ON CERTAIN ASPECTS OF THE ASYMPTOTICS OF THE HOLOMORPHIC TORSION FORMS

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Zusammenfassung

Das Ziel der vorliegenden Arbeit ist die Untersuchung des asymptotischen Verhaltens der holomorphen analytischen Torsionsformen und ihre äquivariante Version bezüglich höhere Potenzen eines faserweisen positiven Linienbündels. Wir beweisen, dass die asymptotische Entwicklung der holomorphen analytischen Torsionsformen vom Grad $2k$ aus Termen der Form $p^{k+n-i}\log p, p^{k+n-i}$, $i \in \mathbf{N}_0$, und lokale Koeffizienten, wobei n die komplexe Dimension der Fasern ist. Für den Fall, dass das Familienvektorbündel aus einem Prinzipalbündel entsteht, stellen wir eine konkret Formel für die ersten Koeffizienten in der Asymptotik des Wärmeleitungskerns der Krümmung des Bismut-Superzusammenhangs dar. Die angegebenen Ergebnisse sind Familienversionen von Resultaten von Finski. Zusätzlich studieren wir das asymptotische Verhalten der äquivarianten holomorphen analytischen Torsionsformen und verallgemeinern einen Resultat von Puchol für den äquivarianten Fall.

Abstract

The purpose of this thesis is to investigate the asymptotic behavior of the holomorphic analytic torsion forms and its equivariant version associated with increasing powers p of a given fibrewise positive line bundle. We prove that the asymptotic expansion of the holomorphic analytic torsion forms of degree $2k$ consists of terms of the form $p^{k+n-i} \log p, p^{k+n-i}$, $i \in \mathbf{N}_0$, and local coefficients where n is the complex dimension of the fibres. For the case that when the familiy of vector bundles arise from a principle bundle we give concrete formulas for the first coefficients in the asymptotic of the heat kernel of the curvature of the Bismut superconnection. These results are family versions of the results of Finski. We also study the asymptotic behavior of the equivariant holomorphic analytic torsion forms and generalize a result of Puchol for the equivariant case.

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Introduction

The holomorphic analytic torsion was introduced in 1973 by Ray and in Singer in [RS73] as an analogue of the real analytic torsion. It is a positive real number associated with the spectrum of the Kodaira Laplacian of holomorphic vector bundles on a compact complex manifold. In the works of Bismut-Gillet-Soulé ([BGS88a],[BGS88b],[BGS88c]) regarding derterminant bundles they showed that it gives rise to a metric with the desired properties. Another application lies in Arakelov geometry where Gillet-Soulé proved an arithmetic Grothendieck-Riemann-Roch Theorem in which the holomorphic analytic torsion appers ([GS92]). One of the main difficulties was the compatibility with immersions which was provided in [BL91] using heavy analytical techniques. In [BV90] Bismut and Vasserot studied the asymptotic of the holomorphic torsion associated with increasing powers of a positive line bundle and extended it in [BV90] by replacing the line bundle by the symmetric powers of a Griffiths-positive vector bundle. In context to Arakelov geometry Gillet-Soulé used the asymptotic expansion for a result on arithmetic ampleness ([GS92]). The result of [BV90] has been sharpened by Finski in [F18] where he proved a formula for the full asymptotic of the holomorphic analytic torsion. In the same paper Finski also generalized his result for proper orbifold vector bundles over compact effective orbifolds.

In [BK92] Bismut and Köhler introduced the holomorphic analytic torsion form which is an extension of the holomorphic analytic torsion to the family setting. The generalization of the immersion formula for the holomorphic torsion in [BL91] to the family case has been done in [Bi97] which is also referred as Bismut's immersion theorem. With the provided analytical tool of the holomorphic analytic torsion form the arithmetic Grothendieck-Riemann-Roch Theorem for higher degrees was established, see [Kö05] and [GRS08].

In [P16] Puchol gave an asymptotic formula for the holomorphic analytic torsion forms which generalizes the result of Bismut-Vasserot [BV90] for the family setting where the line bundle is fibrewise positive. In contrast to the proof of [Bi87, Theorem 1.5] where probabilistic methods were used and which is cited in [BV90], in [P16] Puchol relies on localisation techniques introduced by Bismut-Lebeau in [BL91] by using finite propagation speed of the wave equation. See also [Ma00, Appendix D].

The holomorphic analytic torsion has an equivariant version in introduced in [Kö93]. In [KR1] Köhler-Rössler used the equivariant holomorphic analytic torsion in their work on a Lefschetz type fixed point formula in equivariant Arakelov geometry. Similary there is a generalizaion for the family case called equivariant holomorphic analytic torsion form introduced in [Ma00]. For its applications in Arakelov Geometry see for instance [Kö05].

In this thesis we study the asymptotics of the holomorphic analytic torsions and its equivariant extension. Let us describe in more detail what new results this study provides.

Our first goal is to sharpen Puchol's asymptotic formula ([P16, Theorem 0.3]) for the holomorphic analytic torsion form in the same fashion as Finksi did in degree zero ([F18]). Let $M \xrightarrow{\pi} B$ be a holomorphic fibre bundle with compact fibre Z. Let ω^M be a real $(1,1)$-form on M such that (π, ω^M) defines a Kähler fibration. Let $(\mathcal{E}, h^{\mathcal{E}})$ respectively $(\mathcal{L}, h^{\mathcal{L}})$ be a holomorphic Hermitian vector respectively line bundle on M. Let $\Omega^{\mathcal{L}}$ denote the curvature of the Hermitian holomorphic connection of $(\mathcal{L}, h^{\mathcal{L}})$. Let $T_{\mathbf{C}}Z$ be the complexification of the real tangent bundle of Z and $T^{1,0}Z \subset T_{\mathbf{C}}Z$ the i-eigenbundle of the complex structure of the fibre. We make the assumption that $i\Omega^{\mathcal{L}}$ is positive along the fibres, that is, for any $0 \neq U \in T^{1,0}Z$, we have

$$\Omega^{\mathcal{L}}(U, \overline{U}) > 0.$$

For $p \in \mathbf{N}$ we write $\mathcal{L}^p := \mathcal{L}^{\otimes p}$ for the p^{th} tensor product of $\mathcal{L}$. Let $h^{\mathcal{E}\otimes\mathcal{L}^p}$ be the metric on $\mathcal{E} \otimes \mathcal{L}^p$ induced by $h^{\mathcal{E}}$ and $h^{\mathcal{L}}$. We make the assumption that the direct image $R^i\pi_*(\mathcal{E} \otimes \mathcal{L}^p)$ is locally free for p large. For $u > 0$ let ψ_u be the linear map which multiplies a section with degree k in $\Lambda^\bullet T^*_{\mathbf{C}}B$ by u^k. Let $B_{p,u}$ be the Bismut superconnection associated to the Kähler fibration (π, ω^M) and the Hermitian vector bundle $(\mathcal{E} \otimes \mathcal{L}^p, h^{\mathcal{E}\otimes\mathcal{L}^p})$. Let $\langle \cdot \,|\exp(-B^2_{p,u})|\, \cdot \rangle$ be its heat kernel with respect to the fibrewise volume forms induced by ω^M. Set $Z_b = \pi^{-1}\{b\}$ and let $\mathcal{E}_b$ be the restriction of $\mathcal{E}$ over Z_b. Our first main result is the full asymptotic expansion of the heat kernel.

Theorem 1. *Let $b \in B$, $z \in Z_b$ and $m \in \mathbf{N}_0$. There exist $a_{i,u} \in \Gamma(Z_b, \Lambda^\bullet(T^*_{\mathbf{R},b}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*Z_b) \otimes \mathcal{E}_b))$ with $i \in \mathbf{N}_0$ such that for every $u > 0$ and $l \in \mathbf{N}_0$ we have as $p \to \infty$*

$$\langle z|\psi_{1/\sqrt{p}} \exp(-B^2_{p,u/p})|z\rangle = \sum_{i=0}^{l} a_{i,u}(z)p^{n-i} + O(p^{n-l-1})$$

for the $\mathscr{C}^m$-norm in the parameter $(b,z) \in M$ and uniform in u as u varies in a compact subset of $]0,\infty[$.

Now let $T(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})$ be the associated holomorphic analytic torsion form of [BK92]. For a differential form α on B we denote by $\alpha^{(k)}$ its component of degree k. By the term "local coefficients" we will mean quantities which can be expressed as an integral of a density defined locally over Z. Our second result is the following.

Theorem 2. *Let $k \in \{0, \ldots, \dim_{\mathbf{C}} B\}$. There are differential forms α_i, β_i on B which are local coefficients such that for any $l \in \mathbf{N}_0$ the component of degree $2k$ of the analytic torsion forms* *has the following asymptotic as $p \to \infty$:*

$$T(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})^{(2k)} = \sum_{i=0}^{l} p^{k+\dim_{\mathbf{C}} Z - i}(\alpha_i \log p + \beta_i)^{(2k)} + o(p^{k+\dim_{\mathbf{C}} Z - l})$$

in the topology of $\mathscr{C}^\infty$ convergence on compact subsets of B.

Theorem 2 has been proven in [F18, Theorem 1.1] when B is a point and in [P16, Theorem 0.3] for $l = 0$ where Puchol gave an explicit formula for α_0 and β_0.

Our next main result is the calculation of the first coefficient $a_{1,u}$ in Theorem 1. We restrict ourself to a situation where the geometry arises from a principal bundle. Let us describe it in more detail.
Let (Z, ω^Z) be a compact Kähler manifold with complex dimension n. Let (E, h^E) (respectively (L, h^L)) be holomorphic Hermitian vector (respectively line) bundle on Z. We make the assumption that (L, h^L) is positive, i.e. for any $0 \neq U \in T^{1,0}Z$, we have $\Omega^L(U, \overline{U}) > 0$ where Ω^L denotes the curvature of the Hermitian holomorphic connection of (L, h^L). Let G be a compact Lie group acting holomorphically on (Z, ω^Z) preserving ω^Z and assume that this action lifts to (E, h^E) and (L, h^L) preserving h^E and h^L. We also assume that the action of G on (Z, ω^Z) is Hamiltonian with moment map μ. The complexification $G_{\mathbf{C}}$ acts holomorphically on Z and the action lifts to an action on E and L. Let $p : P \overset{G_{\mathbf{C}}}{\to} B$ be a $G_{\mathbf{C}}$ principle bundle. Set

$$M := P \times_{G_{\mathbf{C}}} Z, \quad \mathcal{E} := P \times_{G_{\mathbf{C}}} E \qquad \text{and} \qquad \mathcal{L} := P \times_{G_{\mathbf{C}}} L.$$

Let $q : Q \overset{G}{\to} B$ be a G-reduction of P to a G-principle bundle. By this reduction the G-bundle Q is equipped with a cannonical Cartan connection with Cartan curvature Θ. Thus $M = Q \times_G Z$ is equipped with a connection, that is we have a splitting $TM = T^H M \oplus TZ$ and the fibration $\pi : M \to B$ is a Kähler fibration in the sense of [BGS88b]. The bundles $\mathcal{E}$ and $\mathcal{L}$ become Hermitian holomorphic bundles over M with Hermitian metric $h^{\mathcal{E}}$ and $h^{\mathcal{L}}$ induced by h^E, h^L and Θ. One can then construct the analytic torsion form $T(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})$ to these data. We make the additional assumption that the Kähler form equals the representative of the first Chern of the line bundle

$$\omega^M = -\frac{1}{2\pi i}\Omega^{\mathcal{L}} \quad \text{and} \quad \omega^Z = -\frac{1}{2\pi i}\Omega^L.$$

Let m^{TZ} respectively m^E and m^L denote the moment relative to the Hermitian holomorphic connection of (Z, ω^Z) respectively (E, h^E) and (L, h^L). For $m \in \mathbf{N}_0$ we will write

$\omega^{Z,m} := \left(\omega^Z\right)^{\wedge m}$. By [P16, Theorem 0.3] under the described assumptions the differential forms α_0 and β_0 on B in Theorem 2 are given by

$$\alpha_0 = \frac{n\mathrm{rk}(E)}{2}\int_Z e^{-\frac{1}{2\pi i}m^L(\Theta)}\frac{\omega^{Z,n}}{n!} \qquad \text{and} \qquad \beta_0 = 0.$$

When B is a point in [F18, Theorem 1.3] Finski gave explicit formulas for α_1 and β_1 for which he had to calculate $a_{1,u}$ ([F18, Lemma 4.5]). We extend his formula for $a_{1,u}$ in the family setting descibed above as our third main result. Let $z \in \mathbf{Z}$ and $(z_1 \dots z_n)$ be complex coordinates on $T_{\mathbf{R},z}Z \cong \mathbf{C}^n$. Let Ω^{TZ} and Ω^E be the curvatues of the Hermitian holomorphic connections ∇^{TZ}, ∇^E on (TZ, h^{TZ}) and (E, h^E). Let Θ_Z be the corresponding fundamental vector field of Θ with differential form value and let $\Theta^\flat$ be its metric dual. Put

$$\Omega^{TZ}_{i\bar{j}k\bar{l}} = \Big\langle \Omega^{TZ}_z\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\Big)\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\Big\rangle, \quad \Omega^E_{i\bar{j}} = \Omega^E_z\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\Big),$$
$$\Omega^{\Lambda}_{i\bar{j}} = \Omega^{\Lambda^\bullet(T^{*0,1}Z)}_z\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\Big), \qquad \Omega^{\det}_{i\bar{j}} = \Omega^{\det}_z\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\Big) \quad \text{and} \quad d\Theta^\flat_{i\bar{j}} = d\Theta^\flat_{Z,z}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\Big)$$

where $\Omega^{\det}$ is the curvature of the Hermitian holomorphic connection of $(\det T^{1,0}Z, h^{\det T^{1,0}Z})$ with $h^{\det T^{1,0}Z}$ induced by $h^{T^{1,0}Z}$ and $\Omega^{\Lambda^\bullet(T^{*0,1}Z)}$ is the curvature of the connection on $\Lambda^\bullet(T^{*0,1}Z)$ induced by ∇^{TZ}.

Theorem 3. *The coefficient $a_{1,u}$ in Theorem 1 is given by*

$$\begin{aligned}
a_{1,u}(z) = \Big[&-\frac{4}{3}\Omega^{TZ}_{i\bar{i}j\bar{j}}(1-e^{-2\pi u})^{-2}\Big(\frac{u}{2}(1+4e^{-2\pi u}+e^{-4\pi u}) - \frac{3}{4\pi}(1-e^{-4\pi u})\Big) \\
&+ \frac{4}{6}u\Omega^{TZ}_{i\bar{i}j\bar{j}} + u\Omega^E_{i\bar{i}} - 2\Omega^E_{i\bar{i}}(1-e^{-2\pi u})^{-1}\Big(\frac{u}{2}+\frac{u}{2}e^{-2\pi u} - \frac{1}{2\pi}(1-e^{-2\pi u})\Big) \\
&- 2\Omega^{\Lambda}_{i\bar{i}}(1-e^{-2\pi u})^{-1}\Big(\frac{u}{2}+\frac{u}{2}e^{-2\pi u} - \frac{1}{2\pi}(1-e^{-2\pi u})\Big) \\
&- \Big(\Omega^{\det}_{i\bar{j}}\overline{w}^j \wedge \iota_{\overline{w}_i} + 2\Omega^E_{i\bar{j}}\overline{w}^j \wedge \iota_{\overline{w}_i}\Big)u \\
&- \Big(\frac{d\Theta^\flat_{i\bar{i}}}{2}(1-e^{-2\pi u})^{-1}\Big(\frac{1}{2}+\frac{1}{2}e^{-2\pi u} - \frac{1}{2\pi u}(1-e^{-2\pi u})\Big) \\
&+ \big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\big)_z\Big)\Big]\frac{e^{-2\pi u N}}{(1-e^{-2\pi u})^n}e^{-m^L(\Theta)}.
\end{aligned}$$

Our last result concerns the equivariant holomorphic analytic torsion form. Let us also here describe briefly the situation first.

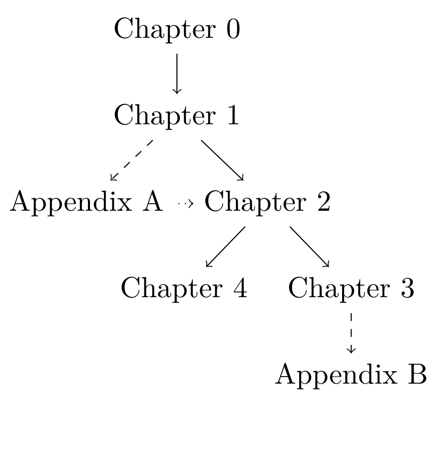
The preliminaries and terminologies are the same as in Theorem 1 and Theorem 2 with additional equivariant structures: Let G be a compact Lie group acting holomorphically on M such that this action lifts to an action on $\mathcal{E}$ and $\mathcal{L}$ so that they become G-equivariant bundles over M. We assume that G acts on B, too, so that M becomes a G-equivariant bundle over B. We also require that ω^M, $h^{\mathcal{E}}$ and $h^{\mathcal{L}}$ are G-invariant. For $\gamma \in G$ and $p \in \mathbf{N}$ one can construct the equivariant analytic torsion form $T_\gamma(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})$ which coincides with $T(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})$ if γ is the neutral element. Let $B_\gamma = \{b \in B |\ \gamma \cdot b = b\}$ be the fixed-point sets of γ with complex dimension $\dim_{\mathbf{C}} B_\gamma$. Then $T_\gamma(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})$ is a differential form on B_γ. In the same way let $Z_\gamma = \{z \in Z |\ \gamma \cdot z = z\}$ with complex dimension $\dim_{\mathbf{C}} Z_\gamma$. The asymptotic behavior of the equivariant torsion form for $p \to \infty$ is given by our fourth and last main result:

Theorem 4. *Let $k \in \{0, \dots, \dim_{\mathbf{C}} B_\gamma\}$. Assume the action of γ on $\mathcal{L}$ is given by $e^{i\varphi}$. Then there are differential forms $\alpha_\gamma(e^{ip\varphi}), \beta_\gamma(e^{ip\varphi})$ on B_γ which are local coefficients such that the component of degree $2k$ of the equivariant holomorphic analytic torsion forms has the following asymptotic as $p \to \infty$:*

$$T_\gamma(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})^{(2k)} = p^{\dim_{\mathbf{C}} Z_\gamma + k}\Big(\alpha_\gamma(e^{ip\varphi}) \log p + \beta_\gamma(e^{ip\varphi})\Big)^{(2k)} + o(p^{\dim_{\mathbf{C}} Z_\gamma + k})$$

in the topology of $\mathscr{C}^\infty$ convergence on compact subsets of B_γ.

This thesis is organized as follows. In chapter 0 we state our notations and give necessary background information on global analysis. In chapter 1 we summerize the construction of the equivariant analytic torsion form. In chapter 2, respectively 3 and 4, we proof Theorem 1, respectively 2 and 3. Appendix A contains a few additional informations and results concerning spectral analysis and wave operators. Appendix B gives insight on the Lie algebraic equivariant torsion and its relationship to the other torsions we have already met. For the reader who are not familiar with the localization technique using finite propagation speed of the wave equation we recommand to read Appendix A between chapter 1 and chapter 2. The dependence of the chapters on each other can be seen in the following figure.



Chapter 0

General Notations and Background on Families of Operators

The purpose of this chapter is to provide an uniform notation and background on families of operators. We follow strictly [BGV92, chapters 9 and 10]. Furthur treatments regarding spectrum and the wave equation will be dealt separately in the appendix of this thesis. We will assume that the reader is familiar with Quillen's formalism of superbundles, graded products, supertraces and supercommutator ([Q85]). See also [BGV92].

In this thesis, when in a formula a subscript index appears two times and there is no sum sign, then it will be sum with this index.

The natural numbers and the natural numbers including zero, respectively, will be denoted by $\mathbf{N} = \{1, 2, \dots\}$ and $\mathbf{N}_0 = \{0, 1, 2, \dots\}$, respectively.

Let Z be a compact complex manifold with complex dimension $\dim_{\mathbf{C}} Z = n \in \mathbf{N}$. We will denote by $T_{\mathbf{R}}Z$ the real tangent bundle and by $T_h Z$ the holomorphic tangent bundle of Z. The complexification $T_{\mathbf{R}}Z \otimes_{\mathbf{R}} \mathbf{C}$ of the real tangent bundle will be denoted by $T_{\mathbf{C}}Z$. Let $J^{T_{\mathbf{R}}Z}$ be the complex structure on $T_{\mathbf{R}}Z$ and $J^{T_{\mathbf{C}}Z}$ its complex linear extension to $T_{\mathbf{C}}Z$. Let $T^{1,0}Z$ and $T^{0,1}Z$ be the i and $-i$ eigenbundles of $J^{T_{\mathbf{C}}Z}$. For a complex vector bundle $E \overset{\pi_E}{\to} Z$ over Z we will use the following notations for the space of complex differential forms on Z with coefficients in E:

$$\mathfrak{A}^k(Z,E) = \Gamma^\infty(Z, \Lambda^k(T^*_{\mathbf{C}}Z) \otimes E), \quad \mathfrak{A}^\bullet(Z,E) = \bigoplus_{k\geq 0} \mathfrak{A}^k(Z,E) \qquad (k \in \mathbf{N}_0),$$
$$\mathfrak{A}^{p,q}(Z,E) = \Gamma^\infty(Z, \Lambda^p(T^{*1,0}Z) \wedge \Lambda^q(T^{*0,1}Z) \otimes E) \qquad (p, q \in \mathbf{N}_0).$$

If not other stated all sections in a vector bundle in this thesis will be considered as smooth so that we will simply write Γ instead of Γ^∞. The space of sections with compact support will be denoted by Γ_c.

Let $P^Z = \oplus_{p\geq 0}\mathfrak{A}^{p,p}(Z,\mathbf{C})$ be the vector space of smooth forms on Z which are sums of forms of type (p,p). Let $P^{Z,0}$ be the vector space of the forms $\alpha \in P^Z$ such that there exist smooth forms β and γ on Z for which $\alpha = \partial\beta + \overline{\partial}\gamma$. For $\alpha \in P^Z$ its class in $P^Z/P^{Z,0}$ is called Bott-Chern class of α. One has to distinguish the Bott-Chern classes which are secondary classes from the classes in the Bott-Chern cohomology which are cohomology classes. The Bott-Chern cohomology groups $H^{p,q}_{BC}(Z,\mathbf{C})$ are defined by $H^{p,q}_{BC}(Z,\mathbf{C}) := \left(\mathfrak{A}^{p,q}(Z,\mathbf{C})\cap \ker d\right)/\overline{\partial}\partial\mathfrak{A}^{p-1,q-1}(Z,\mathbf{C})$. For more background on this topic see [Bi13, chapter 4.1]

If E, F are vector bundles over Z which have the structures of superalgebras then in the whole thesis we will denote the $\mathbf{Z}_2$-graded tensor product by $E\otimes F$ instead of the also common notation $E\hat{\otimes}F$ since the ungraded tensor product will never be used.

If $Y \in \Gamma(Z, T_{\mathbf{C}}Z)$ is a vector field on Z the interior product with Y will be denoted by ι_Y.

The filtered algebra of differential operators on a vector bundle $E \overset{\pi_E}{\to} Z$ will be denoted by $\mathrm{Op}(Z,E) \subset \mathrm{End}(\Gamma(Z,E))$. By $\mathscr{K}(Z,E) \subset \mathrm{Op}(Z,E)$ we will denote the set of smoothing operators.

Now let Z be equipped with a Riemannian metric $g^{T_{\mathbf{R}}Z}$ and let $\nabla^{T_{\mathbf{R}}Z,LC}$ be the Levi-Civita connection on $(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$. Let (E,h^E) be a Hermitian vector bundle on Z with Hermitian metric h^E and let ∇^E be a Hermitian connection on (E,h^E). Let $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}$ be the Riemannian volume form of $(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$. For $l \in \mathbf{N}_0$ the metrics $g^{T_{\mathbf{R}}Z}$ and h^E induce a pointwise norm on $(T^*_{\mathbf{R}}Z)^{\otimes l}\otimes E$ and its subbundles denoted by $\|\cdot\|_{g^{T_{\mathbf{R}}Z},h^E}$. The L^2-norm $\|\cdot\|_{L^2(Z,E)}$ induced by $g^{T_{\mathbf{R}}Z}$ and h^E is defined for $s \in \Gamma(Z,(T^*_{\mathbf{R}}Z)^{\otimes l}\otimes E)$ by

$$\|s\|_{L^2(Z,E)} := \frac{1}{(2\pi)^{\dim_{\mathbf{C}} Z}} \int_Z \|s\|_{g^{T_{\mathbf{R}}Z},h^E} d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}.$$

For $m \in \mathbf{N}_0$ and $s \in \Gamma(Z,E)$ the $\mathscr{C}^m(Z,E)$-norm $\|\cdot\|_{\mathscr{C}^m(Z,E)}$ and the Sobolev norm $\|\cdot\|_{\mathbf{H}^m(Z,E)}$ are defined by

$$\|s\|_{\mathscr{C}^m(Z,E)} := \sum_{l=0}^{m} \sup_{z\in Z} \left\|\nabla^{(T^*_{\mathbf{R}}Z)^{\otimes l}\otimes E} \ldots \nabla^E s\right\|_{g^{T_{\mathbf{R}}Z},h^E}(z) \qquad \text{and}$$

$$\|s\|^2_{\mathbf{H}^m(Z,E)} := \sum_{l=0}^{m} \|\nabla^{(T^*_{\mathbf{R}}Z)^{\otimes l}\otimes E} \dots \nabla^E s\|_{L^2(Z,E)}.$$

We will often refer $\|\cdot\|_{\mathscr{C}^m(Z,E)}$ as "the $\mathscr{C}^m$-norm induced by h^E and ∇^E". Occasionally, for the sake of transparency, we will not write down the function spaces beside the norm when it is clear.

If G and H are two Hilbert spaces we denote by $\mathscr{L}(G,H)$ the set of bounded linear operators and $\mathscr{L}(G) := \mathscr{L}(G,G)$. The corresponding operator norm will, if not otherwise stated, be denoted by $\|\cdot\|_\infty$. We will denote the spectrum of an operator A by $\mathrm{Spec}(A)$.

For a normed vector space $(V, \|\cdot\|)$ and $r > 0$ we denote by $B^V_r(0)$ the subset of vectors with norm lesser r.

Let pr_1 respectively pr_2 be the projection of $Z \times Z$ onto the first respectively second component. For two vector bundles E_1 and E_2 on Z set

$$E_1 \boxtimes E_2 := \mathrm{pr}_1^* E_1 \otimes \mathrm{pr}_2^* E_2.$$

If P is a smoothing operator acting on $\Gamma(Z,E)$ given by a kernel $p \in \Gamma(Z \times Z, E \boxtimes E^*)$ with respect to the Riemannian volume form of $(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$ then we will use Dirac's notation $\langle \cdot | P | \cdot \rangle$ for the kernel instead: For $(x,y) \in Z \times Z$ and $s \in \Gamma(Z,E)$,

$$(Ps)(x) = \int_{y\in Z} \langle x|P|y\rangle s(y) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(y) = \int_{y \in Z} p(x,y)s(y) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(y).$$

Given two operators $P(t)$ and $Q(t)$ on $L^2(Z,E)$ depending on $t > 0$ their convolution is given by

$$(P * Q)(t) = \int_0^t P(t-s)Q(s)ds.$$

The λ-fold product $P(t) * \cdots * P(t)$ will be denoted by $P(t)^{*\lambda}$ and set $P(t)^{*1} = P(t)$. If both $P(t)$ and $Q(t)$ are smoothing operators with kernels $\langle \cdot |P(t)| \cdot \rangle, \langle \cdot |Q(t)| \cdot \rangle \in \Gamma(Z \times Z, E \boxtimes E^*)$ then $(P * Q)(t)$ is smooth, too, with kernel

$$\langle x|(P * Q)(t)|y\rangle = \int_0^t \int_Z \langle x|P(s)|z\rangle \cdot \langle z|Q(t-s)|y\rangle d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z)ds. \tag{0.0.1}$$

Now let M, B be complex manifolds and $M \xrightarrow{\pi} B$ be a holomorphic fibre bundle with compact fibre Z. Let n, m and $d = n + m$ be the complex dimension of Z, B and M. Let $\mathcal{E}$ be a complex manifold and $\mathcal{E} \overset{\pi_{\mathcal{E}}}{\to} M$ a holomorphic fibre bundle over M. For $b \in B$ set

$$Z_b := \pi^{-1}\{b\} \quad \text{and} \quad \mathcal{E}_b := \mathcal{E}_{|Z_b} := \Big((\pi_{\mathcal{E}})_{Z_b} : \pi_{\mathcal{E}}^{-1}(Z_b) \to Z_b\Big).$$

Then we call $\{\mathcal{E}_b\}_{b\in B}$ a family of vector bundles if the restriction $\mathcal{E}_b$ is a vector bundle over Z_b for each $b \in B$. We will often simply write $\mathcal{E}$ instead of $\{\mathcal{E}_b\}_{b\in B}$. To a given family of vector bundles $\{\mathcal{E}_b\}_{b\in B}$ we can associate an infinite dimensional bundle $\mathbf{E}$ over B whose fibre is given by $\mathbf{E}_b = \Gamma(Z_b, \mathcal{E}_b)$. The space of complex differential forms on B with values in $\mathbf{E}$ is defined to be

$$\mathfrak{A}^\bullet(B, \mathbf{E}) := \Gamma(M, \pi^*(\Lambda^\bullet T^*_{\mathbf{C}}B) \otimes \mathcal{E}).$$

In particular the set of smooth section of $\mathbf{E}$ over B is given by $\Gamma(B, \mathbf{E}) = \Gamma(M, \mathcal{E})$. Recall that if $\mathcal{E}$ has the structure of a superalgebra the product in $\pi^*(\Lambda^\bullet T^*_{\mathbf{C}}B) \otimes \mathcal{E}$ is the graded product.

Assume $\{Z_b\}_{b\in B}$ is a family of Riemannian manifolds that is the vertical bundle $T^V M = \ker T\pi$ is equipped with a metric such that its restriction on each fibre Z_b defines Riemannian structures $g^{T_{\mathbf{R}}Z_b}$ on each Z_b. The associated Riemannian volume form on the fibre will be denoted by $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_b}}$.

By $\mathrm{Op}(\mathcal{E})$ respectively $\mathscr{K}(\mathcal{E})$ we will denote the bundle over B whose fibre at $b \in B$ is given by $\mathrm{Op}(Z_b, \mathcal{E}_b)$ respectively $\mathscr{K}(Z_b, \mathcal{E}_b)$. One has to distinguish between $\mathrm{Op}(M, \mathcal{E})$ the set of differential operators on $\mathcal{E} \overset{\pi_{\mathcal{E}}}{\to} M$ and $\mathrm{Op}(\mathcal{E})$ the set of families of vertical operators parameterized by B. Set

$$\mathfrak{A}^\bullet(B, \mathrm{Op}(\mathcal{E})) := \Gamma(B, \Lambda^\bullet T^*_{\mathbf{C}}B \otimes \mathrm{Op}(\mathcal{E})) \quad \text{and} \quad \mathfrak{A}^\bullet(B, \mathscr{K}(\mathcal{E})) := \Gamma(B, \Lambda^\bullet T^*_{\mathbf{C}}B \otimes \mathscr{K}(\mathcal{E}))$$

which are the spaces of families of vertical respectively smoothing operators with differential form coefficients. Next we are going to give a short overview on kernels of families of smoothing operators. Denote by $M \times_\pi M$ the fibre-product

$$M \times_\pi M := \{(x, y) \in M \times M |\ \pi(x) = \pi(y)\}$$

which is a fibre bundle over B with fibre at $b \in B$ equals to $Z_b \times Z_b$. We will also use the letter π for the submersion $\pi : M \times_\pi M \to B$. In particular $\pi^*\Lambda^\bullet T^*_{\mathbf{C}}B$ is a vector bundle over $M \times_\pi M$. Let pr_1 respectively pr_2 be the projection of $M \times_\pi M$ onto the first respectively second component. For two vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on M set

$$\mathcal{E}_1 \boxtimes_\pi \mathcal{E}_2 := \mathrm{pr}_1^*\mathcal{E}_1 \otimes \mathrm{pr}_2^*\mathcal{E}_2$$

which is a vector bundle over $M \times_\pi M$. Consider a section

$$k \in \Gamma(M \times_\pi M, \pi^* \Lambda^\bullet T^*_{\mathbf{C}} B \otimes (\mathcal{E} \boxtimes_\pi \mathcal{E}^*)).$$

When restricted to a fibre $Z_b \times Z_b$ it gives rise to a kernel

$$k_b \in \Gamma(Z_b \times Z_b, \Lambda^\bullet T^*_{\mathbf{C},b} B \otimes (\mathcal{E}_b \boxtimes \mathcal{E}^*_b)) \qquad \text{(where } \Lambda^\bullet T^*_{\mathbf{C},b} B \text{ is the trivial bundle over } Z_b \times Z_b\text{)}$$

and with respect to the Riemannian volume form of the fibre it defines an operator K_b with kernel k_b. Thus such a section k defines a family of smoothing operators $K = \{K_b\}_{b \in B} \in \mathfrak{A}^\bullet(B, \mathscr{K}(\mathcal{E}))$ and the mapping k will also be refered as a kernel.

Suppose $K \in \mathfrak{A}^\bullet(B, \mathscr{K}(\mathcal{E}))$. When restricted to the diagonal the kernel $\langle z|K_b|z\rangle$ is a smooth section of $\pi^* \Lambda^\bullet T^*_{\mathbf{C}} B \otimes \mathrm{End}(\mathcal{E})$ over M where M is identified with its embedding in $M \times_\pi M$ as the diagonal. The $\mathfrak{A}^\bullet(B, \mathbf{C})$-valued supertrace $\mathrm{Tr}_s : \mathfrak{A}^\bullet(B, \mathscr{K}(\mathcal{E})) \to \mathfrak{A}^\bullet(B, \mathbf{C})$ of the family of operators K is the differential form on B given by

$$b \mapsto \int_{Z_b} \mathrm{Tr}_s\, k_b(z,z) d\mathrm{vol}_{g^{T_{\mathbf{R}} Z_b}}(z) = \int_{Z_b} \mathrm{Tr}_s \, \langle z|K_b|z\rangle d\mathrm{vol}_{g^{T_{\mathbf{R}} Z_b}}(z).$$

For this differential form we will write $\int_Z \mathrm{Tr}_s \, \langle z|K|z\rangle d\mathrm{vol}_{g^{T_{\mathbf{R}} Z}}(z) \in \mathfrak{A}^\bullet(B, \mathbf{C})$.

Let $D = \{D_b\}_{b \in B} \in \Gamma(B, \mathrm{Op}(\mathcal{E}))$ be a family of Dirac operators on $\mathcal{E}$. A differential operator $\mathbb{A} \in \mathrm{Op}(M, \pi^*(\Lambda^\bullet T^*_{\mathbf{C}} B) \otimes \mathcal{E})$ is called a superconnection adapted to D if it satifies the following conditions:

- 1.) the operator is of odd parity,
- 2.) it satisfies the Leibniz's rule

$$\mathbb{A}(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbb{A}\beta, \quad \forall \alpha \in \mathfrak{A}^k(B), \beta \in \mathfrak{A}^\bullet(B, \mathbf{E}),$$

- 3.) $\mathbb{A} = \sum_{k=0}^{\dim_{\mathbf{R}} B} \mathbb{A}^{(k)}$ with $\mathbb{A}^{(0)} = D$ and $\mathbb{A}^{(k)} : \mathfrak{A}^\bullet(B, \mathbf{E}) \to \mathfrak{A}^{\bullet+k}(B, \mathbf{E})$ for $k \geq 1$.

The curvature $\mathbb{A}^2$ lies in $\mathfrak{A}^\bullet(B, \mathrm{Op}(\mathcal{E}))$ since it supercommutes with $\mathfrak{A}^\bullet(B, \mathbf{C})$ and it has a decompositon $\mathbb{A}^2 = D^2 + \mathbb{A}^{2,(+)}$ where the operator $\mathbb{A}^{2,(+)}$ raises the exterior degree in $\Lambda^\bullet T^*_{\mathbf{C},b} B \otimes \Gamma(M_b, \mathcal{E}_b)$. By [BGV92, Appendix 1] for $t > 0$ there exists a unique smooth family of heat kernels for $\mathbb{A}^2$ with corresponding family of smoothing operators denoted by $e^{-t\mathbb{A}^2} \in \mathfrak{A}^\bullet(B, \mathscr{K}(\mathcal{E}))$ given by

$$e^{-t\mathbb{A}^2} = e^{-tD^2} + \sum_{k>0} (-t)^k I_{k,t} \tag{0.0.2}$$

where

$$I_{k,t} = \int_{t\Delta_k} e^{-(t-t_k)D^2} \mathbb{A}^{2,(+)} e^{-(t_k - t_{k-1})D^2} \mathbb{A}^{2,(+)} \dots \mathbb{A}^{2,(+)} e^{-t_1 D^2} dt_1 \dots dt_k$$

and $t\Delta_k = \{(t_1, \dots, t_k) \in \mathbf{R}^k |\ 0 \le t_1 \le \dots \le t_k \le t\}$ is the rescaled simplex. Note that the sum in (0.0.2) is a finite sum since $\mathbb{A}^{2,(+)}$ has positive degree in $\Lambda^\bullet T^*_{\mathbf{C},b} B$. We can also write $I_{k,t}$ as

$$I_{k,t} = e^{-tD} * (\mathbb{A}^{2,(+)} e^{-tD})^{*k}.$$

For the precise definition of a kernel to be a heat kernel in the family setting we refer [BGV92, p.304].

For $u > 0$ let ψ_u be the automorphism of $\mathfrak{A}^\bullet(B, \mathbf{E})$ which multiplies $\mathfrak{A}^k(B, \mathbf{E})$ by u^k. It satisfies $\psi_u^{-1} = \psi_{1/u}$ and $\psi_u \cdot \psi_{u'} = \psi_{uu'},\ (u' > 0)$. The rescaled superconnection

$$\mathbb{A}_u := \sqrt{u}\psi_{1/\sqrt{u}} \mathbb{A} \psi_{\sqrt{u}} = \sqrt{u} D + \mathbb{A}^{(1)} + u^{-1/2} \mathbb{A}^{(2)} + \dots$$

is a superconnection adapted to $u^{1/2}D$ with curvature $\mathbb{A}_u^2 = u\psi_{1/\sqrt{u}} \cdot \mathbb{A}^2 \cdot \psi_{1/\sqrt{u}}^{-1}$ and heat kernel $e^{-\mathbb{A}_u^2} = \psi_{1/\sqrt{u}}(e^{-u\mathbb{A}^2})$.

The most important superconnection in our context will be the Bismut superconnection which we will encounter in the next chapter.

Chapter 1

The Equivariant Holomorphic Analytic Torsion Form

In this chapter we give the definiton and construction of the equivariant holomorphic analytic torsion forms for Kähler fibrations and state the curvature and anomaly formula. The chapter is organized as follows. In section 1.1 we recall the definition of Kähler fibrations following [BGS88b]. In section 1.2 we describe the Bismut superconnection followed by the Lichnerowicz formula which we will need several times in the later chapters. In section 1.3 we give the definition of the equivariant holomorphic analytic torsion form by summerizing the results established in [Ma00].

1.1 Kähler Fibration

Let M and B be complex manifolds and $M \overset{\pi}{\to} B$ be a holomorphic fibre bundle with compact fibre Z. Let n, m and $d = n + m$ be the complex dimensions of Z, B and M. Let TZ be the vertical holomorphic subbundle of $T_h M$ whose fibre is given by $T_p Z = (T_h Z_{\pi(p)})_{|_p}$ and denote by $T_{\mathbf{R}} Z$ the underlying real vector bundle which is equipped with a complex structure $J^{T_{\mathbf{R}} Z} \in \Gamma(M, \mathrm{End}(T_{\mathbf{R}} Z))$. Let $T^{1,0} Z$ and $T^{0,1} Z$ be the i and $-i$ eigenbundle of its complex linear extension $J^{T_{\mathbf{C}} Z}$ to $T_{\mathbf{C}} Z = T_{\mathbf{R}} Z \otimes_{\mathbf{R}} \mathbf{C}$. Let $\omega^M \in \mathfrak{A}^{1,1}(M, \mathbf{R})$ be a smooth real $(1,1)$-form on M. Set

$$\omega^Z := \omega^M_{|_{T_{\mathbf{R}} Z \times T_{\mathbf{R}} Z}}.$$

We now give the definition of a Kähler fibration established in [BGS88b, Def 1.4, Theorem 1.5]:

Definition 1.1.1. *We say that the tuple (π, ω^M) defines a Kähler fibration if the following conditions hold:*

- *a) ω^M is closed.*

- *b) the bilinear map*

$$\Gamma(M, T_{\mathbf{R}}Z) \ni X, Y \mapsto g^{T_{\mathbf{R}}Z}(X,Y) := \omega^Z(J^{T_{\mathbf{R}}Z}X, Y)$$

defines a metric on $T_{\mathbf{R}}Z$*.*

We will now assume that (π, ω^M) is a Kähler fibration. By this each fibre Z_b is equipped with a Riemannian metric denoted by $g^{T_{\mathbf{R}}Z_b}$ which are Kähler with Kähler form obtained from ω^M by restricting on that fibre. Let $T^HM \subset T_hM$ be the orthogonal bundle to TZ in T_hM with respect to ω^M and $T^H_{\mathbf{R}}M$ its underlying real vector bundle. We get a decomposition of smooth bundles

$$T_hM = T^HM \oplus TZ, \qquad T_{\mathbf{R}}M = T^H_{\mathbf{R}}M \oplus T_{\mathbf{R}}Z$$

in horizontal and vertical parts. Note that in general T^HM is not a holomorphic subbundle of T_hM. We also have the isomorphism of smooth vector bundles

$$T^HM \cong \pi^*T_hB, \qquad \text{and} \qquad \Lambda^\bullet(T^*_{\mathbf{R}}M) \cong \pi^*\Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \Lambda^\bullet(T^*_{\mathbf{R}}Z).$$

Set

$$\omega^H := \omega^M_{|_{T^H_{\mathbf{R}}M \times T^H_{\mathbf{R}}M}}$$

and extend ω^Z and ω^H by zero to $T^H_{\mathbf{R}}M \oplus T_{\mathbf{R}}Z$ so that we have $\omega^M = \omega^Z + \omega^H$.

Let $\mathcal{E} \to M$ be a holomorphic vector bundle over M. Then as explained in chapter 0 we can associate to the family of vector bundles $\{(\Lambda^{0,k}(T^*Z) \otimes \mathcal{E})_b\}_{b \in B},\ k \in \mathbf{N}_0$, an infinite bundle $\mathbf{E}^k$ whose fibre is given by $\mathbf{E}^k_b = \Gamma(Z_b, (\Lambda^{0,k}(T^*Z) \otimes \mathcal{E})_{|_{Z_b}})$. Let $\mathbf{E}$ be the bundle over B whose fibre at $b \in B$ is

$$\mathbf{E}_b = \bigoplus_{k=0}^{\dim_{\mathbf{C}} Z} \mathbf{E}^k_b.$$

Let h^{TZ} be the Hermitian metric along the fibres obtained from $g^{T_{\mathbf{R}}Z}$ and $h^{T^{1,0}Z}$ the Hermitian metric on $T^{1,0}Z$ induced by h^{TZ} via the isomorphism $(T_{\mathbf{R}}Z, J^{T_{\mathbf{R}}Z}) \cong T^{1,0}Z$.

Let $h^{\mathcal{E}}$ be a Hermitian metric on $\mathcal{E}$. Recall that the Riemannian volume form of the fibre $(Z_b, g^{T_{\mathbf{R}}Z_b})$ was denoted by $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_b}}$. Then we can define a Hermitian product $\langle \cdot, \cdot \rangle$ on $\mathbf{E}$ associated to h^{TZ} and $h^{\mathcal{E}}$ as the following:

$$\langle s_1, s_2 \rangle_b := \frac{1}{(2\pi)^{\dim_{\mathbf{C}} Z}} \int_{Z_b} h^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}(s_1, s_2) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_b}} \tag{1.1.1}$$

where $s_1, s_2 \in \Gamma(B, \mathbf{E})$ and $h^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$ is the Hermitian metric on $\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}$ induced by

h^{TZ} and $h^{\mathcal{E}}$. For an arbitrary Riemannian metric $g^{T_{\mathbf{R}}B}$ on B set

$$g^{T_{\mathbf{R}}M} := g^{T_{\mathbf{R}}Z} + \pi^* g^{T_{\mathbf{R}}B}.$$

Before we are heading to the definitions of certain connections on the considered bundles we will introduce some notation regarding local basis. Let $\{w_i\}_i$ be an local orthonormal basis of $(T^{1,0}Z, h^{T^{1,0}Z})$ with dual basis $\{w^i\}_i$. We get a local orthonormal base $\{e_i\}_i$ of $(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$ by setting

$$e_{2i-1} := \frac{1}{\sqrt{2}}(w_i + \overline{w}_i) \quad \text{and} \quad e_{2i} := \frac{1}{\sqrt{2}}(w_i - \overline{w}_i).$$

The local dual basis will be denoted by $\{e^i\}_i$. Let $\{f_\alpha\}_\alpha$ be a basis of $T_{\mathbf{R}}B$ with dual basis $\{f^\alpha\}_\alpha$. These bases will be identfied with bases of $T^H_{\mathbf{R}}M$ and $(T^H_{\mathbf{R}}M)^*$. For any $(k,0)$-tensor A we will denote by $A_{a_1,\dots,a_k} = A(\mathfrak{e}_{a_1},\dots,\mathfrak{e}_{a_k})$ where $\mathfrak{e}_{a_j} \in \{e_i\}_i \cup \{f_\alpha\}_\alpha$. Latin indices $i, j \dots$ will be used for vertical variables, greek indices $\alpha, \beta, \dots$ for horizontals variables. These notations will be used throughout the upcomming sections.

1.2 The Bismut Superconnection

Let ∇^{TZ} and $\nabla^{\mathcal{E}}$ be the Hermitian holomorphic connections on (TZ, h^{TZ}) and $(\mathcal{E}, h^{\mathcal{E}})$. Let $\nabla^{\Lambda^{0,\bullet}(T^*Z)}$ be the connection on $\Lambda^{0,\bullet}(T^*Z)$ induced by ∇^{TZ} and $\nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$ be the connection on $\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}$ induced by $\nabla^{\Lambda^{0,\bullet}(T^*Z)}$ and $\nabla^{\mathcal{E}}$.

Definition 1.2.1. *For $U \in \Gamma(B, T_{\mathbf{R}}B)$ let U^H be its horizontal lift to $\Gamma(M, T^H_{\mathbf{R}}M)$. The connection $\nabla^{\mathbf{E}}$ on $\mathbf{E}$ is defined to be*

$$\nabla^{\mathbf{E}}_U s := \nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}_{U^H} s$$

for $s \in \Gamma(B, \mathbf{E}) = \Gamma(M, \Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E})$.

This connection can be extended to a connection on $\Gamma(M, \pi^*\Lambda^\bullet(T^*_{\mathbf{R}}B)\otimes\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E})$ by the de Rham operator d_B on B and Leibniz rule. This extension will still be denoted by $\nabla^{\mathbf{E}}$.

Let $\mathbf{Cl}(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$ be the Clifford algebra of $(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$. For any $X \in \Gamma(M, TZ) \subset \Gamma(M, T_{\mathbf{C}}Z)$ with decomposition $X = X^{1,0} + X^{0,1} \in \Gamma(M, T^{1,0}Z \oplus T^{0,1}Z)$ let $X^{1,0\flat} \in \Gamma(M, T^{*0,1}Z)$ be the metric dual of $X^{1,0}$ with respect to h^{TZ},

$$X^{1,0\flat} = h^{TZ}(X^{1,0}, \cdot).$$

Then the bundle $\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}$ becomes a $\mathbf{Cl}(T_{\mathbf{R}}Z, g^{T_{\mathbf{R}}Z})$-module with Clifford action induced by

$$c(X)(\alpha \otimes s) = \sqrt{2}(X^{1,0\flat} \wedge \alpha - \iota_{X^{0,1}}\alpha) \otimes s$$

for $\alpha \in \Gamma(M, \Lambda^{0,\bullet}(T^*Z))$ and $s \in \Gamma(M, \mathcal{E})$.

For $b \in B$ let $\overline{\partial}^{Z_b}$ be the Dolbeault operator acting on $\mathbf{E}_b$ and let $\overline{\partial}^{Z_b *}$ be its formal adjoint with respect to the Hermitian product (1.1.1). Put

$$D^{Z_b} := \overline{\partial}^{Z_b} + \overline{\partial}^{Z_b *}.$$

Then $D^Z = \{D^{Z_b}\}_b \in \mathrm{Op}((\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E})$ is a smooth family of Dirac operators. Since each fibre Z_b is Kähler $\sqrt{2}D^{Z_b}$ is a Dirac operator which satisfies

$$\sqrt{2}D^{Z_b} = \sum_{i=1}^{2n} c(e_i)\nabla_{e_i}^{\Lambda^{0,\bullet}(T^*Z_b)\otimes\mathcal{E}}.$$

Let $\nabla^{T_{\mathbf{R}}B,LC}$ be the Levi-Civita connection on $(T_{\mathbf{R}}B, g^{T_{\mathbf{R}}B})$ which lifts to a connection $\nabla^{T^H_{\mathbf{R}}M}$ on $T^H_{\mathbf{R}}M$. Let $\nabla^{T_{\mathbf{R}}Z}$ be the connection on $T_{\mathbf{R}}Z$ induced by ∇^{TZ}. Set $\nabla^{T_{\mathbf{R}}M,\oplus} := \nabla^{T^H_{\mathbf{R}}M} \oplus \nabla^{T_{\mathbf{R}}Z}$ which is a connection on $T_{\mathbf{R}}M = T^H_{\mathbf{R}}M \oplus T_{\mathbf{R}}Z$. Let $T \in \mathfrak{A}^2(M, T_{\mathbf{R}}M)$ be the torsion of $\nabla^{T_{\mathbf{R}}M,\oplus}$. For $U, V \in \Gamma(B, T_{\mathbf{R}}B)$ set

$$T^H(U,V) := T(U^H, V^H).$$

If $P^{T_{\mathbf{R}}Z}$ denotes the projection $\pi^*T_{\mathbf{R}}B \oplus T_{\mathbf{R}}Z \to T_{\mathbf{R}}Z$ to the vertical part then T^H is given by

$$T^H(U,V) = -P^{T_{\mathbf{R}}Z}[U^H, V^H].$$

By [BGS88b, Theorem 1.7] T takes value in $T_{\mathbf{R}}Z$ and its complex linear extension is of type $(1,1)$ that is $T \in \mathfrak{A}^{1,1}(M, T_{\mathbf{R}}Z)$.

Definition 1.2.2. *Let $c(T^H)$ be the section of $\left(\Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E})\right)^-$ given by*

$$c(T^H) := \frac{1}{2} \sum_{1 \le \alpha,\beta \le 2m} f^\alpha f^\beta c(T^H(f_\alpha, f_\beta)).$$

Let N_V be the number operator defining the $\mathbf{Z}$-grading on $\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}$ and on $\mathbf{E}$. For $X, Y \in \Gamma(B, T_{\mathbf{R}}B)$ set

$$\omega^{H\overline{H}}(X,Y) := \omega^M(X^H, Y^H).$$

Definition 1.2.3. *For $u > 0$ the number operator $N_u \in \Gamma(B, \Lambda^\bullet(T^*_{\mathbf{C}}B) \otimes \operatorname{End}\mathbf{E})$ is defined by*

$$N_u := N_V + \frac{i\omega^{H\overline{H}}}{u}.$$

As differential forms on the base are identified with horizontal forms in the rest of this thesis we will not distinguish between ω^H and $\omega^{H\overline{H}}$.

Definition 1.2.4. *For $u > 0$ the Bismut superconnection B_u on $\mathbf{E}$ is the superconnection*

$$B_u := \sqrt{u}(\overline{\partial}^Z + \overline{\partial}^{Z*}) + \nabla^{\mathbf{E}} - \frac{c(T^H)}{2\sqrt{2u}}.$$

Recall for $a > 0$ the the automorphism ψ_a of $\Lambda^\bullet(T^*_{\mathbf{R}}B)$ was such that if $\alpha \in \Lambda^k(T^*_{\mathbf{R}}B)$, then $\psi_a\alpha = a^q\alpha$. If we set $B = B_1$ we see

$$B_u = \sqrt{u}\psi_{1/\sqrt{u}}B\psi_{\sqrt{u}}$$

i.e. B_u is the rescaled connection of B. We have $B = D^Z + B^{(1)} + B^{(2)}$ with $B^{(1)} = \nabla^{\mathbf{E}}$ and $B^{(2)} = -\frac{c(T^H)}{2\sqrt{2}}$. The curvature $B^2 = \frac{1}{2}[B, B]$ has the decomposition $B^2 = D^{Z,2} + B^{2,(+)}$ with $B^{2,(+)} = \left(\nabla^{\mathbf{E}} - \frac{c(T^H)}{2\sqrt{2}}\right)^2 + \left[\nabla^{\mathbf{E}} - \frac{c(T^H)}{2\sqrt{2}}, D^Z\right]$. Recall that $[\cdot,\cdot]$ is the supercommutator.

Let $\nabla^{T_{\mathbf{R}}M,LC}$ be the Levi-Civita connection on $(T_{\mathbf{R}}M, g^{T_{\mathbf{R}}M})$. Define the tensor S on M by

$$S := \nabla^{T_{\mathbf{R}}M,LC} - \nabla^{T_{\mathbf{R}}M,\oplus}$$

which takes values in the antisymmetric elements of $\operatorname{End}(T_{\mathbf{R}}M)$. If the mean curvature $k := -\frac{1}{2}\sum_{i=1}^{2n} S(e_i)e_i$ vanishes it was proven in [BF86, Proposition 1.4] that $\nabla^{\mathbf{E}}$ is a Hermitian connection with respect to (1.1.1). As it was shown in [BGS88b, Theorem 1.14] k always vanishes on a Kähler fibration from which in this case $\nabla^{\mathbf{E}}$ is Hermitian. This does not have to be true if ω^M is not closed (which in this case is called Hermitian fibration). By [Bi86, Theorem 1.9] the $(3,0)$-tensor

$$\mathcal{S}(\cdot,\cdot,\cdot) := \langle S(\cdot)\cdot,\cdot\rangle_{g^{T_{\mathbf{R}}M}}$$

does not depend on choice of the metric chosen on $T_{\mathbf{R}}B$. For $u > 0$ define

$$\nabla_{u,e_i} := \nabla_{e_i}^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}} + \frac{1}{\sqrt{2u}}\mathcal{S}_{i,j,\alpha}c(e_j)f^\alpha + \frac{1}{2u}\mathcal{S}_{i,\alpha,\beta}f^\alpha f^\beta \tag{1.2.1}$$

which is a fibrewise connection on $\pi^*\Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}$.

In the whole thesis we will use the following notation: if $C \in \Gamma(M, \mathrm{T}^*_{\mathbf{R}}Z \otimes \operatorname{End}(\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E})$

then

$$\Big(\nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}_{e_i} + C(e_i)\Big)^2 := \sum_i \Big(\nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}_{e_i} + C(e_i)\Big)^2 - \nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}_{\sum_i \nabla^{TZ}_{e_i} e_i} - C\Big(\sum_i \nabla^{TZ}_{e_i} e_i\Big).$$

Let Ω^{TZ} and $\Omega^{\mathcal{E}}$ denote the curvatures of ∇^{TZ} and $\nabla^{\mathcal{E}}$. Let s^Z be the scalar curvature of Z. The next Theorem is the Lichnerowicz formula established in [Bi86, Theorem 3.6]. See also [BGS88b, eq. (2.13)].

Theorem 1.2.5. *For $u > 0$ the curvature of the Bismut superconnection satisfies the formula*

$$\begin{aligned} B_u^2 = &-\frac{u}{2}(\nabla_{u,e_i})^2 + u\frac{s^Z}{8} + \frac{u}{4}c(e_i)c(e_j)\Big[\Omega^{\mathcal{E}} + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big](e_i,e_j) \\ &+ \sqrt{\frac{u}{2}}c(e_i)f^\alpha\Big[\Omega^{\mathcal{E}} + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big](e_i,f_\alpha) + \frac{f^\alpha f^\beta}{2}\Big[\Omega^{\mathcal{E}} + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big](f_\alpha,f_\beta). \end{aligned}$$

Recall that by (0.0.2) B_u^2 has a heat kernel with corresponding heat operator which was denoted by $e^{-B_u^2} \in \mathfrak{A}^\bullet(B, \mathscr{K}(\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}))$. We will frequently make use of the notation $\exp(-B_u^2)$ as well.

1.3 The Equivariant Holomorphic Analytic Torsion Form

Let G be a compact Lie group acting holomorphically on M such that this action lifts to an action on $\mathcal{E}$ from which $\mathcal{E}$ becomes a G-equivariant bundle over M. Each $\gamma \in G$ induces a fibre bundle isomorphism $\gamma^{\mathcal{E}}$ so that $\gamma \circ \pi^{\mathcal{E}} = \pi^{\mathcal{E}} \circ \gamma^{\mathcal{E}}$. We assume that G acts on B, too, for which M becomes a G-equivariant bundle over B. We also require that ω^M and $h^{\mathcal{E}}$ are both G-invariant.

The actions of γ on these manifolds and the natural induced action of γ on the various tensor bundles and sections will be all if not other stated simply denoted by γ.

For $\gamma \in G$ let

$$M_\gamma := \{x \in M |\ \gamma \cdot x = x\}, \quad B_\gamma := \{b \in B |\ \gamma \cdot b = b\} \qquad \text{and} \qquad Z_\gamma := \{z \in Z |\ \gamma \cdot z = z\}$$

be the fixed-point sets of γ. Then we have a holomorphic fibration $\pi_\gamma : M_\gamma \to B_\gamma$ with compact fibre Z_γ, ([Ma00, p.1550]).

Let $\gamma \in G$ be given and let $N_{Z_\gamma/Z}$ be the normal bundle along Z_γ. Then γ acts on $N_{Z_\gamma/Z}$ and we have a holomorphic orthogonal splitting

$$T_hZ_{|Z_\gamma} = T_hZ_\gamma \oplus N_{Z_\gamma/Z}$$

which is preserved by γ. Since ω^M is γ-invariant h^{TZ} is also γ-invariant and the distinct eigenvalues $1, e^{i\theta_1} \dots e^{i\theta_q}$ $(0 < \theta_j < 2\pi)$ of γ are locally constant. T_hZ_γ is exactly the eigenbundle with eigenvalue 1 of γ. Let $N^{i\theta_1}_{Z_\gamma/Z}, \dots, N^{i\theta_q}_{Z_\gamma/Z}$ be the eigenbundles corresponding to the other eigenvalues. Let $h^{TZ_\gamma}, h^{N_{Z_\gamma/Z}}, h^{N^{i\theta_1}_{Z_\gamma/Z}}, \dots, h^{N^{i\theta_q}_{Z_\gamma/Z}}$ be the Hermitian metrics on $T_hZ_\gamma, N_{Z_\gamma/Z}, N^{i\theta_1}_{Z_\gamma/Z}, \dots, N^{i\theta_q}_{Z_\gamma/Z}$ induced by h^{TZ}. Then $\nabla^{TZ}_{|Z_\gamma}$ induces the holomorphic Hermitian connections $\nabla^{TZ_\gamma}, \nabla^{N_{Z_\gamma/Z}}, \nabla^{N^{i\theta_1}_{Z_\gamma/Z}}, \dots, \nabla^{N^{i\theta_q}_{Z_\gamma/Z}}$. Let $\Omega^{TZ_\gamma}, \Omega^{N_{Z_\gamma/Z}}, \Omega^{N^{i\theta_1}_{Z_\gamma/Z}}, \dots, \Omega^{N^{i\theta_q}_{Z_\gamma/Z}}$ be their curvatures.

For a (q, q) matrix A put

$$\mathrm{Td}(A) = \det\Big(\frac{A}{1 - e^{-A}}\Big), \;\; \mathrm{ch}(A) = \mathrm{Tr}\,(e^A), \quad c_{\max}(A) = \det A.$$

The genera associated to Td, ch and $c_{\max}$ are called the Todd genus, the Chern character and the maximal Chern class or Euler genus.

Definition 1.3.1. *Let (F, h^F) be an arbitrary Hermitian holomorphic vector bundle over Z with Hermitian holomorphic connection ∇^F and curvature Ω^F. Put*

$$\mathrm{Td}_\gamma(TZ, h^{TZ}) = \mathrm{Td}\Big(\frac{-\Omega^{TZ_\gamma}}{2\pi i}\Big) \prod_{j=1}^q \Big(\frac{\mathrm{Td}}{c_{\max}}\Big)\Big(\frac{-\Omega^{N^{\theta_j}_{Z_\gamma/Z}}}{2\pi i} + i\theta_j\Big),$$

$$\mathrm{Td}'_\gamma(TZ, h^{TZ}) = \frac{\partial}{\partial b}\Big[\mathrm{Td}\Big(\frac{-\Omega^{TZ_\gamma}}{2\pi i} + b\Big) \prod_{j=1}^q \Big(\frac{\mathrm{Td}}{c_{\max}}\Big)\Big(\frac{-\Omega^{N^{\theta_j}_{Z_\gamma/Z}}}{2\pi i} + i\theta_j + b\Big)\Big]_{|b=0},$$

$$(\mathrm{Td}^{-1}_\gamma)'(TZ, h^{TZ}) = \frac{\partial}{\partial b}\Big[\mathrm{Td}^{-1}\Big(\frac{-\Omega^{TZ_\gamma}}{2\pi i} + b\Big) \prod_{j=1}^q \Big(\frac{\mathrm{Td}}{c_{\max}}\Big)^{-1}\Big(\frac{-\Omega^{N^{\theta_j}_{Z_\gamma/Z}}}{2\pi i} + i\theta_j + b\Big)\Big]_{|b=0} \quad \textit{and}$$

$$\mathrm{ch}_\gamma(F, h^F) = \mathrm{Tr}\Big[\gamma \exp\Big(\frac{-\Omega^F_{|Z_\gamma}}{2\pi i}\Big)\Big].$$

These are closed differential forms on Z_γ and their cohomology class does not depend on the metric. The cohomology classes will be denoted by $\mathrm{Td}_\gamma(TZ), \mathrm{Td}'_\gamma(TZ), (\mathrm{Td}^{-1}_\gamma)'(TZ)$ and $\mathrm{ch}_\gamma(F)$.

We make the assumption that the direct image $R^\bullet\pi_*\mathcal{E}$ of $\mathcal{E}$ by π is locally free, i.e. the $R^k\pi_*\mathcal{E}$, $0 \le k \le n$ are locally free. For each base point $b \in B$, let $H^\bullet(Z_b, \mathcal{E}_b)$ be the cohomology of the sheaf of holomorphic sections of $\mathcal{E}$ over the fibre Z_b. Then by the assumption the $H^\bullet(Z_b, \mathcal{E}_b)$ form a $\mathbf{Z}$-graded holomorphic vector bundle $H(Z, \mathcal{E}_{|Z})$ on B and $R^\bullet\pi_*\mathcal{E} = H(Z, \mathcal{E}_{|Z})$.
For $b \in B$ let $K(Z_b, \mathcal{E}_b) := \ker(D^{Z_b})$. By Hodge theory we know that for every $b \in B$

$$H^\bullet(Z_b, \mathcal{E}_b) \cong K(Z_b, \mathcal{E}_b). \tag{1.3.1}$$

The Hermitian product on $\mathcal{E}_b$ restricts to the right side so that by the isomorphism above h^{TZ} and $h^{\mathcal{E}}$ induces a G-invariant metric $h^{H(Z,\mathcal{E}_{|Z})}$ on the holomorphic vector bundle $H(Z, \mathcal{E}_{|Z})$ for which the $H^k(Z, \mathcal{E}_{|Z})$ are mutually orthogonal.

Definition 1.3.2. *Let $\nabla^{H(Z,\mathcal{E}_{|Z})}$ be the Hermitian holomorphic connection on $\left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right)$.*

Let P^{K_b} be the orthogonal projection from $\mathbf{E}_b$ to $K(Z_b, \mathcal{E}_{|Z_b})$. We define the connection $\nabla^{K(Z,\mathcal{E}_{|Z})}$ on $K(Z, \mathcal{E}_{|Z})$ by

$$\nabla^{K(Z,\mathcal{E}_{|Z})} := P^K \nabla^{\mathbf{E}} P^K.$$

We quote the following Proposition which has been proven in [BK92, Theorem 3.2].

Proposition 1.3.3. *Under the identification (1.3.1) the connections $\nabla^{H(Z,\mathcal{E}_{|Z})}$ and $\nabla^{K(Z,\mathcal{E}_{|Z})}$ agree.*

Definition 1.3.4. *Set*

$$\begin{aligned} \mathrm{ch}_\gamma\left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right) &= \sum_{k=0}^{n} (-1)^k \,\mathrm{ch}_\gamma\left(H^k(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right) \qquad \textit{and} \\ \mathrm{ch}'_\gamma\left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right) &= \sum_{k=0}^{n} (-1)^k k \,\mathrm{ch}_\gamma\left(H^k(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right). \end{aligned}$$

Before we come to the next result which involves these classes we introduce some notations again. Let $(\alpha_u)_{u>0}$ and α be smooth differential forms on B_γ. We say that as $u \to \infty$ (resp. $u \to 0$), $\alpha_u = \alpha + O(f(u))$, if and only if for any compact set K in B_γ and any $k \in \mathbf{N}_0$ there exists $C > 0$ such that for every $u \ge 1$ (resp. $u \le 1$) the norm of all derivatives of order lesser equal k of $\alpha_u - \alpha$ over K is bounded by $Cf(u)$.

Recall that the definition of the vector spaces P^B and $P^{B,0}$ was given in chapter 0. The spaces $P^{M_\gamma}, P^{M_\gamma,0}, P^{B_\gamma}, P^{B_\gamma,0}$ are defined in the same manner.

Let $\Phi \in \Gamma(B, \mathrm{End}(\Lambda^{\mathrm{even}} T^*_{\mathbf{C}} B))$ be the endomorphism defined by

$$\Phi : \alpha \mapsto (2\pi i)^{-\deg\alpha/2}\alpha.$$

Now we can state the following Theorem established in [Ma00, Theorem 2.10].

Theorem 1.3.5. *As $u \to 0$*

$$\Phi \operatorname{Tr}_s\left[\gamma \exp(-B_u^2)\right] = \int_{Z_\gamma} \mathrm{Td}_\gamma(TZ, h^{TZ})\, \mathrm{ch}_\gamma(\mathcal{E}, h^{\mathcal{E}}) + O(u).$$

There are differential forms $C_{j,\gamma} \in P^{B_\gamma}$, $j \geq -1$, such that for $k \in \mathbf{N}_0$ as $u \to 0$

$$\Phi \operatorname{Tr}_s\left[\gamma N_u \exp(-B_u^2)\right] = \sum_{j=-1}^{k} C_{j,\gamma} u^j + O(u^{k+1}). \tag{1.3.2}$$

Furthermore in $P^{B_\gamma}/P^{B_\gamma,0}$ the first coefficients in the asymptotic expansion are given by

$$C_{-1,\gamma} = \int_{Z_\gamma} \frac{\omega^M}{2\pi} \mathrm{Td}_\gamma(TZ, h^{TZ})\, \mathrm{ch}_\gamma(\mathcal{E}, h^{\mathcal{E}}) \quad \text{in } P^{B_\gamma}/P^{B_\gamma,0} \quad \text{and} \tag{1.3.3}$$

$$C_{0,\gamma} = \int_{Z_\gamma} \Big(\dim_{\mathbf{C}} Z \cdot \mathrm{Td}_\gamma(TZ, h^{TZ}) - \mathrm{Td}'_\gamma(TZ, h^{TZ})\Big)\, \mathrm{ch}_\gamma(\mathcal{E}, h^{\mathcal{E}}) \quad \text{in } P^{B_\gamma}/P^{B_\gamma,0}. \tag{1.3.4}$$

As $u \to \infty$

$$\Phi \operatorname{Tr}_s\left[\gamma \exp(-B_u^2)\right] = \mathrm{ch}_\gamma\left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right) + O\Big(\frac{1}{\sqrt{u}}\Big) \quad \text{and}$$

$$\Phi \operatorname{Tr}_s\left[\gamma N_u \exp(-B_u^2)\right] = \mathrm{ch}'_\gamma\left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right) + O\Big(\frac{1}{\sqrt{u}}\Big).$$

For $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$ by Theorem 1.3.5 one can set

$$\zeta_{\gamma,1}(s) := -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1} \Phi \operatorname{Tr}_s\left[\gamma N_u \exp(-B_u^2)\right] - \mathrm{ch}'_\gamma\left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right) du.$$

The function $\zeta_{\gamma,1}(s)$ extends to a holomorphic function of $s \in \mathbf{C}$ on $\{|\mathrm{Re}(s)| < \frac{1}{2}\}$.

For $s \in \mathbf{C}$ with $\mathrm{Re}(s) < \frac{1}{2}$ set

$$\zeta_{\gamma,2}(s) := -\frac{1}{\Gamma(s)} \int_1^\infty u^{s-1} \Phi \operatorname{Tr}_s\left[\gamma N_u \exp(-B_u^2)\right] - \mathrm{ch}'_\gamma\left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})}\right) du.$$

Then $\zeta_{\gamma,2}(s)$ also extends to a holomorphic function on $\{|\mathrm{Re}(s)| < \frac{1}{2}\}$.

Definition 1.3.6. *For $s \in \mathbf{C}$ with $|\mathrm{Re}(s)| < \frac{1}{2}$ the zeta function ζ_γ is defined by*

$$\zeta_\gamma(s) := \zeta_{\gamma,1}(s) + \zeta_{\gamma,2}(s).$$

Set

$$T_\gamma(\omega^M, h^{\mathcal{E}}) := \zeta'_\gamma(0)$$

The function $\gamma \mapsto T_\gamma(\omega^M, h^{\mathcal{E}})$ is called equivariant holomorphic analytic torsion form. It is a smooth form on B_γ. The components in the different degrees of $T_\gamma(\omega^M, h^{\mathcal{E}})$ are referred to as equivariant holomorphic analytic torsion forms. If $\gamma = e$ is the neutral element we will write $T(\omega^M, h^{\mathcal{E}})$ instead of $T_e(\omega^M, h^{\mathcal{E}})$ which is called holomorphic analytic torsion form.

Using (1.3.2), (1.3.3), (1.3.4) we see

$$\begin{aligned} T_\gamma(\omega^M, h^{\mathcal{E}}) = &- \int_0^1 \Big(\Phi \operatorname{Tr}_s \left[\gamma N_u \exp(-B_u^2) \right] - \frac{C_{-1,\gamma}}{u} - C_{0,\gamma} \Big) \frac{du}{u} \\ &- \int_0^1 \Big(\Phi \operatorname{Tr}_s \left[\gamma N_u \exp(-B_u^2) \right] - \operatorname{ch}'_\gamma \left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})} \right) \Big) \frac{du}{u} \\ &+ C_{-1,\gamma} + \Gamma'(1) \Big(C_{0,\gamma} - \operatorname{ch}'_\gamma \left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})} \right) \Big). \end{aligned}$$

The equivariant holomorphic analytic torsion form satisfies the following crucial equation known as the curvature formula:

Theorem 1.3.7. *[Ma00, Theorem 2.12] The form $T_\gamma(\omega^M, h^{\mathcal{E}})$ lies in P^{B_γ}. Moreover,*

$$\frac{\bar{\partial}\partial}{2\pi i} T_\gamma(\omega^M, h^{\mathcal{E}}) = \operatorname{ch}_\gamma \left(H(Z, \mathcal{E}_{|Z}), h^{H(Z,\mathcal{E}_{|Z})} \right) - \int_{Z_\gamma} \operatorname{Td}_\gamma(TZ, h^{TZ}) \operatorname{ch}_\gamma(\mathcal{E}, h^{\mathcal{E}}).$$

Now let $(\omega_0^M, h_0^{\mathcal{E}})$ and $(\omega_1^M, h_1^{\mathcal{E}})$ be two couple of G-invariant data. By [BGS88a, §1(f)] there are Bott-Chern classes

$$\begin{aligned} &\widetilde{\operatorname{Td}}_\gamma(TZ, h_0^{TZ}, h_1^{TZ}), \quad \widetilde{\operatorname{ch}}_\gamma(\mathcal{E}, h_0^{\mathcal{E}}, h_1^{\mathcal{E}}) \in P^{M_\gamma}/P^{M_\gamma,0} \\ &\widetilde{\operatorname{ch}}_\gamma(H(Z, \mathcal{E}_{|Z}), h_0^{H(Z,\mathcal{E}_{|Z})}, h_1^{H(Z,\mathcal{E}_{|Z})}) \in P^{B_\gamma}/P^{B_\gamma,0}, \end{aligned}$$

which are defined uniquely axiomatic by [BGS88a, Theorem 1.29]. They fulfill

$$\begin{aligned} &\frac{\bar{\partial}\partial}{2\pi i} \widetilde{\operatorname{Td}}_\gamma(TZ, h_0^{TZ}, h_1^{TZ}) = \operatorname{Td}_\gamma(TZ, h_1^{TZ}) - \operatorname{Td}_\gamma(TZ, h_0^{TZ}), \\ &\frac{\bar{\partial}\partial}{2\pi i} \widetilde{\operatorname{ch}}_\gamma(\mathcal{E}, h_0^{\mathcal{E}}, h_1^{\mathcal{E}}) = \operatorname{ch}_\gamma(\mathcal{E}, h_1^{\mathcal{E}}) - \operatorname{ch}_\gamma(\mathcal{E}, h_0^{\mathcal{E}}) \qquad \text{and} \end{aligned}$$

$$\frac{\bar{\partial}\partial}{2\pi i}\widetilde{\mathrm{ch}}_\gamma(H(Z,\mathcal{E}_{|Z}),h_0^{H(Z,\mathcal{E}_{|Z})},h_1^{H(Z,\mathcal{E}_{|Z})})$$
$$=\mathrm{ch}_\gamma\left(H(Z,\mathcal{E}_{|Z}),h_1^{H(Z,\mathcal{E}_{|Z})}\right)-\mathrm{ch}_\gamma\left(H(Z,\mathcal{E}_{|Z}),h_0^{H(Z,\mathcal{E}_{|Z})}\right).$$

The equivariant holomorphic analytic torsion form verifies, ([Ma00, Theorem 2.13]), the anomaly formula

$$T_\gamma(\omega_1^M,h_1^{\mathcal{E}})-T_\gamma(\omega_0^M,h_0^{\mathcal{E}})=\widetilde{\mathrm{ch}}_\gamma(H(Z,\mathcal{E}_{|Z}),h_0^{H(Z,\mathcal{E}_{|Z})},h_1^{H(Z,\mathcal{E}_{|Z})})$$
$$-\int_{Z_\gamma}\widetilde{\mathrm{Td}}_\gamma(TZ,h_0^{TZ},h_1^{TZ})\,\mathrm{ch}_\gamma(\mathcal{E},h_0^{\mathcal{E}})-\int_{Z_\gamma}\mathrm{Td}_\gamma(TZ,h_1^{TZ})\widetilde{\mathrm{ch}}_\gamma(\mathcal{E},h_0^{\mathcal{E}},h_1^{\mathcal{E}}).$$

In particular the class of $T_\gamma(\omega^M,h^{\mathcal{E}})$ in $P^{B_\gamma}/P^{B_\gamma,0}$ only depends on $(h^{TZ},h^{\mathcal{E}})$.

Remark 1.3.8. *At this point we want to point out some of the different conventions in the cited papers in this thesis. The fibrewise L^2-norm (1.1.1) is not always defined with with the factor $1/(2\pi)^{\dim}$ (eg. [BGS88b],[MM07]). For more background on this factor see [BL91]. In the works of Bismut and* [*P16*] *the Kähler from ω^M differs from ours by a minus sign. In [Bi13, (3.1.1)] another convention for the Clifford action has been used. In comparison with [F18, (2.8)] where B is a point our definition of D^Z differs by a factor $\sqrt{2}$ and thus other appearing operators differs as well. The reader has to be aware what impact these changes has on the resulting formulas.*

Chapter 2

The Asymptotic of the Holomorphic Analytic Torsion Forms

The aim of this chapter is the proof of Theorem 1 and Theorem 2, the full asymptotic of the holomorphic analytic torsion forms. The techniques we are using here are very close to [DLM06], [MM07, chapters 4.1 and 4.2] , [F18] and [P16]. Let us describe in more detail how this chapter is organized. After introducing the higher power of a line bundle with the for our purpose desired properties we start in section 2.1 which is divided in subparts a) - d), with the localization technique of Bismut-Lebeau, ([BL91]), using the finite propagation speed of the wave equation. The threatment of how to use it with the operators in our case has been shown in [P16] and we merely recall his results in a). In b) - d) we follow closely the ideas of [DLM06] and [MM07, Chapters 4.1 and 4.2] where additional adjustments have to be make regarding the structure of the Bismut superconnection. One of more difficult task we will be confronted is the large time behavior of its curvature which will get an own section 2.4. In section 2.2, respectively section 2.3, we will proof Theorem 1, respectively Theorem 2. We will catch up the postponed proofs from section 2.2 in section 2.4.

The objects are the same as in chapter 1. Now let $(\mathcal{L}, h^{\mathcal{L}})$ be a holomorphic Hermitian line bundle on M. We denote the curvature of the Hermitian holomorphic connection $\nabla^{\mathcal{L}}$ of $(\mathcal{L}, h^{\mathcal{L}})$ by $\Omega^{\mathcal{L}}$. We make the assumption that $i\Omega^{\mathcal{L}}$ is positive along the fibres, that is, for any $0 \neq U \in T^{1,0}Z$, we have

$$\Omega^{\mathcal{L}}(U, \overline{U}) > 0.$$

Such holomorphic Hermitian line bundles with this property will be called positive. Let $\dot{\Omega}^{Z,\mathcal{L}} \in$

$\mathrm{End}(T^{1,0}Z)$ be the Hermitian matrix such that for $V, W \in T^{1,0}Z$

$$\Omega^{\mathcal{L}}(V, \overline{W}) = \langle \dot{\Omega}^{Z,\mathcal{L}} V, W \rangle_{h^{T^{1,0}Z}}.$$

By our assumption $\dot{\Omega}^{Z,\mathcal{L}} \in \mathrm{End}(T^{1,0}Z)$ is positive definite. For $p \in \mathbf{N}$ set $\mathcal{L}^p := \mathcal{L}^{\otimes p}$. We will assume that the direct image $R^i\pi_*(\mathcal{E} \otimes \mathcal{L}^p)$ is locally free for p large. Now all the construction from chapter 1 will be used here for $(\mathcal{E} \otimes \mathcal{L}^p, h^{\mathcal{E}\otimes\mathcal{L}^p})$ instead of $(\mathcal{E}, h^{\mathcal{E}})$ and we consider the case $\gamma = e$. In this case the torsion form is real ([BK92, Theorem 3.9]) and the coefficients $C_{j,e}$ in (1.3.2) are real forms as well. In the non-equivariant case when γ is absent we are dealing with $\Lambda^\bullet T^*_{\mathbf{R}}B$ instead of $\Lambda^\bullet T^*_{\mathbf{C}}B$. The corresponding spaces and operators will be denoted by

$$\begin{aligned}
&\mathbf{E}^k_{p,b} = \Gamma(Z_b, (\Lambda^{0,k}(T^*Z) \otimes \mathcal{E} \otimes \mathcal{L}^p)_{|Z_b}),\\
&\bar{\partial}^p = \text{Dolbeault operator of } \mathbf{E}_p\ ,\\
&D_p = \bar{\partial}^p + \bar{\partial}^{p,*},\\
&B_{p,u} = \text{corresponding rescaled Bismut superconnection with } B_p := B_{p,1},\\
&\zeta_{\gamma,p} = \text{corresponding zeta function with } \zeta_p := \zeta_{e,p}.
\end{aligned}$$

By [MM07, Theorem 1.5.8] the operator D_p^2 has a spectral gap property that is there exists a constant $C_{\mathcal{L}} > 0$ depending on $\mathcal{L}$ and $\mu_0 > 0$ such that

$$\mathrm{Spec}(D_p^2) \subset \{0\} \cup]2p\mu_0 - C_{\mathcal{L}}, \infty[. \tag{2.0.1}$$

In particular since by [P16, (2.22)] (or Theorem A.0.3 of Appendix A) we have $\mathrm{Spec}(B_p^2) = \mathrm{Spec}(D_p^2)$, the operator B_p^2 has this property as well which plays a crucial part in the approach for the asymptotic. Note that the spectral gap does not hold if the line bundle is only semipositive ([Do03]).

2.1 Localization

Let $b_0 \in B$ be a given base point and $z_0 \in Z_{b_0}$. Since in this section we will work along the fibres we will denote all the fibres Z_{b_0} simply by Z. In particular $g^{T_{\mathbf{R}}Z_{b_0}}$ will be denoted by $g^{T_{\mathbf{R}}Z}$ as well as the corresponding Riemannian volume form.

---

a) Normal coordinate and localization

For $\varepsilon > 0$ and $z_0 \in Z$ let $B_\varepsilon^Z(z_0)$ and $B_\varepsilon^{T_{\mathbf{R},z_0}Z}(0)$ be the open balls in Z and $T_{\mathbf{R},z_0}Z$ with radius ε and center z_0 and 0 respectively. For ε sufficiently small the exponential map $\exp_{z_0}^Z : B_\varepsilon^{T_{\mathbf{R},z_0}Z}(0) \to B_\varepsilon^Z(z_0), V \mapsto \exp_{z_0}^Z V$ is a diffeomorphism. $B_\varepsilon^Z(z_0)$ and $B_\varepsilon^{T_{\mathbf{R},z_0}Z}(0)$ will be identified by this. In particular $0 \in T_{\mathbf{R},z_0}Z$ will represents z_0.
Let inj^Z be the injectivity radius of Z and let $\varepsilon \in]0, \mathrm{inj}^Z/4[$. Such an ε can be chosen uniformly for b_0 varying in a compact subset of B. Because we are often going to work on $T_{\mathbf{R},z_0}Z$ set

$$Z_0 := T_{\mathbf{R},z_0}Z.$$

The notation of the orthonormal basis $\{e_i\}_i$ of chapter 1 will also be used for Z_0 (instead of $\{e_{i,z_0}\}_i$) without pointing out its dependence on z_0. We will identify $\mathbf{R}^{2n}$ with Z_0 by the isomorphism

$$\mathbf{R}^{2n} \ni (V_1, \ldots, V_{2n}) \mapsto \sum_{i=1}^{2n} V_i e_i \in T_{\mathbf{R},z_0}Z.$$

Let $g^{T_\mathbf{R}Z_0}$ be a Riemannian metric on Z_0 with

$$g^{T_\mathbf{R}Z_0} := \begin{cases} g^{T_\mathbf{R}Z} & \text{on } B_{2\varepsilon}^{T_{\mathbf{R},z_0}Z}(0) \\ g_{z_0}^{T_\mathbf{R}Z} & \text{on } T_{\mathbf{R},z_0}Z \setminus B_{4\varepsilon}^{T_{\mathbf{R},z_0}Z}(0). \end{cases}$$

Let $d\mathrm{vol}_{g^{T_\mathbf{R}Z_0}}$ denote the associated volume form. Let $d\mathrm{vol}_{g_{z_0}^{T_\mathbf{R}Z}}$ be the Riemannian volume form of $(Z_0, g_{z_0}^{T_\mathbf{R}Z})$ and let $\kappa(V)$ be the smooth positive function on Z_0 defined by the equations $\kappa(0) = 1$ and

$$d\mathrm{vol}_{g^{T_\mathbf{R}Z_0}}(V) = \kappa(V) d\mathrm{vol}_{g_{z_0}^{T_\mathbf{R}Z}}(V), \qquad V \in Z_0.$$

We identify $(\mathcal{E}_V, h_V^\mathcal{E})$, $(\mathcal{L}_V, h_V^\mathcal{L})$ and $(\Lambda_V^{0,\bullet}(T^*Z), h_V^{\Lambda^{0,\bullet}(T^*Z)})$ with $(\mathcal{E}_{z_0}, h^{\mathcal{E}_{z_0}})$, $(\mathcal{L}_{z_0}, h^{\mathcal{L}_{z_0}})$ and $(\Lambda^{0,\bullet}(T_{z_0}^*Z), h^{\Lambda^{0,\bullet}(T_{z_0}^*Z)})$ by parallel transport along the geodesic ray $[0,1] \ni t \mapsto tV$ with respect to the connections ∇_1 and $\nabla^\mathcal{L}$. Recall that ∇_1 was defined in (1.2.1) with $u = 1$. Let ϑ_1 and $\vartheta^\mathcal{L}$ be the corresponding connection forms.

Let $\rho : \mathbf{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = \begin{cases} 1\ , & |v| < 2, \\ 0\ , & |v| > 4. \end{cases}$$

Let d_V be the ordinary differentiation operator in direction V on $T_{z_0}Z$. On the trivial bundle

$$\mathbb{E}_{p,z_0} := \Lambda^\bullet(T_{\mathbf{R},b_0}^*B) \otimes (\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E} \otimes \mathcal{L}^p)_{z_0}$$

over $T_{z_0}Z$ define the Hermitian connection

$$\nabla^{\mathbb{E}_{p,z_0}} := d + \rho\Big(\frac{\|V\|}{\varepsilon}\Big)(p\vartheta^{\mathcal{L}} + \vartheta^1).$$

Let $\Delta^{\mathbb{E}_{p,z_0}}$ be the Bochner Laplacian associated with $\nabla^{\mathbb{E}_{p,z_0}}$ and $g^{T_{\mathbf{R}}Z_0}$. Let $\nabla^{T_{\mathbf{R}}Z_0}$ be the Levi-Civita connection on $(Z_0, g^{T_{\mathbf{R}}Z_0})$. Let $\tilde{e}_i(V)$ be the parallel transport of e_i with respect to $\nabla^{T_{\mathbf{R}}Z_0}$ along the curve $[0,1] \ni t \to tV$. Set

$$\begin{aligned}\Psi :=& \frac{s^Z}{8} + \frac{1}{4}c(\tilde{e}_i)c(\tilde{e}_j)\Big[\Omega^{\mathcal{E}} + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big] + \frac{1}{\sqrt{2}}c(\tilde{e}_i)f^\alpha\Big[\Omega^{\mathcal{E}} + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big](\tilde{e}_i, f_\alpha)\\ &+ \frac{f^\alpha f^\beta}{2}\Big[\Omega^{\mathcal{E}} + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big](f_\alpha, f_\beta).\end{aligned} \tag{2.1.1}$$

Define the operator

$$\begin{aligned}M_{p,z_0} :=& \frac{1}{2}\Delta^{\mathbb{E}_{p,z_0}} + \rho\Big(\frac{\|V\|}{\varepsilon}\Big)\Psi + p\rho\Big(\frac{\|V\|}{\varepsilon}\Big)\Big(\frac{1}{4}c(\tilde{e}_i)c(\tilde{e}_j)\Omega^{\mathcal{L}}(\tilde{e}_i,\tilde{e}_j)\\ &+ \frac{1}{\sqrt{2}}c(\tilde{e}_i)f^\alpha\Omega^{\mathcal{L}}(\tilde{e}_i, f_\alpha) + \frac{f^\alpha f^\beta}{2}\Omega^{\mathcal{L}}(f_\alpha, f_\beta)\Big) \quad \in \operatorname{Op}(\mathbb{E}_{p,z_0})\end{aligned}$$

which is a second order elliptic differential operator acting on $\Gamma(Z_0, \mathbb{E}_{p,z_0})$ and coincides with B_p^2 over $B_\varepsilon^{TZ}(0)$ by the Lichnerowicz formula. Let $\langle V|e^{-M_{p,z_0}}|V'\rangle$ be the smooth kernel of the operator M_{p,z_0} with respect to $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_0}}(V')$. Set

$$\mathbb{E}_{z_0} := \Lambda^\bullet(T^*_{\mathbf{R},b_0}B) \otimes (\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E})_{z_0}.$$

$\mathbb{E}$ will be equipped with a connection $\nabla^{\mathbb{E}}$ induced by $\nabla^{T_{\mathbf{R}}B,LC}$ and $\nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$ and with metric $h^{\mathbb{E}}$ induced by $g^{T_{\mathbf{R}}B}$ and $h^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$.

An unit vector of $\mathcal{L}_{z_0}$ gives an isometry $\mathcal{L}^p_{z_0} \cong \mathbf{C}$ and therefore

$$\mathbb{E}_{p,z_0} \cong \mathbb{E}_{z_0}.$$

With this trivialization B_p^2 acts on $\mathbb{E}_{z_0}$ and we will consider M_{p,z_0} as an operator acting on $\Gamma(Z_0, \mathbb{E}_{z_0})$. Because the kernel restricted to the diagonal has its coefficients in $\operatorname{End}(\mathbb{E}_{p,z_0}) \overset{\text{can.}}{\cong} \operatorname{End}(\mathbb{E}_{z_0})$ the formulas do not depend on the choice of the unit vectors.

In [P16, Lemma 2.7] Puchol proved by using the localization technique of Bismut-Lebeau relying on finite propagation speed of the wave equation (cf.[BL91] or [MM07]) that $\forall m \in$

$\mathbf{N}_0, \varepsilon > 0\ \exists C > 0, N \in \mathbf{N} : \forall p \in \mathbf{N}$

$$\Big\| \Big\langle z_0 \Big| \exp\big(-\frac{u}{p}B_p^2\big)\Big|z_0\Big\rangle - \Big\langle 0\Big| \exp\big(-\frac{u}{p}M_{p,z_0}\big)\Big|0\Big\rangle \Big\|_{\mathscr{C}^m(M,\mathrm{End}(\mathbb{E}))} \le Cp^N \exp\Big(-\frac{\varepsilon^2 p}{16u}\Big) \tag{2.1.2}$$

where $\|\cdot\|_{\mathscr{C}^m(M,\mathrm{End}(\mathbb{E}))}$ is the $\mathscr{C}^m$-norm in the parameters $b_0 \in B$ and $z_0 \in Z_{b_0}$ induced by $\nabla^{\mathrm{End}\,\mathbb{E}}$ and $h^{\mathrm{End}\,\mathbb{E}}$. By this localization result one is able to replace the manifold by Z_0 and consider the operator M_{p,z_0} instead.

---

b) Rescaling and Taylor expansion of the rescaled operator

We now do a change of the parameter. Set $t := \frac{1}{\sqrt{p}} \in]0,1]$. The idea behind is to work with an operator depending smoothly on $t \in [0,1]$ so that the Taylor series can be applied at zero from which we get an asymptotic expansion of the kernel.

Definition 2.1.1. *For $s \in \Gamma(Z_0, \mathbb{E}_{z_0})$ and $V \in Z_0$ set*

$$(S_t s)(V) := s(V/t), \qquad \nabla_{t,z_0} := tS_t^{-1}\kappa^{1/2}\nabla^{\mathbb{E}_{p,z_0}}\kappa^{-1/2}S_t,$$
$$\nabla_{0|_V} := d_V + \frac{1}{2}\Omega^{\mathcal{L}}_{z_0}(V,\cdot) \quad \text{and} \quad L_{t,z_0} := t^2 S_t^{-1}\kappa^{1/2}M_{p,z_0}\kappa^{-1/2}S_t.$$

Note that in the definition we have used the mentioned identification $\mathbb{E}_{p,z_0} \cong \mathbb{E}_{z_0}$. For simplicity we will often omit the point z_0 in the notation and write L_t, ∇_t etc. Nonetheless one has to keep the dependence on z_0 in mind.

Lemma 2.1.2. *There exist polynomials $\mathcal{A}_{i,j,r}$ (resp. $\mathcal{B}_{i,r}, \mathcal{C}_r$) in $V \in Z_0$, where $r \in \mathbf{N}, i, j \in \{1, \ldots, 2n\}$, with the following properties:*

1.) their coefficients are polynomials in Ω^{TZ} (resp. $\Omega^1, \Psi, \Omega^{\mathcal{L}}$) and their derivatives at z_0 up to order $r-2$ (resp. $r-2, r-2, r$),

2.) $\mathcal{A}_{i,j,r}$ is a homogenous polynomial in V of degree r, the degree in V of $\mathcal{B}_{i,r}$ is $\le r+1$ (resp. $\mathcal{C}_r$ is $\le r+2$) and has the same parity with $r-1$ (resp. r),

3.) if we set

$$\mathcal{O}_r := \mathcal{A}_{i,j,r}\nabla_{e_i}\nabla_{e_i} + \mathcal{B}_{i,r}\nabla_{e_i} + \mathcal{C}_r$$

and

$$L_0 := -\frac{1}{2}\sum_i (\nabla_{0,e_i})^2 + \frac{1}{4}c(e_i)c(e_j)\Omega^{\mathcal{L}}(e_i,e_j), + \frac{1}{\sqrt{2}}c(e_i)f^\alpha\Omega^{\mathcal{L}}(e_i,f_\alpha) + \frac{f^\alpha f^\beta}{2}\Omega^{\mathcal{L}}(f_\alpha,f_\beta)$$

then the operator L_t has an expansion of the form

$$L_t = L_0 + \sum_{r=1}^{m} \mathcal{O}_r t^r + O(t^{m+1}).$$

Moreover there exists $m' \in \mathbf{N}_0$ so that for every $k \in \mathbf{N}_0$ the derivatives up to order k of the coefficients of the operator $O(t^{m+1})$ are dominated by $C(1+\|V\|)^{m'} t^{m+1}$.

Proof. Lemma 2.1.2 has proven in [P16, Proposition. 2.9] for the case $m = 0$ including the formula for L_0. For arbitary $m \geq 0$ Lemma 2.1.2 has already been proven in [MM07, Theorem 4.1.7] when B is a point and we will proceed in the same way with the present of additional horizontal and coupled terms. From the definition of ∇_t we have

$$\nabla_{t,e_i|_V} = \kappa^{1/2}(tV)\Big\{\nabla_{e_i} + \rho\Big(t\frac{\|V\|}{\varepsilon}\Big)\big(t^{-1}\vartheta^{\mathcal{L}}_{tV}(e_i) + t\vartheta^{1}_{tV}(e_i)\big)\Big\}\kappa^{-1/2}(tV). \tag{2.1.3}$$

Set $g_{ij}(V) := g^{T_{\mathbf{R}}Z_0}(e_i, e_j)_{|_V}$ and let $(g^{ij}(V))_{ij}$ be the inverse of the matrix $(g_{ij}(V))_{ij}$. Let ∇^{TZ_0} be the Levi-Civita connction associated to $g^{T_{\mathbf{R}}Z_0}$. Since in normal coordinates the $c(\tilde{e}_i)_{|_V} = c(e)$ are constant we have

$$\begin{aligned} L_{t|_V} &= -\frac{1}{2}g^{ij}(tV)\Big(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TZ_0}_{e_i}e_j}\Big) \\ &+ \rho(t|V|/\varepsilon)\Big(t^2\Psi_{z_0} + \frac{1}{4}c(e_i)c(e_j)\Omega^{\mathcal{L}}(\tilde{e}_i, \tilde{e}_j) + \frac{1}{\sqrt{2}}c(e_i)f^{\alpha}\Omega^{\mathcal{L}}(\tilde{e}_i, f_\alpha) + \frac{f^\alpha f^\beta}{2}\Omega^{\mathcal{L}}(f_\alpha, f_\beta)\Big)_{tV}. \end{aligned} \tag{2.1.4}$$

Classically, see for instance [MM07, Lemma 1.24], with $\bullet \in \{\mathcal{L}, 1\}$ we have the expansions

$$\sum_{|\alpha|=r} (\partial^\alpha \vartheta^\bullet)_{z_0}(e_j)\frac{V^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha\Omega^\bullet)_{z_0}(V, e_j)\frac{Z^\alpha}{\alpha!}. \tag{2.1.5}$$

Because of

$$\kappa(V) = \sqrt{|\det(g_{ij})(V)|} \tag{2.1.6}$$

Lemma 2.1.2 follows from (2.1.4), (2.1.5) and [MM07, Lemma 1.2.3]. □

---

c) Parameter depended norms and the calculation of $\frac{\partial^r}{\partial t^r}_{|t=0} e^{-uL_t}$

By (2.1.3) and (2.1.4) we see that the operator L_t can be extended smoothly to all $t \in]0, 1]$. Before we can apply Taylor series at zero to its kernel we have to show that it is smooth at zero.

This is the goal of this part of the section.

In the sequel to the objects which have coefficients in $\Lambda^\bullet(T^*_{\mathbf{R},b_0}B)$ an attached superscripted (0) will be meant their part of degree zero in $\Lambda^\bullet(T^*_{\mathbf{R},b_0}B)$.

Let $h^{\mathbb{E}_{z_0}}$ be the metric on $\mathbb{E}_{z_0}$ induced by $h_{z_0}^{\Lambda^\bullet(T^*_{\mathbf{R}}B)}$, $h_{z_0}^{\Lambda^{0,\bullet}}$, $h_{z_0}^{\mathcal{E}}$ and let $\|\cdot\|_{h^{\mathbb{E}_{z_0}}}$ denote the pointwise norm. For $s \in \Gamma(Z_0, \mathbb{E}_{z_0})$ let

$$\|s\|_0 := \|s\|_{L^2(Z_0,\mathbb{E}_{z_0})} = \frac{1}{(2\pi)^n} \int\limits_{Z_0} \|s(V)\|^2_{h^{\mathbb{E}_{z_0}}} d\mathrm{vol}_{g_{z_0}^{T_{\mathbf{R}}Z}}(V)$$

be the L^2-norm on $\Gamma(Z_0, \mathbb{E}_{z_0})$ induced by $h^{\mathbb{E}_{z_0}}$ and the volume form $d\mathrm{vol}_{g_{z_0}^{T_{\mathbf{R}}Z}}$. For $m \in \mathbf{N}$ and $t \geq 0$ set

$$\begin{aligned} &\|s\|^2_{t,0} := \|s\|^2_0 \qquad \text{and} \\ &\|s\|^2_{t,m} := \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l=1}^{2n} \|\nabla^{(0)}_{t,e_{i_1}} \cdots \nabla^{(0)}_{t,e_{i_l}} s\|^2_0. \end{aligned} \tag{2.1.7}$$

In degree zero the elliptic operator $L_t^{(0)}$ is formal self-adjoint with respect to $\|\cdot\|_{t,0}$ while L_t itself does not has to be. Denote by $\mathbf{H}_t^m$ the Sobolev space $\mathbf{H}^m(Z_0, \mathbb{E}_{z_0})$ of order m with the norm $\|\cdot\|_{t,m}$ and by $\mathbf{H}_t^{-1}$ the Sobolev space of order -1 with the norm defined by

$$\|s\|_{t,-1} := \sup_{s' \in \mathbf{H}_t^1 \setminus \{0\}} \frac{\langle s, s' \rangle_{t,0}}{\|s'\|_{t,0}}.$$

For an operater $A \in \mathscr{L}(\mathbf{H}_t^k, \mathbf{H}_t^m)$ we denote by $\|A\|_t^{k,m}$ the operator norm of A associated with $\|s\|_{t,k}$ and $\|s\|_{t,m}$.

By the spectral gap property (2.0.1) of D_p^2 and $\mathrm{Spec}(D_p^2) = \mathrm{Spec}(B_p^2)$ there exists $\vartheta > 0$ such that for p suffieciently large

$$\mathrm{Spec}(\frac{1}{p} B_p^2) \subset \{0\} \cup [\vartheta, \infty[. \tag{2.1.8}$$

By the definition of L_t and since $t^2 = \frac{1}{p}$ we can also find $t_0 \in]0,1]$ sufficiently small such that for all $t \in]0, t_0]$,

$$\mathrm{Spec}(L_t) \subset \{0\} \cup [\vartheta, \infty[.$$

Let $\delta + \Delta$ be the contour in $\mathbf{C}$ indicated by figure 2.1. Then the resolvent $(\lambda - L_t)^{-1}$ exists for $\lambda \in \delta + \Delta$. The boundedness will be given in Lemma 2.1.3 in the next page.

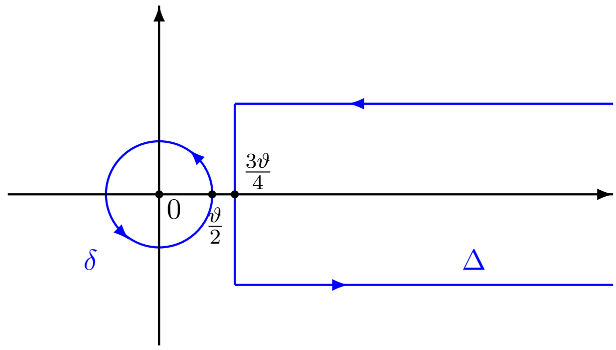


Figure 2.1: Contour $\delta + \Delta$

In [P16, Proposition 2.15 and Proposition 2.17] the countor Γ from figure A.1 in the appendix was considered. But since L_t as no eigenvalues in $]0, \vartheta[$ we have the following analogue result of [P16, Proposition 2.15] for the contour $\delta + \Delta$ which can be proved exactly in the same way making use of that the statement in degree zero which has already been shown in [MM07, Theorem 4.1.10]:

Lemma 2.1.3. *There exsit $C > 0$ and $a, b \in \mathbf{N}_0$ such that for $t \in]0, t_0]$ and $\lambda \in \delta + \Delta$,*

$$\|(\lambda - L_t)^{-1}\|_t^{0,0} \leq C(1 + |\lambda|^2)^a \qquad \text{and}$$
$$\|(\lambda - L_t)^{-1}\|_t^{-1,1} \leq C(1 + |\lambda|^2)^b.$$

The next result is the analogue of [P16, Proposition 2.17] which follows from Lemma 2.1.3 above and [P16, Proposition 2.17] exacatly like [P16, Proposition 2.17] follows from [P16, Proposition 2.15 and 2.16]

Lemma 2.1.4. *For any $t \in]0, t_0]$, $\lambda \in \delta + \Delta$ and $m \in \mathbf{N}_0$ the resolvent $(\lambda - L_t)^{-1}$ maps $\mathbf{H}_t^m$ into $\mathbf{H}_t^{m+1}$,*

$$(\lambda - L_t)^{-1}(\mathbf{H}_t^m) \subset \mathbf{H}_t^{m+1}.$$

Moreover for any multiindex $\alpha \in \mathbf{N}_0^{2n}$ there exist $N \in \mathbf{N}_0$ and $C_{\alpha,m} > 0$ such that for any $t \in]0, t_0], \lambda \in \delta + \Delta$ and $s \in \Gamma_c(Z_0, \mathbb{E}_{z_0})$,

$$\|V^\alpha(\lambda - L_t)^{-1}s\|_{t,m+1} \leq C_{\alpha,m}(1 + |\lambda|^2)^N \sum_{\alpha' \leq \alpha} \|V^{\alpha'}s\|_{t,m}.$$

We will use the same notation as in [MM07, page 184]. For $m \in \mathbf{N}_0$ let $\mathcal{Q}^m$ be the set of operators $\{\nabla^{(0)}_{t,e_{i_1}} \dots \nabla^{(0)}_{t,e_{i_j}}\}_{j\leq m}$. For $k,r\in\mathbf{N}$ set

$$I_{k,r} := \Big\{(\mathbf{k},\mathbf{r}) = (k_i,r_i)_i \Big| \sum_{i=0}^{j} k_i = k+j, \sum_{i=1}^{j} r_i = r,\ k_i,r_i\in\mathbf{N}\Big\}.$$

For $(\mathbf{k},\mathbf{r})\in I_{k,r}$, $\lambda\in\delta+\Delta$ and $t\in]0,t_0]$, set

$$A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t) := (\lambda-L_t)^{-k_0}\frac{\partial^{r_1}L_t}{\partial t^{r_1}}(\lambda-L_t)^{-k_1}\dots\frac{\partial^{r_j}L_t}{\partial t^{r_j}}(\lambda-L_t)^{-k_j}.$$

By induction there exist $a^{\mathbf{k}}_{\mathbf{r}}\in\mathbf{R}$ such that

$$\frac{\partial^r}{\partial t^r}(\lambda-L_t)^{-k} = \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a^{\mathbf{k}}_{\mathbf{r}}A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t). \tag{2.1.9}$$

Lemma 2.1.5. *For any $m\in\mathbf{N}_0$, $k,r\in\mathbf{N}_0$ with $k>2(m+r+1)$ and $(\mathbf{k},\mathbf{r})\in I_{k,r}$ there exists $C>0$ and $N\in\mathbf{N}_0$ such that for any $\lambda\in\delta+\Delta, t\in]0,t_0]$ and operators $Q,Q'\in\mathcal{Q}^m$,*

$$\|QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q's\|_{t,0}\leq(1+|\lambda|)^N\sum_{|\beta|\leq 2r}\|V^\beta s\|_{t,0}.$$

Proof. Lemma 2.1.5 has been proven in [MM07, Theorem 4.1.13] for the case where B is a point. Unlike its part in degree zero $L^{(0)}_t$ the operator L_t is not self-adjoint with respect to $\langle\cdot,\cdot\rangle_{t,0}$. Nonetheless L^*_t has the same structure as L_t and the horizontal parts will not disturb when it comes down to derivations in t. Therefore the main arguments in the proof of [MM07, Theorem 4.1.13] are still valid here which we will now explain in our setting.

From Lemma 2.1.4 we see that there are $N\in\mathbf{N}_0$ and $C_{0,m}>0$ such that for any $\lambda\in\delta+\Delta$ we have $\|(\lambda-L_t)^{-1}\|^{0,1}_t\leq C_{0,m}(1+|\lambda|^2)^N$. We deduce from the definition of the Sobolev-norm (2.1.7) and the set $\mathcal{Q}^m$ that if $Q\in\mathcal{Q}^m$ there is $C_m>0$ with

$$\|Q(\lambda-L_t)^{-m}\|^{0,0}_t\leq C_m(1+|\lambda|^2)^N.$$

By (2.1.4) (see also [P16, (2.75)]) we see that the structure of L_t is of the type

$$\sum_{i,j}a_{i,j}(t,tV)\nabla^{(0)}_{t,e_i}\nabla^{(0)}_{t,e_i}+\sum_i b_i(t,tV)\nabla^{(0)}_{t,e_i}+c(t,tV) \tag{2.1.10}$$

where $a_{i,j},b_i$ and c are polynomials in the first variable and have all their derivatives in the second variable uniformly bounded for $V\in Z_0$ and $t\in[0,1]$. The adjoint connection $(\nabla^{(0)}_t)^*$ of

$(\nabla_t^{(0)})$ with respect to $\langle\cdot,\cdot\rangle_{t,0}$ is given by

$$(\nabla_t^{(0)})^* = -(\nabla_t^{(0)}) - t(\kappa^{-1}d\kappa)(tV).$$

We know that $L_t^{(0)}$ is self-adjoint with respect to $\|\cdot\|_{t,0}$ and since $t(\kappa^{-1}\nabla\kappa)(tV)$ with all its derivates in V are uniformly bounded for $V \in Z_0$ and $t \in [0,1]$ the formal adjoint of L_t with respect to $\|\cdot\|_{t,0}$ has the same structure as the operator L_t in the sense of (2.1.10) and the properties mentioned after. With his the arguments of the proofs in [P16, Proposition 2,16 and 2.17] can be applied to the formal adjoint L_t^* such that we get an analogue result of Lemma 2.1.4 for the adjoint. Especially there is $\tilde{C}_m > 0$ and $\tilde{N} \in N$ such that for any $\lambda \in \delta + \Delta$ and $Q \in \mathcal{Q}^m$

$$\|Q(\lambda - L_t^*)^{-m}\|_t^{0,0} \leq \tilde{C}_m(1+|\lambda|^2)^N.$$

Taking the adjoint we get

$$\|(\lambda - L_t)^{-m}Q\|_t^{0,0} \leq \tilde{C}_m(1+|\lambda|^2)^{\tilde{N}}.$$

We have proven the Lemma for $r=0$. Now consider $r>0$. From the structure of L_t in (2.1.10) the derivative $\frac{\partial^r L_t}{\partial t^r}$ is a linear combination of

$$\frac{\partial^{r_1}}{\partial t^{r_1}}(g^{ij}(tV))\Big(\frac{\partial^{r_2}}{\partial t^{2r}}\nabla_{t,e_i}\Big)\Big(\frac{\partial^{r_3}}{\partial t^{r_3}}\nabla_{t,e_j}\Big), \quad \frac{\partial^{r_1}}{\partial t^{r_1}}(b(tV)), \quad \frac{\partial^{r_1}}{\partial t^{r_1}}(b_i(tV))\Big(\frac{\partial^{r_2}}{\partial t^{2r}}\nabla_{t,e_i}\Big)$$

where $b(V), b_i(V)$ and their derivatives in V are uniformly bounded for $V \in \mathbf{R}^{2n}(\cong Z_0)$. Now $\frac{\partial^{r_1}}{\partial t^{r_1}}(b(tV))$ (resp. $\frac{\partial^{r_1}}{\partial t^{r_1}})\nabla_{t,e_i})$ $(r_1 \geq 1)$, are functions of the type $b'(tV)V^\beta, |\beta| \leq r_1$ (resp. r_1+1) and $b'(V)$ and its derivatives in V are bounded smooth functions of V. Applying the same commutator trick as in the proof of [MM07, Theorem 1.6.10 and Theorem 4.1.12] the operator $QA_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)Q'$ can be rewritten as a linear combination of operators of the form

$$\begin{aligned} &Q(\lambda - L_t)^{-k_0'}R_1(\lambda - L_t)^{-k_1'}R_2\dots R_i(\lambda - L_t)^{-k_i''} \quad \text{and} \\ &(\lambda - L_t)^{-k_i'-k_i''}\dots R_{l'}(\lambda - L_t)^{-k_{l'}'}Q'''Q'', \end{aligned} \tag{2.1.11}$$

with $R_1 \dots R_{l'} \in \mathscr{R}_t := \{[f_{j_1}Q_{j_1},[f_{j_2}Q_{j_2},\dots[f_{j_l}Q_{j_l},L_t]\dots]]\}$ where f_{j_i} are smooth bounded (with its derivatives) functions and $Q_{j_i} \in \{\nabla_{t,e_i}^{(0)}, Z_l\}_{l=1\dots 2n}$. By [P16, Lemma 2.16.] and Lemma 2.1.4 the norm $\|\cdot\|_t^{0,0}$ of each of these operators are bounded by $C(1+|\lambda|^2)^N$ for a $N \in \mathbf{N}_0$. Lemma 2.1.4 follows. □

Let $\langle V|e^{-L_t}|V'\rangle$ be the smooth kernel of the operator e^{-L_t} with respect to $d\mathrm{vol}_{g_{z_0}^{T_{\mathbf{R}}Z}}(V')$. The existence follows from the existence for its degree zero part as $L_t^{(0)}$ is self-adjoint and using

(0.0.2). Denote by $\mathrm{pr}_M : T_{\mathbf{R}}Z \to M$ the submersion of the vector bundle $T_{\mathbf{R}}Z$ over M. As in chapter 0 pr_M will also denote the submersion from the fibre-product $T_{\mathbf{R}}Z \times_{\mathrm{pr}_M} T_{\mathbf{R}}Z$ onto M. Then $\langle \cdot |e^{-L_t}| \cdot \rangle$ is a section of $\mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E})$ over $T_{\mathbf{R}}Z \times_{\mathrm{pr}_M} T_{\mathbf{R}}Z$. Let $\nabla^{\mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E})}$ resp. $h^{\mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E})}$ be the connection resp. metric on $\mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E})$ induced by $\nabla^{\mathbb{E}}$ resp. $h^{\mathbb{E}}$.

Recall the notation that $B_1^{T_{z_0}Z}(0)$ was the set of vectors of norm lesser one.

Lemma 2.1.6. *Let $u > 0$ be fixed. For any $m, m', r \in \mathbf{N}_0$ there exists $C > 0$ such that for any $t \in]0, t_0]$ and $V, V' \in B_1^{T_{z_0}Z}(0)$,*

$$\sup_{|\alpha|,|\alpha'|\leq m} \left\| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial V^{\alpha} \partial V'^{\alpha'}} \frac{\partial^r}{\partial t^r} \langle V|e^{-uL_t}|V'\rangle \right\|_{\mathscr{C}^{m'}(T_{\mathbf{R}}Z\times_{\mathrm{pr}_M} T_{\mathbf{R}}Z, \mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E}))} \leq C$$

where $\|\cdot\|_{\mathscr{C}^{m'}(T_{\mathbf{R}}Z\times_{\mathrm{pr}_M} T_{\mathbf{R}}Z, \mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E}))}$ is the $\mathscr{C}^{m'}$-norm with respect to the parameters b_0 and $z_0 \in Z_{b_0}$ induced by $\nabla^{\mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E})}$ and $h^{\mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E})}$.

Proof. Lemma 2.1.6 has been proven in [P16, Theorem 2.18] for the case $r = 0$. In [P16, (2.80)] it was shown that for any $k \in \mathbf{N}$ we have the identity

$$e^{-uL_t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}} \int_{\delta+\Delta} e^{-u\lambda}(\lambda - L_t)^{-k} d\lambda.$$

By (2.1.9) and Lemma 2.1.5 we can differentiate under the integral to get for any $r \in \mathbf{N}_0$

$$\frac{\partial^r}{\partial t^r} e^{-uL_t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}} \int_{\delta+\Delta} e^{-u\lambda} \frac{\partial^r}{\partial t^r} (\lambda - L_t)^{-k} d\lambda. \tag{2.1.12}$$

With this Lemma 2.1.6 follows from (2.1.9) and Lemma 2.1.5 for the case $m' = 0$. For $m' > 0$ the statement follows with the same argument as in degree zero, [MM07, after (4.2.20)]. □

One can be more precise about the dependence of C on V, V' in (2.1.6) as is it done in [MM07, Theorem 4.2.5] by using the techniques of finite propagation speed. In our case we will not need such improvements.

Definition 2.1.7. *For $u > 0$ and $t \in]0, t_0]$ put*

$$K_u(L_t) := \frac{1}{2\pi i} \int_{\Delta} e^{-u\lambda}(\lambda - L_t)^{-1} d\lambda.$$

For later purposes (section 2.4) we will need an asymptotic expansion of $K_u(L_t)$ and therefore a similar estimate as in Lemma 2.1.6 for $K_u(L_t)$ but this time also the dependence of C on u without fixing u. Let $\langle V|K_u(L_t)|V'\rangle$ be the smooth kernel of $K_u(L_t)$ with respect to $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}_{z_0}}(V')$.

Lemma 2.1.8. *For $u > 0$ and for any $m, m', r \in \mathbf{N}_0$ there are constants $C, c > 0$ such that for any $t \in]0, t_0]$ and $V, V' \in B_1^{T_{z_0}Z}(0)$,*

$$\sup_{|\alpha|,|\alpha'|\leq m} \Big\| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial V^\alpha \partial V'^{\alpha'}} \frac{\partial^r}{\partial t^r} \langle V|K_u(L_t)|V'\rangle \Big\|_{\mathscr{C}^{m'}(T_\mathbf{R}Z\times_{\mathrm{pr}_M} T_\mathbf{R}Z, \mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E}))} \leq Ce^{-cu}.$$

Proof. The proof is the same as in Lemma 2.1.6. Since for $\lambda \in \Delta$ there exists $K > 0$ with $\mathrm{Re}(\lambda) > K$ we have the exponentiell decay. □

Lemma 2.1.9. *For any $r \geq 0$, $k > 0$ there exists $C > 0$, $N \in \mathbf{N}_0$ such that for any $t \in [0, t_0]$ and $\lambda \in \delta + \Delta$,*

$$\Big\|\Big(\frac{\partial^r L_t}{\partial t^r} - \frac{\partial^r L_t}{\partial t^r}_{|t=0}\Big)s\Big\|_{t,-1} \leq Ct \sum_{|\alpha|\leq r+3} \|V^\alpha s\|_{0,1} \qquad \textit{and}$$

$$\Big\|\Big(\frac{\partial^r}{\partial t^r}(\lambda - L_t)^{-k} - \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_\mathbf{r}^\mathbf{k} A_\mathbf{r}^\mathbf{k}(\lambda, 0)\Big)s\Big\|_{0,0} \leq Ct(1+|\lambda|^2)^N \sum_{|\alpha|\leq 4r+3} \|V^\alpha s\|_{0,0}.$$

Proof. By [P16, (2.58)] for $t \in [0,1], k \geq 1$

$$\|s\|_{t,k} \leq C \sum_{|\alpha|\leq 4r+3} \|V^\alpha s\|_{0,1}.$$

Applying Taylor expansion for [P16, (2.58)] we get for s, s' with compact support

$$\Big\langle\Big(\frac{\partial^r L_t}{\partial t^r} - \frac{\partial^r L_t}{\partial t^r}_{|t=0}\Big)s, s'\Big\rangle_{0,0} \leq Ct\|s'\|_{t,1} \sum_{|\alpha|\leq r+3} \|V^\alpha s\|_{0,1}$$

from which we get the first inequality of Lemma 2.1.9. By passing to the limit we obtain that [P16, Propositions 2.15-2.17] still hold for $t = 0$. From [P16, Proposition 2.15], [P16, Prop.2.17], the first inequality of Lemma 2.1.9 and from the fact

$$(\lambda - L_t)^{-1} - (\lambda - L_0)^{-1} = (\lambda - L_t)^{-1}(L_t - L_0)(\lambda - L_0)^{-1}$$

we obtain

$$\|((\lambda - L_t)^{-1} - (\lambda - L_0)^{-1})s\|_{0,0} \leq Ct(1+|\lambda|)^N \sum_{|\alpha\leq 4r} \|V^\alpha s\|_{0,0}. \tag{2.1.13}$$

Only in this we indroduce the notation

$$L_{\lambda,t} := \lambda - L_t.$$

Then we have

$$A_r^k(\lambda,t)-A_r^k(\lambda,0)=\sum_{i=1}^{j}L_{\lambda,t}^{-k_0}\cdots\Big(\frac{\partial^{r_i}L_t}{\partial t^{r_i}}-\frac{\partial^{r_i}L_t}{\partial t^{r_i}}_{|t=0}\Big)L_{\lambda,0}^{-k_i}\cdots L_{\lambda,0}^{-k_j}$$
$$+\sum_{i=1}^{j}(L_{\lambda,t}^{-k_i}-L_{\lambda,0}^{-k_i})\Big(\frac{\partial^{r_{i+1}}L_t}{\partial t^{r_{i+1}}}_{|t=0}\Big)\cdots L_{\lambda,0}^{-k_j}.$$

Together with the first inequality of Lemma 2.1.9 and (2.1.13) we get the second inequality in the claim. □

Definition 2.1.10. *For $r\in\mathbf{N}_0$ and $k\in\mathbf{N}_0$ sufficiently large define the operators*

$$J_{r,u}:=\frac{(-1)^{k-1}(k-1)!}{2\pi i r! u^{k-1}}\int_{\delta+\Delta}e^{-u\lambda}\sum_{(k,r)\in I_{k,r}}a_r^kA_r^k(\lambda,0)d\lambda,$$
$$J_{r,u,t}:=\frac{1}{r!}\frac{\partial^r}{\partial t^r}e^{-uL_t}-J_{r,u},$$
$$K_{r,u}:=\frac{(-1)^{k-1}(k-1)!}{2\pi i r! u^{k-1}}\int_{\Delta}e^{-u\lambda}\sum_{(k,r)\in I_{k,r}}a_r^kA_r^k(\lambda,0)d\lambda \qquad \textit{and}$$
$$K_{r,u,t}:=\frac{1}{r!}\frac{\partial^r}{\partial t^r}K_u(L_t)-K_{r,u}.$$

For the case where B is a point these operators were already introduced in [MM07, (4.2.21),(4.2.25)]. By the Schwartz kernel theorem the operators $J_{r,u,t}$ and $K_{r,u,t}$ are represented by smooth kernels $\langle V|J_{r,u,t}|V'\rangle$ and $\langle V|K_{r,u,t}|V'\rangle$ with respect to $d\mathrm{vol}_{g_{z_0}^{T_\mathbf{R}Z}}(V')$.

Lemma 2.1.11. *Let $u>0$ be fixed. There exist $C,C'>0$, $N\in\mathbf{N}_0$ such that for $t>0$, $q\in\mathbf{N}$, $V,V'\in Z_0$ with $\|V\|,\|V'\|\le q$,*

$$\|\langle V|J_{r,u,t}|V'\rangle\|\le Ct^{1/2n+1}(1+q)^N \qquad \textit{and} \qquad \|\langle V|K_{r,u,t}|V'\rangle\|\le C't^{1/2n+1}(1+q)^N.$$

Proof. Let $J^0_{z_0,q}$ be the vector space of square integrable sections of $\mathbb{E}_{z_0}$ over $\{V\in T_{z_0}Z|\ \|V\|\le q+1\}$. Let $|\cdot|_{(q),m}$ be the usual Sobolev norm on $\Gamma(B_q^{T_{z_0}Z}(0),\mathbb{E}_{z_0})$ induced by $h^{\mathbb{E}_{z_0}}$ and the volume form $d\mathrm{vol}_{g_{z_0}^{T_\mathbf{R}Z}}$ as in (2.1.7). Let $\|A\|_{(q)}$ be the operator norm of $A\in\mathcal{L}(J^0_{z_0,q})$ with respect to $\|\ \|_{(q),0}$. By Lemma 2.1.9, (2.1.12) and the definition of $J_{u,r}$ there exists $C>0$ and $N\in\mathbf{N}$ such that for $t>0$ and $q>1$,

$$\|J_{r,u,t}\|_{(q)}\le Ct(1+q)^N \quad \text{and} \quad \|K_{r,u,t}\|_{(q)}\le C't(1+q)^N.$$

Let $\varphi:\mathbf{R}^{2n}\to[0,1]$ be a smooth function with compact support, equal 1 near 0, such that $\int_{T_{z_0}Z}\varphi(V)d\mathrm{vol}_{g_{z_0}^{T_\mathbf{R}Z}}(V)=1$. Take $\epsilon\in]0,1]$. By Lemma 2.1.6 there exists $C>0$ such that if

$\|V\|, \|V'\| \leq q$, $U, U' \in \mathbb{E}_{z_0}$,

$$\Big|\Big\langle \langle V|J_{r,u,t}|V'\rangle U, U'\Big\rangle - \int_{T_{z_0}Z\times T_{z_0}Z} \Big\langle \langle V-W|J_{r,u,t}|V'-W'\rangle U, U'\Big\rangle \cdot \frac{1}{\epsilon^{4n}}\varphi\Big(\frac{W}{\epsilon}\Big)\varphi\Big(\frac{W'}{\epsilon}\Big) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}_{z_0}}(W) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}_{z_0}}(W')\Big| \leq C\epsilon(1+q)^N\|U\|\|U'\|.$$

On the other hand from $\|J_{r,u,t}\|_{(q)} \leq Ct(1+q)^N$ we have for $\|V\|, \|V'\| \leq q$

$$\Big|\int_{T_{z_0}Z\times T_{z_0}Z} \Big\langle \langle V-W|J_{r,u,t}|V'-W'\rangle U, U'\Big\rangle \frac{1}{\epsilon^{4n}}\varphi\Big(\frac{W}{\epsilon}\Big)\varphi\Big(\frac{W'}{\epsilon}\Big) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}_{z_0}}(W) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}_{z_0}}(W')\Big| \leq Ct\frac{1}{\epsilon^{2n}}(1+q)^N\|U\|\|U'\|.$$

By taking $\epsilon = t^{1/2n+1}$ we get the first inequality. In the same way we obtain the second one for $K_{r,u,t}$ by using Lemma 2.1.8 □

As a direct consequence of Lemma 2.1.11 we have the following result.

Proposition 2.1.12. *The functions* $]0,1] \ni t \mapsto \langle 0|e^{-uL_t}|0\rangle$ *and* $]0,1] \ni t \mapsto \langle 0|K_u(L_t)|0\rangle$ *extend smoothly to* $[0,1]$ *with values at* $t=0$ *given by*

$$\frac{1}{r!}\frac{\partial^r}{\partial t^r}_{|t=0}\langle 0|e^{-uL_t}|0\rangle = \langle 0|J_{r,u}|0\rangle \quad \textit{and} \quad \frac{1}{r!}\frac{\partial^r}{\partial t^r}_{|t=0}\langle 0|K_u(L_t)|0\rangle = \langle 0|K_{r,u}|0\rangle.$$

Moreover all the derivatives are uniformly bounded on $z \in Z$.

Proposition 2.1.13. *For any* $l, m, m' \in \mathbf{N}_0$ *there exsits* $C, C' > 0$ *such that if* $t \in]0, t_0]$, $V, V' \in B_1^{T_{z_0}Z}(0)$,

$$\sup_{|\alpha|,|\alpha'|\leq m}\Big\|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial V^\alpha \partial V^{\alpha'}}\Big(\langle V|e^{-uL_t} - \sum_{r=0}^{l} J_{r,u}t^r|V'\rangle\Big)\Big\|_{\mathscr{C}^{m'}(T_{\mathbf{R}}Z\times_{\mathrm{pr}_M}T_{\mathbf{R}}Z, \mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E}))} \leq Ct^{k+1} \quad \textit{and}$$

$$\sup_{|\alpha|,|\alpha'|\leq m}\Big\|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial V^\alpha \partial V^{\alpha'}}\Big(\langle V|K_u(L_t) - \sum_{r=0}^{l} K_{r,u}t^r|V'\rangle\Big)\Big\|_{\mathscr{C}^{m'}(T_{\mathbf{R}}Z\times_{\mathrm{pr}_M}T_{\mathbf{R}}Z, \mathrm{pr}_M^*(\mathrm{End}\,\mathbb{E}))} \leq C't^{k+1}.$$

Proof. By Taylor expansion

$$A(t) - \sum_{r=0}^{l}\frac{1}{r!}\frac{\partial^r A}{\partial t^r}(0)t^r = \frac{1}{l!}\int_0^t (t-s)^l\frac{\partial^{l+1}A}{\partial t^{l+1}}(s)ds \tag{2.1.14}$$

together with the identities of the derivatives in Proposition 2.1.12 and their estimates from

Lemma 2.1.6 and Lemma 2.1.8 the statements follow. □

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d) Volterra series

For explicit calculation of the first coefficients in the asymptotic of the torsion forms as we will explain in chapter 3 we need to know how the $J_{r,u}$ look like. For $u > 0$ let $u\Delta_j$ be the rescaled simplex

$$u\Delta_j = \{(u_1, \dots, u_j) | 0 \leq u_1 \leq u_2 \leq \dots \leq u_j \leq u\}$$

as below of (0.0.2). Recall the operators L_0 and $\mathcal{O}_r$ were defined in Lemma 2.1.2.

Lemma 2.1.14. *For $r \geq 0$ we have*

$$J_{r,u} = \sum_{\sum_i^j r_i = r, r_i \geq 1} (-1)^j \int_{u\Delta_j} e^{-(u-u_j)L_0} \mathcal{O}_{r_j} e^{-(u_j - u_{j-1})L_0} \dots \mathcal{O}_{r_1} e^{-u_1 L_0} du_1 \dots du_j.$$

Furthermore

$$\langle 0 | J_{2r+1,u} | 0 \rangle = 0.$$

Proof. Since L_0 is a generalized Laplacian, Lemma 2.1.14 can be proved exactly as [DLM06, Theorem 4.17, (4.107), (4.108)] using the Volterra series from [BGV92, chapter 2] and Lemma 2.1.2. □

Lemma 2.1.14 is referred in [F18, (4.10)] as Duhamel's formula. In this thesis we will call it Volterra series (or expansion) of $J_{r,u}$ since Lemma 2.1.14 is a direct consequence of it ([DLM06, (4.109)]).

2.2 Asymptotics of the Kernel

In [P16, Theorem 2.21] it has been shown that there exist $b_{p,j} \in \Gamma(Z, \Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}))$ such that for any $k, m \in \mathbf{N}_0$ there exists $C > 0$ such that for any $u \in]0,1]$ and $p \in \mathbf{N}$

$$\left\| p^{-n} \psi_{1/\sqrt{p}} \langle z | \exp(-B^2_{p,u/p}) | z \rangle - \sum_{j=-d}^{k} b_{p,j}(z) u^j \right\|_{\mathscr{C}^m(M, \mathrm{End}(\mathbb{E}))} \leq C u^{k+1} \tag{2.2.1}$$

where $\mathscr{C}^m(M, \mathrm{End}(\mathbb{E}))$ denotes the $\mathscr{C}^m$-norm in the parameter $(b, z) \in M$.

Recall the Landau symbol O was defined after Definition 1.3.4.

Theorem 2.2.1. *Let $m \in \mathbf{N}_0$. There exist $a_{i,u} \in \Gamma(Z, \Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}))$ with $i \in \mathbf{N}_0$ such that for every $u > 0$ and $l \in \mathbf{N}_0$ we have as $p \to \infty$*

$$\langle z|\psi_{1/\sqrt{p}} \exp(-B^2_{p,u/p})|z\rangle = \sum_{i=0}^{l} a_{i,u}(z)p^{n-i} + O(p^{n-l-1})$$

for the $\mathscr{C}^m$-norm on $\Gamma(M, \mathrm{End}(\mathbb{E}))$ in the parameter $(b,z) \in M$ and uniform in u as u varies in a compact subset of $]0,\infty[$.

Proof. Recall that we have set $t = \frac{1}{\sqrt{p}}$. For $s \in \Gamma(Z_0, \mathbb{E}_{z_0})$ and $V \in Z_0$ we have

$$\begin{aligned}(e^{-uL_t}s)(V) &= (S_t^{-1}\kappa^{1/2}e^{-\frac{u}{p}M_{p,z_0}}\kappa^{-1/2S_ts})(V)\\ &= \kappa^{1/2}(tV)\int_{Z_0}\langle tV|e^{-\frac{u}{p}M_{p,z_0}}|V'\rangle(S_ts)(V')\kappa^{1/2}(V')d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}_{z_0}}(V').\end{aligned}$$

Since $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_0}}(V) = \kappa(V)d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}_{z_0}}$ we get for $V, V' \in Z_0$

$$\langle V|e^{-uL_t}|V'\rangle = p^{-n}\langle tV|e^{-\frac{u}{p}M_{p,z_0}}|V'\rangle\kappa^{1/2}(tV)\kappa^{1/2}(tV'). \tag{2.2.2}$$

In particular as $\kappa(0) = 1$ we have

$$\langle 0|e^{-uL_t}|0\rangle = p^{-n}\langle 0|e^{-\frac{u}{p}M_{p,z_0}}|0\rangle.$$

Using this identity, Taylor expansion for $t \mapsto \langle 0|e^{-uL_t}|0\rangle$ at 0 and substitute $t = \frac{1}{\sqrt{p}}$ there exits $C > 0$ such that for $z_0 \in Z$,

$$\left\|p^{-n}\langle 0|e^{-\frac{u}{p}M_{p,z_0}}|0\rangle - \sum_{r=0}^{l}\langle 0|J_{r,u}|0\rangle p^{-\frac{r}{2}}\right\|_{\mathscr{C}^m(M)} \leq Cp^{-\frac{l+1}{2}}.$$

Now by Lemma 2.1.14 we have $\langle 0|J_{2r+1,u}|0\rangle = 0$ and thus by (2.1.2) we get the statement with

$$a_{i,u} := \psi_{1/\sqrt{u}}\langle 0|J_{2i,u}|0\rangle.$$

□

As in [P16, (2.144)] we define the following operator,

$$\mathbb{K}_{p,u} := \frac{1}{2\pi i}\psi_{1/\sqrt{u}}\int_\Delta e^{-u\lambda}(\lambda - B_p^2/p)^{-1}d\lambda.$$

The well-definedness is given by [P16, (2.175)]. Let $\langle z|\mathbb{K}_{p,u}|z'\rangle$ be the smooth kernel associated to the operator $\mathbb{K}_{p,u}$ with respect to $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_0}}(z')$. Our next goal is to have an asymptotic expansion as in Theorem 2.2.1 for $\langle z|\mathbb{K}_{p,u}|z'\rangle$. But first we need some preperations.

The notations are the same as in section 2.1. Let $z_1, \ldots, z_N$ be points of Z such that $\{U_{z_k} := B^Z_\varepsilon(z_k)\}_{k=1,\ldots,N}$ is an open covering of Z. For each k we fix an orthonormal basis $\{e_i\}_i$ without for simplicity mentioning its dependence on the point z_k. On U_{z_k} we identify $\mathcal{E}_V$ and $\Lambda^{0,\bullet}(T^*_V Z)$ with $\mathcal{E}_{z_k}$ and $\Lambda^{0,\bullet}(T^*_{z_k} Z)$ by parallel transport via $\nabla^{\mathbb{E}}$ and $\nabla^{\Lambda^{0,\bullet}}$ along the geodesic ray $[0,1] \ni t \mapsto tV$. Define the vector bundle $\mathbb{E}_p$ over Z by

$$\mathbb{E}_p := \Lambda^\bullet(T^*_{\mathbf{R},b_0} B) \otimes (\Lambda^{0,\bullet}(T^* Z) \otimes \mathcal{E} \otimes \mathcal{L}^p)$$

where $\Lambda^\bullet(T^*_{\mathbf{R},b_0} B)$ is a trivial bundle over Z. Let d_U be the ordinary differentiation operator in the direction U on $T_{z_k} Z$. Let $\|\cdot\|^2_{\mathbf{H}^m(p)}$ be the Sobolev-norm on $\mathbf{H}^m(Z, \mathbb{E}_p)$ with respect to the partition of unity subordinate to to $\{U_{z_k}\}_{k=1,\ldots,N}$ given by

$$\|s\|^2_{\mathbf{H}^m(p)} := \sum_k \sum_{d=0}^m \sum_{i_1,\ldots,i_d=1} \|d_{e_{i_1}} \ldots d_{e_{i_d}} (\varphi_k s)\|^2_{L^2}.$$

Lemma 2.2.2. *[P16, Lemma 2.1] For any $m \in \mathbf{N}_0$ there exists $C_m > 0$ such that for any $p \in \mathbf{N}$, $u > 0$ and $s \in \mathbf{H}^{2m+2}(Z, \mathbb{E}_p)$,*

$$\|s\|^2_{\mathbf{H}^{2m+2}(p)} \leq C_m p^{4m+4} \sum_{j=0}^{m+1} p^{-4j} \|B_p^{2j} s\|_{L^2}.$$

We will now use the same notations as in [P16, p.15-18] (or see Appendix A). Let $f : \mathbf{R} \to [0,1]$ be a smooth even function with

$$f(t) = \begin{cases} 1\ , & |t| < \frac{\varepsilon}{2} \\ 0\ , & |t| > \varepsilon \end{cases}$$

with the same ε as in section 2.1. For $a \in \mathbf{C}$ and $u > 0$ set

$$F_u(a) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iv\sqrt{2}a} \exp(-v^2/2) f(\sqrt{u}v) dv \qquad \text{and}$$
$$G_u(a) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iv\sqrt{2}a} \exp(-v^2/2)(1-) f(\sqrt{u}v) dv.$$

These are even holomorphic functions, thus there exist holomorphic functions $\tilde{F}_u$ and $\tilde{G}_u$ with $\tilde{F}_u(a^2) = F_u(a)$ and $\tilde{G}_u(a^2) = G_u(a)$. Furthermore one has for $v > 0$

$$\tilde{F}_u(va^2) + \tilde{G}_u(va^2) = e^{-va^2}. \tag{2.2.3}$$

Let $\langle z|\tilde{G}_u(vB_p^2)|z'\rangle$ and $\langle z|\tilde{F}_u(vB_p^2)|z'\rangle$be the smooth kernel of $\tilde{G}_u(vB_p^2)$ and $\tilde{F}_u(vB_p^2)$ with respect to $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z')$.

$\mathbb{E}_p$ will be equipped with the connection $\nabla^{\mathbb{E}_p}$ induced by $\nabla^{T_{\mathbf{R}}B,LC}$, $\nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$, $\nabla^{\mathcal{L}}$ and with the metric $h_p^{\mathbb{E}}$ induced by $g^{T_{\mathbf{R}}B}$, $h^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$ and $h^{\mathcal{L}}$. Let $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ and $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ be the induced connection and metric on $\mathbb{E}_p\boxtimes\mathbb{E}_p^*$. In [P16, Proposition 2.2] it was shown that for any $m\in\mathbf{N}_0$ and $\varepsilon>0$ there exists $C>0$ and $N\in\mathbf{N}$ such that

$$\Big\|\langle\,\cdot\,|\tilde{G}_{\frac{u}{p}}(uC_p)|\,\cdot\,\rangle\Big\|_{\mathscr{C}^m(M\times_\pi M,\mathbb{E}_p\boxtimes\mathbb{E}_p^*)}\le Cp^N\exp\Big(-\frac{\varepsilon^2p}{16u}\Big) \tag{2.2.4}$$

where the $\mathscr{C}^m$-norm is induced by $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ and $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$.

Lemma 2.2.3. *For any $l,m\in\mathbf{N}_0$ there exist $C_{m,l,u},C'_{m,l,u}>0$ such that for $p\in\mathbf{N}$ large such that $t=\frac{1}{\sqrt{p}}\in]0,t_0]$*

$$\Big\|\langle z|\mathbb{K}_{p,u}|z\rangle\Big\|_{\mathscr{C}^m(M,\mathrm{End}(\mathbb{E}_p))}\le C_{m,l,u}p^{-l}\quad\textit{and}\quad p^n\Big\|\langle V|K_u(L_t)|V\rangle\Big\|_{\mathscr{C}^m(T_{\mathbf{R}}Z,\mathrm{pr}_M^*(\mathrm{End}(\mathbb{E}_p))}\le C'_{m,l,u}p^{-l}.$$

Proof. For a set A let $\mathbf{1}_A$ be its indicator function. For $a\in\mathbf{C}$ and $v>0$ set

$$\phi_v(a):=\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}(\mathrm{Re}(a))\cdot e^{-va}.$$

Moreover put

$$(\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{u,v})(a^2):=\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\Big(\mathrm{Re}(\frac{a^2}{v})\Big)\cdot\tilde{F}_u(a^2)\qquad\text{and}$$
$$(\mathbf{1}_{[\vartheta,\infty[}\tilde{G}_{u,v})(a^2):=\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\Big(\mathrm{Re}(\frac{a^2}{v})\Big)\cdot\tilde{G}_u(a^2).$$

Then from (2.2.3) we have

$$\begin{aligned}(\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},v})(va^2)+(\mathbf{1}_{[\frac{5\vartheta}{8},\infty[})\tilde{G}_{\frac{u}{p},v})(va^2)&=\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}(\mathrm{Re}(a^2))\cdot(\tilde{F}_{\frac{u}{p}}(va^2)+\tilde{G}_{\frac{u}{p}}(va^2))\\&=\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}(\mathrm{Re}(a^2))\cdot e^{-va^2}=\phi_v(a^2).\end{aligned}$$

Because $\mathrm{Re}(\lambda)<\frac{5\vartheta}{8}$ for $\lambda\in\delta$ we can rewrite the operator $\mathbb{K}_{p,u}$ as

$$\begin{aligned}\mathbb{K}_{p,u}&=\frac{1}{2\pi i}\psi_{1/\sqrt{u}}\int_\Delta e^{-u\lambda}(\lambda-C_p)^{-1}d\lambda\\&=\frac{1}{2\pi i}\psi_{1/\sqrt{u}}\int_{\delta+\Delta}\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}(\mathrm{Re}(\lambda))e^{-u\lambda}(\lambda-C_p)^{-1}d\lambda\\&=\frac{1}{2\pi i}\psi_{1/\sqrt{u}}\int_{\delta+\Delta}\phi_u(\lambda)(\lambda-C_p)^{-1}d\lambda\\&=\psi_{1/\sqrt{u}}(\phi_u(C_p)).\end{aligned}$$

Thus it follows

$$\mathbb{K}_{p,u} = \psi_{1/\sqrt{u}}(\phi_u(C_p)) = \psi_{1/\sqrt{u}}(\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},u})(uC_p) + \psi_{1/\sqrt{u}}(\mathbf{1}_{[\frac{5\vartheta}{8},\infty[})\tilde{G}_{\frac{u}{p},u})(uC_p). \tag{2.2.5}$$

Now we have a similar estimate as in (2.2.4) for $(\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{G}_{\frac{u}{p},u})$ instead of $\tilde{G}_{\frac{u}{p}}$ since the indicator function has no effect on the exponentiell decay, $|(\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{G}_{\frac{u}{p},u})(x)| \leq |\tilde{G}_{\frac{u}{p}}(x)|$, i.e. there exist $\tilde{C} > 0$ and $\tilde{N} \in \mathbf{N}$ such that

$$\Big\| \langle \cdot | (\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{G}_{\frac{u}{p},u})(uC_p) | \cdot \rangle \Big\|_{\mathscr{C}^m(M\times_\pi M, \mathbb{E}_p \boxtimes \mathbb{E}_p^*)} \leq \tilde{C} p^{\tilde{N}} \exp\big(-\frac{\varepsilon^2 p}{16u}\big). \tag{2.2.6}$$

By Lemma 2.2.2 and

$$\begin{aligned}\sup_{a\geq 0} |a^m (\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},u})(a^2u/p)| = \sup_{a\geq \sqrt{\frac{5}{8}\vartheta p}} |a^m \tilde{F}_{\frac{u}{p}}(a^2u/p)| \leq \sup_{a\geq\sqrt{\vartheta}\sqrt{p}} |a^m F_{\frac{u}{p}}(a\sqrt{u/p})| \\ \leq C_{m,l,u} p^{-l} \end{aligned} \tag{2.2.7}$$

(using $i^m a^m e^{iva} = \frac{\partial^m}{\partial v^m}(e^{iva})$ and integration by parts) we find that if Q_1, Q_2 are differential operators of order $2m', 2m$ respectively and with compact support in U_{z_i}, U_{z_j} respectively, then there is a positive constant $C_{m,m'}$ such that

$$\Big\| Q_1 (\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},u})\Big(uC_p\Big) Q_2 s \Big\|_{L^2} \leq C_{m,m'} p^{-l} \|s\|_{L^2}.$$

By the Sobolev inequality we get

$$\Big\| \langle \cdot | (\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},u})(uC_p) | \cdot \rangle \Big\|_{\mathscr{C}^m(Z\times Z)} \leq C_{m,l,u} p^{-l}.$$

For the derivaties in the directions of the base we have for $U \in T_{\mathbf{R}}B$ and any $q, k \in \mathbf{N}$

$$\Big(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}\Big)^q (\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},u})(uC_p) = \frac{1}{2\pi i} \int_{\delta+\Delta} (\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},u})\Big(\frac{u}{p}\lambda\Big) \Big(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}\Big)^q (\lambda - B_p^2)^{-k} d\lambda.$$

By [P16, (2.41)] there exist $c, d \geq 0$ and $C > 0$ such that

$$\|B_p^{2m} \Big(\nabla_{U^H}^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}\Big)^q (\lambda - B_p^2)^{-k} B_p^{2m'} \|_{L^2} \leq C|\lambda|^c p^d \|s\|_{L^2}$$

from which together with (2.2.7) we get

$$\Big\| \langle \cdot | (\mathbf{1}_{[\frac{5\vartheta}{8},\infty[}\tilde{F}_{\frac{u}{p},u})(uC_p) | \cdot \rangle \Big\|_{\mathscr{C}^m(M\times_\pi M, \mathbb{E}_p \boxtimes \mathbb{E}_p^*)} \leq C_{m,l,u} p^{-l}. \tag{2.2.8}$$

From (2.2.5) (2.2.6) and (2.2.8) we conclude the first statement in Lemma 2.2.3. The second statement follows in the same way by replacing C_p with $\frac{1}{p}M_p$ as both have the same structure. □

Proposition 2.2.4. *Let $m \in \mathbf{N}_0$. There exist $a^{\mathbb{K}}_{i,u} \in \Gamma(Z, \Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}))$, $i \in \mathbf{N}_0$, such that for every $u > 0$ we have as $p \to \infty$*

$$\langle z|\mathbb{K}_{p,u}|z\rangle = \sum_{i=0}^{l} a^{\mathbb{K}}_{i,u}(z)p^{n-i} + O(p^{n-l-1})$$

for the $\mathscr{C}^m$- norm on $\Gamma(M, \mathrm{End}(\mathbb{E}))$ in the parameter $(b, z) \in M$ and uniform in u as u varies in a compact subset of $]0, \infty[$.

Proof. The proof of Proposition 2.2.4 follows the same arguments as the proof of Theorem 2.2.1 where $e^{-B^2_{p,u/p}}$ respectively e^{-uL_t} are replaced by $\mathbb{K}_{p,u}$ respectively $K_u(L_t)$. The only major different is that (2.1.2) has to be substituted by the following: By Lemma 2.2.3 for any $l, m \in \mathbf{N}_0$ there exists $C_{m,l,u}$ such that

$$\left\| \langle z|\mathbb{K}_{p,u}|z\rangle - p^n \psi_{1/\sqrt{u}} \langle 0|K_u(L_t)|0\rangle \right\|_{\mathscr{C}^m} \leq C_{m,l,u}p^{-l}.$$

Thus we get the statement with

$$a^{\mathbb{K}}_{i,u} := \psi_{1/\sqrt{u}} \langle 0|K_{2i,u}|0\rangle.$$

□

Recall that d was the dimension of M.

Proposition 2.2.5. *For any $k \in \mathbf{N}_0$ there exist $a^{[j]}_i(z) \in \Gamma(Z, \Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}))$, $-d \leq j \leq k$ such that as $u \to 0$*

$$a_{i,u}(z) = \sum_{j=-d}^{k} a^{[j]}_i(z)u^j + O(u^k).$$

Proof. From $\langle 0|e^{-uL_t}|0\rangle = p^{-n}\langle 0|e^{-\frac{u}{p}M_{p,z_0}}|0\rangle$ and (2.2.1) we have that for $t \in [0, 1]$, there are $\tilde{b}_{t,r} \in \Gamma(Z, \Lambda^\bullet(T^*_{\mathbf{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E})), r \in \mathbf{Z}$ with $r \geq -d$, such that for any $k, m \in \mathbf{N}_0$, $u_0 > 0$ there is $C > 0$ such that for any $u \in]0, u_0]$

$$\left\| \langle 0|e^{-uL_t}|0\rangle - \sum_{r=-d}^{k} \tilde{b}_{t,r}(z)u^r \right\|_{\mathscr{C}^m(M \times [0,t_0])} \leq Cu^{k+1} \tag{2.2.9}$$

where the second coordinate of $M \times [0, t_0]$ represents t. Since $\frac{\partial^{2i+1}}{\partial t^{2i+1}}_{|t=0} \langle 0|e^{-uL_t}|0\rangle = 0$ we have

$$\frac{\partial^{2i+1}}{\partial t^{2i+1}}_{|t=0} \tilde{b}_{t,r}(z) = 0. \tag{2.2.10}$$

Thus the statement follows with

$$a_i^{[j]}(z) := \frac{1}{(2k)!} \frac{\partial^{2i}}{\partial t^{2i}}_{|t=0} \tilde{b}_{t,j}(z).$$

□

Set $B_{p,-d-1} := 0$ and for $j \geq -d-1$ set

$$B_{p,j} := \int_Z \mathrm{Tr}_s \left[N_V b_{p,j}(z) + i\omega^H b_{p,j+1}(z) \right] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z).$$

In [P16, Corollary 2.22] as a corollary from (2.2.1) it has been shown that for any $k, m \in \mathbf{N}_0$ there exists $C > 0$ such that for any $u \in]0,1]$ and $p \in \mathbf{N}$

$$\left\| p^{-n} \psi_{1/\sqrt{p}} \mathrm{Tr}_s \left[N_{u/p} \exp(-B_{p,u/p}^2) \right] - \sum_{j=-d-1}^{k} B_{p,j} u^j \right\|_{\mathscr{C}^m(B, \Lambda^\bullet T_{\mathbf{R}}^* B)} \leq C u^{k+1}.$$

For $j \geq -d$ set $B_j^{[-d-1]} := 0$ and for $i \geq -d-1$ set

$$B_j^{[i]} := \int_Z \mathrm{Tr}_s \left[N_V a_i^{[j]}(z) + i\omega^H a_i^{[j+1]}(z) \right] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z).$$

Then by Proposition (2.2.5) we have for any $k \in \mathbf{N}_0$ as $u \to 0$

$$\int_Z \mathrm{Tr}_s \left[N_u a_{i,u}(z) \right] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) = \sum_{j=-d-1}^{k} B_j^{[i]} u^j + O(u^k). \tag{2.2.11}$$

Proposition 2.2.6. *As $p \to \infty$ the following expansion holds for any $k \in \mathbf{N}_0$*

$$B_{p,j} = \sum_{i=0}^{k} B_j^{[i]} p^{-i} + O(p^{-k}).$$

Proof. By the proof of Proposition 2.2.5 we know

$$\left\| p^{-n} \psi_{1/\sqrt{p}} \mathrm{Tr}_s \left[N_{u/p} (\exp - B_{p,u/p}^2) \right] - \sum_{j=-d-1}^{k} \tilde{b}_{t,j} u^j \right\|_{\mathscr{C}^m(B, \Lambda^\bullet T_{\mathbf{R}}^* B)} \leq C u^{k+1}$$

and $\left\| p^{-n}\langle z|\psi_{1/\sqrt{p}}\exp(-B^2_{p,u/p})|z\rangle - \sum_{j=-d}^{k} b_{p,j}(z)u^j \right\|_{\mathscr{C}^m(M)} \leq Cu^{k+1}$. Thus we have

$$B_{p,j} = \int_Z \mathrm{Tr}_s\left[N_V\tilde{b}_{t,j}(x) + i\omega^H\tilde{b}_{t,j+1}(z)\right]d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z). \tag{2.2.12}$$

Using $a_i^{[j]}(z) = \frac{1}{(2i)!}\frac{\partial^i}{\partial t^k}{}_{|t=0}\tilde{b}_{t,j}(z)$ Proposition 2.2.6 follows with

$$B_j^{[i]} = \frac{1}{(2i)!}\frac{\partial^{2i}}{\partial t^{2i}}{}_{|t=0}\int_Z \mathrm{Tr}_s\left[N_V\tilde{b}_{t,j}(x) + i\omega^H\tilde{b}_{t,j+1}(z)\right]d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z). \tag{2.2.13}$$

□

Proposition 2.2.7. *For any $k, m \in \mathbf{N}_0$ there exists $C > 0$ such that for any $u \in]0,1]$ and $p \in \mathbf{N}$:*

$$\begin{aligned} p^k \Big\| &\Big(p^{-n}\psi_{1/\sqrt{p}}\mathrm{Tr}_s\left[N_{u/p}\exp(-B^2_{p,u/p})\right] - \sum_{j=-d-1}^{0} B_{p,j}u^j\Big) \\ &- \sum_{i=0}^{k-1} p^{-i}\Big(\int_Z \mathrm{Tr}_s\left[N_V a_{i,u}(x) + i\omega^H a_{i+1,u}(z)\right]d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) - \sum_{j=-d-1}^{0} B_j^{[i]}u^j\Big)\Big\|_{\mathscr{C}^m(B,\Lambda^\bullet T^*_{\mathbf{R}}B)} \leq Cu. \end{aligned}$$

Proof. By (2.2.9) and (2.2.10) it follows that for any $k \in \mathbf{N}$, $u_0, t_0 > 0$ there exists $C > 0$ such that for any $u \in]0, u_0]$, $t \in]0, t_0]$ and $z \in Z$

$$\begin{aligned} \Big\| t^{\frac{1}{2k}} &\Big(\langle 0|e^{-uL_t}|0\rangle - \sum_{r=-d}^{0}\tilde{b}_{t,r}(z)u^r \\ &- \sum_{i=0}^{k-1}\frac{t^{2i}}{(2i)!}\frac{\partial^{2i}}{\partial t^{2i}}(\langle 0|e^{-uL_t}|0\rangle - \sum_{r=-d}^{0}\tilde{b}_{t,r}(z)u^r)_{|t=0})\Big)\Big\|_{\mathscr{C}^m(M)} \leq Cu. \end{aligned}$$

By this the claimed inequality follows from (2.1.2), (2.2.2), (2.2.12) and (2.2.13). □

Theorem 2.2.8. *For any $k, m \in \mathbf{N}_0$, there exists $C > 0$ such that for $u \geq 1$ and $p \in \mathbf{N}$:*

$$\begin{aligned} p^k \Big\| \Big(p^{-n}\psi_{1/\sqrt{p}}\mathrm{Tr}_s\left[N_{u/p}\exp(-B^2_{p,u/p})\right] - \sum_{j=0}^{k-1} p^{-j}\int_Z \mathrm{Tr}_s\Big[N_V a_{i,u}(z) \\ + i\omega^H a_{i+1,u}(z)\Big]d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z)\Big\|_{\mathscr{C}^m(B,\Lambda^\bullet T^*_{\mathbf{R}}B)} \leq \frac{C}{\sqrt{u}}. \end{aligned} \tag{2.2.14}$$

Moreover there is $C' > 0$ such that

$$\Big\| \int_Z \mathrm{Tr}[N_u a_{i,u}(z)d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z)]\Big\|_{\mathscr{C}^m(B,\Lambda^\bullet T^*_{\mathbf{R}}B)} \leq \frac{C'}{\sqrt{u}}. \tag{2.2.15}$$

The proof of Theorem 2.2.8 will be postponed to section 2.4.

2.3 The Full Asymptotic of the Holomorphic Analytic Torsion Forms

By our assumption there is a $p_0 \in \mathbf{N}$ such that the direct image $R^i\pi_*(\mathcal{E} \otimes \mathcal{L}^p)$ is locally free for all $p \geq p_0$ and $i \in \{1, \ldots, n\}$, and vanishes for $i > 0$. In particular for $p \geq p_0$

$$H^i(Z, (\mathcal{E} \otimes \mathcal{L}^p)_{|Z}) = 0 \quad \text{for} \quad i > 0.$$

For $p \geq p_0$ set

$$\tilde{\zeta}_{1,p}(s) = -\frac{p^{-n}}{\Gamma(s)} \int_0^1 u^{s-1}\psi_{1/\sqrt{p}}\Phi\big(\operatorname{Tr}_s\big[N_{u/p}\exp(-B^2_{p,u/p})\big]\big)du \qquad \text{and}$$
$$\tilde{\zeta}_{2,p}(s) = -\frac{p^{-n}}{\Gamma(s)} \int_1^\infty u^{s-1}\psi_{1/\sqrt{p}}\Phi\big(\operatorname{Tr}_s\big[N_{u/p}\exp(-B^2_{p,u/p})\big]\big)du.$$

In the same fashion as in Definition 1.3.6 both $\tilde{\zeta}_{1,p}$ and $\tilde{\zeta}_{1,p}$ has a holomorphic extension near zero and we define

$$\tilde{\zeta}_p := \tilde{\zeta}_{1,p} + \tilde{\zeta}_{1,p}.$$

Clearly we have

$$p^{-n}\psi_{1/\sqrt{p}}\zeta_p(s) = p^{-s}\tilde{\zeta}_p(s).$$

From this we see

$$p^{-n}\psi_{1/\sqrt{p}}\zeta'_p(0) = -\log(p)\tilde{\zeta}_p(0) + \tilde{\zeta}'_p(0).$$

On the other hand we have for $p \geq p_0$

$$\begin{aligned}
\tilde{\zeta}'_p(0) =& -\int_0^1 p^{-n}\Phi\Big(\psi_{1/\sqrt{p}}\operatorname{Tr}_s\big[N_{u/p}\exp(-B^2_{p,u/p})\big] - \sum_{j=-d-1}^{0} B_{p,j}u^j\Big)\frac{du}{u} \\
&- \int_1^\infty p^{-n}\Phi\psi_{1/\sqrt{p}}\operatorname{Tr}_s\big[N_{u/p}\exp(-B^2_{p,u/p})\big]\frac{du}{u} - \sum_{j=-d-1}^{-1}\frac{B_{p,j}}{j} + \Gamma'(1)B_{p,0}, \\
\tilde{\zeta}_p(0) =& -\Phi B_{p,0}.
\end{aligned}$$

To prove Theorem 1 we will need the following result which is a consequence of the Arzelà-Ascoli Theorem.

Lemma 2.3.1 ([P16, Lemma 2.13]). *Let Y be a compact manfold and let (F, h^F) be a Hermitian* *bundle on Y with a connection ∇^F. Let $\{s_n\}_n \subset \Gamma(Y,F)$ be a sequence converging weakly to some distribution s. If for any $k \in \mathbf{N}_0$ there is $C_k > 0$ such that $\sup_n \|s_n\|_{\mathscr{C}^k(Y,F)} \leq C_k$ then s is smooth and s_n converges in the $\mathscr{C}^\infty$ topology to s.*

For the sake of clarity we restate Theorem 1.

Theorem 2.3.2. *Let $k \in \{0,\dots,\dim_{\mathbf{C}} B\}$. There are local coefficients $\alpha_i, \beta_i \in \Gamma(B, \Lambda^\bullet(T^*_{\mathbf{R}}B))$ such that for any $l \in \mathbf{N}_0$ the component of degree $2k$ of the analytic torsion forms has the following asymptotic as $p \to \infty$:*

$$T(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})^{(2k)} = \sum_{i=0}^{l} p^{k+n-i}(\alpha_i \log p + \beta_i)^{(2k)} + o(p^{k+n-l})$$

in the topology of $\mathscr{C}^\infty$ convergence on compact subsets of B.

Proof. By (2.2.11) the following form $\beta_{i,1} \in \Gamma(B, \Lambda^\bullet(T^*_{\mathbf{R}}B))$ is well-defined:

$$\beta_{i,1} := \frac{d}{ds}_{|s=0}\Big(- \frac{1}{\Gamma(s)} \int_0^1 u^{s-1} \Phi\Big(\int_Z \mathrm{Tr}_s \left[N_u a_{i,u}(z) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) \right] \Big) du \Big).$$

Also by Theorem 2.2.8 we can define the differential form $\beta_{i,2} \in \Gamma(B, \Lambda^\bullet(T^*_{\mathbf{R}}B))$ with

$$\beta_{i,2} := \frac{d}{ds}_{|s=0}\Big(- \frac{1}{\Gamma(s)} \int_1^\infty u^{s-1} \Phi\Big(\int_Z \mathrm{Tr}_s \left[N_u a_{i,u}(z) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) \right] \Big) du \Big).$$

Set

$$\beta_i := \beta_{i,1} + \beta_{i,2}. \tag{2.3.1}$$

Then we have

$$\begin{aligned} \beta_i = &- \int_0^1 \Phi\Big(\int_Z \mathrm{Tr}_s \left[N_u a_{i,u}(z) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) \right] - \sum_{j=-d-1}^{0} B_j^{[i]} u^j \Big) \frac{du}{u} \\ &- \int_1^\infty \Phi\Big(\int_Z \mathrm{Tr}_s \left[N_u a_{i,u}(z) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) \right] \Big) \frac{du}{u} - \sum_{j=-d-1}^{-1} \frac{B_j^{[i]}}{j} + \Gamma'(1) B_0^{[i]}. \end{aligned} \tag{2.3.2}$$

From Lebesgue dominated convergence theorem and Lemma 2.2.7, Theorem 2.2.8, we have for $k \in \mathbf{N}_0$, as $p \to \infty$,

- 1) $$\int_0^1 p^k \Big(\Big(p^{-n} \psi_{1/\sqrt{p}} \mathrm{Tr}_s \left[N_{u/p} \exp(-B^2_{p,u/p}) \right] - \sum_{j=-d-1}^{0} B_{p,j} u^j \Big) - \sum_{i=0}^{k-1} p^{-i} \Big(\int_Z \mathrm{Tr}_s \left[N_V a_{i,u}(x) + i\omega^H a_{i+1,u}(z) \right] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) - \sum_{j=-d-1}^{0} B_j^{[i]} u^j \Big) \Big) \frac{du}{u}$$

$$\longrightarrow \int_0^1 \Big(\int_Z \mathrm{Tr}_s \left[N_V a_{i,u}(x) + i\omega^H a_{i+1,u}(z) \right] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) - \sum_{j=-d-1}^{0} B_j^{[k]} u^j \Big) \Big) \frac{du}{u},$$

- 2) $\displaystyle\int_1^\infty p^k \Big(p^{-n} \psi_{1/\sqrt{p}} \mathrm{Tr}_s \left[N_{u/p} \exp(-B^2_{p,u/p}) \right] - \sum_{j=0}^{k-1} p^{-j} \int_Z \mathrm{Tr}_s \left[N_V a_{j,u}(x) + i\omega^H a_{j+1,u}(z) \right]$

$$\times\, d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) \Big) \longrightarrow \int_1^\infty \int_Z \mathrm{Tr}_s \left[N_V a_{j,u}(x) + i\omega^H a_{j+1,u}(z) \right] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) \Big) \frac{du}{u}.$$

By Proposition 2.2.6 we further have, as $p \to \infty$,

- 3) $\quad p^k \Big(B_{p,j} - \sum_{i=0}^{k-1} B_j^{[i]} p^{-i} \Big) \longrightarrow B_j^{[k]}$.

By 1) – 3) we have that for any $k \in \mathbf{N}$

$$\lim_{p\to\infty} p^k \Big(\tilde{\zeta}_p'(0) - \sum_{i=0}^{k-1} \beta_i p^{-i} \Big) = \beta_k$$

from which we get

$$p^n \tilde{\zeta}_p'(0) = \sum_{i=0}^{k} \beta_i p^{n-i} + o(p^{n-k}).$$

Now set

$$\alpha_i := \Phi B_0^{[i]}. \tag{2.3.3}$$

Then by Proposition 2.2.6 we have

$$-\tilde{\zeta}_p(0) = \Phi B_{p,0} = \sum_{i=0}^{k} \alpha_i p^i + o(p^{-k}).$$

Putting all the pieces now together we get

$$\begin{aligned} \psi_{1/\sqrt{p}} \zeta_p'(0) &= -p^n \log(p) \tilde{\zeta}_p(0) + p^n \tilde{\zeta}_p'(0) \\ &= \sum_{i=0}^{k} p^{n-i} (\alpha_i \log p + \beta_i) + o(p^{n-k}). \end{aligned}$$

Finally the statement now follows from Lemma 2.3.1. □

The terms α_0 and β_0 have already been calculated explicitly by [P16] in the more general case of Hermitian fibrations. Let $T^{H'}M$ be the orthogormal complement of TZ with respect to

$\Omega^{\mathcal{L}}$ and set

$$\Omega^{\mathcal{L},H'} := \Omega^{\mathcal{L}}_{|_{T^{H'}_{\mathbf{R}}M\times T^{H'}_{\mathbf{R}}M}}.$$

Define

$$\Theta^M := -\frac{1}{2\pi i}\Omega^{\mathcal{L}} \quad \text{and} \quad \Theta^Z := -\frac{1}{2\pi i}\Omega^{\mathcal{L}}_{|_{T_{\mathbf{R}}Z\times T_{\mathbf{R}}Z}}.$$

Let $(\Theta^Z)^{\perp,\Theta^M}$ be the orthogonal complement of Θ^Z with respect to Θ^M. Then Θ^Z will be extended to $T_{\mathbf{R}}M = \Theta^Z \oplus (\Theta^Z)^{\perp,\Theta^M}$ by zero. Put $\Theta^{Z,n} := (\Theta^Z)^{\wedge n}$. By [P16, (2.140)] the forms α_0, β_0 and $p^n(\alpha_0\log p+\beta_0)$ are given by

$$\begin{aligned}
\alpha_0 &= \frac{n\mathrm{rk}(\mathcal{E})}{2}\int_Z \frac{\Theta^{Z,n}}{n!}e^{-\frac{\Omega^{L,H'}}{2\pi i}},\\
\beta_0 &= \frac{\mathrm{rk}(\mathcal{E})}{2}\int_Z \det\left(\frac{\dot{\Omega}^{Z,\mathcal{L}}}{2\pi}\right)\log\left[\det\left(\frac{\dot{\Omega}^{Z,\mathcal{L}}}{2\pi}\right)\right]e^{-\frac{\Omega^{L,H'}}{2\pi i}}d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z) \qquad \text{and}\\
p^n(\alpha_0\log p+\beta_0) &= \frac{\mathrm{rk}(\mathcal{E})}{2}\int_Z \log\left[\det\left(\frac{p\dot{\Omega}^{Z,\mathcal{L}}}{2\pi}\right)\right]e^{-\frac{\Omega^{L,H'}}{2\pi i}+p\Theta^Z}d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z).
\end{aligned} \tag{2.3.4}$$

2.4 Proof of Theorem 2.2.8

The goal of this section is to prove Theorem 2.2.8. The case $k=0$ has already been proved in [P16, Theorem 2.23] and we will make use of results developed there. We consider the same contour integrals and study the structure of the upcoming operators.

Set

$$C_p := \frac{1}{p}B_p^2 = \frac{1}{p}(D_p^2 + R_p)$$

with $R_p := B_p^{2,(+)}$. Since $B_{p,u/p} = \frac{\sqrt{u}}{\sqrt{p}}\psi_{1/\sqrt{u/p}}B_p\psi_{\sqrt{u/p}}$ and $\psi_{1/\sqrt{p}}N_{u/p} = N_u$ we have

$$p^{-n}\psi_{1/\sqrt{p}}\operatorname{Tr}_s\left[N_{u/p}e^{-B^2_{p,u/p}}\right] = p^{-n}\operatorname{Tr}_s\left[N_u\psi_{1/\sqrt{u}}(e^{-uC_p})\right] \tag{2.4.1}$$

Furtheremore as in (2.1.8) for p sufficient large

$$\mathrm{Spec}(C_p) \subset \{0\}\cup[\vartheta,\infty].$$

For the proof of Theorem 2.2.8 we will assume without loss of generality that in the sequel that $\mathrm{Spec}(C_p)\subset\{0\}\cup[\vartheta,\infty]$ holds for $p\in\mathbf{N}$.

Recall that the contour $\delta+\Delta$ was given by figure 1 after (2.1.8) and the operator $\mathbb{K}_{p,u}$ was given by $\mathbb{K}_{p,u} = \frac{1}{2\pi i}\psi_{1/\sqrt{u}}\int_\Delta e^{-u\lambda}(\lambda - C_p)^{-1}d\lambda$. Now define the operator

$$\mathbb{P}_{p,u} := \frac{1}{2\pi i}\psi_{1/\sqrt{u}}\int_\delta e^{-u\lambda}(\lambda - C_p)^{-1}d\lambda.$$

From (2.4.1) we have

$$p^{-n}\psi_{1/\sqrt{p}}\operatorname{Tr}_s\left[N_{u/p}\exp(-B^2_{p,u/p})\right] = p^{-n}\operatorname{Tr}_s\left[N_u(\mathbb{P}_{p,u}+\mathbb{K}_{p,u})\right]. \tag{2.4.2}$$

The two operators $\mathbb{P}_{p,u}$ and $\mathbb{K}_{p,u}$ were introduced in [P16, section 2.5] and studied for the proof of Theorem 2.2.8 for the case $k=0$ by showing the inequality for $\mathbb{P}_{p,u}$ and $\mathbb{K}_{p,u}$ seperately and using (2.4.2). This is what we also going to do for $k \geq 1$ dealing with $\mathbb{K}_{p,u}$ first.

---

i) The operator $\mathbb{K}_{p,u}$

Recall from the proof of Theorem 2.2.1 that $a^{\mathbb{K}}_{i,u} \in \Gamma(Z, \Lambda^\bullet(T^*_{\mathbf{R}}B)\otimes \operatorname{End}(\Lambda^{0,\bullet}(^*TZ)\otimes\mathcal{E}))$ was given by

$$a^{\mathbb{K}}_{i,u} = \psi_{1/\sqrt{u}}\langle 0|K_{2i,u}|0\rangle = \psi_{1/\sqrt{u}}\frac{1}{i!}\frac{\partial^i}{\partial t^i}_{|t=0}\langle 0|K_u(L_t)|0\rangle. \tag{2.4.3}$$

Set

$$A^{\mathbb{K}}_{i,u} := \int_Z \operatorname{Tr}_s[N_u a^{\mathbb{K}}_{i,u}(z)]d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z).$$

Then from Theorem 2.2.1 as $p\to\infty$ we have an asymptotic expansion

$$\operatorname{Tr}_s\left[N_u\mathbb{K}_{p,u}\right] = \sum_{i=0}^k A^{\mathbb{K}}_{i,u}p^{n-i} + O(p^{n-k-1}). \tag{2.4.4}$$

In [P16, Proposition 2.28] it has been proven that for any $m\in\mathbf{N}_0$, there exist $a, C>0$ such that for $u\geq 1$ and $p\in\mathbf{N}$,

$$\left\|p^{-n}\operatorname{Tr}_s\left[N_u\mathbb{K}_{p,u}\right]\right\|_{\mathscr{C}^m(B,\Lambda^\bullet T^*_{\mathbf{R}}B)} \leq Ce^{-au}. \tag{2.4.5}$$

Now we are going to sharpen this result but without the exponential decay within the meaning of the upcoming Lemma.

Lemma 2.4.1. *For any $k, m \in \mathbf{N}_0$, there is $C > 0$ such that for $u \geq 1$ and $p \in \mathbf{N}$*

$$p^k \Big\| p^{-n} \operatorname{Tr}_s [N_u \mathbb{K}_{p,u}] - \sum_{j=0}^{k-1} A_{i,u}^{\mathbb{K}} p^{-j} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq \frac{C}{\sqrt{u}}.$$

Proof. Since the large time behavior of $\mathbb{K}_{p,u}$ is the same as its component in degree zero the proof of this Lemma is similar to the proof of [F18, Proposition 2.10].

First case: $u > \sqrt{p}$. Then by (2.4.5) we have the estimates

$$p^k \Big\| p^{-n} \operatorname{Tr}_s [N_u \mathbb{K}_{p,u}] \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq C p^k e^{-au} \leq C u^{2k} e^{-au} \leq C' e^{-bu} \tag{2.4.6}$$

for some constants $b, C' > 0$. By Lemma 2.1.8, Proposition 2.1.13 and (2.4.3) there exist $C_i, c_i > 0$ such that for any $u > 0$

$$\Big\| A_{i,u}^{\mathbb{K}} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq C_i e^{-c_i u}. \tag{2.4.7}$$

Thus for $j \geq 0$ we find $d, C', C'' > 0$ such that

$$p^k \Big\| A_{i,u}^{\mathbb{K}} p^{-j} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq C_i p^{k-j} e^{-c_i u} \leq C_i u^{2k-2j} e^{-c_i u} \leq C' e^{-du} \leq \frac{C''}{\sqrt{u}}. \tag{2.4.8}$$

The Lemma now follows from (2.4.6) and (2.4.8) for the case $u > \sqrt{p}$.
Second case: $u \leq \sqrt{p}$. Here we write $t = \frac{1}{\sqrt{p}}$ again. By Lemma 2.1.8, Proposition 2.1.13 and Taylor expansion (2.1.14) we have for some $C, c > 0$

$$p^k \Big\| \operatorname{Tr}_s [N_u K(L_t)] - \sum_{j=0}^{k-1} A_{i,u}^{\mathbb{K}} p^{-j} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq C e^{-cu}$$

By Lemma 2.2.3 (with $l = 0$) , $p^{-1} \leq u^{-2}$ and $u \geq 1$ we find $C' > 0$ such that

$$p^k \Big\| p^{-n} \operatorname{Tr}_s [N_u \mathbb{K}_{p,u}] - \operatorname{Tr}_s [N_u K(L_t)] \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq C' p^{-(n-k)} \leq C' u^{-2(n-k)} \leq \frac{C'}{\sqrt{u}}.$$

Thus we find $C'' > 0$ with

$$\begin{aligned} &p^k \Big\| p^{-n} \operatorname{Tr}_s [N_u \mathbb{K}_{p,u}] - \sum_{j=0}^{k-1} A_{i,u}^{\mathbb{K}} p^{-j} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \\ &\leq p^k \Big\| p^{-n} \operatorname{Tr}_s [N_u \mathbb{K}_{p,u}] - \operatorname{Tr}_s [N_u K(L_t)] \Big\| + p^k \Big\| \operatorname{Tr}_s [N_u K(L_t)] - \sum_{j=0}^{k-1} A_{i,u}^{\mathbb{K}} p^{-j} \Big\| \leq \frac{C''}{\sqrt{u}} \end{aligned}$$

which finishes the proof. □

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ii) The operator $\mathbb{P}_{p,u}$

Our goal here is to an analoge result for for $\mathbb{P}_{p,u}$. The methods we are using here are similar to that of [Bi97, Section 9.13] and [P16, Proposition 2.29]. First we will need some preperations.

Let $p_0 \gg p$ be given. The precise value of p_0 will be specified later. Locally we will consider $\mathcal{L}^p$ as a subset of $\mathcal{L}^{p_0} = \mathcal{L}^p \otimes \mathcal{L}^{p_0-p}$ by fixing a non vanishing local section of $\mathcal{L}$. Thus when working locally we view B_p as an operator acting on $\Gamma(M, \pi^*(\Lambda^\bullet T^*_{\mathbf{R}}B) \otimes \mathcal{E} \otimes \mathcal{L}^{p_0})$ by setting $B_p(s_1 \otimes s_2) := B_p(s_1) \otimes s_2$ for $s_1 \in \Gamma(M, \pi^*(\Lambda^\bullet T^*_{\mathbf{R}}B) \otimes \mathcal{E} \otimes \mathcal{L}^p)$ and $s_2 \in \Gamma(M, \mathcal{L}^{p_0-p})$. Because $\mathrm{End}(\mathcal{L}^{p_0})$ is the trivial bundle our calculations on kernels will not depend on the choices we made.

Over $U_{z_k} = B^Z_\varepsilon(z_k)$ by [P16, (2.7)] which follows from the Lichnerowicz formula Theorem 1.2.5 the operator B_p^2 has locally the form

$$B_p^2 = D^{Z,2} + R + p\mathcal{O}_1 + p\mathcal{O}_0^1 + p^2\mathcal{O}_0^1 \tag{2.4.9}$$

where $R, \mathcal{O}_1$ (resp. $\mathcal{O}_0^1, \mathcal{O}_0^2$ are operators of order 1 (resp. 0). Set $t = \frac{1}{\sqrt{p}}$. Using the right hand side of (2.4.9) and partition of unity subordinate to $\{U_{z_k}\}_{k=1,\dots,N}$ we extend $B^2_{t^{-2}}$ for all $0 < t < 1$ and consider it as an operator acting fibrewise on $\Gamma(Z, \Lambda^\bullet(T^*_{\mathbf{R},b_0}B) \otimes (\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E} \otimes \mathcal{L}^{p_0})$ where $\Lambda^\bullet(T^*_{\mathbf{R},b_0}B)$ is a trivial bundle over Z. We will also write $B^2_{t^{-2}}$ for the extended operator. Here this extension does depend on the choice of the partition of unity and on the non vanishing local sections of $\mathcal{L}$. The mapping $t \to t^{2n}\psi_t \operatorname{Tr}_s\left[N_{ut^2} \exp\left(B^2_{t^{-2},u}\right)\right]$ is smooth on $]0,1]$ and by Theorem 2.2.1 we have the asymptotic expansion

$$t^{2n}\psi_t \operatorname{Tr}_s\left[N_{ut^2} \exp\left(B_{t^{-2},t^2u}\right)\right] = \sum_{i=0}^{k} A_{i,u} t^{2i} + O(t^{2k+2})$$

from which see that $t^{2n}\psi_t \operatorname{Tr}_s\left[N_{ut^2} \exp\left(B_{t^{-2},u}\right)\right]$ has a continuous extenstion at $t = 0$ with value $A_{0,u}$. Our first goal is to show that the function is smooth on $[0,1]$. To archieve this we need a converse of Taylor's Theorem and cite a result from [AR67, 2.1] regarding Peano derivates.

Let G and H be Banach spaces. Denote by $\mathscr{L}^k(G,H)$ the Banach space of bounded k-multilinear maps from G to H, i.e. $\mathscr{L}^0(G,H) = H$ and $\mathscr{L}^{k+1}(G,H) = \mathscr{L}(G, \mathscr{L}^k(G,H))$. Let $\mathscr{L}^k_s(G,H)$ be the Banach space of bounded symmetric k-multilinear maps from G to H. Let

$U \subset G$ an open convex set, $f : U \to H$, $\varphi_k : U \to \mathscr{L}_s^k(G, H)$ for $k = 0, 1, \dots r$. For $a \in U$ and $t \in U$ with $t - a$ small enough define $\rho(a, t) \in H$ by

$$f(t) = \sum_{k=0}^{r} \frac{\varphi_k(a)}{k!}(t-a)^k + \rho(a,t).$$

The φ_k are called kth Peano derivatives of f if $\frac{\|\rho(a,t)\|}{\|t-a\|^r} \to 0$ as $t \to a$.

Theorem 2.4.2. *[AR67, 2.1] If each φ_k , $k = 0, 1, \dots r$, are continous then f is of class $\mathscr{C}^r$ with $D^k f = \varphi_k$ for $k = 0, 1, \dots r$.*

For special cases the condition of continuity can we weakened. If $G = [a, b]$ is an interval and $H = \mathbf{R}$ then only boundedness on G is required ([O54, Theorem 3]).

Lemma 2.4.3. *The function $t \mapsto t^{2n}\psi_t \operatorname{Tr}_s \left[N_{ut^2} \exp\left(B^2_{t^{-2},u}\right)\right]$ is smooth on $[0, 1]$ with derivatives at 0 given by*

$$\frac{\partial^{2r}}{\partial t^{2r}}_{|t=0} t^{2n}\psi_t \operatorname{Tr}_s \left[N_{ut^2} \exp\left(B^2_{t^{-2},t^2u}\right)\right] = (2r)!A_{r,u} \qquad \textit{and}$$
$$\frac{\partial^{2r+1}}{\partial t^{2r+1}}_{|t=0} t^{2n}\psi_t \operatorname{Tr}_s \left[N_{ut^2} \exp\left(B^2_{t^{-2},t^2u}\right)\right] = 0$$

for $r \geq 0$.

Proof. As $\psi_{1/\sqrt{u}}\langle 0|\exp\left(-uL_t\right)|0\rangle = \psi_{1/\sqrt{u}}\langle 0|t^{2n}\exp\left(-ut^2M_{t^{-2},z}\right)|0\rangle$ is smooth in $t \in [0, 1]$ we have

$$\psi_{1/\sqrt{u}}\langle 0|\exp\left(-uL_t\right)|0\rangle = \sum_{k=0}^{r} \frac{\varphi_k(a)}{k!}(t-a)^k + o(|t-a|)^{r+1}$$

with continuous φ_k given by the derivatives of $\psi_{1/\sqrt{u}}\langle 0|\exp\left(-uL_t\right)|0\rangle$. With the same constants as in (2.1.2) we can find $0 < t_a < 1$ small enough such that $t^{2n}Ct^{-2N}\exp\left(-\frac{\varepsilon^2}{16ut^2}\right) \leq C|t-a|^{r+1}$ for all $t \leq t_a$ and a constant $C > 0$. Thus we get

$$\begin{aligned}
&\left\|\psi_{1/\sqrt{u}}t^{2n}\langle z|\exp\left(-ut^2B^2_{t^{-2}}\right)|z\rangle - \sum_{k=0}^{r} \frac{\varphi_k(a)}{k!}(t-a)^k\right\|_{\mathscr{C}^m} \\
&\leq \left\|\psi_{1/\sqrt{u}}t^{2n}\langle z|\exp\left(-ut^2B^2_{t^{-2}}\right)|z\rangle - \psi_{1/\sqrt{u}}\langle 0|\exp\left(-uL_t\right)|0\rangle\right\|_{\mathscr{C}^m} \\
&\quad + \left\|\psi_{1/\sqrt{u}}\langle 0|\exp\left(-uL_t\right)|0\rangle - \sum_{k=0}^{r} \frac{\varphi_k(a)}{k!}(t-a)^k\right\|_{\mathscr{C}^m} \\
&\leq C'|t-a|^{r+1}
\end{aligned}$$

for a constant $C' > 0$. Since the φ_k are continuous we conclude the Lemma by Theorem 2.4.2. $\square$

Next we will need an analogue result for $t^{2n}\,\mathrm{Tr}_s\left[N_u \mathbb{K}_{t^{-2},u}\right]$ and $t^{2n}\,\mathrm{Tr}_s\left[N_u \mathbb{P}_{t^{-2},u}\right]$. Put

$$J^{\mathbb{P}}_{r,u} := J_{r,u} - K_{r,u}.$$

Then we have an asymptotic expansion of $\mathrm{Tr}_s\left[N_u \mathbb{P}_{p,u}\right]$ as $p \to \infty$,

$$\mathrm{Tr}_s\left[N_u \mathbb{P}_{p,u}\right] = \sum_{i=0}^{k} A^{\mathbb{P}}_{i,u} p^{n-i} + O(p^{n-k-1})$$

with $A^{\mathbb{P}}_{i,u} := \psi_{1/\sqrt{u}} \int_Z \mathrm{Tr}_s\left[N_u \langle 0|J^{\mathbb{P}}_{z,2i,u}|0\rangle\right] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z)$. In particular

$$A_{i,u} = A^{\mathbb{K}}_{i,u} + A^{\mathbb{P}}_{i,u}.$$

Lemma 2.4.4. *The function $t \mapsto t^{2n}\,\mathrm{Tr}_s\left[N_u \mathbb{K}_{t^{-2},u}\right]$ is smooth on $[0,1]$ with derivatives at 0 given by*

$$\frac{\partial^{2r}}{\partial t^{2r}}_{|t=0} t^{2n}\,\mathrm{Tr}_s\left[N_u \mathbb{K}_{t^{-2},u}\right] = (2r)! A^{\mathbb{K}}_{i,u} \quad \textit{and} \quad \frac{\partial^{2r+1}}{\partial t^{2r+1}}_{|t=0} t^{2n}\,\mathrm{Tr}_s\left[N_u \mathbb{K}_{t^{-2},u}\right] = 0 \qquad (2.4.10)$$

for $r \geq 0$. Same holds if $\mathbb{K}$ is replaced by $\mathbb{P}$.

Proof. As $\langle 0|K_u(L_t)|0\rangle$ is smooth in $t \in [0,1]$ (Proposition 2.1.13) the proof is the same as in Lemma 2.4.3 with one slight different where

$$\left\| \psi_{1/\sqrt{u}} t^{2n} \langle z| \exp\left(-ut^2 B^2_{t^{-2}}\right)|z\rangle - \psi_{1/\sqrt{u}} \langle 0| \exp\left(-uL_t\right)|0\rangle \right\|_{\mathscr{C}^m} \leq C(|t-a|^{r+1})$$

in the proof of Lemma 2.4.3 has to be changed by the following: By Lemma 2.2.3 and continuity in t we choose $0 < t_a < 1$ und l large enough such that

$$\left\| t^{2n} \langle z|\mathbb{K}_{t^{-2},u}|z\rangle - \psi_{1/\sqrt{u}} \langle 0|K_u(L_t)|0\rangle \right\|_{\mathscr{C}^m} \leq Ct^{2l} \leq C_1(|t-a|^{r+1})$$

for $0 < t < t_a$ and constants $C, C_1 > 0$. The other arguments are the same. By (2.4.1) and Lemma 2.4.3 the statement holds if $\mathbb{K}$ is replaced by $\mathbb{P}$. $\square$

By our assumption on ampleness we have locally on $U \subset Z$ for $p_1 \geq p$

$$\ker D^2_{p|U} = \Gamma^{\mathrm{hol}}(U, \mathcal{E}_{|U} \otimes \mathcal{L}^p_{|U}) \subset \Gamma^{\mathrm{hol}}(U, \mathcal{E}_{|U} \otimes \mathcal{L}^{p_1}_{|U}) = \ker D^2_{p_1|U} \qquad (2.4.11)$$

as subspaces of $\mathfrak{A}^{0,\bullet}(U, \mathcal{E} \otimes \mathcal{L}^{p_0})$. The kernel of the extension $D_{t^{-2}}$ $(t \in]0,1])$ are subspaces of it as well. Let P_p be the orthogonal projection from $\mathfrak{A}^{0,\bullet}(U, \mathcal{E}_{|U} \otimes \mathcal{L}^{p_0}_{|U})$ onto the kernel of D_p^2. We have as in [MM07, (4.1.59)] for any $k \in \mathbf{N}$ the integral representation for the spectral projection,

$$P_p = \frac{1}{2\pi} \int_\delta \lambda^{k-1} (\lambda - D_p^2)^{-k} d\lambda. \tag{2.4.12}$$

Let $P_p^\perp := 1 - P_p$ be the orthogonal projection onto the complement of the kernel. For $p_1 \geq p$ we have

$$\begin{aligned}
&P_{p_1} P_p = P_p P_{p_1} = P_p, \\
&P_{p_1}^\perp P_p^\perp = 1 - P_p - P_{p_1} + P_{p_1} P_p = 1 - P_{p_1} = P_{p_1}^\perp, \\
&P_p^\perp P_{p_1}^\perp = 1 - P_p - P_{p_1} + P_p P_{p_1} = 1 - P_{p_1} = P_{p_1}^\perp.
\end{aligned} \tag{2.4.13}$$

Lemma 2.4.5. *For any $k, m \in \mathbf{N}_0$ there exists $C > 0$ such that for $u \geq 1$ and $p \in \mathbf{N}$:*

$$p^k \Big\| p^{-n} \operatorname{Tr}_s [N_u \mathbb{P}_{p,u}] - \sum_{j=0}^{k-1} A^{\mathbb{P}}_{i,u} p^{-j} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq \frac{C}{\sqrt{u}}.$$

Proof. Throughout the proof let $p \in \mathbf{N}$ be fixed while $t \in]0, \frac{1}{\sqrt{p}}[$ varies. By Taylor's formula we have for some $\tau \in]0, 1[$.

$$\begin{aligned}
&p^k \Big\| p^{-n} \operatorname{Tr}_s [N_u \mathbb{P}_{p,u}] - \sum_{j=0}^{k-1} A^{\mathbb{P}}_{i,u} p^{-j} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \\
&= p^k \Big\| t^{2n} \operatorname{Tr}_s [N_u \mathbb{P}_{t^{-2},u}] - \sum_{j=0}^{2k-1} \frac{1}{j!} \big(\frac{\partial^j}{\partial t^j}_{|t=0} t^{2n} \operatorname{Tr}_s [N_u \mathbb{P}_{t^{-2},u}] \big) t^j \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \\
&= p^k \Big\| \frac{1}{2k!} \frac{\partial^{2k}}{\partial t^{2k}}_{|t=\tau} t^{2n} \operatorname{Tr}_s [N_u \mathbb{P}_{t^{-2},u}] t^{2k} \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \\
&\leq \Big\| \big(\frac{\partial^{2k}}{\partial t^{2k}}_{|t=\tau} t^{2n} \operatorname{Tr}_s [N_u \mathbb{P}_{t^{-2},u}] \big) \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)}.
\end{aligned}$$

Thus our goal here will be to show

$$\Big\| \big(\frac{\partial^{2k}}{\partial t^{2k}}_{|t=\tau} t^{2n} \operatorname{Tr}_s [N_u \mathbb{P}_{t^{-2},u}] \big) \Big\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{R}} B)} \leq \frac{C}{\sqrt{u}}. \tag{2.4.14}$$

For $k = 0$ this inequality has been shown in [P16, Proposition 2.28] and we use the techniques developed and notations there with additional adaption regarding derivations in t.

Choose p_0 large enough such that $\tau \in [\frac{1}{\sqrt{p_0}}, 1]$. Set

$$C_{t^{-2}} := t^2 B_{t^{-2}}^2, \quad C_{t^{-2}}^{(0)} := t^2 D_{t^{-2}}^2, \quad \text{and} \quad \tilde{R}_{t^{-2}} := t^2 R_{t^{-2}} = C_{t^{-2}} - C_{t^{-2}}^{(0)}.$$

$C_{t^{-2}}^{(0)}$ is just the degree zero part of $C_{t^{-2}}$ and it is injective on $\ker(D_{t^{-2}}^2)$. By an abuse of notation we write $(C_{t^{-2}}^{(0)})^{-1}$ for the Green operator of $C_{t^{-2}}^{(0)}$ which vanishes on $\ker(D_{t^{-2}}^2)$ and equals $(C_{t^{-2}}^{(0)})^{-1}$ on $\ker(D_{t^{-2}}^2)^{\perp}$, i.e.

$$(C_{t^{-2}}^{(0)})^{-1} := P_{t^{-2}}^{\perp} (C_{t^{-2}}^{(0)})^{-1} P_{t^{-2}}^{\perp}.$$

By (2.0.1) we have for $p_1 \geq p$

$$\operatorname{Spec}(D_{p_1}^2) \subset \{0\} \cup]2p_1\mu_0 - C_{\mathcal{L}}, \infty[\subset \{0\} \cup]2p\mu_0 - C_{\mathcal{L}}, \infty[.$$

In particular $\lambda - uC_{t^{-2}}^{(0)}$ is invertible for $\lambda \in \delta$ since $t^{-2} \geq p$ and its inverse is equal to its right inverse. As $\ker D_p^2 \subset \ker D_{t^{-2}}^2$ (see 2.4.11) we have

$$\Big(\lambda - uC_{t^{-2}}^{(0)}\Big)\Big(\frac{1}{\lambda} P_p\Big) = P_p - \frac{1}{\lambda} uC_{t^{-2}}^{(0)} P_p = P_p.$$

Therefore we deduce

$$\begin{aligned}
&(\lambda - uC_{t^{-2}}^{(0)})\Big(\frac{1}{\lambda} P_p + (\lambda - uC_{t^{-2}}^{(0)})^{-1} P_p^{\perp}\Big) = P_p + P_p^{\perp} = 1 \\
\Rightarrow &(\lambda - uC_{t^{-2}}^{(0)})^{-1} = \frac{1}{\lambda} P_p + (\lambda - uC_{t^{-2}}^{(0)})^{-1} P_p^{\perp}.
\end{aligned} \tag{2.4.15}$$

The function $\lambda \mapsto (\lambda - uC_{t^{-2}}^{(0)})^{-1} P_p^{\perp}$ is a holomorphic on $B_{\frac{\varrho}{2}}(0) \setminus \{0\}$ and for $\lambda = 0$ we get from (2.4.13) and our notation

$$(uC_{t^{-2}}^{(0)})^{-1} P_p^{\perp} = P_{t^{-2}}^{\perp} (uC_{t^{-2}}^{(0)})^{-1} P_{t^{-2}}^{\perp} P_p^{\perp} \overset{(2.4.13)}{=} P_{t^{-2}}^{\perp} (uC_{t^{-2}}^{(0)})^{-1} P_{t^{-2}}^{\perp} = (uC_{t^{-2}}^{(0)})^{-1}.$$

Thus $\lambda \mapsto (\lambda - uC_{t^{-2}}^{(0)})^{-1} P_p^{\perp}$ is a holomorphic function on the interior of δ. Since $C_{t^{-2}}$ has no eigenvalues between the two circles δ and δ/u we have

$$\begin{aligned}
\mathbb{P}_{t^{-2},u} &= \frac{1}{2\pi i} \psi_{1/\sqrt{u}} \int_{\delta/u} e^{-u\lambda} (\lambda - C_{t^{-2}})^{-1} d\lambda \\
&= \frac{1}{2\pi i} \psi_{1/\sqrt{u}} \int_{\delta} e^{-\lambda} (\lambda - uC_{t^{-2}})^{-1} d\lambda.
\end{aligned}$$

Therefore we deduce the same way as in [P16, (2.188)] using (2.4.15) and

$$e^{-\lambda}(\lambda - uC_{t^{-2}})^{-1} = \Big(\sum_{k\geq 0}\frac{(-1)^k}{k!}\lambda^k\Big)\Big(\sum_{l\geq 0}(\lambda - uC^{(0)}_{t^{-2}})^{-1}(u\tilde{R}_{t^{-2}})\cdots(u\tilde{R}_{t^{-2}})(\lambda - uC^{(0)}_{t^{-2}})^{-1}\Big)$$

that the operator $\mathbb{P}_{t^{-2},u}$ is given by

$$\mathbb{P}_{t^{-2},u} = \psi_{1/\sqrt{u}}\sum_{l=0}^{\dim_{\mathbf{R}} B}\sum_{1\leq i_0\leq l+1}\frac{(-1)^{l-\sum_m j_m}}{(i_0 - 1 - \sum_m j_m)!}T_{p,t^{-2},1}(u\tilde{R}_{t^{-2}})T_{p,t^{-2},2}\dots(u\tilde{R}_{t^{-2}})T_{p,t^{-2},l+1} \tag{2.4.16}$$

where P_p appears i_0 times among the $T_{p,t^{-2},j}$ and the other terms are given respectively by $(uC^{(0)}_{t^{-2}})^{-(1+j_1)},\dots,(uC^{(0)}_{t^{-2}})^{-(1+j_{l+1-i_0})}$. From [P16, (2.189),(2.190)] the sum (2.4.16) can be written as

$$\mathbb{P}_{t^{-2},u} = \sum_{l=0}^{\dim_{\mathbf{R}} B}\prod_j A_{j,t,p,u}$$

with

$$A_{j,t,p,u} \in \Big\{A_1(u\psi_{1/\sqrt{u}})\tilde{R}^{(1)}_t A_2, A_1(u\psi_{1/\sqrt{u}})\tilde{R}^{(\geq 2)}_t A_2 | A_i \in \{P_p, (uC^{(0)}_t)^{-(1+j)}, (uC^{(0)}_t)^{-(1+j)/2}\}\Big\}$$

where $R_{t^{-2}} = R^{(1)}_{t^{-2}} + R^{(\geq 2)}_{t^{-2}}$ is the decomposition of $R_{t^{-2}}$ with respect to the degree in $\Lambda^\bullet(T^*_{\mathbf{R}}B)$. By [P16, (2.195)] it has been shown that for $r, r' \geq \frac{1}{2}$ and if $t = \frac{1}{\sqrt{p}}$ there exist $C, C', C'' > 0$ independent of p such that

$$\begin{aligned} \|f_{1,p}(t)\|_\infty &:= \|P_p\tilde{R}_{t^{-2}}P_p\|_\infty \leq C, & \|f_{2,p}(t)\|_\infty &:= \|P_p\tilde{R}_t(C^{(0)}_t)^{-r}\|_\infty \leq C', \\ \|f_{3,p}(t)\|_\infty &:= \|\tilde{R}_t(C^{(0)}_t)^{-r}P_p\|_\infty \leq C', & \|f_{4,p}(t)\|_\infty &:= \|(C^{(0)}_t)^{-r}\tilde{R}_t(C^{(0)}_t)^{-r}\|_\infty \leq C''. \end{aligned} \tag{2.4.17}$$

Thus for $t = \frac{1}{\sqrt{p}}$ each term in the sum (2.4.16) is a product of uniformly bounded terms in which P_p appears since $i_0 \geq 1$. We want to show that (2.4.17) holds for the higher derivatives in t,too. To archieve this we are left to show that the smooth functions $f_{i,p}$ are bounded on the compact set $[\frac{1}{\sqrt{p_0}}, 1]$ by constants independet of p. Then their derivations will be it as well.

For any operator A it is equivalent to show that $\|As\|_{L^2} \leq C\|s\|_{L^2}$ for any section s or for any section with support in a ball of radius $\varepsilon > 0$. Thus we can work locally and make use of (2.4.11), (2.4.13). As an orthogonal projection P_p has norm one, $\|P_p\|_\infty = 1$. From (2.4.13) and (2.4.17) we infer

$$\|P_p\tilde{R}_{t^{-2}}P_p\|_\infty \overset{(2.4.13)}{=} \|P_pP_{t^{-2}}\tilde{R}_{t^{-2}}P_{t^{-2}}P_p\|_\infty \leq \|P_p\|_\infty\|P_{t^{-2}}\tilde{R}_{t^{-2}}P_{t^{-2}}\|_\infty\|P_p\|_\infty$$

$$= \|P_{t^{-2}}\tilde{R}_{t^{-2}}P_{t^{-2}}\|_\infty \overset{(2.4.17)}{\leq} C.$$

With the same method substituting P_p by $P_{t^{-2}}$ we accomplish that the other $\|f_{i,p}(t)\|_\infty$ are bounded by constans independed of p, too. Thus we conclude

$$\left\|\frac{\partial^r f_{1,p}}{\partial t^r}(t)\right\|_\infty \leq C, \quad \left\|\frac{\partial^r f_{2,p}}{\partial t^r}(t)\right\|_\infty \leq C',$$
$$\left\|\frac{\partial^r f_{3,p}}{\partial t^r}(t)\right\|_\infty \leq C'', \quad \left\|\frac{\partial^r f_{4,p}}{\partial t^r}(t)\right\|_\infty \leq C'''$$

for some constants $C, C', C'', C''' > 0$. By ([P16, p.38]) $\mathbb{P}_{t^{-2},u}$ is a polynom in $\sqrt{u}$, $\mathbb{P}_{t^{-2},u} \in \mathbf{C}_N\left[\frac{1}{\sqrt{u}}\right]$. Differentiating in t does not change this fact. Therfore $\frac{\partial^r}{\partial t^r}\mathbb{P}_{t^{-2},u}$ is a polynom in $\frac{1}{\sqrt{u}}$ with coefficients consist of bounded operators in which P_p appears at least once. It follows

$$\frac{\partial^r}{\partial t^r}p^{-n}\operatorname{Tr}_s\left[N_u\mathbb{P}_{t^{-2},u}\right] = \sum_{k=0}^{K} c_k(p)u^{-k/2}$$

with $c_k(p) \in \mathfrak{A}^\bullet(B)$ satisfying

$$\|c_k(p)\|_{\mathscr{C}^0(B,\Lambda^\bullet T^*_{\mathbf{R}}B)} \leq p^{-n}C\operatorname{Tr}P_p = p^{-n}C\dim\ker(D_p^2) = p^{-n}C\dim H^0(Z,\mathcal{E}\otimes\mathcal{L}^p) \leq C.$$

From [Bi13, Theorem 4.10.4] and the general condition on ampleness we have for $t = \frac{1}{\sqrt{p}}$

$$\lim_{u\to\infty} t^{2n}\operatorname{Tr}_s\left[N_u\mathbb{P}_{t^{-2},u}\right] = t^{-2n}\psi_{1/\sqrt{p}}\operatorname{Tr}_s\left[N_V\exp(-(\nabla^{H(Z,\mathcal{E}\otimes\mathcal{L}^p)})\right) = 0.$$

which implies $\lim_{u\to\infty}\frac{\partial^r}{\partial t^r}p^{-n}\operatorname{Tr}_s\left[N_u\mathbb{P}_{t^{-2},u}\right] = 0$. Thus $c_0(p) = 0$ from which we have

$$\left\|\frac{\partial^r}{\partial t^r}p^{-n}\operatorname{Tr}_s\left[N_u\mathbb{P}_{t^{-2},u}\right]\right\|_{\mathscr{C}^0(B,\Lambda^\bullet T^*_{\mathbf{R}}B)} \leq \frac{C}{\sqrt{u}}.$$

which shows the statement for the case $l = 0$. To show the gerneral case for $l \geq 0$ we can proceed in the same way as in [P16, (2.198)] since differentation in t does not change the fact that $\frac{\partial^{2k}}{\partial t^{2k}}\nabla_U^{\operatorname{End}(\mathbb{E})}\mathbb{P}_{t,u}$ is a sum of product of polynomial in $\frac{1}{\sqrt{u}}$. □

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iii) Proof of Theorem 2.2.8

Now we can complete the proof of Theorem 2.2.8. Because of

$$p^{-n}\psi_{1/\sqrt{p}}\operatorname{Tr}_s\left[N_{u/p}\exp(-B^2_{p,u/p})\right] = p^{-n}\operatorname{Tr}_s\left[N_u(\mathbb{P}_{p,u} + \mathbb{K}_{p,u})\right]$$

the first statement in Theorem 2.2.8 follows from Lemma 2.4.1 and Lemma 2.4.5. The second statement follows from $A_{i,u} = A^{\mathbb{K}}_{i,u} + A^{\mathbb{P}}_{i,u}$, (2.4.7), (2.4.10), Lemma 2.4.4 and (2.4.14). □

Chapter 3

Computation of the First Coefficient associated with a Principle Bundle

In this chapter we consider the case where the family of vector bundles and the analytic torsion forms arise from the geometry of a principle bundle. This setting was studied in [BG00] where a comparison formula between the equivariant torsion form and a Lie algebraic equivariant analytic torsion was established. The main benefit of this special case is the absent of terms coupling horizontal and vertical variables appearing in the Lichnerowicz formula and operators which simplifies certain calculations significantly, see for instance the proof of [P16, Theorem 2.24]. In section 3.1 we begin with summarizing the settings from [BG00, section 2] and [BGV92, chapters 7.6 and 10.7] which we will work with. In section 3.2 we move on to study the (covariant) connections and curvatues in more detail with the goal to translate particular results from chapter 2 in terms of principle connections and curvatures. After that we make in section 3.3 the assumption $\omega^M = -\frac{1}{2\pi i}\Omega^{\mathcal{L}}$ and look how the the objects then looks like, including α_0 and β_0. Their evaluations on $\mathbf{P}^1\mathbf{C}$-bundles will also be studied. Finally in section 3.4 we will compute the coefficient $a_{1,u}$, i.e. proving Theorem 3.

3.1 Analytic Torsion Forms associated to a Principle Bundle

Let Z be a complex manifold with complex dimension n. Let $E \overset{\pi_E}{\to} Z$ be a holomorphic vector bundle on Z. Let G be a compact connected Lie group acting holomorphically on the left on Z. We assume that this action lifts holomorphically to an action on E so that $E \overset{\pi_E}{\to} Z$ becomes a G-equivariant holomorphic vector bundle. The action of G on functions $\mathscr{C}^\infty(Z)$ is given by $(\gamma \cdot f)(z) = f(\gamma^{-1}z)$. Let h^{TZ} and h^E be G-invariant Hermitian metrics on TZ and E. We assume that (Z, h^{TZ}) is a Kähler manifold with Kähler form ω^Z.

For $K \in \mathfrak{g}$ let K_Z be the corresponding fundamental vector field on Z satisfying for $f \in \mathscr{C}^\infty(Z)$

$$(K_Z.f)(z) = \frac{d}{dt} f(\exp(-tK)z)_{|t=0}.$$

The assignment $K \to K_Z$ becomes a Lie algebra homomorphism,

$$[K_Z, K'_Z] = [K, K']_Z, \quad K, K' \in \mathfrak{g}. \tag{3.1.1}$$

The action of G on $\Gamma(Z,E)$ is given by $(\gamma \cdot s)(z) = \gamma^E \cdot s(\gamma^{-1} z)$ and in the same way K induces a vector field K_E on E by replacing f with s above. Our definitions differs from [BG00] where the minus sign in $\frac{d}{dt} f(\exp(-tK)z)_{|t=0}$ is absent so that $K \to K_Z$ is a Lie algebra antihomomorphism ([BG00, (2.2)]).

We assume that for the action of G on Z we have given a smooth moment map $\mu : Z \to \mathfrak{g}^*$, that is μ satisfies the following: For all $\gamma \in G$ and $z \in Z$ the mapping is equivariant,

$$\mu(\gamma z) = \gamma \cdot \mu(z),$$

and for any $K \in \mathfrak{g}$ the vector field K_Z is the Hamiltonian vector field generated by $\langle \mu, K \rangle$, i.e.

$$d\langle \mu, K \rangle - \iota_{K_Z} \omega^Z = 0$$

where $\langle \cdot, \cdot \rangle$ denotes the dual product. In general the moment map does not have to be unique but there are criterion for it, e.g. the uniqueness is provided if G is semisimple.

Let ∇^{TZ} and ∇^E be the holomorphic Hermitian connections on (TZ, h^{TZ}) and (E, h^E).

Definition 3.1.1. *The moment $m^{TZ}(K)$ of $K \in \mathfrak{g}$ relative to the connection ∇^{TZ} is given by*

$$m^{TZ}(K) := \nabla^{TZ} K_Z.$$

Since K_Z is a Killing vector field and ω^Z is a Kähler form $m^{TZ}(K)$ is a skew-adjoint section of $\mathrm{End}(TZ)$. In the same we can define $m^E(K) := \nabla^E K_E$. Let Ω^{TZ} and Ω^E be the curvatures of ∇^{TZ} and ∇^E.

Now we are going to define a family of vector bundles. As shown in [GS82, Proposition 4.1] G has a unique complexification $G_{\mathbf{C}}$ with the properties that its Lie algebra $\mathfrak{g}_{\mathbf{C}}$ is the complexification of $\mathfrak{g}$ and that G is a maximal compact subgroup of $G_{\mathbf{C}}$. The holomorphic action of G on Z extends to a holomorphic action of $G_{\mathbf{C}}$ on Z ([GS82, Theorem 4.4]). It has been proven in [GS82, Theorem 5.1] that for the case if E is a line bundle over Z the action of G on E

can be canonically extended to an action of $G_{\mathbf{C}}$. According to [BG00, p.1314] by proceeding similar as in the proof of [GS82, Theorem 5.1] this is also true for arbitrary G-equivariant bundle.

Let $P \overset{G_{\mathbf{C}}}{\to} B$ be a holomorphic principle bundle with structure group $G_{\mathbf{C}}$. As mentioned above G is a maximal compact subgroup of $G_{\mathbf{C}}$. Let

$$G_{\mathbf{C}} = PG$$

be the Cartan decomposition of $G_{\mathbf{C}}$. The exponential map $\exp : \mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \oplus i\mathfrak{g} \to G_{\mathbf{C}}$ maps $i\mathfrak{g}$ bijectively onto P and $G_{\mathbf{C}}/G$ is contractible. By [KN63, Proposition 5.6 and Theorem 5.7] the $G_{\mathbf{C}}$-bundle P can be reduced to a G-bundle Q. By [Si59, page 586] (see also [A57, Proposition 5]) the G-bundle Q is equipped with a cannonical Cartan connection form θ of type $(1,0)$ associated to its complex structure. More precisely if J denotes the complex structure of $P \overset{G_{\mathbf{C}}}{\to} B$ then the distribution $Q \ni u \to T_uQ \cap J(T_uQ)$ defines a connection on $Q \overset{G}{\to} B$. Moreover the Cartan curvature Θ of θ is a $(1,1)$ form, $\Theta \in \mathfrak{A}^{1,1}(B, P \times_G \mathfrak{g})$ where G acts on $\mathfrak{g}$ by the adjoint representation. Depending on a a basis $\{X_i\}_{1 \leq i \leq m}$ of $\mathfrak{g}$ one can write

$$\theta = \sum_{i=1}^{m} \theta^i X_i \quad \text{and} \quad \Theta = \sum_{i=1}^{m} \Theta^i X_i$$

with 1-forms θ^i on P and horizontal 2-forms Θ^i by the identification $\Theta \in \mathfrak{A}^{1,1}(B, P \times_G \mathfrak{g}) \cong \mathfrak{A}^{1,1}(P, \mathfrak{g})_{\text{bas}}$ (see for instance [BGV92, Definition 1.8] for the definition of basic differential forms or section 3.1 and [BGV92, Proposition 1.9] for the isomorphism)

We can form the associated bundle

$$M := P \times_{G_{\mathbf{C}}} Z$$

which is a holomorphic fibration $M \overset{\pi}{\to} B$ over B with compact fibre Z. By the reduction M is also given by $Q \times_G Z$. Since Q is equipped with the Cartan connection θ the associated bundle $M = Q \times_G Z$ has an Ehresmann connection, that is we have a splitting

$$TM = T^H M \oplus TZ$$

where $T^H M$ is the image of $T^H P = \ker \theta$ unter the projection $Q \times Z \twoheadrightarrow Q \times_G Z$. Also put

$$\mathcal{E} := P \times_{G_{\mathbf{C}}} E = Q \times_G E$$

so that $\pi_{\mathcal{E}} : \mathcal{E} \to M$ with $\pi_{\mathcal{E}}[(p,e)] := [(p, \pi_E(e))]$ is a holomorphic fibre bundle. For $b \in B$ we have

$$M_b = \{[(p,z)] | \pi(p) = b, z \in Z\} \cong Z \quad \text{and} \quad \mathcal{E}_b = \{[(p,e)] | \pi(p) = b, e \in Z\} \cong E$$

from which we see that $\mathcal{E}_b$ is a vector bundle over M_b with $\operatorname{rk}(\mathcal{E}_b) = \operatorname{rk}(E)$. To the family of vector bundles $\{(\Lambda^{0,k}(T^*Z) \otimes \mathcal{E})_b\}_{b \in B}$, $k \in \mathbf{N}_0$, we associate an infinite dimensional bundle $\mathbf{E}$ over B as described in chapter 1. In this case we can identify $\mathbf{E}$ with $Q \times_G \mathfrak{A}^{(0,\bullet)}(Z, E)$.

Now $\Theta \in \mathfrak{A}^{1,1}(B, P \times_G \mathfrak{g}) \cong \mathfrak{A}^{1,1}(P, \mathfrak{g})_{\text{bas}}$. Since the moment map μ is equivariant the product $\langle \mu, \Theta \rangle$ is well defined and lies in $\mathfrak{A}^{1,1}(P)$. Since it is a horizontal form, i.e. $\iota_X(\langle \mu, \Theta \rangle) = 0$ for every vertical vector field X on P, it can be viewed as a differential form on B, $\langle \mu, \Theta \rangle \in \mathfrak{A}^{1,1}(B)$.

Definition 3.1.2. *Let ω^M be the 2-form on M given by*

$$\omega^M := \omega^Z + \pi^* \langle \mu, \Theta \rangle.$$

The 2-form ω^M is a real closed $(1,1)$ form on M. Furthermore one has that the restriction of ω^M to the fibres Z is the Kähler form along the fibres and the vector bundle $T^H M$ is the orhtogonal bundle with respect to ω^M. By this we see that (π, ω^M) is a Kähler fibration. If ω^H denotes the restriction of ω^M to $T^H M$ then we have

$$\omega^H = \pi^* \langle \mu, \Theta \rangle.$$

The vector bundle $\mathcal{E} = Q \times_G E$ over M is equipped with a Hermitian metric $h^{\mathcal{E}}$ induced from h^E and the reduction. Let $\nabla^{\mathcal{E}}$ be the Hermitian holomorphic connection of $(\mathcal{E}, h^{\mathcal{E}})$. We can now define the same objects as in section 1 for this kind of family of vector bundles and we use the same notation. As it is shown in [BGV92, Theorem 10.38] the connection $\nabla^{\mathbf{E}}$ on the infinite bundle $\mathbf{E}$ acting on $\mathfrak{A}^\bullet(B, \mathbf{E}) \cong \left(\mathfrak{A}^\bullet(Q) \otimes \Gamma(Z, E)\right)_{\text{bas}}$ satisfies

$$\nabla^{\mathbf{E}} = d_Q + \sum_{i=1}^m \theta^i L^E(X_i) =: d_Q + L^E(\theta)$$

and its curvature is given by

$$(\nabla^{\mathbf{E}})^2 = \sum_{i=1}^m \Theta^i L^E_{X_i} =: L^E_\Theta$$

where L^E denotes the Lie derivative on E.

The analytic torsion form depends on the reduction of P to Q but its cohomology class is independed of it [BG00, Theorem 2.20]. In the principal bundle setting we will use another notation for the holomorphic analytic torsion form to make its dependence on the geometric quantities more clearer.

$$T_{-\frac{\Theta}{2\pi i}}(\omega^Z, h^E) := T(\omega^M, h^{\mathcal{E}}).$$

Another reasoning for this notation is the because of the relationship between the torsion form and the Lie algebraic equivariant analytic torsion. For more information on this topic see appendix B.

3.2 Connections and Operators associated to the Cartan Curvature

In this section we will first start to transfer the results of [BGV92, chapter 7.6] to our situation. Let $\pi_2 : Q \times Z \to Z$ denote the projection onto the second component Z. The action of G on E induces an action on the pullback $\pi_2^* E$ and we have

$$\begin{aligned}\Gamma(B, \mathbf{E}) = \Gamma(M, \Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E}) &= \Gamma(Q \times_G Z, Q \times_G \Lambda^{0,\bullet}(Z, E)) \\ &\cong \Gamma(Q \times Z, \Lambda^{0,\bullet}(Q \times Z, \pi_2^* E))^G\end{aligned}$$

where $\Gamma(Q \times Z, \Lambda^{0,\bullet}(Q \times Z, \pi_2^* E))^G$ denotes the set of G-invariant section of $\Lambda^{0,\bullet}(Q \times Z, \pi_2^* E)$ that is $\alpha \in \Gamma(Q \times Z, \Lambda^{0,\bullet}(Q \times Z, \pi_2^* E))^G$ which satisfies $\gamma \cdot \alpha = \alpha$, i.e.

$$\alpha(p\gamma, \gamma^{-1} z) = \gamma^{-1} \cdot \alpha(p, z).$$

Let

$$\mathfrak{A}^\bullet(Q \times Z, \pi_2^* E)_{\text{hor}} := \{\alpha \in \mathfrak{A}^\bullet(Q \times Z, \pi_2^* E) |\ \iota_X \alpha = 0\ \forall X \in \mathfrak{g}\}$$

be the space of horizontal differential forms on $Q \times Z$ with coefficients in $\pi_2^* E$ and

$$\mathfrak{A}^\bullet(Q \times Z, \pi_2^* E)_{\text{bas}} = \{\alpha \in \mathfrak{A}^\bullet(Q \times Z, \pi_2^* E)_{\text{hor}} |\ \gamma \cdot \alpha = \alpha\ \forall \gamma \in G\}$$

be the space of horizontal G-invariant differential forms i.e. basic differential forms. M is the quotient of $Q \times Z$ by a free action of G so that $Q \times Z \to (Q \times Z)/G = M$ is a G-principle bundle. There is an natural isomorphism (see e.g. [BGV92, Proposition 1.9]

$$\mathfrak{A}^\bullet(M) \cong \mathfrak{A}^\bullet(Q \times Z)_{\text{bas}}.$$

In similar manner there is the identification

$$\mathfrak{A}^\bullet(M,\mathcal{E}) \cong \mathfrak{A}^\bullet(Q\times Z, \pi_2^*E)_{\mathrm{bas}}. \tag{3.2.1}$$

The connection form θ determines a projection j^E from $\mathfrak{A}^\bullet(Q\times Z,\pi_2^*E)$ onto $\mathfrak{A}^\bullet(Q\times Z,\pi_2^*E)_{\mathrm{hor}}$ given by

$$j^E := \prod_{i=1}^m (\mathrm{id} - \theta^i \iota_{X_i})$$

with $\iota_{X_i} = \iota_{X_{i,Q\times Z}} = \iota_{X_{i,Q}} + \iota_{X_{i,Z}}$. The Hermitian holomorphic connection ∇^E on (E,h^E) and the connection form induces a connection ∇^{θ,h^E} on $\mathfrak{A}^\bullet(Q\times Z,\pi_2^*E)_{\mathrm{bas}}$ given by

$$\nabla^{\theta,h^E} := \left(j^E \circ (d_Q \otimes 1 + 1\otimes \nabla^E)\circ j^E\right)_{|\mathfrak{A}^\bullet(Q\times Z,\pi_2^*E)_{\mathrm{bas}}}$$

(see [BGV92, Def. 7.36]). By the isomorphism (3.2.1) we get a connection on $\mathfrak{A}^\bullet(M,\mathcal{E})$ which will be still denoted by ∇^{θ,h^E}.

Lemma 3.2.1. *The connection ∇^{θ,h^E} is the holomorphic Hermitian connection on $\mathcal{E}$, that is $\nabla^{\theta,h^E} = \nabla^{\mathcal{E}}$.*

Proof. The metric h^E induces a metric on π_2^*E by $\pi_2^*h^E$ and therefore a metric $h^{\theta,E}$ on $\mathcal{E}$ by the identification $\Gamma(M,\mathcal{E}) \cong \Gamma(Q\times Z,\pi_2^*E)_{\mathrm{bas}}$. Since ∇^E is the holomorphic Hermitian connection on (E,h^E) we see that ∇^{θ,h^E} is the holomorphic Hermitian connection on $(\mathcal{E},h^{\theta,E})$. Let $x=[p,\pi^E(e)]\in M = Q\times_G Z$ with $p\in Q$ and $e\in E$. Let $s^1,s^2\in\Gamma(Z,E)$ be G-equivariant sections in E and s_M^1, s_M^2 the corresponding sections in $\mathcal{E}$ with $s_M^i(x) = [(p,s^i(p))]$, $i=1,2$, where $p\in Q\times Z$ with $\pi(p)=x$. Then we have

$$\begin{aligned} &h_x^{\theta,E}(s_M^1(x),s_M^2(x)) = h_x^{\theta,E}([p,e_1],[p,e_2]) = h_p^E(e_1,e_2)\\ &= h_p^E(s_1(p),s_2(p)) = h_x^{\mathcal{E}}([p,e_1],[p,e_2]), \end{aligned}$$

that is $h^{\theta,E} = h^{\mathcal{E}}$. By the uniqueness of the Hermitian holomorphic connection we have $\nabla^{\theta,h^E} = \nabla^{\mathcal{E}}$. □

The curvature of ∇^{θ,h^E} can be written in terms of the Cartan curvature and the curvature Ω^E. This has been done in [BGV92]:

Lemma 3.2.2. *[BGV92, Lemma 7.37] The curvature of the connection $\nabla^{\mathcal{E}}$ is given by*

$$\Omega^{\mathcal{E}} = \Omega^E + \sum_{i=1}^m \Theta^i m^E(X_i) =: \Omega^E + m^E(\Theta).$$

Definition 3.2.3. *For $u > 0$ define*

$$\nabla_{u,\Theta,e_i} := \nabla_{e_i}^{\Lambda^{0,\bullet}(T^*Z)\otimes E} + \frac{\langle \Theta_Z, e_i \rangle}{2u}.$$

By [BG00, Proposition 7.18] the following identity holds,

$$B_u^2 = -\frac{u}{2}(\nabla_{u,\Theta,e_i})^2 + u\frac{s^Z}{8} + \frac{u}{4}c(e_i)c(e_j)\Big(\Omega^E + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big)(e_i,e_j) + \big(m^E(\Theta) + \frac{1}{2}\operatorname{Tr}\left[m^{TZ}(\Theta)\right]\big). \tag{3.2.2}$$

Observe once again the sign difference comming from (3.1.1).

3.3 The Condition $\omega^M = -\frac{1}{2\pi i}\Omega^{\mathcal{L}}$ and $\mathbf{P}^1\mathbf{C}$-bundles

Let (L, h^L) be a holomorphic Hermitian line bundle on Z. We assume that the action of G lifts holomorphically to an action on L so shat L becomes a G-equivariant line bundle over Z. We further assume that h^L is G-invariant. Denote the curvature of the Hermitian holomorphic connection ∇^L of (L, h^L) by Ω^L. We make the assumption that Ω^L is positive that is for any $0 \neq U \in T^{1,0}Z$ we have

$$\Omega^L(U, \overline{U}) > 0.$$

Let $\dot{\Omega}^{Z,L} \in \operatorname{End}(T^{1,0}Z)$ be the Hermitian matrix such that for $V, W \in T^{1,0}Z$

$$\Omega^L(V, \overline{W}) = \langle \dot{\Omega}^{Z,L} V, W \rangle_{h^{T^{1,0}Z}}.$$

By our assumption $\dot{\Omega}^{Z,L} \in \operatorname{End}(T^{1,0}Z)$ is positive definite. For $p \in \mathbf{N}$ put $L^p := L^{\otimes p}$. Set

$$\mathcal{L} := P \times_{G_{\mathbf{C}}} L = Q \times_G L.$$

Then we have an isomorphism of equivariant vector bundles over M,

$$\mathcal{E} \otimes \mathcal{L}^p \cong P \times_{G_{\mathbf{C}}} (E \otimes L^p) = Q \times_G (E \otimes L^p).$$

To these data we can define the torsion form $T_{-\frac{\Theta}{2\pi i}}(\omega^Z, h^{E\otimes L^p}) = T(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})$ from section 3.2.

Remark 3.3.1. *The condition from chapter 2 that the direct image $R^\bullet\pi_*(\mathcal{E} \otimes \mathcal{L}^p)$ is locally free for p large is equivalent that the dimension of $H^k(Z, \mathcal{E} \otimes \mathcal{L}^p_{|Z})$, $0 \le k \le n$, is locally constant over B. This condition will not be needed here. G_C acts on $H^\bullet(Z, E \otimes L^p)$ and G acts isometrically on $(H^\bullet(Z, E\otimes L^p), h^{H^\bullet(Z,E\otimes L^p)})$. $H^\bullet(Z, \mathcal{E}\otimes\mathcal{L}^p_{|Z})$ is given by $P \times_{G_{\mathbf{C}}} H^\bullet(Z, E\otimes L^p) =$*

$Q \times_G H^\bullet(Z, E \otimes L^p)$ which is automatically a vector bundle over B. By [BG00, (2.64)] the last equation in Theorem 1.3.5 reads in this case as, for $u \to \infty$,

$$\Phi\,\mathrm{Tr}_s\left[\gamma N_u \exp(-B_{p,u}^2)\right] = \mathrm{Tr}_s^{H^\bullet(Z,E\otimes L^p)}\left[\gamma N \exp\left(-\frac{\Theta}{2\pi i}\right)\right] + O\left(\frac{1}{\sqrt{u}}\right).$$

We make the general condition that the Kähler form equals the representative of the first Chern of the line bundle, that is

$$\omega^M = -\frac{1}{2\pi i}\Omega^{\mathcal{L}}$$

Now we are going to take a look and understand how the various objects look like under the assumption we made. We have the identities

$$\omega^Z + \pi^*\langle\mu,\Theta\rangle = \omega^M = -\frac{1}{2\pi i}\Omega^{\mathcal{L}} = -\frac{1}{2\pi i}\Omega^L + \Big(-\frac{1}{2\pi i}m^L(\Theta)\Big).$$

that is

$$\omega^Z = -\frac{1}{2\pi i}\Omega^L \quad \text{and} \quad \langle\mu,\Theta\rangle = i m^L\Big(\frac{\Theta}{2\pi}\Big).$$

We can choose $\{w_i\}_i$ to be an orthonormal basis of $T_z^{1,0}Z$ such that

$$\dot{\Omega}_z^{Z,L} = 2\pi\cdot \mathrm{id}_{T_z^{1,0}Z}$$

and

$$\sum_j \Omega^L(w_j,\overline{w}_j) = 2\pi^{\dim_{\mathbf{C}} Z}, \qquad \sum_{l,m}\Omega^L(w_l,\overline{w}_m)\overline{w}^m\wedge\iota_{\overline{w}_l} = 2\pi\sum_j \overline{w}^j\wedge\iota_{\overline{w}_j} = 2\pi N_V.$$

For simplicity we will write $N := N_V$. The objects at the end of section 2.3 read as

$$T^{H'}M = T^H M, \qquad \Omega^{\mathcal{L},H'} = m^L(\Theta), \qquad \Theta^M = \omega^M \quad \text{and} \quad \Theta^Z = \omega^Z.$$

By Lemma 3.2.2 and 2.3.4 the forms α_0 and β_0 are given by

$$\alpha_0 = \frac{n\mathrm{rk}(E)}{2}\int_Z e^{-\frac{1}{2\pi i}m^L(\Theta)}d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}} \qquad \text{and} \qquad \beta_0 = 0. \tag{3.3.1}$$

Example. Let $Z = \mathbf{P}^1\mathbf{C}$. Here E be the trivial bundle and we will use only for this example the letter E for another bundle given in the next page. Put $L = \mathcal{O}(1)$ and with the Hermitian metric induced by the standard metric on $\mathbf{C}^2$. By assumption the Kähler form $\omega^{\mathbf{P}^1\mathbf{C}}$ on $\mathbf{P}^1\mathbf{C}$ is $c_1(\nabla^{\mathcal{O}(1)})$, that is the induced metric is the Fubini-Study metric on $\mathbf{P}^1\mathbf{C}$. Note that every vector bundle on $\mathbf{P}^n\mathbf{C}$ is isomorphic to sums of $\mathcal{O}(k)$, $k \in \mathbf{Z}$. Consider the chart

$$\begin{aligned} \psi :]0, 2\pi[\times] - \frac{\pi}{2}, \frac{\pi}{2}[&\to \mathbf{P}^1\mathbf{C} \subset \mathbf{R}^3 \\ (v, u) &\mapsto \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}. \end{aligned}$$

In this local coordinate we have

$$\omega^{\mathbf{P}^1\mathbf{C}}(\frac{\partial}{\partial v}, \frac{\partial}{\partial u}) = \frac{1}{2\pi}\frac{\cos u}{2}, \quad m^{T\mathbf{P}^1\mathbf{C}}(X) = \sin u \cdot J^{T\mathbf{P}^1\mathbf{C}}, \quad \text{and} \quad -2\pi i \mu\Big(\frac{\partial}{\partial v}\Big) = -\frac{i}{2}\sin u$$

Recall the convention that in this thesis we have $\omega^Z = -\frac{1}{2\pi i}\Omega^L$ whereas for example in [BGV92, end of chapter 7.1] they have $\omega^Z = i\Omega^L$.

Let $P \to B$ be a $\mathbf{U}(2)$ principal bundle. Put $E := P \times_{\mathbf{U}(2)} \mathbf{C}^2$. We evaluate the first coefficients for the $\mathbf{P}^1\mathbf{C}$-bundle $\mathbf{P}(E)$. Since an element $Y \in \mathfrak{u}(2)$ is a skew-Hermitian matrix it can be decomposite as $Y = \mathrm{Ad}_\gamma\big(\mathrm{diag}(i\alpha, i\beta)\big)$ with $\gamma \in \mathbf{U}(2)$ and $\alpha, \beta \in \mathbf{R}$. The induced vector field $Y_{0,\mathbf{P}^1\mathbf{C}}$ of $Y_0 = \mathrm{diag}(i\alpha, i\beta)$ is given by $Y_{0,\mathbf{P}^1\mathbf{C}} = (\alpha - \beta)\frac{\partial}{\partial v}$. Thus

$$-2\pi i\mu(Y_0) = -(\alpha-\beta)2\pi i\mu\Big(\frac{\partial}{\partial v}\Big) = -(\alpha-\beta)\frac{i}{2}\sin u = \frac{1}{2}\sqrt{(\mathrm{Tr}\, Y_0)^2 - 4\det Y_0} \cdot \sin u.$$

The Cartan curvature Θ is $\mathbf{U}(2)$-invariant therefore we have

$$m^{\mathcal{O}(1)}(\Theta) = -2\pi i\mu(\Theta_{\mathbf{P}^1\mathbf{C}}) = \frac{1}{2}\sqrt{(\mathrm{Tr}\,\Theta)^2 - 4\det\Theta} \cdot \sin u.$$

Since $\Omega^E = \rho(\Theta)$ where $\rho : \mathfrak{u}(2) \to \mathrm{End}\,\mathbf{C}^2$ is the standard representation we simply write $\Theta = \Omega^E$. Thus as a cohomology element

$$\Big[m^{\mathcal{O}(1)}\Big(\frac{-\Theta}{2\pi i}\Big)\Big] = \frac{1}{2}\sqrt{c_1(E)^2 - 4c_2(E)} \cdot \sin u.$$

In the same manner we have

$$\begin{aligned} \mathrm{Tr}\Big[m^{T\mathbf{P}^1\mathbf{C}}(\Theta)\Big] =& i\sqrt{(\mathrm{Tr}\,\Theta)^2 - 4\det\Theta} \cdot \mathrm{Tr}\Big[m^{T\mathbf{P}^1\mathbf{C}}\Big(\frac{\partial}{\partial v}\Big)\Big] \\ =& i\sqrt{(\mathrm{Tr}\,\Theta)^2 - 4\det\Theta} \cdot \sin u \cdot \mathrm{Tr}(J^{T\mathbf{P}^1\mathbf{C}}) = -\sqrt{(\mathrm{Tr}\,\Theta)^2 - 4\det\Theta} \cdot \sin u. \end{aligned}$$

By (3.3.1) α_0 is given by

$$\alpha_0 = \frac{1}{2}\int_{\mathbf{P}^1\mathbf{C}} \omega^{\mathbf{P}^1\mathbf{C}} e^{-\frac{1}{2\pi i}m^L(\Theta)}.$$

We calculate

$$\omega^{\mathbf{P}^1\mathbf{C}} e^{-\frac{1}{2\pi i}m^L(\Theta)} = \omega^{\mathbf{P}^1\mathbf{C}} e^{\frac{1}{2}\sqrt{(c_1(E)^2-4c_2(E)}\cdot\sin u} = \frac{1}{2\pi}\frac{\cos u}{2} e^{\frac{1}{2}\sqrt{c_1(E)^2-4c_2(E)}\cdot\sin u} dv\wedge du$$
$$= \frac{1}{2\pi}\frac{\cos u}{2}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}\sqrt{c_1(E)^2-4c_2(E)}\cdot\sin u)^k}{k!} dv\wedge du.$$

It follows

$$\int_{\mathbf{P}^1\mathbf{C}} \omega^{\mathbf{P}^1\mathbf{C}} e^{-\frac{1}{2\pi i}m^L(\Theta)} = \frac{1}{2\pi}\frac{1}{2}\sum_{k=0}\frac{1}{k!}(\sqrt{c_1(E)^2-4c_2(E)}\frac{1}{2})^k \int_0^{2\pi} dv \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^k(u)\cos(u)du$$
$$= \sum_{k=0,k \text{ even}} \frac{1}{k+1!}(\sqrt{c_1(E)^2-4c_2(E)}\frac{1}{2})^k = \sum_{l=0}\frac{1}{2k+1!}(\sqrt{c_1(E)^2-4c_2(E)}\frac{1}{2})^{2k}$$
$$= \sum_{l=0}\frac{1}{2k+1!}\Big(\frac{c_1(E)^2-4c_2(E)}{4}\Big)^k.$$

Thus

$$\alpha_0^{(4k)} = \frac{1}{2(2k+1)!}\Big(\frac{c_1(E)^2-4c_2(E)}{4}\Big)^k.$$

and $\beta_0 = 0$, by (3.3.1). We conclude: Let $\pi : E \to B$ be a holomorphic vector bundle of rank 2. For $k \geq 0$ the asymptotic of the analytic torsion form of degree $4k$ for $\mathcal{O}(p)$ on the $\mathbf{P}^1\mathbf{C}$-bundle $\mathbf{P}(E)$ as $p \to \infty$ is given by

$$T_{-\frac{\Theta}{2\pi i}}(\omega^{\mathbf{P}^1\mathbf{C}}, h^{\mathcal{O}(p)})^{(4k)} = p^{2k+1}(\alpha_0^{(4k)}\log p + \beta_0^{(4k)}) + o(p^{2k+1})$$
$$= \Big[\frac{1}{2^{2k+1}(2k+1)!}p^{2k+1}\log p\Big]\cdot\Big(c_1(E)^2-4c_2(E)\Big)^k + o(p^{2k+1})$$

in the topology of $\mathscr{C}^\infty$ convergence on compact subsets of B.

Remark. The analytic torsion form for $\mathbf{P}(E)$ was already calculated explicitely by K. Köhler (unpublished) using the comparison formula of Bismut-Goette (Apendix B). The calculations in this example is heavenly inspired by his work and we do not claim originality here.

3.4 Computation of the Coefficient $a_{1,u}$

Let $b_0 \in B$ be a given base point and $z_0 \in Z_{b_0}$. Recall that Z_0 was defined as $Z_0 = T_{\mathbf{R},z_0} Z$. We repeat the localization procedure from section 2.1 a) with the Lichnerowicz formula (3.2.2) to get

$$\Psi = \frac{s^Z}{8} + \frac{1}{4} c(\tilde{e}_i)c(\tilde{e}_j)\Big[\Omega^E + \frac{1}{2}\operatorname{Tr}\Omega^{TZ}\Big](\tilde{e}_i, \tilde{e}_j) + (m^E(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]) \tag{3.4.1}$$

and

$$M_{p,z_0} = \frac{1}{2}\Delta^{\mathbb{E}_{p,z_0}} + \rho\Big(\frac{\|V\|}{\varepsilon}\Big)\Psi + p\rho\Big(\frac{\|V\|}{\varepsilon}\Big)\Big(\frac{1}{4}c(\tilde{e}_i)c(\tilde{e}_j)\Omega^L(\tilde{e}_i, \tilde{e}_j) + m^L(\Theta)\Big).$$

Here $\Delta^{\mathbb{E}_{p,z_0}}$ is the Bochner Laplacian associated with $d + \rho\Big(\frac{\|V\|}{\varepsilon}\Big)(p\vartheta^L + \vartheta^1_\Theta)$ and $g^{T_\mathbf{R} Z_0}$ with the connection form ϑ^1_Θ of $\nabla_{1,\Theta}$.

In the same way as in [MM07, section 4.1.6], [F18, section 4.2] we will introduce complex coordinates $(z_1 \dots z_n)$ on $\mathbf{C}^n \cong \mathbf{R}^{2n} \cong Z_0$ such that $V = z - \overline{z}$ and

$$w_i = \sqrt{2}\frac{\partial}{\partial z_i}, \qquad \overline{w}_i = \sqrt{2}\frac{\partial}{\partial \overline{z}_i}$$

where $\{w_i\}_i$ was the orthonormal basis of $T^{1,0}_{z_0}Z$. We have

$$e_{2i-1} = \frac{1}{\sqrt{2}}(w_i + \overline{w}_i) = \frac{\partial}{\partial z_i} + \frac{\partial}{\partial \overline{z}_i} \quad \text{and} \quad e_{2i} = \frac{1}{\sqrt{2}}(w_i - \overline{w}_i) = \frac{\partial}{\partial z_i} - \frac{\partial}{\partial \overline{z}_i}.$$

We will also identify z to $\sum_i z_i \frac{\partial}{\partial z_i}$ and $\overline{z}$ to $\sum_i \overline{z}_i \frac{\partial}{\partial \overline{z}_i}$ and regarding z and $\overline{z}$ as vector fields. Note that

$$\Big|\frac{\partial}{\partial z_i}\Big| = \Big|\frac{\partial}{\partial \overline{z}_i}\Big| = \frac{1}{2}, \qquad |z|^2 = |\overline{z}|^2 = \frac{1}{2}\|V\|^2.$$

Now define the creation and annihilation operators (see [MM07, (4.1.73)])

$$b_j := -2\nabla_{0,\frac{\partial}{\partial z_j}} \quad \text{and} \quad b_j^+ := 2\nabla_{0,\frac{\partial}{\partial \overline{z}_j}}$$

where $\nabla_{0|_V} = d_V + \frac{1}{2}\Omega^{\mathcal{L}}_{z_0}(V, \cdot)$ with the ordinary differentation operator d_V in direction V on Z_0.

Let $\langle \cdot, \cdot \rangle$ denote the $\mathbf{C}$-bilinear extension of g^{TZ}. As usual the superscript (0) attached to an object means its degree 0 part in $\Lambda^\bullet T^*_{\mathbf{R},b_0}B$ and we denote by $(>)$ the parts of degree higher than zero. Recall that the operators $\mathcal{O}_r$ are from Lemma 2.1.2. $\mathcal{O}_1^{(0)}$ and $\mathcal{O}_2^{(0)}$ are given in [F18, Theorem 4.2] with $\mathcal{O}_1^{(0)} = 0$ and

$$\begin{aligned}
2\mathcal{O}_2^{(0)} =& \frac{1}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(\overline{z}, \frac{\partial}{\partial z_j}\Big)\overline{z}, \frac{\partial}{\partial z_j}\Big\rangle b_i^+ b_j^+ + \frac{1}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(z, \frac{\partial}{\partial z_j}\Big)z, \frac{\partial}{\partial z_j}\Big\rangle b_i b_j - \frac{1}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(z, \frac{\partial}{\partial \overline{z}_j}\Big)\overline{z}, \frac{\partial}{\partial z_j}\Big\rangle b_i b_j^+ \\
& - \frac{1}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(\overline{z}, \frac{\partial}{\partial z_j}\Big)z, \frac{\partial}{\partial \overline{z}_j}\Big\rangle b_i^+ b_j - 2\Omega^E_{z_0}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}\Big) - \frac{s^Z_{z_0}}{6} \\
& + \frac{2}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(\overline{z}, \frac{\partial}{\partial z_j}\Big)\frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial z_j}\Big\rangle b_j^+ - \frac{2}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(z, \frac{\partial}{\partial \overline{z}_j}\Big)\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}\Big\rangle b_j + \frac{\pi}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(z, \overline{z}\Big)\overline{z}, \frac{\partial}{\partial z_i}\Big\rangle b_j^+ \\
& - \frac{\pi}{3}\Big\langle \Omega^{TZ}_{z_0}\Big(z, \overline{z}\Big)z, \frac{\partial}{\partial \overline{z}_i}\Big\rangle b_i - \Omega^E_{z_0}\Big(\overline{z}, \frac{\partial}{\partial z_i}\Big)b_i^+ + \Omega^E_{z_0}\Big(z, \frac{\partial}{\partial \overline{z}_i}\Big)b_i \\
& + 2\Omega^{\det}_{z_0}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})\overline{w}^j \wedge \iota_{\overline{w}_i} + 4\Omega^E_{z_0}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})\overline{w}^j \wedge \iota_{\overline{w}_i}.
\end{aligned}$$

where $\Omega^{\det}$ is the curvature of the Hermitian holomorphic connection of $(\det T^{1,0}Z, h^{\det T^{1,0}Z})$ with $h^{\det T^{1,0}Z}$ induced by $h^{T^{1,0}Z}$. See also [MM06, Theorem 2.2]. Recall from Remark 1.3.8 that the factor 2 appears here from the different convention.

Lemma 3.4.1. *The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ from Lemma 2.1.2 are given by*

$$\begin{aligned}
&\mathcal{O}_1 = 0 \qquad \textit{and} \\
&\mathcal{O}_2 = \mathcal{O}_2^{(0)} - \frac{1}{4}d\Theta^\flat_{Z,z_0}\Big(\overline{z}, \frac{\partial}{\partial z_i}\Big)b_i^+ + \frac{1}{4}d\Theta^\flat_{Z,z_0}\Big(z, \frac{\partial}{\partial \overline{z}_i}\Big)b_i + \Big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\Big)_{z_0}.
\end{aligned}$$

Proof. We will use the same notation as in the proof of Lemma 2.1.2. We have the Taylor expansions

$$g_{ij}(V) = \delta_{ij} + \frac{1}{3}\langle \Omega^{TZ}(V, e_i)V, e_j\rangle_{z_0} + O(\|V\|^3)$$

and

$$\kappa(V) = \sqrt{|\det(g_{ij})(V)|} = 1 + \frac{1}{6}\langle \Omega^{TZ}(V, e_i)V, e_i\rangle_{z_0} + O(\|V\|^3).$$

If ϑ^l_{ij} is the connection form of ∇^{TZ_0} with respect to the basis $\{e_i\}_i$ then $\nabla^{TZ_0}_{e_i} e_{j_{|V}} = \vartheta^l_{ij}(V)e_l$. Let Ω^{TZ_0} be the curvature of ∇^{TZ_0}. Note that for $\|tV\| < 2\varepsilon$ we have $\rho\Big(t\frac{\|V\|}{\varepsilon}\Big) = 1$ and $\Omega^{TZ_0} = \Omega^{TZ}$ on $B^{T_{\mathbf{R},z_0}Z}_{2\varepsilon}(0)$. As in [MM07, (4.1.102)] we have

$$\begin{aligned}
\vartheta^l_{ij}(V) &= \frac{1}{2}g^{lk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}), \\
&= \frac{1}{3}(\langle \Omega^{TZ}(V, e_j)e_i, e_l\rangle_{z_0} + \langle \Omega^{TZ}(V, e_i)e_j, e_l\rangle_{z_0}) + O(\|V\|^2).
\end{aligned}$$

By [BG00, (7.109), (7.110)]

$$\vartheta^1_{\Theta|V} = \vartheta^E_{|V} + \vartheta^{\Lambda^\bullet(T^{*(0,1)}Z)}_{|V} + \frac{1}{4}d\Theta^\flat_{Z,z_0}(V,\cdot) + O(|V|^2).$$

Hence together with (2.1.3) and (2.1.5) we get on $B^{T_{\mathbf{R},z_0}Z}_{\varepsilon/t}(0)$ the identity

$$\begin{aligned}
L_{t|_V} = &-\frac{1}{2}(\delta_{ij} - \frac{t^2}{3}\langle\Omega^{TZ}(V,e_i)V,e_j\rangle_{z_0} + O(t^3))\kappa^{1/2}(tV) \\
&\times\Big((\nabla_{e_i} + \frac{1}{2}\Omega^L_{z_0} + \frac{t}{3}(\partial_k\Omega^L)_{z_0})V_k + \frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha\Omega^L)_{z_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}\Omega^1_{z_0})(V,e_i) + O(t^3)) \\
&\times(\nabla_{e_i} + \frac{1}{2}\Omega^L_{z_0} + \frac{t}{3}(\partial_k\Omega^L)_{z_0})V_k + \frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha\Omega^L)_{z_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}\Omega^1_{z_0})(V,e_i) + O(t^3)) \\
&- t\Gamma^l_{ij}(tV)(\nabla_{e_l} + \frac{1}{2})\Omega^L_{z_0}(Z,e_l)) + O(t)\Big)\kappa^{-1/2}(tV) \\
&+\Big(t^2\Psi_{z_0} + \frac{1}{4}c(e_i)c(e_j)\Omega^L(e_i,e_j) + m^L(\Theta)\Big)_{tV} + O(t^3).
\end{aligned} \tag{3.4.2}$$

Since $\omega^Z = -\frac{1}{2\pi i}\Omega^L$, $\partial^\alpha\Omega^L$ vanishes. In particular we have $\mathcal{O}_1 = 0$ and

$$\begin{aligned}
2\mathcal{O}_2 = &\frac{1}{3}\langle\Omega^{TZ}(V,e_i)V,e_j\rangle_{z_0}\Big(\nabla_{e_i} + \frac{1}{2}\Omega^L(V,e_i)\Big)(\nabla_{e_j} + \frac{1}{2}\Omega^L(V,e_j)) \\
&+\Big(\frac{2}{3}\langle\Omega^{TZ}(V,e_j)e_j,e_i\rangle_{z_0} - \Omega^E_{z_0}(V,e_i) - \frac{1}{4}d\Theta^\flat_{Z,z_0}(V,e_i)\Big)\Big(\nabla_{e_i} + \frac{1}{2}\Omega^L(V,e_i)\Big) \\
&+\Big(\Psi_{z_0} + \frac{1}{4}c(e_i)c(e_j)\Omega^L(e_i,e_j) + m^L(\Theta)\Big)_V.
\end{aligned}$$

The equalitiy in the formulas in degree zero has been as mentioned already shown in [F18, Theorem 4.2] while the equalitiy in higher degree follows from (3.4.1) we conclude. □

Afer these preperations let us now begin with the calculation of $a_{1,u}$. Recall from the proof of Theorem 2.2.1 that we have

$$a_{1,u} = \psi_{1/\sqrt{u}}\langle 0|J_{2,u}|0\rangle = \psi_{1/\sqrt{u}}\frac{1}{2}\frac{\partial^2}{\partial t^2}_{|t=0}\langle 0|e^{-uL_t}|0\rangle.$$

By Lemma 2.1.14 and since $\mathcal{O}_1 = 0$ we have

$$J_{2,u} = -\int_{u\Delta_1} e^{-(u-v)L_0}\mathcal{O}_2 e^{-vL_0}dv = -e^{-uL_0} * \mathcal{O}_2 e^{-uL_0}.$$

Thus by (0.0.1) $a_{1,u}$ is given by

$$a_{1,u} = -\int_0^u \int_{Z_0} \psi_{1/\sqrt{u}} \langle 0|e^{-vL_0}|V\rangle \cdot \langle V|\mathcal{O}_2 e^{-(u-v)L_0}|0\rangle dV dv.$$

Note that the term $\psi_{1/\sqrt{u}}\langle 0|e^{-vL_0}|0\rangle$ was already computed in [P16, (2.88)].

Lemma 3.4.2. *For $u > 0$ the following identity holds.*

$$\langle V|e^{-uL_0}|0\rangle = e^{-2\pi uN - um^L(\Theta)} \frac{1}{(1-e^{-2\pi u})^n} \exp\Big(-\frac{\pi\|V\|^2}{2\tanh(2\pi u)}\Big) \otimes \mathrm{id}_E.$$

Proof. Set

$$\Omega_1 := \Omega^{\mathcal{L}}(w_k, \overline{w}_l)\overline{w}^l \wedge \iota_{\overline{w}_k} + \frac{1}{\sqrt{2}} c(e_i) f^\alpha \Omega^{\mathcal{L}}_{i,\alpha} + \frac{f^\alpha f^\beta}{2}\Omega^{\mathcal{L}}_{\alpha,\beta}.$$

Then by [P16, (2.87)], uL_0 can be rewritten as

$$uL_0 = -\frac{u}{2}\sum_i \Big(d + \frac{1}{2}\langle \dot{\Omega}^{Z,\mathcal{L}}_{z_0} V, e_i\rangle\Big)^2 + u\Omega_1(z_0) - \frac{u}{2}\operatorname{Tr}(\dot{\Omega}^{Z,\mathcal{L}}_{z_0}).$$

The formula for the heat kernel of a harmonic oscillator (see [MM07, (E.24)]) yields

$$\langle V|e^{-uL_0}|0\rangle = e^{-u\Omega_1} \frac{1}{(1-e^{-2\pi u})^n} \exp\Big(-\frac{\pi\|V\|^2}{2\tanh(2\pi u)}\Big) \otimes \mathrm{id}_E$$

By our general condition on the Kähler form we have

$$\sum_{k,l} \Omega^{\mathcal{L}}(w_k, \overline{w}_l)\overline{w}^l \wedge \iota_{\overline{w}_k} = 2\pi \sum_j \overline{w}^l \wedge \iota_{\overline{w}_j} = 2\pi N,$$

$$\Omega^{\mathcal{L}}_{i,\alpha} = -2\pi i \omega^M(e_i, f_\alpha) = 0 \quad \text{and} \quad \frac{f^\alpha f^\beta}{2}\Omega^{\mathcal{L}}_{\alpha,\beta} = \frac{f^\alpha f^\beta}{2}\Omega^{\mathcal{L},H'}_{\alpha,\beta} = m^L(\Theta).$$

Thus $u\Omega_1$ is given by

$$u\Omega_1 = 2\pi uN + um^L(\Theta)$$

from which the statement follows. □

We use the similar notation as in [F18, (4.11)]. Set

$$\Omega^{TZ}_{i\bar{j}k\bar{l}} := \Big\langle \Omega^{TZ}_{z_0}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j}\Big)\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z}_l}\Big\rangle, \quad \Omega^E_{i\bar{j}} := \Omega^E_{z_0}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j}\Big),$$

$$\Omega^\Lambda_{i\bar{j}} := \Omega^{\Lambda^\bullet(T^{*0,1}Z)}_{z_0}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j}\Big), \quad \Omega^{\det}_{i\bar{j}} := \Omega^{\det}_{z_0}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j}\Big) \quad \text{and} \quad d\Theta^\flat_{i\bar{j}} := d\Theta^\flat_{Z,z_0}\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j}\Big).$$

With this notation and as we identify z to $\sum_i z_i \frac{\partial}{\partial z_i}$ and $\overline{z}$ to $\sum_i \overline{z}_i \frac{\partial}{\partial \overline{z}_i}$ we have in particular

$$d\Theta^\flat_{i\bar{i}}(\overline{z}_i b_i^+ + z_i b_i) = -d\Theta^\flat_{i\bar{i}}\overline{z}_i b_i^+ + d\Theta^\flat_{i\bar{i}} z_i b_i = -d\Theta^\flat_{Z,z_0}\Big(\overline{z}, \frac{\partial}{\partial \overline{z}_i}\Big)b^+ + d\Theta^\flat_{Z,z_0}\Big(z, \frac{\partial}{\partial z_i}\Big)b_i. \quad (3.4.3)$$

The term $a^{(0)}_{1,u}$ has been computed in [F18, Lemma 4.5] and is given by

$$\begin{aligned} a^{(0)}_{1,u}(z_0) =& \Big[-\frac{4}{3}\Omega^{TZ}_{i\bar{i}j\bar{j}}(1-e^{-2\pi u})^{-2}\Big(\frac{u}{2}(1+4e^{-2\pi u}+e^{-4\pi u}) - \frac{3}{4\pi}(1-e^{-4\pi u})\Big) \\ &+ \frac{4}{6}u\Omega^{TZ}_{i\bar{i}j\bar{j}} + u\Omega^{E}_{i\bar{i}} - 2\Omega^{E}_{i\bar{i}}(1-e^{-2\pi u})^{-1}\Big(\frac{u}{2}+\frac{u}{2}e^{-2\pi u} - \frac{1}{2\pi}(1-e^{-2\pi u})\Big) \\ &- 2\Omega^{\Lambda}_{i\bar{i}}(1-e^{-2\pi u})^{-1}\Big(\frac{u}{2}+\frac{u}{2}e^{-2\pi u} - \frac{1}{2\pi}(1-e^{-2\pi u})\Big) \\ &- \Big(\Omega^{\det}_{i\bar{j}}\overline{w}^j \wedge \iota_{\overline{w}_i} + 2\Omega^{E}_{i\bar{j}}\overline{w}^j \wedge \iota_{\overline{w}_i}\Big)u\Big] \frac{e^{-2\pi u N}}{(1-e^{-2\pi u})^n}. \end{aligned} \quad (3.4.4)$$

We now restate Theorem 3 and give the proof.

Theorem 3.4.3. *For $u > 0$ the following identity holds.*

$$\begin{aligned} a_{1,u}(z_0) = a^{(0)}_{1,u}(z_0)e^{-m^L(\Theta)} - &\Big[\frac{d\Theta^\flat_{i\bar{i}}}{2}(1-e^{-2\pi u})^{-1}\Big(\frac{1}{2}+\frac{1}{2}e^{-2\pi u} - \frac{1}{2\pi u}(1-e^{-2\pi u})\Big) \\ &+ \Big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\Big)_{z_0}\Big] \frac{e^{-2\pi u N}}{(1-e^{-2\pi u})^n} e^{-m^L(\Theta)}. \end{aligned}$$

Proof. We have to evaluate $-\int_0^u \int_{Z_0} \psi_{1/\sqrt{u}} \langle 0|e^{-vL_0}|V\rangle \cdot \langle V|\mathcal{O}_2 e^{-(u-v)L_0}|0\rangle dV dv$. Since e^{-uL_0}, $u > 0$, is a semigroup and because even forms commute with each other we have with Lemma 3.4.2

$$\begin{aligned} &\langle 0|e^{-vL_0}|V\rangle \cdot \langle V|\mathcal{O}_2 e^{-(u-v)L_0}|0\rangle = \langle 0|e^{-vL_0^{(0)}}|V\rangle \cdot \langle V|\mathcal{O}_2 e^{-(u-v)L_0^{(0)}}|0\rangle \cdot e^{-um^L(\Theta)} \\ =&\langle 0|e^{-vL_0^{(0)}}|V\rangle \cdot \langle V|\mathcal{O}_2^{(0)} e^{-(u-v)L_0^{(0)}}|0\rangle e^{-um^L(\Theta)} + \langle 0|e^{-vL_0^{(0)}}|V\rangle \cdot \langle V|\mathcal{O}_2^{(>)} e^{-(u-v)L_0^{(0)}}|0\rangle e^{-um^L(\Theta)}. \end{aligned}$$

The first summand has been handled by [F18, chapter 4] and we are dealing with the second summand. By Lemma 3.4.1 and (3.4.3) we have

$$\begin{aligned} &\langle V|\mathcal{O}_2^{(>)} e^{-uL_0^{(0)}}|0\rangle \\ =&\langle V|\Big(\frac{1}{4}d\Theta^\flat_{i\bar{i}}(\overline{z}_i b_i^+ + z_i b_i) + \big(m^E(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\big)_{z_0}\Big)e^{-uL_0^{(0)}}|0\rangle. \end{aligned}$$

From the following identites established in [F18, (4.15)],

$$\langle V|b_i e^{-uL_0^{(0)}}|0\rangle = \Big(\pi + \frac{2\pi}{2\tanh(2\pi u)}\Big)\overline{z}_i \langle V|e^{-uL_0^{(0)}}|0\rangle \quad \text{and}$$

$$\langle V|b_i^+ e^{-uL_0^{(0)}}|0\rangle = \Big(\pi - \frac{2\pi}{2\tanh(2\pi u)}\Big) z_i \langle V|e^{-uL_0^{(0)}}|0\rangle,$$

we conclude

$$\begin{aligned}&\langle V|\mathcal{O}_2^{(>)} e^{-uL_0^{(0)}}|0\rangle = \Big(2\pi \cdot d\Theta^\flat_{i\bar{i}}|z_i|^2 + \big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\big)_{z_0}\Big)\langle V|e^{-uL_0^{(0)}}|0\rangle \\ \Rightarrow\ &\langle 0|e^{-vL_0^{(0)}}|V\rangle \cdot \langle V|\mathcal{O}_2^{(>)} e^{-(u-v)L_0^{(0)}}|0\rangle \\ &= \Big(\frac{\pi}{2}\cdot d\Theta^\flat_{i\bar{i}}|z_i|^2 + \big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\big)_{z_0}\Big)\langle V|e^{-uL_0^{(0)}}|0\rangle.\end{aligned}$$

Therefore with Lemma 3.4.2 we have

$$\begin{aligned}&-\int_0^u \int_{Z_0} \langle 0|e^{-vL_0^{(0)}}|V\rangle \cdot \langle V|\mathcal{O}_2^{(>)} e^{-(u-v)L_0^{(0)}}|0\rangle dV dv \\ &= -\Big[\frac{d\Theta^\flat_{i\bar{i}}}{2}(1-e^{-2\pi u})^{-1}\Big(\frac{u}{2} + \frac{u}{2}e^{-2\pi u} - \frac{1}{2\pi}(1-e^{-2\pi u})\Big) \\ &\quad + u\big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\big)_{z_0}\Big]\frac{e^{-2\pi u N}}{(1-e^{-2\pi u})^n}.\end{aligned}$$

Now since

$$\begin{aligned}&\psi_{1/\sqrt{u}} d\Theta^\flat_{i\bar{i}} = \frac{1}{u} d\Theta^\flat_{i\bar{i}}, \qquad \psi_{1/\sqrt{u}} e^{-um^L(\Theta)} = e^{-m^L(\Theta)} \qquad \text{and} \\ &\psi_{1/\sqrt{u}}\Big(u\big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\big)_{z_0}\Big) = \big(m^E(\Theta) + m^L(\Theta) + \frac{1}{2}\operatorname{Tr}[m^{TZ}(\Theta)]\big)_{z_0}\end{aligned}$$

the claimed formula follows. □

If f is a smooth function on $]0,\infty[$ with asymptotic expansion $f(u) = \sum_j^k f_j u^j + o(u^k)$ as $u \to 0$ write $f^{[j]} := f_j$. Then by (2.2.11) and (2.3.3) the form α_1 in Theorem 2 is given by

$$\alpha_1 = \Big(\Phi \int_Z \operatorname{Tr}_s\left[N_u a_{1,u}(z) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z)\right] du\Big)^{[0]}$$

while by (2.3.1) the form β_1 is given by

$$\beta_1 = \frac{d}{ds}_{|s=0}\Big[-\frac{1}{\Gamma(s)}\Big(\int_0^1 + \int_1^\infty\Big) u^{s-1}\Phi\Big(\int_Z \operatorname{Tr}_s\left[N_u a_{1,u}(z) d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z)\right]\Big) du\Big].$$

As mentioned in the introduction Finski calculated α_1 and β_1 in [F18, Theorem 1.3] when

B is a point. Now exponential of terms coupling horizontal forms and vertical Clifford variables which causes difficulties in calculation of supertraces (see for instance the proof of [P16, Theorem 2.24]) are absent here. Thus with the methods of [F18, section 4] one should be able to compute the forms α_1 and β_1 in the principle setting explicitly. In particular the result in the example regarding $\mathbf{P}^1\mathbf{C}$-bundles from section 3.3 can be more specified. All these calculations will not be part of this thesis.

Chapter 4

The Asymptotic of the Equivariant Holomorphic Analytic Torsion Forms

In this chapter we compute the asymptotic of the equivariant analytic torsion forms. We consider the situation as in the beginning of chapter 2 but now with an arbitrary $\gamma \in G$. In addition the holomorphic Hermitian line bundle $(\mathcal{L}, h^{\mathcal{L}})$ is supposed to be an G-equivariant bundle with G-invariant metric $h^{\mathcal{L}}$.

4.1 Localization near the Fixed-point Manifold

Recall from section 2.1 a) that inj^Z was the injectivity radius of Z. Let $\varrho \in]0, \mathrm{inj}^Z/8[$. The precise value of ϱ will be fixed later. Also recall that by finite propagation speed the map $z' \to \langle z|\tilde{F}_u(uB_p^2)|z'\rangle$ vanishes on the complement of $B_\varrho^Z(z)$ and depends for any $z \in Z$ only on the restriction of the operator B_p^2 to the ball $B_\varrho^Z(z)$. In particular, if dist^Z denotes the distance function on Z, $\langle \gamma^{-1}z|\tilde{F}_u(uB_p)|z\rangle$ vanishes if $\mathrm{dist}^Z(\gamma^{-1}z, z) \geq \varrho$.

Now we explain the choice of ϱ. For $\epsilon > 0$ let U_ϵ be the ϵ-neighbourhood of Z_γ in $N_{Z_\gamma/Z}$. Here $N_{Z_\gamma/Z}$ is identified with the orthogonal bundle of TZ_γ in $TZ_{|Z_\gamma}$ and Z_γ is identified with the set of zero sections. There exist $\epsilon_0 \in]0, \mathrm{inj}^Z/32]$ such that if $\epsilon \in]0, 16\epsilon_0]$ the map $N_{Z_\gamma/Z} \ni (z, V) \mapsto \exp_z^Z(V)$ is a diffeomorphism of U_ϵ on the tubular neighbourhood V_ϵ of Z_γ in Z. In the sequel we will identify U_ϵ and V_ϵ.
We now assume that $\varrho \in]0, \epsilon_0]$ is small enough such that if $z \in Z$, $d^Z(\gamma^{-1}z, z) \leq \varrho$, then $z \in V_{\epsilon_0}$. By [P16, Proposition 2.2 and (2.55)] (see also Appendix A) we see that (2.1.2) still holds outside the diagonal and one has

$$\Big\|\big\langle \exp_z^Z \gamma^{-1}V\big|\exp\big(-\frac{u}{p}B_p^2\big)\big|\exp_z^Z V\big\rangle - \big\langle \gamma^{-1}V\big|\exp\big(-\frac{u}{p}M_{p,z}\big)\big|V\big\rangle\Big\|_{\mathscr{C}^m(M\times_\pi M, \mathbb{E}_p\boxtimes\mathbb{E}_p^*)}$$

$$\leq Cp^N \exp\Big(-\frac{\epsilon_0^2 p}{16u}\Big) \tag{4.1.1}$$

so that we can study the kernel $\langle \gamma^{-1}V|\exp\big(-\frac{u}{p}M_{p,z}\big)|V\rangle$ instead of $\big\langle \exp_z^Z \gamma^{-1}V\big|\exp\big(-\frac{u}{p}B_p^2\big)\big|\exp_z^Z V\big\rangle$. By this the problem is localized on the ϵ_0-neighbourhood V_{ϵ_0}.

Let $d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}$ and $d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}$ be the volume forms on TZ_γ and $N_{Z_\gamma/Z}$ induced by h^{TZ}. Let $\kappa(z,V)$ be be the smooth function on U_{ϵ_0} defined by

$$d\mathrm{vol}_{g^{T_{\mathbf{R}}Z}}(z,V) = \kappa(z,V)d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}(z)d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V).$$

In particular it satisfies

$$\kappa_{|Z_\gamma} = 1.$$

For $u>0$ define as in [P16, (2.63)]

$$\Omega_u := u\Omega^{\mathcal{L}}(w_k,\overline{w}_l)\overline{w}^l \wedge \iota_{\overline{w}^k} + \sqrt{\frac{u}{2}}c(e_i)f^\alpha\Omega^{\mathcal{L}}_{i,\alpha} + \frac{f^\alpha f^\beta}{2}\Omega^{\mathcal{L}}_{\alpha,\beta}.$$

For $z \in Z_\gamma$ and $V \in N_{z,Z_\gamma/Z,\mathbf{R}}$ set

$$\begin{aligned} D_u(z,V) &:= \frac{1}{(2\pi)^n}\exp(-\Omega_{u,(z,V)})\frac{\det(\dot{\Omega}^{\mathcal{L}}_{(z,V)})}{\det(1-\exp(-u\dot{\Omega}^{\mathcal{L}}_{(z,V)}))} \qquad \text{and} \\ e_{\gamma,u}(z,V) &:= \exp\Big(-\Big\langle\frac{\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2}{\tanh(u\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2)}(z,V),(z,V)\Big\rangle \\ &\qquad + \Big\langle\frac{\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2}{\sinh(u\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2)}e^{u\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2}(z,\gamma^{-1}V),(z,V)\Big\rangle\Big). \end{aligned}$$

$D_u(z,V)$ already appeared in [P16, (2.89)]. For sake of convenience put

$$n_\gamma = \dim_{\mathbf{C}} Z_\gamma.$$

Let the action of γ on $\mathcal{L}$ given by multiplication with $e^{i\varphi}$. As usual $\gamma^{\mathcal{E}}$ denotes the action of γ on $\mathcal{E}$.

Theorem 4.1.1. *Let $m \in \mathbf{N}_0$. Then there exists $\delta > 0$ such that as $p \to \infty$, uniformly as u varies in a compact subset of $\mathbf{R}_{>0}$, the following asymptotic for the $\mathscr{C}^m(B)$-norm holds:*

$$p^{-n_\gamma}\psi_{1/\sqrt{p}}\operatorname{Tr}_s\Big[\gamma N_{u/p}\exp\big(-B^2_{p,u/p}\big)\Big]$$

$$= \operatorname{rk}(\mathcal{E}) \int_{z\in Z_\gamma} \int_{V\in N_{z,Z_\gamma/Z,\mathbf{R}}} e^{ip\varphi} e_{\gamma,u}(z,V) \operatorname{Tr}_s \Big[\gamma^{\mathcal{E}} N_u D_u(z,V)\Big] d\mathrm{vol}_{g^{T_\mathbf{R}Z_\gamma}}(z) d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V)$$
$$+ o(p^{-\delta}).$$

Proof. For the kernel of the curvature of the rescaled superconnection we have

$$\psi_{1/\sqrt{p}} e^{-B^2_{p,u/p}} = \psi_{1/\sqrt{p}} e^{-\frac{u}{p}\psi_{1/\sqrt{u/p}} B_p^2 \psi_{\sqrt{u/p}}} = \psi_{1/\sqrt{p}} \psi_{1/\sqrt{u/p}} \Big(e^{-\frac{u}{p}B_p^2}\Big) = \psi_{1/\sqrt{u}} e^{-\frac{u}{p}B_p^2}.$$

It follows together with $\psi_{1/\sqrt{p}} N_{u/p} = N_u$ that

$$\psi_{1/\sqrt{p}} \operatorname{Tr}_s \Big[\gamma N_{u/p} \exp\big(-B^2_{p,u/p}\big)\Big] = \int_Z \operatorname{Tr}_s \Big[\gamma N_u \psi_{1/\sqrt{u}} \langle \gamma^{-1} z | \exp\Big(-\frac{u}{p}B_p^2\Big)|z\rangle\Big] d\mathrm{vol}_{g^{T_\mathbf{R}Z}}(z).$$

By [P16, Proposition 2.2] there exists $C > 0$ such that we have the estimate

$$\Big\|\langle \gamma^{-1} z | \exp\Big(-\frac{u}{p}B_p^2\Big)|z\rangle\Big\|_{\mathscr{C}^m} \le \Big\|\langle \gamma^{-1} z | \tilde{F}\Big(-\frac{u}{p}B_p^2\Big)|z\rangle\Big\|_{\mathscr{C}^m} + Cp^N \exp\big(-\frac{\epsilon_0^2 p}{16u}\big).$$

Because $\langle \gamma^{-1} z | \tilde{F}\big(-\frac{u}{p}B_p^2\big)|z\rangle$ vanishes if $d^Z(\gamma^{-1}z, z) \ge \varrho$ and $\Big(d^Z(\gamma^{-1}z,z) \le \varrho \Rightarrow z \in V_{\epsilon_0}\Big)$ we deduce for any $l \in \mathbf{N}$

$$\psi_{1/\sqrt{p}} \operatorname{Tr}_s \Big[\gamma N_{u/p} \exp\big(-B^2_{p,u/p}\big)\Big] = \int_{V_{\epsilon_0}} \operatorname{Tr}_s \Big[\gamma N_u \psi_{1/\sqrt{u}} \langle \gamma^{-1} z | \tilde{F}\Big(-\frac{u}{p}B_p^2\Big)|z\rangle\Big] d\mathrm{vol}_{g^{T_\mathbf{R}Z}}(z)$$
$$+ o(p^{-l}).$$

Let $\{B_\epsilon^{V_{\epsilon_0}}(z_j)\}_j$ be an open covering of V_{ϵ_0} and $\{\tau_j\}_j$ a partition of unity subordinate to $\{B_\epsilon^{V_{\epsilon_0}}(z_j)\}_j$. By the identification of V_{ϵ_0} with U_{ϵ_0} we get an open covering $\{B_\epsilon^{U_{\epsilon_0}}((z_j,0))\}_j$ of U_{ϵ_0} and a partition of unity subordinate it, still denoted by τ_j. It follows

$$\sum_j \int_{B_\epsilon^{V_{\epsilon_0}}(z_j)} \tau_j(z) \operatorname{Tr}_s \Big[\gamma N_u \psi_{1/\sqrt{u}} \langle \gamma^{-1} z | \tilde{F}\Big(-\frac{u}{p}B_p^2\Big)|z\rangle\Big] d\mathrm{vol}_{g^{T_\mathbf{R}Z}}(z)$$
$$= \sum_j \int_{B_\epsilon^{U_{\epsilon_0}}((z_j,0))} \tau_j\big((z,V)\big) \operatorname{Tr}_s \Big[\gamma N_u \psi_{1/\sqrt{u}} \langle \gamma^{-1}(z,V) | \tilde{F}\Big(-\frac{u}{p}B_p^2\Big)|(z,V)\rangle\Big]$$
$$\times \kappa_{z_j}(z,V) d\mathrm{vol}_{g^{T_\mathbf{R}Z_\gamma}}(z) d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V)$$
$$= \sum_j \int_{B_\epsilon^{U_{\epsilon_0}}((z_j,0))} \tau_j\big((z,V)\big) \operatorname{Tr}_s \Big[\gamma N_u \psi_{1/\sqrt{u}} \langle \gamma^{-1}(z,V) | \tilde{F}\Big(-\frac{u}{p}M_{p,z_j}\Big)|(z,V)\rangle\Big]$$
$$\times \kappa_{z_j}(z,V) d\mathrm{vol}_{g^{T_\mathbf{R}Z_\gamma}}(z) d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V).$$

Again by [P16, Proposition 2.2] which is also true for M_{p,z_j} we have

$$\begin{aligned}\Big\|\langle\cdot|\tilde{F}_{\frac{u}{p}}\Big(-\frac{u}{p}M_{p,z_j}\Big)|\cdot\rangle\Big\|_{\mathscr{C}^m} =&\Big\|\langle\cdot|\exp\Big(-\frac{u}{p}M_{p,z_j}\Big)+\tilde{G}_{\frac{u}{p}}\Big(-\frac{u}{p}M_{p,z_j}\Big)|\cdot\rangle\Big\|_{\mathscr{C}^m}\\ \leq&\Big\|\langle\cdot|\exp\Big(-\frac{u}{p}M_{p,z_j}\Big)|\cdot\rangle\Big\|_{\mathscr{C}^m}+Cp^N\exp\big(-\frac{\epsilon_0^2p}{16u}\big).\end{aligned}$$

Furthermore the following identity holds,

$$\langle(z,V)|e^{-\frac{u}{p}M_p}|(z,V')\rangle = p^n\langle(z,V/t)|e^{-uL_t}|(z,V'/t)\rangle\kappa^{-1/2}(z,V)\kappa^{-1/2}(z,V').$$

Thus for any $l\in\mathbf{N}$ we get

$$\begin{aligned}&\psi_{1/\sqrt{p}}\operatorname{Tr}_s\Big[\gamma N_{u/p}\exp\big(-B^2_{p,u/p}\big)\Big]\\ =&\int_{z\in Z_\gamma}\int_{\substack{V\in N_{z,Z_\gamma/Z,\mathbf{R}}\\ \|V\|\leq\epsilon_0}}\operatorname{Tr}_s\Big[\gamma N_u\psi_{1/\sqrt{u}}\Big\langle(z,\gamma^{-1}V)\Big|\exp\Big(-\frac{u}{p}M_{p,z_j}\Big)\Big|(z,V)\Big\rangle\Big]\\ &\times\kappa(z,V)d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}(z)d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V)+o(p^{-l})\\ =&p^n\int_{z\in Z_\gamma}\int_{\substack{V\in N_{z,Z_\gamma/Z,\mathbf{R}}\\ \|V\|\leq\epsilon_0}}\operatorname{Tr}_s\Big[\gamma N_u\psi_{1/\sqrt{u}}\Big\langle(z,\gamma^{-1}V/t)\Big|e^{-uL_{t,z}}\Big|(z,V/t)\Big\rangle\Big|\Big]\\ &\times\kappa_z^{-1/2}(\gamma^{-1}V)\kappa_z^{-1/2}\kappa(z,V)d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}(z)d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}+o(p^{-l})\\ =&p^{\dim_{\mathbf{C}}Z_\gamma}\int_{z\in Z_\gamma}\int_{\substack{V\in N_{z,Z_\gamma/Z,\mathbf{R}}\\ \|tV\|\leq\epsilon_0}}\operatorname{Tr}_s\Big[\gamma N_u\psi_{1/\sqrt{u}}\Big\langle(z,\gamma^{-1}V)\Big|e^{-uL_{t,z}}\Big|(z,V)\Big\rangle\Big]\\ &\times\kappa_z^{-1/2}(\gamma^{-1}tV)\kappa_z^{-1/2}\kappa(z,tV)d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}(z)d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V)+o(p^{-l})\end{aligned}$$

where in the last step we used the transformation $V\mapsto tV$. By [P16, Theorem 2.20] for $u>0$ fixed there exsists $C>0$ such that for $t>0$ and $V,V'\in B_1^{T_{z_0}Z}(0)$

$$\Big\|\Big\langle V\Big|e^{-uL_t}-e^{-uL_0}\Big|V'\Big\rangle\Big\|\leq Ct^{1/(2n+1)}.$$

Define the rescaled version of L_0 which already appeard in [P16, (2.84)],

$$L_{0,u}:=u\psi_{1/\sqrt{u}}L_0\psi_{\sqrt{u}}.$$

Then it follows for the $\mathscr{C}^m$-norm on $\Gamma(M\times_\pi M,\mathbb{E}_p\boxtimes\mathbb{E}_p^*)$ that

$$\psi_{1/\sqrt{u}}\Big\langle(z,\gamma^{-1}V)\Big|e^{-uL_{t,z}}\Big|(z,V)\Big\rangle=\Big\langle(z,\gamma^{-1}V)\Big|e^{-u\psi_{1/\sqrt{u}}L_{t,z}\psi_{\sqrt{u}}}\Big|(z,V)\Big\rangle$$

$$= \Big\langle (z,\gamma^{-1}V) \Big| e^{-L_{0,u}} \Big| (z,V) \Big\rangle + O(p^{-1/(4n+2)}). \tag{4.1.2}$$

The operator $L_{0,u}$ is a harmonic oscillator with, see [P16, (2.87)],

$$L_{0,u} = \frac{u}{2}\sum_i \Big(d + \frac{1}{2}\langle \dot{\Omega}_z^{Z,\mathcal{L}} V, e_i\rangle \Big)^2 + \Omega_u(z) - \frac{u}{2}\operatorname{Tr}(\dot{\Omega}_z^{T,\mathcal{L}}).$$

Since γ is an isometry the formula for the heat kernel of a harmonic oscillator ([MM07, E.2.4]) gives

$$\begin{aligned}
&\Big\langle (z,\gamma^{-1}V) \Big| e^{-L_{0,u}} \Big| (z,V) \Big\rangle = \frac{1}{(2\pi)^n}\exp(-\Omega_{u,(z,V)})\frac{\det(\dot{\Omega}^{\mathcal{L}}_{(z,V)})}{\det(1-\exp(-u\dot{\Omega}^{\mathcal{L}}_{(z,V)}))} \\
&\times \exp\Big(-\Big\langle \frac{\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2}{\tanh(u\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2)}(z,V),(z,V)\Big\rangle + \Big\langle \frac{\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2}{\sinh(u\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2)} e^{u\dot{\Omega}^{\mathcal{L}}_z/2}(z,\gamma^{-1}V),(z,V)\Big\rangle\Big) \otimes \mathrm{id}_{\mathcal{E}} \\
&= e_{\gamma,u}(z,V)\cdot D_u(z,V)\otimes \mathrm{id}_{\mathcal{E}}.
\end{aligned} \tag{4.1.3}$$

By Taylor expansion of (2.1.6) of $\kappa(z,tV)$ and because $e_{\gamma,u}(z,V)$ decays exponentially as $\|V\| \to \infty$ we conclude Theorem 4.1.1. □

4.2 The Asymptotic of the Equivariant Holomorphic Analytic Torsion Forms

The function $D_u(z,V)$ has an asymptotic expansion as $u \to 0$ ([P16, (2.90)]) and because $e_{\gamma,u}(z,V)$ is analytic in $u=0$ we find, for $j \geq -d$, $a_\gamma^{[j]} \in \Gamma(Z, \operatorname{End}(\Lambda^\bullet(T^*_{\mathbf{R},b_0}B)\otimes \Lambda^{0,\bullet}(T^*Z)))$ such as $u \to 0$

$$(e_{\gamma,u}\cdot D_u)(z,V) = \sum_{j=-d}^{k} a_\gamma^{[j]}(z,V)u^j + O(u^{k+1}).$$

Set $a_\gamma^{[-d-1]} := 0$ and for $j \geq -d-1$ set

$$\begin{aligned}
B_{j,e^{ip\varphi},\gamma} := \int_{z\in Z_\gamma}\int_{V\in N_{z,Z_\gamma/Z,\mathbf{R}}} & e^{ip\varphi}\operatorname{Tr}_s\Big[\gamma(N_V a_\gamma^{[j]}(z,V) + i\omega^H a_\gamma^{[j+1]}(z,V))\Big] \\
& \times d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}(z)\, d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V).
\end{aligned}$$

Then for any $k \in \mathbf{N}_0$ as $u \to 0$

$$\int_{z\in Z_\gamma}\int_{V\in N_{z,Z_\gamma/Z,\mathbf{R}}} e^{ip\varphi} e_{\gamma,u}(z,V)\operatorname{Tr}_s\Big[\gamma^{\mathcal{E}} N_u D_u(z,V)\Big] d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}(z) d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V)$$
$$= \sum_{j=-d-1}^{k} B_{j,e^{ip\varphi},\gamma} u^j + O(u^{k+1}). \tag{4.2.1}$$

Proposition 4.2.1. *There exist forms $B_{p,j,\gamma} \in \mathfrak{A}^\bullet(B_\gamma,\mathbf{C})$ such that for any $k,m \in \mathbf{N}_0$ there exists $C>0$ such that for any $u \in]0,1]$ and $p \in \mathbf{N}$*

$$\Big\| p^{-n_\gamma}\psi_{1/\sqrt{p}} \operatorname{Tr}_s\Big[\gamma N_{u/p}\exp\big(-B^2_{p,u/p}\big)\Big] - \sum_{j=-d-1}^{k} B_{p,j,\gamma}u^j\Big\|_{\mathscr{C}^m(B_\gamma,\Lambda^\bullet T^*_{\mathbf{C}}B_\gamma)} \leq C u^{k+1}.$$

Moreover as $p \to \infty$ for any $j \geq -n_\gamma$

$$B_{p,j,\gamma} = \operatorname{rk}(\mathcal{E}) B_{j,e^{i\varphi p},\gamma} + O\Big(\frac{1}{\sqrt{p}}\Big)$$

where the convergence is in the in the $\mathscr{C}^\infty$ topology on B_γ.

Proof. The proof of Proposition 4.2.1 follows the same techniques as in [P16, Theorem 2.21, Corollary 2.22] and [MM07, Theorem 5.5.9] with off-diagonal adjustments. For more transparancy we recall their results.

As in section 2.1 b) the problem can be localized near $z_0 \in Z$ and we rescale the superconnection as in Definition 2.1.1 to optrain the operator L_{t,z_0}. By the finite propagation speed of the wave operator [MM07, Theorem D.2.1], for t small, $\langle 0|\tilde{F}_u(uL_{t,z_0}|\cdot\rangle$ only depend on the restriction of L_{t,z_0} on $B_{2\epsilon}^{T_{\mathbf{R},z_0}Z}(0)$ and is supported in $B_{2\epsilon}^{T_{\mathbf{R},z_0}Z}(0)$. Consider the sphere bundle

$$\mathbb{S} := \{(V,c) \in T_{\mathbf{R}}Z \times \mathbf{R} |\ \|V\|^2 + c^2 = 1\}$$

over Z. $B_{2\epsilon}^{T_{\mathbf{R},z_0}Z}(0)$ will be embeded in $\mathbb{S}_{z_0}$ by the map

$$V \mapsto (V, \sqrt{1-\|V\|^2})$$

and the operator L_{t,z_0} will be extended to a geralized Laplacian $\tilde{L}_{t,z_0}$ on $\mathbb{S}_{z_0}$ with values in $\mathrm{pr}_M^*(\operatorname{End}\mathbb{E})$. By [P16, Proposition 2.2, (2.54)] we see that [P16, (2.93)] still holds outside of the diagonal, i.e. we have

$$\Big\|\langle \gamma^{-1}V|e^{-uL_{t,z}} - e^{-u\tilde{L}_{t,z}}|V\rangle\Big\|_{\mathscr{C}^m(M\times[0,1],\operatorname{End}(\mathbb{E}))} \leq Cp^N \exp\Big(-\frac{\epsilon_0^2}{32u}\Big).$$

As the total space of $\mathbb{S}$ is compact the heat kernel $\langle \gamma^{-1}V|e^{-u\tilde{L}_t}|V\rangle$ has an asymptotic expansion as $u \to 0$ starting with u^{-n_γ} which depends smoothly on the parameters z_0 and t ([BGV92, chapter 6]). With (4.1.1) we get the first statement and thus the second statement follows from (4.1.2), (4.1.3) and (4.2.1). $\square$

Theorem 4.2.2. *For any $k, m \in \mathbf{N}_0$ there is $C > 0$ such that for $u \geq 1$ and $p \in \mathbf{N}$*

$$\left\| p^{-n_\gamma}\psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[\gamma N_{u/p} \exp(-B^2_{p,u/p})\right]\right\|_{\mathscr{C}^m(B,\Lambda^\bullet T^*_{\mathbf{C}}B)} \leq \frac{C}{\sqrt{u}}.$$

The proof of Theorem 4.2.2 will be postponed to section 4.3.

For the sake of clarity we restate Theorem 4.

Theorem 4.2.3. *Let $k \in \{0, \ldots, \dim_{\mathbf{C}} B_\gamma\}$. Assume the action of γ on $\mathcal{L}$ is given by $e^{i\varphi}$. Then there are differential forms $\alpha_\gamma(e^{ip\varphi}), \beta_\gamma(e^{ip\varphi})$ on B_γ which are local coefficients such that the component of degree $2k$ of the equivariant holomorphic analytic torsion forms has the following asymptotic as $p \to \infty$:*

$$T_\gamma(\omega^M, h^{\mathcal{E}\otimes\mathcal{L}^p})^{(2k)} = p^{n_\gamma+k}\Big(\alpha_\gamma(e^{ip\varphi})\log p + \beta_\gamma(e^{ip\varphi})\Big)^{(2k)} + o(p^{n_\gamma+k})$$

in the topology of $\mathscr{C}^\infty$ convergence on compact subsets of B_γ.

Proof. Recall that by our assumption there is a $p_0 \in \mathbf{N}$ such that the direct image $R^i\pi_*(\mathcal{E}\otimes\mathcal{L}^p)$ is locally free for all $p \geq p_0$ and $i \in \{1, \ldots, n\}$, and vanishes for $i > 0$. In particular, for $p \geq p_0$. $H^i(Z, (\mathcal{E}\otimes\mathcal{L}^p)_{|Z}) = 0 \quad$ for $\quad i > 0$. For $p \geq p_0$ set

$$\tilde{\zeta}_{1,\gamma,p}(s) := -\frac{p^{-n_\gamma}}{\Gamma(s)}\int_0^1 u^{s-1}\psi_{1/\sqrt{p}}\Phi\big(\operatorname{Tr}_s\left[\gamma N_{u/p}\exp(-B^2_{p,u/p})\right]\big)du \qquad \text{and}$$
$$\tilde{\zeta}_{2,\gamma,p}(s) := -\frac{p^{-n_\gamma}}{\Gamma(s)}\int_1^\infty u^{s-1}\psi_{1/\sqrt{p}}\Phi\big(\operatorname{Tr}_s\left[\gamma N_{u/p}\exp(-B^2_{p,u/p})\right]\big)du.$$

In the same fashion as in Definition 1.3.6 both $\tilde{\zeta}_{1,\gamma,p}$ and $\tilde{\zeta}_{1,\gamma,p}$ have a holomorphic extension near zero and we define

$$\tilde{\zeta}_{\gamma,p} := \tilde{\zeta}_{1,\gamma,p} + \tilde{\zeta}_{1,\gamma,p}.$$

It satisfies

$$p^{-n_j}\psi_{1/\sqrt{p}}\zeta_{\gamma,p}(s) = p^{-s}\tilde{\zeta}_{\gamma,p}(s)$$

from which we see

$$p^{-n_\gamma}\psi_{1/\sqrt{p}}\zeta'_{\gamma,p}(0) = \log(p)\tilde{\zeta}_{\gamma,p}(0) + \tilde{\zeta}'_{\gamma,p}(0).$$

On the other hand we have for $p \geq p_0$

$$\begin{aligned}\tilde{\zeta}'_{\gamma,p}(0) =& -\int_0^1 p^{-n_\gamma}\Phi\Big(\psi_{1/\sqrt{p}}\operatorname{Tr}_s\left[\gamma N_{u/p}\exp(-B^2_{p,u/p})\right] - \sum_{j=-d-1}^{0} B_{p,j,\gamma}u^j\Big)\frac{du}{u}\\ &-\int_1^\infty p^{-n_\gamma}\Phi\psi_{1/\sqrt{p}}\operatorname{Tr}_s\left[\gamma N_{u/p}\exp(-B^2_{p,u/p})\right]\frac{du}{u} - \sum_{j=-d-1}^{-1}\frac{B_{p,j,\gamma}}{j} + \Gamma'(1)B_{p,0,\gamma},\\ \tilde{\zeta}_{\gamma,p}(0) =& -\Phi B_{p,0,\gamma}.\end{aligned}$$

Set

$$\theta_{u,\gamma}(e^{ip\varphi}) := \int_{z\in Z_\gamma}\int_{V\in N_{z,Z_\gamma/Z,\mathbf{R}}} e^{ip\varphi}e_{\gamma,u}(z,V)\operatorname{Tr}_s\left[\gamma^{\mathcal{E}}N_uD_u(z,V)\right]d\mathrm{vol}_{g^{T_{\mathbf{R}}Z_\gamma}}(z)d\mathrm{vol}_{g^{N_{Z_\gamma/Z,\mathbf{R}}}}(V).$$

Let $\tilde{\zeta}_{e^{ip\varphi},\gamma}(s)$ be the Mellin transform of $u\mapsto\theta_{u,\gamma}(e^{ip\varphi})$, i.e.

$$\tilde{\zeta}_{e^{ip\varphi},\gamma}(s) := -\frac{1}{\Gamma(s)}\int_0^\infty \theta_{u,\gamma}(e^{ip\varphi})u^{s-1}du.$$

By (4.2.1) we see

$$\begin{aligned}\tilde{\zeta}'_{e^{ip\varphi},\gamma}(0) =& -\int_0^1 \theta_{u,\gamma}(e^{ip\varphi}) - \sum_{j=-d-1}^{0} B_{j,e^{ip\varphi},\gamma}u^j\Big)\frac{du}{u}\\ &-\int_1^\infty \theta_{u,\gamma}(e^{ip\varphi})\frac{du}{u} - \sum_{j=-d-1}^{-1}\frac{B_{j,e^{ip\varphi},\gamma}}{j} + \Gamma'(1)B_{0,e^{ip\varphi},\gamma}.\end{aligned}$$

From Theorem 4.1.1, Corollary 4.2.1 and Theorem 4.2.2 we get

$$\psi_{1/\sqrt{p}}\zeta'_{p,\gamma}(0) = \log(p)p^{n_\gamma}\Phi B_{0,e^{ip\varphi},\gamma} + p^{n_\gamma}\operatorname{rk}(\mathcal{E})\Phi\tilde{\zeta}'_{\gamma,e^{ip\varphi}}(0) + o(p^{n_\gamma}).$$

Thus the statement now follows from Lemma 2.3.1 with

$$\alpha_\gamma(e^{ip\varphi}) := \Phi B_{0,e^{ip\varphi},\gamma} \qquad\text{and}\qquad \beta_\gamma(e^{ip\varphi}) := \operatorname{rk}(\mathcal{E})\Phi\tilde{\zeta}'_{\gamma,e^{ip\varphi}}(0).$$

□

4.3 Proof of Theorem 4.2.2

The strategy will be the same as in [P16, section 2.5] i.e. showing the inequality for $\mathbb{P}_{p,u}$ and $\mathbb{K}_{p,u}$ seperately and using (2.4.2) with n_γ instead.

For $q \geq 1$ let $\|\cdot\|_q$ denote the Schatten q-norm given by

$$\|A\|_q = \Big(\operatorname{Tr}\Big[(A^*A)^{q/2}\Big]\Big)^{1/q}$$

for $A \in \mathfrak{A}^\bullet(B, \operatorname{Op}(\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}\otimes\mathcal{L}^p))$. Since γ is an unitary operator and the Schatten norms are unitary invariant we have for any operator A that $\|\gamma A\|_q{=}\|A\|_q$. Moreover as the operators we are considering commutes with γ the arguments showing the boundedness will be the same as in the non-equivariant case and we will not repeat those arguments if not necessary. The rather unclear steps will be estimates where p^{-n} are replaced by p^{-n_γ}.

Lemma 4.3.1. *Let $\lambda_0 \in \mathbf{R}_+^*$. Then there exists $q_0 \in \mathbf{N}_0$ such that for $q \geq q_0$, for $U \in T_{\mathbf{R}}B$ and $l \in \mathbf{N}_0$, there exists $C > 0$ such that for $p \in \mathbf{N}$*

$$p^{-n_\gamma}\Big\|(\nabla_U^{\operatorname{End}(\mathbb{E})_p})^l\gamma(\lambda_0 - C_p)^{-q}\Big\|_1 \leq C.$$

Proof. Set

$$H_p := D_p^2/p - \lambda_0.$$

By [P16, (2.150)(2.151)] for $k \gg 1$ high enough

$$\operatorname{Tr}\big[\gamma H_p^{-k}\big] = -\frac{1}{(k-1)!}\int_0^\infty \operatorname{Tr}\big[\gamma e^{-tH_p}\big]t^{k-1}dt. \tag{4.3.1}$$

By Theorem 4.1.1 in degree zero $p^{-n_\gamma}\operatorname{Tr}\big[\gamma e^{D_p^2/p}\big]$ and its derivatives are bounded. Thus as in [P16, (2.152)] for $m \in \mathbf{N}_0$ there is $C > 0$ such that for $t \geq 1$ and $p \in \mathbf{N}$,

$$p^{-n_\gamma}\Big\|\operatorname{Tr}\big[\gamma H_p^{-k}\big]\Big\|_{\mathscr{C}^m(B_\gamma,\mathbf{C})} \leq p^{-n_\gamma}\Big\|\operatorname{Tr}\big[\gamma e^{D_p^2/p}\big]\Big\|_{\mathscr{C}^m(B_\gamma,\mathbf{C})} e^{\lambda_0 t} \leq Ce^{\lambda_0 t}. \tag{4.3.2}$$

In the same way as in [P16, (2.153)] we find by using Proposition 4.2.1 in degree 0 that for any $k, m \in \mathbf{N}_0$ there exist $a_{p,j,\gamma} \in \mathbf{R}$ and $C > 0$ such that for any $t \in]0,1]$ and $p \in \mathbf{N}$,

$$\Big\|p^{-n_\gamma}\operatorname{Tr}_s\Big[\gamma\exp\big(-\frac{t}{p}D_p^2\big)\Big] - \sum_{j=-n_\gamma-1}^{k} a_{p,j,\gamma}t^j\Big\|_{\mathscr{C}^m(B_\gamma,\mathbf{C})} \leq Ct^{k+1}. \tag{4.3.3}$$

Splitting the integral in 4.3.1 at $t = 1$ and using (4.3.2) and (4.3.3) we get fo k large enough

$$p^{-n_\gamma} \left\| \operatorname{Tr}\left[\gamma H_p^{-k}\right] \right\|_{\mathscr{C}^m(B_\gamma, \mathbf{C})} \leq C.$$

Thus there exists $q_0 \in \mathbf{N}_0$ such that for $q > q_0$ there is $C > 0$ such that

$$p^{-n_\gamma} \left\| \gamma(\lambda_0 - C_p)^{-q} \right\|_1 \leq p^{-n_\gamma} \left\| \operatorname{Tr}\left[\gamma H_p^{-q}\right] \right\|_{\mathscr{C}^m(B_\gamma, \mathbf{C})} \leq C.$$

By [P16, (2.160)-(2.161)] the Lemma follows for $l = 0$. Since $\nabla_U^{\operatorname{End}(\mathbb{E})_p}$ is a G-invariant connection the case $l \geq 1$ follows with the same arguments as in the proof of [P16, Lemma 2.27]. □

Lemma 4.3.2. *For any $k, m \in \mathbf{N}_0$ there are $a, C > 0$ such that for $u \geq 1$ and $p \in \mathbf{N}$:*

$$\left\| p^{-n_\gamma} \operatorname{Tr}_s \left[\gamma N_u \mathbb{K}_{p,u}\right] \right\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{C}} B)} \leq C e^{-au}.$$

Proof. The proof is the same as in [P16, Proposition 2.28] with using Lemma 4.3.1 instead of [P16, Lemma 2.27]. □

Lemma 4.3.3. *For any $k, m \in \mathbf{N}_0 > 0$ there is $C > 0$ such that for $u \geq 1$ and $p \in \mathbf{N}$:*

$$\left\| p^{-n_\gamma} \operatorname{Tr}_s \left[\gamma N_u \mathbb{P}_{p,u}\right] \right\|_{\mathscr{C}^m(B, \Lambda^\bullet T^*_{\mathbf{C}} B)} \leq \frac{C}{\sqrt{u}}.$$

Proof. We have

$$\left(\lambda - u C_p^{(0)}\right)\left(\frac{1}{\lambda} P_p\right) = P_p - \frac{1}{\lambda} u C_p^{(0)} P_p = P_p.$$

Thus it follows

$$\begin{aligned} &(\lambda - u C_p^{(0)})\left(\frac{1}{\lambda} P_p + (\lambda - u C_p^{(0)})^{-1} P_p^\perp\right) = P_p + P_p^\perp = 1 \\ \Rightarrow\ &\gamma(\lambda - u C_p^{(0)})^{-1} = \frac{1}{\lambda} \gamma P_p + \gamma(\lambda - u C_p^{(0)})^{-1} P_p^\perp. \end{aligned} \tag{4.3.4}$$

The function $\lambda \mapsto (\lambda - u C_p^{(0)})^{-1} P_p^\perp$ is a holomorphic on $B_{\frac{\vartheta}{2}}(0) \setminus \{0\}$ and for $\lambda = 0$ we get from (2.4.13) and our notation

$$(u C_p^{(0)})^{-1} P_p^\perp = P_p^\perp (u C_p^{(0)})^{-1} P_p^\perp P_p^\perp \overset{(2.4.13)}{=} P_p^\perp (u C_p^{(0)})^{-1} P_p^\perp = (u C_p^{(0)})^{-1}.$$

Thus $\lambda \mapsto (\lambda - u C_p^{(0)})^{-1} P_p^\perp$ is a holomorphic function on the interior of δ. Since C_p has no eigenvalues between the two circles δ and δ/u we have

$$\gamma \mathbb{P}_{p,u} = \frac{1}{2\pi i} \psi_{1/\sqrt{u}} \int_{\delta/u} \gamma e^{-u\lambda} (\lambda - C_p)^{-1} d\lambda$$

$$= \frac{1}{2\pi i}\psi_{1/\sqrt{u}}\int_\delta \gamma e^{-\lambda}(\lambda - uC_p)^{-1}d\lambda.$$

Therefore we deduce in the same way as in [P16, (2.188)] using (4.3.4) and

$$\gamma e^{-\lambda}(\lambda - uC_p)^{-1} = \Big(\sum_{k\geq 0}\frac{(-1)^k}{k!}\lambda^k\Big)\Big(\sum_{l\geq 0}\gamma(\lambda - uC_p^{(0)})^{-1}(u\tilde{R}_p)\cdots(u\tilde{R}_p\gamma^{-1})\gamma(\lambda - uC_p^{(0)})^{-1}\Big)$$

that the operator $\gamma\mathbb{P}_{p,u}$ is given by

$$\gamma\mathbb{P}_{p,u} = \psi_{1/\sqrt{u}}\sum_{l=0}^{\dim_{\mathbf{R}} B}\sum_{1\leq i_0\leq l+1}\frac{(-1)^{l-\sum_m j_m}}{(i_0 - 1 - \sum_m j_m)!}T_{p,1}(u\tilde{R}_p)T_{p,2}\dots(u\tilde{R}_p)T_{p,l+1} \tag{4.3.5}$$

where γP_p appears i_0 times among the $T_{p,j}$ and the other terms are given respectively by $(uC_p^{(0)})^{-(1+j_1)},\dots,(uC_p^{(0)})^{-(1+j_{l+1-i_0})}$. Each term in the sum (4.3.5) is a product of uniformly bounded terms in which γP_p appears since $i_0\geq 1$. Using the Atiyah-Segal-Singer index formula in [P16, (2.196)] instead and proceeding in the same way the claim follows for $m = 0$. By the equivariance of the connections the case $m\geq 1$ follows from the same reasoning as in the non-equivariant case. □

With Lemma 4.3.2 and Lemma 4.3.3 the proof of Theorem 4.2.2 is complete.

4.4 Remarks towards Generalizations and Arakelov Geometry

The holomorphic analytic torsion forms can be defined for a more gerneralized class of fibration called Hermitian fibration where the 2-form ω^M does not need to be closed. For the precise definition of the Bismut superconnection and torsion form in the non-Kähler case see [Bi13] or [P16]. A full asymptotic of the analytic torsion forms for Hermitian fibration should be obtainable as well with the present in Lemma 2.1.2 of additional terms of the form $(\bar{\partial}^M\partial^M i\omega)^c - \frac{1}{16}\|(\bar{\partial}^Z - \partial^Z)i\omega^Z\|^2_{\Lambda^\bullet(T^*_{\mathbf{R}}Z)}$ coming from the Lichnerowicz formula [Bi13, Theorem 3.9.3]. This should not disturb the calculation but rather one has to modify Ψ. Since we have restricted ourselves in chapter 3 on families of vector bundles arising from a principle bundle where the form ω^M is closed we have neglected Hermitian fibrations from the beginning to keep it transparent.

An interesting but rather difficult generalization is to obtain a full asymptotic of the equivariant holomorphic analytic torsion forms which generalizes Theorem 2. It is not clear if the methods in the proof of Theorem 2.2.8 are compatibel with that from Theorem 4.2.2 as for example the identification $\mathrm{End}(\mathcal{L}^p) \cong \mathbf{C}$ were used when restricted the kernel to the diagonal. While an explicit formula for α_0 and β_0 has been calculated by [P16] the equivariant case is more complicated since the Mehler formula for the harmonic oscillator gave

$$e^{-L_{0,u}}\Big((z,\gamma^{-1}V),(z,V)\Big) = \frac{1}{(2\pi)^n}\exp(-\Omega_{u,(z,V)})\frac{\det(\dot{\Omega}^{\mathcal{L}}_{(z,V)})}{\det(1-\exp(-u\dot{\Omega}^{\mathcal{L}}_{(z,V)}))}$$
$$\times \exp\Big(-\Big\langle\frac{\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2}{\tanh(u\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2)}(z,V),(z,V)\Big\rangle + \Big\langle\frac{\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2}{\sinh(u\dot{\Omega}^{\mathcal{L}}_{(z,V)}/2)}e^{u\Omega^{\mathcal{L}}_z/2}(z,\gamma^{-1}V),(z,V)\Big\rangle\Big)\otimes \mathrm{id}_{\mathcal{E}}$$

which complicates the calculation of the derivative of the Mellin transform. Even in the principal bundle case concrete formulas for $\alpha_{\gamma,0}$, $\beta_{\gamma,0}$, $\alpha_{\gamma,1}$ and $\beta_{\gamma,1}$ are missing here. Nontheless the asymptotic behavior of the holomorphic torsion alone already found applications in Arakelov Geometry for which we will now give a short overview. This will be held very briefly as it is not our main research area.

For the precise definitions of the objects we refer to [S92] and [KR1]. Let $f : X \to \mathrm{Spec}\,\mathbf{Z}$ be an arithmetic variety, that is a regular scheme where the map of definition h is flat and projective over Spec $\mathbf{Z}$. Let $X(\mathbf{C})$ denote its complex points which is a complex manifold. Let $\overline{E} = (E,h)$ be a Hermitian vector bundle over X. To the arithmetic variety X one can associate arithmetic Chow groups $\widehat{\mathrm{CH}}^p(X)$ and define an arithmetic Chern character $\widehat{\mathrm{ch}}(\overline{E}) \in \widehat{\mathrm{CH}}^p(X)_{\mathbf{Q}}$. There is a natural isomorphism $\widehat{\deg} : \widehat{\mathrm{CH}}(\mathrm{Spec}\,\mathbf{Z}) \overset{\cong}{\to} \mathbf{R}$ called the arithmetic degree. The first arithmetic Chern class $\hat{c}_1(E) \in \widehat{\mathrm{CH}}^1(X)$ is the degree one part of $\widehat{\mathrm{ch}}(\overline{E})$. If $\overline{V}$ is a a finitely generated free $\mathbf{Z}$-module with a Hermitian metric on $V \otimes \mathbf{C}$ then

$$\widehat{\deg}(\hat{c}_1(\overline{V})) = -\log(\mathrm{covol}(\overline{V})).$$

Let $\overline{L}$ be an Hermitian ample line bundle over X. If n denotes the dimension of X then by [S92, Theorem 2']

$$\log\#\{s \in H^0(X, E\otimes L^p) | \|s\|_{L^2}\} \geq \mathrm{rk}(E)\frac{p^{n+1}}{(n+1)!}f_*(\hat{c}_1(\overline{L})^{n+1}) + O(p^n \log p) \tag{4.4.1}$$

where # denotes the cardinality of a set. For its proof Gillet-Soulé used the asymptotic behavior $T(\omega^{X(\mathbf{C})}, h^{E\otimes L^p}) = O(p^n \log p)$ of the holomorphic torsion provided by Bismut-Vasserot. Their concrete formula for the top term in the asymptotic of the torsion was not needed here, however its application can be found in [GS92, Theorem 8] a more precise version of (4.4.1).

Let $\mu_N = \mathrm{Spec}\ {}^{\mathbf{Z}[T]}/_{T^N-1} \to \mathrm{Spec}\,\mathbf{Z}$ be the group scheme of N-th root of unity. We assume that X is endowed with an group action of μ_N and that this action lifts to an action on E which

is compatible with the metric. Then E is a μ_N-equivariant Hermitian vector bundles.
Let X_{μ_N} be the fixed point scheme. Fix a primitive N-th complex root of unity ζ_N. Then it induces an automorphism γ on $X(\mathbf{C})$ with $X_{\mu_N}(\mathbf{C}) \cong X(\mathbf{C})_\gamma$. The μ_N-actions induces a $\mathbf{Z}/N$-grading $E_{|X_{\mu_N}} \cong \oplus_{k\in\mathbf{Z}/N} E_k$. For an abelian group S let S_{Tors} denote the torsion subgroup. Then by [KR1, Theorem 7.14] the following equalitiy holds:

$$\begin{aligned} &-\sum_{q\geq 0}(-1)^q\Big(\sum_{k\in\mathbf{Z}/N}\zeta_N^k\cdot\big(\log(\mathrm{covol}(\overline{H^q(Z,E)}_k)-\log(\#H^q(X,E)_{k,\mathrm{Tors}})\big)\Big)\\ &=\frac{1}{2}T_\gamma(\omega^{X(\mathbf{C})},h^E)-\frac{1}{2}\int_{X_{\mu_N}(\mathbf{C})}\mathrm{Td}_\gamma(TX_{\mathbf{C}})\,\mathrm{ch}_\gamma(E_{\mathbf{C}})R_\gamma(TX_{\mathbf{C}})+\widehat{\deg}(f_*(\widehat{\mathrm{Td}}_{\mu_N}(\overline{Tf})\widehat{\mathrm{ch}}_{\mu_N}(\overline{E}))) \end{aligned}$$

(see [KR1, Definition 3.5 and Definition 7.13] for the definitions of $R_\gamma, \widehat{\mathrm{ch}}_{\mu_N}, \widehat{\mathrm{Td}}_{\mu_N}$).

Now let $\overline{L}$ be a μ_N-equivariant Hermitian ample line bundle on X. By the degree zero part of Theorem 4.2.2 we have $T_\gamma(\omega^{X(\mathbf{C})},h^E) = O(p^{n_\gamma}\log p)$. We hope that similar to the non-equivariant case this provides the necessary analytical part for proving an asymptotic formula for the quantity $-\sum_{q\geq 0}(-1)^q(\sum_{k\in\mathbf{Z}/N}\zeta_N^k\cdot\log\frac{\mathrm{covol}(\overline{H^q(Z,E\otimes L^p)}_k)}{\#H^q(X,E\otimes L^p)_{k,\mathrm{Tors}}}$ as p tends to infinity.

Appendix A

Spectrum and Finite Propagation Speed

In this appendix we continue the treatment on families of operators of chapter 1. We start with studying the spectrum of the curvature and then move on to the wave equation. For the latter topic we retrict ourselfs to the Bismut superconnection. But first we will cite two theorems from [MM07] whch will be used.

Let $(Z, g^{T_{\mathbf{R}}Z})$ be a complete orientable Riemannian manifold with boundary ∂Z. Let (E, h^E) be a Hermitian vector bundle on Z with connection ∇^E. Let e_n be the inward pointing unit normal at any boundary point of Z. Then $s \in \Gamma(Z, E)$ satisfies the Dirichlet boundary condition if

$$s = 0 \ \text{ on } \partial Z.$$

$s \in \Gamma(Z, E)$ satisfies the Neumann boundary condition if

$$\nabla^E_{e_n} s = 0 \ \text{ on } \partial Z.$$

Let H be a positive generalized Laplacian on E with domain $\mathrm{Dom}(H) = \{s \in \Gamma(Z, E) \mid s = 0 \text{ or } \nabla^E_{e_n} s = 0 \ \text{ on } \partial Z\}$. Its Friedrichs extension will still denoted by H which is positive.

Theorem A.0.1 ([MM07, Theorem D.2.1])**.** *For $\omega(t, x)$, $t \in \mathbf{R}$, $z \in Z$ we consider the wave equation*

$$\Big(\frac{\partial^2}{\partial t^2} + H\Big)\omega$$

with the Dirichlet or Neumann boundary condition. Then for any $s_0, s_1 \in \Gamma(Z, E)$ verifying the corresponding boundary condition, there exists a unique solution ω for the equation with initial *conditions* $\omega(0,z) = s_0$, $\frac{\partial}{\partial t}\omega(0,z) = s_1$. *The solution* ω *is given by*

$$w(t,x) = \cos(t\sqrt{H})s_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}s_1.$$

It satisfies

$$\operatorname{supp}(w(t,\cdot)) \subset \{x \in Z \mid d(x,y) \leq t, \text{ for some } y \in \operatorname{supp}(s_0) \cup \operatorname{supp}(s_1)\}.$$

Furtheremore one has the following identity which relate the heat kernel and the wave equation:

$$e^{-t^2H/2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \cos(st\sqrt{H})e^{-s^2/2}ds.$$

Theorem A.0.2 ([MM07, Theorem A.3.2])**.** *Let* $K \subset Z$ *be compact and* H *a generalized Laplacian. For any* $m \in \mathbf{N}_0$ *there exist* $C_1, C_2 > 0$ *such that for any* $s \in \mathbf{H}^{m+2}(Z,E)$ *with* $\operatorname{supp}(s) \subset K$ *we have*

$$\|s\|^2_{\mathbf{H}^{m+2}(Z,E)} \leq C_1\|Hs\|^2_{\mathbf{H}^m(Z,E)} + C_2\|s\|^2_{L^2(Z,E)}.$$

Now assume we are in the situation of chapter 1 with the same notations. The following theorem has been proven for a special case in [Bi97] which plays a keyrole in the proof of Bismut's immersion Theorem. The statement is still valid for a wider class of operators without any changes of the original proof. The idea is to use the formal identity "$(a-b)\cdot\sum_{n=1}^{\infty}\frac{b^{n-1}}{a^n} = 1$" and that the sum is finite for nilpotent b. For the sake of completeness we reproduce its proof.

Theorem A.0.3. *Assume* $\mathbb{A}^{2,(+)} \in \mathfrak{A}^\bullet(B, \operatorname{Op}(\mathcal{E}))$ *is a operator of order less than or equal to* 1. *Then*

$$\operatorname{Spec}(\mathbb{A}^2) = \operatorname{Spec}(D^2).$$

Proof. Take $\lambda \notin \operatorname{Spec}(D^2)$. Then we have the formal identity

$$\begin{aligned}(\lambda - \mathbb{A}^2)^{-1} =& \sum_{l\geq 0}(\lambda - D^2)^{-1}\mathbb{A}^{2,(+)}\cdots\mathbb{A}^{2,(+)}(\lambda - D^2)^{-1}\\ =&(\lambda - D^2)^{-1} + (\lambda - D^2)^{-1}\mathbb{A}^{2,(+)}(\lambda - D^2)^{-1} + \dots\end{aligned}$$

Because $\mathbb{A}^{2,(+)}$ has positive degree in $\Lambda^\bullet T^*_{\mathbf{C}}B$ it is a nilpotent operator and the sum contains only a finite number of terms. By Theorem A.0.2 together with the assumption that $\mathbb{A}^{2,(+)}$ is of order lesser equal 1 we deduce

$$\begin{aligned}
&\|\mathbb{A}^{2,(+)}(\lambda - D^2)^{-1}s\|_{\mathbf{H}^0(M,\pi^*(\Lambda^\bullet T^*_{\mathbf{C}}B)\otimes\mathcal{E})} \leq C_1\|(\lambda - D^2)^{-1}s\|_{\mathbf{H}^2(M,\pi^*(\Lambda^\bullet T^*_{\mathbf{C}}B)\otimes\mathcal{E})} \\
&\leq C_2\|s\|_{\mathbf{H}^0(M,\pi^*(\Lambda^\bullet T^*_{\mathbf{C}}B)\otimes\mathcal{E})} + C_3\|(\lambda - D^2)^{-1}s\|_{\mathbf{H}^0(M,\pi^*(\Lambda^\bullet T^*_{\mathbf{C}}B)\otimes\mathcal{E})} \\
&\leq C_4\|s\|_{\mathbf{H}^0(M,\pi^*(\Lambda^\bullet T^*_{\mathbf{C}}B)\otimes\mathcal{E})}.
\end{aligned}$$

Therefore the operator $(\lambda - \mathbb{A}^2)^{-1}$ is a bounded operator on the Sobolev space of order 0 and we conclude $\lambda \notin \operatorname{Spec}(\mathbb{A}^2)$. By exchanging the roles of D^2 and $\mathbb{A}^2$ we find that if $\lambda \notin \operatorname{Spec}(\mathbb{A}^2)$ then $\lambda \notin \operatorname{Spec}(D^2)$. The claim follows. □

For $\alpha > 0$ let $f : \mathbf{R} \to [0,1]$ be a smooth even function with

$$f(t) = \begin{cases} 1 \ , \ |t| < \frac{\alpha}{2} \\ 0 \ , \ |t| > \alpha. \end{cases}$$

For $a \in \mathbf{C}$ and $u > 0$ define the functions

$$\begin{aligned}
F_u(a) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{is\sqrt{2}a} \exp(-s^2/2) f(\sqrt{u}s) ds \quad \text{and} \\
G_u(a) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{is\sqrt{2}a} \exp(-s^2/2)(1-) f(\sqrt{u}s) ds
\end{aligned}$$

These are even holomorphic functions, thus there exist holomorphic functions $\tilde{F}_u$, $\tilde{G}_u$ with $\tilde{F}_u(a^2) = F_u(a)$ and $\tilde{G}_u(a^2) = G_u(a)$. For $C > 0$ let Γ be the contour indicated in the following figure:

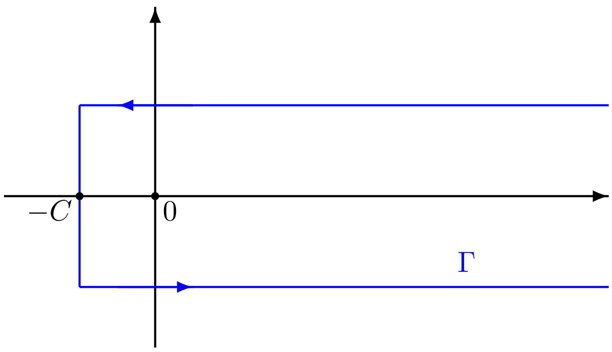

Figure A.1: Contour Γ

Since $\operatorname{Spec}(B^2) = \operatorname{Spec}(D^2) \subset [0,\infty[$ the resolvent $(\lambda - B^2)^{-1}$ exsists for $\lambda \in \Gamma$ and by [P16, (2.27)] there is $k \geq 0$ such that

$$\|(\lambda - B^2)^{-1}\|_\infty \leq |\lambda|^k. \tag{A.0.1}$$

The restriction of $\tilde{F}_u$ and $\tilde{G}_u$ lies in the Schwartze space $\mathcal{S}(\mathbf{R})$. In particular $\tilde{F}_u(B^2)$ is well-defined,

$$\tilde{F}_u(B^2) = \int_\Gamma \tilde{F}_u(\lambda)(\lambda - B^2)^{-1} d\lambda.$$

Same goes for $\tilde{G}_u(B^2)$ and for $v > 0$ one has

$$\tilde{F}_u(vB^2) + \tilde{G}_u(vB^2) = e^{-vB^2}. \tag{A.0.2}$$

Because f is an even function and $f(\sqrt{u}s)$ vanishes for $|\sqrt{u}s| \geq \alpha$ we can write $\tilde{F}_u(uB^2)$ as

$$\tilde{F}_u(uB^2) = \frac{2}{\sqrt{2\pi}} \int_0^{\alpha/\sqrt{u}} \cos(s\sqrt{2}\sqrt{uB^2}) \exp(-s^2/2) f(\sqrt{u}s) ds. \tag{A.0.3}$$

Recall for a distribution $T \in \mathcal{D}'(Z)$ on Z its support is defined as $\operatorname{supp}(T) = \{z \in Z|$ for all neighborhood U of z there exsists a testfunction $\varphi \in \mathcal{D}(U)$ with $\langle T, \varphi \rangle \neq 0\}$. This definition extends to vector valued distributions. For $z \in Z$ let $\delta_{\{z\}}$ be the Dirac delta distribution. Then $\operatorname{supp}(\delta_{\{z\}}) = \{z\}$. For $b \in B$ let $V \in \Lambda^\bullet(T^*_{\mathbf{C},b}B) \otimes (\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E})_z$ be a given vector. Then $V \cdot \delta_{\{z\}}$ is a vector valued distribution given by for any $f \in \mathcal{D}(Z)$

$$\langle V \cdot \delta_{\{z\}}, f \rangle = f(z) \cdot V \in \Lambda^\bullet(T^*_{\mathbf{C},b}B) \otimes (\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E})_z.$$

By the finite propagation speed of the wave equation from Theorem A.0.1 we have

$$\operatorname{supp}\Big(\cos(s\sqrt{2}\sqrt{uB^2}) V \delta_{\{z\}}\Big) \subset B^Z_{\sqrt{u}s}(z),$$

thus if $\sqrt{u}s \leq \alpha$ we have

$$\operatorname{supp}\Big(\cos(s\sqrt{2}\sqrt{uB^2}) V \delta_{\{z\}}\Big) \subset B^Z_\alpha(z). \tag{A.0.4}$$

By the Schwartz kernel theorem $\tilde{G}_u(vB^2)$ and $\tilde{F}_u(vB^2)$ are represented by the smooth kernels $\langle z|\tilde{G}_u(vB^2)|z'\rangle$ and $\langle z|\tilde{F}_u(vB^2)|z'\rangle$ with respect to $d\mathrm{vol}_{g^{T_\mathbf{R}Z}}(z')$. Thus from (A.0.3) and (A.0.4) the map

$$z' \mapsto \langle z|\tilde{F}_u(uB^2)|z'\rangle \tag{A.0.5}$$

depends only on the restriction of the operator B^2 to the ball $B^Z_\alpha(z)$ and if $z' \notin B^Z_\alpha(z)$ then

$\langle z|\tilde{F}_u(uB^2)|z'\rangle$ vanishes.

Let $(\mathcal{L}, h^{\mathcal{L}})$ be a holomorphic Hermitian line bundle on M and for $p \in \mathbf{N}$ put $\mathcal{L}^p := \mathcal{L}^{\otimes p}$. The Bismut superconnection can then be defined for $(\mathcal{E} \otimes \mathcal{L}^p, h^{\mathcal{E}\otimes\mathcal{L}^p})$ instead of $(\mathcal{E}, h^{\mathcal{E}})$ and will denoted by B_p. Set

$$\mathbb{E}_p := \Lambda^\bullet(T^*_{\mathbf{R},b_0}B) \otimes (\Lambda^{0,\bullet}(T^*Z) \otimes \mathcal{E} \otimes \mathcal{L}^p)$$

where $\Lambda^\bullet(T^*_{\mathbf{R},b_0}B)$ is a trivial bundle over Z. $\mathbb{E}_p$ will be equipped with the connection $\nabla^{\mathbb{E}_p}$ induced by $\nabla^{T_\mathbf{R}B,LC}$, $\nabla^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$, $\nabla^{\mathcal{L}}$ and with the metric $h_p^{\mathbb{E}}$ induced by $g^{T_\mathbf{R}B}$, $h^{\Lambda^{0,\bullet}(T^*Z)\otimes\mathcal{E}}$ and $h^{\mathcal{L}}$. Let $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ and $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ be the induced connection and metric on $\mathbb{E}_p \boxtimes \mathbb{E}_p^*$. Let inj^Z be the injectivity radius of Z and let $\varepsilon \in]0, \frac{\mathrm{inj}^Z}{4}[$. In [P16, Proposition 2.2] it was shown that for any $m \in N$ and $\varepsilon > 0$ there exist $C > 0$ and $N \in \mathbf{N}$ such that

$$\Big\|\langle \cdot |\tilde{G}_{\frac{u}{p}}(\frac{u}{p}B_p^2)| \cdot \rangle\Big\|_{\mathscr{C}^m(M\times_\pi M,\mathbb{E}_p\boxtimes\mathbb{E}_p^*)} \leq Cp^N \exp\big(-\frac{\varepsilon^2 p}{16u}\big) \tag{A.0.6}$$

where the $\mathscr{C}^m$-norm is induced by $\nabla^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$ and $h^{\mathbb{E}_p\boxtimes\mathbb{E}_p^*}$. By (A.0.2) and (A.0.6) if one wants to study the behavior of $\Big\|\langle z| \exp\big(-\frac{u}{p}B_p^2\big)|z'\rangle$ as $p \to \infty$ or $u \to 0$ then $\exp\big(-\frac{u}{p}B_p^2\big)$ can be replaced by $\tilde{F}_{\frac{u}{p}}(\frac{u}{p}B_p^2)$.

We now assume the reader have read section 2.1 a). The operator $M_{p,z}$ has the same structure as B_p^2 thus (A.0.6) holds if B_p^2 is replaced by $M_{p,z}$. Because $M_{p,z}$ coincides with B_p^2 over $B_\varepsilon^Z(0)$ by the comment after (A.0.5) with $\alpha = \varepsilon$ we have

$$\langle z|\tilde{F}_{\frac{u}{p}}(\frac{u}{p}B_p^2)| \cdot \rangle = \langle z|\tilde{F}_{\frac{u}{p}}(\frac{u}{p}M_{p,z}| \cdot \rangle.$$

from which (2.1.2) follows. Furthermore we see that (2.1.2) is valid outside the diagonal as well.

This localization technique can not only be applied to the Bismut superconnection but also to general superconnections $\mathbb{A}$ such that $\mathbb{A}^{2,(+)}$ is a differential operator of order 1 because the proof of (A.0.1) and (A.0.6) uses only the structure of B_p^2 (see [P16, (2.7)]) and not how the operator exactly looks like. Instead of copying the proof in section 2.1 of [P16] and replace B_p^2 by $\mathbb{A}_p^2$ with using [Bi86, (2.25),(2.68)] for its local structure we leave the details for the reader.

Appendix B

Lie Algebraic Equivariant Holomorphic Analytic Torsion

In this appendix we want to give the reader some insight of an infinitesimal equivariant analytic torsion defined in [BG00]. It relates the analytic torsion form with the equivariant torsion when the geometry comes from a principle bundle as in chapter 3. The notation here will be the same as in chapter 3.1. Be careful from the different sign convention between (1.1.1), (3.1.1) and [BG00, (1.16),(2.2)]. The formulas in [BG00] are changed due to $\omega^Z \rightsquigarrow -\omega^Z$ and $K_Z \rightsquigarrow -K_Z$.

Let $\gamma \in G$ be given. Let $Z(\gamma)_{\mathbf{C}}$ be the centralizer of γ in $G_{\mathbf{C}}$. Let $P \overset{Z(\gamma)_{\mathbf{C}}}{\to} B$ be a holomorphic principle bundle with structure group $Z(\gamma)_{\mathbf{C}}$. All the consideration from section 3.1 will be now applied to $Z(\gamma)_{\mathbf{C}}, Z(\gamma), \mathfrak{z}(\gamma)$ instead of $G_{\mathbf{C}}, G, \mathfrak{g}$ and we assume that $Z(\gamma)$ is connected. In particular $\Theta \in \mathfrak{A}^{1,1}(B, P \times_G \mathfrak{z}(\gamma))$ and M is given by $Q \times_{\mathfrak{z}(\gamma)} Z$.

Let $K \in \mathfrak{g}$ and K_Z its corresponding vector field on Z. Let $K_Z^{1,0} \in \Gamma(Z, T^{1,0}Z)$ and $K_Z^{0,1} \in \Gamma(Z, T^{0,1}Z)$ denote the induced vector fields. Set

$$\begin{aligned} d_K &:= d + 2\pi i \iota_{K_Z}, \\ \partial_K &:= \partial + 2\pi i \iota_{K_Z^{0,1}} \qquad \text{and} \\ \overline{\partial}_K &:= \overline{\partial} + 2\pi i \iota_{K_Z^{1,0}}. \end{aligned}$$

The K-equivariant curvatures Ω_K^{TZ} and Ω_K^{TZ} are defined as

$$\Omega_K^{TZ} := \Omega^{TZ} + 2\pi i m^{TZ}(K) \quad \text{and} \quad \Omega_K^{TZ} := \Omega^E + 2\pi i m^E(K).$$

Note as mentioned that the sign difference in [BG00, (2.25),(2.30)] comes from (3.1.1). As in chapter 1 we define for $\gamma \in G$ the bundles $N^{i\theta_j}_{Z_\gamma/Z}$ and the corresponding connections and curvatures. The restriction $m^{TZ}(K)_{|Z_\gamma}$ preserves the splitting of TZ_γ in its eigenbundles and let $m^{TZ_\gamma}(K)$ and $m^{N^{i\theta_j}_{Z_\gamma/Z}}$ be the restriction of $m^{TZ}(K)_{|Z_\gamma}$ to TZ_γ and $N^{i\theta_j}_{Z_\gamma/Z}$ $(1 \leq j \leq q)$. The corresponding equivariant curvatures $\Omega^{TZ_\gamma}_K, \Omega^{N^{i\theta_j}}_K$ are then to defined in the same manner as above.

Definition B.0.4. *For $K \in \mathfrak{z}(\gamma)$ with $|K|$ small enough define*

$$\mathrm{Td}_{\gamma,K}(TZ, h^{TZ}) = \mathrm{Td}\Big(\frac{-\Omega^{TZ_\gamma}_K}{2\pi i}\Big)\prod_{j=1}^{q}\Big(\frac{\mathrm{Td}}{c_{\max}}\Big)\Big(\frac{-\Omega^{N^{\theta_j}_{Z_\gamma/Z}}_K}{2\pi i} + i\theta_j\Big),$$

$$\mathrm{Td}'_{\gamma,K}(TZ, h^{TZ}) = \frac{\partial}{\partial b}\Big[\mathrm{Td}\Big(\frac{-\Omega^{TZ_\gamma}_K}{2\pi i} + b\Big)\prod_{j=1}^{q}\Big(\frac{\mathrm{Td}}{c_{\max}}\Big)\Big(\frac{-\Omega^{N^{\theta_j}_{Z_\gamma/Z}}_K}{2\pi i} + i\theta_j + b\Big)\Big]_{|b=0} \quad \textit{and}$$

$$\mathrm{ch}_{\gamma,K}(E, h^E) = \mathrm{Tr}\Big[\gamma\exp\Big(\frac{-\Omega^E_{K|Z_\gamma}}{2\pi i}\Big)\Big].$$

These forms $\mathrm{Td}_{\gamma,K}(TZ, h^{TZ})$ and $\mathrm{ch}_{\gamma,K}(E, h^E)$ are G-invariant forms lying in P^{Z_γ}. The requirement on $|K|$ is because $\mathrm{Td}(x)$ vanishes for $x \in 2i\pi\mathbf{Z}$.

Definition B.0.5. *For $u > 0$ and $K \in \mathfrak{z}(\gamma)$ put*

$$\beta_u(\gamma, K) := \mathrm{Tr}_s\Big[\Big(N - i\frac{\langle\mu, K\rangle}{u}\Big)\gamma\exp\Big(-(\sqrt{u}D^Z - \frac{c(K_Z)}{2\sqrt{2u}})^2 - L_K\Big)\Big].$$

The above definition can be extended to the case where $K \in \mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \oplus i\mathfrak{g}$.

Theorem B.0.6 ([BG00, Theorem 2.22])**.** *For $K \in \mathfrak{z}(\gamma)$ with $|K|$ sufficiently small there exists $\delta \in]0,1[$ and complex numbers $C_{-1}(\gamma, K), C_0(\gamma, K) \in \mathbf{C}$ such that for $u \in]0,1]$*

$$\beta_u(\gamma, K) = \frac{C{-1}(\gamma, K)}{u} + C_0(\gamma, K) + O(u^\delta).$$

Moreover

$$C_{-1}(\gamma, K) = \int_Z \Big(-\frac{\omega^Z}{2\pi} - i\langle\mu, K\rangle\Big)\mathrm{Td}_{\gamma,K}(TZ, h^{TZ})\,\mathrm{ch}_{\gamma,K}(E, h^E) \quad \textit{and}$$

$$C_0(\gamma, K) = \int_Z \mathrm{Td}_{\gamma,K}(TZ, h^{TZ})\Big(\dim_{\mathbf{C}} Z - \Big(\frac{\mathrm{Td}'}{\mathrm{Td}}\Big)_{\gamma,K}(TZ, h^{TZ})\Big)\mathrm{ch}_{\gamma,K}(E, h^E).$$

For $K \in \mathfrak{z}(\gamma)$, as $u \to \infty$,

$$\beta_u(\gamma, K) = \mathrm{Tr}_s^{H^\bullet(Z,E)}[N\gamma e^K] + O\Big(\frac{1}{\sqrt{u}}\Big).$$

Definition B.0.7. *For $K \in \mathfrak{z}(\gamma)$ with $|K|$ sufficiently small and for $s \in \mathbf{C}$ with $0 < \mathrm{Re}(s) < \frac{1}{2}$ set*

$$\zeta_{\gamma,K}(s) := -\frac{1}{\Gamma(s)} \int_0^\infty u^{s-1}\Big(\beta_u(\gamma, K) - \mathrm{Tr}_s^{H^\bullet(Z,\mathcal{E})}[N\gamma e^K]\Big) du.$$

By Theorem B.0.6 the function $\zeta_{\gamma,K}$ is well defined and extends to a holomorphic function near $s = 0$.

Definition B.0.8. *The Lie algebraic equivariant analytic torsion form $T_{\gamma,K}(\omega^Z, h^E)$ is defined as*

$$T_{\gamma,K}(\omega^Z, h^E) := \zeta'_{\gamma,K}(0).$$

For $\gamma = e$ and with the notation from section 3.1 $T_{e,K}(\omega^Z, h^E)$ is related with the torsion form $T_{-\frac{\Theta}{2\pi i}}(\omega^Z, h^E)$ by replacing K with $-\frac{\Theta}{2\pi i}$ (see [BG00, (2.74)]),

$$\begin{aligned} &T_{e,-\frac{\Theta}{2\pi i}}(\omega^Z, h^E) && \text{(Definition B.0.8)} \\ =\ &T_{-\frac{\Theta}{2\pi i}}(\omega^Z, h^E) && \text{(notation at the end of section 3.1).} \end{aligned}$$

The following result has been proven in [BG00, Proposition 2.25] which follows from Theorem B.0.6.

Proposition B.0.9. *For $K \in \mathfrak{z}(\gamma)$ and $|K|$ small enough*

$$\begin{aligned} \zeta'_{\gamma,K}(0) = &-\int_0^1 \Big(\beta_u(\gamma, K) - \frac{C_{-1}(\gamma, K)}{u} - C_0(\gamma, K)\Big)\frac{du}{u} \\ &- \int_0^\infty (\beta_u(\gamma, K) - \mathrm{Tr}_s^{H^\bullet(Z,\mathcal{E})}[N\gamma e^K])\frac{du}{u} \\ &+ C_{-1}(\gamma, K) + \Gamma'(1)(C_0(\gamma, K) - \mathrm{Tr}_s^{H^\bullet(Z,E)}[N\gamma e^K]). \end{aligned}$$

Definition B.0.10. *For $u > 0$ let d_u be the even form on Z given by*

$$d_u := \frac{\omega^Z}{2\pi u} \exp\Big(\frac{\overline{\partial}_K \partial_K}{2\pi i u}\frac{-\omega^Z}{2\pi}\Big).$$

Define the set

$$Z_K := \{z \in Z |\ K_Z(z) = 0\}$$

which is a complex totally geodesic submanifold of Z.

Definition B.0.11. *Let P^Z_{K,Z_K} be the set of K_Z-invariant currents on Z which are sums of currents of type (p,p) whose wave-front set is included in $N^*_{Z_K/Z,\mathbf{R}}$.*

The current of integration on Z will be denoted by δ_Z. Let $h^{N_{Z_K/Z}}$ be the Hermitian metric on $N_{Z_K/Z}$ induced by ω^Z. Then there are currents $\delta_Z \wedge \rho_1, \dots \delta_Z \wedge \rho_k, \dots$ in P^Z_{K,Z_K} where ρ_1 are (p,p) forms such that as $u \to 0$

$$\delta_Z \wedge d_u = \delta_{Z_K} \wedge \frac{\omega^Z/2\pi}{c_{\max,K}(N_{Z_k/Z}, h^{N_{Z_k/Z}})} \frac{1}{u} + \delta_Z \wedge \sum_{j=0}^{k} \frac{\omega^Z}{2\pi} \rho_{j+1} u^j + o(u^k), \tag{B.0.1}$$

see [BG00, (3.8)] and the literatures noted there. For an arbitrary smooth from η on Z and for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$ let F^1_η be the function

$$F^1_\eta(s) := \frac{1}{\Gamma(s)} \int_0^1 u^{s-1} \langle \delta_Z \wedge d_u, \eta \rangle du.$$

By equation (4.2) F^1_η esxtends to a meromorphic function which is holomorphic at $s = 0$. For $s \in \mathbf{C}$ with $\mathrm{Re}(s) < 1$ the function

$$F^2_\eta(s) := \frac{1}{\Gamma(s)} \int_1^{+\infty} u^{s-1} \langle \delta_Z \wedge d_u, \eta \rangle du$$

is holomorphic at $s = 0$.

Definition B.0.12. *Let $S_K(Z, -\omega^Z)$ be the current on Z given by*

$$\langle S_K(Z, -\omega^Z), \eta \rangle := \frac{d}{ds}_{|s=0} (F^1_\eta + F^2_\eta).$$

This current will appear in the comparison formula which relates the equivariant holomorphic torsion with the torsion form.

Now we recall the construction of the genus I from [BG00, section 4].

Definition B.0.13. *For $\theta, \theta' \in \mathbf{R}$ with $|\theta'|$ small enough and $x \in \mathbf{C}$ with $|x|$ small enough, set*

$$I(\theta, \theta', x) := \sum_{\substack{k \in \boldsymbol{Z} \\ 2k\pi + \theta \neq 0}} \frac{\log(1 + \frac{\theta'}{2k\pi + \theta})}{i(2k\pi + \theta + \theta') + x}.$$

Let (N, h^N) be a equivariant Hermitian holomorphic vector bundle on Z with G-invariant Hermitian metric h^N. Let $e^{i\theta}, \dots, e^{i\theta_q}, 0 \leq \theta_j < 2\pi$, be the distinct eigenvalues of γ on N with corresponding eigenbundles N^{θ_j}. Let B be a holomorphic skew-adjoint section of $\mathrm{End}(N)$ which commutes with γ and let $i\theta'_1, \ldots, i\theta'_q$ with $\theta_j \in \mathbf{R}$ be the locally constant eigenvalues of the action of B on N with corresponding eigenbundles $N^{\theta'_j}$. Then there are bundles N^{θ_j, θ'_j} so that N splits orthogonally as

$$N = \oplus_{\substack{1 \le j \le q \\ 1 \le j' \le q}} N^{\theta_j, \theta'_j}$$

such that on N^{θ_j, θ'_j}, γ acts by multiplication by $e^{i\theta_j}$ and B by multiplication with $i\theta'_j$. Let $h^{N^{\theta_j, \theta'_j}}$ be the induced metric on N^{θ_j, θ'_j} and $\nabla^{N^{\theta_j, \theta'_j}}$ the holomorphic Hermitian connection on $N^{\theta_j, \theta'_j}, h^{N^{\theta_j, \theta'_j}}$ with curvature $\Omega^{N^{\theta_j, \theta'_j}}$. $I(\theta, \theta', \cdot)$ will be identified with the corresponding additive genus.

Definition B.0.14. *Define*

$$I_{\gamma, B}(N, h^N) := \sum_{\substack{1 \le j \le q \\ 1 \le j' \le q}} \mathrm{Tr}\left[I\Big(\theta_j, \theta'_j, -\frac{\Omega^{N^{\theta_j, \theta'_j}}}{2\pi i}\Big)\right]$$

Take $z \in \mathbf{R}^*$, $K_0 \in \mathfrak{z}(\gamma)$ and $K = zK_0$. Put

$$Z_{\gamma, K} := Z_\gamma \cap Z_K.$$

Then for z small enough, $Z_{\gamma, K} = Z_{\gamma e^K}$. The operator γ acts on $N_{Z_K/Z_{|Z_{\gamma,K}}}$ with locally constant distinct eigenvalues $e^{i\theta}, \ldots, e^{i\theta_q}$. We have that $-\nabla^{TZ} K_Z$ acts as a skew-adjoint morphism of $N_{Z_K/Z}$ which commutes with γ over $Z_{\gamma, K}$.

For $z \in \mathbf{R}^*$ close enough to 0 the characteristic class $I_{\gamma, K}(N_{Z_K/Z})$ on $Z_{\gamma, K}$ is defined to be

$$I_{\gamma, K}(N_{Z_K/Z}) := [I_{\gamma, -\nabla^{TZ} K_Z}(N_{Z_K/Z}, h^{N_{Z_K/Z}})].$$

With the constructed current $S_K(Z_\gamma, -\omega^{Z_\gamma})$ and the genus I all the data are collected to state the comparison formula of Bismut-Goette, the main result of [BG00].

Theorem B.0.15 ([BG00, Theorem 5.1])**.** *Let $z \in \mathbf{R}^*$, $K_0 \in \mathfrak{z}(\gamma)$ and $K = zK_0$. For $|z|$ sufficiently small the following identity holds:*

$$\begin{aligned} T_{\gamma e^K}(\omega^Z, h^E) - T_{\gamma, K}(\omega^Z, h^E) = & \int_{Z_\gamma} \mathrm{Td}_{\gamma, K}(TZ, h^{TZ}) \mathrm{ch}_{\gamma, K}(E, h^E) S_K(Z_\gamma, -\omega^{Z_\gamma}) \\ & - \int_{Z_{\gamma, K}} \mathrm{Td}_{\gamma e^K}(TZ, h^{TZ}) I_{\gamma, K}(N_{Z_K/Z}) \mathrm{ch}_{\gamma e^K}(E). \end{aligned}$$

Once the equivariant analytic torsion is known Theorem B.0.15 can pave the way for the calculation of the equivariant torsion form by replacing K with $-\frac{\Theta}{2\pi i}$. Still one has to respect that K is only to be taken from $\mathfrak{z}(\gamma)$ which is not a minor restriction. Also nonetheless to deal with the S-current which is a difficult object to compute is not an easy task at all.

For now we have only given the necessary definitions of the current S_K and the genus I. For more background and properties of these objects see [BG00, chapters 3-5] and [KR2].

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Erklärung

Ich versichere an Eides Statt, dass die Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“erstellt worden ist.

Die Dissertation wurde bisher noch an keiner anderen Fakultät vorgelegt.

Ich habe bisher noch keine Promotionsversuche unternommen.

Düsseldorf, den

(Pascal Teßmer)