for every $i \in [0, d-1]$ and $j \in [2, s-1]$

$$\mathrm{id} \equiv_{\psi^{-1}(\gamma_3(B)^m)} [[a_{i,j}, a_{0,0}], [a_{0,1}, a_{0,0}]] = ([a_{i,j-1}^{-1}, a_{0,0}^{-1}], [a_{i,j-1}, a_{0,0}], \mathrm{id}^{*(m-2)}).$$

This implies $[a_{i,j-1}, a_{0,0}] \equiv_{\gamma_3(B)} \mathrm{id}$, which is a contradiction to Theorem 8.7.2(ii). □

8.8 Congruence properties of the generalised Basilica groups

Here we prove that the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_p^d)$ has the p-CSP for d, $s \in \mathbb{N}_+$ with $s > 2$ and p a prime. We follow the strategy from [46], where it is proved that the original Basilica group $\mathcal{B} = \mathrm{Bas}_2(\mathcal{O}_2)$ has the 2-congruence subgroup property. However, on account of Theorem 8.7.2 and Lemma 8.7.3, our reasoning must be different, and we will use Theorem 8.5.1.

Let G be a subgroup of the automorphism group of the p-regular rooted tree T and let $\mathcal{C}$ be the class of all finite p-groups.

Definition 8.8.1 ([46, Definition 5])**.** A subgroup G of $\mathrm{Aut}\,T$ has the p-congruence subgroup property (p-CSP) if every normal subgroup $N \trianglelefteq G$ satisfying $G/N \in \mathcal{C}$ contains some layer stabiliser in G. The group G has the p-CSP modulo a normal subgroup $M \trianglelefteq G$ if every normal subgroup $N \trianglelefteq G$ satisfying $G/N \in \mathcal{C}$ and $M \leqslant N$ contains some layer stabiliser in G.

By setting $\mathcal{C}$ as the class of all finite p-groups in [46, Lemma 6], we obtain the following result:

Lemma 8.8.2. *Let G be a subgroup of* $\mathrm{Aut}\,T$ *and $N \trianglelefteq M \trianglelefteq G$. If G has the p-CSP modulo M and M has the p-CSP modulo N then G has the p-CSP modulo N.*

Let d, $s \in \mathbb{N}_+$ with $s > 2$ and let p be a prime. Set $B = \mathrm{Bas}_s(\mathcal{O}_p^d)$. From Theorem 8.1.6(vi) B is weakly regular branch over its commutator subgroup B' and from Lemma 8.7.3

$$B' \geqslant \gamma_3(B) \geqslant B'' > \psi^{-1}(\gamma_3(B)^p).$$

We will prove that

1. B has the p-CSP modulo $\gamma_3(B)$, and,

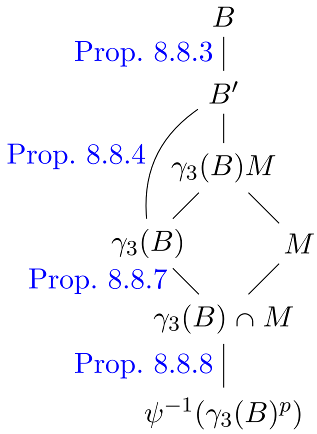

Figure 8.5: The steps of the proof of Theorem 8.1.10, where $M := \psi^{-1}((B')^p)$

2. $\gamma_3(G)$ has the p-CSP modulo $\psi^{-1}(\gamma_3(B)^p)$.

Then Theorem 8.1.10 follows by a direct application of [46, Theorem 1]. Applying Lemma 8.8.2 to Proposition 8.8.3 and Proposition 8.8.4 we will obtain step (1). Similarly, by applying Lemma 8.8.2 to Proposition 8.8.7 and Proposition 8.8.8 yields step (2). Now, set $M := \psi^{-1}((B')^p)$ and $N := \psi^{-1}(\gamma_3(B)^p)$. Considering Proposition 8.7.1, Theorem 8.7.2 and Lemma 8.7.3, we summarise the proof of Theorem 8.1.10 in Figure 8.5.

Proposition 8.8.3. *The group B has the p-CSP modulo B'.*

Proof. Set $b_{is+j} = a_{i,j}$ for all $i \in [0, d-1]$ and $j \in [0, s-1]$. Define, for $r \in [0, ds-1]$, $A_r = \langle b_r, \dots, b_{ds-1}\rangle B'$ and set $A_{ds} = B'$. We will prove that A_r has the p-CSP modulo A_{r+1} for all $r \in [0, ds-1]$. Then the result follows from the Lemma 8.8.2.

Clearly, $A_r/A_{r+1}\operatorname{St}_{A_r}(n) \in \mathcal{C}$ and by Theorem 8.1.6(iv) we have $A_r/A_{r+1} = \langle b_r\rangle \cong \mathbb{Z}$. In $\mathbb{Z}$, the subgroups of index a power of p are totally ordered, whence it suffices to prove that $|A_r : A_{r+1}\operatorname{St}_{A_r}(n)|$ tends to infinity when n tends to infinity. In fact, we prove that $b_r^{p^n} \notin A_{r+1}\operatorname{St}_{A_r}(nds+r+1)$ for $n \in \mathbb{N}_0$. Assume to the contrary that $b_r^{p^n} \in A_{r+1}\operatorname{St}_{A_r}(nds+r+1)$. In particular, $b_r^{p^n} \in A_{r+1}\operatorname{St}_B(nds+r+1)$. Thanks to Theorem 8.5.1, we have $\operatorname{St}_B(nds+r+1) = \langle b_0^{p^{n+1}}, \dots, b_r^{p^{n+1}}, b_{r+1}^{p^n}, \dots, b_{ds-1}^{p^n}\rangle^B$. Thus, there exists $x_0, \dots, x_{ds-1} \in \mathbb{Z}$ such that

$$b_r^{p^n} \equiv_{B'} b_0^{x_0 p^{n+1}} \cdots b_r^{x_r p^{n+1}} b_{r+1}^{x_{r+1}} \cdots b_{ds-1}^{x_{ds-1}},$$

contradicting Theorem 8.1.6(iv). □

Proposition 8.8.4. *The group B' has the p-CSP modulo $\gamma_3(B)$.*

Proof. Notice from Theorem 8.7.2(ii) that $\gamma_3(B)$ is a subgroup of index a power of p in B' and hence it suffices to prove that $\operatorname{St}_{B'}(n)$ is contained in $\gamma_3(B)$ for some n, equivalently

$|B'/\gamma_3(G)\,\mathrm{St}_{B'}(n)| = |B'/\gamma_3(B)|$. Observe that,

$$B'/\gamma_3(B)\,\mathrm{St}_{B'}(n) \cong B'\,\mathrm{St}_B(n)/\gamma_3(B)\,\mathrm{St}_B(n).$$

Now, in light of Theorem 8.7.2(ii), we choose $n \in \mathbb{N}_+$ such that the set

$$\{[a_{i,j}, a_{0,0}] \mid i \in [0, d-1],\ j \in [1, s-1]\} \cup \{[a_{0,1}, a_{i',0}] \mid i' \in [1, d-1]\} \cup \{[a_{0,1}, a_{0,0}]^p\},$$

has trivial intersection with $\mathrm{St}_B(n)$. One can easily compute from the description of the stabilisers in Theorem 8.5.1 that $n = ds + 2$ is the smallest number with this property. We construct a group H which admits an epimorphism from the group $B/\gamma_3(B)\,\mathrm{St}_B(ds+2)$ and see that commutator subgroup H' has the desired size.

Now fix $n = ds + 2$ and set $\Gamma = B/\gamma_3(B)\,\mathrm{St}_B(n)$. Again from Theorem 8.5.1 we have $\mathrm{St}_B(n) = \langle b_0^{p^2}, b_1^{p^2}, b_2^p, \ldots, b_{ds-1}^p \rangle^B$, where $b_{is+j} = a_{i,j}$ as in the proof of Proposition 8.8.3. By a straightforward calculation using the presentation of $B/\gamma_3(B)$, given in the proof of Theorem 8.7.2(ii), we obtain the following presentation for Γ:

$$\langle\, \mathcal{S} \mid \mathcal{R} \,\rangle, \tag{8.10}$$

where $\mathcal{S} = \{b_r \mid r \in [0, ds-1]\}$ and

$$\mathcal{R} = \left\langle \begin{array}{l|l} b_0^{p^2}, b_1^{p^2}, b_t^p, [b_t, b_{t'}], & t, t' \in [2, ds-1] \\ [b_1, b_{t''}], & t'' \in [2, ds-1],\ \text{not a multiple of } s \\ [b_0, b_{is}], \gamma_3(F) & i \in [1, d-1] \end{array} \right\rangle,$$

where F is the free group on the set of generators of Γ.

Let R be the ring $\mathbb{Z}/p^2\mathbb{Z}$. Let $\mathrm{UT}_{ds+1}(R) \leqslant \mathrm{GL}_{ds+1}(R)$ be the group of all upper triangular matrices over R with entries 1 along the diagonal. Denote by $E_{i,j}(\ell)$ the element of $\mathrm{UT}_{ds+1}(R)$ with the entry $\ell \in R$ at the position (i, j). For $i \in [1, d(s-1)-1]$ and $j \in [1, d-1]$, define

$$x_i = E_{i,ds-1}(p), \qquad y_j = E_{d(s-1)+j,ds}(p),$$
$$y = E_{ds-1,ds}(1), \qquad z = E_{ds,ds+1}(1),$$

and define $\mathcal{H}$ to be the subgroup of $\mathrm{UT}_{ds+1}(R)$ generated by the set $\{x_i, y_j, y, z\}$. By abuse of notation denote the image of the set of generators of $\mathcal{H}$ in the quotient group $\mathcal{H}/\gamma_3(\mathcal{H})$ by the same symbols and set $H = \mathcal{H}/\gamma_3(\mathcal{H})$. By an easy computation, we obtain

$$x_i^p = y_j^p = y^{p^2} = z^{p^2} = [x_i, x_{i'}] = [y_j, y_{j'}] = [y, y_j] = [x_i, y_j] = [x_i, z] = \mathrm{id},$$

for all $i, i' \in [1, d(s-1)-1]$ and $j, j' \in [1, d-1]$. Now, fix a bijection α from the set $\{b_r \mid r \in [2, ds-1] \backslash \{s, 2s, \ldots, (d-1)s\}$ to the set $\{x_i \mid i \in [1, d(s-1)-1]\}$. Define a map φ from the set of generators of Γ to the set of generators of H by

$$\begin{aligned} \varphi(b_0) &= y & \varphi(b_1) &= z \\ \varphi(b_{js}) &= y_j \text{ for } j \in [1, d-1] & \varphi(b_r) &= \alpha(b_r), \text{ otherwise.} \end{aligned}$$

Then φ extends to an epimorphism $\Gamma \to H$, since as seen above, $\varphi(b_r)$ satisfies all the relations of the given presentation (8.10) of the group Γ. Furthermore, observe that the commutator subgroup of H is generated by the union of the sets

$$\{ [x_i, y] \mid i \in [1, d(s-1)-1] \} \cup \{ [y_j, z] \mid j \in [1, d-1] \} \cup \{ [y, z] \}.$$

Hence,

$$|\Gamma'| \geqslant |\varphi(\Gamma')| = |H'| = p^{d(s-1)-1} p^{d-1} p^2 = p^{ds}.$$

Indeed $|\Gamma'| \leqslant |B'/\gamma_3(B)| = p^{ds}$, and thus $|\Gamma'| = p^{ds}$, completing the proof. □

We now need two general lemmata.

Lemma 8.8.5. *Let $H \leqslant \operatorname{Aut} T$ and $L, K \trianglelefteq H$ with $L \leqslant K$ and let $\mathcal{C}$ be the class of all finite p-groups. Assume further that $H/K \in \mathcal{C}$ and H/L is abelian. If H has the p-CSP modulo L, then K has the p-CSP property modulo L.*

Proof. Let $\tilde{K}$ be a normal subgroup of K satisfying $L \leqslant \tilde{K}$ and $K/\tilde{K} \in \mathcal{C}$. Since H/L is abelian, $\tilde{K}/L$ is normal in H/L and hence $\tilde{K}$ is normal in H. Also notice that $H/\tilde{K} \in \mathcal{C}$. As H has the p-CSP there exists $n \in \mathbb{N}_0$ such that $\mathrm{St}_H(n) \leqslant \tilde{K}$. In particular $\mathrm{St}_K(n) = \mathrm{St}_H(n) \cap K \leqslant \mathrm{St}_H(n) \leqslant \tilde{K}$, completing the proof. □

Lemma 8.8.6. *Let $H \leqslant \operatorname{Aut} T$ and $L, K \trianglelefteq H$. If KL has the p-CSP modulo L, then K has the p-CSP property modulo $K \cap L$.*

Proof. Choose $\tilde{K} \trianglelefteq K$ with $K \cap L \leqslant \tilde{K}$ and $K/\tilde{K} \in \mathcal{C}$. Then, $\tilde{K}L \trianglelefteq KL$ and $KL/\tilde{K}L \cong K/\tilde{K} \in \mathcal{C}$. As KL has the p-CSP property modulo L, it holds that $\mathrm{St}_{KL}(n) \leqslant \tilde{K}L$ for some n. Thus, $\mathrm{St}_K(n) = \mathrm{St}_{KL}(n) \cap K \leqslant \tilde{K}L \cap K = \tilde{K}$. □

Proposition 8.8.7. *The group $\gamma_3(B)$ has the p-CSP modulo $\gamma_3(B) \cap M$.*

Proof. We prove that $\gamma_3(B)M$ has the p-CSP modulo M. Then by Lemma 8.8.6 we obtain the result. It follows from Proposition 8.7.1 and Theorem 8.7.2(ii) that B'/M is abelian and that $B'/\gamma_3(B)M \in \mathcal{C}$, respectively. Thanks to Lemma 8.8.5, it is enough to prove that B' has the p-CSP modulo M.

Let $i \in [0, d-1], j \in [1, s-1]$ and $k \in [1, p-1]$. Define $c_{i(s-1)+j} := b_{is+j} := a_{i,j}$. Set $t = i(s-1)+j$ and $r = is+j$ and note that c_t is a relabeling of the elements b_r (defined in the proof of Proposition 8.8.3) by excluding the elements of the form b_{is} for $i \in [0, d-1]$. From Proposition 8.7.1, we have

$$B'/M = \langle\, [a_{i,j}, a_{0,0}^k] \mid i \in [0, d-1],\ j \in [1, s-1],\ k \in [1, p-1] \,\rangle.$$

Set $\ell = (k-1)(ds-d)+t$ and $e_\ell = [c_t, a_{0,0}^k]$. Then,

$$\psi(e_\ell) = \psi([c_t, a_{0,0}^k]) = \psi([b_r, a_{0,0}^k]) = (b_{r-1}^{-1}, \mathrm{id}^{*(k-1)}, b_{r-1}, \mathrm{id}^{*(p-k-1)}).$$

For $\ell \in [1, (p-1)(ds-d)]$, set $M_\ell = \langle e_\ell, \ldots, e_{(p-1)(ds-d)} \rangle M$ and $M_{(p-1)(ds-d)+1} = M$. It follows from Theorem 8.1.6(iv) that $M_\ell / M_{\ell+1} = \langle e_\ell \rangle \cong \mathbb{Z}$. We will prove that $|M_\ell : M_{\ell+1} \operatorname{St}_{M_\ell}(n)|$ tends to infinity as n tends to infinity. Assume to the contrary that there are $n, n' \in \mathbb{N}_+$ such that for all $\tilde{n} \geqslant n'$, $e_\ell^{p^n} \in M_{\ell+1} \operatorname{St}_{M_\ell}(\tilde{n})$. There exist $x_{\ell+1}, \ldots, x_{(p-1)(ds-d)} \in \mathbb{Z}$ such that

$$e_\ell^{p^n} e_{\ell+1}^{x_1} \cdots e_{(p-1)(ds-d)}^{x_{(p-1)(ds-d)}} \in M \operatorname{St}_{M_\ell}(\tilde{n}) \leqslant M \operatorname{St}_B(\tilde{n}),$$

hence

$$\psi(e_\ell^{p^n} e_{\ell+1}^{x_1} \cdots e_{(p-1)(ds-d)}^{x_{(p-1)(ds-d)}}) \in (B')^p \cdot (\operatorname{St}_B(\tilde{n}-1))^p.$$

Consider the k-th coordinate, $x b_{r-1}^{p^n} \in B' \operatorname{St}_B(\tilde{n}-1)$, where x is a product of elements of the form $b_{r'}$ such that $r' > r-1$. Then $x \in A_r$, where A_r is defined as in the proof of Proposition 8.8.3. This implies $b_{r-1}^{p^n} \in A_r \operatorname{St}_B(\tilde{n})$ for all $\tilde{n} \geqslant n'-1$, which contradicts Proposition 8.8.3. □

Proposition 8.8.8. *The group $\gamma_3(B) \cap M$ has the p-CSP modulo N.*

Proof. It is straightforward from Theorem 8.7.2(ii) that the group M/N is a finite abelian and $M/N \in \mathcal{C}$. By Lemma 8.8.5, it suffices to prove that M has the p-CSP modulo N. From Proposition 8.8.4, it follows that $\operatorname{St}_{B'}(n) \leqslant \gamma_3(B)$ for some n. Therefore,

$$\psi(\operatorname{St}_M(n+1)) \leqslant (\operatorname{St}_{B'}(n))^p \leqslant \gamma_3(B)^p,$$

and hence $\operatorname{St}_M(n+1) \leqslant \psi^{-1}((\operatorname{St}_{B'}(n))^p) \leqslant N$. □

Proof of Theorem 8.1.10. By applying Lemma 8.8.2 to Proposition 8.8.3 and Proposition 8.8.4 we obtain that the group B has the p-CSP modulo $\gamma_3(B)$. Further application of Lemma 8.8.2 to Proposition 8.8.7 and Proposition 8.8.8 yields that $\gamma_3(G)$ has the p-CSP modulo N. Now, the result follows by [46, Theorem 1]. □

Chapter 9

Maximal subgroups of generalised Basilica Groups

9.1 Introduction

Groups acting on rooted trees have drawn a great deal of attention over the last couple of decades because they exhibit prominent features and solve several long-standing problems in group theory. The initial examples studied were Grigorchuk's groups of intermediate word growth ([51]; answering Milnor's question) and Gupta and Sidki's examples of finitely generated infinite torsion p-groups ([63]; providing an explicit family of 2-generated counterexamples to the general Burnside problem). Ever since, attempts have been made to characterise and generalise the groups of automorphisms of rooted trees. Today, the Grigorchuk groups and the Gupta–Sidki groups are known as the first examples of groups in the family of *branch groups*. Branch groups are groups acting spherically transitively on a spherically homogeneous rooted tree T and having subnormal subgroups similar to that of the full automorphism group $\operatorname{Aut} T$ of the tree T, see Section 9.2 for definitions. The groups studied in this paper belong to a more general class of groups, the *weakly branch groups*, obtained by weakening some of the algebraic properties of the branch groups; cf. [18].

The Basilica group is a 2-generated weakly branch, but not branch, group acting on the binary rooted tree, which was introduced by Grigorchuk and Żuk in [59] and [58]. It is the first known example of an amenable [24] but not sub-exponentially amenable group [59]. In contrast to the Grigorchuk and the Gupta–Sidki groups, the Basilica group is torsion-free and has exponential word growth [59]. Moreover, it is the iterated monodromy group of the complex polynomial $z^2 - 1$; [76, Section 6.12.1]. The generators of the Basilica group are recursively defined as follows:

$$a = (1, b) \quad \text{and} \quad b = (1, a)\sigma,$$

where σ is the cyclic permutation which swaps the subtrees rooted at the first level of the binary rooted tree, and (x, y) represents the independent action on the two maximal subtrees, where $x, y \in \operatorname{Aut} T$. Recently, Petschick and Rajeev [92] introduced a construction which relates the Basilica group and the one-generated dyadic odometer $\mathcal{O}_2$ (also known as the adding machine). Let m, $s \geqslant 2$ be integers and let G be a subgroup of the automorphism group $\operatorname{Aut} T$ of the m-adic tree T. The sth Basilica group of G is given by

$$\operatorname{Bas}_s(G) = \langle \beta_i^s(g) \mid g \in G,\, i \in \{0, 1, \ldots, s-1\} \rangle,$$

where $\beta_i^s : \operatorname{Aut} T \to \operatorname{Aut} T$ are monomorphisms given by

$$\begin{aligned} &\beta_i^s(g) = (1, \ldots, 1, \beta_{i-1}^s(g)) && \text{for } i \in \{1, \ldots, s-1\}, \\ &\beta_0^s(g) = (\beta_{s-1}^s(g_0), \ldots, \beta_{s-1}^s(g_{m-1}))g^\epsilon, \end{aligned}$$

where g_x is the restriction of g to the subtree rooted at a first-level vertex $x \in \{0, \ldots, m-1\}$, and g^ϵ is the local action of the element g at the root of T (in [92] the generators $\beta_i^s(g)$, for $i \in \{1, \ldots, s-1\}$, are defined along the left-most spine and the element g^ϵ is acting from the left, which is equivalent to the definition above). We obtain the classical Basilica group by applying the operator Bas_2 to the dyadic odometer as follows: let $c = (1, c)\sigma$ be the automorphism of the binary rooted tree generating the dyadic odometer. Then the generators of the Basilica group are given by

$$a = \beta_1^2(c) \quad \text{and} \quad b = \beta_0^2(c).$$

This gives a natural generalisation of the Basilica group given by $\operatorname{Bas}_s(\mathcal{O}_m)$ for every pair of integers $m\,, s \geqslant 2$. Here $\mathcal{O}_m$ is the m-adic odometer, which is an embedding of the infinite cyclic group into the automorphism group of the m-adic tree T, and is generated by

$$c = (1, \overset{m-1}{\ldots}, 1, c)\sigma$$

where $\sigma = (0\ 1\ \cdots\ m-1)$ is the m-cycle that cyclically permutes the m subtrees rooted at the first level of T. The generalised Basilica groups $\operatorname{Bas}_s(\mathcal{O}_m)$ resemble the classical Basilica group, as they are weakly branch, but not branch, torsion-free groups of exponential word growth [92, Theorem 1.6]. They are also weakly regular branch over their derived subgroup.

In this paper, we study the maximal subgroups of the generalised Basilica groups $\operatorname{Bas}_s(\mathcal{O}_m)$. The study of maximal subgroups of branch groups was initiated by Pervova [86], where she proved that the torsion Grigorchuk groups do not contain maximal subgroups of infinite index. Thenceforth, attempts have been made to generalise the results and techniques from [86], for instance see [5], [70], and [40]. Among which, our interest lies in the work of Francoeur [40] (or see [39, Section 8.4]), who provided a strategy to study the maximal subgroups of weakly branch groups. In particular, he proved that the classical Basilica group does not contain maximal subgroups of infinite index. Following this technique we prove that the generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_m)$ do not admit maximal subgroups of infinite index.

Theorem 9.1.1. *Let m and s be positive integers such that $m, s \geqslant 2$. Then the generalised Basilica group $\mathrm{Bas}_s(\mathcal{O}_m)$ does not admit a maximal subgroup of infinite index.*

Since we are considering generalised Basilica groups $\mathrm{Bas}_s(\mathcal{O}_m)$ for an arbitrary $s \geqslant 2$, the final stages of our proof differ from previously seen results; compare Theorem 9.4.6. This is also the first time that maximal subgroups of a weakly branch, but not branch, group G have been considered for a group G with more than 2 generators.

It is interesting to note that there are currently no examples of finitely generated weakly branch, but not branch, groups with maximal subgroups of infinite index. There are only examples of finitely generated branch groups with maximal subgroups of infinite index; see [27] and [41]. It remains to be seen whether being a finitely generated weakly branch group with maximal subgroups of infinite index implies the group is branch.

Furthermore, in all known examples of finitely generated weakly branch, but not branch, groups with maximal subgroups only of finite index, these groups have maximal subgroups that are not normal; compare Remark 9.4 and [33, 42]. Therefore it is also natural to ask if there exists a finitely generated weakly branch, but not branch, group with all maximal subgroups of finite index and normal.

Organisation. Section 9.2 contains preliminary material on groups acting on the m-adic tree. In Section 9.3, we record some length reducing properties of generalised Basilica groups, and in Section 9.4 we prove Theorem 9.1.1.

9.2 Preliminaries

By $\mathbb{N}$ we denote the set of positive integers, and by $\mathbb{N}_0$ the set of non-negative integers.

Let $m \in \mathbb{N}_{\geqslant 2}$ and let $T = T_m$ be the m-adic tree, that is, a rooted tree where all vertices have m children. Using the alphabet $X = \{0, 1, \ldots, m-1\}$, the vertices u_ω of T are labelled bijectively by the elements ω of the free monoid X^* in the following natural way: the root of T is labelled by the empty word, and is denoted by ϵ, and for each word $\omega \in X^*$ and letter $x \in X$ there is an edge connecting u_ω to $u_{\omega x}$. More generally, we say that u_ω precedes u_λ whenever ω is a prefix of λ.

There is a natural length function on X^*, which is defined as follows: the words ω of length $|\omega| = n$, representing vertices u_ω that are at distance n from the root, are the nth level vertices and constitute the *nth layer* of the tree.

We denote by T_u the full rooted subtree of T that has its root at a vertex u and includes all vertices succeeding u. For any two vertices $u = u_\omega$ and $v = u_\lambda$, the map $u_{\omega\tau} \mapsto u_{\lambda\tau}$, induced by replacing the prefix ω by λ, yields an isomorphism between the subtrees T_u and T_v.

Now each $f \in \operatorname{Aut} T$ fixes the root, and the orbits of $\operatorname{Aut} T$ on the vertices of the tree T are the layers of the tree T. The image of a vertex u under f will be denoted by $f(u)$. The automorphism f induces a faithful action on X^* given by $f(u_\omega) = u_{f(\omega)}$. For $\omega \in X^*$ and $x \in X$ we have $f(\omega x) = f(\omega)x'$, for $x' \in X$ uniquely determined by ω and f. This induces a permutation f^ω of X which satisfies

$$f(\omega x) = f(\omega) f^\omega(x), \quad \text{and consequently} \quad f(u_{\omega x}) = u_{f(\omega) f^\omega(x)}.$$

More generally, for an automorphism f of T, since the layers are invariant under f, for $u, v \in X^*$, the equation

$$f(uv) = f(u) f_u(v)$$

defines a unique automorphism f_u of T called the *section of f at u*. This automorphism can be viewed as the automorphism of T induced by f upon identifying the rooted subtrees of T at the vertices u and $f(u)$ with the tree T. As seen here, we often do not differentiate between X^* and vertices of T.

9.2.1 Subgroups of $\operatorname{Aut} T$

Let G be a subgroup of $\operatorname{Aut} T$ acting *spherically transitively*, that is, transitively on every layer of T. The *vertex stabiliser* $\operatorname{st}_G(u)$ is the subgroup consisting of elements in G that fix the vertex u. For $n \in \mathbb{N}$, the *nth level stabiliser* $\operatorname{St}_G(n) = \bigcap_{|\omega|=n} \operatorname{st}_G(u_\omega)$ is the subgroup consisting of automorphisms that fix all vertices at level n.

Each $g \in \operatorname{St}_{\operatorname{Aut} T}(n)$ can be completely determined in terms of its restrictions $g_1, \ldots, g_{m^n}$ to the subtrees rooted at vertices at level n. There is a natural isomorphism

$$\psi_n \colon \operatorname{St}_{\operatorname{Aut} T}(n) \to \prod\nolimits_{|\omega|=n} \operatorname{Aut} T_{u_\omega} \cong \operatorname{Aut} T \times \overset{m^n}{\cdots} \times \operatorname{Aut} T$$

defined by sending $g \in \mathrm{St}_{\mathrm{Aut}\, T}(n)$ to its tuple of sections $(g_1, \ldots, g_{m^n})$. For conciseness, we will omit the use of ψ_1, and simply write $g = (g_1, \ldots, g_m)$ for $g \in \mathrm{St}_{\mathrm{Aut}\, T}(1)$.

Let $\omega \in X^n$ be of length n. We further define

$$\varphi_\omega : \mathrm{st}_{\mathrm{Aut}\, T}(u_\omega) \to \mathrm{Aut}\, T_{u_\omega} \cong \mathrm{Aut}\, T$$

to be the map sending $f \in \mathrm{st}_{\mathrm{Aut}\, T}(u_\omega)$ to the section f_{u_ω}.

A group $G \leqslant \mathrm{Aut}\, T$ is said to be *self-similar* if for all $f \in G$ and all $\omega \in X^*$ the section f_{u_ω} belongs to G. We will denote G_ω to be the subgroup $\varphi_\omega(\mathrm{st}_G(u_\omega))$.

Let G be a subgroup of $\mathrm{Aut}\, T$ acting spherically transitively. Here the vertex stabilisers at every level are conjugate under G. We say that the group G is *fractal* if $G_\omega = \varphi_\omega(\mathrm{st}_G(u_\omega)) = G$ for every $\omega \in X^*$, after the natural identification of subtrees.

The *rigid vertex stabiliser* of u in G is the subgroup $\mathrm{rist}_G(u)$ consisting of all automorphisms in G that fix all vertices of T not succeeding u. The *rigid nth level stabiliser* is the direct product of the rigid vertex stabilisers of the vertices at level n:

$$\mathrm{Rist}_G(n) = \prod\nolimits_{|\omega|=n} \mathrm{rist}_G(u_\omega) \trianglelefteq G.$$

We recall that a spherically transitive group G is a *branch group* if $\mathrm{Rist}_G(n)$ has finite index in G for every $n \in \mathbb{N}$; and G is *weakly branch* if $\mathrm{Rist}_G(n)$ is non-trivial for every $n \in \mathbb{N}$. If, in addition, the group G is self-similar and there exists a subgroup $1 \neq K \leqslant G$ with $K \times \overset{m}{\cdots} \times K \subseteq \psi_1(K \cap \mathrm{St}_G(1))$ and $|G : K| < \infty$, then G is said to be *regular branch over* K. If in the previous definition the condition $|G : K| < \infty$ is omitted, then G is said to be *weakly regular branch over* K.

9.2.2 A basic result

Here we record a general result that will be useful in the sequel. For $g \in \mathrm{Aut}\, T$, recall that g^ϵ denotes the action induced by g at the root of T.

Lemma 9.2.1. *For a self-similar group $G \leqslant \mathrm{Aut}\, T$, let $z = (z_0, \ldots, z_{m-1})z^\epsilon \in G'$. Then $z_0 \cdots z_{m-1} \in G'$.*

Proof. It suffices to prove the result for a basic commutator $[g, h]$, where $g, h \in G$. Write $g = (g_0, \ldots, g_{m-1})g^\epsilon$ and $h = (h_0, \ldots, h_{m-1})h^\epsilon$. For notational convenience, let us write $\tau = (g^\epsilon)^{-1}$ and $\kappa = (h^\epsilon)^{-1}$, and for $\alpha \in \mathrm{Sym}(X)$ and $x \in X$ we write x^α for $\alpha(x)$. As

$$\begin{aligned}
&[g, h] \\
&= \tau(g_0^{-1}, \ldots, g_{m-1}^{-1})\kappa(h_0^{-1}, \ldots, h_{m-1}^{-1})(g_0, \ldots, g_{m-1})g^\epsilon(h_0, \ldots, h_{m-1})h^\epsilon \\
&= (g_{0^\tau}^{-1}, \ldots, g_{(m-1)^\tau}^{-1})(h_{0^{\tau\kappa}}^{-1}, \ldots, h_{(m-1)^{\tau\kappa}}^{-1})(g_{0^{\tau\kappa}}, \ldots, g_{(m-1)^{\tau\kappa}})(h_{0^{\tau\kappa g^\epsilon}}, \ldots, h_{(m-1)^{\tau\kappa g^\epsilon}})\tau\kappa g^\epsilon h^\epsilon,
\end{aligned}$$

the result follows. $\square$

9.3 Length reducing properties

For any two integers i, j, let $[i, j]$ denote the set $\{i, i+1, \ldots, j-1, j\}$. In the following sections, we fix $m, s \in \mathbb{N}_{\geqslant 2}$. For convenience, write $G = \mathrm{Bas}_s(\mathcal{O}_m)$ for the remainder of this paper. Then G is generated by the elements

$$\beta_0^s(c) = (1, \overset{m-1}{\ldots}, 1, \beta_{s-1}(c))\sigma,$$
$$\beta_1^s(c) = (1, \overset{m-1}{\ldots}, 1, \beta_0(c)),$$
$$\vdots$$
$$\beta_{s-1}^s(c) = (1, \overset{m-1}{\ldots}, 1, \beta_{s-2}(c)),$$

where $c = (1, \overset{m-1}{\ldots}, 1, c)\sigma$ is the generator of the m-adic odometer $\mathcal{O}_m$ acting on the m-adic tree T and σ is the permutation $(0\,1\,\cdots\,m-1)$ which cyclically permutes the subtrees rooted at the first level of T. We refer the reader to [92] for a detailed study of these groups.

Denote by $\beta_i^s(c) = a_i$, for every $i \in [0, s-1]$. We shall adopt the convention that the subscripts of the a_i's are taken modulo s. Set $S = \{a_i^{\pm 1} \mid i \in [0, s-1]\}$ and then $G = \langle S \rangle$. For each word $w \in S^*$, the length $|w|$ is the usual word length of w over the alphabet S. If $g \in G$ then $|g|$ denotes the minimal length of all words in the alphabet S representing g. A word $w \in S^*$ is called a *geodesic word* if $|w| = |g|$, where g is the image of the word w in G. Notice that for every $g \in G$, the local action g^ϵ of g at the root is an element of $\langle \sigma \rangle$. Hence, for conciseness, we denote g^ϵ by σ_g.

Lemma 9.3.1. *Let $g = (g_0, \ldots, g_{m-1})\sigma_g \in G$. Then $\sum_{k=0}^{m-1} |g_k| \leqslant |g|$.*

Proof. The proof proceeds by induction on the length of g. Clearly, the result is true if $|g| = 0$ and $|g| = 1$. Assume that $|g| > 1$. Let $w \in S^*$ be a geodesic word representing g. The word w can be written as $w = xw'$ for some $x \in S$ and $w' \in S^*$ such that w' is reduced. Then $|w'| < |w|$ and w' does not represent g in G. Denote by g' the corresponding element in G. Then $|g'| \leqslant |w'| < |w| = |g|$. We obtain

$$\begin{aligned}(g_0, \ldots, g_{m-1})\sigma_g = g = xg' &= (x_0, \ldots, x_{m-1})\sigma_x (g'_0, \ldots, g'_{m-1})\sigma_{g'} \\ &= (x_0 g'_{0^{\sigma_x}}, \ldots, x_{m-1} g'_{(m-1)^{\sigma_x}})\sigma_x \sigma_{g'},\end{aligned}$$

which implies $g_k = x_k g'_{k^{\sigma_x}}$ for all $k \in [0, m-1]$. It follows by induction that,

$$\sum_{k=0}^{m-1} |g_k| = \sum_{k=0}^{m-1} |x_k g'_{k^{\sigma_x}}| \leqslant \sum_{k=0}^{m-1} |x_k| + \sum_{k=0}^{m-1} |g'_{k^{\sigma_x}}| \leqslant |x| + |g'| \leqslant |x| + |w'| = |w| = |g|. \quad \square$$

Lemma 9.3.2. *Let $g = (g_0, \ldots, g_{m-1})\sigma_g \in G$ with $\sigma_g = \sigma^i$ for some $i \in [1, m-1]$ such that* $\gcd(i, m) = 1$. *Let $\alpha_0, \ldots, \alpha_{m-1} \in G$ be such that $g^m = (\alpha_0, \ldots, \alpha_{m-1})$. Then $|\alpha_k| \leqslant \sum_{\ell=0}^{m-1} |g_\ell| \leqslant |g|$ for all $k \in [0, m-1]$.*

Proof. Observe that

$$g^m = \left(g_0 g_{0^{\sigma^i}} g_{0^{\sigma^{2i}}} \cdots g_{0^{\sigma^{(m-1)i}}}, \ \ldots, \ g_{m-1} g_{(m-1)^{\sigma^i}} g_{(m-1)^{\sigma^{2i}}} \cdots g_{(m-1)^{\sigma^{(m-1)i}}}\right).$$

By setting $\alpha_k = g_k g_{k^{\sigma^i}} \cdots g_{k^{\sigma^{(m-1)i}}}$, for each $k \in [0, m-1]$, we obtain

$$|\alpha_k| = |g_k g_{k^{\sigma^i}} \cdots g_{k^{\sigma^{(m-1)i}}}| \leqslant \sum_{\ell=0}^{m-1} |g_\ell| \leqslant |g|,$$

where the last inequality follows from Lemma 9.3.1. □

Lemma 9.3.3. *Let $g = (g_0, \ldots, g_{m-1})\sigma_g \in G$ and let $x_1 \cdots x_\ell \in S^*$ be a geodesic word representing g. If there exist $1 \leqslant r < r' \leqslant \ell$ such that $x_r = a_0$, $x_{r'} = a_0^{-1}$, then $\sum_{k=0}^{m-1} |g_k| < |g|$.*

Proof. By assumption, the word $x_1 \cdots x_\ell$ contains a subword of the form $a_0 w a_0^{-1}$, where w is a non-trivial reduced word in the alphabet S. We assume, without loss of generality, that w is a reduced word in the alphabet $S \backslash \{a_0^{\pm 1}\}$. Let w represent an element h in G. Since $x_1 \cdots x_\ell \in S^*$ is a geodesic word, the word w is also geodesic and so $|h| = |w|$. Notice that $|a_0 w a_0^{-1}| = |w| + 2$. Realising the word $a_0 w a_0^{-1}$ in G gives

$$a_0 w a_0^{-1} = (1, \ldots, 1, \varphi_{m-1}(h), 1).$$

Also, we have

$$h = (1, \ldots, 1, \varphi_{m-1}(h)).$$

By Lemma 9.3.1, we get $|\varphi_{m-1}(h)| \leqslant |h| = |w|$. Therefore we conclude that

$$\sum_{k=0}^{m-1} |g_k| \leqslant |g| - 2 < |g|.$$

□

9.4 Maximal subgroups

Recall that we write $G = \mathrm{Bas}_s(\mathcal{O}_m)$. It follows from Proposition 9.4.1 below together with [40, Proposition 2.21] that the group G admits maximal subgroups of infinite index if and only if it admits a proper subgroup $H < G$ such that $HN = G$ for every non-trivial normal subgroup $N \trianglelefteq G$. A subgroup $H \leqslant G$ satisfying the above condition is called a *prodense subgroup*. As seen below, we prove that G does not admit any proper prodense subgroup, which proves Theorem 9.1.1.

Proposition 9.4.1. *The group G is just non-(virtually nilpotent). Hence, maximal subgroups of proper quotients of G are of finite index.*

Proof. As G has exponential word growth, it follows from Bass [8] and Guivarc'h [62] that G is not virtually nilpotent. To see that every proper quotient of G is virtually nilpotent, by [40, Theorem 4.10], it suffices to prove that G/G'' is virtually nilpotent. Set $N = \psi_1^{-1}(\gamma_3(G)\times\cdots\times\gamma_3(G))$. From [92, Lemma 7.3], we have $N \leqslant G'' < \mathrm{St}_G(1) < G$. Therefore ψ_1 induces a homomorphism

$$\widetilde{\psi}_1 : \mathrm{St}_G(1)/N \to G/\gamma_3(G) \times \overset{m}{\cdots} \times G/\gamma_3(G).$$

Since $\widetilde{\psi}_1$ is injective and $\widetilde{\psi}_1(\mathrm{St}_G(1)/N)$ is nilpotent (being a subgroup of a nilpotent group), we obtain that $\mathrm{St}_G(1)/N$ is nilpotent. This implies that $\mathrm{St}_G(1)/G''$ is nilpotent as it is a quotient of $\mathrm{St}_G(1)/N$. As the subgroup $\mathrm{St}_G(1)$ has finite index in G, the group $\mathrm{St}_G(1)/G''$ has finite index in G/G'' and hence G/G'' is virtually nilpotent. The last part of the result follows from [39, Corollary 5.1.3]. □

Hereafter, for $g, h \in G$, the equivalence $g \equiv h \mod G'$ will simply be denoted by $g \equiv h$. Notice that for every $z \in G'$, we have $\sigma_z = 1$ and $G' \leqslant \mathrm{St}_G(1)$.

Lemma 9.4.2. *Let $g \in G$ be such that $g \equiv a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$, where $\epsilon_i \in \{\pm 1\}$. Let $j \in \mathbb{N}_0$ be such that $j = \ell s + r$, where $\ell \in \mathbb{N}_0$ and $0 \leqslant r < s$. If $\psi_j(g^{m^j}) = (g_0, \ldots, g_{m^j-1})$ then $g_k \equiv a_{s-1-r}^{\epsilon_{s-1}} \cdots a_{0-r}^{\epsilon_0}$ for all $k \in [0, m^j - 1]$.*

Proof. Since $g \equiv a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$, there exists an element $(z_0, \ldots, z_{m-1}) = z \in G'$ such that $g = a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0} z$. We have

$$\begin{aligned}(g_0, \ldots, g_{m-1}) &= g^m = (a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0} z)^m \\ &= \begin{cases} ((z_1, \ldots, z_{m-1}, a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} a_{s-1} z_0)\sigma)^m & \text{if } \epsilon_0 = 1, \\ ((a_{s-1}^{-1} z_{m-1}, z_0, \ldots, z_{m-3}, a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} z_{m-2})\sigma^{-1})^m & \text{if } \epsilon_0 = -1, \end{cases}\end{aligned}$$

which equals

$$(\, z_1 z_2 \cdots z_{m-1} a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} a_{s-1} z_0 \,, \ \ldots\ , \ a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} a_{s-1} z_0 z_1 z_2 \cdots z_{m-1} \,)$$

if $\epsilon_0 = 1$, and

$$(\, a_{s-1}^{-1} z_{m-1} a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} z_{m-2} z_{m-3} \cdots z_0 \,, \ \ldots\ , \ a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} z_{m-2} z_{m-3} \cdots z_0 a_{s-1}^{-1} z_{m-1} \,)$$

if $\epsilon_0 = -1$.

Therefore $g_k \equiv a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} a_{s-1}^{\epsilon_0}$ for all $k \in [0, m-1]$, since $z_0 \cdots z_{m-1} \in G'$ by Lemma 9.2.1. The result then follows upon repeating the above process. □

In the following, we denote by $\mathrm{Sym}(s)$ the symmetric group on $\{0, 1, \ldots, s-1\}$. Recall also from Subsection 9.2.1 the map φ_u for $u \in X^*$.

Lemma 9.4.3. *Let $g = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \in G$ where $\pi \in \mathrm{Sym}(s)$ and $\epsilon_i \in \{\pm 1\}$. Let $j \in \mathbb{N}_0$ be such that $j = \ell s + r$, where $\ell \in \mathbb{N}_0$ and $0 \leqslant r < s$. Then $\varphi_{(m-1)^j}(g^{m^j}) = a_{(s-1)^\pi - r}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi - r}^{\epsilon_{0^\pi}}$.*

Proof. Let $i \in [0, s-1]$ be such that $0 = i^\pi$. Then

$$g = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi}^{\epsilon_{(i+1)^\pi}} a_0^{\epsilon_0} a_{(i-1)^\pi}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}.$$

By taking the mth power of the element g we get

$$g^m = (*, \ldots, *, a_{(s-1)^\pi - 1}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi - 1}^{\epsilon_{(i+1)^\pi}} a_{s-1}^{\epsilon_0} a_{(i-1)^\pi - 1}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi - 1}^{\epsilon_{0^\pi}})$$

where

$$* = \begin{cases} a_{(i-1)^\pi - 1}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi - 1}^{\epsilon_{0^\pi}} a_{(s-1)^\pi - 1}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi - 1}^{\epsilon_{(i+1)^\pi}} a_{s-1} & \text{if } \epsilon_0 = 1, \\ a_{s-1}^{-1} a_{(i-1)^\pi - 1}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi - 1}^{\epsilon_{0^\pi}} a_{(s-1)^\pi - 1}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi - 1}^{\epsilon_{(i+1)^\pi}} & \text{if } \epsilon_0 = -1. \end{cases}$$

In particular, we have $\varphi_{m-1}(g^m) = a_{(s-1)^\pi - 1}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi - 1}^{\epsilon_{0^\pi}}$, and the result follows recursively. □

We recall that H_u denotes the subgroup $\varphi_u(\mathrm{st}_H(u))$ for a vertex $u \in X^*$. By [92, Theorem 1.6(ii)], the group G is fractal, so $G_u = G$ for all $u \in X^*$.

Lemma 9.4.4. *Let H be a subgroup of G. Assume that $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \in H$ for some $\pi \in \mathrm{Sym}(s)$, where $\epsilon_i \in \{\pm 1\}$. Then the following assertions hold.*

(i) For each $n \in \mathbb{N}$ and vertex u of level ns, the subgroup H_u contains a cyclic permutation of the word $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}$.

(ii) Furthermore, if $\epsilon_i = 1$ for some $i \in [0, s]$, then for each $n \in \mathbb{N}$, there is a vertex u of level ns such that the cyclic permutation of $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}$ contained in H_u ends with a_i on the right.

Proof. (i) Let $g = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \in H$. Then $0 = i^\pi$ for some $i \in [0, s-1]$. Observe from the proof Lemma 9.4.3 of that

$$\varphi_{m-1}(g^m) = a_{(s-1)^\pi - 1}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi - 1}^{\epsilon_{(i+1)^\pi}} a_{s-1}^{\epsilon_0} a_{(i-1)^\pi - 1}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi - 1}^{\epsilon_{0^\pi}}$$

and that $\varphi_j(g^m)$ is a cyclic permutation of $\varphi_{m-1}(g^m)$ for every $j \in [0, m-2]$. By repeating the process of taking powers we get that $\psi_s(g^{m^s}) = (g_0, \ldots, g_{m^s - 1})$ with $g_{m^s-1} = g$ and g_k is a cyclic permutation of the word $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}$ for $k \in [0, m^s - 2]$.

(ii) In particular, if $\epsilon_0 = 1$, we note from the proof of Lemma 9.4.3 that

$$\varphi_{m-2}(g^m) = a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi-1}^{\epsilon_{0^\pi}} a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi}} a_{s-1},$$

and by Lemma 9.4.3 we see that

$$\varphi_{(m-2)(m-1)^{s-1}}(g^m) = a_{(i-1)^\pi}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi}^{\epsilon_{(i+1)^\pi}} a_0.$$

More generally, suppose that $\epsilon_j = 1$ for $j \in [0, s]$ and let $\ell \in [0, s]$ be such that $\ell^\pi = j$. Writing $v_j = (m-1)\overset{j}{\cdots}(m-1)$ and $w_j = (m-2)(m-1)\overset{s-j-1}{\cdots}(m-1)$, we recall from Lemma 9.4.3 that

$$\varphi_{v_j}(g^{m^j}) = a_{(s-1)^\pi-j}^{\epsilon_{(s-1)^\pi}} \cdots a_{(\ell+1)^\pi-j}^{\epsilon_{(\ell+1)^\pi}} a_0 a_{(\ell-1)^\pi-j}^{\epsilon_{(\ell-1)^\pi}} \cdots a_{0^\pi-j}^{\epsilon_{0^\pi}}.$$

Then similar to the above we see that

$$\varphi_{v_j w_j}(g^m) = a_{(\ell-1)^\pi}^{\epsilon_{(\ell-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{(\ell+1)^\pi}^{\epsilon_{(\ell+1)^\pi}} a_j,$$

and as $u_j := v_j w_j$ is a vertex of level s, we have that H_{u_j} contains a cyclic permutation of $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}$ that ends with a_j on the right.

Now, by using Lemma 9.4.3 repeatedly, one can see that the result holds for level ns of T, for $n > 1$. □

Proposition 9.4.5. *Let $g \in G$ be such that $g \equiv a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$, where $\epsilon_i \in \{\pm 1\}$. Then there exists a vertex u of level ns in T, for some $n \in \mathbb{N}_0$, and an element $g' \in \mathrm{st}_{\langle g \rangle}(u)$ such that $\varphi_u(g') = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}$ for some $\pi \in \mathrm{Sym}(s)$.*

Proof. The proof proceeds by induction on the length of g. Recall from [92, Theorem 1.6(iv)] that $G/G' = \langle a_0 G', \ldots, a_{s-1}G' \rangle \cong \mathbb{Z}^s$. Hence if g is equivalent to $a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$ then $|g| > s-1$, since any word containing each of the distinct generators of G has length at least s. Assume that $|g| = s$. Then

$$g \in \{a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \mid \pi \in \mathrm{Sym}(s)\}$$

and the result follows trivially by choosing u as the root vertex. Now, assume that $|g| > s$. Since the exponent sum of a_0 in any word representing g is ϵ_0, we can write $g = (g_0, \ldots, g_{m-1})\sigma^{\epsilon_0}$ with $g_0, \ldots, g_{m-1} \in G$. We get

$$g^m = \begin{cases} (g_0 \cdots g_{m-1},\ g_1 \cdots g_{m-1} g_0\,,\ \ldots\ ,\ g_{m-1} g_0 \cdots g_{m-2}) & \text{if } \epsilon_0 = 1, \\ (g_0 g_{m-1} g_{m-2} \cdots g_1,\ g_1 g_0 g_{m-1} \cdots g_2\,,\ \ldots\ ,\ g_{m-1} g_{m-2} \cdots g_0) & \text{if } \epsilon_0 = -1. \end{cases}$$

For every $k \in [0, m-1]$, we set $\alpha_k = \varphi_k(g^m)$. It follows from Lemma 9.4.2 that

$$\alpha_k \equiv a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} a_{s-1}^{\epsilon_0}$$

for all $k \in [0, m-1]$. Furthermore $|\alpha_k| \leqslant |g|$ for all $k \in [0, m-1]$ by Lemma 9.3.2. If there exists $k \in [0, m-1]$ such that $|\alpha_k| < |g|$, then it follows by induction that there exist a vertex u of level ns in T, for some $n \in \mathbb{N}_0$, and $g' \in \mathrm{st}_{\langle \alpha_k \rangle}(u)$ such that

$$\varphi_u(g') = a_{(s-1)^\pi - 1}^{\epsilon_{(s-1)^\pi}} a_{(s-2)^\pi - 1}^{\epsilon_{(s-2)^\pi}} \cdots a_{0^\pi - 1}^{\epsilon_{0^\pi}}$$

for some $\pi \in \mathrm{Sym}(s)$. Using Lemma 9.4.3, we get that

$$\varphi_{(m-1)^{s-1}}((g')^{m^{s-1}}) = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} a_{(s-2)^\pi}^{\epsilon_{(s-2)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}},$$

and hence the result follows.

Assume that $|\alpha_k| = |g|$ for all $k \in [0, m-1]$. Since $|\alpha_k| \leqslant \sum_{\ell=0}^{m-1} |g_\ell|$, in particular, we get

$$\sum_{\ell=0}^{m-1} |g_\ell| = |g|.$$

Let $w_g \in S^*$ be a geodesic word representing g. Since for each $i \in [0, s-1]$ the element $a_i^{\epsilon_i}$ contributes $a_{i-1}^{\epsilon_i}$ in exactly one component, we can obtain words w_{g_k} representing g_k by substituting $a_i^{\epsilon_i}$ in w_g with $a_{i-1}^{\epsilon_i}$ in the appropriate component. Notice that $|g_k| \leqslant |w_{g_k}|$ for every $k \in [0, m-1]$. Moreover, the words w_{g_k} are geodesic. Indeed,

$$\sum_{k=0}^{m-1} |g_k| \leqslant \sum_{k=0}^{m-1} |w_{g_k}| \leqslant |w_g| = |g| = \sum_{\ell=0}^{m-1} |g_\ell|,$$

which forces that $|w_{g_k}| = |g_k|$. Now, set

$$w_{\alpha_k} = \begin{cases} w_{g_k} w_{g_{k+1}} \cdots w_{g_{k+m-1}} & \text{if } \epsilon_0 = 1, \\ w_{g_k} w_{g_{k-1}} \cdots w_{g_{k-(m-1)}} & \text{if } \epsilon_0 = -1. \end{cases}$$

Clearly w_{α_k} represents α_k. Therefore $|\alpha_k| \leqslant |w_{\alpha_k}|$. Furthermore,

$$|w_{\alpha_k}| \leqslant \sum_{\ell=0}^{m-1} |w_{g_\ell}| = \sum_{\ell=0}^{m-1} |g_\ell| = |g| = |\alpha_k|.$$

Thus $|\alpha_k| = |w_{\alpha_k}|$ and w_{α_k} is a geodesic word.

Now, we claim that in order to prove the result, it suffices to consider the situation in which for every $i \in [0, s-1]$ there exists a unique $k \in [0, m-1]$ such that w_{g_k} contains a non-trivial power of a_i. First we consider the case when $i = 0$. Assume to the contrary that there exist distinct $k_1, k_2 \in [0, m-1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain non-trivial powers of a_0. We can reduce to the following two cases.

Case 1: Suppose that there exist distinct $k_1, k_2 \in [0, m-1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain a_0 and a_0^{-1} respectively. Then for some $k \in [0, m-1]$, the word w_{α_k} contains a subword of the form $a_0 w a_0^{-1}$ with $w \in S^*$. If $\alpha_k^m = (\beta_0, \dots, \beta_{m-1})$ then, by Lemma 9.3.2 and Lemma 9.3.3, we obtain that $|\beta_\ell| < |\alpha|$ for every $\ell \in [0, m-1]$. Again, the result follows by induction.

Case 2: Suppose there exist distinct $k_1, k_2 \in [0, m-1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain a_0. Recall that $G/G' = \langle a_0 G', \dots, a_{s-1} G' \rangle \cong \mathbb{Z}^s$. Hence, as the exponent sum of a_1 in any word representing g is ϵ_1, the exponent sum of a_0 in w_{α_k} is equal to ϵ_1 for all $k \in [0, m-1]$. This implies that there exists $k_3 \in [0, m-1]$ such that $w_{g_{k_3}}$ contains a_0^{-1}, and we are in the previous case. Analogously, the same argument works if both $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain a_0^{-1}.

We reduce to the case such that there exists a unique $k \in [0, m-1]$ such that w_{g_k} contains a non-trivial power of a_0. By inducting on $i \in [0, s-1]$, assume that there exists a unique $k \in [0, m-1]$ such that w_{g_k} contains a non-trivial power of a_{i-1}. Suppose that there exist distinct $k_1, k_2 \in [0, m-1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain non-trivial powers of a_i. We can find $k_3 \in [0, m-1]$ such that $w_{\alpha_{k_3}}$ contains a subword of the form $a_i^{\ell_1} w a_i^{\ell_2}$, where $\ell_1, \ell_2 \in \mathbb{Z}\backslash\{0\}$ and $w \in S^*$ with exponent sum of a_0 in w is not equal to $0 \bmod m$. Thanks to Lemma 9.4.2, we may replace g with α_{k_3}. Then we find more than one w_{g_k} containing non-trivial powers of a_{i-1}, contradicting the assumption and hence proving the claim.

Thus, we reduce to the situation in which for every $i \in [0, s-1]$ there exists a unique $k \in [0, m-1]$ such that w_{g_k} contains a non-trivial power of a_i. An easy computation yields that w_g does not contain a subword of the form $a_i^{\ell_1} w a_i^{\ell_2}$, for some $i \in [0, s-1]$ where $\ell_1, \ell_2 \in \mathbb{Z}\backslash\{0\}$ and $w \in S^*$ with the exponent sum of a_0 in w is not equal to $0 \bmod m$. Hence, we conclude that w_g must be of the form

$$w_1(a_{i_1}, \dots, a_{i_r}) a_0^{\epsilon_0} w_2(a_{i_{r+1}}, \dots, a_{i_{s-1}}).$$

where w_1 and w_2 are words in the given elements, and $\{i_1, \dots, i_r, i_{r+1}, \dots, i_{s-1}\} = [1, s-1]$ such that the intersection $\{i_1, \dots, i_r\} \cap \{i_{r+1}, \dots, i_{s-1}\}$ is empty. Consider the element α_{m-1} obtained from the element g above. Then the corresponding w_{α_k} has the form

$$w_1(a_{i_1-1}, \dots, a_{i_r-1}) a_{s-1}^{\epsilon_0} w_2(a_{i_{r+1}-1}, \dots, a_{i_{s-1}-1}),$$

and continuing the above procedure with this word, yields the element $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \in H_u$ for some u of level ns in T, for some $n \in \mathbb{N}_0$. □

Theorem 9.4.6. *If H is a prodense subgroup of G then $H = G$.*

Proof. Note that $HG' = G$ as H is a prodense subgroup. Therefore there exists an element $z \in G'$ such that $a_{s-1} \cdots a_0 z \in H$. By an application of Proposition 9.4.5, we can find $u \in T$ such that H_u contains $a_{(s-1)^\pi} \cdots a_{0^\pi}$ for some $\pi \in \mathrm{Sym}(s)$. We set $g = a_{(s-1)^\pi} \cdots a_{0^\pi}$.

Thanks to [40, Lemma 3.1], the subgroup H_u is again a prodense subgroup of G. Without loss of generality, we replace H with H_u.

Again, as H is prodense, for some $\tilde{z} \in G'$ we similarly have $a_{s-1} \cdots a_1 a_0^{-1} \tilde{z} \in H$. By Proposition 9.4.5, there exists a vertex u at level ns, for some $n \in \mathbb{N}$ such that H_u contains an element h_0 of the form

$$h_0 = a_{(s-1)^{\tau_0}}^{\epsilon_{(s-1)^{\tau_0}}} \cdots a_{0^{\tau_0}}^{\epsilon_{0^{\tau_0}}},$$

where $\tau_0 \in \mathrm{Sym}(s)$, with $\epsilon_{i^{\tau_0}} = -1$ if $i^{\tau_0} = 0$ and $\epsilon_{i^{\tau_0}} = 1$ otherwise. Now, by Lemma 9.4.4(i), the subgroup H_u also contains some cyclic permutation of the element g. By abuse of notation, we replace g with this cyclic permutation of g. We again replace H with H_u. Now H contains the elements g and h_0. Repeating this argument $s-1$ times, we may assume that H contains the elements $g, h_0, \ldots, h_{s-1}$, where

$$h_j = a_{(s-1)^{\tau_j}}^{\epsilon_{(s-1)^{\tau_j}}} \cdots a_{0^{\tau_j}}^{\epsilon_{0^{\tau_j}}},$$

where $\tau_j \in \mathrm{Sym}(s)$ with $\epsilon_{i^{\tau_j}} = -1$ if $i^{\tau_j} = j$ and $\epsilon_{i^{\tau_j}} = 1$ otherwise. Appealing to Lemma 9.4.4(ii), we now choose a vertex v, with v of level $\tilde{n}s$ for some $\tilde{n} \in \mathbb{N}$, such that the cyclic permutation of g that is contained in H_v ends with a_0 on the right. We rename this element g. So we have $g \in H_v$ and by Lemma 9.4.4(i) we have a cyclic permutation of each of the elements $h_0, \ldots, h_{s-1}$ in H_v. By abuse of notation, we rename these cyclic permutations $h_0, \ldots, h_{s-1}$ respectively. As before we replace H with H_v. Now H contains the elements $g, h_0, \ldots, h_{s-1}$, where g ends with a_0 on the right.

For each $n \in \mathbb{N}_0$, let $v_n = (m-1) \overset{n}{\cdots} (m-1)$ denote the right-most vertex at level n. It follows from Lemma 9.4.3 that for $d \in \mathbb{N}$ we have $\varphi_{v_{ds}}(g^{m^{ds}}) = g$ and $\varphi_{v_{ds}}(h_i^{m^{ds}}) = h_i$ where $i \in [0, s-1]$. Furthermore, for any element $f \in G$ of the form

$$f = a_{\iota_1}^{\epsilon_1} \cdots a_{\iota_t}^{\epsilon_t},$$

for pairwise distinct $\iota_1, \ldots, \iota_t \in [0, s-1]$ with $t \in [1, s]$ and $\epsilon_i \in \{\pm 1\}$, we can consider its contribution to H_{v_n}. Specifically, if $f \in \mathrm{St}_G(1)$, we simply consider its image under φ_{m-1}. If $f \notin \mathrm{St}_G(1)$, then we consider $\varphi_{m-1}(f^m)$. We refer to this general process as projecting along the right-most path. By projecting along the right-most path, we observe that if $f \in H_{v_j}$, for $j \in \mathbb{N}$, then $f \in H_{v_{j+s}}$; compare the proof of Lemma 9.4.3. This observation will be used repeatedly throughout the proof without special mention.

The strategy of the proof is now to consider the contributions from

$$\langle g \rangle, \langle h_0 \rangle, \ldots, \langle h_{s-1} \rangle$$

to H_{v_n}, and to multiply them appropriately to separate the generators $a_0, \ldots, a_{s-1}$. More specifically, if for some $n \in \mathbb{N}$, suppose we have non-trivial elements $\alpha, \beta \in H_{v_n}$ of the form

$$\alpha = a_{i_1}^{\epsilon_1} \cdots a_{i_q}^{\epsilon_q}, \qquad \beta = a_{j_1}^{\delta_1} \cdots a_{j_r}^{\delta_r}$$

where $\epsilon_i, \delta_j \in \{\pm 1\}$ and $2 \leqslant q, r \leqslant s$, with $i_1, \ldots, i_q \in [0, s-1]$ pairwise distinct, and also $j_1, \ldots, j_r \in [0, s-1]$ pairwise distinct. We consider two situations below, where we assume always that $\widetilde{\alpha}, \widetilde{\beta}$ are non-trivial.

(i) If $\alpha = \widetilde{\alpha} a_0$ and $\beta = \widetilde{\beta} a_0^{-1} \widehat{\beta}$, then

$$\beta\alpha = \widetilde{\beta} a_0^{-1} \widehat{\beta} \widetilde{\alpha} a_0$$

yields

$$\varphi_{m-1}(\widetilde{\beta}) \in H_{v_{n+1}},$$

and hence

$$\widetilde{\beta} \in H_{v_{n+s}} \qquad \text{and} \qquad a_0^{-1} \widehat{\beta} \in H_{v_{n+s}}.$$

(ii) If $\alpha = a_0 \widetilde{\alpha}$ and $\beta = \widehat{\beta} a_0^{-1} \widetilde{\beta}$, from

$$\alpha\beta = a_0 \widetilde{\alpha} \widehat{\beta} a_0^{-1} \widetilde{\beta},$$

we obtain

$$\varphi_{m-1}(\widetilde{\beta}) \in H_{v_{n+1}},$$

and similarly,

$$\widetilde{\beta} \in H_{v_{n+s}} \qquad \text{and} \qquad \widehat{\beta} a_0^{-1} \in H_{v_{n+s}}.$$

In other words, upon replacing H_{v_n} with $H_{v_{n+s}}$ we have split $\beta \in H_{v_{n+s}}$ into two non-trivial parts. The plan is to repeatedly perform such operations as in (i) and (ii) above to keep splitting products of generators. Eventually we will end up with $a_0, \ldots, a_{s-1} \in H_u$ for some u, which gives $H_u = G$ and equivalently that $H = G$, as required.

We begin by first considering the contributions from $\langle g \rangle$ and $\langle h_0 \rangle$ along the right-most path of the tree. For convenience, write

$$g = a_{i_1} \cdots a_{i_{s-1}} a_0 \qquad \text{and} \qquad h_0 = a_{j_1} \cdots a_{j_{d-1}} a_0^{-1} a_{j_{d+1}} \cdots a_{j_s},$$

for some $d \in [1, s]$, where $\{i_1, \ldots, i_{s-1}\} = \{j_1, \ldots, j_{d-1}, j_{d+1}, \ldots, j_s\} = [1, s-1]$.

<u>Case 1:</u> Suppose $1 < d < s$. Then we are in situation (i) from above, and it follows that

$$a_{j_1 - 1} \cdots a_{j_{d-1} - 1} \in H_{v_1} \quad \text{and} \quad a_{s-1}^{-1} a_{j_{d+1} - 1} \cdots a_{j_s - 1} \in H_{v_1}.$$

We will now use (the projections of) these two parts of h_0 to split g into two non-trivial parts.

Let $j := j_{d-1}$. We consider the contribution of $a_{j_1} \cdots a_{j_{d-1}}$ to H_{v_j}. In other words, we project along the right-most path down to level j, which gives

$$a_{j_1-j} \cdots a_{j_{d-2}-j} a_0 \in H_{v_j}.$$

Recalling that $g = a_{i_1} \cdots a_{i_{s-1}} a_0$, we have that $i_r = j$ for some $r \in [1, s-1]$. Then setting

$$\beta^{-1} := \varphi_{v_j}(g^{m^j}) = a_{i_1-j} \cdots a_{i_{r-1}-j} a_0 a_{i_{r+1}-j} \cdots a_{i_{s-1}-j} a_{s-j} \in H_{v_j}$$

and

$$\alpha := a_{j_1-j} \cdots a_{j_{d-2}-j} a_0 \in H_{v_j},$$

it follows from situation (i) that

$$a_{i_{r+1}-j-1} \cdots a_{i_{s-1}-j-1} a_{s-j-1} \in H_{v_{j+1}} \quad \text{and} \quad a_{i_1-j-1} \cdots a_{i_{r-1}-j-1} a_{s-1} \in H_{v_{j+1}},$$

so we have split g into two non-trivial parts.

We now use the two parts of g to split the parts of h_0 further. For clarity, let us first project to v_s. Here in H_{v_s} we have the elements

$$a_{j_1} \cdots a_{j_{d-2}} a_j, \qquad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \qquad a_{i_1} \cdots a_{i_{r-1}} a_j, \qquad a_{i_{r+1}} \cdots a_{i_{s-1}} a_0.$$

The left two elements are the two parts of h_0, and the right two are those of g. Without loss of generality, we replace H with H_{v_s}.

Subcase (a): Suppose $1 < r < s-1$. Let $k := i_1$. Then either $k = j_q$ for $q \in [1, d-2]$ or $k = j_q$ for $q \in [d+1, s]$. Suppose the former; a similar argument works for the latter. If $q > 1$, we let β^{-1} be the kth level projection of $a_{j_1} \cdots a_{j_{d-2}} a_j$ (as usual along the right-most path) and α be that of $a_k a_{i_2} \cdots a_{i_{r-1}} a_j$, which by (ii) gives, upon replacing H with H_{v_s}, the following elements in H:

$$a_{j_1} \cdots a_{j_{q-1}}, \quad a_k a_{j_{q+1}} \cdots a_{j_{d-2}} a_j, \quad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \quad a_k a_{i_2} \cdots a_{i_{r-1}} a_j, \quad a_{i_{r+1}} \cdots a_{i_{s-1}} a_0.$$

If $q = 1$, we have instead the following elements in H:

$$a_k a_{j_2} \cdots a_{j_{d-2}} a_j, \qquad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \qquad a_k a_{i_2} \cdots a_{i_{r-1}} a_j, \qquad a_{i_{r+1}} \cdots a_{i_{s-1}} a_0$$

Hence we let $\ell := i_{r+1}$ and let $c \in [2, d-2] \cup [d+1, s]$ be such that $j_c = \ell$. We consider the ℓth projection of $a_\ell a_{i_{r+2}} \cdots a_{i_{s-1}} a_0$ multiplied accordingly with that of $a_j^{-1} a_{j_{d-2}}^{-1} \cdots a_{j_2}^{-1} a_k^{-1}$ or $a_{j_s}^{-1} \cdots a_{j_{d+1}}^{-1} a_0$. This is situation (ii).

Subcase (b): Suppose $r = 1$. Then we have the following elements in H:

$$a_{j_1} \cdots a_{j_{d-2}}, \qquad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \qquad a_j, \qquad a_{i_2} \cdots a_{i_{s-1}} a_0.$$

If $i_2 \neq j_1$, let $k := j_1$, and we proceed according to (ii), with α being the kth projection of $a_k a_{j_2} \cdots a_{j_{d-2}}$ and β that of $(a_{i_2} \cdots a_{i_{s-1}} a_0)^{-1}$. If $i_2 = j_1$, we let instead $k := j_s$ and consider the kth projection of $(a_{i_2} \cdots a_{i_{s-1}} a_0)^{-1}$ multiplied with that of $a_0^{-1} a_{j_{d+1}} \cdots a_{j_{s-1}} a_k$; that is, situation (i).

Subcase (c): Suppose $r = s - 1$. Here we have the following elements in H:

$$a_{j_1} \cdots a_{j_{d-2}} a_j, \qquad a_{j_{d+1}} \cdots a_{j_s}, \qquad a_{i_1} \cdots a_{i_{s-2}} a_j, \qquad a_0.$$

If $i_1 \neq j_1$, we let $k := j_1$, and proceed as in (ii), taking α to be the kth projection of $a_k a_{j_2} \cdots a_{j_{d-2}} a_j$ and β that of $(a_{i_1} \cdots a_{i_{s-1}} a_0)^{-1}$. If $i_1 = j_1$, we instead let $k := j_{d+1}$ and likewise following (ii) we consider the kth level projection of $a_k a_{j_{d+2}} \cdots a_{j_s}$ multiplied with that of $\big(a_{i_1} \cdots a_{i_{s-2}} a_j\big)^{-1}$.

We aim to continue in this manner, using newly-formed parts of g to split the existing parts of h_0, and then using the newly-formed parts of h_0 to split the existing parts of g. Observe also that if a_i, for some $i \in [0, s-1]$, is an isolated part of g (that is, a part of g of length one), then using (i) or (ii), one can further split the parts of h_0 to isolate a_i from the parts of h_0. Indeed, if a_i or a_i^{-1} occurs as an endpoint of a part of h_0, then it is clear. If a_i is an interior point of a part $a_{r_1} \cdots a_{r_\xi} a_i a_{r_{\xi+1}} \cdots a_{r_{\xi+z}}$ of h_0, then projecting to the ith level, we have

$$\big(a_{r_1 - i} \cdots a_{r_\xi - i} a_0 a_{r_{\xi+1} - i} \cdots a_{r_{\xi+z} - i}\big) a_0^{-1} \in H_{v_i},$$

and thus

$$a_{s-1}, \qquad a_{r_1 - i - 1} \cdots a_{r_\xi - i - 1}, \qquad a_{r_{\xi+1} - i - 1} \cdots a_{r_{\xi+z} - i - 1}$$

are elements of $H_{v_{i+1}}$, giving

$$a_i, \qquad a_{r_1} \cdots a_{r_\xi}, \qquad a_{r_{\xi+1}} \cdots a_{r_{\xi+z}}$$

in $H_{v_{i+s}}$. As usual, we then replace H with $H_{v_{i+s}}$. We proceed similarly in the case when a_i^{-1} is an interior point in a part of h_0.

Hence we may assume that the set of length one parts of g is equal to the set of length one parts of h_0. Equivalently, the set of parts of g of length at least two involve the same generators that appear in the parts of h_0 of length at least two.

If there are no parts of length at least two, then all generators have been isolated, and we are done, so assume otherwise. Suppose for now that the parts of g of length at least two are labelled as follows:

$$a_{e_1} * \cdots * a_{f_1}, \quad a_{e_2} * \cdots * a_{f_2}, \quad \ldots \quad , \quad a_{e_\mu} * \cdots * a_{f_\mu},$$

for some $1 \leqslant \mu < s$, and similarly for h_0:

$$a_{p_1}^{\gamma_1} * \cdots * a_{q_1}^{\lambda_1}, \quad a_{p_2}^{\gamma_2} * \cdots * a_{q_2}^{\lambda_2}, \quad \ldots \quad , \quad a_{p_\nu}^{\gamma_\nu} * \cdots * a_{q_\nu}^{\lambda_\nu},$$

for some $1 \leqslant \nu < s$, with $\gamma_j = 1$ if $j \in [1, s-1]$ and $\gamma_j = -1$ if $j = 0$ and similarly for λ_j. Here $*$ stands for unspecified elements in the alphabet S. Write

$$\mathcal{E}_g = \{(a_{e_1}, a_{f_1}), \ldots, (a_{e_\mu}, a_{f_\mu})\}$$

for the set of ordered pairs of the so-called endpoint generators. If a_0 has not been isolated, it follows that the corresponding set $\mathcal{E}_{h_0}$ of endpoint generator pairs for h_0 is of the form

$$\mathcal{E}_{h_0} = \{(a_0^{-1}, a_{q_1}), (a_{p_2}, a_{q_2}), \ldots, (a_{p_\nu}, a_{q_\nu})\},$$

subject to reordering the parts of h_0. Indeed, else we may separate the parts further using (i). Without loss of generality, write

$$\mathcal{E}_g = \{(a_{e_1}, a_0), (a_{e_2}, a_{f_2}), \ldots, (a_{e_\mu}, a_{f_\mu})\}.$$

Note that if

$$\{p_2, \ldots, p_\nu\} \cup \{q_1, \ldots, q_\nu\} \neq \{e_1, \ldots, e_\mu\} \cup \{f_2, \ldots, f_\mu\},$$

we may proceed as in (i) or (ii), since then an endpoint from a part of g is an interior point in a part of h_0, or vice versa. Hence $\mu = \nu$ and

$$\{p_2, \ldots, p_\mu\} \cup \{q_1, \ldots, q_\mu\} = \{e_1, \ldots, e_\mu\} \cup \{f_2, \ldots, f_\mu\},$$

Since $\{p_2, \ldots, p_\mu\}$ has less elements than $\{e_1, \ldots, e_\mu\}$, it follows that $e_i \in \{q_1, \ldots, q_\mu\}$ for some $i \in [1, \mu]$. Then we proceed as in (ii). Hence, if a_0 is not an isolated part of g (equivalently of h_0), then we can continue splitting the parts of g and h_0.

So suppose now that a_0 has been isolated. As reasoned above, we have

$$\mathcal{E}_g = \{(a_{e_1}, a_{f_1}), \ldots, (a_{e_\mu}, a_{f_\mu})\}$$

and

$$\mathcal{E}_{h_0} = \{(a_{p_1}, a_{q_1}), \ldots, (a_{p_\mu}, a_{q_\mu})\}$$

with

$$\{e_1, \ldots, e_\mu\} \cup \{f_1, \ldots, f_\mu\} = \{p_1, \ldots, p_\mu\} \cup \{q_1, \ldots, q_\mu\}.$$

Similarly if

$$\{e_1,\ldots,e_\mu\}\cap\{q_1,\ldots,q_\mu\}\neq\varnothing,$$

we proceed as in (ii). So we assume that

$$\{e_1,\ldots,e_\mu\}=\{p_1,\ldots,p_\mu\}\quad\text{and}\quad\{f_1,\ldots,f_\mu\}=\{q_1,\ldots,q_\mu\}.$$

To proceed, we now consider the element h_{e_1} defined at the beginning of the proof. Proceeding as in (i) and (ii), we use the parts of g and h_0 to split h_{e_1} into parts, and if possible, we likewise use the parts of h_{e_1} to further split the parts of g and h_0. We claim that a_{e_1} has been isolated through this process. Indeed, analogously to the considerations above for when a_0 was assumed to be an endpoint in $\mathcal{E}_g$, if we have

$$\mathcal{E}_{h_{e_1}}=\{(a_{k_1},a_{e_1}^{-1}),(a_{k_2},a_{\ell_2}),\ldots,(a_{k_\eta},a_{\ell_\eta})\}$$

and

$$\mathcal{E}_g=\{(a_{e_1},a_{f_1}),(a_{e_2},a_{f_2}),\ldots,(a_{e_\eta},a_{f_\eta})\},$$

where here $\eta\geqslant\mu$, and by abuse of notation we still write e_i for the left endpoints and f_i for the right endpoints for the parts of g. Then, as seen before, there is some $f_i\in\{k_1,\ldots,k_\eta\}$ for $i\in[1,\eta]$, and we can proceed as in (i) or (ii). If instead $(a_{e_1}^{-1},a_{\ell_1})\in\mathcal{E}_{h_{e_1}}$ then we multiply the e_1th projection of $a_{e_1}*\cdots*a_{f_1}$ with that of $a_{e_1}^{-1}*\cdots*a_{\ell_1}$ as in (ii). Lastly, if $a_{e_1}^{-1}$ is an interior point in $\mathcal{E}_{h_{e_1}}$, then we proceed as in (ii). In other words, if a_{e_1} is not an isolated part of g (equivalently of h_0 and of h_{e_1}), then we can always continue splitting.

By abuse of notation, we redefine $\mathcal{E}_g$ to be the new set of endpoint pairs, after this further splitting of the parts of g. If $\mathcal{E}_g\neq\varnothing$, pick a left endpoint a_e for some $e\in\mathcal{E}_g$. From working in a similar manner with the element h_e, we can isolate a_e.

Proceeding in this manner, we will end up with all individual generators.

Case 2: Suppose $d=1$. Thus we have

$$g=a_{i_1}\cdots a_{i_{s-1}}a_0\quad\text{and}\quad h_0=a_0^{-1}a_{j_2}\cdots a_{j_s}.$$

Write $i:=i_1$ and let $r\in[2,s]$ be such that $j_r=i$. As in situation (ii), we consider instead the ith projection of g multiplied with that of h_0^{-1}. We now proceed as in Case 1 with the argument using the pairs of endpoints $\mathcal{E}_g$.

Case 3: Suppose $d=s-1$. Here we proceed first using (i), and then following the argument laid out in Case 1. □

Akin to [33, Proposition 6.6], one can show that the group G has a non-normal maximal subgroup of index q, for infinitely many primes q. Indeed, the group G has a proper quotient isomorphic to $W_m(\mathbb{Z})$, where for $\mathcal{G}$ a group and $m \in \mathbb{N}_{\geqslant 2}$, we write $W_m(\mathcal{G})$ for the wreath product of $\mathcal{G}$ with a cyclic group of order m. Writing $L = \psi_1^{-1}(G' \times \cdots \times G')$ and $N = L\langle a_0^m \rangle \trianglelefteq G$, analogous to [33, Lemma 6.4] we have that $G/N \cong W_m(\mathbb{Z}^{s-1})$, which has $W_m(\mathbb{Z})$ as a quotient group; compare also [92, Theorem 1.6(iv)].

Chapter 10

Appendix

Here we give a MAGMA code that produce first 500 terms of the representation zeta function of G_3. Furthermore, the MAGMA code computes a conjectural approximation to the true abscissa of convergence of the representation zeta function of G_3 based on the truncated representation zeta function of G'_3 with 500 terms.

```
clear;

Q := Rationals();

R<x> := PolynomialRing(Q);

a := R!6;    // alpha(s)

b := R!6;    // beta(s)

t := R!6;    // tau(s)

h := R!54;   // xi(s)

z := 3 + a + 2*b + t + h;

N := 500;

for i in [1..N] do

b1 := 6 + 3*x*Evaluate(t,x^3);

a1 := b1 + x^2*Evaluate(a + 2*b + h,x^3);

```

```
t1 := 6 + x*(18 + 9*t + 9*a + 18*b + 3*b^2)
        + x^2*(3^(-1)*2*b^3 + 3*b^2 + 3^(-1)*a*b^2 + 2*a*b + a^2
      + h*(3^(-1)*b^2 + 2*b + 3)
    + t*(3^(-1)*2*b^2 + 4*b + 2*a + 3^(-1)*a^2));
h1 := 54 + x*(t^3 + 9*t^2 + 18*t + 3*a*b^2 +18*a*b + 6*b^2+ 18*a + 36*b)
       + x^2 *( 9^(-1)*h^3 +  3^(-1)*h^2*a + 3^(-1)*2*h^2*b+ 3^(-1)*h^2*t + h^2
       + 3^(-1)*h*a^2 + 3^(-1)*4*h*a*b + 3^(-1)*2*h*a*t + 2*h*a + h*b^2 + 3^(-1)*4*h*b*t
       + 2*h*b +  3^(-1)*h*t^2 + 2*h*t + 9^(-1)*a^3 + 3^(-1)*2*a^2*b + 3^(-1)*2*a*b^2
       + 3^(-1)*4*a*b*t + 3^(-1)*a*t^2 + 9^(-1)*2*b^3 + 3^(-1)*2*b^2*t + 3^(-1)*2*b*t^2)
      - x*9*Evaluate(t,x^3)  -  x^2*Evaluate(a+2*b+h,x^3);
a := a1 mod x^(i+1);
b := b1 mod x^(i+1);
t := t1 mod x^(i+1);
h := h1 mod x^(i+1);
z := 3 + a + 2*b + t + h;
end for;
print "Log-Coefficients of zeta:";
C := Coefficients(z);
for i in [1..#C] do
```

```
 if i mod 50 eq 0 then
    s := 0;
     for j in [1..i] do
     s := s + C[i];
     end for;
  print i, Log(3^(i-1),s);
 end if;
end for;
// zeta function for the Gupta-Sidki 3-group:
Z := 9 + 2*x + (a + 2*b + t)*x + h*1/9*x^2;
print "Zeta function for the Gupta-Sidki 3-group:";
print Coefficients(Z);
```

In the table below (Table 10.1; see next page), we record the conjectural approximation to the true abscissa of convergence of the representation zeta function of G_3 obtained from the above MAGMA code. Let $C[i]$ be the i-th coefficient of the truncated representation zeta function Z of the Gupta–Sidki 3-group obtained by the above MAGMA code. We set

$$R_N = \sum_{i=1}^{N} C[i], \qquad \text{and} \qquad \alpha_N = \frac{\log R_N}{\log 3^{N-1}},$$

for $n \in \mathbb{N}$.

N = No. coefficients	α_N
50	4.2880948264420582761842700l965
100	4.2658264169513132083710567l922
150	4.2594112249288642787580557391 5
200	4.2564944917571904544034440719 6
250	4.2548688041259991980478480589 3
300	4.2538496680367445673457160928 5
350	4.2531596297289368048361452940 2
400	4.2526662410008054123409900995 7
450	4.2522988127702424621699439953 8
500	4.2520164176494705125318443887 9

Table 10.1: Conjectural approximation to the true abscissa of convergence of the representation zeta function of G_3.

Bibliography

[1] A. G. Abercrombie, *Subgroups and subrings of profinite rings*, Math. Proc. Cambridge Philos. Soc. **116** (1994), no. 2, 209–222, DOI 10.1017/S0305004100072522. MR1281541

[2] Miklós Abért, *Representing graphs by the non-commuting relation*, Publ. Math. Debrecen **69** (2006), no. 3, 261–269. MR2273978

[3] Miklós Abért and Bálint Virág, *Dimension and randomness in groups acting on rooted trees*, J. Amer. Math. Soc. **18** (2005), no. 1, 157–192, DOI 10.1090/S0894-0347-04-00467-9. MR2114819

[4] S. V. Alešin, *Finite automata and the Burnside problem for periodic groups*, Mat. Zametki **11** (1972), 319–328 (Russian). MR301107

[5] Theofanis Alexoudas, Benjamin Klopsch, and Anitha Thillaisundaram, *Maximal subgroups of multi-edge spinal groups*, Groups Geom. Dyn. **10** (2016), no. 2, 619–648, DOI 10.4171/GGD/359. MR3513111

[6] Nir Avni, Benjamin Klopsch, Uri Onn, and Christopher Voll, *Representation zeta functions of some compact p-adic analytic groups*, Zeta functions in algebra and geometry, Contemp. Math., vol. 566, Amer. Math. Soc., Providence, RI, 2012, pp. 295–330, DOI 10.1090/conm/566/11226. MR2858928

[7] ———, *Representation zeta functions of compact p-adic analytic groups and arithmetic groups*, Duke Math. J. **162** (2013), no. 1, 111–197, DOI 10.1215/00127094-1959198. MR3011874

[8] H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. (3) **25** (1972), 603–614, DOI 10.1112/plms/s3-25.4.603. MR379672

[9] Yiftach Barnea and Aner Shalev, *Hausdorff dimension, pro-p groups, and Kac-Moody algebras*, Trans. Amer. Math. Soc. **349** (1997), no. 12, 5073–5091, DOI 10.1090/S0002-9947-97-01918-1. MR1422889

[10] Laurent Bartholdi, *Croissance de groupes agissant sur des arbres*, PhD Thesis, University of Geneva, 2000.

[11] ———, *Endomorphic presentations of branch groups*, J. Algebra **268** (2003), no. 2, 419–443, DOI 10.1016/S0021-8693(03)00268-0. MR2009317

[12] ———, *Branch rings, thinned rings, tree enveloping rings*, Israel J. Math. **154** (2006), 93–139, DOI 10.1007/BF02773601. MR2254535

[13] Laurent Bartholdi and Floriane Pochon, *On growth and torsion of groups*, Groups Geom. Dyn. **3** (2009), no. 4, 525–539, DOI 10.4171/GGD/68. MR2529946

[14] Laurent Bartholdi, *Representation zeta functions of self-similar branched groups*, Groups Geom. Dyn. **11** (2017), no. 1, 29–56, DOI 10.4171/GGD/386. MR3641831

[15] Laurent Bartholdi, Bettina Eick, and René Hartung, *A nilpotent quotient algorithm for certain infinitely presented groups and its applications*, Internat. J. Algebra Comput. **18** (2008), no. 8, 1321–1344, DOI 10.1142/S0218196708004871. MR2483125

[16] Laurent Bartholdi and Rostislav I. Grigorchuk, *On the spectrum of Hecke type operators related to some fractal groups*, Tr. Mat. Inst. Steklova **231** (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 5–45; English transl., Proc. Steklov Inst. Math. **4(231)** (2000), 1–41. MR1841750

[17] ———, *On parabolic subgroups and Hecke algebras of some fractal groups*, Serdica Math. J. **28** (2002), no. 1, 47–90. MR1899368

[18] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić, *Branch groups*, Handbook of algebra, Vol. 3, Handb. Algebr., vol. 3, Elsevier/North-Holland, Amsterdam, 2003, pp. 989–1112, DOI 10.1016/S1570-7954(03)80078-5. MR2035113

[19] Laurent Bartholdi and Pierre de la Harpe, *Representation zeta functions of wreath products with finite groups*, Groups Geom. Dyn. **4** (2010), no. 2, 209–249, DOI 10.4171/GGD/81. MR2595090

[20] Laurent Bartholdi, Vadim A. Kaimanovich, and Volodymyr V. Nekrashevych, *On amenability of automata groups*, Duke Math. J. **154** (2010), no. 3, 575–598, DOI 10.1215/00127094-2010-046. MR2730578

[21] Laurent Bartholdi and Volodymyr V. Nekrashevych, *Iterated monodromy groups of quadratic polynomials. I*, Groups Geom. Dyn. **2** (2008), no. 3, 309–336, DOI 10.4171/GGD/42. MR2415302

[22] Laurent Bartholdi, Olivier Siegenthaler, and Pavel Zalesskii, *The congruence subgroup problem for branch groups*, Israel J. Math. **187** (2012), 419–450, DOI 10.1007/s11856-011-0086-5. MR2891709

[23] Laurent Bartholdi and Zoran Šunić, *On the word and period growth of some groups of tree automorphisms*, Comm. Algebra **29** (2001), no. 11, 4923–4964, DOI 0.1081/AGB-100106794. MR1856923

[24] Laurent Bartholdi and Bálint Virág, *Amenability via random walks*, Duke Math. J. **130** (2005), no. 1, 39–56, DOI 10.1215/S0012-7094-05-13012-5. MR2176547

[25] H. Bass, M. Lazard, and J.-P. Serre, *Sous-groupes d'indice fini dans* $\mathbf{SL}(n, \mathbf{Z})$, Bull. Amer. Math. Soc. **70** (1964), 385–392, DOI 10.1090/S0002-9904-1964-11107-1 (French). MR161913

[26] Hyman Bass, Alexander Lubotzky, Andy R. Magid, and Shahar Mozes, *The proalgebraic completion of rigid groups*, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000), 2002, pp. 19–58, DOI 10.1023/A:1021221727311. MR1950883

[27] Ievgen V. Bondarenko, *Finite generation of iterated wreath products*, Arch. Math. (Basel) **95** (2010), no. 4, 301–308, DOI 10.1007/s00013-010-0169-2. MR2727305

[28] Jérémie Brieussel, *Croissance et moyennabilité de certains groupes d'automorphismes d'un arbre enraciné*, PhD thesis, Université Paris Diderot - Paris 7, 2008 (English).

[29] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) **42** (1996), no. 3-4, 215–256. MR1426438

[30] Ching Chou, *Elementary amenable groups*, Illinois J. Math. **24** (1980), no. 3, 396–407. MR573475

[31] Mahlon M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544. MR92128

[32] Daniele D'Angeli, Alfredo Donno, Michel Matter, and Tatiana Nagnibeda, *Schreier graphs of the Basilica group*, J. Mod. Dyn. **4** (2010), no. 1, 167–205, DOI 10.3934/jmd.2010.4.167. MR2643891

[33] Elena Di Domenico, Gustavo A. Fernández-Alcober, Marialaura Noce, and Anitha Thillaisundaram, *p-Basilica groups* (2021), available at arXiv:2105.12443[math.GR].

[34] Artem Dudko and Rostislav I. Grigorchuk, *On irreducibility and disjointness of Koopman and quasi-regular representations of weakly branch groups*, Modern theory of dynamical systems, Contemp. Math., vol. 692, Amer. Math. Soc., Providence, RI, 2017, pp. 51–66, DOI 10.1090/conm/692. MR3666066

[35] Jacek Fabrykowski and Narain Gupta, *On groups with sub-exponential growth functions*, J. Indian Math. Soc. (N.S.) **49** (1985), no. 3-4, 249–256 (1987). MR942349

[36] Kenneth Falconer, *Fractal geometry*, John Wiley & Sons, Ltd., Chichester, 1990. Mathematical foundations and applications. MR1102677

[37] Gustavo A. Fernández-Alcober, Alejandra Garrido, and Jone Uria-Albizuri, *On the congruence subgroup property for GGS-groups*, Proc. Amer. Math. Soc. **145** (2017), no. 8, 3311–3322, DOI 10.1090/proc/13499. MR3652785

[38] Gustavo A. Fernández-Alcober and Amaia Zugadi-Reizabal, *GGS-groups: order of congruence quotients and Hausdorff dimension*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 1993–2017, DOI 10.1090/S0002-9947-2013-05908-9. MR3152720

[39] Dominik Francoeur, *On maximal subgroups and other aspects of branch groups*, PhD thesis, University of Geneva, 2019.

[40] ———, *On maximal subgroups of infinite index in branch and weakly branch groups*, J. Algebra **560** (2020), 818–851, DOI 10.1016/j.jalgebra.2020.06.005. MR4114190

[41] Dominik Francoeur and Alejandra Garrido, *Maximal subgroups of groups of intermediate growth*, Adv. Math. **340** (2018), 1067–1107, DOI 10.1016/j.aim.2018.10.026. MR3886188

[42] Dominik Francoeur and Anitha Thillaisundaram, *Maximal subgroups of non-torsion Grigorchuk-Gupta-Sidki groups*, Canad. Math. Bull., to appear, available at arXiv:2005.02346[math.GR].

[43] Alejandra Garrido, *Aspects of branch groups*, PhD thesis, University of Oxford, 2015.

[44] ———, *Abstract commensurability and the Gupta-Sidki group*, Groups Geom. Dyn. **10** (2016), no. 2, 523–543, DOI 10.4171/GGD/355. MR3513107

[45] ———, *On the congruence subgroup problem for branch groups*, Israel J. Math. **216** (2016), no. 1, 1–13, DOI 10.1007/s11856-016-1402-x. MR3556961

[46] Alejandra Garrido and Jone Uria-Albizuri, *Pro-$\mathcal{C}$ congruence properties for groups of rooted tree automorphisms*, Arch. Math. (Basel) **112** (2019), no. 2, 123–137, DOI 10.1007/s00013-018-1278-6. MR3908831

[47] Piotr W. Gawron, Volodymyr V. Nekrashevych, and Vitaly I. Sushchansky, *Conjugation in tree automorphism groups*, Internat. J. Algebra Comput. **11** (2001), no. 5, 529–547, DOI 10.1142/S021819670100070X. MR1869230

[48] E. S. Golod, *On nil-algebras and finitely approximable p-groups*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 273–276 (Russian). MR0161878

[49] E. S. Golod and I. R. Šafarevič, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 261–272 (Russian). MR0161852

[50] Rostislav I. Grigorchuk, *Symmetric random walks on discrete groups*, Uspehi Mat. Nauk **32** (1977), no. 6(198), 217–218 (Russian). MR0474511

[51] ———, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 53–54 (Russian). MR565099

[52] ———, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985 (Russian). MR764305

[53] ———, *An example of a finitely presented amenable group that does not belong to the class EG*, Mat. Sb. **189** (1998), no. 1, 79–100, DOI 10.1070/SM1998v189n01ABEH000293 (Russian, with Russian summary); English transl., Sb. Math. **189** (1998), no. 1-2, 75–95. MR1616436

[54] ———, *Just infinite branch groups*, New horizons in pro-p groups, Progr. Math., vol. 184, Birkhäuser Boston, Boston, MA, 2000, pp. 121–179. MR1765119

[55] Rostislav I. Grigorchuk and Dima Grigoriev, *Key agreement based on automaton groups*, Groups Complex. Cryptol. **11** (2019), no. 2, 77–81, DOI 10.1515/gcc-2019-2012. MR4026474

[56] Rostislav I. Grigorchuk and John S. Wilson, *The uniqueness of the actions of certain branch groups on rooted trees*, Geom. Dedicata **100** (2003), 103–116, DOI 10.1023/A:1025851804561. MR2011117

[57] Rostislav I. Grigorchuk and Andrzej Żuk, *The lamplighter group as a group generated by a 2-state automaton, and its spectrum*, Geom. Dedicata **87** (2001), no. 1-3, 209–244, DOI 10.1023/A:1012061801279. MR1866850

[58] ———, *Spectral properties of a torsion-free weakly branch group defined by a three state automaton*, Computational and statistical group theory (Las Vegas, NV/Hoboken, NJ, 2001), Contemp. Math., vol. 298, Amer. Math. Soc., Providence, RI, 2002, pp. 57–82, DOI 10.1090/conm/298/05114. MR1929716

[59] ———, *On a torsion-free weakly branch group defined by a three state automaton*, Internat. J. Algebra Comput. **12** (2002), no. 1-2, 223–246, DOI 10.1142/S0218196702001000. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000). MR1902367

[60] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73. MR623534

[61] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), no. 1, 185–223, DOI 10.1007/BF01393692. MR943928

[62] Yves Guivarc'h, *Groupes de Lie à croissance polynomiale*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A1695–A1696 (French). MR302819

[63] Narain Gupta and Saïd Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), no. 3, 385–388, DOI 10.1007/BF01179757. MR696534

[64] Peter Hilton and Jean Pedersen, *Catalan numbers, their generalization, and their uses*, Math. Intelligencer **13** (1991), no. 2, 64–75, DOI 10.1007/BF03024089. MR1098222

[65] I. Martin Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)]. MR1280461

[66] A. Jaikin-Zapirain, *Zeta function of representations of compact p-adic analytic groups*, J. Amer. Math. Soc. **19** (2006), no. 1, 91–118, DOI 10.1090/S0894-0347-05-00501-1. MR2169043

[67] Irving Kaplansky, *"Problems in the theory of rings" revisited*, Amer. Math. Monthly **77** (1970), 445–454, DOI 10.2307/2317376. MR258865

[68] Steffen Kionke, *Groups acting on rooted trees and their representations on the boundary*, J. Algebra **528** (2019), 260–284, DOI 10.1016/j.jalgebra.2019.03.023. MR3933948

[69] Benjamin Klopsch and Claas Röver, *Representation growth of fractal branch groups*. Unpublished manuscript.

[70] Benjamin Klopsch and Anitha Thillaisundaram, *Maximal subgroups and irreducible representations of generalized multi-edge spinal groups*, Proc. Edinb. Math. Soc. (2) **61** (2018), no. 3, 673–703, DOI 10.1017/s0013091517000451. MR3834728

[71] T. Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 1991. MR1125071

[72] Alexander Lubotzky, Avinoam Mann, and Dan Segal, *Finitely generated groups of polynomial subgroup growth*, Israel J. Math. **82** (1993), no. 1-3, 363–371, DOI 10.1007/BF02808118. MR1239055

[73] Alexander Lubotzky and Benjamin Martin, *Polynomial representation growth and the congruence subgroup problem*, Israel J. Math. **144** (2004), 293–316, DOI 10.1007/BF02916715. MR2121543

[74] I. G. Lysënok, *A set of defining relations for the Grigorchuk group*, Mat. Zametki **38** (1985), no. 4, 503–516, 634 (Russian). MR819415

[75] Jens L. Mennicke, *Finite factor groups of the unimodular group*, Ann. of Math. (2) **81** (1965), 31–37, DOI 10.2307/1970380. MR171856

[76] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005. MR2162164

[77] ———, *Iterated monodromy groups*, Groups St Andrews 2009 in Bath. Volume 1, London Math. Soc. Lecture Note Ser., vol. 387, Cambridge Univ. Press, Cambridge, 2011, pp. 41–93. MR2858850

[78] ———, *Palindromic subshifts and simple periodic groups of intermediate growth*, Ann. of Math. (2) **187** (2018), no. 3, 667–719, DOI 10.4007/annals.2018.187.3.2. MR3779956

[79] J. Neumann, *Zur allgemeinen Theorie des Masses*, Fundamenta Mathematicae **13** (1929), no. 1, 73-116 (German).

[80] A. Yu. Ol′shanskii, *An infinite simple torsion-free Noetherian group*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 6, 1328–1393 (Russian). MR567039

[81] ———, *On the question of the existence of an invariant mean on a group*, Uspekhi Mat. Nauk **35** (1980), no. 4(214), 199–200 (Russian). MR586204

[82] ———, *An infinite group with subgroups of prime orders*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 2, 309–321, 479 (Russian). MR571100

[83] A. Yu. Ol′shanskii and Mark V. Sapir, *Non-amenable finitely presented torsion-by-cyclic groups*, Publ. Math. Inst. Hautes Études Sci. **96** (2002), 43–169 (2003). MR1985031

[84] D. S. Passman, *The semiprimitivity of group algebras*, Methods in ring theory (Levico Terme, 1997), Lecture Notes in Pure and Appl. Math., vol. 198, Dekker, New York, 1998, pp. 199–211. MR1767979

[85] D. S. Passman and W. V. Temple, *Representations of the Gupta-Sidki group*, Proc. Amer. Math. Soc. **124** (1996), no. 5, 1403–1410, DOI 10.1090/S0002-9939-96-03241-8. MR1307556

[86] E. L. Pervova, *Everywhere dense subgroups of a group of tree automorphisms*, Tr. Mat. Inst. Steklova **231** (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 356–367 (Russian, with Russian summary); English transl., Proc. Steklov Inst. Math. **4(231)** (2000), 339–350. MR1841763

[87] ______, *Maximal subgroups of some non locally finite p-groups*, Internat. J. Algebra Comput. **15** (2005), no. 5-6, 1129–1150, DOI 10.1142/S0218196705002803. MR2197824

[88] ______, *Profinite completions of some groups acting on trees*, J. Algebra **310** (2007), no. 2, 858–879, DOI 10.1016/j.jalgebra.2006.11.023. MR2308183

[89] Jan Moritz Petschick, *On conjugacy of GGS-groups*, J. Group Theory **22** (2019), no. 3, 347–358, DOI 10.1515/jgth-2018-0137. MR3943343

[90] ______, *On the derived subgroups of* GGS*-groups*. In preparation.

[91] J. Moritz Petschick, *Two periodicity conditions for spinal groups* (2022), available at arXiv:2112.12428[math.GR].

[92] Jan Moritz Petschick and Karthika Rajeev, *On the Basilica operation*, Groups Geom. Dyn., to appear, available at arXiv:2103.05452[math.GR].

[93] Richard Pink, *Profinite iterated monodromy groups arising from quadratic polynomials* (2013), available at arXiv:1307.5678[math.GR].

[94] Karthika Rajeev and Anitha Thillaisundaram, *Maximal subgroups of generalised Basilica groups*, available at arXiv:2109.12503[math.GR].

[95] Derek J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996. MR1357169

[96] Hanna Sasse, *Basilica-Gruppen und ihre Wirkung auf p-regulären Bäumen*, Master Thesis, Heinrich-Heine-Universität Düsseldorf, 2018 (German).

[97] Said Sidki, *A primitive ring associated to a Burnside* 3*-group*, J. London Math. Soc. (2) **55** (1997), no. 1, 55–64, DOI 10.1112/S0024610796004644. MR1423285

[98] Olivier Siegenthaler, *Hausdorff dimension of some groups acting on the binary tree*, J. Group Theory **11** (2008), no. 4, 555–567, DOI 10.1515/JGT.2008.034. MR2429355

[99] Rachel Skipper, *The congruence subgroup problem for a family of branch groups*, Internat. J. Algebra Comput. **30** (2020), no. 2, 397–418, DOI 10.1142/S0218196720500046. MR4077417

[100] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282

[101] Zoran Šunić, *Hausdorff dimension in a family of self-similar groups*, Geom. Dedicata **124** (2007), 213–236, DOI 10.1007/s10711-006-9106-8. MR2318546

[102] B. Sury, *The congruence subgroup problem*, Texts and Readings in Mathematics, vol. 24, Hindustan Book Agency, New Delhi, 2003. An elementary approach aimed at applications. MR1978430

[103] Jone Uria-Albizuri, *On the concept of fractality for groups of automorphisms of a regular rooted tree*, Reports@SCM **2** (2016), no. 1, 33–44.

[104] Ana Cristina Vieira, *On the lower central series and the derived series of the Gupta-Sidki* 3*-group*, Comm. Algebra **26** (1998), no. 4, 1319–1333, DOI 10.1080/00927879808826201. MR1612177

[105] Taras Vovkivsky, *Infinite torsion groups arising as generalizations of the second Grigorchuk group*, Algebra (Moscow, 1998), de Gruyter, Berlin, 2000, pp. 357–377. MR1754681

[106] Herbert S. Wilf, *generatingfunctionology*, 2nd ed., Academic Press, Inc., Boston, MA, 1994. MR1277813

[107] John S. Wilson, *Groups with every proper quotient finite*, Proc. Cambridge Philos. Soc. **69** (1971), 373–391, DOI 10.1017/s0305004100046818. MR274575

[108] ———, *Structure theory for branch groups*, Geometric and cohomological methods in group theory, London Math. Soc. Lecture Note Ser., vol. 358, Cambridge Univ. Press, Cambridge, 2009, pp. 306–320. MR2605182

Eidesstattliche Erklärung

Ich versichere an Eides statt, dass diese Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der “Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Univeristät Düsseldorf” erstellt worden ist.

Karthika Rajeev

Düsseldorf, Februar 2022