Pinching Azumaya algebras

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Zusammenfassung

Meine Motivation für diese Arbeit sind die folgenden Fragen. Sei $f : X \to Y$ ein Morphismus von Schemata und $\beta \in \mathrm{Br}'(Y)$ eine kohomologische Brauer-Klasse so, dass $f^*\beta \in \mathrm{Br}'(X)$ von einer Azumaya-Algebra $\mathscr{A}$ auf X repräsentiert wird. Gibt es eine Azumaya-Algebra $\mathscr{B}$ auf Y so, dass $f^*\mathscr{B} \simeq \mathscr{A}$? Und repräsentiert $\mathscr{B}$ dann auch β?

Sei $\mathrm{Az}(X)$ die Kategorie der Azumaya Algebren auf X. Zu einem gegebenen kartesischem und kokartesischem Quadrat

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y, \end{tikzcd}$$

wobei f ein affin und u eine abgeschlossene Einbettung sei, konstruieren wir eine adjungierte Äquivalenz von Kategorien

$$\mathrm{Az}(Y) \underset{F}{\overset{G}{\rightleftarrows}} \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X),$$

wobei die rechte Seite das kategorielle Faserprodukt, konstruiert via Pullbacks, bezeichnet. Für die Konstruktion verwenden wir ein Ergebnis von Ferrand [21], wo er eine derartige adjungierte Äquivalenz für die Kategorien der endlichen lokal freien Garben konstruiert.

Unter Nutzung der ersten Resultats zeigen wir, dass, wenn $f : X \to Y$ eine endliche Modifikation in endlichen vielen abgeschlossenen Punkten ist, dann wird eine kohomologische Brauer Klasse $\beta \in \mathrm{Br}'(Y)$ genau dann von einer Azumaya Algebra auf Y repräsentiert, wenn ihr Pullback $f^*\beta$ von einer Azumaya Algebra auf X repräsentiert wird.

Im ersten Kapitel der Dissertation werden Azumaya Algebren und Brauer Gruppen eingeführt. Wir geben Beispiele und definieren die Brauer Abbildung. Im zweite Kapitel wird das Ergebnis von Ferrand diskutiert. Schließlich werden im dritten Kapitel die Hauptresultate bewiesen.

Summary

My motivation for this thesis are the following questions: For a morphism of schemes $f : X \to Y$, given a cohomological Brauer class $\beta \in \mathrm{Br}'(Y)$ such, that $f^*\beta \in \mathrm{Br}'(X)$ is represented by an Azumaya Algebra $\mathscr{A}$ on X. Does there exist an Azumaya Algebra $\mathscr{B}$ on Y such that $f^*\mathscr{B} \simeq \mathscr{A}$? And does $\mathscr{B}$ then also represent β?

Let $\mathrm{Az}(X)$ be the category of Azumaya algebras on X. Given a cartesian and cocartesian square of schemes

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y, \end{tikzcd}$$

where f is affine and u a closed immersion, we will construct an adjoint equivalence of categories

$$\mathrm{Az}(Y) \underset{F}{\overset{G}{\rightleftarrows}} \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X),$$

where the right hand side denotes the fiber product category constructed via pullbacks. For this construction we use a similar result by Ferrand [21], where he constructs such an adjoint equivalence for the categories of finite locally free sheaf.

Using the first result, we show, that if $f : X \to Y$ is a finite modification in finitely many closed points, then a cohomological Brauer class $\beta \in \mathrm{Br}'(Y)$ is represented by an Azumaya algebra on Y if and only if the pullback $f^*\beta$ is represented by an Azumaya algebra on X.

The first chapter of this thesis give an introduction to Azumaya algebras and Brauer groups. We give examples and define the Brauer map. The second discusses the result by Ferrand. Finally, the third chapter presents the proofs of the main results.

Contents

Introduction

What we today know as the Brauer group was introduced by Richard Brauer in [11]. He showed that it is a torsion abelian group. The term Brauer group was first coined by Hasse as "R. Brauersche Algebrenklassengruppe" in a paper dedicated to Emmy Noether's fiftieth birthday [37]. An interesting survey on the early theory of the Brauer group as developed by Albert, Brauer, Hasse and Noether is [61].

Azumaya generalized central simple algebras and the Brauer group to local rings [6]. Today, these generalizations and their successors are called Azumaya algebras in his honor. Auslander and Goldman further generalized this ideas to arbitrary rings in [4]. In "Le groupe de Brauer I, II, III" [29],[30],[31], Grothendieck developed the theory of Azumaya algebras and Brauer groups over schemes. Grothendieck generalized the Brauer group in two ways: Once as equivalence classes of Azumaya algebras over a scheme X, defining the Brauer group $\mathrm{Br}(X)$, and secondly with cohomology theory, defining the cohomological Brauer group $\mathrm{Br}'(X) = H^2(X_{et}, \mathbb{G}_m)_{tors}$. These two groups are connected by an injective map called the Brauer map.

The Brauer group has a variety of applications. Noted is for example the Brauer-Manin obstruction to the Hasse principle. Given some Diophantine equation every rational solution also yields a real and p-adic solution. The Hasse principle asks whether the reverse can be done, or more accurate, what the obstruction to the reverse is. When can real and p-adic solutions be patched to a rational one? In 1970, Manin [52] showed that the obstruction for all know counter examples to the Hasse principle at that time could be expressed in terms of Brauer classes. When X is a separated scheme of finite type over $\mathbb{Q}$, that is given by a Diophantine equation, the obstruction of X to have a $\mathbb{Q}$-rational point can be described as an element in the Brauer group.

The Brauer group can also be used to describe an obstruction in relation to the Picard scheme. Let X be a proper k-scheme that is geometrically reduced and geometrically connected, and $\mathrm{Pic}_{X/k}$ its Picard scheme. Using the Leray-Serre spectral sequence, one obtains an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/k}(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X).$$

For a k rational point of the Picard scheme to come from a line bundle in $\mathrm{Pic}(X)$ it needs to be mapped to zero in $\mathrm{Br}(k)$. The obstruction lies in the Brauer group. There is a good survey by Kleiman [44] on the Picard group, which treats the more general case of this obstruction with regard to the relative Picard functor.

Another application is due to Artin and Mumford. They used the Brauer group to construct an example of a unirational but not rational variety over the complex numbers [2]. Here a variety means an separated scheme X of finite type over some field. Such a scheme is called unirational if it is defined over an algebraically closed field k and there exist an embedding $k(X) \subset k(X_1, \ldots, X_n)$. It is called rational if this embedding is an isomorphism. The question whether these two cases are equivalent is know as the Lüroth problem and the Brauer group gives one way to construct a counterexample.

The Brauer group $\mathrm{Br}(k)$ of a field k consists of equivalence classes of central simple algebras over k. Two such algebras are seen as equivalent if they are isomorphic after tensoring with some matrix algebra $M_n(k)$. The group operation is given by the tensor product and the class of the matrix algebras themselves is the trivial class. Every central simple algebra is isomorphic to the matrix algebra of some central division ring, so it is equally possible to define the Brauer group as equivalence classes of such. Furthermore, a central simple algebra becomes isomorphic to a matrix algebra $M_n(K)$ after some field extension K/k, so one can view them as twisted forms of matrix algebras. The best know non-trivial example of a central simple algebra is probably the Hamilton quaternions $\mathbb{H}$ over the real number, which where discovered by Hamilton in 1843 [33], [34]. In fact over $\mathbb{R}$ there exist no other central simple algebra that is not a matrix algebra.

An Azumaya algebra over a local ring R is an associative R-algebra with identity, that is free of finite rank as an R-module, such that the canonical morphism

$$A \otimes_R A^\circ \longrightarrow \mathrm{End}_{R\text{-mdl}}(A)$$

is an isomorphism. This holds if and only if A becomes a central simple algebra over the residue field of R, which gives the connection to the Brauer group of a field. As with central simple algebras two Azumaya algebras are declared to be equivalent if they are isomorphic after tensoring with some matrix algebra $M_n(R)$. Again, the Brauer group $\mathrm{Br}(R)$ is defined with the tensor product as the group operation, and the matrix algebras determine the trivial class. An Azumaya algebra $\mathscr{A}$ on some scheme X is sheaf of

$\mathcal{O}_X$-algebras, that is finite locally free as a $\mathcal{O}_X$-module, such that $\mathscr{A}_x$ is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$ for every point. Following the theme, the equivalence relation is given by tensoring with endomorphism sheaves $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})$ of some locally free sheaf $\mathscr{E}$. The equivalence classes of Azumaya algebras form the Brauer group $\mathrm{Br}(X)$; group operation is again given by the tensor product, and the trivial class consists of sheaves of the form $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})$.

One of Grothendieck's insights was to link Azumaya algebras and Brauer groups to cohomology theory. The link is possible since the Brauer group of a strictly Henselian local ring is trivial. This implies that every Azumaya algebra $\mathscr{A}$ on a scheme X is trivialized by some étale cover $(U_i \to X)$ such that $\mathscr{A}_{|U_i} \simeq M_{n_i}(\mathcal{O}_{U_i})$. An Azumaya algebra is thus a twisted form of $M_{n_i}(\mathcal{O}_{U_i})$ and these can be described by classes of 1-cocycles in $\check{H}^1(X_{et}, PGL_n)$ with $PGL_n(U) = \mathcal{A}ut(M_n(\mathcal{O}_U))$. The connection of the cohomological Brauer group

$$\mathrm{Br}'(X) = H^2(X_{et}, \mathbb{G}_m)_{tors}$$

to the "classical" Brauer group $\mathrm{Br}(X)$ arises from the short exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1.$$

Using non abelian cohomology gives maps $\check{H}^1(X_{et}, PGL_n) \to H^2(X_{et}, \mathbb{G}_m)$ which combine to an injective map, the Brauer map

$$\delta : \mathrm{Br}(X) \to \mathrm{Br}'(X).$$

We say that a cohomological Brauer class is represented by an Azumaya algebra, if the Brauer class of the Azumaya algebra is mapped to it.

Grothendieck asked whether the Brauer map is surjective. The question gains relevance since it is usually quite hard to calculate a Brauer group. On the other hand, for the cohomological Brauer group one can use the whole machinery of cohomology theory. The question whether the Brauer map is bijective is still open in the general case. Positive answers where given by Grothendieck himself for schemes of dimension ≤ 1 and for regular surfaces [30]. The result which is perhaps best known, is due to Gabber. He proves $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for schemes that admit an ample invertible sheaf. Gabber's proof is unpublished but there exists a proof by de Jong [13]. However the Brauer map is not always surjective. A counterexample for a non-separated normal surface was given by Edidin, Hassett, Kresch and Vistoli ([18] Tag 3.11).

For a morphism of schemes $f : X \to Y$ the pullback of an Azumaya algebra $\mathscr{B}$ on Y is an Azumaya algebra on X. One wonders when the reverse is true. For a morphism of schemes $f : X \to Y$, given a cohomological Brauer class $\beta \in \mathrm{Br}'(Y)$ such, that $f^*\beta \in \mathrm{Br}'(X)$ is represented by an Azumaya Algebra $\mathscr{A}$ on X. Does there exist an Azumaya Algebra $\mathscr{B}$ on Y such that $f^*\mathscr{B} \simeq \mathscr{A}$? And does $\mathscr{B}$ then also represent β?

For the first question there are positive answers if f is a flat and surjective morphism. We will discuss results of this kind. In the general case it is hard to answer the question. We will show the following:

Denote by $\mathrm{Az}(X)$ the category of Azumaya algebras on a scheme X. Given a cartesian and cocartesian square of schemes

$$\begin{array}{ccc} X' & \xrightarrow{w} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y, \end{array}$$

where f is affine and u a closed immersion, we will construct functors

$$\mathrm{Az}(Y) \underset{F}{\overset{G}{\rightleftarrows}} \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X),$$

where the right hand side denotes the fiber product category constructed via pullbacks. This means, that the objects of this category are triples $(\mathscr{B}', \tau, \mathscr{A})$, with $\mathscr{B}' \in \mathrm{Az}(Y')$, $\mathscr{A} \in \mathrm{Az}(X)$ and $\tau : g^*\mathscr{B}' \to w^*\mathscr{A}$ is an isomorphism.

This construction leads to the first main result of this thesis:

Theorem. *(Theorem 3.2) The functors F and G are an adjoint equivalence of categories*

$$\mathrm{Az}(Y) \underset{F}{\overset{G}{\rightleftarrows}} \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X).$$

Furthermore, a quasicoherent $\mathcal{O}_Y$-algebra $\mathscr{B}$ is an Azumaya algebra on Y if and only if $u^*\mathscr{B} \in \mathrm{Az}(Y')$ and $f^*\mathscr{B} \in \mathrm{Az}(X)$.

Since pullbacks of Azumaya algebras are Azumaya algebras we see that $G(\mathscr{B}) = (u^*\mathscr{B}, \sigma_{\mathscr{B}}, f^*\mathscr{B})$, with $\sigma_{\mathscr{B}} : g^*u^*\mathscr{B} \to w^*f^*\mathscr{B}$ an isomorphism, defines an element in the fiber product category. The trouble lies in the other direction. We use a result by Ferrand [21] to map a triple $(\mathscr{B}', \tau, \mathscr{A})$ to a finite locally free sheaf on Y. Then we show that there is a canonical way to equip this sheaf with an algebra structure. It is possible to check locally that this algebra is an Azumaya algebra. The way we construct F and G furthermore implies that they are an adjoint equivalence of categories.

The theorem allows us to construct an Azumaya algebra on Y by constructing Azumaya algebras on Y' and X that agree after pullback to X'. So given some Azumaya algebra $\mathscr{A}$ on X, it is enough to find an Azumaya algebra $\mathscr{B}'$ on Y' that is isomorphic to $\mathscr{A}$ after pullback to X', so $g^*\mathscr{B}' \simeq w^*\mathscr{A}$. Then the adjoint equivalence of categories determines an Azumaya algebra on Y.

The proof of the theorem relies on the fact that Azumaya algebras are finite locally free sheaves. In his work, Ferrand constructs an adjoint equivalence of categories

$$\mathrm{Loc}(Y) \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{Loc}(Y') \times_{\mathrm{Loc}(X')} \mathrm{Loc}(X),$$

for finite locally free sheaves. Ferrand starts by constructing an adjunction between the categories of quasicoherent sheaves and the respective fiber product category. Calculating the fixed points of this adjunction, we will point out why flat $\mathcal{O}_X$-modules, and then of course finite locally free sheaves, are a natural choice of subcategories, when one wants to construct an adjoint equivalence of categories. Ferrand's original statement is about the affine case (Theorem 2.13) but since all involved morphism are affine it is clear how to prove the statement for schemes.

For the second question, we can give the following answer, which constitute the second main result of this thesis:

Theorem. *(Theorem 3.3) Let $f : X \to Y$ be finite modification in finitely many closed points. A cohomological Brauer class $\beta \in \mathrm{Br}'(Y)$ is represented by an Azumaya algebra on Y if and only if the pullback $f^*\beta$ is represented by an Azumaya algebra on X.*

A finite modification $f : X \to Y$ in finitely many closed points means, that f is finite, there are dense open subsets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$, $f_{|U} : U \to V$ is an isomorphism and $Y' = Y \setminus V$ consists of only finitely many closed points $Y' = \{y_1, \cdots, y_n\}$. Furthermore, the image of f shall be schematically dense, i.e. the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective.

So we have $Y' = \mathrm{Spec}(S)$ for some Artin ring S. The finite fibers $f^{-1}(y_i) = \mathrm{Spec}(R_i)$ have to be Artin rings R_i. Set $R = \bigoplus R_i$. This allows us to construct a cartesian and cocartesian square

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(S) \arrow[r, "u"'] & Y.
\end{tikzcd}$$

Here u and w are closed embeddings. We call such a square a conductor square, since the closed embeddings are both determined by the conductor ideal $\mathcal{C} = \mathrm{Ann}_{\mathcal{O}_Y}(f_*\mathcal{O}_X/\mathcal{O}_Y)$.

We can apply Theorem 3.2 to this square. Given a cohomological Brauer class $\alpha = f^*\beta$, that is represented by an Azumaya algebra on X, we explicitly construct an Azumaya algebra $\mathscr{B}$ on Y, such that $f^*\mathscr{B}$ represents α. Thanks to the unique structure of the commutative square we can use étale cohomology to show that the pullback map $f^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. This shows that the Azumaya algebra $\mathscr{B}$ does represent the cohomological Brauer class β.

Outline of structure. The first chapter gives an introduction to Brauer groups. We introduce central simple algebras and their Brauer groups. Then, we continue with Azumaya algebras over local rings and their close connection to central simple algebras. In particular, this includes the fact that a Brauer group of a strictly Henselian local ring is zero (Corollary 1.20). We also show how Azumaya algebras behave under restriction and extension of scalars for a morphism of rings $R \to S$. We define Azumaya algebras for schemes and generalize facts about Azumaya algebras on local rings. We define the Brauer group of a scheme and show that the pullback of an Azumaya algebra is an Azumaya algebra (Lemma 1.30). We also give some answers for the other direction. We will not treat Azumaya algebras on general rings separately but view them as a special case of Azumaya algebras over an affine scheme. Finally we connect Azumaya algebras to étale cohomology and define the Brauer map. We include a discussion on what is know on the bijectivity of the Brauer map.

The second chapter is a discussion of Ferrand's results in [21]. We start by stating the necessary category theory, with a focus on adjunctions and how to construct an adjoint equivalence of categories from them. We continue by showing some facts about fiber products of rings and modules. After this preparation we thoroughly discuss Ferrand's results ([21] Théorème 2.2) in the affine case (Theorem 2.13). Via the fixed points of the adjunction, we explain why the category of flat modules seem to be a natural choice of subcategory, if one wants to obtain an adjoint equivalence of categories (Proposition 2.12). We explain how to check whether a commutative square of schemes is cartesian as well as cocartesian, and show that a cartesian and cocartesian square of rings gives one of affine schemes (Proposition 2.14). Furthermore, we define finite modifications (Definition 2.15) and explain how to attach a cartesian and cocartesian square of schemes, which we call a conductor square, to them. In the last section, we show how to generalize the affine results to schemes and sheaves, and prove that we also have an adjoint equivalence of categories in the case of schemes (Theorem 2.18 and Theorem 2.20).

The third chapter contains the main results. Having done thorough groundwork, we can construct an adjoint equivalence of categories for categories of Azumaya algebras and prove Theorem 3.2. Then we move to the case where $f : X \to Y$ is finite modification in finitely many closed points. We discuss the structure of Artin rings and their Brauer groups, which play an important part in the proof of Theorem 3.3. The proof gives an explicit construction of an Azumaya algebra $\mathscr{B}$ that represents β. Finally, we show how the theorem can be applied to the S_2-ization of a surface.

The Appendix contains a summary of (étale) cohomology theory of group schemes and twisted forms.

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Chapter 1

Brauer groups

In this chapter we will discuss Azumaya algebras and Brauer groups. The presented facts can be found in most textbooks. A good introduction to the subject can be found in the book "Étale cohomology" by Milne [55]. As the name suggests Milne's book also includes all required background on étale cohomology. For central simple algebras most facts can be found in "Central simple algebras and Galois cohomology" by Gille and Szamuely [24]. We will quote the standard results from these two books. Other sources include books by Colliot-Thélène and Skorobogatov [12], Jahnel [42], Knus and Ojanguren [45]; and of course the work of Grothendieck [29],[30],[31]. For a view on Azumaya algebras as central separable algebras over a ring refer to Auslander and Goldman [4] and DeMeyer and Ingraham [15].

1.1 Central simple algebras

We give a short reminder about facts on central simple algebras. Let k be a field. In the following a *k-algebra* A shall always be an associative algebra with an identity element. This means, that A is a k-vector space, equipped with a k-bilinear map $A \times A \to A$, defining a multiplication on A, which is associative and has an identity element. The center

$$Z(A) = \{x \in A \mid ax = xa \text{ for all } a \in A\}$$

denotes the subring of A whose elements commute with all elements in A.

Definition 1.1. Let k be a field and A a k-algebra. The algebra A is called *simple* if the only two-sided ideals in A are the trivial ideals 0 and A. We say that A is *central* if its center $Z(A)$ equals k. A k-algebra A is called a *central simple algebra* if it is finite dimensional as a k-vector space, and both central and simple over k.

Every field is a central simple algebra over itself. Every non trivial field extension is simple but not central. Over any field k the matrix algebras $M_n(k)$ are an example of a central simple algebras and we have $M_n(k) \otimes_k M_m(k) \simeq M_{nm}(k)$. Of course not all central simple algebras are matrix algebras.

For example over the real numbers $\mathbb{R}$ we have the Hamilton quaternions

$$\mathbb{H} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k,$$

with relation $ij = -ji = k$, and $i^2 = j^2 = -1$. They are a central simple algebra over $\mathbb{R}$, but they are not isomorphic to a matrix algebra. For $a, b, c, d \in \mathbb{R}$, we have

$$(a1 + bi + cj + dk)(a1 - bi - ci - dk) = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}_{\geq 0},$$

which shows that non-zero elements of the Hamilton quaternions have a multiplicative inverse. So $\mathbb{H} \neq M_n(\mathbb{R})$. Note that even though we can define an inclusion $\mathbb{C} \to \mathbb{H}$, $(a + bi) \mapsto a + bi$, the relation $ij = -ji$ ensures that $\mathbb{C}$ is not in the center.

This construction can be generalized to every field k with $\mathrm{char}(k) \neq 2$: A *quaternion algebra* $Q_k(a, b)$ is generated over k by the basis $1, i, j, ij$ with relations

$$i^2 = a, \quad j^2 = b, \quad ij = -ji,$$

where $a, b \in k^\times$. Quaternion algebras are central simple over k. As a k-vector space they are 4-dimensional. Note that in general quaternion algebras can be isomorphic to matrix algebras.

Quaternion algebras are a special case of *cyclic algebras*. Such a algebras are constructed as follows: Let k be a field and K/k be a cyclic Galois field extension; such an extension has a cyclic Galois group $G = \mathrm{Gal}(K/k)$ of order n. We fix a group isomorphism $\chi : \mathbb{Z}/n \to G$, which in turn fixes $\sigma = \chi(1)$. This σ generates G; note that there exist some isomorphism $\chi : \mathbb{Z}/n \to G$ with $\chi(1) = \sigma$ for every generator $\sigma \in G$. Finally we choose an $x \in k^\times$.

Now we can construct a k-algebra, which we denote by $D_k(\chi, x)$. For the additive group we define an n-dimensional K vector space with basis $e^0 = 1, e, e^2, \ldots, e^{n-1}$ by

$$D_k(\chi, x) = \bigoplus_{i=0}^{n-1} Ke^i.$$

We define multiplication with the relations

$$e^n = x \quad \text{and} \quad \lambda e = e\sigma(\lambda), \quad \text{for all } \lambda \in K.$$

In other words $D_k(\chi, x)$ is generated as a k-algebra by K and e subjected to the relations above. It is straightforward to check that this in fact defines an associative k-algebra, that is then a central simple k-algebra. We call such an algebra a cyclic algebra. Note that $D_k(\chi, x)$ has dimension n^2 as a k-vector space.

For example we choose $k = \mathbb{R}$, $K = \mathbb{C}$, $\chi : \mathbb{Z}/2 \to G$ such that $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is the conjugation automorphism $\sigma(z) = \bar{z}$ for all $z \in \mathbb{C}$, and $x \in \mathbb{R}^\times$. Then

$$D_k(\chi, x) = \{z + we \mid z, w \in \mathbb{C}\},$$

with $e^2 = x$ and $ze = e\bar{z}$. In particular let $z + we$ be non trivial, we have

$$(z + we)(z - we) = |z|^2 - x|w|^2 \in \mathbb{R}.$$

For $x < 0$ the right side is positive and we can divide by it, to get $1 \in \mathbb{R}$. So

$$(z + we)^{-1} = (z - we)(|z|^2 - x|w|^2)^{-1}.$$

Thus every non zero element of the algebra is invertible. We will discuss such algebras, called division algebras, at the end of the section. For now we see that $D_k(\chi, x) \simeq \mathbb{H}$. To see this extend the inclusion $\mathbb{C} \to \mathbb{H}$, to a map $\phi : D_k(\chi, x) \to \mathbb{H}$ by setting $\phi(e) = \sqrt{|x|}j$ and check that this is an isomorphism.

For $x > 0$ not every element of $D_k(\chi, x)$ is invertible. Instead we have an isomorphism $D_k(\chi, x) \simeq M_2(\mathbb{R})$. For every element $z + we \in D_k(\chi, x)$ one writes $z = a+bi$ and $w = c+di$ with $a, b, c, d \in \mathbb{R}$. As before one extends the usual embedding $\mathbb{C} \to M_2(\mathbb{R})$ to a map of $\mathbb{R}$-algebras $\phi : D_k(\chi, x) \to M_2(\mathbb{R})$ by setting

$$\phi(e) = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & -\sqrt{x} \end{pmatrix}, \quad \text{so} \quad \phi(a+bi+\gamma e+\gamma ie) = \begin{pmatrix} a + c\sqrt{x} & b - d\sqrt{x} \\ -b - d\sqrt{x} & a - c\sqrt{x} \end{pmatrix}.$$

Now ϕ is injective and $\dim_k(D_k(\chi, x)) = 4 = \dim_k(M_2(\mathbb{R}))$, which shows that ϕ is an isomorphism (for details, see [67] Example 2.1).

We would like to see that the tensor product $A \otimes_k B$ of two central simple algebras A and B is again a central simple algebra over k. This is a direct consequence of the following theorem.

Theorem 1.2. *([24] Theorem 2.2.1) Let k be a field and A a finite dimensional k-algebra. Then A is a central simple algebra if and only if there exist a finite field extension K/k and an integer $n > 0$ so that $A \otimes_k K \simeq M_n(K)$.*

We call such a finite field extension K/k a *splitting field*, which *splits* the central simple algebra A. This theorem also shows that the dimension of a central simple algebra A as a vector space is always a square of some positive integer d. We call d the *degree* of A. Furthermore, over an algebraically closed field $\bar{k}$ every central simple algebra has to be of the form $M_n(\bar{k})$.

We can now give anther important example of a central simple algebra. Let A be any k-algebra. Define the *opposite algebra* A° of A as a k-algebra which has the same underling vector space, but the multiplication is defined as the opposite of the multiplication in A, i.e. $x *_{A^\circ} y = y *_A x$. If A is a central simple algebra the opposite algebra A° is also central simple. The algebra $A \otimes A^\circ$ is central simple, more so:

Proposition 1.3. *For a central simple algebra A of degree d the morphism of k-algebras*

$$A \otimes_k A^\circ \longrightarrow \mathrm{End}_k(A), \quad \sum a_i \otimes b_i \longmapsto (x \mapsto \sum a_i x b_i)$$

is an isomorphism. Consequently $A \otimes_k A^\circ \simeq M_d(k)$.

Note that the multiplication $a_i x b_i$ is defined in A.

Proof. The map is non-zero, take for example $a_i = b_i = 1$. Since $A \otimes A^\circ$ is simple it has trivial kernel and is injective. Since both sides have the same dimension as a k-vector space this is enough to show that it is an isomorphism. $\square$

This isomorphism will play a central part in the definition of Azumaya algebras in the next section. Keep in mind that $A \otimes_k A^\circ \simeq \mathrm{End}_k(A) \simeq M_d(k)$.

We have defined a composition of two central simple algebras via tensor product and an inverse with opposite algebra. We want to use this to define a group. We declare two central simple algebras over a field k as *equivalent* $A \sim B$ if there exist $n, m \geq 1$ such that there exist a morphism of k-algebras $A \otimes M_n(k) \simeq B \otimes M_m(k)$. One can see that this is an equivalence relation. For transitivity one checks that if A_1, A_2, B_1 and B_2 are central simple algebras over k with $A_1 \sim B_1$ and $A_2 \sim B_2$ then $A_1 \otimes A_2 \sim B_1 \otimes B_2$. This holds true because $M_n(k) \otimes M_m(k) = M_{nm}(k)$. Write $[A]$ for the equivalence class.

Theorem 1.4. *(Brauer) The set of all equivalence classes equipped with the tensor product is an abelian group.*

Proof. The neutral element is the class $[k]$, the class consists of all matrix algebras $M_n(k)$. The group is associative and commutative since the tensor product is. An inverse element to a class $[A]$ is given by the class of the opposite algebra $[A^\circ]$. $\square$

This group is called the *Brauer group* of k and denoted as $\mathrm{Br}(k)$. We call the class of the neutral element $[k]$ the *trivial class*. $\mathrm{Br}(-)$ can be seen as a functor from the category of fields to $(\mathbf{Ab})$.

So far almost everything we have said about Brauer groups can be generalized to rings. There is another useful description due to Wedderburn that can not be generalized but will nevertheless be useful later on.

Definition 1.5. Let D be a non-zero algebra over a field k. If every non-zero element of D has a multiplicative inverse we call D a *division algebra*.

The center of a division algebra is a field and D is then a central simple algebra over $K = Z(D)$. More so $M_n(D)$ is also a central simple algebra over K. Take note that in general it is possible that $k \neq Z(D)$, for example every finite field extension of k is a division algebra. Division algebras are thus examples of simple algebras that are not necessarily central. But k lies in the center of D, so K/k is a finite field extension. We are of course mostly concerned with division algebras viewed as central simple algebras over their center, i.e. central division algebras over some field k.

A useful way to construct division algebras is the following: Let M be a simple R-module. By Schur's lemma ([24] Lemma 2.1.5), the endomorphism ring $D = \mathrm{End}_R(M)$ is a division ring. The proof of Schur's lemma uses the fact that the kernel and the image of some endomorphism $f : M \to M$ is a submodule of M. Since M is simple it is either $\ker(f) = M$, so $f = 0$; or $\ker(f) = 0$ and $\mathrm{im}(f) = M$, hence f is an isomorphism. So each non zero element in D has an inverse. Then D is central simple over the field $K = Z(D)$. This links central simple algebras to representation theory.

For another example we have already seen that every non-zero element in the Hamilton quaternions $\mathbb{H}$ is invertible, so the quaternions $\mathbb{H}$ form a central division algebra over $\mathbb{R}$. On can show that a quaternion algebra $Q_k(a, b)$ over a field k is either a division algebra or isomorphic to $M_2(k)$ ([24] Proposition 1.1.7).

Central simple algebras can be characterized as following:

Theorem 1.6. *(Wedderburn)([24] Theorem 2.1.3) Let A be a central simple algebra over a field k. There exist a central division algebra D over k such that $A \simeq D \otimes M_n(k) \simeq M_n(D)$. The division algebra is unique up to isomorphism.*

Obviously A and D are in the same Brauer class. So this theorem implies that every Brauer class is represented by a central division algebra. In fact one could define the Brauer group using this theorem. Furthermore, it follows that two central simple algebras A and B, that have the same class in $\mathrm{Br}(k)$

and have the same rank, are already isomorphic to each other; since then $A \simeq M_n(D) \simeq B$. Moreover, if for three central simple algebras it holds that $A \otimes C \simeq B \otimes C$, this implies $A \simeq B$.

We can use this to describe the Brauer group over $\mathbb{R}$. The Frobenius theorem ([38] Chapter 4 Section 3.(3)) shows that the only division algebras over $\mathbb{R}$ are $\mathbb{R}$ itself, the complex numbers $\mathbb{C}$ and the Hamilton quaternions $\mathbb{H}$. The complex number are not central over $\mathbb{R}$, so there exist only two different Brauer classes, which shows that $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$.

Since over an algebraically closed field there exist no finite dimensional division algebra other than the field itself, this again shows that the Brauer group of an algebraically closed field is zero ([24] Corollary 2.1.7). We can improve this with the following result:

Theorem 1.7. *([24] Theorem 2.2.7) Let k be an infinite field and D a central division algebra over k of degree d. Then D is split by a finite separable field extension K/k of degree d. Moreover, K can be chosen in the center of D and is thus a k-subalgebra of D.*

For a finite field every finite field extension is separable, so in that case every central division algebra is split by a finite separable extension as well. In fact by a theorem of Wedderburn every finite division algebra over a finite field is a field itself ([24] Remark 6.2.7), so the Brauer group of a finite field is trivial.

Important for us is that every central division algebra over k is split by the separable closure k^{sep}. As a direct consequence this also shows:

Corollary 1.8. *The Brauer group of a separably closed field is trivial.*

Now every finite separable field extension embeds into finite Galois extension. Together with the fact that a central simple algebra is isomorphic to $M_n(D)$ for some division algebra and Theorem 1.2 we get:

Corollary 1.9. *Let k be a field and A a finite dimensional k-algebra. Then A is a central simple algebra if and only if there exist a finite Galois extension K/k and an integer $n > 0$ so that $A \otimes_k K \simeq M_n(K)$.*

One can use this describe the Brauer group as isomorphism classes of twisted forms of central simple algebras over k. In this sense two central simple algebras are equivalent if the have the same finite Galois extension as a splitting field; then one central simple algebra is called a twisted form of the other. This is yet another way to see that the Brauer group of an algebraically closed field is trivial. More importantly, this opens up the machinery Galois cohomology theory. It is possible to describe Twisted forms in terms of first Galois cohomology groups. We will return to and give some details on this point of view when talking about Azumaya algebras over schemes.

1.2 Azumaya algebras over local rings

Let R be any (commutative) ring. An *algebra* A over R shall be an associative algebra with an identity element. The only part that changes from the definition to an associative algebra over a field, is that we allow a ring R instead of only fields and A shall be an R-module. Of course an algebra over a field is also an associative algebra in this sense. This means that A is an R-module with an R-bilinear map $A \times A \to A$, called the multiplication, such that the multiplication is associative and there is an identity element. The multiplication does not need to be commutative. The opposite algebra of A, where multiplication is reversed, is denoted by A°. We note that A and A° do have the same center.

For the rest of this section let R be a local ring with maximal ideal $\mathfrak{m}$ and A an R-algebra.

Definition 1.10. An algebra A over a local ring R is called an *Azumaya Algebra* if it is free of finite rank as an R-module and if the R-algebra homomorphism

$$A \otimes_R A^\circ \longrightarrow \mathrm{End}_{R\text{-mod}}(A), \quad \sum a_i \otimes b_i \longmapsto (x \mapsto \sum a_i x b_i)$$

is an isomorphism.

Note that the multiplication $a_i x b_i$ is defined in A. This homomorphism always exists. It simply fails to be an isomorphism if A is not an Azumaya algebra. Furthermore, since an Azumaya algebra A is a free R-module of finite rank, the map $R \to A$, $r \mapsto 1r$ is an injection and we can identify R with a subring in the center of A. We have $\mathrm{End}_{R\text{-mod}}(A) = M_n(R)$.

Proposition 1.3 tells us immediately that all central simple algebras are Azumaya algebras. The obvious question is if the reverse is true? If $R = k$ is a field, are all Azumaya algebras then central simple algebras? The answer is positive. An Azumaya algebra over a field is a central simple algebra and we can use these properties interchangeable. We show even more by describing the center and the ideals of an Azumaya algebra.

Proposition 1.11. *Let A be an Azumaya algebra over a local ring R. Then R is the center of A. Furthermore, for any ideal $I \subset R$ it holds that $I = (IA) \cap R$. And for any ideal $J \subset A$ it holds that $J = (J \cap R)A$. Finally the map $J \mapsto J \cap R$ gives a bijection between ideals of A and ideals of R.*

Proof. Let $\varphi \in \mathrm{End}_{R\text{-mod}}(A)$. Then by the definition of Azumaya algebras $\varphi(x) = \sum a_i x b_i$. Choose a basis $x_1, x_2, \ldots, x_n$ of the free R-module A with

$x_1 = 1$. Define R-linear endomorphisms $\varphi_1, \ldots, \varphi_n$ of A by

$$\varphi_i(x_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

For any $x \in A$ we have

$$x = \sum r_j x_j, \text{ with } r_j \in R \quad \text{and} \quad \varphi_i(x) = \varphi_i(\sum r_j x_j) = r_i.$$

Let $c \in Z(A) = Z(A^\circ)$ then $\varphi(xc) = \sum a_i x c b_i = \varphi(x)c$. Write $c = \sum r_i x_i$, $r_i \in R$ with respect to the basis of A. Then in particular

$$c = \varphi_1(x_1)c = \varphi_1(x_1 c) = \varphi_1(c) = r_1$$

and thus $c \in R$.

Let $I \subset R$ be an ideal and $x \in IA$. Then $x = \sum r_i x_i$ with $r_i \in I$. We have $x \in R$ if and only if $r_i = 0$ for $i > 1$. This shows that $IA \cap R = I$.

Let $J \subset A$ be an ideal. It holds that $\varphi(J) \subset J$. Let $x \in J$ and write it with respect to the basis as $x = \sum r_i x_i$, with $r_i \in R$. Then $\varphi_i(x) = r_i$ but since $\varphi_i(x) \in J$ this shows $r_i \in J$. Thus $x \in (J \cap R)A$.

The map $J \mapsto J \cap R$ is a bijection since $I \mapsto IA$ is an inverse. □

As a next step, we wish to understand how Azumaya algebras behave under base change. In particular base change to the residue field. Let R' a local rings with a homomorphism $f : R \to R'$ that makes R' into a commutative R-algebra. Note that f is not required to be a local morphism. The base change to the residue field is included since we can choose $R' = R/\mathfrak{m}$. We can say the following

Proposition 1.12. *Let A be an Azumaya algebra over R and let R' be as above. Then $A \otimes_R R'$ is an Azumaya algebra over R'*

Proof. Obviously $R' \otimes_R A$ is an R'-algebra, that has R' as a subring in its center, and is free and of finite ranks as an R'-module. We get a commutative diagram

$$\begin{array}{ccc} \mathrm{id}_{R'} \otimes_R \Phi : R' \otimes_R (A \otimes_R A^\circ) & \longrightarrow & R' \otimes_R \mathrm{End}_{R\text{-mod}}(A) \\ \downarrow \simeq & & \downarrow \simeq \\ \Phi' : (R' \otimes_R A) \otimes_{R'} (R' \otimes_R A^\circ) & \longrightarrow & \mathrm{End}_{R\text{-mod}}(R' \otimes_R A). \end{array}$$

Here Φ is the isomorphism in the definition of an Azumaya algebra. This directly implies that $\mathrm{id}_{R'} \otimes_R \Phi$ is an isomorphism. Since the vertical arrows are isomorphisms Φ' is also an isomorphism. □

We ask ourselves if this can be done in the other direction.

Lemma 1.13. *Let $f : R \to R'$ be faithfully flat morphism of rings and A a R-algebra that is finite as an R-module. Then A is an Azumaya algebra if and only if $R' \otimes_R A$ is.*

Proof. One direction has already been shown in Proposition 1.12. So let $R' \otimes_R A$ be an Azumaya algebra. In particular it is free of finite rank as an R'-module. Since R' is a faithfully flat R-module this directly implies that A is free and of finite rank as an R-module. We have an isomorphism

$$R' \otimes_R (A \otimes_R A^\circ) \longrightarrow R' \otimes_R \mathrm{End}_{R\text{-mod}}(A)$$

and since f is faithfully flat this is equivalent to

$$A \otimes_R A^\circ \longrightarrow \mathrm{End}_{R\text{-mod}}(A)$$

being an isomorphism. □

An important special case occurs if $R = k$, $R' = K$ are fields and K/k is a field extension. Since field extensions are faithfully flat this lemma always holds for central simple algebras. Continuing this line of thought, what happens if we set $R' = R/\mathfrak{m}$ as the residue field of the local ring R. For this to work we need a way to "lift" isomorphisms.

Before we start. Since we will apply it at several places we repeat the statement of Nakayama's Lemma. Note that this statement does not require that the ring is local.

Theorem 1.14. *(Nakayama's Lemma) ([54] Chapter I Theorem 2.2) Let S be a ring, M be a finite S-module and $I \subset S$ an ideal that is contained in the Jacobson radical $I \subset \mathrm{rad}(S)$. If $M = IM$, or equivalent $M/IM = 0$, then $M = 0$.*

Lemma 1.15. *Let M be a finite R-module, N a free R-module, $\mathfrak{m} \subset R$ the maximal ideal and $\Phi : M \to N$ a R-linear map. If the induced map $\bar{\Phi} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ of $R/\mathfrak{m}$-modules is an isomorphism then Φ is an isomorphism as well.*

Proof. We have an exact sequence $0 \to K \to M \xrightarrow{\Phi} N \to C \to 0$, with $K := \ker(\Phi)$ and $C := \mathrm{coker}(\Phi)$. Tensoring with $R/\mathfrak{m}$ is right exact so $C/\mathfrak{m}C = \mathrm{coker}(\bar{\Phi}) = 0$. By Nakayama's Lemma is $C = 0$. We get a short exact sequence $0 \to K \to M \to N \to 0$. Since N is free it is also flat and tensoring with $R/\mathfrak{m}$ gives the short exact sequence $0 \to K/\mathfrak{m}K \to M/\mathfrak{m}M \xrightarrow{\bar{\Phi}} N/\mathfrak{m}N \to 0$ (see [50] Chapter 1 Proposition 2.6). So $K/\mathfrak{m}K = \ker(\bar{\Phi}) = 0$ and Nakayama's Lemma again gives $K = 0$. □

Now we will see what can be said of Azumaya algebras under base change to the residue field. And we will see that Azumaya algebra over a local ring corresponds to an Azumaya algebra, or better a central simple algebra, over its residue field.

Proposition 1.16. *Let A be R-algebra that is free and of finite rank as a R-module and let $\mathfrak{m} \subset R$ be the maximal ideal. Then A is an Azumaya algebra if and only if $A/\mathfrak{m}A$ is one over $R/\mathfrak{m}$.*

Proof. On direction has already been shown in Proposition 1.12. So let $A/\mathfrak{m}$ be an Azumaya algebra. Then it is a central simple algebra. Also, we have a commutative diagram

$$\begin{tikzcd} A \otimes A^\circ \arrow[r] \arrow[d] & \mathrm{End}_{R\text{-mod}}(A) \arrow[d] \\ A/\mathfrak{m} \otimes (A/\mathfrak{m})^\circ \arrow[r] & \mathrm{End}_{(R/\mathfrak{m})\text{-mod}}(A/\mathfrak{m}) \end{tikzcd}$$

The lower arrow is an isomorphism according to Proposition 1.3, and since A is a free R-module so is $\mathrm{End}_{R\text{-mod}}(A)$. Now Lemma 1.15 tells us that the upper arrow is an isomorphism as well. $\square$

Since the dimension of a central simple algebra as a vector space is always a square, the lemma also implies that the rank of an Azumaya algebra as a free module is a square.

Corollary 1.17. *If A and B are two Azumaya algebras over R then their tensor product $A \otimes B$ is also an Azumaya algebra over R. The Matrix algebra $M_n(R)$ is an Azumaya algebra over R.*

Proof. We have seen in the last section that both statements hold for central simple algebras, so Proposition 1.16 directly proves the result. $\square$

Next we define the Brauer group of a ring. The definition is a generalization of the one for central simple algebras. Let A and B two Azumaya algebras over R. They are *equivalent* $A \sim B$ if there exist $n, m \geq 1$ such that there exists a morphism of R-algebras

$$A \otimes_R M_n(R) \simeq B \otimes_R M_m(R).$$

One sees that this is an equivalence relation. For transitivity one has to check that if A_1, A_2, B_1 and B_2 are Azumaya algebras over R with $A_1 \sim B_1$ and $A_2 \sim B_2$ then $A_1 \otimes A_2 \sim B_1 \otimes B_2$. As for fields, this is the case because $M_n(R) \otimes M_m(R) = M_{nm}(R)$. Write $[A]$ for the equivalence class.

Lemma 1.18. *The set of all equivalence classes equipped with the tensor product $[A][A'] = [A \otimes A']$ as an operation is an abelian group.*

Proof. The neutral element is the class $[R]$, this class consists of all matrix algebras $M_n(R)$. The group is associative and commutative since the tensor product is. An inverse element to a class $[A]$ is given by the class of the opposite algebra $[A^\circ]$. $\square$

This group is called the *Brauer group* of R and denoted as $\mathrm{Br}(R)$. Again we call the neutral element $[R]$ the *trivial class*. If $R = k$ is a field the definition agrees with the definition of the Brauer group of a field. $\mathrm{Br}(-)$ can again be seen as a functor from the category of local rings to $(\mathbf{Ab})$.

A map of rings $f : R \to R'$ induces a canonical map of Brauer groups

$$\mathrm{Br}(R) \longrightarrow \mathrm{Br}(R'), \quad [A] \longmapsto [R' \otimes A].$$

This map is well defined, as the base change of an Azumaya algebra is still an Azumaya algebra, and base change respects isomorphism so equivalent Azumaya algebras stay equivalent. Keep in mind that this map is in general neither surjective not injective. For example $f : \mathbb{R} \to \mathbb{C}$ gives a non injective map $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \to \mathrm{Br}(\mathbb{C}) = 0$. A non surjective map can be constructed by taking any morphism $f : k \to R$ where R has a non trivial Brauer group and k is separably closed, which implies $\mathrm{Br}(k) = 0$. We are particularly interested in the following case:

Proposition 1.19. *Let R be an Henselian local ring and $R/\mathfrak{m}$ the residue field. Then the map $\mathrm{Br}(R) \to \mathrm{Br}(R/\mathfrak{m})$ is injective.*

Proof. Let A be an Azumaya algebra over R with $A \otimes_R R/\mathfrak{m} \simeq M_n(R/\mathfrak{m})$. We have to show that then A is itself trivial.

Set $k = R/\mathfrak{m}$. Choose an idempotent matrix $\epsilon \in M_n(k)$ of rank 1. We simply take the Matrix with 1 at the $(1, 1)$ position and 0 otherwise. Let $a \in A$ such that $\bar{a} \in A \otimes_R k$ is mapped to ϵ under the isomorphism. Then $R[a]$ is a finite commutative R-algebra, and such an algebra over a Henselian ring is a direct product of local rings ([55] Chapter I Theorem 4.2). Since $R[a] \otimes_R k \simeq k[\epsilon] \simeq k \oplus k$ this direct product has to consist exactly of two local rings $R[a] \simeq R/I \oplus R/J$. Let ϵ correspond to the element $(1, 0) \in k \oplus k$. This lifts to an idempotent $(1, 0) \in R/I \oplus R/J$, which in turn determines an idempotent $e \in R[a]$.

Since $A = Ae \oplus A(1 - e)$ we see that Ae is a finite and free R-module. In particular this implies $\mathrm{End}_R(Ae) \simeq M_m(R)$ for some $m > 0$. We then define a homomorphism of R-algebras

$$\Phi : A \longrightarrow \mathrm{End}_R(Ae), \quad a \longmapsto (xe \mapsto axe).$$

It is left to check that this map is an isomorphism. The kernel of Φ is an ideal in A and its intersection with R is zero, since Ae is a free R-module. But then the kernel has to already be zero itself by Proposition 1.11. Using the same argument we show that $\bar{\Phi} : A \otimes_R k \to \mathrm{End}_k((A \otimes_R k)\epsilon)$ is injective. Both sides have the same dimension as k vector spaces, so $\bar{\Phi}$ is an isomorphism. By Nakayama's Lemma, this implies that Φ is also surjective

□

We will later show in Corollary 1.40 that this map is actually an isomorphism.

Corollary 1.20. *For any strictly Henselian local ring R it holds that*

$$\mathrm{Br}(R) = 0.$$

Proof. The residue field of a strictly Henselian local ring is separably closed and the Brauer group of such a field is zero by Corollary 1.8 □

As the last part in this section we describe the automorphism group of an Azumaya algebra. This is a generalization of the Skolem-Noether Theorem.

Proposition 1.21. *(Skolem-Noether) Let A be an Azumaya algebra on R. Every R-algebra automorphism of A is of the form*

$$a \longmapsto uau^{-1},$$

where $u \in A$ is a unit, i.e. every automorphism of A is inner.

Proof. Let $\Phi : A \to A$ be an R-algebra automorphism. We equip A with two different left $A \otimes A^{\circ}$-module structure by defining two different scalar multiplication

$$\begin{aligned}(a \otimes b) *_1 x &= axb,\\ (a \otimes b) *_2 x &= \Phi(a)xb,\end{aligned}$$

for $a \otimes b \in A \otimes A^{\circ}$ and $x \in A$. The multiplication on the right hand side is defined in A. We denote the resulting modules A_1 and A_2.

Since $A \otimes A^{\circ} \simeq \mathrm{End}_{R-mod}(A)$ they are also left $\mathrm{End}_{R-mod}(A)$-modules. We show that A_1 is projective. Since $A_1 = A$ as an R-module and A is a free R-module, there exist a homomorphism of R-modules $g : A \to R$ with $g(r) = r$ for all $r \in R$. Now the morphism of $\mathrm{End}_{R-mod}(A)$-modules $\mathrm{End}_{R-mod}(A) \to A_1$, $f \mapsto f(1)$ is surjective and $a \mapsto (a' \mapsto g(a')a)$ defines a section. This shows that A_1 is projective.

Base change to the residue field makes both $A_1 \otimes R/\mathfrak{m}$ and $A_2 \otimes R/\mathfrak{m}$ into simple left modules over $\bar{A} = (A \otimes R/\mathfrak{m}) \otimes (A \otimes R/\mathfrak{m})^{\circ} = M_n(A \otimes R/\mathfrak{m})$, that do have the same rank. And $\bar{A}$ is a central simple algebra over the field $R/\mathfrak{m}$. This in particular implies that $\bar{A}$ is a simple Artin ring and over such a ring all simple modules are isomorphic ([38] Lemma 4.3.2). There exists an isomorphism $\bar{\Psi} : A_1 \otimes R/\mathfrak{m} \xrightarrow{\sim} A_2 \otimes R/\mathfrak{m}$ of $\bar{A}$-modules. This isomorphism can be lifted to a morphism of $A \otimes A^\circ$-modules

$$\begin{tikzcd} A_1 \arrow[r, "\Psi"] \arrow[d] & A_2 \arrow[d] \\ A_1 \otimes R/\mathfrak{m} \arrow[r, "\bar{\Psi}"'] & A_2 \otimes R/\mathfrak{m}. \end{tikzcd}$$

The lift Ψ exist since the vertical arrows are surjective and A_1 is projective. We can view everything as a map of R-modules, then Nakayama's lemma shows that Ψ is surjective. Actually, Lemma 1.15 implies that Ψ is an isomorphism.

Set $u = \Psi(1)$. The $A \otimes A^\circ$-linearity of Ψ shows that for any $a \in A_1$

$$\Psi(a) = \Psi((a \otimes 1) *_1 1) = (a \otimes 1) *_2 \Psi(1) = (a \otimes 1) *_2 u = \Phi(a)u,$$

and

$$\Psi(a) = \Psi((1 \otimes a) *_1 1) = (1 \otimes a) *_2 \Psi(1) = (1 \otimes a) *_2 u = ua.$$

So $\Phi(a)u = ua$ holds in A_2 and as an R-module we have $A = A_2$. It remains to show that u is a unit. Since Ψ is surjective there exist $a_1 \in A_1$ with $\Psi(a_1) = 1$, and we get

$$1 = \Psi(a_1) = \Psi((a_1 \otimes 1) *_1 1) = \Phi(a_1)u.$$

So we have a left inverse $u^{-1} = \Phi(a_1)$. Since also $1 = ua_1$, left composition with u^{-1} gives $u^{-1} = a_1$ and so $\Phi(a_1)$ also defines a right inverse. Note that this also shows $\Phi(a_1) = a_1$. $\square$

Corollary 1.22. *The group* $\mathrm{Aut}(M_n(R))$ *of automorphisms of* $M_n(R)$ *as an* R*-algebra is* $PGL_n(R) = GL_n(R)/R^*$

Proof. $M_n(R)$ is an Azumaya algebra so every automorphism is given by conjugation with a unit. The group of units is $GL_n(R)$ and conjugation by an element U of this group is the identity map if and only if U is in the center R^* of $M_n(R)$. $\square$

1.3 Azumaya algebras on schemes

Before we come to the definition of Azumaya algebras on schemes we give a short reminder on properties of locally free sheaves.

Definition 1.23. Let X be a scheme and $\mathscr{E}$ be an $\mathcal{O}_X$-module.

(i) We call $\mathscr{E}$ *locally free* if for every point $x \in X$ there exist a open neighborhood $U \subset X$ of x and a set I such that $\mathscr{E}_{|U} \simeq \bigoplus_{i \in I} \mathcal{O}_U$ is an isomorphism of $\mathcal{O}_U$-modules.

(ii) We call $\mathscr{E}$ *finite locally free* or *locally free of finite type* if for all U the index sets I can be chosen to be finite.

(iii) We call $\mathscr{E}$ *locally free of finite rank* r if for all U the index sets I can be chosen to have cardinality r.

Note that a locally free sheaf is quasicoherent. Furthermore, if X is a connected scheme, than a finite locally free sheaf is of finite rank. Otherwise the rank may vary between different connected components. All locally free sheaves we work with will at least be assumed to be finite. These sheaves can be described by the following lemma.

Lemma 1.24. *([5] Tag 05P2 and Tag 00NX) Let $\mathscr{F}$ be a quasi-coherent sheaf on a scheme X. Then $\mathscr{F}$ is a finite locally free $\mathcal{O}_X$-module if and only if it is a flat $\mathcal{O}_X$-module of finite presentation. If $X = \operatorname{Spec}(R)$ is additionally an affine scheme and $\mathscr{F} = \tilde{M}$ this is also equivalent to M being a finite projective R-module.*

This allows us to implicitly use the following at several places. Let $\mathscr{E}$ be a finite locally free sheaf, so in particular it is of finite presentation. Then for all $x \in X$ the canonical homomorphism

$$\varphi : \mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})_x \longrightarrow \mathcal{E}nd_{\mathcal{O}_{X,x}}(\mathscr{E}_x)$$

is an isomorphism of $\mathcal{O}_{X,x}$-modules. Furthermore, there exist an open neighborhood $U \subset X$ of x such that

$$\phi : \mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})_{|U} \longrightarrow \mathcal{E}nd_{\mathcal{O}_U}(\mathscr{E}_{|U})$$

is an isomorphism and $\phi_x = \varphi$ (see [26] Proposition 7.27). This in particular implies that $\mathcal{E}nd(\mathscr{E})$ is finite locally free.

Now let us get back to Azumaya algebras. Let X be any scheme and $\mathscr{A}$ an $\mathcal{O}_X$-algebra, or more precisely a sheaf of $\mathcal{O}_X$-algebras. Note that again we do not require an $\mathcal{O}_X$-algebra to be commutative. In particular for every $x \in X$ the stalk $\mathscr{A}_x$ is an algebra over the local ring $\mathcal{O}_{X,x}$. We denote the sheaf of opposite algebras by $\mathscr{A}^\circ$.

Proposition 1.25. *Let $\mathscr{A}$ be an $\mathcal{O}_X$-algebra, that is finite locally free as an $\mathcal{O}_X$-module. The following are equivalent:*

(i) *For all $x \in X$ the stalk $\mathscr{A}_x$ is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$.*

(ii) *For all $x \in X$ the algebra $\mathscr{A}_x \otimes_{\mathcal{O}_X} \kappa(x)$ is a central simple algebra over the residue field $\kappa(x)$.*

(iii) *The canonical homomorphism*

$$\mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A}^\circ \longrightarrow \mathcal{E}nd_{\mathcal{O}_X\text{-}mod}(\mathscr{A})$$

is an isomorphism.

(iv) *$\mathscr{A}$ is étale locally isomorphic to an matrix algebra sheaf, i.e. there exist an étale covering $(U_i \to X)$ and positive integers n_i, with $\mathscr{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \simeq M_{n_i}(\mathcal{O}_{U_i})$.*

(v) *$\mathscr{A}$ is isomorphic to a matrix algebra sheaf with respect to the fppf topology, i.e. there exist an flat covering $(U_i \to X)$ and positive integers n_i, with $\mathscr{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \simeq M_{n_i}(\mathcal{O}_{U_i})$.*

Proof. $(i) \Leftrightarrow (ii)$ Follows directly with Proposition 1.16.

$(i) \Leftrightarrow (iii)$ Since $\mathscr{A}$ is locally free we have $(\mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A}^\circ)_x \simeq \mathscr{A}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{A}_x^\circ$ and $(\mathrm{End}_{\mathcal{O}_X}(\mathscr{A}))_x \simeq \mathrm{End}_{\mathcal{O}_{X,x}}(\mathscr{A}_x)$. From the definition of Azumaya algebras over local ring we get an isomorphism $\mathscr{A}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{A}_x^\circ \simeq \mathrm{End}_{\mathcal{O}_{X,x}}(\mathscr{A}_x)$ which shows the equivalence.

$(i) \Rightarrow (iv)$ Let $x \in X$ and $\bar{x} \in X$ be the a geometric point over x, then $\mathcal{O}_{X,\bar{x}}$ is a strictly Henselian local ring. By Corollary 1.20 every Azumaya algebra over this ring is trivial, in particular $\mathscr{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\bar{x}} \simeq M_n(\mathcal{O}_{X,\bar{x}})$. There has to exist an étale morphism $U \to X$ whose image contains x such that $\mathscr{A} \otimes_{\mathcal{O}_X} \mathcal{O}_U \simeq M_n(\mathcal{O}_U)$.

$(iv) \Rightarrow (v)$ Any étale covering is also a flat covering.

$(v) \Rightarrow (ii)$ Let $x \in X$. There exists a U_i, with $\mathscr{A} \otimes \mathcal{O}_{U_i} \simeq M_{n_i}(\mathcal{O}_{U_i})$, such that x is in the image of $U_i \to X$. Also there is an $x_i \in U_i$ that is mapped to x. So we have a field extension $\kappa(x_i)/\kappa(x)$ for the residue fields. The fact that $\mathscr{A} \otimes \mathcal{O}_{U_i} \simeq M_{n_i}(\mathcal{O}_{U_i})$ implies that $\mathscr{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \otimes_{\mathcal{O}_{U_i}} \kappa(x_i) \simeq M_{n_i}(\kappa(x_i))$. By construction this is the same as the base change of $\mathscr{A} \otimes_{\mathcal{O}_X} \kappa(x)$ along the field extension $\kappa(x_i)/\kappa(x)$. By Lemma 1.13 $\mathscr{A} \otimes_{\mathcal{O}_X} \kappa(x)$ is then a central simple algebra. $\square$

Note that Proposition 1.12 implies that we could demand condition (i) only for closed points.

Definition 1.26. A finite locally free $\mathcal{O}_X$-algebra $\mathscr{A}$ is called *Azumaya algebra* over X if it fulfills the equivalent conditions of Proposition 1.25.

Two Azumaya algebras are isomorphic to each other if they are isomorphic as $\mathcal{O}_X$-algebras. Such an isomorphism automatically respects the structure of an Azumaya algebra. The fact that the rank of an Azumaya algebra over a local ring is always a square implies that the rank of $\mathscr{A}$ as a locally free sheaf is a square when restricted to each of the connected components of X. Of course the rank may vary on different connected components. If an Azumaya algebra has constant rank n^2 we call n the *degree* of $\mathscr{A}$, even though we are mostly concerned with the rank. Furthermore, Lemma 1.24 immediately shows that an Azumaya algebra is flat as an $\mathcal{O}_X$-module and corresponds to a finite projective module in the affine case.

Let $X = \operatorname{Spec}(R)$ be an affine scheme. Then an Azumaya algebra $\mathscr{A}$ corresponds to an associative R-algebra with identity, that we denote by A. This algebra is finite locally free as an R-module, and for every prime $\mathfrak{p} \in \operatorname{Spec}(R)$ is $A_\mathfrak{p}$ an Azumaya algebra over the local ring $R_\mathfrak{p}$. We call A then an *Azumaya algebra* over the ring R. As first example, $M_n(R)$ is an Azumaya algebra over R.

We can "generalize" the quaternion algebras to $\mathbb{Z}[\frac{1}{n}]$. Define an algebra

$$A_{a,b} = \mathbb{Z}[\tfrac{1}{n}]\langle i, j, t\rangle/(i^2 - a, j^2 - b, t - ij, t + ji).$$

Here a, b and n shall have the following properties: First $a, b \neq 0$. Next $2 \mid n$, and for every prime p, that divides either a or b, we demand $p \mid n$. This ensures that for all primes (p) in $\mathbb{Z}[\frac{1}{n}]$ we have $a, b \in \mathbb{F}_p^\times$. So $A_{a,b} \otimes \mathbb{F}_p$ is a quaternion algebra over $\mathbb{F}_p$. Note that this algebra is trivial, since the Brauer group of $\mathbb{F}_p$ is trivial. This construction is not possible over the whole integers, since there always exist finitely many primes p with $a \bmod p \equiv 0$. In fact it can be shown that $\operatorname{Br}(\mathbb{Z}) = 0$ ([42] Chapter III Example 8.1.ii).

Part (iii) of Proposition 1.25 together with Lemma 1.24 shows that an algebra A over a ring R is an Azumaya algebra if and only if A is finite and projective as an R-module and $A \otimes_R A^\circ \to \operatorname{End}_R(A)$ is an isomorphism. Proposition 1.11 also implies that R is the center of A. This shows that our definition of Azumaya algebras over rings agrees with that of central separable algebras by Auslander and Goldman in [4].

A *separable* algebra is a R-algebra A, which is projective as an $A \otimes_R A^\circ$-module; the scalar multiplication is given by $(a \otimes b)x = axb$ for $x \in A$ and

$(a \otimes b) \in A \otimes_R A^\circ$. Even if A is not projective we have an $A \otimes_R A^\circ$-module homomorphism $\mu : A \otimes_R A^\circ \to A$ by $\mu(\sum a_i \otimes b_i) = \sum a_i b_i$. This gives a short exact sequence

$$0 \longrightarrow J \longrightarrow A \otimes_R A^\circ \xrightarrow{\mu} A \longrightarrow 0.$$

A is projective if and only if μ has a section. If it exists, such a section can always be given by an element $e \in A \otimes_R A^\circ$ with $\mu(e) = 1$ and $Je = 0$, by defining a homomorphism $\psi : A \to A \otimes_R A^\circ$ with $\psi(a) = (a \otimes 1)e$. The condition $Je = 0$ essentially means $(a \otimes 1 - 1 \otimes a)e = 0$ for all $a \in A$. Such an e is necessarily an idempotent since $e^2 - e = (e - 1 \otimes 1)e \in eJ = 0$ ([15] Chapter II Proposition 1.1). This is a useful criterion to check for separability. We remark, that if $R = k$ is a field, this definition of separability agrees with the classic definition of a separable algebra over a field ([15] Chapter II Corollary 2.4).

The scalar multiplication commutes with the center of A, so there exist an inclusion $\eta : A \otimes_R A^\circ \to \mathrm{End}_{Z(A)}(A)$, given by $\eta(a, b)(x) = axb$. For a central separable algebra A, so $Z(A) = R$ over R, the map η is an isomorphism and A is a finite and projective R-module ([4] Theorem 2.1). Since furthermore any separable R-algebra A is separable over its center ([4] Theorem 2.3), every separable algebra is an Azumaya algebra over its center.

Let us consider an example. Let R be a ring and G be a finite group of order n such that n is invertible in R. Then the group algebra $R[G]$ is a separable algebra. We set

$$e = \frac{1}{n} \sum_{g \in G} g \otimes g^{-1} \in R[G] \otimes R[G]^\circ.$$

Then

$$\mu(e) = 1 \quad \text{and} \quad (x \otimes 1)e = \frac{1}{n} \sum_{g \in G} xg \otimes g^{-1} = \frac{1}{n} \sum_{y \in G} y \otimes y^{-1}x = (1 \otimes x)e$$

for any $x \in G$ as required. The center of $R(G)$ is spanned by the conjugacy classes of G ([43] Proposition 3.1.1), and $R(G)$ is an Azumaya algebra over it.

Another example can be given by Weyl algebras over fields k of characteristic p. Let $A_n(k) = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with non commuting indeterminates, which have the relations $[x_i, y_i] = x_i y_i - y_i x_i = 1$ and $[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 0$ for all $i \neq j$. It is shown by Revoy [58] that these algebras are central separable over their center, with the center of $A_n(k)$ being isomorphic to the ring of polynomials in $2n$ variables $k[x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p]$.

We aim to define the Brauer group of a scheme.

Corollary 1.27. *Let $\mathscr{A}$ and $\mathscr{B}$ be two Azumaya algebras on a scheme X, then their tensor product $\mathscr{A} \otimes \mathscr{B}$ is also an Azumaya algebra on X.*

Proof. $\mathscr{A} \otimes \mathscr{B}$ is locally free and is an $\mathcal{O}_X$-algebra. The fact that it is an Azumaya algebra can be checked on stalks. Let $x \in X$. The stalk at x is $(\mathscr{A} \otimes \mathscr{B})_x = \mathscr{A}_x \otimes \mathscr{B}_x$, and over a local ring $\mathcal{O}_{X,x}$ we already know that product of two Azumaya algebras is an Azumaya algebra by Corollary 1.17. □

Corollary 1.28. *Let $\mathscr{E}$ be a finite locally free sheaf on X, then $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})$ is an Azumaya algebra on X.*

Proof. $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})$ is finite locally free and a sheaf of algebras. Since $\mathscr{E}$ is finite locally free there exist a Zariski open cover U_i such that $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})_{|U_i} \simeq M_{n_i}(\mathcal{O}_{U_i})$. Since a Zariski cover is also an étale cover $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})$ is an Azumaya algebra. □

Two Azumaya algebras $\mathscr{A}$ and $\mathscr{A}'$ are *equivalent* if there exist locally free $\mathcal{O}_X$-modules $\mathscr{E}$ and $\mathscr{E}'$ such that

$$\mathscr{A} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E}) \simeq \mathscr{A}' \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E}).$$

Note, as before, that the relation is an equivalence relation. We check only transitivity. If $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{B}_1$ and $\mathscr{B}_2$ are Azumaya algebras on a scheme X with $\mathscr{A}_1 \sim \mathscr{B}_1$ and $\mathscr{A}_2 \sim \mathscr{B}_2$ then $\mathscr{A}_1 \otimes \mathscr{A}_2 \sim \mathscr{B}_1 \otimes \mathscr{B}_2$, since $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E}) \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E}') \simeq \mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{E}')$.

Lemma 1.29. *The set of all equivalence classes equipped with the tensor product $[\mathscr{A}][\mathscr{A}'] = [\mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A}']$ as an operation is an abelian group.*

Proof. The neutral element is the class $[\mathcal{O}_X]$, the class consists of all sheaves of the form $\mathcal{E}nd_{\mathcal{O}_X}(\mathscr{E})$ where $\mathscr{E}$ is locally free. The group is associative and commutative since the tensor product is. An inverse element to a class $[\mathscr{A}]$ is given by the class of the opposite algebra $[\mathscr{A}^\circ]$. □

This group is called the *Brauer group* of X and denoted as $\mathrm{Br}(X)$. Again, we call the neutral element $[\mathcal{O}_X]$ the *trivial class*. In particular the sheaves $\mathcal{E}nd(\bigoplus_{i=1}^n \mathcal{O}_X) \simeq M_n(\mathcal{O}_X)$ are representatives of this class. $\mathrm{Br}(-)$ can be seen as a functor from the category of schemes to $(\mathbf{Ab})$. For the Brauer group of an affine scheme $\mathrm{Spec}(R)$, we usually write $\mathrm{Br}(R)$. For a local ring this is the same group as the Brauer group of a local ring defined in the last section, since the definitions of Azumaya algebras agree. Some authors calls $\mathrm{Br}(X)$ the geometric Brauer group to distinguish it from the cohomological Brauer group, which we will define in the next section.

Lemma 1.30. *Let $f : X \to Y$ be a morphism of schemes and $\mathscr{B}$ an Azumaya algebra on Y. Then the pullback $f^*\mathscr{B}$ is an Azumaya algebra on X.*

Proof. The pullback of a finite locally free sheaf is finite locally free and the pullback of a sheaf of algebras has a canonical algebra structure. The Azumaya property can be checked locally and thus follows from Proposition 1.12. □

A morphism of schemes $f : X \to Y$ now induces a canonical map of Brauer groups

$$\mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X), \quad [\mathscr{B}] \longmapsto [f^*\mathscr{B}].$$

The map is well defined, as the pullback of an Azumaya algebra is an Azumaya algebra, and pullbacks respect isomorphism, so equivalent Azumaya algebras stay equivalent. As already seen in the last section this map is in general neither injective nor surjective.

Lemma 1.31. *Let $f : X \to Y$ be a surjective morphism of schemes and $\mathscr{B}$ be an $\mathcal{O}_Y$-algebra, that is finite locally free as an $\mathcal{O}_Y$-module. Then $\mathscr{B}$ is an Azumaya algebra on Y if and only if $\mathscr{A} = f^*\mathscr{B}$ is an Azumaya algebra on X.*

Proof. Again, one direction has already been shown in Lemma 1.30. So let $\mathscr{A}$ be an Azumaya algebra. By Proposition 1.25 it is enough to check that $\mathscr{B} \otimes_{\mathcal{O}_Y} \kappa(y)$ is an Azumaya algebra over $\kappa(y)$ for every $y \in Y$. Since f is surjective there always exists an $x \in X$ with $f(x) = y$ and $\mathscr{A} \otimes_{\mathcal{O}_X} \kappa(x)$ is an Azumaya algebra over $\kappa(x)$. Now

$$\mathscr{A} \otimes_{\mathcal{O}_X} \kappa(x) = \mathscr{B} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \otimes_{\mathcal{O}_X} \kappa(x) = \mathscr{B} \otimes_{\mathcal{O}_Y} \kappa(x).$$

And $\kappa(x)/\kappa(y)$ is a field extension and thus faithfully flat. Then $\mathscr{B} \otimes_{\mathcal{O}_Y} \kappa(y)$ is an Azumaya algebra over $\kappa(y)$ by Lemma 1.13. □

Lemma 1.32. *Let $f : X \to Y$ be flat and surjective morphism of schemes and $\mathscr{B}$ be a quasicoherent $\mathcal{O}_Y$-algebra. Then $\mathscr{B}$ is an Azumaya algebra on Y if and only if $\mathscr{A} = f^*\mathscr{B}$ is an Azumaya algebra on X.*

Proof. One direction has already been shown in Lemma 1.30. So let $\mathscr{A}$ be an Azumaya algebra. We show $\mathscr{B}$ is finite locally free. Let $y \in Y$ and choose $x \in X$ with $f(x) = y$. Since f is flat and surjective we know that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a faithfully flat morphism of rings (see [5] Tag 00HQ). Also $\mathscr{A}_x = (f^*\mathscr{B})_x = \mathscr{B}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$ is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$. In particular it is free of finite rank, hence $\mathscr{B}_y$ has to be free and of finite rank as an $\mathcal{O}_{Y,y}$-module. This in turn shows that $\mathscr{B}$ is finite locally free. The fact that $\mathscr{B}$ is Azumaya follows from Lemma 1.31. □

Next we generalize Skolem-Noether to Azumaya algebra over schemes.

Proposition 1.33. *Let $\mathscr{A}$ be an Azumaya algebra on a scheme X and let $\Phi : \mathscr{A} \to \mathscr{A}$ be an automorphism. Then there exists a Zariski covering $X = \bigcup U_i$ of open sets such that $\Phi_{|U_i}$ is given by $a \mapsto uau^{-1}$ where $u \in \Gamma(U_i, \mathscr{A})$ is a unit.*

Proof. Let $x \in X$, then the automorphism Φ induces an automorphism $\Phi_x : \mathscr{A}_x \to \mathscr{A}_x$. By Skolem-Noether (Proposition 1.21) there is a unit $u_x \in \mathscr{A}_x$ such that $\Phi_x(a_x) = u_x^{-1} a_x u_x$ for all $a_x \in A_x$. Then x has some open neighborhood U such that $u \in \Gamma(U, \mathscr{A})$ is a unit, which localizes to u_x. Now define a map $\mathscr{A}_{|U} \to \mathscr{A}_{|U}$, $(a \mapsto u^{-1}au)$, which localizes to Φ_x. The map $\Phi_{|U} : \mathscr{A}_{|U} \to \mathscr{A}_{|U}$ also localizes to Φ_x, so there exist a open neighborhood $V \subset U$ of x on which these maps agree. □

So far we have not made any demands on the rank of an Azumaya algebra as a finite locally free sheave. If X is connected, this implies that an Azumaya algebra $\mathscr{A}$ has constant rank n^2. Under this assumption the matrix algebras $M_{n_i}(\mathcal{O}_X)$ in Proposition 1.25 are of rank n^2 with $n_i = n$.

Lemma 1.34. *Let X be a scheme with at most finitely many connected components. Then every Azumaya algebra is equivalent to an Azumaya algebra of constant rank. In particular every Brauer class is represented by an Azumaya algebra of constant rank.*

Proof. We write $X = \bigsqcup_{i=1}^{l} X_i$ as the finite union of its connected components X_i. On each of the connected components the Azumaya algebra $\mathscr{A}_i = \mathscr{A}_{|X_i}$, has constant rank n_i and $\mathscr{A} = \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_l$. For the Brauer group we have $\mathrm{Br}(X) = \mathrm{Br}(X_1) \oplus \cdots \oplus \mathrm{Br}(X_l)$. Set

$$r_i = \sum_{j=1, j\neq i}^{l} \sqrt{n_i}.$$

For $i = 1, \ldots, l$ we replace $\mathscr{A}_i$ by

$$\mathscr{A}_i \otimes (End(\bigoplus_{j=1}^{r_i} \mathcal{O}_{X_i})).$$

This does not change the Brauer class of $\mathscr{A}_i$ in $\mathrm{Br}(X_i)$ and thereby does not change the Brauer class of $\mathscr{A}$ in $\mathrm{Br}(X)$. Now every $\mathscr{A}_i$ has the same rank and thus $\mathscr{A}$ has constant rank $n = \prod_{i=1}^{l} n_i$. □

If X has infinitely many connected components we could have an Azumaya algebra whose rank increases on each connected component and thus increases infinitely. In particular the Lemma does not hold true. In many ways one wants only to work with Azumaya algebras of constant rank. We will be exclusively concerned with schemes with finitely many connected components and can assume that a Brauer class is represented by an Azumaya algebra of constant rank. For the second main result we need to allow schemes with finitely many connected components, which is the main reason why we do not restrict to connected schemes.

1.4 The cohomological Brauer group

We aim to connect Azumaya algebras and Brauer groups to cohomology theory. This makes it possible to use cohomology theory to show facts about the Brauer group.

Let $\mathscr{A}$ be an Azumaya algebra over a scheme X of constant rank n^2. Proposition 1.25 tells us, that there exists a cover $(U_i \to X)$ for the flat (fppf) and étale topology, respectively, such that the Azumaya algebra $\mathscr{A}$ and the sheaf $M_n(\mathcal{O}_X)$ become isomorphic when restricted to this cover. So an Azumaya algebra of rank n is a twisted form of $M_n(\mathcal{O}_X)$, which determines the class of a 1-cocycle in the pointed set $\check{H}^1(X, PGL_n)$. Here PGL_n is defined as the sheaf $PGL_n(U) = \mathrm{Aut}(M_n(\mathcal{O}_U))$. A short discussion on this non standard definition of PGL_n, the definition of twisted forms, and the necessary theory for sheaf of groups can be found in Appendix A. We work with the étale topology and write $c(\mathscr{A})$ for the class in $\check{H}^1_{et}(X, PGL_n)$.

Proposition 1.35. *Let X be a scheme. The set of isomorphism classes of Azumaya algebras of rank n^2 over X is isomorphic to $\check{H}^1_{et}(X, PGL_n)$.*

Proof. We already discussed that every Azumaya algebra of rank n^2 determines the class of a 1-cocycle in $\check{H}^1_{et}(X, PGL_n)$. We need to show that every 1-cocycle does come from an Azumaya algebra. A section of PGL_n is an automorphism of M_n as an $\mathcal{O}_X$-algebra. It determines an endomorphism of M_n as an $\mathcal{O}_X$-module, which is a section of GL_{n^2}. So every PGL_n 1-cocycle gives rise to a GL_{n^2} 1-cocycle. Furthermore, we know that every GL_{n^2} 1-cocycle determines a locally free $\mathcal{O}_X$-module of rank n^2 (Appendix A). Since the 1-cocycle that determines the locally free $\mathcal{O}_X$-module comes from a PGL_n 1-cocycle, this locally free $\mathcal{O}_X$-module has the structure of an Azumaya algebra. □

Using Lemma 1.34 we see that if X has at most finitely many connected components, then every Brauer class is representable by an Azumaya algebra of constant rank. This shows that every Brauer class is connected to $\check{H}^1_{et}(X, PGL_n)$ for some n. Of course there are always Azumaya algebras that have different rank in the same Brauer class. We can construct simple examples by tensoring an Azumaya algebra with a trivial one of rank ≥ 2, this increases the rank but does not change the Brauer class.

The proposition connects Azumaya algebras to Brauer-Severi schemes. A scheme P over X is a *Brauer-Severi scheme* of relative dimension $n-1$ if there exist an étale covering $(U_i \to X)$, such that $P \otimes \mathcal{O}_{U_i} \simeq \mathbb{P}^{n-1}_{U_i}$. Such a scheme is by definition a twisted form of $\mathbb{P}^{n-1}_X$ for the étale topology. It is know that

$$\mathcal{A}ut(\mathbb{P}^{n-1}_X) = PGL_n = \mathcal{A}ut(M_n(\mathcal{O}_X))$$

(see [57] Chapter 0 §5.b), so we get an injection from the set of isomorphism classes of Brauer-Severi schemes of relative dimension $n-1$ into $\check{H}^1(X_{et}, PGL_n)$. According to [32] Chapter VIII Proposition 7.8 every 1-cocycle in turn determines a Brauer-Severi scheme, so this is an isomorphism. The argument uses decent theory to show that the gluing data in a 1-cocycle determines a Brauer-Severi scheme. This gives a connection between isomorphism classes of Azumaya algebras and Brauer-Severi scheme. Even more, the data in a 1-cocycle determines an Azumaya algebra as well as a Brauer-Severi scheme, and vice versa both objects give a 1-cocycle. So there exist a bijection between Azumaya algebras of rank n^2, or better here, of degree n, and Brauer-Severi schemes of relative dimension $n-1$ via this 1-cocycle.

Brauer-Severi schemes over k are called *Brauer-Severi varieties*. For example, set $k = \mathbb{R}$. Then a quadric curve $X \subset \mathbb{P}^2_{\mathbb{R}}$ given by a homogeneous polynomial

$$X_0^2 + X_1^2 + X_2^2 = 0$$

has no rational point over $\mathbb{R}$, and thus cannot be isomorphic to $\mathbb{P}^1_{\mathbb{R}}$. But over $\mathbb{C}$ this is not the case anymore, and we have $X \otimes \mathbb{C} \simeq \mathbb{P}^1_{\mathbb{C}}$. In fact this is a necessary and sufficient condition; a Brauer-Severi variety over k is trivial if and only if it has a k-rational point (see [12] Proposition 7.1.5).

Lemma 1.36. *Let X be a scheme. The sequence*

$$1 \longrightarrow \mathbb{G}_m \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1$$

is exact with respect to the flat, big and small étale and Zariski topologies on X.

Proof. Let $U \to X$ be any object in the site. We need to check that $\mathbb{G}_m(U)$ is in the kernel of $GL_n(U) \to PGL_n(U)$ and that every section $s \in PGL_n(U)$

is locally liftable to a section of $GL_n(U)$. This is a direct consequence of the Skolem-Noether theorem, namely Corollary 1.22. For every local ring R the sequence

$$1 \longrightarrow R^* \longrightarrow GL_n(R) \longrightarrow PGL_n(R) \longrightarrow 1$$

is exact as a sequence of R-modules, and this shows the sought-after properties. $\square$

This exact sequence induces a long exact sequence of pointed sets. Since the sheaf $\mathbb{G}_m$ is abelian and is mapped to the center of GL_n we have the boundary map δ constructed by Giraud to the second cohomology group of $\mathbb{G}_m$ (see A.1). We obtain a long exact sequence where we are interested in the terms

$$\check{H}^1_{et}(X, GL_n) \xrightarrow{\iota} \check{H}^1_{et}(X, PGL_n) \xrightarrow{d} H^2_{et}(X, \mathbb{G}_m).$$

We want to determine the kernel of d. Therefore we determine the image of ι. From Appendix A we know that $\check{H}^1_{et}(X, GL_n)$ is isomorphic to the set of isomorphism classes of locally free sheaves $\mathscr{E}$ of rank n over X.

Lemma 1.37. *The map ι sends the class of a locally free sheaf $\mathscr{E}$ of rank n to the class of the Azumaya algebra $\mathcal{E}nd(\mathscr{E})$.*

Proof. We have to show that $\iota c(\mathscr{E}) = c(\mathcal{E}nd(\mathscr{E}))$. Choose a Zariski covering $\mathscr{U} = (U_i)$ of X that trivializes $\mathscr{E}$. So there exist isomorphisms $\phi_i : \mathcal{O}^n_{U_i} \to \mathscr{E}_{|U_i}$ and the locally free sheaf $\mathscr{E}$ is mapped to the 1-cocycle $(\phi_i^{-1}\phi_j)$. Set $\mathscr{A} = \mathcal{E}nd(\mathscr{E})$. By construction there exist isomorphisms $\psi_i : M_n(\mathcal{O}_{U_i}) \to \mathscr{A}_{|U_i}$, which have the property that $\psi_i(x) = \phi_i a \phi_i^{-1}$, for all $x \in M_n(\mathcal{O}_{U_i})$. Thus $\mathscr{A}$ corresponds to the 1-cocycle $\alpha_{ij} = (\psi_i^{-1}\psi_j)$ in PGL_n, with $\alpha_{ij}(x) = \phi_i^{-1}\phi_j x \phi_j^{-1}\phi_i$ for $x \in M_n(\mathcal{O}_{U_{ij}})$. Then (α_{ij}) is the image of $(\phi_i^{-1}\phi_j)$ under ι, since $GL_n \to PGL_n$ is determined by $u \mapsto (x \mapsto uau^{-1})$. $\square$

To determine the Brauer map we need one last fact, namely that boundary maps d are compatible for varying n. That means if $\mathscr{A}$ is an Azumaya algebra of rank n^2 and $\mathscr{B}$ is an Azumaya algebra of rank m^2, then $dc(\mathscr{A} \otimes \mathscr{B}) = dc(\mathscr{A})dc(\mathscr{B})$. The verification of this fact is purely technical. For details see [25] Chapter IV §3.

We finally have every ingredient to define the Brauer map

Theorem 1.38. *Let X be a scheme with at most finitely many connected components. There exist a canonical injective homomorphism $\delta : \mathrm{Br}(X) \to H^2_{et}(X, \mathbb{G}_m)$. We call this morphism the Brauer map.*

Proof. By Lemma 1.34 every Brauer class is represented by an Azumaya algebra $\mathscr{A}$ of constant rank n^2. This determines a class $[\mathscr{A}] \in \mathrm{Br}(X)$ and $c(\mathscr{A}) \in \check{H}^1_{et}(X, PGL_n)$. We define the Brauer map by

$$\delta : \mathrm{Br}(X) \longrightarrow H^2_{et}(X, \mathbb{G}_m), \quad [\mathscr{A}] \longmapsto dc(\mathscr{A}),$$

for the appropriate boundary map d.

We check that this map is well defined. To do this, we use the fact that by Lemma 1.37 $dc(\mathscr{A}) = 1$ maps to the neutral element in the group if and only if $\mathscr{A} = \mathcal{E}nd(E)$ for some locally free sheaf $\mathscr{E}$. If $\mathscr{B}$ is an Azumaya algebra with constant rank m^2 that is another representative of the Brauer class $[\mathscr{A}]$, there exist locally free sheaves $\mathscr{E}$ and $\mathscr{F}$ with $\mathscr{A} \otimes \mathcal{E}nd(\mathscr{E}) \simeq \mathscr{B} \otimes \mathcal{E}nd(\mathscr{F})$. Both sides are Azumaya algebras and they are isomorphic so

$$dc(\mathscr{A} \otimes \mathcal{E}nd(\mathscr{E})) = dc(\mathscr{B} \otimes \mathcal{E}nd(\mathscr{F})).$$

Then

$$\begin{aligned} dc(\mathscr{A} \otimes \mathcal{E}nd(\mathscr{E})) &= dc(\mathscr{A})dc(\mathcal{E}nd(\mathscr{E})) = dc(\mathscr{A}), \\ dc(\mathscr{B} \otimes \mathcal{E}nd(\mathscr{F})) &= dc(\mathscr{B})dc(\mathcal{E}nd(\mathscr{F})) = dc(\mathscr{B}), \end{aligned}$$

which shows $dc(\mathscr{A}) = dc(\mathscr{B})$. The map respects the group structure of $\mathrm{Br}(X)$ since d fulfills the compatibility condition $dc(\mathscr{A} \otimes \mathscr{B}) = dc(\mathscr{A})dc(\mathscr{B})$. The map is injective since $dc(\mathscr{A}) = 1$ if and only of $\mathscr{A}$ is a trivial Azumaya algebra (see also [55] Chapter IV Theorem 2.5). □

Though we are only concerned with schemes that have at most finitely many connected components it is nevertheless possible to define a Brauer map, even if X does have infinitely many connected components (see [55] Chapter IV Theorem 2.5 or [12] Theorem 4.2.1). The proof relies on Giraud's description of the second cohomology group via gerbes. Basically one maps an Azumaya algebra to its $\mathbb{G}_m$-gerbe of trivializations (see [25] Chapter V §4).

Using the Kummer exact sequence $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ and a diagram chase, one can show that $d : \check{H}^1_{et}(X, PGL_n) \to H^2_{et}(X, \mathbb{G}_m)$ factors through $H^2(X, \mu_n)$ which implies that the image of d is torsion and that any Azumaya algebra of rank n^2 is annihilated by n. Furthermore, if X has at most finitely many connected components this implies that $\mathrm{Br}(X)$ is torsion ([55] IV Proposition 2.7.). The fact that an Azumaya algebra of constant rank n^2 is annihilated by n is also directly proven by Saltman [62] in the affine case. His argument can be extended to arbitrary schemes ([5] Tag 0A2L).

Let X be a scheme with at most finitely many connected components. So the image of δ always lies in a torsion subgroup. We call the whole torsion subgroup $\mathrm{Br}'(X) = H^2_{et}(X, \mathbb{G}_m)_{tors}$ the *cohomological Brauer group*. The Brauer map is then an injective map

$$\delta : \mathrm{Br}(X) \longrightarrow \mathrm{Br}'(X).$$

There exist schemes where $H^2_{et}(X, \mathbb{G}_m)$ is not a torsion group, so the choice of $\mathrm{Br}'(X)$ as the torsion subgroup is important. A first example of a normal surface with this property was given by Grothendieck using an example of Mumford ([30] Remarques 11.1.b).

Philosophically the Brauer group is in line with the lower cohomological groups of $\mathbb{G}_m$. We have

$$H^0_{et}(X, \mathbb{G}_m) = \mathcal{O}_X^\times, \quad H^1_{et}(X, \mathbb{G}_m) = \mathrm{Pic}(X), \quad H^2_{et}(X, \mathbb{G}_m)_{tors} = \mathrm{Br}(X).$$

For a morphism of schemes $f : X \to Y$ we have pullback morphism $f^* : H^2_{et}(Y, \mathbb{G}_m) \to H^2_{et}(X, \mathbb{G}_m)$. The pullback of a torsion element is a torsion element, so we get pullback maps

$$f^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X).$$

We have already seen that we can pullback pullback Brauer classes. This gives a diagram

$$\begin{tikzcd} \mathrm{Br}(X) \arrow[r, "\delta"] & \mathrm{Br}'(X) \\ \mathrm{Br}(Y) \arrow[r, "\delta"'] \arrow[u, "f^*"] & \mathrm{Br}'(Y) \arrow[u, "f^*"'] \end{tikzcd}$$

Lemma 1.39. *The Brauer map is functorial, i.e. the diagram above is commutative.*

Proof. Since the boundary map d is functorial it is enough to show that the pullback of an Azumaya algebra $\mathscr{B}$ on Y of rank n^2 has the same class in $\check{H}_{et},^1(XPGL_n)$ as the pullback of the class of $\mathscr{B}$ in $\check{H}^1_{et}(Y, PGL_n)$. We have to show that $c(f^*\mathscr{B}) = f^*(c(\mathscr{B}))$. Choose some trivialization cover $\mathscr{V} = (V_i \to Y)$ of $\mathscr{B}$ with isomorphisms $\phi_i : \mathcal{O}_{Y|U_i} \to \mathscr{B}_{|U_i}$. This defines a 1-cocycle (β_{ij}) with $\beta_{ij} = (\phi_i \mid V_{ij})^{-1}(\phi_j \mid V_{ij})$ that represents the class $c(\mathscr{B})$. The lift of the cover $f^*\mathscr{U} = (V_i \times X \to X)$ trivializes $f^*\mathscr{B}$. Set $U_i = V_i \times X$. The lifts of the ϕ_i define isomorphism $\phi_i' : \mathcal{O}_{X|U_i} \to f^*\mathscr{B}_{|U_i}$. They determine a 1-cocycle (α_{ij}) with $\alpha_{ij} = (\phi_i' \mid U_{ij})^{-1}(\phi_j' \mid U_{ij})$, which represents $c(f^*\mathscr{B})$. This cocycle is a lift of the cocylce (β_{ij}), so we are done. $\square$

We call an Azumaya algebra $\mathscr{A}$ whose class $[\mathscr{A}] \in \mathrm{Br}(X)$ is mapped to a cohomological Brauer class $\alpha \in \mathrm{Br}(X)$ a *representative* of α, and we say that $\mathscr{A}$ represents α.

The Brauer map connects Brauer groups to cohomology theory which is useful in many ways. This makes it possible to calculate the Brauer group using technique from cohomology theory. For example:

Corollary 1.40. *Let R be a Henselian local ring with maximal ideal R. Then* $\mathrm{Br}(R) = \mathrm{Br}(R/\mathfrak{m})$.

Proof. For the étale cohomology groups of the sheaf of groups $\mathbb{G}_m$ over a Henselian local ring it holds that

$$H^i_{et}(\mathrm{Spec}(R), \mathbb{G}_m) \simeq H^i_{et}(\mathrm{Spec}(R/\mathfrak{m}), \mathbb{G}_m),$$

for all $i \geq 1$ ([55] Chapter III Remark 3.11.a)). So this particularly holds true for $i = 2$ and then $\mathrm{Br}'(R) \simeq \mathrm{Br}'(R/\mathfrak{m})$. And for affine schemes the Brauer map is an isomorphism. □

In fact we have not stated yet that the Brauer map is an isomorphism for affine schemes. The question whether the Brauer map is an isomorphism, i.e. whether over a given scheme every cohomological Brauer class is represented by an Azumaya algebra was asked by Grothendieck in [30].

For smooth algebraic surfaces a positive answer was essentially know to Auslander and Goldman [4], even before Grothendieck's invention of the cohomological Brauer group for schemes. Grothendieck himself gave first positive answers for schemes of dimension ≤ 1 and for regular surfaces. He also showed that the Brauer map is surjective for a local Henselian ring [30]; this would be enough for the proof of Corollary 1.40. Another early positive result is due to Hoobler for Abelian Varieties [39].

Gabber then proved in [23], that $\mathrm{Br}(X) = \mathrm{Br}'(X)$, in the case where X is either an affine scheme or the union of an affine scheme, whose intersection is again an affine scheme. Other proofs of this result can be found in [40], [46] and [49]. Gabber later improved his result to the case where there exists an ample invertible sheaf on X. In particular this means that X is quasicompact and separated. For example a quasiprojective scheme over a base scheme has an ample invertible sheaf. Gabber's proof is unpublished but there exist a different proof by DeJong [13].

DeMeyer and Ford proved a positive result for a toric variety over an algebraically closed field of characteristic 0 [17]. Schröer [63] showed that the Brauer map is an isomorphism, when X is a separated geometrically normal algebraic surface. A recent preprint of Mathur shows $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for a separated surface [53].

However the Brauer map is not always an isomorphism. There is a counterexample by Edidin, Hassett, Kresch, Vistoli ([18] Corollary 3.11). They construct a scheme X, as the union of two copies of $\mathrm{Spec}(\mathbb{C}[x,y,z]/(xy-z^2))$, glued along the non-singular locus, and show $\mathrm{Br}(X) \neq \mathrm{Br}'(X)$. Here X is a non-separated normal surface. A further discussion of this counterexample can also be found in paper by Bertuccioni [8], where the different Brauer groups of X are explicitly calculated as $\mathrm{Br}(X) = 0$ and $\mathrm{Br}'(X) = \mathbb{Z}/2\mathbb{Z}$. In general Grothendieck's question remains open to this day. In particular not much is know about higher dimensional schemes that do not admit an ample invertible sheaf.

Our definition of the cohomological Brauer group of course also works for fields and there we have $\mathrm{Br}(k) = \mathrm{Br}'(k)$. The usual way to define this group for fields uses Galois cohomology. For details on the construction see [24]. We have seen that every central simple algebras is trivialized by a finite Galois extension and that the Brauer group of a separably closed field is trivial. For a Galois extension K/k with Galois group $\mathrm{Gal}(K/k)$ the isomorphism classes of central simple algebras of rank n, which are split by K, are the same as 1-cocyles in $H^1(\mathrm{Gal}(K/k), \mathrm{PGL}_n(k))$. One uses the exact sequence

$$1 \longrightarrow k^\times \longrightarrow \mathrm{GL}_n(k) \longrightarrow \mathrm{PGL}_n(k) \longrightarrow 1$$

to obtain a map $H^1(\mathrm{Gal}(K/k), \mathrm{PGL}_n(K)) \to H^2(\mathrm{Gal}(K/k), K^\times)$. Hilbert's Theorem 90 ([24] Example 2.3.4) shows that the map is injective. Taking limits over all n gives a map $\mathrm{Br}(K/k) \to H^2(\mathrm{Gal}(K/k), K^\times)$, which can be shown to be an isomorphism. Here $\mathrm{Br}(K/k)$ denotes the equivalence classes of all central simple algebras split by K. Since all central simple algebras are split by a separable closure one can fix a separable closure k_{sep} and take the limit over all finite Galois extension contained in it. For the Galois group this gives the absolute Galois group $\mathrm{Gal}(k_{sep}/k)$ and an isomorphism $\mathrm{Br}(k) \to H^2(\mathrm{Gal}(k_{sep}/k), k_{sep}^\times)$. For the étale topology we have an isomorphism

$$H^2_{et}(\mathrm{Spec}(k), \mathbb{G}_m) \simeq H^2(\mathrm{Gal}(k_{sep}/k), k_{sep}^\times)$$

(see [12] 2.2.3), so everything works out fine.

Chapter 2

Pinching of sheaves

In this chapter we discuss results by Ferrand [21] on pinching of schemes and sheaves. For the convenience of the reader we give some more details and examples. In particular, we explicitly describe the fixed points of the adjunction used in the main theorem, to motivate why flat modules are a natural choice of subcategory to obtain an adjoint equivalence of categories. Then we describe how to generalizes this results to affine morphism of schemes. Additionally, we introduce what we understand under a finite modification.

2.1 Category theory

Before we start on Ferrand's results, we recall some general definitions of category theory, which can be found in most textbooks on the subject (see for example [51] or [60]). Note that a category has a set of objects $\mathrm{ob}(\mathcal{C})$. So we tacitly assume to be working in a Grothendieck universe (see for example [64] Section 1).

We keep in mind that the following categories are abelian: the category of (finite) modules over a ring, and (quasicoherent) sheaves over a scheme. On the other hand, the categories of locally free sheaves over a given scheme is not abelian, though it is an additive category.

Let $\mathcal{C}$ be a category and

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

a commutative square. Note that if the category is abelian, every square, with notation as above, defines a sequence

$$0 \longrightarrow P \xrightarrow{(p_1,p_2)} X \oplus Y \xrightarrow{(f,-g)} Z \longrightarrow 0.$$

The square is commutative if and only if the sequence is a complex.

A commutative square is called *cartesian* if for any triple (T, q_1, q_2), with $q_1 : T \to X$, $q_2 : T \to Y$ and $fq_1 = gq_2$, there exist a unique morphism $t : T \to P$ so that $p_1 t = q_1$ and $p_2 t = q_2$. The situation is illustrated in the following diagram:

$$\begin{tikzcd}
T \arrow[dr, dotted, "t"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"'] \\
& X \arrow[r, "f"] & Z.
\end{tikzcd}$$

If a square is cartesian we write $P = X \times_Z Y$. This is also called a *pullback* or *fiber product*. Given two morphism $f : X \to Z$ and $g : Y \to Z$ we ask, whether there exists a fiber product $X \times_Z Y$ such that we obtain a cartesian square. If a fiber product exists, it is uniquely defined up to unique isomorphism, so we can talk about the fiber product.

In an abelian category the fiber product always exists. It is the kernel of the morphism

$$(f, -g) : X \oplus Y \longrightarrow Z.$$

A given commutative square is cartesian, if and only if the complex

$$0 \longrightarrow P \xrightarrow{(p_1,p_2)} X \oplus Y \xrightarrow{(f,-g)} Z \longrightarrow 0$$

is exact at P and $X \oplus Y$.

For example, in the abelian category of R-modules the fiber products is defined as follows: Given homomorphism of R-modules $f : X \to Z$ and $g : Y \to Z$ the fiber product is defined by $X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$.

One notes that the fiber product in the category of rings is also the fiber product of sets, when we forget the ring structure. In many categories, whose objects are sets endowed with some algebraic structure, the fiber product in the category agrees with the fiber product of sets, when equipped with a suitable algebraic structure.

A commutative square is called *cocartesian* if for any triple (U, j_1, j_2) with $j_1 : X \to U$, $j_2 : Y \to U$ and $j_1 p_1 = j_2 p_2$, there exist a unique map $u : Z \to U$

so that $uf = j_1$ and $ug = j_2$. The situation is illustrated in the following diagram:

$$\begin{tikzcd}
P \arrow[r, "p_2"'] \arrow[d, "p_1"] & Y \arrow[d, "g"'] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "f"] \arrow[drr, bend right, "j_1"'] & Z \arrow[dr, dotted, "u"] & \\
& & U.
\end{tikzcd}$$

If a square is cocartesian we write $Z = X \sqcup_P Y$. This is also called a *pushout* or *fibered coproduct*. Given two morphism $p_1 : P \to X$ and $p_2 : P \to Y$ we ask, whether there exist a pushout $X \sqcup_P Y$, such that we obtain a cocartesian square. If a pushout exist, it is uniquely defined up to unique isomorphism, so we can talk about the pushout.

In an abelian category the pushout always exists. The pushout is the cokernel of of the morphism

$$(p_1, p_2) : P \longrightarrow X \oplus Y.$$

A given commutative square is cocartesian, if and only if the complex

$$0 \longrightarrow P \xrightarrow{(p_1,p_2)} X \oplus Y \xrightarrow{(f,-g)} Z \longrightarrow 0$$

is exact at $X \oplus Y$ and Z.

For example in the category of commutative rings the pushout is given by the tensor product of rings.

For another example, let P, X and Y be topological spaces and p_1, p_2 continuous maps. Then the pushout is defined by an equivalence relation. It is $Z = X \sqcup Y/\sim$, where the relation identifies a point in $x \in X$ with a point in $y \in Y$ if there exists a point in $p \in P$, with $p_1(p) = x$ and $p_2(p) = y$. A subset V of Z is open if and only if $f^{-1}(V)$ is open in X and $g^{-1}(V)$ is open in Y.

Note that the pushouts of rings does not agree with the pushout of sets, contrary to the case for fiber products. For example as a tensor product of rings we have $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, while the pushout in the category of sets is non zero.

We will mostly be interested in commutative squares that are cartesian as well as cocartesian. For an abelian category this is the case if and only if the complex

$$0 \longrightarrow P \xrightarrow{(p_1,p_2)} X \oplus Y \xrightarrow{(f,-g)} Z \longrightarrow 0$$

is exact.

In the following it will be assumed that functors are covariant, mostly for ease of notation in some commutative diagrams. The definitions do hold for contravariant functors, just with inverted arrows at some points. One can also use opposite categories, to make the definitions work for contravariant functors.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and F and G functors $\mathcal{C} \to \mathcal{D}$. A *natural transformation* $\eta : F \to G$ is a family $\eta = (\eta_X)_{X \in \mathcal{C}}$ of morphisms

$$\eta_X : F(X) \to G(X)$$

in $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$ it holds that

$$\eta_Y \circ F(f) = G(f) \circ \eta_X.$$

This property is visualized by the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ Y & F(Y) \arrow[r, "\eta_Y"'] & G(Y). \end{tikzcd}$$

If for every object X in $\mathcal{C}$ the morphism η_X is an isomorphism in $\mathcal{D}$, then η is called a *natural isomorphism* or *isomorphism of functors*.

Next we define fiber products of categories themselves (see for example [7] Chapter VII §3). Let $\mathcal{A}', \mathcal{B}, \mathcal{B}'$ be categories with functors $G : \mathcal{B} \to \mathcal{B}'$ and $P' : \mathcal{A}' \to \mathcal{B}'$. We define the *fiber product category*

$$\mathcal{A} = \mathcal{B} \times_{\mathcal{B}'} \mathcal{A}'$$

as follows:

An objects in $\mathcal{A}$ is a triple (N, s, M'), where N is an object of $\mathcal{B}$, M' is an object of $\mathcal{A}'$ and

$$s : G(N) \xrightarrow{\sim} P'(M')$$

is an isomorphism in $\mathcal{B}'$.

A morphism $(N_1, s_1, M_1') \to (N_2, s_2, M_2')$ in $\mathcal{A}$ is a pair (b, a'), where $b : N_1 \to N_2$ is a morphism in $\mathcal{B}$ and $a' : M_1' \to M_2'$ is a morphism in $\mathcal{A}'$, such that the following diagram commutes:

$$\begin{tikzcd} G(N_1) \arrow[r, "s_1"] \arrow[d, "G(b)"'] & P'(M_1') \arrow[d, "P'(a')"] \\ G(N_2) \arrow[r, "s_2"'] & P'(M_2'). \end{tikzcd}$$

There are canonical forgetful functors $F : \mathcal{A} \to \mathcal{A}'$ given by

$$\begin{aligned}(N, s, M') &\longmapsto M',\\ (b, a') &\longmapsto a',\end{aligned}$$

and $P : \mathcal{A} \to \mathcal{B}$ given by

$$\begin{aligned}(N, s, M') &\longmapsto N,\\ (b, a') &\longmapsto b.\end{aligned}$$

The square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "P"'] & \mathcal{A}' \arrow[d, "P'"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{B}'. \end{tikzcd}$$

is commutative up to the isomorphism of functors (or natural isomorphism)

$$\sigma : GP \longrightarrow P'F$$

which maps $GP(N, s, M') = G(N)$ to $P'(M') = P'F(N, s, M')$ via s.

This construction has the following universal property, which is similar to the universal property of pullbacks. Given a square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "Q"'] & \mathcal{A}' \arrow[d, "P'"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{B}'. \end{tikzcd}$$

and a isomorphism of functors $\tau : GQ \to P'H$, there exist a unique functor $T : \mathcal{C} \to \mathcal{A}$ such that $H = FT$, $Q = PT$ and $\tau = \sigma * T$. Concretely, for an object C of $\mathcal{C}$ we have $T(C) = (Q(C), \tau_C, H(C))$ and for a morphism f in $\mathcal{C}$ we have $T(f) = (Q(f), H(f))$

Note that the fiber product category can be equally defined for contravariant functors.

If the given categories and functors are additive so is the fiber product category. However if we additionally demand, that the categories are abelian, this does not necessarily mean that the fiber product category is abelian. For a morphism (b, a') we would like to define the kernel as $(\ker(b), s, \ker(a'))$; unfortunately there is no reason that such an s exist. It is possible that $G(\ker(b))$ and $P'(\ker(a'))$ are not isomorphic. Same is true for the cokernel. If we additionally demand that our functors are exact, an argument using the five lemma can show that now $G(\ker(b)) \simeq P'(\ker(a'))$, and the same is true for the cokernels. It is then straightforward to check that the fiber product category is abelian. However we will not be working with exact functors.

Our main interest in the following sections will be questions of the following kind: Suppose we have morphisms of rings $g : B \to B'$ and $p' : A' \to B'$, then the universal property of the fiber product category defines a unique morphism

$$\mathrm{Mod}(B \times_{B'} A) \to \mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A').$$

We wish to understand this morphism.

To do this we construct a functor in the other direction, which will define an adjunction. For more details on adjunctions than given here see [51] Chapter IV, especially Section 1, Theorem 1 and Theorem 2.

Definition 2.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. An *adjunction* from $\mathscr{C}$ to $\mathscr{D}$ is a triple (S, T, φ) consisting of two functors $T : \mathscr{C} \to \mathscr{D}$ and $S : \mathscr{D} \to \mathscr{C}$, while φ is a function, which assigns to each pair of objects $c \in \mathscr{C}$ and $d \in \mathscr{D}$ a bijection of sets

$$\varphi_{c,d} : \mathrm{Mor}_{\mathscr{D}}(Tc, d) \xrightarrow{\simeq} \mathrm{Mor}_{\mathscr{C}}(c, Sd), \tag{2.1.1}$$

which is natural in c and d. The functor T is called a *left adjoint* to S, while S is the *right adjoint* to T.

The condition of φ being natural in c and d means, that for every morphism $v : Tc \to d$ holds

$$\varphi_{c',d}(v \circ T(g)) = \varphi_{c,d}(v) \circ g \quad \text{and} \quad \varphi_{c,d'}(h \circ v) = S(h) \circ \varphi_{c,d}(v)$$

for all morphisms $g : c' \to c$ and $h : d \to d'$. Illustrated this means that the following diagrams do commute:

$$\begin{tikzcd} \mathrm{Mor}_{\mathscr{D}}(Tc, d) \arrow[r, "\varphi_{c,d}"] \arrow[d, "T(g)"'] & \mathrm{Mor}_{\mathscr{C}}(c, Sd) \arrow[d, "g"] \\ \mathrm{Mor}_{\mathscr{D}}(Tc', d) \arrow[r, "\varphi_{c',d}"'] & \mathrm{Mor}_{\mathscr{C}}(c', Sd) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Mor}_{\mathscr{D}}(Tc, d) \arrow[r, "\varphi_{c,d}"] \arrow[d, "h"'] & \mathrm{Mor}_{\mathscr{C}}(c, Sd) \arrow[d, "S(h)"] \\ \mathrm{Mor}_{\mathscr{D}}(Tc, d') \arrow[r, "\varphi_{c,d'}"'] & \mathrm{Mor}_{\mathscr{C}}(c, Sd'). \end{tikzcd}$$

Let us begin with an example. Let $f : A \to B$ be a morphism of rings, M be an A-module, and N a B-module. Then the extension of scalars functor

$$f^* : \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B), \quad M \longmapsto B \otimes_A M$$

is left adjoint to the restriction of scalars functor

$$f_* : \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(A), \quad N \longmapsto N_A.$$

The latter restricts N to an A-module N_A by defining multiplication $a \cdot n = f(a)n$. We will often say, that we treat N as an A-module, instead of writing N_A.

We define

$$\varphi_{M,N} : \mathrm{Hom}_{B\text{-mod}}(B \otimes_A M, N) \longrightarrow \mathrm{Hom}_{A\text{-mod}}(M, N_A),$$

$$v \longmapsto v \circ (f \otimes \mathrm{id}_M),$$

where $f \otimes \mathrm{id}_M : M \simeq A \otimes_A M \to B \otimes_A M$, so $\varphi_{M,N}(v)(m) = v(1 \otimes m)$. To see that it is an isomorphism we give an inverse. For $w : M \to N_A$ we set

$$\varphi^{-1}_{M,N}(w) : B \otimes_A M \xrightarrow{\mathrm{id}_B \otimes w} B \otimes_A N_A \longrightarrow N,$$

where the second homomorphism is defined by $b \otimes n \mapsto bn$. So we have $\varphi^{-1}_{M,N}(w)(b \otimes m) = bw(m)$.

To be thorough, we check that φ is natural in M and N. So let $v : B \otimes_A M \to N$, $g : M' \to M$ and $h : N \to N'$. We have

$$\varphi_{M,N'}(h \circ v) = f_* h \circ \varphi_{M,N}(v).$$

The equality of

$$\varphi_{M',N}(v \circ f^* g) = f_*(v \circ f^* g \circ (f \otimes \mathrm{id}_{M'}))$$

and

$$\varphi(v)_{M,N} \circ g = f_*(v \circ (f \otimes \mathrm{id}_M)) \circ g = f_*(v \circ (f \otimes \mathrm{id}_M) \circ g),$$

follows, since the square

$$\begin{tikzcd} M' \arrow[r, "g"] \arrow[d, "f\otimes \mathrm{id}'_M"'] & M \arrow[d, "f\otimes \mathrm{id}_M"] \\ B \otimes_A M' \arrow[r, "f^* g"'] & B \otimes_A M \end{tikzcd}$$

is commutative.

This adjunction generalizes to schemes. Let $f : X \to Y$ be a morphism of schemes. Then the pullback functor f^* is left adjoint to the pushforward functor f_*. For every $\mathcal{O}_X$-module $\mathscr{F}$ and every $\mathcal{O}_Y$-module $\mathscr{G}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(f^* \mathscr{G}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{G}, f_* \mathscr{F})$$

(see for example [26] Proposition 7.11).

When working with adjoint functors other points of view are useful. If we set $d = Tc$ in (2.1.1) the identity $\mathrm{id}_{Tc} \in \mathrm{Mor}_{\mathscr{D}}(Tc, Tc)$ is sent to a morphism $\eta_c \in \mathrm{Mor}_{\mathscr{C}}(c, STc)$. We get such a morphism for every object of $\mathscr{C}$ and the function $c \mapsto \eta_c$ defines a natural transformation $\eta : \mathrm{id}_{\mathscr{C}} \to ST$, because every diagram

$$\begin{tikzcd} c' \arrow[r, "\eta_{c'}"] \arrow[d, "g"'] & STc' \arrow[d, "ST(g)"] \\ c \arrow[r, "\eta_c"'] & STc \end{tikzcd}$$

is commutative. This follows from the fact that φ is natural.

More so each η_c is *universal* from c to S. This means that for each morphism $w : c \to Sd$ there exist a unique morphism $v : Tc \to d$ such that the following diagram commutes

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, "w"'] & STc \arrow[d, "S(v)"] \\ & Sd. \end{tikzcd}$$

To check that η_c actually is universal set $v = \varphi_{c,d}^{-1}(w)$. Then

$$w = \varphi_{c,d}(v \circ \mathrm{id}_{Tc}) = S(v) \circ \varphi_{c,Tc}(\mathrm{id}_{Tc}) = S(v) \circ \eta_c.$$

Given functors S, T and a universal natural transformation η as above we can recover the adjunction. For every $v : Tc \to d$ set

$$\varphi_{c,d}(v) = S(v) \circ \eta_c. \tag{2.1.2}$$

We have recovered φ, so this datum defines an adjunction.

Let $f : A \to B$ again be a morphism of rings. For the adjunction, which is given by extension and restriction of scalar functors, we get $\eta_M = f_*(f \otimes \mathrm{id}_M)$ and this gives the universal property for extension of scalars. For a homomorphism of A-modules $w : M \to N_A$ there is a unique homomorphism of B-modules $v : B \otimes_A M \to N$ such that $f_* v$ makes the following diagram commute

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "w"'] & (B \otimes_A M)_A \arrow[d, "f_* v"] \\ & N_A. \end{tikzcd}$$

A similar construction can be done in the other direction. This time we set $c = Sd$ in (2.1.1) and obtain a universal natural transformation

$$\epsilon_d = \varphi^{-1}(\mathrm{id}_{Sd}), \quad \epsilon_d : TSd \to d, \quad d \in \mathscr{D}.$$

Again for a morphism $v : Tc \to d$ we can recover $\varphi^{-1}(v) = \epsilon_d \circ T(v)$. We call η the *unit* and ϵ the *counit* of the adjunction.

Finally we again set $d = Tc$ in (2.1.1). Using the definition of φ via unit and counit we have

$$\mathrm{id}_{Tc} = \varphi^{-1}(\eta_c) = \epsilon_{Tc} \circ T(\eta_c).$$

This assures that the composition

$$T \xrightarrow{T\eta} TST \xrightarrow{\epsilon T} T \tag{2.1.3}$$

is the identity. If we instead set $c = Sd$ the same holds for

$$S \xrightarrow{\eta S} STS \xrightarrow{S\epsilon} S. \tag{2.1.4}$$

Given two natural transformations such that the identities above hold we can recover φ. In conclusion we have the following theorem:

Theorem 2.2. *([51] Chapter IV.1 Theorem 2) An adjunction $(S, T, \varphi) : \mathscr{C} \to \mathscr{D}$ is completely determined by any of the following data:*

(i) *Functors S, T and a natural transformation $\eta : id_{\mathscr{C}} \to ST$ such that for every $c \in \mathscr{C}$ the morphism $\eta_c : c \to STc$ is universal from c to S.*

(ii) *Functors S, T and a natural transformation $\epsilon : TS \to id_{\mathscr{D}}$ such that for every $d \in \mathscr{D}$ the morphism $\epsilon_d : TSd \to d$ is universal from T to d.*

(iii) *Functors S, T as well as natural transformations $\eta : \mathrm{id} \to ST$ and $\epsilon : TS \to \mathrm{id}$, such that both composites (2.1.3) and (2.1.4) are the identity transformations.*

For two contravariant functors S and T it is often useful to use opposite categories. If these two functors define an adjunction between $\mathscr{C}$ and $\mathscr{D}^{op}$ we get a *dual adjunction* between $\mathscr{C}$ and $\mathscr{D}$. The dualization causes the counit to be $\epsilon : id_{\mathscr{D}} \to TS$ and (2.1.3) reads as $T \xrightarrow{\epsilon T} TST \xrightarrow{T\eta} T$. Furthermore, $\varphi_{c,d} : \mathrm{Mor}_{\mathscr{D}}(d, Tc) \xrightarrow{\cong} \mathrm{Mor}_{\mathscr{C}}(c, Sd)$.

Let us make an example, by viewing a classical construction as an adjunction. Let R be a ring and

$$\mathrm{Spec}(R) = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ prime ideal }\}$$

the spectrum, viewed solely as a set without any additional structure. For every subset $M \subset R$ we define

$$V(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M \subset \mathfrak{p}\}.$$

This defines a inclusion reversing map from the subsets of R to the subsets of $\mathrm{Spec}(R)$. We define a map in the other direction. For a subset $X \subset \mathrm{Spec}(R)$ define

$$I(X) = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}.$$

One immediately sees that this map is also inclusion reversing. Furthermore, we have inclusions

$$M \subset IV(M) \text{ and } X \subset VI(X),$$

for any subsets $M \subset R$ and $X \subset \mathrm{Spec}(R)$ (see for example [9] Chapter II §4.3). For the sake of making an example we can view this construction as an adjunction. Treat the sets $\mathscr{C} = \{M \mid M \subset R\}$ and $\mathscr{D} = \{X \mid X \subset \mathrm{Spec}(R)\}$ as categories where the only morphisms are inclusions. The inclusion reversing maps I and V are treated as contravariant functors between these categories

$$\mathscr{C} \underset{I}{\overset{V}{\rightleftarrows}} \mathscr{D}.$$

We show that they define an adjunction. Unit and counit are given by the inclusions $X \subset VI(X)$ and $M \subset IV(M)$. We check property (i) of Theorem 2.2. Let $M \subset R$ and $X \subset \mathrm{Spec}(R)$ so that there exists an inclusion $w : M \subset I(X)$. We have to find a inclusion $v : X \to V(M)$ so that the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[dr] & IV(M) \arrow[d, "I(v)"] \\
& I(X).
\end{tikzcd}$$

But that is just the inclusion $X \subset VI(X) \subset V(M)$. We do not need to check uniqueness since our inclusions are unique by definition. There are simply no other morphisms.

Of course in Algebraic Geometry the set on both sides are then restricted in a certain way such that the maps I and V become inclusion reversing bijections; we will sketch this construction later. From a functorial point of view this means that the adjunction defined by V and I can be restricted to an adjoint equivalence of categories.

This is a general fact. One can always obtain an adjoint equivalence of categories from an adjunction.

Definition 2.3. Let $\mathscr{C}$, $\mathscr{D}$ be categories. An *adjoint equivalence* of categories between this categories is an adjunction (S, T, φ) between them, in which both the unit $\eta : \mathrm{id}_{\mathscr{C}} \to ST$ and the counit $\eta : \mathrm{id}_{\mathscr{D}} \to ST$ are natural isomorphism.

If (S, T, φ) is an adjoint equivalence of categories, then both S and T, respectively, are an equivalence of categories in the usual sense. In reverse, if some functor $T : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories then it is part of an adjoint equivalence of categories (S, T, φ) ([51] Chapter IV.4 Theorem 1).

We will mostly construct adjunctions via Theorem 2.2, by constructing two functors S and T and then give their unit and/or counit, without specifying φ. If both unit and counit are isomorphisms of functors we say that S and T are an adjoint equivalence of categories

$$\mathscr{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathscr{D}.$$

Note that unit and counit are natural transformations by definition, to check whether a given adjunction defines an adjoint equivalence, we need to check whether they become isomorphisms for every object in the category.

Definition 2.4. Let $\mathscr{C}$, $\mathscr{D}$ be categories and (S, T, φ) an adjunction between them. An object $c \in \mathscr{C}$ is a *fixed point* of the adjunction if its unit

$$\eta_c : c \xrightarrow{\simeq} STc$$

is an isomorphism. We write $\mathscr{C}_{fix} \hookrightarrow \mathscr{C}$ for the full subcategory of these objects. An object $d \in \mathscr{D}$ is a fixed point of the adjunction if its counit

$$\epsilon_d : TSd \xrightarrow{\simeq} d$$

is an isomorphism. We write $\mathscr{D}_{fix} \hookrightarrow \mathscr{D}$ for the full subcategory of these objects.

Proposition 2.5. *Let $\mathscr{C}$, $\mathscr{D}$ be categories and (S, T, φ) an adjunction from $\mathscr{C}$ to $\mathscr{D}$. Then for every $c \in \mathscr{C}_{fix}$ we have $Tc \in \mathscr{D}_{fix}$, and for every $d \in \mathscr{D}_{fix}$ it holds that $Sd \in \mathscr{C}_{fix}$. This gives an adjoint equivalence of categories*

$$\mathscr{C}_{fix} \underset{S}{\overset{T}{\rightleftarrows}} \mathscr{D}_{fix}.$$

Proof. Let $c \in \mathscr{C}_{fix}$. By definition $c \simeq STc$, which- implies $Tc \simeq TSTc$ and thus $Tc \in \mathscr{D}_{fix}$. The other direction is equivalent. Now the unit and counit, respectively, are isomorphisms for every object in the subcategories of fixed points, and hence the functors are an adjoint equivalence of categories. $\square$

Back to our example. First we equip $\mathrm{Spec}(R)$ with a topology. We define closed sets as $V(M)$ where M runs through all subsets of R. This topology is called the *Zariski topology.* To see that it is a topology one checks that the $V(M)$ fulfill the axioms for closed sets:

$$V(0) = \mathrm{Spec}(R), \quad V(1) = \emptyset \quad \text{and}$$

$$V(\bigcup M_i) = V(\sum M_i) = \bigcap V(M_i),$$

for every family $(M_i)_{i \in I}$ of subsets of R. Finite unions are still missing. One shows that for any pair of ideals $\mathfrak{a}, \mathfrak{a}'$ in R

$$V(\mathfrak{a} \cap \mathfrak{a}') = V(\mathfrak{a}\mathfrak{a}') = V(\mathfrak{a}) \cup V(\mathfrak{a}'),$$

and uses the fact that if $\mathfrak{a}$ is the ideal generated by a subset M then $V(\mathfrak{a}) = V(M)$ (see for example [9] Chapter II §4.3).

The last fact suggests that we should replace the set of all subsets of R with the set of ideals. This does not change the closed sets in the Zariski topology though we write them as $V(\mathfrak{a})$ now. We have now inclusion reversing maps

$$\{\mathfrak{a} \mid \mathfrak{a} \subset R \text{ ideal } \} \underset{I}{\overset{V}{\rightleftarrows}} \mathrm{Spec}(R).$$

From a functorial point of view, this is an adjunction and we want to determine the fixed points.

Denote the *radical* of $\mathfrak{a}$ by

$$\sqrt{\mathfrak{a}} = \{f \in A \mid \exists r \in \mathbb{N} : f^r \in \mathfrak{a}\}.$$

An ideal is called a *radical ideal* if $\mathfrak{a} = \sqrt{\mathfrak{a}}$. One sees that prime ideals are radical ideals. An important result from commutative algebra is that

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}$$

with $\mathfrak{p} \in \mathrm{Spec}(R)$ ([9] Chapter II §2.6 Corollaire 1). Then for any ideal $\mathfrak{a} \subset R$ we have

$$IV(\mathfrak{a}) = I(\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subset \mathfrak{p}\}) = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p} = \sqrt{\mathfrak{a}}.$$

The fixed points in $\mathcal{C} = \{\mathfrak{a} \mid \mathfrak{a} \subset R \text{ ideal } \}$ are exactly the radical ideals.

The fixed points in $\mathrm{Spec}(R)$ are the closed sets. Let $X \subset \mathrm{Spec}(R)$ be a subset. We determine the closure of X. The set X is contained in any closed set $X \subset V(\mathfrak{a})$ if and only of for all $\mathfrak{p} \in X$ holds $\mathfrak{p} \subset \mathfrak{a}$ i.e. $\mathfrak{a} \subset I(X)$. This implies that $\bar{X} = VI(X)$ is the smallest closed subset that contains I. So the closed subsets are the fixed points of the adjunction. Actually from a certain perspective the closed sets have been chosen exactly as the fixed points of the adjunction. We get mutually inverse bijections

$$\{\mathfrak{a} \mid \mathfrak{a} \subset R \text{ radical ideal } \} \underset{I}{\overset{V}{\rightleftarrows}} \{ \text{ closed subsets in } \operatorname{Spec}(R)\},$$

and this is an adjoint equivalence of categories between the subcategories of fixed points by Proposition 2.5 (for more details, see [9] Chapter II §4.3 or [26] Proposition 2.3).

Another example occurs in Galois theory. Let E/F be a Galois extension, i.e. it is algebraic, separable, and normal. Denote the Galois group by $G = \operatorname{Gal}(E/F)$. We have sets

$$\begin{aligned} \mathscr{C} &= \{L \mid F \subset L \subset E \text{ subextension } \} \\ \mathscr{D} &= \{H \mid H \subset G \text{ subgroup } \}. \end{aligned}$$

Both sets are ordered by inclusion and have G-action $\sigma \cdot L = \sigma(L)$ and $\sigma \cdot H = \sigma H \sigma^{-1}$, respectively. There are inclusion reversing maps

$$T : \mathscr{C} \longrightarrow \mathscr{D}, \quad L \longmapsto \operatorname{Gal}(E/L)$$

and

$$S : \mathscr{D} \longrightarrow \mathscr{C}, \quad H \longmapsto E^H$$

between these sets.

If the extension E/F is finite the main theorem of Galois theory tells us these maps are bijections who are inverse to each other. Furthermore, normal subgroups $N \subset G$ correspond exactly to the subextensions $L = E^N$ where L/K is a Galois extension (see for example [41] Theorem 8.5).

For an infinite Galois extension E/F the theorem fails. Though we retain the maps S and T, they fail to be a bijection. The first counter example was given by Dedekind in 1901 ([14] §6). A solution involving a topology on the Galois group was proposed by Krull in 1928 ([47]).

The main problem is that there are to many subgroups in the Galois group. We give an example. The idea for this construction can be found in [41] Chapter 8.6. Let p be a prime and $\mathbb{F}_p$ the finite field with p elements. Any finite Galois extension of $\mathbb{F}_p$ has the form $\mathbb{F}_{p^n}$. The algebraic closure $\overline{\mathbb{F}_p}$ is an infinite Galois extension of $\mathbb{F}_p$. Denote $G = \operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$. Clearly the Frobenius homomorphism $\pi : x \to x^p$ is an element in the Galois group with fixed field $\mathbb{F}_p$. The powers of π generate a subgroup $H = \langle \pi \rangle$ of G and

$S(H) = \overline{\mathbb{F}_p}^H = \mathbb{F}_p$. Indeed, for any finite Galois extension of $\mathbb{F}_p$ the Frobenius generates the Galois group. This is not the case here. We construct an element $s \in G$ that is not a power of π. Then $H' = \langle \pi, s \rangle$ is a subgroup of G with $S(H') = \mathbb{F}_p$ but is clearly larger than H. Take any infinite field extension with $\mathbb{F}_p \subsetneq L \subsetneq \overline{\mathbb{F}_p}$. For example let q be a prime and set $L = \bigcup \mathbb{F}_{p^m}$, where $m = q^r$, $r = 1, 2, \dots$. Since this set of subfields is totally ordered, their union L is a subfield and it is an infinite field extension of $\mathbb{F}_p$. Take an element $x \in \overline{\mathbb{F}_p} \backslash L$. Let f be the minimal polynomial of x over L. Since $x \notin L$ we have $\deg(f) > 1$ and f has a second root $b \neq a$ in $\overline{\mathbb{F}_p}$. Since $\overline{\mathbb{F}_p}$ is a splitting field of $\{x^{p^n} - x \mid n = 1, 2, \dots\}$ over the fields $L(a)$ and $L(b)$, Galois theory tells us that there exist a non trivial automorphisms $s \in \mathrm{Gal}(\overline{\mathbb{F}_p}/L)$ with $s(a) = b$ ([41] Theorem 8.2). Clearly $s \in G$, and the infinite field L is fixed by s. But the fixed points under π^k are the roots of $x^{p^k} - x$ and there are only finitely many of these. So $s \neq \pi^k$, and thus $s \notin \langle \pi \rangle$. If we want to build examples with another fixed field instead of $\mathbb{F}_p$, we can use the subgroup $\langle \pi^m \rangle$ with fixed field $\mathbb{F}_{p^m}$ instead; the construction works for any finite field $\mathbb{F}_{p^m}$ which is a subfields $\mathbb{F}_{p^m} \subset L$.

Again we can view $\mathscr{C}$ and $\mathscr{D}$ as categories whose only morphism are inclusions. The maps S and T can be seen as contravariant functors between these categories. One checks that

$$L = E^{\mathrm{Gal}(E/L)} = ST(L) \text{ for any } L \in \mathscr{C},$$

$$H \subset \mathrm{Gal}(E/E^H) = TS(H) \text{ for any } H \in \mathscr{D}.$$

This defines unit and counit, thus we have an adjunction. We check property (i) of Theorem 2.2. Let $L \in \mathscr{C}$ and $H \in \mathscr{D}$ such that there exists an inclusion $w : L \subset S(H) = E^H$. We have to find a inclusion $v : H \to \mathrm{Gal}(E/L)$ such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "\eta_L"] \arrow[dr] & L \arrow[d, "S(v)"] \\ & E^H. \end{tikzcd}$$

But since $L \subset E^H$ is a subfield that is just the inclusion $H \subset \mathrm{Gal}(E/E^H) \subset \mathrm{Gal}(E/L)$.

So how can one fix the Galois correspondence, or, from our point of view what are the fixed points of this adjunction? Since $L = E^{\mathrm{Gal}(E/L)}$ we have $\mathscr{C} = \mathscr{C}_{fix}$, we only need to find fixed points in $\mathscr{D}$. From this perspective the fixed points are simply all subgroups of the form $\mathrm{Gal}(E/L)$ where L is some subextension. But this is not a very useful description.

The solution is again to equip $\mathrm{Gal}(E/F)$ with a topology, where the subgroups $\mathrm{Gal}(E/L)$ are the closed subgroups. This topology is called the Krull topology. Set $\mathcal{K} = \{K_i \mid i \in I\}$ the collection of all intermediate fields $F \subset K_i \subset E$, where K_i/F is a finite Galois extension. Set $G = \mathrm{Gal}(E/F)$ and $U_i = \mathrm{Gal}(E/K_i)$. Then

$$E = \bigcup_{i \in I} K_i.$$

One checks that the collection $\{U_i \mid i \in I\}$ is a fundamental system of neighborhoods of the identity element $1 \in G$. For then the fundamental system defines a unique topology on G, compatible with the group structure (see [10] Chapter III §1 Proposition 1). Basically one has to check that

- U_i is a subgroup in G and $G/U_i = \mathrm{Gal}(K_i/F)$ is finite for all $i \in I$;
- For $i, j \in I$ then there exist an $h \in I$ such that $U_h \leq U_i \cap U_j$;
- $\bigcap_{i \in I} = \{1\}$.

This can be done by applying standard facts from Galois theory. Then one proves ([59] Theorem 2.11.1) that the Galois group equipped with Krull topology is a profinite group with

$$\mathrm{Gal}(E/F) = \varprojlim_{i \in I} \mathrm{Gal}(K_i/F).$$

This topology is discrete if E/F is a finite extension. Now for any subextension L we have

$$L = \bigcup_{i \in I'} K_i'$$

where K_i' are all intermediate extensions $F \subset K_i' \subset L$ with K_i'/F a finite Galois extension. So

$$\mathrm{Gal}(L/F) = \varprojlim_{i \in I'} \mathrm{Gal}(K_i'/F)$$

is a closed subgroup. Furthermore, all closed subgroups of the profinite group G are of this form ([59] Proposition 2.2.1(a)). This shows that the subgroups $\mathrm{Gal}(L/F)$ are exactly the closed subgroups in the Krull topology (for details, see [59] Chapter 2.11)

2.2 Fiber products of rings and modules

We start with the affine case and give some facts about fiber products respectively cartesian squares of rings and modules. Fiber products as well as coproducts exist in both categories. If $\mathcal{C}$ is the category of modules over a fixed ring, or if it is the category of commutative rings, then a commutative square is cartesian if and only if the underlying square in the category of sets is. As noted this does not hold for cocartesian squares.

We prove some facts on cartesian squares.

Lemma 2.6. *Let A be a ring and let the following diagram be a commutative square in the category of A-modules*

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "w"'] & M' \arrow[d, "w'"] \\ N \arrow[r, "v"'] & N'. \end{tikzcd}$$

It is a cartesian square if and only if the by u induced homomorphism

$$\tilde{u} : \ker(w) \to \ker(w')$$

is an isomorphism, and the by v induced homomorphism

$$\tilde{v} : \operatorname{coker}(w) \to \operatorname{coker}(w')$$

is an injection.

Proof. For $x \in N$ we denote its image in the cokernel by $\tilde{x} \in \operatorname{coker}(w)$.

Suppose the square is cartesian, so

$$M \simeq N \times_{N'} M' = \{(x, y) \in N \times M' \mid v(x) = w'(y)\}.$$

It is $w(x, y) = x$ and $u(x, y) = y$.

The map $\tilde{u}$ is surjective. If $y \in \ker(w')$, then $(0, y) \in M$ and also $(0, y) \in \ker(w)$ with $\tilde{u}(0, y) = y$.

The map $\tilde{u}$ is also injective. It holds that $w(x, y) = x = 0$ for all $(x, y) \in \ker(w)$. So for two elements $(0, y), (0, y') \in \ker(w)$ we have $\tilde{u}(0, y) = \tilde{u}(0, y')$, which implies $y = y'$.

The map $\tilde{v}$ is injective. If $\tilde{x} \in \operatorname{coker}(w)$, with $\tilde{v}(\tilde{x}) = 0$, this implies that $v(x) \in \operatorname{im}(w')$. So there exists a $y \in M'$ with $w'(y) = v(x)$. Then by assumption $(x, y) \in M$. Thus $x \in \operatorname{im}(w)$ and $\tilde{x} = 0$.

In the converse case we assume that $\tilde{u}$ is an isomorphism and $\tilde{v}$ is an injection. We have to show that $M \simeq N \times_{N'} M'$. The fiber product $N \times_{N'} M'$

induced by v and w' exists in the category of modules. By definition there exist projections $pr_1 : N \times_{N'} M' \to N$, $pr_2 : N \times_{N'} M' \to M'$ and a unique homomorphism $t : M \to N \times_{N'} M'$ with $pr_1 t = w$ and $pr_2 t = u$. We show that t is an isomorphism.

The map t is injective. Let $m \in M$, with $t(m) = 0$. This implies that $w(m) = 0$ and $u(m) = 0$. The first gives $m \in \ker(w)$. The second shows that $\tilde{u}(m) = 0$ and since $\tilde{u}$ is an isomorphism this implies $m = 0$.

The map t is surjective. Let $(x, y) \in N \times_{N'} M$. We have to find an element $m \in M$, with $w(m) = x$ and $u(m) = y$, since this would imply $t(m) = (x, y)$. Since (x, y) is an element of the fiber product we have $v(x) = w'(y)$; in particular, $v(x) \in \mathrm{im}(w')$. It follows $\tilde{v}(\tilde{x}) = 0$ for $\tilde{x} \in \mathrm{coker}(w)$. Now by assumption $\tilde{v}$ is injective so $\tilde{x} = 0$ in $\mathrm{coker}(w')$. This implies $x \in \mathrm{im}(w)$. So there exist an element $m' \in M$ with $w(m') = x$. Set $y' = u(m')$. The commutativity of the diagram gives $w'(y') = w'u(m') = vw(m') = v(x) = w'(y)$, so $y - y' \in \ker(w')$. Since $\tilde{u}$ is an isomorphism, there exists an $m'' \in \ker(w)$ with $u(m'') = y - y'$. Set $m = m' + m''$, then $w(m) = x$ and $u(m) = y$ as demanded. □

Lemma 2.7. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & A' \arrow[d, "p'"] \\ B \arrow[r, "g"'] & B'. \end{tikzcd}$$

be a cartesian square in the category of rings and let p' be surjective. Then the following hold:

(i) *The homomorphism p is surjective.*

(ii) *Set $I = \ker(p)$, the map f induces a bijection between I and $f(I) = \ker(p')$, in particular we have $f(I)A' = f(I)$.*

(iii) *Via the homomorphism g and p' we can identify B' with $B \otimes_A A'$, i.e. the square is cocartesian.*

Proof.

(i) Let $b \in B$. Since p' is surjective there exists an $a' \in A'$ with $p'(a') = g(b)$. The square is cartesian, so there exists an $a \in A$ with $f(a) = a'$ and in particular $p(a) = b$.

(ii) This follows directly from the previous lemma.

(iii) Since in the category of ring coproducts always exist and are given by tensor products, we can use their universal property. There exists a unique map $u : B \otimes_A A' \to B'$, with maps $j_1 : B \to B \otimes_A A'$ and $j_2 : A' \to B \otimes_A A'$, such that $g = uj_1$ and $p' = uj_2$. The map u is surjective since p' is. We show that u is invective and thus an isomorphism.

Let $m \in \ker(u)$. If $m \in \operatorname{im}(j_1)$ or $m \in \operatorname{im}(j_2)$ then m is trivial. For the first case suppose there exists $b \in B$ with $j_1(b) = m$. Then $g(b) = uj_1(b) = 0$. Since the square is cartesian $g(b) = 0 = p'(0)$ defines an element $a \in A$ with $p(a) = b$ and $f(a) = 0$. The commutativity of the diagram gives $m = j_1(b) = j_2(0) = 0$. The same construction can be done if there exists $a' \in A$ with $j_2(a') = m$.

Since any possible non trivial element of $\ker(u)$ is in the image of neither j_1 nor j_2 the composition maps uj_1 and uj_2 can be factored through $(B \otimes_A A')/\ker(u)$. So if the kernel is non trivial this is a contradiction to uniqueness of the coproduct.

□

In the situation of the lemma the commutative square is cartesian as well as cocartesian. This is the situation that is of most interest to us.

We will work with cartesian squares of modules which lie over cartesian squares of rings. Therefore it is useful to understand what happens to ideals under this maps.

Lemma 2.8. *Let $f : A \to A'$ be a morphism of rings and M an A-module. Let M' be an A'-module which is a quotient of $A' \otimes_A M$ and $u : M \to A' \otimes_A M \twoheadrightarrow M'$ the resulting composition map. Furthermore, let I be an ideal in A whose image $f(I)$ is an ideal in A'. Then $f(I)M' = u(IM)$. So the image of the A-module IM under the map u is an A'-module.*

Proof. Since M' is a quotient of $A' \otimes_A M$ any element of M' can be written as a finite sum of elements $a'u(x)$, with $a' \in A'$ and $x \in M$. Since $f(I)$ is an ideal of A' the elements of the module $f(I)M'$ can also be written as finite sums of elements $a'u(x)$, this time with $a' \in f(I)$ and $x \in M$. Since $a' \in f(I)$, there exists an $a \in I$ with $f(a) = a'$, which implies $a'u(x) = u(ax)$ and elements of IM are exactly of this form. □

One has to be careful here. In the above notation the module IM is not necessarily an A'-module; at least when u is not injective. Only the image $u(IM)$ is an A'-module.

Definition 2.9. Let $A \to A'$ be an injective morphism of rings. The *conductor* of A in A' is the ideal $\mathrm{Ann}_A(A'/A) = \{a \in A \mid aA' \subset A\}$.

For example, let A be an integral ring and A' a finite commutative A-algebra which is contained in the field of fractions of A but not A itself. The conductor I of $A \to A'$ is not trivial and can be rewritten as

$$I = \{0\} \cup \{a \in A - \{0\} \mid A' \subset \tfrac{1}{a}A\},$$

where $\frac{1}{a}A$ is considered as a subset of the fraction field. For each non trivial element $t \in I$ we have strict inclusions $tA \subsetneq tA' \subsetneq A$, so tA is not an A'-module. Define an A-module $M = A/tA$, then $IM = I/tA$ and this is also not an A'-module.

To make this more concrete we take a look at a cusp singularity. Set

$$A = k[x, y]/(x^2 - y^3) \text{ and } A' = k[T].$$

The inclusion $f : A \to A'$ is defined by $f(x) = T^3$ and $f(y) = T^2$. The conductor ideal in A is $I = (x, y)$ and in A' we get $I' = (T^2, T^3) = f(I)$, so we can simply write I instead of I'. We have $M = A/(y) = k[x]/(x^2)$ and $IM = x(k[x]/(x^2)) = k + kx$. This is not a $k[T]$-module. To see this concretely we write $A = k[T^2, T^3]$ and IM as a $k[T^2, T^3]$-module, so $IM = T^3k[T^2, T^3]/(T^2) = T^3k[T^3]/(T^6)$. This module is not a $k[T]$-module, since multiplication by T is not defined. For example $T^4 \notin IM$.

2.3 Affine Pinching

For the next part suppose we have homomorphism of rings $g : B \to B'$ and $p' : A' \to B'$. We construct first a fiber product category $\mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A')$. Then we construct an adjunction from the category $\mathrm{Mod}(B \times_{B'} A)$ to this fiber product category. We study this adjunction.

We start by constructing the fiber product category. We have the cartesian square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & A' \arrow[d, "p'"] \\ B \arrow[r, "g"'] & B'. \end{tikzcd} \tag{2.3.1}$$

with $A = B \times_{B'} A'$. Using the extension of scalars functors

$$g^* : \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(B'), \quad N \longmapsto B' \otimes_B N$$

and

$$p'^* : \mathrm{Mod}(A') \longrightarrow \mathrm{Mod}(B'), \quad M \longmapsto B' \otimes_B M$$

we can define the fiber product category and get a diagram

$$\begin{tikzcd}
\mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A') \arrow[r] \arrow[d] & \mathrm{Mod}(A') \arrow[d, "p'^*"] \\
\mathrm{Mod}(B) \arrow[r, "g^*"'] & \mathrm{Mod}(B').
\end{tikzcd} \tag{2.3.2}$$

Recall from Section 2.1, that the objects of the fiber product category are triples (N, s, M'), where N is an B-module and M' an A'-module, and $s : g^*(N) \to p'^*(M')$, or better $s : B' \otimes_B N \to B' \otimes_{A'} M'$, is an isomorphism of B'-modules.

The equality $gp = p'f$ from (2.3.1) induces an isomorphism of functors

$$\sigma : g^* p^* \xrightarrow{\sim} p'^* f^*.$$

Thus for each A-module M we have an isomorphism

$$\sigma_M : g^* p^*(M) \longrightarrow p'^* f^*(M).$$

This is the last datum we need to define the additive covariant functor

$$T : \mathrm{Mod}(B \times_{B'} A) \longrightarrow \mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A'),$$

$$M \longmapsto (B \otimes_A M, \sigma_M, A' \otimes_A M),$$

which is induced by the universal property of the fiber product category.

We define an additive functor in the other direction

$$S : \mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A') \longrightarrow \mathrm{Mod}(B \times_{B'} A),$$

$$(N, s, M') \longmapsto \{(y, x') \in N \times M' \text{ with } s(1 \otimes y) = 1 \otimes x'\},$$

and get the following cartesian square of A-modules

$$\begin{tikzcd}
S(N, s, M') \arrow[rr] \arrow[d] & & M' \arrow[d] \\
N \arrow[r] & B' \otimes_B N \arrow[r, "s"'] & B' \otimes_{A'} M'.
\end{tikzcd} \tag{2.3.3}$$

Abusing notation one could also write $S(N, s, M') = N \times_s M'$. The lower morphism should be read as the composition of the canonical homomorphism $N \to B' \otimes_B N$ with s.

Keep in mind that the canonical morphism is given by the unit of the adjunction between restriction and extension of scalars functors and can be written as $N \to g_* g^* N$. Or to be completely precise as $p_* N \to p_* g_* g^* N$. We technically used the restriction of scalars functors here, to treat all modules and homomorphisms as objects respectively morphisms in $\mathrm{Mod}(A)$. Since all our rings are over A by construction, we can always do this. For ease of notation we do not write N_A for restricted modules. Nevertheless, we will often use the fact that our modules respectively homomorphism have additional structure over A', B and B'.

Let M be an A-module and $T(M) = (B \otimes_A M, \sigma_M, A' \otimes_A M)$. Since

$$B' \otimes_B B \otimes M \simeq B' \otimes_A M \simeq B' \otimes_{A'} A' \otimes_A M,$$

the isomorphism σ_M is also encoded by homomorphisms

$$B \otimes_A M \to B' \otimes_A M \text{ and } A' \otimes_A M \to B' \otimes_A M.$$

We can write

$$ST(M) = (B \otimes_A M) \times_{(B' \otimes_A M)} (A' \otimes_A M). \tag{2.3.4}$$

The universal property for the fiber product of modules gives a morphism $\eta_M : M \to ST(M)$. The situation is displayed in the commutative diagram

$$\begin{tikzcd}
M \arrow[dr, "\eta_M"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& B \otimes_A M \times_{B' \otimes_A M} A' \otimes_A M \arrow[r] \arrow[d] & A' \otimes_A M \arrow[d] \\
& B \otimes_A M \arrow[r] & B' \otimes_A M.
\end{tikzcd} \tag{2.3.5}$$

Since η_M exist for every M we have constructed a natural transformation

$$\eta : \mathrm{id}_{\mathrm{Mod}(A)} \to ST.$$

Lemma 2.10. *The functors S and T, together with natural transformation η, determine an adjunction, i.e. the functor S is right adjoint to T and η is the unit of the adjunction.*

Proof. According to Theorem 2.2 it is left to check that for every A-module M, the homomorphism $\eta_M : M \to ST(M)$ is universal from M to S. So let $(N, s, M') \in \mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A')$ be an object of the fiber product category and $g : M \to S(N, s, M')$ a homomorphism. We need to construct a unique homomorphism $f : T(M) \to (N, s, M')$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "g"'] & ST(M) \arrow[d, "S(f)"] \\ & S(N, s, M'). \end{tikzcd} \tag{2.3.6}$$

Composition of g with the homomorphisms in the cartesian square (2.3.3) defines homomorphism $u : M \to M'$ and $w : M \to N$. Vice versa the universal property of cartesian squares ensures that these morphism define g. The situation is displayed in the diagram

$$\begin{tikzcd} M \arrow[dr, "g"] \arrow[drrr, bend left, "u"] \arrow[ddr, bend right, "w"'] & & & \\ & S(N, s, M') \arrow[rr] \arrow[d] & & M' \arrow[d] \\ & N \arrow[r] & B' \otimes_B N \arrow[r, "s"'] & B' \otimes_{A'} M'. \end{tikzcd}$$

We view both M' and N as A-modules induced by the restriction of scalar functors f_* and p_*, to which the respective extension of scalar functors f^* and p^* are left adjoint (see first example to Definition 2.1). These adjunctions applied to u and w induce canonical maps

$$\bar{u} : A' \otimes_A M \longrightarrow M' \quad \text{and} \quad \bar{w} : B \otimes_A M \longrightarrow N,$$

of A' respectively of B-modules. These maps now define a morphism $(\bar{w}, \bar{u}) : T(M) \to (N, s, M')$ and we set $f = (\bar{w}, \bar{u})$.

It is left to check that $S(f)$ makes (2.3.6) commutative. But this can be seen by the universal property of the extension of scalar functor, which says that the diagrams

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "u"'] & A' \otimes_A M \arrow[d, "\bar{u}"] \\ & M' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r] \arrow[dr, "w"'] & B \otimes_A M \arrow[d, "\bar{w}"] \\ & N \end{tikzcd}$$

commute over A. Here the horizontal arrows, from M to the tensor products, are induced by those in (2.3.5) that define η_M uniquely by the universal property of cartesian squares. $\square$

It is useful to know the counit of the adjunction $\epsilon : TS \to \mathrm{id}$. Therefore let (N, s, M') be an object of the fiber product category. We construct

$$\epsilon_{(N,s,M')} : TS(N, s, M') \longrightarrow (N, s, M').$$

Set $N' = B' \otimes_{A'} M'$. We get a cartesian square of A-modules as in (2.3.3), which we relabel as

$$\begin{tikzcd} S(N,s,M') \arrow[r, "u"] \arrow[d, "w"'] & M' \arrow[d, "w'"] \\ N \arrow[r, "v"'] & N', \end{tikzcd} \tag{2.3.7}$$

where $w' : M' \to B' \otimes_{A'} M'$, $v : N \to B' \otimes_B N \xrightarrow{s} B' \otimes_{A'} M'$. We can again use the adjunction between restriction and extension of scalars functors to obtain canonical maps

$$\bar{u} : A' \otimes_A S(N, s, M') \longrightarrow M' \quad \text{and} \quad \bar{w} : B \otimes_A S(N, s, M') \longrightarrow N, \tag{2.3.8}$$

of A' and B-modules, respectively. The pair $(\bar{w}, \bar{u})$ now defines a homomorphism $\epsilon_{(N,s,M')} : TS(N, s, M') \to (N, s, M')$.

To check whether ϵ actually is the counit of the adjunction, we check whether $\epsilon_{(N,s,M')} = (\bar{w}, \bar{u})$ is mapped to $\mathrm{id}_{S(N,s,M')}$ under the adjunction map. We recall, that the adjunction is also defined by

$$\varphi(\epsilon_{(N,s,M')}) = S(\epsilon_{(N,s,M')}) \circ \eta_{S(N,s,M')}$$

following (2.1.2). Since we have constructed $(\bar{w}, \bar{u})$ the same way as in the proof of Lemma 2.10 the diagram

$$\begin{tikzcd} S(N,s,M') \arrow[r, "\eta_{(N,s,M')}"] \arrow[dr, "\mathrm{id}_{S(N,s,M')}"'] & STS(N,s,M') \arrow[d, "S(\bar{w},\bar{u})"] \\ & S(N,s,M') \end{tikzcd}$$

commutes.

We now have the necessary tools to formulate the main result of this section. For the remainder of this section we use the following setting: Let $g : B \to B'$ and $p' : A' \to B'$ be homomorphisms of rings. Additionally, we demand that p' is a surjection. Let $A = B \times_{B'} A'$ be the fiber product with cartesian square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & A' \arrow[d, "p'"] \\ B \arrow[r, "g"'] & B'. \end{tikzcd} \tag{2.3.9}$$

Then according to Lemma 2.7 the map p is a surjection and f induces an A-linear bijection $\tilde{f} : \ker(p) \to \ker(p')$. We set $I = \ker(p) = \ker(p')$ with this identification, so I is an ideal in A as well as in A'. Furthermore, the cartesian square is also cocartesian, i.e. B' is isomorphic to $A' \otimes_A B$. Note that $B = A/I$ and $B' = A'/I$.

We have an adjunction

$$\mathrm{Mod}(B \times_{B'} A') \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A')$$

as already defined, with unit η and counit ϵ. We can show the following:

Theorem 2.11. *([21] Théorème 2.2) In the setting above it holds that:*

(i) *The counit $\epsilon : TS \to \mathrm{id}$ is an isomorphism of functors.*

(ii) *Let M be an A-module, then $M = 0$ if and only if $T(M) = 0$.*

(iii) *For every A-module M, the homomorphism $\eta_M : M \to ST(M)$ is surjective. Its kernel $\ker(\eta_M)$ is annihilated by $I = \ker(p)$ and $\ker(\eta_M)$ is a subset of IM.*

(iv) *Furthermore, let $\bar{A}$ be a commutative A-algebra. The homomorphism*

$$\eta_{\bar{A}} : \bar{A} \longrightarrow (\bar{A} \otimes_A A') \times_{(\bar{A} \otimes_A B')} (\bar{A} \otimes_A B)$$

is surjective and its kernel has square zero.

Proof.

(i) Let (N, s, M') be an element of the fiber product category. To prove the statement, we have to show that the homomorphisms

$$\epsilon_{(N,s,M')} : TS(N, s, M') \to (N, s, M')$$

are isomorphisms. Set $M = S(N, sM')$. With (2.3.7) we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "w"'] & M' \arrow[d, "w'"] \\ N \arrow[r, "v"'] & N', \end{tikzcd} \qquad (2.3.10)$$

It is enough to show that the canonical homomorphisms

$$\bar{u} : A' \otimes_A M \to M' \quad \text{and} \quad \bar{w} : B \otimes_A M \to N,$$

from (2.3.8), which determine $\epsilon_{(N,s,M')}$, are isomorphisms. We have to check whether they are bijective. But to check bijectivity we can also treat them as homomorphism of A-modules.

We start by noting that v induces a homomorphism $\bar{v} : A' \otimes_A N \to N'$ of A'-modules. This is an isomorphism. Of course to check bijectivity we can again treat $\bar{v}$ as a homomorphism of A-modules. The homomorphism $\bar{v}$ can be viewed as a composition of the following isomorphisms of A-modules:

$$A' \otimes_A N \xrightarrow{\sim} A' \otimes_A B \otimes_B N \xrightarrow{\sim} B' \otimes_B N \underset{s}{\xrightarrow{\sim}} N'.$$

This works since v is a composition $v : N \to B' \otimes_B N \xrightarrow{s} N'$, $B' \simeq A' \otimes_A B$ and N is also a B-module.

In the following part of the proof we treat everything as an A-module respectively as a homomorphism of such.

Since p' is surjective, so is $w' = p' \otimes 1_{M'}$. Now Lemma 2.7 i) also holds for cartesian square of modules, which tells us that w is also surjective. This implies that $\bar{w}$ is surjective.

The homomorphism $u : M \to M'$ factors as $u : M \to A' \otimes_A M \xrightarrow{\bar{u}} M'$, by the universal property of extension of scalars functor; similar for v. Thus from the cartesian square (2.3.10) we can construct the following commutative diagram:

$$\begin{tikzcd}
\ker(w) \arrow[r] \arrow[d] \arrow[rr, bend left, "\sim"] & \ker(1 \otimes w) \arrow[r] \arrow[d] & \ker(w') \arrow[d] \\
M \arrow[r] \arrow[d, "w"'] & A' \otimes_A M \arrow[r, "\bar{u}"] \arrow[d, "1\otimes w"'] & M' \arrow[d, "w'"] \\
N \arrow[r] & A' \otimes_A N \arrow[r, "\bar{v}"] & N'.
\end{tikzcd}$$

The isomorphism from $\ker(w)$ to $\ker(w')$ is induced by u following Lemma 2.6. The surjectivity of w implies the surjectivity of $1 \otimes w$. This allows us to apply the snake lemma (see [9] Chapter I §1 Proposition 2) to the right hand side of the diagram. Now since $\bar{v}$ is an isomorphism and the homomorphism $\ker(1 \otimes w) \to \ker(w')$ is at least surjective, both have an empty cokernel. By the snake lemma the same holds for $\operatorname{coker}(\bar{u}) = 0$, thus $\bar{u}$ is surjective.

The module IM is a submodule of $\ker(w)$. Furthermore, Lemma 2.8 tells us that $u(IM) = I'M' = \ker(w')$. Since u induces an isomorphism between these two kernels, it follows that

$$\ker(w) = IM.$$

Since $B \otimes_A M = A/I \otimes_A M = M/IM = M/\ker(w)$, the homomorphism $\bar{w}$ is injective and thus an isomorphism.

It is left to show that $\bar{u}$ is injective. Using a variant of the five lemma (see [9] Chapter I Exercises 1.4) on the right hand side of the diagram above, we can instead show that the morphism $\ker(1 \otimes w) \to \ker(w')$ is injective. Since $\ker(w) \simeq \ker(w')$, we can instead show that the morphism $\ker(w) \to \ker(1 \otimes w)$ is surjective. Now, since w is surjective $\ker(1 \otimes w)$ is the image of the homomorphism $A' \otimes_A \ker(w) \to A' \otimes_A M$. Since $\ker(w) = IM$ and $f(I)A' = f(I)$ this image equals $f(I)(A' \otimes_A M)$. By Lemma 2.8 this is exactly the image of $\ker(w) = IM$ under the homomorphism $M \to A' \otimes_A M$.

(ii) Obviously $T(0) = 0$ if $M = 0$. So let M be an A-module with $T(M) = 0$, we have to show that M is trivial. By assumption $B \otimes_A M = 0$ and $A' \otimes_A M = 0$. The first statement gives $B \otimes_A M = M/IM = 0$, which implies that the canonical homomorphism $I \otimes_A M \to M$ is surjective. Now, f induces an isomorphism of A-modules from I to $f(I)$, which in turn gives us isomorphisms $I \otimes_A M \simeq f(I) \otimes_{A'} (A' \otimes_A M) \simeq 0$.

(iii) Let M be an A-module. To prove that $\eta_M : M \to ST(M)$ is surjective, we use the fact that the functor T is right exact and gives a homomorphism $T(\eta_M) : T(M) \to TST(M)$. If we can show that $T(\eta_M)$ is surjective it will imply $T(\mathrm{coker}(\eta_M)) = 0$ and by (ii) then $\mathrm{coker}(\eta_M) = 0$. By the property (2.1.3) of adjoint functors the composition

$$T(M) \xrightarrow{T(\eta_M)} TST(M) \xrightarrow{\epsilon_{T(M)}} T(M)$$

is an isomorphism, and we know from (i) that $\epsilon_{T(M)} : TST(M) \to T(M)$ is already an isomorphism; so $T(\eta_M)$ has to be an isomorphism.

We give a second argument which allows us to get a handle on $M_0 = \ker(\eta_M)$. Since the commutative square of rings is cartesian as well as cocartesian we have an exact sequence of A-modules

$$0 \longrightarrow A \longrightarrow A' \oplus B \xrightarrow{q} B' \longrightarrow 0,$$

where q is defined as $q(a', b) = p'(a') - g(b)$. Tensoring with M gives an exact sequence

$$\mathrm{Tor}_1^A(M, B') \longrightarrow M \xrightarrow{h} (A' \otimes_A M) \oplus (B \otimes_A M) \xrightarrow{q \otimes 1} B' \otimes_A M \longrightarrow 0,$$

and by construction $ST(M) = \ker(q \otimes 1)$. The exactness of this sequence again shows that $\eta_M : M \to ST(M)$ is surjective. The kernel M_0 of η_M is the same as the kernel of h. By exactness of the sequence it is then the image of $\mathrm{Tor}_1^A(M, B') \to M$. Since I annihilates B' the same holds for $\mathrm{Tor}_1^A(M, B')$ and so also for M_0. Furthermore, for an element of M to be in this kernel it has to at least be zero in $B \otimes_A M$. Since $B = A/I$, this is only the case for elements of the submodule $IM \subset M$, which implies $M_0 \subset IM$.

(iv) Let $\bar{A}$ be a commutative A-algebra. Since fiber product exists for algebras, the homomorphism

$$\eta_{\bar{A}} : \bar{A} \to ST(\bar{A})$$

is

$$\bar{A} \longrightarrow \bar{A} \otimes_A A' \times_{\bar{A} \otimes_A B'} \bar{A} \otimes_A B.$$

With part (iii) the kernel of this map is contained in $I\bar{A}$ and is annihilated by I, so its square is zero.

□

Unfortunately, in the general case, the surjections $\eta_M : M \to ST(M)$ fail to be injective. So the unit is not necessarily a isomorphism of functors, and S and T are not necessarily an adjoint equivalence of categories.

We give examples where η_M is not injective. The idea is to use the conductor I of an injection $f : A \to A'$ to construct a cartesian square. By its very definition the conductor I is both an ideal in A as well as in A'. So f restricted to I is an isomorphism. Then Lemma 2.7 tells us that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ A/I \arrow[r, "g"'] & A'/I. \end{tikzcd} \tag{2.3.11}$$

is a cartesian and cocartesian square as in the setting of Theorem 2.11. We call such a commutative diagram a *conductor square*. We will later generalize this construction to schemes, where it will play quite an important role.

Here the morphism g is injective by construction. It is important to note that the generators of I in A and A' might vary and we use this fact to produce examples. Since

$$ST(M) = (A/I \otimes_A M) \times_{(A'/I \otimes_A M)} (A' \otimes_A M)$$

by (2.3.4) we have to construct an A-module M such that there exist $m \in M$ that is zero when mapped to $A/I \otimes_A M$ and $A' \otimes_A M$.

We continue the example to Definition 2.9 of conductor ideals. In this setting we have

$$\begin{aligned} ST(A/tA) &= (A/I \otimes_A A/tA) \times_{(A'/I \otimes_A A/tA)} (A' \otimes_A A/tA) \\ &= A/I \times_{A'/I} A'/tA' \\ &= A/tA'. \end{aligned}$$

But the surjection $A/tA \to A/tA'$ is in general not injective (see [21] Remarques 2.3.a). We construct more concrete examples where the rings a noetherian and the modules are of finite presentation. Even in this setting η_M fails to be injective.

First, we continue our example with a cusp singularity, so

$$A = k[x, y]/(x^2 - y^3) \text{ and } A' = k[T].$$

The inclusion $f : A \to A'$ is defined by $f(x) = T^3$ and $f(y) = T^2$, with conductor ideal $I = (x, y)$. Now the element y generates a smaller ideal $(y) \subsetneq I$ in A, but for $f(y) = T^2$ we have $(T^2) = I$ in A'. Then

$$A/I = k \text{ and } A'/I = k[T]/(T^2).$$

Furthermore, $A/I \to A'/I$ is injective. So we have a cartesian square like in (2.3.11):

$$\begin{tikzcd} k[x,y]/(x^2-y^3) \arrow[r, "f"] \arrow[d] & k[T] \arrow[d] \\ k \arrow[r] & k[T]/(T^2). \end{tikzcd} \tag{2.3.12}$$

We define a commutative square of A-modules of finite presentation by

$$\begin{aligned} M &= A/(y) &&= k[x]/(x^2), \\ M' &= A' \otimes_A M &&= A'/f(y)A' &&= k[T]/(T^2), \\ N &= A/I \otimes_A M &&= A/I \otimes_A A/(T^2) &&= A/I = k, \\ N' &= A'/I \otimes_A M &&= A'/I &&= k[T]/(T^2). \end{aligned}$$

So the cartesian square of modules is

$$\begin{array}{ccc} k[x]/(x^2) & \longrightarrow & k[T]/(T^2) \\ \downarrow & & \downarrow \\ k & \longrightarrow & k[T]/(T^2). \end{array}$$

We study the homomorphism $\eta_M : M \to ST(M)$. We have

$$ST(k[x]/(x^2)) = k \times_{k[T]/(T^2)} k[T]/(T^2) = k$$

and $\eta_M : k[x]/(x^2) \to k$ is definitely not an isomorphism. Concretely, for the non zero element $x \in M$ we have $f(x) = T^3 = 0$ in $k[T]/(T^2)$. Since $x \in I$ it is also mapped to zero in $N = A/I$. Thus $\eta_M(x) = 0$. The element x even generates the kernel of η_M.

For another example set

$$A = \mathbb{Z}[2i] = \mathbb{Z} + 2i\mathbb{Z} \text{ and } A' = \mathbb{Z}[i].$$

Or written differently we have $A = \mathbb{Z}[T]/(T^2 + 4)$ and $A' = \mathbb{Z}[T]/(T^2 + 1)$. The conductor of the inclusion $A \to A'$ is

$$I = \{2m + 2ni \mid m, n \in \mathbb{Z}\}.$$

The element $2 \in I$ generates an ideal in A as well as in A', but these ideals are different. Indeed $2\mathbb{Z}[2i] = 2\mathbb{Z} + 4i\mathbb{Z}$ and $2\mathbb{Z}[i] = 2\mathbb{Z} + 2i\mathbb{Z} = I$. For example $2i \notin 2\mathbb{Z}[2i]$ but $2i \in 2\mathbb{Z}[i]$. By construction we have a cartesian square as in (2.3.11) with

$$A/I = \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2 \text{ and } A'/I = \mathbb{Z}[i]/2\mathbb{Z}[i] = \mathbb{F}_2[i].$$

Here $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}[i]$ is injective. We define a commutative square of A-modules of finite presentation by

$$\begin{aligned} M &= \mathbb{Z}[2i]/2\mathbb{Z}[2i] &&= \mathbb{Z}/2\mathbb{Z} + 2i\mathbb{Z}/4i\mathbb{Z}, && \\ M' &= A' \otimes_A M &&= \mathbb{Z}[i]/2\mathbb{Z}[i] &&= \mathbb{F}_2[i], \\ N &= A/I \otimes_A A/2A &&= A/I &&= \mathbb{F}_2, \\ N' &= A'/I \otimes_A A/2A &&= A'/I &&= \mathbb{F}_2[i]. \end{aligned}$$

The last two equations hold since $2 \in I$. Then

$$ST(M) = N \times_{N'} M = \mathbb{F}_2 \times_{\mathbb{F}_2[i]} \mathbb{F}_2[i] \simeq \mathbb{F}_2.$$

So the homomorphism $\eta_M : M \to ST(M)$ is

$$\mathbb{Z}[2i]/2\mathbb{Z}[2i] \to \mathbb{F}_2$$

and has kernel $2i\mathbb{Z}/4i\mathbb{Z} \simeq i\mathbb{F}_2$.

Of course, we would like to have an adjoint equivalence of categories at least between suitable subcategories of $\mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A')$ and $\mathrm{Mod}(B \times_{B'} A')$. Let our setting be as in Theorem 2.11. We know from Proposition 2.5 that S and T are an adjoint equivalence of categories between the subcategories of fixed points of the adjunction. Since ϵ is an isomorphisms of functors the fixed points of $\mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A')$ are the whole category. We ask ourselves, what are the fixed points in $\mathrm{Mod}(B \times_{B'} A')$? Or formulated differently, for which modules M is

$$\eta_M : M \longrightarrow ST(M)$$

an isomorphism?

In the proof of Theorem 2.11 (iii) we already noted that we can describe the kernel of η_M in the following way: Since (2.3.9) is cartesian as well as cocartesian we have an exact sequence of A-modules

$$0 \longrightarrow A \longrightarrow A' \oplus B \xrightarrow{q} B' \longrightarrow 0,$$

where q is defined as $q(a', b) = p'(a') - g(b)$. Tensoring with M gives a exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(M, A' \oplus B) \xrightarrow{g} \mathrm{Tor}_1^A(M, B') \longrightarrow M \xrightarrow{h} (A' \otimes_A M) \oplus (B \otimes_A M)$$
$$\xrightarrow{q\otimes 1} B' \otimes_A M \longrightarrow 0.$$

And $ST(M) = \ker(q \otimes 1)$. The kernel M_0 of η_M is same as the kernel of h.

To get a handle on M_0 we extended the exact sequence to the left. The exactness of the sequence tells us that $M_0 = 0$ if and only if g is surjective. Since the Tor functor respects direct sums and the exact sequence shows that g is always injective, we can say the following:

Proposition 2.12. *The subcategory of fixed points of* $\mathrm{Mod}(A)$*, i.e. the subcategory of A-modules, for which $\eta_M : M \to ST(M)$ is an isomorphism, consists exactly of the A-modules M for which the injection*

$$g : \mathrm{Tor}_1^A(M, A') \oplus \mathrm{Tor}_1^A(M, B) \longrightarrow \mathrm{Tor}_1^A(M, B')$$

is an isomorphism.

So S and T are an adjoint equivalence of Categories

$$\mathrm{Mod}(A)_{fix} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{Mod}(B) \times_{\mathrm{Mod}(B')} \mathrm{Mod}(A')$$

according to Proposition 2.5. We ask ourselves if there are some nice subcategories, for which S and T still are an adjoint equivalence of categories. The natural choice are the respective subcategories of flat modules. An A-module M is flat if and only if $\mathrm{Tor}_1^A(M, N) = 0$ for any A-module N (see for example [9] Chapter I §4 Proposition 1). So if M is flat then g is an isomorphism and this shows that flat A-modules are fixed points. We check whether S and T give an adjoint equivalence of categories between the subcategory of flat A-modules and the fiber product category of flat modules. The problem is to prove that S and T map modules in the correct way.

Theorem 2.13. *([21] Théorème 2.2) Let R be a ring and $\mathcal{C}(R)$ category of*

(i) *flat R-modules,*

(ii) *flat R-modules of finite type,*

(iii) *flat R-modules of finite presentation.*

In the setting of Theorem 2.11 the following holds: For every $M \in \mathcal{C}(A)$ we have $T(M) \in \mathcal{C}(B) \times_{\mathcal{C}(B')} \mathcal{C}(A')$, and for every $(N, s, M') \in \mathcal{C}(B) \times_{\mathcal{C}(B')} \mathcal{C}(A')$ it holds that $S(N, s, M') \in \mathcal{C}(A)$. Furthermore, these functors are an adjoint equivalence of categories

$$\mathcal{C}(A) \underset{S}{\overset{T}{\rightleftarrows}} \mathcal{C}(B) \times_{\mathcal{C}(B')} \mathcal{C}(A').$$

Moreover, for any A-module M it holds that $M \in \mathcal{C}(A)$ if and only if $A' \otimes_A M \in \mathcal{C}(A')$ and $B \otimes_A M \in \mathcal{C}(B)$.

The result for (i) regarding flat modules can already be found in an earlier paper of Ferrand (see [20] Lemme (ii)). A result like in the theorem regarding projective modules of finite type can be found in early work of Milnor (see [56] §2 or [7] Chapter IX Theorem 5.1). According to Lemma 1.24 this is equivalent to the statement of (iii); a R-module M is projective and of finite type if and only if it is flat and of finite presentation. It is important to note that condition (iii) is also equivalent to M being finite locally free.

Before we prove the theorem we give examples of the cases (i)-(iii).

(i) Take the rational numbers $\mathbb{Q}$ as a $\mathbb{Z}$-module. This module is flat but not of finite type.

(ii) We construct a flat and finite R-module, which is not of finite presentation. This is only possible if R is not noetherian, since over a noetherian ring every finite module is already of finite presentation. The following nice example can be found in a paper by Vasconcelos ([68] Example 3.2.). Set $R = \mathbb{Z} \oplus \mathfrak{a}$, with $\mathfrak{a} = \bigoplus_\alpha \mathbb{F}_2$ as an infinite sum. We define addition componentwise, then R becomes a commutative group. We define a multiplication by $(n, a)(n', a') = (nn', na' + n'a + aa')$, where the multiplication aa' is defined by a componentwise paring $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$. This makes R into a commutative ring with multiplicative identity $(1, 0)$. Set $f = (2, 0)$. This is a non zero element in the ring. The ideal $(f) = Rf$ is flat and of finite type but not of finite presentation.

Obviously Rf is of finite type. To see that it is not of finite presentation consider the exact sequence

$$0 \longrightarrow \operatorname{ann}(f) \longrightarrow R \longrightarrow Rf \longrightarrow 0.$$

Here $\operatorname{ann}(f) = \mathfrak{a}$ and this is not a finitely generated ideal in R, thus Rf is not of finite presentation.

It is left to show that $M = Rf$ is flat. It is enough to check, that $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for any maximal ideal $\mathfrak{m} \in \operatorname{Spec}(R)$ (see [9] Chapter II §3.4 Corollary). If $f \notin \mathfrak{m}$ we have $M_\mathfrak{m} = R_\mathfrak{m}$, so it is left to check this for ideals with $f \in \mathfrak{m}$. First we assume $\mathfrak{a} \not\subset \mathfrak{m}$. Then there exist a non zero element $a \in \mathfrak{a} \setminus \mathfrak{m}$ and $af = 0$. Thus $M_\mathfrak{m} = 0$ is free. Now let $\mathfrak{a} \subset \mathfrak{m}$. For any element $a \in \mathfrak{a}$ we have $a^2 = a$ hence $(1 - a)a = 0$. Since $1 - a \notin \mathfrak{m}$ the element a becomes zero in $R_\mathfrak{m}$. As this holds for every element of $\mathfrak{a}$, f is not a zero divisor in $R_\mathfrak{m}$ and $M_\mathfrak{m} = fR_\mathfrak{m}$ is free as desired.

(iii) Set $R = \mathbb{Z}/6\mathbb{Z}$. Since $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ the module $\mathbb{Z}/3\mathbb{Z}$ is not free, but it is a finite projective module, and thus also flat, and of finite presentation (Lemma 1.24). Since every Azumaya algebra over a ring R is finite locally free, Azumaya algebras give another example of flat R-modules of finite presentation.

Proof of Theorem 2.13. We use the notation of Theorem 2.11. We know that $\epsilon : TS \to \mathrm{id}$ is an isomorphism of functors by Theorem 2.11 (i). Furthermore, Proposition 2.12 tells us that $\eta : \mathrm{id} \to TS$ is an isomorphism of functors for the category of flat modules and thus also for all subcategories. It is left to show, that S and T do map modules as described in the theorem. Indeed then they are an adjoint equivalence of categories.

The functor T works by tensoring, so it automatically maps as demanded. If M is a flat module so are $B \otimes_A M$ and $A' \otimes_A M$. The same holds if M is finite or of finite presentation. Thus

$$(B \otimes_A M, \sigma_M, A' \otimes_A M) \in \mathcal{C}(B) \times_{\mathcal{C}(B')} \mathcal{C}(A')$$

in all three cases.

We have to show that S maps as demanded. Let

$$(N, s, M') \in \mathcal{C}(B) \times_{\mathcal{C}(B')} \mathcal{C}(A')$$

be an object in the fiber product category. Set $M = S(N, s, M')$. We have to show that $M \in \mathcal{C}(A)$. Since ϵ is an isomorphism of functors we know that $T(M) \simeq \mathrm{id}(N, s, M')$ is in $\mathcal{C}(B) \times_{\mathcal{C}(B')} \mathcal{C}(A')$ and thus $B \otimes_A M \simeq N$ is in $\mathcal{C}(B)$ and $A' \otimes_A M \simeq M'$ is in $\mathcal{C}(A')$.

(i) Let $\mathcal{C}(R)$ denote the category of flat R-modules.

For the proof we can replace A' by $A' \oplus B$ and B' by $B' \oplus B$. This is possible since one of the two squares

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[g]{} & B' \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{(f,p)} & A' \oplus B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p' \oplus 1} \\ B & \xrightarrow[(g,1)]{} & B' \oplus B \end{array}$$

is cartesian if and only if the other is. Furthermore, the same holds for the commutative squares of modules

$$\begin{array}{ccc} M & \xrightarrow{u} & M' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w'} \\ N & \xrightarrow[v]{} & N', \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{(u,w)} & M' \oplus N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle w' \oplus 1} \\ N & \xrightarrow[(v,1)]{} & N' \oplus N. \end{array}$$

And $M' \oplus N$ is flat if and only of M' and N are flat.

In the following we can therefore assume that g is injective. This also implies that f is injective. For $f(a_1) = f(a_2)$ we have $p'f(a_1) = p'f(a_2) = gp(a_1) = gp(a_2)$ and since g is injective $p(a_1) = p(a_2)$. Since the square is cartesian this implies $a_1 = a_2$.

By assumption $B \otimes_A M$ and $A' \otimes_A M$ are flat and we use this fact to show that M is also flat. First we show that $\mathrm{Tor}_1^A(M, A/I) = 0$, or equivalently that $M \otimes_A I \to M$ is injective. Since I is an ideal of A', the left vertical arrow of the commutative diagram

$$\begin{tikzcd} M \otimes_A I \arrow[r] \arrow[d] & M \arrow[d] \\ (M \otimes_A A') \otimes_{A'} I \arrow[r] & M \otimes_A A' \end{tikzcd}$$

is an isomorphism. Since $M \otimes_A A'$ is a flat A' module, the lower arrow is injective. Then the upper one has to be injective as well.

According to [9] Chapter I §4 Proposition 2, the last paragraph implies, that $\mathrm{Tor}_1^A(M, M_0) = 0$ for every A-module M_0 that is annihilated by I. We construct such a module. Let $\mathfrak{a} \subset A$ be an ideal. Set $M_0 = \mathfrak{a}A'/\mathfrak{a}$. This module lies in the exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a}A' \longrightarrow M_0 \longrightarrow 0.$$

Since I is also an ideal of A' with $IA' = I$, we have $IM_0 = 0$. Furthermore, since $\mathrm{Tor}_1^A(M, M_0) = 0$, the morphism $M \otimes_A \mathfrak{a} \to M \otimes_A \mathfrak{a}A'$ is injective. As $M \otimes_A A'$ is a flat A'-module, it is also flat as an A-module. So the injective morphism $\mathfrak{a} \to A$ induces an injective morphism $M \otimes_A \mathfrak{a}A' \to M \otimes_A A'$. Composition gives an injective morphism $M \otimes_A \mathfrak{a} \to M \otimes_A A'$, which factors as

$$M \otimes_A \mathfrak{a} \to M \to M \otimes_A A'.$$

So the morphism $M \otimes_A \mathfrak{a} \to M$ has to be injective as well, hence M is a flat A-module, according to [9] Chapter I §2 Proposition 1.

(ii) Now let $\mathcal{C}(R)$ denote the category of flat R-modules of finite type. Let $B \otimes_A M$ and $A' \otimes_A M$ be flat and of finite type. We can lift their generators to construct a sub-module of finite type $M_0 \subset M$ such that $B \otimes_A M/M_0 = 0$ and $A' \otimes_A M/M_0 = 0$. This implies $T(M/M_0) = 0$ and then $M = M_0$ by Theorem 2.11 (ii). By (i) we already know that M is flat.

(iii) Let $\mathcal{C}(R)$ is the category of flat R-modules of finite presentation. So let M be an A-module and both $B \otimes_A M$ and $A' \otimes_A M$ are flat and of finite presentation. We have already seen in (ii) that M is then flat and of finite type. So there exist an exact sequence of A-modules $0 \to K \to L \to M \to 0$, where L is of finite type. All that is left to show is that K is also of finite type. Since M is flat we get exact sequences

$$0 \longrightarrow B \otimes_A K \longrightarrow B \otimes_A L \longrightarrow B \otimes_A M \longrightarrow 0,$$

and

$$0 \longrightarrow A' \otimes_A K \longrightarrow A' \otimes_A L \longrightarrow A' \otimes_A M \longrightarrow 0.$$

The right side of both sequences is of finite presentation, so the left side has to be of finite type. And we have already seen in (ii) that then K is of finite type itself.

$\square$

In the proof one could also solve the case (iii) by using the fact that M being flat and of finite presentation is that same as M being projective and of finite type (Lemma 1.24). We have an isomorphism $A \to ST(A)$. The functor is additive and thus an isomorphism for free modules. Then using additivity again this also holds for projective modules of finite type (for this argument, see [7] Chapter IX proof of Theorem 5.1).

In his paper Farrand also claimed that Theorem 2.13 holds when $\mathcal{C}(R)$ is the category of R-modules of finite type. His proof does not check that the unit is an isomorphism of functors. And indeed the statement is wrong (see [66] Remark 4.3.8 and Example 4.3.4. for a non Noetherian counterexample). We have already given counterexample for in case where the rings are Noetherian and the modules of finite presentation right after the proof of Theorem 2.11. Take note that we have shown in part (ii) of the proof that S and T do define maps

$$\mathcal{C}(A) \underset{S}{\overset{T}{\rightleftarrows}} \mathcal{C}(B) \times_{\mathcal{C}(B')} \mathcal{C}(A'),$$

when $\mathcal{C}(R)$ denotes the category of R-modules of finite type. This simply fails to be an adjoint equivalence of categories.

2.4 Pinching of schemes

We want to transfer the results from the last section to schemes. To have nice diagrams of schemes we will write commutative diagrams of rings in the other direction from now on. The motivational idea goes like this. If we have a cusp singularity like in (2.3.12) the Spec functor gives a commutative diagram of affine schemes

$$\begin{tikzcd} \mathrm{Spec}(k[T]/(T^2)) \arrow[r, "w"] \arrow[d, "g"'] & \mathrm{Spec}(k[T]) \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "u"'] & \mathrm{Spec}(k[x,y]/(x^2-y^3)). \end{tikzcd}$$

and f resolves the singularity of the curve. The closed embedding u maps to the singularity and f is a blow up along this closed point. The closed embedding w maps to the exceptional divisor of the blow up and g is a finite morphism. Now Theorem 2.13 can be interpreted as saying, that a quasicoherent sheaf on $\mathrm{Spec}(k[x,y]/(x^2-y^3))$ is flat if and only if its pullbacks along f and u are flat. Furthermore, one can construct a flat sheaf on $\mathrm{Spec}(k[x,y]/(x^2-y^3))$ by constructing flat shaves on $\mathrm{Spec}(k)$ and $\mathrm{Spec}(k[T]/(T^2))$ that become isomorphic after pullback along g respectively w.

More generally let

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y. \end{tikzcd}$$

be a commutative square of schemes, where w and u are closed immersions. The square is cartesian if $X' = Y' \times_Y X$. If furthermore the square is cocartesian, i.e. Y is the pushout, the scheme Y can be seen as the "pinching" of X in the closed subscheme X' along g.

To be able to apply Theorem 2.11 and Theorem 2.13 to such a diagram we need some affine covering of Y, where the preimages of the covering are affine as well. So we will demand that f and thus also g are affine morphism.

So when is a commutative square of schemes

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y. \end{tikzcd}$$

cocartesian? First Y needs to be the pushout in the category of topological spaces, i.e. Y needs to be a pushout in the category of sets and a set $V \subset Y$ is open if and only if $u^{-1}(V)$ and $f^{-1}(V)$ are open. Secondly the structure sheaf $\mathcal{O}_Y$ needs to agree with this structure. This means that the commutative square of $\mathcal{O}_Y$-modules

$$\begin{tikzcd} h_*\mathcal{O}_{X'} & f_*\mathcal{O}_X \arrow[l] \\ u_*\mathcal{O}_{Y'} \arrow[u] & \mathcal{O}_Y \arrow[l] \arrow[u] \end{tikzcd}$$

with $h = ug = fv$ has to be cartesian (see [21] Scolie 4.3). So we can check whether

$$\mathcal{O}_Y = u_*\mathcal{O}_{Y'} \times_{h_*\mathcal{O}_{X'}} f_*\mathcal{O}_X.$$

Usually we are interested in commutative squares of schemes which are cartesian as well as cocartesian. The commutative square of $\mathcal{O}_Y$-modules above defines a complex

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow u_*\mathcal{O}_{Y'} \oplus f_*\mathcal{O}_X \longrightarrow h_*\mathcal{O}_{X'} \longrightarrow 0. \tag{2.4.1}$$

And the square is cartesian and cocartesian if and only if this complex is exact.

We check whether this agrees in the affine case.

Proposition 2.14. *([21] Théorème 5.1) Let*

$$\begin{tikzcd} A'/I & A' \arrow[l] \\ A/I \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

be a cartesian and cocartesian square of rings as in the setting of Theorem 2.11. Then the commutative square of affine schemes

$$\begin{tikzcd} \mathrm{Spec}(A'/I) \arrow[r, "w"] \arrow[d, "g"'] & \mathrm{Spec}(A') \arrow[d, "f"] \\ \mathrm{Spec}(A/I) \arrow[r, "u"'] & \mathrm{Spec}(A) \end{tikzcd}$$

is cartesian as well as cocartesian in the category of schemes.

Proof. We set $Y = \mathrm{Spec}(A)$, $Y' = \mathrm{Spec}(A/I)$, $X = \mathrm{Spec}(A')$, and $X' = \mathrm{Spec}(A'/I)$. Then we have $A'/I = A/I \otimes_A A'$ according to Lemma 2.7. So $\mathrm{Spec}(A'/I)$ is exactly the fiber product of schemes (see, for example [35] Chapter II Theorem 3.3) and thus the diagram is cartesian in the category of schemes.

We first show that the diagram is cocartesian in the category of sets. Denote $h : A \to A'$. By Lemma 2.7 we have $h(I) = I$. For every $t \in I$ we have a homomorphism

$$\varphi_t : A_t \longrightarrow A'_{h(t)}$$

and we show that this is an isomorphism.

Any element in the kernel can be written as at^m with $h(at^m) = 0$ for $a \in A$ and some $m \geq 0$. We have $at^{m+1} \in I \cap \ker(h)$. Since $h(I) \simeq I$ this intersection is zero. So $at^{m+1} = 0$ and thus a is zero in A_t. So the kernel of φ_t is trivial.

Any element in $A'_{h(t)}$ can be written as $a'h(t)^{-m}$, with $a' \in A$ and some $m \geq 0$. Now $a'h(t)$ is an element of the ideal $h(I)$, so there exist an $a \in I \subset A$ with $h(a) = a'h(t)$. Thus we get

$$a'h(t)^{-m} = a'h(t)h(t)^{-(m+1)} = h(a)h(t)^{-(m+1)} = h(at^{-(m+1)})$$

with $at^{-(m+1)} \in A_t$. So φ_t is surjective.

The fact that φ_t is an isomorphism for every $t \in I$ implies that we have an isomorphism of affine schemes

$$f_{|D(I)} : D_{A'}(I) \to D_A(I),$$

where $D_{A'}(I) = X - X'$ and $D_A(I) = Y - Y'$. Since $u : Y' \to Y$ is a closed immersion it is injective as a morphism of sets. So $u(Y') \sqcup f(X - X') = Y$ and both maps are injective. It follows with the universal property that the canonical morphism of the cocartesian product of sets $Y' \sqcup_{X'} X \to Y$ has to be bijective. Thus the diagram is cocartesian in the category of sets.

Next we show that the diagram is cocartesian in the category of topological spaces. We have to show that a subset $U \subset Y$ is open if and only if $u^{-1}(U)$ and $f^{-1}(U)$ are open. One direction is trivial. For the other let $u^{-1}(U)$ and $f^{-1}(U)$ be open and $y \in U$. If $y \in D(I)$ the isomorphism $f_{|D(I)}$ shows that U is open. If $y \notin D(I)$ the argument is more involved:

Let $y \in V(I)$. We show that there exist a basic open neighborhood of y that is contained in U, which implies that U is open. Since $u^{-1}(U)$ is open there exists a $\bar{g} \in A/I$ such that $u^{-1}(y) \in D(\bar{g}) \subset u^{-1}(U)$. Let $\bar{f} \in A'/I$ be the image of $\bar{g}$, and let $f \in A'$ be a preimage of $\bar{f}$. This gives an element $g = (\bar{g}, f)$ of $A = A/I \times_{A'/I} A'$ and a point $y \in D(g) \subset Y$. Unfortunately this basic open is not necessarily contained in U, only its intersection with $V(I)$ is by construction; so $D(g) \cap V(I) \subset U$. We change g so that the basic open is contained in U. We know that the preimages of $D(g)$ in the diagram are the basic open $D(\bar{g}) \subset Y', D(\bar{f}) \subset X'$ and $D(f) \subset X$. There exists an ideal $J \subset A'$ with $D(J) = f^{-1}(U)$ in X. And $D(f) \cap D(J) \cap V(I) = D(f) \cap V(I)$ in X. This shows that $Z = D(f) \cap V(I)$ and $Z' = D(f) \cap V(J)$ are disjoint closed subsets of $D(f) = \mathrm{Spec}(A'_f)$. So the closed embedding $Z \sqcup Z' \to \mathrm{Spec}(A'_f)$ gives a surjection $A'_f \to A'_f/IA'_f \times A'_f/JA'_f$. Let a'' be the preimage of $(1, 0)$, so after clearing denominators we have an element $a' \in J$ such that $\bar{a'} = \bar{f}^n$

in A'/I, and $D(a'f) \subset D(J)$. Set $h = (\bar{g}^{n+1}, a'f)$ in A. By construction $y \in D(h)$ and $D(h) \subset U$ (see also [5] Tag 0B7J).

It left to check that the cocartesian square

$$\begin{tikzcd} h_*\mathcal{O}_{X'} & f_*\mathcal{O}_X \arrow[l] \\ u_*\mathcal{O}_{Y'} \arrow[u] & \mathcal{O}_Y \arrow[l] \arrow[u] \end{tikzcd}$$

is also cartesian. But the global sections just give our initial cartesian and cocartesian square of rings. This is equivalent to the complex

$$0 \longrightarrow A \longrightarrow A/I \oplus A' \longrightarrow A'/I \longrightarrow 0$$

being exact And since localization is exact the square stays cartesian and cocartesian under localization. $\square$

Note that when we start with a cartesian and cocartesian square of affine schemes as in the proposition the global sections immediately give a cartesian and cocartesian square of rings.

This proposition is useful for checking, whether a commutative square of schemes, where all morphism are affine, is cartesian and cocartesian. We mostly need it for the cocartesian part. Basically we take an affine open cover of Y. Then the preimages of each element of this cover form a commutative square of affine schemes. Next the Proposition tells us that it is enough to check whether the underling commutative squares of rings are cocartesian.

Lets make this more concrete.

Definition 2.15. We call a morphism of schemes $f : X \to Y$ a *finite modification* if the following holds true: First, f is finite. Second, there exist dense open subsets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f_{|U} : U \to V$ is an isomorphism. Third, the image of f is schematically dense, i.e. the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective.

For a finite modification we can define the *conductor ideal*

$$\mathcal{C} = \mathrm{Ann}_{\mathcal{O}_Y}(f_*\mathcal{O}_X/\mathcal{O}_Y).$$

The conductor ideal defines a closed subscheme $u : Y' \to Y$. The inverse image $\mathcal{C}' = \mathcal{C}\mathcal{O}_X$ defines a closed subscheme $w : X' \to X$, which is the base change of the closed immersion u, i.e. $X' = Y' \times_Y X$. So we get a cartesian square of schemes

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y. \end{tikzcd} \tag{2.4.2}$$

Again we call such a construction a *conductor square*. It generalizes the conductor square of rings (2.3.11).

Lemma 2.16. *The conductor square is cartesian and cocartesian in the category of schemes.*

Proof. The conductor square is cartesian by construction, we have to show that it is cocartesian.

Choose any affine cover $(V_j)_{j\in J}$ of Y. Since all morphisms in the square are finite, the preimages

$$U_j = f^{-1}(V_j), \quad u^{-1}(V_j), \quad \text{and} \quad g^{-1}(u^{-1}(V_j)) = w^{-1}(U_j),$$

are again affine open and form a cover of X, Y' and X', respectively.

For each $j \in J$ we have a cartesian square

$$\begin{tikzcd} w^{-1}(U_j) \arrow[r] \arrow[d] & U_j \arrow[d] \\ u^{-1}(V_j) \arrow[r] & V_j. \end{tikzcd}$$

We show that these are also cocartesian.

Locally on the V_j the conductor $\mathcal{C}$ is the conductor ideal I_j of a ring homomorphism $\mathcal{O}_{V_j} \to f_*\mathcal{O}_{U_j}$. So with $V_j = \mathrm{Spec}(A_j)$, $U_j = \mathrm{Spec}(A'_j)$, we have $u^{-1}(V_j) = \mathrm{Spec}(A_i/I_j)$ and $w^{-1}(U_j) = \mathrm{Spec}(A'_j/I_j)$. These rings form a cartesian and cocartesian square as (2.3.11). According to Proposition 2.14 the cartesian square of affine schemes above is then also cocartesian.

This implies that the conductor square is cocartesian. To see that the complex

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow u_*\mathcal{O}_{Y'} \oplus f_*\mathcal{O}_X \longrightarrow h_*\mathcal{O}_{X'} \longrightarrow 0$$

is exact, we use that it is exact after localization. $\square$

We will later be mostly concerned with finite modification in finitely many closed points. This means that, for the open dense subset $V \subset Y$ from Definition 2.15, the complement $Y' = Y \setminus V$, where we "modify" our scheme, consists of only finitely many closed points. Since f is finite, this implies that $X' = X \setminus U$ also consists of only finitely many points. In this situation we have $Y' = \mathrm{Spec}(S)$, $X' = \mathrm{Spec}(R)$, and both rings are Artin rings. Furthermore, the lemma above ensures that this conductor square is cartesian and cocartesian.

While we are mostly concerned with properties of cartesian and cocartesian squares of schemes, in his paper Ferrand is also interested in the construction of pushouts. He shows that given a closed immersion $w : X' \to X$ and integral morphism $g : X' \to Y'$, with the property, that for any $y' \in Y'$ there exist an affine open $U \subset X$, such that $g^{-1}(\{y'\}) \subset w^{-1}(U)$, then there exist a pushout in the categories of schemes $Y = Y' \sqcup_{X'} X$. And this pushout gives a cartesian and cocartesian square of schemes

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y, \end{tikzcd}$$

where f is integral and u a closed immersion. ([21] Théorème 7.1).

2.5 Pinching sheaves

We aim to generalize the statements of Theorem 2.11 and Theorem 2.13. A short summary of this generalization can also be found in [27] Appendix A.

Let

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y. \end{tikzcd}$$

be a cartesian and cocartesian square of schemes. Locally our sheaves shall look like modules, so quasicoherent sheaves are the right choice (see for example [35] Chapter II Proposition 5.4). The pullback of a quasicoherent sheaf is quasicoherent ([35] Chapter II Proposition 5.8.a). We require the same for pushforwards. Furthermore, the pushforward of an isomorphism of sheaves should be an isomorphism. To achieve the second, we demand that all morphism in the square are affine. The diagram is cartesian, so it is enough if f and u are affine. Since affine morphisms are stable under base change this implies that g and w are also affine ([26] Proposition 12.3). Under an affine morphism the pushforward of a quasicoherent sheaf is again quasicoherent ([35] Proposition 5.8.c). Finally, while the pushforward f_* is in general only left exact, under an affine morphism the higher direct images of quasicoherent sheaves vanish, i.e. $R^i f_* \mathscr{F} = 0$, for all $i > 0$ ([5] Tag 01XC). So in our case the pushforward is an exact functor.

The constructions are generalizations of the ones for modules in the last section. We define the fiber product category through pullbacks

$$\begin{tikzcd}
\mathrm{QCoh}(X') & & \mathrm{QCoh}(X) \arrow[ll, "w^*"'] \\
\mathrm{QCoh}(Y') \arrow[u, "g^*"] & \mathrm{QCoh}(Y') \times_{\mathrm{QCoh}(X')} \mathrm{QCoh}(X) \arrow[l] \arrow[ur]
\end{tikzcd}$$

The objects of the fiber product category are triples $(\mathscr{N}, \tau, \mathscr{F})$, with $\mathscr{N} \in \mathrm{QCoh}(Y')$, $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\tau : g^*\mathscr{N} \to w^*\mathscr{F}$ is an isomorphism of quasicoherent $\mathcal{O}'_X$-modules.

The equality $h = ug = fw$ induces an isomorphism of functors

$$\sigma : g^*u^* \xrightarrow{\simeq} w^*f^*.$$

So, for each quasicoherent $\mathcal{O}_Y$-module $\mathscr{G}$ we have an isomorphism

$$\sigma_{\mathscr{G}} : g^*u^*\mathscr{G} \to w^*f^*\mathscr{G}.$$

The universal property of fiber products, together with this data, gives an additive covariant functor

$$T : \mathrm{QCoh}(Y) \longrightarrow \mathrm{QCoh}(Y') \times_{\mathrm{QCoh}(X')} \mathrm{QCoh}(X),$$

$$\mathscr{G} \longmapsto (u^*\mathscr{G}, \sigma_{\mathscr{G}}, f^*\mathscr{G}).$$

We want to make this into an adjunction, so we define an additive functor in the other direction

$$S : \mathrm{QCoh}(Y') \times_{\mathrm{QCoh}(X')} \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(Y).$$

This functor should map a triple $(\mathscr{N}, \tau, \mathscr{F})$ to a fiber product $u_*\mathscr{N} \times_{h_*\tau} f_*\mathscr{F}$. So we have to construct this fiber product in such a way that $S(\mathscr{N}, \tau, \mathscr{F})$ makes the diagram

$$\begin{tikzcd}
S(\mathscr{N}, \tau, \mathscr{F}) \arrow[rr] \arrow[d] & & f_*\mathscr{F} \arrow[d] \\
u_*\mathscr{N} \arrow[r] & u_*g_*g^*\mathscr{N} \arrow[r, "h_*\tau"'] & f_*w_*w^*\mathscr{F}
\end{tikzcd} \tag{2.5.1}$$

cartesian. This diagram is a generalization of (2.3.3) to schemes. We know that for a morphism of schemes $f : X \to Y$ we have an adjunction

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathscr{G}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

between pullbacks and pushforwards. The unit of this adjunction gives a canonical morphism $\mathscr{F} \to f_* f^* \mathscr{F}$. This generalizes the canonical morphisms of the form $M' \to B' \otimes_{A'} M'$, which come from the unit of the adjunction between extension and restriction of scalars functors, to morphisms of sheaves. Since $\tau : g^* \mathscr{N} \to w^* \mathscr{F}$ is a isomorphisms the same holds for the pushforward $h_* \tau : h_* g^* \mathscr{N} \to h_* w^* \mathscr{F}$, which then allows us to define the fiber product. We define $S(\mathscr{N}, \tau, \mathscr{F})$ as the kernel of

$$u_* \mathscr{N} \oplus f_* \mathscr{F} \longrightarrow h_* g^* \mathscr{N} \oplus h_* w^* \mathscr{F} \xrightarrow{h_* \tau - \mathrm{id}} h_* w^* \mathscr{F}.$$

This gives a well defined fiber product. Abusing notation we can now write

$$S(\mathscr{N}, \tau, \mathscr{F}) = u_* \mathscr{N} \times_{h_* \tau} f_* \mathscr{F}.$$

For any quasicoherent $\mathcal{O}_Y$-module $\mathscr{G}$ we get

$$ST(\mathscr{G}) = u_* u^* \mathscr{G} \times_{h_* \sigma_{\mathscr{G}}} f_* f^* \mathscr{G}.$$

Then the universal property of the fiber product gives a morphism

$$\eta_{\mathscr{G}} : \mathscr{G} \to ST(\mathscr{G}),$$

which defines a natural transformation

$$\eta : \mathrm{id}_{\mathrm{QCoh(Y)}} \longrightarrow ST.$$

Lemma 2.17. *The functors S and T, together with natural transformation η, determine an adjunction, i.e. the functor S is right adjoint to T and η is the unit of the adjunction.*

Proof. The proof is similar to the proof of Lemma 2.10. According to Theorem 2.2 we have to check that for every quasicoherent $\mathcal{O}_Y$-module $\mathscr{G}$, the homomorphism $\eta_{\mathscr{G}} : \mathscr{G} \to ST(\mathscr{G})$ is universal from $\mathscr{G}$ to S. So let $(\mathscr{N}, \tau, \mathscr{F})$ be an object of the fiber product category and $q : \mathscr{G} \to S(\mathscr{N}, \tau, \mathscr{F})$ a homomorphism. We need to construct a unique homomorphism $p : T(\mathscr{G}) \to (\mathscr{N}, \tau, \mathscr{F})$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\eta_{\mathscr{G}}"] \arrow[dr, "q"'] & ST(\mathscr{G}) \arrow[d, "S(p)"] \\ & S(\mathscr{N}, \tau, \mathscr{F}). \end{tikzcd}$$

By the universal property of the fiber product the morphism q is uniquely determined by two maps $\mathscr{G} \to u_* \mathscr{N}$ and $\mathscr{G} \to f_* \mathscr{F}$, which come from (2.5.1).

The adjunction between pushforward and pullback gives unique maps $u^*\mathscr{G} \to \mathscr{N}$ and $f^*\mathscr{G} \to \mathscr{F}$, which in turn determine a unique morphism

$$p : (u^*\mathscr{G}, \sigma_{\mathscr{G}}, f^*\mathscr{G}) \to (\mathscr{N}, \tau, \mathscr{F})$$

as required.

Commutativity of the diagram, i.e. that $S(p) \circ \eta_{\mathscr{G}} = q$, follows again with the universal properties of cartesian squares and the fact that the diagrams

$$\begin{tikzcd} \mathscr{G} \arrow[r] \arrow[dr] & u_*u^*\mathscr{G} \arrow[d] \\ & u_*\mathscr{N} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{G} \arrow[r] \arrow[dr] & f_*f^*\mathscr{G} \arrow[d] \\ & f_*\mathscr{F} \end{tikzcd}$$

are commutative.

$\square$

We want to find the counit $\epsilon : TS \to \mathrm{id}$ of this adjunction. Let $(\mathscr{N}, \tau, \mathscr{F})$ be an object of the fiber product category. We need to define

$$\epsilon_{(\mathscr{N},\tau,\mathscr{F})} : TS(\mathscr{N}, \tau, \mathscr{F}) \longrightarrow (\mathscr{N}, \tau, \mathscr{F}),$$

so that $S(\epsilon_{(\mathscr{N},\tau,\mathscr{F})}) = \mathrm{id}_{S(\mathscr{N},\tau,\mathscr{F})}$. In the construction we do in the proof of Lemma 2.17 above we simply set $\mathscr{G} = S(\mathscr{N}, \tau, \mathscr{F})$ and $q = \mathrm{id}_{S(\mathscr{N},\tau,\mathscr{F})}$. Then $\epsilon_{(\mathscr{N},\tau,\mathscr{F})} = p$ is the counit.

To translate the setting of Theorem 2.11 let

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y, \end{tikzcd} \tag{2.5.2}$$

be a cartesian and cocartesian square, where f is affine and u a closed immersions. Denote the quasicoherent ideal that defines the closed subschemes Y' by $\mathcal{I}$. The diagram is cartesian, and, as mentioned, this implies that g is affine. The morphism w is also a closed immersions, since closed immersions are also stable under base change ([26] Proposition 4.32).

We have already constructed an adjunction

$$\mathrm{QCoh}(Y) \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{QCoh}(Y') \times_{\mathrm{QCoh}(X')} \mathrm{QCoh}(X),$$

with unit η and counit ϵ. We can now state the theorem in the language of schemes:

Theorem 2.18. *([21] Théorème 2.2) In the setting above it holds that:*

(i) *The counit $\epsilon : TS \to \mathrm{id}$ is an isomorphism of functors.*

(ii) *Let $\mathscr{G}$ be a quasicoherent $\mathcal{O}_Y$-module, then $\mathscr{G} = 0$ if and only if $T(\mathscr{G}) = 0$.*

(iii) *For every quasicoherent $\mathcal{O}_Y$-module $\mathscr{G}$, the homomorphism $\eta_{\mathscr{G}} : \mathscr{G} \to ST(\mathscr{G})$ is surjective. Its kernel $\ker(\eta_{\mathscr{G}})$ is annihilated by $\mathcal{I}$ and contained in $\mathcal{I}\mathscr{G}$.*

Proof. Choose any affine cover $(V_j)_{j\in J}$ of Y. Since all morphism are affine, the preimages $U_j = f^{-1}(V_j)$, $u^{-1}(V_j)$ and $g^{-1}(u^{-1}(V_j)) = w^{-1}(f^{-1}(V_j))$ are again affine open and form a cover of X, Y' and X' respectively. For each $j \in J$ we have a commutative square of affine schemes

$$\begin{tikzcd} w^{-1}(U_j) \arrow[r] \arrow[d] & U_j \arrow[d] \\ u^{-1}(V_j) \arrow[r] & V_j \end{tikzcd}$$

that fulfills all the properties in the setting of the theorem. This induces cartesian and cocartesian squares of rings as in the setting of Theorem 2.11. Applying Theorem 2.11 to these squares shows that the statements (i)-(iii) hold for a covering of Y. This finishes the proof. □

We will use this technique, where we "cover" the diagram of schemes by a diagram of affine schemes and then prove the statement with respect to this covering, for the rest of this section.

We have already seen that this adjunction is not an adjoint equivalence of categories. Again, we determine the fixed points of the adjunction. Since the counit ϵ is an isomorphism the fixed points are the whole category. What are the fixed points of η, i.e. when is $\eta_{\mathscr{G}} : \mathscr{G} \to ST(\mathscr{G})$ an isomorphism?

Once more we can generalize the affine case of Proposition 2.12. Since diagram (2.5.2) is cartesian and cocartesian we have a cartesian and cocartesian square of $\mathcal{O}_Y$-modules

$$\begin{tikzcd} h_*\mathcal{O}_{X'} & f_*\mathcal{O}_X \arrow[l, "w^\#"'] \\ u_*\mathcal{O}_{Y'} \arrow[u, "g^\#"] & \mathcal{O}_Y, \arrow[l, "u^\#"] \arrow[u, "f^\#"'] \end{tikzcd}$$

with $h = ug = fv$. So we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow u_*\mathcal{O}_{Y'} \oplus f_*\mathcal{O}_X \xrightarrow{q} h_*\mathcal{O}_{X'} \longrightarrow 0,$$

where $q = w^{\#} - g^{\#}$. Tensoring with a quasicoherent $\mathcal{O}_Y$-module $\mathcal{G}$ gives an exact sequence

$$0 \longrightarrow \mathrm{Tor}_{\mathcal{O}_Y}(\mathcal{G}, u_*\mathcal{O}_{Y'} \oplus f_*\mathcal{O}_X) \longrightarrow \mathrm{Tor}_{\mathcal{O}_Y}(\mathcal{G}, h_*\mathcal{O}_{X'}) \longrightarrow \mathcal{G}$$

$$\longrightarrow (u_*\mathcal{O}_{Y'} \otimes \mathcal{G}) \oplus (f_*\mathcal{O}_X \otimes \mathcal{G}) \xrightarrow{q \otimes 1} h_*\mathcal{O}_{X'} \otimes \mathcal{G} \longrightarrow 0.$$

Now, if all involved schemes would be affine, we could use that for an morphism of affine schemes $f : X \to Y$ we have $f_*\mathcal{O}_X \otimes \mathcal{G} = f_*f^*\mathcal{G}$. Conveniently, we can use the same construction as in the proof of Theorem 2.18. The exact sequence becomes

$$0 \longrightarrow \mathrm{Tor}^1_{\mathcal{O}_Y}(\mathcal{G}, u_*\mathcal{O}_{Y'} \oplus f_*\mathcal{O}_X) \xrightarrow{\Phi} \mathrm{Tor}^1_{\mathcal{O}_Y}(\mathcal{G}, h_*\mathcal{O}_{X'}) \longrightarrow \mathcal{G}$$

$$\xrightarrow{\varphi} (u_*u^*\mathcal{G}) \oplus (f_*f^*\mathcal{G}) \xrightarrow{q \otimes 1} h_*h^*\mathcal{G} \longrightarrow 0$$

and with (2.5.1) we see that $ST(M) = \ker(q \otimes 1)$. So the kernel of $\eta_{\mathcal{G}}$ is the same as the kernel of φ. The exact sequence tells us, that this kernel is zero if and only if Φ is surjective. Since Φ is already injective and since the Tor functor respects direct sums we can state the following:

Proposition 2.19. *The subcategory of fixed points of* $\mathrm{QCoh}(Y)$*, i.e. the subcategory of quasicoherent* $\mathcal{O}_Y$*-modules for which* $\eta_{\mathcal{G}} : \mathcal{G} \to ST(\mathcal{G})$ *is an isomorphism, consists exactly of those quasicoherent* $\mathcal{O}_Y$*-modules* $\mathcal{G}$ *for which the injection*

$$\Phi : \mathrm{Tor}^1_{\mathcal{O}_Y}(\mathcal{G}, u_*\mathcal{O}_{Y'}) \oplus \mathrm{Tor}^1_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{O}_X) \longrightarrow \mathrm{Tor}^1_{\mathcal{O}_Y}(\mathcal{G}, h_*\mathcal{O}_{X'})$$

is an isomorphism.

Now S and T are an adjoint equivalence of Categories

$$\mathrm{QCoh}(Y)_{fix} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{QCoh}(Y') \times_{\mathrm{QCoh}(X')} \mathrm{QCoh}(X)$$

according to Proposition 2.5. Again flat modules are an obvious choice for subcategories, where S and T still are an adjoint equivalence of categories.

Theorem 2.20. *([21] Théorème 2.2) Let* Z *be a scheme and* $\mathcal{C}(Z)$ *the category of*

(i) *flat* $\mathcal{O}_Z$*-modules,*

(ii) *flat* $\mathcal{O}_Z$*-modules of finite type,*

(iii) *finite locally free $\mathcal{O}_Z$-modules.*

In the setting of Theorem 2.18 the following holds true: For every $\mathscr{G} \in \mathcal{C}(Y)$ we have $T(\mathscr{G}) \in \mathcal{C}(Y') \times_{\mathcal{C}(X')} \mathcal{C}(X)$, and for every $(\mathscr{N}, \tau, \mathscr{F}) \in \mathcal{C}(Y') \times_{\mathcal{C}(X')} \mathcal{C}(X)$ it holds that $S(\mathscr{N}, \tau, \mathscr{F}) \in \mathcal{C}(Y)$. Furthermore, these functors are an adjoint equivalence of categories

$$\mathcal{C}(Y) \underset{S}{\overset{T}{\rightleftarrows}} \mathcal{C}(Y') \times_{\mathcal{C}(X')} \mathcal{C}(X).$$

Moreover, for any quasicoherent $\mathcal{O}_Y$-module $\mathscr{G}$ it holds that $\mathscr{G} \in \mathcal{C}(Y)$ if and only if $u^*\mathscr{G} \in \mathcal{C}(Y')$ and $f^*\mathscr{G} \in \mathcal{C}(X)$.

Proof. Do the same construction as in the proof of Theorem 2.18 and just apply Theorem 2.13 to the affine squares instead. For part (*iii*) note that according to Lemma 1.24 a quasicoherent sheaf is finite locally free if and only if it is flat and of finite presentation. □

One could of course also use Proposition 2.19 in this proof, but since the affine version of the proposition is used in the proof of Theorem 2.13 this is not necessary.

Chapter 3

Pinching Azumaya algebras

In this chapter we will show how to apply the theorems of the last chapter to Azumaya algebras and then use this results to prove the second main theorem. We also give a short application.

3.1 Pinching Azumaya algebras

Let $f : X \to Y$ be a morphism of schemes. Given an Azumaya algebra $\mathscr{A}$ on X, we ask whether there exist an Azumaya algebra $\mathscr{B}$ on Y with $f^*\mathscr{B} \simeq \mathscr{A}$?

If f is flat and surjective, Lemma 1.32 gives an answer to this question if we can find some quasicoherent $\mathcal{O}_Y$-algebra $\mathscr{B}$ with $f^*\mathscr{B} = \mathscr{A}$. In general, this question is quite hard to answer. The trouble lies in constructing a locally free sheaf $\mathscr{B}$ with an algebra structure such that $f^*\mathscr{B} = \mathscr{A}$. Lemma 1.31 then tells us that such a $\mathscr{B}$ is an Azumaya algebra, at least when f is surjective. Unfortunately, it is quite possible that we find some algebra with $f^*\mathscr{B} \simeq \mathscr{A}$ which is not an Azumaya algebra. This can be seen in the affine case.

We have already given an example. In Section 1.3 we saw that $A_{a,b} = \mathbb{Z}[i,j,k]/(i^2 - a, j^2 - b, ij - k, ji + k)$ is not an Azumaya algebra on $\mathbb{Z}$ but becomes one after base change to $\mathbb{Z}[\frac{1}{n}]$ for certain n. $A_{a,b}$ also becomes a quaternion algebra after base change to $\mathbb{F}_p$, with, $p \nmid a$, $p \nmid b$ and $p \neq 2$.

Over $\mathbb{Z}$, the module $\mathbb{Z}/n\mathbb{Z}$ has a canonical algebra structure but it is not locally free. The module vanishes after base change to $\mathbb{Q}$. For $n = p$ we get a free module after base change to $\mathbb{F}_p$.

We want to "pinch" Azumaya algebras. Our aim is to construct an adjoint equivalence of categories for Azumaya algebras like in Theorem 2.20. The construction of the functors runs along the same lines as the construction in

Section 2.5. Let

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y, \end{tikzcd} \tag{3.1.1}$$

be a cartesian and cocartesian square of schemes, with f affine and u a closed immersion, like in the setting of Theorem 2.18. Let S and T denote the functors which are the adjoint equivalence of categories from the theorem

$$\mathrm{Loc}(Y) \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{Loc}(Y') \times_{\mathrm{Loc}(X')} \mathrm{Loc}(X).$$

Here $\mathrm{Loc}(X)$ denotes the category of finite locally free sheaves.

We denote the category of Azumaya algebras with $\mathrm{Az}(X)$. The pullback of an Azumaya algebra is an Azumaya algebra (Lemma 1.30). We can again define the fiber product category through pullbacks

$$\begin{tikzcd} \mathrm{Az}(X') & & \mathrm{Az}(X) \arrow[ll, "w^*"'] \\ \mathrm{Az}(Y') \arrow[u, "g^*"] & \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X) \arrow[l] \arrow[ur] & \end{tikzcd}$$

The objects of the fiber product category are triples $(\mathscr{B}', \tau, \mathscr{A})$, where $\mathscr{B}' \in \mathrm{Az}(Y')$, $\mathscr{A} \in \mathrm{Az}(X)$ and $\tau : g^*\mathscr{B}' \to w^*\mathscr{A}$ is an isomorphism.

The equality $h = ug = fw$ induces an isomorphism of functors

$$\sigma : g^*u^* \xrightarrow{\simeq} w^*f^*.$$

So for each Azumaya algebra $\mathscr{B}$ on Y we have an isomorphism

$$\sigma_{\mathscr{B}} : g^*u^*\mathscr{A} \to w^*f^*\mathscr{B}.$$

The universal property of fiber products, together with this data, gives a covariant functor

$$G : \mathrm{Az}(Y) \longrightarrow \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X),$$

$$\mathscr{B} \longmapsto (u^*\mathscr{B}, \sigma_{\mathscr{B}}, f^*\mathscr{B}).$$

Note that every Azumaya algebra is also a finite locally free sheave. And we have constructed G in such a way that $G(\mathscr{B}) = T(\mathscr{B})$.

It is a bit harder do define a functor in the other direction. We wish do define a functor

$$F : \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X) \longrightarrow \mathrm{Az}(Y).$$

Let $(\mathscr{B}', \tau, \mathscr{A}) \in \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X)$. We can view $(\mathscr{B}', \tau, \mathscr{A})$ as an element in $\mathrm{Loc}(Y') \times_{\mathrm{Loc}(X')} \mathrm{Loc}(X)$. Then

$$\mathscr{B} = u_*\mathscr{B}' \times_{h_*\tau} f_*\mathscr{A} = S(\mathscr{B}', \tau, \mathscr{A})$$

is a finite locally free sheaf.

We equip $\mathscr{B}$ with the structure of an Azumaya algebra. From the construction of S, we remember, that the pushforward of a quasicoherent sheaf along an affine morphism is exact, since $R^i f_*\mathscr{F} = 0$, for all $i > 0$ ([5] Tag 01XC). This implies that $u_*\mathscr{B}'$ and $f_*\mathscr{A}$ are both $\mathcal{O}_Y$-algebras, and $h_*\tau$ is an isomorphism of $\mathcal{O}_Y$-algebras between them. So $u_*\mathscr{B}' \times_{h_*\tau} f_*\mathscr{A}$ can be viewed as a fiber product of $\mathcal{O}_Y$-algebras, with the induced algebra structure. This gives us a canonical way to equip $\mathscr{B}$ with an $\mathcal{O}_Y$-algebra structure.

Lemma 3.1. *This $\mathcal{O}_Y$-algebra $\mathscr{B}$ is an Azumaya algebra.*

Proof. We know that $\mathscr{B}$ is finite locally free. By Theorem 2.20 we have $T(\mathscr{B}) \simeq (\mathscr{B}', \tau, \mathscr{A}))$ as locally free sheaves. And thus $u^*\mathscr{B} \simeq \mathscr{B}'$ and $f^*\mathscr{B} \simeq \mathscr{A}$. No,w $\mathscr{B}'$ and $\mathscr{A}$ are Azumaya algebras and the algebra structure on $\mathscr{B}$ was constructed in such a way that this isomorphisms respect it. This implies that $u^*\mathscr{B}$ and $f^*\mathscr{B}$ are Azumaya algebras.

To show that $\mathscr{B}$ is an Azumaya algebra we have to show that the stalk $\mathscr{B}_y$ is an Azumaya algebra over the local Ring $\mathcal{O}_{Y,y}$, for every $y \in Y$ (Proposition 1.25). If y is in the image of the closed immersion $y \in u(Y')$, there exist an $y' \in Y'$ with $u(y') = y$. The fact that $u^*\mathscr{B}_{y'}$ is an Azumaya algebra over $\mathcal{O}_{Y',y'}$ implies that $\mathscr{B}_y$ is an Azumaya algebra over the local ring $\mathcal{O}_{Y,y}$, by the same argument as used in the proof of Lemma 1.31. If $y \notin u(Y')$ there exist an $x \in X$ with $f(x) = y$, since the diagram is cocartesian. Since $f^*\mathscr{B}_x$ is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$ the same has to be true for $\mathscr{B}_y$ over the local ring $\mathcal{O}_{Y,y}$, again by the same argument as used in the proof of Lemma 1.31. □

We set $F(\mathscr{B}', \tau, \mathscr{A}) = \mathscr{B}$. We then have $F(\mathscr{B}', \tau, \mathscr{A}) = S(\mathscr{B}', \tau, \mathscr{A})$, if we view $\mathscr{B}'$ and $\mathscr{A}$ as locally free sheaves on the right side.

Now we have defined two functors F and G. We have

$$FG(\mathscr{B}) = ST(\mathscr{B}) \quad \text{and} \quad GF(\mathscr{B}', \tau, \mathscr{A}) = TS(\mathscr{B}', \tau, \mathscr{A}),$$

as an equality of finite locally free sheaves. By Theorem 2.18 we have isomorphisms of finite locally free sheaves $ST(\mathscr{B}) \simeq \mathscr{B}$ and $TS(\mathscr{B}', \tau, \mathscr{A}) \simeq (\mathscr{B}', \tau, \mathscr{A})$. From the way we have constructed F and G, we obtain isomorphisms of Azumaya algebras $\mathscr{B} \simeq FG(\mathscr{B})$ and $GF(\mathscr{B}', \tau, \mathscr{A}) \simeq (\mathscr{B}', \tau, \mathscr{A})$. These isomorphism in turn determine unit and counit, which are isomorphism of functors. We have shown:

Theorem 3.2. *In the setting above the functors F and G are an adjoint equivalence of categories*

$$\mathrm{Az}(Y) \underset{F}{\overset{G}{\rightleftarrows}} \mathrm{Az}(Y') \times_{\mathrm{Az}(X')} \mathrm{Az}(X).$$

Furthermore, a quasicoherent $\mathcal{O}_Y$-algebra $\mathscr{B}$ is an Azumaya algebra on Y if and only if $u^*\mathscr{B} \in \mathrm{Az}(Y')$ and $f^*\mathscr{B} \in \mathrm{Az}(X)$.

Basically, we have used Ferrand's results to construct an adjoint equivalence of categories for finite locally free sheaves that are also algebras. The question whether such an algebra is Azumaya can then be answered locally. Here the fact that the diagram is cocartesian ensures that every $y \in Y$ is an the image of u or f.

This theorem allows us to reduce our motivational question. Given some Azumaya algebra $\mathscr{A}$ on X the question whether there exist an Azumaya algebra $\mathscr{B}$ on Y with $f^*\mathscr{B} = \mathscr{A}$ can be reduced to the question whether there is some Azumaya algebra $\mathscr{B}'$ on Y' with $g^*\mathscr{B}' \simeq w^*\mathscr{A}$. In fact this property describes all Azumaya algebras on X, that are the pullback of some Azumaya algebra on Y.

3.2 Pinching along finite morphisms

Let $f : X \to Y$ be finite modification in finitely many closed points, in the sense of Definition 2.15. We fix dense open subsets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$, $f_{|U} : U \to V$ is an isomorphism and $Y' = X \setminus U$ consists of only finitely many closed points. We take the conductor ideal $\mathcal{C} = \mathrm{Ann}_{\mathcal{O}_Y}(f_*\mathcal{O}_X/\mathcal{O}_Y)$ to construct a conductor square

$$\begin{tikzcd} X' \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y. \end{tikzcd}$$

like in (2.4.2). So g is also finite and u and w are closed embeddings. By Lemma 2.16 this cartesian square is also cocartesian. This allows us to apply Theorem 3.2 to it.

Now, f can be written as a finite composition of finite modifications in one closed point. So we can do the construction for each part of this composition separately. We can assume that Y' consists of only one closed point. Furthermore, we can restrict Y to the connected component of this point. So we can assume that Y is connected. Since f is finite we can then assume that X has only finitely many connected components.

We purposely do not assume that X is also connected. For example, let Y be the intersection of two curves, or more generally two schemes, in one closed point $Y' = \{y\}$ and f the resolution of this singularity. Then X has at least two connected components.

Now the fact that Y' consists of only one closed point implies that we have $Y' = \mathrm{Spec}(S)$, where S is a local Artin ring, that is, a ring with only one prime ideal, namely the maximal ideal $\mathfrak{m}$. Of course, every field is a local Artin ring. The most prominent example, which is not a field, is the ring of dual numbers $k[\epsilon]/(\epsilon^2)$ with maximal ideal (ϵ). This can be generalized, as the ring $k[\epsilon]/(\epsilon^n)$ is a local Artin ring. Other examples include $k[x,y]/(x^2, y^3, xy^2)$ with maximal ideal (x,y), or $k[T^2,T^3]/(T^4)$ with maximal ideal (T^2,T^3).

Since g is finite, $X' = \mathrm{Spec}(R)$ has to be the spectrum of a ring R, which is finitely generated, not only as an S-algebra, but also as a S-module. And every finite module over an Artin ring is Artin itself (see [54] Chapter I Theorem 3.1) so R is an Artin ring, i.e. there is no infinite descending chain of ideals in R. Though R is not necessarily local.

Thus R has finitely many prime ideals $\mathfrak{m}_i$ and all of those are maximal ideals (see [3] Proposition 8.1 and Proposition 8.3). The structure theorem for Artin rings ([3] Theorem 8.7) tells us that an Artin ring R is uniquely, up to isomorphism, a finite direct product of local Artin rings R_i. It is

$$R = \prod_{i=1}^{l} R_i.$$

So for examples of Artin rings we simply take products of the examples for local Artin rings. This also shows that the Zariski topology on an Artin ring agrees with the discrete topology. Set $U_i = \mathrm{Spec}(R_i)$, then $\mathrm{Spec}(R) = \bigsqcup_{i=1}^{l} U_i$ and this is a finite and disjoint open cover that consist of local rings.

The structure theorem, together with the fact that every finite algebra over a local Artin ring is an Artin ring, has another consequence for us. It shows that every finite algebra over a local Artin ring is a direct product of local rings. This property implies that a local Artin ring is a Henselian local ring ([55] Chapter I Theorem 4.2). For such rings the Brauer group is wholly described by the Brauer group of the residue field, as we have shown in Corollary 1.40. Thus

$$\mathrm{Br}(S) = \mathrm{Br}(k) \quad \text{and} \quad \mathrm{Br}(R) = \bigoplus_{i=1}^{l} \mathrm{Br}(R_i) = \bigoplus_{i=1}^{l} \mathrm{Br}(k_i),$$

with $k = S/\mathfrak{m}$ and $k_i = R/\mathfrak{m}_i$. Note that the k_i are finite field extensions of k, since g is finite. We remark that we could also have used that $\mathrm{Br}(R) = \mathrm{Br}(R/I)$ for a nilpotent ideal ([16] Theorem 1). If we set I as the nilradical of R, the Chinese remainder theorem (see [48] Chapter II Theorem 2.1.) gives $R/I = \prod_{i=1}^{l} R_i/\mathfrak{m}_i = \prod_{i=1}^{l} k_i$.

Theorem 3.3. *Let $f : X \to Y$ be finite modification in finitely many closed points. A cohomological Brauer class $\beta \in \mathrm{Br}'(Y)$ is represented by an Azumaya algebra on Y if and only if the pullback $f^*\beta$ is represented by an Azumaya algebra on X.*

Proof. As we have discussed, we can assume that f gives a cartesian and cocartesian square

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r, "w"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(S) \arrow[r, "u"'] & Y. \end{tikzcd} \tag{3.2.1}$$

where $u : \mathrm{Spec}(S) \to Y$ determines a closed point, so S is a local Artin ring. Furthermore, Y is connected, X has only finitely many connected components, and R is an Artin ring, with $R = \prod_{i=1}^{l} R_i$ fore some local Artin rings R_i. Set $h = u \circ g = f \circ w$.

The Brauer maps δ is injective. By Lemma 1.39 we have a commutative diagram

$$\begin{tikzcd} \mathrm{Br}(X) \arrow[r, "\delta_X"] & \mathrm{Br}'(X) \\ \mathrm{Br}(Y) \arrow[r, "\delta_Y"'] \arrow[u, "f^*"] & \mathrm{Br}'(Y). \arrow[u, "f^*"'] \end{tikzcd} \tag{3.2.2}$$

If β is represented by an Azumaya algebra the commutativity of the diagram ensures that the same holds for $f^*\beta$. The problem lies in the other direction.

So let $\beta \in \mathrm{Br}'(Y)$ be a cohomological Brauer class whose pullback $f^*\beta \in \mathrm{Br}'(X)$ is represented by an Azumaya algebra $\mathscr{A}$ on X.

If we replace X by $Y' = \mathrm{Spec}(S)$, and the maps accordingly, in the commutative square (3.2.2), we receive another commutative square. Combined these give the following commutative square:

$$\begin{tikzcd} \mathrm{Br}(S) \oplus \mathrm{Br}(X) \arrow[r, "\delta_S \oplus \delta_X"] & \mathrm{Br}'(S) \oplus \mathrm{Br}'(X) \\ \mathrm{Br}(Y) \arrow[r, "\delta_Y"'] \arrow[u, "{(u^*, f^*)}"] & \mathrm{Br}'(Y). \arrow[u, "{(u^*, f^*)}"'] \end{tikzcd} \tag{3.2.3}$$

First, we show that the pullback map $(u^*, f^*) : \mathrm{Br}'(Y) \to \mathrm{Br}'(S) \oplus \mathrm{Br}'(X)$ is injective. This implies, that every morphism in the commutative square (3.2.3) is injective. Let $T \in \{X, \mathrm{Spec}(R), \mathrm{Spec}(S)\}$ and $\varphi : T \to Y$ the corresponding finite morphism $\varphi \in \{f, u, h\}$. From Appendix A we know that the pullback map $\varphi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(T)$ is induced by

$$\varphi^* : H^2_{et}(Y, \mathbb{G}_{m,Y}) \xrightarrow{\varphi_1} H^2_{et}(Y, \varphi_*\mathbb{G}_{m,T}) \xrightarrow{\varphi_2} H^2_{et}(T, \mathbb{G}_{m,T}).$$

We show that

$$\begin{aligned} (u^*, f^*) : H^2_{et}(Y, \mathbb{G}_{m,Y}) &\xrightarrow{(u_1,f_1)} H^2_{et}(Y, \varphi_*\mathbb{G}_{m,S}) \oplus H^2_{et}(Y, \varphi_*\mathbb{G}_{m,X}) \\ &\xrightarrow{(u_2,f_2)} H^2_{et}(S, \mathbb{G}_{m,S}) \oplus H^2_{et}(X, \mathbb{G}_{m,X}) \end{aligned}$$

is injective, by showing that (u_1, f_1) and (u_2, f_2) are injective.

The Leray-Serre spectral sequence (see [5] Tag 03QC) gives the five term exact sequence

$$\begin{aligned} 0 \longrightarrow H^1_{et}(Y, \varphi_*\mathbb{G}_{m,T}) \longrightarrow H^1_{et}(T, \mathbb{G}_{m,T}) \longrightarrow H^0_{et}(Y, R^1\varphi_*\mathbb{G}_{m,T}) \\ \longrightarrow H^2_{et}(Y, \varphi_*\mathbb{G}_{m,T}) \xrightarrow{\varphi_2} H^2_{et}(T, \mathbb{G}_{m,T}). \end{aligned}$$

Now, higher direct images of a finite morphism of schemes vanish ([5] Tag 03QP), so $R^1\varphi_*\mathbb{G}_{m,T} = 0$, and thus also $H^0_{et}(Y, R^1\varphi_*\mathbb{G}_{m,T}) = 0$. Now the exactness of the sequence implies first, that $H^1_{et}(Y, \varphi_*\mathbb{G}_{m,T}) \simeq H^1_{et}(T, \mathbb{G}_{m,T})$, and second, that φ_2 is injective. In particular, for $T = Y$ and $\varphi = u$ this shows that u_2 is injective, and for $T = X$ and $\varphi = f$ also that f_2 is injective. Thus (u_2, f_2) is injective.

The cartesian and cocartesian square of schemes (3.2.1) gives us a short exact sequence like (2.4.1):

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow u_*\mathcal{O}_S \oplus f_*\mathcal{O}_X \longrightarrow h_*\mathcal{O}_R \longrightarrow 0.$$

This exact sequence induces a short exact sequence

$$1 \longrightarrow \mathcal{O}_Y^\times \longrightarrow u_*\mathcal{O}_S^\times \oplus f_*\mathcal{O}_X^\times \longrightarrow h_*\mathcal{O}_R^\times \longrightarrow 1.$$

Now, we could use pullback via a morphism of sites to obtain a short exact sequence of étale sheaves. Instead, we argue more concretely that base change by any étale morphism $U \to Y$ is exact, since étale morphisms are in particular flat. So the exact sequence of Zariski sheaves above induces an exact sequence of étale sheaves and, by definition of $\mathbb{G}_m$, this is the short exact sequence

$$1 \longrightarrow \mathbb{G}_{m,Y} \longrightarrow u_*\mathbb{G}_{m,S} \oplus f_*\mathbb{G}_{m,X} \longrightarrow h_*\mathbb{G}_{m,R} \longrightarrow 1.$$

Taking cohomology we obtain an exact sequence

$$H^1_{et}(Y, h_*\mathbb{G}_{m,R}) \longrightarrow H^2_{et}(Y, \mathbb{G}_{m,Y}) \longrightarrow H^2_{et}(Y, u_*\mathbb{G}_{m,S}) \oplus H^2_{et}(Y, f_*\mathbb{G}_{m,X}).$$

Using the five term exact sequence above for $h : \mathrm{Spec}(R) \to Y$ we see that $H^1_{et}(Y, h_*\mathbb{G}_{m,R}) \simeq H^1_{et}(\mathrm{Spec}(R), \mathbb{G}_{m,R})$. Now $H^1_{et}(\mathrm{Spec}(R), \mathbb{G}_{m,R}) = \mathrm{Pic}(R)$ and for an Artin ring is $\mathrm{Pic}(R) = 0$. This shows that

$$(u_1, f_1) : H^2_{et}(Y, \mathbb{G}_{m,Y}) \longrightarrow H^2_{et}(Y, u_*\mathbb{G}_{m,S}) \oplus H^2_{et}(Y, f_*\mathbb{G}_{m,X})$$

is injective.

Now by assumption we have an Azumaya algebra $\mathscr{A}$ on X with $\delta_X([\mathscr{A}]) = f^*\beta$. Since $\mathrm{Spec}(S)$ is affine there exist an Azumaya algebra $\mathscr{B}'$ on it with $\delta_S([\mathscr{B}']) = u^*\beta$. Then $(u^*\beta, f^*\beta) \in \mathrm{Br}'(S) \oplus \mathrm{Br}'(X)$ is in the image of $\delta_S \oplus \delta_X$ in the commutative diagram (3.2.3). Since all maps in the diagram are injective it is enough to construct an Azumaya algebra $\mathscr{B}$ on Y with $[u^*\mathscr{B}] = [\mathscr{B}']$ and $[f^*\mathscr{B}] = [\mathscr{A}]$. For then $\delta([\mathscr{B}]) = \beta$.

We can apply Theorem 3.2 to the the cartesian and cocartesian square of schemes (3.2.1). Unfortunately, it could be that $g^*\mathscr{B}' \not\simeq w^*\mathscr{A}$. We will modify $\mathscr{B}'$ and $\mathscr{A}$, without changing their Brauer class, in such a way that their pullbacks to $\mathrm{Spec}(R)$ become isomorphic and we can apply Theorem 3.2.

If $\mathscr{A}$ does not have constant rank n replace it by an equivalent Azumaya algebra that does. This is possible thanks to Lemma 1.34. Since $\mathscr{B}'$ is defined over the local Artin ring S it has constant rank m. The rank of an Azumaya algebra is a square, so there always exist a trivial Azumaya algebra $End(\bigoplus \mathcal{O}_X)$ of the same rank. We replace $\mathscr{A}$ and $\mathscr{B}'$ by

$$\mathscr{A} \otimes End(\bigoplus_{j=1}^{\sqrt{m}} \mathcal{O}_X) \quad \text{and} \quad \mathscr{B}' \otimes End(\bigoplus_{j=1}^{\sqrt{n}} \mathcal{O}_{Y'})$$

respectively. This does not change their Brauer class, but they now both have the same rank, namely nm. So $g^*\mathscr{B}'$ and $w^*\mathscr{A}$ are also of the same rank nm. By construction they still represent the same cohomological Brauer class in $\mathrm{Br}'(X')$, so they are in the same Brauer class in $\mathrm{Br}(X')$.

Over the Artin ring R this is enough to show that they are isomorphic. Write B' and A for Azumaya algebras of rank nm over R, that correspond to the pullbacks $g^*\mathscr{B}'$ respectively $w^*\mathscr{A}$. Now $[B'] = [A]$ have the same Brauer class in $\mathrm{Br}(R) = \bigoplus_{i=1}^{l} \mathrm{Br}(k_i)$. Then $k_i \otimes B'$ and $k_i \otimes A$ also have the same Brauer class in $\mathrm{Br}(k_i)$. And, they have the same rank. For central simple algebras this implies that they are isomorphic, which we explained in the discussion following Wedderburn's theorem (Theorem 1.6). But if they are isomorphic over each k_i then Lemma 1.15 tells us that they are also isomorphic over R_i, so $R_i \otimes B' \simeq R_i \otimes A$. Since $R = \prod_{i=1}^{l} R_i$ this shows $B' \simeq A$.

We have constructed an isomorphism $\tau : g^*\mathscr{B}' \simeq w^*\mathscr{A}$, so $(\mathscr{B}', \tau, \mathscr{A})$ is an element in the fiber product category. With Theorem 3.2 we set $\mathscr{B} = F(\mathscr{B}', \tau, \mathscr{A})$ and this Azumaya algebra satisfies $u^*\mathscr{B} \simeq \mathscr{B}'$ and $f^*\mathscr{B} \simeq \mathscr{A}$.

□

The proof also gives a concrete way of constructing an Azumaya algebra $\mathscr{B}$, that represents a cohomological Brauer class $\beta \in \mathrm{Br}'(Y)$, from an Azumaya algebra $\mathscr{A}$, that represents $f^*\beta \in \mathrm{Br}'(X)$.

3.3 Application: S_2-ization

We give a short recollection of the notion of depth and the meaning of Serre's criterion. These notions can be found in most books on commutative algebra (see for example [54] Chapter 6, [19] Section 18) or in algebraic geometry textbooks ([35] pp. 184-187, [26] Appendix B.12).

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M an R-module.

Definition 3.4. A finite sequence of elements $(r_1, \ldots, r_d)$ of elements $r_i \in R$ is called *regular* or *regular sequence* if for all $i = 1, \ldots, d$ the image of r_i in $R/(r_1, \ldots, r_{i-1})R$ is regular, i.e. not a zero divisor, and $R/(r_1, \ldots, r_d)R$ is non zero. We define the *depth* of R as $\mathrm{depth}(R) = d$, where d is the maximal length of a regular sequence.

Since R is Noetherian $\mathrm{depth}(R)$ is well defined. All maximal regular sequences have the same length. Note that the elements in a regular sequence are always contained in the maximal ideal $\mathfrak{m}$.

It holds $\mathrm{depth}(R) \leq \dim(R)$. An arbitrary noetherian ring A is called *Cohen-Macaulay* if $\mathrm{depth}(A_\mathfrak{p}) = \dim(A_\mathfrak{p})$ for all ideals $\mathfrak{p} \subset A$. One says that locally Noetherian scheme is *Cohen-Macaulay* if all of it local rings are Cohen-Macaulay. We use a slightly different condition.

A Noetherian ring A fulfills *Serre's Criterion* (S_n) if

$$\mathrm{depth}(A_\mathfrak{p}) \geq \min(\mathrm{ht}(\mathfrak{p}), n),$$

for all prime ideals $\mathfrak{p} \subset A$. A locally Noetherian scheme X is (S_n) if all local rings $\mathcal{O}_{X,x}$ are. If $\dim(X) = n$ Serre's Criterion (S_n) is equivalent to the scheme being Cohen-Macaulay.

On noted application of Serre's criterion is presented in the following.

Theorem 3.5. *(Serre's normality criterion)([28] Théorème 5.8.6) A noetherian ring A is normal if and only if*

- (R_1) $A_{\mathfrak{p}}$ *is regular for all prime ideal* $\mathfrak{p}$ *with* $ht(\mathfrak{p}) \leq 1$
- (S_2) $depth(A_{\mathfrak{p}}) \geq min(ht(\mathfrak{p}), 2)$*, for all prime ideals* $\mathfrak{p}$*.*

Lemma 3.6. *([5] Tag 02OL) Let Y be a locally noetherian scheme. The set of ideals $\{\mathscr{I} \subset \mathcal{O}_Y \mid \operatorname{supp}(\mathscr{I})$ is nowhere dense in $Y\}$ has a maximal element $\mathscr{I}$. This ideal defines a closed subscheme Y_0 with the following properties:*

(i) *As topological spaces is $Y = Y_0$;*

(ii) *Y_0 has no embedded points;*

(iii) *There exists a dense open $U \subset Y$ such that U is dense in Y_0 and $\mathcal{O}_{Y_0|U} \simeq \mathcal{O}_{Y|U}$.*

Since Y_0 has no embedded points, it fulfills Serre's criterion (S_1). We call Y_0 the S_1*-ization* of Y.

If Y is a Noetherian scheme the ideal $\mathscr{I}$ defining an (S_1)-ization is nilpotent. Since Y is Noetherian we can check this locally. So let $Y = \operatorname{Spec}(R)$ with a Noetherian ring R, and $I \subset R$ the ideal corresponding to $\mathscr{I}$. For $f \in I$ the prime ideal $\mathfrak{p} = \operatorname{ann}(f)$ is an embedded prime, with $\operatorname{ht}\mathfrak{p} \geq 1$. So, for every generic point η_i there exists an $h \in \mathfrak{p}$ with $h \notin \eta_i$. It holds that $hf = 0 \in \eta_i$ so $f \in \eta_i$. Thus the element f is contained in every generic point, which implies $f \in \mathfrak{p}$ and thus $f^2 = 0$.

An affine scheme that is (S_1) but not (S_2), is for example given by the rings

$$\{f \in \mathbb{C}[X, Y] \mid f(0,0) = f(0,1))\}$$

and

$$k[u, v, w, z]/(u^2w - v^2, u^3z - v^3, w^3 - z^2) \simeq k[x, xy, y^2, y^3].$$

If Y is a surface, that is, a 2-dimensional scheme of finite type over some field k, we can go further. First, note that S_1-ization does not change the cohomological Brauer group. The closed embedding $Y_0 \subset Y$ gives an exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y_0} \longrightarrow 0.$$

Like in the proof of Theorem 3.3 this induces an exact sequence of étale sheaves

$$1 \longrightarrow \mathscr{I} \longrightarrow \mathbb{G}_{m,Y} \longrightarrow \mathbb{G}_{m,Y_0} \longrightarrow 1.$$

From this we get a long exact étale cohomology sequence

$$H^2_{et}(Y_0, \mathscr{I}) \longrightarrow H^2_{et}(Y, \mathbb{G}_{m,Y}) \longrightarrow H^2_{et}(Y_0, \mathbb{G}_{m,Y_0}) \longrightarrow H^3_{et}(Y_0, \mathscr{I}).$$

Now $\mathscr{I}$ can be at most one dimensional, by construction. Thus we have $H^2_{et}(Y_0, \mathscr{I}) = 0$ and $H^3_{et}(Y_0, \mathscr{I}) = 0$. This implies $\mathrm{Br}'(Y) = \mathrm{Br}'(Y_0)$.

We define an S_2*-ization* $f : X \to Y_0$ where X is (S_2) and f shall be a finite modification in finitely many closed points. Set

$$X = \mathrm{Spec}(\mathscr{H}^0_{X/Z^{(2)}}(\mathcal{O}_{Y_0})).$$

The general definition of the right hand side can be found in [28] Chapter IV §5.9. There $\mathscr{H}^0_{X/Z}(\mathscr{F})$ is defined as a direct limit of an inductive system which is defined via a family open sets; this family depends on a subset $Z \subset X$ that is stable under specialization.

For our purpose it is enough to know that $\mathscr{H}^0_{X/Z^{(2)}}(\mathcal{O}_{Y_0})$ is a quasicoherent $\mathcal{O}_{Y_0}$-algebra whose affine Y_0-scheme $f_*\mathcal{O}_X$ is determined by

$$f_*(\mathcal{O}_X)_y = \bigcap_{\substack{x \in \mathrm{Spec}(\mathcal{O}_{Y_0,y}) \\ ht(x)=1}} \mathcal{O}_{Y_0,x}.$$

The intersection takes place in the ring of total fractions of $\mathcal{O}_{Y_0,y}$. Since Y_0 is also of finite type over k, [28] Proposition 5.11.1 ensures that $\mathscr{H}^0_{X/Z^{(2)}}(\mathcal{O}_{Y_0})$ is coherent and f is finite. In this case, [28] Proposition 5.10.16 tells us, that X is (S_2), that there exist an open subscheme $V \subset Y_0$ such that the restriction of f to $f^{-1}(V) \to V$ is an isomorphism, that this V is (S_2), and that $\mathrm{codim}(Y_0 - V, Y_0) \geq 2$.

The S_2-ization $f : X \to Y_0$ is a finite modification in the finitely many closed points $y \in Y_0 \setminus V$ where $\mathcal{O}_{Y_0,y}$ is not (S_2). This allows us to apply Theorem 3.3. Furthermore, we have $\mathrm{Br}(Y) = \mathrm{Br}(Y_0)$ for the S_1-ization. So whenever we want to construct an Azumaya algebra that represents some cohomological Brauer class $\beta \in \mathrm{Br}(Y)$ we can instead construct one that represents $f^*\beta$ on the S_2-ization X. This for example allows one to use methods from [36]. Furthermore, as a short corollary we see the following:

Corollary 3.7. *Let Y be a separated surface that is regular in codimension one and whose normalization is geometrically normal. Then* $\mathrm{Br}(Y) = \mathrm{Br}'(Y)$*.*

Proof. Since Y is already regular in codimension one, it is S_1 and the S_2-ization $f : X \to Y$ is the normalization. We already mentioned in the discussion of the Brauer map that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for a separated geometrically normal scheme by [63] Theorem 3.1. Then the result follows from Theorem 3.3 $\square$

Appendix A

Cohomology of group schemes

We repeat some facts about group schemes and cohomology. Most statements can be found in the book of Milne [55] (in particular Chapter II and III). Other sources are the books by Tamme [65], Fu [22], or Colliot-Thélène and Skorobogatov [12]. For details on non abelian sheaves and their cohomology we refer to the book of Giraud [25].

Let X be a scheme. Milne works on the big flat site X_{fl}, by which he means the fppf site. Though this implies that results hold over the big étale site X_{Et}, with which we will work with initially. We then further restrict to the small étale site X_{et}. The small Zariski site is denoted by X_{Zar}. We write X_E for an arbitrary flat, étale or Zariski site. A covering $\mathscr{U} = (U_i \to X)$ is always a covering with respect to some site.

Let Y be a scheme with site Y_E. We want to work with sheaves of groups on Y, i.e. functors from the category of Y-schemes to the category of groups. On way to check if some presheaf of groups is a sheaf is to check if it is representable by a group scheme G on Y.

Lets give some examples. Actually these examples will be the only group schemes that are of relevance to us. With U we denote some Y-scheme, that is an object $U \to Y$ in the site.

- The sheaf $\mathbb{G}_a$ is defined by $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$. It is representable by $\mathrm{Spec}(\mathbb{Z}[T])$.
- The sheaf $\mathbb{G}_m$ is defined by $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^\times$. It is representable by $\mathrm{Spec}(\mathbb{Z}[T, T^{-1}])$.
- The sheaf GL_n is defined by

$$GL_n(U) = GL_n(\Gamma(U, \mathcal{O}_U)) = M_n(\Gamma(U, \mathcal{O}_U))^\times.$$

It is representable by

$$\operatorname{Spec}\left(\frac{\mathbb{Z}[T_{11},\dots,T_{nn},T]}{(T\det(T_{ij})-1)}\right).$$

- The sheaf PGL_n is defined by $PGL_n(U) = \operatorname{Aut}(M_n(\mathcal{O}_U))$, here the automorphism group is the one of $M_n(\mathcal{O}_U)$ as an $\mathcal{O}_U$-algebra, such an algebra automorphism can also be seen as an endomorphism of $M_n(\mathcal{O}_U)$ as an $\mathcal{O}_U$-module. We can view PGL_n as a subfunctor of M_{n^2}. The condition that some endomorphism is an automorphism of algebras is described by polynomials, hence a scheme representing PGL_n has to be a closed subscheme of the one representing M_{n^2}. Let $S \subset \mathbb{Z}[T_{11},\dots,T_{nn},\det(T_{ij})^{-1}]$ be the subring of elements of degree zero, then PGL_n is represented by $\operatorname{Spec}(S)$ (see [55] p.142).

Note that we at first defined presheaves above. The fact that these are actually sheaves is then shown by giving a representation.

We defined PGL_n as functor $(Sch) \to (Gp)$ such that $PGL_n(U) = \operatorname{Aut}(M_n(\mathcal{O}_U))$. Usually the sheaf PGL_n is defined different. For a field it is $\mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^\times$, then Skolem-Noether (Proposition 1.21) shows $\operatorname{Aut}(M_n(k)) = \mathrm{PGL}_n(k)$. For rings the quotient $GL_n(R)/R^\times$ does not necessarily define a sheaf but only a presheaf. The same of course holds then for $U \mapsto GL_n(U)/\mathcal{O}_U^\times$ One usually defines PGL_n as the sheafification of this presheaf. This gives an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1.$$

Lemma 1.36 then implies, that this more standard definition of PGL_n is isomorphic to the one we use.

The main reason why we initially work on a big site or alternative flat site is the following: Let Y be a scheme, with flat or big étale site. We denote this by Y_{big}. Let G a group scheme over Y, that is not necessarily commutative which defines a sheaf of groups $\mathscr{G}_Y$ on Y.

For a morphism of schemes $f : X \to Y$ the scheme X is an object in the site. This induces a morphism of sites $f : X_{big} \to Y_{big}$, and any covering $(U_i \to Y)$ gives a covering $(U_i \times X \to X)$. The group scheme G can be restricted to $G \times X \to X$ and so defines a group scheme on X_{big}, which in turn determines a sheaf of groups $\mathscr{G}_X$. In this situation the pullback functor f^* is exact and $f^*\mathscr{G}_Y = \mathscr{G}_X$. In fact, it simply is the restriction functor $f^*\mathscr{G}_Y = \mathscr{G}_{Y|X}$ and the restriction induces a map $\mathscr{G}_Y \to \mathscr{G}_X$.

On the other hand the small étale site is a subcategory of the big étale site; it is also a subcategory of the flat site, so we could use the flat site instead of the big étale site. There exist a morphism of sites $\pi : Y_{Et} \to Y_{et}$ defined by the identity on Y. In this case both π^* and π_* are exact. Then $\pi_* \mathscr{G}_Y$ is the sheaf on Y_{et} which is the restriction of $\mathscr{G}_Y$ to the small étale site. It is represented by G on Y_{et}. So let $U \to Y$ be étale over Y an object in the small étale site. Then $U \to Y$ is also an object in the big étale site and $\Gamma(U, \mathscr{G}_Y) \simeq \Gamma(U, \pi_* \mathscr{G}_Y)$. In general we will denote the sheaves on the small site also by $\mathscr{G}_Y$. Note that the morphism $\mathscr{G}_Y \to \mathscr{G}_X$ on the big étale site does restrict to a morphism on the small étale site.

If $\mathscr{G}_Y$ is a sheaf of commutative groups we can extend this to cohomology groups. For this short part we write $\mathscr{G}_{Y_{Et}}$ and $\mathscr{G}_{Y_{et}}$ if it is necessary to differentiate these sheaves on the big or small site. Let $f : X \to Y$ be a morphism of schemes. There the pullback induces a "pullback" of cohomology groups,

$$f^* : H^i_{Et}(Y, \mathscr{G}_{Y_{Et}}) \longrightarrow H^i_{Et}(Y, f_* f^* \mathscr{G}_{Y_{Et}}) \longrightarrow H^i_{Et}(X, f^* \mathscr{G}_{Y_{Et}}).$$

The first map is given by the adjunction $\mathscr{G}_{Y_{Et}} \to f_* f^* \mathscr{G}_{Y_{Et}}$. The second comes from the Leray spectral sequence:

$$H^p_{Et}(Y, R^q f_* \mathscr{G}_{X_{Et}}) \Rightarrow H^{p+q}_{Et}(X, \mathscr{G}_{X_{Et}}),$$

which exists for any continuous morphism of sites $f : X'_{Et} \to X_{Et}$ and any sheaf $\mathscr{F}'$ on X'_{Et} ([55] Chapter III Theorem 1.18). Since $f^* \mathscr{G}_{Y_{Et}} = \mathscr{G}_{X_{Et}}$ we have a pullback map

$$f^* : H^i_{Et}(Y, \mathscr{G}_{Y_{Et}}) \longrightarrow H^i_{Et}(Y, f_* \mathscr{G}_{X_{Et}}) \longrightarrow H^i_{Et}(X, \mathscr{G}_{X_{Et}}).$$

Let $\pi : Y_{Et} \to Y_{et}$ again be the morphism of sites defined by the identity on Y. There exist canonical isomorphisms

$$H^i_{et}(Y, \pi_* \mathscr{G}_{Y_{Et}}) \longrightarrow H^i_{Et}(Y, \mathscr{G}_{Y_{Et}}) \quad \text{and} \quad H^i_{et}(Y, \mathscr{G}_{Y_{et}}) \longrightarrow H^i_{Et}(Y, \pi^* \mathscr{G}_{Y_{et}})$$

for all $i \geq 0$ ([55] Chapter III Proposition 3.1). These isomorphisms allow us to replace the big étale site by the small étale site. The pullback for the big site induces a pullback map for the small site

$$f^* : H^i_{et}(Y, \mathscr{G}_{Y_{et}}) \longrightarrow H^i_{et}(Y, f_* \mathscr{G}_{X_{et}}) \longrightarrow H^i_{et}(X, \mathscr{G}_{X_{Et}}).$$

and we can stop worrying about the difference between $\mathscr{G}_{Y_{Et}}$ and $\mathscr{G}_{Y_{et}}$.

Now let G be a possible non abelian group scheme on X_{Et}, which defines a sheaf of groups $\mathscr{G}_X$. The cohomology theory we have used above is defined for abelian sheaves. So we need a different approach. We can at least construct first cohomology classes. The construction can be done for the big and small étale site, or even for the flat site. Since this is the case of most interest to us we will describe the construction for the small étale site. Let $\mathscr{U} = (U_i \to X)$ denote an étale covering. The construction is similar to the definition of Čech cohomology. A 1-*cocycle* for $\mathscr{U}$ with values in $\mathscr{G}_X$ is a family $(g_{ij})_{I\times I}$, with $g_{ij} \in \mathscr{G}_X(U_{ij})$ such that

$$(g_{ij} \mid U_{ijk})(g_{jk} \mid U_{ijk}) = (g_{ik} \mid U_{ijk}).$$

Let g and g' be two cocycles. They are *cohomologous* if there is a family $(h_i)_I$, with $h_i \in \mathscr{G}_X(U_i)$, such that $g'_{ij} = (h_i \mid U_{ij})g_{ij}(h_j \mid U_{ij})^{-1}$. This defines an equivalence relation. We write $\check{H}^1(\mathscr{U}/X, \mathscr{G}_X)$ for the set of cohomology classes. This set has a distinguished element $(1)_{I\times I}$, by which we mean the element $(g_{ij})_{I\times I}$, where all $g_{ij} = 1$. Define the set

$$\check{H}^1_{et}(X, \mathscr{G}_X) = \varinjlim \check{H}^1(\mathscr{U}/X, \mathscr{G}_X),$$

where the limit is taken over all étale coverings of X like in the construction of Čech cohomology (see [55] Chapter III §2). When $\mathscr{G}_X$ is abelian this definition gives the first Čech cohomology group, which is why the notation is the same. For an abelian sheaf we even have $\check{H}^1_{et}(X, \mathscr{G}_X) \simeq H^1_{et}(X, \mathscr{G}_X)$ ([55] Chapter III Corollary 2.10). In general though $\check{H}^1_{et}(X, \mathscr{G}_X)$ is not a group but a pointed set.

We define pullbacks $f^* : \check{H}^1_{et}(Y, \mathscr{G}_Y) \to \check{H}^1_{et}(X, \mathscr{G}_X)$ along a morphism of schemes $f : X \to Y$ as a pullback of a 1-cocycle. For a cover $(U_i \to Y)$ we get a cover $(U_i \times X \to X)$ and we have restriction maps $\mathscr{G}_Y(U_i) \to \mathscr{G}_X(U_i \times X)$. Two cohomologous 1-cocycles stay so under this maps, since we can also pull back the family (h_i) which defines the relation.

The next step is of course cohomology of a short exact sequences. A sequence of sheaf of groups

$$1 \longrightarrow \mathscr{G}_X \longrightarrow \mathscr{H}_X \longrightarrow \mathscr{K}_X \longrightarrow 1$$

is called *exact*, if $\mathscr{G}_X(U)$ is the kernel of the homomorphism $\mathscr{H}_X(U) \to \mathscr{K}_X(U)$ for every étale morphism $U \to X$, and every section $s \in \mathscr{K}_X(U)$ is locally liftable to a section of $\mathscr{H}_X$.

For such an exact sequence of groups there exists an associated sequence of pointed sets

$$1 \longrightarrow \mathscr{G}_X(X) \longrightarrow \mathscr{H}_X(X) \longrightarrow \mathscr{K}_X(X)$$

$$\xrightarrow{d} \check{H}^1_{et}(X, \mathscr{G}_X) \longrightarrow \check{H}^1_{et}(X, \mathscr{H}_X) \longrightarrow \check{H}^1_{et}(X, \mathscr{K}_X).$$

Let $k \in \mathscr{K}_X(X)$, we describe the image under d. Choose an étale covering $(U_i \to X)$ such that there exist $h_i \in \mathscr{H}_X(U_i)$ that are mapped to $k \mid U_i$ under $\mathscr{H}_X(U_i) \to \mathscr{K}_X(U_i)$. Then

$$d(k)_{ij} = (h_i \mid U_{ij})^{-1}(h_j \mid U_{ij})$$

(see [55] Chapter III Proposition 4.5 and [25] Chapter III §3.6).

In the case where $\mathscr{G}_X$ is an abelian sheaf and for every étale morphism $U \to X$ the group $\mathscr{G}_X(U)$ is mapped into the center of $\mathscr{H}_X(U)$, there exists a boundary map

$$d : \check{H}^1_{et}(X, \mathscr{K}_X) \longrightarrow H^2_{et}(X, \mathscr{G}_X), \tag{A.1}$$

which continues the exact sequence. This important construction is due to Giraud ([25] Chapter IV §3.4). Note that in general this construction is described in terms of torsors and gerbes.

If X is a quasicompact scheme where every finite subset is contained in an affine open set, we can use Čech cohomology to describe this boundary map. For example a quasiprojective scheme over some base ring has this property. Under the assumption we have $\check{H}^2_{et}(X, \mathscr{G}_X) \simeq H^2_{et}(X, \mathscr{G}_X)$, where the left side describes the second Čech cohomology group ([55] Chapter III Theorem 2.17). So we can attempt to describe the image of the boundary map by the class of a 2-cocycle.

Let $\gamma \in \check{H}^1_{et}(X, \mathscr{K}_X)$ be a class that is represented by a cocylce (k_{ij}) for the covering $(U_i \to X)$. Since every finite subset is contained in an affine open it is possible to apply a theorem of Artin on the joins of Henselian rings ([1] Theorem 3.4.(iii)). This allows us to refine the cover in such a way, that under the maps $\mathscr{H}_X(U_{ij}) \to \mathscr{K}_X(U_{ij})$, each k_{ij} is the image of some $h_{ij} \in \mathscr{H}_X(U_{ij})$ ([55] Chapter III Lemma 2.19). Then the image $d(\gamma)$ is given by the class of the 2-cocycle (g_{ijk}) with $g_{ijk} \in \mathscr{G}_X(U_{ijk})$ and

$$(g_{ijk}) = h_{jk}(h_{ik})^{-1}h_{ij}.$$

The boundary map d is functorial in the following sense: Let $f : X \to Y$ be a morphism of schemes. We have already seen that there exist pullback morphisms between the cohomology groups. The boundary map is functorial with respect to this pullbacks, so the diagram

$$\begin{tikzcd}
\check{H}^1_{et}(X, \mathscr{K}_X) \arrow[r, "d"] & H^2_{et}(X, \mathscr{G}_X) \\
\check{H}^1_{et}(Y, \mathscr{K}_Y) \arrow[r, "d"'] \arrow[u, "f^*"] & H^2_{et}(Y, \mathscr{G}_Y) \arrow[u, "f^*"']
\end{tikzcd}$$

is commutative ([25] Chapter IV 3.4.1.1.)

The reason why we are interested in cohomology of non abelian sheaves of groups is that it can be used to describe twisted forms. Let Y be a scheme and X an object over Y. Think of X as a scheme, a sheaf, a group scheme, or, most important for us, an Azumaya algebra. Another object X' of the same type over Y is a *twisted-form* of X for the étale topology if there exists an étale covering $\mathscr{U} = (U_i \to Y)$ such that there are isomorphisms

$$\phi_i : X \times_Y U_i \xrightarrow{\sim} X' \times_Y U_i.$$

We call such a covering $\mathscr{U}$, with isomorphisms ϕ_i, a *trivialization cover* of the twisted form X'. X is a trivial twisted form too itself. If two twisted forms X' and X'' of X are isomorphic over Y this implies that we can choose the same trivialization cover $\mathscr{U}$ for X' and X''. We equivalently define twisted forms with respect to the flat or Zariski topology, though we are mostly interested in the étale case.

A prime example of twisted forms on for the Zariski topology are locally free sheaves $\mathscr{F}$ of a fixed rank n. Each locally free sheaf of rank n is by definition isomorphic to $\mathscr{F}_{|U_i} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{U_i}$ with respect to a Zariski covering $(U_i \to Y)$.

Let X' be a twisted form of X with fixed trivialization cover $\mathscr{U}$ and isomorphism ϕ_i. Furthermore, let $\mathcal{A}ut(X)$ be the sheaf of automorphism on X, i.e. the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Aut}_U(X \times_Y U).$$

This is not necessarily an abelian sheaf, so we need non abelian cohomology. We can associated X' to a class in $\check{H}^1(\mathscr{U}/X, \mathcal{A}ut(X))$ in the following way: If we restrict the isomorphism ϕ_i to U_{ij} and compose them as $\alpha_{ij} = (\phi_i \mid U_{ij})^{-1}(\phi_j \mid U_{ij})$ we get an automorphism on U_{ij} over X, so an element $\alpha_{ij} \in \mathcal{A}ut(U_{ij})$. Then (α_{ij}) defines an 1-cocycle. Two twisted forms have cohomologous 1-cocycles if and only if they are isomorphic over Y. So we have a well defined injective map from isomorphism classes of twisted forms trivialized by $\mathscr{U}$ to the pointed set $\check{H}^1(\mathscr{U}/X, \mathcal{A}ut(X))$. We write $[X']$ for the class. Using the limit construction

$$\check{H}^1(X, \mathcal{A}ut(X)) = \varinjlim \check{H}^1(\mathscr{U}/X, \mathcal{A}ut(X)),$$

where the limit is taken over all coverings of X, we get classes with respect to any covering. We have thus constructed an injection from the set of isomorphism classes of twisted forms to $\check{H}^1(X, \mathcal{A}ut(X))$.

In general, this map does not need to be surjective. Of course one is usually interested in the case where this map is surjective, i.e. where every

1-cocycle does describe a twisted form. Assume for a moment that we work in the flat topology. Then every element of $\check{H}^1(\mathscr{U}/X, \mathcal{A}ut(X))$ defines a descent datum on $X \times U$, where $U = \bigsqcup U_i$ (see [55] Chapter I Theorem 2.23). The map $X' \mapsto [X']$ is therefore surjective if every descent datum on $X \times U$ comes from a twisted form that is trivialized by $\mathscr{U}$. The statement for flat topology implies the same for étale and Zariski topology.

For example, let us classify locally free sheaves of rank n on some scheme Y. By a locally free sheaf we mean a $\mathcal{O}_Y$-module that is isomorphic to $\mathcal{O}_{U_i}^n$ when restricted to a covering $(U_i \to Y)$ with respect to Y_E, which denotes the flat, étale or Zariski site. So locally free sheaves are twisted forms of $\mathcal{O}_Y^n$. The automorphism sheaf is $GL_n = \mathcal{A}ut_{\mathcal{O}_Y}(\mathcal{O}_Y^n)$. Now descent theory shows, that in this case every 1-cocycle, for a cover $\mathscr{U}$ with values in GL_n, defines a locally free sheaf, which is trivial with respect to the cover. Why, in the Zariski case the 1-cocycles give gluing data $a_{ij} : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$. Descent theory further shows that a locally free $\mathcal{O}_Y$-module of rank n in the flat or étale topology is locally free on Y_{Zar}. Taking limits over all coverings we see that $\check{H}^1(Y, GL_n)$ does classify isomorphism classes of locally free $\mathcal{O}_Y$-modules of rank n in the flat, étale or Zariski topology. If we further restrict to the case of locally free sheaves of rank 1, i.e. invertible sheaves, we have the automorphism sheaf $\mathbb{G}_m = GL_1 = \mathcal{A}ut_{\mathcal{O}_Y}(\mathcal{O}_Y)$. Indeed, Hilbert's Theorem 90 ([55] Chapter III Proposition 4.9.) tells us that

$$\check{H}^1_{Zar}(Y, \mathcal{O}_X^\times) \simeq \check{H}^1_{et}(Y, \mathbb{G}_m) \simeq \check{H}^1_{fl}(Y, \mathbb{G}_m)$$

and then these cohomology classes are all isomorphic to $\mathrm{Pic}(X)$.

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Johannes Fischer, Düsseldorf, November 2021