Uniform Rationality for Compact p-adic Analytic Groups

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Summary

Given a group, it is natural to ask how many irreducible complex representations it has. We here focus on p-adic analytic groups and, for $n \in \mathbb{N}$, encode the number of isomorphism classes of n-dimensional irreducible complex representation in a Dirichlet series. To ensure that the numbers of these isomorphism classes are finite (*representation rigidity*), we proceed with FAb compact p-adic analytic groups, which provide the proper condition. Then we call this Dirichlet generating series *representation zeta function*, which is a function in s, and a great tool to investigate the distribution of character degrees. Stasinski and Zordan in [SZ20] proved that this series is essentially (or virtually, as it is called in the literature) a rational function in p^{-s}; this can be considered as a strong relation between the number of irreducible representations of different dimension. The (virtual) rationality of such representation zeta functions is obtained by the rationality of a reduced zeta series called *partial zeta series*.

In this work, we consider these partial zeta series for a family of FAb compact p-adic analytic groups. We impose the condition that there exists an analytic formula uniformly defining the family of FAb compact p-adic analytic groups, and first show how to obtain a *uniformly powerful pro-p* subgroup of a given p-adic analytic group in a uniformly definable way for $p > 2$. Following this, we prove that the partial zeta series are uniformly rational. The technical term *uniform rationality* is a way to control the p-dependence of these rational functions.

Adapting some ideas from [SZ20], we then obtain a family of uniformly definable equivalence relations on $\mathbb{Q}_p^m$ as p varies, which allows us to express partial zeta series as generating functions enumerating the equivalence classes in this uniformly definable family of equivalence relations. To do so, we describe an expansion of the analytic language, which is conventionally used for studying valued fields in model theory. Uniform rationality then follows by a result of Nguyen, [Ngu19].

Contents

Chapter 1

Preface

1.1 Acknowledgments

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1.2 Chapter Progression

The 2nd chapter (**Introduction**) provides a humble survey of the historical progress of results and methodologies around zeta functions. The chapter begins with the rationality of Igusa's zeta functions and is followed by Denef's work broadening his ideas via model theoretic methods. Throughout this chapter, we introduce notions such as subgroup growth and representation growth leading to different zeta functions to highlight the use of model theory while naming some results that influence our work. We finish the chapter by introducing our main theorem and describing our framework.

In Chapter 3 (**Background**), we give an overview of key notions on a broad spectrum from the main object, p-adic analytic groups, to the model theory of valued fields. While getting familiar with the nature of mathematical objects of this work, we also fix our model theoretic setting. Sections 3.3 and 3.4 are worth explicitly mentioning as they explain vital notions such as uniform definability and rationality and precisely display our main result. The rest of this text aims to find equivalence classes mentioned in 3.4.

Chapter 4 (**Good Bases**) is devoted to du Sautoy's parametrization of open subgroups of a uniform pro-p group, namely *good bases*. It is a key ingredient in many works concerning p-adic analytic groups, including this one. The chapter progresses by examining its features and finally obtaining the set of good bases in a uniformly definable way. Later on, we give an example covering some of the key notions appearing in previous chapters as well as the good basis.

In the 5th chapter (**Projective Representations**), we introduce basic concepts and ideas from the theory of projective representations along with a few noteworthy results from ordinary representation theory. The last section in this chapter discloses a concise report on the cohomology of finite groups due to their relation to projective representations.

In Chapter 6 (**Partial Zeta Series**), we develop Clifford Theory for projective representations. It continues by showing how *partial zeta series* occur in the study of the representation zeta function of FAb compact p-adic analytic groups. Besides its motivational purposes, it is helpful to highlight what to bear in mind to apply our methods to obtain uniform virtual rationality of a family of FAb compact p-adic analytic groups, which is the most natural direction to take after this work.

In the 7th chapter (**Tools for Constructing Equivalence Classes**), we bring in all the tools to obtain equivalences classes, which we will use to describe partial zeta series. In the first section, we show how to reduce the problem to the case of linear characters. The following section discusses the uniform definability of this reduction while presenting two interludes on the key components of uniform definability: 1) the group $\mathbb{Q}_p/\mathbb{Z}_p$ exploring its relation with *Prüfer p-group*, 2) another parametrization based on good bases.

Chapter 8 (**Main Theorem**) is where we combine all our findings in the right order. The main goal of this chapter is to show how to interpret the sets we want to count uniformly and definably in $\mathbb{Q}_p$. To this end, we describe a family of uniformly definable subsets of $\mathbb{Q}_p^m$ for some m, and specify uniformly definable equivalence relations on these subsets.

As a word of caution, we want to mention that the word *uniform* appears in two different ways in this text; its model theoretic meaning should be understood as defined by a formula not depending on p. Also, in the theory of p-adic Lie groups, there is a central notion called *uniform pro-p groups* which is also fundamental to this work. A characterization can be given as follows: A pro-p group is uniform if and only if it is finitely generated, powerful and torsion-free.

Chapter 2

Introduction

Counting solutions. Let X be a system of polynomial equations in $\mathbb{Z}[x_1, \ldots, x_n]$. For a prime p, consider the set $X(\mathbb{Z}/p^k\mathbb{Z})$ of solutions of X in $\mathbb{Z}/p^k\mathbb{Z}$. Define N_k to be the number of elements of $X(\mathbb{Z}/p^k\mathbb{Z})$ and associate the following Poincaré series

$$\mathrm{P}_X(T) = \sum_{k \geq 0} N_k \cdot T^k.$$

Theorem 2.0.1. *[Igu00] The series* $\mathrm{P}_X(T)$ *is a rational function in* T.

To describe Igusa's framework, we assume X is given by $F(x_1, \ldots, x_n) = 0$. Recall the generalized residue map $\pi_k : \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$, and let B_k be the ball (centered at 0) of radius p^{-k}; $B_k = \{(x_1, \ldots, x_n) \in \mathbb{Z}_p^n : |x_i| \leq p^{-k}\}$. Then

$$\begin{aligned} N_k =& \#(\{(x_1, \ldots, x_n) \in \mathbb{Z}_p^n : F(\pi_k(x_1, \ldots, x_n)) = 0\} \mod B_k) \\ =& \#(\{(x_1, \ldots, x_n) \in \mathbb{Z}_p^n : \pi_k(F(x_1, \ldots, x_n)) = 0\} \mod B_k) \\ =& \#(\{(x_1, \ldots, x_n) \in \mathbb{Z}_p^n : \ |F(x_1, \ldots, x_n)| \leq p^{-k}\} \mod B_k). \end{aligned}$$

Let $X_k = \{(x_1, \ldots, x_n) : \ |F(x_1, \ldots, x_n)| \leq p^{-k}\}$. Recall there exists a unique Haar measure μ on $\mathbb{Q}_p^n$ that is translation invariant, and $\mu(B_0) = \mu(\mathbb{Z}_p^n) = 1$. Therefore $\mu(B_k) = p^{-nk}$. Then

$$\begin{aligned} \int_{B_0} |F(x_1, \ldots, x_n)|^s \, d\mu =& \sum_{k \geq 0} \mu(\{(x_1, \ldots, x_n) \in B_0 : |F(x_1, \ldots, x_n)| = p^{-k}\}) p^{-ks} \\ =& \sum_{k \geq 0} p^{-ks}(\mu(X_k) - \mu(X_{k+1})) \\ =& 1 + (1 - p^s) \sum_{k \geq 1} N_k p^{k(-n-s)}. \end{aligned}$$

The above function is called *Igusa's zeta function*, and denoted by $Z_F(s)$. So one obtains that

$$Z_F(s) = p^s + (1 - p^s) P_X(p^{-n-s}).$$

Integration on semi-algebraic sets. Denef generalized the rationality of Igusa's local zeta function in his paper [Den84]. Let $\varphi(\mathbf{x})$ be an $\mathcal{L}_{ring}$ formula, and $n \in \mathbb{N}$. Set $N_{k,\varphi}$ and $\tilde{N}_{k,\varphi}$ as follows:

$$\begin{aligned} N_{k,\varphi} =& \#\{a \in (\mathbb{Z}/p^k\mathbb{Z})^m : \varphi(a) \text{ holds in } \mathbb{Z}/p^k\mathbb{Z}\}, \\ \tilde{N}_{k,\varphi} =& \#\{a \bmod p^k : a \in {\mathbb{Z}_p}^m, \varphi(a) \text{ holds in } \mathbb{Z}_p\}. \end{aligned}$$

He showed that $\mathrm{P}_{k,\varphi}(T) = \sum_k N_{k,\varphi} \cdot T^k$ and $\tilde{P}_{k,\varphi}(T) = \sum_k \tilde{N}_{k,\varphi} \cdot T^k$ are rational. Note that if you let $\varphi(\mathbf{x})$ to be the system of polynomials X given above, then Igusa's rationality result follows. The first key component of Denef's work is Macintyre's quantifier elimination.

Theorem 2.0.2. *[Mac76] $\mathbb{Q}_p$ admits elimination of quantifiers in $\mathcal{L}_{Mac}$* [1].

To highlight the importance of this result, we first need to mention another important ingredient called p-adic cell decomposition which was used in place of *Hironaka's resolution of singularities* in Igusa's proof. This method analyzes definable sets systematically in terms of controlled definable functions, and the cells admit nice geometric properties. Denef's proof uses Macintyre's quantifier elimination theorem to show the set $\{a \mod p^k : a \in {\mathbb{Z}_p}^m, \varphi(a) \text{ holds in } \mathbb{Z}_p\}$ is not extremely complex, so that we have a control on its measure. Consequently, the cardinalities $N_{k,\varphi}$ and $\tilde{N}_{k,\varphi}$ can be expressed as measures of definable sets (e.g. semi-algebraic sets, subanalytic sets) and the rationality of the power series reduces to showing the rationality of related p-adic integrals - p-adic integration then evolved to motivic integration. Later on, this method was extended by Denef and van den Dries to a larger category of definable sets and functions in [DvdD88], where they prove similar rationality results.

Subgroup growth. One of the notable topics in geometric group theory is counting the number of subgroups of finite index in a given group. Assume G to be a finitely generated group; hence it has finitely many subgroups of finite index of n for any natural number $n \geq 1$. Set $a_n(G)$ to be the number of subgroups of index n in G. The asymptotic behaviour of the sequence $\{a_n\}_n$ indicates the *subgroup growth of G*. A far-reaching theory of subgroup growth has been established, and it is thoroughly presented in [Lub95].

[1] It means that each formula in $\mathcal{L}_{Mac}$ is equivalent to a formula without quantifiers in $\mathbb{Q}_p$. See Section 3.1 for the language $\mathcal{L}_{Mac}$

As a tool to study subgroup growth, we now introduce how to encode the counting sequence as a generating function. To this aim, we consider the following Dirichlet series called the *subgroup zeta function of* G; $s \in \mathbb{C}$

$$\zeta_{(a_n(G))}(s) := \sum_{n \geq 1} a_n(G) \cdot n^{-s} = \sum_{H \leq_f G} (G : H)^{-s}.$$

This analytic function is a direct analogue of *Dedekind zeta function* of a number field encoding the number of ideals of index n in a ring of algebraic integers. Moreover, if we suppose $G = \mathbb{Z}$, then $\{a_n\}_n = \{1\}_n$. Correspondingly, we obtain nothing but the *Riemann zeta function*

$$\zeta_G(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Parametrizing open subgroups. Here we bring in a remarkable example of counting subgroups - the subgroup growth of compact p-adic analytic groups - which vividly portraits the use of model theory of valued fields. Recall first Lazard's purely algebraic characterization of p-adic analytic groups: a compact topological group G is a p-adic analytic group if and only if it has an open uniform pro-p subgroup. [2] du Sautoy, in [dS93], established that $\sum_k a_{p^k}(G) \cdot T^k$ is rational where G is a compact p-adic analytic group.

In this work, du Sautoy applies the extended rationality result from [DvdD88] by describing how to interpret group-theoretic statements in the analytic language of Denef and van den Dries' work. To this end, he introduces new generating sets for open subgroups called *good bases* which are essential to describe analytic structure of uniform pro-p groups. The notion of good basis plays a crucial role in many works studying zeta functions in the framework of model theory as well as this work.

Representation growth. A plausible modification of subgroup growth would be enumerating the finite dimensional irreducible representations of a given group. Let G be a group. We now consider, for $n \geq 1$, the set of n-dimensional irreducible complex representations of G up to isomorphism. [3] Let $r_n(G)$ denote the number of isomorphism classes of complex irreducible n-dimensional representations of G. In a similar way to the subgroup growth, we shall explore the asymptotic behavior of $r_n(G)$ - the *representation growth* of G. For an introductive survey on this subject, see [Klo13].

[2] A pro-p group is uniform if and only if it is finitely generated, torsion-free and powerful.

[3] If G admits additional structure, it is conventional to proceed respectively, e.g. for a topological group G, one should take into account only continuous representations.

For a finite G, it is clear that $r_n(G) < \infty$. In this case, only finitely many of the terms of $\{r_n(G)\}_n$ are non-zero, and they reflect the degrees of irreducible characters of G. The group G is called *representation rigid* or *rigid* for short, if for each n, $r_n(G)$ is finite. In this case, we write another Dirichlet generating series called the *representation zeta function* of G; $s \in \mathbb{C}$ and $\zeta_G(s) = \zeta_{(r_n(G))}(s)$

$$\zeta_G(s) := \sum_{n \geq 1} r_n(G) \cdot n^{-s} = \sum_{\rho} (\dim(\rho))^{-s},$$

where ρ varies over the isomorphism classes described above. We can further modify the above expression if we can establish a bijection between the irreducible characters of G and the isomorphism classes of irreducible representations of G. Let $\mathrm{Irr}(G)$ be the set of irreducible characters of G. Then we obtain

$$\zeta_G(s) := \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s}.$$

Kirillov orbit method. A group G is called *FAb*, or has the *finite abelianization property*, if $H^{ab} = H/[H,H]$ is finite for every subgroup H of finite index in G. If G is finitely generated profinite, this conditions reads as H^{ab} is finite for every open subgroup H of G. Recall that the *derived series* of a finitely generated profinite group G can be given as a series of closed normal subgroups $\{G_i\}_i$ such that

$$G_0 = G \geqslant G_1 \geqslant \ldots \geqslant G_{i+1} = [G_i, G_i] \geqslant \ldots \quad .$$

When G is assumed to be a finitely generated pro-p group, the FAbness property becomes that the factors G_i's of the derived series of G are all open in G.

Proposition 2.0.3. *[[BLMM02], Proposition 2] If G is finitely generated profinite, then $r_n(G) < \infty$ for all n if and only if G is FAb.*

Let G be a FAb compact p-adic analytic group. Considering a uniform pro-p subgroup N of G, we can associate a $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \log(N)$ to G. Then G is FAb if and only if $[\mathcal{L}(G), \mathcal{L}(G)] = \mathcal{L}(G)$. In [JZ06], Jaikin-Zapirain showed the (virtual) rationality of the representation zeta function of a FAb compact p-adic analytic group for $p > 2$. His work is based on Kirillov orbit method, which provides a correspondence between the characters of a uniform pro-p group and the orbits of the co-adjoint action of G. This allows him to "linearize" the problem, and proceed with p-adic integration using the model theoretic work [DvdD88] , as in [dS93].

Uniform rationality - semi-algebraic setting. Regarding the rationality of Poincaré series of equivalence relations, the main issue is that the set of equivalence classes can not always be expressed bv a definable set. One solution to that problem is *elimination of imaginaries*, that can be described as definably associating a point to every equivalence classes, namely the imaginary elements. In [HMR18], Hrushovski, Martin and Rideau showed that the theory of the p-adics $\mathbb{Q}_p$ admits elimination of imaginaries in the geometric language $\mathcal{L}_{\mathcal{G}}$ [4], and it is uniform in p. They also applied it to show the uniform rationality of representation zeta functions $\sum a_{p^n} p^{-ns}$ counting twist isomorphism classes of a finitely generated nilpotent groupl while they proved the rationality of subgroup zeta functions (of various kinds) of such a nilpotent group uniformly in p.

The theory does not always eliminate imaginaries, for example, in the subanalytic language on $\mathbb{Q}_p$ the topic is not yet fully established. To tackle this problem regarding generating power series, in the appendix of [HMR18], Cluckers generalized the rationality results of the main body, for a fixed p, to the analytic setting by using cell decomposition. More precisely, he codes an imaginary element by a definable set whose volume can be computed easily instead of coding it by a point.

In [SZ20], they gave a new proof of the virtual rationality of representation zeta function of FAb compact p-adic analytic groups by applying Cluckers' result. They followed Jaikin-Zapirain's idea to reduce the virtual rationality to the rationality of what we will call *partial zeta series*, see the following section. However, to show that this partial zeta series is rational, they use the theory of projective representations avoiding Kirillov orbit method thanks to the idea from [HMR18] of parametrizing $\mathrm{Irr}(N)$ by certain pairs (H, χ) where N is an open normal uniform subgroup of G, $H \leqslant N$, and $\chi \in \mathrm{Irr}(H)$. They also obtained analogous results for twist zeta functions[5] of compact p-adic analytic groups.

Uniform rationality - subanalytic setting. To generalize the idea of p-adic integration, Cluckers and Loeser provided a theory of motivic functions in the language Denef-Pas $\mathcal{L}_{DP}$ via uniform cell decomposition theorem, [CL08]. This motivic integration theory brings us a new machinery to study the p dependence of the rationality of Poincaré series.

In [Ngu19], Nguyen developed Cluckers' idea in the Appendix of [HMR18] by introducing rational motivic constructible functions, and their motivic integrals to show the p-uniform rationality of Poincaré series associated with definable family of equivalence relations. This can be seen as a generalization of the rationality

[4]They add a sort S_n called *geometric imaginaries*, for each n, for the family of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^n$

[5]They count the irreducible complex representations up to one-dimensional twists as in [HMR18]

result of Hrushovski-Martin-Rideau to the analytic setting (in an expansion of the language Denef-Pas $\mathcal{L}_{DP}$) as well as Cluckers' result since it provides uniformity in p. This result of Nguyen will become central in this work; we will describe how below.

Main theorem and framework. In this work, we will study representation zeta function of FAb compact p-adic analytic groups. As stated in [AKOV13], the key examples of such groups are the special linear groups $\mathrm{SL}_n(A)$ and their principal congruence subgroups $\mathrm{SL}_n^m(A)$, where A is a compact discrete valuation ring of characteristic 0 and residue field characteristic p. In particular, one can consider any open pro-p-subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$. Let G be such a group and let $\zeta_G(s)$ be the corresponding representation zeta function. We say $\zeta_G(s)$ is *virtually rational* in p^{-s} if it is of the following form

$$\sum_{i=1}^{k} n_i^{-s} f_i(p^{-s}),$$

for $n_i \in \mathbb{N}$, $f_i(T) \in \mathbb{Q}(T)$. For a fixed p, in [JZ06] and [SZ20], the virtual rationality of $\zeta_{G_p}(s)$ is reduced to the rationality of the following *partial zeta series*

$$\zeta^{(N_p,K_p,c)}(s) = \sum_{\theta \in \mathrm{Irr}^c_{K_p}(N_p)} \theta(1)^{-s},$$

where N_p is an open normal uniform subgroup of G_p and K_p is a subgroup of G_p containing N_p with Sylow pro-p subgroup P_p, and $\mathrm{Irr}^c_{K_p}(N_p)$ is the set of irreducible characters of N_p giving the cohomology class c in $\mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$ with stabilizer K_p, see Corollary 7.1.4 and Section 6.2 for details.

We will show the uniform rationality of partial zeta series of FAb compact p-adic analytic groups. Broadly speaking, by saying *uniformly rational*, we mean having rational functions over $\mathbb{Q}$ and sets whose cardinalities forming the denominators and the numerators are uniformly definable in p (and in appropriate subgroups K_p and corresponding cohomology classes c). To this end, we first ensure that the family $\{G_p\}_p$ is uniformly definable in p; we define a property ($\diamond$) for a family FAb compact p-adic analytic groups in Section 3.2, and proceed with the families satisfying the property ($\diamond$). Our main result is the following - see Theorem 3.4.3 for the precise formulation:

Main Theorem: The partial zeta functions $\zeta^{(N_p,K_p,c)}(s)$ are uniformly rational for families of FAb compact p-adic analytic groups satisfying the property ($\diamond$) for sufficiently large p.

To see the uniform rationality of this partial zeta series, we show that enumerating characters in $\mathrm{Irr}^{c}_{K_p}(N_p)$ is corresponding to enumerating the classes of a uniformly definable family of equivalence relations on a uniformly definable subset of $\mathbb{Q}_p^m$ for some m. Following this, we conclude the uniform rationality by the result of Nguyen given in previous section.

We now detail the process of obtaining such a correspondence. One of the key ingredients is to obtain that a character triple (P_p, N_p, θ) can be replaced by a character triple $(N_p, N_p \cap H, \chi)$, where H is an open subgroup of G_p such that $P_p = HN_p$, and χ is a linear character as we can recover the cohomology class in $\mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$ related with (P_p, N_p, θ) by χ, following [SZ20]. Another key idea is to use the fact that any irreducible character of a finite p-group is induced from a linear character of a subgroup. We parametrize irreducible characters of N_p fixed by K_p by pairs (H, χ) modulo a uniformly definable equivalence relation;

$$(H, \chi) \rightsquigarrow \mathrm{Ind}_{N_p \cap H}^{N_p}(\chi).$$

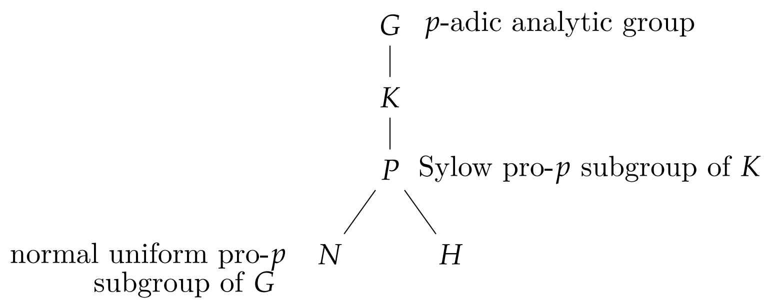
Following this, we introduce a function $\mathcal{C}$ from the set of pairs (H, χ) to the cohomology group $\mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$, which parametrizes $\mathrm{Irr}^{c}_{K_p}(N_p)$ by its fibers modulo the uniformly definable equivalence relation mentioned above.

We conclude this section with a notational remark to avoid any confusion. In the construction of equivalence classes mentioned above, we will consider a family of FAb compact p-adic analytic groups $\{G_p\}_p$ and a subfamily of normal uniform pro-p subgroups $\{N_p\}_p$. In the view of model theory of valued fields, it is beneficial to state that we work in an analytic expansion of the Denef-Pas language as suggested in [Ngu19]. We further expand this language with some constant symbols as this work progresses. For instance, in order to obtain such normal uniform pro-p subgroups in a uniformly definable way, we add constant symbols $a_1 \ldots, a_k$, call $\bar{a}$ and achieve uniformity in p and $\bar{a}$. The natural indexing would be $N_{(p,\bar{a})}$; nevertheless, we omit constants for the sake of notational simplicity.

Once we get a uniformly definable family of normal uniform pro-p subgroups $\{N_p\}_p$ of $\{G_p\}_p$, we then work with subgroups K_p of G_p containing N_p while dealing with Sylow pro-p subgroups P_p of K_p. In addition, we study the elements of

$$\mathcal{H}(P_p) = \{H_p \leq P_p : H_p \text{ open in } P_p, P_p = H_p N_p\}.$$

To simplify the notation, we name all these sets as G, N, K, P, H unless we aim to treat uniform definability. The relations explained above can be summarized by the following diagram;



Chapter 3

Background

This chapter introduces the required tools from model theory of valued fields and the theory of analytic pro-p groups and, at the same time, presents our results, which are core to this work. In the first section, we first deliver a quick summary of how the analytic language of p-adic numbers evolved, providing descriptions of noteworthy languages. Then we introduce the language we use here. A recap about uniform pro-p groups follows it, highlighting their key features. We then discuss its model theoretic properties while expanding our language to describe uniform pro-p subgroups of FAb compact p-adic analytic groups in a uniformly definable way. We conclude this chapter with a section devoted to explaining uniform rationality; we also fix our framework and present our main theorem.

3.1 The analytic language of p-adic numbers

Recall first that a *valued field* K is a field with a *valuation map* $v : K \to \Gamma \bigcup \{\infty\}$ where $(\Gamma; +, 0, <)$ [1] is an ordered abelian group such that

(i) $v(ab) = v(a) + v(b)$

(ii) $v(a + b) \geqslant \min(v(a), v(b))$

(iii) $v(a) = \infty \Leftrightarrow a = 0$

One can study the model theory of valued fields via different languages. The most basic language to examine fields $\mathcal{L}_{ring}$ [2] can be combined with a divisibility predicate $\mathcal{D} : K \times K \to K$ defined as follows:

[1] Note that we will assume that v is surjective; hence the *value group* is Γ.

[2] $\mathcal{L}_{ring} = \{+, -, \cdot, 0, 1\}$ where $-$ is a unary function symbol.

$$(x,y) \mapsto \begin{cases} x \mid y, & \text{if } \ v(x) \geq v(y) \ \text{ and } \ y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then the *valuation ring* $\mathcal{O}_K = \{a \in K : v(a) \geq 0\}$ of K, as well as its unique maximal ideal $\mathcal{M}_K = \{a \in K : v(a) > 0\}$ are definable in this extended language. Also, the *value group* Γ is interpretable in $(K; +, -, \cdot, 0, 1, \mathcal{D})$ as Γ is isomorphic to $K^*/\mathcal{O}_K{}^*$ while the order is determined by $\mathcal{D}$. Moreover the *residue field* $k = \mathcal{O}_K/\mathcal{M}_K$ and the *residue map* $\mathcal{O}_K \to k$ are interpretable in this language.

Instead of the divisibility predicate $\mathcal{D}$, we can add unary predicates P_n to $\mathcal{L}_{ring}$, which is the set of n^{th} powers in K^*. In other words, for each $n > 1$, P_n is interpreted by

$$P_n(x) : \exists y \ \ y^n = x \wedge x \neq 0.$$

This extended language is called the Macintyre's language $\mathcal{L}_{Mac}$, and p-adic fields eliminate quantifiers in $\mathcal{L}_{Mac}$, [Mac76].

We now enhance the above languages by an *angular component* map. On a given valued field K, one could define the map $\mathsf{ac} : K^* \longrightarrow k^*$ which is a group homomorphism (hence multiplicative) extended by $\mathsf{ac}(0) = 0$, and agrees with the residue map on $\mathcal{O}_K^*$. If $K = \mathbb{Q}_p$, we define the angular component map by $\mathsf{ac}(p) = 1$, so that $\mathsf{ac}(\sum_{i \geq n} a_i p^i) = a_n$ if $a_n \neq 0$. Similarly, if $K = k((t))$, we define ac by $\mathsf{ac}(0) = 0$ and $\mathsf{ac}(t) = 1$. Note that ac is definable in the valued field $\mathbb{Q}_p$ as it is equal to 1 on the $(p-1)^{th}$ powers.

Another natural approach to study valued fields would be thinking of them as three sorted structures: a *valued field* sort VF, a *value group* sort VG and a *residue field* sort RF as the work of Ax and Kochen suggests. This yields to the *language $\mathcal{L}_{DP}$ of Denef-Pas*. More precisely, $\mathcal{L}_{DP}$ is a three sorted language in which we have two copies of $\mathcal{L}_{ring}$ for the valued field sort and the residue field sort and the language $\mathcal{L}_{oag}$ of ordered abelian groups for the value group sort combined with the valuation map $v : VF \to VG$ and the angular component map $\mathsf{ac} : VF \to RF$. So a Denef-Pas language can be given in the following form

$$(\mathcal{L}_{ring}, \mathcal{L}_{ring}, \mathcal{L}_{oag}, v, \mathsf{ac}).$$

3.1.1 An Analytic Expansion of $\mathcal{L}_{DP}$

Throughout this thesis, we consider an analytic expansion of $\mathcal{L}_{DP}$ which enables us to apply the uniform rationality result given in [Ngu19]. Prior to introducing the formalism, we provide some background on the notion of a valued field with an analytic structure via the following example $\mathbb{Z}[\![t]\!]$.

Consider the ring of formal power series $\mathbb{Z}[\![t]\!]$ over $\mathbb{Z}$ and equip it with the t-adic topology. Then $a(t) \mapsto a(p)$ gives a continuous homomorphism between $\mathbb{Z}[\![t]\!]$ and the ring of p-adic integers $\mathbb{Z}_p$ with the kernel $(t-p)\mathbb{Z}[\![t]\!]$.

Consequently, for each power series $\sum a_i \cdot \mathbf{X}^i$ over $\mathbb{Z}[\![t]\!]$ whose coefficients converging to zero t-adically as $\mid t \mid \rightarrow \infty$ defines an n-ary function on $\mathbb{Z}_p^m$. Precisely, we have m commuting indeterminates constituting $\mathbf{X} = (X_1, \ldots, X_m)$ with $a_i \in \mathbb{Z}[\![t]\!]$ such that $a_i(t) \rightarrow 0$ as $i_1 + \ldots + i_m \rightarrow \infty$ and the power series $\sum a_i \cdot \mathbf{X}^i$ gives rise to a function $F : \mathbb{Z}_p^m \rightarrow \mathbb{Z}_p$

$$\mathbf{X} = (X_1, \ldots, X_m) \mapsto \sum_{i \in \mathbb{N}^m} a_i(p) X_1^{i_1} \cdot \ldots \cdot X_m^{i_m} \,.$$

Note that the above power series are strictly convergent power series over $\mathbb{Z}[\![t]\!]$. They form a ring called *the ring of restricted power series in X over $\mathbb{Z}[\![t]\!]$* and it is the t-adic completion of the polynomial ring $\mathbb{Z}[\![t]\!][\mathbf{X}]$.

We provide the following definitions from [Ngu19] axiomatizing the analytic properties of $\mathbb{Z}[\![t]\!]$ and introducing an expansion of Denef-Pas language accordingly. Consider a commutative Noetherian unital ring A and fix a proper ideal I of A such that A is complete for the I-adic topology. (In the above example, we have $A = \mathbb{Z}[\![t]\!]$ and $I = t\mathbb{Z}[\![t]\!]$) We write A_m for the I-adic completion of the polynomial ring $A[x_1, \ldots, x_m]$ for each $m \in \mathbb{N}$ and denote the family $(A_m)_{m \in \mathbb{N}}$ by $\mathcal{A}$.

Definition 3.1.1. [[Ngu19], Definition 1.2.8] For a valued field K, an *analytic $\mathcal{A}$-structure* on K is defined as a collection of ring homomorphisms

$$\sigma_m : A_m \rightarrow \{f : \mathcal{O}_K^m \rightarrow \mathcal{O}_K\}$$

such that

1. $I \subset \sigma_0^{-1}(\mathcal{M}_K)$
2. $\sigma_m(x_i)$: the ith coordinate function on $\mathcal{O}_K^m$
3. σ_m extends to σ_{m+1} in the most obvious way as we identify the functions on $\mathcal{O}_K^m$ with the functions on $\mathcal{O}_K^{m+1}$ independent of the last coordinate

Definition 3.1.2. [[Ngu19], Definition 1.2.9] The *$\mathcal{A}$-analytic language $\mathcal{L}_{\mathcal{A}}$* is defined as $\mathcal{L}_{DP} \cup (A_m)_{m \in \mathbb{N}}$, where elements of A_m are function symbols. An analytic $\mathcal{A}$- structure on K turns K into an $\mathcal{L}_{\mathcal{A}}$-structure.

Following [Ngu19], we shall use the language $\mathcal{L}_{\mathbb{Z}[\![t]\!]} = \mathcal{L}_{DP} \cup (A_m)_{m \in \mathbb{N}}$ where $A = \mathbb{Z}[\![t]\!]$. In this case, we have $I = t\mathbb{Z}[\![t]\!]$ and the t-adic completion of the polynomial rings $\mathbb{Z}[\![t]\!][X_1, \ldots, X_m]$ as A_m, for each m.

From now on, we are concerned about an $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$-structure with underlying sets $\mathbb{Q}_p$ as the valued field sort, $\mathbb{Z}$ as the value group sort and $\mathbb{F}_p$ as the residue field sort. We have all constants, functions and relations of ring language for the valued field and the residue field, and all constants, functions and relations of ordered abelian group language for the value group sort. In addition, we interpret v as the valuation map on $\mathbb{Q}_p$ and $ac : \mathbb{Q}_p \to \mathbb{F}_p$ as $ac(x) = xp^{-v(x)} \mod p$ with $0 \mapsto 0$.

To understand the analytic $\mathcal{A}$-structure on $\mathbb{Q}_p$, it is enough to see that the ring of strictly convergent power series over $\mathbb{Q}_p$ is a homomorphic image of the ring of strictly convergent power series over $\mathbb{Z}[\![t]\!]$ by extending the homomorphism $\varphi : \mathbb{Z}[\![t]\!] \to \mathbb{Q}_p$, $t \mapsto p$. Accordingly we interpret the analytic structure on $\mathbb{Q}_p$ via φ:

- A is interpreted as $\varphi(\mathbb{Z}[\![t]\!]) = \mathbb{Z}_p$ (hence I as $p\mathbb{Z}_p$)
- for each m, A_m is interpreted as the p-adic completion of the polynomial ring $\mathbb{Z}_p[X_1, \ldots, X_m] = \mathbb{Z}_p[\mathbf{X}]$ which is the ring of formal series with coefficients in $\mathbb{Z}_p$, namely $\mathbb{Z}_p[\![\mathbf{X}]\!]$.
- Consequently, we interpret, for each element $\sum_i a_i X^i$ of A_m, the corresponding $\mathcal{O}_K$-valued function symbols f as the restricted analytic functions $f : \mathbb{Z}_p^m \to \mathbb{Z}_p$ given by the corresponding power series

$$\mathbf{X} \mapsto \sum_{i \in \mathbb{N}^m} a_i \cdot X_1^{i_1} \cdot \ldots \cdot X_m^{i_m}.$$

We therefore work in the structure $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}_{\mathbb{Z}[\![t]\!]})$. In Section 3.3, we expand the language $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$ by some constant symbols to the language $\mathcal{L}'$ and we add another constant symbols in Section 3.4, and call the language $\mathcal{L}''$ for our uniformity purposes and notational convenience.

3.2 Uniform pro-p groups

Uniform pro-p groups play a central role in this work; identifying a uniform pro-p subgroup with $\mathbb{Z}_p^d$, for some d, allows us defining du Sautoy's good basis, see Chapter 4. This will be crucial to treat open subgroups uniformly definable in p. To this end, we present here brief yet informative summary of relevant parts of [DdSMS99], [RZ10], [Wil98].

3.2.1 An Overview

Recall first that a *profinite group* is an inverse limit of inverse systems of finite groups. Considering finite groups as topological groups with the discrete topology, one can obtain the inverse limit topology on profinite groups; hence profinite groups can be defined as compact, Hausdorff, totally disconnected, topological groups.

A topological group G is called *topologically generated* by a subset X of G if the subgroup generated by X is dense in G. Accordingly, G is called *finitely generated* if X is finite, and we denote the minimal number of elements of a topological generating by $d(G)$.

For a profinite group G, we recall some basic topological properties below;

- Any open subgroup of G has finite index in G, and contains an open normal subgroup of G. In addition, open subgroups of G are closed. A closed subgroup of G is open if and only if it has finite index in G.
- For a subset $X \subseteq G$, its closure $\overline{X}$ is given by $\bigcap_{N \trianglelefteq_o G} XN$.
- For a closed normal subgroup N of G, G/N is profinite, and the quotient map $G \to G/N$ is an open and closed continuous homomorphism.
- Let H be a (normal) closed subgroup of G. Then H is the intersection of all open (normal) subgroups of G containing H. A subset $X \subseteq H$ (topologically) generates H if and only if XN/N generates HN/N for all open normal subgroups $N \trianglelefteq_o G$.
- If G is finitely generated, then G has only finitely many open subgroups of given finite index, and every open subgroup of G is finitely generated,

Let G be a profinite group with a closed subgroup H. Then one can generalize the index notion by the fact that H is the intersection of the open subgroups of G containing H. To this end, recall first that the notion of *Steinitz number* which is a formal infinite product $n = \prod_{p \text{ prime}} p^{n(p)}$ where $n(p)$ is a non-negative integer or infinity. Then the least common multiple of a given family $\{n_i\}_{i \in I}$ is defined as

$$lcm\{n_i\}_{i \in I} = \prod_{p \text{ prime}} p^{n(p)} \quad \text{where} \quad n(p) = \sup_{i \in I}\{n_i(p)\}.$$

And the *index of H in G* denoted by $(G : H)$ is defined to be the least common multiple of the indices of the open subgroups of G containing H. Consequently, we obtain Lagrange's theorem for profinite groups. Let H, K be subgroups of G such that $K \leq H \leq G$. Then

$$(G : K) = (G : H)(H : K).$$

A p-Sylow subgroup P of G is then defined as a (possibly infinite) subgroup satisfying that $(P:1)$ is a p-power, and $(G:P)$ is coprime to p.

For a fixed prime p, a profinite group G is called *pro-p group* if every open normal subgroup of G has index equal to some power of p. Then p-Sylow subgroups are maximal pro-p subgroups of G. Following this, we will call them *Sylow pro-p subgroups* of G. The Sylow theorem generalizes to profinite groups via inverse limits; for a profinite group G and a prime p, $i)$ G has a Sylow pro-p subgroup, $ii)$ Any two Sylow pro-p subgroups of G are conjugate.

Recall now that the *Frattini subgroup* $\Phi(G)$ of a profinite group G, which is the intersection of all maximal open subgroups of G. It is a topologically characteristic subgroup of G, and, as in the case of finite groups, it consists of all non-generators [3].

For a pro-p group G, one can define the *lower p-series* in G as follows:

$$P_{i+1}(G) = \overline{P_i(G)^p[P_i(G), G]},$$

where $P_1(G) = G$. Note that if G is finitely generated pro-p group, every subgroup of finite index is open in G, and the lower p-series is well-behaved consisting of open subgroups.

A pro-p group G is called *powerful* if p is odd and $G/\overline{G^p}$ is abelian, or if $p = 2$ and $G/\overline{G^4}$ is abelian. If G is assumed to be powerful finitely generated pro-p group, then G^p becomes the set of pth powers, and $G^p = \Phi(G)$.

A *uniform pro-p group* N, or uniform group for short, is a pro-p group which is finitely generated, powerful and we have $|P_i(N) : P_{i+1}(N)| = |N : P_2(N)|$ for all i. The *dimension* of a uniform pro-p group N is defined to be the size of a minimal topological generating set, and denoted by $d(N)$. If $\{a_1, \ldots, a_d\}$ is a minimal (topological) generating set of N, we have the following identities for the above filtration

$$P_{i+1}(N) = P_i(N)^p = \{x^{p^{i-1}} : x \in N\} = \langle a_1^{p^{i-1}}, \ldots, a_d^{p^{i-1}} \rangle.$$

Note that a subgroup H of N is open if and only if it contains $P_m(N)$ for some m. Furthermore, the p^{th} power map $x \mapsto x^p$ induces an isomorphism

$$P_i(N)/P_{i+1}(N) \to P_{i+1}(N)/P_{i+2}(N).$$

Thus, $P_i(N)/P_{i+1}(N)$ is an $\mathbb{F}_p$-vector space and we have $d = \dim_{\mathbb{F}_p}(N/P_2(N))$ where d is the cardinality of a minimal topological generating set for N. We will continue writing $N_i = P_i(N)$ for $i \in \mathbb{N}$.

[3] An element $g \in G$ is called *non-generator* if $G = \langle X, g \rangle$ implies $G = \langle X \rangle$ for any $X \subseteq G$

3.2.2 $\mathbb{Z}_p$-coordinates: a multiplicative coordinate systems

In this section, we will define the p-adic exponentiation and obtain a homeomorphism between $\mathbb{Z}_p^d$ and a uniform pro-p group of dimension d.

Definition 3.2.1. Suppose G is a pro-p group, $g \in G$ and $\lambda \in \mathbb{Z}_p$. The *p-adic exponentiation* is defined by

$$g^{\lambda} = \lim_{n \to \infty} g^{a_n} \quad \text{where} \quad \lim_{n \to \infty} a_n = \lambda, a_n \in \mathbb{Z}.$$

Let $\{a_n\}_n$ and $\{b_n\}_n$ be two sequences from $\mathbb{Z}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ in $\mathbb{Z}_p$. Then, for any $g \in G$, the convergent sequences $\{g^{a_n}\}_n$ and $\{g^{b_n}\}_n$ have the same limit in G; hence the p-adic exponentiation is well-defined. Moreover, it behaves well so that the above definition gives rise a natural action of the ring $\mathbb{Z}_p$ on G and consequently turns G into a (possibly non-commutative) topological $\mathbb{Z}_p$-module.

Proposition 3.2.2. *Let G be a pro-p group, $g, h \in G$ and $\lambda, \mu \in \mathbb{Z}_p$,*

(i) $g^{\lambda+\mu} = g^{\lambda} g^{\mu}$ *and* $g^{\lambda\mu} = (g^{\lambda})^{\mu}$.

(ii) If $gh = hg$, *then* $(gh)^{\lambda} = g^{\lambda} h^{\lambda}$.

(iii) The map $\mathbb{Z}_p \to G$, $v \mapsto g^{v}$ *is a continuous homomorphism whose image is the closure of* $\langle g \rangle$ *in* G.

We now introduce the multiplicative coordinate systems of a given uniform pro-p group. First recall that finitely generated powerful pro-p groups can be given as the product of some of its pro-cyclic subgroups.

Proposition 3.2.3. *[[DdSMS99], Proposition 3.7] Let* $G = \overline{\langle a_1, \ldots, a_d \rangle}$ *be a powerful pro-p group, then*

$$G = \overline{\langle a_1 \rangle} \ldots \overline{\langle a_d \rangle}.$$

Accordingly, for any $x \in G = \overline{\langle a_1 \rangle} \ldots \overline{\langle a_d \rangle}$, there are $\lambda_1, \ldots, \lambda_d$ such that $x = a_1^{\lambda_1} \ldots a_d^{\lambda_d}$. Furthermore, if G is uniform pro-p, we can obtain a homeomorphism between $\mathbb{Z}_p^d$ and N.

Theorem 3.2.4. *[[DdSMS99], Theorem. 4.9] Suppose N is a uniform pro-p group, and let* $\{a_1, \ldots, a_d\}$ *be a minimal topological generating set for N. The following map is a homeomorphism*

$$\mathbb{Z}_p^d \to N$$
$$(\lambda_1, \ldots, \lambda_d) \mapsto a_1^{\lambda_1} \ldots a_d^{\lambda_d}$$

As a corollary of Theorem 3.2.4, for each $x \in N$, there are unique $\lambda_1, \dots, \lambda_d \in \mathbb{Z}_p$ such that $x = a_1^{\lambda_1} \dots a_d^{\lambda_d}$. We call this tuple $(\lambda_1, \dots, \lambda_d)$ $\mathbb{Z}_p$-*coordinates* of x, and we will denote $x = a_1^{\lambda_1} \dots a_d^{\lambda_d}$ by $x(\lambda)$ to emphasize this system of coordinates.

Definition 3.2.5. Let G be a pro-p group. We define $\omega : N \to \mathbb{N} \cup \{\infty\}$ by $\omega(g) = n$ if $g \in P_n(N) \setminus P_{n+1}(N)$ and put $w(1) = \infty$.

One can immediately see the following holds $\omega(gh) \geqslant \min\{\omega(g), \omega(h)\}$ and $\omega(gh) = \omega(g)$ if $\omega(g) < \omega(h)$. Moreover, ω is compatible with the usual valuation v on $\mathbb{Z}_p$ if G is a uniform pro-p group.

Proposition 3.2.6. *Let N be a uniform pro-p group and the set $\{a_1, \dots, a_d\}$ be a topological generating set for N. Then the following holds*

(i) For $x \in N$ and $\lambda \in \mathbb{Z}_p$, $\omega(x^\lambda) = \omega(x) + v(\lambda)$.

(ii) [[dS93], Theorem 1.18 (iv)] Setting $x = x(\lambda)$, we have

$$\omega(x) = \min\{v(\lambda_i) + 1 : i \in \{1, \dots, d\}\}.$$

Proof. (i) We begin with assuming that $x \neq 1, \lambda = 0$; otherwise it is trivial. Let $\alpha = v(\lambda)$ and $n = \omega(x)$; we therefore aim to see that $x^\lambda \in N_{n+\alpha} \setminus N_{n+\alpha+1}$.

As $\mathbb{Z}_p$ is a Euclidean domain with the p-adic norm, there exist q, r such that

$$\lambda = p^\alpha.q + r,$$

$0 \leq q < p$ and $\alpha < v(r)$. To see that $x^\lambda \equiv x^{p^\alpha.q} \mod N_{n+\alpha+1}$, it is enough to note that $x^{p^{\alpha+1}} \in N_{n+\alpha+1}$ and $v(r) > \alpha$.

Then the statement reduces to show that $x^{p^\alpha q} \in N_{n+\alpha} \setminus N_{n+\alpha+1}$. It follows from the fact that the $x \mapsto x^{p^\alpha}$ induces an isomorphism between $N_n/N_{n+1} \to N_{n+\alpha}/N_{n+\alpha+1}$. Since $x \in N_n \setminus N_{n+1}$, $x^{p^\alpha} \in N_{n+\alpha} \setminus N_{n+\alpha+1}$. Therefore $x^{p^\alpha q} \in N_{n+\alpha} \setminus N_{n+\alpha+1}$ as $q \nmid p^d$ where p^d is the number of elements of N_n/N_{n+1} for all $n \geq 1$; in particular it equals to $(N_{n+\alpha} : N_{n+\alpha+1})$.

(ii) We first obtain that $\omega(a_i) = 1$, for all $i \in \{1, \dots, d\}$. This is a direct consequence of that N_2 (the Frattini subgroup of N) consists of all non-generators of N; thus $a_i \notin N_2$ for any i.

Consider $x = x(\lambda) = a_1^{\lambda_1} \dots a_d^{\lambda_d}$, by the previous argument, we have

$$\omega(a_i^{\lambda_i}) = \omega(a_i) + v(\lambda_i) \quad = 1 + v(\lambda_i)$$

Set $m = \min\{v(\lambda_i) + 1 : i \in \{1, \dots, d\}$. The rest is straightforward; we will see $x(\lambda) \in N_m \setminus N_{m+1}$. It is obvious that $x(\lambda) \in N_m$, since $a_i^{\lambda_i} \in N_m$.

To see the latter, we will examine $a_i^{\lambda_i} \mod N_{m+1}$, say $a_i^{\lambda_i} \equiv a_i^{p^{m-1}.q_i}$. The minimality of m dictates that there exists an $i \in \{1,\ldots,d\}$ such that $q_i \neq 0$. Then $\prod_{i=1}^{d} a_i^{p^{m-1}.q_i}$ gives a non-trivial linear combination of $a_i^{p^{m-1}}$, which constitutes a basis for the vector space N_m/N_{m+1}. In particular, $\prod_{i=1}^{d} a_i^{p^{m-1}.q_i}$ is not trivial in N_m/N_{m+1}; hence $x(\lambda) \notin N_{m+1}$. □

3.2.3 Associated additive structure

We now define an abelian group structure on a uniform pro-p group N of dimension d, and report that it is isomorphic to $\mathbb{Z}_p^d$.

Lemma 3.2.7. *[[DdSMS99], Lemma 4.10] The map $N \to P_{n+1}(N), x \mapsto x^{p^n}$ is a homeomorphism*

(i) restricting to a bijection $P_k(N) \to P_{k+n}(N)$,

(ii) inducing a bijection $P_k(N)/P_{k+m}(N) \to P_{n+k}(N)/P_{n+k+m}(N)$

Then one can easily see that every element $x \in N_n$ has a unique p^nth root which we will denote by $x^{p^{-n}}$. Hence we can define an additive structure on N via the addition defined by

$$x + y = \lim_{n\to\infty} (x^{p^n} \cdot y^{p^n})^{p^{-n}}$$

Note that the sequence $(x^{p^n} \cdot y^{p^n})^{p^{-n}}$ is a Cauchy sequence; hence the above limit exists.

Proposition 3.2.8. *[[dS93], Section 4.3] $(N,+)$ constitutes an abelian group. For all $x, y \in N$, we have the following;*

(i) If $xy = yx$, then $x + y = xy$.

(ii) $mx = x^m$ for all integer m.

(iii) If $x, y \in N_m$, then $x + y \equiv xy \mod N_{m+1}$.

Remark 3.2.9. *The above proposition indicates that inverses with respect to $+$ are the same as multiplicative inverses and p-adic exponentiation becomes scalar multiplication. In other words, for $x \in N$ and $\lambda \in \mathbb{Z}_p$, we have $x^\lambda = \lambda x$.*

Theorem 3.2.10. *[[dS93], Proposition 4.16, Theorem 4.17] Let N be a uniform pro-p group of dimension d, and let $\{a_1, \ldots, a_d\}$ be a minimal (topological) generating set for N. Then* *(i)* N_i *is a subgroup of* $(N, +)$*, for each* i*. Moreover, the terms of lower* p*-series of* $(N, +)$ *is exactly the terms of lower* p*-series of* N *with respect to the original multiplication.*

(ii) $(N, +)$ *is a uniform pro-*p *group of dimension* d *with the original topology of* N*.*

(iii) $(N, +)$ *is a free* $\mathbb{Z}_p$*-module on the basis* $\{a_1, \ldots, a_d\}$*, and there is an isomorphism of* $\mathbb{Z}_p$*-modules* $\psi : N \to \mathbb{Z}_p^d$

$$\lambda_1 a_1 + \ldots + \lambda_d a_d \mapsto (\lambda_1, \ldots, \lambda_d)$$

3.2.4 Analytic structure on uniform pro-p groups

The p-adic analytic groups are the best known class of pro-p groups. This section primarily concerns the historic result of Lazard characterizing p-adic analytic groups in a purely algebraic way. We will also explain how to obtain a p-adic analytic structure on a given uniform pro-p group.

Recall first that *a basis for the topology on* $\mathbb{Z}_p^r$ induced by the p-adic metric, for a fixed r, is given by the following balls

$$\begin{aligned} B(y, p^{-h}) =& \{z \in \mathbb{Z}_p^r : \mid z_i - y_i \mid \leqslant p^{-h}, i \in \{1, \ldots, r\}\} \\ =& \{y + p^h x : x \in \mathbb{Z}_p^r\} \end{aligned}$$

where $y \in \mathbb{Z}_p^r$ and $h \in \mathbb{N}$.

Definition 3.2.11. Suppose V is a non-empty subset of $\mathbb{Z}_p^r$ and let

$$f = (f_1, \ldots, f_s) : V \subseteq \mathbb{Z}_p^r \to \mathbb{Z}_p^s$$

be a function on $\mathbb{Z}_p^r$.

(i) f is *analytic at* $y \in V$ if there exist $h \in \mathbb{N}$ such that $B(y, p^{-h}) \subseteq V$ and formal power series $F_i(X)$ in $\mathbb{Q}_p[\![X]\!]$ satisfying

$$f_i(y + p^h x) = F_i(x)$$

for all $x \in \mathbb{Z}_p^r$.

(ii) The function f is *analytic* on V if it is analytic at each $y \in V$.

Lemma 3.2.12. *1. Let* $f : U \subseteq \mathbb{Z}_p^r \to V \subseteq \mathbb{Z}_p^s$ *and* $g : V \subseteq \mathbb{Z}_p^s \to W \subseteq \mathbb{Z}_p^t$ *be two analytic functions, where* U, V *and* W *and open subsets. Then* $g \circ f$ *is analytic on* U*.*

2. *Suppose that* $F(X) = \sum_{i \in \mathbb{N}^r} a_i \cdot X_1^{i_1} \cdot \ldots \cdot X_r^{i_r}$ *converges on an open subset* $U \subseteq \mathbb{Z}_p^r$. *Then there exists* $k \in \mathbb{N}$ *such that* $p^{k(i_1+\ldots+i_r)} a_i \in \mathbb{Z}_p$.

Recall the following topological definition

Definition 3.2.13. 1. Let X be a topological space. A *chart* on X is a triple (U, φ, n) where U is a non-empty open subset of X and $\varphi : U \to \mathbb{Z}_p^n$ is a homeomorphism onto an open subset of $\mathbb{Z}_p^n$.

2. Two charts (U, φ, n) and (V, ψ, m) on X are *compatible* if $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are analytic functions on $\psi(U \cap V)$ and $\varphi(U \cap V)$ respectively.

3. An *atlas* on X is a family $\{(U_i, \varphi_i, n_i)\}$ consisting of pairwise compatible charts satisfying $X = \bigcup_{i \in I} U_i$. Note that two atlases A and B on X are *compatible* if every chart in A is compatible with every chart in B; and this defines an equivalence relation on the set of atlases on X.

4. A *p-adic analytic manifold* structure on a topological space X is an equivalence class of compatible atlases on X. Accordingly, a function $f : X \to Y$ between two p-adic analytic manifolds is *analytic* if for every pair of charts (U, φ, n) on X and (V, ψ, m) on Y, we have the following

   (a) $f^{-1}(V)$ is open in X,

   (b) $\psi \circ f \circ \varphi^{-1}$ is analytic on $\varphi(U \cap f^{-1}(V))$.

Remark 3.2.14. *An analytic function between two* p*-adic analytic manifolds is continuous. Moreover, the product of two* p*-adic analytic manifolds carries a* p*-adic analytic manifold structure.*

Definition 3.2.15. A *p-adic analytic group* G is a topological group which carries a p-adic analytic manifold structure such that group operation and inversion

$$\begin{array}{ll} G \times G \to G & G \to G \\ (x, y) \mapsto x \cdot y & x \mapsto x^{-1} \end{array}$$

are analytic functions.

Definition 3.2.16. [[DdSMS99], Theorem 8.36, Definition 8.37] For a p-adic analytic group G, there exists a (unique) non-negative integer n which is called the *dimension* of G satisfying the following:

- every chart belonging to an atlas defining the manifold structure on G has dimension n,
- every open uniform pro-p subgroup of G has finite dimension n

Let N be a uniform pro-p group of dimension d, generated topologically by $\{a_1, \ldots, a_d\}$. Recall the homeomorphism

$$\phi : \mathbb{Z}_p^d \to N$$
$$(\lambda_1, \ldots, \lambda_d) \mapsto a_1^{\lambda_1} \ldots a_d^{\lambda_d}$$

defined in Theorem 3.2.4. This homeomorphism gives us a global atlas (N, ϕ, d) on N; hence we may consider N as a compact p-adic analytic manifold of dimension d.

Theorem 3.2.17. *[[DdSMS99], Theorem 8.32] Let G be a topological group. Then G has the structure of a p-adic analytic group if and only if G contains an open subgroup that is a uniform pro-p group.*

3.3 Model theory of uniform groups

Now we associate a two-sorted language to a normal pro-p subgroup N of a compact p-adic analytic group G following [dS93] in order to study the group theoretic properties.

Definition 3.3.1. The language $\mathcal{L}_N$ has two sorts, namely the group sort M_1 and M_2 with the following

(i) all function symbols of the group language on the sort M_1

(ii) a unary function symbol, for each $g \in G$, φ_g on the sort M_1

(iii) a binary relation symbol $x \mid y$ on the sort M_1

(iv) a binary function symbol: $x^\lambda : M_1 \times M_2 \to M_1$

We see a normal pro-p subgroup N of a compact p-adic analytic group G as $\mathcal{L}_N$-structure $\mathcal{M}_N$ by having the domain N for the first sort M_1 and $\mathbb{Z}_p$ for the second sort M_2. Hence we obtain all the functions of the group language in N with the interpretation of φ_g as the conjugation by g, for each $g \in G$,

$$\varphi_g : N \to N, x \mapsto gxg^{-1}.$$

To see the interpretation of the binary relation symbol $x \mid y$, recall the function ω given in Definition 3.2.5. Accordingly, we interpret $x \mid y$ as $\omega(x) \geq \omega(y)$. In addition, the binary function symbol: x^λ is interpreted as the p-adic exponentiation as in Definition 3.2.1.

In case N is a normal uniform pro-p subgroup of a compact p-adic analytic group G, we can interpret $(N, \mathcal{L}_N)$ definably in $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}_{\mathbb{Z}[\![t]\!]})$ by passing to the $\mathbb{Z}_p$-coordinates. To this aim, fix a minimal topological generating set $\{a_1, \ldots, a_d\}$ for N. Recall that, for each $x \in N$, there are unique $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}_p$ such that $x = a_1^{\lambda_1} \ldots a_d^{\lambda_d} = x(\lambda)$. In this way, we obtain constants (the elements of N) as the tuples in $\mathbb{Z}_p$.

Theorem 3.3.2. *[[dS93], Theorem 1.18, Lemma 1.19]*

1. *The function $f : \mathbb{Z}_p^d \times \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ defined by $x(\lambda)(x(\mu))^{-1} = x(f(\lambda, \mu))$ is analytic.*
2. *For $\phi \in Aut(N)$, the function $\Phi : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ defined by $\phi(x(\lambda)) = x(\Phi(\lambda))$ is analytic.*
3. *The function $f : \mathbb{Z}_p^d \times \mathbb{Z}_p \to \mathbb{Z}_p^d$ defined by $x(\lambda)^\mu = x(f(\lambda, \mu))$ is analytic.*

Recall that $\mathbb{Z}_p$-coordinates provide a system of coordinates for N, then by (1) above, the group operation is $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$-definable if we identify N with $\mathbb{Z}_p^d$ using $\mathbb{Z}_p$-coordinates. The above theorem also shows that conjugation by the elements of G and the p-adic exponentiation can be given by analytic functions; hence they are all interpretable in the analytic language. $(N, \mathcal{L}_N)$ is therefore definably interpretable in $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}_{\mathbb{Z}[\![t]\!]})$.

Interpreting uniform pro-p subgroups of p-adic analytic groups has been influential not only for zeta functions but also for the classification of full profinite NIP groups. However, this is not enough for us; we want to be able to interpret the group structure of N in this structure uniformly in p. This means that we need to establish a systematic (p-independent) way to obtain a subfamily $\{N_p\}_p$ of uniform pro-p subgroups of a given family of compact p-adic analytic groups $\{G_p\}_p$. To this end, we introduce the following criteria on the family of compact p-adic analytic groups $\{G_p\}_p$.

Definition 3.3.3. We say that a family of FAb compact p-adic analytic groups $\{G_p\}_p$ indexed by the primes $p > 2$ has the property $(\diamond)$ if there exists an $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$-formula ϕ defining the family $\{G_p\}_p$ with its p-adic analytic structure [4] uniformly in p.

[4] The formula ϕ defines the sets G_p with the group operation $G_p \times G_p \to G_p$ and the p-adic exponentiation $G_p \times \mathbb{Z}_p \to G_p$ uniformly in p.

We now show that if this condition is satisfied, we can recover the subfamily of uniform pro-p subgroups with their p-adic analytic structure. To this end, we will add to the language $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$ new symbols of constants $a_1, \dots, a_k$ in Lemma 3.3.4, and call this language $\mathcal{L}'$. Recall first that

$$\omega(x) = min\{v(\lambda_i) + 1 : i \in \{1, \dots, d\}\}$$

where $x = a_1^{\lambda_1} \dots a_d^{\lambda_d}$; we will use this relation to define ω in the following.

Lemma 3.3.4. *Suppose that $\{G_p\}_p$ is a family of FAb compact p-adic analytic groups satisfying the property $(\diamond)$. Then there exists a uniformly definable (with parameters) family of uniform pro-p subgroups of the family $\{G_p\}_p$ in the structure $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}')$. In other words, the property $(\diamond)$ ensures that there are $\mathcal{L}'$-formulas φ_j, ψ_j such that,*

- *φ_j defines the finitely generated pro-p subgroup N_p of G_p;*

$$\varphi_j(x) \equiv \exists \lambda_1 \dots \lambda_j \quad x = \prod_i a_i^{\lambda_i}.$$

- *$\psi_j(r, x)$ defines the lower p-series of N_p, namely $N_{(p,r)}$;*

$$\psi_j(r, x) \equiv \omega(x) \geq r.$$

Then there exists an $\mathcal{L}'$-formula Ψ such that $\mathbb{Q}_p \models \Psi \Leftrightarrow \exists d \leq k \quad N_p = \langle a_1, \dots, a_d \rangle$ is uniform pro-p, and, for each p, there exists an interpretation of the constant symbols $a_1, \dots, a_k$ such that Ψ holds.

Proof. We now see how to interpret the above formulas to get the subfamily $\{N_p\}_p$ of uniform pro-p subgroups of $\{G_p\}_p$. We assume that there exists an $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$-formula (hence $\mathcal{L}'$-formula) ϕ defining the family of p-adic analytic groups $\{G_p\}_p$ uniform in p. Then the number of variables in ϕ gives an upper bound for the (topological) dimension of any subgroup of G_p. Call the number of such variables k.

For the following, when we write $\mathbb{Q}_p$, we consider it as an $\mathcal{L}'$-structure with some interpretation of the constant symbols a_i. Let φ_j be the following

$$\varphi_j(x) \equiv \exists \lambda_1 \dots \lambda_j \quad x = \prod_i a_i^{\lambda_i}.$$

To ensure that one gets unique $\lambda_1, \dots, \lambda_j$, we put the sentence χ_j while forming Ψ.

$$\chi_j \equiv \forall \lambda, \lambda' : \prod_i a_i^{\lambda_i} = \prod_i a_i^{\lambda'_i} \to \lambda = \lambda'.$$

Then $\varphi_j(\mathbb{Q}_p)$ becomes a subgroup of G_p which has dimension $\leq j$ if and only if

$$\mathbb{Q}_p \models \Gamma(\varphi_j(\cdot)),$$

where $\Gamma(\varphi_j) : (\exists x, \varphi_j(x)) \wedge \forall x, y\big(\varphi_j(x) \wedge \varphi_j(y) \to \varphi_j(xy^{-1})\big)$.

Let $r \in \mathbb{N}_{>0}$, and let ψ_j be the $\mathcal{L}'$-formula choosing the elements of $\varphi_j(\mathbb{Q}_p)$ whose values are bigger than r under the map ω.

$$\psi_j(x, r) \equiv \omega(x) \geq r.$$

Note that we let r, as a parameter, run over the value group. Since ω is compatible with the usual valuation v on $\mathbb{Z}_p$, this does not lead to any issues with definability. In a similar fashion $\psi_j(\mathbb{Q}_p, r)$ becomes a subgroup of a group G_p if and only if $\mathbb{Q}_p \models \Gamma(\psi_j(\cdot, r))$.

Finally we construe γ_j to be

$$\forall x, y \in \varphi_j(\mathbb{Q}_p) \;\; \exists z \in \varphi_j(\mathbb{Q}_p) \;\; [x, y] = z \wedge w(z) \geqslant 2.$$

Note that the elements satisfying $w(z) \geqslant 2$ are the pth powers. Following this, let

$$\Psi \equiv \bigvee_{j=1}^{k} \Big(\big(\Gamma(\varphi_j(\cdot)) \wedge \chi_j \wedge \gamma_j \wedge \big(\forall r : \Gamma(\psi_j(\cdot, r))\big)\Big).$$

Then $\mathbb{Q}_p \models \Psi$ implies that there is a finitely generated powerful pro-p subgroup $N_p = \overline{\langle a_1, \ldots, a_d \rangle}$ of G_p, for some $d \leq k$ with the filtration $N_{(p,r)} = \{x \in N_p : \omega(x) \geqslant r\}$. Consequently, for any r, the quotient $N_{(p,r)}/N_{(p,r+1)}$ is an elementary abelian group; hence a d-dimensional $\mathbb{F}_p$-vector space. Moreover, for any $\mu \in p + p^2\mathbb{Z}_p = \{\mu : v(\mu) = 1, ac(\mu) = 1\}$, the map $x \mapsto x^{\mu}$ defines an isomorphism between the quotients $N_{(p,r)}/N_{(p,r+1)}$ and $N_{(p,r+1)}/N_{(p,r+2)}$. Therefore N_p turns out to be a uniform pro-p group, and

$$\mathbb{Q}_p \models \Psi \Leftrightarrow \exists d \leq k \;\; N_p = \overline{\langle a_1, \ldots, a_d \rangle} \;\text{ is uniform pro-}p.$$

□

Remark 3.3.5. *In the following chapters, we repeatedly exploit the fact that we can define the uniform pro-p subgroups in the structure $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}')$ uniformly. By Lemma 3.3.4, we obtain each uniform pro-p subgroup N_p of G_p with its group structure in our analytic language. In the setting of [dS93], this implies that, for each p, $(N_p, \mathcal{L}_{N_p})$ is definably interpretable in $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}')$. The main contribution here is that the construction in the proof of Lemma 3.3.4 provides a machinery to study the group theoretic properties of uniform pro-p subgroups uniformly in the analytic language.*

We conclude this section with a critical note; throughout the rest of this thesis, *uniform definability* always means that it is uniformly definable in p, as well as in $a_1, \dots, a_k$ and other constant symbols introduced whenever needed.

3.4 Uniform rationality in p

We finalize the background section by a result from [Ngu19] which enables us to prove rationality of a Poincaré series enumerating uniformly definable families of equivalence classes. In [Ngu19], a theory $\mathcal{T}$ which satisfies certain properties (*) and (**) in a language $\mathcal{L}$ extending $\mathcal{L}_{DP}$ is considered. We are only concerned about the family of local fields $\{\mathbb{Q}_p\}_{p\in\mathbb{P}}$, and the theory of $\mathbb{Q}_p$ in the language $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$ expanded by some constant symbols satisfies (*) and (**) as indicated in Section 1.2.4 of [Ngu19]. Hence, we omit introducing the properties (*) and (**) which can be found in Section 1.2.3, [Ngu19].

Let $\varphi(x, y, n)$ be an $\mathcal{L}$-formula with free variables x and y running over K^m and n running over $\mathbb{N}$. Suppose that for each local field K and $n \in \mathbb{N}$, $\varphi(x, y, n)$ gives an equivalence relation $\sim_{K,n}$ on K^m with finitely many, say, $a_{\varphi,K,n}$, equivalence classes. We consider the following Poincaré series, for each local field K,

$$\mathrm{P}_{\varphi,K}(T) = \sum_{n\geqslant 0} a_{\varphi,K,n} T^n.$$

Prior to Nguyen's result on uniform rationality, we will see how Igusa's result mentioned at the beginning of the introduction, Chapter 2, appears in this formalism. Let $F(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$, and define $\sim_{p,n}$ on the vanishing set of $F(x_1, \dots, x_m)$ (a uniformly definable subset of $\mathbb{Q}_p^m$ in $\mathcal{L}_{DP}$) as follows:

$$x \sim_{p,n} y \Leftrightarrow v_p(x - y) \geqslant n.$$

Let $\varphi(x, y, n)$ be the $\mathcal{L}_{DP}$ formula defining $\sim_{p,n}$, and let $a_{\varphi,p,n}$ be number of equivalence classes of $\sim_{p,n}$. Then

$$\begin{aligned} \mathrm{P}_{\varphi,p}(T) &= \sum_{n\geqslant 0} a_{\varphi,p,n} T^n \\ &= \sum_{n\geqslant 0} N_n T^n \\ &= \mathrm{P}_X(T) \end{aligned}$$

where X is given by $F(x_1, \dots, x_m) = 0$, and N_n is the number of elements of $X(\mathbb{Z}/p^n\mathbb{Z})$.

Theorem 3.4.1. *[[Ngu19], Theorem 1.4.1] Let φ and $P_{\varphi,K}(T)$ be as above. There exists a positive integer M such that the power series $P_{\varphi,K}(T)$ is rational in T for each local field K whose residue field has characteristic at least M. Moreover, for such K, the series $P_{\varphi,K}(T)$ only depends on the $\mathcal{L}$-structure induced on the residue field sort k_K.*

More precisely, there exist non-negative integers N, k, b_j, e_i, integers a_j and $\mathcal{L}$-formulas X_i and Y for subsets of some power of the residue field for $i = 0, \ldots, N$ and $j = 0, \ldots, k$ such that

- *(i) for each j, a_j and b_j are not both zero, and*
- *(ii) for all local fields K with residue field k_K with q_K elements and characteristic at least M, $Y(K)$ is non-empty and*

$$\mathrm{P}_{\varphi,K}(T) = \frac{\sum_{i=0}^{N}(-1)^{e_i}\#\mathrm{X}_i(K)T^i}{\#Y(K)\prod_{j=1}^{k}(1-q_K^{a_j}T^{b_j})} \tag{3.1}$$

Recall that we work in the language $\mathcal{L}'$ expanding $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$ with constant symbols $a_1, \ldots, a_k$. We now formally define what uniform rationality means in this language.

Definition 3.4.2. Let $\bar{a} = a_1, \ldots, a_k$. For a given Poincaré series $\mathrm{P}_{p,\bar{a}}(T) = \sum_{n\geqslant 0} a_{p,\bar{a}}T^n$ for each $\mathbb{Q}_p$ and each interpretation of $a_1, \ldots, a_k$, we call the series $\mathrm{P}_{p,\bar{a}}(T)$ *uniformly rational in* p if there exists $M > 0$ such that the power series $\mathrm{P}_{p,\bar{a}}(T)$ are of the form in (3.1), where K consists of $\mathbb{Q}_p$ such that $p > M$ with the given interpretation of $\bar{a}$.

Note that there will be more constants later on; we will then implement Definition 3.4.2 in a similar way. Now we apply Nguyen's result to the partial zeta series. Recall that we work with a family of FAb compact p-adic analytic groups G_p satisfying the property ($\diamond$). In the previous section, we saw how to obtain a uniformly definable family of uniform pro-p subgroups N_p of G_p. We now consider subgroups K_p of G_p such that $N_p \leqslant K_p$ with fixed Sylow pro-p subgroups P_p of K_p. Let $r = (P_p : N_p)$, $u = (K_p : N_p)$ and fix a set of (left) coset representatives $y_1, \ldots, y_r$ for N_p in P_p, namely a *left transversal* $(y_1, \ldots, y_r)$. We extend it to a (left) transversal for N_p in K_p; set $y_{r+1}, \ldots, y_u \in K_p$ such that $(y_1, \ldots, y_u)$ is a (left) transversal of N_p in K_p, see Section 7.2.2 for a detailed discussion.

Recall that we aim to see the uniform rationality of the following partial zeta series

$$\zeta^{(N_p,K_p,c)}(s) = \sum_{\theta\in\mathrm{Irr}^c_{K_p}(N_p)} \theta(1)^{-s},$$

where N_p is an open normal uniform subgroup of G_p and K_p is a subgroup of G_p containing N_p, and $\mathrm{Irr}^c_{K_p}(N_p)$ is a subset of the set of irreducible characters of N_p whose stabilizer (under the conjugation action) is K_p, see Section 6.2.

We will later see in Chapter 8 that, for each $c \in \mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$, there is a one-to-one correspondence between the set of characters $\mathrm{Irr}^c_{K_p}(N_p)$ and a set of equivalence classes of a uniformly definable equivalence relation. To obtain c in a uniformly definable way, we introduce parameters b_{ij} in the proof of Lemma 8.1.1. We now add to our language $\mathcal{L}'$ new constant symbols $y_1, \ldots, y_u$ for the (left) transversal of N_p in K_p such that $y_1, \ldots, y_r$ is left transversal of N_p in P_p and b_{ij} yielding to $c \in \mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$, and call this new language $\mathcal{L}''$. From now on, our work will be in the language $\mathcal{L}''$.

Following this, in the language $\mathcal{L}''$, we describe a uniformly definable subset $\mathcal{D}^c_{p,n}$ of $\mathbb{Q}_p^m$ for some m, and uniformly definable equivalence relations $\sim_{p,n}$ on $\mathcal{D}^c_{p,n}$ such that

$$\zeta^{(N_p,K_p,c)}(s) = \sum_{n>0} |\mathcal{D}^c_{p,n}/\sim_{p,n}| p^{-ns}.$$

This means that there exists a p-independent $\mathcal{L}''$-formula $\varphi(x, y, n)$ defining the equivalence relation $\sim_{p,n}$ and $|\mathcal{D}^c_{p,n}/\sim_{p,n}|$ is finite. We therefore apply the above result of Nguyen, and conclude that $\zeta^{(N_p,K_p,c)}(s)$ is uniformly rational for large enough p whenever the family $\{G_p\}_p$ of FAb compact p-adic analytic groups satisfies the property $(\diamond)$. More precisely,

Theorem 3.4.3. *There exist $\mathcal{L}''$-formulas defining subsets X_i and Y of some power of the residue field such that the following holds: for every sufficiently large p and every interpretation of the constant symbols a_i, y_j, b_{ij} in the structure $\mathbb{Q}_p$, if the a_i's yield a uniform pro-p subgroup N_p of G_p, the y_j's give transversals for N_p in P_p and in K_p respectively, and b_{ij} give $c \in \mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$, then*

$$\zeta^{(N_p,K_p,c)}(s) = \frac{\sum_{i=0}^{N} (-1)^{e_i} \#(X_i(\mathbb{Q}_p)) p^{-si}}{\#(Y(\mathbb{Q}_p)) \prod_{j=1}^{k} (1 - p^{a_j - sb_j})}.$$

Note that, for each r and u, we obtain uniformity for all K_p for which the index $(P_p : N_p)$ is equal to r, and the index $(K_p : N_p)$ is u. We want to conclude this section with a result that can be deduced from various arguments presented in different places in this text. Our main aim in giving such a statement is to highlight that one certainly needs to parametrize all the possible K_p and c when intended to deal with the representation zeta function of FAb compact p-adic analytic groups.

Theorem 3.4.4. *There exists an $\mathcal{L}''$-sentence Ξ such that $\mathbb{Q}_p$ (with some interpretations of the constant symbols a_i, y_j, b_{ij}) satisfies Ξ if and only if the tuple $a_1, \ldots, a_k$ generates a uniform pro-p subgroup N_p of G_p, $y_1, \ldots, y_u$ gives transversals for N_p in P_p and K_p, and b_{ij} yield to $c \in \mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$.*

Theorem 3.4.3 and Theorem 3.4.4 originate simultaneously in this work. The proof of the latter can be recovered from the construction of the equivalence relations $\sim_{p,n}$. The argument for the a_i's is already given in Lemma 3.3.4. To express that the y_j's form transversals for N_p in P_p in K_p, one can follow similar steps to the proof of Lemma 7.2.6. Finally, the assertion that b_{ij}'s yield a cohomology class $c \in \mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$ is contained in the proof of Lemma 8.1.1 (*iv*).

Chapter 4

Good Basis

In this chapter, we introduce du Sautoy's parametrization of finite index subgroups of a uniform pro-p group. Throughout, assume that N is a uniform pro-p subgroup of a p-adic analytic group G, and $\{a_1, \dots, a_d\}$ is a minimal (topological) generating set of N.

We begin with presenting the following construction from [dS93]. Let $x = x(\lambda) \in N_m = P_m(N)$. Then we know

$$m \leq \omega(x(\lambda)) = \min\{v(\lambda_i) + 1 : i \in \{1, \dots, d\}\}.$$

Consequently, $v(\lambda_i) \geq m - 1$ and $p^{-(m-1)}\lambda_i \in \mathbb{Z}_p$ for each $i \in \{1, \dots, d\}$. Consider the residue map $\pi : \mathbb{Z}_p \to \mathbb{F}_p$, and accordingly define $\pi_m : N_m \to \mathbb{F}_p^d$;

$$x(\lambda) \mapsto (\pi(p^{-(m-1)}\lambda_1), \dots, \pi(p^{-(m-1)}\lambda_d)).$$

The map π_m gives a homomorphism of groups, to see this, pick two elements x, x' from N_m such that $x = x(\lambda)$ and $x' = x(\mu)$, and let ρ be a d-tuple from $\mathbb{Z}_p$ such that $x(\rho) = x \cdot x'$. One can immediately see that

$$x(\lambda + \mu - \rho) \equiv 1 \mod N_{m+1}$$

as the quotient N_m/N_{m+1} is abelian. This implies that $v(\lambda_i + \mu_i - \rho_i) \geq m$, for each $1 \leq i \leq d$; in particular $\pi_m(x(\rho)) = \pi_m(x(\lambda + \mu)) = \pi_m(x).\pi_m(x')$. And the kernel of this homomorphism is exactly N_m; hence π_m is an $\mathbb{F}_p$-vector space isomorphism between N_m/N_{m+1} and $\mathbb{F}_p^d$.

Definition 4.0.1. Let $H \leq N$ be an open subgroup. A tuple $(h_1, \ldots, h_d)$ in H is called a *good basis* for H if

(i) $\omega(h_i) \leq \omega(h_j)$ if $i \leq j$

(ii) $\{\pi_m(h_j) : j \in I_m\}$, where $I_m = \{j : \omega(h_j) = m\}$, extends the linearly independent set

$$\{\pi_m(h_j{}^{p^{m-\omega(h_j)}}) : j \in I_1 \cup \ldots \cup I_{m-1}\}$$

to a basis for $\pi_m(H \cap N_m)$.

Note that a good basis for N is an ordered minimal set of topological generators of N. Also, this constructive definition ensures that a good basis for an open subgroup H of N always exists. To see this, one can consider a tuple $(h_1, \ldots, h_k)$ for some $k < d$ fulfilling (i) while assuming that (ii) holds for all $m \leq l$ for some l. Then the minimality of some $q > l$ such that $\mathbf{dim}(\pi_l(H \cap N_q)) > \mathbf{dim}(\pi_l(H \cap N_l))$ gives us the elements $h_{k+1}, \ldots, h_u$ in $(H \cap G_q) \setminus G_{q+1}$, and the set $\{\pi_q(h_{k+1}), \ldots, \pi_q(h_u)\}$ extends the following linearly independent set

$$\{\pi_q(h_j{}^{p^{n-\omega(h_j)}}) : j \in I_1 \cup \ldots \cup I_{q-1}\}$$

to a basis for $\pi_q(H \cap N_q)$.

Lemma 4.0.2. *[[dS93], Lemma 2.4] Let H be an open subgroup of N with a good basis $(h_1, \ldots, h_d)$. Then for each $h \in H$, there exist $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}_p$ such that*

$$h = h_1^{\lambda_1} \ldots h_d^{\lambda_d}.$$

Moreover, if $h = h_1^{\lambda_1} \ldots h_d^{\lambda_d}$, then $\omega(h) = \mathbf{min}\{\omega(h_i) + v(\lambda_i) : i \in \{1, \ldots, d\}\}$

The recursive construction given in the proof of the above lemma in [dS93] indicates that such a tuple of λ_i is unique. And the second assertion shows that ω and p-adic valuation are compatible with respect to good bases. With the above result, one can obtain du Sautoy's characterization of good bases.

Lemma 4.0.3. *[[dS93], Lemma 2.5] Let N be a uniform pro-p group and $d = d(N)$. Then $(h_1, \ldots, h_d)$ is a good basis for some open subgroup of N if and only if*

(1) $\omega(h_i) \leq \omega(h_j)$ whenever $i \leq j$,

(2) $h_i \neq 1$ for $i \in \{1, \ldots, d\}$,

(3) the set $\{h_1^{\lambda_1} \dots h_d^{\lambda_d} : \lambda_i \in \mathbb{Z}_p\}$ is a subgroup of N,

(4) for all $\lambda_1, \cdots, \lambda_d \in \mathbb{Z}_p$, $\omega(h(\lambda)) = \min\{\omega(h_i) + v(\lambda_i) : i \in \{1, \dots, d\}\}$

The notion of good bases allows us to give a many-to-one parametrization of the set of finite index subgroups of N in terms of p-adic analytic coordinates. Considering this characterization of good basis, one can obtain that the set of all good bases is definable in $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}_{\mathbb{Z}[\![t]\!]})$.

Recall that we are given a uniformly definable family of p-adic analytic groups $\{G_p\}_p$ in the language $\mathcal{L}_{\mathbb{Z}[\![t]\!]}$. In Section 3.3, we showed how to obtain a subfamily $\{N_p\}_p$ of uniform pro-p groups of these p-adic analytic groups in a uniform way by using $\mathcal{L}''$-formulas. We rewrite the definition of good basis to highlight that the set of good bases for an open subgroup of N_p, for any $p > 2$, is uniformly definable. To this aim, we consider the map $x \mapsto x^\mu$ where $\mu \in \{\mu : v(\mu) = 1, ac(\mu) = 1\}$, instead of the pth power map between the quotients as we did in Section 3.3. We have

$$h_i{}^{\mu^{n-\omega(h_i)}} \equiv h_i{}^{p^{n-\omega(h_i)}} \mod P_{n+1}(N_p)$$

in the quotients of the lower p-series of N_p, we therefore obtain the same set of good bases for a given open subgroup of N_p if we replace p by μ. So the following is just a reformulation for the purposes of our work.

Definition 4.0.4. [**good basis**, *revisited*] Let $H_p \leq N_p$ be open with $P_m(N_p) = P_m \leq H_p$ for some m. A tuple $(h_1, \dots, h_d)$ from H_p is called a *good basis* for H_p if

(i) $\omega(h_i) \leq \omega(h_j)$ if $i \leq j$

(ii) for each $n \leq m$, the following set

$$\{h_i{}^{\mu^{n-\omega(h_i)}} P_{n+1} : 1 \leq i \leq d, \omega(h_i) \leq n\}$$

is a basis for the $\mathbb{F}_p$ vector space $(P_n \cap H_p) \cdot P_{n+1}/P_{n+1}$ for any $\mu \in \{\mu : v(\mu) = 1, ac(\mu) = 1\}$.

Lemma 4.0.5. *The set of good bases is uniformly definable.*

Proof. Let $N_p = \Psi(\mathbb{Q}_p)$, and $(h_1, \dots, h_d) \in N_p$. We keep the notation $h_j = x(\lambda_j) = a_1^{\lambda_{1j}} \dots a_d^{\lambda_{dj}}$, and show that the conditions in Lemma 4.0.3 can be given uniformly.

(1) Recall that $\omega(h_i) = \min\{v(\lambda_{ki}) + 1 : k \in \{1, \dots, d\}\}$. Accordingly, $\omega(h_i) \leq \omega(h_j)$ if and only if

$$\min\{v(\lambda_{ki}) + 1 : k \in \{1, \dots, d\}\} \leqslant \min\{v(\lambda_{kj}) + 1 : k \in \{1, \dots, d\}\}.$$

This is uniformly definable as v is definable in the language $\mathcal{L}''$.

(2) It is enough to obtain that $h_i = 1$ if and only if $\lambda_{ki} = 0$ for all $k \in \{1, \ldots, d\}$.

(3) As described in the Section 3.3, one can confirm if the set $\{h_1^{\lambda_1} \ldots h_d^{\lambda_d} : \lambda_i \in \mathbb{Z}_p\}$ is a subgroup of N_p in a uniformly definable way.

(4) Since $\mathbb{Z}_p$-coordinates can be given uniformly in p, this reduces to the uniform definability of the valuation v, which is clear. □

4.1 An Example

We now provide a panorama of concepts introduced in Section 3.2 with a concrete example aiming to create a better intuition on the notion of good basis. For the details of the following, see Chapter 5, [DdSMS99].

For a positive integer n and a prime $p \geq 3$, we consider the general linear group $\Gamma = \mathrm{GL}_n(\mathbb{Z}_p)$, which is a compact Hausdorff topological group with respect to the subspace topology induced from the topology on the space $M_n(\mathbb{Z}_p)$ of $n \times n$ matrices over $\mathbb{Z}_p$. Moreover, a base of the open neighborhoods of the identity element is given by the principal congruence subgroups of $\mathrm{GL}_n(\mathbb{Z}_p)$ defined as follows:

$$\begin{aligned}\Gamma_i = \mathrm{GL}_n^i(\mathbb{Z}_p) &= \{g \in \mathrm{GL}_n(\mathbb{Z}_p) : g \equiv \mathrm{Id} \mod p^i\} \\ &= 1 + p^i M_n(\mathbb{Z}_p)\end{aligned}$$

Consequently, this natural filtration fully determines the topology on $\mathrm{GL}_n(\mathbb{Z}_p)$. Note that for each $i \in \mathbb{N}$, Γ_i can be regarded as the kernel of the projections $\mathrm{GL}_n(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{Z}/p^i\mathbb{Z})$. Therefore, we have

$$\begin{aligned}(\Gamma : \Gamma_i) &= (p^n - 1)(p^n - p) \ldots (p^n - p^{n-1}) \\ (\Gamma_1 : \Gamma_i) &= p^{n^2(i-1)}\end{aligned}$$

Following this, we conclude that Γ is a profinite group, and Γ_1 is a pro-p group. Moreover, $\Gamma = \mathrm{GL}_n(\mathbb{Z}_p)$ is a p-adic analytic manifold with the global atlas

$$\{(\mathrm{GL}_n(\mathbb{Z}_p), \varphi\restriction_{\mathrm{GL}_n(\mathbb{Z}_p)}, n^2)\},$$

where $\varphi : M_n(\mathbb{Z}_p) \to \mathbb{Z}_p^{n^2}$ is the natural homeomorphism. The group operation in $\mathrm{GL}_n(\mathbb{Z}_p)$ are given by the following analytic functions $\mathbb{Z}_p^{n^2} \times \mathbb{Z}_p^{n^2} \to \mathbb{Z}_p$

$$(g_{11}, \ldots, g_{nn}, h_{11}, \ldots, h_{nn}) \mapsto g_{i1}h_{1j} + \ldots + g_{in}h_{nj}.$$

Similarly, the inversion in Γ can be described by analytic functions; consider the functions $\mathbb{Z}_p^{n^2} \to \mathbb{Z}_p$

$$(g_{11}, \ldots, g_{nn}) \mapsto h_{ij} = \frac{\det(g(i,j))}{\det(g)},$$

where $g(i,j)$ is the matrix formed by replacing jth column of g with the ith column of the identity matrix. Then Cramer's rule tells that h_{ij}'s constitute the inverse of the matrix g, and Leibniz formula secures that these functions are analytic.

The Theorem 5.2, [DdSMS99] states that $\Gamma_1 = \mathrm{GL}_n^1(\mathbb{Z}_p)$ is a uniform pro-p group, and the principal congruence subgroups of Γ_1 coincide with its lower p-series,

$$P_i(\Gamma_1) = \Gamma_i = \mathrm{GL}_n^i(\mathbb{Z}_p) = 1 + p^i M_n(\mathbb{Z}_p).$$

With all being said, we now consider the uniform pro-p group $\mathrm{GL}_2^1(\mathbb{Z}_3)$,

$$G := \{A \in \mathrm{GL}_2(\mathbb{Z}_3) : A \equiv \mathrm{Id} \mod 3)\}.$$

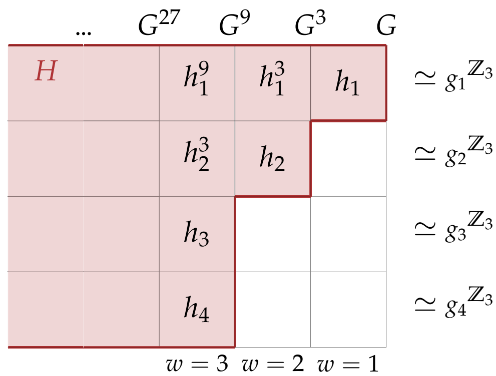
Then the lower p-series $\ldots \subseteq G^{3^{(n-1)}} \subseteq \ldots \subseteq G^3 \subseteq G$ in G can be given as follows: $P_n(G) = G^{3^{(n-1)}} = \{\begin{bmatrix} 1+3^n\mathbb{Z}_3 & 3^n\mathbb{Z}_3 \\ 3^n\mathbb{Z}_3 & 1+3^n\mathbb{Z}_3 \end{bmatrix}\}$. As $(P_i(G) : P_{i+1}(G)) = (G : P_2(G)) = 3^4$, we have $d(G) = 4$. Then the following gives a minimal (topological) topological generating set for G

$$\{g_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, g_2 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, g_3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, g_4 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\}.$$

As an example, consider now the subgroup $H = \{\begin{bmatrix} 1+3\mathbb{Z}_3 & 3^2\mathbb{Z}_3 \\ 3^3\mathbb{Z}_3 & 1+3^3\mathbb{Z}_3 \end{bmatrix}\}$ of G. It is obvious that $P_3(G) \leqslant H$. Then a good basis for H can be given as follows:

$$\{h_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, h_2 = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}, h_3 = \begin{bmatrix} 1 & 0 \\ 27 & 1 \end{bmatrix}, h_4 = \begin{bmatrix} 1 & 0 \\ 0 & 28 \end{bmatrix}\}.$$

The following image illustrates all the above-mentioned groups, relations and elements; the entire picture stands for the group G. The principal subgroups decrease to the left, i.e. G^3 is everything to the left of the line denoted by G^3, etc. In addition, the rows symbolize the groups generated by each of g_i, and the image shows how much of these subgroups is contained in H and where the elements h_i and their powers live.



Chapter 5

Projective Representations

The following chapter covers the most of the representation theoretic part of our work. We heavily use projective characters of profinite groups. To this end, we provide a brief survey of definitions and results from the theory of projective representation, which was founded and improved by Schur in [Sch04], [Sch07], [Sch11], and show how they become handy for us. An interested reader shall find it beneficial to check the following references; [Kar94], [Isa06] and [Hup98].

From now on, we only consider closed subgroups (denoted by $\leqslant$), continuous representations and their characters. To this end, we view the general linear groups over $\mathbb{C}$ with the discrete topology. Before going any further, let us have a quick recall on some basic notions from representation theory whilst fixing notations.

Definition 5.0.1. Let G be a profinite group and, let N be an open normal subgroup. Note that we can generalize the statements about finite groups to this setting as we work with continuous representations and finite index subgroups of G. As a rule of thumb, one can recover the statements about profinite groups by pulling back the data from the finite quotients to the inverse limit within this setting.

- $\mathrm{Irr}(G)$ is defined to be the set of characters of continuous irreducible complex representations of G.
- For any subgroup $K \leqslant G$ and $\theta \in \mathrm{Irr}(K)$,

$$\mathrm{Irr}(G \mid \theta) = \{\chi \mid \chi \in \mathrm{Irr}(G), \langle \mathrm{Res}^G_K(\chi), \theta \rangle > 0\}.$$

- For any $\theta \in \mathrm{Irr}(N)$, we define the *conjugate character* ${}^g\theta : N \to \mathbb{C}$ of θ by $h \mapsto \theta(ghg^{-1})$. Accordingly, we write the *conjugation action* of G on $\mathrm{Irr}(N)$ as follows:

$$\begin{array}{ccc} G \times \mathrm{Irr}(N) & \to & \mathrm{Irr}(N) \\ (g, \theta) & \mapsto & {}^{g}\theta \end{array}$$

Hence the stabilizer of θ under this action is given by

$$\mathsf{stab}_G(\theta) = \{g \in G : {}^{g}\theta = \theta\}.$$

Note that this stabilizer group is often called the *inertia group* of θ in G, and $(G : \mathsf{stab}_G(\theta))$ is the size of G-orbit of θ.

- For any $K \leqslant G$, we write $\mathrm{Irr}_K(N) = \{\theta \in \mathrm{Irr}(N) : \mathsf{stab}_G(\theta) = K\}$ for the irreducible characters of N whose stabilizer is K.

- We call (K, N, θ) a *character triple* if $\theta \in \mathrm{Irr}(N)$ and K fixes θ, i.e. $K \leqslant \mathsf{stab}_G(\theta)$. Hence, $\mathrm{Irr}_G(N)$ can be regarded as the set of character triples (G, N, θ), see [Isa06], later parts of Chapter 11.

We begin with recalling Clifford's theorem that helps us to connect representations of N with representations of G, and build up representations of G from representations of N. Later on, we extend Clifford's theory (actually his viewpoint of reducing the problem) to projective representations and characters, see Section 6.1.

Theorem 5.0.2. *[Clifford] Let $\theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G \mid \theta)$. Consider the G-orbit (under the conjugation action) $\{\theta = \theta_1, \ldots, \theta_m\}$ of θ where $m = (G : \mathsf{stab}_G(\theta))$. Then we have the following*

(i) $\mathrm{Res}^G_N(\mathrm{Ind}^G_N(\theta)) = (\mathsf{stab}_G(\theta) : N) \cdot \sum_{i=1}^{m} \theta_i$

(ii) $\langle \mathrm{Ind}^G_N(\theta), \mathrm{Ind}^G_N(\theta)\rangle = (\mathsf{stab}_G(\theta) : N)$. *In particular,* $\mathrm{Ind}^G_N(\theta) \in \mathrm{Irr}(G)$ *if and only if* $\mathsf{stab}_G(\theta) = N$.

(iii) Call $\langle \mathrm{Res}^G_N(\chi), \theta\rangle = e$ *and note that* $e > 0$. *Then* $\mathrm{Res}^G_N(\chi) = e \cdot \sum_{i=1}^{m} \theta_i$. *Equivalently, one can say that the irreducible constituents of* $\mathrm{Res}^G_N(\chi)$ *are all of the same multiplicity and form the G-orbit of θ.*

(iv) $\theta(1)$ *divides* $\chi(1)$. *Consequently, given a character triple* (K, N, θ), *we obtain that* $\chi(1)/\theta(1)$ *is an integer for any* $\chi \in \mathrm{Irr}(K \mid \theta)$.

Theorem 5.0.3. *[[BKZ18], Chp. VII, Theorem 2.2] For any $\theta \in \mathrm{Irr}(N)$, the following statements hold;*

(i) For any $\varphi \in \mathrm{Irr}(\mathsf{stab}_G(\theta) \mid \theta)$, $\mathrm{Ind}^G_{\mathsf{stab}_G(\theta)}(\varphi) \in \mathrm{Irr}_\theta(G)$.

(ii) *Let* $\chi \in \operatorname{Irr}(G \mid \theta)$ *and* $\varphi \in \operatorname{Irr}(\operatorname{stab}_G(\theta) \mid \theta)$ *be such that* $\operatorname{Ind}^G_{\operatorname{stab}_G(\theta)} \varphi = \chi$. *Then*

$$\langle \operatorname{Res}^G_{\operatorname{stab}_G(\theta)}(\chi), \theta \rangle = 1.$$

(iii) *There is a bijection* $\operatorname{Irr}(\operatorname{stab}_G(\theta) \mid \theta) \to \operatorname{Irr}(G \mid \theta)$ *given by* $\varphi \mapsto \operatorname{Ind}^G_{\operatorname{stab}_G(\theta)}(\varphi)$; *in particular,* $|\operatorname{Irr}(\operatorname{stab}_G(\theta) \mid \theta)| = |\operatorname{Irr}(G \mid \theta)|$

Definition 5.0.4. A map $\alpha : G \times G \to \mathbb{C}^*$ is called a *2-cocycle* (or a *factor set*) on G if for all $g, h, k \in G$,

$$\alpha(gh, k)\alpha(g, h) = \alpha(g, hk)\alpha(h, k).$$

The set of 2-cocycles on G has an abelian group structure under pointwise multiplication. This group is denoted by $\mathrm{Z}^2(G, \mathbb{C}^*)$. We shall take a closer look to a special subgroup $\mathrm{B}^2(G, \mathbb{C}^*)$ of $\mathrm{Z}^2(G, \mathbb{C}^*)$ consisting of elements α called *2-coboundary* in the following form;

$$\alpha(g, h) = \frac{\mu(gh)}{\mu(g)\mu(h)},$$

where $\mu : G \to \mathbb{C}^*$ is an arbitrary function which sends 1 to 1. Following this, we consider the quotient group $\mathrm{Z}^2(G, \mathbb{C}^*)/\mathrm{B}^2(G, \mathbb{C}^*) = \{[\alpha] : \alpha \in \mathrm{Z}^2(G, \mathbb{C}^*)\}$ and denote this by $\mathrm{H}^2(G, \mathbb{C}^*)$. Note that this special cohomology group $\mathrm{H}^2(G, \mathbb{C}^*)$ is also called the *Schur multiplier* of G.

Definition 5.0.5. Let V be an n-dimensional vector space over $\mathbb{C}$ where $n < \infty$. A continuous function $\rho : G \to \mathrm{GL}(V)$ is called a *projective representation* of G over V if there exists a continuous function $\alpha : G \times G \to \mathbb{C}^*$ such that

$$\rho(g)\rho(h) = \rho(gh)\alpha(g, h)$$

for all $g, h \in G$, and the associated function α is said to be the *factor set of* ρ.

Note that the factor set α of a projective representation ρ is uniquely determined by ρ taking non-zero values, and lies in $\mathrm{Z}^2(G, \mathbb{C}^*)$. Moreover, for any $\alpha \in \mathrm{Z}^2(G, \mathbb{C}^*)$, there exists a projective representation of G with factor set α. Notice that a projective representation with a trivial factor set, i.e. $\alpha = 1$, is an ordinary representation. As in the case of ordinary representation theory, we call the function $G \to \mathbb{C}$ given by $g \mapsto tr(\rho(g))$ the *projective character* of ρ.

5.1 Twisted Group Algebra

It is well known that ordinary representations of groups can be seen as modules over group algebras. One can say more on projective characters by observing a similar correspondence for projective representations considering the twisted group algebra instead of the group algebra. To this end, we introduce the notion of twisted group algebra. Let G be a profinite group with an open normal subgroup N.

For a fixed 2-cocycle $\alpha \in Z^2(G, \mathbb{C}^*)$, we denote the (α)-*twisted group algebra* over $\mathbb{C}$ by $\mathbb{C}^\alpha[G]$. This algebra has a basis $\{\bar{g} : g \in G\}$, consequently each element of $\mathbb{C}^\alpha[G]$ can be uniquely given as

$$\sum_{g \in G} x_g.\bar{g},$$

where $x_g \in \mathbb{C}$. The multiplication in $\mathbb{C}^\alpha[G]$ is given by $\bar{g}\bar{h} = \overline{gh}\alpha(g,h)$ and extended via the distributive law.

For a given factor set α of G, consider an arbitrary (ordinary) representation π of $\mathbb{C}^\alpha[G]$. Set $\Theta(g) = \pi(\bar{g})$. Then one can see that Θ is a projective representation of G with the factor set α by the following;

$$\Theta(g)\Theta(h) = \pi(\bar{g})\pi(\bar{h}) = \pi(\bar{g}.\bar{h}) = \pi(\overline{gh}\alpha(g,h)) = \Theta(gh)\alpha(g,h).$$

Conversely, if Θ is a projective representation of G with a factor set α, one can define a representation π of $\mathbb{C}^\alpha[G]$ by letting $\pi(\bar{g}) = \Theta(g)$ and extending by linearity. The projective characters of G with the factor set α are therefore in a one-to-one correspondence with the representations of the twisted group algebra $\mathbb{C}^\alpha[G]$.

Two projective representations ρ and σ are called *similar* if there exists an invertible matrix P (over $\mathbb{C}$) satisfying $\rho(g) = P\sigma(g)P^{-1}$ for all $g \in G$. Two projective representations have the same (projective character) if and only if they are similar. Also, a projective representation Θ is called *irreducible* if it is not similar to a projective representation in the form

$$\begin{bmatrix} * & \cdots & * & \cdots & * \\ \vdots & * & \vdots & * & \vdots \\ * & \cdots & * & \cdots & * \\ & 0 & \vdots & * & \vdots \\ & & * & \cdots & * \end{bmatrix}$$

In other words, a projective representation Θ with factor set α and the character it affords are called *irreducible* if Θ corresponds to a simple $\mathbb{C}^\alpha[G]$-module.

Let ρ_1 and ρ_2 be two projective representations of G with factor sets α_1, α_2 respectively. For $g \in G$, the *tensor product* $\rho_1 \otimes \rho_2$ of ρ_1 and ρ_2 is given by

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g).$$

Correspondingly, we have the following relation for any $g, h \in G$,

$$\begin{aligned}(\rho_1(g) \otimes \rho_2(g))(\rho_1(h) \otimes \rho_2(h)) =& (\rho_1(g)\rho_1(h)) \otimes (\rho_2(g)\rho_2(h)) \\ =& (\rho_1(gh)\alpha_1(g,h)) \otimes (\rho_2(gh)\alpha_2(g,h)) \\ =& (\rho_1(gh) \otimes (\rho_2(gh))\alpha_1(g,h)\alpha_2(g,h)) \\ =& (\rho_1 \otimes \rho_2)(gh)\alpha_1(g,h)\alpha_2(g,h))\end{aligned}$$

$\rho_1 \otimes \rho_2$ is therefore a projective representation of G with factor set $\alpha_1\alpha_2$. In terms of characters, we also get what one expects; let χ_1, χ_2 be the (projective) characters of ρ_1 and ρ_2 respectively, then the character $\chi = \chi_{\rho_1 \otimes \rho_2}$ of $\rho_1 \otimes \rho_2$ equals to $\chi_1\chi_2$.

A projective representation ρ of G with factor set α is called *equivalent* to an ordinary representation if there exists $\delta : G \to \mathbb{C}^*$ such that

$$\alpha(g_1, g_2) = \frac{\delta(g_1)\delta(g_2)}{\delta(g_1 \cdot g_2)},$$

for all $g_1, g_2 \in G$.

5.2 Strong extensions

Now we will define another notion relating ordinary representations with projective ones as in [SZ20] following [Isa06]. We still assume G to be profinite with an open normal subgroup N.

Definition 5.2.1. Let Θ be an (ordinary) irreducible representation of N fixed by a subgroup K of G. We say that a projective representation Π of K *strongly extends* Θ if for all $g \in K$ and $n \in N$, we have

(i) $\Pi(n) = \Theta(n)$,
(ii) $\Pi(ng) = \Pi(n)\Pi(g)$,
(iii) $\Pi(gn) = \Pi(g)\Pi(n)$.

It immediately follows from the definition that a projective representation Π of G with factor set α satisfying $\Pi(n) = \Theta(n)$ for all $n \in N$ gives a strong extension of Θ if and only if for all $g \in K$ and $n \in N$,

$$\alpha(g, n) = \alpha(n, g) = 1.$$

Theorem 5.2.2. *For an (ordinary) irreducible representation Θ of N fixed by $K \leqslant G$, we have the following:*

(i) There exists a projective representation Π of K which strongly extends Θ.

(ii) Let α' be the factor set of Π. There exists a well-defined element α of $Z^2(K/N, \mathbb{C}^*)$ given by

$$\alpha(gN, hN) = \alpha'(g, h).$$

Proof. We begin with noting an argument that will be repeatedly used in this work. As N is open in K, it has a finite index in K. Recall also that N is an open (so closed) subgroup of the profinite group G. N is therefore profinite, and any continuous complex finite dimensional representation of N factors through a finite quotient. Consequently, we may consider the case of finite groups.

We write ${}^g\Theta(n) = \Theta(gng^{-1})$ for $g \in K$ and $n \in N$. Note that Θ and ${}^g\Theta$ are similar representations of K as Θ is fixed by K. To define Π, we consider a transversal T for N in K. For each $t \in T$, we choose an invertible matrix P_t such that

$$P_t \Theta P_t^{-1} = {}^t\Theta,$$

and we fix $P_1 = Id$.

Now we define $\Pi(nt) := \Theta(n)P_t$ as each element of K can be given uniquely in the form nt for some $n \in N$ and $t \in T$. Then, we have the following:

$$\begin{aligned}
\Pi(nt)\Pi(m) &= \Theta(n)P_t\Theta(m) = \Theta(n)^t\Theta P_t \\
&= \Theta(ntmt^{-1})P_t = \Pi(ntmt^{-1}.t) \\
&= \Pi(nt.m)
\end{aligned}$$

Therefore the properties $(i), (ii), (iii)$ given in Definition 5.2.1 immediately follow. Henceforth, by combining these properties, we obtain, for all $g \in K$, $n \in N$

$$\begin{aligned}
\Pi(g)\Theta(n) &= \Pi(gn) = \Pi(gng^{-1}.g) \\
&= \Theta(gng^{-1})\Pi(g)
\end{aligned} \tag{5.1}$$

Consequently, $\Pi(g)\Theta(n)\Pi(g)^{-1} = \Theta(gng^{-1})$. Similarly, we have

$$\begin{aligned}
\Pi(g)\Pi(h)\Theta(n)\Pi(h)^{-1}\Pi(g)^{-1} &= \Pi(g)\Theta(hnh^{-1})\Pi(g)^{-1} \\
&= \Theta(ghnh^{-1}g^{-1})
\end{aligned}$$

By combining these two, we obtain

$$\Pi(gh)\Theta(n)\Pi(gh)^{-1} = \Pi(g)\Pi(h)\Theta(n)\Pi(h)^{-1}\Pi(g)^{-1}$$

for all $g, h \in K$, $n \in N$. Then by Schur's lemma, $\Pi(gh)\Pi(h)^{-1}\Pi(g)^{-1}$ needs to be a scalar, so set $\alpha'(g,h) = \Pi(gh)\Pi(h)^{-1}\Pi(g)^{-1}$. We have

$$\Pi(g)\Pi(h) = \Pi(gh)\alpha'(g,h)$$

for some $\alpha' : K \times K \to \mathbb{C}^*$. This finishes the first part of the proof. Now we see that α' is constant on cosets of N in K; hence we have a well-defined element $\alpha \in \mathrm{Z}^2(K/N, \mathbb{C}^*)$. For arbitrary $n, m \in N$ and $g, h \in K$, we have the following

$$\begin{aligned}
\alpha'(gn, hm)\Pi(gnhm) &= \Pi(gn)\Pi(hm) = \Pi(g)\Pi(nhm) \\
&= \Pi(g)\Pi(h(h^{-1}nh)m) = \Pi(g)\Pi(h)\Pi(hnh^{-1}m) \\
&= \alpha'(g,h)\Pi(gh)\Pi(hnh^{-1}m) \\
&= \alpha'(g,h)\Pi(ghhnh^{-1}m)
\end{aligned}$$

As $\Pi(gnhm)$ is invertible and $\Pi(gnhm) = \Pi(ghhnh^{-1}m)$, we get $\alpha'(gn, hm) = \alpha'(g,h)$. Therefore, α is well defined. □

Corollary 5.2.3.

(i) Let Π' be another projective representation of K strongly extending θ. Then there exists a function $\mu : K \to \mathbb{C}^*$, that is constant on cosets of N in K such that, for all $g \in K$,

$$\Pi'(g) = \Pi(g)\mu(g).$$

(ii) [[SZ20], Theorem 3.4.] There exists a well-defined function mapping the irreducible characters of N fixed by $K \leqslant G$ to the cosets of $\mathrm{B}^2(K/N, \mathbb{C}^*)$ in $\mathrm{Z}^2(K/N, \mathbb{C}^*)$; $\theta \mapsto [\alpha]$

$$\mathcal{C}_K : \{\theta \in \mathrm{Irr}(N) : K \leq \mathrm{stab}_G(\theta)\} \to \mathrm{H}^2(K/N, \mathbb{C}^*).$$

Proof. (i) This immediately follows from the proof of Theorem 5.2.2. Since Π' is another strong extension of θ, we also have

$$\Pi'(g)\theta(n)\Pi'(g)^{-1} = \theta(gng^{-1})$$

for all $n \in N$, $g \in K$ as in Equation 5.1. Then one can see $\Pi'(g)^{-1}\Pi(g)$ commutes with $\theta(n)$ for all n by following the steps above. So we conclude that $\Pi'(g)^{-1}\Pi(g)$ is scalar. Hence $\Pi'(g) = \Pi(g)\mu(g)$ for some $\mu : K \to \mathbb{C}^*$. To see it is constant on the cosets, it is enough to state, for all $n \in N$ and $g \in K$,

$$\Pi(n)\Pi(g)\mu(g) = \Pi'(n)\Pi'(g) = \Pi'(ng) = \Pi(ng)\mu(ng).$$

(ii) We now explain the construction briefly to make it clear that this is a direct consequence of the Theorem 5.2.2. For an irreducible character θ of N fixed by K, by Theorem 5.2.2, we can find a projective representation Π with factor set $\alpha' \in \mathrm{Z}^2(K, \mathbb{C}^*)$. This factor set induces $\alpha \in \mathrm{Z}^2(K/N, \mathbb{C}^*)$, and we map θ to the class of α in $\mathrm{H}^2(K/N, \mathbb{C}^*)$. And the first assertion ensures that every two strong extensions of θ to K give the same element $\mathcal{C}_K(\theta)$, as their factor sets are congruent mod $\mathrm{B}^2(K/N, \mathbb{C}^*)$. So $\mathcal{C}_K$ is well defined. □

5.3 Induced Projective Representations

Let G be a profinite group and let $H \leqslant G$. For any 2-cocycle $\alpha \in \mathrm{Z}^2(G, \mathbb{C}^*)$, we denote the restriction of α to $H \times H$ by α_H; hence $\alpha_H \in \mathrm{Z}^2(H, \mathbb{C}^*)$. Following this, we see $\mathbb{C}^{\alpha_H}[H]$ as the sub-algebra of $\mathbb{C}^{\alpha}[G]$ consisting of $\mathbb{C}$-linear combinations of the elements $\bar{h}$ for all $h \in H$. Furthermore, we denote the $\mathbb{C}^{\alpha_H}[H]$-module by V_H in case V is an $\mathbb{C}^{\alpha}[G]$ module. This fundamental construction is called *restriction* and we denote the corresponding projective representation by $\mathrm{Res}^G_{H,\alpha_H}(V)$.

If χ is the projective character of G afforded by V, then we denote the projective character of H afforded by V_H by $\mathrm{Res}^G_{H,\alpha_H}(\chi)$ accordingly. For given two subgroups K, K' of G such that $H \leqslant K' \leqslant K$, suppose that a projective representation Π of K is a strong extension of an irreducible representation Θ of H. Note that $\mathrm{Res}^K_{K',\alpha_{K'}}(\Pi)$ also strongly extends Θ.

As in the case of ordinary representations, we also have a dual notion called *induction*. In the following definition, we will provide two descriptions of induced projective representations.

Definition 5.3.1. With the above setting, suppose that (ρ, W) is a projective representation of H with factor set α_H. Let V' be the following vector space;

$$V' = \{f : G \to W : f(hg) = \alpha(hg, g^{-1})\rho(h)f(g) \ \text{ for all } h \in H, g \in G\}.$$

Accordingly, let $\Theta : G \to \mathrm{GL}(V')$ to be the map given by

$$(\Theta(g)(f))(g') = \alpha(g', g)f(g'g).$$

Then Θ defines a projective representation of G with the associated factor set α which will be denoted by $\mathrm{Ind}^G_{H,\alpha}(W)$.

If one wants to follow a more module theoretic approach, it can be proceeded by considering W as a $\mathbb{C}^{\alpha_H}[H]$-module. Following this, we define an $\mathbb{C}^{\alpha}[G]$-module structure on the tensor product $\mathbb{C}^{\alpha}[G] \otimes_{\mathbb{C}^{\alpha_H}[H]} W$ as $\mathbb{C}^{\alpha_H}[H]$ can be considered as a subalgebra of $\mathbb{C}^{\alpha}[G]$. This will be called the *induced module* and denoted by W^G. Accordingly, the *induced projective character* $\mathrm{Ind}^G_{H,\alpha}\,\chi$ is the character of the induced projective representation of the tensor product $\mathbb{C}^{\alpha}[G] \otimes_{\mathbb{C}^{\alpha_H}[H]} W$ where χ is a projective character of H with factor set α_H. $\mathrm{Ind}_{H,\alpha}^{G}(\chi)$ is therefore a projective character of G, and its factor set is α.

Suppose that $\alpha \in Z^2(G, \mathbb{C}^*)$ and H is a subgroup of G. If χ is the projective character of H with factor set α_H afforded by an $\mathbb{C}^{\alpha_H}[H]$-module V, then we define the g-conjugate ${}^{g}\chi$ of χ by ${}^{g}\chi(h) = \chi(ghg^{-1})$ for any $g \in G$. As in the case of ordinary representations, ${}^{g}\chi$ is a projective character of ${}^{g}H = gHg^{-1}$ with factor set α_H which is afforded by ${}^{g}V$.

Now, we introduce important and relevant facts on projective representations which are straightforward analogues of established results for ordinary representations.

We begin with recalling the notion of double cosets. Let G be a group having two subgroups H and K. For each $g \in G$, the *double coset* KgH is given as

$$KgH = \{kgh : h \in H, k \in K\}.$$

We will denote the set of double cosets by $K \backslash G / H$ and write $\bar{g} \in K \backslash G / H$ for the double coset representatives.

Also, recall that *the intertwining number* $i(V, W)$ of two finitely generated R-modules V, W where R is a finite dimensional algebra over a field F is defined as $\dim_F(\mathrm{Hom}_R(V, W))$. As in ordinary representation theory, the intertwining number of $\mathbb{C}[G]^{\alpha}$-modules can be given by the inner product of corresponding projective characters.

Lemma 5.3.2. *[[Kar94], Chp. 1, Lemma 11.6] Let χ and θ be projective characters of G with factor set α. Suppose that V and W are $\mathbb{C}[G]^{\alpha}$-modules affording χ and θ respectively. Then we have,*

$$i(V, W) = \langle \chi, \theta \rangle_G.$$

Theorem 5.3.3. *[Mackey's Formula, [[Kar94], Chp. 1, Theorem. 8.6] Fix 2-cocycle $\alpha \in Z^2(G, \mathbb{C}^*)$ and let H and K be two subgroups of G. Suppose that V and W are finitely generated $\mathbb{C}^{\alpha_H}[H]$ and $\mathbb{C}^{\alpha_K}[K]$-modules respectively. Then we have*

$$i(V^G, W^G) = \sum_{\bar{g} \in K \backslash G / H} i({}^{g}V_{{}^{g}H \cap K}, W_{{}^{g}H \cap K}).$$

In terms of characters, the theorem reads: For a projective character χ of H,

$$\mathrm{Res}_K^G(\mathrm{Ind}_H^G(\chi)) = \sum_{\bar{g} \in K \backslash G / H} \mathrm{Ind}_{{}^{g}H \cap K}^{K}\, {}^{g}\chi.$$

Proposition 5.3.4. *[[Kar94], Chp. 1, Lemma 8.10 (iii)] Let χ, θ be two projective characters of G with the same factor set α. Assume that χ is irreducible. Then the multiplicity of χ as an irreducible constituent of θ is given by $\langle \chi, \theta \rangle_G$.*

Theorem 5.3.5. *[Frobenius Reciprocity, [Kar94], Chp. 1, Proposition 9.18] Let $\alpha \in Z^2(G, \mathbb{C}^*)$ and let H be a subgroup of G. Suppose that χ is a projective character of H with factor set α_H and that θ is an irreducible projective character of G with factor set α. Then the multiplicity of χ in $\mathrm{Res}_H^G \theta$ is equal to the multiplicity of θ in $\mathrm{Ind}_H^G(\chi)$. In other words,*

$$\langle \mathrm{Ind}_H^G(\chi), \theta \rangle_G = \langle \chi, \mathrm{Res}_H^G \theta \rangle_H.$$

Definition 5.3.6. A projective representation $\rho : G \to \mathrm{GL}(V)$ of G is called *monomial* if there exists a subgroup $H \leq G$ and a one-dimensional projective representation Θ of H such that $\rho = \mathrm{Ind}_H^G(\Theta)$.

Theorem 5.3.7. *[[Kar93], Chp. 3, Theorem 11.2] Let G be a supersolvable [1] group. Then every irreducible projective representation of G over $\mathbb{C}$ is monomial.*

Corollary 5.3.8. *Projective representations of pro-p groups are induced from a one-dimensional projective representation of some open subgroup.*

5.4 Cohomology of finite groups

We now provide notable results from the cohomology of finite groups which are relevant to this work. We shall start with generalizing Definition 5.0.4. To this aim, we assume G to be a finite group, A to be an abelian group (both written multiplicatively), and that G acts on A. Recall first that an *i-cochain* of G with a coefficient in A is a function $f : G^i \to A$. Note that they form an abelian group under the multiplication

$$(fg)(x_1, \ldots, x_i) = f(x_1, \ldots, x_i)g(x_1, \ldots, x_i),$$

where $G^0 := \{1\}$. We denote this group by $C^i(G, A)$. For each i, the coboundary homomorphisms are given as follows:

$$\delta^i : C^i(G, A) \to C^{i+1}(G, A).$$

$$f \mapsto \delta^i(f)$$

[1] A group G is supersolvable if there is a normal series $\{1\} \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$ such that. for all i, $G_i \triangleleft G$, and every quotient group G_{i+1}/G_i is cyclic

for $f \in C^i(G,A)$ and $(x_1,\dots,x_{i+1}) \in C^{i+1}(G,A)$ such that

$$\delta^i(f)(x_1,\dots,x_{i+1}) = {}^{x_1}f(x_2,\dots,x_{i+1})f(x_1,\dots,x_i)^{(-1)^{i+1}}\prod_{j=1}^{i} f(x_1,\dots,x_{j-1},x_jx_{j+1},\dots,x_{i+1})^{(-1)^j}.$$

Following this, we define

$$\mathrm{Z}^i(G,A) := \ker(\delta^i)$$
$$\mathrm{B}^i(G,A) := \begin{cases} \mathrm{im}(\delta^{(i-1)}), & i \geqslant 0 \\ 1, & \text{otherwise} \end{cases}$$

We call the elements of $\mathrm{Z}^i(G,A)$ and $\mathrm{B}^i(G,A)$ *i-cocycles* and *i-coboundaries* respectively. One can see that $\mathrm{B}^i(G,A)$ is a subgroup of $\mathrm{Z}^i(G,A)$. Following this, we define the *i-th cohomology group* $\mathrm{H}^i(G,A)$ of G as the quotient group $\mathrm{Z}^i(G,A)/\mathrm{B}^i(G,A)$. Note that, in Definition 5.0.4, we considered the trivial action of G on $A = \mathbb{C}^*$ in order to define the second cohomology group $\mathrm{H}^i(G,\mathbb{C}^*)$.

Proposition 5.4.1. *In addition to the above setting, let $|G| = m$. For any integer $i \geqslant 1$, we have the following*

(i) [[Lan96], Section 2.2] $\mathrm{H}^i(G,A)$ *is a torsion group. In particular $\alpha^m = 1$ for all $\alpha \in \mathrm{H}^i(G,A)$.*

(ii) [[Lan96], Section 2.2] Let $\mathrm{H}^i(G,A)_{(p)}$ be the p-primary part of $\mathrm{H}^i(G,A)$, that is, the subgroup of $\mathrm{H}^i(G,A)$ consisting of all elements whose order is a power of p. Then

$$\mathrm{H}^i(G,A) = \bigoplus_{p|m} \mathrm{H}^i(G,A)_{(p)}.$$

It follows that, any $\alpha \in \mathrm{H}^i(G,A)$ can be written as $\alpha = \prod_{p|m} \alpha_{(p)}$ where $\alpha_{(p)}$ is the p-part of α, i.e. $\alpha = \alpha_{(p)}.\alpha_{(p')}$ where $\alpha_{(p)} \in \mathrm{H}^i(G,A)_{(p)}$ and $\alpha_{(p')}$ has order not divisible by p. In particular, if G is a p-group, then $\mathrm{H}^i(G,A) = \mathrm{H}^i(G,A)_{(p)}$.

(iii) [[Lan96], Section 2.2, Theorem 2.1] Let P be a Sylow p-subgroup of G. Then the restriction map $\mathrm{res}^G_P : \mathrm{H}^i(G,A) \to \mathrm{H}^i(P,A)$ gives an injection

$$\mathrm{res}_p : \mathrm{H}^i(G,A)_{(p)} \to \mathrm{H}^i(P,A).$$

We finalize this section by going back to the 2-cocycles and 2-coboundaries on a finite group G, giving a smaller version of Lemma 3.10, [SZ20].

Lemma 5.4.2. *Let $|G| = m$, and let U^2 be the following set of 2-cocycles on G*

$$\{\alpha \in \mathrm{Z}^2(G, \mathbb{C}^*) : \alpha^m = 1\}.$$

Then $\mathrm{B}^2(G, \mathbb{C}^*)$ is complemented in $\mathrm{Z}^2(G, \mathbb{C}^*)$ by U^2. Consequently, $\mathrm{H}^2(G, \mathbb{C}^*)$ is finite.

Proof. First of all, we observe that $\mathrm{B}^2(G, \mathbb{C}^*)$ is divisible. Let $\beta \in \mathrm{B}^2(G, \mathbb{C}^*)$. Then there exists $\mu : G \to \mathbb{C}^*$ such that

$$\beta(g, h) = \frac{\mu(gh)}{\mu(g)\mu(h)}.$$

Let $n \in \mathbb{N}$, and for $g \in G$, let $\gamma : G \to \mathbb{C}^*$ be such that $\gamma(g)^n = \mu(g)$. Then

$$\beta(g, h) = \frac{\mu(gh)}{\mu(g)\mu(h)} = \Big(\frac{\gamma(gh)}{\gamma(g)\gamma(h)}\Big)^n.$$

Now we can move to our main claim. It is obvious that $\mathrm{B}^2(G, \mathbb{C}^*)U^2 \subseteq \mathrm{Z}^2(G, \mathbb{C}^*)$. To see that $\mathrm{Z}^2(G, \mathbb{C}^*))$ is contained in $\mathrm{B}^2(G, \mathbb{C}^*)U^2$, suppose that $\alpha \in \mathrm{Z}^2(G, \mathbb{C}^*)$. By Proposition 5.4.1 (i), we have $\alpha^m \in \mathrm{B}^2(G, \mathbb{C}^*)$. As we recently observed, $\mathrm{B}^2(G, \mathbb{C}^*)$ is divisible, so there exists $\beta \in \mathrm{B}^2(G, \mathbb{C}^*)$ such that $\alpha^m = \beta^m$, and hence $\alpha\beta^{-1} \in U_2$. That means, we have $\alpha \in \mathrm{B}^2(G, \mathbb{C}^*)U^2$ and this finishes the proof.

So $U_2 = \{\alpha : G \to \{a \in A : a^m = 1\}\}$. Since $\{a \in \mathbb{C} : a^m = 1\}$ is finite; U^2 is finite. This yields to the fact that $\mathrm{H}^2(G, \mathbb{C}^*)$ is finite as it embeds in U^2. □

Chapter 6

Partial Zeta Series

We do not need this chapter in such detail to prove any of our claims. We nevertheless present it to keep the completeness, more importantly, to explain how partial zeta series arise when the representation zeta function of FAb compact p-adic analytic groups is concerned and to motivate their definition. As their names suggest, the first section is dedicated to Clifford theory for projective representations, whilst we treat the reduction steps to the partial zeta series in the second section.

This reduction idea originated from [JZ06], and it also allows us to work with the cohomology classes over quotients formed by Sylow pro-p groups, which is crucial to obtain linear characters while studying character triples. In [SZ20], the authors give a different proof with additional steps; we present their formulation as it is the base of this work.

6.1 Clifford Theory

Schur studied projective representations of finite groups thoroughly; however its connection with Clifford theory was first obtained by Clifford, [Cli37]. In this section, we introduce some parts of his work, which enable us to find a bijection between $\mathrm{Irr}(K|\psi)$ and $\mathrm{Irr}(K|v)$ for two character triples (K, N, ψ) and (K, N, v) where $\mathcal{C}_K(\psi) = \mathcal{C}_K(v)$ in $\mathrm{H}^2(K/N, \mathbb{C}^*)$. The concise exposition below follows [BKZ18].

Let G be a finite group, $\psi \in \mathrm{Irr}(N)$ where $N \triangleleft G$. Set $H = \mathrm{stab}_G(\theta)$, the inertia subgroup of ψ in G. If Ψ is an (ordinary) irreducible representation of N

affording ψ, then for any $h \in H$, $x \in N$,

$$\Psi^h(x) = P(h)\Psi(x)P(h)^{-1} \tag{6.1}$$

where $P(h) \in \mathrm{GL}_{\deg(\Psi)}(\mathbb{C})$. Then P is an irreducible projective representation of H, call its factor set α^{-1} (for notational convenience). Note that P is uniquely determined up to a factor $\mu : N \to \mathbb{C}^*$; therefore the equivalence class of the factor set α is uniquely determined by ψ. And the restriction to N, $\mathrm{Res}^H_{N,\alpha_N^{-1}}(P)$, can be given as $\lambda^{-1}\Psi$ for some $\lambda : N \to \mathbb{C}^*$.

Lemma 6.1.1. *[[BKZ18], Chp. VII, Lemma 3.3] In addition to the above setting, let θ be an irreducible character of H such that $\langle \mathrm{Res}^H_N, \psi \rangle > 0$; i.e. $\theta \in \mathrm{Irr}(H|\psi)$.*

(i) There exists an (ordinary) irreducible representation Δ of H affording θ such that $\mathrm{Res}^H_N(\Delta) = e\Psi$ where e is the ramification of θ over N, and an (ordinary) irreducible representation Ψ with the character ψ.

(ii) There exists an irreducible projective representation Ω of H with factor set α (of degree e) such that

$$\Delta = \Omega \otimes P.$$

(iii) Ω is uniquely determined by Δ, and for all $x \in N$, $\Omega(x) = \lambda(x)Id$ for some $\lambda : N \to \mathbb{C}^*$.

Remark 6.1.2. *One can assume that $\alpha(g_1, h_1) = \alpha(g_2, h_2)$ if $g_1N = g_2N, h_1N = h_2N$, and $\alpha(h, n) = \alpha(n, h) = 1$ for all $g_i, h_i, h \in H$ and $n \in N$ by multiplying P with $\mu : N \to \mathbb{C}^*$, if required. For the rest, we proceed with that assumption.*

Proposition 6.1.3. *[[BKZ18], Chp. VII, Lemma 3.5] Let $I_{\alpha,\lambda}(H)$ be the set of irreducible projective representations Ω of H with factor set α such that $\lambda^{-1}\mathrm{Res}^H_N(\Omega)$ is the identity representation of N for some $\lambda : N \to \mathbb{C}^*$.*

(i) If $\Omega \in I_{\alpha,\lambda}(H)$, then for all $x \in N, h \in H$, $\lambda(x)\Omega(h) = \Omega(xh)$. Moreover, λ is a linear character of N, and an H-invariant map.

(ii) Let $H = \bigcup_{a \in H/N} Nh_a$ be such that $h_1 = 1, h_ah_b = f(a,b)h_{ab}$ for some $f : H/N \times H/N \to N$. For $\Omega \in I_{\alpha,\lambda}(H)$ with $\deg(\Omega) = e$, define

$$\begin{aligned} \overline{\Omega} : H/N \to\ & \mathrm{GL}_e(\mathbb{C}) \\ a \mapsto\ & \Omega(h_a) \end{aligned}$$

Then $\overline{\Omega}$ is an irreducible projective representation (of degree e) of H/N with the following factor set $\bar{\alpha}$

$$\bar{\alpha}(a, b) = \alpha(h_a, h_b)\lambda(f(a, b)).$$

Proposition 6.1.4. *[[BKZ18], Chp. VII, Lemma 3.6] Let* $\mathrm{PIrr}_{\bar{\alpha}}(H/N)$ *denote the set of irreducible projective characters of* H/N *with factor set* $\bar{\alpha}$*. Then the following map is bijective*

$$\begin{aligned} I_{\alpha,\lambda}(H) &\to \mathrm{PIrr}_{\bar{\alpha}}(H/N) \\ \Omega &\mapsto \overline{\Omega} \end{aligned}$$

Let $\theta \in \mathrm{Irr}(H|\psi)$ and Δ_θ be an (ordinary) irreducible representation of H satisfying condition (i) of Lemma 6.1.1, i.e. $\mathrm{Res}_N^H(\Delta_\theta) = e_\delta \Psi$. Then there exists a unique projective representation Ω_θ such that $\Delta_\theta = \Omega_\theta \otimes P$. Let χ_θ denote the corresponding character; hence $\theta = \chi_\theta.\gamma$ where γ is the character of P. Accordingly, we denote the character of the representation of $\overline{\Omega}_\theta$ by $\overline{\chi}_\theta$. If $x \in N$, $a \in H/N$ and $xh_a \in Nh_a = a$, then

$$\lambda(x)\overline{\chi}_\theta(a) = \chi_\theta(xh_a).$$

Theorem 6.1.5. *[[BKZ18], Chp. VII, Theorem 3.10] There is a bijection of* $\mathrm{Irr}(H|\psi)$ *onto* $\mathrm{PIrr}_{\bar{\alpha}}(H/N)$ *given by* $\theta \mapsto \overline{\chi}_\theta$*.*

Suppose that G is a profinite group with an open normal subgroup N, and let (K, N, ψ) be a character triple. Then $\mathsf{stab}_K(\psi) = K$, since $K \le stab_G(\psi)$. Recall that ψ factors through a finite group, so we assume K to be H in the above setting, and that N is finite. Then the following corollary follows. Note that we go back to our original notation for factor sets, i.e. for a given factor set $\alpha' \in \mathrm{Z}^2(K, \mathbb{C}^*)$, we denote the well-defined element $\alpha \in \mathrm{Z}^2(K/N, \mathbb{C}^*)$ given by $\alpha(gN, hN) = \alpha'(g, h)$. Also α and α^{-1} in Theorem 6.1.5 will be swapped in the corollary below.

Corollary 6.1.6. *For a character triple* (K, N, ψ)*, let* ψ' *be a strong extension of* ψ *with factor set* $\alpha' \in \mathrm{Z}^2(K, \mathbb{C}^*)$ *such that* $\mathcal{C}_K(\psi) = [\alpha]$*.*

(i) [[SZ20], Lemma 3.7.] There is a one-to-one correspondence between $\mathrm{PIrr}_{\alpha^{-1}}(K/N)$ *and* $\mathrm{Irr}(K|\psi)$ *given by* $\pi' \mapsto \psi'\pi$*, where* π *is the pull-back of* π' *along the quotient map* $K \to K/N$ *(with factor set* $(\alpha')^{-1}$*).*

(ii) [[SZ20], Lemma 3.8.] Let v *be another irreducible character of* N *such that* $\mathsf{stab}_G(v) = K$ *and* $\mathcal{C}_K(\psi) = \mathcal{C}_K(v) = [\alpha]$ *for some* $\alpha \in \mathrm{Z}^2(K/N, \mathbb{C}^*)$*. Let* ψ', v' *be strong extensions of* ψ *and* v *respectively with a common factor set* $\alpha' \in \mathrm{Z}^2(K, \mathbb{C}^*)$*. Then the map* $\psi'\pi \mapsto v'\pi$*, for* π *as in* (i)*, gives a bijection* $\sigma : \mathrm{Irr}(K|\psi) \to \mathrm{Irr}(K|v)$ *such that*

$$\frac{(\psi'\pi)(1)}{\psi(1)} = \frac{(\sigma(\psi'\pi))(1)}{v(1)}.$$

Proof. (i) We begin with obtaining that P in Equation (6.1) on page 56 is a strong extension of ψ; in particular, one may assume $\gamma = \psi'$. Recall that P is uniquely determined up to a factor $\mu : N \to \mathbb{C}^*$; we may therefore require $P(x) = \Psi(x)$ for $x \in N$. Note also that

$$\pi(h, n) = \pi(n, h) = 1$$

for all $g \in K, n \in N$ by the remark following Lemma 6.1.1. Then P strongly extends Ψ as discussed after the definition of strong extension. Accordingly the following map gives the identity map on $\mathrm{PIrr}_{\alpha^{-1}}(K/N)$;

$$\begin{array}{ccccc} \sigma : \mathrm{Irr}(K|\psi) & \xrightarrow{\text{bij.}} & \mathrm{PIrr}_{\alpha^{-1}}(K/N) & \longrightarrow & \mathrm{Irr}(K|\psi) \\ \theta & \longmapsto & \overline{\chi}_\theta & \longmapsto & \gamma\chi_\theta \end{array}$$

since $\theta = \chi_\theta \cdot \gamma$, and the claimed bijection follows.

(ii) Consequently, we construct σ as follows:

$$\begin{array}{ccccc} \sigma : \mathrm{Irr}(K|\psi) & \xrightarrow{\text{bij.}} & \mathrm{PIrr}_{\alpha^{-1}}(K/N) & \xrightarrow{\text{bij.}} & \mathrm{Irr}(K|v) \\ \psi'\pi & \longmapsto & \pi' & \longmapsto & v'\pi \end{array}$$

The rest is to manipulate the degrees of the representations to get the claimed equality. Recall that $(\psi'\pi)(1) = \psi'(1)\pi(1)$ and $\psi'(n) = \psi(n)$ for all $n \in N$; hence we have $\frac{(\psi'\pi)(1)}{\psi(1)} = \pi(1)$. Same applies to v and v', we therefore obtain $\frac{(v'\pi)(1)}{v(1)} = \pi(1)$. □

6.2 Reduction to Partial Zeta Series

Let G be a finite group, $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. Suppose $\mathrm{stab}_G(\theta) = G$, and write $\mathrm{Ind}_N^G(\theta) = \sum e_i\chi_i$ for $\chi_i \in \mathrm{Irr}(G)$ and $e_i \geqslant 1$. Then we have

$$\mathrm{Res}_N^G(\chi_i) = e_i\theta.$$

We say that θ *allows an extension to* G if $e_i = 1$ for some i, or equivalently $\mathrm{Res}_N^G(\chi_i) = \theta$. In the following, we present a theorem collecting extension results but first we need to introduce the notion of *determinantal order* of a character.

Let $\chi \in \mathrm{Irr}(G)$, and let D be an (ordinary) representation of G affording χ. We define the *determinant of the character* χ, $\det(\chi) : G \to \mathbb{C}^*$ as follows:

$$(\det(\chi))(g) = \det(D(g)).$$

So $\det(\chi)$ becomes a linear character of G in $\mathrm{Irr}(G/[G,G])$. Then the *determinantal order* $o(\chi)$ of χ is defined as the order of the linear character $\mathrm{ord}(\det \chi)$ in the group of linear characters of G;

$$o(\chi) = \mathrm{ord}(\det \chi).$$

Theorem 6.2.1. *[[Hup98], Theorem 22.3] With the setting introduced at the beginning of the section, θ allows an extension χ to G if at least one of the following is satisfied:*

(i) *Any projective representation of the quotient group G/N is equivalent to an ordinary representation of G/N.*

(ii) *G/N is cyclic.*

(iii) *$((G : N), \theta(1)) = 1$ and $\det(\theta)$ allows an extension to G.*

(iv) *$((G : N), \theta(1)o(\theta)) = 1$.*

(v) *$((G : N), |N|) = 1$.*

Let G be a FAb compact p-adic analytic group with its open normal uniform pro-p subgroup N. For any subgroup K of G containing N, let P be a Sylow pro-p subgroup of K. Following this, we have $N \leq P$. We now introduce a construction from [SZ20], that adopts the idea of Jaikin-Zapirain, [JZ06] to reduce the rationality problem. Recall the function $\mathcal{C}_K$ introduced in Corollary 5.2.3 (*ii*). For $c \in \mathrm{H}^2(P/N, \mathbb{C}^*)$, now define the set

$$\mathrm{Irr}_K^c(N) = \{\theta \in \mathrm{Irr}_K(N) | \mathcal{C}_P(\theta) = c\}.$$

Observe that the set $\mathrm{Irr}_K^c(N)$ is independent of the choice of Sylow pro-p subgroup P as they are all G-conjugate.

Consider a character triple (K, N, θ), by Clifford's theorem 5.0.2, $\chi(1)/\theta(1)$ is an integer for each $\chi \in \mathrm{Irr}(K \mid \theta)$. Thus it makes sense to define the following (finite) Dirichlet series

$$f_{(K,N,\theta)}(s) = \sum_{\chi \in \mathrm{Irr}(K|\theta)} \left(\frac{\chi(1)}{\theta(1)}\right)^{-s}.$$

We first observe that we obtain the same series for two character triples (K, N, θ) and (K, N, v).

Proposition 6.2.2. *[[SZ20], Lemma 4.1.] Let (K, N, θ) and (K, N, v) be character triples for a finite index pro-p subgroup N of K. Then $\mathcal{C}_P(\theta) = \mathcal{C}_P(v)$ implies $\mathcal{C}_K(\theta) = \mathcal{C}_K(v)$, and $f_{(K,N,\theta)}(s) = f_{(K,N,v)}(s)$.*

Proof. Recall that, for any $c \in \mathrm{H}^2(K/N, \mathbb{C}^*)$, we have

$$c = \prod_{\substack{q|(K:N) \\ q \text{ prime}}} c_{(q)},$$

where $c_{(q)}$ is the q-part of c. Now let K_q be a subgroup of K such that K_q/N is a Sylow q-subgroup of K/N. Then, by Proposition 5.4.1 - (*iii*), we have the following injection

$$\mathbf{res}_q : \mathrm{H}^2(K/N, \mathbb{C}^*)_{(q)} \to \mathrm{H}^2(K_q/N, \mathbb{C}^*).$$

We first observe that $\mathbf{res}_q$ maps $\mathcal{C}_K(\theta)_{(q)}$ to $\mathcal{C}_{K_q}(\theta)$ in $\mathrm{H}^2(K_q/N, \mathbb{C}^*)$. To this aim, recall the restriction map

$$\mathbf{res}^{K/N}_{K_q/N} : \mathrm{H}^2(K/N, \mathbb{C}^*) \to \mathrm{H}^2(K_q/N, \mathbb{C}^*).$$

Our claim is that $\mathbf{res}^{K/N}_{K_q/N}(\mathcal{C}_K(\theta)) = \mathcal{C}_{K_q}(\theta)$. Note that $\mathrm{Res}^K_{K_q,\alpha_{K_q}}(\theta')$ strongly extends θ as θ' is a strong extension of θ with factor set α. Therefore α_{K_q} determines the element $\mathbf{res}^{K/N}_{K_q/N}(\mathcal{C}_K(\theta))$ in $\mathrm{H}^2(K_q/N, \mathbb{C}^*)$, and the claim follows.

On the other hand, $\mathrm{H}^2(K_q/N, \mathbb{C}^*)$ is a q-group; i.e. the map $\mathbf{res}^{K/N}_{K_q/N}$ is only non-trivial on $\mathrm{H}^2(K/N, \mathbb{C}^*)_{(q)}$. Consequently, for any $c \in \mathrm{H}^2(K/N, \mathbb{C}^*)$,

$$\mathbf{res}^{K/N}_{K_q/N}(c) = \mathbf{res}^{K/N}_{K_q/N}(c_{(q)}) = \mathbf{res}_q(c_{(q)}).$$

In particular, $\mathbf{res}_q(\mathcal{C}_K(\theta)_{(q)}) = \mathbf{res}^{K/N}_{K_q/N}(\mathcal{C}_K(\theta)) = \mathcal{C}_{K_q}(\theta)$.

Now we apply Theorem 6.2.1 (iii) to see that θ extends to K_q. To this end, note that $\theta(1)$ is a p-power since N is a pro-p group. And one can easily obtain that $o(\theta)$ is also a p-power since θ factors through a finite quotient of order p-power. Recall that K_q/N is chosen as a Sylow q-subgroup of K/N, so $(K_q : N)$ must be a q-power. Therefore, when we assume $q \neq p$, $p \nmid (K_q : N)$, consequently θ extends to K_q. Then θ gets mapped to the class of 1; $\mathcal{C}_{K_q}(\theta) = [1]$. Then, combining with $\mathbf{res}_q$ being injective, we have

$$\mathbf{res}_q(\mathcal{C}_K(\theta)_{(q)}) = 1 \Longrightarrow \mathcal{C}_K(\theta)_{(q)} = 1.$$

Therefore, by writing $\mathcal{C}_K(\theta) = \mathcal{C}_K(\theta)_{(p)}\mathcal{C}_K(\theta)_{(p')}$ where $\mathcal{C}_K(\theta)_{(p')} = \prod_{q \neq p} \mathcal{C}_K(\theta)_{(q)}$ as given in Proposition 5.4.1 (*iii*), we obtain $\mathcal{C}_K(\theta) = \mathcal{C}_K(\theta)_{(p)}$. The same argument applies to v; hence $\mathcal{C}_K(v) = \mathcal{C}_K(v)_{(p)}$. So for $q = p$, we get

$$\mathbf{res}_p(\mathcal{C}_K(\theta)_{(p)}) = \mathcal{C}_P(\theta) = \mathcal{C}_P(v) = \mathbf{res}_p(\mathcal{C}_K(v)_{(p)}).$$

Hence we have $\mathcal{C}_K(\theta)_{(p)} = \mathcal{C}_K(v)_{(p)}$, and thus $\mathcal{C}_K(\theta) = \mathcal{C}_K(v)$.

For the second part, we combine what we obtained above with Corollary 6.1.6; there is a bijection $\sigma : \mathrm{Irr}(K|\theta) \to \mathrm{Irr}(K|v)$ such that $\frac{\chi(1)}{\theta(1)} = \frac{\sigma(\chi)(1)}{v(1)}$. Following this, we have

$$f_{(K,N,\theta)}(s) = \sum_{\chi\in\mathrm{Irr}(K|\theta)} \left(\frac{\chi(1)}{\theta(1)}\right)^{-s} = \sum_{\sigma(\chi)\in\mathrm{Irr}(K|v)} \left(\frac{\sigma(\chi)(1)}{v(1)}\right)^{-s} = f_{(K,N,v)}(s). \quad \square$$

By the Clifford theorem (5.0.2), for each $\rho \in \mathrm{Irr}(G)$, there are $(G : \mathrm{stab}_G(\theta))$ distinct characters $\theta \in \mathrm{Irr}(N)$ such that $\rho \in \mathrm{Irr}(G \mid \theta)$. Therefore, one obtains easily that

$$\zeta^G(s) = \sum_{\rho\in\mathrm{Irr}(G)} \rho(1)^{-s} = \sum_{\theta\in\mathrm{Irr}(N)} \frac{1}{(G : \mathrm{stab}_G(\theta))} \sum_{\rho\in\mathrm{Irr}(G|\theta)} \rho(1)^{-s}.$$

As we pointed out in Theorem 5.0.3 (*iii*), $|\mathrm{Irr}(G \mid \theta)| = |\mathrm{Irr}(\mathrm{stab}_G(\theta) \mid \theta)|$ for any $\theta \in \mathrm{Irr}(N)$. So the above equation becomes

$$\begin{aligned}
\zeta^G(s) &= \sum_{\theta\in\mathrm{Irr}(N)} \frac{1}{(G : \mathrm{stab}_G(\theta))} \sum_{\lambda\in\mathrm{Irr}(\mathrm{stab}_G(\theta)|\theta)} (\lambda(1)\cdot(G : \mathrm{stab}_G(\theta)))^{-s} \\
&= \sum_{\theta\in\mathrm{Irr}(N)} (G : \mathrm{stab}_G(\theta))^{-s-1} \sum_{\lambda\in\mathrm{Irr}(\mathrm{stab}_G(\theta)|\theta)} \theta(1)^{-s}\cdot\left(\frac{\lambda(1)}{\theta(1)}\right)^{-s} \\
&= \sum_{\theta\in\mathrm{Irr}(N)} (G : \mathrm{stab}_G(\theta))^{-s-1}\theta(1)^{-s}\cdot f_{(\mathrm{stab}_G(\theta),N,\theta)}(s)
\end{aligned}$$

Now consider the set of subgroups K of G such that $N \leq K \leq G$ and $\mathrm{stab}_G(\theta) = K$ for some $\theta \in \mathrm{Irr}(N)$, and call this set $\mathcal{S}$. Then the last terms of the equation can be written as

$$\begin{aligned}
\zeta^G(s) &= \sum_{K\in\mathcal{S}} (G : K)^{-s-1} \sum_{\theta\in\mathrm{Irr}_K(N)} \theta(1)^{-s}\cdot f_{(K,N,\theta)}(s) \\
&= \sum_{K\in\mathcal{S}} (G : K)^{-s-1} \sum_{c\in\mathrm{H}^2(P/N)} f_{(K,N,\theta)}(s) \sum_{\theta\in\mathrm{Irr}_K^c(N)} \theta(1)^{-s}
\end{aligned}$$

We will call the part $\sum_{\theta\in\mathrm{Irr}_K^c(N)} \theta(1)^{-s}$ *partial zeta series* and denote by $\zeta^{(N,K,c)}(s)$. As the set $\mathcal{S}$ and the group $\mathrm{H}^2(P/N)$ are finite, the virtual rationality of $\zeta^G(s)$ follows from the rationality of the partial zeta series $\zeta^{(N,K,c)}(s)$.

Chapter 7

Tools for constructing equivalence classes

In the following chapter, we will provide machinery to construct uniformly definable equivalence classes that give rise to the partial zeta series following [SZ20]. As in the previous chapter, we follow their proofs for the cited statements to keep the text self-contained.

The first section discusses how linear characters come into play precisely, which is essential in terms of definability as we can only proceed by degree one characters. These types of arguments are also central for other works on the representation (twist) zeta function employing model theory, such as [HMR18], [JZ06].

The second section initiates with a discussion about reducing the limit of a given cohomology class $c \in \mathrm{H}^2(P, \mathbb{C}^*)$ and outlines how to describe the fibres of the map assigning the character triples with linear characters to the elements of $\mathrm{H}^2(P, \mathbb{C}^*)$. We then present two interludes: one for the group $\mathbb{Q}_p/\mathbb{Z}_p$ and the other one for extending good basis and showing that it is uniformly definable. We then improve the ideas from [SZ20] on describing such fibers to make their construction uniform.

7.1 Classes in $\mathrm{H}^2(P, \mathbb{C}^*)$ and Linear Characters

Let G be a profinite group having a finite index normal pro-p subgroup $N \leqslant G$. For any $K \leqslant G$ such that $N \leqslant K$, we will define the set

$$\mathcal{H}(K) = \{H \leq K : H \text{ open in } K, K = HN\}.$$

In the following, we consider a Sylow subgroup pro-p P of K. As we have a normal pro-p subgroup N of G contained in K, $N \leq P$ immediately follows.

For $H \leq P$ such that $P = HN$, one can obtain a one-to-one correspondence between 2-cocycles of P/N and $H/(N \cap H)$. To this end, note first that for each coset gN in P/N, there exists a unique coset $h(N \cap H)$ in $H/(N \cap H)$ such that $h(N \cap H) \subseteq gN$.

The isomorphism $P/N \to H/(N \cap H)$ induces an isomorphism between $\mathrm{Z}^2(H/(N \cap H), \mathbb{C}^*)$ and $\mathrm{Z}^2(P/N, \mathbb{C}^*)$ by pulling back 2-cocycles;

$$\widetilde{f_H} : \mathrm{Z}^2(H/(N \cap H), \mathbb{C}^*) \to \mathrm{Z}^2(P/N, \mathbb{C}^*) \tag{7.1}$$

For $\alpha \in \mathrm{Z}^2(H/(N \cap H), \mathbb{C}^*)$ and $g, g' \in P$, $\widetilde{f_H}$ is defined by

$$\widetilde{f_H}(\alpha)(gN, g'N) = \alpha(h(N \cap H), h'(N \cap H)),$$

where $h(N \cap H) \subseteq gN$ and $h'(N \cap H) \subseteq g'N$. Furthermore, for $\beta \in \mathrm{Z}^2(P/N, \mathbb{C}^*)$ and $h, h' \in H$, we have

$$\widetilde{f_H}^{-1}(\beta)(h(N \cap H), h'(N \cap H)) = \beta(hN, h'N).$$

Therefore, $\widetilde{f_H}$ induces an isomorphism f_H between the cohomology groups

$$f_H : \mathrm{H}^2(H/(N \cap H), \mathbb{C}^*) \to \mathrm{H}^2(P/N, \mathbb{C}^*) \ \text{ given by } \ f_H([\alpha]) = [\widetilde{f_H}(\alpha)].$$

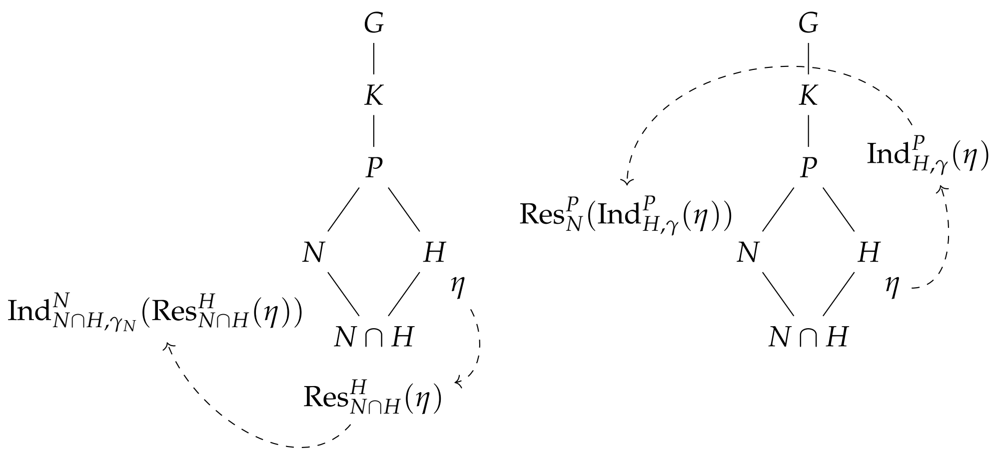
In this section, for a given character triple (K, N, θ), we will obtain a character triple $(H, N \cap H, \chi)$ with a linear character χ such that

$$\mathcal{C}_P(\theta) = f_H(\mathcal{C}_H(\chi)).$$

Lemma 7.1.1. *[[SZ20], Lemma 5.1.] Let γ be a 2-cocycle in $\mathrm{Z}^2(P, \mathbb{C}^*)$. Suppose that $H \in \mathcal{H}(P)$ and that η is a linear projective character of H with the factor set γ_H. If one of $\mathrm{Ind}_{N \cap H, \gamma_N}^{N}(\mathrm{Res}_{N \cap H}^{H}(\eta))$ and $\mathrm{Res}_{N}^{P}(\mathrm{Ind}_{H,\gamma}^{P}(\eta))$ is irreducible, then*

$$\mathrm{Ind}_{N \cap H, \gamma_N}^{N}(\mathrm{Res}_{N \cap H}^{H}(\eta)) = \mathrm{Res}_{N}^{P}(\mathrm{Ind}_{H,\gamma}^{P}(\eta)).$$

The groups and corresponding characters given above can be illustrated by the following:



Proof. We first apply Frobenius Reciprocity to obtain

$$\left\langle \operatorname{Ind}^{N}_{N\cap H,\gamma_N}(\operatorname{Res}^{H}_{N\cap H}(\eta)), \operatorname{Res}^{P}_{N}(\operatorname{Ind}^{P}_{H,\gamma}(\eta)) \right\rangle_N = \left\langle \operatorname{Res}^{H}_{N\cap H}(\eta), \underbrace{\operatorname{Res}^{N}_{N\cap H}(\operatorname{Res}^{P}_{N}}_{\operatorname{Res}^{P}_{N\cap H}}(\operatorname{Ind}^{P}_{H,\gamma}\,\eta)) \right\rangle_{N\cap H}.$$

Also, Mackey's theorem gives, for any (double coset) representative g of $\bar{g} \in (N \cap H)\backslash P/H$

$$\operatorname{Res}^{P}_{N\cap H}(\operatorname{Ind}^{P}_{H,\gamma}(\eta)) = \sum_{\bar{g}\in(N\cap H)\backslash P/H} \operatorname{Ind}^{N\cap H}_{N\cap H\cap {}^gH,\gamma_{N\cap H}}(\operatorname{Res}^{{}^gH}_{N\cap H\cap {}^gH}({}^g\eta)).$$

Following this, we have

$$\begin{aligned}\left\langle \operatorname{Res}^{H}_{N\cap H}(\eta), \operatorname{Res}^{P}_{N\cap H}(\operatorname{Ind}^{P}_{H,\gamma}(\eta)) \right\rangle_{N\cap H} &= \sum_{\bar{g}\in(N\cap H)\backslash P/H} \left\langle \operatorname{Res}^{H}_{N\cap H}(\eta), \operatorname{Ind}^{N\cap H}_{N\cap H\cap {}^gH,\gamma_{N\cap H}}(\operatorname{Res}^{{}^gH}_{N\cap H\cap {}^gH}\,{}^g(\eta)) \right\rangle_{N\cap H} \\ &\geq \left\langle \operatorname{Res}^{H}_{N\cap H}(\eta), \operatorname{Res}^{H}_{N\cap H}(\eta) \right\rangle_{N\cap H} = 1\end{aligned}$$

Recall that $(P : N) \cdot (N : N \cap H) = (P : H) \cdot (H : N \cap H) = (P : H) \cdot (HN : N)$ and $P = HN$, then $(P : N) = (N : N \cap H)$ follows. This yields to that the degrees of $\operatorname{Ind}^{N}_{N\cap H,\gamma N} \operatorname{Res}^{H}_{N\cap H}\,\eta$ and $\operatorname{Res}^{P}_{N} \operatorname{Ind}^{P}_{H,\gamma}\,\eta$ are the same; hence irreducibility of one of them implies that they are equal. □

Proposition 7.1.2. *[[SZ20], Proposition 5.2.]*

(1) *For any character triple (K, N, θ), there exists an $H \in \mathcal{H}(P)$ and a character triple $(H, N \cap H, \chi)$ such that:*

(i) *χ is of degree one,*

(ii) *$\theta = \mathrm{Ind}_{N\cap H}^{N} \chi$*

(iii) *$\mathcal{C}_P(\theta) = f_H(\mathcal{C}_H(\chi))$*

(2) *Let $H \in \mathcal{H}(P)$ be such that $(H, N \cap H, \chi)$ is a character triple with a linear character χ such that, $(K, N, \mathrm{Ind}_{N\cap H}^{N} \chi)$ is a character triple. Then*

$$\mathcal{C}_P(\theta) = f_H(\mathcal{C}_H(\chi)).$$

Proof. Suppose that (K, N, θ) is a character triple. Then, by Corollary 5.2.3 (*ii*), one can find a 2-cocycle $\alpha \in Z^2(P/N, \mathbb{C}^*)$ such that $[\alpha] = \mathcal{C}_P(\theta)$. Moreover there exists an irreducible projective character θ' of P with factor set α' strongly extending θ by Theorem 5.2.2.

By Corollary 5.3.8, we have an open subgroup H of P and a linear projective character η of H with factor set α'_H such that $\theta' = \mathrm{Ind}_{H,\alpha'}^{P}(\eta)$. Then restricting the projective representation θ' to N, we get $\theta = \mathrm{Res}_N^P(\mathrm{Ind}_{H,\alpha'}^{P}(\eta))$. Then

$$\begin{aligned}
1 &= \left\langle \theta, \theta \right\rangle \\
&= \left\langle \mathrm{Res}_N^P(\mathrm{Ind}_{H,\alpha'}^{P}(\eta)), \mathrm{Res}_N^P(\mathrm{Ind}_{H,\alpha'}^{P}(\eta)) \right\rangle \\
&= \sum_{\bar{g}\in N\backslash P/H} \sum_{\bar{h}\in N\backslash P/H} \left\langle \mathrm{Ind}_{N\cap {}^gH}^{N}(\mathrm{Res}_{N\cap {}^gH}^{{}^gH}({}^g\eta)), \mathrm{Ind}_{N\cap {}^hH}^{N}(\mathrm{Res}_{N\cap {}^hH}^{{}^hH}({}^h\eta)) \right\rangle \\
&= \sum_{\bar{g}\in P/HN} \sum_{\bar{h}\in P/HN} \left\langle \mathrm{Ind}_{N\cap {}^gH}^{N}(\mathrm{Res}_{N\cap {}^gH}^{{}^gH}({}^g\eta)), \mathrm{Ind}_{N\cap {}^hH}^{N}(\mathrm{Res}_{N\cap {}^hH}^{{}^hH}({}^h\eta)) \right\rangle \\
&\geq \sum_{\bar{g}\in P/HN} \left\langle \mathrm{Ind}_{N\cap {}^gH}^{N}(\mathrm{Res}_{N\cap {}^gH}^{{}^gH}({}^g\eta)), \mathrm{Ind}_{N\cap {}^gH}^{N}(\mathrm{Res}_{N\cap {}^gH}^{{}^gH}({}^g\eta)) \right\rangle \\
&\geq (P : HN)
\end{aligned}$$

This implies that $(P : HN) = 1$, i.e. $P = HN$; hence $H \in \mathcal{H}(P)$.

As $\theta = \mathrm{Res}_N^P(\mathrm{Ind}_{H,\alpha'}^{P}(\eta))$ is irreducible, by Lemma 7.1.1, we have $\theta = \mathrm{Ind}_{N\cap H}^{H}(\mathrm{Res}_{N\cap H}^{H}(\eta))$. Set $\chi = \mathrm{Res}_{N\cap H}^{H}(\eta)$; hence χ is fixed by H. In addition, we obtain a 2-cocycle $\alpha_H \in Z^2(H/(N\cap H), \mathbb{C}^*)$ defined by

$$\alpha_H(h(N\cap H), h'(N\cap H)) = \alpha'_H(h, h').$$

By (7.1) on page 63, we get $f_H([\alpha_H]) = [\widetilde{f_H}(\alpha_H)] = [\alpha] = \mathcal{C}_P(\theta)$. Note that η strongly extends χ; hence the function $\mathcal{C}_H$ defined in Corollary 5.2.3 (*ii*)

$$\mathcal{C}_H : \{\chi \in \mathrm{Irr}(N \cap H) : H \leq \mathsf{stab}_G(\chi)\} \to \mathrm{H}^2(H/(N \cap H), \mathbb{C}^*)$$

maps χ to $[\alpha_H]$. Thus we have $\mathcal{C}_P(\theta) = f_H(\mathcal{C}_H(\chi))$. This finishes the first part of the proof.

Choose now a subgroup $H \in \mathcal{H}(P)$ such that $(H, N \cap H, \chi)$ is a character triple with a linear character χ and $(K, N, \mathrm{Ind}_{N\cap H}^N \chi)$ is a character triple. Let $\theta = \mathrm{Ind}_{N\cap H}^N(\chi)$. Considering that $(H, N \cap H, \chi)$ is a character triple, we have $\beta \in \mathrm{Z}^2(H/(N \cap H), \mathbb{C}^*)$. So there exists a projective character χ' of H with factor set β' strongly extending χ such that $\beta'(g, h) = \beta(g(N \cap H), h(N \cap H))$ by Theorem 5.2.2. According to Corollary 5.2.3 (*ii*), we have $[\beta] = \mathcal{C}_H(\chi)$. Recall that $\widetilde{f_H}(\beta) \in \mathrm{Z}^2(P/N, \mathbb{C}^*)$. Let $\gamma \in \mathrm{Z}^2(P, \mathbb{C}^*)$ be such that, for any $g, g' \in P$,

$$\gamma(g, g') = \widetilde{f_H}(\beta)(gN, g'N).$$

For any $h, h' \in H$, we have

$$\gamma_H(h, h') = \gamma(h, h') = \widetilde{f_H}(\beta)(hN, h'N) = \beta(h(N \cap H), h'(N \cap H)) = \beta'(h, h').$$

Thus $\gamma_H = \beta'$. Recall now that the projective character χ' of H strongly extends χ with factor set β'. Therefore we have the following by Lemma 7.1.1 as θ is irreducible

$$\begin{aligned}\theta = \mathrm{Ind}_{N\cap H}^N(\chi) &= \mathrm{Ind}_{N\cap H,\gamma_N}^N(\mathrm{Res}_{N\cap H}^H(\chi')) \\ &= \mathrm{Res}_N^P(\mathrm{Ind}_{H,\gamma}^P(\chi'))\end{aligned}$$

Hence the projective representation $\mathrm{Ind}_{H,\gamma}^P(\chi')$ of P is an extension of θ. We now verify that $\mathrm{Ind}_{H,\gamma}^P(\chi')$ strongly extends θ with factor set γ. Recall the discussion following Definition 5.2.1, that is $\gamma(x, n) = \gamma(n, x) = 1$ for all $x \in P$ and $n \in N$. By definition γ is constant on the cosets of N in P, so we write $x = hn'$ with $h \in H, n' \in N$ and obtain the following

$$\gamma(x, n) = \gamma(hn', n) = \gamma(h, 1) = \gamma_H(h, 1) = \beta'(h, 1).$$

Moreover we know that $\beta'(h, 1) = 1$ as β' is the factor set of a strong extension of χ; hence $\gamma(x, n) = 1$. In a similar way, one can see that $\gamma(n, x) = 1$; we conclude that $\mathrm{Ind}_{H,\gamma}^P(\chi')$ strongly extends θ. Since $\mathrm{Ind}_{H,\gamma}^P(\chi')$ has factor set γ that is given by $\gamma(g, g') = \widetilde{f_H}(\beta)(gN, g'N)$, we get

$$\mathcal{C}_P(\theta) = [\widetilde{f_H}(\beta)] = f_H([\beta]) = f_H(\mathcal{C}_H(\chi)). \qquad \square$$

Remark 7.1.3. *Let X_K be the set of pairs (H, χ) with $H \in \mathcal{H}(P)$ where*

(i) $(H, N \cap H, \chi)$ *is a character triple,*

(ii) χ *is of degree one,*

(iii) $\mathrm{Ind}_{N \cap H}^{N} \chi \in \mathrm{Irr}_K(N)$.

One can define the following function $\mathcal{C}$ to assign the pairs (H, χ) to the elements of $\mathrm{H}^2(P/N, \mathbb{C}^*)$

$$\begin{aligned} \mathcal{C} : & X_K \to \mathrm{H}^2(P/N, \mathbb{C}^*) \\ & (H, \chi) \mapsto f_H(\mathcal{C}_H(\chi)) \end{aligned}$$

Corollary 7.1.4. *[[SZ20], Corollary 5.3.] We have a surjective function $X_K \to$* $\mathrm{Irr}_K(N)$ *and the following commutative diagram*

$$\begin{array}{ccl} X_K & \longrightarrow & \mathrm{Irr}_K(N); (H, \chi) \mapsto \mathrm{Ind}_{N \cap H}^{N}(\chi) \\ & \searrow^{\mathcal{C}} & \downarrow \mathcal{C}_P \\ & & \mathrm{H}^2(P/N, \mathbb{C}^*) \end{array}$$

Proof. By definition, each $\theta \in \mathrm{Irr}_K(N)$ produces a character triple (K, N, θ). Hence the first part of Proposition 7.1.2 gives the surjectivity. The commutativity of the diagram, i.e. for any $(H, \chi) \in X_k$, $\mathcal{C}_P(\mathrm{Ind}_{N \cap H}^{N}(\chi)) = f_H(\mathcal{C}_H(\chi)) = \mathcal{C}(H, \chi)$ follows from the second part of Proposition 7.1.2. □

We finalize this section with a lemma, which will be used to express $K = \mathsf{stab}_G(\theta)$, for some $\theta \in \mathrm{Irr}(N)$, in a uniformly definable way.

Lemma 7.1.5. *[[SZ20], Lemma 6.8.] Let M be a finite index subgroup of N, and let χ be a linear character of M.*

(i) For all $g \in G$, we have

$$^{g}(\mathrm{Ind}_{M}^{N}(\chi)) = \mathrm{Ind}_{^{g}M}^{N}(^{g}\chi).$$

(ii) Let M' be also a finite index subgroup of N, and suppose that χ, χ' are linear characters of M and M' respectively, satisfying that $\mathrm{Ind}_{M}^{N} \chi$ *and* $\mathrm{Ind}_{M'}^{N} \chi'$ *are irreducible. Then the following holds:*

$$\mathrm{Ind}_{M}^{N}(\chi) = \mathrm{Ind}_{M'}^{N}(\chi') \Leftrightarrow \exists g \in N : \mathrm{Res}_{^{g}M \cap M'}^{^{g}M}(^{g}\chi) = \mathrm{Res}_{^{g}M \cap M'}^{M'}(\chi').$$

Proof. By Frobenius Reciprocity for ordinary characters, we have

$$\left\langle \mathrm{Ind}_M^N(\chi), \mathrm{Ind}_{M'}^N(\chi') \right\rangle_N = \left\langle \mathrm{Res}_{M'}^N(\mathrm{Ind}_M^N(\chi)), \chi' \right\rangle_N.$$

Now we apply Mackey's formula for ordinary characters this time

$$\mathrm{Res}_{M'}^N(\mathrm{Ind}_M^N(\chi)) = \sum_{\bar{g} \in M' \backslash N / M} \mathrm{Ind}_{{}^gM \cap M'}^{M'}(\mathrm{Res}_{{}^gM \cap M'}^{{}^gM}({}^g\chi)).$$

So we have

$$\begin{aligned}\left\langle \mathrm{Ind}_M^N(\chi), \mathrm{Ind}_{M'}^N(\chi') \right\rangle_N &= \sum_{\bar{g} \in M' \backslash N / M} \left\langle \mathrm{Ind}_{{}^gM \cap M'}^{M'}(\mathrm{Res}_{{}^gM \cap M'}^{{}^gM}({}^g\chi)), \chi' \right\rangle_N \\ &= \sum_{\bar{g} \in M' \backslash N / M} \left\langle (\mathrm{Res}_{{}^gM \cap M'}^{{}^gM}({}^g\chi)), \mathrm{Res}_{{}^gM \cap M'}^{M'} \chi' \right\rangle_{{}^gM \cap M'}.\end{aligned}$$

Note that the last equation follows from Frobenius Reciprocity again. Then, this vanishes if and only if each of the summands vanishes. This holds if and only if $\mathrm{Res}_{{}^gM \cap M'}^{{}^gM}({}^g\chi) \neq \mathrm{Res}_{{}^gM \cap M'}^{M'}(\chi')$ for each $g \in G$ as the characters we worked with are linear. □

7.2 Describing the fibres in terms of linear characters

We assume G to be a FAb compact p-adic analytic group with a normal uniform pro-p subgroup $N \leq G$. Let K be a subgroup of G such that $N \leq K$ with a Sylow pro-p subgroup P of K. In this section, we aim to describe the set $\mathcal{C}^{-1}(c)$ for a fixed $c \in \mathrm{H}^2(P/N, \mathbb{C}^*)$ by the elements of N and linear characters of finite index subgroups of N.

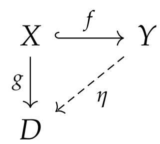
First we see how to narrow the range for c to $\mathrm{H}^2(P/N, \Omega_{(p)})$ where $\Omega_{(p)}$ is the group of roots of unity of a power of p. It is followed by two short interludes explaining the group $\mathbb{Q}_p/\mathbb{Z}_p$ and how to give a parametrization of the set $\mathcal{H}(P)$. We conclude the section by providing a criteria for $\mathcal{C}(H, \chi) = c$ involving the coboundaries with values in $\mathrm{B}^2(P/N, \Omega_{(p)})$ with a uniformly definable parametrization of $\mathrm{Z}^2(P/N, \Omega_{(p)})$ and $\mathrm{B}^2(P/N, \Omega_{(p)})$.

Now consider the group $\Omega \leqslant \mathbb{C}^*$ of all complex roots of unity. It is torsion, and, for a prime p, its p-primary part is the subgroup of pth roots of unity, call it $\Omega_{(p)}$. Then

$$\Omega = \bigoplus_{p \text{ prime}} \Omega_{(p)}.$$

We now see how $\mathrm{Z}^2(P/N, \Omega_{(p)})$ and $\mathrm{B}^2(P/N, \Omega_{(p)})$ relate to $\mathrm{Z}^2(P/N, \mathbb{C}^*)$ and $\mathrm{B}^2(P/N, \mathbb{C}^*)$ In order to restrict the range of functions whose image is contained in $\mathbb{C}^*$ to $\Omega_{(p)}$, we begin with the notion of injective group .

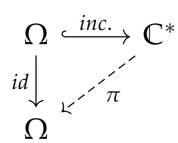
Definition 7.2.1. A group D is called *injective*, if for every diagram



consisting a monomorphism $f : X \to Y$ and a homomorphism $g : X \to D$, there exists a homomorphism $\eta : Y \to D$ such that the diagram commutes; i.e. $g = \eta \circ f$.

Theorem 7.2.2. *[[Fuc03], Theorem. 21.1] Divisible groups are injective.*

To see how injective groups arise in our setting, recall first that Ω is a divisible group. It is therefore injective by the previous result. We use the above diagram to produce a map $\mathbb{C}^* \to \Omega$ which enables us to define the restriction map mentioned above. Given the inclusion map $\Omega \to \mathbb{C}^*$ and the identity map on Ω, we have $\pi : \mathbb{C}^* \to \Omega$ such that the following diagram commutes



For each prime p, we define $\pi_{(p)} : \mathbb{C}^* \to \Omega_{(p)}$ as the composition of π and the projection $\Omega \to \Omega_{(p)}$. Accordingly, for a function f which has values in $\mathbb{C}^*$, we define

$$f_{(p)} := \pi_{(p)} \circ f.$$

As $\pi_{(p)}$ is a homomorphism, for any f and f' which have image in $\mathbb{C}^*$, we have

$$(ff')_{(p)} = f_{(p)} f'_{(p)}.$$

Proposition 7.2.3. $\mathrm{B}^2(P/N, \mathbb{C}^*) \cap \mathrm{Z}^2(P/N, \Omega_{(p)}) = \mathrm{B}^2(P/N, \Omega_{(p)})$, *and we have the following isomorphism*

$$\mathrm{H}^2(P/N, \mathbb{C}^*) \simeq \mathrm{Z}^2(P/N, \Omega_{(p)}) / \mathrm{B}^2(P/N, \Omega_{(p)}).$$

Proof. As the Lemma 5.4.2 indicates, we have

$$\mathrm{Z}^2(P/N,\mathbb{C}^*) = \mathrm{B}^2(P/N,\mathbb{C}^*).U^2,$$

where $U^2 = \{\alpha \in \mathrm{Z}^2(P/N,\mathbb{C}^*) : \alpha^r = 1\}$ and $r = (P : N)$. Therefore each class in $\mathrm{H}^2(P/N,\mathbb{C}^*)$ has a representative in $\mathrm{Z}^2(P/N,\Omega_{(p)})$. So $\mathrm{B}^2(P/N,\Omega_{(p)})$ lies in the intersection $\mathrm{B}^2(P/N,\mathbb{C}^*) \cap \mathrm{Z}^2(P/N,\Omega_{(p)})$.

In addition, for any $\delta \in \mathrm{B}^2(P/N,\mathbb{C}^*) \cap \mathrm{Z}^2(P/N,\Omega_{(p)})$, we have a function $\mu : P/N \to \mathbb{C}^*$ such that for all $a, b \in P/N$,

$$\delta(a,b) = \frac{\mu(ab)}{\mu(a)\mu(b)}.$$

On the other hand, as $\delta \in \mathrm{Z}^2(P/N,\Omega_{(p)})$, we know $\delta : P/N \to \Omega_{(p)}$. Therefore, for all $a, b \in P/N$, we have

$$\delta(a,b) = \delta_{(p)}(a,b) = \frac{\mu_{(p)}(ab)}{\mu_{(p)}(a)\mu_{(p)}(b)}.$$

Hence $\delta \in \mathrm{B}^2(P/N,\Omega_{(p)})$, and $\mathrm{B}^2(P/N,\mathbb{C}^*) \cap \mathrm{Z}^2(P/N,\Omega_{(p)}) = \mathrm{B}^2(P/N,\Omega_{(p)})$. □

7.2.1 Interlude # 1: The group $\mathbb{Q}_p/\mathbb{Z}_p$.

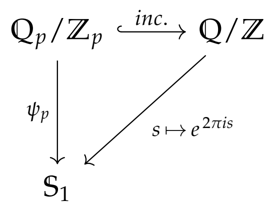
Any p-adic number $x \in \mathbb{Q}_p$ can be written (uniquely) as a sum of a p-adic integer and its fractional part that is a rational number $0 \leq q < 1$ whose denominator a power of p. Accordingly we define *the p-adic fractional part* $\{x\}_p$ of $x = p^{-n}\sum_{i\geq 0} x_i p^i$ as follows:

$$\{x\}_p = \begin{cases} p^{-n}(x_0 + x_1 p + x_2 p^2 + \ldots + x_{n-1}p^{n-1}), & \text{if } \ n > 0 \\ 0, & \text{if } \ n \leq 0 \ \text{ or } \ x = 0 \end{cases}$$

Then, for any x, we have $\{x\}_p = \frac{a}{p^n} \in [0,1)$, i.e. $0 \leqslant a < p^n$. We now define

$$\begin{aligned} \psi_p : \mathbb{Q}_p &\to \mathbb{S}_1 \subseteq \mathbb{C}^* \\ x &\mapsto e^{2\pi i \{x\}_p}, \end{aligned}$$

where $\mathbb{S}_1 = \{z \in \mathbb{C}^* : |z| = 1\}$ is the unit circle. The function ψ_p is a group homomorphism as the difference $\{x\}_p + \{y\}_p - \{x+y\}_p$ is an integer. And we have the following commutative diagram:



The image of ψ_p is exactly the subgroup $\Omega_{(p)} \leqslant \Omega$ of roots of unity of a power of p - it is also called *Prüfer p-group*. Therefore

$$\begin{aligned}\Omega_{(p)} &= \{z \in \mathbb{C}^* : z^{p^n} = 1 \text{ for some } \; n \in \mathbb{Z}^+\} \\ &= \{e^{2\pi i a/p^n} : 0 \leqslant a < p^n, \; n \in \mathbb{Z}^+\}.\end{aligned}$$

To describe the kernel of ψ_p, recall that $x - \{x\}_p \in \mathbb{Z}_p$. It immediately follows that $\{x\}_p = 0 \Leftrightarrow x \in \mathbb{Z}_p$, in particular $\ker(\psi_p) = \mathbb{Z}_p$. Consequently, the groups $\mathbb{Q}_p/\mathbb{Z}_p$ and $\Omega_{(p)}$ are isomorphic via the map

$$\begin{aligned}\iota : \mathbb{Q}_p/\mathbb{Z}_p &\to \; \Omega_{(p)} \\ \frac{a}{p^n} \; &\mapsto \; e^{2\pi i a/p^n}.\end{aligned}$$

7.2.2 Interlude # 2: Extending good bases

We inherit the setting given at the beginning of the section; as in good basis, we again work with a family of FAb compact p-adic analytic groups G_p indexed by primes $p > 2$. Correspondingly, we consider normal uniform pro-p subgroups $N_p \leq G_p$, and subgroups K_p of G_p such that $N_p \leqslant K_p$ with fixed Sylow pro-p subgroups P_p of K_p. We can parametrize $\mathcal{H}(P_p)$ by extending the parametrization formed by good basis, see Definition 4.0.4.

$$\mathcal{H}(P_p) = \{H_p \leq P_p : H_p \;\text{ open in } P_p, P_p = N_p H_p\}.$$

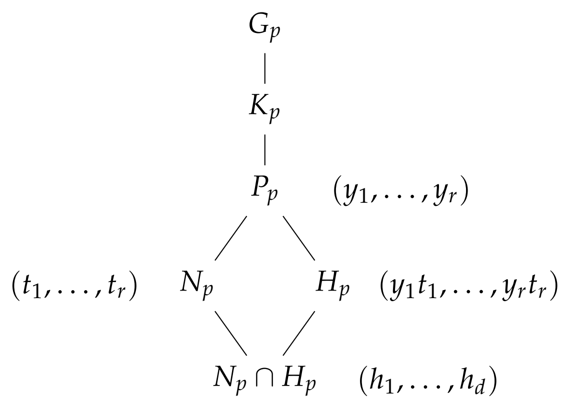
Recall that $r = (P_p : N_p)$, and that we fixed a (left) transversal $(y_1, \ldots, y_r)$ for N_p in P_p with $y_1 = 1$. Then for each y_i, we have $y_i.t_i \in H_p$ for some $t_i \in N_p$ since each (left) coset $y_i N_p$ in K_p contains a unique (left) coset of $N_p \cap H_p$. Therefore we can find $t_1, \ldots, t_r$ in N_p such that $(y_1 t_1, \ldots, y_r t_r)$ gives a (left) transversal to the cosets of $N_p \cap H_p$ in H_p. We now see how to use such a tuple to extend the good basis of $N_p \cap H_p$ to parametrize $H_p \in \mathcal{H}(P_p)$ by following Definition 2.10, [dS93] .

Definition 7.2.4. Let $H_p \in \mathcal{H}(P_p)$ and $(y_1, \ldots, y_r)$ be a left transversal for N_p in P_p. We call a tuple $(h_1, \ldots, h_d, t_1, \ldots, t_r)$ in N_p a *basis* for H_p if

(i) $(h_1, \ldots, h_d)$ is a good basis for $N_p \cap H_p$

(ii) $(y_1 t_1, \ldots, y_r t_r)$ is a left transversal for $N_p \cap H_p$ in H_p

Then the relations explained above can be summarized as follows:



Remark 7.2.5. *The existence of good basis of $N_p \cap H_p$ ensures the existence of such basis for $H_p \in \mathcal{H}(P_p)$ as $P_p = N_p H_p$. A basis is not necessarily a topological generating set for H_p. Nevertheless we have, for a given basis $(h_1, \ldots, h_d, t_1, \ldots, t_r)$ for H_p,*

$$H_p = \overline{\langle h_1, \ldots, h_d, y_1 t_1, \ldots, y_r t_r \rangle}.$$

Recall that uniform definability should be understood as uniformly definable in p, and the constant symbols a_i, y_j, b_{ij}.

Lemma 7.2.6. *The set of bases is uniformly definable.*

Proof. The tuple $(h_1, \ldots, h_d, t_1, \ldots, t_r)$ is a basis for some $H_p \in \mathcal{H}(P_p)$ if and only if $(h_1, \ldots, h_d)$ gives a good basis for $N_p \cap H_p$, and

$$H_p = \bigcup_{i=1}^{r} y_i t_i (N_p \cap H_p).$$

As we know the set of good basis is defined uniformly, it is enough to show that the latter condition can be expressed by an $\mathcal{L}''$-formula independent from p. First note that it is equivalent to say that

$$y_i t_i h_1 (y_j t_j h_2)^{-1} \in H_p$$

for all $h_1, h_2 \in N_p \cap H_p$ and $i, j \in \{1, \dots, r\}$. To see this, we now define $a_{ij}, b_i \in N_p$ and

$$\gamma : \{1, \dots, r\} \times \{1, \dots, r\} \to \{1, \dots, r\} \text{ by } \; y_i y_j = y_{\gamma(i,j)} a_{ij}$$
$$\delta : \{1, \dots, r\} \to \{1, \dots, r\} \; \text{ by } \; y_i^{-1} = y_{\delta(i)} b_i.$$

Then the following suffices to finish the proof since we have the conjugation map on N_p uniformly definable in $\mathcal{L}''$. The expression $y_i t_i h_1 (y_j t_j h_2)^{-1} = y_k t_k h_3$ for some $h_3 \in H_p$ and $k \in \{1, \dots, r\}$ can be given as follows:

$$\exists h_3 \; : \; y_{\delta(j)}^{-1} (y_i^{-1} a_{i\delta(j)} y_i t_i h_1 h_2^{-1} t_j^{-1}) y_{\delta(j)} b_j = t_{\gamma(i,\delta(j))} h_3. \quad \square$$

7.2.3 Back to describing the fibres of $\mathcal{C}$

We again assume that G is a FAb compact p-adic analytic group with its normal uniform pro-p subgroup N. We also let K be a subgroup of G such that $N \leq K$, and fix a Sylow pro-p subgroup P of K.

Recall the set X_K given in Corollary 7.1.4. Let $H \in \mathcal{H}(P)$ be such that $(H, \chi) \in X_K$. Fix $t_1, \dots, t_r \in N$ such that $(y_1 t_1, \dots, y_r t_r)$ gives a left transversal for $N \cap H$ in H. Recall also the maps given in the above proof, for some $a_{ij}, b_i \in N$,

$$\gamma : \{1, \dots, r\} \times \{1, \dots, r\} \to \{1, \dots, r\} \text{ by } \; y_i y_j = y_{\gamma(i,j)} a_{ij}$$
$$\delta : \{1, \dots, r\} \to \{1, \dots, r\} \; \text{ by } \; y_i^{-1} = y_{\delta(i)} b_i$$

In addition, we also define the automorphisms of G given by the elements of left transversal $(y_1, \dots, y_r)$

$$\varphi_i : G \to G \text{ by } \; g \mapsto y_i g y_i^{-1}.$$

Lemma 7.2.7. *[[SZ20], Lemma 6.5.] For a given $\alpha \in \mathrm{Z}^2(P/N, \Omega_{(p)})$ such that $[\alpha] = c$, $\mathcal{C}(H, \chi) = c$ if and only if*

$$\exists \delta \in \mathrm{B}^2(P/N, \Omega_{(p)}), \forall n, n' \in N \cap H, \forall i, j \in \{1, \dots, r\} :$$

$$\chi(t_{\gamma(i,j)}^{-1} a_{ij} \varphi_j^{-1}(t_i n) t_j n') \alpha(y_i N, y_j N) \delta(y_i N, y_j N) = \chi(n n').$$

Proof. As obtained in the previous subsection, we have

$$\mathcal{C}(H,\chi) = [\alpha]$$

if and only if there exists an irreducible projective character χ' of H with factor set β' such that $f_H([\beta]) = [\alpha]$, which strongly extends χ. Notice that β' lies in $\mathrm{Z}^2(H, \mathbb{C}^*)$; hence $\beta \in \mathrm{Z}^2(H/(N \cap H), \mathbb{C}^*)$ as indicated in Theorem 5.2.2. Therefore $\mathcal{C}(H,\chi) = [\alpha]$ if and only if there is $\beta \in \mathrm{Z}^2(H/(N\cap H), \mathbb{C}^*)$ such that $f_H([\beta]) = [\alpha]$, so one gets

$$\chi'(y_i t_i n.y_j t_j n')\beta'(y_i t_i n, y_j t_j n') = \chi'(y_i t_i n)\chi'(y_j t_j n'). \tag{7.2}$$

for all $n, n' \in N \cap H$ and all $i, j \in \{1, \ldots, r\}$. Recall that any pair of strong extensions of χ to H give the same element $\mathcal{C}_H(\chi) \in \mathrm{H}^2(H/(N \cap H), \mathbb{C}^*)$. So assume that χ' is given by

$$\chi'(y_i t_i n) = \chi(n)$$

for each $n \in N \cap H$ and $i \in \{1, \ldots, r\}$. Consequently, the LHS of the Equation (7.2) becomes

$$\chi'(y_i t_i n)\chi'(y_j t_j n') = \chi(n)\chi(n') = \chi(n.n').$$

Note that χ' has values in $\Omega_{(p)}$; hence we may assume $\beta' \in \mathrm{Z}^2(H, \Omega_p)$. Therefore $\beta \in \mathrm{Z}^2(P/N, \Omega_{(p)})$ and the equation (7.2) turns into

$$\chi'(y_i t_i n y_j t_j n')\beta(y_i t_i (N \cap H), y_j t_j (N \cap H)) = \chi(nn').$$

Recall that $y_i t_i (N \cap H) \subseteq y_i N$. Then $f_H([\beta]) = [\alpha]$ if and only if there is $\delta \in \mathrm{B}^2(P/N, \Omega_{(p)})$ satisfying

$$\beta(y_i t_i (N \cap H), y_j t_j (N \cap H)) = \alpha(y_i N, y_j N)\delta(y_i N, y_j N). \tag{7.3}$$

for all $i, j \in \{1, \ldots, r\}$. We now combine the equations (7.2) and (7.3) to obtain that $\mathcal{C}(H,\chi) = [\alpha]$ if and only if there exists $\delta \in \mathrm{B}^2(P/N, \Omega_{(p)})$ such that for all $n, n' \in N \cap H$ and for all $i, j \in \{1, \ldots, r\}$, we have

$$\chi'(y_i t_i n y_j t_j n')\alpha(y_i N, y_j N)\delta(y_i N, y_j N) = \chi(nn').$$

In order to finalize the proof, the last equality that we need to see is the following:

$$\chi'(y_i t_i n y_j t_j n') = \chi(t^{-1}_{\gamma(i,j)} a_{ij} \varphi_j^{-1}(t_i n) t_j n').$$

Note that $y_i t_i n y_j t_j n'$ and $y_{\gamma(i,j)} t_{\gamma(i,j)}$ are the elements of H; hence $t^{-1}_{\gamma(i,j)} a_{ij} \varphi_j^{-1}(t_i n) t_j n' \in N \cap H$. Then we have the following identities

$$\begin{aligned} y_i t_i n y_j t_j n' =& y_i y_j y_j^{-1} t_i n y_j t_j n' \\ =& y_i y_j \varphi_j^{-1}(t_i n) t_j n' \\ =& y_{\gamma(i,j)} a_{ij} \varphi_j^{-1}(t_i n) t_j n' \\ =& y_{\gamma(i,j)} t_{\gamma(i,j)} t^{-1}_{\gamma(i,j)} a_{ij} \varphi_j^{-1}(t_i n) t_j n'. \end{aligned} \tag{7.4}$$

As we assumed $\chi'(y_i t_i n) = \chi(n)$ for each $n \in N \cap H$ and $i \in \{1, \ldots, r\}$, we have

$$\chi(t^{-1}_{\gamma(i,j)} a_{ij} \varphi_j^{-1}(t_i n) t_j n') = \chi'(y_i t_i n y_j t_j n'). \quad \square$$

We conclude this section by definably parametrizing $\mathrm{Z}^2(P_p/N_p, \Omega_{(p)})$ and $\mathrm{B}^2(P_p/N_p, \Omega_{(p)})$ in a p-independent way. Recall the isomorphism ι of $\mathbb{Q}_p/\mathbb{Z}_p$ onto $\Omega_{(p)}$. Let $z \in M_r(\mathbb{Q}_p)$ where $M_r(\mathbb{Q}_p)$ is the $r \times r$-matrices over $\mathbb{Q}_p$, consequently $\iota(z_{ij} + \mathbb{Z}_p) \in \Omega_{(p)}$. Consider the following map

$$\begin{aligned} \delta : P_p/N_p \times P_p/N_p \to& \ \Omega_{(p)} \\ \delta(y_i N_p, y_j N_p) \ \mapsto& \ \iota(z_{ij} + \mathbb{Z}_p) \end{aligned}$$

We now collect the matrices $(z_{ij}) \in M_r(\mathbb{Q}_p)$ such that the map δ lie in $\mathrm{Z}^2(P_p/N_p, \Omega_{(p)})$ and $\mathrm{B}^2(P_p/N_p, \Omega_{(p)})$, call them $\mathcal{Z}_p$ and $\mathcal{B}_p$ respectively.

Lemma 7.2.8. *The sets $\mathcal{Z}_p$ and $\mathcal{B}_p$ are uniformly definable subsets of $\mathbb{Q}_p^{r^2}$.*

Proof. Let $(z_{ij}) \in M_r(\mathbb{Q}_p)$ and let $\delta \in \mathrm{Z}^2(P_p/N_p, \Omega_{(p)})$ be the map $P_p/N_p \times P_p/N_p \to \mathbb{Q}_p/\mathbb{Z}_p$ defined as

$$\delta(y_i N_p, y_j N_p) \mapsto \iota(z_{ij} + \mathbb{Z}_p).$$

Then δ satisfies the identity given below

$$\delta(y_i N_p y_j N_p, y_k N_p) \delta(y_i N_p, y_j N_p) = \delta(y_i N_p, y_j N_p y_k N_p) \delta(y_j N_p, y_k N_p).$$

To see what it means, recall the map γ given by $y_i y_j = y_{\gamma(i,j)} a_{ij}$. Then the above identity holds if and only if

$$\delta(y_{\gamma(i,j)} N_p, y_k N_p) \delta(y_i N_p, y_j N_p) = \delta(y_i N_p, y_{\gamma(j,k)} N_p) \delta(y_j N_p, y_k N_p).$$

Therefore $(z_{ij}) \in \mathcal{Z}_p$ if and only if for all $i, j, k \in \{1, \ldots, r\}$, we have

$$z_{\gamma(i,j)k} + z_{ij} = z_{i\gamma(j,k)} + z_{jk} \mod \mathbb{Z}_p.$$

It is obvious that $\mathbb{Z}_p$ is uniformly definable; hence equivalence modulo $\mathbb{Z}_p$ is also uniformly definable. Consequently, the set $\mathcal{Z}_p \subseteq \mathbb{Q}_p^{r^2}$ is uniformly definable.

Now we prove that set $\mathcal{B}_p$ is also uniformly definable. Recall that $\delta \in \mathrm{B}^2(P_p/N_p, \Omega_{(p)})$ if and only if

$$\delta(x_1N_p, x_2N_p) = \varphi(x_1N_p)\varphi(x_2N_p)\varphi(x_1N_px_2N_p)^{-1},$$

for some function $\varphi : P_p/N_p \to \mathbb{Q}_p/\mathbb{Z}_p$ since $\mathbb{Q}_p/\mathbb{Z}_p \simeq \Omega_{(p)}$. We now construct a parametrization for such φ exploiting the fact that $(y_1, \ldots, y_r)$ is a left transversal for N_p in P_p.

$$\begin{aligned} \varphi : P_p/N_p &\to \mathbb{Q}_p/\mathbb{Z}_p \\ y_iN_p &\mapsto b_i + \mathbb{Z}_p \end{aligned}$$

Accordingly, $\delta \in \mathrm{B}^2(P_p/N_p, \Omega_{(p)})$ if and only if there are $b_1, \ldots, b_r \in \mathbb{Q}_p$ such that for all $1 \leq i, j \leq r$

$$\delta(y_iN_p, y_jN_p) = \varphi(y_iN_p)\varphi(y_jN_p)\varphi(y_1iN_py_jN_p)^{-1}.$$

By using γ again, one can see that this holds if and only if

$$z_{ij} = b_i + b_j - b_{\gamma(i,j)} \mod \mathbb{Z}_p.$$

Hence $\mathcal{B}_p$ is uniformly definable. □

Chapter 8

Towards the Main Theorem

In this chapter, we realize our aim to parametrize $\mathrm{Irr}^c_{K_p}(N_p)$ in a uniformly definable way. To this end, we first present a construction of a subset $\mathcal{D}^c_p$ of $\mathbb{Q}^m$ for some m, from [SZ20], which is a variation of Lemma 8.8, [HMR18]. We then see that the sets $\{\mathcal{D}^c_p\}_p$ are uniformly definable. Following this, we establish uniformly definable equivalence relations on $\{\mathcal{D}^c_p\}_p$ with classes corresponding to the elements of $\mathrm{Irr}^c_{K_p}(N_p)$, and conclude by applying the uniform rationality result of Nguyen, given in Section 3.3.

8.1 Uniformly definable parametrization of fibres.

Let G_p be a uniformly definable FAb compact p-adic analytic group in in the structure $(\mathbb{Q}_p, \mathbb{Z}, \mathbb{F}_p, \mathcal{L}_{\mathbb{Z}[\![t]\!]})$. Let N_p be a normal uniform subgroup $N_p \leq G_p$. We showed that N_p is also uniformly definable in p using additional constants $a_1 \ldots, a_k$. Let K_p be a subgroup of G_p such that $N_p \leq K_p$, and fix a Sylow pro-p subgroup P_p of K_p. In Section 7.1, we use the fact that finite p-groups are monomial to define the set X_{K_p} of tuples (H, χ) with $H \in \mathcal{H}(P_p)$ and a linear character χ enabling us to work with linear characters instead of arbitrary ones.

We now see that the parametrization, given in [SZ20], Proposition 6.9., of the fibres of the following map is actually uniformly definable;

$$\begin{aligned} \mathcal{C}: \ X_{K_p} &\longrightarrow \mathrm{H}^2(P_p/N_p, \mathbb{C}^*) \\ (H, \chi) &\mapsto f_H(\mathcal{C}_H(\chi)) \end{aligned}$$

Lemma 8.1.1. *For a fixed $c \in \mathrm{H}^2(P_p/N_p, \mathbb{C}^*)$, define $\mathcal{D}_p^c$ to be the set of tuples $(\lambda, \xi) := (\lambda, (\xi_1, \ldots, \xi_d)) \in M_{d\times(d+r)}(\mathbb{Z}_p) \times \mathbb{Q}_p^d$ such that:*

(i) For $1 \leq j \leq d + r$, the columns $(\lambda_{1j}, \ldots, \lambda_{dj})$ of λ give the $\mathbb{Z}_p$-coordinates of a basis $(h_1, \ldots, h_d, t_1, \ldots, t_r)$ for some subgroup $H \in \mathcal{H}(P_p)$ with respect to the generating set $\{a_1, \ldots, a_d\}$ and the (left) transversal $y_1, \ldots, y_r$,

(ii) The mapping from the set $\{h_1, \ldots, h_d\}$ to the quotient $\mathbb{Q}_p/\mathbb{Z}_p$; $h_i \mapsto \xi_i + \mathbb{Z}_p$ induces a continuous H-invariant homomorphism

$$\chi : N_p \cap H \to \mathbb{Q}_p/\mathbb{Z}_p,$$

(iii) The induced character $\mathrm{Ind}_{N_p\cap H}^{N_p}(\iota \circ \chi) \in \mathrm{Irr}_{K_p}(N_p)$,

(iv) $\mathcal{C}(H, (\iota \circ \chi)) = c$.

Then the sets $\{\mathcal{D}_p^c\}_p \subseteq \mathbb{Q}_p^{d\times(d+r)}$ are uniformly definable.

Proof. As we obtained in the Lemma 7.2.6, the first condition is uniformly definable. To see that condition (ii) can be expressible uniformly in p, we will follow the steps of the proof of Lemma 8.8, [HMR18]; we see that $(i) \Rightarrow (ii)$ if and only if:

(1) there exists $(\mu_{ij}) \in M_d(\mathbb{Z}_p)$, and its columns give the $\mathbb{Z}_p$-coordinates of a good basis for some finite index normal subgroup M of $N_p \cap H$;

(2) there exist $\xi \in \mathbb{Q}_p, r_1 \ldots, r_d \in \mathbb{Z}_p$, and $h \in N_p \cap H$ such that the order of ξ in $\mathbb{Q}_p/\mathbb{Z}_p$ is $(N_p \cap H : M)$, and for every $i, j \in \{1, \ldots, d\}$ we get

$$h_j = t_i^{-1}\varphi_i^{-1}(h^{r_j})t_i \mod M \quad \text{and} \quad r_i\xi = \xi_i \mod \mathbb{Z}_p.$$

In particular, we have $h_j = h^{r_j} \mod M$ as $y_1 = 1$.

We begin with assuming that the conditions (i) and (ii) hold. Then the continuity of χ, together with the fact that $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to $\Omega_{(p)}$, implies that χ factors through a finite quotient of $N_p \cap H$. Therefore $\ker(\chi)$ is of finite index in $N_p \cap H$, so we set M in the condition (1) as $\ker(\chi)$, and choose $(\mu_{ij}) \in M_d(\mathbb{Z}_p)$ such that its columns are the $\mathbb{Z}_p$-coordinates of a good basis of M as required in the condition (1).

To find $\xi \in \mathbb{Q}_p, r_1 \ldots, r_d \in \mathbb{Z}_p$ and $h \in N_p \cap H$ in the condition (2), we first note that $(N_p \cap H)/M$ is cyclic as it is isomorphic to a subgroup of $\mathbb{C}^*$, let

$$(N_p \cap H)/M = \langle hM \rangle$$

for some $h \in N_p \cap H$. Set $\xi = \chi(h)$, then its order is $(N_p \cap H : M)$. Recall that $h_1, \ldots, h_d \in N_p \cap H$, accordingly set $r_1, \ldots, r_d \in \mathbb{Z}$ such that

$$h_i M = h^{r_i} M$$

for each $1 \leq i \leq d$; hence $\xi_i = \chi(h_i) = \chi(h^{r_i}) = r_i \xi \mod \mathbb{Z}_p$.

We assumed that $(h_1, \ldots, h_d, t_1, \ldots, t_r)$ is a basis for H, so $(y_1 t_1, \ldots, y_r t_r)$ is a left transversal for $N_p \cap H$ in H. Since χ is assumed to be an H-invariant map, for any $1 \leq j \leq d$,

$$\chi(h^{r_j}) = \chi(h_j) = \chi(y_i t_i h_j t_i^{-1} y_i^{-1}),$$

for all $1 \leq i \leq d$. Therefore $h_j = t_i^{-1} \varphi_i^{-1}(h^{r_j}) t_i \mod M$, and the condition (2) follows.

To see the other direction, we first assume that (i) holds; $(h_1, \ldots, h_d, t_1, \ldots, t_r)$ is a basis for H. We also assume that there are $(\mu_{ij}) \in M_d(\mathbb{Z}_p)$, $h \in H$, and $\xi \in \mathbb{Q}_p$ such that (1) and (2) hold. To define a continuous homomorphism $\chi : N_p \cap H \to \mathbb{Q}_p/\mathbb{Z}_p$, we first recall that, by Theorem 3.3.2 (3), the map $\lambda \mapsto h^\lambda$ between $\mathbb{Z}_p$ and $N_p \cap H$ is analytic in the $\mathbb{Z}_p$-coordinates of N_p, and it is therefore continuous. M is an open subgroup of $N_p \cap H$ as a finite index subgroup, so one can find a neighborhood U of 0 such that $h^\lambda \in M$ for all $\lambda \in U$. The fact that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ implies that there exists $s_i \in (r_i + U) \cap \mathbb{Z}$ for all $1 < i < d$; hence

$$h^{s_i} M = h^{r_i} M = h_i M.$$

Therefore, $(N_p \cap H)/M$ is cyclic, and generated by hM. To define χ given (ii), consider the following map

$$\begin{aligned} \beta : (N_p \cap H)/M &\to \mathbb{Q}_p/\mathbb{Z}_p \\ hM \quad &\mapsto \xi + \mathbb{Z}_p \end{aligned}$$

Then β gives an injective homomorphism since the order of $\xi + \mathbb{Z}_p$ in $\mathbb{Q}_p/\mathbb{Z}_p$ is equal to $(N_p \cap H : M)$ which is the order of hM in $(N_p \cap H)/M$. Then β induces a continuous homomorphism

$$\chi : N_p \cap H \to \mathbb{Q}_p/\mathbb{Z}_p$$

given by $x \mapsto \beta(xM)$ as the quotient map $N_p \cap H \to (N_p \cap H)/M$ is continuous.

Now we will see that χ is H-invariant. For any $1 < j < d$, then

$$\chi(\ t_1^{-1} h^{r_j} t_1) = \beta(t_1^{-1} h^{r_j} t_1 M) = \beta(h_j M) = \chi(h_j) = r_j \xi + \mathbb{Z}_p = \xi_j + \mathbb{Z}_p.$$

By assumption $y_1 = 1$, so $t_1 \in N \cap H$. Thus, for all $1 < j < d$,

$$\chi(h_j) = \chi(t_i^{-1}(h^{r_j}) t_i) = r_j \xi + \mathbb{Z}_p.$$

Similarly, for $1 < i, j < d$, one obtains $\chi(t_i^{-1}\varphi_i^{-1}(h^{r_j})t_i) = r_j\xi + \mathbb{Z}_p = \xi_j + \mathbb{Z}_p$ proving our claim.

To see how to express conditions (1) and (2) uniformly definably, it is enough to pass to $\mathbb{Z}_p$-coordinates. For the condition (1), we add the following formulae

$$(\exists \Lambda_{1j}, \ldots, \Lambda_{dj} \in \mathbb{Z}_p) \;\; h_j = a_1^{\Lambda_{1j}} \ldots a_d^{\Lambda_{dj}}$$

for $1 \leq j \leq d$, to the formulae defining the set of good bases. Recall that equivalence modulo $\mathbb{Z}_p$ is uniformly definable. Working in modulo M is also uniformly definable, since we have a good basis for M. The remaining ingredient is the condition on the order of ξ which can be given as follows:

$$\Big(h^{(\xi^{-1})} \in M\Big) \wedge \Big(\forall \eta \in \mathbb{Q}_p \; \big(v(\eta) > v(\xi)\big) \Rightarrow h^{(\eta^{-1})} \notin M\Big).$$

We can move to the condition (iii); to see that it is expressible in a uniformly definable way, we first rewrite Mackey's irreducibility criterion and then show how to recover having K as a stabilizer in a uniformly definable way. Note first that $\iota \circ \chi : N_p \cap H \to \Omega_{(p)} \subseteq \mathbb{C}^*$. To avoid heavy notation, we now identify the group $\mathbb{Q}_p/\mathbb{Z}_p$ with $\Omega_{(p)}$ via the isomorphism ι; accordingly we impose $\chi = \iota \circ \chi$.

Recall that Mackey's irreducibility criterion indicates that $\mathrm{Ind}_{N_p \cap H}^{N_p}(\chi)$ is irreducible if and only if for each $g \in N_p \setminus (N_p \cap H)$,

$$\langle \mathrm{Res}^H_{{}^g(N_p\cap H)\cap(N_p\cap H)}(\chi), \mathrm{Res}^H_{{}^g(N_p\cap H)\cap(N_p\cap H)}({}^g\chi)\rangle = 0.$$

Therefore $\mathrm{Ind}_{N_p \cap H}^{N_p}(\chi)$ is irreducible if and only if

$$\forall g \in N_p : \big(\forall h \in N_p \cap H, \; \chi({}^g h) = \chi(h) \Rightarrow g \in H\big).$$

Writing the formula above in terms of $\mathbb{Z}_p$-coordinates in N_p and λ, ξ, by the first two assertion, we see that the irreducibility statement in condition (iii) is uniformly definable.

As the final step towards obtaining that (iii) is uniformly definable, we see how to express K_p-stability (under the conjugation action). By Lemma 7.1.5 (i), $\mathsf{stab}_{G_p}(\mathrm{Ind}_{N_p \cap H}^{N_p}(\chi))$ can be given as

$$\{g \in G : \mathrm{Ind}_{N_p \cap H}^{N_p}(\chi) = \mathrm{Ind}_{{}^g(N_p \cap H)}^{N_p}({}^g\chi)\}.$$

Then $\mathsf{stab}_{G_p}(\mathrm{Ind}_{N_p \cap H}^{N_p}(\chi)) = K_p$ if and only if the following statement holds:

$$\forall g \in G_p : \Big(\mathrm{Ind}_{N_p \cap H}^{N_p}(\chi) = \mathrm{Ind}_{{}^g(N_p \cap H)}^{N_p}(\chi) \Leftrightarrow g \in K_p\Big).$$

Recall that we extended $(y_1, \ldots, y_r)$ to the (left) transversal $(y_1, \ldots, y_u)$ for N_p in K_p in Section 3.4. By using the transversals y_i and that N_p is uniformly definable, we write the following expression uniformly defining K_p:

$$g \in K_p \leftrightarrow g \in y_1 N_p \vee \ldots \vee g \in y_u N_p.$$

Now we proceed by examinining the identity $\mathrm{Ind}_{N_p \cap H}^{N_p}(\chi) = \mathrm{Ind}_{{}^g(N_p \cap H)}^{N_p}(\chi)$. To this end, in Lemma 7.1.5 (ii), we put the following

$$M = N_p \cap H, \; M' = {}^g(N_p \cap H), \; \chi' = {}^g(\chi).$$

Then it follows that $\mathrm{Ind}_{N_p \cap H}^{N_p}(\chi) = \mathrm{Ind}_{{}^g(N_p \cap H)}^{N_p}({}^g\chi)$ if and only if

$$\exists n \in N_p, \forall h \in N_p \cap H : \big({}^n h \in {}^g(N_p \cap H) \Rightarrow \chi(h) = {}^g\chi({}^n h) \big).$$

By a similar argument, one can write the above formula in terms of $\mathbb{Z}_p$-coordinates in N_p and λ, ξ; so we are done.

Now we will show that the condition (iv) is expressible uniformly in p. In Lemma 7.2.7, we already saw when $\mathcal{C}(H, \chi) = c$ holds. So fix $\alpha \in \mathrm{Z}^2(P_p/N_p, \Omega_{(p)})$ such that $[\alpha] = c$. Then the condition $iv)$ is equivalent to

$$\exists \delta \in \mathrm{B}^2(P_p/N_p, \Omega_{(p)}) :$$
$$\Big(\bigwedge_{i,j \in \{1,\ldots,r\}} \forall n, n' \in N_p \cap H_p \;\; \big(\chi(t^{-1}_{\gamma(i,j)} a_{ij} \varphi_j^{-1}(t_i n) t_j n') \alpha(y_i N_p, y_j N_p) \delta(y_i N_p, y_j N_p) = \chi(nn')\big)\Big).$$

We now parametrize α and δ by elements of $\mathcal{Z}_p$ as suggested in Lemma 7.2.8. Therefore $\exists \delta \in \mathrm{B}^2(P_p/N_p, \Omega_{(p)})$ in the formula above can be replaced by $\exists (d_{ij}) \in \mathcal{B}_p$ and similarly we can put $(b_{ij}) \in \mathcal{Z}_p$ in place of α providing the following

$$\alpha(y_i N_p, y_j N_p) = b_{ij} + \mathbb{Z}_p$$
$$\delta(y_i N_p, y_j N_p) = d_{ij} + \mathbb{Z}_p$$

for all $i, j \in \{1, \ldots, r\}$. Using (i) and (ii) once more, we get the uniform definability of (iv) in p with parameters b_{ij}, since we recover the equalities in the above formula as equalities modulo $\mathbb{Z}_p$ involving λ, ξ and of the $\mathbb{Z}_p$-coordinates in N_p. □

8.2 Construction of a uniformly definable equivalence relation on $\mathcal{D}_p^c$.

We now describe our uniformly definable equivalence relation on $\mathcal{D}_p^c$; the classes will be in one-to-one correspondence with $\mathrm{Irr}_{K_p}^c(N_p)$. To this end, we first define $\Psi : \mathcal{D}_p^c \to \mathcal{C}^{-1}(c)$ by

$$\Psi\big((\lambda,\xi)\big) = (H,\chi)$$

where $H \in \mathcal{H}(P)$ is given by the basis $(h_1,\dots,h_d,t_1,\dots,t_r)$ and $\chi : N_p \cap H \to \mathbb{Q}_p/\mathbb{Z}_p$ is the homomorphism as in previous lemma. The map Ψ is therefore surjective. Following this, we can define the following equivalence relation on $\mathcal{D}_p^c$. Let $(\lambda,\xi),(\lambda',\xi')$ be such that $(H,\chi) = \Psi(\lambda,\xi)$ and $(H',\chi') = \Psi(\lambda',\xi')$. Define $\mathcal{E}_p$ by

$$(\lambda,\xi) \sim_p (\lambda',\xi') \Leftrightarrow \mathrm{Ind}_{N_p\cap H}^{N_p}(\chi) = \mathrm{Ind}_{N_p\cap H'}^{N_p}(\chi').$$

Proposition 8.2.1. *The equivalence relations $\{\mathcal{E}_p\}_p$ are uniformly definable.*

Proof. Let $(\lambda,\xi),(\lambda',\xi')$ be such that $(H,\chi) = \Psi(\lambda,\xi)$ and $(H',\chi') = \Psi(\lambda',\xi')$. As obtained in the Lemma 7.1.5,

$$\mathrm{Ind}_{N_p\cap H}^{N_p}(\chi) = \mathrm{Ind}_{N_p\cap H}^{N_p}(\chi') \Leftrightarrow \exists g \in N_p : \mathrm{Res}^{^g(N_p\cap H)}_{^g(N_p\cap H)\cap(N_p\cap H')}(^g\chi) = \mathrm{Res}^{(N_p\cap H)'}_{^g(N_p\cap H)\cap(N_p\cap H')}(\chi').$$

Accordingly, we obtain

$$(\lambda,\xi) \sim (\lambda',\xi') \Leftrightarrow \exists g \in N : \forall h \in N_p \cap H \ \ (^gh \in N_p \cap H' \Rightarrow \chi(h) = \chi'(^gh)).$$

By describing the above in terms of $\mathbb{Z}_p$-coordinates of N, we have an $\mathcal{L}''$-formula independent of p. And by realizing such a formula over the sets $\{\mathcal{D}_p^c\}_p$, we can define the relations $\{\mathcal{E}_p\}_p$ uniformly. □

We now give a uniform definable enumeration to apply the Nguyen's result given in [Ngu19]. First recall that we established a surjective function

$$X_K \to \mathrm{Irr}_{K_p}(N_p); (H,\chi) \mapsto \mathrm{Ind}_{N_p\cap H}^{N_p}(\chi)$$

in Corollary 7.1.4. Compose this function with Ψ,

$$\begin{array}{ccccc} \mathcal{D}_p^c & \xrightarrow{\Psi} & \mathcal{C}^{-1}(c) \subseteq X_{K_p} & \xrightarrow{\text{surj.}} & \mathrm{Irr}_{K_p}(N_p) \\ (\lambda,\xi) & \mapsto & (H,\chi) & \mapsto & \mathrm{Ind}_{N_p\cap H}^{N_p}(\chi) \end{array}$$

Write $c = f_H(c_H(\chi))$, then $\mathcal{C}_{P_p}(\mathrm{Ind}_{N_p\cap H}^{N_p}\chi) = f_H(\mathcal{C}_H(\chi))$ by Proposition 7.1.2 as $(H,\chi) \in \mathcal{C}^{-1}(c)$. Therefore $\mathrm{Ind}_{N_p\cap H}^{N_p}\chi \in \mathrm{Irr}_{K_p}^c(N_p)$ and the following composition map is surjective $\mathcal{D}_p^c \to \mathcal{C}^{-1}(c) \to \mathrm{Irr}_{K_p}^c(N_p)$. Moreover, we obtain a bijection between the set of equivalence classes and $\mathrm{Irr}_{K_p}^c(N_p)$ when we quotient $\mathcal{D}_p^c$ out by the equivalence relation $\mathcal{E}_p$.

We now focus on producing a new definable family of equivalence relation by using this bijection in order to work within Nguyen's framework. For the tuples $(\lambda, \xi) \in \mathcal{D}_p^c$, we write $(h_1(\lambda), \ldots, h_d(\lambda))$ for the corresponding good basis given in Lemma 8.1.1,(i). Define $f_p : \mathcal{D}_p^c \to \mathbb{Z}$ by

$$(\lambda, \xi) \mapsto \sum_{i=1}^{d} \omega(h_i(\lambda)) - 1.$$

Note that, if $\Psi\big((\lambda, \xi)\big) = (H, \chi)$, then $p^{f_p(\lambda,\xi)}$ equals to the index of $N_p \cap H$ in N_p, which is exactly the degree of $\mathrm{Ind}_{N_p \cap H}^{N_p} \chi$. Consequently, if $((\lambda, \xi), (\lambda', \xi')) \in \mathcal{E}_p$, $f_p((\lambda, \xi)) = f_p((\lambda', \xi'))$ as the degrees of the associated induced characters $\mathrm{Ind}_{N_p \cap H}^{N_p} \chi$ and $\mathrm{Ind}_{N_p \cap H'}^{N_p} \chi'$ are equal.

We now aim to define an equivalence relation on the following set

$$\mathcal{D}_{p,n}^c = \{(\lambda, \xi) \in \mathcal{D}_p^c : f_p(\lambda, \xi) = n\}.$$

Let $(\lambda, \xi), (\lambda', \xi') \in \mathcal{D}_p^c$, then define $\mathcal{E}_{p,n}$ by

$$(\lambda, \xi) \sim_{p,n} (\lambda', \xi') \Leftrightarrow \mathrm{Ind}_{N \cap H}^{N}(\chi) = \mathrm{Ind}_{N_p \cap H'}^{N_p}(\chi').$$

Let $F_p : \mathcal{E}_p \to \mathbb{Z}$ be the function given by $((\lambda, \xi), (\lambda', \xi')) \mapsto f_p(\lambda, \xi)$. Then we can write $\mathcal{E}_p$ as the fibres of F_p at n, $\mathcal{E}_{p,n} = F_p^{-1}(n)$, for any natural number n;

$$\mathcal{E}_{p,n} = \mathcal{E}_p \cap (\mathcal{D}_{p,n}^c \times \mathcal{D}_{p,n}^c).$$

Then we have a uniformly definable family of equivalence relations $\{\mathcal{E}_{p,n}\}_{p,n}$ on a uniformly definable subset $\mathcal{D}_{p,n}^c$ of $\mathbb{Q}_p^{d \times (d+r)}$. And for all n, there is a bijection of the set $\mathcal{D}_{p,n}^c / \mathcal{E}_{p,n}$ onto the subset of characters of degree p^n in $\mathrm{Irr}_{K_p}^c(N_p)$. Accordingly, we have

$$\zeta^{(N_p, K_p, c)}(s) = \sum_{n} |\mathcal{D}_{p,n}^c / \mathcal{E}_{p,n}| p^{-ns}.$$

One can then conclude the main theorem as explained in Section 3.4.

Chapter 9

What is next?

The most natural next step is to apply this work to obtain uniform (in a sense given in Section 3.3) virtual rationality of the representation zeta function of FAb compact p-adic analytic groups. In [SZ20], the virtual rationality is reduced to the rationality of the partial zeta series as explained in Section 6.2;

$$\zeta^{G_p}(s) = \sum_{K_p \in \mathcal{S}_p} (G_p : K_p)^{-s-1} \sum_{c \in \mathrm{H}^2(P_p/N_p)} f_{(K_p, N_p, \theta)}(s) \sum_{\theta \in \mathrm{Irr}^c_{K_p}(N_p)} \theta(1)^{-s}.$$

With the tools provided in this work, we know how to uniformly definably parametrize $\mathrm{Irr}^c_{K_p}(N_p)$, and obtain uniform rationality by the result of Nguyen, [Ngu19], presented in Section 3.3. To recover the virtual rationality, one needs to obtain a uniformly definable parametrization of all the subgroups K_p such that $N_p \leqslant K_p \leqslant G_p$, and $\mathrm{stab}_{G_p}(\theta) = K_p$ for some $\theta \in \mathrm{Irr}(N_p)$, in other words, uniformly definable parametrization of $\mathcal{S}_p$ and the cohomology classes $c \in \mathrm{H}(P_p/N_p, \mathbb{C}^*)$ such that $\mathcal{C}_{P_p}(\theta) = c$.

As repeatedly mentioned, this work generalizes some ideas inheriting their framework from the first part of [SZ20] to a uniform setting. Stasinski and Zordan also deliver corresponding results for twist zeta functions of compact p-adic analytic groups in the second part and reduce the problem to *partial twist zeta series* with similar but more sophisticated methods this time. Another natural direction would be applying our findings to partial twist zeta series and investigating uniform rationality.

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I declare under oath that I have produced my thesis independently and without any undue assistance by third parties under consideration of the 'Principles for the Safeguarding of Good Scientific Practice at Heinrich Heine University Düsseldorf.'

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