Witt groups of maximal isotropic Grassmann bundles

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Zusammenfassung

Bereits im frühen 20. Jahrhundert untersuchte Ernst Witt symmetrische und antisymmetrische Bilinearformen auf Vektorräumen, deren Klassifizierung schließlich zur Definition der Wittgruppe über einem Körper führte. Erst wesentlich später definierte Balmer *derivierte Wittgruppen* von Schemata, welche nun auch in verschiedenen Graden und Twists auftreten. Dies motiviert die *totale Wittgruppe* eines Schemas,

$$W^{\mathrm{tot}}(X) = \bigoplus_{i \in \mathbb{Z}/4} \bigoplus_{L \in \mathrm{Pic}(X)/2} W^i(X, L), \tag{1}$$

in welcher die für Körper bereits bekannte Wittgruppe im Grad Null und trivialem Twist auftritt, d.h. $W(X) = W^0(X, \mathcal{O}_X)$ für $X = \mathrm{Spec}(k)$. Seitdem wurden viele Methoden für die Berechnung der Wittgruppen entwickelt, wie zum Beispiel die lange exakte Lokalisierungssequenz von Balmer ([Bal00]).

Maximale isotrope Grassmannbündel parametrisieren die bezüglich einer gegebenen symmetrischen oder antisymmetrischen Bilinearform isotropen Unterbündel maximalen Rangs eines fixierten Vektorbündels und die Berechnung deren Wittgruppen ist das Ziel dieser Arbeit. Den Fall gewöhnlicher Grassmannschen haben Balmer und Calmès bereits behandelt ([BC12a]), indem sie die Randabbildung der Lokalisierungssequenz untersucht haben. Es stellte sich heraus, dass die totale Wittgruppe mithilfe *gerader Young-Diagramme* beschrieben werden kann und, dass sogar die Grade und Twists an den Diagrammen direkt abgelesen werden können.

Auch wenn diese Methode auf keinen der beiden zu untersuchenden Fälle uneingeschränkt übertragen werden kann, beweisen wir ein analoges Resultat über die totale Wittgruppe, diesmal mithilfe *gerader* und *fast gerader versetzter Young-Diagramme*. Allerdings fällt die Aussage im antisymmetrischen Fall etwas schwächer aus, da hier keine explizite Basis angegeben werden kann.

Das Vorgehen von Balmer-Calmès kann auch direkt auf Quadriken angewandt werden. Daher führt diese Arbeit zur vollständigen Beschreibung der Wittgruppen von Klassen homogener Varietäten gewöhnlichen Lie-Typs, welche als *minuscule* und *cominuscule* bezeichnet werden. Es ist wahrscheinlich, dass auch für die beiden exzeptionellen (co-)minuscule Varietäten, nämlich die Cayley-Ebene sowie die Freudenthal-Varietät, eine Beschreibung der Wittgruppen durch spezielle Young-Diagramme möglich ist, was zum jetzigen Zeitpunkt jedoch unklar ist.

Abstract

Named after Ernst Witt, Witt groups were first introduced for fields in the earlier 20th century to classify symmetric and or antisymmetric bilinear forms over them. Much later, Balmer introduced derived Witt groups of schemes ([Bal99]), including shifted and twisted Witt groups. This yields the notion of the *total Witt group* of a scheme X,

$$W^{\mathrm{tot}}(X) = \bigoplus_{i \in \mathbb{Z}/4} \bigoplus_{L \in \mathrm{Pic}(X)/2} W^i(X, L), \tag{2}$$

which includes the well-known Witt group $W(X) = W^0(X, \mathcal{O}_X)$ for a field $X = \mathrm{Spec}(k)$. Since then, many strong tools for the computation of Witt groups have been developed, such as the long exact localization sequence by Balmer [Bal00].

The goal of this work is to compute the Witt groups of maximal isotropic Grassmann bundles, that is, schemes parametrizing subbundles of maximal rank of a fixed vector bundle which are isotropic with respect to a given symmetric or antisymmetric bilinear form. The case of non-isotropic Grassmannians has been accomplished by Balmer and Calmès ([BC12a]) by investigating the boundary map in the localization long exact sequence of Witt groups. It turns out that the total Witt group (2) is indexed by *even Young diagrams* and the twists and shifts can easily be read off the tableaux.

Although the ordinary approach cannot be applied entirely to neither of the isotropic cases, we manage to describe the total Witt group in these cases by using *even* and *almost even shifted Young diagrams*. At this point we remark that the result for the antisymmetric case is weaker in the sense that no explicit basis has been computed.

It is known that the same procedure can be applied to quadrics. Hence, this thesis will lead to a complete description of the Witt groups of minuscule and cominuscule homogeneous varieties of ordinary Lie types. We believe, that similar results can be obtained for the Cayley plane and the Freudenthal variety, the only exceptional minuscule varieties, but this remains open.

Contents

Introduction

Witt groups of fields have been known for almost one hundred years and introduced by Ernst Witt. Over a field, any non-degenerate symmetric bilinear form is uniquely determined by a symmetric, diagonalizible matrix and two forms are isometric if and only if their matrices A, A' are congruent, i.e. we have $A' = M^T AM$ for some invertible matrix M. The collection of isometry classes forms a monoid via orthogonal sums and its Grothendieck-construction is called the *Grothendieck-Witt group of the ground field* and denoted by $GW(-)$. For example, over the field $\mathbb{R}$, any bilinear form can be represented by $\mathrm{diag}(\pm 1)$ and over $\mathbb{C}$, or in general over any quadratically closed field, any form is represented by the unit matrix $\mathrm{diag}(1)$. Hence, the only further parameter is the rank and we obtain $GW(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$ and $GW(\mathbb{C}) = \mathbb{Z}$. A two-dimensional quadratic vector space is called a *hyperbolic plane*, denoted by $\mathbb{H}$, if its form diagonalizes to $\mathrm{diag}(1, -1)$. Finally, the *Witt group*, denoted by $W(-)$, is the quotient of the Grothendieck-Witt group by hyperbolic spaces, i.e. orthogonal sums of copies of hyperbolic planes. For instance $W(\mathbb{R}) = \mathbb{Z}$ and $W(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$.

For a general scheme the category $\mathrm{Vect}(-)$ of vector bundles is an example of a *triangulated category*, when equipped with a duality corresponding to a line bundle and a double dual identification. For those, Balmer ([Bal00], [Bal01]) developed a non-oriented cohomology theory of *triangular Witt groups* and we can define the Witt group of a scheme as the Witt group of the induced triangulated category. Varying over all possible dualities and double dual identifications gives us a variety of different Witt groups, the i-th shifted Witt groups twisted by a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$, denoted by $W^i(X, \mathcal{L})$. It turns out that these groups are 4-periodic in the shifts i and square-periodic in the twists $\mathcal{L}$, so the term

$$W^{\mathrm{tot}}(X) := \bigoplus_{[i] \in \mathbb{Z}/4} \ \bigoplus_{[\mathcal{L}] \in \mathrm{Pic}(X)/2} W^i(X, \mathcal{L}) \tag{3}$$

covers all groups and is called the *total Witt group of* X. For $X = \mathrm{Spec}(k)$ we have $W^0(X, \mathcal{O}_X) = W(k)$ as introduced above. The big advantage of the new setting is that the classical Witt group occurs in a cohomology theory which comes with a variety of tools, allowing further computations. For example, Nenashev ([Nen09]) and Walter ([Wal03]) simultaneously gave a complete description of the Witt groups of projective bundles over schemes. If $\mathbb{P}(E)$ is a projective space of dimension r over k, where $\mathrm{char}(k) \neq 2$, we have $\mathrm{Pic}(\mathbb{P}(E))/2 = \mathbb{Z}/2\mathbb{Z}$ with $\mathcal{O}(1)$ the only non-trivial class.

Theorem ([Nen09]**,** [Wal03]**).** *For $i \in \mathbb{Z}/4$ and $l \in \mathbb{Z}/2$ the i-th shifted Witt group of $\mathbb{P}(E)$ twisted by $\mathcal{O}(l)$ is given by*

$$W^i(\mathbb{P}(E), \mathcal{O}(l)) \cong \bigoplus_{\substack{\lambda \in \{0,r\} \\ t(\lambda) \equiv l(2)}} W^{i-\lambda}(k). \tag{4}$$

where the twist is given by $t(\lambda) := \frac{\lambda(r+1)}{r}$.

Another very important tool throughout this thesis is *localization* ([Bal00]). If $Z \hookrightarrow X$ is a regular closed embedding of a subscheme and the open complement is denoted by $U := X \setminus Z$, there is a long exact sequence of Witt groups

$$\cdots \to W^{i-1}(U, \mathcal{L}|_U) \xrightarrow{\partial} W^i_Z(X, \mathcal{L}) \xrightarrow{\iota_*} W^i(X, \mathcal{L}) \xrightarrow{v^*} W^i(U, \mathcal{L}|_U) \xrightarrow{\partial} \ldots \tag{5}$$

which is 12-periodic by 4-periodicity of the shifts. Note that the boundary map in general does not vanish. But if that is the case, this long exact sequence breaks down into split short exact sequences, which allows us to compute the Witt groups of the scheme of interest in terms of Witt groups that are smaller in some sense. But this is not satisfactory, since in general this map does not vanish and in this case it is not clear how to describe it.

However in a special setting, Balmer and Calmès ([BC09]) were able to explicitly describe the boundary map in terms of pull-backs and pushfowards along projective bundles and blow-ups.

Hypothesis (Balmer-Calmès hypothesis, [BC09, 1.2]**).** *Given a regular closed embedding $\iota : Z \hookrightarrow X$ of sufficiently nice schemes with open complement $v : U \hookrightarrow X$, assume that U is an affine bundle over a scheme Y admiting a morphism starting from the blow-up $\mathrm{BL}_Z(X)$ of X along Z such that the diagram*

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "v"'] \arrow[dl] \arrow[d, "\alpha"] \\ E \arrow[u] \arrow[r, hook] & \mathrm{Bl}_Z(X) \arrow[u] \arrow[r, "\tilde{\alpha}"'] & Y \end{tikzcd} \tag{6}$$

commutes.

Under these assumptions the boundary map $\partial : W^i(U, \mathcal{L}|_U) \to W^{i+1}_Z(X, \mathcal{L})$ in the localization sequence either vanishes or is a sequence of push-forwards and pull-backs along maps in (6).

Fix a ground field k with $\mathrm{char}(k) \neq 2$ and denote by $\mathrm{Gr}(d,n)$ the ordinary Grassmannian, i.e. the scheme parametrizing d-dimensional subspaces of k^n. For integers $d,e > 0$ the embedding $Z := \mathrm{Gr}(d,d+e-1) \subset \mathrm{Gr}(d,d+e) =: X$ satisfies all assumptions above. By homotopy invariance and dévissage the boundary map in the corresponding long exact sequence is a map

$$\partial : W^i(\mathrm{Gr}(d-1,d+e-1),(\alpha^*)^{-1}\mathcal{L}|_U) \to W^{i-d}(\mathrm{Gr}(d,d+e-1),\omega_\iota \otimes \mathcal{L}|_Z). \qquad (7)$$

Recall that, for any Young diagram with d rows and e columns, there is a (not necessarily smooth) Schubert cell inside $\mathrm{Gr}(d,n)$ and we can construct a family of elements inside the total Witt group of the Grassmannian by pushing forward the unit form along suitable resolutions of these Schubert varieties. Note that push-forwards of Witt groups, in particular starting from $W^0(-,\mathcal{O})$, do not necessarily exist, since in general a push-forward along a map $f : X \to Y$ is of the form

$$f_* : W^i(X,\omega_f \otimes f^*\mathcal{L}) \to W^{i-\dim f}(Y,\mathcal{L})$$

and there is no reason for the existence of a line bundle $\mathcal{L}$ satisfying $\omega_f \otimes f^*\mathcal{L} \equiv \mathcal{O}_X$ modulo squares. After all it turns out that for *even Yong diagrams*, that is, Young diagrams with inner segments of even length, all required push-forwards exist and the constructed elements are compatible with the boundary map in a certain sense. Further, $\mathrm{Pic}(\mathrm{Gr}(d,d+e))/2 = \mathbb{Z}/2$ where $\mathcal{O}(1)$ is the only non-trivial twist and we get the following description of the total Witt group:

Theorem (Witt groups of Grassmannians, [BC12a, 7.1]**).** *There is an isomorphism*

$$\bigoplus_{\substack{\Lambda \text{ even s.t.} \\ t(\Lambda)\equiv l(2)}} W^{i-|\Lambda|}(k) \xrightarrow{\sim} W^i(\mathrm{Gr}(d,d+e),\mathcal{O}(l)) \qquad (8)$$

where for a Young diagram Λ we denote by $|\Lambda|$ its weight and by $t(\Lambda)$ its twist, i.e. the half perimeter.

Note that this theorem is proved more general for Grassmann bundles over a base scheme which requires more assumptions on the underlying vector bundle.

The purpose of this thesis is to investigate the total Witt group of maximal isotropic Grassmannians $\mathrm{IG}(n)$, i.e. subschemes of $\mathrm{Gr}(n,2n)$ of subspaces which are isotropic with respect to a given symmetric or antisymmetric bilinear form. If the form is symmetric, we write $\mathrm{OG}(n)$ and it turns out that the embedding $\mathrm{OG}(n-1) \hookrightarrow \mathrm{OG}(n)$ satisfies the hypothesis above. In the antisymmeric setting we write $\mathrm{LG}(n) = \mathrm{IG}(n)$, but the hypothesis does no hold anymore and more work needs to be done.

Hypothesis (3.2.1, 3.2.7). *Let $\iota : Z \hookrightarrow X$ be a regular closed embedding of sufficiently nice schemes with open complement $v : U \hookrightarrow X$. Assume that U is an affine bundle over a scheme Y admitting an incoming morphism from the* double blow-up $\mathrm{Bl}_{\tilde{Z}'}(\mathrm{Bl}_Z(X))$ *where $Z' \subset E \subset$ $\mathrm{Bl}_Z(X)$ is a regular embedding and the diagram*

$$\begin{tikzcd}
 & Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "v"'] \arrow[dl] \arrow[ddl] \arrow[dd, "\alpha"] \\
Z' \arrow[r, hook] & E \arrow[r, hook] \arrow[u] & \mathrm{Bl}_Z(X) \arrow[u] & \\
E' \arrow[rr, hook] \arrow[u] & & \mathrm{Bl}_{Z'}(\mathrm{Bl}_Z(X)) \arrow[r, "\tilde{\alpha}"'] \arrow[u] & Y
\end{tikzcd} \tag{9}$$

commutes.

As a consequence, we again get a description of the boundary map in the localization sequence (section 2.3 for the symmetric and Theorem 3.3.1 for the antisymmetric case).

A *shifted Young diagram* for $\mathrm{OG}(n+1)$ or $\mathrm{LG}(n)$ is a Young diagram for $\mathrm{Gr}(n, 2n)$, i.e. with n rows and n columns where the lengths of the rows are strictly decreasing. We can right align these diagrams and recover *shifted Young-diagrams*, which are known to index the Chow group of $\mathrm{OG}(n+1)$ resp. $\mathrm{LG}(n)$ the same way as for the ordinary Grassmannian. In addition to the well-known even diagrams, we will also need *almost-even* diagrams, in which all, except for the last inner segments, are of even length and the last segment is of odd length.

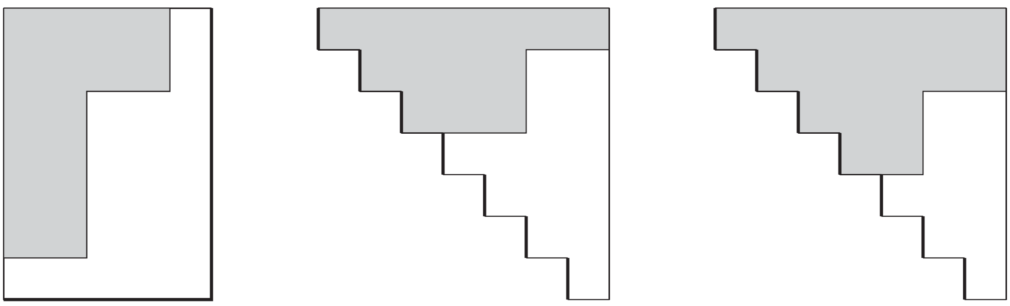

FIGURE 1. From left to right: An even Young diagram, an even shifted Young diagram and an almost even shifted Young diagram. The weight of any diagram is the area of the shaded part and denoted by $|\Lambda|$ and (in most cases) the twist $t(\Lambda)$ is the length of the boundary Λ along the thick part except that for the symmetric case this number has to be doubled (and hence is always even).

Grassmannian $\mathrm{Gr}(d,n) = A_{n-1}/P_d$:

Odd quadric $Q^{2n-1} = B_n/P_1$:

Lagrangian Grassmannian $\mathrm{LG}(n,2n) = C_n/P_n$:

Even quadric $Q^{2n} = D_n/P_1$:

Max. orth. Grassmannian $\mathrm{OG}(n) = D_n/P_n \cong D_n/P_{n-1}$:

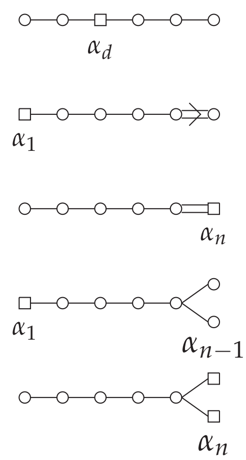

FIGURE 2. Over $\mathbb{C}$ the cominuscule varieties are precisely the irreducible compact hermitian symmetric spaces

The main result is the following condensed version of Theorems 4.2.1 and 4.3.2:

Theorem. *There is an isomorphism*

$$\bigoplus_{\substack{\Lambda \text{ s.t.} \\ t(\Lambda)\equiv l(2)}} W^{i-|\Lambda|}(k) \xrightarrow{\sim} W^i(\mathrm{IG}(n), \mathcal{O}(l)), \tag{10}$$

where the direct sums runs over even shifted diagrams in the symmetric case and over almost even shifted diagrams in the antisymmetric case.

Finally, it is worth mentioning that, together with the projective bundle theorem, the work of Balmer and Calmès on Grassmannians and numerous computations on quadrics such as by Xie ([Xie19]), this work completes the computation of Witt groups of *cominuscule varieties*, see Figure 2 for a list of these spaces. The only minuscule but not cominuscule varieties are $B_n/P_n \cong D_n/P_n$ and $C_n/P_1 \cong A_{2n}/P_1$, both of which are isomorphic to some cominuscule variety; hence we get these varieties for free.

Theorem. *Assume the homogeneous space $X = G/P$ is minuscule and cominuscule and of ordinary Lie type. Then the total Witt group of X is indexed by even Young diagrams of shape $\Lambda_{G/P}$, where the shape is as in* [BS16], *see also Figure 4.8.*

From [IM05, 4.1] we can construct a similar blow-up setup as for the Lagrangian Grassmannian, which leads to the following conjecture:

Conjecture. *The preceeding theorem also holds in exceptional types, that is, for the Cayley plane and the Freudenthal variety.*

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Chapter 1

Preliminaries

1.1 Isotropic vector bundles

For a smooth connected scheme X containing $\frac{1}{2}$ an *isotropic vector bundle* $\mathcal{V}$ over X is a vector bundle equipped with a symmetric or antisymmetric non-degenerate bilinear form ω on $\mathcal{V} \times \mathcal{V}$ with values in a line bundle $\mathcal{L}$ over X. If ω is symmetric, $(\mathcal{V}, \omega)$ is called *orthogonal*, whereas if ω is antisymmetric the pair is called *symplectic*. This form induces an isomorphism $\phi : \mathcal{V} \longrightarrow \mathrm{Hom}(\mathcal{V}, \mathcal{L}) = \mathcal{V}^\vee \otimes \mathcal{L}$. In particular, if $\mathcal{V}$ has rank n,

$$(\det V)^{\otimes 2} \cong \mathcal{L}^{\otimes n}. \tag{1.1}$$

Definition 1.1.1 (Subbundles, [BC12a, 1.1]**).** A subbundle $\mathcal{W} \subset \mathcal{V}$ is an $\mathcal{O}_X$-submodule of $\mathcal{V}$ s.t. $\mathcal{W}$ and $\mathcal{V}/\mathcal{W}$ are vector bundles.

For a subbundle $\mathcal{W} \subset \mathcal{V}$ of an isotropic vector bundle denote by $\mathcal{W}^\perp \subset \mathcal{V}$ the orthogonal complement of $\mathcal{W}$ defined by the short exact sequence

$$0 \to W^\perp \to \mathcal{V} \to \mathrm{Hom}(\mathcal{W}, \mathcal{L}) \to 0, \tag{1.2}$$

where the epimorphism is defined by $\omega|_{\mathcal{W}}$. Then $\mathcal{W}$ is called *isotropic*, if $\mathcal{W} \subset \mathcal{W}^\perp$, i.e. if the restriction of ω to $\mathcal{W}$ vanishes. Clearly $\mathrm{rk}(\mathcal{W}) \leq n$ and we say that $\mathcal{W}$ is *maximal isotropic*, if $\mathrm{rk}(\mathcal{W}) = n$. Maximal isotropic subbundles do not necessarily exist; however in case such a bundle $\mathcal{W}$ exists, the extension in 1.2 reads

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{V} \longrightarrow \mathrm{Hom}(\mathcal{W}, \mathcal{L}) \longrightarrow 0. \tag{1.3}$$

In this case we can reduce equation (1.1) to $\det \mathcal{V} = \mathcal{L}^{\otimes n}$.

Denote by $\mathrm{Vect}(X)$ the category of vector bundles over X. Then for any $\mathcal{V} \in \mathrm{Vect}(X)$ the dual is the vector bundle $\mathcal{V}^* := \mathrm{Hom}(\mathcal{V}, \mathcal{O}_X)$ and there is a natural isomorphism $\omega_{\mathcal{V}} : \mathcal{V} \cong \mathcal{V}^{**}$ which is compatible with the duality in the sense that there is a commutative diagram

$$\begin{tikzcd} \mathcal{V}^{***} \arrow[r, "\omega_{\mathcal{V}}^*"] & \mathcal{V}^* \\ \mathcal{V}^* \arrow[u, "\omega_{\mathcal{V}^*}"] \arrow[ur, equal] & \end{tikzcd}$$

Definition 1.1.2 (Symmetric bundles and isometries). (i) A *symmetric bundle* is an orthogonal vector bundle $(\mathcal{V}, \omega)$ with values in $\mathcal{O}_X$ such that the isomorphism $\phi : \mathcal{V} \to \mathcal{V}^*$ induced by ω is compatible with the chosen double-dual identification in the sense that the square

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\phi"] \arrow[d, "\omega"', "\sim"] & \mathcal{V}^* \arrow[d, equal] \\ \mathcal{V}^{**} \arrow[r, "\phi^*"'] & \mathcal{V}^* \end{tikzcd}$$

commutes. An *isometry* of symmetric bundles is an isomorphism $\psi : \mathcal{V}_1 \to \mathcal{V}_2$ which respects the duality in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, "\psi"] \arrow[d, "\phi_1"'] & \mathcal{V}_2 \arrow[d, "\phi_2"] \\ \mathcal{V}_1^* & \mathcal{V}_2^* \arrow[l, "\psi^*"] \end{tikzcd}$$

(ii) Let $\mathrm{MW}(X, *, \omega)$ be the set of isometry classes of symmetric bundles over X. There is a monoidal structure on this set by taking orthogonal sums in the obvious way.

1.2 Witt groups

1.2.1 Witt groups of exact categories

In the previous section we constructed the monoid MW of isometry classes of symmetric bundles. Now the Witt group of $\mathrm{Vect}(X)$ will essentially be this monoid, where we mod out those bundles admitting maximal *lagrangians*, i.e. subbundles of half the rank on which the symmetry vanishes. In the following we discuss this in more detail; see also [Bal05].

Definition 1.2.1 (Metabolic bundles). (i) A *sublagrangian* of the symmetric bundle $(\mathcal{V}, \phi)$ is an isotropic subbundle $\iota : \mathcal{W} \hookrightarrow \mathcal{V}$. That is, it is a subbundle which satisfies $\phi|_{\mathcal{W}} = 0$, i.e. $\mathcal{W} \subset \mathcal{W}^{\perp}$, where

$$\mathcal{W}^{\perp} = \ker(\mathcal{V} \xrightarrow{\phi} \mathcal{V}^{*} \xrightarrow{\iota^{*}} \mathcal{W}^{*})$$

denotes the orthogonal complement of $\mathcal{W}$ in $\mathcal{V}$. Moreover, $\mathcal{W}$ is called *lagrangian* if $\mathcal{W}$ is maximal, i.e. if $\mathcal{W} = \mathcal{W}^{\perp}$. Note that in this case $\mathrm{rk}(\mathcal{V}) = 2\mathrm{rk}(\mathcal{W})$. Finally, $(\mathcal{V}, \phi)$ is called *metabolic*, if it contains a lagrangian.

(ii) Let $\mathrm{NW}(X, *, \omega)$ be the subset of $\mathrm{MW}(X, *, \omega)$ consisting of the isometry classes of metabolic bundles.

In this setting Knebusch ([Kne77]) defined the Witt group as follows:

$$W(X) := W(X, *, \omega) := \frac{\mathrm{MW}(X, *, \omega)}{\mathrm{NW}(X, *, \omega)}.$$

Example 1.2.2. For any vector bundle $\mathcal{V}$, the *hyperbolic space* $\mathcal{H}(\mathcal{V})$ given by $\mathcal{V} \oplus \mathcal{V}^{*}$ with the symmetry isomorphism

$$\begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix} : \mathcal{V} \oplus \mathcal{V}^{*} \to \mathcal{V} \oplus \mathcal{V}^{*} \cong \mathcal{V}^{*} \oplus \mathcal{V}^{**}$$

is clearly metabolic; the converse is not necessarily true.

Example 1.2.3. If $X = \mathrm{Spec}(R)$ with $\frac{1}{2} \in R^{\times}$ the converse in (i) is true so in particular we recover $W(\mathrm{Spec}(k), *, \omega) = W(k)$, where $W(k)$ is the classical Witt group of equivalence classes of non-degenerate symmetric bilinear forms.

Example 1.2.4. Let $X = \mathrm{Spec}(k)$ where k is a quadratically closed field. Then a symmetric bundle is given by the data of a symmetric invertible matrix. Two such symmetric bundles are isometric if and only if the corresponding matrices are congruent, i.e. if one obtains one from another by conjugation. Recall from linear algebra that any symmetric invertible matrix is diagonalizable with non-zero eigenvalues. Moreover, since in $k^{\times}$ every element is a square, we conclude that any symmetric bundle is isometric to the symmetric bundle associated with the unit matrix $\mathbb{1}_n$, i.e. $\mathrm{MW}(X, *, \omega) = \mathbb{Z}$. By (i), any symmetric bundle is equivalent to some symmetric bundle of rank at most one in the Witt group. In other words, $\mathcal{H}(\mathbb{1}_n) \cong \mathbb{1}_{2n}$ implies $\mathrm{NW} = 2\mathrm{MW}$, so we conclude that $W(\mathrm{Spec}(k)) = \mathbb{Z}/2\mathbb{Z}$.

Remark 1.2.5 (Shifts and twists). Changing the duality or the double-dual identification on $\mathrm{Vect}(X)$, we obtain *shifted and twisted Witt groups* as follows:

- Any line bundle $\mathcal{L}$ on X gives us a duality $\mathcal{V}^{*_\mathcal{L}} = \mathrm{Hom}(\mathcal{V}, \mathcal{L})$ which in turn defines the twisted Witt group
$$W(X, \mathcal{L}) := W(X, *_\mathcal{L}, \omega_\mathcal{L}).$$
Note that this twisted Witt group depends only on the class of $\mathcal{L}$ in $\mathrm{Pic}(X)/2$. Indeed, if $\mathcal{L}' = \mathcal{L} \otimes \mathcal{M}^{\otimes 2}$ an isomorphism $\mathcal{V} \cong \mathcal{V}^{*_{\mathcal{L}'}} = \mathcal{V}^* \otimes \mathcal{L}'$ induces an isomorphism $\mathcal{V} \otimes \mathcal{M}^* \cong \mathcal{V}^* \otimes \mathcal{L}' \otimes \mathcal{M}^* = \mathcal{V}^* \otimes \mathcal{L} \otimes \mathcal{M} = (\mathcal{V} \otimes \mathcal{M}^*)^{*_\mathcal{L}}$ and vice versa. Hence the Witt groups for both dualities are isomorphic.

- If ω satisfies the compatibility condition above then so does $-\omega$, and we can consider the Witt group
$$W^2(X) := W(X, *, -\omega),$$
which this time describes *anti-symmetric bundles*. The meaning of the shift will become clear later, when we develop all possible shifts $W^i(X)$.

More generally, for an arbitrary category other than $\mathrm{Vect}(X)$, a *duality* consists of a contravariant functor $*$ and an equivalence $\omega : \mathrm{id} \cong * \circ *$ such that $(\omega_A)^* \circ (\omega_{A^*}) = \mathrm{id}_{A^*}$ for any object A. The triple $(\mathcal{C}, *, \omega)$ is then called a *category with duality*. We will also assume additivity, i.e. that direct sums are compatible with the duality such that $(A \oplus B)^* = A^* \oplus B^*$. Then we can define the monoid $\mathrm{MW}(\mathcal{C}, *, \omega)$ as above. For an adequate definition of $\mathrm{NW}(\mathcal{C}, *, \omega)$ we need more structure on our category; otherwise the considerations in the discussion above cannot be adopted. A subbundle $\mathcal{M}$ is a metabolic subspace of $(\mathcal{V}, \phi)$, if it fits into a short exact sequence $0 \to \mathcal{M} \to \mathcal{V} \to \mathcal{M}^* \to 0$. This leads to the notion of *exact categories*. Roughly speaking an additive category is *exact*, if it is equipped with a class $\mathcal{E}$ of short exact sequences (the admissible short exact sequences) subject to some compatibility conditions (see for instance [Qui73]). Finally an *exact category with duality* is an addtive exact category equipped with a duality such that the dual of an admissible short exact sequence is again admissible. In this more general setting one is now able to define $\mathrm{NW}(\mathcal{C}, *, \omega)$ and hence

$$W(\mathcal{C}, *, \omega) := \frac{\mathrm{MW}(\mathcal{C}, *, \omega)}{\mathrm{NW}(\mathcal{C}, *, \omega)}. \tag{1.4}$$

As mentioned above taking $\mathcal{C} = \mathrm{Vect}(X)$ one recovers Knebusch's classical Witt group.

1.2.2 Triangular Witt groups: Derived Witt groups of schemes

So far we have introduced several Witt groups in a geometric setting, but they still lack a cohomological flavour. Ideally, we would like to have a cohomology theory for Witt groups, in which the Witt groups defined above appear in certain degrees. Having this, one is provided many well-known tools for computations of Witt groups as we know them, for instance, from algebraic geometry or the theory of Chow groups. Precisely such a cohomology theory was constructed by Balmer in [Bal00] and [Bal01]: For the derived Witt groups, starting with an exact category with duality, we consider the derived category of chain complexes which is not exact, but it is still triangulated. Adapting the notion of metabolic spaces, one can define Witt groups of triangulated categories and these will be exactly the objects we want to have.

Let $(\mathcal{K}, *, \omega)$ be an additive category with duality. Recall that in an exact category with duality we have a collection of short exact sequences such that all dual sequences are also short exact. In a triangulated category we instead have a collection of sequences

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A), \tag{1.5}$$

the *distinguished* or *exact triangles*, where T is some fixed endofunctor, called the *translation functor*. There is an obvious notion of morphisms between triangles and there are certain conditions (T1)-(T4) (see e.g. [Ver77]) which essentially are adapted versions of the conditions on exact categories that the triangles need to satisfy. We say that $(\mathcal{K}, *, \omega)$ is a *triangulated category with δ-duality* for $\delta = \pm 1$ if $T(A^*) = T^{-1}(A)^*$, the functor T commutes with the double-dual isomorphism ω, and the "δ-dual" triangle

$$C^* \xrightarrow{v^*} B^* \xrightarrow{u^*} A^* \xrightarrow{\delta \cdot T(w^*)} T(C^*) \tag{1.6}$$

of any distinguished triangle as in (1.5) is again distinguished. In this context, we can define the monoid $\mathrm{MW}(\mathcal{K}, *, \omega)$ of symmetric spaces exactly as before. However, since the definition of metabolics involved short exact sequences, a triangular version of metabolic spaces needs to be established (see [Bal01, §2]). At the end the triangular Witt group is again defined as

$$W(\mathcal{K}, *, \omega) := \frac{\mathrm{MW}(\mathcal{K}, *, \omega)}{\mathrm{NW}(\mathcal{K}, *, \omega)}. \tag{1.7}$$

This leads us to the notion of *shifted* Witt groups:

Remark 1.2.6 (Shifted Witt groups). Let $(\mathcal{K}, *, \omega)$ be a triangulated category with δ-exact duality. Then the *n-shifted duality on* $\mathcal{K}$ is defined as

$$T^n(\mathcal{K}, *, \omega) := \left(\mathcal{K}, T^n \circ *, (-1)^{\frac{n(n+1)}{2}} \cdot \delta^n \cdot \omega\right)$$

and it is again a triangulated category, but this time $\delta_n := (-1)^n \cdot \delta$-exact (see [Bal00], in particular, for the cone consruction in §1). We then define the shifted Witt groups as

$$W^n(\mathcal{K}, *, \omega) := W(T^n(\mathcal{K}, *, \omega)). \tag{1.8}$$

Now given a scheme (as usual over $\mathbb{Z}[\frac{1}{2}]$) or more generally an exact category with duality $(\mathcal{E}, *, \omega)$, we want to construct a triangulated category such that the triangular Witt groups coincide with the usual Witt groups. This can be done by considering chain complexes with the right notion of morphisms.

Definition 1.2.7. Let $(\mathcal{E}, *, \omega)$ be an exact category with duality and denote by $K(\mathcal{E})$ the category of chain complexes over $\mathcal{E}$ with morphisms up to chain homotopies. Let further $K^b(\mathcal{E})$ be the subcategory of bounded chain complexes. Then the duality on $\mathcal{E}$ induces a (canonical) duality on $K^b(\mathcal{E})$ by

$$\left(\cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} \cdots\right)^* = \left(\cdots \to C_{-1}^* \xrightarrow{d_0^*} C_0^* \xrightarrow{d_1^*} C_1^* \xrightarrow{d_2^*} \cdots\right),$$

turning $K^b(\mathcal{E})$ into a triangulated category with duality, so technically we can compute the Witt group of this category (the shifts again given by moving all terms by one to the left and triangles are defined using the mapping cone construction (cf. [Bal01, §2])). Now one could define the triangular Witt groups of $\mathcal{E}$ by $W(\mathcal{E}) := W(K^b(\mathcal{E}))$, but since the functor $\mathcal{E} \to K^b(\mathcal{E})$, which sends an object C to the complex $\cdots \to 0 \to C \to 0 \to \ldots$, concentrated in degree zero, does not send short exact sequences to distinguished triangles in general (but only to triangles which are *quasi-isomorphic* to a distinguished one), we cannot expect to end up with a nice generalization of the usual Witt groups. Instead, we need to turn quasi-isomorphisms into isomorphisms by passing to the *derived category* $D^b(\mathcal{E})$ by localizing the morphisms at the class of quasi-isomorphisms. The triangulated structure survives this localization and we can finally define the *derived Witt group*

$$W_{\mathrm{der}}(\mathcal{E}, *, \omega) := W(D^b(\mathcal{E}), *, \omega) \tag{1.9}$$

as the triangulated Wtt group of the derived category $D^b(\mathcal{E})$ with induced duality; this is discussed in much more detail in [Bal01] to which we refer the reader.

Theorem 1.2.8 (Usual=derived, [Bal01, 4.3]**).** *Assume that $\frac{1}{2} \in \mathcal{E}$. Then the canonical functor $\mathcal{E} \to D^b(\mathcal{E})$ induces a group isomorphism*

$$W_{\text{usual}}(\mathcal{E}, *, \omega) \longrightarrow W_{\text{der}}(\mathcal{E}, *, \omega).$$

In particular, for a scheme X over $\mathbb{Z}[\frac{1}{2}]$, the Witt group

$$W(X) := W^0(X, \mathcal{O}_X) := W_{\text{usual}}(\text{Vect}(X), \text{Hom}(-, \mathcal{O}_X), \omega)$$

with the canonical double-dual identification ω satisfies

$$W(X) \cong W_{\text{der}}(D^b(\text{Vect}(X)), \text{Hom}(-, \mathcal{O}_X), \omega).$$

As in Remark 1.2.6, we can define shifted Witt groups by considering shifted dualities. In $D^b(\mathcal{E})$ the shift is given by moving the terms in a complex to the left, so the *n-th shifted duality* is given by

$$\left(\cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \to \dots\right)^{*_n} = \left(\cdots \to C^*_{-1+n} \xrightarrow{d^*_{-1+n}} C^*_n \xrightarrow{d^*_n} C^*_{1+n} \to \dots\right).$$

The double-dual identification ω on $\mathcal{E}$ induces one on $D^b(\mathcal{E})$ and we define (see 1.2.6)

$$W^n_{\text{der}}(\mathcal{E}, *, \omega) := W\left(D^b(\mathcal{E}), *_n, (-1)^{\frac{n(n+1)}{2}} \omega_n\right). \tag{1.10}$$

Definition 1.2.9. For a scheme X over $\mathbb{Z}[\frac{1}{2}]$ and a line bundle $\mathcal{L} \in \text{Pic}(X)$ we define the *n-th shifted Witt group of X twisted by $\mathcal{L}$* by

$$W^n(X, \mathcal{L}) := W\left(D^b(\text{Vect}(X)), (\text{Hom}(-, \mathcal{L}))_n, (-1)^{\frac{n(n+1)}{2}} \omega_n\right),$$

where we denote by $(\text{Hom}(-, \mathcal{L}))_n$ the n-th shifted duality induced on the derived category by the duality $\text{Hom}(-, \mathcal{L})$ on $\text{Vect}(X)$.

Remark 1.2.10 (Two periodicities). The Witt groups are four-periodic in the shifts. From $T^n(\mathcal{K}, *, \omega) = T^{n+2}(\mathcal{K}, *, -\omega)$ one deduces $W^{n+4}(X, \mathcal{L}) \cong W^n(X, \mathcal{L})$ for any $n \in \mathbb{Z}$ and $\mathcal{L} \in \text{Pic}(X)$. For simplicity, we denote the four different Witt groups by W^0, W^1, W^2 and W^3. There is a second periodicity, the *square periodicity* on the twists: As already seen in remark 1.2.5, the Witt group shifted by a line bundle $\mathcal{L}$ only depends on the class of the line bundle in $\text{Pic}(X)/2$. However, the computation of $W^n(X, \mathcal{L})$ involves the choice of a representative of such a class which leads to the theory of alignments and lax-similtude. This will be discussed in sections 1.2.9, 1.2.10.

1.2.3 Coherent Witt groups

With a scheme one can associate triangulated categories other than the derived one and hence construct other Witt groups. Consider the category $\mathcal{M}_\mathrm{c}(X)$ of coherent modules over X and the associated derived category $D^b(\mathcal{M}_\mathrm{c}(X))$. Again we have dualities on $D^b(\mathcal{M}(X))$, which however are not necessarily induced by line bundles. We have the more abstract notion of a *dualizing complex* which is an object $K \in D^b(\mathcal{M}(X))$ such that the right derived functor $\mathrm{RHom}(-, K)$ induces an equivalence $D^b(\mathcal{M}_\mathrm{c}(X))^{\mathrm{op}} \cong D^b(\mathcal{M}_\mathrm{c}(X))$. We then define

$$W_{\mathrm{coh}}(X, K) := W(D^b(\mathcal{M}_\mathrm{c}(X)), \mathrm{RHom}(-, K), \omega)$$

where ω is a canonical double-dual identification (see e.g. [Gil03]). There is again a notion of twists and shifts as for derived Witt groups. We collect some properties of coherent Witt groups and dualizing complexes:

Remark 1.2.11.
- Two dualizing complexes differ only by a shifted line bundle, i.e. given dualizing complexes K, K' there is a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ and $n \in \mathbb{Z}$ s.t. $K \cong K' \otimes \mathcal{L}[n]$ in the derived category, i.e. the two complexes are quasi-isomorphic ([Nee10, 3.9]). Here, a shifted line bundle is the complex concentrated in degree n.
- If X is Gorenstein, $\mathcal{O}_X$ itself, concentrated in degree zero, is a dualizing complex and hence dualizing complexes are simply shifted line bundles. In this case the canonical functor $D^b(\mathrm{Vect}(X)) \to D^b(\mathcal{M}_\mathrm{c}(X))$ (think of $\mathrm{Vect}(X) \subset \mathcal{M}_c(X)$ to be the locally free sheaves) induces homomorphisms $W^n_{\mathrm{der}}(X) \to W^n_{\mathrm{coh}}(X)$.
- This homomorphism is an isomorphism for smooth schemes ([Gil02, 2.13]).
- From now on we write $W(-)$ for derived and $\tilde{W}(-)$ for coherent Witt groups.

1.2.4 Pull-backs, push-forwards, base change

Let $f : X \to Y$ be a morphism of separated and noetherian schemes. Then the functor f^* admits a left derived functor $\mathrm{L}f^*$ on $D^b(\mathrm{Vect}(X))$ which, under certain conditions ([CH11]) and in particular, when X and Y are regular, restricts to a functor $\mathrm{L}f^* : \mathcal{M}_c(Y) \to \mathcal{M}_c(X)$ with the notation from above. If moreover for a dualizing complex K_Y on Y the complex $\mathrm{L}f^*(K_Y)$ is again dualizing, then this functor induces a map

$$f^* : \tilde{W}^n(Y, K) \to \tilde{W}^n(X, \mathrm{L}f^*(K)) \tag{1.11}$$

on coherent Witt groups, called the *pull-back*. If X, Y are regular, we can replace dualizing complexes by (shifted) line bundles and coherent by derived Witt groups and the pull-back map reads

$$f^* : W^n(Y, \mathcal{L}) \to W^n(X, f^*\mathcal{L}). \tag{1.12}$$

Similarly, f admits a right derived functor $\mathrm{R}f_*$ on the derived category of quasi-coherent $\mathcal{O}_X$-modules which, under certain conditions, restricts to a functor $\mathrm{R}f_* : D^b(\mathcal{M}(X)) \to D^b(\mathcal{M}(Y))$. Then $\mathrm{R}f_*$ admits a right adjoint functor $f^!$. If additionally for a dualizing complex K_Y on Y the complex $f^!K_X$ is dualizing, then this functor induces a map

$$f_* : \tilde{W}^n(X, f^!K_Y) \to \tilde{W}^n(Y, K_Y) \tag{1.13}$$

on coherent Witt groups, called the *push-forward*. If f is equidimensional, we define $\omega_f := f^!\mathcal{O}_Y[d]$ where d is the relative dimension. If f is proper and a locally complete intersection (e.g. smooth), ω_f is a line bundle and called the *relative canonical line bundle*. Denote by $\otimes^L$ the left derived functor on the derived category of the tensor product. Then, if Y is Gorenstein, we can rewrite the push-forward map as

$$f_* : \tilde{W}^{n+d}(X, \omega_f \otimes^{\mathrm{L}} \mathrm{L}f^*K_Y) \to \tilde{W}^n(Y, K_Y)$$

and as

$$f_* : W^{n+d}(X, \omega_f \otimes f^*\mathcal{L}) \to W^n(Y, \mathcal{L}), \tag{1.14}$$

if both X, Y are smooth. In the following cases we are able to describe the relative canonical line bundle:

- If $f : X \to Y$ is smooth and proper, the relative canonical bundle is given by $\omega_f = \det T^*_{X/Y}$, where $T^*_{X/Y}$ denotes the dual of the relative tangent bundle. In particular, if E is a vector bundle of rank $r+1$ over X (see section 1.2.13), then we have for the associated projective bundle $p : \mathbb{P}(E) \to X$ that

$$\omega_p = \mathcal{O}_{\mathbb{P}(E)}(-1)^{\otimes r+1} \otimes p^*(\det E). \tag{1.15}$$

  Similarly, if $\pi : \mathrm{Gr}_X(d, \mathcal{V}) \to X$ is the Grassmann bundle parametrizing rank d subbundles of a vector bundle $\mathcal{V}$ over X of rank $d+e$, we have ([BC09, A.9(i)] and [BC12a, 1.5])

$$\omega_\pi = \mathcal{O}_{\mathrm{Gr}_X(d,\mathcal{V})}(-1)^{\otimes(d+e)} \otimes \pi^*(\det \mathcal{V})^{\otimes(-d)}. \tag{1.16}$$

- If $\iota : Z \hookrightarrow X$ is a regular closed immersion, the relative canonical bundle is $\omega_\iota = \det \mathcal{N}_{Z/X}$, the determinant of the normal bundle ([BC09, A.9(ii)]). In particular, if $\iota : D \hookrightarrow X$ is the inclusion of a Cartier divisor, $\omega_\iota = \mathcal{O}(E)|_E = \mathcal{O}_E(-1)$.

Pull-backs and push-forwards are compatible in the following sense:

Theorem 1.2.12 (Base-change, [CH11, Thm. 5.5]**).** *Let*

$$\begin{tikzcd} W \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a pull-back diagram of Noetherian, separated schemes such that push-forwards along horizontal and pull-backs along vertical maps exist. Let K_Z be a dualizing complex on Z. Then there is map $\epsilon : \mathrm{L}\overline{g}^* f^! K_Z \to \overline{f}^! \mathrm{L}g^* K_Z$ of dualizing complexes on W. Moreover, if the square is flat (i.e. f is flat), then ϵ is an isomorphism and the induced square on Witt groups commutes, i.e. we have $g^* \circ f_* = \overline{f}_* \circ \epsilon \circ \overline{g}^*$.

1.2.5 Witt groups with support

Let $Z \subset X$ be a closed subscheme. An element in $D^b(\mathrm{Vect}(X))$ is a complex and we say it has *support* in Z, if the stalks of all its homology modules vanish outside of Z. Then the full subcategory $D^b_Z(\mathrm{Vect}(X), *, \omega)$ of those complexes is triangulated with duality and double-dual identification inherited from $D^b(\mathrm{Vect}(X)) = D^b(\mathcal{P}(X))$. Its Witt groups are called the *Witt groups with support in Z* and denoted by $W^n_Z(X, \mathcal{L}) := W^n_{\mathrm{der},Z}(X, \mathcal{L})$. In the extreme case, we of course have $W^n_X(X, \mathcal{L}) = W^n(X, \mathcal{L})$ and there is a natural map $e : W^n_Z(X, \mathcal{L}) \to W^n(X, \mathcal{L})$, referred to as the *extension of support*. In the same manner we can define the *coherent Witt groups with support*. We still have an isomorphism between the usual and derived Witt groups: The canonical functor in Remark 1.2.11 restricts to a homomorphism on the supported derived categories and induces an isomorphism in the regular case.

Given a closed embedding $\iota : Z \hookrightarrow X$, it is natural to ask whether the Witt groups with support may be written in terms of the Witt groups of Z. The canonical functor of triangulated categories $D^b(\mathcal{P}(Z)) \to D^b(\mathcal{P}(X))$ (with the induced duality) clearly factors through the supported category $D^b_Z(\mathcal{P}(X))$. If Z and X are smooth, the push-forward induces a *dévissage isomorphism*

$$\iota_* : W^n(Z, \iota^! \mathcal{L}) = W^{n-d}(Z, \omega_\iota \otimes \mathcal{L}|_Z) \xrightarrow{\sim} W^n_Z(X, \mathcal{L}). \tag{1.17}$$

A good reference for more details on supported Witt groups is [Gil03].

1.2.6 Localization

One of the benefits of the theory of derived and coherent Witt groups compared to the classical Witt groups is the repertoire of cohomological tools. One of the most important properties for us will be localization, which we describe in the following.

The key observation is that from a short exact sequence of triangulated categories we are able to construct a 12-periodic exact sequence as follows ([Bal00]): Given a triangulated category with duality $\mathcal{K}$ with $\frac{1}{2} \in \mathcal{K}$ and a system S of morphisms in $\mathcal{K}$, which is closed under duality, we can *localize* $\mathcal{K}$ by S to obtain another triangulated category with duality $S^{-1}\mathcal{K}$ together with a map $W(\mathcal{K}) \to W(S^{-1}\mathcal{K})$. Denoting by $J(S)$ the kernel category of this map, one shows that this gives rise to a short exact sequence of triangulated categories

$$0 \to J(S) \to \mathcal{K} \to S^{-1}\mathcal{K} \to 0.$$

This in turn will induce a long exact sequence

$$\cdots \to W^n(J(S)) \to W^n(\mathcal{K}) \to W^n(S^{-1}\mathcal{K}) \xrightarrow{\partial} W^{n+1}(J(S)) \to W^{n+1}(\mathcal{K}) \to \ldots, \tag{1.18}$$

where, of course, the construction of the *connecting homomorphism* ∂ is the interesting part.

Let us now apply this to our derived and coherent Witt groups. Let X be a Noetherian and separated scheme, let $v : U \hookrightarrow X$ be an open subscheme such that the inclusion of the closed complement of codimension d is a regular closed embedding $\iota : Z \hookrightarrow X$. Let $\mathcal{K} = D^b(\mathrm{Vect}(X))$. If X is smooth, $D^b(\mathrm{Vect}(U))$ is the localization of $\mathcal{K}$ by some multiplactive system S (see e.g. [Bal99, §5]). Hence, by dévissage (1.17), the localization long exact sequence in the regular case takes the form

$$\cdots \to W^{n-1}(U, \mathcal{L}|_U) \xrightarrow{\partial} W^{n-d}(Z, \omega_\iota \otimes \mathcal{L}|_Z) \xrightarrow{\iota_*} W^n(X, \mathcal{L}) \xrightarrow{v^*} W^n(U, \mathcal{L}|_U) \xrightarrow{\partial} \ldots, \tag{1.19}$$

where we denote by $\mathcal{L}|_U$ the restriction $v^*\mathcal{L}$ of the line bundle $\mathcal{L}$ to U and similarly for $\mathcal{L}|_Z$. If the schemes are not smooth, starting with $\mathcal{K} = D^b(\mathcal{M}_c(X))$, the sequence becomes

$$\cdots \to \tilde{W}^{n-1}(U, K|_U) \xrightarrow{\partial} \tilde{W}^n_Z(X, K) \xrightarrow{e} \tilde{W}^n(X, K) \xrightarrow{v^*} \tilde{W}^n(U, K|_U) \xrightarrow{\partial} \ldots \tag{1.20}$$

for any dualizing complex K on X, where we denote by e the *extension of support map*, and again by $K|_U$ the restricted dualizing complex on U. Note that we still have a map

$$\iota_* : \tilde{W}^n(Z, \iota^! K) \to \tilde{W}^n_Z(X, K),$$

but it is not necessarily an isomorphism.

1.2.7 Projective bundles

For a vector bundle E of rank $r+1$ over a scheme X we consider the corresponding projective bundle $p : \mathbb{P}(E) \to X$ of rank r. For any *oriented* cohomology theory H one has

$$H^*(\mathbb{P}(E)) \cong H^*(X)^{\oplus(r+1)}$$

simply by definition (e.g. [LM01]). Most of the classical theories are known to be oriented, such as K-theory and Chow theory. However, in the 80s, Arason showed $W^0(\mathbb{P}^n_k) = W(k)$, proving that (at least the untwisted and unshifted) Witt groups cannot form an oriented cohomology theory. Only about twenty years later, Walter ([Wal03]) and Nenashev ([Nen09]) independently computed the Witt groups for arbitrary shifts and twists, i.e. the groups $W^n(\mathbb{P}(E), \mathcal{L})$ for any $n \in \mathbb{Z}/4$ and line bundle $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}(E))/2$; recently Rohrbach generalized these results to Grothendieck-Witt spectra ([Roh20]). Note that the Picard group has one generator $\mathcal{O}_{\mathbb{P}(E)}(1)$ over the base, so line bundles over $\mathbb{P}(E)$ are of the form

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(E)}(1)^{\otimes m} \otimes p^*\mathcal{M} = \mathcal{O}_{\mathbb{P}(E)}(m) \otimes p^*\mathcal{M} \tag{1.21}$$

for some $\mathcal{M} \in \mathrm{Pic}(X)$ and $m \in \mathbb{Z}$. The projective bundle formula for Witt groups reads as follows:

Theorem 1.2.13 (Projective bundle formula). *Depending on m and n in the notation above we have:*

(i) If r is even, then the pull-back along p induces an isomorphism

$$W^n(\mathbb{P}(E), p^*\mathcal{M}) \xleftarrow{p^*} W^n(X, \mathcal{M}) \tag{1.22}$$

and the push-forward along p induces an isomorphism

$$W^n(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^*\mathcal{M}) \xrightarrow{p_*} W^{n-r}(X, \mathcal{M} \otimes \det E^*) \tag{1.23}$$

where in the second isomorphism we suppressed the periodicity isomorphism associated with

$$\omega_p \otimes p^*(\mathcal{M} \otimes \det E^*) \cong \mathcal{O}_{\mathbb{P}(E)}(-r-1) \otimes p^*(\det E) \otimes p^*(\mathcal{M} \otimes \det E^*).$$

(ii) If r is odd, the twisted Witt groups vanish:

$$W^n(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^*\mathcal{M}) = 0. \tag{1.24}$$

(iii) If r is odd and the twist is trivial, the pull-back and push-forward fit into a (not necessarily

split) long exact sequence

$$\cdots \to W^n(X,\mathcal{M}) \xrightarrow{p^*} W^n(\mathbb{P}(E),p^*\mathcal{M}) \xrightarrow{p_*} W^{n-r}(X,\mathcal{M}\otimes\det E^*) \xrightarrow{\theta} \dots \tag{1.25}$$

(where again in p_* is suppressed a periodicity isomorphism). If E admits a subbundle of even rank over X, the sequence splits, so in this case

$$W^n(\mathbb{P}(E),p^*\mathcal{M}) \cong W^n(X,\mathcal{M}) \oplus W^{n-r}(X,\mathcal{M}\otimes\det E^*). \tag{1.26}$$

1.2.8 The Balmer-Calmès setup

Given a regular closed embedding $\iota : Z \hookrightarrow X$ of smooth schemes with open complement $v : U \hookrightarrow X$, we can consider the corresponding localization long exact sequence (1.19)

$$\cdots \to W^{n-1}(U,\mathcal{L}|_U) \xrightarrow{\partial} W^{n-d}(Z,\omega_\iota\otimes\mathcal{L}|_Z) \to W^n(X,\mathcal{L}) \to W^n(U,\mathcal{L}|_U) \xrightarrow{\partial} \dots.$$

The computation of the Witt groups of X becomes a lot easier, when the connecting homomorphism vanishes; in this case, the long exact sequence splits into short exact sequences. If the Witt groups of the "smaller" schemes Z and U are known, then we can obtain a complete description of the Witt groups of X. Balmer and Calmès have developed a setup in which we can describe this map as a composition of push-forwards and pull-backs along certain involved maps.

Setup 1.2.14 (Balmer-Calmès setup, [BC09, 1.1]**).** Given a regular closed embedding of schemes $\iota : Z \hookrightarrow X$ with open complement $v : U \hookrightarrow X$, let $\pi : \mathrm{Bl}_Z(X) \to X$ be the blow-up of X along Z and denote by $\tilde{\iota} : E \hookrightarrow \mathrm{Bl}_Z(X)$ the inclusion of the exceptional divisor which is the pojective bundle $\pi' : E = \mathbb{P}(\mathcal{N}_{Z/X}) \to Z$ over Z. Hence we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "v"'] \arrow[dl, hook, "\tilde{v}"] \\ E \arrow[u, "\pi'"] \arrow[r, hook, "\tilde{\iota}"'] & \mathrm{Bl}_Z(X) \arrow[u, "\pi"'] & \end{tikzcd} \tag{1.27}$$

If the open complement U is an affine bundle over some scheme and the bundle map factors through the blow-up of X along Z, then for an oriented and homotopy invariant cohomology theory this is already sufficient to show that the connecting homomorphism splits. This is not necessarily the case for Witt groups, but a first good step.

Hypothesis 1.2.15 (Balmer-Calmès hypothesis, [BC09, 1.2]**).** In Setup 1.2.14 assume that there is a scheme Y and map $\tilde{\alpha} : \mathrm{Bl}_Z(X) \to Y$ such that $\alpha := \tilde{\alpha}|_U = \tilde{\alpha} \circ \tilde{v} : U \to Y$ is an affine bundle, i.e. the diagram

$$\begin{tikzcd}
Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "v"'] \arrow[dl, hook', "\tilde{v}"'] \arrow[d, "\alpha"] \\
E \arrow[u, "\pi'"] \arrow[r, hook, "\tilde{\iota}"'] & \mathrm{Bl}_Z(X) \arrow[u, "\pi"'] \arrow[r, "\tilde{\alpha}"'] & Y
\end{tikzcd} \tag{1.28}$$

commutes.

Under this hypothesis, for an oriented and homotopy invariant cohomology theory, the composition $\pi_* \circ \tilde{\alpha}^* \circ (\alpha^*)^{-1}$ is a splitting of v^* in the localization sequence. However, for Witt groups this hypothesis does not guarantee the vanishing of the boundary map since the push-forward π_* does not always exist. Nevertheless, we still can describe the map in all cases which we briefly discuss in the following.

For the Picard group of the blow-up we have $\mathrm{Pic}(\mathrm{Bl}_Z(X)) = \mathrm{Pic}(X) \oplus \mathbb{Z}\mathcal{O}(E)$ where $\mathcal{O}(E)$ denotes the class of the exceptional divisor E in the Picard group. Given a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ over X, the hypothesis above gives us another line bundle

$$\mathcal{L}' := (\tilde{\alpha}^* \circ (\alpha^*)^{-1} \circ v^*)(\mathcal{L}) \in \mathrm{Pic}(\mathrm{Bl}_Z(X)) = \mathrm{Pic}(X) \oplus \mathbb{Z}\mathcal{O}(E).$$

Define $\lambda(\mathcal{L}) \in \mathbb{Z}$ such that

$$\mathcal{L}' = \pi^*(\mathcal{L}) \otimes \mathcal{O}(E)^{\otimes \lambda(\mathcal{L})}.$$

From (1.14) we see that a push-forward along some map f starting at a Witt group with a certain twist only exists if the twist lies in the image of $f^!$. In our situation, it turns out that fixing the twist $\mathcal{L}'$ we can *either* push-forward along π *or* along π' after restricting to the exceptional divisor. This is determined by the parity of $\lambda(\mathcal{L})$ which in the end decides whether the localization long exact sequence splits or not:

Theorem 1.2.16 (Connecting homomorphism, [BC09, 2.3, 2.6]**).** *(i) If $\lambda(\mathcal{L}) \not\equiv d \mod 2$ we can push-forward along π and the connecting homomorphism vanishes, i.e. the localization sequence splits.*

(ii) If $\lambda(\mathcal{L}) \equiv d \mod 2$ we can push-forward along π' and the connecting homomorphism is given by

$$\partial = \iota_* \circ \pi'_* \circ (\tilde{\alpha} \circ \tilde{\iota})^* \circ (\alpha^*)^{-1} : W^n(U, \mathcal{L}|_U) \to W^{n-d+1}(Z, \omega_\iota \otimes \mathcal{L}|_Z).$$

Example 1.2.17. If $X = \mathrm{Gr}(d, V)$ is the Grassmannian parametrizing d-dimensional subspaces of a given $(d+e)$-dimensional vector space V, $V^1 \subset V$ a subspace of codimension one and $Z = \mathrm{Gr}(d, V^1)$, then we can choose $Y = \mathrm{Gr}(d-1, V^1)$ and this setup satisfies 1.2.15 above with $Z \subset X$ of codimension d. One checks that one has projective bundles π' of rank $d-1$ and $\tilde{\alpha} \circ \tilde{\iota}$ of rank $e-1$. If d and e are odd, by the projective bundle formula all factors of ∂ for $\lambda(\mathcal{L}) \equiv 0 \mod 2$ are isomorphisms, so $W^n(X, \mathcal{L}) = 0$ for this twist $\mathcal{L}$. Since d is odd, this twist is just $\mathcal{L} = \mathcal{O}_{\mathrm{Gr}(d,V)}(1) =: \mathcal{O}(1)$, so we conclude that the twisted Witt groups vanish:

$$W^n(\mathrm{Gr}(d, V), \mathcal{O}(1)) = 0.$$

Remark: For $\mathcal{L} = \mathcal{O}$ (d and e still odd), the connecting homomorphism vanishes. Hence, since $\omega_\iota = \mathcal{O}(-1)$, we have a splitting

$$W^n(\mathrm{Gr}(d, V), \mathcal{O}) \cong W^n(\mathrm{Gr}(d-1, V^1), \mathcal{O}) \oplus W^{n-d}(\mathrm{Gr}(d, V^1), \mathcal{O}(1)).$$

Even if we have not computed the groups on the right hand side, these are the Witt groups of smaller Grassmannians, so one can obtain a complete description using an inductive approach. This has been carried out for orthogonal Grassmannians in [HMX21].

1.2.9 Lax-similtude

In the next two sections we define the total Witt group and enlighten the notion of a total basis of the total Witt group. This is all discussed in detail in [BC12b] to which we refer. Let

$$W^{\mathrm{tot}}(X) := \bigoplus_{\substack{i \in \mathbb{Z}/4\mathbb{Z}, \\ [\mathcal{L}] \in \mathrm{Pic}(X)/2}} W^i(X, \mathcal{L}) \tag{1.29}$$

be the *total Witt group* of X. We already pointed out in Remark 1.2.10 that this total Witt group is not defined canonically, since the choice of a representative in a class of a line bundle is required. This is the main disruption and we need to make sure that in the end this choice does not affect us in what we do.

The main idea of solving this issue is to identify in a Witt group those elements which are mapped onto each other by a possible isomorphism between the Witt group, which itself is induced by a square periodicty isomorphism

$$\phi : \mathcal{L} \otimes \mathcal{M}^{\otimes 2} \xrightarrow{\sim} \mathcal{L}' \tag{1.30}$$

of line bundles over X of the same class in $\mathrm{Pic}(X)$. Such an isomorphism is called an *alignment* and denoted by $A : \mathcal{L} \rightsquigarrow \mathcal{L}'$; it induces an isomorphism

$$W^n(X,\mathcal{L}) \xrightarrow{\text{periodicity}} W^n(X,\mathcal{L}\otimes\mathcal{M}^2) \xrightarrow{\text{induced}} W^n(X,\mathcal{L}') \tag{1.31}$$

on Witt groups (this remains true for supported Witt groups), denoted by $A^{\circlearrowleft}$. In this context, two elements $w \in W^n(X,\mathcal{L}')$ and $w' \in W(X,\mathcal{L}')$ are called *lax-similar*, written $w \leftrightsquigarrow w'$, if $A^{\circlearrowleft}(w) = w'$ for some alignment $A : \mathcal{L} \rightsquigarrow \mathcal{L}'$ and this defines an equivalence relation. There is also a relative version: If the scheme X is defined over a base S, an *S-alignment* $\mathcal{L} \rightsquigarrow \mathcal{L}'$ is again an isomorphism as in (1.30), but this time in the *relative Picard group* $\mathrm{Pix}_S(X)/2 = (\mathrm{Pic}(X)/\mathrm{Pic}(S))\,/2$. In this case it follows that there is a well-defined $W^{\mathrm{tot}}(S)$-module structure on $W^{\mathrm{tot}}(X)$ which we call the *lax-structure*.

Recall from section 1.2.4 that, given a proper map $f : X \to Y$ of relative dimension d of smooth schemes and $\mathcal{L} \in \mathrm{Pic}(Y)$, we have induced maps

$$f^* : W^n(Y,\mathcal{L}) \to W^n(X,f^*\mathcal{L}), \qquad f_* : W^n(X,\omega_f\otimes f^*\mathcal{L}) \to W^{n-d}(Y,\mathcal{L}),$$

the *pull-back* and *push-forward*. It is immediate that these maps behave well with alignments. So in the target of the f^* we can allow instead of the twist $f^*\mathcal{L}$ any twist $\mathcal{L}'$ aligned to $f^*\mathcal{L}$ and similarly, in the domain of f_* we allow any line bundle $\mathcal{L}'$ aligned to $\omega_f \otimes f^*\mathcal{L}$. This gives us the notion of the *lax-pull-back* and *lax-push-forward*

$$(f^*)_{\mathrm{lax}} : W^n(Y,\mathcal{L}) \to W^n(X,\mathcal{L}'), \qquad (f_*)_{\mathrm{lax}} : W^n(X,\mathcal{L}') \to W^{n-d}(Y,\mathcal{L}) \tag{1.32}$$

and these maps behave well under composition. Moreover, alignment isomorphisms are combatible with localization.

So what are these alignments and lax-similitude good for? The answer is: As long as we do only care for Witt classes *up to lax-similtude*, we do not have to care about the exact twists, but only about the twists *up to alignments*. This really allows to simply choose representatives of classes in $\mathrm{Pic}(X)/2$ in (1.29). Similarly, the domain of a lax-pull-back and the target of a lax-push-forward only matter *up to alignment*.

1.2.10 Total bases

Given Witt classes, each of them in a certain shift and twist, in order to make use of the notion of a basis, we need to know what a linear combination is. Having this, a family of Witt classes forms a basis if, as usual, the family is "linear independent" and "generating". In this section we make this more precise and state a useful method how to inductively obtain such bases from the localization sequence. This will cover [BC12b, §5- §7].

Let $\mathcal{S}_S$ be the category of smooth, noetherian schemes over the base S subject to the conditions (I)-(III) in [BC12b, 4.1]. Roughly these assumptions make sure that given a map $X \to Y$, the pull-backs of two line bundles cannot be aligned over X without being so over Y. As pointed out in the previous section, there is a $W^{\mathrm{tot}}(S)$-module structure on $W^{\mathrm{tot}}(S)$. A *linear combination* of Witt classes $w_i \in W^{n_i}(X, \mathcal{L}_i)$ is an expression

$$\lambda_1 \cdot w_1 + \ldots + \lambda_k \cdot w_k \tag{1.33}$$

with suitable elements $\lambda_i \in W^{\mathrm{tot}}(S)$ and a suitable scalar multiplication $\cdot$. Both can be extracted from the module structure which can be made explicit in shifts and twists. The requirement on the coefficients λ_i then is that all the products $\lambda_i \cdot w_i$ should all land in some common $W^k(X, \mathcal{L})$, see [BC12b, 5.2 and 6.2].

Definition 1.2.18 (Total bases, [BC12b, 6.3-6.6]**).** In the setting above, the family of Witt classes $\{w_1, \ldots, w_k\}$ is called:

(i) *totally independent over* S, if the equality $\lambda_1 \cdot w_1 + \ldots + \lambda_k \cdot w_k = 0$ induces $\lambda_i = 0$ for all i;

(ii) *totally generating* $W^{\mathrm{tot}}(X)$ *over* S, if any element in the total Witt group can be written as in (1.33);

(iii) *a total basis of* $W^{\mathrm{tot}}(X)$ *over* S, if it is both totally independent over S and totally generating.

Consider now the localization long exact sequence

$$\cdots \to W^{n-1}(U, \mathcal{L}|_U) \xrightarrow{\partial} W^n_Z(X, \mathcal{L}) \xrightarrow{\mathrm{e}} W^n(X, \mathcal{L}) \xrightarrow{v^*} W^n(U, \mathcal{L}|_U) \xrightarrow{\partial} \ldots$$

where the scheme of interest X is defined over the smooth, separated, noetherian scheme S over $\mathbb{Z}[\frac{1}{2}]$. Assume, that

$$v^* : \mathrm{Pic}_S(X)/2 \to \mathrm{Pic}_S(U)/2 \tag{1.34}$$

is *injective*. Note that this is true in many, but not in all cases. Then we can construct a total basis of $W^{\mathrm{tot}}(X)$ out of total bases of the total Witt group of the "smaller" scheme U and the supported Witt group $W^{\mathrm{tot}}_Z(X)$, which by dévissage (1.17) itself can also be identified with the total Witt group of the "smaller" Z in the following way (we call U and Z smaller as in the case of Grassmannians these subschemes are again Grassmannians of lower dimensions).

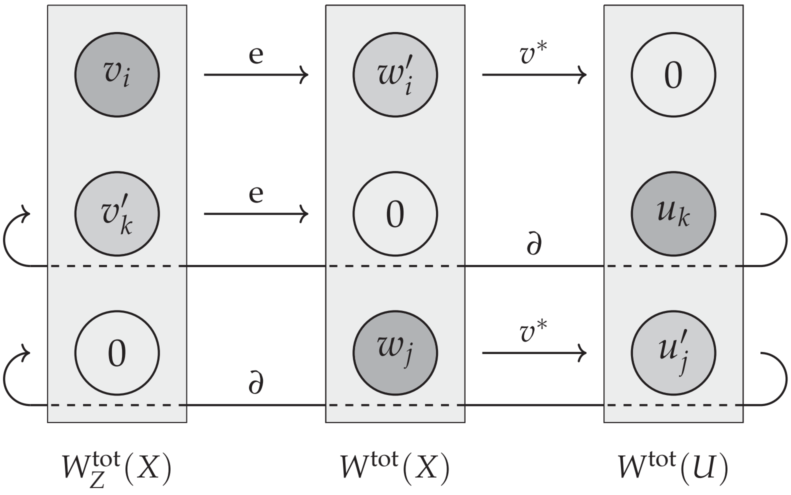


FIGURE 1.1. Illustration of Theorem 1.2.19, see [BC12b, Fig. 1]. Essentially it says that the localization sequence is still exact, if we replace equalities by lax-similitudes.

Theorem 1.2.19 ([BC12b, 7.1]**).** *With notation above let $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ be index sets and*

$$v'_k, v_i \in W^{\mathrm{tot}}_Z(X), \qquad w'_i, w_j \in W^{\mathrm{tot}}(X), \qquad u'_j, u_k \in W^{\mathrm{tot}}(U)$$

be Witt classes of the respective total Witt groups for all $i \in \mathcal{I}, j \in \mathcal{J}$ and $k \in \mathcal{K}$. Further, assume that

$$\mathrm{e}(v_i) \leftrightsquigarrow w'_i, \qquad v^*(w_j) \leftrightsquigarrow u'_j, \qquad \partial(u_k) \leftrightsquigarrow v'_k$$

for all i, j, k. Then

$$\mathrm{e}(v'_k) = 0, \qquad v^*(w'_j) = 0, \qquad \partial(u'_k) = 0.$$

Moreover, if any two of the sets $\{v'_k, v_i\}$, $\{w'_i, w_j\}$ or $\{u'_j, u_k\}$ form a total basis of $W^{\mathrm{tot}}_Z(X)$, $W^{\mathrm{tot}}(X)$ or $W^{\mathrm{tot}}(U)$, then so does the third.

Remark 1.2.20. The injectivity assumption in (1.34) often fails., e.g. in the case of Grassmannians ([BC12a]). In this case, one needs to consider a subset $P \subset \mathrm{Pic}_S(X)/2$ such that the restriction of v^* is injective. One then ends up with a totally generating set and total basis *only for the* P-part of the total Witt group, i.e. for those Witt groups whose twists are in P. This is not as much as an obstruction as one might think: Such a subset can always be found and one may "glue" together bases of different parts together as explained in [BC12b, 6.10] and as it was necessary for the ordinary Grassmannian. However in our applications (1.34) is always satisfied, so we do not include this discussion here.

Using dévissage, we can "transmit" total bases of the supported total Witt group $W_Z^{\mathrm{tot}}(X)$ to total bases of $W^{\mathrm{tot}}(Z)$. Similarly, due to homotopy invariance, affine bundles come with a similar property.

Lemma 1.2.21 ([BC12b, 6.16]). *(i) Let $\iota : Z \hookrightarrow X$ be a closed immersion of schemes in $\mathcal{S}_S$ of constant codimension d s.t. $\iota^* : \mathrm{Pic}_S(X)/2 \to \mathrm{Pic}_S(Z)/2$ is injective. Then a family $\{v_i\}$ of Witt classes in $W^{\mathrm{tot}}(Z)$ is a total basis if and only if the family $\{\iota_*(v_i)\}$ is a total basis of $W_Z^{\mathrm{tot}}(X)$.*

(ii) Let $\alpha : U \to Y$ be an affine bundle. Then a family $\{v_i\}$ of Witt classes in $W^{\mathrm{tot}}(Y)$ is a total basis if and only if the family $\{\alpha^*(v_i)\}$ is a total basis of $W^{\mathrm{tot}}(U)$.

1.3 Functors of points

In this section we discuss the language of functors of points which we will use later on for intuitive descriptions of the considered schemes. See [Kar01] for more details.

For a scheme X, the *functor of points* is the functor

$$\mathcal{F} : \mathrm{Aff} \to \mathrm{Set}, \qquad \mathcal{F}(\mathrm{Spec}(A)) = \mathrm{Mor}(\mathrm{Spec}(A), X).$$

If X is defined over a commutative unitary ring R, we define $\mathcal{F}(\mathrm{Spec}(A))$ as above for any R-algebra A. If X is defined over a scheme S, we define $\mathcal{F}(\mathrm{Spec}(A))$ as above for any affine scheme $\mathrm{Spec}(A) \to S$ over S. If on the other hand, for a functor Aff $\to$ Set (resp. a R-functor R-Alg $\to$ Set) there is a scheme X (resp. a scheme over R) such that $\mathcal{F}$ is isomorphic to the functor of points of X, we call $\mathcal{F}$ *representable*. The functor $\mathcal{F}$ is called *local* if it is a sheaf in the Zariski-toplology on Aff (resp. the right category). There is a Yoneda-like equivalence between these objects: Any representable R-functor determines a unique scheme X over R, the *geometric realization of* $\mathcal{F}$.

Definition 1.3.1 (Subfunctors). Let $\mathcal{F}, \mathcal{G}$ be functors as above.

(i) $\mathcal{G}$ is called a *subfunctor* of $\mathcal{F}$, if $\mathcal{G}(A) \subset \mathcal{F}(A)$ for all suitable A (i.e. $\mathrm{Spec}(A)$ affine, A an R-algebra or $\mathrm{Spec}(A)$ affine over S) and

$$\mathcal{G}(f : A \to B) = \mathcal{F}(f)|_{\mathcal{G}(A)}.$$

(ii) If $\mathcal{G} \subset \mathcal{F}$ is a subfunctor and $f : \mathcal{F}' \to \mathcal{F}$ is a morphism, the *inverse image of $\mathcal{G}$ w.r.t the morphism f* is the subfunctor $\mathcal{G}' \subset \mathcal{F}'$ defined by $\mathcal{G}'(A) = f(A)^{-1}(\mathcal{G}(A))$.

Assume now that $\mathcal{G} \subset \mathcal{F}$ is a subfunctor and let $\mathcal{F}' = \mathrm{Spec}(R)$, i.e. $\mathcal{F}(A) = \mathrm{Mor}(R, A)$. Then we also define the following:

(iii) The subfunctor $\mathcal{G} \subset \mathcal{F}$ is called *open*, if for any A and any morphism of schemes $f : \mathrm{Spec}(R) \to \mathcal{F}$ there is an ideal $I \subset R$ such that

$$\mathcal{G}'(A) = \left\{\phi \in \mathcal{F}'(A) = \mathrm{Mor}(R, A) \mid \phi(I) \cdot A = A\right\}.$$

If $\mathcal{F}$ is representable, so is $\mathcal{G}$ by a unique open subscheme.

(iv) The subfunctor $\mathcal{G} \subset \mathcal{F}$ is called *closed*, if for any A and any morphism of schemes $f : \mathrm{Spec}(R) \to \mathcal{F}$ there is an ideal $I \subset R$ such that

$$\mathcal{G}'(A) = \left\{\phi \in \mathcal{F}'(A) = \mathrm{Mor}(R, A) \mid \phi(I) = 0\right\}.$$

If $\mathcal{F}$ is representable, so is $\mathcal{G}$ by a unique closed subscheme.

Remark 1.3.2 (Criteria for coverings and representability). Let $\mathcal{F}$ be a functor and let $\{\mathcal{G}_i\}_{i\in I}$ be a family of subfunctors of $\mathcal{F}$. Then we say that the family $\{\mathcal{G}_i\}$ *covers* $\mathcal{F}$, if for any field A the subsets $\mathcal{G}_i(A)$ cover $\mathcal{F}(A)$, i.e. if $\mathcal{F}(A) = \bigcup \mathcal{G}_i(A)$ as a set. Moreover we have the following:

Representability. A functor $\mathcal{F}$ is representable if and only if it is local and covered by finitely many open subfunctors.

Coverings. If $\mathcal{F}$ is representable, a family $\{\mathcal{G}_i\}$ of subfunctors of $\mathcal{F}$ such that each $\mathcal{G}_i$ is open or closed covers $\mathcal{F}$, if and only the scheme associated with $\mathcal{F}$ is covered by the corresponding subschemes.

Example 1.3.3 (Affine line). Fix a field k and consider the functor $\mathbb{A}^1$ given by

$$\mathbb{A}^1(\mathrm{Spec}(A)) = \mathrm{Hom}_{\mathrm{Aff}}(\mathrm{Spec}(A), \mathrm{Spec}(k[t])) = \mathrm{Hom}_{k-\mathrm{alg}}(k[t], A)$$

for any k-algebra A, where as a set $\mathbb{A}^1(\mathrm{Spec}(A)) = A$. Then the functor $\mathbb{G}_m$ given by

$$\mathbb{G}_m(A) = \mathrm{Hom}(\mathrm{Spec}(k[t, t^{-1}], A)$$

is called the *multiplicative group* of the affine line $\mathbb{A}^1$. It is a subfunctor of $\mathbb{A}^1$, since as sets we have $\mathbb{G}_m(A) = A^\times \subset A$ for any k-algebra A. Of course in general $A^\times \cup \{0\} \subsetneq A$, but at least for fields we have $\mathbb{A}^1(K) = \mathbb{G}_m(K) \cup \{0\}$ which gives us the well-known decomposition $\mathbb{A}^1 = \mathbb{G}_m \cup \{0\}$.

1.4 Grassmann and flag bundles

Example 1.4.1 (Grassmann bundles). Fix a vector bundle $\mathcal{V}$ of rank $d+e$ over a smooth scheme X over $\mathbb{Z}$. Then the functor

$$\begin{aligned} \mathcal{F} : \mathrm{Aff}_X &\longrightarrow \mathrm{Set}, \\ \mathcal{F}(f : \mathrm{Spec}(R) \to X) &= \left\{ P \subset f^*\mathcal{V} \,\middle|\, \begin{array}{c} P, f^*\mathcal{V}/P \text{ are direct} \\ \text{summands of rank } d \text{ and } e \end{array} \right\} \end{aligned}$$

is called the Grassmann functor and is denoted by $\mathrm{Gr}_X(d, \mathcal{V})$. This functor is representable: For its localness see [Kar01, proof of 9.2]. We can obtain an open cover as follows: Fix an epimorphism $p : \mathcal{V} \to \mathcal{U}$ for some $\mathcal{U}$ over X which for any $f : \mathrm{Spec}(R) \to X$ induces a map $p_R : f^*\mathcal{V} \to f^*\mathcal{U}$. Define a subfunctor $\mathcal{G}_{\mathcal{U}} \subset \mathrm{Gr}_X(d, \mathcal{V})$ by

$$\mathcal{G}_{\mathcal{U}}(f : \mathrm{Spec}(R) \to X) = \{P \in \mathrm{Gr}_X(d, \mathcal{V})(R) \mid p_R|_P : P \to f^*\mathcal{U} \text{ is an isomorphism}\}.$$

One checks that $\mathcal{G}_{\mathcal{U}}$ is open and that finitely many of them cover $\mathrm{Gr}_X(d, \mathcal{V})$. Hence, we conclude that $\mathrm{Gr}_X(d, \mathcal{V})$ is representable by a scheme which we also denote by $\mathrm{Gr}_X(d, \mathcal{V})$. In the following we interpret the setup of [BC12a, §5]. Fix a subbundle $\mathcal{V}^1 \subset \mathcal{V}$ of rank $d+e-1$. Recall the strong definition of "non-subbundles" ([BC12a, 5.2]): For a subbundle $P \subset \mathcal{V}$ we write $P \dot{\not\subset} \mathcal{V}^1$ if

(i) $P \not\subset \mathcal{V}^1$ and

(ii) $P \cap \mathcal{V}^1$ is a subbundle of P (in the sense of Definition 1.1.1). This is equivalent to saying that the induced map $P/(P \cap \mathcal{V}^1) = (P + \mathcal{V}^1)/\mathcal{V}^1 \to \mathcal{V}/\mathcal{V}^1$ is an isomorphism.

Now consider the functors $Z := Z_X(d, \mathcal{V})$ and $U := U_X(d, \mathcal{V})$ given by

$$\begin{aligned} Z_X(d, \mathcal{V})(f : \mathrm{Spec}(R) \to X) &= \left\{ P \in \mathrm{Gr}_X(d, \mathcal{V})(R) \mid P \subset f^*\mathcal{V}^1 \right\}, \\ U_X(d, \mathcal{V})(f : \mathrm{Spec}(R) \to X) &= \left\{ P \in \mathrm{Gr}_X(d, \mathcal{V})(R) \mid P \dot{\not\subset} f^*\mathcal{V}^1 \right\}. \end{aligned}$$

Then Z is closed and U is open. Indeed, for the epimorphism $p : \mathcal{V} \to \mathcal{V}/\mathcal{V}^1 =: \mathcal{U}$ given by $p(P) = (P + \mathcal{V}^1)/\mathcal{V}^1$ we see that $U_X(d, \mathcal{V}) = \mathcal{G}_{\mathcal{U}}$ as above which was proved to be open. Now for the same epimorphism consider the functor

$$\mathrm{Gr}_X(d, \mathcal{V})_{(0)}(R) = \{P \in \mathrm{Gr}_X(d, \mathcal{V})(R) \mid p_R(P) = 0\}.$$

Then one easily verifies $Z = \mathrm{Gr}_X(d, \mathcal{V})_{(0)}$ and by [Kar01, 9.7] this functor is closed.

We now claim that the Grassmann bundle as a scheme is covered by the subschemes (again denoted by $U_X(d,\mathcal{V})$ and $Z_X(d,\mathcal{V})$) defined by these two functors. This is true by the remark above: Since $\mathrm{Gr}_X(d,\mathcal{V})$ is representable, Z is closed and U is open, we only need to show that the functors Z and U cover $\mathrm{Gr}_X(d,\mathcal{V})$, i.e. that for any field R we have $\mathrm{Gr}_X(d,\mathcal{V})(R) = Z(R) \cup U(R)$. But this follows from the fact that $\dot{\not\subset}$ and $\subset$ are equivalent over a field, so the conditions for Z and U in this case are complementary. Note that in general

$$Z_X(d,\mathcal{V})(R) \cup U_X(d,\mathcal{V})(R) \subsetneq \mathrm{Gr}_X(d,\mathcal{V})(R).$$

In the following we introduce some preliminaries about isotropic Grassmann bundles. For details of the following discussion we refer to [FP98], [PR97] and [Eis06].

For this we are going to use the language of functors of points. Let $(\mathcal{V},\omega)$ be an isotropic vector bundle of rank $2n$. Denote by $\mathrm{Gr}_X(\mathcal{V})$ the scheme representing the functor of points (which is also denoted by $\mathrm{Gr}_X(\mathcal{V})$) whose set $\mathrm{Gr}_X(\mathcal{V})(R)$ consists of all direct summands of $f^*(\mathcal{V})$ for $f : \mathrm{Spec}(R) \longrightarrow X$ (i.e. $\mathcal{O}_X$-submodules $\mathcal{W}$ of $f^*(\mathcal{V})$ s.t. $\mathcal{W}$ and $f^*(\mathcal{V})/\mathcal{W}$ are vector bundles). Then $\mathrm{Gr}_X(d,\mathcal{V})$ is the closed subscheme of $\mathrm{Gr}_X(\mathcal{V})$ representing the closed subfunctor (denoted by $\mathrm{Gr}_X(d,\mathcal{V})$) for which $\mathrm{Gr}_X(d,\mathcal{V})(R)$ consists of the submodules in $\mathrm{Gr}_X(\mathcal{V})(R)$ of constant rank d. We now define the functor $\mathrm{IG}_X(\mathcal{V},\omega)$ by

$$\mathrm{IG}_X(\mathcal{V},\omega)(R) := \{\mathcal{W} \in \mathrm{Gr}_X(\mathcal{V})(R) \mid \mathcal{W} \text{ is isotropic}\}$$

which by [Kar01, 13.1] defines a closed subscheme $\mathrm{IG}_X(\mathcal{V},\omega)$ of $\mathrm{Gr}_X(\mathcal{V})$. Analogously from [Kar01, 13.6, 13.7] we find the scheme $\mathrm{IG}_X(n,(\mathcal{V},\omega))$.

- If $(\mathcal{V},\omega)$ is orthogonal, we will denote this scheme by $\mathrm{OG}_X(n,\mathcal{V})$.
- If $(\mathcal{V},\omega)$ is symplectic, we will denote it by $\mathrm{LG}_X(n,\mathcal{V})$.

Proposition 1.4.2 ([EPW01, 1.1]). *Let $(\mathcal{V},\omega)$ be an orthogonal vector bundle over a connected scheme X and $\mathcal{W},\mathcal{W}' \in \mathrm{OG}_X(n,\mathcal{V})$. Then $\dim_{k(x)}[\mathcal{W}(x) \cap \mathcal{W}'(x)]$ is constant modulo 2.*

From now on we require the existence of a maximal isotropic subbundle E_n of $(\mathcal{V},\omega)$. In particular, the scheme parametrizing maximal isotropic subbundles of $\mathcal{V}$ consists of two connected components which are denoted by $\mathrm{OG}^+_X(n,(\mathcal{V},\omega),E_n)$ (which is the one containing E_n) and $\mathrm{OG}^-_X(n,(\mathcal{V},\omega),E_n)$ (the other one). They are given by

$$\begin{aligned}
\mathrm{OG}^+_X(n,(\mathcal{V},\omega),E_n) &= \Big\{\mathcal{W} \in \mathrm{OG}_X(n,\mathcal{V}) \mid \dim_{k(x)}[W(x) \cap E_n(x)] \equiv n \bmod 2\Big\},\\
\mathrm{OG}^-_X(n,(\mathcal{V},\omega),E_n) &= \Big\{\mathcal{W} \in \mathrm{OG}_X(n,\mathcal{V}) \mid \dim_{k(x)}[W(x) \cap E_n(x)] \equiv n-1 \bmod 2\Big\}.
\end{aligned}$$

If there is no ambiguity we will simply write $\mathrm{OG}_X(n,\mathcal{V}) := \mathrm{OG}^+_X(n,(\mathcal{V},\omega),E_n)$ and $\mathrm{OG}^-_X(n,\mathcal{V}) := \mathrm{OG}^-_X(n,(\mathcal{V},\omega),E_n)$.

The scheme $\mathrm{IG}_X(n,\mathcal{V})$ comes with a structural morphism $\pi : \mathrm{IG}_X(n,\mathcal{V}) \longrightarrow X$ and the *tautological orthogonal subbundle* $\mathcal{S} \subset \pi^*\mathcal{V}$ of rank n, where the latter is the restriction of the tautological subbundle $\mathcal{S}_{\mathrm{Gr}}$ of the ordinary Grassmannian $\mathrm{Gr}_X(n,\mathcal{V})$.

Recall that we have inclusions $\mathrm{IG}_X(n,\mathcal{V}) \overset{j}{\hookrightarrow} \mathrm{Gr}_X(n,\mathcal{V}) \overset{i}{\hookrightarrow} \mathbb{P}(\wedge^n\mathcal{V})$. Then it is well-known that $\mathrm{Pic}(\mathrm{Gr}_X(n,\mathcal{V})) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}[\det \mathcal{S}_{\mathrm{Gr}}]$ and $\det(\mathcal{S}^\vee_{\mathrm{Gr}}) = i^*\mathcal{O}_{\mathbb{P}(\wedge^n\mathcal{V})}(1)$. This remains valid in the symplectic setting but is not true anymore in the orthogonal one. Instead this pull-back admits a square root and we have the following:

Theorem 1.4.3. *(i) If $(\mathcal{V},\omega)$ is orthogonal, the pull-back $(i\circ j)^*\mathcal{O}_{\mathbb{P}(\wedge^n\mathcal{V})}(1) = \det(\mathcal{S}^\vee)$ has a unique ample square root denoted by $\mathcal{O}(1)$.*

(ii) If $(\mathcal{V},\omega)$ is symplectic, denote the pull-back $(i\circ j)^*\mathcal{O}_{\mathbb{P}(\wedge^n\mathcal{V})}(1) = \det(\mathcal{S}^\vee)$ by $\mathcal{O}(1)$.

In both cases the scheme $\mathrm{IG}_X(n,\mathcal{V})$ is smooth over X with Picard group

$$\mathrm{Pic}(X) \oplus \mathbb{Z} \cong \mathrm{Pic}(\mathrm{IG}_X(n,\mathcal{V})), \qquad (\mathcal{M},n) \mapsto \pi^*(\mathcal{M}) \otimes \mathcal{O}(1)^{\otimes n}.$$

Moreover, writing $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$, the class of the relative canonical bundle $\omega_{\mathrm{IG}_X(n,\mathcal{V})/X}$ in the Picard group is given by

$$\omega_{\mathrm{OG}_X(n,\mathcal{V})/X} = \mathcal{O}(-2n+2) \otimes \mathcal{L}^{-\frac{n(n-1)}{2}}, \tag{1.35}$$

$$\omega_{\mathrm{LG}_X(n,\mathcal{V})/X} = \mathcal{O}(-n-1) \otimes \mathcal{L}^{-\frac{n(n+1)}{2}}. \tag{1.36}$$

Proof. See [EPW00, §1] and [Pra91, 6.1(i)] for the assertions on the Picard groups. The relative canonical bundle is given by

$$\omega_{\mathrm{IG}_X(n,\mathcal{V})/X} := \det\left[T\mathrm{IG}_X(n,\mathcal{V})\right]^\vee, \tag{1.37}$$

the determinant of the cotangent bundle. Recall that for the Grassmannian the tangent bundle is given by

$$T\mathrm{Gr}_X(n,\mathcal{V}) = \mathrm{Hom}(\mathcal{S},\mathcal{Q}) = \mathcal{S}^\vee \otimes \mathcal{Q}, \tag{1.38}$$

where $\mathcal{S}$ and $\mathcal{Q}$, respectively, denote the tautological and the universal quotient bundle, respectively, which appear in the short exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{V} \to \mathcal{Q} \to 0. \tag{1.39}$$

The bilinear form on $\mathcal{V}$ induces an isomorphism $\phi : \mathcal{V} \to \mathcal{V}^\vee \otimes \mathcal{L}$, which clearly remains an isomorphism when pulled back along the structure morphism of the Grassmannian.

Denote this pullback again by ϕ. Dualizing (1.39) and tensoring with $\mathcal{L}$ we obtain

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] & \pi^*\mathcal{V} \arrow[r] \arrow[d, "\phi"] & \mathcal{Q} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{Q}^\vee \otimes \mathcal{L} \arrow[r] & (\pi^*\mathcal{V})^\vee \otimes \mathcal{L} \arrow[r] & \mathcal{S}^\vee \otimes \mathcal{L} \arrow[r] & 0. \end{tikzcd} \tag{1.40}$$

By definition the form ϕ vanishes on $\mathcal{S}$, which gives us vertical maps

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d, dashed, "\psi"] & \pi^*\mathcal{V} \arrow[r] \arrow[d, "\phi"] & \mathcal{Q} \arrow[r] \arrow[d, dashed, "\psi'"] & 0 \\ 0 \arrow[r] & \mathcal{Q}^\vee \otimes \mathcal{L} \arrow[r] & (\pi^*\mathcal{V})^\vee \otimes \mathcal{L} \arrow[r] & \mathcal{S}^\vee \otimes \mathcal{L} \arrow[r] & 0, \end{tikzcd} \tag{1.41}$$

which are almost dual to each other in the sense that $\psi^\vee = \psi' \otimes \mathcal{L}^\vee$ and $\psi'^\vee = \psi \otimes \mathcal{L}^\vee$. Comparing the ranks of $\mathcal{Q}$ and $\mathcal{S}^\vee \otimes \mathcal{L}$ and using [Wei13, Ex. 1.3] we see that ψ' is an isomorphism, yielding an isomorphism $\mathcal{Q} \cong \mathcal{S}^\vee \otimes \mathcal{L}$. Then

$$T\mathrm{Gr}_X(n, \mathcal{V})|_{\mathrm{IG}_X(n,\mathcal{V})} = \mathrm{Hom}(\mathcal{S}, \mathcal{Q}) = \mathrm{Hom}(\mathcal{S}, \mathcal{S}^\vee \otimes \mathcal{L}) \cong T^2(\mathcal{S}^\vee) \otimes \mathcal{L},$$

where we use the same notation for the bundles over $\mathrm{Gr}_X(n, \mathcal{V})$ restricted to $\mathrm{IG}_X(n, \mathcal{V})$. Now clearly $T\mathrm{IG}_X(n, \mathcal{V}) \subset T\mathrm{Gr}_X(n, \mathcal{V})|_{\mathrm{OG}_X(n,\mathcal{V})}$. For some $\phi : \mathcal{S} \longrightarrow \mathcal{Q}$ to be in $T\mathrm{IG}_X(n, \mathcal{V})$ means to satisfy some isotropy conditions. If ω is symmetric we obtain

$$T\mathrm{OG}_X(n, \mathcal{V}) = \wedge^2(\mathcal{S}^\vee) \otimes \mathcal{L} \tag{1.42}$$

(e.g. [HCC20, proof of Lemma 3.2]). Recall that for bundles $\mathcal{M}, \mathcal{N}$ of ranks m, n we have

$$\det[\mathcal{M} \otimes \mathcal{N}] = [\det \mathcal{M}]^{\otimes n} \otimes [\det \mathcal{N}]^{\otimes m}, \tag{1.43}$$
$$\det \wedge^k(\mathcal{M}) = [\det \mathcal{M}]^{\otimes \binom{m-1}{k-1}}, \tag{1.44}$$
$$\det S^k(\mathcal{M}) = [\det \mathcal{M}]^{\otimes \binom{m+k-1}{m}}. \tag{1.45}$$

(see [Tor52], [Mar73, §2]). Dualizing (1.42) and taking the top exterior power gives us the desired result, where $\Delta = \mathcal{O}(-2)$ as before. In particular, the class of $\omega_{\mathrm{OG}_X(n,\mathcal{V})/X}$ in $\mathrm{Pic}_X(\mathrm{OG}_X(n, \mathcal{V}))$ is $\mathcal{O}(-2n + 2)$, which is consistent with the case $X = \mathrm{Spec}(k)$, cf. [HCC20, proof of Lemma 3.2]. If ω is antisymmetric, similar calculations show

$$T\mathrm{LG}_X(n, \mathcal{V}) = S^2(\mathcal{S}^\vee) \otimes \mathcal{L}, \tag{1.46}$$

proving (ii). □

Example 1.4.4 (Maximal orthogonal Grassmannians). (i) If $X = \mathrm{Spec}(k)$ and V is a k-vector space of dimension 2, we have $\mathrm{OG}_X(1, V) = \{\mathrm{pt}\}$. Indeed, if ω is the symmetric form defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the lines generated by $(1,0)^\top$ and $(0,1)^\top$ are the unique elements of $\mathrm{OG}_X(1, V)$ and $\mathrm{OG}_X^-(1, V)$. More generally, $\mathrm{OG}_X(1, \mathcal{V}) = X$, if the vector bundle $\mathcal{V}$ of rank two admits a maximal isotropic subbundle. Thus, any submaximal isotropic subbundle $\mathcal{W}_{n-1} \subset \mathcal{V}$ defines 2 unique maximal isotropic subbundles $\mathcal{W} \in \mathrm{OG}_X(n, \mathcal{V})$ and $\mathcal{W}^- \in \mathrm{OG}_X(n, \mathcal{V})$ containing $\mathcal{W}_{n-1}$, one in each component.

(ii) If $\mathcal{V}$ is of rank 4 then $\mathrm{OG}_X^-(2, \mathcal{V}) \cong \mathbb{P}(E_2)$. Indeed, for any 1-dimensional subspace $W \subset E_2$ there is a unique maximal isotropic subspace in $\mathrm{OG}_X^-(2, \mathcal{V})$ by the previous discussion.

(iii) If $X = \mathrm{Spec}(k)$ for a field k, then the maximal orthogonal Grassmannian is the homogeneous space D_n/P_n, where P_n denotes a maximal parabolic subgroup associated with one of the two right-end simple roots. In this setting, maximal isotropic subspaces always exist. More generally, such a subbundle always exists if the vector bundle $\mathcal{V}$ is free; in this case it even admits a full flag (this will be discussed later in more detail).

(iv) If $X = \mathrm{Spec}(R)$ for a local ring R, then $\mathrm{Pic}(X) = 0$, so $\mathrm{Pic}(\mathrm{OG}_X(n, \mathcal{V})) \cong \mathbb{Z}$ has a single generator and no line bundles coming from the base scheme.

(v) Let X be an elliptic curve over a field k, i.e. a smooth, projective algebraic curve of genus 1 with one point at infinity denoted by 0. Let $P \in X$ be a point of order 2 with respect to the group law, i.e. $P + P = 2P = 0$. If X is the solution set of an equation $y^2 = f(x)$, then such a point is specified by the condition $y = 0$. Consider the line bundle $\mathcal{L} := \mathcal{O}_X(P)$ associated with the divisor $D = P$. Then there is an isomorphism $\phi : \mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$. Now define a vector bundle on X by $\mathcal{V} := \mathcal{O}_X \oplus \mathcal{L}$. By [Har77, Exercise IV.2.7] the line bundle $\mathcal{L}$ defines an $\mathcal{O}_X$-algebra structure on $\mathcal{V}$ by

$$(o, l) \cdot (p, m) = (op + \phi(l \otimes m), om + pl)$$

and an unramified cover $\pi : Y \to X$ of degree 2, where $Y := \mathbf{Spec}(\mathcal{V})$ (see [Har77, Exercise II.5.17]). By Hurwitz's Theorem, Y is again an elliptic curve and hence, in particular, connected as a scheme. On the other hand, the algebra structure on $\mathcal{V}$ and the projection to $\mathcal{O}_X$ define a symmetric form $\mathcal{V} \otimes \mathcal{V} \to \mathcal{O}_X$ and we can consider the orthogonal Grassmann bundle $Y' := \mathrm{OG}_X(1, \mathcal{V})$ which comes with a projection $\pi' : Y' \to X$. This map is finite and hence affine in the sense of [Har77, Exercise II.5.17(d)] which gives us $Y' \cong \mathbf{Spec}(\pi'_* \mathcal{O}_{Y'}) \cong \mathbf{Spec}(\mathcal{V})$ and hence $Y \cong Y'$, where the isomorphism $\pi'_* \mathcal{O}_{Y'} \cong \mathcal{V}$ follows from [Har77, Exercise IV.2.7(c)]. In particular, in this case the maximal orthogonal Grassmannian does not admit two connected components as remarked above due to the lack of a maximal isotropic subbundle of $\mathcal{V}$.

Example 1.4.5 (Lagrangian Grassmannians). If V is a vector space equipped with an antisymmetric form ω, it is clear that any line $L = \langle l \rangle$ is isotropic with respect to ω, since $\omega(v, v) = 0$ for all $v \in V$ and, in particular, for the generator $l \in V$ of the line. Hence we conclude $L \subset L^{\perp}$. This directly applies to vector bundles and we obtain identifications $\mathrm{LG}_X(1, \mathcal{V}) = \mathrm{Gr}_X(1, \mathcal{V}) \cong \mathbb{P}(\mathcal{V})$ for any vector bundle $\mathcal{V}$ of rank 2. Of course, our formula for the relative canonical bundle yields the same result as the one from [BC12a]

$$\omega_{\mathrm{LG}_X(1,\mathcal{V})/X)} = \mathcal{O}(-2) \otimes \mathcal{L}^{-1} = \mathcal{O}(-2) \otimes [\det \mathcal{V}]^{-1} = \omega_{\mathrm{Gr}_X(1,\mathcal{V})/X}$$

and the one for projective bundles in (1.15).

Remark 1.4.6 (Non-maximal isotropic Grassmannians). Clearly there are Grassmannians parametrizing isotropic subbundles of any rank up to n of a given isotropic vector bundle of rank $2n$. We denote by $\pi : \mathrm{IG}_X(k, \mathcal{V}) \to X$ the scheme over X parametrizing subbundles of rank k and we write

(i) $\mathrm{OG}_X(k, \mathcal{V})$ if $(\mathcal{V}, \omega)$ is orthogonal and

(ii) $\mathrm{SG}_X(k, \mathcal{V})$ if $(\mathcal{V}, \omega)$ is symplectic.

Note that both schemes are connected when $k \leq n - 1$. The scheme $\mathrm{IG}_X(k, \mathcal{V})$ again comes equipped with a tautological subbundle $\mathcal{S}_k \subset \pi^* \mathcal{V}$ of rank k and we have

$$\mathrm{Pic}_X(\mathrm{SG}_X(k, \mathcal{V})) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}[\det \mathcal{S}_k].$$

The tangent bundle is given by

$$T\mathrm{IG}_X(k, \mathcal{V}) \;\cong\; \begin{cases} \left(\mathcal{S}^\vee \otimes (\mathcal{S}^\perp/\mathcal{S})\right) \oplus \left(\Lambda^2 \mathcal{S}^\vee \otimes \mathcal{L}\right), & \text{if } \omega \text{ is symmetric,} \\ \left(\mathcal{S}^\vee \otimes (\mathcal{S}^\perp/\mathcal{S})\right) \oplus \left(S^2 \mathcal{S}^\vee \otimes \mathcal{L}\right), & \text{if } \omega \text{ is antisymmetric,} \end{cases}$$

which yields

$$\omega_{\mathrm{IG}_X(k,\mathcal{V})/X} = \begin{cases} \mathcal{O}(-2n+k+1) \otimes \mathcal{L}^{-\left(\frac{k(k-1)}{2}+k(n-k)\right)}, & \text{if } \omega \text{ is symmetric, } k \leq n-1, \\ \mathcal{O}(-2n+k-1) \otimes \mathcal{L}^{-\left(\frac{k(k+1)}{2}+k(n-k)\right)}, & \text{if } \omega \text{ is antisymmetric} \end{cases}$$

with $\mathcal{O}(1) := \det(\mathcal{S}^\vee)$. To reveal the formula for $k = n$ in the orthogonal case, we need to take the generator twice since in this case $\det \mathcal{S}_n$ has a square root in the Picard group as discussed above.

Definition 1.4.7 (Isotropic flag bundles). Let $(\mathcal{V}, \omega)$ be an isotropic vector bundle, $k \geq 1$ and $\underline{d}$, $\underline{e}$ two k-tuples of non-negative integers satisfying

$$0 < d_1 < \ldots < d_k \leq n, \qquad 0 < d_1 + e_1 \leq \ldots \leq d_k + e_k \leq n.$$

Then let $\mathrm{IFl}_X(\underline{d}, \underline{e}, E_\bullet)$ be the scheme over X which parametrizes flags of subbundles (always as in Definition 1.1.1) $P_{d_1} \subset P_{d_2} \subset \ldots \subset P_{d_k} \subset P_n$ over X s.t. $\mathrm{rk}(P_{d_i}) = d_i$, $P_{d_i} \subset E_{d_i+e_i}$ and moreover $P_n \subset \mathcal{V}$ is maximal isotropic. If ω is symmetric, we require $d_k \leq n-1$ and P_n to be in the same component as E_n.

(i) If ω is symmetric, this scheme will be denoted by $\mathrm{OFl}_X(\underline{d}, \underline{e}, E_\bullet)$, the *orthogonal flag bundle*.

(ii) If ω is antisymmetric, we write $\mathrm{SFl}_X(\underline{d}, \underline{e}, E_\bullet)$ for the *symplectic flag bundle*.

There is an obvious projection $f_{\underline{d},\underline{e}} : \mathrm{IFl}_X(\underline{d}, \underline{e}, E_\bullet) \longrightarrow \mathrm{IG}_X(n, \mathcal{V})$ sending a flag to its highest rank subbundle. If there is no confusion, we omit $E_\bullet$ in the notation and simply write $\mathrm{OFl}_X(\underline{d}, \underline{e})$. In the orthogonal case, we can analogously define the opposite bundle $f^-_{\underline{d},\underline{e}} : \mathrm{OFl}^-_X(\underline{d}, \underline{e},) \longrightarrow \mathrm{OG}^-_X(n, \mathcal{V})$. If moreover $d_k = n-1$ there are projections to the other components

$$g_{\underline{d},\underline{e}} : \mathrm{OFl}_X(\underline{d}, \underline{e}) \longrightarrow \mathrm{OG}^-_X(n, \mathcal{V}), \qquad g^-_{\underline{d},\underline{e}} : \mathrm{OFl}^-_X(\underline{d}, \underline{e}) \longrightarrow \mathrm{OG}_X(n, \mathcal{V})$$

by sending a flag to the unique maximal isotropic subbundle other than P_n containing P_{n-1}, as explained in Example 1.4.4(i).

As usual, the isotropic flag bundle comes equipped with several tautological subbundles $\mathcal{S}_{d_i}, \mathcal{S}_n \subset \pi^*\mathcal{V}$ of ranks d_i and n. Just as in the case for the ordinary Grassmannians ([BC12a, lemma 1.11]), we can describe the orthogonal flag bundle as a tower of ordinary flag bundles followed by an orthogonal Grassmannian:

Lemma 1.4.8. *For any k-tuples $\underline{d}, \underline{e}$ as above let $Y := \mathrm{Fl}_X(\underline{d}, \underline{e}, E_\bullet)$ be as in* [BC12a]. *Then* $\mathrm{IFl}_X(\underline{d}, \underline{e}) = \mathrm{IG}_Y\left(n - d_k, (\mathcal{S}_{d_k}^{\perp}/\mathcal{S}_{d_k}, \omega), \mathcal{S}_n/\mathcal{S}_{d_k}\right)$. *In other words the isotropic flag bundle is a tower*

$$\mathrm{IFl}_X(\underline{d}, \underline{e}, E_\bullet) \longrightarrow \mathrm{Fl}_X(\underline{d}, \underline{e}, E_\bullet) \longrightarrow X$$

of an isotropic Grassmann bundle over an ordinary flag bundle.

This lemma (including its proof) is entirely analogous to [BC12a, Lemma 1.11] and allows us to compute the Picard group. According to the notation in [BC12a] let us write $\Delta_n := \mathcal{O}_{\mathrm{OG}_X(n,\mathcal{V})}(-1)$ for the dual of the generator of $\mathrm{Pic}(\mathrm{OG}_X(n, \mathcal{V}))$ which satisfies

$$\det \mathcal{S}_n \;=\; \begin{cases} \Delta_n^{\otimes 2}, & \text{if } \omega \text{ is symmetric,} \\ \Delta_n, & \text{if } \omega \text{ is antisymmetric.} \end{cases}$$

In $\mathrm{Fl}_X(\underline{d}, \underline{e})$ we write Δ_{d_i} for the class of the determinant bundle $\mathcal{S}_{d_i}$.

Proposition 1.4.9. *With the notation above we have:*

(i) The scheme $\mathrm{OFl}_X(\underline{d}, \underline{e})$ is smooth over X of relative dimension

$$\dim \mathrm{OFl}_X(\underline{d}, \underline{e}) = \left[\sum_{i=1}^{k} (d_i - d_{i-1}) e_i\right] + \frac{(n - d_k)(n - 1 - d_k)}{2}$$

and Picard group

$$\mathrm{Pic}(\mathrm{OFl}_X(\underline{d}, \underline{e})) \cong \mathrm{Pic}(X) \oplus \left[\bigoplus_{i=1,\ e_i \neq 0}^{k} \mathbb{Z}\Delta_{d_i}\right] \oplus \mathbb{Z}\sqrt{\Delta_n}.$$

Further the classes of the relative canonical bundles of the maps $\pi : \mathrm{OFL}_X(\underline{d}, \underline{e}) \longrightarrow X$ and $f_{\underline{d},\underline{e}} : \mathrm{OFl}_X(\underline{d}, \underline{e}) \longrightarrow \mathrm{OG}_X(n, \mathcal{V})$ are given by

$$[\omega_\pi] = \mathcal{L}^{l_-(\underline{\lambda})} \cdot \prod_{i=1}^{k} [\det E_{d_i + e_i}]^{d_{i-1} - d_i} \cdot \prod_{i=1}^{k} \Delta_{d_i}^{d_i - d_{i-1} + e_i - e_{i+1}} \cdot \Delta_n^{n - d_k - 1}$$

and

$$\left[\omega_{f_{\underline{d},\underline{e}}}\right] = \mathcal{L}^{l'_-(\underline{\lambda})} \cdot \prod_{i=1}^{k} [\det E_{d_i + e_i}]^{d_{i-1} - d_i} \cdot \prod_{i=1}^{k} \Delta_{d_i}^{d_i - d_{i-1} + e_i - e_{i+1}} \cdot \Delta_n^{-d_k},$$

where $\Delta_{d_i} = \det(E_{d_i + e_i})$ if $e_i = 0$ and

$$l_-(\underline{\lambda}) = -\frac{(n - d_k)(n - d_k - 1)}{2}, \qquad l'_-(\underline{\lambda}) = \frac{d_k(2n - d_k - 1)}{2}.$$

(ii) The scheme $\mathrm{SFl}_X(\underline{d},\underline{e})$ *is smooth of relative dimension*

$$\dim \mathrm{SFl}_X(\underline{d},\underline{e}) = \left[\sum_{i=1}^{k}(d_i - d_{i-1})e_i\right] + \frac{(n-d_k)(n-d_k+1)}{2}$$

and Picard group

$$\mathrm{Pic}(\mathrm{SFl}_X(\underline{d},\underline{e})) \cong \mathrm{Pic}(X) \oplus \left[\bigoplus_{i=1,\ e_i\neq 0}^{k} \mathbb{Z}\Delta_{d_i}\right] \oplus \mathbb{Z}\Delta_n.$$

Further the classes of the relative canonical bundles of the maps $\pi : \mathrm{SFL}_X(\underline{d},\underline{e}) \to X$ *and* $f_{\underline{d},\underline{e}} : \mathrm{SFl}_X(\underline{d},\underline{e}) \to \mathrm{OG}_X(n,\mathcal{V})$ *are given by*

$$[\omega_\pi] = \mathcal{L}^{l_+(\underline{\lambda})} \cdot \prod_{i=1}^{k}[E_{d_i+e_i}]^{d_{i-1}-d_i} \cdot \prod_{i=1}^{k-1} \Delta_{d_i}^{d_i-d_{i-1}+e_i-e_{i+1}} \cdot \Delta_{d_k}^{d_k-d_{k-1}+e_k-e_{k+1}-1} \cdot \Delta_n^{n-d_k+1}$$

and

$$\left[\omega_{f_{\underline{d},\underline{e}}}\right] = \mathcal{L}^{l'_+(\underline{\lambda})} \cdot \prod_{i=1}^{k}[\det E_{d_i+e_i}]^{d_{i-1}-d_i} \cdot \prod_{i=1}^{k-1} \Delta_{d_i}^{d_i-d_{i-1}+e_i-e_{i+1}} \cdot \Delta_{d_k}^{d_k-d_{k-1}+e_k-e_{k+1}-1} \cdot \Delta_n^{-d_k},$$

where $\Delta_i = \det(E_{d_i+e_i})$ *if* $e_i = 0$ *and*

$$l_+(\underline{\lambda}) = -\frac{(n-d_k)(n-d_k+1)}{2}, \qquad l'_+(\underline{\lambda}) = \frac{d_k(2n-d_k+1)}{2}.$$

Proof. Use Lemma 1.4.8, [BC12a, 1.13] and the description of the canonical bundle ω_{IG} above. Note that $\mathcal{S}_{d_k}^{\perp}/\mathcal{S}_{d_k}$ is of rank $2(n-d_k)$ and equipped with the induced form and the tautological bundle on $\mathrm{IG}_{\mathrm{Fl}_X(\underline{d},\underline{e})}(\mathcal{S}_{d_k}^{\perp}/\mathcal{S}_{d_k})$ is identified with $\mathcal{S}_n/p^*S_{d_k}$ for the structure map $p : \mathrm{IG}_{\mathrm{Fl}_X(\underline{d},\underline{e})}(S_{d_k}^{\perp}/S_{d_k}) \to \mathrm{Fl}_X(\underline{d},\underline{e})$.

Example 1.4.10. For $k = 0$ we have $\mathrm{OFl}_X(\emptyset) = \mathrm{OG}_X(n,\mathcal{V})$. For $k = 1$ and $e_1 = 0$ we may identify $\Phi : \mathrm{OFl}_X(d,0) \cong \mathrm{OG}_X(n-d, E_d^{\perp}/E_d)$ via $(E_d \subset P_n) \mapsto P_n/E_d$. Under this isomorphism, the relative canonical bundle becomes

$$\begin{aligned}
\omega_{\mathrm{OFl}_X(d,0)} &= \mathcal{L}^{-\frac{(n-d)(n-d-1)}{2}} \cdot [\det E_d]^{-(n-1-d)} \cdot \Delta_n^{n-1-d} \\
&= \Phi^*\left(\mathcal{L}^{-\frac{(n-d)(n-d-1)}{2}} \cdot \Delta_{n-d}^{n-d-1}\right) \\
&= \Phi^*(\omega_{\mathrm{OG}_X(n-d,E_d^{\perp}/E_d)}).
\end{aligned}$$

1.5 Schubert calculus

Schubert varieties appear as special subvarieties of Grassmannians which can be described by incidence relations with a given fixed flag of the underlying vector space. Their classes turn out to be very useful. For example, they form an additive basis of the Chow ring and, as we will see later, a suitable choice of them forms an additive basis of the total Witt group. In this section we summarize well-known facts about Schubert varieties by giving the definition and establishing geometric visualizations in terms of Young diagrams and quivers.

1.5.1 Homogeneous spaces

In this subsection we give a very brief introduction to our objects of interest, namely homogeneous spaces of the form $X = G/P$. For more details and proofs see [Hum72] and [Hum75].

An *algebraic group* is a variety (i.e. a regular reduced scheme of finite type over an algebraically closed field k) together with a group structure. The general linear group $G = \mathrm{GL}_n(k)$ is an algebraic group in the obvious way and any algebraic group which is isomorphic to a subgroup of $\mathrm{GL}_n(k)$ is called *linear*. Since G is a group, there is an identity element and the tangent space to it is called the *Lie algebra* of G and denoted by $\mathfrak{g}$. Under certain conditions there is a one-to-one correspondance between algebraic groups and Lie algebras, so one rather investigates Lie algebras which admit the structure of a vector space. Usually, one is mainly interested in *semisimple* Lie algebras, i.e. those not admitting any non-zero abelian ideals. These can be classified as follows:

- type A_n. Let V be a vector space of dimension $n+1$. Then the group defined by $\mathrm{SL}(V) = \mathrm{SL}_{n+1}(k) = \{f \in \mathrm{End}(V) \mid \det f = 1\}$ is the *special linear group*. Its Lie algebra is the *special linear algebra*

$$\mathfrak{g} = \mathfrak{sl}(V) = \mathfrak{sl}_{n+1}(k) = \{f \in \mathrm{End}(V) \mid \mathrm{trace}(f) = 0\}.$$

- type C_n. Let V be a vector space of dimension $2n$ equipped with a nondegenerate, antisymmetric, bilinear form ω represented by the matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in k^{2n \times 2n}.$$

  Then $\mathrm{Sp}(V) = \{f \in \mathrm{SL}(V) \mid f^\top J f = J\}$ is the *symplectic group*. Its Lie algebra is the *symplectic algebra*

$$\mathfrak{sp}(V) = \mathfrak{sp}_{2n}(k) = \{f \in \operatorname{End}(V) \mid \omega(f(v), w) = -\omega(v, f(w)) \text{ for all } v, w \in V\}.$$

- type D_n. As in type C_n, but ω is symmetric and none of the I_n in J admits a sign (denote this matrix again by J). Then $\mathrm{SO}(V) = \{f \in \mathrm{SL}(V) \mid f^\top J f = J\}$ is the *even special orthogonal group*. Its Lie algebra is the *even special orthogonal algebra*

$$\mathfrak{so}(V) = \mathfrak{so}_{2n}(k) = \{f \in \operatorname{End}(V) \mid \omega(f(v), w) = -\omega(v, f(w)) \text{ for all } v, w \in V\}.$$

- type B_n: As in type D_n with the matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

- exceptional types E_6, E_7, E_8, F_4, and G_2 which for our purposes are irrelevant.

The classification relies on the classification of *root data*: A compact, connected, abelian Lie subgroup $T \subset G$ is called torus and it is called maximal, if it is maximal among all such tori. If T is a maximal torus, it is well-known that

$$X^*(T) := \operatorname{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^{\operatorname{rk}(T)}$$

is a lattice. Then a root datum is a quadruple $(X^*(T), \Phi, X^*(T)^\vee, \Phi^\vee)$ where the elements of $\Phi \subset X^*(T)$ are called *roots*, $X^*(T)^\vee$ is the dual lattice and $\Phi^\vee \subset X^*(T)^\vee$ is the set of coroots, subject to some angle, length and pairing conditions (e.g. [Spr09, §7]). In particular, Φ and $\Phi^\vee$ form root systems inside $\langle\Phi\rangle \otimes \mathbb{R}$ and $\langle\Phi^\vee\rangle \otimes \mathbb{R}$, respectively, in the usual sense and these root systems may be classified by Dynkin diagrams.

In the case of Lie algebras, roughly speaking, we can find a simultaneously diagonazible subset of $\mathfrak{g}$ which gives us a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi \cup \{0\}} \mathfrak{g}_\alpha,$$

where the sum runs over a root system associated with $\mathfrak{g}$ and this root system is uniquely determined by the Lie algebra structure of $\mathfrak{g}$. A subset $\Delta \subset \Phi$ is called a *basis* of the root system (and its elements *simple roots*) if any other root can be written as a linear combination with entirely positive or negative integer coefficients.

Arranging the data of the root system in a graph, we obtain the notion of a *Dynkin diagram*, as explained in Figure 1.2.

Definition 1.5.1 (Borel and parabolic subgroups). Let G be a linear algebraic group.

(i) A maximal closed, connected, solvable subgroup $B \subset G$ is called *Borel subgroup*.

(ii) A subgroup $B \subset P \subset G$ is called *parabolic subgroup*. In this case the quotient G/P is a projective algebraic variety. If P is *maximal*, we call G/P a *Grassmannian*. If P is *minimal* (i.e. a Borel subgroup) we call G/P a *flag variety*. In the other cases, we call G/P a *partial flag variety*.

(iii) For any Borel subgroup B and any subset $I \subset \Delta$ of simple roots there is a unique *standard parabolic subgroup* $B \subset P_I := BW_{P_I}B \subset G$ where $W_{P_I} = \{s_\alpha \mid \alpha \in I\}$. For $I = \emptyset$ we obtain the Borel subgroup, for $I = \Delta$ the whole group G and if $I = \Delta \setminus \{\alpha_i\}$ for some i we call P_i the *maximal standard parabolic subgroup associated with the simple root* α_i. Note that any parabolic subgroup is isomorphic to some standard parabolic subgroup.

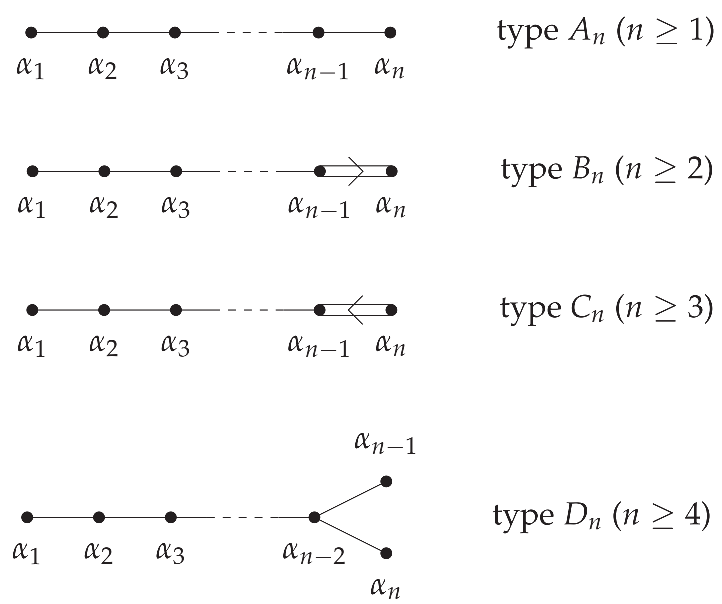


FIGURE 1.2. The Dynkin diagrams of the classical Lie algebras. Each vertice represents a simple root and two vertices are connected by an edge if the associated simple roots are not orthogonal. Otherwise there are one, two or three edges depending on the angle (120, 135 or 150 degree). Moreover, if two roots do not have the same length, we draw an arrow on the edge pointing from the longer to the shorter root.

Example 1.5.2 (Grassmannians as homogeneous spaces). (i) Let V be a vector space of dimension $d+e$. Then a Borel subgroup is given by the subgroup of upper triangular $(d+e)\times(d+e)$-matrices and the standard parabolic subgroup associated with a simple root α_d of A_{d+e-1} is the subgroup of upper triangular block matrices of sizes d and e. Then the homogeneous space A_{d+e-1}/P_d is precisely the Grassmannian $\mathrm{Gr}(d,V)=\{P\subset V \mid \dim P = d\}$.

(ii) Let similarly V be a vector space of dimension $2n$ and ω an antisymmetric form ω on V. Then for $1\leq k\leq n$ the homogeneous space C_n/P_k is the symplectic Grassmannian $\mathrm{LG}(k,V)=\{P\subset V \mid \dim P = k,\ P \text{ is isotropic w.r.t. } \omega\}$.

(iii) If in (ii) ω is symmetric, the homogeneous space D_n/P_k for $1\leq k\leq n-2$ is the orthogonal Grassmannian $\mathrm{OG}(k,V)$ of k-dimensional isotropic subspaces of V. Note that for $k\in\{n-1,n\}$ the homogeneous spaces D_n/P_{n-1} and D_n/P_n are isomorphic to each other and describe the two connected components of the maximal orthogonal Grassmannian $\mathrm{OG}(n,V)$.

1.5.2 Schubert varieties

Fix a Borel and parabolic subgroup $B\subset P\subset G$ of G and consider the homogeneous space $X=G/P$. Let Φ be the associated root system with simple roots $\Delta=\{\alpha_1,\dots,\alpha_n\}$ generating the Euclidean vector space $(E,\langle -,-\rangle)$. The *Weyl group of* Φ is the finite subgroup of $\mathrm{Sym}(E)$ generated by the *simple reflections*

$$s_{\alpha_i}: E\to E, \qquad s_{\alpha_i}(v)=v-2\frac{\langle v,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle}\alpha_i$$

for every simple root $\alpha_i\in\Delta$.

Definition 1.5.3. The Weyl group of X is defined as the Weyl group of the underlying root system Φ and denoted by W. Hence, each element $w\in W$ can be written as a product $w=s_{\alpha_{i_1}}\cdots s_{\alpha_{i_k}}$; the *length of* w is the minimal number of required simple reflections and denoted by $l(w)$.

Theorem 1.5.4. *With the notation above, we have:*

(i) For $w\in W$ the double coset $BwB\subset G$ is called Bruhat cell.

(ii) Bruhat-decomposition: *We have $G=\coprod_{w\in W} BwB$.*

(iii) If P_I is a standard parabolic subgroup, then the image of the Bruhat cell along the projection $G\to G/P_I$ is called Schubert cell *and denoted by $X^o(w)$. It is isomorphic to the affine*

space $\mathbb{A}_k^{l(w)}$. If W_{P_I} denotes the Weyl group of the root system of P_I as above, then

$$G/P_I = \coprod_{\bar{w} \in W/W_{P_I}} X^o(w) = \coprod_{\bar{w} \in W/W_{P_I}} BwP_I/P_I,$$

i.e. G/P_I is the union of its Schubert cells.

(iv) *The closure of a Schubert cell is called* Schubert variety *and denoted by $X(w)$. It is a closed irreducible subvariety of dimension $l(w)$.*

Before discussing more intuitive descriptions of Schubert varieties in the upcoming sections, we state a well-known theorem to emphasize the importance of them:

Theorem 1.5.5. *Recall that the* Chow ring *$A(G/P)$ of G/P is the graded ring of cycles, i.e. subvarieties modulo rational equivalence (e.g.* [Eis06, Ch.1]*) where $A^i(G/P)$ contains the cycles of codimension i. Then the classes of the Schubert varieties form an additive basis of $A(G/P)$.*

1.5.3 Shifted, even and almost even Young diagrams

In this subsection we investigate Schubert varieties in the homogeneous spaces discussed in Example 1.5.2.

Let $d, e, n \geq 1$ be integers. Then a $(d \times e)$*-frame* is a rectangle of height d and width e. A *shifted n-frame* consists of those boxes in an $(n \times n)$-frame which are on or above the diagonal pointing from north-west to south-east:

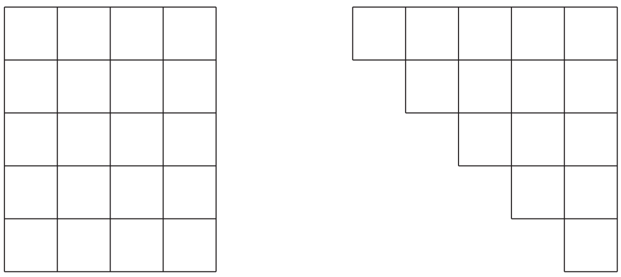


FIGURE 1.3. A (5×4)-frame and a shifted 5-frame.

Given a decreasing sequence of integers $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s)$ and a (shifted) frame let us fill the leftmost λ_i boxes in the i-th row (assume that the frame is large enough). We call the resulting picture the *(shifted) Young-diagram associated with $\underline{\lambda}$* and denote it by $\Lambda(\underline{\lambda})$. The *boundary* of $\Lambda(\underline{\lambda})$ is given by all the boundary segments of $\Lambda(\underline{\lambda})$ which lie on the right vertical or the lower horizontal segment of the frame. A segment in the boundary is called an *inner segment* if it does not lie on the boundary of the underlying (shifted) frame. The inner segments form the *inner boundary*. We call $\underline{\lambda}$ a

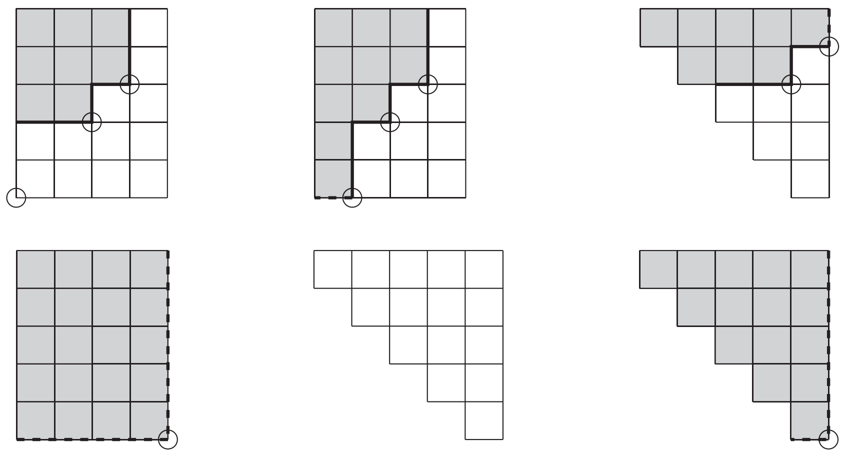


FIGURE 1.4. The sequences $(3,3,2)$ and $(3,3,2,1,1)$ are 5-partitions of 4 (upper left and upper middle). The sequence $(5,3)$ is a shifted partition of 5 (upper right). We drew the inner segments with thick lines. The inner segments together with the dashed thick lines form the *boundary* of the partition. We also marked the corners of the partitions by circles. In the second line we drew the special full and empty partitions together with their corners.

(i) *d-partition of e*, if $s \leq d$ and $e \geq \lambda_1 \geq \ldots \geq \lambda_s \geq 0$. In other words the horizontal inner segments of $\Lambda(\underline{\lambda})$ point from east to west when following the boundary from the north-east to the south-west corner. Denote by $\mathfrak{P}_{d,e}$ the set of d-partitions of e.

(ii) *strict shifted partition of n*, if $s \leq n$ and $n \geq \lambda_1 > \ldots > \lambda_s > 0$. In other words, the horizontal inner segments of $\Lambda(\underline{\lambda})$ point from east to west when following the boundary from the north-east corner to the south-west. Denote by $\mathfrak{S}_n$ the set of strict partitions of n.

For later computations it is convenient to characterize partitions as in [BC12a] by different parameters. A *corner* of a (shifted) partition is a point where the boundary of $\Lambda(\underline{\lambda})$ bends from vertical to horizontal. We have the following special cases:

(i) If $\lambda_s = 0$ the south-west corner of the non-shifted frame also is a corner by default (see Figure 1.4).

(ii) If $\underline{\lambda}$ is full (i.e. $\lambda_s = e$ in the unshifted case and $\lambda_s = 1$ in the shifted case), the south-east corner of the frame is a corner (see Figure 1.4).

(iii) In particular, in the unshifted case there are no partitions without any corners where as the empty shifted partition is the only shifted partition without any corners.

Let $c_1, \dots, c_k$ be the corners of $\underline{\lambda}$ where we start enumerating in the north-east. Then we define the k-tuples $\underline{d}$ and $\underline{e}$ as follows:

- d_i is the distance of c_i to the upper horizontal segment of the frame.
- e_i is the distance of c_i to the right vertical segment of the frame.

Example 1.5.6. The tuples for the partitions illustrated in Figure 1.4 are given by

$$\begin{aligned}
(\underline{d}, \underline{e}) &= ((2,3,5),(1,2,4)),\\
(\underline{d}, \underline{e}) &= ((2,3,5),(1,2,3)),\\
(\underline{d}, \underline{e}) &= ((1,2),(0,1)).
\end{aligned}$$

Convention 1.5.7. Sometimes we will need to extend the tuples by the entries $d_0 = 0 = e_0$ for easier notation. We also let $d_{k+1} = d_k$ if $d_k \equiv n+1(2)$ and $d_{k+1} = d_k + 1$ otherwise and finally $e_{k+1} = n - d_k$. We denote by $\underline{d}^*$ and $\underline{e}^*$ the extended $(k+1)$-tuples

$$\begin{aligned}
\underline{d}^* &= (d_i)_{1 \le i \le k+1},\\
\underline{e}^* &= (e_i)_{1 \le i \le k+1}.
\end{aligned}$$

Proposition 1.5.8. *Let $k \geq 1$.*

(i) There is a bijection

$$\left\{\begin{matrix}\text{strict } d\text{-partitions of } e\\ \text{with } k \text{ corners}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}k\text{-tuples } \underline{d}, \underline{e} \text{ with}\\ d = d_k > d_{k-1} > \cdots > d_1 > 0 \text{ and}\\ e \geq e_k > e_{k-1} > \ldots > e_1 \geq 0\end{matrix}\right\}.$$

(ii) There is a bijection

$$\left\{\begin{matrix}\text{strict shifted partitions of } n\\ \text{with } k \text{ corners}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}k\text{-tuples } \underline{d}, \underline{e} \text{ with}\\ n \geq d_k > d_{k-1} > \cdots > d_1 > 0,\\ n > e_k > e_{k-1} > \ldots > e_1 \geq 0 \text{ and}\\ d_k + e_k \leq n\end{matrix}\right\}.$$

By abuse of notation we associate the 1-tuples $\underline{d} = (0)$ and $\underline{e} = (n)$ with the empty partition.

Proof. We show (ii). First let $\underline{\lambda}$ be a non-zero shifted partition of n. Let $\underline{\tilde{e}}$ be given by $\tilde{e}_i = n + 1 - i - \lambda_i$, $(1 \leq i \leq s)$, where $\tilde{e}_{s+1} = \infty$. Then

$$\underline{e} = (\tilde{e}_i)_{\substack{1 \le i \le s,\\ \tilde{e}_{i+1} > \tilde{e}_i}}, \qquad \underline{d} = (i)_{\substack{1 \le i \le s,\\ \tilde{e}_{i+1} > \tilde{e}_i}}$$

are k-tuples for $k = s - \operatorname{card}(\{i \mid \lambda_{i+1} = \lambda_i - 1\})$. Note that k is exactly the number of corners: In the i-th row there is no corner if and only if $\lambda_{i+1} = \lambda_i - 1$ and there is a corner if $\lambda_{i+1} < \lambda_i - 1$. The conditions for $\underline{d}, \underline{e}$ are automatically satisfied. On the other hand, if we are given k-tuples $\underline{d}, \underline{e}$ subject to these conditions, we define

$$\lambda_i = \begin{cases} n + 1 - d_j - e_j, & \text{if } i = d_j \text{ for some } 1 \leq j \leq k, \\ \lambda_{d_j} + (d_j - i), & \text{if } j \text{ is minimal with } i \leq d_j. \end{cases}$$

One checks that this defines a decreasing sequence $n \geq \lambda_1 > \ldots > \lambda_{d_k} > 0$ and the two constructions are inverse to each other.

So associated with a partition $\underline{\lambda}$ we always have some k-tuples $\underline{d}, \underline{e}$. A special class of (shifted) Young diagrams are the even and almost even (shifted) Young diagrams:

Definition 1.5.9 (Even and almost even (shifted) partitions and Young diagrams). Let $\underline{\lambda}$ be a (shifted) partition. Then we call $\underline{\lambda}$ *even* if all inner segments of the corresponding Young diagram $\Lambda(\underline{\lambda})$ are of even length. We call $\underline{\lambda}$ *almost even* if all inner segments except of the last one (i.e. the most south-west) are of even length and the last inner segment is of odd length, see Figure 1.5 for an illustration. Denote by $\mathfrak{E}_{d,e}$ the subset of $\mathfrak{P}_{d,e}$ of even d-partitions of e and by $\mathfrak{E}_n$ and $\mathfrak{F}_n$ the subsets of $\mathfrak{S}_n$ containing even and almost even shifted partitions of n.

We can write this in terms of the k-tuples $\underline{d}$ and $\underline{e}$. For unshifted partitions this is [BC12a, 2.7]. If $\underline{\lambda}$ is shifted, we call $\underline{\lambda}$ *even* if

(E1) $d_{i+1} - d_i$ is even for all $1 \leq i \leq k-2$,

(E2) $e_{i+1} - e_i$ is even for all $1 \leq i \leq k-1$,

(E3) d_1 is even if $e_1 > 0$,

(E4) $d_k - d_{k-1}$ is even and

(E5) $n - d_k - e_k =: e_{k+1} - e_k$ is even if $e_k < n - d_k$.

Further $\underline{\lambda}$ is *almost even* if (E1)-(E3) hold and instead of (E4) and (E5) we have

(O4) If $e_k = n - d_k$ then $e_{k+1} - e_k = 0$, in particular even, and $d_k - d_{k-1}$ is odd.

(O5) If $e_k < n - d_k$ then $e_{k+1} - e_k$ is odd and $d_k - d_{k-1}$ is even.

A shifted partition is both even and almost even if and only if it is empty or full.

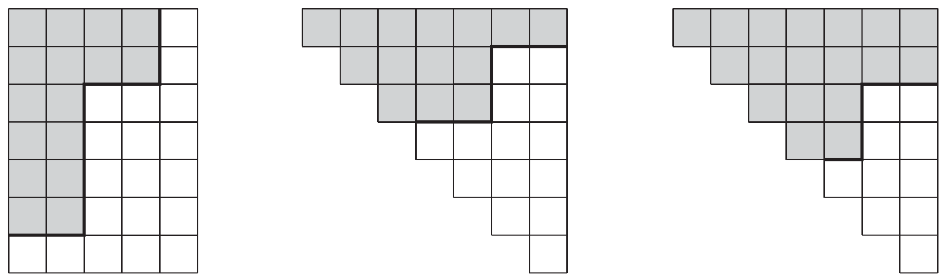


FIGURE 1.5. From left to right: An even Young diagram, an even shifted Young diagram and an almost even shifted Young diagram.

Example 1.5.10. The empty and full (shifted) partition is always even and almost even since it does not have inner segments and hence any condition on inner segments is satisfied.

The following lemma will be needed later. It states that even and almost even diagrams of a special shape do not occur in all sizes:

Lemma 1.5.11. *Let λ be a shifted partition of n with associated k-tuples $\underline{d}, \underline{e}$. Assume that d_1 is odd.*

(i) If $\underline{\lambda}$ is even, then n is odd.

(ii) If $\underline{\lambda}$ is almost even, then n is even.

Proof. All equivalences are modulo 2. We have $d_{k-1} \equiv d_1$ and $e_k \equiv e_1 \equiv 0$ by (E1)-(E3).

(i) Let $\underline{\lambda}$ first be even, in which case $d_k \equiv d_{k-1} \equiv 1$. If $e_k < n - d_k$, by (E5) we have $1 \equiv n - d_k - e_k \equiv n$. Otherwise $e_k = n - d_k$ which implies $n \equiv e_k + d_k \equiv 1$, so in any case n is odd.

(ii) Assume now that $\underline{\lambda}$ is almost even. Suppose first that $e_k < n - d_k$, i.e. $d_k - d_{k-1}$ is even and $n - d_k - e_k$ is odd by (O5). Then as above $d_k \equiv 1$ and we get that $n \equiv 1 + d_k + e_k \equiv 0$. If on the other hand $e_k = n - d_k$ we have that

$$d_k = d_k - d_{k-1} + (d_{k-1} - d_{k-2}) + \ldots + d_1 \equiv 0$$

and we conclude that $n = d_k + e_k$ is even.

Notation 1.5.12. In the following, in order to avoid confusions, we will write $\underline{\lambda}$ for partitions in $\mathfrak{S}_n$ and $\underline{\lambda}', \underline{\lambda}''$ for partitions in $\mathfrak{S}_{n-1}$.

Definition 1.5.13. (i) Let $\underline{\lambda}' \in \mathfrak{S}_{n-1}$. We define the induced partition $\bar{\imath}(\underline{\lambda}') \in \mathfrak{S}_n$ by $\bar{\imath}(\underline{\lambda}') = (n, \lambda'_1, \ldots, \lambda'_s)$.

(ii) Let $\underline{\lambda} \in \mathfrak{S}_n$ such that $\lambda_1 \leq n-1$. We define the induced partition $\overline{v}(\underline{\lambda}) \in \mathfrak{S}_{n-1}$ by $\overline{v}(\underline{\lambda}) = (\lambda_1, \ldots, \lambda_s) = \underline{\lambda}$.

(iii) Let $\underline{\lambda}'' \in \mathfrak{S}_{n-1}$ such that $\lambda''_1 = n-1$. We define the induced partition $\overline{\partial}(\underline{\lambda}'') \in \mathfrak{S}_{n-1}$ by $\overline{\partial}(\underline{\lambda}'') = (\lambda''_2, \ldots, \lambda''_s)$.

Proposition 1.5.14. *Let $n \geq 3$. Then we have the following:*

(i) The map $\bar{\imath}$ induces bijections

$$\begin{array}{rcl}
\left\{ \begin{matrix} \underline{\lambda}' \in \mathfrak{F}_{n-1} \text{ almost even with} \\ e'_1 < n-1 \text{ even or } e'_1 = n-1 \text{ odd} \end{matrix} \right\} & \longleftrightarrow & \left\{ \underline{\lambda} \in \mathfrak{F}_n \text{ almost even with } e_1 = 0 \right\}, \\
\left\{ \underline{\lambda}' \in \mathfrak{E}_{n-1} \text{ even with } e'_1 \text{ even} \right\} & \longleftrightarrow & \left\{ \underline{\lambda} \in \mathfrak{E}_n \text{ even with } e_1 = 0 \right\}, \\
\underline{\lambda}' & \overset{\bar{\imath}}{\mapsto} & (n, \lambda'_1, \ldots, \lambda'_s), \\
(\lambda_2, \ldots, \lambda_s) & \leftarrow\!\shortmid & \underline{\lambda}.
\end{array}$$

(ii) The map $\overline{v}$ induces a bijection:

$$\begin{array}{rcl}
\left\{ \underline{\lambda} \in \mathfrak{F}_n \text{ almost even with } e_1 > 0 \right\} & \longleftrightarrow & \left\{ \begin{matrix} \underline{\lambda}'' \in \mathfrak{F}_{n-1} \text{ almost even with} \\ d''_1 < n-1 \text{ even or } d''_1 = n-1 \text{ odd} \end{matrix} \right\}, \\
\left\{ \underline{\lambda} \in \mathfrak{E}_n \text{ even with } e_1 > 0 \right\} & \longleftrightarrow & \left\{ \underline{\lambda}'' \in \mathfrak{E}_{n-1} \text{ even with } d''_1 \text{ even} \right\}, \\
\underline{\lambda} & \overset{\overline{v}}{\mapsto} & (\lambda_1, \ldots, \lambda_s), \\
(\lambda''_1, \ldots, \lambda''_s) & \leftarrow\!\shortmid & \underline{\lambda}.''
\end{array}$$

(iii) The map $\overline{\partial}$ induces a bijection:

$$\begin{array}{rcl}
\left\{ \begin{matrix} \underline{\lambda}'' \in \mathfrak{F}_{n-1} \text{ almost even with} \\ d''_1 < n-1 \text{ odd or } d''_1 = n-1 \text{ even} \end{matrix} \right\} & \longleftrightarrow & \left\{ \begin{matrix} \underline{\lambda}' \in \mathfrak{F}_{n-1} \text{ almost even with} \\ e'_1 < n-1 \text{ odd or } e'_1 = n-1 \text{ even} \end{matrix} \right\}, \\
\left\{ \underline{\lambda}'' \in \mathfrak{E}_{n-1} \text{ even with } d''_1 \text{ odd} \right\} & \longleftrightarrow & \left\{ \underline{\lambda}' \in \mathfrak{E}_{n-1} \text{ even with } e'_1 \text{ odd} \right\}, \\
\underline{\lambda}'' & \overset{\overline{\partial}}{\mapsto} & (\lambda''_2, \ldots, \lambda''_s), \\
(n-1, \lambda'_1, \ldots, \lambda'_s) & \leftarrow\!\shortmid & \underline{\lambda}'.
\end{array}$$

Proof. Check that the maps $\bar{\imath}, \overline{v}$ and $\overline{\partial}$ preserve even and almost even shifted partitions and that the assignments are well-defined. $\square$

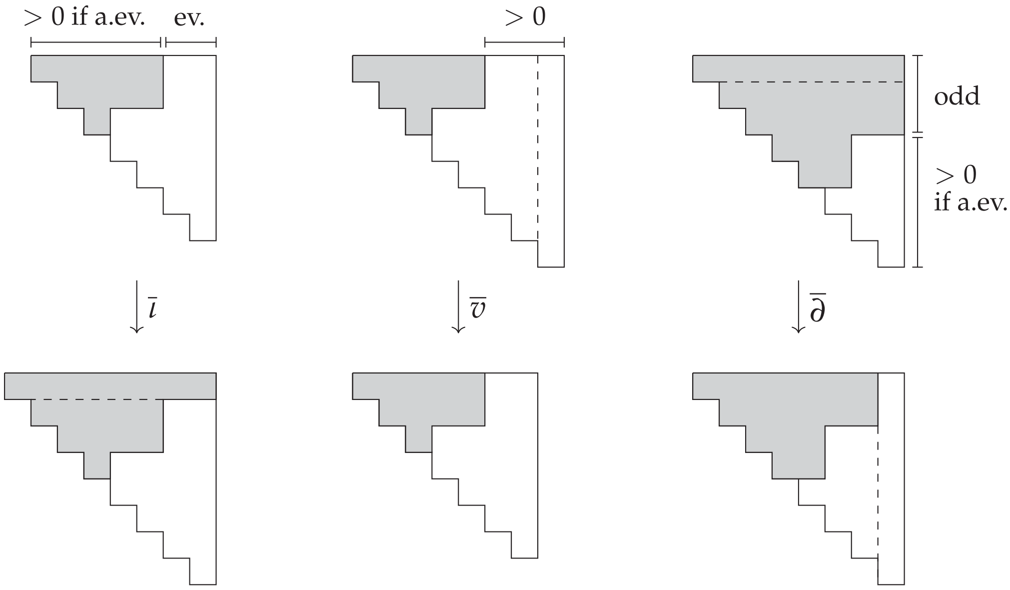


FIGURE 1.6. Illustration of the introduced maps in the even and almost even (a.ev.) case. The map $\bar{\imath}$ adds a row on top of an (almost) even partition if the resulting partition is (almost) even, $\overline{v}$ forgets the rightmost column if it's empty and $\overline{\partial}$ adds an empty column on the right and drops the first row of an (almost) even patition if the resulting partition is (almost) even. Note that we were inconsistent with the sizes since never all three cases occur for fixed n, see Lemma 1.5.11.

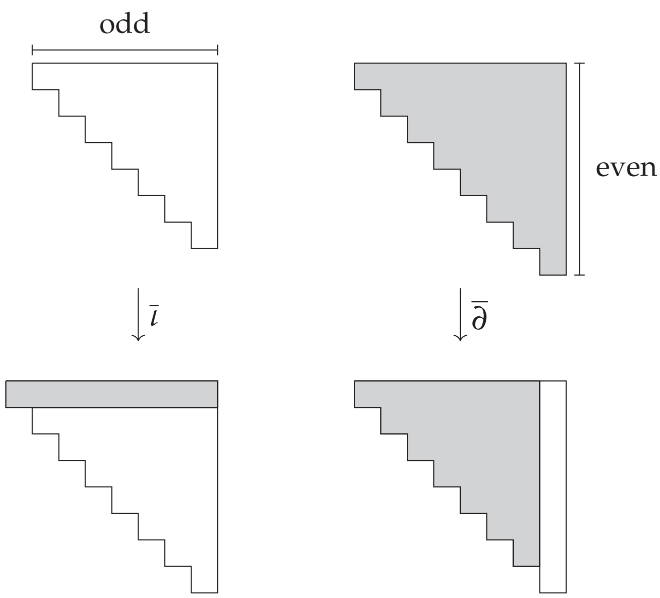


FIGURE 1.7. The maps $\bar{\imath}$ and $\overline{\partial}$ for the two special cases in the almost even case, i.e. when $e_1' = n-1$ is odd and $d_1'' = n-1$ is even.

Definition 1.5.15. Define the maps $\bar{\iota}_e, \overline{v}_e$ and $\overline{\partial}_e$ by

$$\bar{\iota}_e(\underline{\lambda}') = \begin{cases} \bar{\iota}(\underline{\lambda}'), & \text{if } e_1' \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\overline{v}_e(\underline{\lambda}) = \begin{cases} \overline{v}(\underline{\lambda}), & \text{if } e_1 > 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\overline{\partial}_e(\underline{\lambda}'') = \begin{cases} \overline{\partial}(\underline{\lambda}''), & \text{if } d_1'' \text{ is odd ,} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we define $\bar{\iota}_o, \overline{v}_o$ and $\overline{\partial}_o$. If it is clear from the context whether it is about even or almost even shifted partitions, by abuse of notation we just write $\bar{\iota}, \overline{v}$ and $\overline{\partial}$. See Figures 1.6 and 1.7 for illustrations.

1.5.4 Young diagrams parametrize Schubert varieties

It is well-known that Schubert varieties of the ordinary Grassmannian are indexed by Young diagrams (i.e. partitions) and Schubert varieties of maximal orthogonal and Lagrangian Grassmannians are indexed by shifted partitions. Let V be a vector space of dimension $d+e$ over some field k. Fix a complete flag of V, i.e. a filtration

$$0 = E_0 \subset E_1 \subset \ldots E_{n-1} \subset E_n = V$$

of subspaces E_i of dimension $\dim(E_i) = i$. Recall that the ordinary Grassmannian $\mathrm{Gr}(d, V)$ is the homogeneous space $X = A_{d+e-1}/P_d$ and inside this variety for each element $w \in W = W(A_{d+e+1})/W_{P_d}$ there is a Schubert cell $X^o(w)$ whose closure is the Schubert variety $X(w)$. Then there is a bijection $W \cong \mathfrak{P}_{d,e}$ (cf. section 1.5.5) and the Schubert variety associated with a d-partition $\underline{\lambda}$ of e is given by

$$X(\underline{\lambda}) = \{W \in \mathrm{Gr}(d, V) \mid \dim(W \cap E_{d_i+e_i}) \geq d_i \text{ for } 1 \leq i \leq k\}. \tag{1.47}$$

A similar description exists for the isotropic setting. Let (V, ω) be a vector space of dimension $2n$ equipped with a non-degenerate bilinear form and fix a complete flag

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n$$

of isotropic subspaces E_i of dimension $\dim(E_i) = i$ of V. If ω is symmetric, the Schubert varieties in $\mathrm{OG}(n, V)$ are indexed by $W(D_n/P_n) \cong \mathfrak{S}_{n-1}$ (cf. section 1.5.5) and the Schubert variety associated with a shifted partition $\underline{\lambda}$ is described as follows: If $s \equiv n(2)$ we have

$$X(\underline{\lambda}) = \{W \in \mathrm{OG}(n, V) \mid \dim(W \cap E_{d_i+e_i}) \geq d_i \text{ for } 1 \leq i \leq k\} \tag{1.48}$$

and otherwise

$$X(\underline{\lambda}) = \left\{ W \in \mathrm{OG}(n, V) \;\middle|\; \begin{array}{l} \dim(W \cap E_{d_i+e_i}) \geq d_i \text{ for } 1 \leq i \leq k-1 \\ \dim(W \cap E_{d_k+e_k+1}) \geq d_k + 1 \end{array} \right\}. \tag{1.49}$$

If ω is antisymmetric, the Schubert varieties in $\mathrm{LG}(n, V)$ are indexed by $W(C_n/P_n) \cong \mathfrak{S}_n$ (cf. section 1.5.5) and the Schubert variety associated with a shifted partition $\underline{\lambda}$ is given by

$$X(\underline{\lambda}) = \{W \in \mathrm{LG}(n, V) \mid \dim(W \cap E_{d_i+e_i}) \geq d_i \text{ for } 1 \leq i \leq k\}. \tag{1.50}$$

For ordinary Grassmannians the equivalence of the two different notions has been shown in [LB15, Ch. 5]. A subgroup of $G = \mathrm{SL}_{d+e}(k)$ of type A_{d+e-1} is Borel if it is conjugate to the subgroup $T_{d+e}(k)$ of upper triangular matrices and a maximal parabolic subgroup is a closed subgroup of G containing a Borel subgroup, i.e. conjugate to a subset of the form

$$P_d = \left\{ \begin{pmatrix} A_{d\times d} & B \\ 0 & C_{e\times e} \end{pmatrix} \right\}.$$

Without loss let $B = T_{d+e}(k)$ with respect to the basis associated with the fixed flag $E_\bullet$. Recall that $\mathrm{Gr}(d, V) = G/P_d$. Now the Schubert variety as defined in (1.47) is stable under the action of the Borel subgroup: By definition the flag is invariant under B, so $\dim(W \cap E_{d_i+e_i}) \geq d_i$ implies

$$\dim(b \cdot W \cap E_{d_i+e_i}) = \dim(b \cdot (W \cap E_{d_i+e_i})) = \dim(W \cap E_{d_i+e_i}) \geq d_i,$$

i.e. $b \cdot W \in X(\underline{\lambda})$. By considering the Plücker embedding $\mathrm{Gr}(d, V) \hookrightarrow \mathbb{P}_k^{\binom{d+e}{d}-1}$ one finally concludes that the two notions coincide.

This procedure can immediately be adapted to isotropic Grassmannians by considering $G = \mathrm{Sp}(V)$ (and $G = \mathrm{SO}(V)$, respectively) and an isotropic flag

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E_n^\perp \subset E_{n+1} = E_{n-1}^\perp \subset \ldots \subset E_{2n-1} = E_1^\perp \subset E_{2n} = V$$

corresponding to a Borel subgroup, i.e. a conjugate of the upper triangular special orthogonal (and symplectic, respectively) group. Throughout, for Schubert varieties we will always use the descriptions (1.47)-(1.50).

1.5.5 Quivers of minuscule and cominuscule varieties

There is another method to visualize Schubert varieties of Grassmannians, which not only allows us to read directly the intersection conditions as in (1.47)-(1.50), but also enables us to determine the Weyl group element (a representative in the quotient W/W_P, respectively) associated with this Schubert variety. This is done by *quivers*. These have been intensively studied by Perrin (e.g. [Per07]) for *minuscule* and *cominuscule* varieties. In this section we first remark that our objects of interest are minuscule or cominuscule before discussing basic facts about quivers and giving some examples.

Let G be a semisimple linear algebraic group (for example $\mathrm{SL}_{d+e}(k), \mathrm{SO}_n(k), \mathrm{Sp}_n(k)$) and $B \subset P \subset G$ a Borel and parabolic subgroup. Assume that $P = P_{\alpha_i}$ is standard and maximal, i.e. associated with a single simple root $\alpha_i \in \Delta$ in the root system Φ. Recall that a root is called *positive* if all its coefficients, when written as a sum of simple roots, are positive. The *dual root system* $\Phi^\vee$ is the set of coroots $\alpha^\vee$ of roots α in Φ where

$$\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle}\alpha$$

(here $\langle -, - \rangle$ denotes the form on the Euclidean space E associated with the root system Φ. In this setting the form is induced by the so-called *Killing form*). It is immediate that $\Phi^\vee$ itself is a root system. The *highest root* is the unique (positive) root $\gamma \in \Phi$ such that $\gamma - \alpha$ is positive for any root $\alpha \in \Phi$.

Definition 1.5.16 (Minuscule and cominuscule varieties). For $\alpha \in \Phi$ and a simple root $\beta \in \Delta$ denote by $\alpha(\beta)$ the coefficient of β when α is expressed as a sum of simple roots.

(i) The root $\beta \in \Delta$ is called *cominuscule* if $|\alpha(\beta)| \leq 1$ for all $\alpha \in \Phi$. This is equivalent to $\gamma(\beta) = 1$ where γ denotes the highest root.

(ii) The root β is called *minuscule* if $\beta^\vee$ is cominuscule in the dual root system $\Phi^\vee$.

(iii) The homogeneous space $X = G/P_\beta$ is (co-)minuscule if β is (co-)minuscule.

At the end of this thesis, in section 4.5, we give an overview of all minuscule and cominuscule varieties in ordinary and exceptional types. In particular any minuscule variety is cominuscule, possibly for a different Lie type. Hence, it suffices to focus on the cominuscule varieties and we will see that the description of the Witt groups of all of them will be covered by the works of Balmer and Calmès on ordinary Grassmannians ([BC12a]), Nenashev and Walter on quadrics ([Nen09], [Wal03]) and the present work on maximal isotropic Grassmannians. In the following, we illustrate how to view these spaces as cominuscule varieties.

Remark 1.5.17 (e.g. [Spr66]**).** (i) The Grassmannian $\mathrm{Gr}(d,V) = \mathrm{SL}_{d+e}(k)/P_d$ is minuscule and cominuscule for any $1 \leq d \leq d+e-1$: The roots are given by vectors in k^{d+e} of length $\sqrt{2}$ with integer entries summing up to zero, i.e. by $e_i - e_j$ for some $i \neq j$. A choice of simple roots is given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq d+e-1$. The highest root is given by $\gamma = \alpha_1 + \ldots + \alpha_{d+e-1}$, so all simple roots are cominuscule. Since all the roots have the same length, $\Phi^\vee \cong \Phi$ and all simple roots are minuscule as well.

(ii) Similarly, the orthogonal Grassmannian $\mathrm{OG}(n,V) = \mathrm{SO}_n(k)/P_n$ is minuscule and cominuscule: The roots are given by integer vectors in k^n of length $\sqrt{2}$, i.e. by $r = \pm e_i \pm e_j$ for some $i \neq j$. A choice of simple roots is given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = e_{n-1} + e_n$ in k^n. The highest root is given by

$$\gamma = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,$$

i.e. α_1, α_{n-1} and α_n are cominuscule. Again $\alpha_i^\vee = \alpha_i$, so these roots are also minuscule.

(iii) The Lagrangian Grassmannian $\mathrm{LG}(n,V) = \mathrm{Sp}_n(k)/P_n$ is cominuscule but not minuscule: The roots are given by the roots in type D (i.e. integer vectors of length $\sqrt{2}$) and all doubled integer vectors of length 1. A choice of simple roots is given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = 2e_n$. The highest root is given by

$$\gamma = 2\alpha_1 + \ldots + 2\alpha_{n-1} + \alpha_n,$$

so α_n is the only cominuscule root. Now $\alpha_i^\vee = \alpha_{n+1-i}$, so α_1 is the only minuscule root.

Note: Even if the chosen description of the (simple) roots is not unique, the numbers $\langle \alpha_i, \alpha_j \rangle$ are. Arranged in a matrix, they completely determine the root system. The matrix is also referred to as the *Cartan-matrix of the root system* Φ or *of the Lie algebra* $\mathfrak{sl}_{d+e}(k)$.

Let now $X(\underline{\lambda}) \subset G/P$ be a Schubert variety associated with a Weyl group element $\overline{w} \in W/W_P$ whose shortest representative in W (in the sense of Definition 1.5.3) is denoted by w. We use minusculeness and cominusculeness as follows:

Theorem 1.5.18 ([Ste97]**).** *If X is (co-)minuscule, $w \in W$ as above has a unique reduced presentation (modulo commuting relations of the form $s_\alpha s_\beta = s_\beta s_\alpha$ whenever $\alpha \perp \beta$) $w = s_{\beta_1} \cdots s_{\beta_r}$ with simple roots $\beta_i \in \Delta$.*

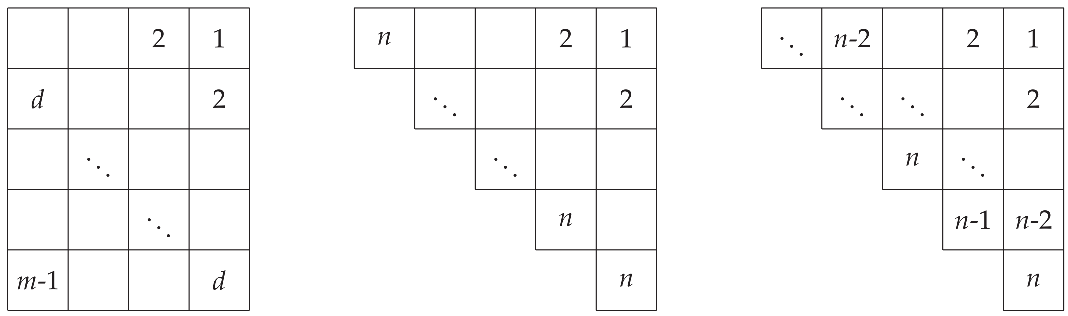


FIGURE 1.8. Coloration in type A (left, write $m := d + e$), C (middle) and D (right). The entries are beginning with a 1 in the north-east box, constant along north-west to south-east diagonals and increasing by one each step. In type D, the entries on the last diagonal are alternating, beginning with n in the south-east.

This allows us to assign to a Schubert variety a unique reduced expression of the shortest representative $w \in W$ of the corresponding $\overline{w} \in W/W_P$. To such an expression $w = s_{\beta_1} \cdots s_{\beta_r}$ we assign the following *quiver* $Q_w := Q_{\underline{\lambda}}$.

- For each reflection s_{β_i} there is a node i.
- Two nodes $i < j$ are linked by an arrow, pointing from i to j if the corresponding simple roots are not orthogonal and $\beta_k \neq \beta_i$ for all $i < k \leq j$.
- The quiver comes equipped with a coloration of its nodes $\beta : \{1, \ldots, r\} \longrightarrow \Delta$ given by $\beta(i) = \beta_i$. In other words, every node is associated with a particular simple root.

Let us first describe the quivers of Grassmannians, i.e. the case where the Schubert variety is the whole Grassmannian. We therefore chose a shortest representative of the longest element in the Weyl group which is given by

$$w_0 = \prod_{k=1}^{d+e-1} \left(\prod_{i=1}^{k} s_{\alpha_{d+e-k+i}} \right) \qquad \text{for } \mathrm{Gr}(d, V), \tag{1.51}$$

$$w_0 = \prod_{k=1}^{n} \left(\prod_{i=1}^{k} s_{\alpha_{n-k+i}} \prod_{i=1}^{k-1} s_{\alpha_{n-i}} \right) \qquad \text{for } \mathrm{LG}(n, V), \tag{1.52}$$

$$w_0 = \prod_{i=1}^{n-1} \left(\prod_{j=n-i,\ j\neq n-n_i}^{n} s_{\alpha_j} \right) \qquad \text{for } \mathrm{OG}(n, V), \tag{1.53}$$

where $n_i \in \{0, 1\}$ is a representative of $n + i(2)$ (e.g. [BKOP14, §2]). See Figure 1.10 for the ordinary and Figure 1.11 for the maximal isotropic Grassmannians.

Definition 1.5.19 (Subquivers). (i) A *subquiver* Q' of a quiver Q consists of a subset of the nodes and arrows of Q such that from each node i

- no arrow is starting if the node is the last node $i = r$,
- one arrow is starting if for any $j > i$ in the subquiver we have $\beta_j \neq \beta_i$,
- and two arrows are starting otherwise.

(ii) Subquivers bijectively correspond to the quivers of Schubert varieties. Hence we write $Q' = Q_{\underline{\lambda}} = Q_w$ for the quiver associated with a (shifted) partition $\underline{\lambda}$ or some $w \in W/W_P$.

(iii) A *peak* of the subquiver Q' is a node i with no incoming arrow.

(iv) A *non-virtual hole* of q is a node i with exactly two incoming arrows and such that $\beta_j \neq \beta_i$ for any $j < i$ in Q'.

(v) A *virtual hole* of the subquiver Q' is a node i of $Q \setminus Q'$ such that s_{β_i} does not commute with w.

(vi) We can view the quiver of a Schubert variety as the complement of the corresponding (shifted) Young diagram. Figure 1.8.

Theorem 1.5.20 ([Per07], [Per09]). *Let $X = G/P$ be minuscule or cominuscule and $X(\underline{\lambda}) \subset X$ a Schubert variety.*

(i) The dimension of a Schubert variety is the number of nodes in the subquiver.

(ii) If X is minuscule, $X(\underline{\lambda})$ is Gorenstein if and only if all peaks have the same height.

(iii) If X is cominuscule but not minuscule, $X(\underline{\lambda})$ is Gorenstein if and only if all peaks colored with a short root have the same height, the peak colored with a long root (if it exists) has height one more and the least occuring height is even.

(iv) A Schubert variety is locally factorial if and only if it has a unique peak.

(v) If X is (co-)minuscule, $X(\underline{\lambda})$ is smooth if and only if it has a unique peak whose color is (co-)minuscule.

Example 1.5.21. Consider the Schubert variety $X(\underline{\lambda}) \subset \mathrm{OG}(8)$ given by the reduced expression

$$w = s_{\alpha_5}s_{\alpha_6}s_{\alpha_8}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_8}.$$

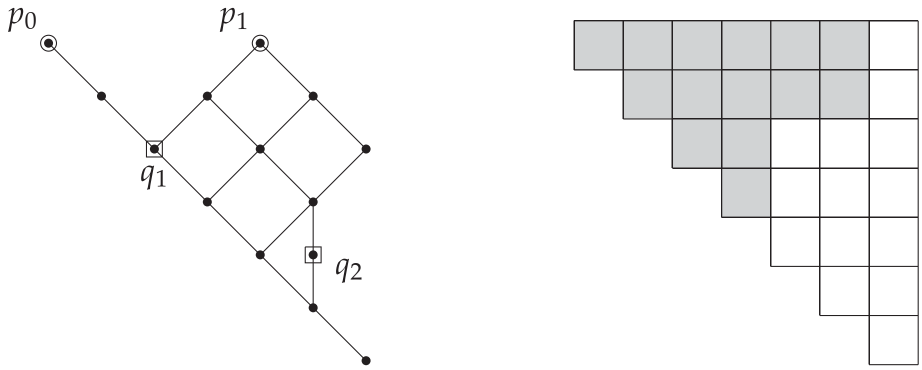


FIGURE 1.9. The quiver and the shifted Young diagram for the Schubert variety $X(\underline{\lambda}) \subset \mathrm{OG}(8)$ in Example 1.5.21. It is given by the 2-tuples $\underline{d} = (2,4)$, $\underline{e} = (1,3)$ and is of type II, i.e. we have two peaks labeled by p_0, p_1 and $k = 2$ holes labeled by q_1, q_2.

The corresponding quiver and the shifted partition of $X(w)$ are illustrated in Figure 1.9. We can describe the Schubert variety by

$$X(\underline{\lambda}) = \left\{ W \in \mathrm{OG}(8) \;\middle|\; \begin{array}{c} \dim(W \cap E_3) \geq 2, \\ \dim(W \cap E_7) \geq 4 \end{array} \right\}.$$

It is a non-smooth and non-Gorenstein subvariety of $\mathrm{OG}(8)$ of dimension 14.

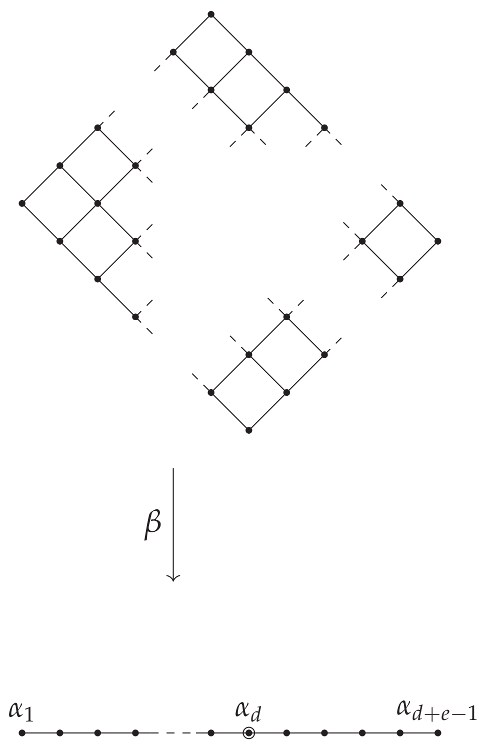


FIGURE 1.10. The quiver of the ordinary Grassmannian $\mathrm{Gr}(d, V)$.

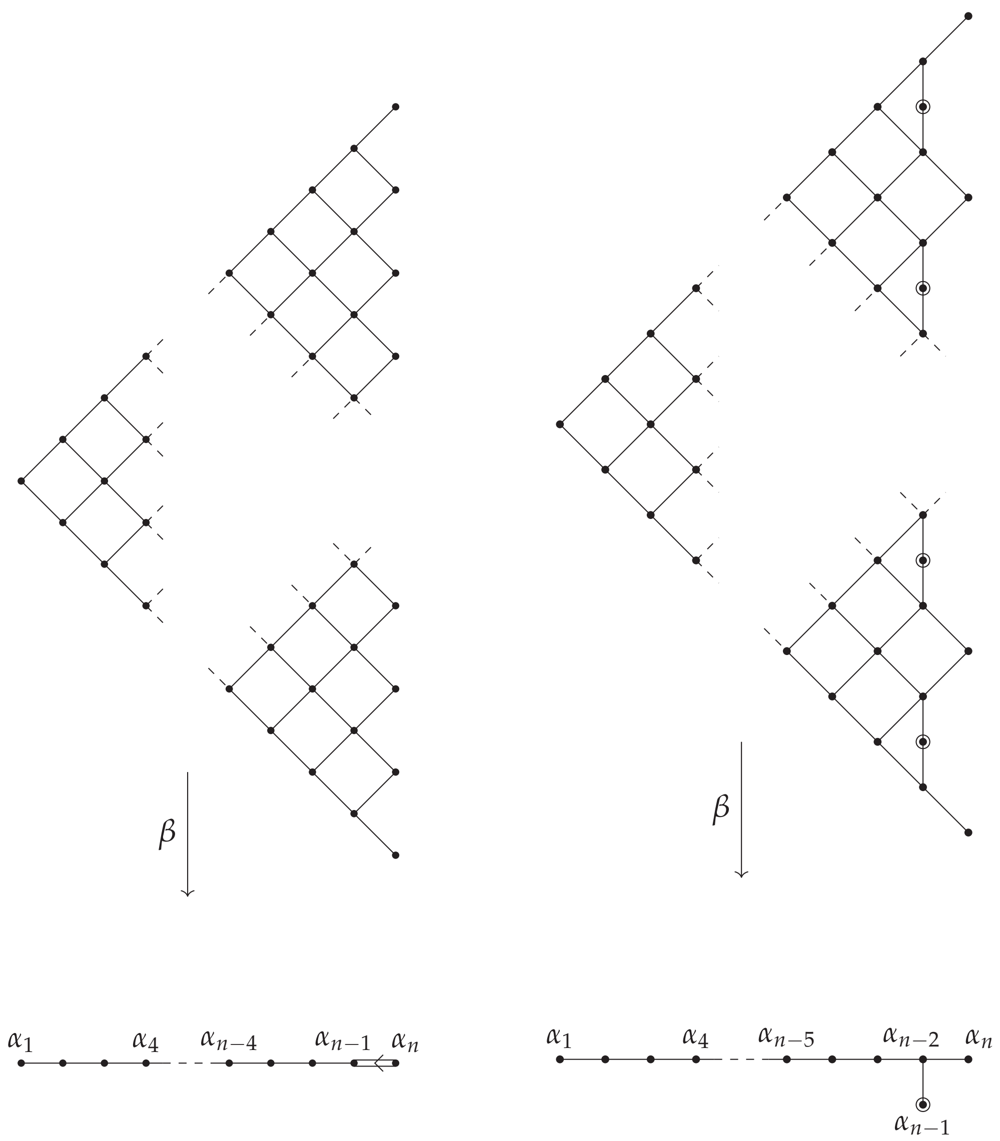


FIGURE 1.11. The quiver of the Lagrangian Grassmannian $\mathrm{LG}(n, V)$ (left) and the orthogonal Grassmannian $\mathrm{OG}(n, V)$ (right). By abuse of notation we will use this slightly different illustration (for odd n the uppermost node will belong to α_{n-1} instead) for OG. The nodes with circles correspond to α_{n-1} and are usually omitted. Note that we also omitted the arrows of large slope since they will not be relevant to us.

Chapter 2

Witt groups of maximal orthogonal Grassmannian bundles

Let $(\mathcal{V}, \omega)$ be an orthogonal vector bundle of rank $2n$ over a smooth, noetherian scheme X over $\mathbb{Z}[\frac{1}{2}]$ where ω admits values in the line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. From now on we require the existence of an isotropic flag $E_\bullet : 0 = E_0 \subset E_1 \subset \ldots \subset E_n \subset \mathcal{V}$ of $(\mathcal{V}, \omega)$ which extends to a complete flag of $\mathcal{V}$ by setting $E_{n+i} = E_{n-i}^\perp$ for $1 \leq i \leq n$. We denote by $\mathrm{OG}_X(n, \mathcal{V})$ the connected component of the maximal orthogonal Grassmann bundle containing E_n and by $\mathrm{OG}_X^-(n, \mathcal{V})$ the opposite one. The flag $E_\bullet$ induces a complete flag $E'_\bullet$ for $E_1^\perp / E_1$ via $E'_i := E_{i+1}/E_1$ for $0 \leq i \leq n-1$ with $\dim E'_i = i$ and this flag is isotropic since $E_\bullet$ is.

2.1 Resolutions of degeneracy loci

Recall that a *(not necessarily strict) shifted partition* of $n-1$ is an s-tuple $\underline{\lambda}$ of decreasing non-negative integers $n-1 \geqslant \lambda_1 > \lambda_2 > \ldots > \lambda_s \geq 0$ for some $s \geq 0$. We may arrange $\underline{\lambda}$ in the shifted $(n-1)$-frame, see Figure 2.1.

If $s \equiv n \mod 2$ there is an associated *Schubert cell*

$$Y_X(\underline{\lambda})^o = \{\mathcal{W} \in \mathrm{OG}_X(n, \mathcal{V}) \mid \mathrm{rk}(\mathcal{W} \cap E_{n-\lambda_i}) = i\}$$

and a *degeneracy locus* or *Schubert subscheme*

$$Y_X(\underline{\lambda}) = \{\mathcal{W} \in \mathrm{OG}_X(n, \mathcal{V}) \mid \mathrm{rk}(\mathcal{W} \cap E_{n-\lambda_i}) \geq i\}$$

inside $\mathrm{OG}_X(n, \mathcal{V})$ and all degeneracy loci can be described this way (we change the pre-

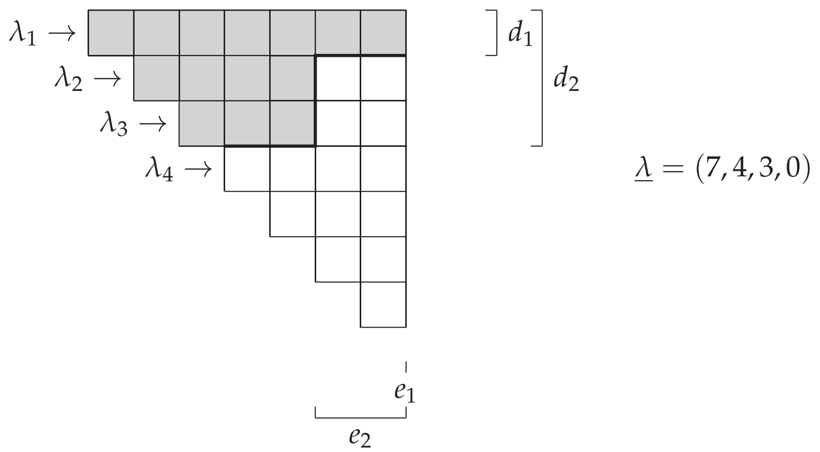


FIGURE 2.1. A shifted partition with $4 = s \equiv 8 = n \mod 2$, hence defining a Schubert subscheme inside $\mathrm{OG}_X(8, \mathcal{V})$. On the other hand, the truncated partition $\underline{\lambda}' = (7,4,3)$ gives us a Schubert subscheme in the opposite component.

viously introduced notation due to our fixed base scheme X). Note that the assumption on s guarantees that $\mathcal{W}$ lies in the right connected component, which is determined by the parity of $\mathrm{rk}(\mathcal{W} \cap E_n)$. In Figure 2.1, the essential intersection conditions are given by

$$\mathrm{rk}(\mathcal{W} \cap E_1) \geq 1, \qquad \mathrm{rk}(\mathcal{W} \cap E_5) \geq 3.$$

For $\mathcal{W}$ to be in $\mathrm{OG}_X(n, \mathcal{V})$ we hence need $\mathrm{rk}[\mathcal{W} \cap E_8] \geq 4$: The intersection is of rank at least 3 and if it was of rank exactly 3 the subbundle $\mathcal{W}$ would be contained in OG^-.

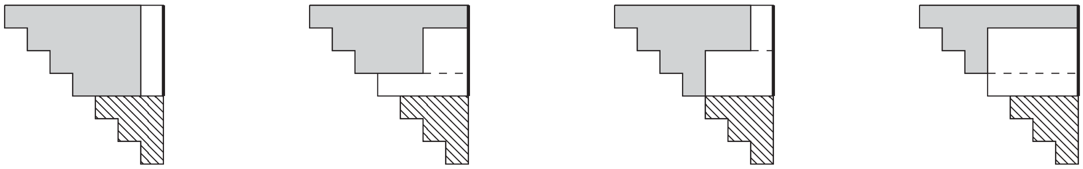
A shifted partition $\underline{\lambda}$ is called *strict*, if $\lambda_s \geq 1$. If $s \equiv n \mod 2$, it defines a Schubert subscheme for OG whereas the partition $(\underline{\lambda}, 0)$ defines one for OG^-. Analoguously, if $s \equiv n+1 \mod 2$, the partition $\underline{\lambda}$ defines a Schubert subscheme for OG^- and $(\underline{\lambda}, 0)$ one for OG. It follows, that Schubert subschemes of $\mathrm{OG}_X(n, \mathcal{V})$ (as well as those of $\mathrm{OG}_X^-(n, \mathcal{V})$) are indexed by $\mathfrak{S}_{n-1}$, the set of strict shifted partitions of $n-1$. From now on, if not mentioned otherwise, all shifted partitions are strict and occasionally we will call them simply partitions, if no ambiguity can occur.

Example 2.1.1. Let $\underline{\lambda} \in \mathfrak{E}_{n-1}$ and $\underline{\lambda}', \underline{\lambda}'' \in \mathfrak{E}_{n-2}$ be strict partitions. Recall the maps $\bar{\iota}, \bar{v}$ and $\bar{\partial}$ from Definition 1.5.13.

(i) If $Y_X(\underline{\lambda}') \subset \mathrm{OG}_X(n-1, E_1^{\perp}/E_1)$, then $Y_X(\bar{\iota}(\underline{\lambda}')) \subset \mathrm{OG}_X(n, \mathcal{V})$.

(ii) If $Y_X(\underline{\lambda}) \subset \mathrm{OG}_X(n, \mathcal{V})$, then $Y_X(\bar{v}(\underline{\lambda})) \subset \mathrm{OG}_X^-(n-1, E_1^{\perp}/E_1)$.

(iii) If $Y_X(\underline{\lambda}'') \subset \mathrm{OG}_X^-(n-1, E_1^{\perp}/E_1)$, then $Y_X(\bar{\partial}(\underline{\lambda}'')) \subset \mathrm{OG}_X(n-1, E_1^{\perp}/E_1)$.



$d_k \equiv n$ $\quad$ $d_k + e_k < n-1$ $\qquad$ $d_k \equiv n+1$ $\quad$ $d_k + e_k < n-1$ $\qquad$ $d_k \equiv n$ $\quad$ $d_k + e_k = n-1$ $\qquad$ $d_k \equiv n+1$ $\quad$ $d_k + e_k = n-1$

FIGURE 2.2. The dashed lines separate the white part of each partition into rectangles (Grassmannians) and triangles (smaller orthogonal Grassmannians). Roughly speaking, these correspond to the ordinary and orthogonal Grassmannians in the flag bundle since the full Young diagrams of these Grassmannians are exactly of this shape. The numbers $l(\underline{\lambda})$ and $t(\underline{\lambda})$ introduced in Definition 2.2.5 can be visualized as the area of the complement of the hatched triangle inside the frame and twice the height of this complement, respectively.

By Proposition 1.5.8 there is a bijection between $\mathfrak{S}_{n-1}$ and pairs of tuples $(\underline{d}, \underline{e})$ satisfying

$$\left\{ \begin{pmatrix} 0 < d_1 < d_2 < \ldots < d_k \leq n-1 \\ 0 \leq e_1 < e_2 < \ldots < e_k < n-1 \end{pmatrix}, \quad k \geq 0,\ e_k + d_k \leq n-1 \right\}$$

together with the pair $((0), (n-1))$. Recall from Convention 1.5.7 that, whenever needed, we extend the tuples with entries d_0, e_0, d_{k+1} and e_{k+1} and denote by $\underline{d}^*$ and $\underline{e}^*$ the extended $(k+1)$-tuples $\underline{d}^* = (d_i)_{1 \leq i \leq k+1}$ and $\underline{e}^* = (e_i)_{1 \leq i \leq k+1}$.

Let now $\underline{\lambda}$ be a strict partition with k-tuples $\underline{d}, \underline{e}$. If $d_k \equiv n \bmod 2$, we define

$$\mathrm{OFl}_X(\underline{\lambda}, E_\bullet) := \mathrm{OFl}_X(\underline{d}, \underline{e}, E_\bullet) = \left\{ \begin{array}{lllll} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_n, & \mathrm{rk}(P_{d_i}) = d_i \\ \cap & & \cap & \Cap & \\ E_{d_1+e_1} & \subset \ldots \subset & E_{d_k+e_k} & \mathrm{OG} & \end{array} \right\} \tag{2.1}$$

where we write $\mathrm{OG} := \mathrm{OG}_X(n, \mathcal{V})$. If $d_k \equiv n+1 \bmod 2$, we define

$$\mathrm{OFl}_X(\underline{\lambda}, E_\bullet) := \mathrm{OFl}_X(\underline{d}^*, \underline{e}^*, E_\bullet) = \left\{ \begin{array}{lllll} P_{d_1} & \subset \ldots \subset & P_{d_{k+1}} & \subset P_n, & \mathrm{rk}(P_{d_i}) = d_i \\ \cap & & \cap & \Cap & \\ E_{d_1+e_1} & \subset \ldots \subset & E_{d_{k+1}+e_{k+1}} & \mathrm{OG} & \end{array} \right\} \tag{2.2}$$

Denote by $f_{\underline{\lambda}}$ the projection onto $\mathrm{OG}_X(n, \mathcal{V})$ which maps all members of the flag to the highest rank bundle and observe that it maps birationally onto the Schubert cell $Y_X(\underline{\lambda})^{\mathrm{o}}$. See Figure 2.2 for an illustration of the resolutions.

Notation 2.1.2. As already seen, any strict partition $\underline{\lambda} \in \mathfrak{S}_{n-1}$ also defines a degeneracy locus inside $\mathrm{OG}^-_X(n,\mathcal{V})$ and we denote by $f^-_{\underline{\lambda}} : \mathrm{OFl}^-_X(\underline{\lambda}, E_\bullet) \to \mathrm{OG}^-_X(n,\mathcal{V})$ the corresponding projection. Here $\mathrm{OFl}^-_X(\underline{\lambda}, E_\bullet)$ is defined as above with the parity assumptions on d_k reversed, i.e. as in (2.1), if $d_k \equiv n+1$ and as in (2.2), if $d_k \equiv n \bmod 2$.

2.2 Construction of the basis

The aim ist to assign Witt classes in the total Witt group of $\mathrm{OG}_X(n,\mathcal{V})$ associated with certain shifted partitions $\underline{\lambda} \in \mathfrak{S}_{n-1}$. The map $f_{\underline{\lambda}} : \mathrm{OFl}_X(\underline{\lambda}) \longrightarrow \mathrm{OG}_X(n,\mathcal{V})$ constructed in the previous section is birational whose image is the Schubert subscheme associated with $\underline{\lambda}$ and we know how to compute the relative canonical bundle of this map. Recall that by [CH11] we have push-forwards

$$(f_{\underline{\lambda}})_* : W^i\left(\mathrm{OFl}_X(\underline{\lambda}), \omega_{f_{\underline{\lambda}}} \otimes f^*_{\underline{\lambda}}(\mathcal{M})\right) \longrightarrow W^{i-\dim f_{\underline{\lambda}}}(\mathrm{OG}_X(n,\mathcal{V}), \mathcal{M})$$

for any line bundle $\mathcal{M} \in \mathrm{Pic}(\mathrm{OG}_X(n,\mathcal{V}))$, where $\dim f_{\underline{\lambda}}$ denotes the constant relative dimension of $f_{\underline{\lambda}}$. We want to push forward the unit form $1 \in W^0(\mathrm{OFl}_X(\underline{\lambda}))$ along $f_{\underline{\lambda}}$, for which by section 2.4.3 we need an alignment $\omega_{f_{\underline{\lambda}}} \otimes f^*_{\underline{\lambda}}(\mathcal{M}) \rightsquigarrow \mathcal{O}_{\mathrm{OFl}_X(\underline{\lambda})}$. Following the discussion in section 1.2.9, this implies the relation

$$\omega_{f_{\underline{\lambda}}} \otimes f^*_{\underline{\lambda}}(\mathcal{M}) \equiv \mathcal{O}_{\mathrm{OFl}_X(\underline{\lambda})} \tag{2.3}$$

in $\mathrm{Pic}(\mathrm{OFl}_X(\underline{\lambda}))/2$. This, of course, cannot be true for all shifted partitions. If, for example, n is odd, choosing the non-even partition $\underline{\lambda}$ with 1-tuples $\underline{d} = (d_1) = (n-2)$ and $\underline{e} = (e_1) = (1)$, we have

$$f_{\underline{\lambda}} : \mathrm{OFl}_X(d,e) = \left\{ \begin{matrix} P_{n-2} \subset P_n \in \mathrm{OG}_X(n,\mathcal{V}) \\ \cap \\ E_{n-1} \end{matrix} \right\} \longrightarrow \mathrm{OG}_X(n,\mathcal{V})$$

and by Proposition 1.4.9, the class of the relative canonical bundle in $\mathrm{Pic}(\mathrm{OFl}_X(d,e))/2$ is given by

$$\begin{aligned} \omega_{f_{\underline{\lambda}}} &= [\det E_{n-1}]^{-(n-2)} \cdot \mathcal{L}^{\frac{n(n-1)}{2}-1} \cdot \Delta_{n-2}^{n-2} \cdot \Delta_n^{-(n-2)} \\ &\equiv [\det E_{n-1}] \cdot \mathcal{L}^{\frac{n+1}{2}} \cdot \Delta_{n-2} \cdot \Delta_n \quad \bmod 2 \end{aligned}$$

which cannot be "cancelled" by a pull-back.

In this section, we not only show that even shifted partitions are a good choice in the sense that (2.3) holds for some line bundle $\mathcal{M}$, but in Lemma 2.2.3 we also prove that even partitions are exactly those partitions s.t. (2.3) is satisfied for both the partition itself and its dual, subject to some parity assumption on the heights of the peaks. Note that the authors of [BC12a] have remarked that in type A more choices than even Young diagrams allow us to push forward the unit form, but it remains unclear in loc. cit., why the remaining ones should not be considered. To prove this, we will need quivers and some related results introduced in section 1.5.5.

Lemma 2.2.1. *Let $\underline{\lambda} \in \mathfrak{S}_{n-1}$ be a strict shifted partition. Then the following are equivalent:*

(i) The numbers $d_i - d_{i-1} + e_i - e_{i+1}$ are even for all $2 \leq i \leq k-1$ (and also for $i = 1$, if $n-1 > e_1 > 0$ and for $i = k$ if $e_k < n-1-d_k$).

(ii) All peaks of the quiver $Q_{\underline{\lambda}}$ have the same height modulo 2.

(iii) There is a line bundle $\mathcal{M} \in \operatorname{Pic}(\mathrm{OG}_X(n, \mathcal{V}))$ satisfying (2.3), i.e. we can push-forward the unit form along $f_{\underline{\lambda}}$.

This is, in particular, true for even shifted partitions.

Proof. If $d_k \equiv n \mod 2$ we have by Proposition 1.4.9

$$\left[\omega_{f_{\underline{\lambda}}}\right] = \mathcal{L}^{\frac{d_k(2n-1-d_k)}{2}} \cdot \prod_{i=1}^{k} [\det E_{d_i+e_i}]^{d_{i-1}-d_i} \cdot \prod_{i=1}^{k} \Delta_{d_i}^{d_i-d_{i-1}+e_i-e_{i+1}} \cdot \Delta_n^{-d_k}$$

and there is an additional factor

$$[\det E_n]^{-1} \cdot \mathcal{L}^{n-d_k-1} \cdot \Delta_{d_{k+1}}^2 \cdot \Delta_n^{-1},$$

if $d_k \equiv n+1 \mod 2$. Then the equivalence (i) $\Leftrightarrow$ (iii) follows by requiring all the terms which do not lie in the image of the pull-back, i.e. for Δ_i for $1 \leq i \leq k$ (resp. $1 \leq i \leq k+1$), to have even exponents - this is precisely stated in (i). The peaks of $Q_{\underline{\lambda}}$ are given by

$$\operatorname{peaks}(Q_{\underline{\lambda}}) = \begin{cases} \{p_1, \ldots, p_{k-1}\}, & \text{for } \underline{\lambda} \text{ of type I,} \\ \{p_0, \ldots, p_{k-1}\}, & \text{for } \underline{\lambda} \text{ of type II,} \\ \{p_1, \ldots, p_k\}, & \text{for } \underline{\lambda} \text{ of type III,} \\ \{p_0, \ldots, p_k\}, & \text{for } \underline{\lambda} \text{ of type IV} \end{cases} \tag{2.4}$$

with $h(p_i) = n - 2 - d_i + e_{i+1}$. Since $h(p_i) - h(p_{i-1}) = d_{i-1} - d_i + e_{i+1} - e_i$ for $1 \leq i \leq k$ we see (i) $\Leftrightarrow$ (ii). $\square$

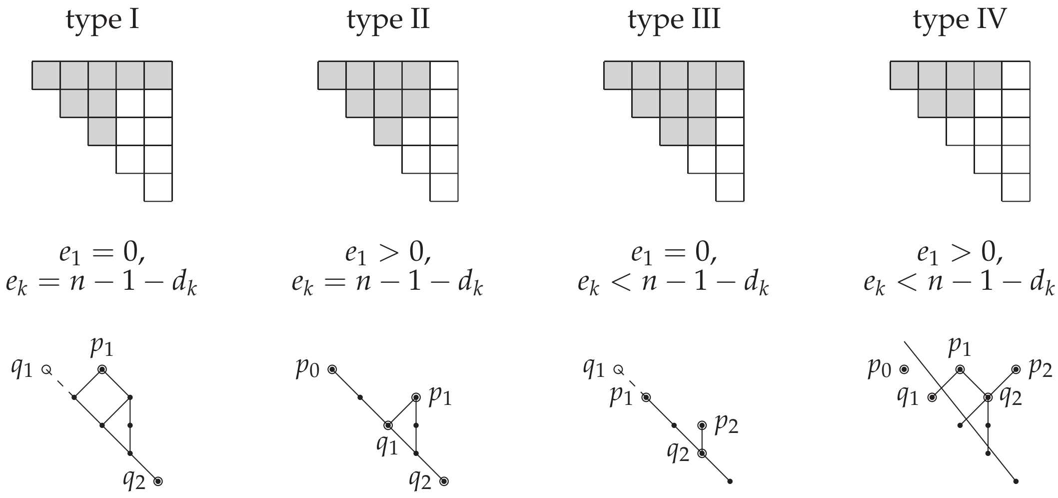


FIGURE 2.3. The 4 different types of strict partitions. If $p := |\text{peaks}(Q)|$, we have $p = k-1$, in type I, $p = k+1$ in type IV and $p = k$ in types II, III.

Shifted partitions are a special case of *straight shapes* which exist for all (co-)minuscule varieties. These shapes index Schubert varieties, i.e. they are in bijection with representatives $\overline{w}$ of W/W_P. If w_0 denotes the longest element of W and w_P the longest element of W_P, the element $w_{\underline{\lambda}^\vee} = w_0 w_{\underline{\lambda}} w_P$ is the *Poincaré dual* of $w_{\underline{\lambda}}$ and its partition is called the *dual partition* of $\underline{\lambda}$, denoted by $\underline{\lambda}^\vee$. For the maximal orthogonal Grassmannian, this duality is obtained by reflection on shifted Young diagrams. See [BS16, §2] for more details.

Definition 2.2.2 (Dual partitions). Consider the classification of shifted partitions into 4 types as in Figure 2.3. Let $\underline{\lambda} \in \mathfrak{S}_{n-1}$ be a partition with k-tuples $\underline{d}$ and $\underline{e}$. With the usual convention $d_0 = 0$ and $e_{k+1} = n-1-d_k$ the dual $\underline{\lambda}^\vee$ of $\underline{\lambda}$ is given by the tuples

$$(\underline{d}, \underline{e})^\vee = \begin{cases} (e_{i+1}, d_i)_{1 \le i \le k-1} & \text{for } \underline{\lambda} \text{ of type I,} \\ (e_{i+1}, d_i)_{0 \le i \le k-1}, & \text{for } \underline{\lambda} \text{ of type II,} \\ (e_{i+1}, d_i)_{1 \le i \le k}, & \text{for } \underline{\lambda} \text{ of type III,} \\ (e_{i+1}, d_i)_{0 \le i \le k}, & \text{for } \underline{\lambda} \text{ of type IV.} \end{cases}$$

Lemma 2.2.3. *Let $\underline{\lambda} \in \mathfrak{S}_{n-1}$ with k-tuples $\underline{d}$ and $\underline{e}$. Then the following are equivalent:*

(i) $\underline{\lambda}$ is even.

(ii) $\underline{\lambda}^\vee$ is even.

(iii) The push-forwards of the unit form along both $f_{\underline{\lambda}}$ and $f_{\underline{\lambda}^\vee}$ exist and for any $p \in peaks(Q_{\underline{\lambda}})$ and $p' \in peaks(Q_{\underline{\lambda}^\vee})$ we have $h(p) \equiv h(p') \mod 2$.

Proof. Clearly (i) $\Leftrightarrow$ (ii). Assume now that $\underline{\lambda}$ is of type IV, i.e. that $\underline{\lambda}^\vee$ is of type I. If $\underline{\lambda}$ is even, then all the terms $d_i - d_{i-1} + e_i - e_{i+1}$ for $1 \leq i \leq k$ are even, so by Lemma 2.2.1 there is a push-forward along $f_{\underline{\lambda}}$ starting from $W^0(-,\mathcal{O})$. By the same lemma we can push-forward the unit form along $f_{\underline{\lambda}^\vee}$ if and only if $e_{i+1} - e_i + d_i - d_{i+1}$ is even for $1 \leq i \leq k-1$, which is also true and proves (ii) $\Rightarrow$ (iii). Now assume (iii), i.e. that

(a) $d_i - d_{i-1} + e_i - e_{i+1}$ is even for $1 \leq i \leq k$,

(b) $e_i - e_{i-1} + d_{i-1} - d_i$ is even for $2 \leq i \leq k$,

(c) $(n-2-d_i+e_{i+1}) - (n-2-e_j+d_j)$ is even for $0 \leq i \leq k$ and $1 \leq j \leq k-1$.

We need to show (E1)-(E5) for $\underline{\lambda}$. Adding equations (a) and (b) suitably, we see

$$d_1 \equiv d_2 - d_1 \equiv \ldots \equiv d_{k-1} - d_{k-2} \equiv d_k - d_{k-1} \mod 2$$

and similarly

$$e_2 - e_1 \equiv e_3 - e_2 \equiv \ldots \equiv e_k - e_{k-1} \equiv e_{k+1} - e_k \mod 2.$$

By (a) for $i = 1$ we even see that all these difference are the same and putting $i = 1$ and $j = 2$ in (c) shows that these differences are all even. The arguments for the other types are the same, so we do not include them. $\square$

Remark 2.2.4. One can show, that the previous lemma is also valid in type A, where in (iii) the statement needs to be replaced by $h(p) - h(p') \equiv d + e \mod 2$.

Definition 2.2.5. For an even strict shifted partition $\underline{\lambda} \in \Lambda_n$ we define the *twist*

$$T(\underline{\lambda}) := \mathcal{L}^{-l(\underline{\lambda})} \cdot [\det E_n]^{-(d_{k+1}-d_k)} \cdot \mathcal{O}(1)^{t(\underline{\lambda})} \in \mathrm{Pic}(\mathrm{OFl}_X(\underline{\lambda}))/2, \tag{2.5}$$

where the numbers $t(\underline{\lambda})$ and $l(\underline{\lambda})$ are defined as

$$t(\underline{\lambda}) := 2d_{k+1} \in \mathbb{Z}, \tag{2.6}$$

$$l(\underline{\lambda}) := \frac{d_{k+1}(2n-1-d_{k+1})}{2} \in \mathbb{Z}. \tag{2.7}$$

Note that, in particular, the class of $T(\underline{\lambda})$ in $\mathrm{Pic}_X(-)/2$ is trivial, since the interesting twist $t(\underline{\lambda})$ is always even. The number $l(\underline{\lambda})$ can be visualized as the dimension of the complement of the orthogonal part in the resolution, i.e. the complement of the hatched area in Figure 2.2 and $t(\underline{\lambda})$ is twice the vertical length of this complement.

Theorem 2.2.6. *We have* $\left[\omega_{f_{\underline{\lambda}}}\right] \cdot f_{\underline{\lambda}}^*(T(\underline{\lambda})) \equiv 1$ *in* $\operatorname{Pic}(\mathrm{OFl}_X(\underline{\lambda}))/2$ *for any* $\underline{\lambda} \in \mathfrak{E}_{n-1}$.

Proof. Consider the case $d_k \equiv n \mod 2$; the other case follows similarly. All, except for the first and the last exponent, in

$$\left[\omega_{\mathrm{OFl}_X(\underline{\lambda})/\mathrm{OG}_X(n,\mathcal{V})}\right] = \mathcal{L}^{\frac{d_k(2n-1-d_k)}{2}} \cdot \prod_{i=1}^{k} [\det E_{d_i+e_i}]^{d_{i-1}-d_i} \cdot \prod_{i=1}^{k} \Delta_{d_i}^{d_i-d_{i-1}+e_i-e_{i+1}} \cdot \Delta_n^{-d_k}$$

are even by [BC12a, 4.8]. Note that $\Delta_n^{d_k} = \sqrt{\Delta_n}^{2d_k} = \sqrt{\Delta_n}^{t(\underline{\lambda}}$ is always a square. $\square$

Definition 2.2.7. For $\underline{\lambda} \in \mathfrak{E}_{n-1}$ let $\mathcal{L}_{\underline{\lambda}}$ be a line bundle on $\mathrm{OG}_X(n,\mathcal{V})$ of class $T(\underline{\lambda})$ in $\operatorname{Pic}(\mathrm{OG}_X(n,\mathcal{V}))/2$. By Theorem 2.2.6 we can choose a line bundle $\mathcal{M}_{\underline{\lambda}} \in \operatorname{Pic}(\mathrm{OFl}_X(\underline{\lambda}))$ and an isomorphism $\psi_{\underline{\lambda}} : \mathcal{M}_{\underline{\lambda}}^{\otimes 2} \longrightarrow f_{\underline{\lambda}}^*(\mathcal{L}_{\underline{\lambda}}) \otimes \omega_{f_{\underline{\lambda}}}$, i.e. an alignment $\mathcal{O} \rightsquigarrow f_{\underline{\lambda}}^*(\mathcal{L}_{\underline{\lambda}}) \otimes \omega_{f_{\underline{\lambda}}}$ which itself induces a lax-push-forward along $f_{\underline{\lambda}}$. We define

$$\phi_n(\underline{\lambda}) \in W^{|\underline{\lambda}|}\left(\mathrm{OG}_X(n,\mathcal{V}), \mathcal{L}_{\underline{\lambda}}\right) \tag{2.8}$$

to be the image of the unit form $1 \in W^0\left(\mathrm{OFl}_X(\underline{\lambda}), \mathcal{O}\right)$ under this lax-push-forward. The elements $\phi_n^-(\underline{\lambda}) \in W^{|\underline{\lambda}|}(\mathrm{OG}_X^-(n,\mathcal{V}), \mathcal{L}_{\underline{\lambda}}^-)$ are defined accordingly.

2.3 The blow-up setting

We will now apply the methods discussed in section 1.2.8 to the maximal orthogonal Grassmannian. Throughout, we are going to use the language of functors of points explained in section 1.3, i.e. we will describe all schemes on points. Let $(\mathcal{V},\omega)$ be an orthogonal vector bundle over a smooth connected scheme X of rank $2n$. As usual, we require the existence of a complete isotropic flag $E_\bullet$ of V. Then E_n is a maximal isotropic subbundle of $\mathcal{V}$ and we denote by $\mathrm{OG}(n) := \mathrm{OG}_X(n,\mathcal{V})$ the connected component of maximal isotropic subbundles of $\mathcal{V}$ containing E_n. Consider the closed subscheme $Z \subset \mathrm{OG}(n)$ given by

$$Z = \{P_n \in \mathrm{OG}(n) \mid E_1 \subset P_n \subset E_1^\perp\} \cong \mathrm{OG}_X(n-1, E_1^\perp/E_1) =: \mathrm{OG}(n-1)$$

and denote by U the open complement $U = \mathrm{OG}(n) \setminus Z$. Again we are going to use the language of functor of points as described in [Kar01]. Note that, in general, we only have $\mathrm{OG}(n)(R) \supsetneq U(R) \cup Z(R)$ (equality holds for fields, as in this case the second condition in Notation 2.3.1 is redundant). But since U (resp. Z) are open (resp. closed) subfunctors of OG we have that the associated open (resp. closed) subschemes cover OG.

Notation 2.3.1. [cf. [BC12a, 5.2]] For an isotropic subbundle $\mathcal{W} \subset \mathcal{V}$ write $\mathcal{W} \dot{\not\subset} E_1^\perp$ if $\mathcal{W} \not\subset E_1^\perp$ (i.e. is not a subbundle) and moreover $\mathcal{W} \cap E_1^\perp$ is a subbundle of $\mathcal{W}$ which is equivalent to the natural map $\mathcal{W}/(\mathcal{W} \cap E_1^\perp) \longrightarrow \mathcal{V}/E_1^\perp$ being an isomorphism. This additional condition is automatically satisfied over fields, but not in general.

Definition 2.3.2. Consider the strict shifted partition $\underline{\lambda} = (n-1)$ and the corresponding smooth degeneracy locus inside $\mathrm{OG}_X^-(n, \mathcal{V})$, denoted by Y. Denote by $\mathcal{S}_Y$ the restriction of the tautological bundle over $\mathrm{OG}^-(n)$ to Y. Note, that $Y \cong \mathrm{OG}_X^-(n-1, E_1^\perp/E_1)$ and under this isomorphism, the tautological bundle is identified with $\mathcal{S}_Y/E_1$. Finally, consider the scheme

$$\mathrm{Gr}_Y(n-1, \mathcal{S}_Y). \tag{2.9}$$

It is smooth and given on points by $\{P_{n-1} \subset P_n^- \subset E_1^\perp\}$.

Now consider the diagram

$$\begin{tikzcd}
Z \arrow[r, hook, "\iota"] & \mathrm{OG}_X(n) & & U \arrow[ll, hook', "v"'] \arrow[d, "\alpha"] \arrow[dll, "\tilde{v}"] \\
\mathrm{OG}_X(n-2, E_1^\perp/E_1) \arrow[u, "\tilde{\pi}"] \arrow[r, hook, "\tilde{\iota}"'] & \mathrm{Gr}_Y(n-1, \mathcal{S}_Y) \arrow[u, "\tilde{\pi}"] \arrow[rr, "\tilde{\alpha}"'] & & \mathrm{OG}^-(n-1, E_1^\perp/E_1) =: Y
\end{tikzcd} \tag{2.10}$$

Let us fix the following notation:

- We write $P_n := P_n^+$ and P_n^- in order to express whether a maximal isotropic subbundle is contained in $\mathrm{OG}(n)$ or $\mathrm{OG}^-(n)$ (and similar in other dimensions).
- The scheme $\mathrm{OG}_X(n-2, E_1^\perp/E_1)$ parametrizes isotropic subbundles of rank $n-2$ of $E_1^\perp/E_1$. Any such P_{n-2} defines a unique bundle $E_1 \subset P_{n-1}$ of rank $n-1$ and this bundle is contained in exactly two maximal isotropic subbundles of $\mathcal{V}$, one in each component, which are denoted by $P_{n-1}^e \in \mathrm{OG}(n)$ and $P_{n-1}^o \in \mathrm{OG}^-(n)$.

Then diagram (2.10) is given on points by

$$\begin{tikzcd}
\{E_1 \subset P_n \subset E_1^\perp\} \arrow[r, hook, "\iota"] & \{P_n \subset \mathcal{V}\} & & \{P_n \dot{\not\subset} E_1^\perp\} \arrow[ll, hook', "v"'] \arrow[d, "\alpha"] \arrow[dll, "\tilde{v}"] \\
\{E_1 \subset P_{n-1} \subset E_1^\perp\} \arrow[u, "\tilde{\pi}"] \arrow[r, hook, "\tilde{\iota}"'] & \{P_{n-1} \subset P_n^- \subset E_1^\perp\} \arrow[u, "\tilde{\pi}"] \arrow[rr, "\tilde{\alpha}"'] & & \{E_1 \subset P_n^- \subset E_1^\perp\}
\end{tikzcd} \tag{2.11}$$

and the involved maps are as follows:

(i) The maps ι, v and $\tilde{\alpha}$ are the obvious ones.

(ii) The map $\tilde{\pi}$ maps P_{n-1} to P^e_{n-1}.

(iii) The inclusion $\tilde{\imath}$ maps P_{n-1} to $P_{n-1} \subset P^o_{n-1}$.

(iv) We let $\bar{\pi}(P_{n-1} \subset P_n^-) = P^e_{n-1}$ (note that $P_n^- = P^o_{n-1}$).

(v) Further $\tilde{v}(P_n) = (P_n \cap E_1^\perp \subset (P_n \cap E_1^\perp) + E_1)$.

(vi) Finally $\alpha := \tilde{\alpha} \circ \tilde{v}$.

Theorem 2.3.3. *The left hand square is a blow-up diagram, i.e. $\bar{\pi} : \mathrm{Gr}_Y(n-1, \mathcal{S}) \longrightarrow \mathrm{OG}(n)$ is isomorphic to the blow-up $\pi : \mathrm{Bl}_Z(\mathrm{OG}(n)) \longrightarrow \mathrm{OG}(n)$ of $\mathrm{OG}(n)$ along Z and the exceptional divisor $i : E \hookrightarrow \mathrm{Bl}_Z(\mathrm{OG}(n))$ identifies with $\tilde{\imath} : \mathrm{OG}_X(n-2, E_1^\perp / E_1)) \hookrightarrow \mathrm{Gr}_Y(n-1, \mathcal{S}_Y)$.*

Proof. We have

$$\operatorname{codim}(Z \hookrightarrow \mathrm{OG}(n)) = \dim \mathrm{OG}(n) - \dim Z = \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1.$$

If $P_n \not\subset E_1^\perp$ we let $P_{n-1} = P_n \cap E_1^\perp$ and $P_n^- = P_{n-1} + E_1$ and this is unique, so $\bar{\pi}$ is an isomorphism over the open complement U. Further, the map $\bar{\pi}|_{\bar{\pi}^{-1}(Z)} : \bar{\pi}^{-1}(Z) \longrightarrow Z$ is a projective bundle of rank $n-2$ since the left hand square is cartesian. Indeed it clearly is cartesian on points which is sufficient by the arguments in [Kar01, p.25]. Moreover, the map $\tilde{\alpha}$ is a $\mathbb{P}^{n-1}$-bundle.

Clearly, $\mathrm{Gr}_Y(n-1, \mathcal{S}_Y)$ is smooth. The preimage $\pi^{-1}(Z) \subset \mathrm{Gr}_Y(n-1, \mathcal{S}_Y)$ has codimension 1, i.e. is a Cartier divisor, which gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_Y(n-1, \mathcal{S}_Y) \arrow[rr, "\phi"] \arrow[dr, "\bar{\pi}"'] & & \mathrm{Bl}_Z(\mathrm{OG}(n)) \arrow[dl, "\pi"] \\
& \mathrm{OG}(n) &
\end{tikzcd}$$

where ϕ is birational over U, since π and p are. Over Z, ϕ restricts to a surjective map $\mathbb{P}^{n-2} \longrightarrow \mathbb{P}^{n-2}$, which necessarily needs to be an isomorphism, making ϕ bijective. Since the blow-up is normal, ϕ is an isomorphism. $\square$

Definition 2.3.4. Denote by $\mathrm{B}_X(n, E_\bullet) := \mathrm{Gr}_Y(n-1, \mathcal{S}_Y)$ the blow-up of $\mathrm{OG}_X(n, \mathcal{V})$ along Z and by $E_X(n, E_\bullet) := \mathrm{OG}_X(n-2, E_1^\perp / E_1))$ the exceptional divisor.

Remark 2.3.5. On points the exceptional divisor can also be described as

$$\{(P_{n-1}, P_{n-1}^-) \in \mathrm{OG}_X(n-1, E_1^\perp/E_1) \times \mathrm{OG}_X^-(n-1, E_1^\perp/E_1) \mid \mathrm{rk}(P_{n-1} \cap P_{n-1}^-) = n-2\}$$

via the isomorphism $P_{n-1} \mapsto (P_{n-1}^e/E_1, P_{n-1}^o/E_1)$. Similarly, the blow up can alternatively be described by

$$\{(P_n, P_n^-) \in \mathrm{OG}_X(n, \mathcal{V}) \times \mathrm{OG}_X^-(n, \mathcal{V}) \mid \mathrm{rk}(P_n \cap P_n^-) = n-1,\ P_n^- \subset E_1^\perp\} \tag{2.12}$$

via $(P_{n-1} \subset P_n^-) \mapsto (P_{n-1}^e, P_n^-)$.

Theorem 2.3.6. *The map $\alpha : U \longrightarrow Y$ is an affine bundle. In particular the diagram above satisfies* [BC09, Hypothesis 1.2].

Proof. Given $P_n \dot{\not\subset} E_1^\perp$ we have $\alpha(P_n) = ((P_n \cap E_1^\perp) + E_1)$. This defines (as in [BC12a]) an $\mathbb{A}^{n-1}$-bundle. Indeed, we can imitate the arguments in loc. cit. as follows. Using isotropicness of all involved bundles one verifies that the blow-up and the exceptional divisor are given by

$$\begin{aligned} B_X(n, E_\bullet) &= \mathrm{Gr}_Y(n-1, \mathcal{S}_Y) \cong \mathbb{P}_Y(\mathcal{V}/\mathcal{S}_Y) && (2.13)\\ E_X(n, E_\bullet) &= \mathrm{Gr}_Y(n-2, \mathcal{S}_Y/E_1) \cong \mathbb{P}_Y(E_1^\perp/\mathcal{S}_Y) && (2.14)\end{aligned}$$

and under the isomorphism

$$U \cong B_X(n, E_\bullet) \setminus E_X(n, E_\bullet) \cong \mathbb{P}_Y(\mathcal{V}/\mathcal{S}_Y) \setminus \mathbb{P}_Y(E_1^\perp/\mathcal{S}_Y)$$

α corresponds to the structure moprhism to Y. Hence, α is an $\mathbb{A}^{n-1}$-bundle. □

Lemma 2.3.7. *Let $n \geq 2$ and $\underline{\lambda} \in \mathfrak{E}_{n-1}$ such that $\lambda_1 < n-1$, i.e. the last column is empty and thus $\overline{v}$ is well-defined on $\underline{\lambda}$. Then the base-changes to U of $f_{\underline{\lambda}}$ and $f^-_{\overline{v}(\underline{\lambda})}$ coincide, in the sense that in the diagram*

$$\begin{tikzcd} \mathrm{OG}_X(n, \mathcal{V}) & U \arrow[l, hook', "v"'] \arrow[r, "\alpha"] & \mathrm{OG}_X^-(n-1, E_1^\perp/E_1) \\ \mathrm{OFl}_X(\underline{\lambda}, E_\bullet) \arrow[u, "f_{\underline{\lambda}}"] & \tilde{U} \arrow[l, hook', "v'"] \arrow[r, "\alpha'"'] \arrow[u, "f"] & \mathrm{OFl}_X^-(\overline{v}(\underline{\lambda}), E'_\bullet) \arrow[u, "f^-_{\overline{v}(\underline{\lambda})}"'] \end{tikzcd} \tag{2.15}$$

both squares are cartesian.

Proof. First assume that $d_k \equiv n \mod 2$ and $\underline{\lambda}$ comes with k-tuples $\underline{d}$ and $\underline{e}$. Associated with $\overline{v}(\underline{\lambda})$ we have the k-tuples $\underline{d}$ and $\underline{e}-1$, where $(\underline{e}-1)_i = e_i - 1$. In Example 2.1.1 we already saw that this partition $\overline{v}(\underline{\lambda})$ defines a Schubert variety in the opposite component $\mathrm{OG}^-_X(n-1, E_1^\perp/E_1)$. Also note that by assumption $e_i \geq 1$ for all i. The left hand square from (2.15) is given on points by

$$\begin{tikzcd}
\{P_n \subset \mathcal{V}\} & & \{P_n \dot{\not\subset} E_1^\perp\} \arrow[ll, hook', "v"'] \\
\left\{\begin{matrix} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_n \\ \cap & & \cap & \\ E_{d_1+e_1} & \subset \ldots \subset & E_{d_k+e_k} & \end{matrix}\right\} \arrow[u, "f_{\underline{\lambda}'}"] & & \left\{\begin{matrix} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_n \\ \cap & & \cap & \dot{\not\cap} \\ E_{d_1+e_1} & \subset \ldots \subset & E_{d_k+e_k} & E_1^\perp \end{matrix}\right\} =: \tilde{U} \arrow[ll, hook', "v'"] \arrow[u, "f"']
\end{tikzcd}$$

and clearly is cartesian. Let us compute the pullback on the left hand side and compare it with $\tilde{U}$. We have

$$\begin{tikzcd}
\{P_n \dot{\not\subset} E_1^\perp\} \arrow[rr, "\alpha"] & & \{P^-_{n-1} \subset E_1^\perp/E_1\} \\
\tilde{U}' := \left\{\begin{matrix} P_{d_1} & \subset \ldots \subset P_{d_k} & \subset P^-_{n-1}, & P_n \\ \cap & & \cap & \dot{\not\cap} \\ E'_{d_1+e_1-1} & \subset \ldots \subset & E'_{d_k+e_k-1} & E_1^\perp \end{matrix}\right\} \arrow[u, "f'"] \arrow[rr, "\alpha'"] & & \left\{\begin{matrix} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P^-_{n-1} \\ \cap & & \cap & \\ E'_{d_1+e_1-1} & \subset \ldots \subset & E'_{d_k+e_k-1} & \end{matrix}\right\} \arrow[u, "f^-_{\overline{v}(\underline{\lambda})}"']
\end{tikzcd}$$

and it remains to show that $\tilde{U} \cong \tilde{U}'$. Note that $P_{d_i} \subset E_{d_i+e_i} \subset E_1^\perp$ for all $i \leq i \leq k$ and $E_1 \dot{\not\subset} P_{d_i}$ for none of the subbundles in $\tilde{U}$ since this would contradict the condition on P_n. Hence,

$$\begin{aligned}
\Phi : \tilde{U} \;&\longrightarrow \tilde{U}' \\
((P_{d_i}), P_n) \;&\mapsto\; (((P_{d_i}+E_1)/E_1), ((P_n \cap E_1^\perp)+E_1)/E_1, P_n), \\
(((P_{d_i}+E_1)\cap P_n), P_n) \;&\leftarrow\!\shortmid\; ((P_{d_i}), P^-_{n-1}, P_n)
\end{aligned}$$

is the desired isomorphism where, by abuse of notation, for a subbundle $P \subset E_1^\perp/E_1$ we again denote by P its preimage under the projection $E_1^\perp \to E_1^\perp/E_1$. If $d_k \equiv n+1 \mod 2$, apply the proof to the extended tuples $\underline{d}^*, \underline{e}^*$. $\square$

Lemma 2.3.8. *Let $n \geq 3$ and $\underline{\lambda}'' \in \mathfrak{S}_{n-2}$ even such that d_1'' is odd, i.e. $\overline{\partial}$ is well-defined on $\underline{\lambda}''$. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{OG}_X(n-1, E_1^{\perp}/E_1) & E_X(n, E_\bullet) \arrow[l, "\tilde{\pi}"'] \arrow[r, "\tilde{\alpha}\tilde{\iota}"] & \mathrm{OG}_X^-(n-1, E_1^{\perp}/E_1) \\
\mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''), E'_\bullet) \arrow[u, "f_{\overline{\partial}(\underline{\lambda}'')}"] & \tilde{E} \arrow[u, "\tilde{f}"] \arrow[r, "\tilde{\alpha}'"'] & \mathrm{OFl}_X^-(\underline{\lambda}'', E'_\bullet) \arrow[u, "f^-_{\underline{\lambda}''}"'] \\
& F \arrow[u, "q"] \arrow[ul, "p"] &
\end{tikzcd} \tag{2.16}$$

where the right hand square is cartesian and the lax push-forwards along both p and q preserve the unit form up to lax-similitude.

Proof. This lemma will be essential in the proof of the Main Theorem. The statement is similar to [BC12a, 5.8], but unlike in the preceeding lemma, we can not imitate the proof, since $\overline{\partial}$ acts differently on shifted Young diagrams than on unshifted Young diagrams.

First, observe that we only need to deal with the first kind of resolution in Figure 2.2. Any strict partition of $n-2$ with odd d_1 automatically satisfies that d_k is odd and $n-1$ is even by Lemma 1.5.11. Hence the corresponding Schubert cell lies in $\mathrm{OG}^-(n-1)$. Let now $\underline{d}$ and $\underline{e}$ be the k-tuples associated with $\underline{\lambda}''$. On points, the right hand square of (2.16) becomes

$$\begin{tikzcd}
\left\{\begin{matrix} P_{n-1}, P_{n-1}^- \subset E_1^{\perp}/E_1, \\ \mathrm{rk}(P_{n-1} \cap P_{n-1}^-) = n-2 \end{matrix}\right\} \arrow[r, "\tilde{\alpha}\tilde{\iota}"] & \left\{P_{n-1}^- \subset E_1^{\perp}/E_1\right\} \\
\left\{\begin{matrix} \left(\left(\begin{matrix} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_{n-1}^- \\ \cap & & \cap & \\ E'_{d_1+e_1} & \subset \ldots \subset & E'_{d_k+e_k} & \end{matrix}\right), P_{n-1}\right), \\ \mathrm{rk}(P_{n-1} \cap P_{n-1}^-) = n-2 \end{matrix}\right\} \arrow[u, "\tilde{f}"] \arrow[r, "\tilde{\alpha}'"'] & \left\{\begin{matrix} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_{n-1}^- \\ \cap & & \cap & \\ E'_{d_1+e_1} & \subset \ldots \subset & E'_{d_k+e_k} & \end{matrix}\right\} \arrow[u, "f_{\underline{\lambda}''}"']
\end{tikzcd}$$

where all the maps are the obvious ones. Here we used the characterization of the exceptional fiber as given in Remark 2.3.5. Let us first explain Figure 2.4, which illustrates the strategy of the proof. The upper left and right pictures O and O' describe the resolutions of the Schubert schemes corresponding to $\overline{\partial}(\underline{\lambda}'')$ and $\underline{\lambda}''$, where we simplified the quivers for the reader's convenience. The aim is to construct a common resolution of O and the pull-back $\tilde{E}$ of O' along $\tilde{\alpha}\tilde{\iota}$ and this resolution should be a composition of maps, each of which preserves the unit form when pushed forward along. Note that in type A, there is a direct map $\tilde{E} \to O$ ([BC12a, 5.8]) but this is not true anymore in the orthogonal case.

Write $\tilde{E} = \mathrm{Gr}_{\mathrm{OFl}_X^-(\underline{\lambda}'')}(n-2, \mathcal{S}_{n-1}^-)$ where $\mathcal{S}_{n-1}^-$ denotes the top tautological bundle on $\mathrm{OFl}_X^-(\underline{\lambda}'')$. Consider the locus $Z_k := \mathrm{Gr}_{\mathrm{OFl}_X(\underline{\lambda})}(n-2-d_k, \mathcal{S}_{n-1}^-/\mathcal{S}_{d_k}) \subset \tilde{E}$, which on points is given by

$$Z_k = \left\{ \begin{matrix} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_{n-2} \\ \cap & & \cap & \cap \\ E'_{d_1+e_1} & \subset \ldots \subset & E'_{d_k+e_k} & P_{n-1}^- \end{matrix} \right\} = \left\{ (P_{d_1}, \ldots, P_{d_k}, P_{n-1}^-, P_{n-1}) \in \tilde{E} \mid P_{d_k} \subset P_{n-1} \right\}.$$

Let $q_k : F_k := \mathrm{Bl}_{Z_k}(\tilde{E}) \longrightarrow \tilde{E}$ be the blow-up of $\tilde{E}$ along Z_k. On points F_k is given by

$$F_k = \left\{ \begin{matrix} & & Q_{d_k-1} & \subset P_{n-1} \\ & & \cap & \\ P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_{n-1}^- \\ \cap & & \cap & \\ E'_{d_1+e_1} & \subset \ldots \subset & E'_{d_k+e_k} & \end{matrix} \;\middle|\; \mathrm{rk}(P_{n-1} \cap P_{n-1}^-) = n-2 \right\}$$

To see this, observe that $q_k|_{F_k \setminus Z_k}$ is an isomorphism, since outside the center the preimage $Q_{d_k-1} = P_{d_k} \cap P_n - 1$ is uniquely determined. Over the center q_k is a $\mathbb{P}^{d_k-1}$ bundle and we have $\mathrm{codim}(Z_k \hookrightarrow \tilde{E}) = d_k$. In particular, q_k identifies with a blow-up along a center of odd codimension d_k. Let Z_{k-1} be the closed subscheme of F_k defined by the tower

$$\begin{aligned} Z_{k-1} \quad = \quad & \mathrm{Gr}_{F_k^1}(n-1-d_k, \mathcal{S}_{n-1}^-/\mathcal{T}'_{d_k-1}) \\ \longrightarrow \; & F_k^1 := \mathrm{Gr}_{\mathrm{OFl}_X^-(\underline{\lambda}'')}(d_k - d_{k-1} - 1, \mathcal{S}_{d_k}/\mathcal{S}_{d_{k-1}}) \\ \longrightarrow \; & \mathrm{OFl}_X^-(\underline{\lambda}''), \end{aligned}$$

where in the first line we used $\mathcal{T}_{d_k-d_{k-1}-1} = \mathcal{S}_{d_k}/\mathcal{S}_{d_{k-1}}$ and wrote $\mathcal{T}'_{d_k-1}$ for the pull-back of $\mathcal{T}_{d_k-d_{k-1}-1}$ under

$$\left\{ \mathcal{W} \in \mathrm{Gr}_{\mathrm{OFl}_X^-(\underline{\lambda}'')}(d_k - 1, \mathcal{S}_{d_k}) \;\middle|\; \mathcal{S}_{d_{k-1}} \subset \mathcal{W} \right\} \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{OFl}_X^-(\underline{\lambda}'')}(d_k - d_{k-1} - 1, \mathcal{S}_{d_k}/\mathcal{S}_{d_{k-1}})$$

(on points, this precisely means that $P_{d_{k-1}} \subset Q_{d_k-1}$). Let

$$q_{k-1} : F_{k-1} := \mathrm{Bl}_{Z_{k-1}}(F_k) \longrightarrow F_k$$

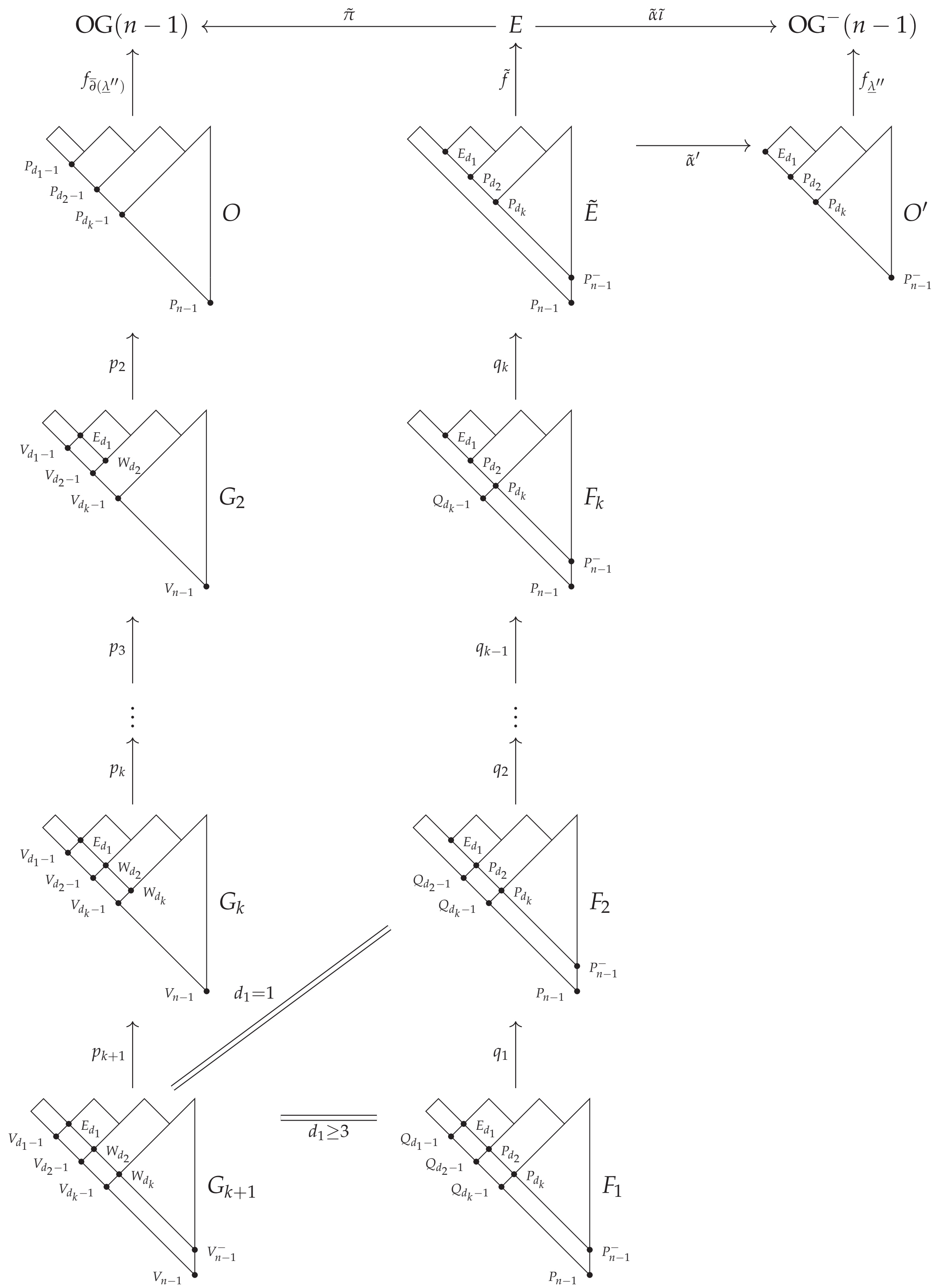


FIGURE 2.4. Sketch of proof. Write $O := \mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''))$ and $O' = \mathrm{OFl}_X(\underline{\lambda}'')$. From the common refined resolution we have two towers of blow-ups, where the centers of the blow-ups q_i have odd codimension d_i and of the blow-ups p_i have odd codimension $e_i + 1$.

be the blow-up of F_k along Z_{k-1}. As before we see that, on points, F_{k-1} is given by

$$F_{k-1} = \left\{ \begin{array}{ccccc|c} & & Q_{d_{k-1}-1} & \subset Q_{d_k-1} \subset P_{n-1} & & \\ & & \cap & \cap & & \\ P_{d_1} \subset \ldots \subset & & P_{d_{k-1}} & \subset \quad P_{d_k} \quad \subset P^-_{n-1} & & \mathrm{rk}(P_{n-1} \cap P^-_{n-1}) = n-2 \\ \| & & \cap & \cap & & \\ E'_{d_1} \subset \ldots \subset & & E'_{d_{k-1}+e_{k-1}} & \subset E'_{d_k+e_k} & & \end{array} \right\}$$

and $\mathrm{codim}(Z_{k-1} \hookrightarrow F_k) = d_{k-1}$.

It is immediate, that $\mathrm{codim}(Z_{i-1} \hookrightarrow F_i) = d_i$. Finally let

$$F := \begin{cases} F_1, & \text{if } d_1 \geq 3, \\ F_2, & \text{if } d_1 = 1. \end{cases}$$

and $q : F \longrightarrow \tilde{E}$ be the composition of all the q_i. Since q is the composition of several blow-ups, all of which are along centers of odd codimension, the lax push-forward along q preserves the unit form by [BC12a, 3.15 (b)].

Let us now switch to the other side. The tuples of $\overline{\partial}(\underline{\lambda}'')$ are given by $\underline{d} - 1$ and $\underline{e} + 1$, where both are truncated from their first entry in the case $d_1 = 1$. The Schubert cell and its resolution are given by

$$Y_X(\overline{\partial}(\underline{\lambda}''), E'_\bullet) = \left\{ \begin{array}{c} V_{n-1} \in \mathrm{OG}(n-1), \text{ s.t.} \\ \dim(V_{n-1} \cap E'_{d_i+e_i}) \geq d_i - 1 \end{array} \right\},$$

$$\mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''), E'_\bullet) = \left\{ \begin{array}{ccc} V_{d_1-1} \subset \ldots \subset & V_{d_k-1} & \subset V_{n-1} \\ \cap & & \cap \\ E'_{d_1} & \subset \ldots \subset & E'_{d_k+e_k} \end{array} \right\}$$

and over $\mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''), E'_\bullet)$ there are tautological subbundles $\mathcal{S}_{d_1-1}, \ldots, \mathcal{S}_{d_k-1}, \mathcal{S}_{n-1}$ (where $\mathcal{S}_{d_1-1}$ does not appear in the case $d_1 = 1$). Now while in the type A setting in [BC12a] we would be able to construct a morphism $\tilde{E} \longrightarrow \mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''), E'_\bullet)$ this is not possible in this setting. Instead we will have a map $p : F \longrightarrow \mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''), E'_\bullet)$.

We will now describe a similar construction as seen before, this time over $\mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''))$ in order to write p as consecutive blow-ups. Note that after change of letters F is given on points by

$$G_{k+1} := F = \left\{ \begin{array}{ccccc|c} V_{d_1-1} \subset \ldots \subset & V_{d_{k-1}-1} & \subset V_{d_k-1} & \subset V_{n-1} & & \\ \cap & \cap & \cap & & & \\ W_{d_1} \subset \ldots \subset & W_{d_{k-1}} & \subset W_{d_k} & \subset V^{-}_{n-1} & & \mathrm{rk}(V_{n-1} \cap V^{-}_{n-1}) = n-2 \\ \| & \cap & \cap & & & \\ E'_{d_1} \subset \ldots \subset & E'_{d_{k-1}+e_{k-1}} & \subset E'_{d_k+e_k} & & & \end{array} \right\}$$

(if $d_1 = 1$ we need to omit V_{d_1-1} and we obtain $G_k = F$). Forgetting all the W_{d_i} and V^{-}_{n-1} gives us a map to $\mathrm{OFl}_X(\overline{\partial}(\underline{\lambda}''))$. We now show that each step is a blow-up along a suitably chosen center of odd codimension.

- Dropping V^{-}_{n-1} identifies with the blow-up along the locus $\{W_{d_k} \subset V_{n-1}\}$. This is the same setup as in the beginning of the section (this time over a different base), namely the inclusion $\mathrm{OG}(n-1-d_k) \subset \mathrm{OG}(n-d_k)$ which is of odd codimension $n - d_k - 1 = e_{k+1} + 1$. Denote this blow-up by $p_{k+1} : G_{k+1} \longrightarrow G_k$.
- Similarly with no new arguments one shows that $p_{i+1} : G_{i+1} \longrightarrow G_i$ which drops $W_{d_{i+1}}$ (for $1 \leq i \leq k-1$) identifies with the blow-up of G_i along the closed locus $\{W_{d_i} \subset V_{d_{i+1}-1}\}$ which is of odd codimension $e_i + 1$.

Hence the lemma is proven. □

2.4 Main Theorem

Theorem 2.4.1 (Main Theorem). *Let $n \geq 1$ and $\mathcal{V}$ be an orthogonal vector bundle of rank $2n$. Assume that $\mathcal{V}$ admits a complete isotropic flag $E_\bullet$. Then the elements $\{\phi_n(\underline{\lambda})\}_{\underline{\lambda} \in \mathfrak{S}_{n-1}}$ defined in Definition 2.2.7 form a total basis of the Witt groups of $\mathrm{OG}_X(n, \mathcal{V})$ and the generators have degree $|\lambda|$ and trivial twist.*

We will prove this theorem, by induction on n, in the same way as in [BC12a] using the long exact localization sequence

$$\ldots \to W^i_Z(\mathrm{OG}(n), \mathcal{L}) \xrightarrow{e} W^i(\mathrm{OG}(n), \mathcal{L}) \xrightarrow{v^*} W^i(U, \mathcal{L}|_U) \xrightarrow{\partial} W^{i+1}_Z(\mathrm{OG}(n), \mathcal{L}) \to \ldots, \tag{2.17}$$

with Z, U and v as introduced in section 2.3, the extension of support map e from section 1.2.5 and Theorem 1.2.19. First, consider some low-dimensional cases.

Example 2.4.2. For $n \in \mathbb{N}$ let $\mathcal{V}$ be an orthogonal vector bundle of rank $2n$ admitting a complete isotropic flag.

(i) For $n = 1$ we have $\mathrm{OG}_X(1, \mathcal{V}_2) = X$ and $\{1_X\}$ is a total basis of the Witt group over X.

(ii) For $n = 2$ we have $\mathrm{OG}_X(2, \mathcal{V}_4) \cong \mathbb{P}(E_2^-)$ as in Example 1.4.4(ii) and the constructed basis (of order two) coincides with the one in [BC12a].

Hence, we are left to prove the induction step, assuming the induction hypothesis is true. A key observation is that all the maps in (2.17), in particular the boundary map, are compatible with the constructed basis, in the sense that the basis elements are mapped to each other (up to lax-similitude).

Proposition 2.4.3 (Analogon of [BC12a, 6.8]**).** *(i) Let $n \geq 3$ and $\underline{\lambda}' \in \mathfrak{E}_{n-2}$ with k-tuples $\underline{d}$ and $\underline{e}$ such that e_1 is even. Then*

$$\iota_*(\phi_{n-1}(\underline{\lambda}')) \leftrightsquigarrow \phi_n(\bar{\iota}(\underline{\lambda}')). \tag{2.18}$$

(ii) Let $n \geq 3$ and $\underline{\lambda} \in \mathfrak{E}_{n-1}$ with k-tuples $\underline{d}$ and $\underline{e}$ such that $e_1 > 0$. Then

$$v^*(\phi_n(\underline{\lambda})) \leftrightsquigarrow \alpha^*(\phi_{n-1}^-(\overline{v}(\underline{\lambda}))). \tag{2.19}$$

(iii) Let $n \geq 3$ and $\underline{\lambda}'' \in \mathfrak{E}_{n-2}$ with k-tuples $\underline{d}$ and $\underline{e}$ such that d_1 is odd. Then

$$\partial(\alpha^*(\phi_{n-1}^-(\underline{\lambda}''))) \leftrightsquigarrow (\iota_Z)_*(\phi_{n-1}(\overline{\partial}(\underline{\lambda}''))). \tag{2.20}$$

Proof. (i) There is an obvious bijection of isotropic subbundles $P_m \subset E'_p = E_{p+1}/E_1$ of $E_1^\perp/E_1$ and isotropic subbundles $E_1 \subset P_{m+1} \subset E_{p+1}$ of $\mathcal{V}$. Hence there is a diagram

$$\begin{tikzcd}
\mathrm{OG}(n-1) = \{P_{n-1} \subset E_1^\perp/E_1\} \arrow[r, hook, "\iota"] & \{P_n \subset \mathcal{V}\} = \mathrm{OG}(n) \\
\left\{\begin{matrix} P_{d_1} & \subset \ldots \subset & P_{d_k} & \subset P_{n-1} \\ \cap & & \cap & \cap \\ E'_{d_1+e_1} & \subset \ldots \subset & E'_{d_k+e_k} & E_1^\perp \end{matrix}\right\} \arrow[u, "f_{\underline{\lambda}'}"] \arrow[r, equal] & \left\{\begin{matrix} P_{d_1+1} & \subset \ldots \subset & P_{d_k+1} & \subset P_n \\ \cap & & \cap & \\ E_{d_1+e_1+1} & \subset \ldots \subset & E_{d_k+e_k+1} & \end{matrix}\right\} \arrow[u, "f_{\bar{\iota}(\underline{\lambda}')}"']
\end{tikzcd}$$

and base-change (Theorem 1.2.12) yields (i).

(ii) This follows directly from Lemma 2.3.7.

(iii) This time use Lemma 2.3.8. Note that by Lemma 1.5.11 for such a partition $n-1 \equiv 0 \mod 2$, i.e. [BC09, 1.4(B)] applies here the same way as in [BC12a, 6.8 (c)]. Explicitely we have

$$\begin{aligned}
\partial(\alpha^*(\phi^-_{n-1}(\underline{\lambda}''))) = \partial(\alpha^*(f^-_{\underline{\lambda}''})_*(1)) &\overset{1.4}{=} ((\iota_Z)_* \circ \tilde{\pi}_* \circ \tilde{\iota}^* \circ \tilde{\alpha}^* \circ (f^-_{\underline{\lambda}''})_*)(1) \\
&\overset{(bc)}{=} ((\iota_Z)_* \circ \tilde{\pi}_* \circ \tilde{f}_*)(1) \\
&\overset{(1)}{=} ((\iota_Z)_* \circ \tilde{\pi}_* \circ \tilde{f}_* \circ q_*)(1) \\
&\overset{(c)}{=} ((\iota_Z)_* \circ (f_{\overline{\partial}(\underline{\lambda}'')})_* \circ p_*)(1) \\
&\overset{(2)}{\rightsquigarrow} ((\iota_Z)_* \circ (f_{\overline{\partial}(\underline{\lambda}'')})_*)(1) \\
&= (\iota_Z)_*(\phi_{n-1}(\overline{\partial}(\underline{\lambda}'')))
\end{aligned}$$

where the unit form 1 is in the right Witt groups respectively. Here (bc) follows by base-change, (c) by commutativity of the diagram in Lemma 2.3.8 and (1) resp. (2) by the push-forward properties of the two maps constructed in the proof of Lemma 2.3.8. □

Lemma 2.4.4. *Let $n \geq 3$ and consider the commutative triangle*

$$\begin{tikzcd}
W^i_Z(\mathrm{OG}_X(n,\mathcal{V}),\mathcal{L}) \arrow[rr, "e"] & & W^i(\mathrm{OG}_X(n,\mathcal{V}),\mathcal{L}) \\
& W^{i+1-n}(Z,\omega_\iota \otimes \mathcal{L}|_Z) \arrow[ul, "(\iota_Z)_*", "\sim"'] \arrow[ur, "\iota_*"'] &
\end{tikzcd} \tag{2.21}$$

Then the elements

$$(\iota_Z)_*(\phi_{n-1}(\underline{\lambda}')), \qquad \underline{\lambda}' \in \mathfrak{E}_{n-2}$$

form a total basis of $W^{\mathrm{tot}}_Z(\mathrm{OG}_X(n,\mathcal{V}))$.

Proof. By induction, the $\phi_{n-1}(\underline{\lambda}')$ for even shifted partitions $\underline{\lambda}' \in \mathfrak{E}_{n-2}$ form a total basis of $W(Z)$. Since $n \geq 3$, the map $\iota^* : \mathrm{Pic}(\mathrm{OG}_X(n,\mathcal{V}))/2 \longrightarrow \mathrm{Pic}(Z)/2$ is an isomorphism, so we can apply Lemma 1.2.21 to obtain the claim. □

proof of Main Theorem. Let again $n \geq 3$. From $\mathrm{codim}(Z \hookrightarrow \mathrm{OG}_X(n,\mathcal{V})) \geq 2$ we see that

$$v^* : \mathrm{Pic}(\mathrm{OG}_X(n,\mathcal{V}))/2 \longrightarrow \mathrm{Pic}(U)/2$$

is injective. By induction and the preceeding lemma, we have that the $\phi_{n-1}(\underline{\lambda}')$ form a basis for $W(Z)$ and by homotopy invariance, the elements $\alpha^*(\phi^-_{n-1}(\underline{\lambda}''))$ form a basis for

$W(U)$ (where $\underline{\lambda}', \underline{\lambda}''$ run over $\mathfrak{E}_{n-2}$). Now apply Theorem 1.2.19 to the following index sets

$$\begin{aligned}
\mathcal{I} = \left\{\underline{\lambda}' \in \mathfrak{E}_{n-2} \text{ even with } e_1' \text{ even}\right\} &\xrightarrow{\bar{\iota}} \left\{\underline{\lambda} \in \mathfrak{E}_{n-1} \text{ even with } e_1 = 0\right\} \\
\mathcal{J} = \left\{\underline{\lambda} \in \mathfrak{E}_{n-1} \text{ even with } e_1 > 0\right\} &\xrightarrow{\bar{v}} \left\{\underline{\lambda}'' \in \mathfrak{E}_{n-2} \text{ even with } d_1'' \text{ even}\right\} \\
\mathcal{K} = \left\{\underline{\lambda}'' \in \mathfrak{E}_{n-2} \text{ even with } d_1'' \text{ odd}\right\} &\xrightarrow{\bar{\partial}} \left\{\underline{\lambda}' \in \mathfrak{E}_{n-2} \text{ even with } e_1' \text{ odd}\right\}
\end{aligned}$$

and the Witt classes

$$\begin{aligned}
W_Z^{\mathrm{tot}}(\mathrm{OG}_X(n,\mathcal{V})) &\ni \begin{cases} v_{\underline{\lambda}'} = (\iota_Z)_*(\phi_{n-1}(\underline{\lambda}')), \\ v'_{\underline{\lambda}''} = (\iota_Z)_*(\phi_{n-1}(\bar{\partial}(\underline{\lambda}''))) \leftrightsquigarrow \partial(\alpha^*(\phi_{n-1}(\underline{\lambda}''))), \end{cases} \\
W^{\mathrm{tot}}(\mathrm{OG}_X(n,\mathcal{V})) &\ni \begin{cases} w_{\underline{\lambda}} = \phi_n(\underline{\lambda}), \\ w'_{\underline{\lambda}'} = \phi_n(\bar{\iota}(\underline{\lambda}')) \leftrightsquigarrow \iota_*(\phi_{n-1}(\underline{\lambda}')), \end{cases} \\
W^{\mathrm{tot}}(U) &\ni \begin{cases} u_{\underline{\lambda}''} = \alpha^*(\phi_{n-1}^-(\underline{\lambda}'')), \\ u'_{\underline{\lambda}} = \alpha^*(\phi_{n-1}^-(\bar{v}(\underline{\lambda}))) \leftrightsquigarrow v^*(\phi_n(\underline{\lambda})) \end{cases}
\end{aligned}$$

for $\underline{\lambda}' \in \mathcal{I}$, $\underline{\lambda} \in \mathcal{J}$ and $\underline{\lambda}'' \in \mathcal{K}$ (see Propositions 1.5.14 and 2.4.3). Then we see that $\mathcal{I} \cup \bar{\partial}(\mathcal{K})$ resp. $\bar{v}(\mathcal{J}) \cup \mathcal{K}$ parametrize a basis of the Witt groups $W(Z)$ resp. $W(U)$. Hence, a basis of $W(\mathrm{OG}_X(n,\mathcal{V})$ is parametrized by $\left\{\underline{\lambda} \in \mathfrak{E}_{n-1} \text{ even }\right\} = \mathcal{J} \cup \bar{\iota}(\mathcal{I})$ which finishes the proof. $\square$

Chapter 3

Witt groups of Lagrangian Grassmann bundles

In this chapter we compute the Witt groups of Lagrangian Grassmann bundles, i.e. maximal isotropic Grassmann bundles with respect to a non-degenerate *antisymmetric* bilinear form. This scheme seems very similar to the orthogonal analogon discussed in the previous chapter, so one might think that calculations go through with minor changes. Unfortunately, this turns out to be false - already the localization sequence is much harder to understand due to the connecting homomorphism which cannot be directly computed using the methods of section 1.2.8. Instead, a second blow-up in this setup is necessary and this allows us again to describe the boundary map as composition of pull-backs and push-forwards along the involved maps. Fortunately, even though the computation requires some work, the map turns out to behave well, as is it is an isomorphism whenever it does not vanish.

In the first three sections of the chapter we establish the new setup and compute the boundary map in all the cases we need. A *recursive* description of the Witt groups is given at the end of section 3.3. In the remaining section we try to imitate the procedure from the last chapter by constructing some explicit elements via lax push-forwards from suitable resolutions of Schubert cells, this time using *almost even shifted partitions*. Due to the more delicate structure of the boundary map, a compatible resolution in the sense that we may apply Theorem 1.2.19 is harder to construct. Unfortunately, it is still unclear to the author how these elements could be compatible, although it is believed to be true.

3.1 The extended blow-up setting

In this section we discuss a decomposition of the Lagrangian Grassmann bundle as it was done in type A ([BC12a]) and D (chapter 2).

Let $(\mathcal{V}, \omega)$ be a symplectic vector bundle of rank $2n$ over a smooth connected scheme X containing $\frac{1}{2}$ where ω admits values in the line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. Denote by $\mathcal{S}_n \subset p^*\mathcal{V}$ the tautological subbundle over $p : \mathrm{LG}_X(n, \mathcal{V}) \longrightarrow X$. Fix an isotropic flag $E_\bullet$ of $\mathcal{V}$ and consider the closed subscheme

$$Z = \{\mathcal{W} \in \mathrm{LG}_X(n, \mathcal{V}) \,|E_1 \subset \mathcal{W} \subset E_1^\perp\} \cong \mathrm{LG}_X(n-1, E_1^\perp/E_1).$$

Note that as in types A and D such a flag does not necessarily exist. Denote by $U := \mathrm{LG}_X(n, \mathcal{V}) \setminus Z$ the open complement of Z.

Definition 3.1.1 (Odd symplectic Grassmannians). The symplectic form ω on $\mathcal{V}$ restricts to an *odd symplectic form* on the codimension one subbundle $E_1^\perp = E_{2n-1}$, i.e. ω restricts to a antisymmetric bilinear form with one-dimensional kernel E_1. For $1 \leq k \leq n$, we define the scheme $\mathrm{SG}_X(k, E_1^\perp)$ parametrizing isotropic subbundles of rank k of $E_1^\perp$. Note that $\mathrm{SG}_X(k, E_1^\perp)$ comes equipped with a tautological bundle $\mathcal{S}_k$ and may be viewed as a smooth Schubert variety inside $\mathrm{LG}_X(k, \mathcal{V})$. For more details on odd symplectic Grassmannians we refer to [Pec13].

Now let $B = \mathrm{SG}_X(n-1, E_1^\perp)$ and define $\tilde{X} := \mathrm{LG}_B(1, S_{n-1}^\perp/S_{n-1})$ (cf. Example 1.4.5). In particular $\tilde{X}$ is a projective bundle of rank 1 over B. Further, let $E := \mathrm{Gr}_Z(n-1, \mathcal{S}_n)$. We can establish a setup as in Setup 1.2.14 given by

$$\begin{tikzcd}
Z \arrow[r, hook, "\iota"] & \mathrm{LG}_X(n) & & U \arrow[ll, hook', "v"'] \arrow[dll, "\tilde{v}"] \arrow[d, "\alpha"] \\
E \arrow[u, "\pi'"] \arrow[r, hook, "\tilde{\iota}"'] & \tilde{X} \arrow[u, "\pi"'] & & \mathrm{LG}_X(n-1, E_1^\perp/E_1) =: Y
\end{tikzcd} \tag{3.1}$$

which on points is given by

$$\begin{tikzcd}
\{E_1 \subset P_n \subset E_1^\perp\} \arrow[r, hook, "\iota"] & \{P_n \subset \mathcal{V}\} & \{P_n \dot{\not\subset} E_1^\perp\} \arrow[l, hook', "v"'] \arrow[dl, "\tilde{v}"'] \arrow[d, "\alpha"] \\
\{P_{n-1} \subset P_n \subset E_1^\perp\} \arrow[u, "\pi'"] \arrow[r, hook, "\tilde{\iota}"'] & \{P_{n-1} \subset P_n \mid P_{n-1} \subset E_1^\perp\} \arrow[u, "\pi"'] & \{E_1 \subset P_n' \subset E_1^\perp\}
\end{tikzcd} \tag{3.2}$$

Here all maps, except for $\tilde{v}$ and α, are the obvious ones and we have

$$\begin{aligned} \tilde{v}(P_n) &= (P_n \cap E_1^{\perp} \subset P_n), \\ \alpha(P_n) &= (P_n \cap E_1^{\perp}) + E_1. \end{aligned}$$

Remark 3.1.2. (i) The left square in the diagram above is a blow-up square, where ι is a regular closed embedding of codimension n; this is shown as in the orthogonal case. Write $\mathrm{B}_X(n, E_\bullet) := \tilde{X}$ and $E_X(n, E_\bullet) := E$.

(ii) The map α is an $\mathbb{A}^n$-bundle and does not canonically extend to a morphism defined on $\tilde{X}$.

Recall that the Lagrangian Grassmannian is described as a homogeneous space by the quotient C_n/P_n, where P_n denotes the maximal parabolic subgroup associated with the right end root. The decomposition $\mathrm{LG}_X(n, \mathcal{V}) = Z \,\dot{\cup}\, U$ corresponds to the decomposition of C_n/P_n into the two orbits for the action of the maximal parabolic subgroup P_1 corresponding to α_1 (see [Pec14]). Now the extension of α to the blow-up fails as the induced action on $\tilde{X}$ admits three orbits, with a closed and smooth "invisible" orbit $Z' \subset E$, which on points is given by

$$Z' = \{E_1 \subset P_{n-1} \subset P_n\}. \tag{3.3}$$

The fact that α does not extend to the blow-up makes the computation of the boundary map in the long exact localization sequence of the embedding ι harder. We will need another blow-up to obtain an auxillary morphism. An intuitive attempt to fix this problem above is to blow up $\tilde{X}$ along Z', where $\iota' := \tilde{\iota} \circ \tilde{\tilde{\iota}} : Z' \hookrightarrow \tilde{X}$ is an embedding of codimension 2. This yields the extended diagram

$$\begin{tikzcd}
 & Z \arrow[r, hook, "\iota"] & \mathrm{LG}_X(n) & U \arrow[l, hook', "v"'] \arrow[dl, "\tilde{v}"] \arrow[d, "u"] \arrow[ddl, "\tilde{\tilde{v}}"] \arrow[dd, bend left, "\alpha"] \\
Z' \arrow[r, hook, "\tilde{\tilde{\iota}}"] \arrow[rr, hook, bend right, "\iota'"'] & E \arrow[u, "\pi'"] \arrow[r, hook, "\tilde{\iota}"] & \tilde{X} \arrow[u, "\pi"] & U' \arrow[l, hook', "v'"'] \arrow[dl, "\tilde{v}'"] \\
E' \arrow[u, "\tilde{\pi}'"] \arrow[rr, hook, "\tilde{\iota}'"'] & & \tilde{\tilde{X}} \arrow[u, "\tilde{\pi}"] \arrow[r, "\tilde{\alpha}"'] & Y
\end{tikzcd} \tag{3.4}$$

with $U' = \tilde{X} \setminus Z$ and the exceptional divisors E, E' of the blow-ups. On points the diagram is given as follows where the new maps are the obvious ones:

$$\begin{tikzcd}
& \{P_n \subset E_1^\perp\} \arrow[r, hook, "\iota"] & \{P_n \subset \mathcal{V}\} & \{P_n \dot{\not\subset} E_1^\perp\} \arrow[l, hook', "v"'] \arrow[dl, hook, "\tilde{v}"'] \arrow[d, hook, "u"] \arrow[ddl, hook, bend right, "\tilde{\tilde{v}}"] \arrow[dd, bend left, "\alpha"] \\
\begin{Bmatrix} P_{n-1} \subset P_n \\ \cup \\ E_1 \end{Bmatrix} \arrow[r, hook, "\tilde{i}"] & \begin{Bmatrix} P_{n-1} \subset P_n \\ \cap \\ E_1^\perp \end{Bmatrix} \arrow[u, "\pi'"] \arrow[r, hook, "\tilde{\imath}"] & \begin{Bmatrix} P_{n-1} \subset P_n \\ \cap \\ E_1^\perp \end{Bmatrix} \arrow[u, "\pi"] & \begin{Bmatrix} E_1 \dot{\not\subset} P_{n-1} \subset P_n \\ \cap \\ E_1^\perp \end{Bmatrix} \arrow[l, hook', "v'"] \arrow[dl, hook, "\tilde{v}'"] \\
\begin{Bmatrix} E_1 \subset P_{n-1} \subset P_n \\ \cap \\ P'_n \end{Bmatrix} \arrow[u, "\tilde{\pi}'"] \arrow[rr, hook, "\tilde{\imath}'"'] & & \begin{Bmatrix} P_{n-1} \subset P_n \\ \cap \\ P'_n \subset E_1^\perp \end{Bmatrix} \arrow[u, "\tilde{\pi}"] \arrow[r, "\tilde{\alpha}"'] & \{P'_n \subset E_1^\perp\}
\end{tikzcd} \tag{3.5}$$

Remark 3.1.3. Recall the non-maximal Lagrangian Grassmannian of Definition 1.4.6. Then

$$\begin{tikzcd}
\mathrm{SG}_X(n-2, E_1^\perp/E_1) = \{E_1 \subset P_{n-1} \subset E_1^\perp\} \arrow[r, hook, "i"] & \{P_{n-1} \subset E_1^\perp\} = \mathrm{SG}_X(n-1, E_1^\perp) \\
Z' = \{E_1 \subset P_{n-1} \subset P_n\} \arrow[u, "\tilde{p}"] \arrow[r, hook, "\tilde{\imath}"'] & \{P_{n-1} \subset P_n \subset E_1^\perp\} = E. \arrow[u, "p"']
\end{tikzcd} \tag{3.6}$$

is a blow-up square and allows computations on Picard groups and canonical bundles for the odd symplectic Grassmannians using the methods in [BC12a, Appendix].

For a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V}))$ we consider the long exact localization sequence

$$\ldots \to W^i_Z(\mathrm{LG}(n), \mathcal{L}) \xrightarrow{e} W^i(\mathrm{LG}(n), \mathcal{L}) \xrightarrow{v^*} W^i(U, \mathcal{L}) \xrightarrow{\partial} W^{i+1}_Z(\mathrm{LG}(n), \mathcal{L}) \to \ldots$$

where we write $\mathrm{LG}(n) := \mathrm{LG}_X(n, \mathcal{V})$ and we have a dévissage isomorphism

$$W^{i-n}(Z, \mathcal{L}|_Z \otimes \omega_\iota) \xrightarrow{\sim} W^i_Z(\mathrm{LG}_X(n, \mathcal{V}), \mathcal{L}).$$

The twist is given by $\omega_\iota = [\mathcal{L}/E_1]^n \Delta_n^{-1}$ where $\Delta_n = \det \mathcal{S}_n = \mathcal{O}(-1)$ as usual. Hence, by homotopy invariance and writing $\mathrm{LG}(n-1) := \mathrm{LG}_X(n-1, E_1^\perp/E_1)$, we end up with two essential sequences

$$\ldots \to W^{i-n}(\mathrm{LG}(n-1), \mathcal{O}) \longrightarrow W^i(\mathrm{LG}(n), \mathcal{O}(1)) \xrightarrow{v^*} W^i(\mathrm{LG}(n-1), \mathcal{O}(1)) \xrightarrow{\partial} \tag{3.7}$$

and

$$\ldots \to W^{i-n}(\mathrm{LG}(n-1),\mathcal{O}(1)) \longrightarrow W^i(\mathrm{LG}(n),\mathcal{O}) \xrightarrow{v^*} W^i(\mathrm{LG}(n-1),\mathcal{O}) \xrightarrow{\partial} \ldots \tag{3.8}$$

modulo $\mathrm{Pic}_X(\mathrm{LG}_X(n,\mathcal{V}))/2$ where $\mathrm{Pic}_X(\mathrm{LG}_X(n,\mathcal{V})) = \mathrm{Pic}(\mathrm{LG}_X(n,\mathcal{V}))/\mathrm{Pic}(X)$. We will now discuss these boundary maps in a more general setting.

3.2 The connecting homomorphism revisited

In this section we establish an anologuous description of the boundary map for a more general setting as in [BC09]. For completeness we state the results for general schemes although all schemes in the application will be smooth.

Setup 3.2.1. Let $\iota : Z \hookrightarrow X$ a regular closed embedding of separated, noetherian, connected schemes over $\mathbb{Z}[\frac{1}{2}]$ of codimension c. Denote by U the open complement of Z in X. Let $\pi : \mathrm{Bl}_Z(X) \longrightarrow X$ be the blow-up of X along Z. Assume further that $\tilde{\imath} \circ \tilde{\tilde{\imath}} =: \iota' : Z' \hookrightarrow \mathrm{Bl}_Z(X)$ is a regular closed embedding of codimension c' and let $\tilde{\pi} : Bl_{Z'}(Bl_Z(X)) \longrightarrow \mathrm{Bl}_Z(X)$ be the blow-up of $\mathrm{Bl}_Z(X)$ along Z'. Denote by E, E' the corresponding exceptional divisors. Then we have a diagram

$$\begin{tikzcd}
 & Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "v"'] \arrow[d, hook, "w"] \arrow[dl, "\tilde{v}"'] \arrow[ddl, "\tilde{\tilde{v}}"] \\
Z' \arrow[r, hook, "\tilde{\tilde{\imath}}"] & E \arrow[u, "\pi'"] \arrow[r, hook, "\tilde{\imath}"] & \mathrm{Bl}_Z(X) \arrow[u, "\pi"] & U' \arrow[l, hook', "v'"] \arrow[dl, "\tilde{v}'"] \\
E' \arrow[u, "\tilde{\pi}'"] \arrow[rr, hook, "\tilde{\iota}'"'] & & \mathrm{Bl}_{Z'}(\mathrm{Bl}_Z(X)) \arrow[u, "\tilde{\pi}"] &
\end{tikzcd} \tag{3.9}$$

In the following we shall write $\tilde{X} := \mathrm{Bl}_Z(X)$ and $\tilde{\tilde{X}} := \mathrm{Bl}_{Z'}(Bl_Z(X))$. Let us fix a dualizing complex K_X on X and denote by K_U the restricted dualizing complex on U (in particular this complex is dualizing). If not mentioned otherwise, ∂ will denote the connecting homomorphism in the long exact sequence

$$\ldots \longrightarrow \tilde{W}^i_Z(X,K_X) \longrightarrow \tilde{W}^i(X,K_X) \longrightarrow \tilde{W}^i(U,K_U) \xrightarrow{\partial} \tilde{W}^{i+1}_Z(X,K_X) \to \ldots$$

as usual.

Lemma 3.2.2 ([BC09, 3.4]). *In Setup 3.2.1 we have*

$$\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}\mathcal{O}(E), \qquad \mathrm{Pic}(\tilde{\tilde{X}}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}\mathcal{O}(E) \oplus \mathbb{Z}\mathcal{O}(E').$$

Moreover, if K_X is a dualizing complex on X, dualizing complexes on $\tilde{X}$ rsp. $\tilde{\tilde{X}}$ are of the form

$$\begin{aligned} \tilde{K} &= \pi^!(K_X \otimes \mathcal{L}) \otimes \mathcal{O}(E)^{\otimes m}, \\ \tilde{\tilde{K}} &= \tilde{\pi}^! \pi^!(K_X \otimes \mathcal{L}) \otimes \tilde{\pi}^* \mathcal{O}(E)^{\otimes m} \otimes \mathcal{O}(E')^{\otimes m'} \end{aligned}$$

with integers $m, m' \in \mathbb{Z}$ and $\mathcal{L} \in \mathrm{Pic}(X)$.

Notation 3.2.3. By the preceeding lemma, a dualizing complex K on X induces several dualizing complexes on the schemes involved in Setup 3.2.1. More precisely

$$\begin{aligned} K_U &:= \mathrm{L}v^* K, \\ \tilde{K} &:= \pi^! K, \\ \tilde{K}_U &:= \mathrm{L}\tilde{v}^* \tilde{K}, \\ \tilde{K}_{U'} &:= \mathrm{L}v'^* \tilde{K}, \\ \tilde{\tilde{K}} &:= \tilde{\pi}^! \pi^! K, \\ \tilde{\tilde{K}}_{U'} &:= \mathrm{L}\tilde{v}'^* \tilde{\tilde{K}}. \end{aligned}$$

Note that by the blow-up property we have $\tilde{K}_U = K_U$, since π restrict to an isomorphism over U. Similarly, $\tilde{\tilde{K}}_{U'} = \tilde{K}_{U'}$.

Lemma 3.2.4. *Let $U_2 \subset U_1 \subset X$ be open subsets of X, K_X a dualizing complex on X and denote by $Z_i := X \setminus U_i$ the closed complements. Then the localization long exact sequences induce a commutative ladder diagram*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \tilde{W}^i_{Z_1}(X, K_X) & \longrightarrow & \tilde{W}^i(X, K_X) & \longrightarrow & \tilde{W}^i(U_1, K_{U_1}) & \longrightarrow & \tilde{W}^{i+1}_{Z_1}(X, K_X) & \longrightarrow \cdots \\ & \downarrow & & \| & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \tilde{W}^i_{Z_2}(X, K_X) & \longrightarrow & \tilde{W}^i(X, K_X) & \longrightarrow & \tilde{W}^i(U_2, K_{U_2}) & \longrightarrow & \tilde{W}^{i+1}_{Z_2}(X, K_X) & \longrightarrow \cdots \end{array} \tag{3.10}$$

with support extensions and restrictions as vertical maps.

Proof. Use [Gil02, 2.9]. By [Bal99, 5.23] we know that the derived category $D^b(\mathrm{Coh}(U_i))$ is the localization of $D^b(\mathrm{Coh}(X))$ with respect to those morphisms in $D^b(\mathrm{Coh}(X))$ which restrict to isomorphisms on $D^b(\mathrm{Coh}(U_i))$; write $D^b(\mathrm{Coh}(U_i)) = S_i^{-1} D^b(\mathrm{Coh}(X))$. The respective kernel categories are given by $\mathcal{I}_{S_i} = D^b_{Z_i}(\mathrm{Coh}(X))$ and the long exact sequences corresponding to the exact sequences

$$0 \longrightarrow D^b_{Z_i}(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(U_i)) \longrightarrow 0$$

yield the horizontal maps of (3.10). Now clearly the identity functor on $D^b(\mathrm{Coh}(X))$ is duality preserving and satisfies $\mathrm{id}(S_1) \subset S_2$, hence it induces duality preserving functors

$$D^b(\mathrm{Coh}(U_1)) \longrightarrow D^b(\mathrm{Coh}(U_2)), \qquad D^b_{Z_1}(\mathrm{Coh}(X)) \longrightarrow D^b_{Z_2}(\mathrm{Coh}(X))$$

which by [Gil02, 2.7] themselves induce functors

$$\tilde{W}^i(U_1, K_{U_1}) \longrightarrow \tilde{W}^i(U_2, K_{U_2}), \qquad \tilde{W}^i_{Z_1}(X, K_X) \longrightarrow \tilde{W}^i_{Z_2}(X, K_X),$$

given by restriction and extension of support. Finally, [Gil02, 2.9] shows commutativity of (3.10). □

Proposition 3.2.5. *In Setup 3.2.1 assume that X admits a dualizing complex K. Denote the occuring extensions of support by* $\mathrm{e}, \mathrm{e}', \epsilon$. *Then we have a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{W}^i_Z(X,K) \arrow[r, "\mathrm{e}"] & \tilde{W}^i(X,K) \arrow[r, "v^*"] & \tilde{W}^i(U,K_U) \arrow[r, "\partial"] \arrow[d, equal] & \tilde{W}^{i+1}_Z(X,K) \arrow[r] & \cdots \\
\cdots \arrow[r] & \tilde{W}^i_E(\tilde{X},\tilde{K}) \arrow[r, "\mathrm{e}'"] \arrow[u, "\pi_*"] & \tilde{W}^i(\tilde{X},\tilde{K}) \arrow[r, "\tilde{v}^*"] \arrow[u, "\pi_*"] & \tilde{W}^i(U,K_U) \arrow[r, "\partial'"] & \tilde{W}^{i+1}_E(\tilde{X},\tilde{K}) \arrow[r] \arrow[u, "\pi_*"] & \cdots \\
\cdots \arrow[r] & \tilde{W}^i_{Z'}(\tilde{X},\tilde{K}) \arrow[r, "\mathrm{e}''"] \arrow[u, "\epsilon"] & \tilde{W}^i(\tilde{X},\tilde{K}) \arrow[r, "v'^*"] \arrow[u, equal] & \tilde{W}^i(U',\tilde{K}_{U'}) \arrow[r, "\partial''"] \arrow[u, "w^*"] \arrow[d, equal] & \tilde{W}^{i+1}_{Z'}(\tilde{X},\tilde{K}) \arrow[r] \arrow[u, "\epsilon"] & \cdots \\
\cdots \arrow[r] & \tilde{W}^i_{E'}(\tilde{\tilde{X}},\tilde{\tilde{K}}) \arrow[r, "\mathrm{e}''"] \arrow[u, "\tilde{\pi}_*"] & \tilde{W}^i(\tilde{\tilde{X}},\tilde{\tilde{K}}) \arrow[r, "\tilde{v}'^*"] \arrow[u, "\tilde{\pi}_*"] \arrow[uuur, "\tilde{\tilde{v}}^*"] & \tilde{W}^i(U',\tilde{K}_{U'}) \arrow[r, "\partial''"] & \tilde{W}^{i+1}_{E'}(\tilde{\tilde{X}},\tilde{\tilde{K}}) \arrow[r] \arrow[u, "\tilde{\pi}_*"] & \cdots
\end{tikzcd}$$

Proof. The horizontal lines form long exact localization sequences. The middle horizontal ladder commutes by the preceeding lemma, the middle vertical ladder commutes by base change (Theorem 1.2.12) and functoriality of pushforwards of Witt groups (e.g. [CH11, 5.2]. The squares in the corners commute by commutativity of push-forwards with connecting maps and extension of support respectively ([BC09, proof of 3.3] and [Nen07, 4.2] respectively). □

A key tool, similar to the Main Lemma in [BC09] will be essential in the proof of the main theorem of the section:

Lemma 3.2.6. *In Setup 3.2.1 assume that X admits a dualizing complex K_X and let*

$$D_{m,m'} := \tilde{\pi}^! \pi^! K_X \otimes \mathcal{O}(E')^{\otimes m'} \otimes \tilde{\pi}^* \mathcal{O}(E)^{\otimes m} \tag{3.11}$$

for $m, m' \in \{0, 1\}$. Consider the composition

$$\tilde{W}^i(\tilde{\tilde{X}}, D_{m,m'}) \xrightarrow{\tilde{\tilde{v}}^*} \tilde{W}^i(U, K_U) \xrightarrow{\partial} \tilde{W}^{i+1}_Z(X, K_X). \tag{3.12}$$

Then we have the following:

(1) *If $m = 0 = m'$ then $\partial \circ \tilde{\tilde{v}}^* = 0$.*

(2) *If $m = 0$ and $m' = 1$ then $\partial \circ \tilde{\tilde{v}}^*$ coincides with*

$$\begin{tikzcd}
 & \tilde{W}^{i+1}_Z(X, K_X) \\
\tilde{W}^i(\tilde{\tilde{X}}, \tilde{\tilde{K}} \otimes \mathcal{O}(E')) \arrow[d, "\tilde{\imath}'^*"'] & \tilde{W}^{i+1}_E(\tilde{X}, \tilde{K}) \arrow[u, "\pi_*"'] \\
\tilde{W}^i\left(E', \mathrm{L}\tilde{\imath}'^*(\tilde{\tilde{K}} \otimes \mathcal{O}(E'))\right) \arrow[d, "\sim"'] & \tilde{W}^{i+1}_{Z'}(\tilde{X}, \tilde{K}) \arrow[u, "e"'] \\
\begin{matrix} \tilde{W}^{i+1}(E', \tilde{\imath}'^! \tilde{\tilde{K}}) \cong \\ \tilde{W}^{i+1}(E', \tilde{\pi}'^! \iota'^! \tilde{K}) \end{matrix} \arrow[r, "\tilde{\pi}'_*"] & \begin{matrix} \tilde{W}^{i+1}(Z', \iota'^! \pi^! K_X) \cong \\ \tilde{W}^{i+1}(Z', \tilde{\tilde{\imath}}^! \tilde{\imath}^! \pi^! K_X) \end{matrix} \arrow[u, "\iota'_*"']
\end{tikzcd}$$

(3) *If $m = 1$ and $m' = 0$ then $\partial \circ \tilde{\tilde{v}}^*$ coincides with*

$$\begin{tikzcd}
\tilde{W}^i(\tilde{\tilde{X}}, \tilde{\tilde{K}} \otimes \tilde{\pi}^* \mathcal{O}(E)) \arrow[d, "\tilde{\pi}_*"'] & & \\
\tilde{W}^i(\tilde{X}, \pi^! K_X \otimes \mathcal{O}(E)) \arrow[d, "\tilde{\imath}^*"'] & & \tilde{W}^{i+1}_Z(X, K_X) \\
\tilde{W}^i(E, L\iota^*(\pi^! K_X \otimes \mathcal{O}(E))) \arrow[r, "\sim"'] & \tilde{W}^{i+1}(E, \pi'^! \iota^! K_X) \arrow[r, "\pi'_*"] & \tilde{W}^{i+1}(Z, \iota^! K_X) \arrow[u, "(\iota)_*"]
\end{tikzcd}$$

Proof. (1) This follows from Proposition 3.2.5 and the fact that the composition of consecutive morphisms in the localization long exact sequence vanishes.

(2) Consider the diagram

$$
\begin{tikzcd}
 & \tilde{W}^i(U, K_U) \arrow[r, "\partial"] \arrow[d, equal] & \tilde{W}^{i+1}_Z(X, K_X) \\
\tilde{W}^i(\tilde{\tilde{X}}, \tilde{\tilde{K}} \otimes \mathcal{O}(E')) \arrow[r] \arrow[ur, "\tilde{\tilde{v}}^*"] \arrow[dr, "\tilde{v}'^*"'] \arrow[ddd, "\tilde{\iota}'^*"'] & \tilde{W}^i(U, K_U) \arrow[r, "\partial'"'] & \tilde{W}^{i+1}_E(\tilde{X}, \tilde{K}) \arrow[u, "\pi_*"'] \\
 & \tilde{W}^i(U', \tilde{K}_{U'}) \arrow[r, "\partial''"'] \arrow[u, "w^*"'] & \tilde{W}^{i+1}_{Z'}(\tilde{X}, \tilde{K}) \arrow[u, "\epsilon"'] \\
 & & \\
\tilde{W}^i(E', \mathrm{L}\tilde{\iota}'^*(\tilde{\tilde{K}} \otimes \mathcal{O}(E'))) \arrow[r, "\sim"'] & \tilde{W}^{i+1}(E', \tilde{\pi}'^! \iota'^! \tilde{K}) \arrow[r, "\tilde{\pi}'_*"'] & \tilde{W}^{i+1}(Z', \iota'^! \tilde{K}) \arrow[uu, "\iota'_*"']
\end{tikzcd}
$$

The upper part commutes by Proposition 3.2.5. The lower diagram commutes by [BC09, 3.5(B)].

(3) This directly follows from [BC09, 3.5 (B)] and $\tilde{\tilde{v}}^* = \tilde{v}^* \circ \tilde{\pi}_*$. □

Now similarly to [BC09, Hypothesis 1.2], we will ask U to be an affine bundle over a scheme Y, coming with an auxiliary morphism from the blow-up. The difference this time will be that we may not necessarily have such a morphism coming directly from $\mathrm{Bl}_Z(X)$, but only from the double blow-up:

Hypothesis 3.2.7. In Setup 3.2.1 let $\tilde{\alpha} : \mathrm{Bl}_{\tilde{Z}'}(\mathrm{Bl}_Z(X)) \longrightarrow Y$ be a morphism such that the composition $\alpha := \tilde{\alpha} \circ \tilde{\tilde{v}} : U \longrightarrow Y$ is an $\mathbb{A}^*$-bundle.

Theorem 3.2.8 (Connecting homomorphism). *In Setup 3.2.1 assume that X admits a dualizing complex K_X. Moreover, assume Hypothesis 3.2.7 and the following:*

(i) There is a dualizing complex K_Y on Y satisfying $\mathrm{L}\alpha^* K_Y = K_U$.

(ii) The bundle α induces an isomorphism $\tilde{W}^i(Y, K_Y) \xrightarrow{\sim} \tilde{W}^i(U, K_U)$.

(iii) The morphism $\tilde{\alpha}$ is of finite Tor dimension and $\mathrm{L}\tilde{\alpha}^*(K_Y)$ is a dualizing complex on $\tilde{\tilde{X}}$.

(iv) The sequences $\mathbb{Z} \to \mathrm{Pic}(\tilde{\tilde{X}}) \xrightarrow{\tilde{v}'^*} \mathrm{Pic}(U')$ and $\mathbb{Z} \to \mathrm{Pic}(\tilde{X}) \xrightarrow{\tilde{v}^*} \mathrm{Pic}(U)$, where the first maps send 1 to $\mathcal{O}(E')$ and $\mathcal{O}(E)$ respectively, are exact.

Then $\mathrm{L}\tilde{\alpha}^*(K_Y) \cong \tilde{\pi}^!\pi^! K_X \otimes \tilde{\pi}^*\mathcal{O}(E)^{\otimes m} \otimes \mathcal{O}(E')^{\otimes m'}$ *for some* $m, m' \in \mathbb{Z}$ *and the following holds:*

(1) *If* m *and* m' *can be chosen even, the composition* $\pi_*\tilde{\pi}_*\tilde{\alpha}^*(\alpha^*)^{-1}$ *is a section of* v^**, so* $\partial = 0$.

(2) *If* m *can be chosen even and* m' *can be chosen odd, the boundary homomorphism* $\partial : \tilde{W}^i(U, K_U) \to \tilde{W}^{i+1}_Z(X, K_X)$ *is given by the composition*

$$\partial = \pi_* \mathrm{e}(\iota')_* \tilde{\pi}'_* \tilde{\iota}'^* \tilde{\alpha}^*(\alpha^*)^{-1}$$

where $\mathrm{e} : \tilde{W}^i_{Z'}(\tilde{X}) \to \tilde{W}^i_E(\tilde{X})$ *denotes the extension of support (for any twist).*

(3) *If* m *can be chosen odd and* m' *can be chosen even, the boundary homomorphism* $\partial : \tilde{W}^i(U, K_U) \to \tilde{W}^{i+1}_Z(X, K_X)$ *is given by the composition*

$$\partial = \iota_* \pi'_* \tilde{\iota}^* \tilde{\pi}_* \tilde{\alpha}^*(\alpha^*)^{-1}.$$

Proof. By (iii) we have two dualizing complexes $\mathrm{L}\tilde{\alpha}^*K_Y$ and $\tilde{\tilde{K}}$ on $\tilde{\tilde{X}}$ and by Remark 1.2.11(i) they differ by a shifted line bundle on $\tilde{\tilde{X}}$, i.e. we have

$$\mathrm{L}\tilde{\alpha}^*K_Y = \tilde{\tilde{K}} \otimes \mathcal{K}[n]$$

for some $\mathcal{K} \in \mathrm{Pic}(\tilde{\tilde{X}})$, $n \in \mathbb{Z}$. Consider the restrictions to U. We have

$$K_U \otimes \tilde{\tilde{v}}^*\mathcal{K}[n] = \tilde{\tilde{v}}^*(\tilde{\pi}^!\pi^!K_X \otimes \mathcal{K}[n]) = \tilde{\tilde{v}}^*\mathrm{L}\tilde{\alpha}^*K_Y = K_U,$$

i.e. $\mathcal{K}|_U$ is trivial and $n = 0$. By (iv) and $\mathrm{Pic}(U') \cong \mathrm{Pic}(\tilde{X})$ we get

$$\mathcal{K} = \tilde{\pi}^*\mathcal{O}(E)^m \otimes \mathcal{O}(E')^{m'}$$

for some $m, m' \in \mathbb{Z}$. Then (1)-(3) follow by base-change and Lemma 3.2.6. □

3.3 The connecting homomorphism for $\mathrm{LG}(2m+1)$

As before we denote by $p : \mathrm{LG}_X(n, \mathcal{V}) \to X$ the projection to the base scheme where the underlying antisymmetric form admits values in the line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. By abuse of notation, we will denote by p any other projection to X as well. Denote from now on by Setup 3.2.1 the diagram (3.4). In this section, we prove the following (see Corollaries 3.3.9, 3.3.10).

Theorem 3.3.1. *Let n be odd and $\mathcal{K} \in \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V}))$ be a line bundle. Then there is a line bundle $\mathcal{K} \in \mathrm{Pic}(X)$ and an integer $p \in \mathbb{Z}$, such that $\mathcal{K} = p^*\mathcal{M} \otimes \mathcal{O}(p)$ and we have the following:*

(i) If p is odd, then the connecting homomorphism

$$\partial : W^i(U, \mathcal{K}|_U) \longrightarrow W^{i-n}(Z, p^*\mathcal{M} \otimes \mathcal{L} \otimes E_1^{-1}) \cong W^{i-n}(Z, \mathcal{K}|_Z \otimes \omega_\iota) \tag{3.13}$$

is an isomorphism.

(ii) If p is even, then the connecting homomorphism

$$\partial : W^i(U, \mathcal{K}|_U) \longrightarrow W^{i-n}(Z, p^*\mathcal{M} \otimes \mathcal{L} \otimes E_1^{-1} \otimes \mathcal{O}(1)) \cong W^{i-n}(Z, \mathcal{K}|_Z \otimes \omega_\iota) \tag{3.14}$$

vanishes.

Theorem 3.3.2. *Let n be even and $\mathcal{K} \in \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V}))$ as above. Then the connecting homomorphism*

$$\partial : W^i(U, \mathcal{K}|_U) \longrightarrow W^{i-n}(Z, \mathcal{K}|_Z \otimes \omega_\iota) \tag{3.15}$$

vanishes.

Before passing to the proof, we want to point out the difference to the orthogonal case.

Remark 3.3.3. Despite the more complicated description of the boundary map due to the extended blow-up setup in the symplectic case, the above theorem states that the map vanishes or is an isomorphism. This only applies partly in the case for maximal orthogonal Grassmannians as has been shown in the joint article [HMX21].

- If n is even, again the connecting homomorphism vanishes.
- The difference arises for odd n. In the Lagrangian case, the boundary map either vanishes or is an isomorphism, depending on the twist (cf. theorem above). Recall that there are no twists for the orthogonal Grassmannians due to the fact, that the tautological determinant bundle is a square in the Picard group. Instead, we divide the origin of ∂ as in the localization sequence and determine the restriction of ∂ on these parts. More precisely, since $n-1$ is even, the localization sequence for the embedding $\mathrm{OG}(n-2) \hookrightarrow \mathrm{OG}(n-1)$ splits and we have a decomposition

$$W^i(\mathrm{OG}(n-1)) \cong W^i(\mathrm{OG}(n-2)) \oplus W^{i-(n-2)}(\mathrm{OG}^-(n-2)).$$

Then the main fact used in [HMX21] is that ∂ is an isomorphism when restricted to the first summand and vanishes when restricted to the second summand.

In order to compute ∂ we need to calculate the relevant canonical bundles. We first examine the first blow-up square. Recall that the embedding $\iota : Z \hookrightarrow \mathrm{LG}_X(n, \mathcal{V})$ is of codimension $c = n$ and that we have $\mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V})) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}\mathcal{O}(1)$, where as usual $\Delta_n = \mathcal{O}(-1)$ for the dual generator. Similarly $\mathrm{Pic}(Z) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}[\Delta_n E_1^{-1}]$; in particular $\mathrm{Pic}(Z) \cong \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V}))$ for $n \geq 2$. By Proposition 1.4.9 $\omega_\iota = \mathcal{L}^n E_1^{-n} \Delta_n^{-1} \in \mathrm{Pic}(Z)$.

Lemma 3.3.4. *We have* $\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V})) \oplus \mathbb{Z}\Delta_{n-1}$ *and in the first blow-up, the class of E in* $\mathrm{Pic}(\tilde{X})$ *is given by* $\mathcal{O}(E) = \mathcal{L}E_1^{-1}\Delta_n^{-1}\Delta_{n-1}$. *In particular,*

$$\omega_{\pi'} = \Delta_{n-1}^{n}\Delta_n^{-n+1}, \tag{3.16}$$

$$\omega_{\pi} = \left[\mathcal{L}E_1^{-1}\right]^{n-1} \Delta_n^{-n+1}\Delta_{n-1}^{n-1}. \tag{3.17}$$

Proof. By assumption $\tilde{\pi}$ is a projective bundle of rank $n-1$ and hence in particular a Grassmann bundle. Then (3.16) follows from (1.15). In the setting of Remark 3.1.3, we have

$$\omega_{\tilde{p}} = \mathcal{L}^{-1}\Delta_n^2\Delta_{n-1}^{-2} = \mathcal{O}(Z')|_{Z'}^{\otimes 2} \otimes \tilde{p}^*\omega_i^\vee, \tag{3.18}$$

$$\omega_{\tilde{i}} = E_1^{-1}\Delta_{n-1}^{-1}\Delta_n^1 = \mathcal{O}(Z')|_{Z'}. \tag{3.19}$$

where the first equality in (3.18) again follows from (1.15) and the second one from [BC09, A.11(ii)]. In (3.19) the first equality follows from $\omega_{\tilde{i}} = \omega_{Z'} \cdot \left(\tilde{i}^*\omega_E\right)^{-1}$ and [BC12a, 1.14] where the second one again comes from [BC12a, A.11(i)]. This gives us $\tilde{p}^*\omega_i^\vee = \mathcal{L}^{-1}E_1^2$ and hence $\omega_i = \mathcal{L}E_1^{-2}$. Using this and Remark 1.4.6, we compute

$$\omega_{\mathrm{SG}_X(n-1, E_1^\perp)} = \mathcal{L}^{-\frac{n(n-1)}{2}} E_1^{-n+1}\Delta_{n-1}^{n+1}.$$

Further

$$\omega_{\tilde{X}} = \mathcal{L}^{-\left(\frac{n(n-1)}{2}+1\right)} E_1^{-n+1}\Delta_{n-1}^{n-1}\Delta_n^2.$$

since $\tilde{X}$ is a $\mathbb{P}^1$-bundle over $\mathrm{SG}_X(n-1, E_1^\perp)$. Subtracting $\omega_{\mathrm{LG}_X(n,\mathcal{V})}$ proves the lemma. $\square$

Now do the same computations one level higher. We have $\mathrm{Pic}(\tilde{\tilde{X}}) \cong \mathrm{Pic}(\tilde{X}) \oplus \mathbb{Z}\mathcal{O}(E')$. Denote by Δ_n' the determinant of the "new" tautological bundle over the double blow-up $\tilde{\tilde{X}}$. Then using the same methods we obtain the following:

Lemma 3.3.5. *In the second blow-up the class of E' in* $\mathrm{Pic}(\tilde{\tilde{X}})$ *is given by* $\mathcal{O}(E') = E_1^{-1}\Delta_n'\Delta_{n-1}^{-1}$. *In particular,*

$$\omega_{\tilde{\pi}'} = \mathcal{L}^{-1}\Delta_n'^2\Delta_{n-1}^{-2}, \tag{3.20}$$

$$\omega_{\tilde{\pi}} = E_1^{-1}\Delta_n'\Delta_{n-1}^{-1}. \tag{3.21}$$

Proposition 3.3.6. *Let $\mathcal{K} \in \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V}))$ be a line bundle. Choose a line bundle $\mathcal{K} \in \mathrm{Pic}(X)$ and an integer $p \in \mathbb{Z}$, such that $\mathcal{K} = p^*\mathcal{M} \otimes \mathcal{O}(p)$. Then in Setup 3.2.1 we have the following:*

(i) If n is even and p is odd, then Theorem 3.2.8 (1) applies.

(ii) If n is odd and p is even, then Theorem 3.2.8 (2) applies.

(iii) If n is odd and p is odd, then Theorem 3.2.8 (3) applies.

Proof. We have $\tilde{\alpha}^*(\alpha^*)^{-1}(\mathcal{K}|_U) = p^*\mathcal{M} \otimes \mathcal{L}^{-p}E_1^{2p} \otimes \Delta_n'^{-p}$. Indeed, by Notation 2.3.1, we have that $v^*(\Delta_n)\alpha^*(\Delta_n')^{-1} = \mathcal{L}E_1^{-2}$, where the sequence $0 \to E_1^\perp \to \mathcal{V} \to E_1^\vee \otimes \mathcal{L} \to 0$ yields $\mathcal{V}/E_1^\perp \cong \mathcal{L}E_1^{-1}$. Using $\mathcal{O}(p) = \Delta_n^{-p} \in \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V}))$ and by [BC09, A.8] we get

$$\begin{aligned} \tilde{\pi}^!\pi^!(\mathcal{K}) &= \tilde{\pi}^!(\omega_\pi \otimes \pi^*\mathcal{K}) = \omega_{\tilde{\pi}} \otimes \tilde{\pi}^*(\omega_\pi) \otimes \tilde{\pi}^*\pi^*(\mathcal{K}) \\ &= p^*\mathcal{M} \otimes \mathcal{L}^{n-1}E_1^{-n} \otimes \Delta_{n-1}^{n-2}\Delta_n^{-n+1-p}\Delta_n'. \end{aligned}$$

Then comparing the terms on both sides we have

$$\tilde{\alpha}^*(\alpha^*)^{-1}(\mathcal{K}|_U) = \tilde{\pi}^!\pi^!(\mathcal{K}) \otimes \tilde{\pi}^*\mathcal{O}(E)^{\otimes n-1+p} \otimes \mathcal{O}(E')^{-\otimes p+1}$$

which shows (i)-(iii) by comparing parities of the exponents. □

Remark 3.3.7. If both n and p are even, no statement can be made so far. Nevertheless we will prove that also in the latter case the boundary map vanishes.

We now have established an explicit description of the boundary map in the two cases we will need to complete the computations of the Witt groups. In the second part of this section we use these formulas to prove Theorem 3.3.1. One of the main tools will be the description of the Witt groups of projective bundles due to Walter and Nenashev ([Wal03], [Nen09]), Theorem 1.2.13.

From here, let n be odd. First, let p be odd, in which case by Proposition 3.3.6(iii)

$$\partial = \iota_*\pi'_*\tilde{\iota}^*\tilde{\pi}_*\tilde{\alpha}^*(\alpha^*)^{-1}. \tag{3.22}$$

Note that

(i) α^* is an isomorphism since α is an $\mathbb{A}^*$-bundle,

(ii) π'_* is an isomorphism by Theorem 1.2.13, since π' is a projective bundle of even rank $n-1$,

(iii) ι_* is an isomorphism into the supported Witt groups by dévissage, see (1.17).

Hence, it suffices to investigate the map $\partial' = \tilde{\iota}^* \tilde{\pi}_* \tilde{\alpha}^*$. In the diagram

$$\begin{tikzcd}
E = \{P_{n-1} \subset P_n \subset E_1^{\perp}\} \arrow[d, "\tilde{\iota}"] \arrow[drr, "p"] & & \\
\{P_{n-1} \subset P_n \mid P_{n-1} \subset E_1^{\perp}\} = \tilde{X} \arrow[rr, "p'"'] & & \mathrm{SG}_X(n-1, E_1^{\perp}) = \{P_{n-1} \subset E_1^{\perp}\} \\
\{E_1^{\perp} \supset P_n' \supset P_{n-1} \subset P_n\} = \tilde{\tilde{X}} \arrow[u, "\tilde{\pi}"] \arrow[rr, "\tilde{\alpha}_1"'] \arrow[drr, "\tilde{\alpha}"'] & & \tilde{E} = \{P_{n-1} \subset P_n' \subset E_1^{\perp}\} \arrow[u, "q"'] \arrow[d, "\tilde{\alpha}_2"] \\
& & Y = \{P_n' \subset E_1^{\perp}\}
\end{tikzcd}$$

the upper and lower triangles are commutative and the square is cartesian. By base-change and commutativity of pull-backs we see $\partial' = p^* q_* \tilde{\alpha}_2^*$. Since $\tilde{\alpha}_2$ is a projective bundle of even rank $n-1$, the first pull-back $\tilde{\alpha}_2^*$ is an isomorphism and it again suffices to investigate the map $\partial'' = p^* q_*$.

Let us write down the maps induced on the Witt groups. Write $\mathcal{K} = p^*\mathcal{M} \otimes \mathcal{O}(p)$ as above for some odd p and $\mathcal{M} \in \mathrm{Pic}(X)$. Then we have

$$\begin{tikzcd}
& W_Z^{i+1}(\mathrm{LG}_X(n, \mathcal{V}), \mathcal{K}) & \\
W^i(U, \mathcal{K}|_U) \arrow[ur, "\partial"] \arrow[d, "\sim"'] & & W^{i+1-n}(Z, \mathcal{M} \otimes \mathcal{L}E_1^{-1}) \arrow[ul, "\iota_*"', "\sim"] \\
W^i(\tilde{E}, \tilde{\alpha}_2^*(\alpha^*)^{-1}\mathcal{K}|_U) \arrow[rr, "\partial''"] \arrow[d, equal] & & W^i(E, \omega_{\pi'} \otimes \pi'^*(\omega_\iota^\vee \otimes \mathcal{K}|_Z)) \arrow[u, "\sim"'] \arrow[d, equal] \\
W^i(\tilde{E}, \mathcal{M} \otimes \mathcal{L}E_1^{-2} \otimes \Delta_n') \arrow[dr, "q_*"'] & & W^i(E, p^*\mathcal{M} \otimes \mathcal{L}E_1^{-1} \otimes \Delta_{n-1}) \\
& W^i(S, \mathcal{M} \otimes \mathcal{L}E_1^{-1} \otimes \Delta_{n-1}) \arrow[ur, "p^*"'] &
\end{tikzcd}$$

where $S := \mathrm{SG}_X(n-1, E_1^{\perp})$ and we used $\omega_q = E_1^{-1}\Delta_{n-1}^{-1}\Delta_n'$, which follows by Remark 3.1.3 and [BC09, A.11 (iii)]. Let $S' := \mathrm{SG}_X(n-2, E_1^{\perp}/E_1)$ and consider the following diagram

$$
\begin{tikzcd}
 & E = \{P_{n-1} \subset P_n \subset E_1^{\perp}\} \arrow[dd, "p"'] & \\
\{E_1 \subset P_{n-1} \subset P_n\} = Z' \arrow[ur, hook, "\tilde{i}"] \arrow[d, "\mathbb{P}^1"] & & V' := \{E_1 \dot{\not\subset} P_{n-1} \subset P_n \subset E_1^{\perp}\} \arrow[ul, hook', "w'"'] \arrow[d, equal, "\wr"] \\
\{E_1 \subset P_{n-1} \subset E_1^{\perp}\} = S' \arrow[r, hook] & S = \{P_{n-1} \subset E_1^{\perp}\} & V := S \setminus S' = \{E_1 \dot{\not\subset} P_{n-1} \subset E_1^{\perp}\} \arrow[l, hook'] \\
\{E_1 \subset P_{n-1} \subset P_n'\} = Z'' \arrow[u, "\mathbb{P}^1"] \arrow[dr, hook, "\tilde{i}'"'] & & V'' := \{E_1 \dot{\not\subset} P_{n-1} \subset P_n' \subset E_1^{\perp}\} \arrow[u, equal, "\wr"] \arrow[dl, hook', "w''"] \\
 & \tilde{E} = \{P_{n-1} \subset P_n' \subset E_1^{\perp}\} \arrow[uu, "q"] &
\end{tikzcd}
$$

Note that over the open complements we indeed have isomorphisms, since over V the bundles $P_n = P_n'$ are uniquely determined by $P_{n-1} + E_1$ (which also induces the relation $\Delta_n = \Delta_{n-1}E_1 = \Delta_n'$ in the Picard groups). Consequently, the composition $\partial'' = p^*q_*$ is an isomorphism when restricted to U', U''. However, the restrictions of ∂'' to Z', Z'' vanish by [HMX21, A.2]. Luckily, this is no obstruction due to the following lemma:

Lemma 3.3.8. *If n is odd, the inclusion w'' induces an isomorphism*

$$w''^* : W^i(\tilde{E}, p^*\mathcal{M}' \otimes \Delta_n') \longrightarrow W^i(U'', p^*\mathcal{M}' \otimes \Delta_n') = W^i(U'', p^*\mathcal{M} \otimes \mathcal{L}E_1^{-1} \otimes \Delta_{n-1})$$

where we write $\mathcal{M}' := \mathcal{M} \otimes \mathcal{L}E_1^{-2}$. The same holds for w'.

Proof. Consider the localization sequence for the decomposition $Z'' \subset \tilde{E} \supset U''$:

$$\ldots \to W^i_{Z''}(\tilde{E}, p^*\mathcal{M}' \otimes \Delta_n') \xrightarrow{\tilde{i}'_*} W^i(\tilde{E}, p^*\mathcal{M}' \otimes \Delta_n') \xrightarrow{v''^*} W^i(U'', p^*\mathcal{M}' \otimes \Delta_n'|_{U''}) \to \ldots$$

The inclusion $\tilde{i}'$ is of codimension one and we have $\omega_{\tilde{i}'} = E_1^{-1}\Delta_{n-1}^{-1}\Delta_n'$ which gives us

$$W^{i-1}(Z'', p^*\mathcal{M} \otimes \mathcal{L}E_1^{-1} \otimes \Delta_{n-1}^{-1}) \xrightarrow{\sim} W^i_{Z''}(\tilde{E}, p^*\mathcal{M}' \otimes \Delta_n') \tag{3.23}$$

by dévissage (by square periodicity we cancelled the terms $E_1^{-2}\Delta_n'^2$). Now Z'' is a projective bundle over Y of odd rank $n-2$ with $\Delta_{n-1} = \mathcal{O}(1)$, i.e the twist is nontrivial. Hence, by Theorem 1.2.13(ii), the twisted Witt group in (3.23) vanishes and v''^* is an isomorphism, as desired. □

Corollary 3.3.9. *The connecting homomorphism in Theorem 3.3.1(i) is an isomorphism.*

We now turn to the case where p is even (and n is still odd); w.l.o.g. let $p = 0$, i.e. $\mathcal{K} = p^*\mathcal{M}$. Here, by Theorem 3.3.6(ii), the connecting homomorphism is given by

$$\partial = \pi_* \mathrm{e}(\iota')_* \tilde{\pi}'_* \tilde{\iota}'^* \tilde{\alpha}^* (\alpha^*)^{-1}.$$

From the commutative square

$$\begin{tikzcd}
\{P'_n \supset P_{n-1} \subset P_n \mid E_1 \subset P_{n-1}\} = E' \arrow[r, hook, "\tilde{\iota}'"] \arrow[d, "q'"', "\mathbb{P}^1"] & \tilde{\tilde{X}} \arrow[d, "\tilde{\alpha}"] \\
\{E_1 \subset P_{n-1} \subset P'_n\} = Z'' \arrow[r, "\mathbb{P}^{n-2}", "q"'] & Y = \{E_1 \subset P'_n\}
\end{tikzcd}$$

we have $\partial = \pi_* \mathrm{e}(\tilde{\iota}')_* \tilde{\pi}'_* p_* q^* (\alpha^*)^{-1}$. We will now focus on the map $\partial' = \tilde{\pi}'_* p^*$ in

$$\begin{tikzcd}
W^i(U, \mathcal{K}|_U) \arrow[r, "q^*(\alpha^*)^{-1}"] \arrow[d, "\partial"'] & W^i(Z'', \mathcal{L}/E_1^2 \otimes p^*\mathcal{M}) \arrow[d, "\partial'"] \\
W^i_Z(\mathrm{LG}_X(n, \mathcal{V}), \mathcal{K}) & W^i(Z', \mathcal{L}/E_1^2 \otimes p^*\mathcal{M}) \arrow[l, "\pi_* \mathrm{e}(\tilde{\iota}')_*"']
\end{tikzcd}$$

and show that it is the zero map. Since $p' : Z'' \longrightarrow S' = \mathrm{SG}_X(n-2, E_1^\perp / E_1)$ is a $\mathbb{P}^1$-bundle we have an isomorphism

$$W^i(Z'', p^*\mathcal{M}') \cong W^i(S', p^*\mathcal{M}') \oplus W^{i-1}(S', p^*\mathcal{M}' \otimes \mathcal{L}^{-1})$$

induced by p'^* and $p'_* \circ \mathrm{per}$ where we write $\mathcal{M}' = \mathcal{L}/E_1^2 \otimes \mathcal{M}$ as above. Denote by Ψ the splitting of $p'_* \circ \mathrm{per}$ as in [Wal03]. Then the restriction of ∂' to both its components is equal to the pull-back followed by the push-forward along a projective bundle of odd rank and hence vanishes by [Wal03, 1.4] (see also [HMX21, A.2]). To see this, consider the diagram

$$\begin{tikzcd}
S' & Z' \arrow[l, "p"'] \arrow[r, "p"] & S' \\
Z' \arrow[u, "p"] \arrow[d, "p"'] & E' \arrow[l, "q"'] \arrow[r, "q"] \arrow[u, "\tilde{\pi}'"] \arrow[d, "q'"'] & Z'' \arrow[u, "p'"'] \arrow[d, "p'"] \\
S' & Z'' \arrow[l, "p'"] \arrow[r, "p'"'] & S'
\end{tikzcd}$$

in which the upper right and lower left squares are cartesian and the remaining ones commute. By Theorem 1.2.13 the composition $p'_* p^*$ vanishes, i.e. $\partial' p'^* = 0$. The same argument applies when we restrict ∂' to the other component via Ψ. This proves:

Corollary 3.3.10. *The connecting homomorphism in Theorem 3.3.1(ii) vanishes.*

proof of Corollary 3.3.2. For odd twists this follows from Proposition 3.3.6(i). For even twists, ∂ lands in the twisted Witt group $W^{i-n}(\mathrm{LG}(n-1), \mathcal{O}(1))$ which vanishes, since the boundary map for $n-1$ odd and non-trivial twist is an isomorphism. □

The results of this section already allow us to describe the Witt groups of symplectic Grassmannians for all twists. To shorten notation we will write $\mathrm{LG}(n) := \mathrm{LG}_X(n, \mathcal{V})$ and similar in other dimensions.

Remark 3.3.11. We have $\mathrm{LG}(1) \cong \mathbb{P}(\mathcal{V})$. In particular, for a line bundle $\mathcal{K} = p^*\mathcal{M} \otimes \mathcal{O}(p)$ on $\mathrm{LG}(1)$ by Theorem 1.2.13(iii) and the identity $\det \mathcal{V} = \mathcal{L}$ we have

$$W^i(\mathrm{LG}(1), \mathcal{K}) = \begin{cases} 0, & \text{if } p \text{ is odd}, \\ W^i(X, \mathcal{M}) \oplus W^{i-1}(X, \mathcal{M} \otimes \mathcal{L}^{-1}), & \text{if } p \text{ is even}. \end{cases}$$

This is a special case of the following recursive description of the Witt groups:

Theorem 3.3.12. *We have the following description of twisted Witt groups (modulo* $\mathrm{Pic}(X)$*)*

$W^i(\mathrm{LG}(n), \mathcal{K})$	$\mathcal{K} = \mathcal{O}$	$\mathcal{K} = \mathcal{O}(1)$
n even	$W^i(\mathrm{LG}(n-1), \mathcal{O})$	$W^{i-n}(\mathrm{LG}(n-1), \mathcal{O})$
n odd	$W^i(\mathrm{LG}(n-2, \mathcal{O}) \oplus W^{i+1-2n}(\mathrm{LG}(n-2), \mathcal{O})$	0

Proof. Apply Theorem 3.3.1 applied to the long exact sequences (3.7) and (3.8). Since ∂ is an isomorphism for odd n, the twisted Witt groups for $\mathrm{LG}(n)$ with n odd vanish. Hence the localization sequence gives us isomorphisms

$$\begin{aligned} v^* : W^i(\mathrm{LG}(n), p^*\mathcal{M}) \xrightarrow{\sim} W^i(U, p^*\mathcal{M}|_U) &\cong W^i(\mathrm{LG}(n-1), \mathcal{L}/E_1^2 \otimes p^*\mathcal{M}|_U) \\ &\cong W^i(\mathrm{LG}(n-1), p^*\mathcal{M}) \end{aligned}$$

with the last isomorphism stemming from periodicity and

$$\begin{aligned} \iota_* : W^{i-n}(\mathrm{LG}(n-1), p^*\mathcal{M}) &\cong W^{i-n}(\mathrm{LG}(n-1), [\mathcal{L}/E_1]^n \otimes p^*\mathcal{M}) \\ &\cong W^i_Z(\mathrm{LG}(n), p^*\mathcal{M} \otimes \mathcal{O}(1)) \\ &\xrightarrow{\sim} W^i(\mathrm{LG}(n), p^*\mathcal{M} \otimes \mathcal{O}(1)) \end{aligned}$$

for n even. If on the other hand n is odd, we only need to compute the untwisted Witt groups. Since ∂ vanishes in the associated localization sequence, the resulting split short exact sequences gives us

$$\begin{aligned} W^i(\mathrm{LG}(n), p^*\mathcal{M}) &\cong W^i_Z(\mathrm{LG}(n), p^*\mathcal{M}) \oplus W^i(U, p^*\mathcal{M}|_U) \\ &\cong W^{i-n}\left(\mathrm{LG}(n-1), [\mathcal{L}/E_1]^n \otimes p^*\mathcal{M} \otimes \mathcal{O}(1)\right) \\ &\quad \oplus W^i\left(\mathrm{LG}(n-1), p^*\mathcal{M}\right) \\ &\cong W^{i-2n+1}(\mathrm{LG}(n-2), \mathcal{L}^{2n-1} \otimes [\det E_2]^{-(n-1)} \otimes p^*\mathcal{M}) \\ &\quad \oplus W^i(\mathrm{LG}(n-2), p^*\mathcal{M}) \\ &\cong W^{i-2n+1}(\mathrm{LG}(n-2), \mathcal{L} \otimes p^*\mathcal{M}) \oplus W^i(\mathrm{LG}(n-2), p^*\mathcal{M}) \end{aligned}$$

where we used the results for the even case above, periodicity isomorphisms to cancel even exponents and

$$\omega_{\mathrm{LG}(n-2)/\mathrm{LG}(n-1)} = \mathcal{L}^{n-1} \otimes [\det E_2]^{-(n-1)} \otimes E_1^n \otimes \mathcal{O}(-1),$$

due to Example 1.4.10. $\square$

3.4 Construction of a basis - an approach

In this section, we will construct an explicit basis for the Witt groups in terms of almost even diagrams of which the twists and shifts of the associated generators can easily be extracted, exactly as in [BC12a] and chapter 2.

The aim is to assign a Witt class in the total Witt group to every almost even partition $\underline{\lambda}$. In order to do so in the usual way, we need to construct a map $f_{\underline{\lambda}} : \tilde{Y}_X(\underline{\lambda}) \longrightarrow \mathrm{LG}_X(n, \mathcal{V})$ mapping birationally into the Schubert subscheme associated with $\underline{\lambda}$, i.e. a resolution. Recall that by [CH11] we then have push-forwards

$$(f_{\underline{\lambda}})_* : W^i\left(\tilde{Y}_X(\underline{\lambda}), \omega_{f_{\underline{\lambda}}} \otimes f_{\underline{\lambda}}^*(\mathcal{M})\right) \longrightarrow W^{i-\dim f_{\underline{\lambda}}}(\mathrm{LG}_X(n, \mathcal{V}), \mathcal{M})$$

for any line bundle $\mathcal{M} \in \mathrm{Pic}(\mathrm{LG}_X(n, \mathcal{V}))$, where $\dim f_{\underline{\lambda}}$ denotes the constant relative dimension of $f_{\underline{\lambda}}$. In the case of ordinary and orthogonal Grassmannians the resolution was $\tilde{Y}_X(\underline{\lambda}) = \mathrm{Fl}_X(\underline{\lambda})$ resp. $\tilde{Y}_X(\underline{\lambda}) = \mathrm{OFl}_X(\underline{\lambda})$ and defined the basis element associated with the even Young diagram resp. even shifted partition $\underline{\lambda}$ as the lax push-forward of $1 \in W^0(\tilde{Y}_X(\underline{\lambda}), \mathcal{O})$ along $(f_{\underline{\lambda}})_*$. This cannot work in the same generality in the Lagrangian case, as the following example shows.

Example 3.4.1. Let $n \geq 3$ be odd and $\underline{\lambda} = (1)$ which is always almost even with tuples $\underline{d} = d = 1$ and $\underline{e} = e = n-1$. Then taking $\tilde{Y}_X(\underline{\lambda}) = \mathrm{IFl}_X(d,e) = \{E_n \supset P_1 \subset P_n\}$ gives us

$$\omega_{f_{\underline{\lambda}}} = [\det E_n]^{-1} \cdot \mathcal{L}^{l(\underline{\lambda})} \cdot \Delta_n^{-1}$$

and pushing forward the unit form produces a class in the twisted Witt group

$$W^1 \left(\mathrm{LG}_X(n,\mathcal{V}), [\det E_n] \cdot \mathcal{L} \cdot \mathcal{O}(1)\right).$$

But this group vanishes by Theorem 3.3.12, so this push-forward cannot be part of a total basis. We also mention that a similar version of Lemma 2.3.8 which we finally need in order to apply Theorem 1.2.19 fails in the Lagrangian case with this choice of resolutions. This is due to *almost* evenness, which causes blow-ups along centers of even codimension and these make it hard to trace push-forwards.

Hence, one might like to replace the pair $(\tilde{Y}_X(\underline{\lambda}), f_{\underline{\lambda}})$ with a different resolution. Actually, there is a priori no reason to take a resolution at all - any push-forward, e.g. along projective bundles, could produce a Witt class. However, intuitive calculations (see chapter 4) concerning twists and shifts and comparisons to results by Zibrowius ([Zib11]) over $\mathrm{Spec}(\mathbb{C})$ make us believe, that maps of relative dimension 0 should lead to the desired result. Moreover, push-forwards of unit forms along projective bundles are zero if they exist at all ([CH11, 7.3]), so at least those cannot be interesting.

Assume that $f_{\underline{\lambda}} : \tilde{Y}_X(\underline{\lambda}) \to Y_X(\underline{\lambda}) \subset \mathrm{LG}_X(n-1,\mathcal{V})$ is a resolution of the Schubert cell associated with some almost even shifted partition $\underline{\lambda} \in \mathfrak{F}_{n-1}$. The aim is to have the following: If $\overline{\partial}(\underline{\lambda})$ exists (that is, if $d_1 < n-1$ is odd or $d_1 = n-1$ is even), in the square

$$\begin{tikzcd}
W^{|\underline{\lambda}|+1-n}\left(Z, \mathcal{O}|_Z\right) & W^{|\lambda|}\left(U, \mathcal{O}(1)|_U\right) \arrow[l, "\partial"'] \\
W^0\left(\tilde{Y}_X(\overline{\partial}(\underline{\lambda}), \mathcal{O}\right) \arrow[u, "f_{\overline{\partial}(\underline{\lambda})}"] & W^0(\tilde{Y}_X(\underline{\lambda}), \mathcal{O}) \arrow[l, "\tilde{\partial}"'] \arrow[u, "f_{\underline{\lambda}}"']
\end{tikzcd}$$

there should be an induced map $\tilde{\partial}$ on the resolutions and the unit form should be preserved, up to lax-similitude, when pushed forward along $\tilde{\partial}$. In the ordinary and orthogonal setting, this has been done by showing that $\tilde{\partial}$ is a sequence of push-forwards along blow-ups along centers of odd codimension and pull-backs (i.e. restrictions). Here, this turns out to be much harder, especially due to the second blow-up which is of codimension 2. Nevertheless, we have that ∂ is an isomorphism and if $\tilde{\partial}$ is an isomorphism as well, which there is no obvious reason for, this would lead to the desired result.

Consider the following example, where all Schubert varieties are smooth, i.e. no resolution is necessary.

Example 3.4.2 (a very smooth case). Let $\underline{\lambda} \in \mathfrak{F}_{n-1}$ be almost even such that $\overline{\partial}(\underline{\lambda})$ exists and both $Y_X(\underline{\lambda})$ and $Y_X(\overline{\partial}(\underline{\lambda}))$ are smooth. One easily checks that in this case n is odd and $\underline{\lambda}$ is given by the tuples $\underline{d} = (1)$ and $\underline{e} = (0)$. Then the associated Schubert subscheme is a smaller Lagrangian Grassmannian

$$Y_X(\underline{\lambda}) = \mathrm{LG}_X(n-2, E_2^{\perp}/E_2) \subset \mathrm{LG}_X(n-1, E_1^{\perp}/E_1). \tag{3.24}$$

Moreover, $\overline{\partial}(\underline{\lambda}) = 0$, i.e.

$$Y_X(\overline{\partial}(\underline{\lambda})) = \mathrm{LG}_X(n-1, E_1^{\perp}/E_1) = Z. \tag{3.25}$$

Let $\tilde{Y}_X = Y_X$ for booth Schubert schemes and denote by $f_{\underline{\lambda}}$ and $f_{\overline{\partial}(\underline{\lambda})}$ the embeddings into Z and Y, respectively. Abbreviating $\mathcal{M} := \mathcal{L}^{n-1} \cdot E_1^n \cdot [\det E_2]^{-(n-1)} \in \mathrm{Pic}(X)$, such that $\omega_{\underline{\lambda}} = \mathcal{M} \cdot \Delta_n^{-1}$, gives us a diagram

$$\begin{tikzcd}
W^{n-1}(U, \mathcal{L}^{-n} \cdot E_1^{-n+2} \cdot [\det E_2]^{n-1} \cdot \Delta_n) \arrow[r, "\partial"] & W^0(Z, E_1^{-2n+2} \cdot [\det E_2]^{n-1}) \\
W^{n-1}(Y, \mathcal{M}^{-1} \cdot \Delta_n) \arrow[u, "\alpha^*"] & W^0(Z, \mathcal{O}) \arrow[u, "\sim"'] \\
W^0(Y_X(\underline{\lambda}), \mathcal{O}) \arrow[u, "f_{\underline{\lambda}}"] & W^0(Y_X(\overline{\partial}(\underline{\lambda})), \mathcal{O}) \arrow[l, "v^*"'] \arrow[u, equal, "f_{\overline{\partial}(\underline{\lambda})}"']
\end{tikzcd} \tag{3.26}$$

where the upper left vertical map is the isomorphism on Witt groups induced by the affine bundle $\alpha : U \to Y$ and the upper right vertical map is a periodicity isomorphism. The map v^* is the restriction map in the localization sequence

$$\ldots \to W^i(\mathrm{LG}(n-1), \mathcal{O}) \xrightarrow{v^*} W^i(\mathrm{LG}(n-2), \mathcal{O}) \to W^{i+2-n}(\mathrm{LG}(n-2), \omega_\iota) \to \ldots$$

for the inclusion $\iota : \mathrm{LG}(n-2) \subset \mathrm{LG}(n-1)$. Since $n-2$ is odd, all the twisted Witt groups for $\mathrm{LG}(n-2)$ vanish and we see, that v^* is an isomorphism. In particular, it preserves the unit form (up to lax similitude). If we denote by $\phi_{n-1}(\underline{\lambda})$ the lax push-forward of the unit form along $f_{\underline{\lambda}}$, as a direct consequence we have

$$\partial(\alpha^*(\phi_{n-1}(\underline{\lambda}))) \leftrightsquigarrow (\iota_Z)_*(\phi_{n-1}(\overline{\partial}(\underline{\lambda}))).$$

In other words, the crucial combatibility condition here is satisfied.

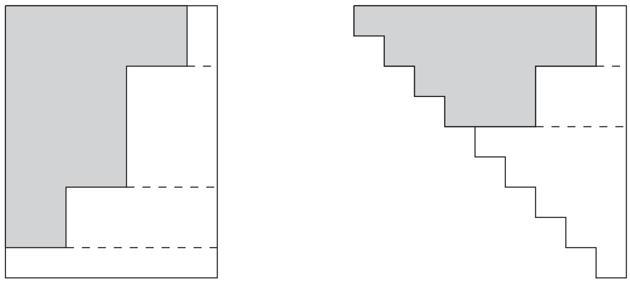


FIGURE 3.1. Standard resolutions for ordinary and maximal orthogonal Grassmannians.

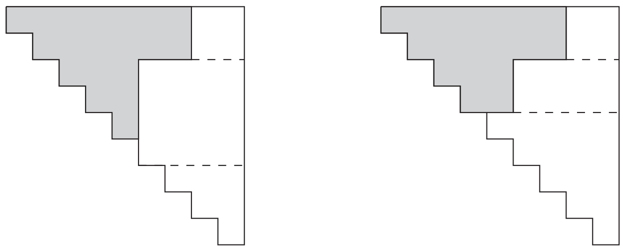


FIGURE 3.2

We now introduce a pseudo resolution of Schubert cells, which produces Witt classes of the right twists and shifts via lax push-forwards. As a motivation, consider the standard resolutions of Schubert varieties of ordinary and orthogonal Grassmannians corresponding to even partitions which we illustrated in Figure 3.1.

In order to be able to push forward the unit form along the resolution, the relative canonical bundle needs to be a square modulo pull-backs. Translated into the language of Young diagrams this means, more or less, that the heights of the rectangles (except for the lowest rectangle for the ordinary and the lowest triangle for the orthogonal Grassmannian) need to be even. The standard resolution does indeed respect this condition for even partitions for obvious reasons but it fails in the case of almost even partitions. A solution to this problem could be a pseudo-resolution, such as drawn in Figure 3.2. Whenever $d_k + e_k < n$ this is just the standard resolution, in particular smooth, but for the cases $d_k + e_k = n$ (to which in particular Example 3.4.1 belongs) it is slightly different and not smooth in general, but at least Gorenstein. This theory is discussed in detail in [Per07] for minuscule varieties and can easily be adapted to cominuscule varieties, such as the Lagrangian Grassmannian, which has been done by Perrin in an unpublished preprint. In the following, we summarize its basics. For this, recall that there is an equivalence between quivers and Young diagrams.

Let V be a symplectic vector space of dimension $2n$ and denote by $\mathrm{LG}(n) = C_n/P_n$ as usual the Lagrangian Grassmannian of maximal isotropic subspaces of V. For each shifted partition $\lambda \in \mathfrak{S}_n$ there is a Schubert variety $Y(\underline{\lambda}) \subset \mathrm{LG}(n)$. Recall that $Y(\underline{\lambda}) = Y(w)$ for the shortest element $w \in W$ (which is unique upto commuting relations) of some class $\overline{w} \in W/W_{P_n}$. The element w has a reduced expression of length $\frac{n(n+1)}{2} - |\lambda|$ with simple reflections as factors and we can associate a quiver $Q_{\underline{\lambda}} = Q_w$. In [Per07] the author describes towers

$$\tilde{Y}(\underline{\lambda}) \to \hat{Y}(\underline{\lambda}) \to Y(\underline{\lambda}) \to \mathrm{LG}(n)$$

of birational maps of varieties, where $\tilde{Y}(\underline{\lambda})$ is the Bott-Samelson resolution of $Y(\underline{\lambda})$ and $\hat{Y}(\underline{\lambda})$ is some "intermediate" or "pseudo" resolution which can be constructed for any partition of the quivers into subquivers. In this sense, the Bott-Samelson resolution is constructed as the finest possible partition of $Q_{\underline{\lambda}}$ into subquivers consisting of one vertex each. However, in many cases $\tilde{Y}(\underline{\lambda})$ is too fine and one is interested in corser resolutions. The partition we aim for is drawn in Figure 3.2 (recall that the quiver is given by the white complement together with the coloring by roots) and we denote it again by $\hat{Y}(\underline{\lambda})$. We also mention that many properties such as smoothness and Gorensteinness of the resulting resolution can be directly read off the quiver and its subquivers as in Theorem 1.5.20.

We now want to compute the canonical divisor of $\hat{Y}(\underline{\lambda})$. Assume that $Y(\underline{\lambda})$ has dimension $\frac{n(n+1)}{2} - |\lambda| = r$, i.e. the quiver $Q_{\underline{\lambda}}$ has r vertices. Then we can choose special divisors $Z_1, \ldots, Z_r$ in $\tilde{Y}(\underline{\lambda})$ such that their classes in the Chow ring, denoted by $\xi_1, \ldots, \xi_r$, form a basis of $A^*(\tilde{Y}(\underline{\lambda}))$. Generators of the divisor class group of $\hat{Y}(\underline{\lambda})$ now are obtained pushing forward the generators ξ_i to $\hat{Y}(\underline{\lambda})$ whenever i is a *peak* of a subquiver. Finally the canonical divisor of $\hat{Y}(\underline{\lambda})$ is obtained by pushing forward the canonical divisor of the Bott-Samelson resolution. We can parametrize all peaks in terms of our k-tuples $\underline{d}, \underline{e}$. A calculation shows the following:

Theorem 3.4.3. *Let $\underline{\lambda} \in \mathfrak{F}_n$. Denote by $p_0, \ldots, p_k$ be the peaks of the partition of the associated quiver as above where p_0 only exists if $e_1 > 0$ and p_k only exists if $d_1 \leq n-2$. As usual, we number them from left to right in the quiver, i.e. in a way that p_k is the unique peak associated with the long simple root α_n. Denote by $D_{p_c}, \ldots, D_{p_l}$ the push-forwards of the associated generators of the Chow ring of the Bott-Samelson resolution.Then these divisors form a basis of the divisor class group of $\hat{Y}(\underline{\lambda})$ and the canonical divisor is given by*

$$-K_{\hat{Y}(\underline{\lambda})} = (n+2)D_{p_0}$$

if $d_1 = n-1$ and by

$$-K_{\hat{Y}(\underline{\lambda})} = \sum_{i=c}^{k-1} (h(p_i)+2)\, D_{p_i} + \left(\frac{h(p_k)+3}{2}\right) D_{p_k} \tag{3.27}$$

$$= \sum_{i=c}^{k-1} (n+1-d_i+e_{i+1}) D_{p_i} + \left(\frac{n+2-2d_{k+1}+d_k+e_{k+1}}{2}\right) D_{p_k}. \tag{3.28}$$

otherwise. Denote by $D \in \operatorname{Pic}(\mathrm{LG}(n))$ *the generator of the Picard group such that the canonical divisor of* $\mathrm{LG}(n)$ *is given by* $-K_{\mathrm{LG}(n)} = (n+1)D$*. Then its pull-back to* $\hat{Y}(\underline{\lambda})$ *along the projection* $\hat{\pi} : \hat{Y}(\underline{\lambda}) \to \mathrm{LG}(n)$ *is given by*

$$\hat{\pi}^*(D) = \left(\sum_{i=c}^{k-1} 2D_{p_i}\right) + D_{p_k} \tag{3.29}$$

and hence the relative canonical divisor is given by

$$-K_{\hat{Y}(\underline{\lambda})/\mathrm{LG}(n)} = \begin{cases} -nD_{p_0}, & \text{if } d_1 = n-1, \\ \sum_{i=c}^{k-1}(-n-3-d_i+e_{i+1})D_{p_i} + (-d_{k+1})D_{p_k}, & \text{otherwise,} \end{cases}$$

where we used $n - d_k = e_{k+1}$*.*

proof sketch. This has been proved in [Per07, 4.15] for minuscule varieties. Without giving a complete proof we highlight the differences in the cominuscule case. Recall that in type C we have short and long roots. Then in [Per07, Proposition 2.16] we need to adjust the formula which gives

$$\mathcal{L}_r = \xi_r + \sum_{k=1}^{r-1} 2\xi_k \tag{3.30}$$

and, writing $|\cdot|$ for the length of a root, [Per02, Corollary 2.18] becomes

$$\sum_{j=k+1, \beta_j=\beta_r}^{r} \langle \gamma_k^\vee, \gamma_j \rangle = \begin{cases} 1, & \text{if } \beta_k \neq \beta_r \text{ and } |\beta_k| \geq |\beta_r|, \\ 2, & \text{if } \beta_k \neq \beta_r \text{ and } |\beta_k| < |\beta_r|, \\ 0, & \text{if } \beta_k = \beta_r. \end{cases} \tag{3.31}$$

Finally, it follows that [Per07, Lemma 4.16] modifies to

$$\sum_{k=i}^{r} \langle \alpha_i^\vee, \alpha_k \rangle = \begin{cases} \frac{h(i)+3}{2}, & \text{if } \beta(i) = \alpha_n \text{ is long,} \\ h(i)+2, & \text{if } \beta(i) \in \{\alpha_1, \dots, \alpha_{n-1}\} \text{ is short.} \end{cases} \tag{3.32}$$

Then (3.27) follows by pushing forward the canonical divisor of the Bott-Samelson resolution into $\hat{Y}(\underline{\lambda})$. To obtain (3.28) one writes the heights of the peaks in terms of the k-tuples as in the proof of Lemma 2.2.1 for the orthogonal case. □

Lemma 3.4.4. *Let $\underline{\lambda} \in \mathfrak{F}_n$ be almost even. Then the following holds for the coefficients above:*

(i) If $d_1 = n-1$, then $-n$ is even.

(ii) For all $c \leq i \leq k-1$ we have that $-n-3-d_i+e_{i+1}$ is even.

(iii) If $e_1 > 0$, then $-d_{k+1}$ is even.

(iv) If $e_1 = 0$, then $-d_{k+1} - d_1$ is even.

Proof. (i) is clear, (iii) follows from Definition 1.5.9 and (iv) holds since $d_{k+1} - d_{k-1}$ is even by definition. It remains to show (ii). Assume $k \geq 2$, otherwise nothing is to show. Then

$$n-1-d_{k-1}+e_k \equiv -n-3-d_i+e_{i+1} \equiv n-1-d_1+e_2 \mod 2$$

for all $1 \leq i \leq k-1$. If d_1 is odd, then $e_1 = 0$ and by Lemma 1.5.11 n is even, which shows (i). Let now d_1 be even. If d_k is odd (i.e. the odd inner segment is vertical), then $n - d_k = e_k$ is odd and hence (i). If, on the other hand d_k is even, then $e_{k+1} - e_k = n - d_k - e_k$ is odd and again we conclude (ii).

Theorem 3.4.5. *Let $\underline{\lambda} \in \mathfrak{F}_n$ be almost even and let $T(\underline{\lambda}) = \mathcal{O}(t(\underline{\lambda})) \in \mathrm{Pic}(\mathrm{LG}(n))/2$ where*

$$t(\underline{\lambda}) = \begin{cases} 0, & \text{if } e_1 > 0, \\ [d_1], & \text{if } e_1 = 0 \text{ and } d_1 < n, \\ [n+1], & \text{if } d_1 = n. \end{cases}$$

Then $\mathcal{O}(K_{\hat{Y}(\underline{\lambda})/\mathrm{LG}(n)}) \cdot \hat{\pi}^* (T(\underline{\lambda})) \cong \mathcal{O}$ and there is a lax push-forward

$$f_{\underline{\lambda}} : W^0(\hat{Y}(\underline{\lambda}), \mathcal{O}) \to W^{|\lambda|}(\mathrm{LG}(n), \mathcal{O}(t(\underline{\lambda}))).$$

Define $\phi_n(\underline{\lambda})$ to be the image of the unit form under this push-forward.

Remark 3.4.6. In order to show that the obtained elements $\phi_n(\underline{\lambda})$ form a total basis of the total Witt group of $\mathrm{LG}(n)$, one needs to apply Theorem 1.2.19 as it has been done for the ordinary and orthogonal Grassmannians. The crucial part here is to prove compatibility with the boundary map $\partial' : W^i(U, \mathcal{O}(1)|_U) \to W^{i+1-n}(Z, \mathcal{O})$, i.e. to show that

$$\partial(\phi_{n-1}(\underline{\lambda})) \leftrightsquigarrow \phi_{n-1}(\overline{\partial}(\underline{\lambda})).$$

For the accomplished cases, the main tools where [BC12a, 5.8] and Lemma 2.3.8 of which the author could not provide an analogon.

Chapter 4

Conclusions and examples

In this chapter, we state the results from chapters 2 and 3 in a more classical way by writing the Witt groups as modules over $W^0(X, \mathcal{O}_X)$. This is done in section 4.2 for the orthogonal and in 4.3 for the symplectic case. In section 4.4, we explicitely compute the ranks of these modules by showing, that these numbers coincide with the ranks of certain exterior algebras. Finally, in the last section we give a summary of the thesis. For easier notation let us write $\mathrm{IG}(n) := \mathrm{IG}_X(n, \mathcal{V})$ whenever no confusion can occur. Also, if not mentioned otherwise, all partitions are shifted and strict. Moreover, we simultaneously use the notions of shifted partitions and shifted Young diagrams and denote by $\mathfrak{S}_n, \mathfrak{E}_n$ and $\mathfrak{F}_n$ the set of strict shifted, even strict shifted and almost even strict shifted partitions of n or shifted Young diagrams in the shifted $n \times n$ frame.

4.1 Combinatorics on shifted Young diagrams

Lemma 4.1.1. *(i) There are 2^{n-1} partitions in $\mathfrak{S}_{n-1}$ (for $n \geq 2$).*

(ii) There are $2^{n'}$ even partitions in $\mathfrak{E}_{n-1}$ where $n' = \lfloor \frac{n}{2} \rfloor$ (for $n \geq 2$).

(iii) There are $2^{n'+1}$ almost even partitions in $\mathfrak{F}_n$ (for $n \geq 1$).

Proof. (i) By induction. For $n = 2$ there are two partitions. Now suppose that there are 2^{n-1} partitions in $\mathfrak{E}_{n-1}$. We want to extend them by a column on the right as follows: For $0 \leq x \leq n-1$, denote by $S_{n-1}(x)$ the number of partitions in $\mathfrak{S}_{n-1}$ satisfying $d_1 = x$. By induction hypothesis we have

$$S_{n-1}(x) = \begin{cases} 2^{n-2-x}, & \text{for } 0 \leq x \leq n-1, \\ 1, & \text{for } x = n-1. \end{cases}$$

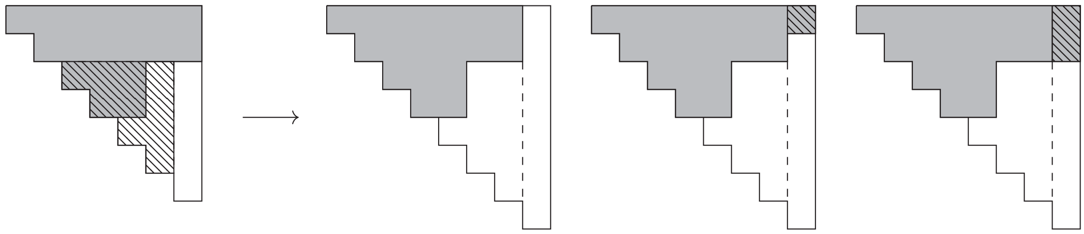


FIGURE 4.1. On the left a partition $\underline{\lambda}$ in $\mathfrak{S}_7$ with $d_1 = 2$ where we cut off the first d_1 rows and then the rightmost column. Partitions in the resulting hatched frame (which is $\mathfrak{S}_4$) give rise to all partitions in $\mathfrak{S}_7$ satisfying $d_1 = 2$. Now $\underline{\lambda}$ induces $d_1 + 1 = 3$ partitions in $\mathfrak{S}_8$ by filling up the last column with at most 2 boxes for which there are three possibilities.

Indeed, for a partition to satisfy $d_1 \geq x$ means that the first x rows are fixed and the variable part is a partition in $\mathfrak{S}_{n-1-x}$ in the lower part. Moreover, $d_1 \not> x$ implies, that the rightmost column of this smaller partition is empty (otherwise $d_1 > x$) and hence we end up with a partition in $\mathfrak{S}_{n-2-x}$, see Figure 4.1. For a partition $\underline{\lambda} \in \mathfrak{S}_{n-1}$ with $d_1 = x$ there are $x + 1$ partitions in $\mathfrak{S}_n$ via specifying the number of boxes in the last column from 0 to x. Clearly all partitions in $\mathfrak{S}_n$ can be obtained with this construction and they are distinct. Hence, there are

$$\begin{aligned} 1 + \sum_{x=0}^{n-1} S_n(x)(x+1) &= 1 + n + \sum_{i=0}^{n-2} 2^{n-2-i}(1+i) \\ &= 1 + n + 2^{n-1} \cdot \left(\sum_{i=0}^{n-1} i \cdot \left(\frac{1}{2}\right)^i \right) = 2^n \end{aligned}$$

partitions in $\mathfrak{S}_n$ where we used

$$\sum_{k=0}^{n-1} kz^k = \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) \sum_{k=0}^{n-1} z^k = \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\frac{z^n - 1}{z - 1}\right) = \frac{nz^n(z-1) - z(z^n - 1)}{(z-1)^2}.$$

(ii) First, let n be odd. Then the partitions in $\mathfrak{E}_{n-1}$ are in bijection with (not necessarily even) partitions in $\mathfrak{S}_{\frac{n-1}{2}}$ and by (i) we obtain the claim. Let now n be even. The partitions with odd d_1 bijectively correspond to even partitions in $\mathfrak{E}_{n-2}$ via $(\bar{\iota})^{-1}$, i.e. via cutting off the first row and the number of those partitions is $2^{(n-1)'}$ by (i). Similarly the partitions with d_1 even (and in which case necessarily $e_1 \geq 1$ is odd, since $e_1 \equiv e_{k+1} = n - 1 - d_k \equiv 1 \bmod 2$) bijectively correspond to even partitions in $\mathfrak{E}_{n-2}$ via $\overline{v}$ of which there are also $2^{(n-1)'}$ by (i). We conclude that there are precisely $2^{(n-1)'} \cdot 2 = 2^{(n-1)'+1} = 2^{\frac{n}{2}} = 2^{n'}$ even partitions in $\mathfrak{E}_{n-1}$.

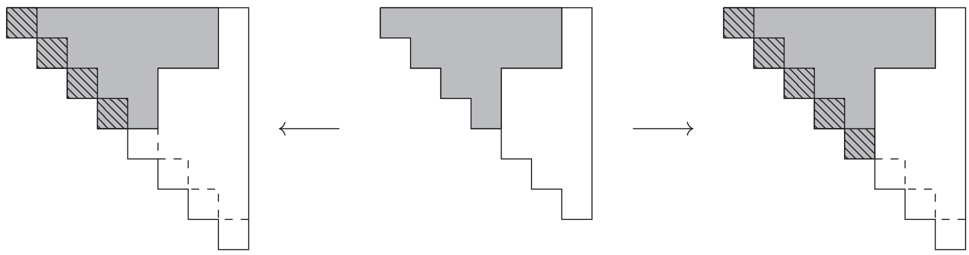


FIGURE 4.2. In the middle we drew an even shifted partition in $\mathfrak{E}_7$ which gives rise to exactly two almost even shifted partitions in $\mathfrak{F}_8$.

(iii) This follows from (ii) and the fact that each even partition in $\mathfrak{E}_{n-1}$ induces two unique distinct almost even partitions in $\mathfrak{F}_n$ as illustrated in Figure 4.2 and all elements in $\mathfrak{F}_n$ can be obtained this way. □

The following two remarks are essential for linking the recursive results on Witt groups with (almost) even partitions:

Remark 4.1.2. Let n be odd and $\underline{\lambda} \in \mathfrak{S}_{n-1}$ be even with k-tuples $\underline{d}, \underline{e}$. Then d_1 is even by 1.5.11 and e_1 is even since $e_1 + d_1 \equiv e_k + d_k \equiv n - 1$ is even. Hence, adding a full row on top and an empty column on the right are allowed operations in the sense that the resulting partition is still even and we have the following:

(i) $\underline{\lambda}$ induces two even partitions in $\mathfrak{E}_n$ via $\bar{\imath}$ and $\bar{v}^{-1}$, i.e. by either adding a full row on top or an empty column on the right and all partitions in $\mathfrak{E}_n$ are obtained this way.

(ii) $\underline{\lambda}$ induces two even partitions in $\mathfrak{E}_{n+1}$ via $\bar{\imath}^2$ and $(\bar{v}^{-1})^2$, i.e. by either adding two full rows on top or two empty columns on the right and all partitions in $\mathfrak{E}_{n+1}$ are obtained this way.

Remark 4.1.3. Let n be odd and $\underline{\lambda} \in \mathfrak{F}_n$ be almost even with k-tuples $\underline{d}, \underline{e}$. Then if $d_1 \neq n$ then d_1 is even and moreover e_1 is even if $k \geq 2$. Hence, adding empty coumns on the right and full rows on top are allowed operations in the sense that the resulting partition is still almost even. Then we have the following:

(i) $\underline{\lambda}$ induces two almost even partitions in $\mathfrak{F}_{n+1}$ via $\bar{\imath}$ and $\bar{v}^{-1}$, i.e. by either adding a full row on top or an empty column on the right and all partitions in $\mathfrak{F}_{n+1}$ arise this way.

(ii) $\underline{\lambda}$ induces two almost even partitions in $\mathfrak{F}_{n+2}$ via $\bar{\imath}^2$ and $(\bar{v}^{-1})^2$, i.e. by either adding two full rows on top or two empty columns on the right and all partitions in $\mathfrak{F}_{n+2}$ arise this way.

4.2 Reformulation for the maximal orthogonal case

Recall that a line bundle on $\mathrm{OG}(n)$ is of the form $\mathcal{K} = \pi^*\mathcal{M} \otimes \mathcal{O}(1)^{\otimes l}$ for some line bundle $\mathcal{M} \in \mathrm{Pic}(X)$ and $l \in \mathbb{Z}$, where $\mathcal{O}(1)$ denotes the ample square root of $\det \mathcal{S}^\vee$. For an even strict partition $\underline{\lambda} \in \mathfrak{E}_{n-1}$ we have $\phi_n(\underline{\lambda}) \in W^{|\lambda|}(\mathrm{OG}(n), \mathcal{L}_{\underline{\lambda}})$ for some line bundle $\mathcal{L}_{\underline{\lambda}} \in \mathrm{Pic}(\mathrm{OG}(n))$ with $\mathcal{L}_{\underline{\lambda}} \equiv T(\underline{\lambda})$ in $\mathrm{Pic}(\mathrm{OG}(n))/2$.

Theorem 4.2.1. *Let X be a smooth scheme over $\mathbb{Z}[\frac{1}{2}]$ and $(\mathcal{V}, \omega)$ be an orthogonal vector bundle of rank $2n$ over X where ω admits values in a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. Let further $\mathcal{M}$ be another line bundle over X and $l \in \mathbb{Z}$ an integer. For all even strict partitions $\underline{\lambda} \in \mathfrak{E}_{n-1}$ choose a line bundle $N_{\underline{\lambda}}$ together with an isomorphism*

$$N_{\underline{\lambda}}^{\otimes 2} \otimes \pi^*(\mathcal{M} \otimes \mathcal{L}^{l(\underline{\lambda})} \otimes [\det E_n]^{d_{k+1}-d_k}) \otimes \mathcal{L}_{\underline{\lambda}} \cong \pi^*\mathcal{M}.$$

Then there is an isomorphism of $W^0(X, \mathcal{O}_X)$-modules

$$\bigoplus_{\substack{\underline{\lambda} \in \mathfrak{E}_{n-1}, \\ t(\underline{\lambda}) = [l]_2}} W^{i-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{l(\underline{\lambda})} \otimes [\det E_n]^{d_{k+1}-d_k}) \xrightarrow{\sim} W^i(\mathrm{OG}_X(n, \mathcal{V}), \pi^*\mathcal{M} \otimes \mathcal{O}(l)) \quad (4.1)$$

sending $(x_{\underline{\lambda}})$ to $\sum x_{\underline{\lambda}} \cdot \phi_n(\underline{\lambda})$. In particular, since $t(\underline{\lambda})$ is even, for l odd we have

$$W^i(\mathrm{OG}_X(n, \mathcal{V}), \pi^*\mathcal{M} \otimes \mathcal{O}(l)) = 0. \quad (4.2)$$

whereas for l even

$$W^i(\mathrm{OG}_X(n, \mathcal{V}), \pi^*\mathcal{M} \otimes \mathcal{O}(l)) \cong \bigoplus_{\underline{\lambda} \in \mathfrak{E}_{n-1}} W^{i-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{l(\underline{\lambda})} \otimes [\det E_n]^{d_{k+1}-d_k}) \quad (4.3)$$

Proof. This is [BC12b, 6.9] applied to the computed basis. The result for the twisted Witt groups follows by the fact that all occuring twists are even. □

Hence, whenever we talk about orthogonal Grassmannians, we will restrict from now on to the untwisted Witt groups and write $W^i(\mathrm{OG}(n)) := W^i(\mathrm{OG}(n), \mathcal{O})$, since the twisted groups vanish.

Remark 4.2.2. Calmès-Fasel ([CF12]) have already developed an easy criterion for the vanishing of twisted Witt groups via Dynkin diagrams, but it does not apply to the maximal orthogonal Grassmannian. However, Zibrowius generalized this ([Zib14]) via marking schemes and from this it indeed follows that the twisted Witt groups vanish.

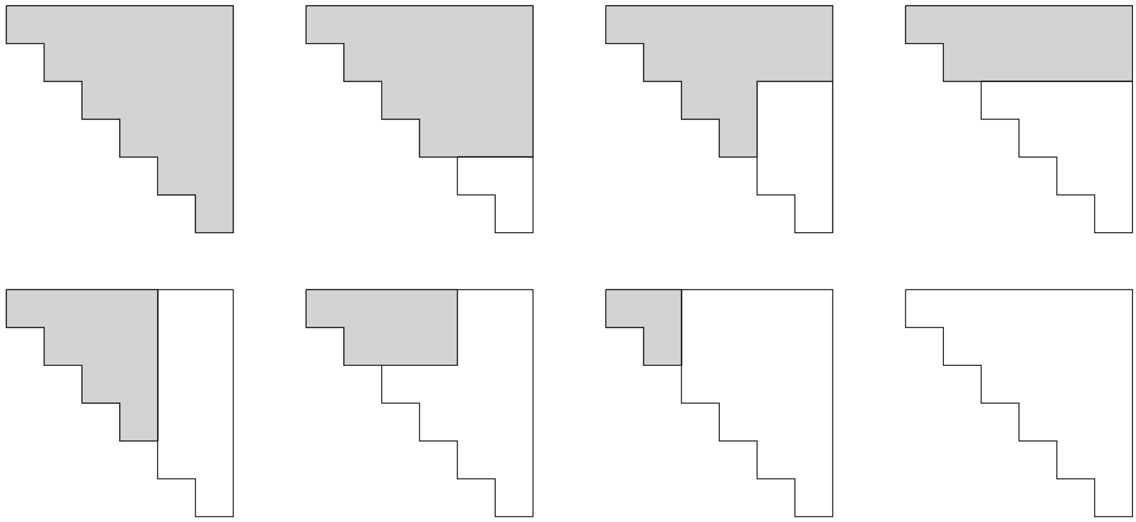


FIGURE 4.3. The eight even strict partitions in 6×6-frame, parametrizing a total basis for $W(\mathrm{OG}(7))$.

Example 4.2.3. For $\mathrm{OG}(7)$ we have eight strict partitions, see Figure 4.3. We can read the degrees of the basis elements from the weight of the corresponding partition. Here the degrees are $1, 2, 2, 3$ in the first line and $2, 3, 3, 0$ in the second line. Further all twists are trivial. Note that in the first line, chopping off the first two full rows yields a total basis of $W(\mathrm{OG}(5))$. The same occurs when chopping off the two rightmost empty columns in the second line. This is precisely a visualization of Remark 4.1.2(ii).

Using the combinatorics of section 4.1 we can describe the orthogonal Witt groups recursively:

Corollary 4.2.4. *(i) For n even we have*

$$W^i(\mathrm{OG}(n)) \cong W^{i-(n-1)}(\mathrm{OG}(n-1)) \oplus W^i(\mathrm{OG}(n-1))$$

where the isomorphism is induced by v and ι.

(ii) For n odd we have

$$W^i(\mathrm{OG}(n)) \cong W^{i-(2n-3)}(\mathrm{OG}(n-2)) \oplus W^i(\mathrm{OG}(n-2))$$

where the isomorphism is induced by v^2 and ι^2.

Proof. This follows from Theorem 4.2.1 and Remark 4.1.2. □

4.3 Reformulation for the Lagrangian case

For the Lagrangian Grassmannian, we have a similar result as in the preceeding section. Let us introduce the following "artificial" definition of twists of almost even partitions. Artificial here means that this time the definition of the twist does not come from the behaviour of the push-forward from the corresponding resolution into the Grassmannian (although we believe that one could obtain it by the old means as well). At this point we observe that this newly introduced twist coincides with the one computed in Theorem 3.4.5.

Definition 4.3.1. For an almost even shifted partition $\underline{\lambda} \in \mathfrak{F}_n$ let $t(\underline{\lambda}) \in \mathbb{Z}/2$ be defined as

$$t(\underline{\lambda}) = \begin{cases} 0, & \text{if } e_1 > 0, \\ [d_1]_2, & \text{if } e_1 = 0 \text{ and } d_1 < n, \\ [n+1]_2, & \text{if } d_1 = n. \end{cases} \tag{4.4}$$

Moreover define $l(\underline{\lambda}) \in \mathbb{Z}/2$ as

$$l(\underline{\lambda}) := \begin{cases} l'_+(\underline{\lambda}) = \frac{d_k(2n-d_k+1)}{2}, & \text{if } d_k + e_k < n, \\ l'_+(\underline{\lambda}, 0) = \frac{(d_k+1)(2n-(d_k+1)+1)}{2}, & \text{if } d_k + e_k = n. \end{cases} \tag{4.5}$$

Theorem 4.3.2. *Let X be a smooth scheme over $\mathbb{Z}[\frac{1}{2}]$ and $(\mathcal{V}, \omega)$ a symplectic vector bundle of rank $2n$ over X, where ω admits values in a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. Let $\mathcal{M}$ a line bundle over X and $l \in \mathbb{Z}$ an integer. Then there is an isomorphism of $W^0(X, \mathcal{O}_X)$-modules*

$$\bigoplus_{\substack{\underline{\lambda} \in \mathfrak{F}_n \text{ s.t.} \\ t(\underline{\lambda}) = [l]_2}} W^{i-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{-l(\underline{\lambda})}) \xrightarrow{\sim} W^i\left(\mathrm{LG}_X(n, \mathcal{V}), \pi^*\mathcal{M} \otimes \mathcal{O}(l)\right). \tag{4.6}$$

Superficially this theorem looks just like the one above for the orthogonal Grassmannians, but it is weaker in the sense that we did not actually produce basis elements. In other words the isomorphism does not come from a map of the form $(x_{\underline{\lambda}}) \mapsto \sum x_{\underline{\lambda}} \cdot \phi_n(\underline{\lambda})$ but rather from the isomorphisms computed in section 3.3.

Corollary 4.3.3. *If n is odd, $W^i(\mathrm{LG}(n), \mathcal{O}(2l+1)) = 0$ for all $i \in \mathbb{Z}$ and $l \in \mathbb{Z}$.*

Proof. This has already been shown earlier, but now we can conclude this combinatorially from the definition of the twist $t(\underline{\lambda})$, Lemma 1.5.11 and Theorem 4.3.2 in terms of almost even Young diagrams. Over $\mathbb{C}$, this again follows from [Zib14]. □

proof of Theorem 4.3.2. First, let n be odd. For $n = 1$ we have the well known isomorphisms

$$\begin{aligned} W^i(\mathrm{LG}(1), p^*\mathcal{M}) &\cong W^i(X, \mathcal{M}) \oplus W^{i-1}(X, \mathcal{M} \otimes \mathcal{L}^{-1}) \\ W^i(\mathrm{LG}(1), p^*\mathcal{M} \otimes \mathcal{O}(1)) &\cong 0 \end{aligned}$$

from Theorem 1.2.13. In the untwisted Witt group, the first summand is represented by the almost even shifted partition $\underline{\lambda}_1 = (0)$ with $l(\underline{\lambda}_1) = 0$ and twist $t(\underline{\lambda}_1) = 0$ whereas the second one is represented by $\underline{\lambda}_2 = (1)$ with $l(\underline{\lambda}_2) = 1$ and twist $t(\underline{\lambda}_2) = 0$. Now suppose, that (4.6) holds for some odd $n - 1 \geq 3$. By Theorem 3.3.12 we have (modulo $\mathrm{Pic}(X)$) $W^i(\mathrm{LG}(n), p^*\mathcal{M} \otimes \mathcal{O}(1)) \cong 0$ and

$$W^i(\mathrm{LG}(n), p^*\mathcal{M}) \cong W^{i-2n+1}(\mathrm{LG}(n-2), \mathcal{L} \otimes p^*\mathcal{M}) \oplus W^i(\mathrm{LG}(n-2), p^*\mathcal{M}).$$

We now want to link these results with almost even shifted partitions by using the combinatorics of the previous section. Note that we did not make use of them for the orthogonal case, since in that case we knew more, namely the basis. Now assume that the Witt groups of $\mathrm{LG}(n-2)$ are indexed by almost even shifted partitions in the described sense. By Remark 4.1.3, any such partition $\underline{\lambda} \in \mathfrak{F}_{n-2}$ determines exactly two almost even shifted partitions $\underline{\lambda}_1, \underline{\lambda}_2$ in $\mathfrak{F}_n$ by either adding two empty columns on the right or by adding two full rows on the top (here we need n odd which implies $d_1 < n$ even or $d_1 = n$). We need to show that the twists and shifts of these newly produced partitions are the right ones.

- The first procedure does not change the weight $|\underline{\lambda}|$ whereas the second one increases it by $2n - 1$.
- We have $t(\underline{\lambda}_1) = t(\underline{\lambda}_2) = [0]$.
- Let $\underline{d}, \underline{e}$ be the k-tuples associated with $\underline{\lambda}$ and $\underline{d}^i, \underline{e}^i$ the ones for $\underline{\lambda}_i$ for $i = 1, 2$. Then $d^1_k = d_k + 2$ and $d^2_k = d_k$ and a direct calculations shows

  $$l(\underline{\lambda}_1) \not\equiv l(\underline{\lambda}), \qquad l(\underline{\lambda}_2) \equiv l(\underline{\lambda}).$$

  In other words, the second procedure requires another factor $\mathcal{L}$.

Putting these observations together we have

$$\begin{aligned} W^i(\mathrm{LG}(n), p^*\mathcal{M}) &\cong W^{i-2n+1}(\mathrm{LG}(n-2), \mathcal{L} \otimes p^*\mathcal{M}) \oplus W^i(\mathrm{LG}(n-2), p^*\mathcal{M}) \\ &\cong \bigoplus_{\underline{\lambda} \in \mathfrak{F}_{n-2}} W^{i-2n+1-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{-l(\underline{\lambda})+1}) \oplus \end{aligned}$$

$$\begin{aligned} & \bigoplus_{\underline{\lambda} \in \mathfrak{F}_{n-2}} W^{i-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{-l(\underline{\lambda})}) \\ \cong & \bigoplus_{\underline{\lambda} \in \mathfrak{F}_{n}} W^{i-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{-l(\underline{\lambda})}) \end{aligned}$$

which proves the theorem for odd n. Let now n be even. Then again by Theorem 3.3.12, we have

$$\begin{aligned} W^i(\mathrm{LG}(n), p^*\mathcal{M}) &\cong W^i(\mathrm{LG}(n-1), p^*\mathcal{M}), \\ W^i(\mathrm{LG}(n), p^*\mathcal{M} \otimes \mathcal{O}(1)) &\cong W^{i-n}(\mathrm{LG}(n-1), p^*\mathcal{M}). \end{aligned}$$

Now the proof works as in the odd case. Note that by Remark 4.1.3 any almost even partition $\underline{\lambda} \in \mathfrak{F}_{n-1}$ again gives rise to exactly two partitions $\underline{\lambda}_1, \underline{\lambda}_2 \in \mathfrak{F}_n$ via adding an empty column and a full row and all partitions in $\mathfrak{F}_n$ arise this way. One easily checks that the first procedure neither changes the twist nor the weight whereas the second one flips the twists and increases the weight by n. Moreover $l(\underline{\lambda}_1) \equiv l(\underline{\lambda}) \equiv l(\underline{\lambda}_2)$. Hence

$$\begin{aligned} W^i(\mathrm{LG}(n), p^*\mathcal{M}) &\cong W^i(\mathrm{LG}(n-1), p^*\mathcal{M}) \\ &\cong \bigoplus_{\underline{\lambda} \in \mathfrak{F}_{n-1}} W^{i-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{-l(\underline{\lambda})}) \\ &\cong \bigoplus_{\underline{\lambda} \in \mathfrak{F}_{n}} W^{i-|\underline{\lambda}|}(X, \mathcal{M} \otimes \mathcal{L}^{-l(\underline{\lambda})}) \end{aligned}$$

and similar for the twisted Witt group. □

4.4 Enumerative results

In this section, we write $\mathrm{IG}(n) = \mathrm{IG}_X(n, \mathcal{V})$ and we consider twists in $\mathrm{Pic}_X(\mathrm{IG}(n))/2$. Hence, we are left with only two distinct twists $\mathcal{O}$ and $\mathcal{O}(1)$. Since no twists occur for OG, we write $W^i(\mathrm{OG}(n)) := W^i(\mathrm{OG}(n), \mathcal{O})$. By *orthogonal* and *symplectic Witt groups* we mean the Witt groups of maximal orthogonal and Lagrangian Grassmannians, respectively.

Notation 4.4.1. Write

$$r_{\mathrm{OG}}(n, i) := \mathrm{rk}(W^i(\mathrm{OG}(n))) \tag{4.7}$$

for the ranks of the orthogonal Witt groups, considered as a $W^{\mathrm{tot}}(X)$-modules.

Notation 4.4.2. For an integer $n \in \mathbb{Z}$ and for $0 \leq i \leq 3$ define the number as in [Zib11]

$$\rho(n, i) := \sum_{m \equiv i \bmod 4} \binom{n}{m}. \tag{4.8}$$

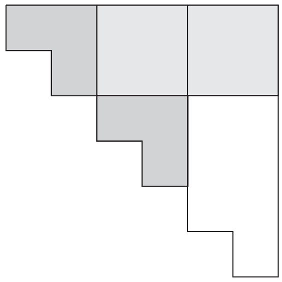


FIGURE 4.4. An even shifted partition for OG(7) consisting of two triangles and two squares. The corresponding basis element is in $W^2(\mathrm{OG}(7))$.

Theorem 4.4.3. *Let $n \geq 2$ and $n' := \lfloor \frac{n}{2} \rfloor$. Then*

$$r_{\mathrm{OG}}(n,i) = \begin{cases} \rho(n', 1-i), & \text{if } n \equiv 2 \mod 4, \\ \rho(n', -i), & \text{otherwise.} \end{cases} \tag{4.9}$$

Proof. First, let $n \geq 3$ be odd. The even partitions are composed by two different types of blocks, namely triangles (containing three boxes, thus reducing the degree of the Witt class by one) and 2×2-blocks (not changing the degree), see Figure 4.4. Now fix $0 \leq i \leq 3$. For the rank of $W^i(\mathrm{OG}(n))$ we need to count for each $m \in \mathbb{N}$ the number of even shifted partitions with $4m - i$ triangles. But these correspond to even Young diagrams in the rectangle on the right of the triangles which is of size $((n-1) - 2(4m-i)) \times 2(4m-i)$, see Figure 4.5. By the well-known fact

$$\#\{\text{even Young diagrams in } d \times e\text{-frame}\} = \binom{d'+e'}{e'}, \qquad d' = \left\lfloor \frac{d}{2} \right\rfloor, e' = \left\lfloor \frac{e}{2} \right\rfloor$$

we conclude

$$r_{\mathrm{OG}}(n,i) = \sum_{m \in \mathbb{N}} \binom{\lfloor \frac{(n-1)-2(4m-i)+2(4m-i)}{2} \rfloor}{4m-i} = \sum_{m \in \mathbb{N}} \binom{n'}{4m-i} = \rho(n', -i).$$

Let now n be even. We have $\mathrm{OG}(2) = \mathbb{P}^1$ and the statement is clear. Let $n \geq 4$. Recall from Remark 4.1.2 that for each even partition in $\mathfrak{E}_{n-2}$ (for $\mathrm{OG}(n-1)$) there are exactly two even partitions for $\mathrm{OG}(n)$ - one with an empty column added to the right and one with a full first row added on top. Adding an empty column does not change the shift whereas adding a full row increases it by $n-1$. We conclude

$$\begin{aligned} r_{\mathrm{OG}}(n,i) &= r_{\mathrm{OG}}(n-1,i) + r_{\mathrm{OG}}(n-1, i-(n-1)) \\ &= \rho\left((n-1)', -i\right) + \rho\left((n-1)', -i + (n-1)\right) \end{aligned}$$

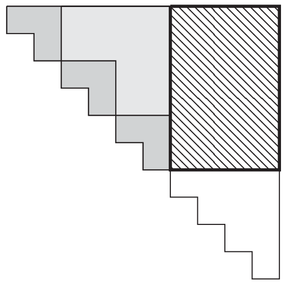


FIGURE 4.5. For OG(11), the number of even shifted partitions with three triangles (yielding the rank of $W^1(\mathrm{OG}(11))$) is given by the number of even young diagrams in the hatched rectangle of size 6×4, i.e. the number of all possible Young diagrams in 3×2-frame, which is given by $\binom{3+2}{2} = 10$.

$$
\begin{aligned}
&= \sum_{\substack{m=0,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m} + \sum_{\substack{m=0,\\ m\equiv -i+(n-1) \bmod 4}}^{n'-1} \binom{(n-1)'}{m} \\
&= \sum_{\substack{m=0,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m} + \sum_{\substack{m=0,\\ m\equiv -i+(n-1) \bmod 4}}^{n'-1} \binom{n'-1}{m}. \qquad (4.10)
\end{aligned}
$$

- Assume $n \equiv 2 \mod 4$, i.e $n' \equiv 1$. Then (4.10) becomes

$$
\begin{aligned}
r_{\mathrm{OG}}(n,i) &= \sum_{\substack{m=0,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m} + \sum_{\substack{m=0,\\ m\equiv -i+1 \bmod 4}}^{n'-1} \binom{n'-1}{m} \\
&= \sum_{\substack{m=0,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m} + \sum_{\substack{m=-1,\\ m\equiv -i \bmod 4}}^{n'-2} \binom{n'-1}{m+1} \\
&= \sum_{\substack{m=-1,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m} + \sum_{\substack{m=-1,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m+1} \\
&= \sum_{\substack{m=-1,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'}{m+1} \\
&= \sum_{\substack{m=0,\\ m\equiv -i+1 \bmod 4}}^{n'} \binom{n'}{m} \\
&= r_{\mathrm{OG}}(n+1,i-1) \\
&= \rho\left(n',1-i\right).
\end{aligned}
$$

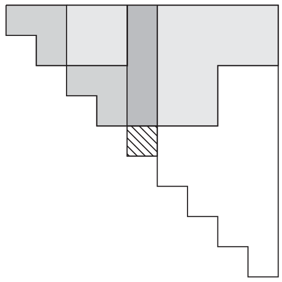


FIGURE 4.6. An almost even shifted partition for LG(9) consisting of two triangles, a 4×1 rectangle, four 2×2 blocks and the optional box.

- Analogously for $n \equiv 0 \mod 4$ we obtain

$$\begin{aligned} r_{\mathrm{OG}}(n,i) &= \sum_{\substack{m=0,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m} + \sum_{\substack{m=0,\\ m\equiv -i-1 \bmod 4}}^{n'-1} \binom{n'-1}{m} \\ &= \sum_{\substack{m=0,\\ m\equiv -i \bmod 4}}^{n'-1} \binom{n'-1}{m} + \sum_{\substack{m=1,\\ m\equiv -i \bmod 4}}^{n'} \binom{n'-1}{m-1} \\ &= \sum_{\substack{m=0,\\ m\equiv -i \bmod 4}}^{n'} \binom{n'}{m} \\ &= \rho(n',-i), \end{aligned}$$

finishing the proof. $\square$

Notation 4.4.4. Write

$$r_{\mathrm{LG}}(n,i,0) := \mathrm{rk}(W^i(\mathrm{LG}(n)), \qquad r_{\mathrm{LG}}(n,i,1) := \mathrm{rk}(W^i(\mathrm{LG}(n),\mathcal{O}(1)).$$

for the ranks of the symplectic Witt groups.

Theorem 4.4.5. *Let $n \geq 1$ and write $(n+1)' = \lfloor \frac{n+1}{2} \rfloor$. Then*

$$r_{\mathrm{LG}}(n,i,t) \begin{cases} 0, & \text{if } n \text{ is odd and } t=1, \\ \rho((n+1)',i-n), & \text{if } n \text{ is even and } t=1, \\ \rho((n+1)',i), & \text{otherwise.} \end{cases}$$

Proof. First, let n be odd. The vanishing of the twisted Witt groups has already been shown in Theorem 3.3.12. However, we want to give an alternative proof by means of

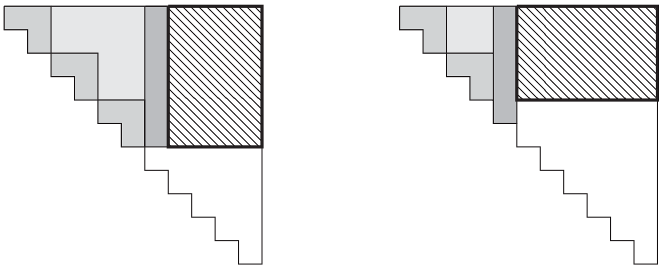


FIGURE 4.7. For LG(11) almost even partitions contributing to W^3 either have 3 triangles without the optional box or 2 triangles with the optional box. The only freedom is the hatched rectangle where we need to fit even Young diagrams of size 6×4 on the left and 4×6 on the right. Hence $W^3(\mathrm{LG}(11))$ has rank $\binom{5}{2} + \binom{5}{3} = 20$ over $W(X)$.

shifted partitions. We first show, that all almost even shifted partitions have trivial twist. Recall that the twist is given by

$$t(\underline{\lambda}) = \begin{cases} 0, & \text{if } e_1 > 0, \\ [d_1], & \text{if } e_1 = 0 \text{ and } d_1 < n, \\ [n+1], & \text{if } d_1 = n. \end{cases}$$

By Lemma 1.5.11 the middle case does not occur, if n is odd and since $[n+1]$ is even, all twists are trivial.

Recall from Lemma 4.1.1(iii), see also Figure 4.2, that almost even partitions in $\mathfrak{F}_n$ arise from even shifted partitions in $\mathfrak{E}_{n-1}$ by adding a diagonal in one of the two possible ways, i.e. for each $\underline{\lambda} \in \mathfrak{E}_{n-1}$ with k-tuples $\underline{d}, \underline{e}$ there are two distinct extensions $\underline{\lambda}_1, \underline{\lambda}_2$ in $\mathfrak{F}_n$. Let us count almost even partitions contributing to $W^i(\mathrm{LG}(n), \mathcal{O})$. Just as in Figure 4.5, we can break down the structure of an almost even partition into the following ingredients:

- t triangles on the diagonal, each of them decreasing the shift by one,
- A $(2t) \times 1$-rectangle next to the last triangle, increasing the shift by $2t$,
- An "optional" 1×1 box which the only variable when extending from $\mathfrak{E}_{n-1}$ to $\mathfrak{F}_n$ as above, increasing the shift by one and
- 2×2 boxes, not changing the shift,

see Figure 4.6. Hence, in order for an almost even partition $\underline{\lambda} \in \mathfrak{F}_n$ to have shift i, we need to have the following:

(i) If $i \equiv 0 \mod 2$, we need $4m - i$ triangles for some $m \geq 1$ (not filling the optional box) or $4m - i - 1$ triangles for some $m \geq 1$ (filling the optional box).

(ii) If $i \equiv 1 \mod 2$ we need $4m + i$ triangles for some $m \geq 0$ (not filling the optional box) or $4m + i - 1$ triangles for some $m \geq 0$ (filling the optional box).

Fixing the number t of triangles, counting the possible partitions works as in the orthogonal case, see also Figure 4.7 for an example. In both cases we are counting even Young diagrams in $(2t) \times (n - 1 - 2t)$ frame, of which there are

$$\binom{\frac{n-1}{2}}{\frac{n-1}{2} - t} = \binom{\frac{n-1}{2}}{t} = \binom{n'}{t}.$$

(i) If $i \equiv 0 \mod 2$,

$$\begin{aligned} r_{\mathrm{LG}}(n,i) &= \sum_{m \in \mathbb{N}} \binom{n'}{4m - i} + \sum_{m \in \mathbb{N}} \binom{n'}{4m - i - 1} \\ &= \sum_{k \equiv -i \mod 4} \binom{n'}{k} + \sum_{k \equiv -(i+1) \mod 4} \binom{n'}{k} \\ &= \sum_{k \equiv -i \mod 4} \binom{n'}{k} + \sum_{k \equiv -i \mod 4} \binom{n'}{k-1} \\ &= \sum_{k \equiv -i \mod 4} \binom{n'+1}{k} \\ &= \sum_{k \equiv i \mod 4} \binom{n'+1}{k} \\ &= \rho(n'+1, i) \end{aligned}$$

where we used $k \equiv i \mod 4$ if and only if $k \equiv -i \mod 4$, if i is even.

(ii) If $i \equiv 1 \mod 2$, similarly

$$\begin{aligned} r_{\mathrm{LG}}(n,i) &= \sum_{m \in \mathbb{N}} \binom{n'}{4m + i} + \sum_{m \in \mathbb{N}} \binom{n'}{4m + i - 1} \\ &= \sum_{k \equiv i \mod 4} \binom{n'}{k} + \sum_{k \equiv i-1 \mod 4} \binom{n'}{k} \\ &= \sum_{k \equiv i \mod 4} \binom{n'+1}{k} \\ &= \rho(n'+1, i). \end{aligned}$$

Since n is odd, we have $n' + 1 = (n+1)'$ which finishes the first part of the proof.

Let now n be even. This works similar as the even part in Theorem 4.4.3. If $\underline{\lambda} \in \mathfrak{F}_n$ is such that $e_1 > 0$, it arises from some $\underline{\lambda}' \in \mathfrak{E}_{n-1}$ by adding an empty column on the right which has the same shift and the resulting partition has trivial twist, so we have

$$r_{\mathrm{LG}}(n,i,0) = r_{\mathrm{LG}}(n-1,i,0) = \rho((n-1+1)',i) = \rho(n',i).$$

If on the other hand $\underline{\lambda}$ is such that $e_1 = 0$, then d_1 is odd and it arises from some $\underline{\lambda}'' \in \mathfrak{E}_{n-1}$ by adding a full row on the top. The resulting partition then has nontrivial twist. Indeed if d_1 was even and $e_1 = 0$, cancelling the uppermost row would give rise to an almost even shifted partition in $\mathfrak{F}_{n-1}$ with d_1 odd, which contradicts Lemma 1.5.11. Since adding the row on top increases the shift by n, we need to start with shift $i-n$, so

$$r_{\mathrm{LG}}(n,i,1) = r_{\mathrm{LG}}(n-1,i-n,0) = \rho(n',i-n)$$

which finishes the proof. □

For a complete and explicit result it remains to compute the numbers $\rho(n',i)$. For this we recall graded exterior algebras:

Remark 4.4.6 (Exterior algebras). Denote by $W := \Lambda(g_1,\ldots,g_n)$ the $\mathbb{Z}$-graded exterior algebra over $\mathbb{Z}/2$ with n generators of degree one. Then clearly the i-graded part of W has rank $\binom{n}{i}$. If we endow W with a $\mathbb{Z}/4$-grading instead, then the rank becomes precisely $\rho(n,i)$ and these numbers have been analyzed by Hemmert in [Hem18]. In the appendix, the author computes the ranks of $\mathbb{Z}_4$-graded exterior algebras over $\mathbb{Z}_2$ with generators in degrees 1 and 3, which are linked to the Witt rings of complex flag varieties. In the maximal case it turns out that only generators of degree one occur. With the usual notation $n' = \lfloor \frac{n}{2} \rfloor$ and $n'' = \lfloor \frac{n'}{2} \rfloor$ we obtain the following table for the numbers $\rho(n',i)$:

$n' \setminus i$	0	1	2	3
0	$a-(-1)^{\frac{n''}{2}-1}\cdot b$	a	$a+(-1)^{\frac{n''}{2}-1}\cdot b$	a
1	$a-(-1)^{\frac{n''}{2}-1}\cdot b$	$a-(-1)^{\frac{n''}{2}-1}\cdot b$	$a+(-1)^{\frac{n''}{2}-1}\cdot b$	$a+(-1)^{\frac{n''}{2}-1}\cdot b$
2	a	$a+(-1)^{\frac{n''-1}{2}}\cdot b$	a	$a-(-1)^{\frac{n''-1}{2}}\cdot b$
3	$a-(-1)^{\frac{n''-1}{2}}\cdot b$	$a+(-1)^{\frac{n''-1}{2}}\cdot b$	$a+(-1)^{\frac{n''-1}{2}}\cdot b$	$a-(-1)^{\frac{n''-1}{2}}\cdot b$

TABLE 4.1 A table for $\rho(n',i)$ with $n',i \in \{0,1,2,3\} \mod 4$ where $a = 2^{n'-2}$ and $b = 2^{n''-1}$. This is a quarter of the table in [Hem18] since with the notation in loc. cit. $f = 0$ and $g = n'$.

Remark 4.4.7 (Ranks of the Witt groups of maximal orthogonal Grassmannians). Combining Table 4.1 with Theorem 4.4.3 we can state the ranks of the Witt groups of OG:

$r(n,i)$	$i=0$	$i=1$	$i=2$	$i=3$
$n=2$	1	1	0	0
$n=3$	1	0	0	1
$n=4$	1	0	1	2
$n=5$	1	0	1	2
$n=6$	3	1	1	3
$n=7$	1	1	3	3
$n=8$	2	4	6	4
$n=9$	2	4	6	4
$n=10$	6	6	10	10
$n=11$	6	10	10	6
$n=12$	20	16	12	16
$n=13$	16	20	16	12
$n=14$	28	36	36	28
$n=15$	36	36	28	28

Remark 4.4.8 (Ranks of the Witt groups of Lagrangian Grassmannians). Combining Table 4.1 with Theorem 4.4.5 we can state the ranks of the Witt groups of LG:

$r(n,i,0)$	$i=0$	$i=1$	$i=2$	$i=3$
$n=1$	1	1	0	0
$n=2$	1	1	0	0
$n=3$	1	2	1	0
$n=4$	1	2	1	0
$n=5$	1	3	3	1
$n=6$	1	3	3	1
$n=7$	2	4	6	4
$n=8$	2	4	6	4
$n=9$	6	6	10	10
$n=10$	6	6	10	10
$n=11$	16	12	16	20
$n=12$	16	12	16	20

$r(n,i,1)$	$i=0$	$i=1$	$i=2$	$i=3$
$n=1$	0	0	0	0
$n=2$	0	0	1	1
$n=3$	0	0	0	0
$n=4$	1	2	1	0
$n=5$	0	0	0	0
$n=6$	3	1	1	3
$n=7$	0	0	0	0
$n=8$	2	4	6	4
$n=9$	0	0	0	0
$n=10$	10	10	6	6
$n=11$	0	0	0	0
$n=12$	16	12	16	20

4.5 Summary and perspectives

Recall from section 1.5.5 that with cominuscule and minuscule varieties $G/P_{\max}$ we can associate a quiver or, equivalently, a "Young diagram" (not necessarily rectangular or shifted) due to the existence of a unique reduced expression of the longest element in the Weyl group. Let us give a complete list of all (co-)minscule varieties in Figure 4.8 as it can be found in [BS16, tables 1, 2], including the exceptional cases.

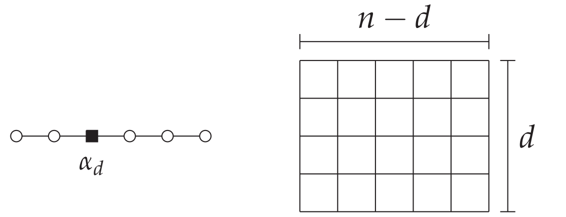
Witt groups of quadrics have already been computed, see e.g. [Xie19]. They can also be computed by constructing an explicit basis as for the Grassmannians. The even dimensional quadrics are minuscule and for them an easy calculation shows that *even diagrams* (that is, all the inner segments have even length) parametrize the total Witt groups of which there are four, regardless the dimension. The odd-dimensional quadrics, are not minuscule and again we might need diagrams other than the even ones. However, Setup 1.2.14 applies and since the localization long exact sequence splits one should be able to construct the basis, also consisting of four elements; this has not been done in detail. Finally, there is a well-known isomorphism $\mathrm{OG}(n, 2n+1) \cong \mathrm{OG}(n+1, 2n+2)$ and hence all ordinary types of minuscule and cominscule varieties are covered.

The two exceptional types are both minuscule and cominuscule, so a good guess for the basis is again to take all the even diagrams which leads to the conjecture

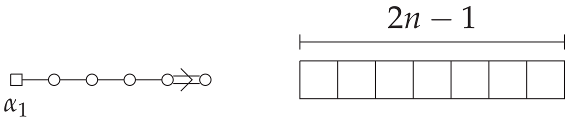
$$\mathrm{W}^{\mathrm{tot}}(\mathbb{OP}_2) \cong \mathrm{W}(X)^{\oplus 6}, \qquad \mathrm{W}^{\mathrm{tot}}(G(\mathbb{O}^3, \mathbb{O}^3)) \cong \mathrm{W}(X)^{\oplus 8}.$$

The shift of a generator should as usual be the weight of the corresponding diagram and these numbers match the results for $X = \mathrm{Spec}(\mathbb{C})$ by Zibrowius in [Zib11, IV.3g]. However, the exceptional types are harder to handle since we cannot describe them by certain subspaces of a vector space as in the ordinary types. By the methods of [IM05] one can make sense of the notion of a full flag in the Cayley plane and we also believe that a similar double-blow-up setup as for the Lagrangian Grassmannian applies for the Cayley plane due to Lemma 4.1 in loc.cit. Thus, alhough we theoretically could compute the boundary map, it is currently unclear to the author how to construct a basis via push-forwards into Schubert cells. No efforts have been made so far concerning the Freudenthal variety.

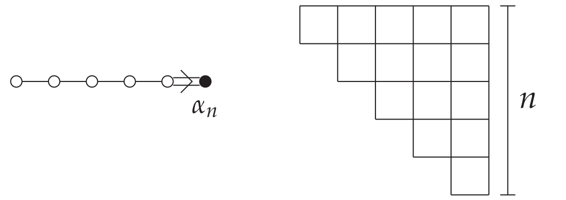
ordinary
Grassmannian
$\mathrm{Gr}(d, n) = A_{n-1}/P_d$
$1 \leq d \leq n-1$



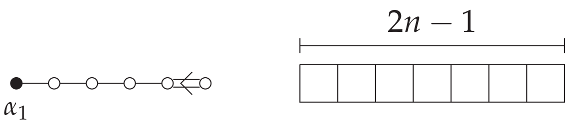
odd quadric
$Q^{2n-1} = B_n/P_1$



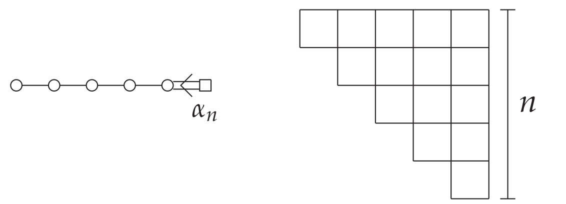
odd orthogonal
Grassmannian
$\mathrm{OG}(n, 2n+1) = B_n/P_n$



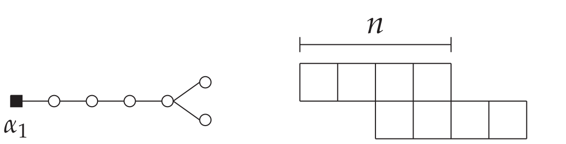
Projective space
$\mathbb{P}^{2n-1} = C_n/P_1$



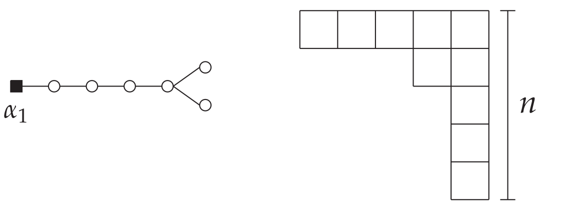
Lagrangian
Grassmannian
$\mathrm{LG}(n, 2n) = C_n/P_n$



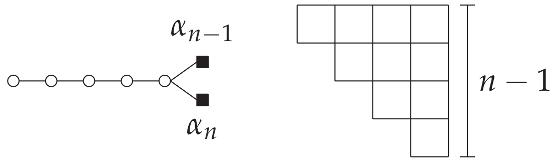
even quadric (n even)
$Q^{2n} = D_n/P_1$



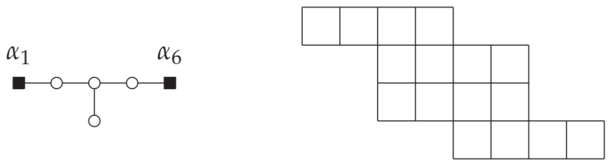
even quadric (n odd)
$Q^{2n} = D_n/P_1$



maximal orthogonal
Grassmannian
$\mathrm{OG}(n) = D_n/P_n \cong D_n/P_{n-1}$



Cayley plane
$\mathbb{OP}^2 = E_6/P_1 \cong E_6/P_6$



Freudenthal variety
$G(\mathbb{O}^3, \mathbb{O}^6) = E_7/P_7$

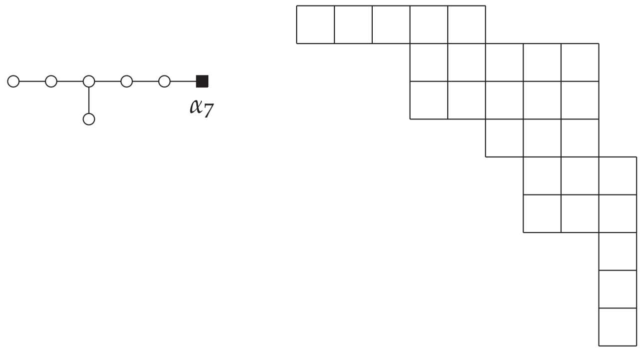


FIGURE 4.8. An overview over cominuscule and minuscule varieties in all types, including the Dynkin diagrams with associated simple roots and the Young diagrams. We marked minuscule roots by $\bullet$, cominuscule roots by $\square$ and by $\blacksquare$ those roots which are both minuscule and cominuscule. Schubert varieties in these spaces are as always given by partitions (that is, for any filled box the box on the left and above are filled and for any filled right end box on its south east is not filled) in the corresponding Young diagrams.

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