Optimal Regularity for the Stokes Equations on a 2D Wedge Domain Subject to Perfect Slip, Dirichlet and Navier Boundary Conditions

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Laura Beatrice Westermann
aus Düsseldorf

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1. Prof. Dr. Jürgen Saal

2. PD Dr. habil. Mathias Wilke

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Zusammenfassung

In der vorliegenden Dissertation untersuchen wir die Stokes Gleichungen in drei verschiedenen Settings auf einem zweidimensionalen Keilgebiet mit Öffnungswinkel kleiner π. Der Grund, Keilgebiete zu betrachten, liegt in der Physik. In der Tat kann man, indem man eine geeignete Hanzawa Transformation anwendet, Kontaktlinienprobleme zu Problemen auf einem Keilgebiet transformieren, was zu Navier-Stokes-Gleichungen mit freiem Rand führt. Da ein analytischer Ansatz, diese Probleme zu lösen, sehr schwierig ist, und da viele Resultate der Navier-Stokes-Gleichungen aus den linearisierten Stokes-Gleichungen folgen, ist es sinnvoll, die Stokes-Gleichungen mit verschiedenen Randbedingungen auf dem Keilgebiet zu betrachten. Bisher sind die instationären Stokes-Gleichungen auf Keilgebieten wenig untersucht. In dieser kumulativen Dissertation werden wir also die instationären Stokes-Gleichungen bzw. ihre zugehörige Resolventengleichung mit Perfect-Slip-, Dirichlet- und Navier-Randbedingungen betrachten. Ziel ist es dann, die $W^{2,p}$-Regularität des jeweiligen Stokes Operator in drei verschiedenen Manuskripten zu zeigen. Während $W^{2,p}$-Regularität für glatte Gebiete schon seit langem wohlbekannt ist, ist dies für Gebiete mit singulären Randanteilen alles andere als offensichtlich und gilt generell als schwieriges Problem. Das erste Manuskript handelt über die Stokes-Gleichung mit Perfect-Slip-Randbedingungen. Da in diesem Setting die Helmholtz-Projektion mit dem Laplace-Operator kommutiert, können wir den Stokes-Operator als Teil des Laplace-Operators im Raum L^p_σ betrachten. Ziel dieses Manuskripts ist, die $W^{2,p}$-Regularität und maximale Regularität auf L^p vom damit verbundenen stationären und instationären Stokes-Problem für alle $p \in (1, \infty)$ zu zeigen. Hinsichtlich Regularität ist dies eine enorme Verbesserung von [15, Theorem 1.1, Corollary 3], indem diese nur für $p \in (1, 1 + \delta)$ für kleine $\delta > 0$ gezeigt wird. $W^{2,p}$-Regularität für den perfect-slip Stokes Operator für alle $p \in (1, \infty)$ ist überraschend, da, wie es ja ebenfalls im Manuskript gezeigt wird, dies für den entsprechenden Laplace Operator nicht gilt. Im zweiten Manuskript behandeln wir das Resolventenproblem für den Stokes-Operator mit Dirichlet-Randbedingungen. Ziel ist es, die $W^{2,p}$-Regularität vom entsprechenden Stokes-Operator für alle p in einem offenen Intervall um $p = 2$ zu zeigen. Indem wir die biharmonische Gleichung betrachten, können wir dank eines Resultats aus [5] die schwache und starke Regularität des biharmonischen Operators und somit des Stokes-Operators für geeignete p zeigen. Dank dieser Resultate folgt die $W^{2,p}$-Regularität für das Resolventenproblem für den Stokes-Operator. Im letzten Manuskript betrachten wir die Stokes-Gleichungen mit inhomogenen Navier-Randbedingungen. Wir zeigen die Existenz und Eindeutigkeit der Lösung mit optimaler Regularität in einem L^p-Setting für alle $p \in (1, \infty) \setminus \{ \frac{2\theta_0}{3\theta_0 - \pi}, \ \frac{2\theta_0}{3\theta_0 - 2\pi}, \ 3/2, \ 2, \ 3 \}$, wobei θ_0 der Öffnungswinkel des Keils ist. Der Beweis basiert unter anderem auf einem Resultat aus dem ersten Manuskript.

Summary

In this thesis we consider the Stokes equations defined on a two-dimensional wedge type domain with opening angle less than π in three different settings. The reason to consider this type of domain is motivated by problems from physics. In fact, by employing a suitable Hanzawa transformation, one may transform a contact line problem onto a problem on a wedge, which leads to the Navier-Stokes equations subject to free boundary conditions. Since an analytical approach of these problems seems to be very difficult and since many results for the Navier-Stokes equations are obtained based on the linearized Stokes equations, it is useful to consider the Stokes equations subject to different boundary conditions on a wedge domain. It seems that results on the instationary Stokes equations on wedge domains are very rare. Hence, in this cumulative thesis we consider the instationary problem or the resolvent problem of the Stokes equations subject to perfect slip, Dirichlet and Navier boundary conditions. The main objective is then $W^{2,p}$-regularity of the corresponding Stokes operator. This is studied in three independent manuscripts of the thesis. Whereas the $W^{2,p}$-regularity on smooth domains is well-known, similar results on domains with singular boundary parts are not obviously available and difficult to prove. In the first manuscript, we consider the Stokes equations subject to perfect slip boundary conditions. There, since the Helmholtz projector and the Laplacian commute in the underlying setting, we can treat the Stokes operator as part of the Laplacian in the space L^p_σ. We show the $W^{2,p}$-optimal regularity and maximal regularity on L^p_σ of the associated stationary and instationary Stokes problem for the whole range of $p \in (1, \infty)$. Concerning regularity, this improves a result of [15, Theorem 1.1, Corollary 3] to a large extend as there it is merely proved for $p \in (1, 1 + \delta)$ and $\delta > 0$ small. $W^{2,p}$-regularity for $p \in (1, \infty)$ is for perfect slip Stokes surprising, since as it is shown in the manuscript as well, this does not hold for the corresponding Laplacian. In the second manuscript we treat the Stokes resolvent problem subject to Dirichlet boundary conditions. The objective of this manuscript is to prove the $W^{2,p}$-regularity of the corresponding Stokes operator for all p defined in an open interval about $p = 2$. Considering the corresponding biharmonic equation, to which the stationary Stokes problem can be transformed, we can prove with the aid of a result of [5] weak and strong regularity of the biharmonic operator and, hence, of the Stokes operator in the L^p-setting for suitable p. Based on these results the $W^{2,p}$-regularity follows for the resolvent problem of the Stokes operator. In the last manuscript we consider the Stokes equations subject to inhomogeneous Navier slip boundary conditions. We prove existence and uniqueness of solutions with optimal regularity in an L^p-setting for all $p \in (1, \infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi},\ \frac{2\theta_0}{3\theta_0 - 2\pi},\ 3/2,\ 2,\ 3\}$ where θ_0 is the opening angle of the wedge. Its proof is based on a result of the first manuscript.

Contents

Chapter 1

Introduction

1.1 General information

Properties of PDEs on non-smooth domains are important for many applications. For example a water drop running down a glass describes a contact line problem, which constitutes a three-phase problem with one solid and two fluid phases. Employing a suitable Hanzawa transformation, see e.g. [2, 18], the contact line problem can be transformed onto a wedge type domain, which leads to the Navier-Stokes equations subject to the free boundary conditions on the liquid-gas interface. An analytical approach of these problems seems to be very difficult. Up to know, results for 0° or 90° contact angle are available, for instance, see [20] (0°) or [21, 22] (90°). Since many results for the Navier-Stokes equations are obtained by properties of the linearized Stokes equations, it is significant to consider the Stokes equations subject to different boundary conditions on a wedge. However concerning regularity, results for the Stokes equations in domains with conical boundary points or non-smooth domains are very rare. One may find some classical regularity results in [11, 9, 1, 14, 5, 4]. We also refer to [6], where an approach to analytic regularity was presented, or to [12], where the Stokes equations subject to no-slip boundary conditions in a cone are studied and to [7] for an overview of the Stokes equations including approaches to non-smooth domains. In this thesis we consider the Stokes equations on a two-dimensional wedge type domain in three different settings, which are the Stokes equations subject to perfect slip, Dirichlet and Navier boundary conditions. These results are contained in three independent manuscripts included in this thesis. They are joint works with Jürgen Saal and Matthias Köhne.

The Navier-Stokes equations are the fundamental equations in fluid dynamics, which describe the flow of incompressible Newtonian fluids. Many results for the Navier-Stokes equations are obtained based on properties of the linear Stokes equations given as

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla p = f \quad & \text{in} \quad J \times G, \\ \operatorname{div} u = 0 \quad & \text{in} \quad J \times G, \\ B(u) = 0 \quad & \text{on} \quad J \times \partial G, \\ u(0) = u_0 \quad & \text{in} \quad G, \end{aligned}\right\} \tag{1.1}$$

where $J = (0, T)$ with $T > 0$ is a time interval, $B(u)$ are certain boundary conditions and $G \subset \mathbb{R}^2$ is a two-dimensional domain. Moreover, by $u = u(t, x)$ we denote the velocity field and by $p = p(t, x)$ the pressure. The given data are described by the external force $f = f(t, x)$, and by the initial velocity field $u_0 = u_0(x)$. In this cumulative thesis the main objective is to find a unique solution (u, p) of system (1.1) in three different settings, which are explained in the following. The thesis is structured in three independent manuscripts. Note that the uniqueness of the pressure is to be understood as uniqueness up to a constant. The boundary conditions defined on the two-dimensional domain G considered in the manuscripts are given by

$$\begin{aligned} B_1(u) \quad &:= \quad \begin{pmatrix} \operatorname{curl} u \\ u \cdot \nu \end{pmatrix} && \text{(perfect slip boundary condition)}, \\ B_2(u) \quad &:= \quad u && \text{(no slip or Dirichlet boundary condition)}, \\ B_3^{\pm}(u) \quad &:= \quad \begin{pmatrix} \alpha u \cdot \tau - \tau^T D_{\pm}(u)\nu \\ u \cdot \nu \end{pmatrix} && \text{(Navier or partial slip boundary condition)}. \end{aligned}$$

Here, we denote by $D_{\pm}(u) := \frac{1}{2}(\nabla u \pm \nabla u^T)$ the rate of the deformation tensor and the rate of rotation tensor, respectively, by ν the outer normal vector and by τ the tangential vector on the boundary ∂G, respectively. The parameter $\alpha \in BUC^1(\partial G)$ is related to the slip length. We notice that the homogeneous boundary condition $B_3^{\pm}(u) = 0$ on ∂G can be reformulated as

$$\alpha u_\tau \pm \frac{1}{2} \operatorname{curl} u = 0, \;\; u \cdot \nu = 0 \quad \text{on } \partial G.$$

Hence, for the special case $\alpha \equiv 0$, the boundary condition $B_3^{\pm}(u) = 0$ on ∂G corresponds to the perfect slip boundary condition $B_1(u) = 0$ on ∂G. If $\alpha \to \infty$, the condition $B_3^{\pm}(u) = 0$ on ∂G would formally approximate the no-slip (or Dirichlet) boundary condition $B_1(u) = 0$ on ∂G. In our setting, however, we always assume $\alpha|_{x=0} = 0$.

Now, let $G \subset \mathbb{R}^2$ be a two-dimensional wedge domain given as

$$G := \left\{(x_1, x_2) \in \mathbb{R}^2 : \; 0 < x_2 < x_1 \tan \theta_0\right\} \tag{1.2}$$

with opening angle $\theta_0 \in (0, \pi)$. Throughout the included manuscripts, we are interested in the best possible regularity for the solution to the Stokes equations subject to the boundary conditions mentioned above posed in the wedge domain G. In the case of perfect slip boundary conditions, we prove that the instationary Stokes equations have optimal regularity in the L^p-setting for all $p \in (1, \infty)$, i.e. the solution of (1.1) is in $W^{2,p}$. For the Dirichlet boundary conditions, we consider the corresponding resolvent problem of (1.1) and show its well-posedness in the L^p-setting for a small range of p in a neighborhood of $p = 2$. Moreover, we show that the corresponding Stokes operator is sectorial with angle equal to zero. Finally, we prove the optimal regularity of (1.1) subject to inhomogeneous Navier boundary conditions with $J = (0, T)$ for a finite $T > 0$ in the L^p-setting for all $p \in (1, \infty) \setminus \{2\theta_0/(3\theta_0 - 2\pi),\ 3/2,\ 2,\ 3\}$.

This thesis is stuctured as follows: In Section 1.2, we give an overview of the three manuscripts contained in this thesis. Section 1.3 is divided in three parts. We first give some basic notation used throughout this thesis. Then we give an introduction into operator classes, $\mathcal{H}^\infty$-calculus, maximal regularity and a result on the operator sum method, which is based on the Kalton-Weis theorem, see [8]. In fact, in two of the manuscripts included in the thesis, which are contained in Chapter 2 and 4, the operator sum method will play an important role to show the invertibility of differential operators defined on a layer domain. Moreover, since in all the three manuscripts we transform some elliptic problems defined on the wedge domain onto a layer domain, we give an introduction to these transformations in the last part of Subsection 1.3. The subsequent three chapters contain the self-contained manuscripts called "Optimal Sobolev regularity for the Stokes equations on a 2D wedge type domain", "The Dirichlet Stokes operator on a 2D wedge domain in L^p: Sectoriality and optimal regularity" and "Optimal regularity of the Stokes equations on a 2D wedge domain subject to Navier boundary conditions".

1.2 Summary

The first manuscript included in Chapter 2 was published in *Mathematische Annalen* online in 2020 and in print in 2021, see [10]. It contains the $W^{2,p}$-optimal regularity and maximal regularity on L^p of the Stokes operator of the associated stationary and instationary Stokes problem on the two-dimensional wedge domain subject to perfect slip boundary conditions. The advantage of the perfect slip conditions is explained by the fact that Helmholtz projector and the Laplacian commute (this has been already utilized in [16, 15]). Hence, we can treat the Stokes operator as part of the Laplacian in the subspace of solenoidal functions. In [15, Theorem 1.1 and Corollary 3] it is already proven that the Laplace and Stokes operators in the underlying setting defined on a three- and two-dimensional wedge domain have maximal regularity on L^p and optimal Sobolev $W^{2,p}$-regularity, but only for a small range of p. This small range of p is restricted to the interval

$1 < p < 1 + \delta$ for a small $\delta > 0$ depending on the opening angle of the wedge. The main objective of the manuscript [10] is to improve these last results for the Stokes operator, that means maximal regularity on L^p and $W^{2,p}$-Sobolev regularity of the Stokes operator in the space L^p_σ for the full range of $1 < p < \infty$ and opening angles less than π for the two-dimensional wedge. For the Laplacian these results only hold on a suitable subspace depending on the opening angle of the wedge and not for every $p \in (1, \infty)$ on the entire L^p-space. However, the problematic subspace is complemented to the space of solenoidal vector fields.

The manuscript in Chapter 3 is on the Stokes resolvent problem on a two-dimensional wedge type domain subject to Dirichtlet boundary conditions. The objective of this manuscript is the $W^{2,p}$-regularity of the corresponding Stokes operator for p in a neighborhood of $p = 2$. A main part of the manuscript is to consider the corresponding biharmonic equation, to which the stationary Stokes problem can be transformed. In [5] there are already results available on the stationary Stokes equations on polygonal domains in the L^p-setting. The corresponding biharmonic problems are also considered there. They are localized on the vertices and transformed from the polygonal domain to a layer domain. In fact transforming the biharmonic equation defined on the wedge onto the layer leads to the same problem as considered in [5] on the layer. Hence, thanks to regularity results of the transformed operator on the layer in [5], we can prove weak and strong optimal regularity of the Stokes operator in the L^p-setting for $p \in (1,2) \cup (2, \infty)$ and $p \in I_\kappa := ((2+\kappa)', 2+\kappa)$ with a small $\kappa > 0$, respectively. Based on these results, the $W^{2,p}$-regularity for the resolvent problem of the Stokes operator follows for all $p \in I_\kappa$.

In Chapter 4 the included manuscript shows the $W^{2,p}$-regularity of the instationary Stokes equations subject to inhomogeneous Navier boundary conditions on the two-dimensional wedge type domain for all $p \in (1, \infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi},\ \frac{2\theta_0}{3\theta_0 - 2\pi},\ 3/2,\ 2,\ 3\}$ where θ_0 is the opening angle of the wedge. The proof of this result is based on a result of the first manuscript included in Chapter 2 (see also [10]). Decomposing the instationary Stokes system into two systems, we obtain one system describing the instationary Stokes equations subject to an inhomogeneous boundary condition $\operatorname{curl} \mathrm{u} = h$. Based on the result of [10] we can derive that it is well-posed in the L^p-setting for all $p \in (1, \infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi},\ \frac{2\theta_0}{3\theta_0 - 2\pi},\ 3/2,\ 2,\ 3\}$ with θ_0 being the opening angle of the wedge. Then, solving the other system, which describes a divergence equation in the wegde subject to inhomogeneous boundary conditions, which is left after decomposing the Stokes system, we can prove the $W^{2,p}$-regularity of the Stokes operator for all $p \in (1, \infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi},\ \frac{2\theta_0}{3\theta_0 - 2\pi},\ 3/2,\ 2,\ 3\}$.

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1.3 Preliminaries

1.3.1 Basic notation

Let $\mathbb{N}$ be set of natural numbers with $\mathbb{N} = \{1,\ 2,\ 3, \dots\}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{R}$ be the real numbers. Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a domain. We employ the usual notation for partial derivatives. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \sum_{k=1}^n \alpha_k$ and a suitable function $u : \Omega \to \mathbb{R}$ we write

$$D^\alpha u(x) := D^{\alpha_1} \cdots D^{\alpha_n} u(x) = \partial^\alpha u(x) := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x), \quad x \in \Omega.$$

We denote the gradient of u by $\nabla u(x) = (\partial_{x_1} u(x), \cdots, \partial_{x_n} u(x))^T$ for $x \in \Omega$ and the Laplacian of u by $\Delta u(x) = \sum_{k=1}^n \partial_{x_k}^2 u(x)$ for $x \in \Omega$. For a suitable vector field $f : \Omega \to \mathbb{R}^n$ we define

$$\operatorname{div} f(x) := \sum_{k=1}^n \partial_{x_k} f_k(x), \quad x \in \Omega$$

and the Laplacian of f by $\Delta f(x) = (\Delta f_1(x), \cdots, \Delta f_n(x))^T$ for $x \in \Omega$. For $n = 2$ the curl of f is given by $\operatorname{curl} f(x) = \partial_{x_1} f_2(x) - \partial_{x_2} f_1(x)$ for $x \in \Omega$.

Let X be a Banach space and $\Omega \subset \mathbb{R}^n$ be a domain. For $1 \leq p \leq \infty$, we denote by $L^p(\Omega, X)$ the X-valued Bochner-Lebesgue space. We denote the Sobolev space of order $k \in \mathbb{N}_0$ as $W^{k,p}(\Omega) := W^{k,p}(\Omega, \mathbb{R})$ and $W^{k,p}(\Omega, \mathbb{R}^n)$, where $W^{0,p} := L^p$. We denote the Kondrat'ev spaces by

$$L^p_\gamma(\Omega) := L^p(\Omega, \rho^\gamma d(x_1, x_2)),\ \rho := |(x_1, x_2)|, \quad \gamma \in \mathbb{R},$$

and we abbreviate $L^p_\gamma(\Omega) := L^p_\gamma(\Omega, \mathbb{R})$. We set

$$\widehat{W}^{k,p}_\gamma(\Omega) := \{u \in L^1_{loc}(\Omega) :\ \partial^\alpha u \in L^p_\gamma(\Omega),\ |\alpha| = k\}.$$

For Banach spaces X, Y the space of bounded linear operators from X to Y is denoted by $\mathscr{L}(X, Y)$, where $\mathscr{L}(X) := \mathscr{L}(X, X)$.

We denote for a linear operator A in X domain and range by $D(A)$ and $R(A)$. Its spectrum, point spectrum, and resolvent set are written as $\sigma(A)$, $\sigma_p(A)$, and $\rho(A)$.

1.3.2 Operator classes, maximal regularity and the operator sum method

We refer to [3, 8, 13, 17] for an introduction to sectorial operators, the $\mathcal{H}^\infty$-calculus, $\mathcal{R}$-bounded operators and the operator sum method.

Let $\phi \in (0, \pi)$ be fixed and $\sum_\phi := \{z \in \mathbb{C} :\ z \neq 0,\ |\arg(z)| < \phi\}$ be the complex sector of angle ϕ. A closed linear operator A in a Banach space X is called sectorial if $\overline{D(A)} = X$, $\overline{R(A)} = X$, $(-\infty, 0) \subset \rho(A)$ and $\|\lambda(\lambda + A)^{-1}\|_{\mathscr{L}(X)} \leq M$ for a constant $M > 0$ and all

$\lambda > 0$. We denote the spectral angle of A by

$$\phi_A := \inf \left\{ \phi : \ \rho(-A) \supset \Sigma_{\pi-\phi}, \ \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\|_{\mathscr{L}(X)} < \infty \right\}.$$

We denote the class of sectorial operators in X by $\mathcal{S}(X)$. If A is sectorial with spectral angle $\phi_A < \pi/2$, then $-A$ generates a bounded holomorphic C_0-semigroup on X. We refer to [3] for a detailed introduction to sectorial operators.

Now, we turn to the $\mathcal{H}^\infty$-calculus. We first introduce the following functional algebras: For a $\sigma \in (0, \pi)$, we define

$$\mathcal{H}^\infty(\Sigma_\sigma) := \{f \in \Sigma_\sigma \to \mathbb{C} : \ f \text{ holomorphic}, \ \|f\|_\infty < \infty\}$$

with $\|f\|_\infty := \sup\{|f(z)| : \ z \in \Sigma_\sigma\}$ and its subalgebra $\mathcal{H}_0(\Sigma_\sigma)$ given by

$$\mathcal{H}_0(\Sigma_\sigma) := \{f \in \mathcal{H}^\infty(\Sigma_\sigma) : \ |f(z)| \le C|g(z)|^\varepsilon \text{ for some } C \ge 0, \ \varepsilon > 0 \text{ and all } z \in \Sigma_\sigma\}$$

with $g(z) = \frac{z}{(1+z)^2}$. Let A be a sectorial operator with spectral angle ϕ_A, let $\phi \in (\phi_A, \pi)$ and let $\theta \in (\phi_A, \phi)$. We define the path $\Gamma := \left\{te^{i\theta} : \infty > t > 0\right\} \cup \left\{te^{-i\theta} : 0 \le t < \infty\right\}$. It passes from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$ and stays in the resolvent set of A with the exception at $t = 0$. Then by the Cauchy integral formula and sectoriality of the operator A we may define the Dunford integral

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(\mu)(\mu - A)^{-1} d\mu,$$

which is well defined for all $f \in \mathcal{H}_0(\Sigma_\phi)$. The above formula defines an algebra homomorphism

$$\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathscr{L}(X), \ f \mapsto f(A)$$

called Dunford calculus. The operator A is said to admit a bounded $\mathcal{H}^\infty$-calculus on X, if there exists a constant $C_\sigma > 0$ satisfying

$$\|f(A)\|_{\mathscr{L}(X)} \le C_\sigma \|f\|_\infty, \qquad f \in \mathcal{H}_0(\Sigma_\sigma). \tag{1.3}$$

We define the class of all operators admitting a bounded $\mathcal{H}^\infty$-calculus on X by $\mathcal{H}^\infty(X)$ and denote the $\mathcal{H}^\infty$-angle of A by $\phi_A^\infty := \inf\{\sigma \in (\phi_A, \pi) : \ (1.3) \text{ is fulfilled}\}$.

Now, we turn to the definition to $\mathcal{R}$ sectorial operators. See e.g. [3, 13] for an introduction of $\mathcal{R}$-boundedness. Let X, Y be Banach spaces. We say that a family of operators $\mathcal{T} \subset \mathscr{L}(X, Y)$ is $\mathcal{R}$-bounded, if there is a constant $C > 0$ and $p \in [1, \infty)$ and a probability space $(\Omega, \mathcal{M}, \mu)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric $\{-1,1\}$-valued random variables ε_j for $j=1,\dots,N$ the inequality

$$\|\sum_{j=1}^{N}\varepsilon_j T_j x_j\|_{L^p(\Omega,Y)} \le C\|\sum_{j=1}^{N}\varepsilon_j x_j\|_{L^p(\Omega,X)} \tag{1.4}$$

is valid. We call $\mathcal{R}(\mathcal{T}) := \inf\{C:\ (1.4)\text{ holds}\}$ $\mathcal{R}$-bound of $\mathcal{T}$. A sectorial operator is then called $\mathcal{R}$-sectorial if there exists an angle $\phi \in (\phi_A,\pi)$ and a constant $C_\phi > 0$ such that

$$\mathcal{R}(\{\lambda(\lambda+A)^{-1}:\ \lambda\in\Sigma_{\pi-\phi}\}) \le C_\phi. \tag{1.5}$$

We denote the class of $\mathcal{R}$-sectorial operators by $\mathcal{RS}(X)$ and the $\mathcal{R}$-angle of A by $\phi_A^{\mathcal{R}} := \inf\{\phi\in(\phi_A,\pi):\ (1.5)\text{ holds}\}$.

Now, we give an introduction to maximal regularity for an operator A. We refer here, e.g., again to [3, 13]. Let X be a Banach space, $A: D(A)\to X$ be a closed densely defined operator. Moreover, let $1<p<\infty$ and $T\le\infty$, and $(\cdot,\cdot)_{\theta,p}$ be the real interpolation space with parameter $\theta\in(0,1)$. We consider the Cauchy problem

$$\left.\begin{aligned} u'+Au &= f \quad \text{in} \quad (0,T),\\ u(0) &= u_0 \end{aligned}\right\} \tag{1.6}$$

with given data f and u_0. Then A has maximal L^p-regularity on X for $(0,T)$ if for each $f\in L^p((0,T),X)$ and each $u_0\in I_p := (X,D(A))_{1-1/p,p}$ there exists a unique solution $u:(0,T)\to D(A)$ of (1.6) satisfying

$$\|u\|_{\widehat{W}^{1,p}((0,T),X)} + \|Au\|_{L^p((0,T),X)} \le C\left(\|f\|_{L^p((0,T),X)} + \|u\|_{I_p(A)}\right)$$

for a constant $C>0$ independent of f and u_0. The following result from [23, Theorem 4.2] gives a characterization of maximal regularity in terms of $\mathcal{R}$-sectoriality. There, the notion of class $\mathcal{HT}$ appears. Hence, we introduce its definition before giving the result of [23, Theorem 4.2]. A Banach space X is said to be of class $\mathcal{HT}$ if the Hilbert transform

$$Hf(t) := \sqrt{\frac{2}{\pi}}\lim_{\varepsilon\to0}\int_{|s|>\varepsilon}\frac{|f(t-s)|}{s}ds, \quad f\in\mathcal{S}(\mathbb{R},X)$$

is bounded on $L^p(\mathbb{R},X)$ for some $p\in(1,\infty)$. Here, $\mathcal{S}(\mathbb{R},X)$ denotes the Schwartz space of rapidly decreasing X-valued functions. Then we have:

Proposition 1.3.1. [23, Theorem 4.2] Let X be a Banach space of class $\mathcal{HT}$, $1<p<\infty$ and let A be a sectorial operator with spectral angle $\phi_A<\pi/2$. Then A admits maximal regularity on X for $(0,\infty)$ if and only if A is $\mathcal{R}$-sectorial with $\phi_A^{\mathcal{R}}<\pi/2$.

The next proposition [17, Proposition 3.5] is on the operator sum method, which is based on the Kalton-Weis theorem, see [8, Corollary 5.4]. Since the result of [17, Proposition 3.5] employs the notion of property (α) we first give its definition and refer to [3, 8, 13] for more details. Let $\mathcal{P}$ be a probability space. $\mathcal{E}_\mathcal{P}$ denotes the set of all independent symmetric $\{-1,1\}$-valued random variables on $\mathcal{P}$. A Banach space X has property (α) if there exist spaces $\mathcal{P}(\Omega,\mathcal{M},\mu)$, $\mathcal{P}'(\Omega',\mathcal{M}',\mu')$, $p\in[1,\infty)$ and a constant $\alpha>0$, such that for all $N\in\mathbb{N}$, $x_{jk}\in X$, $a_{jk}\in\mathbb{C}$, $|a_{jk}|\le 1$ and $(\varepsilon_j)_{j=1,\cdots,N}\subset\mathcal{E}_\mathcal{P}$, $(\varepsilon'_k)_{k=1,\cdots,N}\subset\mathcal{E}_{\mathcal{P}'}$ the estimate

$$\|\sum_{j,k=1}^{N}\varepsilon_j\varepsilon' a_{jk}x_{jk}\|_{L^p(\Omega\times\Omega',X)}\le\alpha\|\sum_{j,k=1}^{N}\varepsilon_j\varepsilon'_k x_{jk}\|_{L^p(\Omega\times\Omega',X)}$$

holds. Summarizing, it is well known that for a Banach space X having property (α) that $\mathcal{H}^\infty(X)\subset\mathcal{RS}(X)\subset\mathcal{S}(X)$ with corresponding angles satisfying $\phi_A\le\phi_A^\mathcal{R}\le\phi_A^\infty$. Furthermore, we remark that $L^p(\Omega)$-spaces, $1\le p<\infty$, enjoy property (α).

Proposition 1.3.2. [17, Proposition 3.5] Let X be a Banach space of class $\mathcal{HT}$ having property (α). Suppose $A,B\in\mathcal{H}^\infty(X)$ with $\phi_A^\infty+\phi_B^\infty<\pi$ be two resolvent commuting operators. Then $A+B\in\mathcal{H}^\infty(X)$ with $\phi_{A+B}^\infty\le\max\{\phi_A^\infty,\phi_B^\infty\}$.

1.3.3 Transformations from the wedge onto the layer domain and the appearing $W^{k,p}$-spaces

In all of the three manuscripts included in this thesis we consider elliptic problems on the two dimensional wedge domain G subject to different boundary conditions. To solve these problems we transform them from the wedge onto a layer domain and solve them at first on the layer. Since the problems on the wedge and layer are equivalent we obtain the solvability of the elliptic problems on the wedge domain. The transformations we apply follow a standard procedure used for example in [5, 15, 19]: we use polar coordinates to transform the problem on a semi-layer and employ Euler transformation to transform the latter problem onto a layer. The corresponding pull-back and push-forward operators on $W^{-k,p}$-spaces depend on k and p, hence weighted function spaces appear in the transformed setting. Choosing the right transformation, roughly speaking the right k, p included in the pull-back and push-forward respectively, we can then work in unweighted $W^{-k,p}$-spaces on the layer for $k\in\mathbb{N}_0$. In this subsection we give an introduction to these transformations. We consider

$$\left.\begin{aligned}\Delta^i u&=f\quad\text{in}\quad G,\\ B(u)&=0\quad\text{on}\quad\partial G,\end{aligned}\right\}\tag{1.7}$$

with $G\subset\mathbb{R}^2$ be the wedge defined in (1.2), $B(u)$ be one of the boundary conditions introduced in Section 1.1, $u=u(x_1,x_2)$, $f=f(x_1,x_2)$ and $i\in\mathbb{N}$. We write the inverse of the transform to polar coordinates as

$$\psi_P : \mathbb{R}_+ \times I \to G, \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta) = (x_1, x_2).$$

Next, we employ the Euler transformation $r = e^x$ in radial direction, where by an abuse of notation we write $x \in \mathbb{R}$ for the new variable. We set

$$\psi_E : \Omega \to \mathbb{R}_+ \times I, \quad (x, \theta) \mapsto (e^x, \theta) =: (r, \theta).$$

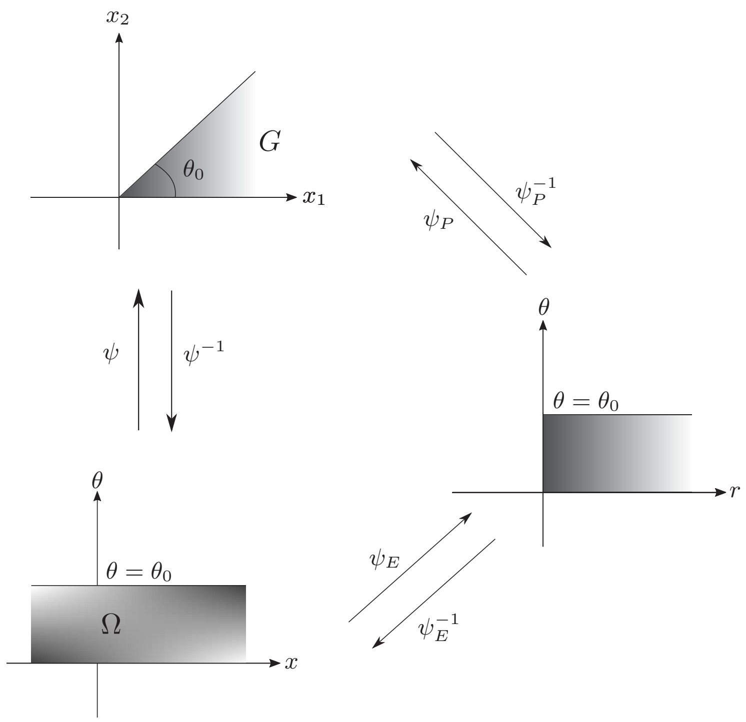
It is not difficult to see that

$$\psi := \psi_P \circ \psi_E : \Omega \to G$$

is a diffeomorphism. For $u : G \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ we set

$$\Psi u := u \circ \psi \quad \text{and} \quad \Psi^{-1} v := v \circ \psi^{-1}.$$

For $\alpha \in \mathbb{R}$ we also denote the multiplication operator by $\mathcal{M}_\alpha v := e^{\alpha x} v$ for all $x \in \mathbb{R}$.



– transformations between the domains –

Let u be the solution of (1.7), $\beta_p \in \mathbb{R}$. Analogous to [15] we define pull-back and push-forward by

$$v := \Theta_p^* u := M_{-\beta_p} \Psi u \quad \text{and} \quad u = \Theta_*^p v = \Psi^{-1} M_{\beta_p} v,$$

respectively.
Now let $i \in \mathbb{N}$, u be the solution of problem (1.7), $\beta_p \in \mathbb{R}$. Then Lemma B.4 of Chapter 3 implies

$$\Theta_p^*(\Delta^i u) = e^{-2ix} \prod_{j=1}^{i} \left(r_{\beta_p - 2(i-j)}(\partial_x) + \partial_\theta^2 \right) v$$

with the polynomial

$$r_a(\partial_x) := (\partial_x + a)^2 \quad (a \in \mathbb{R}).$$

We remark that the order of problem (1.7) is equal to $2i$ for $i \in \mathbb{N}$. Hence we see that the term $2i$ of the factor e^{2ix} defined in $\widetilde{\Theta}_p^*$ is equal to the order of problem (1.7). In the manuscripts included in this thesis we defined $l := 2i$. In the following we continue to use the notation $l := 2i \in \mathbb{N}$. Hence, l will be a fixed value depending on the order of the elliptic problem.

Now, let $1 < p < \infty$. We introduce the role of β_p defined in the pull-back and push-forward, respectively. In fact, β_p depends on $p \in (1, \infty)$, $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Hence, to work in unweighted $W^{-k,p}$-spaces for $k \in \mathbb{N}_0$, we set

$$\beta_p := l - k - \frac{2+\gamma}{p}, \qquad \gamma \in \mathbb{R}. \tag{1.8}$$

By the proof of Lemma B.3 (2) of the manuscript of Chapter 3, we have

$$\sum_{|\alpha|=k} \int_G \left| D^\alpha (\rho^{2k} \widetilde{\Theta}_*^p g(\psi^{-1}(x_1, x_2)) \right|^p \rho^\gamma d(x_1, x_2)$$
$$= \int_\Omega \left| \sum_{m \le \alpha} \binom{\alpha}{m} e^{lx} e^{-(l-k-\frac{2+\gamma}{p})x} e^{-kx} e^{2kx} P(\partial_x,\ \partial_\theta) g(x, \theta) \right|^p e^{(2+\gamma)x} d(x, \theta).$$

There $P(\partial_x,\ \partial_\theta)$ is the product of homogeneous polynomial in $\partial_x, \partial_\theta$ of order k with coefficients depending on $\cos\theta$, $\sin\theta$ functions, and $m = (m_1, m_2) \in \mathbb{N}^2$ such that $m \le \alpha$. See Lemma B1 of Chapter 3 for its precise definition. The determinant of the transform on the right-hand side of the above equation is equal to e^{2x}. The term $\frac{2}{p}$ of β_p absorbs this determinant. Hence, we see that by the choice of β_p defined in (1.8) we can work in the transformed setting in an unweighted $W^{k,p}(\Omega)$-space. Lemma B1 (3) and (5) of the manuscript in Chapter 3 then imply

$$\widetilde{\Theta}_p^* \in \mathscr{L}_{is}\left(\widehat{W}_\gamma^{-k,p}(G), W^{-k,p}(\Omega) \right), \quad k \in \mathbb{N}_0.$$

Furthermore, we remark that in the third manuscript of Chapter 4, we show higher regularity for the Neumann-Laplace problem in $\widehat{W}^{1,p}(G)$ on the wedge. This follows by considering this equation on the layer domain. On the contrary to the other two manuscript, to get higher regularity, we substitute $k := -k'$ with $k' := 1$ in β_p, i.e. $\beta_p := 2+k'-\frac{2+\gamma}{p} = 3-\frac{2+\gamma}{p}$. Hence weighted functions appear. The norms in the corresponding weighted function spaces can be estimated thanks to Hardy's inequality for all $p \in (1,\infty)$ except for $p = 2$, see Lemma A.2 of the manuscript in Chapter 4.

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Chapter 2

Manuscript 1
Optimal Sobolev Regularity for the Stokes Equations on a 2D Wedge Domain

Joint Work with Matthias Köhne and Jürgen Saal.

OPTIMAL SOBOLEV REGULARITY FOR THE STOKES EQUATIONS ON A 2D WEDGE DOMAIN

MATTHIAS KÖHNE, JÜRGEN SAAL, AND LAURA WESTERMANN

Abstract. In this note we prove that the solution of the stationary and the instationary Stokes equations subject to perfect slip boundary conditions on a 2D wedge domain admits optimal regularity in the L^p-setting, in particular it is $W^{2,p}$ in space. This improves known results in the literature to a large extend. For instance, in [17, Theorem 1.1 and Corollary 3] it is proved that the Laplace and the Stokes operator in the underlying setting have maximal regularity in the L^p-setting. In that result the range of p admitting $W^{2,p}$ regularity, however, is restricted to the interval $1 < p < 1 + \delta$ for small $\delta > 0$, depending on the opening angle of the wedge. This note gives a detailed answer to the question, whether the optimal Sobolev regularity extends to the full range $1 < p < \infty$. We will show that for the Laplacian this does only hold on a suitable subspace, but, depending on the opening angle of the wedge domain, not for every $p \in (1, \infty)$ on the entire L^p-space. On the other hand, for the Stokes operator in the space of solenoidal fields L^p_σ we obtain optimal Sobolev regularity for the full range $1 < p < \infty$ and for all opening angles less that π. Roughly speaking, this relies on the fact that an existing "bad" part of L^p for the Laplacian is complementary to the space of solenoidal vector fields.

Contents

1. Introduction and main results

It is well-known that regularity properties for PDE on non-smooth domains are important for many applications. The main objective of this note is to derive best possible

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regularity in the L^p-setting for the instationary Stokes equations

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla\pi = f \quad &\text{in} \quad (0,\infty)\times G,\\ \operatorname{div} u = 0 \quad &\text{in} \quad (0,\infty)\times G,\\ \operatorname{curl} u = 0,\ u\cdot\nu = 0 \quad &\text{on} \quad (0,\infty)\times \partial G,\\ u(0) = u_0 \quad &\text{in} \quad G, \end{aligned}\right\} \tag{1.1}$$

subject to perfect slip boundary conditions on a two-dimensional wedge type domain given as

$$G := \left\{(x_1,x_2)\in\mathbb{R}^2:\ 0 < x_2 < x_1\tan\theta_0\right\}. \tag{1.2}$$

Here ν denotes the outer normal vector at ∂G, $\theta_0\in(0,\pi)$ the opening angle of the wedge, and $\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$.

Whereas the L^p-theory for classical elliptic and parabolic problems on domains with conical boundary points is well developed, see e.g. the classical monographs [7, 18], corresponding results for the Stokes equations are very rare, in particular for the instationary case. For the stationary Stokes equations there are the classical regularity results [15, 14, 3, 18, 7, 5]. For a negative result concerning the generation of an analytic semigroup in three dimensions for the Stokes operator subject to the no-slip condition see [6]. More recently, an approach to analytic regularity was presented in [8]. We also refer to [10] for an overview on the Stokes equations including approaches to non-smooth domains.

It seems that a general approach to the instationary Stokes equations on domains with edges and vertices does not exist in the literature, even for domains having a simple structure such as wedge domains. There is, of course, the Lipschitz approach to even more general non-smooth domains. Existence and analyticity of the Stokes semigroup on L^p_σ on Lipschitz domains is proved, for instance, in [19, 22, 24]. Note that the Lipschitz approach does not provide full $W^{2,p}$ Sobolev regularity which, however, might be crucial for the treatment of related quasilinear problems. Moreover, in the Lipschitz approach the range of available p is restricted in general. Thus, for our purposes this approach seems to be too general. The main objective of this note is $W^{2,p}$ Sobolev regularity for (1.1) for all $p\in(1,\infty)$.

Concerning Stokes the advantage of imposing perfect slip conditions lies in the fact that Helmholtz projector and Laplacian commute, which is not the case in general. Hence the Stokes operator is given as the part of the Laplacian in the solenoidal subspace. Note that this observation has been utilized in [19] and [17] already. In fact, in [17] maximal regularity for (1.1) is proved in two and three dimensional wedges in Kondrat'ev spaces

$$L^p_\gamma(G,\mathbb{R}^2) := L^p(G,\rho^\gamma d(x_1,x_2),\mathbb{R}^2), \quad \rho := |(x_1,x_2)|,\ \gamma\in\mathbb{R}. \tag{1.3}$$

(Note that [17] focuses on the 3D version; the 2D counterpart then is completely analogous.) Optimal regularity in the sense of our main results below, however, could only be established for $1<p<1+\delta$ with $\delta>0$ possibly small, depending on the opening angle θ_0 of the wedge and the Kondrat'ev exponent γ. This shortcoming relies on a spectral constraint that relates to the constraint (1.6) in Theorem 1.3 below. In fact, for $\gamma=0$ under the constraint imposed in [17] we even have $\delta\to 0$ for $\theta_0\to\pi$ such that for angles close to π only a very small interval for p remains.

In this note we will show that in 2D this vast restriction on p can be dropped completely. To be precise, our main result reads as follows (see (3.1) for the definition of the solenoidal subspace $L^p_\sigma(G)$ on a wedge domain).

Theorem 1.1. *Let $1<p<\infty$, $\theta_0\in(0,\pi)$, $\rho=|(x_1,x_2)|$, and $G\subset\mathbb{R}^2$ be defined as in (1.2). Then the Stokes operator subject to perfect slip*

$$A_S u = -\Delta u,$$

$$u \in D(A_S) = \Big\{ v \in W^{2,p}(G,\mathbb{R}^2) \cap L^p_\sigma(G):\ \operatorname{curl} v = 0,\ \nu \cdot v = 0 \text{ on } \partial G,$$
$$\rho^{|\alpha|-2}\partial^\alpha v \in L^p(G,\mathbb{R}^2)\ (|\alpha| \le 2) \Big\}$$

is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^{\mathcal{R}}_{A_S} < \pi/2$, hence has maximal regularity on $L^p_\sigma(G)$.

As an immediate consequence we obtain strong solvability of (1.1).

Corollary 1.2. *Let $1 < p, q < \infty$, $\theta_0 \in (0,\pi)$, $\rho = |(x_1,x_2)|$, and $G \subset \mathbb{R}^2$ be defined as in (1.2). Then for every $f \in L^q((0,\infty), L^p_\sigma(G))$ and $u_0 \in \mathcal{I}_{p,q} := (L^p_\sigma(G), D(A_S))_{1-1/p,q}$ there is a unique solution (u,π) of (1.1) such that $\pi = 0$ and*

$$\|\partial_t u\|_{L^q(\mathbb{R}_+,L^p)} + \sum_{|\alpha|\le 2} \|\rho^{|\alpha|-2}\partial^\alpha u\|_{L^q(\mathbb{R}_+,L^p)} \le C\left(\|f\|_{L^q(\mathbb{R}_+,L^p)} + \|u_0\|_{\mathcal{I}_{p,q}}\right)$$

with $C > 0$ independent of f and u_0.

For the proof of Theorem 1.1 we basically follow the strategy in [17], that is, we first consider the Laplace equation subject to perfect slip conditions. In a standard procedure, by employing polar coordinates and Euler transformation, we reduce the Laplace equation on a wedge to a problem on a layer. On the layer we apply the operator sum method as it is performed originally in [21].

The difference to [17] lies in the fact that here we consider the elliptic problem

$$\left.\begin{aligned} -\Delta u = f \quad &\text{in} \quad G, \\ \operatorname{curl} u = 0,\ u\cdot\nu = 0 \quad &\text{on} \quad \partial G \end{aligned}\right\} \tag{1.4}$$

instead of the corresponding resolvent problem. The advantage is that for the transformed problem we then have precise knowlege on the spectrum. This, in turn, allows to completely characterize the set of p for which optimal regularity for (1.4) is available. We formulate this in our second main result which also represents the basis for Theorem 1.1 and which we even prove in Kondrat'ev spaces.

Theorem 1.3. *Let $1 < p < \infty$, $\theta_0 \in (0,\pi)$, $\gamma \in \mathbb{R}$, and $\rho = |(x_1,x_2)|$. Then equation* (1.4) *is for each $f \in L^p_\gamma(G,\mathbb{R}^2)$ uniquely solvable with a solution u satisfying*

$$\rho^{|\alpha|-2}\partial^\alpha u \in L^p_\gamma(G,\mathbb{R}^2) \quad (|\alpha| \le 2) \tag{1.5}$$

if and only if

$$2 - \frac{2+\gamma}{p} \notin \left\{ \frac{k\pi}{\theta_0} \pm 1 : k \in \mathbb{N} \right\} \cup \{1\}. \tag{1.6}$$

Remark 1.4. (a) For $\gamma = 0$ condition (1.6) reduces to

$$2 - \frac{2}{p} \notin \left\{ 1,\ \frac{\pi}{\theta_0} - 1,\ \frac{2\pi}{\theta_0} - 1 \right\}, \tag{1.7}$$

see Subsection 2.5. From this we see that for each angle $\theta_0 \in (0,\pi)$ the case $p = 2$ is excluded. On the other hand, from the results obtained in [7] one would expect $\partial^\alpha u \in L^2(G,\mathbb{R}^2)$ for $|\alpha| = 2$. Taking into account Hardy's inequality, by which the lower oder terms in (1.5) can be estimated by the second order terms, this looks curious at a first glance. However, $p = 2$ is exactly the case when Hardy's inequality is not valid. Thus, for $p = 2$ (1.5) still can fail for one of the lower order terms, although $\partial^\alpha u \in L^2(G,\mathbb{R}^2)$, $|\alpha| = 2$, might be true. For the excluded $p \neq 2$ (1.5) must fail for at least one of the second order terms, since otherwise Hardy's inequality would yield (1.5) to be valid for all terms, see also Remark 2.7(b).

(b) Another curious looking case is given by $\gamma = 0$ and $\theta_0 = \pi/2$. Then, by reflection arguments the wedge G can be reduced to $-\Delta u = f$ on $\mathbb{R}^2$. This fact implies $\partial^\alpha u \in L^p(G, \mathbb{R}^2)$, $|\alpha| = 2$, to be valid for all $p \in (1, \infty)$. Again this does not contradict the assertion of Theorem 1.3, since in this case (1.7) is reduced to $2 - 2/p \notin \{1\}$. Thus, only $p = 2$ is excluded and we find ourselves in the situation explained in (a).

It seems that Theorem 1.3 is not contained in the previous literature. This might rely on the fact that due to the boundary conditions (1.4) is a system, whereas in previous literature the Laplace equation is preferably considered as a scalar equation.

In contrast to Theorem 1.1, as a first consequence of Theorem 1.3 we obtain that for the instationary diffusion equation subject to perfect slip $W^{2,p}$ regularity is not available if condition (1.6) is not fulfilled, see Theorem 2.19 below. The point why we nevertheless can prove Theorem 1.1 relies on the fact that the part of L^p destroying $W^{2,p}$ regularity is more or less complementary to the space of solenoidal fields $L^p_\sigma(G)$. By this fact we obtain optimal regularity for the stationary Stokes equations, too.

Theorem 1.5. *Let $1 < p < \infty$ and $\theta_0 \in (0, \pi)$. Then for each $f \in L^p_\sigma(G, \mathbb{R}^2)$ there exists a unique solution (u, π) of the stationary version of (1.1) satisfying $\pi = 0$ and*

$$\rho^{|\alpha|-2}\partial^\alpha u \in L^p(G, \mathbb{R}^2) \quad (|\alpha| \leq 2).$$

Of course, the Stokes equations subject to perfect slip in 2D can also be considered without taking the path via the Laplace equation, by utilizing its equivalence to a biharmonic equation. The authors of this note, however, also wanted to compare the two equations concerning regularity. In this regard, we find it most interesting that in the underlying situation the outcome for the Stokes equations is better than for the Laplace or diffusion equation, which usually is vice versa by the fact that the Laplacian enjoys much nicer properties than the Stokes operator.

We outline the strategy of the proofs and the organization of this note. Section 2 contains the approach to the Laplace operator and equation. After fixing notation and transforming from a wedge to a layer, in Subsection 2.3 we establish optimal regularity for the transformed problem. This is based on operator sum methods, that is, Kalton-Weis type theorems. Since the transform from a wedge to a layer is a diffeomorphism, this gives instantly Theorem 1.3, as stated in Subsection 2.4. To carry over regularity from the elliptic problem (1.4) to the instationary diffusion equation, it is enough to show optimal regularity for the resolvent problem

$$\left.\begin{aligned} (1-\Delta)u = h \quad &\text{in} \quad G, \\ \operatorname{curl} u = 0,\ u \cdot \nu = 0 \quad &\text{on} \quad \partial G. \end{aligned}\right\} \tag{1.8}$$

The idea is to regard u as the solution of the elliptic problem (1.4) with right-hand side $f = h - u \in L^p(G, \mathbb{R}^2)$. According to Theorem 1.3 we know that this problem has a solution, say v, with the regularity given in (1.5). It remains to prove $u = v$. By the outcome given in [17] this is valid for $p > 1$ close to 1. This means, if the solution operators to problems (1.4) and (1.8) are consistent on the scale $(L^p(G, \mathbb{R}^2))_{1<p<\infty}$, the regularity in (1.5) transfers to the solution u of (1.8) for all $1 < p < \infty$. By the equivalence in Theorem 1.3, however, consistency for the solution operator of (1.4) cannot hold on the full scale $(L^p(G, \mathbb{R}^2))_{1<p<\infty}$. But, as shown in Subsection 2.5, it is consistent on a suitable scale of “nice” subspaces. This leads in Subsection 2.6 to optimal regularity for the diffusion equation on the subspaces for all $1 < p < \infty$ (see Theorem 2.23).

A major difficulty for the transference of optimal regularity to the Stokes equations is given by the fact that the space of solenoidal fields $L^p_\sigma(G, \mathbb{R}^2)$ is not directly included in the "nice" subspace of L^p. A crucial issue, taking the major part of Section 3, is therefore to prove that it can be isomorphically embedded into this subspace. This isomorphic embedding is also valid for the domains of the involved operators, finally leading to Theorem 1.1 and Theorem 1.5.

2. The Laplace operator on a wedge domain subject to perfect slip

2.1. **Notation.** First we introduce the notation used throughout this note. Let X be a Banach space. For $1 \le p \le \infty$ and a measure space (S, Σ, μ), we denote by $L^p(S, \mu, X)$ the usual Bochner-Lebesgue space. If $1 \le p \le \infty$ and (S, Σ, μ) is a complete measure space, then $L^p(S, \mu, X)$ is a Banach space. If $\Omega \subset \mathbb{R}^n$ is a domain and μ is the (Borel-) Lebesgue measure, we write $L^p(\Omega, X)$. We define the Sobolev space of order $k \in \mathbb{N}_0$ as $W^{k,p}(\Omega, \mathbb{R}^n)$, where $W^{0,p} := L^p$.

Let $G \subset \mathbb{R}^2$ be the wedge domain defined in (1.2) and let $\rho = \rho(x_1, x_2) = |(x_1, x_2)|$. We set

$$K^m_{p,\gamma}(G, \mathbb{R}^2) := \{u \in L^1_{loc}(G, \mathbb{R}^2) : \ \rho^{|\alpha|-m}\partial^\alpha u \in L^p_\gamma(G, \mathbb{R}^2), \ |\alpha| \le m\}$$

where $\alpha \in \mathbb{N}^m$ denotes a multiindex, $\gamma \in \mathbb{R}$, and $L^p_\gamma(G, \mathbb{R}^2)$ is defined as in (1.3). Then $K^m_{p,\gamma}(G, \mathbb{R}^2)$ equipped with

$$\|u\|_{K^m_{p,\gamma}} := \|u\|_{K^m_{p,\gamma}(G,\mathbb{R}^2)} := \left(\sum_{|\alpha| \le m} \|\rho^{|\alpha|-m}\partial^\alpha u\|^p_{L^p_\gamma(G,\mathbb{R}^2)} \right)^{1/p}$$

is a Banach space. We also set $K^m_p(G, \mathbb{R}^2) := K^m_{p,0}(G, \mathbb{R}^2)$. Let $1 < p < \infty$ with $1/p + 1/p' = 1$. If $u \in L^p(\Omega, \mathbb{R}^2)$ and $v \in L^{p'}(\Omega, \mathbb{R}^2)$ we denote the duality pairing by $(u, v) := (u, v)_\Omega := \int_\Omega uv dx$. For a family $(x_j)_{j\ge1}$ of elements in a linear space X, we denote by

$$\langle x_j \rangle_{j\ge1} = \langle x_1, x_2, \dots \rangle$$

its linear hull.

For Banach spaces X, Y the space of bounded linear operators from X to Y is denoted by $\mathscr{L}(X, Y)$, where $\mathscr{L}(X) := \mathscr{L}(X, X)$. The subclass of isomorphisms is denoted by $\mathscr{L}_{is}(X, Y)$ or $\mathscr{L}_{is}(X)$, respectively. If X' is the dual space of X, then we use for the corresponding duality pairing the notation

$$\langle x', x \rangle_{X',X}, \quad x \in X, \ x' \in X'.$$

We denote for a linear operator A in X domain and range by $D(A)$ and $R(A)$. Its spectrum, point spectrum, and resolvent set are written as $\sigma(A)$, $\sigma_p(A)$, and $\rho(A)$. We say that an operator $A : D(A) \subset X \to X$ is sectorial, if $\overline{D(A)} = \overline{R(A)} = X$, $(0, \infty) \subset \rho(-A)$, and the family $((\lambda + A)^{-1})_{\lambda>0}$ is uniformly bounded. If the latter family is $\mathcal{R}$-bounded, then we call A $\mathcal{R}$-sectorial. By ϕ_A and $\phi^{\mathcal{R}}_A$ we denote the corresponding spectral and $\mathcal{R}$-angle, respectively [13, 4, 16].

In this note we also employ elements of the $\mathcal{H}^\infty$-calculus (e.g. in Theorem 2.3). By $\mathcal{H}^\infty(X)$ we denote the class of all operators A in X admitting a bounded $\mathcal{H}^\infty$-calculus on X. The corresponding $\mathcal{H}^\infty$-angle is denoted by ϕ^∞_A. We refer to [13, 4, 16] for an introduction into $\mathcal{H}^\infty$-calculus, $\mathcal{R}$-boundedness, and related notions.

2.2. **Transformation of the elliptic linear problem.** In this section we transform the elliptic linear problem (1.4) on a two-dimensional wedge domain onto a layer domain of the form $\Omega := \mathbb{R} \times I$. If θ_0 denotes the angle of the wedge G we set $I := (0, \theta_0)$. In the first step we introduce polar coordinates whereas in the second step we employ the Euler transformation. Last we rescale the appearing terms such that we can work in the transformed setting in unweighted L^p-spaces.

We write the inverse of the transform to polar coordinates as

$$\psi_P : \mathbb{R}_+ \times I \to G, \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta) = (x_1, x_2)$$

with the associated orthogonal basis

$$e_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

We identify the orthogonal transformation matrix $\mathcal{O}$ of the components of a vector field as

$$\mathcal{O} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Next, we employ Euler transformation $r = e^x$ in radial direction, where by an abuse of notation we write $x \in \mathbb{R}$ for the new variable. We set

$$\psi_E : \Omega \to \mathbb{R}_+ \times I, \quad (x, \theta) \mapsto (e^x, \theta) =: (r, \theta).$$

It is not difficult to see that

$$\psi := \psi_P \circ \psi_E : \Omega \to G$$

is a diffeomorphism. We set

$$\Psi u := u \circ \psi \quad \text{and} \quad \Psi^{-1} v := v \circ \psi^{-1}.$$

For $\alpha \in \mathbb{R}$ we also denote the multiplication operator by

$$M_\alpha v := e^{\alpha x} v.$$

Analogous to [17] we define pull back resp. push forward by

$$v := \Theta_p^* u := M_{-\beta_p} \mathcal{O}^{-1} \Psi u \quad \text{resp.} \quad u = \Theta_*^p v = \Psi^{-1} \mathcal{O} M_{\beta_p} v \tag{2.1}$$

with $\beta_p \in \mathbb{R}$ to be chosen later. Then the transformed Laplacian, computed straight forwardly, is given as

$$\Theta_p^*(\Delta u) = e^{-2x} \begin{pmatrix} r_p(\partial_x) v_x + \partial_\theta^2 v_x - v_x - 2\partial_\theta v_\theta \\ r_p(\partial_x) v_\theta + \partial_\theta^2 v_\theta - v_\theta + 2\partial_\theta v_x \end{pmatrix}$$

with the polynomial

$$r_p(\partial_x) := \partial_x^2 + 2\beta_p \partial_x + \beta_p^2. \tag{2.2}$$

To absorb the factor e^{-2x}, we put

$$g = (g_x, g_\theta) := \widetilde{\Theta}_p^* f := e^{2x} \Theta_p^* f \tag{2.3}$$

so that

$$\int_{\mathbb{R}} |g(x, \theta)|^p dx = \int_0^\infty |r^{2-\beta_p} \mathcal{O}^{-1} f(\psi_p(r, \theta))|^p \frac{dr}{r}.$$

Then by the choice $p(2 - \beta_p) = \gamma + 2$, that is

$$\beta_p = 2 - \frac{2 + \gamma}{p}, \tag{2.4}$$

we see that in the transformed setting we can work in an unweighted L^p-space, see [21, 17]. Notice that by this choice of β_p also pull back and push forward depend on p, i.e., the corresponding families are not consistent in p.

Finally, we transform the boundary conditions $\nu \cdot u = 0$, $\operatorname{curl} u = 0$ on ∂G of the problem (1.4) to the result that

$$\partial_\theta v_x = 0, \quad v_\theta = 0 \quad \text{on } \partial\Omega = \mathbb{R} \times \{0, \theta_0\}.$$

Summarizing, we receive the following transformed problem on $\Omega = \mathbb{R} \times I$:

$$\left.\begin{aligned} r_p(\partial_x)v_x + \partial_\theta^2 v_x - v_x - 2\partial_\theta v_\theta &= g_x \quad \text{in } \Omega, \\ r_p(\partial_x)v_\theta + \partial_\theta^2 v_\theta - v_\theta + 2\partial_\theta v_x &= g_\theta \quad \text{in } \Omega, \\ \partial_\theta v_x = 0, \quad v_\theta &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.5}$$

2.3. **Optimal elliptic regularity for the transformed problem.** Here we consider problem (2.5). In this subsection we frequently identify $L^p(\Omega, \mathbb{R}^2)$ with its isometrically isomorphic version $L^p\big(\mathbb{R}, L^p(I, \mathbb{R}^2)\big)$, often without further notice. We introduce the operators associated to the single parts in (2.5):

(1) Let r_p be the polynomial given in (2.2). We define $\mathcal{T}_{p,x}$ in $L^p(\mathbb{R})$ by means of

$$\mathcal{T}_{p,x}v := r_p(\partial_x)v, \quad v \in D(\mathcal{T}_{p,x}) := W^{2,p}(\mathbb{R}).$$

The spectrum of $\mathcal{T}_{p,x}$ is given by the parabola $r_p(i\mathbb{R})$ which is symmetric about the real axis, open to the left and has its intersection point with the x-axis at β_p^2 with β_p as in (2.4). It is straight forward to show that $-\mathcal{T}_{p,x} + b \in \mathcal{H}^\infty(L^p(\mathbb{R}))$ for $b > \beta_p^2$ with $\phi^\infty_{-\mathcal{T}_{p,x}+b} < \pi/2$, e.g., by the use of Fourier transform, see also [21, 17]. By means of operator-valued Fourier multiplier results [27, 4, 16] these facts obviously transfer to the vector-valued version on $L^p(\mathbb{R}, L^p(I, \mathbb{R}^2))$ given as

$$T_{p,x}v := \mathcal{T}_{p,x}v, \quad v \in D(T_x) := W^{2,p}(\mathbb{R}, L^p(I, \mathbb{R}^2)).$$

(2) We define $\mathcal{T}_{p,\theta}$ in $L^p(I, \mathbb{R}^2)$ by

$$\mathcal{T}_{p,\theta}v := \begin{pmatrix} \partial_\theta^2 - 1 & -2\partial_\theta \\ 2\partial_\theta & \partial_\theta^2 - 1 \end{pmatrix} v$$

on $D(\mathcal{T}_{p,\theta}) := \{v = (v_x, v_\theta) \in W^{2,p}(I, \mathbb{R}^2) : \partial_\theta v_x = 0,\ v_\theta = 0 \text{ on } \partial I\}$. It is also straight forward to identify

$$\sigma(\mathcal{T}_{p,\theta}) = \sigma_p(\mathcal{T}_{p,\theta}) = \left\{ -\left(\frac{k\pi}{\theta_0} \pm 1\right)^2 : k \in \mathbb{N} \right\} \cup \{-1\} \tag{2.6}$$

as its spectrum with corresponding eigenfunctions $(v_x^k, v_\theta^k)^\tau$, where

$$v_x^k(\theta) := \cos\left(\frac{k\pi}{\theta_0}\theta\right), \quad v_\theta^k(\theta) := \pm \sin\left(\frac{k\pi}{\theta_0}\theta\right), \quad k \in \mathbb{N}_0,\ \theta \in I,$$

see also [17]. Note that $\mathcal{T}_{p,\theta}$ is self-adjoint in $L^2(I, \mathbb{R}^2)$. Hence the eigenfunctions represent a basis of $L^2(I, \mathbb{R}^2)$. We denote by $(\lambda_i)_{i\in\mathbb{N}_0}$ the set of eigenvalues, i.e., $(\lambda_i)_{i\in\mathbb{N}_0} = \sigma(\mathcal{T}_{p,\theta})$ such that $\lambda_0 = -1$ and $\lambda_1 > \lambda_2 > \ldots$. Setting $e_0 := \big(1/\sqrt{\theta_0}, 0\big)^\tau$ and $e_i := \frac{\widetilde{e}_i}{\sqrt{\theta_0}}$ for $i \in \mathbb{N}$ where $\widetilde{e}_i$ denotes the eigenfunction to the eigenvalue λ_i, we have

$$(e_i, e_j) = \frac{1}{\theta_0} \int_0^{\theta_0} \widetilde{e}_i \cdot \widetilde{e}_j \, d\theta = \delta_{ij}.$$

By Fourier series techniques it is also standard to prove that $-\mathcal{T}_{p,\theta}$ admits an $\mathcal{H}^\infty$-calculus on $L^p(I, \mathbb{R}^2)$ with $\phi^\infty_{-\mathcal{T}_{p,\theta}} = 0$. The same properties remain valid for the canonical extension to $L^p(\mathbb{R}, L^p(I, \mathbb{R}^2))$ denoted by

$$T_{p,\theta}v := \mathcal{T}_{p,\theta}v, \quad D(T_{p,\theta}) := L^p(\mathbb{R}, D(\mathcal{T}_{p,\theta})).$$

Optimal regularity for (2.5) is then reduced to invertibility of the operator

$$T_p := T_{p,x} + T_{p,\theta} : D(T_{p,x}) \cap D(T_{p,\theta}) \to L^p(\Omega, \mathbb{R}^2), \tag{2.7}$$

if we can also show that

$$\begin{aligned} D(T_p) &:= \{v = (v_x, v_\theta) \in W^{2,p}(\mathbb{R} \times I, \mathbb{R}^2),\ \partial_\theta v_x = v_\theta = 0 \text{ on } \partial\Omega\} \\ &= D(T_{p,x}) \cap D(T_{p,\theta}). \end{aligned} \tag{2.8}$$

The proof of these facts requires some preparation. Let

$$P^c_{m,p} v := \Sigma^m_{i=1}(v, e_i)e_i \tag{2.9}$$

be the projection of $v \in L^p(I, \mathbb{R}^2)$ to $\langle e_1, ..., e_m\rangle$. We set $P_{m,p} := 1 - P^c_{m,p}$ and $E^p_m := P_{m,p}\left(L^p(I, \mathbb{R}^2)\right)$, i.e., E^p_m is the complement to $\langle e_1, ..., e_m\rangle$. Note that $(P_{m,p})_{1<p<\infty}$ is a consistent family. By this fact we omit the index p and write just P_m. We denote the extension of P_m to $L^p(\mathbb{R}, L^p(I, \mathbb{R}^2))$ by $\mathbb{P}_m$. Obviously then $\mathbb{P}_m \in \mathscr{L}(L^p(\Omega, \mathbb{R}^2))$ is a projector as well and we have

$$L^p(\Omega, \mathbb{R}^2) = L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle) \oplus L^p(\mathbb{R}, E^p_m). \tag{2.10}$$

The following properties are obvious.

Lemma 2.1. *Let $T_{p,x}$ and $T_{p,\theta}$ in $L^p(\Omega, \mathbb{R}^2)$ for $1 < p < \infty$ be defined as above and let $b > \beta_p^2$ with β_p as given in (2.4). Then we have*

1. *$\mathbb{P}_m v \in D(T_{p,i})$ and $\mathbb{P}_m T_{p,i} v = T_{p,i}\mathbb{P}_m v$ for $v \in D(T_{p,i})$ and $i \in \{\theta, x\}$;*
2. *$-T_{p,x}+b, -T_{p,\theta} \in \mathcal{H}^\infty(L^p(\mathbb{R}, E^p_m)) \cap \mathcal{H}^\infty(L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle))$ with the corresponding angles $\phi^\infty_{-T_{p,x}+b} < \frac{\pi}{2}$ and $\phi^\infty_{-T_{p,\theta}} = 0$;*
3. *$\mathbb{P}_m R(\lambda, T_{p,i}) = R(\lambda, T_{p,i})\mathbb{P}_m$ for $\lambda \in \rho(T_{p,i})$ and $i \in \{\theta, x\}$;*
4. *$(\lambda - T_{p,x})^{-1}(\mu - T_{p,\theta})^{-1} = (\mu - T_{p,\theta})^{-1}(\lambda - T_{p,x})^{-1}$ for $\lambda \in \rho(T_{p,x})$ and $\mu \in \rho(T_{p,\theta})$.*

The domains of the Operators $T_{p,x}$ and $T_{p,\theta}$ in the subspace $L^p(\mathbb{R}, E^p_m)$ are defined as

$$\begin{aligned} D_m(T_{p,x}) &:= D(T_{p,x}) \cap L^p(\mathbb{R}, E^p_m) \quad \text{and} \\ D_m(T_{p,\theta}) &:= D(T_{p,\theta}) \cap L^p(\mathbb{R}, E^p_m) \end{aligned} \tag{2.11}$$

respectively. The assertions of Lemma 2.1 then easily yield

Corollary 2.2. *The operator $\mathbb{P}_m$ is a projector on $D(T_{p,i})$ and we have*

1. *$D_m(T_{p,i}) = \mathbb{P}_m\left(D(T_{p,i})\right)$,*
2. *$D(T_{p,i}) = D_m(T_{p,i}) \oplus (1 - \mathbb{P}_m)D(T_{p,i})$*

for $i \in \{\theta, x\}$.

We will characterize the invertibility of the operator in (2.7) by employing the operator sum method. More precisely, we apply [20, Proposition 3.5] which is obtained as a consequence of the Kalton-Weis theorem [13, Corollary 5.4].

Theorem 2.3. *Let $1 < p < \infty$ and $\beta_p = 2 - \frac{2+\gamma}{p}$. Then*

$$T_{p,\theta} + T_{p,x} \in \mathscr{L}_{is}\left(D(T_{p,\theta}) \cap D(T_{p,x}), L^p(\Omega, \mathbb{R}^2)\right)$$

if and only if $-\beta_p^2 \notin \sigma(T_{p,\theta})$.

Proof. Assume that $-\beta_p^2 \notin \sigma(T_{p,\theta})$ and that $b > \beta_p^2$. The fact that $-\beta_p^2 \notin \sigma(T_{p,\theta})$ guarantees

$$\sigma(-T_{p,x}) \cap \sigma(T_{p,\theta}) = \emptyset. \tag{2.12}$$

We first show that $-T_{p,\theta} - T_{p,x} - \varepsilon \in \mathcal{H}^\infty(L^p(\mathbb{R}, E^p_m))$ for some $\varepsilon > 0$, which essentially gives the assertion.

To this end, pick $m \in \mathbb{N}_0$ so that $-\lambda_{m+1} > b$ with $\lambda_{m+1} \in \sigma(T_{p,\theta})$. This implies $\sigma(-T_{p,\theta}) \subset (b,\infty)$ on $L^p(\mathbb{R}, E^p_m)$ and hence $0 \in \rho(-T_{p,\theta} - b - \varepsilon)$ for some $\varepsilon > 0$. This fact, Lemma 2.1(2) and a standard perturbation argument for $\mathcal{H}^\infty$-calculus [9, Corollary 5.5.5] yield that the shifted operator $-T_{p,\theta} - b - \varepsilon$ still satisfies

$$-T_{p,\theta} - b - \varepsilon \in \mathcal{H}^\infty(L^p(\mathbb{R}, E^p_m)) \text{ with } \phi^\infty_{-T_{p,\theta}-b-\varepsilon} = 0.$$

Thanks to Lemma 2.1(2), which yields $\phi^\infty_{-T_{p,\theta}-b-\varepsilon} + \phi^\infty_{-T_{p,x}+b} < \pi$, and to Lemma 2.1(4) we may apply [13, Corollary 5.4] (see also [20, Proposition 3.5]) to the result that

$$-T_{p,\theta} - T_{p,x} - \varepsilon = -T_{p,\theta} - b - \varepsilon + (-T_{p,x} + b) \in \mathcal{H}^\infty(L^p(\mathbb{R}, E^p_m))$$

with $\phi^\infty_{-T_{p,\theta}-T_{p,x}-\varepsilon} \le \max\{\phi^\infty_{-T_{p,\theta}-b}, \phi^\infty_{-T_{p,x}+b}\}$. Particularly, we obtain $0 \in \rho(-T_{p,\theta} - T_{p,x})$, hence

$$T_{p,\theta} + T_{p,x} \in \mathscr{L}_{is}\big(D_m(T_{p,x}) \cap D_m(T_{p,\theta}),\ L^p(\mathbb{R}, E^p_m)\big). \tag{2.13}$$

For the invertibility of the operator $T_{p,\theta} + T_{p,x}$ on $L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle)$ observe that due to (2.12) we have $\lambda_i \in \rho(-T_{p,x})$ on $L^p(\Omega, \mathbb{R}^2)$ for each $\lambda_i \in \sigma_p(T_{p,\theta})$. Thus

$$\lambda_i + T_{p,x} : L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle) \cap D(T_{p,x}) \to L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle)$$

is invertible. By the fact that

$$(T_{p,x} + T_{p,\theta})^{-1} f = \sum_{i=1}^m (\lambda_i + T_{p,x})^{-1} (f, e_i) e_i,\ \ f \in L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle),$$

we conclude that

$$T_{p,\theta} + T_{p,x} \in \mathscr{L}_{is}\big(L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle) \cap D(T_{p,x}),\ L^p(\mathbb{R}, \langle e_1, ..., e_m\rangle)\big). \tag{2.14}$$

Gathering (2.10), (2.13), and (2.14) we end up with

$$T_{p,\theta} + T_{p,x} \in \mathscr{L}_{is}\big(D(T_{p,x}) \cap D(T_{p,\theta}),\ L^p(\Omega, \mathbb{R}^2)\big).$$

Now, assume that $-\beta_p^2 \in \sigma(T_{p,\theta})$. Then the symbol $\lambda + r_p(i\xi)$ of the operator $T_{p,\theta} + T_{p,x}$ vanishes exactly at $(\lambda, \xi) = (-\beta_p^2, 0)$, where $\lambda \in \sigma(T_{p,\theta})$. Thus, $(\lambda + r_p(i\cdot))^{-1}$ is not bounded, hence not an $L^p(\mathbb{R}, L^p(I, \mathbb{R}^2))$-multiplier. This gives the assertion. □

Remark 2.4. An inspection of the proof of Theorem 2.3 shows that we even have that $-T_{p,x} - T_{p,\theta} - \varepsilon \in \mathcal{H}^\infty(L^p(\Omega, \mathbb{R}^2))$ with $\phi^\infty_{-T_{p,x}-T_{p,\theta}-\varepsilon} < \pi/2$ for some $\varepsilon > 0$.

To obtain optimal regularity we show (2.8).

Lemma 2.5. *Let $1 < p < \infty$. Then we have*

$$D(T_p) = D(T_{p,\theta}) \cap D(T_{p,x}).$$

Proof. Considering the function $\xi \mapsto \frac{i\xi_i \cdot i\xi_j}{|\xi|^2}|\xi|^2$ for $\xi \in \mathbb{R}^2$ and applying Mihklin's Multiplier Theorem [23] it is not difficult to see that

$$W^{2,p}(\mathbb{R}^2, \mathbb{R}^2) = L^p(\mathbb{R}, W^{2,p}(\mathbb{R}, \mathbb{R}^2)) \cap W^{2,p}(\mathbb{R}, L^p(\mathbb{R}, \mathbb{R}^2))$$

with equivalent norms. The validity of (2.8) is proved via an extension Theorem, i.e., via a bounded operator $E : W^{2,p}(\Omega, \mathbb{R}^2) \to W^{2,p}(\mathbb{R}^2, \mathbb{R}^2)$ with $Ef|_\Omega = f$ for all $f \in W^{2,p}(\Omega, \mathbb{R}^2)$. See [1, Theorem 4.26] for the existence of E. □

2.4. **Optimal elliptic regularity for problem (1.4).** We next consider equivalence of the problems (1.4) and (2.5). The Laplace operator on the wedge domain is defined as

$$B_p u := -\Delta u,\ u \in D(B_p) := \{v \in K^2_{p,\gamma}(G,\mathbb{R}^2):\ \operatorname{curl} v = 0,\ \nu \cdot v = 0 \text{ on } \partial G\}.$$

Observe that the boundary conditions are defined in a local sense. Indeed, each $u \in K^2_{p,\gamma}(G,\mathbb{R}^2)$ is locally away from the vertex $(0,0)$ a $W^{2,p}$-function for which the traces are well-defined.

Lemma 2.6. *Let $1 < p < \infty$. Let $\Theta^p_*, \widetilde{\Theta}^p_*, \Theta^*_p, \widetilde{\Theta}^*_p$ be defined as in Subsection 2.2. Then we have*

$$\widetilde{\Theta}^p_* \in \mathscr{L}_{is}\left(L^p(\Omega,\mathbb{R}^2),\, L^p_\gamma(G,\mathbb{R}^2)\right), \qquad \Theta^p_* \in \mathscr{L}_{is}\left(D(T_p),\, D(B_p)\right) \tag{2.15}$$

where $\|\cdot\|_{D(B_p)} = \|\cdot\|_{K^2_{p,\gamma}(G,\mathbb{R}^2)}$ and $\|\cdot\|_{D(T_p)} = \|\cdot\|_{W^{2,p}(\Omega,\mathbb{R}^2)}$.

In particular, $u \in D(B_p)$ is the unique solution of (1.4) to the right-hand side $f \in L^p_\gamma(G,\mathbb{R}^2)$ if and only if $v = \Theta^*_p u \in D(T_p)$ is the unique solution of (2.5) to the right-hand side $g = \widetilde{\Theta}^*_p f$.

Proof. By utilizing the transformations given in Subsection 2.2 and by the definition of $\widetilde{\Theta}^p_*$ and Θ^p_*, it is straight forward to verify (2.15). Hence problem (1.4) and problem (2.5) are equivalent. □

Since $-\beta_p^2 \notin \sigma(T_{p,\theta})$ is precisely condition (1.6), Theorem 2.3, Lemma 2.5, and Lemma 2.6 now imply our second main result Theorem 1.3.

Remark 2.7. (a) Theorem 1.3 in particular implies that $(B_p^{-1})_{1<p<\infty}$ cannot be a consistent family on the scale $\left(L^p(\Omega,\mathbb{R}^2)\right)_{1<p<\infty}$. Otherwise it would be possible to recover the excluded p subject to condition (1.6) by an interpolation argument. By the equivalence in Theorem 1.3 this, however, is not possible.
(b) Note that for $\gamma = 0$ we have

$$\int_G \left|u(x_1,x_2)/|(x_1,x_2)|^2\right|^p\, dx_1 dx_2 = \int_0^{\theta_0}\int_{\mathbb{R}} \left|e^{-(2-2/p)x} u(\psi(x,\theta))\right|^p\, dx d\theta.$$

Thus, employing twice Hardy's inequality on the x integral, the terms $\rho^{|\alpha|-2}\partial^\alpha u$ for $|\alpha| \le 1$ can be estimated by the second order terms. This, however, does only work provided $2 - |\alpha| - 2/p \neq 0$ which means at the end that $p \neq 2$, since otherwise Hardy's inequality is not applicable. As a consequence, Theorem 1.3 implies that

$$(\partial_j\partial_k u)_{1\le j,k\le 2} \not\subset L^p(G,\mathbb{R}^8),$$

if condition (1.6) is not satisfied and $p \neq 2$. In the case $p = 2$ second order derivatives might belong to $L^2(G,\mathbb{R}^2)$, but then at least one of the terms $\rho^{|\alpha|-2}\partial^\alpha u$, $|\alpha| \le 1$, cannot be in $L^2(G,\mathbb{R}^2)$.

2.5. **Consistency of $(B_p^{-1})_{1<p<\infty}$ on a subscale.** Observe that condition (1.6) is always fulfilled if every eigenvalue λ_i of $T_{p,\theta}$ satisfies

$$\lambda_i < -\left(2 - \frac{2+\gamma}{p}\right)^2. \tag{2.16}$$

As our main interest concerns the Stokes equations in $L^p_\sigma(G)$, from now on we restrict ourselves to the case $\gamma = 0$, i.e., to the case of Kondrat'ev weight $\rho^\gamma \equiv 1$. Then we have

$$-\beta_p^2 = -\left(2 - \frac{2}{p}\right)^2 \ge -4 \quad (1 < p < \infty).$$

From (2.6) it is easily seen that

$$\lambda_i < -4 \quad (i \geq 3).$$

Thus, relation (2.16) remains true for all $\lambda_i \in \sigma(T_{p,\theta})$ with $i \geq 3$.

As we will see later (Proposition 3.2), excluding the eigenfunctions e_0, e_1, e_2 to the eigenvalues λ_0, λ_1, λ_2 of the transformed operator $T_{p,\theta}$, will play no significant role for the Stokes equations. Roughly speaking, this is due to the fact that their linear hull in $L^p(\Omega, \mathbb{R}^2)$ does not contain divergence free vector fields. Hence, from now on we consider

$$L^p(\mathbb{R}, E_3^p) = \mathbb{P}_3 \left(L^p(\Omega, \mathbb{R}^2) \right)$$

as the base space for $T_p : D_3(T_p) \to L^p(\mathbb{R}, E_3^p)$ with the projector $\mathbb{P}_3$ defined in (2.10) and domain

$$D_3(T_p) := D(T_p) \cap L^p(\mathbb{R}, E_3^p) = D_3(T_{p,\theta}) \cap D_3(T_{p,x}),$$

with $D_3(T_{p,\theta})$ and $D_3(T_{p,x})$ as given in (2.11). As an immediate consequence of Theorem 2.3 (and its proof for $m = 3$, i.p. (2.13)) we obtain

Corollary 2.8. *We have $T_p \in \mathscr{L}_{is}\left(D_3(T_p),\, L^p(\mathbb{R}, E_3^p)\right)$ for all $1 < p < \infty$.*

By Lemma 2.6 $\widetilde{\Theta}_*^p$ and Θ_*^p are isomorphisms with inverse $\widetilde{\Theta}_p^*$ and Θ_p^*, respectively. This implies that

$$\begin{aligned} \widetilde{\mathbb{Q}}_p &:= \widetilde{\Theta}_*^p \mathbb{P}_3 \widetilde{\Theta}_p^* \quad \text{and} \\ \mathbb{Q}_p &:= \Theta_*^p \mathbb{P}_3 \Theta_p^* \end{aligned} \tag{2.17}$$

are projectors on $L^p(G, \mathbb{R}^2)$ and $D(B_p)$, respectively. We set

$$\mathbb{L}^p := \widetilde{\mathbb{Q}}_p \left(L^p(G, \mathbb{R}^2) \right) = \widetilde{\Theta}_*^p L^p(\mathbb{R}, E_3^p)$$

and define the restricted operator

$$\mathbb{B}_p := B_p|_{D(\mathbb{B}_p)} \quad \text{with} \quad D(\mathbb{B}_p) := \mathbb{Q}_p \left(D(B_p) \right) = \Theta_*^p D_3(T_p).$$

Notice that, unless its meaning is given otherwise, in what follows we understand the multiplication operator $M_\alpha v := e^{\alpha x} v$ for $\alpha \in \mathbb{R}$ as an operator $M_\alpha : F \to M_\alpha(F)$ for a function space F. It is clear that M_α is injective for all appearing function spaces F. Equipping $M_\alpha(F)$ with its canonical norm, we even have $M_\alpha \in \mathscr{L}_{is}\left(F,\, M_\alpha(F)\right)$ and $M_\alpha^{-1} = M_{-\alpha}$. Furthermore, if $T \in \mathscr{L}(F)$ commutes with M_α, then we also have $T \in \mathscr{L}(M_\alpha(F))$.

By construction it follows

Proposition 2.9. *Let $1 < p < \infty$. Then we have*

1. *The scale $(\widetilde{\mathbb{Q}}_p)_{1<p<\infty}$ is consistent on $(L^p(G, \mathbb{R}^2))_{1<p<\infty}$ and the scale $(\mathbb{Q}_p)_{1<p<\infty}$ on $(D(B_p))_{1<p<\infty}$;*
2. *$\widetilde{\mathbb{Q}}_p v = \mathbb{Q}_p v$ for $v \in D(B_p) \cap L^p(G, \mathbb{R}^2)$;*
3. *$B_p \mathbb{Q}_p = \widetilde{\mathbb{Q}}_p B_p$;*
4. *$\mathbb{B}_p \in \mathscr{L}_{is}\left(D(\mathbb{B}_p),\, \mathbb{L}^p\right)$.*

In particular, for every $f \in \mathbb{L}^p$ there is a unique solution $u \in D(\mathbb{B}_p)$ of (1.4).

Proof. (1) Obviously we have

$$M_\alpha \mathbb{P}_3 v = \mathbb{P}_3 M_\alpha v \quad (v \in C_c^\infty(\mathbb{R}, D(\mathcal{T}_{p,\theta})),\ \alpha \in \mathbb{R}) \tag{2.18}$$

with $\mathcal{T}_{p,\theta}$ as defined in the beginning of Subsection 2.3. From Lemma 2.5 and Lemma A.1 we infer that $C_c^\infty(\mathbb{R}, D(\mathcal{T}_{p,\theta}))$ is dense in $D(T_p)$. Thus equality (2.18) extends to $v \in D(T_p)$. By the definition of Θ_*^p and Θ_p^* (see (2.1)) this implies

$$\mathbb{Q}_p u = \Psi^{-1}\mathcal{O}M_{\beta_p}\mathbb{P}_3 M_{-\beta_p}\mathcal{O}^{-1}\Psi u = \Psi^{-1}\mathcal{O}\mathbb{P}_3\mathcal{O}^{-1}\Psi u \quad (u \in D(B_p)). \tag{2.19}$$

By the fact that all operators on the right-hand side do not depend on p we obtain consistency of $(\mathbb{Q}_p)_{1<p<\infty}$. The consistency of $(\widetilde{\mathbb{Q}}_p)_{1<p<\infty}$ is completely analogous.

(2) For $u \in D(B_p) \cap L^p(\Omega, \mathbb{R}^2)$ we deduce similarly as in (2.19) that

$$\begin{aligned}\mathbb{Q}_p u &= \Psi^{-1}\mathcal{O}M_{\beta_p}\mathbb{P}_3 M_{-\beta_p}\mathcal{O}^{-1}\Psi u = \Psi^{-1}\mathcal{O}\mathbb{P}_3\mathcal{O}^{-1}\Psi u \\ &= \Psi^{-1}\mathcal{O}M_{\beta_p+2}\mathbb{P}_3 M_{-\beta_p-2}\mathcal{O}^{-1}\Psi u = \widetilde{\mathbb{Q}}_p u.\end{aligned}$$

(3) Thanks to Lemma 2.1 we have

$$B_p\mathbb{Q}_p = \widetilde{\Theta}_*^p\, T_p\, \Theta_p^*\, \Theta_*^p\, \mathbb{P}_3\, \Theta_p^* = \widetilde{\Theta}_*^p\, \mathbb{P}_3\, T_p\, \Theta_p^* = \widetilde{\mathbb{Q}}_p B_p.$$

(4) This is a consequence of representation

$$\mathbb{B}_p = \widetilde{\Theta}_*^p\, T_p\, \Theta_p^* \quad \text{on} \quad D(\mathbb{B}_p),$$

Lemma 2.6, Corollary 2.8, and the definition of $\mathbb{L}^p$, $D(\mathbb{B}_p)$. □

As for the projector $\mathbb{P}_3$ before, due to the consistency we write from now on $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$, i.e., we omit the subscript p.

Next, we show consistency of the family $(\mathbb{B}_p^{-1})_{1<p<\infty}$ on the subscale $(\mathbb{L}^p)_{1<p<\infty}$. Observe that the operator $\mathbb{B}_p^{-1}$ is represented as

$$\mathbb{B}_p^{-1} = \Theta_*^p T_p^{-1} \widetilde{\Theta}_p^*\big|_{\mathbb{L}^p}. \tag{2.20}$$

So, for consistency we need to prove that the right-hand side above does not depend on p. Note, however, that the single components Θ_*^p, T_p^{-1}, $\widetilde{\Theta}_p^*$ do depend on p. Merely their combination can be consistent. For this purpose we first show

Lemma 2.10. *Let $1 < p \le q < \infty$ and $\beta_p = 2 - 2/p$. For $f \in C_c^\infty(\mathbb{R}, E_3^q)$ we have*

$$T_p^{-1} e^{(\beta_q - \beta_p)x} f = e^{(\beta_q-\beta_p)x} T_q^{-1} f.$$

Proof. First note that $f \in C_c^\infty(\mathbb{R}, E_3^q)$ and $p \le q$ yield

$$e^{(\beta_q-\beta_p)x} f \in C_c^\infty(\mathbb{R}, E_3^q) \subset L^p(\mathbb{R}, E_3^p). \tag{2.21}$$

Hence the application of T_p^{-1} to this quantity is defined. Also recall that

$$T_p v = T_{p,\theta} v + T_{p,x} v = \mathcal{T}_{p,\theta} v + \mathcal{T}_{p,x} v = \mathcal{T}_{p,\theta} v + (\partial_x + \beta_p)^2 v.$$

We observe that

$$(\partial_x + \beta_q)^2 e^{-(\beta_q-\beta_p)x} = e^{-(\beta_q-\beta_p)x}(\partial_x + \beta_p)^2$$

implies that

$$e^{(\beta_q-\beta_p)x} T_q e^{-(\beta_q-\beta_p)x} v = T_p v \quad (v \in C_c^\infty(\mathbb{R}, D_3(\mathcal{T}_{\theta,q}))), \tag{2.22}$$

as an equality in $C_c^\infty(\mathbb{R}, E_3^p)$. Here we set $D_3(\mathcal{T}_{\theta,q}) = D(\mathcal{T}_{\theta,q}) \cap E_3^q$ and notice that the assertions of Corollary 2.2 also hold for $\mathcal{T}_{\theta,q}$.

For $v \in C_c^\infty(\mathbb{R}, D_3(\mathcal{T}_{\theta,p})) \hookrightarrow C_c^\infty(\mathbb{R}, E_3^q)$ (Sobolev embedding) we set

$$v_k := k(k + T_{q,\theta})^{-1} v \in C_c^\infty(\mathbb{R}, D_3(\mathcal{T}_{\theta,q})), \quad k \in \mathbb{N}. \tag{2.23}$$

By the sectoriality of $T_{q,\theta}$ we obtain $v_k \to v$ in $D_3(T_p)$. Hence equality (2.22) extends to $v \in C_c^\infty(\mathbb{R}, D_3(\mathcal{T}_{\theta,p}))$. Setting $X = D_3(\mathcal{T}_{\theta,p})$, $Y = E_3^p$, $k = 0$, and $\ell = 2$ in Lemma A.1, we see that (2.22) extends to all $v \in D_3(T_p)$.

As before, for $\alpha \in \mathbb{R}$ we set $M_\alpha v = e^{\alpha x} v$. For $\alpha = \beta_q - \beta_p$ relation (2.22) then yields

$$T_q = M_{-\alpha} T_p M_\alpha \in \mathscr{L}_{is}\big(M_{-\alpha}(D_3(T_p)),\, M_{-\alpha}(L^p(\mathbb{R}, E_3^p))\big)$$

with inverse

$$\widetilde{T}_q^{-1} = M_{-\alpha} T_p^{-1} M_\alpha.$$

Thanks to (2.21) we see that

$$f = M_{-\alpha}\left(M_\alpha f\right) \in M_{-\alpha}\left(L^p(\mathbb{R}, E_3^p)\right)$$

for $f \in C_c^\infty(\mathbb{R}, E_3^q)$. Due to this fact it remains to show that $\widetilde{T}_q^{-1}$ is consistent with T_q^{-1} on $C_c^\infty(\mathbb{R}, E_3^q)$.

For $f \in C_c^\infty(\mathbb{R}, D_3(\mathcal{T}_{\theta,q}))$ we have $M_\alpha f \in C_c^\infty(\mathbb{R}, E_3^p)$ and hence $T_p^{-1} M_\alpha f \in D_3(T_p)$. Since (2.22) holds for all $v \in D_3(T_p)$ this yields

$$T_q \widetilde{T}_q^{-1} f = T_q M_{-\alpha} T_p^{-1} M_\alpha f = M_{-\alpha} \underbrace{M_\alpha T_q M_{-\alpha}}_{=T_p} T_p^{-1} M_\alpha f = f.$$

Completely analogous we deduce $\widetilde{T}_q^{-1} T_q f = f$ for $f \in C_c^\infty(\mathbb{R}, D_3(\mathcal{T}_{\theta,q}))$. Hence $\widetilde{T}_q^{-1} = T_q^{-1}$ on the set $C_c^\infty(\mathbb{R}, D_3(\mathcal{T}_{\theta,q}))$. By a similar approximation argument as in (2.23) we see that this consistency extends to $C_c^\infty(\mathbb{R}, E_3^q)$. This finally yields the assertion. □

In the proof of consistency we also employ the following density property.

Lemma 2.11. *Let $1 < p \le q < \infty$. Then we have*

$$\widetilde{\Theta}_*^q\big(C_c^\infty(\mathbb{R}, E_3^q)\big) \overset{d}{\hookrightarrow} \mathbb{L}^q \cap \mathbb{L}^p.$$

Proof. Note that

$$\mathbb{L}^p = \widetilde{\Theta}_*^p\big(L^p(\mathbb{R}, E_3^p)\big) = \widetilde{\Theta}_*^q M_{-\alpha}\big(L^p(\mathbb{R}, E_3^p)\big)$$

with $M_{-\alpha}$ as defined in the proof of Lemma 2.10 and where $M_{-\alpha}\big(L^p(\mathbb{R}, E_3^p)\big)$ is again equipped with its canonical norm. This shows that $\widetilde{\Theta}_*^q \in \mathscr{L}_{is}\left(M_{-\alpha}\big(L^p(\mathbb{R}, E_3^p)\big), \mathbb{L}^p\right)$ with inverse $\widetilde{\Theta}_q^*$. Since $\widetilde{\Theta}_*^q \in \mathscr{L}_{is}\left(L^q(\mathbb{R}, E_3^q), \mathbb{L}^q\right)$ has the same inverse we conclude that

$$\widetilde{\Theta}_*^q \in \mathscr{L}_{is}\Big(L^q(\mathbb{R}, E_3^q) \cap M_{-\alpha}\big(L^p(\mathbb{R}, E_3^p)\big),\, \mathbb{L}^q \cap \mathbb{L}^p\Big).$$

Thus, it suffices to show that

$$C_c^\infty(\mathbb{R}, E_3^q) \overset{d}{\hookrightarrow} L^q(\mathbb{R}, E_3^q) \cap M_{-\alpha}\big(L^p(\mathbb{R}, E_3^p)\big) =: Y.$$

To this end, pick $v \in Y$ and choose a bounded interval $J \subset \mathbb{R}$ such that

$$\|v - \chi_J v\|_Y = \|v - \chi_J v\|_{L^q(\mathbb{R}, E_3^q)} + \|M_\alpha(v - \chi_J v)\|_{L^p(\mathbb{R}, E_3^p)} < \varepsilon/2,$$

where χ_J denotes the characteristic function to J. By the fact that $\chi_J v \in L^q(J, E_3^q)$ we find $(v_k) \subset C_c^\infty(J, E_3^q)$ such that $v_k \to \chi_J v$ in $L^q(\mathbb{R}, E_3^q)$. Note that, thanks to $E_3^q \hookrightarrow E_3^p$, we also have

$$\|M_\alpha(\chi_J v - v_k)\|_{L^p(\mathbb{R}, E_3^p)} \le C(J, \alpha) \|\chi_J v - v_k\|_{L^p(J, E_3^q)} \to 0 \quad (k \to \infty).$$

Consequently, choosing k large enough we can achieve

$$\|v - v_k\|_Y \le \|v - \chi_J v\|_Y + \|\chi_J v - v_k\|_Y < \varepsilon$$

and the assertion is proved. □

Now we are in position to prove the claimed consistency.

Proposition 2.12. *The family $(\mathbb{B}_p^{-1})_{1<p<\infty}$ is consistent on the subscale $(\mathbb{L}^p)_{1<p<\infty}$.*

Proof. Let $p, q \in (1, \infty)$ and without loss of generality $p \le q$. By the definition of Θ_*^p, $\widetilde{\Theta}_p^*$ we have

$$\Theta_*^p = \Theta_*^q e^{-(\beta_q - \beta_p)x} \quad \text{and} \quad \widetilde{\Theta}_p^* = e^{(\beta_q - \beta_p)x} \widetilde{\Theta}_q^*.$$

Now, pick

$$f \in \widetilde{\Theta}_*^q \left(C_c^\infty(\mathbb{R}, E_3^q)\right) \subset \mathbb{L}^p \cap \mathbb{L}^q.$$

From (2.20) and Lemma 2.10 we infer

$$\begin{aligned} \mathbb{B}_p^{-1} f &= \Theta_*^p \, T_p^{-1} \, \widetilde{\Theta}_p^* f \\ &= \Theta_*^q \, e^{-(\beta_q - \beta_p)x} \, T_p^{-1} \, e^{(\beta_q - \beta_p)x} \, \widetilde{\Theta}_q^* f \\ &= \Theta_*^q \, T_q^{-1} \, \widetilde{\Theta}_q^* f = \mathbb{B}_q^{-1} f. \end{aligned}$$

Proposition 2.9(4) and Lemma 2.11 then yield the assertion. □

2.6. **The diffusion equation.** As before let $\theta_0 \in (0, \pi)$ be the opening angle of the wedge G. For $1 < p < \infty$ we define the Laplacian A_p subject to perfect slip boundary conditions in $L^p(G, \mathbb{R}^2)$ by

$$\begin{aligned} A_p u &:= -\Delta u, \\ u \in D(A_p) &:= \left\{ v \in W^{2,p}(G, \mathbb{R}^2) : \ \text{curl } v = 0, \ \nu \cdot v = 0 \text{ on } \partial G \right\} \cap K_p^2(G, \mathbb{R}^2). \end{aligned} \tag{2.24}$$

Now [17, Theorem 1.1 and Corollary 3.15] gives the following result.

Theorem 2.13. *There is a $\delta = \delta(\theta_0)$ such that for $1 < p < 1 + \delta$ the operator A_p as defined in (2.24) has maximal regularity on $L^p(G, \mathbb{R}^2)$.*

Remark 2.14. (a) Note that in [17] the case of a three-dimensional wedge is considered. However, by an inspection of the single steps in the proof it is clear that the case of a two-dimensional wedge is completely analogous.

(b) Also observe that $\delta > 0$ can be very small. In fact, the methods in [17] yield the constraint $2 - 2/p < \min\{1, (\pi/\theta_0 - 1)\}$. Hence we have $\delta(\theta_0) \to 0$ for $\theta_0 \to \pi$.

(c) From the proof of [17, Theorem 1.1 and Corollary 3.15] it also follows that for each $\lambda \in \rho(A_p)$ the family $\left((\lambda - A_p)^{-1}\right)_{1<p<1+\delta}$ is consistent on $\left(L^p(G, \mathbb{R}^2)\right)_{1<p<1+\delta}$.

By a scaling argument we obtain the following estimate in the homogeneous norm.

Lemma 2.15. *Let $1 < p < \infty$ and $\rho(A_p) \neq \emptyset$. Then we have*

$$\|u\|_{K_p^2(G,\mathbb{R}^2)} \le C \|A_p u\|_{L^p(G,\mathbb{R}^2)} \quad (u \in D(A_p)).$$

Proof. We have $\mu - A_p \in \mathscr{L}_{is}\left(D(A_p), L^p(G, \mathbb{R}^2)\right)$ for a $\mu \in \mathbb{C}$. We introduce the rescaled function $J_\lambda u(x) := \lambda^{-2} u(\lambda x)$, $\lambda > 0$, and note that the wedge G is invariant under this scaling. This yields

$$\begin{aligned} \|u\|_{K_p^2(G,\mathbb{R}^2)} &= \lambda^{2/p} \|J_\lambda u\|_{K_p^2(G,\mathbb{R}^2)} \le C \lambda^{2/p} \|(A_p - \mu) J_\lambda u\|_{L^p(G,\mathbb{R}^2)} \\ &\le C \lambda^{2+2/p} \|J_\lambda (A_p - \mu \lambda^{-2}) u\|_{L^p(G,\mathbb{R}^2)} \\ &= C \|(A_p - \mu \lambda^{-2}) u\|_{L^p(G,\mathbb{R}^2)} \quad (\lambda > 0, \ u \in D(A_p)). \end{aligned}$$

Letting $\lambda \to \infty$ yields the assertion. □

Remark 2.16. The estimate in Lemma 2.15 implies that A_p is injective provided that $\rho(A_p) \neq \emptyset$. This implies that A_p is sectorial or $\mathcal{R}$-sectorial, whenever $((\lambda + A_p)^{-1})_{\lambda > 0}$ is uniformly bounded or $\mathcal{R}$-bounded, respectively, see [9].

Next, we show that Theorem 2.13 is still valid on $\mathbb{L}^p$. To this end, for $1 < p < \infty$ we define $\mathbb{A}_p$ as the part of A_p in $\mathbb{L}^p$, that is

$$\mathbb{A}_p u := A_p|_{\mathbb{L}^p} u, \quad u \in D(\mathbb{A}_p) := \{v \in D(A_p) \cap \mathbb{L}^p : \ A_p v \in \mathbb{L}^p\}.$$

With the projectors $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ as defined in (2.17) we obtain

Lemma 2.17. *Let $1 < p < \infty$. We have*

(1) $D(A_p) = D(B_p) \cap L^p(G, \mathbb{R}^2)$ *with equivalent norms as well as* $\mathbb{Q} = \widetilde{\mathbb{Q}}$ *and* $A_p = B_p$ *on* $D(A_p)$. *In particular,* $\widetilde{\mathbb{Q}}$ *on* $L^p(G, \mathbb{R}^2)$ *is the continuous extension of* $\mathbb{Q}$ *regarded as a projector on* $D(A_p)$.
(2) $\widetilde{\mathbb{Q}} A_p u = A_p \mathbb{Q} u$ *for* $u \in D(A_p)$.
(3) $\mathbb{Q}(\lambda - A_p)^{-1} f = (\lambda - A_p)^{-1} \widetilde{\mathbb{Q}} f$ *for* $f \in L^p(G, \mathbb{R}^2)$ *and* $\lambda \in \rho(A_p)$.
(4) $D(\mathbb{A}_p) = D(A_p) \cap \mathbb{L}^p = \mathbb{Q} D(A_p)$.
(5) $(\lambda - \mathbb{A}_p)^{-1} = (\lambda - A_p)^{-1}|_{\mathbb{L}^p}$ *for* $\lambda \in \rho(A_p)$.
(6) $\big((\lambda - \mathbb{A}_p)^{-1}\big)_{1<p<1+\delta}$ *is consistent on* $(\mathbb{L}^p)_{1<p<1+\delta}$ *for* $\lambda \in \rho(\mathbb{A}_p)$.

Proof. (1) Note that $D(A_p) \hookrightarrow D(B_p)$ is an immediate consequence of the definition of $D(A_p)$. This gives $B_p = A_p$ and, by virtue of Proposition 2.9(2), also $\mathbb{Q} = \widetilde{\mathbb{Q}}$ on $D(A_p)$. Furthermore, the Gagliardo-Nirenberg inequality and Young's inequality yield

$$\|\nabla u\|_p \le C \left(\|\nabla^2 u\|_p + \|u\|_p\right) \quad (u \in L^p(G, \mathbb{R}^2) \cap K^2_p(G, \mathbb{R}^2)).$$

Note that the wedge G is an (ε, ∞) domain and on domains of this type the Gagliardo-Nirenberg inequality holds true [17, Section 5] thanks to the extension operator for homogeneous Sobolev spaces constructed in [11, 2]. This implies

$$\|u\|_{W^{2,p}} \le C \left(\|u\|_p + \|\nabla^2 u\|_p\right) \le C \left(\|u\|_p + \|u\|_{K^2_p}\right).$$

Thus $D(A_p) = D(B_p) \cap L^p(G, \mathbb{R}^2)$ with equivalent norms. From this we easily obtain that $\mathbb{Q}$ is also a projector on $D(A_p)$. Since $D(A_p)$ is dense in $L^p(G, \mathbb{R}^2)$, $\widetilde{\mathbb{Q}}$ extends $\mathbb{Q}$ continuously on $L^p(G, \mathbb{R}^2)$.
(2) follows directly from (1) and Proposition 2.9(3).
(3) Let $\lambda \in \rho(A_p)$. From (1) and (2) we obtain

$$(\lambda - A_p)\mathbb{Q}(\lambda - A_p)^{-1} f = \widetilde{\mathbb{Q}} f \quad (f \in L^p(G, \mathbb{R}^2)).$$

Applying $(\lambda - A_p)^{-1}$ on both sides yields (3).
(4) Let $u \in D(A_p) \cap \mathbb{L}^p$. By (1) we obtain $u = \widetilde{\mathbb{Q}} u = \mathbb{Q} u$, hence $u \in \mathbb{Q} D(A_p)$. Conversely, (1) also yields $\mathbb{Q} D(A_p) \subset D(A_p) \cap \mathbb{L}^p$. In view of (2) we next conclude

$$A_p u = A_p \mathbb{Q} u = \widetilde{\mathbb{Q}} A_p u \in \mathbb{L}^p,$$

hence $u \in D(\mathbb{A}_p)$. Since the inclusion $D(\mathbb{A}_p) \subset D(A_p) \cap \mathbb{L}^p$ is trivial, the assertion is proved.
(5) Let $\lambda \in \rho(A_p)$. For $f \in \mathbb{L}^p$ relations (3) and (4) yield

$$(\lambda - A_p)^{-1} f = \mathbb{Q}(\lambda - A_p)^{-1} f \in D(\mathbb{A}_p).$$

Thus,

$$(\lambda - \mathbb{A}_p)(\lambda - A_p)^{-1} f = f$$

which proves (5).
(6) follows from (5) and Remark 2.14(c). □

By combining the well-known equivalence of maximal regularity and $\mathcal{R}$-sectoriality [27, Theorem 4.2] with Theorem 2.13, Remark 2.16, and Lemma 2.17 (especially assertion (5)) we obtain

Theorem 2.18. *Let $1 < p < 1+\delta$ with $\delta > 0$ as in Theorem 2.13. Then $\mathbb{A}_p : D(\mathbb{A}_p) \to \mathbb{L}^p$ with domain*

$$D(\mathbb{A}_p) = \left\{u \in W^{2,p}(G,\mathbb{R}^2) : \operatorname{curl} u = 0,\ \nu \cdot u = 0 \text{ on} \partial G\right\} \cap K^2_p(G,\mathbb{R}^2) \cap \mathbb{L}^p$$

is $\mathcal{R}$-sectorial with $\phi^{\mathcal{R}}_{\mathbb{A}_p} < \pi/2$. Thus, $\mathbb{A}_p$ has maximal regularity on $\mathbb{L}^p$.

Our ultimate aim in this subsection is to show that Theorem 2.18, in particular the optimal Sobolev regularity, is available on the full range $1 < p < \infty$. Note that this is not true for $A_p : D(A_p) \subset L^p(G,\mathbb{R}^2) \to L^p(G,\mathbb{R}^2)$ with $D(A_p)$ given in (2.24) as the next result shows.

Theorem 2.19. *Let $1 < p < \infty$ and $\theta_0 \in (0,\pi)$ such that condition (1.6) (with $\gamma = 0$) is not satisfied. Then $\rho(A_p) = \emptyset$. In other words, in this situation for every $\lambda \in \mathbb{C}$ there is an $f \in L^p(G,\mathbb{R}^2)$ such that there is no solution u of*

$$\left.\begin{aligned} \lambda u - \Delta u &= f \quad \text{in} \quad G, \\ \operatorname{curl} u = 0,\ u\cdot\nu &= 0 \quad \text{on} \quad \partial G \end{aligned}\right\} \tag{2.25}$$

satisfying $u \in K^2_p(G,\mathbb{R}^2)$. More precisely, if $p \neq 2$ then $\partial^\alpha u \in L^p(G,\mathbb{R}^2)$ for all α with $|\alpha| = 2$, while for $p = 2$ we have $\rho^{|\alpha|-2}\partial^\alpha u \notin L^2(G,\mathbb{R}^2)$ for some α with $|\alpha| < 2$.

Proof. Suppose there exists a complex number $\mu \in \rho(A_p)$. We can assume $\mu \neq 0$, since otherwise this would immediately contradict Theorem 1.3.

By the scaling argument used in the proof of Lemma 2.15 it easily follows that $((\lambda - A_p/\mu)^{-1})_{\lambda>0}$ is uniformly bounded. Thanks to Remark 2.16 then A_p/μ is sectorial, see [9], in particular it has dense range. For $f \in L^p(G,\mathbb{R}^2)$ we hence find $(u_k) \subset D(A_p)$ such that $A_p u_k \to f$ in $L^p(G,\mathbb{R}^2)$. Due to Lemma 2.15 (u_k) is a Cauchy sequence in $K^2_p(G,\mathbb{R}^2)$ and its limit $u = \lim u_k$ satisfies equation (1.4). The fact that $u \in K^2_p(G,\mathbb{R}^2)$ then contradicts Theorem 1.3. Thus $\rho(A_p)$ must be empty. The additional statement follows from Remark 2.7(b). □

Next, we show that the resolvent of $\mathbb{A}_p$ in $\mathbb{L}^p$ is consistent with its dual resolvent. For this purpose we first identify $(\mathbb{L}^p)'$. This, in turn, is connected to the identification of $\mathbb{P}_3'$ and $\mathbb{Q}'$. By this fact, just within the following lemma, we write $\mathbb{P}_{3,p}$ and $\mathbb{Q}_p$ again.

Lemma 2.20. *Let $1 < p < \infty$, $\beta_p = 2 - 2/p$, and $1/p + 1/p' = 1$. Let $\widetilde{\Theta}^p_* : L^p(\Omega,\mathbb{R}^2) \to L^p(G,\mathbb{R}^2)$ be defined as in Subsection 2.2 with inverse $\widetilde{\Theta}^*_p$ and the projectors $\mathbb{P}_{3,p}$ and $\widetilde{\mathbb{Q}}_p$ be defined as in (2.9) (and the subsequent lines) and (2.17) respectively. Then we have*

(1) $(\widetilde{\Theta}^p_*)' = \widetilde{\Theta}^*_{p'}$ *and* $(\widetilde{\Theta}^*_p)' = \widetilde{\Theta}^{p'}_*$*; in particular* $\widetilde{\Theta}^p_*$ *is an isometric isomorphism;*
(2) $(\mathbb{P}_{3,p})' = \mathbb{P}_{3,p'}$*;*
(3) $(\widetilde{\mathbb{Q}}_p)' = \widetilde{\mathbb{Q}}_{p'}$*;*
(4) $(\mathbb{L}^p)' = \mathbb{L}^{p'}$ *with respect to* $(u,\,v) = \int_G uv dx$ *in the sense of a Riesz isomorphism.*

Proof. (1) Recall that by (2.1) and (2.3) we have $\widetilde{\Theta}^p_* u = \Psi^{-1}\mathcal{O}M_{\beta_p-2}u$ with Ψ, $\mathcal{O}$, M_{β_p-2} as defined in Subsection 2.2. Thanks to

$$\beta_p = 2 - \frac{2}{p} = -\beta_{p'} + 2$$

we can calculate

$$\begin{aligned}\left(\widetilde{\Theta}_*^p u,\, v\right)_G &= \int_G v(y)(\mathcal{O}M_{\beta_p-2}u)(\psi^{-1}(y))dy \\ &= \int_\Omega v(\psi(x,\theta))(\mathcal{O}M_{\beta_p-2}u)(x,\theta)e^{2x}dxd\theta \\ &= \int_\Omega (M_{-\beta_{p'}+2}\mathcal{O}^{-1}\Psi v)(x,\theta)u(x,\theta)dxd\theta \\ &= \left(u,\, \widetilde{\Theta}^*_{p'}v\right)_\Omega \quad \left(u\in L^p(\Omega,\mathbb{R}^2),\ v\in L^{p'}(G,\mathbb{R}^2)\right).\end{aligned}$$

Relation $(\widetilde{\Theta}^*_p)' = \widetilde{\Theta}^{p'}_*$ then follows since $\widetilde{\Theta}^*_p = (\widetilde{\Theta}^p_*)^{-1}$.

Relation (2) follows immediately by the definition of $\mathbb{P}_{3,p}$ and (3) is a consequence of (1) and (2).

(4) By the fact that $\mathbb{L}^p = \widetilde{\mathbb{Q}}_p L^p(G,\mathbb{R}^2)$ this follows from the symmetry of $\widetilde{\mathbb{Q}}_p$ proved in (3) and since $(L^p(G,\mathbb{R}^2))' = L^{p'}(G,\mathbb{R}^2)$ with respect to $(\cdot,\ \cdot)$. ☐

Now, let

$$\mathbb{A}_p' : D(\mathbb{A}_p') \subset \mathbb{L}^{p'} \to \mathbb{L}^{p'}$$

be the Banach space dual operator to $\mathbb{A}_p$ in $\mathbb{L}^p$ for $1<p<1+\delta$. By permanence properties and Theorem 2.18 it follows that also $\mathbb{A}_p'$ is $\mathcal{R}$-sectorial with $\phi^{\mathcal{R}}_{\mathbb{A}_p'} = \phi^{\mathcal{R}}_{\mathbb{A}_p} < \pi/2$. At this point, however, we do not know how $D(\mathbb{A}_p')$ looks like. On our way to characterize $D(\mathbb{A}_p')$ we next show consistency of $(\lambda-\mathbb{A}_p)^{-1}$ and $(\lambda-\mathbb{A}_p')^{-1}$ on $\mathbb{L}^p\cap\mathbb{L}^{p'}$.

Proposition 2.21. *Let $1<p<1+\delta$ with $\delta>0$ as in Theorem 2.18 and $1/p+1/p'=1$. Then*

$$(\lambda-\mathbb{A}_p)^{-1}f = (\lambda-\mathbb{A}_p')^{-1}f \quad (f\in\mathbb{L}^p\cap\mathbb{L}^{p'},\ \lambda\in\rho(\mathbb{A}_p)\cap\mathbb{R}).$$

Proof. Let $\lambda\in\rho(\mathbb{A}_p)\cap\mathbb{R}$. We intent to apply Lemma A.2. Setting $T=\lambda-\mathbb{A}_p$, we first have to verify that there exists an embedding $J: D(\mathbb{A}_p)\to(\mathbb{L}^p)'$ with dense range. Observe that, since $D(\mathbb{A}_p)\hookrightarrow W^{2,p}(G,\mathbb{R}^2)$ and $G\subset\mathbb{R}^2$, the Sobolev embedding yields

$$D(\mathbb{A}_p) \overset{d}{\hookrightarrow} L^{p'}(G,\mathbb{R}^2)\cap\mathbb{L}^p = \mathbb{L}^{p'}.$$

Thus J can be chosen essentially as the Riesz isomorphism given in Lemma 2.20(4). However, since we identify $(\mathbb{L}^p)'$ with $\mathbb{L}^{p'}$ anyway and $T^\sharp$ with $(\lambda-\mathbb{A}_p)^\sharp$ on $\mathbb{L}^{p'}$, that is, with its dual induced by the Riesz isomorphism, we omit J (and hence also $\widetilde{J}$) in what follows.

By virtue of Lemma A.2 and (A.1) it then remains to prove that

$$\lambda-\mathbb{A}_p \subset (\lambda-\mathbb{A}_p)^\sharp,$$

where $(\lambda-\mathbb{A}_p)^\sharp : \mathbb{L}^{p'}\to D(\mathbb{A}_p)'$ denotes the dual operator of $\lambda-\mathbb{A}_p$ regarded as a bounded operator from $D(\mathbb{A}_p)$ to $\mathbb{L}^p$, see Appendix A. To this end, pick $u,v\in D(\mathbb{A}_p)$. Observe that by the fact that $D(\mathbb{A}_p)\hookrightarrow\mathbb{L}^p\cap\mathbb{L}^{p'}$ all duality pairings appearing below are well-defined. Also note that

$$\Delta u = \nabla\mathrm{div}\ u - \mathrm{curl}'\,\mathrm{curl}\ u,$$

where $\mathrm{curl}'\varphi = (\partial_{x_2},-\partial_{x_1})^T\varphi$ for a scalar function φ. Employing the Gauß theorem and the boundary conditions for u and v we calculate

$$(\nabla\,\mathrm{div}\,u, v) = \int_{\partial G}\nu\cdot v\ \mathrm{div}\,u\,d\sigma - (\mathrm{div}\,u,\ \mathrm{div}\,v) = (u,\ \nabla\,\mathrm{div}\,v)$$

as well as

$$\begin{aligned}\big(\operatorname{curl}'\operatorname{curl} u, v\big) &= -\int_{\partial G}\Big(\big(v^2, -v^1\big)^T\cdot\nu\Big)\operatorname{curl} u\, d\sigma + (\operatorname{curl} u,\, \operatorname{curl} v)\\ &= \big(u,\, \operatorname{curl}'\operatorname{curl} v\big)\,.\end{aligned}$$

This yields

$$\begin{aligned}\langle T^\sharp u,\, v\rangle_{D(\mathbb{A}_p)',D(\mathbb{A}_p)} &= (u,\, (\lambda+\Delta)v) = ((\lambda+\Delta)u,\, v)\\ &= (Tu,\, v) = \langle Tu,\, v\rangle_{D(\mathbb{A}_p)',D(\mathbb{A}_p)}\end{aligned}$$

which proves the claim. □

Now we can characterize $D(\mathbb{A}_p')$.

Theorem 2.22. *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Then we have $\mathbb{A}_p' = \mathbb{A}_{p'}$, i.e., in particular $D(\mathbb{A}_p') = D(\mathbb{A}_{p'})$ with $D(\mathbb{A}_{p'})$ as characterized by (2.24) and Lemma 2.17(4). Furthermore, for $\lambda \in \rho(\mathbb{A}_p)$ the family $\big((\lambda - \mathbb{A}_p)^{-1}\big)_{1<p<\infty}$ is consistent on $(\mathbb{L}^p)_{1<p<\infty}$.*

Proof. By definition it is obvious that $\mathbb{A}_{p'} \subset \mathbb{A}_p'$. It is clear that the converse inclusion, particularly the assertion on $D(\mathbb{A}_p')$, is proved, if we can show that

$$(1+\mathbb{A}_p)^{-1} \in \mathscr{L}_{is}(\mathbb{L}^p,\, D(\mathbb{A}_p)) \tag{2.26}$$

for every $p \in (1,\infty)$. By Theorem 2.18 relation (2.26) holds for every $1 < p < 1+\delta$. We take p out of that interval and consider (2.26) for its Hölder conjugated exponent p'.

Let $f \in \mathbb{L}^p \cap \mathbb{L}^{p'}$. Then there is a $u \in D(\mathbb{A}_p')$ such that

$$(1+\mathbb{A}_p')u = f.$$

By the consistency of the resolvents of $\mathbb{A}_p$ and $\mathbb{A}_p'$ proved in Proposition 2.21 we see that $u \in D(\mathbb{A}_p)$ and that

$$(1+\mathbb{A}_p)u = f \quad\Leftrightarrow\quad \mathbb{A}_p u = f - u =: g \in \mathbb{L}^p \cap \mathbb{L}^{p'}.$$

On the other hand, Proposition 2.9(4) and the consistency of $(\mathbb{B}_p^{-1})_{1<p<\infty}$ established in Proposition 2.12 imply that there is an $v \in D(\mathbb{B}_p) \cap D(\mathbb{B}_{p'})$ such that

$$\mathbb{B}_p v = g.$$

The fact that $D(\mathbb{A}_p) \subset D(\mathbb{B}_p)$ and $\mathbb{A}_p = \mathbb{B}_p$ on $D(\mathbb{A}_p)$ (Lemma 2.17(1),(4)) then gives $u = v$. From this and Lemma 2.17(1) we obtain

$$\begin{aligned}\|(1+\mathbb{A}_p')^{-1}f\|_{D(\mathbb{A}_{p'})} = \|u\|_{D(\mathbb{A}_{p'})} &\le C\left(\|u\|_{p'} + \|v\|_{K^2_{p'}}\right)\\ &\le C\|f\|_{p'} \quad (f \in \mathbb{L}^p \cap \mathbb{L}^{p'}).\end{aligned}$$

Since $\mathbb{L}^p \cap \mathbb{L}^{p'}$ lies dense in $\mathbb{L}^{p'}$, relation (2.26) follows for p'.

According to what we just have proved, Lemma 2.17(6), and Proposition 2.21 the family $\big((1+\mathbb{A}_p)^{-1}\big)_{p\in I}$ is consistent on $(\mathbb{L}^p)_{p\in I}$ for

$$I = (1,\infty) \setminus [1+\delta, (1+\delta)']. \tag{2.27}$$

For the remaining p we interpolate. In fact, since $\mathbb{L}^p = \widetilde{\mathbb{Q}}L^p(G,\mathbb{R}^2)$ complex interpolation and [25, Theorem 1.17.1.1] yield

$$\big[\mathbb{L}^p, \mathbb{L}^{p'}\big]_s = \mathbb{L}^q, \quad \frac{1}{q} = s\frac{1}{p'} + (1-s)\frac{1}{p}.$$

Furthermore, by [25] we also have

$$W^{2,q}(G,\mathbb{R}^2) = \big[W^{2,p}(G,\mathbb{R}^2), W^{2,p'}(G,\mathbb{R}^2)\big]_s,$$

$$K^2_q(G,\mathbb{R}^2) = \left[K^2_p(G,\mathbb{R}^2), K^2_{p'}(G,\mathbb{R}^2)\right]_s.$$

(Note that the second identity above follows, e.g., from

$$W^{2,q}(\Omega,\mathbb{R}^2) = \left[W^{2,p}(\Omega,\mathbb{R}^2), W^{2,p'}(\Omega,\mathbb{R}^2)\right]_s,$$

and an application of Stein's interpolation theorem [26], since the dependence of Θ^q_*, Θ^*_q on $z = 1/q$ is analytic on a suitable strip in the complex plane.) This shows that

$$(1+\mathbb{A}_p)^{-1} \in \mathscr{L}\left(\mathbb{L}^p,\, W^{2,p}(G,\mathbb{R}^2)\cap K^2_p(G,\mathbb{R}^2)\cap \mathbb{L}^p\right)$$

for every $p \in (1,\infty)$. For $f \in \mathbb{L}^p \cap \mathbb{L}^q$ with $q \in I$, we also see that $(1+\mathbb{A}_p)^{-1}f$ satisfies the boundary conditions included in $D(\mathbb{A}_p)$. By a density argument and boundedness of the corresponding trace operators relation (2.26) follows to be valid for all $p \in (1,\infty)$. This completes the proof. □

Thanks to Theorem 2.22 we can generalize Theorem 2.18 to all $p \in (1,\infty)$.

Theorem 2.23. *Let $1 < p < \infty$. Then $\mathbb{A}_p$ with domain*

$$D(\mathbb{A}_p) = \left\{u \in W^{2,p}(G,\mathbb{R}^2) : \operatorname{curl} u = 0,\ \nu\cdot u = 0 \ on\, \partial G\right\} \cap K^2_p(G,\mathbb{R}^2)\cap \mathbb{L}^p$$

is $\mathcal{R}$-sectorial on $\mathbb{L}^p$ with $\phi^{\mathcal{R}}_{\mathbb{A}_p} < \pi/2$, and hence has maximal regularity on $\mathbb{L}^p$.

Proof. Due to $\mathbb{A}'_p = \mathbb{A}_{p'}$ and Theorem 2.18, the operator $\mathbb{A}_p$ with $D(\mathbb{A}_p)$ as stated is $\mathcal{R}$-sectorial with $\phi^{\mathcal{R}}_{\mathbb{A}_p} < \pi/2$ for $p \in I$ with I given in (2.27). Note that injectivity, hence also $\overline{R(\mathbb{A}_p)} = \mathbb{L}^p$, follows from Remark 2.16. Since the property of $\mathcal{R}$-sectoriality is invariant under interpolation [12, Theorem 3.23], the result follows by interpolation and the equivalence of maximal regularity and $\mathcal{R}$-sectoriality [27, Theorem 4.2]. □

In this subsection we have shown by consistency arguments that regularity for the elliptic operator $\mathbb{B}_p$ transfers to the parabolic operator $\partial_t + \mathbb{A}_p$. The next result, which in principle shows that the converse is true as well, we state also for later purposes.

Proposition 2.24. *Let $1 < p < \infty$, then*

$$\lim_{k\to\infty}(1/k - \mathbb{A}_p)^{-1} = \mathbb{B}_p^{-1} \quad in \quad \mathscr{L}\left(\mathbb{L}^p,\, K^2_p(G,\mathbb{R}^2)\right).$$

In particular, $D(\mathbb{A}_p)$ is dense in $D(\mathbb{B}_p)$.

Proof. Pick $f \in \mathbb{L}^p$. For $\ell \in \mathbb{N}$ by the resolvent identity, Lemma 2.15, and since $\mathbb{A}_p$ is sectorial we obtain

$$\begin{aligned}
&\|(1/(k+\ell) - \mathbb{A}_p)^{-1}f - (1/k - \mathbb{A}_p)^{-1}f\|_{K^2_p}\\
&\le C\|(1/(k+\ell) - 1/k)(1/k - \mathbb{A}_p)^{-1}\mathbb{A}_p(1/(k+\ell) - \mathbb{A}_p)^{-1}f\|_p\\
&\le C\|(k/(k+\ell) - 1)\mathbb{A}_p(1/(k+\ell) - \mathbb{A}_p)^{-1}f\|_p\\
&\le C\|(k/(k+\ell) - 1)f\|_p \to 0 \quad (k\to\infty).
\end{aligned}$$

Thus $(1/k - \mathbb{A}_p)^{-1}f \to v$ in $D(\mathbb{B}_p)$. The fact that $\mathbb{B}_p \in \mathscr{L}(D(\mathbb{B}_p), \mathbb{L}^p)$, Lemma 2.17(1), and again sectoriality of $\mathbb{A}_p$ yield

$$\mathbb{B}_p v = \lim_{k\to\infty}\mathbb{A}_p(1/k - \mathbb{A}_p)^{-1}f = f,$$

hence $v = \mathbb{B}_p^{-1}f$. □

3. The Stokes equations

In this section, we consider the Stokes problem (1.1). We introduce the space of solenoidal vector fields. For $1 < p < \infty$ and $1/p + 1/p' = 1$ we set

$$L^p_\sigma(G) := \left\{ u \in L^p(G, \mathbb{R}^2) : \int_G u \cdot \nabla\varphi d(x_1, x_2) = 0 \ \ (\varphi \in \widehat{W}^{1,p'}(G)) \right\}, \tag{3.1}$$

where

$$\widehat{W}^{1,p'}(G) := \left\{ \varphi \in L^1_{loc}(G) : \ \nabla\varphi \in L^{p'}(G, \mathbb{R}^2) \right\}.$$

Since $C_c^\infty(G, \mathbb{R}^2) \subset \widehat{W}^{1,p}(G, \mathbb{R}^2)$, it is evident that $u \in L^p_\sigma(G)$ satisfies the condition $\operatorname{div} u = 0$ in the sence of distributions. Moreover $\nu \cdot u$ is well-defined in the trace space (Slobodeckii space) $W_p^{-1/p}(\mathcal{O})$ for all bounded domains $\mathcal{O}$ with $\overline{\mathcal{O}} \subset \partial G \setminus \{(0,0)\}$. This yields that the boundary condition $u \cdot \nu = 0$ is fulfilled in a local sense away from 0.

We define the Stokes operator A_S as the part of A_p in $L^p_\sigma(G)$, i.e.,

$$\begin{aligned} A_S u &:= A_p|_{L^p_\sigma(G)} u, \quad u \in D(A_S), \\ D(A_S) &:= \{ v \in D(A_p) \cap L^p_\sigma(G) : \ A_p v \in L^p_\sigma(G) \}. \end{aligned} \tag{3.2}$$

The next lemma justifies this definition of the Stokes operator.

Lemma 3.1. *Let $1 < p < \infty$. Then*

$$D(A_S) = D(A_p) \cap L^p_\sigma(G).$$

Proof. We only have to show, that the right-hand side is a subset of $D(A_S)$. To this end, let $u \in D(A_p) \cap L^p_\sigma(G)$ and $f := A_p u$. It remains to show that $f \in L^p_\sigma(G)$. By the fact that $f = A_p u = \operatorname{curl}' \operatorname{curl} u$ and $u \in D(A_p) \cap L^p_\sigma(G)$, the Gauß theorem yields

$$\begin{aligned} \int_G f \cdot \nabla\varphi \, d(x_1, x_2) &= \int_G (\operatorname{curl}' \operatorname{curl} u) \cdot \nabla\varphi \, d(x_1, x_2) \\ &= -\left\langle \operatorname{curl} u, \, \nu \cdot \operatorname{curl}' \varphi \right\rangle_{W_p^{1-1/p}(\partial G), W_{p'}^{-1/p'}(\partial G)} = 0 \end{aligned}$$

for all $\varphi \in \widehat{W}^{1,p'}(G, \mathbb{R}^2)$. Note that $\operatorname{div} \operatorname{curl}' \varphi = 0$, hence the trace $\nu \cdot \operatorname{curl}' \varphi$ is defined in $W_{p'}^{-1/p'}(\partial G)$ in the usual sense. By the fact that $\operatorname{curl} u \in W^{1,p}(G, \mathbb{R}^2)$ therefore the duality pairing on the boundary above is well-defined. The proof is complete. □

Recall from (2.10) that $L^p(\Omega, \mathbb{R}^2)$ is decomposed in $L^p(\mathbb{R}, E^p_3)$ and $L^p(\mathbb{R}, \langle e_0, e_1, e_2 \rangle)$ with E^p_m defined in the lines before (2.10) and e_0, e_1, e_2 the normed eigenfunctions to the first three eigenvalues of the operator $\mathcal{T}_{p,\theta}$ introduced in Subsection 2.3.

In order to transfer the properties of $\mathbb{A}_p$ to the Stokes operator A_S a crucial point is that $\widetilde{\Theta}^p_* L^p(\mathbb{R}, \langle e_0, e_1, e_2 \rangle)$ does not contain non-trivial solenoidal vector fields. To carry over full Sobolev regularity, however, this fact is not enough. This purpose requires stronger properties:

Proposition 3.2. *Let $1 < p < \infty$. Then there exists a $\delta > 0$ such that*

1. $\|\widetilde{\mathbb{Q}} u\|_p \geq \delta \|u\|_p$ *for all* $u \in L^p_\sigma(G)$,
2. $\|\mathbb{Q} u\|_{K^2_p} \geq \delta \|u\|_{K^2_p}$ *for all* $u \in D(B_p)$ *such that* $\operatorname{div} u = 0$, *and*
3. $\|\mathbb{Q} u\|_{D(A_p)} \geq \delta \|u\|_{D(A_p)}$ *for all* $u \in D(A_S)$.

Remark 3.3. Proposition 3.2 relies of course on the specific structure of the solenoidal subspace. In fact, its proof (including the proof of the subsequent Lemma 3.4) shows that the operator 'div' is isomorphic on the complementary space to $\mathbb{L}^p$ and on the corresponding higher order complementary subspaces. Furthermore, it keeps the complementary structure in its image. This essentially can be read off the representations of the transformed 'div' operator applied on elements of the complementary subspaces given in (3.4) and (3.9) below.

Proof of Lemma 3.2(1). **Step 1.** Recall from Subsection 2.3 that the eigenfunctions to the first three eigenvalues $(\lambda_i)_{i\in\{0,1,2\}} \in \sigma(\mathcal{T}_{p,\theta})$ are explicitly given as

- $e_0(\theta) := \frac{1}{\sqrt{\theta_0}}\begin{pmatrix}1\\0\end{pmatrix}$ which corresponds to $\lambda_0 = -1$ and
- $e_k(\theta) := \frac{1}{\sqrt{\theta_0}}\begin{pmatrix}\cos(\frac{k\pi}{\theta_0}\theta)\\ -\sin(\frac{k\pi}{\theta_0}\theta)\end{pmatrix}$ which corresponds to $\lambda_k := -(\frac{k\pi}{\theta_0}-1)^2$ for $k \in \{1,2\}$.

We notice that, depending on the value of the angle θ_0, there might be a doubled eigenvalue. This, however, does not matter for what follows. An element $\varphi \in L^p(\mathbb{R}, \langle e_0, e_1, e_2\rangle)$ is then represented by

$$\varphi(x,\theta) = \varphi_0(x)e_0(\theta) + \varphi_1(x)e_1(\theta) + \varphi_2(x)e_2(\theta) \tag{3.3}$$

with coefficients $\varphi_i \in L^p(\mathbb{R})$ for $i \in \{0,1,2\}$.

Step 2. On our way to show (1) we first derive suitable estimates for $\varphi \in L^p(\mathbb{R}, \langle e_0, e_1, e_2\rangle)$ in terms of the transformed divergence operator. To this end, first observe that

$$\operatorname{div} \widetilde{\Theta}^p_* v \circ \psi = e^{(\beta_p-3)x}\left((\beta_p - 1 + \partial_x)v_x + \partial_\theta v_\theta\right) =: e^{(\beta_p-3)x}\operatorname{div}_{\widetilde{\Theta}} v.$$

Applying the transformed divergence operator $\operatorname{div}_{\widetilde{\Theta}}$ to representation (3.3) yields

$$\begin{aligned}\operatorname{div}_{\widetilde{\Theta}} \varphi = (\beta_p - 1 + \partial_x)\,\varphi_0 b_0 + \left(\beta_p - 1 - \frac{\pi}{\theta_0} + \partial_x\right)\varphi_1 b_1 \\ + \left(\beta_p - 1 - \frac{2\pi}{\theta_0} + \partial_x\right)\varphi_2 b_2\end{aligned} \tag{3.4}$$

where

$$\{b_0, b_1, b_2\} := \left\{\frac{1}{\sqrt{\theta_0}}, \frac{\cos(\frac{\pi}{\theta_0}\cdot)}{\sqrt{\theta_0}}, \frac{\cos(\frac{2\pi}{\theta_0}\cdot)}{\sqrt{\theta_0}}\right\} \tag{3.5}$$

is linearly independent in $L^p(I,\mathbb{R})$. We set

$$F_3^p := \langle b_0, b_1, b_2\rangle.$$

The form of the coefficients in (3.4) is

$$(s_j + \partial_x)\varphi_j, \quad s_j \in \mathbb{R},\ j = 0,1,2.$$

Observe that depending on the values of p and θ_0 it can occur $s_j = 0$. Thus, in order to estimate expression (3.4) by φ_j from below we distinguish two cases: $s_j \neq 0$ for all $j = 0,1,2$ or $s_j = 0$ for one $j \in \{0,1,2\}$.

Step 2.1. The case $s_j \neq 0$ for all $j = 0,1,2$. Then we have

$$s_j + \partial_x \in \mathscr{L}_{is}(L^p(\mathbb{R}), W^{-1,p}(\mathbb{R})). \tag{3.6}$$

Furthermore, since $F_3^{p'}$ is finite dimensional, we observe that $W^{1,p'}(\mathbb{R}, F_3^{p'})$ is isomorphic to the space

$$W^{1,p'}(\mathbb{R}, F_3^{p'}) \cap L^{p'}\big(\mathbb{R}, W^{1,p'}(I,\mathbb{R})\big).$$

This implies that the norm of $W^{1,p'}(\mathbb{R}, F_3^{p'})$ and the norm of $W^{1,p'}(\Omega, \mathbb{R})$ are equivalent on $W^{1,p'}(\mathbb{R}, F_3^{p'})$ and that the latter space can be regarded as a closed subspace of $W^{1,p'}(\Omega, \mathbb{R})$. Utilizing these facts, we can estimate as

$$\begin{aligned}
\|\varphi_j\|_p &\le C\|(s_j + \partial_x)\varphi_j\|_{W^{-1,p}(\mathbb{R})} \\
&\le C\Big\|\sum_{j=0}^{2}(s_j + \partial_x)\varphi_j b_j\Big\|_{W^{-1,p}(\mathbb{R}, F_3^p)} = C\|\mathrm{div}_{\widetilde{\Theta}}\varphi\|_{W^{-1,p}(\mathbb{R}, F_3^p)} \\
&= C \sup_{0\neq h\in W^{1,p'}(\mathbb{R}, F_3^{p'})} \frac{|\langle h, \mathrm{div}_{\widetilde{\Theta}}\varphi\rangle|}{\|h\|_{W^{1,p'}(\mathbb{R}, F_3^{p'})}} \\
&\le C \sup_{0\neq h\in W^{1,p'}(\Omega,\mathbb{R})} \frac{|\langle h, \mathrm{div}_{\widetilde{\Theta}}\varphi\rangle|}{\|h\|_{W^{1,p'}(\Omega,\mathbb{R})}} = \|\mathrm{div}_{\widetilde{\Theta}}\varphi\|_{W_0^{-1,p}(\Omega,\mathbb{R})},
\end{aligned}$$

for $j = 0, 1, 2$ with $C > 0$ independent of φ and where $W_0^{-1,p}(\Omega, \mathbb{R}) = (W^{1,p'}(\Omega, \mathbb{R}))'$.

Step 2.2. The case $s_\ell = 0$ for one $\ell \in \{0, 1, 2\}$. This case is more involved, since here we have

$$s_\ell + \partial_x = \partial_x \in \mathscr{L}_{is}(L^p(\mathbb{R}), \widehat{W}^{-1,p}(\mathbb{R})),$$

whereas for the remaining $j \in \{0, 1, 2\} \setminus \{\ell\}$ we still have (3.6). We set

$$U_j := \begin{cases} \widehat{W}^{-1,p}(\mathbb{R}, \langle b_j\rangle), & \text{if } j = \ell, \\ W^{-1,p}(\mathbb{R}, \langle b_j\rangle), & \text{if } j \in \{0,1,2\} \setminus \{\ell\}, \end{cases} \tag{3.7}$$

and

$$V := \overline{\mathrm{div}_{\widetilde{\Theta}} L^p(\mathbb{R}, E_3^p)}^{W^{-1,p}(\Omega,\mathbb{R})}. \tag{3.8}$$

In Lemma 3.4 below it is proved that the sum of $U_0 \oplus U_1 \oplus U_2$ and V is direct and consequently that

$$U_0 \oplus U_1 \oplus U_2 \oplus V, \quad \|\cdot\|_{U_0\oplus U_1\oplus U_2\oplus V} := \|\cdot\|_{U_0} + \|\cdot\|_{U_1} + \|\cdot\|_{U_2} + \|\cdot\|_V$$

is a Banach space. Then, this time we obtain

$$\begin{aligned}
\|\varphi_j\|_p &\le C\|(s_j + \partial_x)\varphi_j b_j\|_{U_j} \\
&\le C\Big\|\sum_{j=0}^{2}(s_j + \partial_x)\varphi_j b_j\Big\|_{U_0\oplus U_1\oplus U_2} = C\|\mathrm{div}_{\widetilde{\Theta}}\varphi\|_{U_0\oplus U_1\oplus U_2} \\
&\le C\|\mathrm{div}_{\widetilde{\Theta}}\varphi\|_{U_0\oplus U_1\oplus U_2\oplus V}
\end{aligned}$$

for $j = 0, 1, 2$ with $C > 0$ independent of φ.

Step 3. Now, let $u \in L^p_\sigma(G)$ and $\varphi \in L^p(\mathbb{R}, \langle e_0, e_1, e_2\rangle)$ such that $\widetilde{\mathbb{Q}}u = u - \widetilde{\Theta}^p_*\varphi$. Observe that both,

$$\mathrm{div}_{\widetilde{\Theta}} : L^p(\Omega, \mathbb{R}^2) \to W_0^{-1,p}(\Omega, \mathbb{R})$$

and by Lemma 3.4 also

$$\mathrm{div}_{\widetilde{\Theta}} : L^p(\Omega, \mathbb{R}^2) \to U_0 \oplus U_1 \oplus U_2 \oplus V$$

are bounded operators. By the fact that $\mathrm{div}_{\widetilde{\Theta}}\widetilde{\Theta}^*_p u = 0$, we can continue the calculations in steps 2.1 and 2.2 to the result that

$$\begin{aligned}
\|\varphi_j\|_p &\le C\|\mathrm{div}_{\widetilde{\Theta}}\,\varphi\|_{\mathcal{W}} = C\|\mathrm{div}_{\widetilde{\Theta}}(\widetilde{\Theta}^*_p u - \varphi)\|_{\mathcal{W}} \\
&\le C\|u - \widetilde{\Theta}^p_*\varphi\|_{L^p(G,\mathbb{R}^2)} = C\|\widetilde{\mathbb{Q}}u\|_p \quad (j = 0, 1, 2),
\end{aligned}$$

where $\mathcal{W}$ denotes either the space $W_0^{-1,p}(\Omega,\mathbb{R})$ or the space $U_0 \oplus U_1 \oplus U_2 \oplus V$, depending on whether we have $s_j \neq 0$ for all j or $s_j = 0$ for one j. Summing up over j yields

$$\|\varphi\|_p = \|\varphi\|_{L^p(\mathbb{R},\langle e_0,e_1,e_2\rangle)} \leq C \sum_{j=0}^{2} \|\varphi_j\|_p \leq C\|\widetilde{\mathbb{Q}}u\|_p$$

for all $u \in L^p_\sigma(G)$ and $\widetilde{\Theta}^p_* \varphi = (1-\widetilde{\mathbb{Q}})u$. By the fact that

$$C_0\|\varphi\|_{L^p(\Omega,\mathbb{R}^2)} \geq \|\widetilde{\Theta}^p_*\varphi\|_{L^p(G,\mathbb{R}^2)} = \|u - \widetilde{\mathbb{Q}}u\|_p \geq \|u\|_p - \|\widetilde{\mathbb{Q}}u\|_p$$

we arrive at (1) by setting $\delta := 1/(C_0C+1)$.

Proof of (2). The proof of (2) is in large parts similar to the proof of (1). Hence we will be briefer in detail.

Step 1. Again we will first provide estimates for $\varphi \in (1-\mathbb{P}_3)\Theta_p^* D(B_p)$ in terms of the transformed divergence. Note that such a φ is still represented by (3.3), but now with coefficients $\varphi_j \in W^{2,p}(\mathbb{R})$. The transformed divergence operator here is

$$\operatorname{div}\Theta^p_* v \circ \psi = e^{(\beta_p-1)x}\left((\beta_p+1+\partial_x)v_x + \partial_\theta v_\theta\right) =: e^{(\beta_p-1)x}\operatorname{div}_\Theta v.$$

Consequently,

$$\begin{aligned}\operatorname{div}_\Theta \varphi = (\beta_p+1+\partial_x)\,\varphi_0 b_0 + \left(\beta_p + 1 - \frac{\pi}{\theta_0} + \partial_x\right)\varphi_1 b_1 \\ + \left(\beta_p + 1 - \frac{2\pi}{\theta_0} + \partial_x\right)\varphi_2 b_2\end{aligned} \tag{3.9}$$

for $\varphi \in (1-\mathbb{P}_3)\Theta_p^* D(B_p)$. Again we write the coefficients as $(s_j+\partial_x)\varphi_j$. Here still s_1 and s_2 can vanish. Hence we again distinguish the two cases: $s_j \neq 0$ for all $j = 0,1,2$ or $s_j = 0$ for one $j \in \{1,2\}$.

Step 1.1. For the case $s_j \neq 0$ for all $j = 0,1,2$ we use

$$s_j + \partial_x \in \mathscr{L}_{is}(W^{2,p}(\mathbb{R}), W^{1,p}(\mathbb{R}))$$

in order to deduce

$$\begin{aligned}\|\varphi_j\|_{W^{2,p}(\mathbb{R})} &\leq C\|(s_j+\partial_x)\varphi_j\|_{W^{1,p}(\mathbb{R})} \\ &\leq C\Big\|\sum_{j=0}^{2}(s_j+\partial_x)\varphi_j b_j\Big\|_{W^{1,p}(\mathbb{R},F_3^p)} \leq C\|\operatorname{div}_\Theta\varphi\|_{W^{1,p}(\Omega,\mathbb{R})}\end{aligned}$$

for $j = 0,1,2$ with $C > 0$ independent of φ.

Step 1.2. If $s_\ell = 0$ for one $\ell \in \{1,2\}$ we use for that ℓ,

$$s_\ell + \partial_x = \partial_x \in \mathscr{L}_{is}(\widehat{W}^{2,p}(\mathbb{R}), \widehat{W}^{1,p}(\mathbb{R}))$$

to estimate

$$\begin{aligned}\|\varphi_\ell\|_{\widehat{W}^{2,p}(\mathbb{R})} &\leq C\|(s_\ell+\partial_x)\varphi_\ell\|_{\widehat{W}^{1,p}(\mathbb{R})} \leq C\|(s_\ell+\partial_x)\varphi_\ell\|_{W^{1,p}(\mathbb{R})} \\ &\leq C\Big\|\sum_{j=0}^{2}(s_j+\partial_x)\varphi_j b_j\Big\|_{W^{1,p}(\mathbb{R},F_3^p)} \leq C\|\operatorname{div}_\Theta\varphi\|_{W^{1,p}(\Omega,\mathbb{R})}\end{aligned}$$

with $C > 0$ independent of φ. The corresponding estimate for φ in the L^p-norm can be established completely analogous as in step 2.2 of the proof of (1). In this regard, observe that all assertions there as well as of Lemma 3.4 obviously remain true, if we replace $\operatorname{div}_{\widetilde{\Theta}}$ by $\operatorname{div}_\Theta$. Hence we obtain

$$\|\varphi_\ell\|_{L^p(\mathbb{R})} \leq C\|\operatorname{div}_\Theta\varphi\|_{U_0\oplus U_1\oplus U_2\oplus V}.$$

Taking into account the well-known interpolation estimate $\|\nabla v\|_{L^p(\mathbb{R})} \le C(\|\nabla^2 v\|_{L^p(\mathbb{R})} + \|v\|_{L^p(\mathbb{R})})$, altogether we have

$$\|\varphi_j\|_{W^{2,p}(\mathbb{R})} \le C\left(\|\mathrm{div}_\Theta\varphi\|_{W^{1,p}(\Omega,\mathbb{R})} + \|\mathrm{div}_\Theta\varphi\|_{U_0\oplus U_1\oplus U_2\oplus V}\right)$$

for $j = 0, 1, 2$ with $C > 0$ independent of φ.

Step 2. Let $u \in D(B_p)$ with $\mathrm{div}\, u = 0$ and $\varphi \in (1 - \mathbb{P}_3)\Theta_p^* D(B_p)$ such that $\mathbb{Q}u = u - \Theta_*^p\varphi$. Thanks to Lemma 3.4 and since

$$\mathrm{div}_\Theta : W^{2,p}(\Omega, \mathbb{R}^2) \to W^{1,p}(\Omega, \mathbb{R})$$

is bounded, by virtue of $\mathrm{div}_\Theta \Theta_p^* u = 0$ and the estimates in Steps 1.1 and 1.2 we conclude

$$\begin{aligned}\|\varphi_j\|_{W^{2,p}(\mathbb{R})} &\le C\left(\|\mathrm{div}_\Theta\, \varphi\|_{W^{1,p}(\Omega,\mathbb{R})} + \|\mathrm{div}_\Theta\varphi\|_{U_0\oplus U_1\oplus U_2\oplus V}\right)\\ &\le C\|\Theta_p^* u - \varphi\|_{W^{2,p}(\Omega,\mathbb{R})}\\ &\le C\|u - \Theta_*^p\varphi\|_{K_p^2(G,\mathbb{R}^2)} = C\|\mathbb{Q}u\|_{K_p^2(G,\mathbb{R}^2)} \quad (j = 0, 1, 2).\end{aligned}$$

Summing up over j, analogous to step 3 of the proof of (1) we arrive at (2).

Proof of (3). According to Lemma 2.17(1), $\|\cdot\|_p + \|\cdot\|_{K_p^2}$ is an equivalent norm on $D(A_p)$ and we have $\mathbb{Q} = \widetilde{\mathbb{Q}}$ on $D(A_p)$. The estimates proved in (1) and (2) then yield

$$\begin{aligned}\|u\|_{D(A_p)} &\le C\left(\|u\|_p + \|u\|_{K_p^2}\right) \le C\left(\|\mathbb{Q}u\|_p + \|\mathbb{Q}u\|_{K_p^2}\right)\\ &\le C\|\mathbb{Q}u\|_{D(A_p)} \quad (u \in D(A_S)).\end{aligned}$$

The proof is now completed. □

We have used the following facts in the proof of Proposition 3.2.

Lemma 3.4. *Let $1 < p < \infty$. Let U_j, $j = 0, 1, 2$, $\mathrm{div}_{\widetilde{\Theta}}$, and V be as defined in the proof of Proposition 3.2(1). Then U_0, U_1, U_2, V are Banach spaces, their sum is direct, and we have*

$$\mathrm{div}_{\widetilde{\Theta}} \in \mathscr{L}\left(L^p(\Omega, \mathbb{R}^2),\, U_0 \oplus U_1 \oplus U_2 \oplus V\right). \tag{3.10}$$

Proof. By their definition (3.7) and (3.8) it is obvious that U_0, U_1, U_2, V are Banach spaces and that the sum of U_0, U_1, U_2 is direct. Note that

$$L^p(\Omega, \mathbb{R}^2) = L^p(\mathbb{R}, E_3^p) \oplus L^p(\mathbb{R}, \langle e_0\rangle) \oplus L^p(\mathbb{R}, \langle e_1\rangle) \oplus L^p(\mathbb{R}, \langle e_2\rangle).$$

It is also obvious that $\mathrm{div}_{\widetilde{\Theta}} : L^p(\mathbb{R}, \langle e_j\rangle) \to U_j$ and hence also

$$\mathrm{div}_{\widetilde{\Theta}} : L^p(\mathbb{R}, \langle e_0, e_1, e_2\rangle) \to U_0 \oplus U_1 \oplus U_2 \tag{3.11}$$

is bounded (even isomorphic due to the estimates for φ in steps 2.1 and 2.2 of the proof of Proposition 3.2). Due to $\mathrm{div}_{\widetilde{\Theta}} \in \mathscr{L}\big(L^p(\Omega, \mathbb{R}^2), W^{-1,p}(\Omega, \mathbb{R})\big)$ we see that by definition of V the operator

$$\mathrm{div}_{\widetilde{\Theta}} : L^p(\mathbb{R}, E_3^p) \to V \tag{3.12}$$

is bounded too. It remains to prove that the sum of V and $U_0 \oplus U_1 \oplus U_2$ is direct.

To this end, denote by $\mathcal{Q}_3 : W^{1,p'}(\Omega, \mathbb{R}) \to W^{1,p'}(\Omega, \mathbb{R})$ the projector

$$\mathcal{Q}_3 v := \sum_{j=0}^{2} (v, b_j) b_j, \quad v \in W^{1,p'}(\Omega, \mathbb{R})$$

with b_j, $j = 0, 1, 2$, be defined as in (3.5). Writing

$$W^{1,p'}(\Omega, \mathbb{R}) = W^{1,p'}(\mathbb{R}, L^{p'}(I, \mathbb{R})) \cap L^{p'}(\mathbb{R}, W^{1,p'}(I, \mathbb{R}))$$

it is easily seen that $\mathcal{Q}_3$ is a bounded projector onto $W^{1,p'}(\mathbb{R}, F_3^{p'})$. Note that $(b_k)_{k=0}^\infty$ with $b_k(\theta) = \cos(k\pi\theta/\theta_0)/\sqrt{\theta_0}$ as the collection of eigenfunctions of the Neumann-Laplacian on the interval $I = (0, \theta_0)$ forms an orthonormal Hilbert basis of $L^2(I, \mathbb{R})$. This shows that $\mathcal{Q}_3$ is symmetric, hence $\mathcal{Q}_3$ is a bounded projector on

$$W_0^{-1,p}(\Omega, \mathbb{R}) = (W_0^{1,p'}(\Omega, \mathbb{R}))' = W^{-1,p}(\mathbb{R}, L^p(I, \mathbb{R})) + L^p(\mathbb{R}, W_0^{-1,p}(I, \mathbb{R})),$$

too. Since all norms on F_3^p are equivalent, for its image we calculate

$$\begin{aligned} \mathcal{Q}_3 W_0^{-1,p}(\Omega, \mathbb{R}) &= W^{-1,p}(\mathbb{R}, F_3^p) + L^p(\mathbb{R}, F_3^p) = W^{-1,p}(\mathbb{R}, F_3^p) \\ &= W^{-1,p}(\mathbb{R}, \langle b_0 \rangle) \oplus W^{-1,p}(\mathbb{R}, \langle b_1 \rangle) \oplus W^{-1,p}(\mathbb{R}, \langle b_2 \rangle). \end{aligned} \tag{3.13}$$

We next show that $V \subset (1 - \mathcal{Q}_3) W_0^{-1,p}(\Omega, \mathbb{R})$. By the fact that $(e_k)_{k=0}^\infty$ forms a basis of $L^2(I, \mathbb{R}^2)$ (see (2.6) and the subsequent lines), every $v \in L^2(\mathbb{R}, E_3^2)$ is represented as $v = \sum_{k=3}^\infty v_k e_k$ with $(v_k) \subset L^2(\mathbb{R})$. Hence we obtain

$$\begin{aligned} \operatorname{div}_{\widetilde{\Theta}} v &= \sum_{k=3}^\infty (\beta_2 - 1 + \partial_x) v_k e_k^1 + v_k \partial_\theta e_k^2 \\ &= \sum_{k=3}^\infty \left(\beta_2 - 1 \pm \frac{k\pi}{\theta_0} + \partial_x \right) v_k b_k. \end{aligned}$$

This shows that

$$\mathcal{Q}_3 \operatorname{div}_{\widetilde{\Theta}} v = 0 \quad \big(v \in L^p(\mathbb{R}, E_3^p) \cap L^2(\mathbb{R}, E_3^2)\big).$$

The boundedness of the operators $\operatorname{div}_{\widetilde{\Theta}}$, $\mathcal{Q}_3$ and a density argument yield that this identity remains true for all $v \in L^p(\mathbb{R}, E_3^p)$. Once more the boundedness of $\mathcal{Q}_3$ on $W_0^{-1,p}(\Omega, \mathbb{R})$ then gives $V \subset (1 - \mathcal{Q}_3) W_0^{-1,p}(\Omega, \mathbb{R})$.

Finally, $W^{1,p}(\mathbb{R}, \langle b_j \rangle) \overset{d}{\hookrightarrow} \widehat{W}^{1,p}(\mathbb{R}, \langle b_j \rangle)$ implies

$$\widehat{W}^{-1,p}(\mathbb{R}, \langle b_j \rangle) \hookrightarrow W^{-1,p}(\mathbb{R}, \langle b_j \rangle).$$

In combination with (3.13) this gives

$$U_0 \oplus U_1 \oplus U_2 \subset \mathcal{Q}_3 W_0^{-1,p}(\Omega, \mathbb{R}),$$

hence $V \cap \big(U_0 \oplus U_1 \oplus U_2\big) = \{0\}$.

Since we equip $U_0 \oplus U_1 \oplus U_2 \oplus V$ with the norm $\|\cdot\|_{U_0 \oplus U_1 \oplus U_2 \oplus V} := \|\cdot\|_{U_0} + \|\cdot\|_{U_1} + \|\cdot\|_{U_2} + \|\cdot\|_V$, relations (3.11) and (3.12) result in (3.10). Now all assertions are proved. □

Corollary 3.5. *Let $1 < p < \infty$. Then we have that*

1. $\widetilde{\mathbb{Q}} L_\sigma^p(G)$ *is closed in* $\mathbb{L}^p$ *and* $\widetilde{\mathbb{Q}} \in \mathscr{L}_{is}\Big(L_\sigma^p(G), \widetilde{\mathbb{Q}} L_\sigma^p(G)\Big)$,
2. $\mathbb{Q} D_\sigma$ *is closed in* $D(\mathbb{B}_p)$ *and* $\mathbb{Q} \in \mathscr{L}_{is}(D_\sigma, \mathbb{Q} D_\sigma)$, *where* $D_\sigma := \{v \in D(\mathbb{B}_p) : \operatorname{div} v = 0\}$, *and*
3. $\mathbb{Q} D(A_S)$ *is closed in* $D(\mathbb{A}_p)$ *and* $\mathbb{Q} \in \mathscr{L}_{is}(D(A_S), \mathbb{Q} D(A_S))$.

With these facts at hand we can prove our main result on the Stokes operator.

Proof of Theorem 1.1. Assume that $\lambda \in \rho(A_p)$. By the fact that A_S is the part of A_p from Lemma 3.1 we infer that

$$(\lambda - A_S)^{-1} = (\lambda - A_p)^{-1}|_{L_\sigma^p(G)}.$$

In combination with Lemma 2.17(3),(4) this implies

$$\mathbb{Q}(\lambda - A_S)^{-1} u = (\lambda - \mathbb{A}_p)^{-1} \widetilde{\mathbb{Q}} u \quad (u \in D(A_S)).$$

In particular, the above line yields $(\lambda - \mathbb{A}_p)^{-1}\widetilde{\mathbb{Q}}L^p_\sigma(G) \subset \mathbb{Q}D(A_S)$. Thus, thanks to Corollary 3.5 we conclude that

$$(\lambda - A_S)^{-1}f = \mathbb{Q}^{-1}(\lambda - \mathbb{A}_p)^{-1}\widetilde{\mathbb{Q}}f \quad (L^p_\sigma(G)). \tag{3.14}$$

For $1 < p < 1 + \delta$ with $\delta > 0$ given in Theorem 2.13 we know by that result that the resolvent set of A_p contains a suitable sector. For those p the assertion hence follows from Corollary 3.5 and Theorem 2.23. For general $p \in (1, \infty)$ representation (3.14) gives a candidate for the resolvent of A_S. In fact, choosing $1 < q < 1 + \delta$, on $L^p_\sigma(G) \cap L^q_\sigma(G)$ we already know that it is the resolvent. A density argument and again Corollary 3.5 and Theorem 2.23 then yield the assertion. □

Remark 3.6. From Proposition 2.9(1) and Theorem 2.22 it also follows consistency of the resolvent of A_S, that is, for every $\lambda \in \rho(A_S)$ the family $\big((\lambda - A_S)^{-1}\big)_{1<p<\infty}$ is consistent on the scale $(L^p_\sigma(G))_{1<p<\infty}$.

Finally we prove our third main result.

Proof of Theorem 1.5. We follow the strategy in the proof of Theorem 1.1. For $f \in L^p_\sigma(G)$ the candidate for the solution of

$$\left.\begin{aligned} -\Delta u + \nabla\pi = f \quad &\text{in} \quad G, \\ \operatorname{div} u = 0 \quad &\text{in} \quad G, \\ \operatorname{curl} u = 0,\ u\cdot\nu = 0 \quad &\text{on} \quad \partial G \end{aligned}\right\} \tag{3.15}$$

is given as $\pi = 0$ and $u = \mathbb{Q}^{-1}\mathbb{B}_p^{-1}\widetilde{\mathbb{Q}}f$. Thanks to Proposition 2.9 and Corollary 3.5 it remains to show that $\operatorname{div} u = 0$. This, in turn, follows from Proposition 2.24, $\mathbb{Q}^{-1}(\lambda - \mathbb{A}_p)^{-1}\widetilde{\mathbb{Q}}f \subset D(A_S)$, and the fact that the operator div acts continuously on the space $K^2_p(G, \mathbb{R}^2)$. □

Appendix A. Elements from harmonic and functional analysis

The following facts might be well-known. Since we could not find an appropriate reference, we give their proofs here.

Lemma A.1. *Let X, Y be Banach spaces such that $X \hookrightarrow Y$. Then we have*

$$C^\infty_c(\mathbb{R}, X) \overset{d}{\hookrightarrow} W^{k,p}(\mathbb{R}, X) \cap W^{\ell,p}(\mathbb{R}, Y)$$

for every $k, \ell \in \mathbb{N}_0$ and $p \in (1, \infty)$.

Proof. First recall that

$$C^\infty_c(\mathbb{R}, E) \overset{d}{\hookrightarrow} W^{k,p}(\mathbb{R}, E)$$

for every $k \in \mathbb{N}_0$, $p \in (1, \infty)$, and arbitrary Banach space E. In fact, it is standard to construct a (universal) sequence of operators $(\Phi_k)_{k\in\mathbb{N}}$ such that for $u \in W^{k,p}(\mathbb{R}, E)$ we have $(\Phi_k u) \subset C^\infty_c(\mathbb{R}, E)$ and

$$\Phi_k u \to u \quad \text{in } W^{k,p}(\mathbb{R}, E) \quad (k \to \infty)$$

for every $k \in \mathbb{N}_0$, $p \in (1, \infty)$, and arbitrary Banach space E. Since $X \subset Y$, for $u \in W^{k,p}(\mathbb{R}, X) \cap W^{\ell,p}(\mathbb{R}, Y)$ this gives $\Phi_k u \to u$ in $W^{k,p}(\mathbb{R}, X)$ and in $W^{\ell,p}(\mathbb{R}, Y)$. □

Let $T : D(T) \subset X \to X$ be a closed, densely defined operator on a Banach space X. We denote by

$$T^\sharp : X' \to D(T)'$$

the dual operator of T, regarded as a bounded operator from $D(T)$ to X, and by

$$T' : D(T') \subset X' \to X'$$

the usual Banach space dual operator of T. The fact that $D(T) \subset X$ is dense, obviously implies $D(T') \hookrightarrow X' \hookrightarrow D(T)'$ and that

$$T^\sharp|_{D(T')} = T'. \tag{A.1}$$

Furthermore, we have the following lemma on consistency.

Lemma A.2. *Let X be a reflexive Banach space and let $T : D(T) \subset X \to X$ be densely defined such that $T \in \mathscr{L}_{is}(D(T), X)$. Assume there is an embedding (with means i.p. injection) $J : D(T) \to X'$ with dense range. Then there exists an embedding $\widetilde{J} : X \to D(T)'$ such that, if $\widetilde{J} \circ T \subset T^\sharp \circ J$, we have*

$$J \circ T^{-1} \circ \widetilde{J}^{-1}|_{\widetilde{J}X \cap X'} = (T^\sharp)^{-1}|_{\widetilde{J}X \cap X'} = (T')^{-1}|_{\widetilde{J}X \cap X'} \quad \text{in } X'. \tag{A.2}$$

Proof. Since $\overline{D(T)} = X$ we have $X' \hookrightarrow D(T)'$. Reflexivity of X and $J(D(T)) \overset{d}{\hookrightarrow} X'$ further imply that there is an embedding $\widetilde{J} : X \to D(T)'$. Thus, $\widetilde{J}X \cap X'$ is well-defined and due to $T \in \mathscr{L}_{is}(D(T), X)$ which also implies $T^\sharp \in \mathscr{L}_{is}(X', D(T)')$ and $T' \in \mathscr{L}_{is}(D(T'), X')$, line (A.2) is meaningful.

Now, let $z \in \widetilde{J}X \cap X'$ and set $x_1 := JT^{-1}\widetilde{J}^{-1}z \in X'$ and $x_2 := (T^\sharp)^{-1}z \in X'$. Thanks to $\widetilde{J} \circ T \subset T^\sharp \circ J$ we obtain

$$\begin{aligned} T^\sharp(x_1 - x_2) &= T^\sharp \left(JT^{-1}\widetilde{J}^{-1}z - (T^\sharp)^{-1}z \right) \\ &= \widetilde{J}TT^{-1}\widetilde{J}^{-1}z - T^\sharp(T^\sharp)^{-1}z = z - z = 0. \end{aligned}$$

Thus $x_1 = x_2$ in X' and the assertion is proved. The second equality in (A.2) follows in a similar manner from (A.1). □

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Mathematisches Institut, Angewandte Analysis, Heinrich-Heine-Universität Düsseldorf, 40204 Düsseldorf, Germany

E-mail address: matthias.koehne@hhu.de

E-mail address: juergen.saal@hhu.de

E-mail address: laura.westermann@hhu.de

Chapter 3

Manuscript 2
The Dirichlet Stokes Operator on a 2D Wedge Domain in L^p: Sectoriality and Optimal Regularity

Joint Work with Matthias Köhne and Jürgen Saal.

THE DIRICHLET-STOKES OPERATOR ON A 2D WEDGE DOMAIN IN L^P: SECTORIALITY AND OPTIMAL REGULARITY

MATTHIAS KÖHNE, JÜRGEN SAAL, AND LAURA WESTERMANN

ABSTRACT. In this note we prove that the solution of the Stokes equations subject to Dirichlet boundary conditions on a 2D wedge domain admits optimal regularity in the L^p-setting for a small neighborhood of p about 2. Here, optimal regularity means that the domain of the Stokes operator in L^p_σ is embedded in $W^{2,p}$. Furthermore, we obtain sectoriality for the Stokes operator with spectral angle equal to zero for the same range of p.

CONTENTS

1. INTRODUCTION AND MAIN RESULT

The objective of this note is to consider the Stokes resolvent problem on a two-dimensional wedge type domain subject to Dirichlet boundary conditions and to derive best possible regularity in the L^p-setting for p in a small neighborhood of $p = 2$. The problem reads as

$$\left.\begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } G,\\ \operatorname{div} u &= 0 \quad \text{in } G,\\ u &= 0 \quad \text{on } \partial G, \end{aligned}\right\} \tag{1.1}$$

where G represents the wedge domain

$$G := \{(x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_2 < x_1 \tan \theta_0\} \tag{1.2}$$

with opening angle $\theta_0 \in (0, \pi)$.

There still exits a Lipschitz approach to the existence and analyticity of the Stokes semigroup on L^p_σ on Lipschitz domains, see e.g. in [12, 13, 8, 14]. Since in the Lipschitz approach $W^{2,p}$-regularity is not avalaible, this approach seems to be too general for our purpose. Whereas one may find for instance in [5, 6] an L^p-theory for the nonstationary Stokes equations in cone domains, similar results on wedge domains are not obviously available.

The main result of this note, which is formulated in Theorem 8.2, establishes resolvent estimates on $L^p_\sigma(G)$ for the solution $(u, \nabla p) \in W^{2,p}(G, \mathbb{R}^2) \times L^p(G, \mathbb{R}^2)$ of system (1.1) for p in a small neighborhood of $p = 2$.

We outline the strategy of the proof of Theorem 8.2 and the organization of this note. In Section 2 we fix the notation. For the proof of the main theorem, we initially consider the stationary Stokes equations

$$\left.\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } G, \\ \operatorname{div} u &= 0 \quad \text{in } G, \\ u &= 0 \quad \text{on } \partial G. \end{aligned}\right\} \tag{1.3}$$

Then using the stream function $u = \begin{pmatrix} -\partial_2 \phi \\ \partial_1 \phi \end{pmatrix}$ and applying curl $u = \partial_2 u_1 - \partial_1 u_2$ to (1.3) we get the corresponding bi-Laplacian problem

$$\left.\begin{aligned} \Delta^2 \phi &= F \quad \text{in } G, \\ \partial_1 \phi = 0, \ \partial_2 \phi &= 0 \quad \text{on } \partial G, \end{aligned}\right\} \tag{1.4}$$

where $F := \operatorname{curl} f = \partial_2 f_1 - \partial_1 f_2$.

Our first results concern the solvability of (1.4) in two weak settings: $\widehat{W}^{-1,p}(G)$ for all $p \in (1,2) \cup (2,\infty)$ and $\widehat{W}^{-2,p}(G)$ for all $p \in (1,\infty) \setminus N$, where $N \subset (1,\infty)$ is a finite set. For the proof of these two results, we follow the strategy in [4, 7, 9], that is, by employing polar coordinates and Euler transformation we reduce (1.4) on a wedge domain to a problem on a layer, see Section 3. Then, to solve the problem on the layer, we use results given in [4]. In fact, in [4] bi-Laplacian problems on polygonal domains are considered and after localizing the vertices and transforming them to the layer leads exactly to the transformed problem of (1.4). Since problem (1.4) on the wedge and its transformed version on the layer are equivalent, the solvability on the layer implies the well-posedness of (1.4) in $\widehat{W}^{-1,p}(G)$ for all $p \in (1,p) \cup (2,\infty)$ and in $\widehat{W}^{-2,p}(G)$ for all $p \in (1,\infty) \setminus N$, see Proposition 5.3 and Proposition 5.7, respectively. As a consequence, we obtain weak and strong well-posedness of (1.3) in the underlying setting. This is contained in Theorem 5.4 and Theorem 5.8, respectively. Based on these results we then can prove Theorem 8.2. This will be proved in three steps: We first consider the weak formulation of the resolvent problem (1.1) which is given by

$$\lambda(u,v) + (\nabla u, \nabla v) = \langle f, v \rangle_{W^{-1,p}_\sigma, W^{1,p'}_{0,\sigma}} \quad (v \in W^{1,p'}_{0,\sigma}).$$

Making use of functional analytic tools and an extrapolation result due to Sneiberg in the version of [11, Theorem 2.7], we prove that the corresponding Stokes operator A_p is sectorial in L^p_σ for $p \in I_\kappa = ((2+\kappa)', 2+\kappa)$ and $\kappa > 0$ sufficiently small. By this approach we also deduce that the Stokes resolvent is consistent on $(W^{-1,p}_\sigma)_{p \in I_\kappa}$, see Proposition 6.5

and Proposition 6.4. Then, in a second step we consider the weak formulation of problem (1.3) which is

$$(\nabla u, \nabla v) = \langle f, v\rangle_{\widehat{W}_\sigma^{-1,p}, \widehat{W}_{0,\sigma}^{1,p'}} \quad (v \in \widehat{W}_{0,\sigma}^{1,p'}). \tag{1.5}$$

Thanks to Theorem 5.4 and to the consistency of the Stokes resolvent, in Proposition 7.4 we can prove that the solution operator $\widehat{A}_p^{-1}$ of (1.5) is consistent on $(\widehat{W}_\sigma^{1,p})_{p\in(1,\infty)}$. By regarding u as the unique solution of $-\Delta u = f - \lambda u \in L^p_\sigma \cap \widehat{W}_\sigma^{-1,p}$, using consistency of $\widehat{A}_p$ yields that $\widehat{A}_p u = A_p u = f - \lambda u \in L^p_\sigma \cap \widehat{W}_\sigma^{-1,p}$. Since Theorem 5.8 implies the existence of a unique solution v of $-\Delta v = f - \lambda v \in L^p_\sigma$ and since A_p is consistent, we can prove that $u = v$. Then Theorem 8.2 follows.

2. Notation

Throughout this note we will use standard notation. The norm in a Banach space X will be denoted by $\|\cdot\|_X$. Let Y be another Banach space. By $\mathscr{L}(X, Y)$ we denote the class of all bounded linear operators from X to Y, whereas $\mathscr{L}_{is}(X, Y)$ stands for its subclass of isomorphisms, and we write $\mathscr{L}(X)$, $\mathscr{L}_{is}(X)$ in case of $X = Y$. The (abstract) topological dual is defined as $X' = \mathscr{L}(X, \mathbb{C})$. Its elements are given by linear continuous functionals

$$\ell : X \to \mathbb{C}, \quad x \mapsto \ell(x),$$

and the norm on X' is given as

$$\|\ell\|_{X'} = \sup_{0\neq x\in X} \frac{|\ell(x)|}{\|x\|_X}.$$

For a linear operator A in X we denote its domain by $D(A)$. Its spectrum is given as $\sigma(A)$ and its resolvent set as $\rho(A)$. We say that the operator A is sectorial if $\overline{D(A)} = \overline{R(A)} = X$, $(0, \infty) \subset \rho(-A)$ and the family $(\lambda(\lambda + A)^{-1})_{\lambda>0}$ is uniformly bounded. By ϕ_A we denote the corresponding spectral angle.

Let $A : D(A) \subset X \to X$ be a closed, densely defined operator on a Banach space X. Then we denote by

$$A' : D(A') \subset X' \to X'$$

the usual Banach space dual operator of A and by

$$A^\# : X' \to D(A)'$$

the dual operator of A, when A is regarded as a bounded operator from $D(A)$ to X.

Let $\Omega \subset \mathbb{R}^2$ be a domain. We set $C_c^\infty(\Omega) := \{u \in C^\infty(\Omega) : \operatorname{supp}(u) \Subset \Omega\}$ and $C_{c,\sigma}^\infty(\Omega) := \{u \in C_c^\infty(\Omega, \mathbb{R}^2) : \operatorname{div} u = 0\}$ where $\operatorname{supp}(u)$ is the support of u. Let X be a Banach space and let $1 \le p \le \infty$. We denote by $L^p(\Omega, X)$ the X-valued Borel-Lebesgue space. The space of solenoidal fields in $L^p(\Omega)$ is defined by $L^p_\sigma(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}$ for $1 < p < \infty$. Let $n \in \mathbb{N}$, we define $W^{k,p}(\Omega, \mathbb{R}^n)$ to be the Sobolev space of order $k \in \mathbb{N}_0$ and $W^{0,p} := L^p$. We denote by $W_0^{k,p}(\Omega, \mathbb{R}^n)$ the closure of $C_c^\infty(\Omega, \mathbb{R}^n)$ in the space $W^{k,p}(\Omega, \mathbb{R}^n)$. We will also need the homogeneous Sobolev space

$$\widehat{W}^{k,p}(\Omega, \mathbb{R}^n) := \{u \in L^1_{loc}(\Omega, \mathbb{R}^n) : \ \partial^\alpha u \in L^p(\Omega, \mathbb{R}^n), \ |\alpha| = k\}$$

for $n \in \mathbb{N}$, with seminorm

$$\|u\|_{\widehat{W}^{k,p}} := \|u\|_{\widehat{W}^{k,p}(\Omega,\mathbb{R}^n)} := \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p(\Omega,\mathbb{R}^n)}.$$

By $\widehat{W}_0^{k,p}(\Omega,\mathbb{R}^n)$ we denote the completion of $(C_c^\infty(\Omega,\mathbb{R}^n),\ \|\cdot\|_{\widehat{W}^{k,p}(\Omega,\mathbb{R}^n)})$. We set $W_{0,\sigma}^{k,p}(\Omega) := W_0^{k,p}(\Omega,\mathbb{R}^2)\cap L^p_\sigma(\Omega)$ and

$$\widehat{W}_{0,\sigma}^{k,p}(\Omega) := \left\{ v \in \widehat{W}_0^{k,p}(\Omega,\mathbb{R}^2):\ \operatorname{div} v = 0 \right\}.$$

Note that $W_{0,\sigma}^{k,p}(\Omega)$ equals the completion of $(C_{c,\sigma}^\infty(\Omega),\ \|\cdot\|_{W^{k,p}(\Omega,\mathbb{R}^n)})$ and that $\widehat{W}_{0,\sigma}^{k,p}(\Omega)$ equals the completion of $(C_{c,\sigma}^\infty(\Omega),\ \|\cdot\|_{\widehat{W}^{k,p}(\Omega,\mathbb{R}^n)})$.

Now, let $G\subset\mathbb{R}^2$ be the wedge domain defined as in (1.2) and $n\in\{1,2\}$. We define the Kondrat'ev spaces by

$$L^p_\gamma(G,\mathbb{R}^n) := L^p(G,\rho^\gamma d(x_1,x_2),\mathbb{R}^n), \quad \rho = |(x_1,x_2)|,\ \gamma\in\mathbb{R},$$

and

$$K^m_{p,\gamma}(G,\mathbb{R}^n) := \{u\in L^1_{loc}(G,\mathbb{R}^n):\ \rho^{|\alpha|-m}\partial^\alpha u \in L^p_\gamma(G,\mathbb{R}^n),\ |\alpha|\le m\},$$

where $\alpha\in\mathbb{N}^2$ denotes a multiindex, $\gamma\in\mathbb{R}$. Then $K^m_{p,\gamma}(G,\mathbb{R}^n)$ equipped with the norm

$$\|u\|_{K^m_{p,\gamma}} := \|u\|_{K^m_{p,\gamma}(G,\mathbb{R}^n)} := \left(\sum_{|\alpha|\le m} \|\rho^{|\alpha|-m}\partial^\alpha u\|^p_{L^p_\gamma(G,\mathbb{R}^n)}\right)^{1/p}$$

is a Banach space, and we set $K^m_p(G,\mathbb{R}^n) := K^m_{p,0}(G,\mathbb{R}^n)$. The weighted homogeneous Sobolev space is defined by

$$\widehat{W}^{k,p}_\gamma(G) := \{u\in L^1_{loc}(G):\ \partial^\alpha u\in L^p_\gamma(G),\ |\alpha| = k\},$$

with the seminorm

$$\|u\|_{\widehat{W}^{k,p}_\gamma(G,\mathbb{R}^n)} := \|u\|_{\widehat{W}^{k,p}_\gamma} := \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p_\gamma(G,\mathbb{R}^n)}$$

where $\gamma\in\mathbb{R}$. We define $\widehat{W}^{k,p}_{0,\gamma}(G,\mathbb{R}^n)$ to be the completion of $(C_c^\infty(G,\mathbb{R}^n),\ \|\cdot\|_{\widehat{W}^{k,p}_\gamma(G,\mathbb{R}^n)})$ for $n\in\{1,2\}$.

3. Transformation of the problem

In this chapter we consider

$$\left.\begin{aligned} \Delta^2\phi &= F \quad \text{in} \quad G,\\ \partial_{x_1}\phi = 0,\ \partial_{x_2}\phi &= 0 \quad \text{on} \quad \partial G, \end{aligned}\right\} \tag{3.1}$$

on a two-dimensional wedge domain G and transform it onto a layer domain of the form $\Omega := \mathbb{R}\times I$, where $I := (0,\theta_0)$ and θ_0 denotes the angle of the wedge G. To this end, we apply a standard procedure as utilized also in [4, 9, 7]: In the first step we introduce polar coordinates to transform the problem on a semi-layer; by employing Euler transformation the latter problem is transformed on a layer; finally we rescale the appearing terms such that we can work in the transformed setting in unweighted $W^{-k,p}$-spaces for $k\in\{1,2\}$.

We write the inverse of the transform to polar coordinates as

$$\psi_P:\ \mathbb{R}_+\times I\to G, \quad (r,\theta)\mapsto (r\cos\theta, r\sin\theta) = (x_1,x_2).$$

Then we apply the Euler transformation $r = e^x$ in radial direction and write by an abuse of notation $x\in\mathbb{R}$ for the new variable. We set

$$\psi_E:\ \Omega\to\mathbb{R}_+\times I, \quad (x,\theta)\mapsto (e^x,\theta) =: (r,\theta).$$

It is not difficult to see that

$$\psi := \psi_P\circ\psi_E:\ \Omega\to G$$

is a diffeomorphism. We set

$$\Psi\phi := \phi \circ \psi \quad \text{and} \quad \Psi^{-1}\varphi := \varphi \circ \psi^{-1}.$$

For $\alpha \in \mathbb{R}$ we denote the multiplication operator by

$$\mathcal{M}_\alpha \varphi := e^{\alpha x}\varphi. \tag{3.2}$$

Analogously to [9] we define pull-back and push-forward as follows:

Definition 3.1. Let $\beta_p \in \mathbb{R}$ be given, $\mathcal{M}_\alpha$, Ψ defined as above. Suppose ϕ is the solution of (3.1). Then pull-back and its inverse push-forward are defined through

$$\varphi := \Theta_p^* \phi := \mathcal{M}_{-\beta_p}\Psi\phi \quad \text{and} \quad \phi = \Theta_*^p \varphi = \Psi^{-1}\mathcal{M}_{\beta_p}\varphi, \tag{3.3}$$

respectively.

Let ϕ be the solution of problem (3.1), $\beta_p \in \mathbb{R}$. Lemma B.4 implies for $i = 2$ that the transformed bi-Laplacian reads as

$$\Theta_p^*(\Delta^2\phi) = e^{-4x}(r_{\beta_p-2}(\partial_x) + \partial_\theta^2)\left(r_{\beta_p}(\partial_x) + \partial_\theta^2\right)\varphi \tag{3.4}$$

with the polynomial

$$r_a(\partial_x) := (\partial_x + a)^2 \quad (a \in \mathbb{R}). \tag{3.5}$$

In order to absorb the factor e^{-4x} in (3.4), we set

$$g := \widetilde{\Theta}_p^* F := e^{4x}\Theta_p^* F \tag{3.6}$$

with inverse $(\widetilde{\Theta}_p^*)^{-1} = \widetilde{\Theta}_*^p$.

Let $1 < p < \infty$. In this note we will first show that problem (3.1) is well-posed in two weak settings, i.e., we consider (3.1) in $\widehat{W}^{-1,p}(G)$ and $\widehat{W}^{-2,p}(G)$, respectively. Here the choice of β_p plays an important role. We set

$$\beta_p := \beta_{p,-k} = 4 - k - \frac{2+\gamma}{p} \quad (k \in \{1, 2\},\ \gamma \in \mathbb{R}). \tag{3.7}$$

We notice that by this choice of β_p, pull-back and push-forward depend explicitly on p and k, i.e., the corresponding families are neither consistent in p nor in k. To indicate the dependence of $\beta_{p,k}$ on k we put a sub- or superscript on $\widetilde{\Theta}_{p,k}^* := \widetilde{\Theta}_p^*$ and $\widetilde{\Theta}_*^{p,k} := \widetilde{\Theta}_*^p$ and the same for $\Theta_{p,k}^* := \Theta_p^*$ and $\Theta_*^{p,k} := \Theta_*^p$. To work in the unweighted spaces $W^{-1,p}(\Omega)$ and $W^{-2,p}(\Omega)$ we choose $k = 1$ and $k = 2$, respectively. Then Lemma B.3 (5) implies for $k = 1$ that

$$\widetilde{\Theta}_p^* = \widetilde{\Theta}_{p,1}^* \in \mathscr{L}_{is}(\widehat{W}_\gamma^{-1,p}(G), W^{-1,p}(\Omega)),$$

and for $k = 2$ that

$$\widetilde{\Theta}_p^* = \widetilde{\Theta}_{p,2}^* \in \mathscr{L}_{is}(\widehat{W}_\gamma^{-2,p}(G), W^{-2,p}(\Omega)).$$

Next, we transform the boundary conditions of (3.1) from the wedge onto the layer. From

$$\partial_{x_1}\phi = 0, \quad \partial_{x_2}\phi = 0 \quad \text{on } \partial G,$$

we deduce

$$\beta_p\varphi + \partial_x\varphi = 0, \quad \partial_\theta\varphi = 0 \quad \text{on } \partial\Omega = \mathbb{R} \times \{0, \theta_0\}. \tag{3.8}$$

Since the general solution of the ODE $\beta_p\varphi + \partial_x\varphi = 0$ is $\varphi(x, \theta) = \alpha(\theta)e^{-\beta_p x}$ for all $x \in \mathbb{R}$ and $\theta \in \{0, \theta_0\}$ with $\alpha(\theta)$ constant in x, it follows for $\varphi \in L^p(\partial\Omega)$ that $\alpha(\theta) = 0$ for $\theta \in \{0, \theta_0\}$ and hence $\varphi = 0$ on $\partial\Omega$. Then (3.8) is equivalent to

$$\partial_\theta\varphi = 0, \quad \varphi = 0 \quad \text{on} \quad \partial\Omega = \mathbb{R} \times \{0, \theta_0\}.$$

The transformed problem on $\Omega = \mathbb{R} \times I$ is then given as

$$\left.\begin{aligned} \left(r_{\beta_p-2}(\partial_x)+\partial_\theta^2\right)\left(r_{\beta_p}(\partial_x)+\partial_\theta^2\right)\varphi = g \quad &\text{in} \quad \Omega, \\ \partial_\theta\varphi = 0,\ \varphi = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}\right\} \tag{3.9}$$

4. Weak regularity of the transformed problem

Here we consider problem (3.9). Let $1 < p < \infty$. We introduce the operator which is associated to problem (3.9). We define

$$T_{p,-k}\varphi := T_p^{a_{-k}}T_p^{b_{-k}}\varphi := \left(\partial_\theta^2 + r_{a_{-k}}(\partial_x)\right)\left(\partial_\theta^2 + r_{b_{-k}}(\partial_x)\right)\varphi$$

with $a_{-k} := \beta_{p,-k} - 2$, $b_{-k} := \beta_{p,-k}$ and with $\beta_{p,-k}$ be defined as in (3.7) for $k \in \{1,2\}$.

Utilizing results derived in [4], we will prove optimal regularity for problem (3.9) in the two weak settings and for certain ranges of p. This is reduced to invertibility of the operator

$$T_{p,-k}:\ D(T_{p,-k}) \to W^{-k,p}(\Omega) \tag{4.1}$$

for $k \in \{1,2\}$ with

$$D(T_{p,-2}) := W_0^{2,p}(\Omega) \quad \text{and} \quad D(T_{p,-1}) := W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega).$$

4.1. **Weak well-posedness of the transformed problem in $W^{-2,p}(\Omega)$.** Here we show invertibility of (4.1) for $k = 2$. Note that this case is not explicitly included in [4]. However, it can be reduced to results in [4] and a duality and an interpolation argument. In fact, utilizing [4, Lemma 7.3.1.3, Theorem 7.3.1.8] we will first show strong optimal regularity of

$$\left.\begin{aligned} T_p^{\pm a_{-2}}T_p^{\pm b_{-2}}\varphi = g \quad &\text{in} \quad \Omega, \\ \partial_\theta\varphi = 0,\ \varphi = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}\right\} \tag{4.2}$$

To be precise, we prove that $T_p^{\pm a_{-2}}T_p^{\pm b_{-2}}:\ D(T_p^{\pm a_{-2}}T_p^{\pm b_{-2}}) \to L^p(\Omega)$ is isomorphic with

$$D(T_p^{\pm a_{-2}}T_p^{\pm b_{-2}}) = W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$$

for all $p \in (1,\infty)$ such that the condition

$$\left.\begin{array}{c} \text{the characteristic equation} \\ \sinh^2(\lambda\theta_0) = \lambda^2\sin^2(\theta_0) \\ \text{has no solution on the line} \\ \operatorname{Im}\lambda = -(1-(2+\gamma)/p), \quad \lambda\in\mathbb{C},\ \gamma\in\mathbb{R}, \end{array}\right\}$$

is satisfied. Then we show that

$$T_p^{a_{-2}}T_p^{b_{-2}} \subset (T_{p'}^{-a_{-2}}T_{p'}^{-b_{-2}})^{\#} \in \mathscr{L}_{is}\big(L^p(\Omega), D(T_{p'}^{-a_{-2}}T_{p'}^{-b_{-2}})'\big).$$

Finally, we apply an interpolation argument.

We start by applying partial Fourier transform in x to the operator $T_p^{\pm a_{-2}}T_p^{\pm b_{-2}}$ which yields the following representations.

Lemma 4.1. *Let $a_{-2} = \beta_{p,-2} - 2$ and $b_{-2} = \beta_{p,-2}$. Then we have*

$$\mathcal{F}\left(T_p^{a_{-2}}T_p^{b_{-2}}\varphi\right)(\tau,\theta) = \left((\mathcal{E}_+^4 + 2\mathcal{E}_+^2 + 1) + (2 - 2\mathcal{E}_+^2)\partial_\theta^2 + \partial_\theta^4\right)\widehat{\varphi}(\tau,\theta)$$

$$\mathcal{F}\left(T_p^{-a_{-2}}T_p^{-b_{-2}}\varphi\right)(\tau,\theta) = \left((\mathcal{E}_-^4 + 2\mathcal{E}_-^2 + 1) + (2 - 2\mathcal{E}_-^2)\partial_\theta^2 + \partial_\theta^4\right)\widehat{\varphi}(\tau,\theta)$$

with $\mathcal{E}_\pm := \pm\tau + i(-\beta_{p,-2} + 1)$ for $\tau \in \mathbb{R}$.

Proof. Since the following calculation does not depend on $\beta_{p,-2}$, i.e. it is fulfilled for all $\beta_p \in \mathbb{R}$ we write in the following β_p instead of $\beta_{p,-2}$, a for a_{-2} and b for b_{-2}. It is straight forward to compute

$$\begin{aligned} T_p^a T_p^b \varphi &= \left((\partial_x+\beta_p)^4 - 4(\partial_x+\beta_p)^3 + 4(\partial_x+\beta_p)^2\right)\varphi \\ &\quad + \left((\partial_x+\beta_p-2)^2 + (\partial_x+\beta_p)^2\right)\partial_\theta^2\varphi + \partial_\theta^4\varphi. \end{aligned}$$

Applying partial Fourier transform in x to the first term of the $T_p^a T_p^b$ operator in x and substituting $\mathcal{E}_+ = \tau + i(-\beta_p+1) \Leftrightarrow i\tau + \beta_p = i\mathcal{E}_+ + 1$ we get

$$\begin{aligned} &\mathcal{F}\left(\left((\partial_x+\beta_p)^4 - 4(\partial_x+\beta_p)^3 + 4(\partial_x+\beta_p)^2\right)\varphi\right)(\tau,\theta) \\ &\quad = \left((i\tau+\beta_p)^4 - 4(i\tau+\beta_p)^3 + 4(i\tau+\beta_p)^2\right)\widehat{\varphi}(\tau,\theta) \\ &\quad = (\mathcal{E}_+^4 + 2\mathcal{E}_+^2 + 1)\widehat{\varphi}(\tau,\theta). \end{aligned}$$

With the same calculation as above we have for the second term of $T_p^a T_p^b$

$$\begin{aligned} &\mathcal{F}\left(\left((\partial_x+\beta_p-2)^2 + (\partial_x+\beta_p)^2\right)\partial_\theta^2\varphi\right)(\tau,\theta) \\ &\quad = \left(2(i\mathcal{E}_+ + 1)^2 - 4(i\mathcal{E}_+ + 1) + 4\right)\partial_\theta^2\widehat{\varphi}(\tau,\theta) \\ &\quad = (2 - 2\mathcal{E}_+^2)\partial_\theta^2\widehat{\varphi}(\tau,\theta). \end{aligned}$$

Summarizing the computations, the first assertion follows for $\beta_p := \beta_{p,-2}$, $a := a_{-2}$ and $b := b_{-2}$.

The second assertion follows analogously to the first one. Applying the Fourier transform to

$$\begin{aligned} T_p^{-a} T_p^{-b} \varphi &= \left((\partial_x-\beta_p)^4 - 4(\partial_x-\beta_p)^3 + 4(\partial_x-\beta_p)^2\right)\varphi \\ &\quad + \left((\partial_x-\beta_p+2)^2 + (\partial_x-\beta_p)^2\right)\partial_\theta^2\varphi + \partial_\theta^4\varphi \end{aligned}$$

and substituting $\mathcal{E}_- = -\tau + i(-\beta_p+1) \ \Leftrightarrow i\tau - \beta_p = i\mathcal{E}_- - 1$ we get the second assertion for $\beta_p := \beta_{p,-2}$, $a := a_{-2}$ and $b := b_{-2}$. ☐

As a consequence, formally $\widehat{\varphi}(\tau,\theta)$ is a solution of

$$\left.\begin{aligned} \left((\mathcal{E}_\pm^4 + 2\mathcal{E}_\pm^2 + 1) + (2 - 2\mathcal{E}_\pm^2)\partial_\theta^2 + \partial_\theta^4\right)\widehat{\varphi} = \widehat{g} \quad &\text{in} \quad (0,\theta_0), \\ \partial_\theta\widehat{\varphi} = 0, \ \widehat{\varphi} = 0 \quad &\text{on} \quad \{0,\theta_0\} \end{aligned}\right\} \tag{4.3}$$

for $\tau \in \mathbb{R}$ if and only if φ solves (4.2). Now, [4, Lemma 7.3.1.1] states that (4.3) is uniquely solvable if and only if the following condition is satisfied:

$$\left.\begin{array}{c} \text{the characteristic equation} \\ \sinh^2(\lambda\theta_0) = \lambda^2\sin^2(\theta_0) \\ \text{has no solution on the line} \\ \mathrm{Im}\lambda = -(3 - (2+\gamma)/p - k) = -\beta_{p,-(k+1)}, \quad \lambda \in \mathbb{C},\ \gamma \in \mathbb{R}. \end{array}\right\} \tag{4.4}$$

In Appendix C this condition is analyzed for the case $\gamma = 0$, in particular concerning the values of the involved parameters interesting for the purposes considered in this note.

Remark 4.2. a) Note that the k in [4, Lemma 7.3.1.1], which we will denote by k' in the following, corresponds to $-k$ here.

b) Also observe that in $\mathcal{E}_\pm = \pm\tau + i(-\beta_{p,-2} + 1)$ only the sign of the real part changes and that p only enters the imaginary part. Thus, the assumptions of [4, Lemma 7.3.1.1], remain fulfilled for $\mathcal{E}_+$ and $\mathcal{E}_-$ with $\beta_{p,-2} = 2 - \frac{2+\gamma}{p}$ and for all $p \in (1,\infty)$ such that condition (4.4) is satisfied.

c) In [4, Lemma 7.3.1.3, Theorem 7.3.1.8] the k' defined in $\mathrm{Im}\lambda = -(k'+1+2/q) = -(3 - 2/p + k')$ corresponds to the k' of the Sobolev space $W^{k'+4,p}(\Omega)$. Since in our setting, [4, Lemma 7.3.1.1] remains fulfilled for $\mathcal{E}_+$, $\mathcal{E}_-$, all $p \in (1,\infty)$ and $\beta_{p,-2}$ such that condition (4.4) is fulfilled, i.e. for $k = 2$, we can use [4, Lemma 7.3.1.3, Theorem 7.3.1.8] for $\beta_{p,-2}$ and the space $W^{k'+4,p}(\Omega)$ with $k' = 0$.

In [4, Chapter 7] bi-Laplacian problems on polygonal domains are considered. Localizing the vertices and transforming them onto a layer leads exactly to problem (4.3). The invertibility of $T_p^{\pm a_{-2}} T_p^{\pm b_{-2}}$ for all $p \in (1,\infty)$ such that condition (4.4) is satisfied therefore follows by [4, Lemma 7.3.1.3, Theorem 7.3.1.8]:

Proposition 4.3. *Let $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ such that condition (4.4) is satisfied. For $a_{-2} = \beta_{p,-2} - 2$, $b_{-2} = \beta_{p,-2}$ with $\beta_{p,-2} = 2 - \frac{2+\gamma}{p}$ we have*

$$T_p^{\pm a_{-2}} T_p^{\pm b_{-2}} \in \mathscr{L}_{is}\left(W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega), L^p(\Omega)\right).$$

Proof. [4, Theorem 7.3.1.8] and Remark 4.2 imply that

$$T_p^{\pm a_{-2}} T_p^{\pm b_{-2}} \in \mathscr{L}_{is}\left(W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega), L^p(\Omega)\right)$$

if and only if the characteristic equation $\sinh^2(\lambda\theta_0) = \lambda^2 \sin^2(\theta_0)$ has no solution $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda = -(1 - \frac{2+\gamma}{p})$. This equation is part of the solution formula from the operator of problem (4.3) which has been constructed in [4, Chapter 7]. See [4, Lemma 4.2.1.3 and Theorem 7.3.1.8] for details of its proof. □

Lemma 4.4. *Let $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ such that condition (4.4) is satisfied and $1 = \frac{1}{p} + \frac{1}{p'}$. Let $a_{-2} = \beta_{p,-2} - 2$, $b_{-2} = \beta_{p,-2}$. Then*

$$T_p^{a_{-2}} T_p^{b_{-2}} \subset (T_{p'}^{-a_{-2}} T_{p'}^{-b_{-2}})^\# \in \mathscr{L}_{is}\left(L^p(\Omega),\ \left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'\right).$$

Proof. Employing the fact that $C_c^\infty(\Omega) \subset W_0^{2,r}(\Omega)$ once for $r = p$ and once for $r = p'$, we see that

$$\left\langle u, T_{p'}^{-a_{-2}} T_{p'}^{-b_{-2}} v \right\rangle_{p,p'} = \left\langle T_p^{a_{-2}} u, T_{p'}^{-b_{-2}} v \right\rangle_{p,p'} = \left\langle T_p^{b_{-2}} T_p^{a_{-2}} u, v \right\rangle_{p,p'}$$

for all $v \in W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)$ and $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$. Since $T_p^{b_{-2}}$ and $T_p^{a_{-2}}$ commute this yields

$$\left\langle (T_{p'}^{-a_{-2}} T_{p'}^{-b_{-2}})^\# u, v \right\rangle_{p,p'} = \left\langle u, T_{p'}^{-a_{-2}} T_{p'}^{-b_{-2}} v \right\rangle_{p,p'} = \left\langle T_p^{a_{-2}} T_p^{b_{-2}} u, v \right\rangle_{p,p'}.$$

Then Proposition 4.3 gives the assertion. □

Note that $D(T_p^{a_{-2}} T_p^{b_{-2}}) = W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ lies dense in $L^p(\Omega)$. Thus $(T_{p'}^{-a_{-2}} T_{p'}^{-b_{-2}})^\#$ represents the unique extension of $T_p^{a_{-2}} T_p^{b_{-2}}$ to $L^p(\Omega)$. By this fact we write $T_p^{a_{-2}} T_p^{b_{-2}}$ also for the operator $(T_{p'}^{-a_{-2}} T_{p'}^{-b_{-2}})^\#$ in the sequel.

Theorem 4.5. *Let $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ such that condition (4.4) is satisfied, $a_{-2} = \beta_{p,-2} - 2$ and $b_{-2} = \beta_{p,-2}$ with $\beta_{p,-2} = 2 - \frac{2+\gamma}{p}$. Then*

$$T_{p,-2} = T_p^{a_{-2}} T_p^{b_{-2}} \in \mathscr{L}_{is}\left(W_0^{2,p}(\Omega), W^{-2,p}(\Omega)\right).$$

Proof. Proposition D.1 and Corollary D.2 imply that

$$W_0^{2,p}(\Omega) = \left[W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega),\ L^p(\Omega)\right]_{\frac{1}{2}}$$

$$W^{-2,p}(\Omega) = \left[L^p(\Omega),\ \left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'\right]_{\frac{1}{2}}.$$

Proposition 4.3, Lemma 4.4 and complex interpolation then give

$$T_p^{a-2}T_p^{b-2} \in \mathscr{L}_{is}\left(W_0^{2,p}(\Omega), W^{-2,p}(\Omega)\right)$$

for all $p \in (1,\infty)$ such that condition (4.4) is satisfied. □

4.2. **Weak well-posedness of the transformed problem in $W^{-1,p}(\Omega)$.** Here we show invertibility of (4.1) for $k=1$. This follows directly from results given in [4].

Theorem 4.6. *Let $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ such that condition (4.4) is satisfied and $\beta_{p,-1} = 3 - \frac{2+\gamma}{p}$. Then*

$$T_{p,-1} \in \mathscr{L}_{is}\left(W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega), W^{-1,p}(\Omega)\right).$$

Proof. Still solvability of (4.3) has to be ensured, this time for

$$\mathcal{E}_+ = \tau + i(-\beta_{p,-1}+1), \quad \tau \in \mathbb{R}.$$

Again due to [4, Lemma 7.3.1.3, Theorem 7.3.1.8] the assertion is proved. □

5. Well-posedness of the stationary Stokes equations

5.1. **Weak optimal regularity of the stationary Stokes equations in $\widehat{W}^{-1,p}(G,\mathbb{R}^2)$.** First we consider the equivalence of the problems (3.1) and (3.9). To this end, we define the bi-Laplacian on wedge domains as

$$B_{p,2}\phi := \Delta^2\phi,\ \phi \in D(B_{p,2}) := \{\eta \in K^2_{p,\gamma}(G):\ \partial_1\eta = 0,\ \partial_2\eta = 0 \text{ on } \partial G\}.$$

Lemma 5.1. *Let $1<p<\infty$, $\beta_{p,-2} = 2 - \frac{2+\gamma}{p}$ and $\gamma \in \mathbb{R}$. Let $\Theta_*^{p,2}$, $\widetilde{\Theta}_*^{p,2}$, $\Theta^*_{p,2}$, $\widetilde{\Theta}^*_{p,2}$ be defined as in Section 3. Then we have*

$$\widetilde{\Theta}^*_{p,2} \in \mathscr{L}_{is}\left(\widehat{W}_\gamma^{-2,p}(G), W^{-2,p}(\Omega)\right), \quad \Theta^*_{p,2} \in \mathscr{L}_{is}\left(D(B_{p,2}), D(T_{p,-2})\right)$$

where $\|\cdot\|_{D(B_{p,2})} = \|\cdot\|_{K^2_{p,\gamma}(G)}$ and $\|\cdot\|_{D(T_{p,-2})} = \|\cdot\|_{W^{2,p}(\Omega)}$. In particular, $\phi \in D(B_{p,2})$ is the unique solution of (3.1) to the right-hand side $F \in \widehat{W}_\gamma^{-2,p}(G)$ if and only if $\varphi = \Theta^*_{p,2}\phi \in D(T_{p,-2})$ is the unique solution of (3.9) to the right-hand side $g = \widetilde{\Theta}^*_{p,2}F$.

Proof. The assertion for $\widetilde{\Theta}^*_{p,2}$ follows directly from Lemma B.3(5) for $k=2$. Furthermore, Lemma B.3(1) for $k=2$ and $l=4$ in combination with the transformation of the boundary conditions performed at the end of Section 3 show that

$$\Theta^*_{p,2} \in \mathscr{L}\left(D(B_{p,2}), D(T_{p,-2})\right).$$

Analogously, we obtain

$$\Theta_*^{p,2} \in \mathscr{L}\left(D(T_{p,-2}), D(B_{p,2})\right).$$

Since $\Theta^*_{p,2}$ is the inverse of $\Theta_*^{p,2}$ the result is proved. □

Remark 5.2. For $\beta_p = 2 - \frac{2+\gamma}{p}$ with $\gamma \in \mathbb{R}$ condition (4.4) is fulfilled if the characteristic equation $\sinh^2(\lambda\theta_0) = \lambda^2\sin^2(\theta_0)$ has no solution $\lambda \in \mathbb{C}$ on the line $\mathrm{Im}\lambda = -(1-\frac{2+\gamma}{p})$. For the case $\gamma = 0$, i.e. Kondrat'ev weight $\rho^\gamma = 1$, Corollary C.5 implies that condition (4.4) is satisfied for all $p \in (1,2)\cup(2,\infty)$.

Theorem 4.5, Lemma 5.1 and the last remark imply the following result.

Proposition 5.3. *Let $1<p<\infty$, $p \neq 2$, $\theta_0 \in (0,\pi)$ and $\rho = |(x_1,x_2)|$. Then equation (3.1) is for each $F \in \widehat{W}^{-2,p}(G)$ uniquely solvable with a solution ϕ satisfying*

$$\rho^{|\alpha|-2}\partial^\alpha\phi \in L^p(G) \quad (|\alpha| \le 2).$$

As a consequence we obtain weak regularity for the stationary Stokes equation (1.3).

Theorem 5.4. *Let $1 < p < \infty$, $p \neq 2$ and $\theta_0 \in (0, \pi)$. Then for each $f \in \widehat{W}^{-1,p}(G, \mathbb{R}^2)$ there exists a unique solution*

$$(u, \nabla p) \in \left\{ v \in K_p^1(G, \mathbb{R}^2) : \operatorname{div} v = 0 \text{ in } G,\ v = 0 \text{ on } \partial G \right\} \times \widehat{W}^{-1,p}(G, \mathbb{R}^2)$$

of (1.3).

Proof. For $f \in \widehat{W}^{-1,p}(G, \mathbb{R}^2)$ we obviously have $F = \operatorname{curl} f \in \widehat{W}^{-2,p}(G)$. Let $\phi \in D(B_{p,2})$ be the unique solution of (3.1) to the right-hand side F given by Proposition 5.3. Setting $u = \begin{pmatrix} -\partial_2 \phi \\ \partial_1 \phi \end{pmatrix}$, we obtain $\operatorname{div} u = 0$ in G, $u = 0$ on ∂G and

$$\rho^{|\alpha|-1} \partial^\alpha u \in L^p(G, \mathbb{R}^2) \quad (|\alpha| \leq 1).$$

Next, we observe that

$$\operatorname{curl}(-\Delta u - f) = \Delta^2 \phi - F = 0$$

in the sense of distributions. The Poincaré lemma, see e.g. [2, Theorem VIII.3.8], yields that for $\psi \in C_{c,\sigma}^\infty(G, \mathbb{R}^2)$ we find an $\eta \in C_c^\infty(G)$ such that $\operatorname{curl}'\eta = \begin{pmatrix} -\partial_2 \eta \\ \partial_1 \eta \end{pmatrix} = \psi$. This yields

$$\langle -\Delta u - f, \psi \rangle_{\mathscr{D}', \mathscr{D}} = \langle \operatorname{curl}(-\Delta u - f), \eta \rangle_{\mathscr{D}', \mathscr{D}} = 0 \quad (\psi \in C_{c,\sigma}^\infty(G, \mathbb{R}^2)).$$

From the theorem of de Rham (see e.g. [3, Lemma III.1.1]) we obtain a $p \in \mathscr{D}'(G)$ such that

$$-\Delta u - f = \nabla p \quad \text{in } \mathscr{D}'(G, \mathbb{R}^2).$$

Since the left-hand side belongs to $\widehat{W}^{-1,p}(G, \mathbb{R}^2)$ so does the right-hand side. Hence, we proved existence of a solution as claimed.

It remains to prove its uniqueness. So, we assume that $(u, \nabla p)$ in the given regularity classes solves (1.3). Let $(h_k)_{k \in \mathbb{N}}$ be a mollifier and set

$$u_k := h_k * u \in C^\infty(G, \mathbb{R}^2) \cap \widehat{W}^{1,p}(G, \mathbb{R}^2).$$

Then we have $\operatorname{div} u_k = 0$ and $u_k \to u$ in $\widehat{W}^{1,p}(G, \mathbb{R}^2)$. Hence, by the Poincaré lemma, there is a $\phi_k \in C^\infty(G)$ such that $u_k = \begin{pmatrix} -\partial_2 \phi_k \\ \partial_1 \phi_k \end{pmatrix}$. Since u_k converges in $\widehat{W}^{1,p}(G, \mathbb{R}^2)$ we see that ϕ_k converges in $\widehat{W}^{2,p}(G)$. Thus, there is a limit $\phi \in \widehat{W}^{2,p}(G)$ such that $u = \begin{pmatrix} -\partial_2 \phi \\ \partial_1 \phi \end{pmatrix}$. Since u vanishes on the boundary, we also have $\nabla \phi = 0$ on ∂G. So, applying curl to (1.3) we see that ϕ solves (3.1) with homogeneous right-hand side. Since such a solution is unique by Proposition 5.3, we conclude $\phi = 0$. This implies $u = 0$ and then by equation (1.3) also $\nabla p = 0$. Hence, the theorem is proved. □

5.2. **Strong optimal regularity of the stationary Stokes equations in $L^p(G, \mathbb{R}^2)$.** At first we consider equivalences of the problems (3.1) and (3.9) in $W^{-1,p}$. We define $B_{p,1}\phi := \Delta^2 \phi$ on the wedge domain as

$$B_{p,1}\phi := \Delta^2 \phi,\ \phi \in D(B_{p,1}) := \{\eta \in K_{p,\gamma}^3(G) : \ \partial_1 \eta = 0,\ \partial_2 \eta = 0 \text{ on } \partial G\}.$$

Lemma 5.5. *Let $1 < p < \infty$, $\beta_{p,-1} = 3 - \frac{2+\gamma}{p}$ and $\gamma \in \mathbb{R}$. Let $\Theta_*^{p,1}$, $\widetilde{\Theta}_*^{p,1}$, $\Theta_{p,1}^*$, $\widetilde{\Theta}_{p,1}^*$ be defined as in Section 3. Then we have*

$$\widetilde{\Theta}_{p,1}^* \in \mathscr{L}_{is}\left(\widehat{W}_\gamma^{-1,p}(G), W^{-1,p}(\Omega)\right), \quad \Theta_{p,1}^* \in \mathscr{L}_{is}\left(D(B_{p,1}), D(T_{p,-1})\right)$$

where $\|\cdot\|_{D(B_{p,1})} = \|\cdot\|_{K^3_{p,\gamma}(G)}$ *and* $\|\cdot\|_{D(T_{p,-1})} = \|\cdot\|_{W^{3,p}(\Omega)}$.
In particular, $\phi \in D(B_{p,1})$ *is the unique solution of* (3.1) *to the right-hand side* $F \in \widehat{W}^{-1,p}_{\gamma}(G)$ *if and only if* $\varphi = \Theta^*_{p,1}\phi \in D(T_{p,-1})$ *is the unique solution of* (3.9) *to the right-hand side* $g = \widetilde{\Theta}^*_{p,1}F$.

Proof. The fact that $\widetilde{\Theta}^*_{p,1}$ and $\Theta^*_{p,1}$ are isomorphisms as claimed follows directly from Lemma B.3(1) and (5) with $k = 1$ and $l = 4$. The remaining assertions follow by the same arguments as in the proof of Lemma 5.1. □

Remark 5.6. For $\beta_p = 3 - \frac{2+\gamma}{p}$ with $\gamma \in \mathbb{R}$ condition (4.4) is fulfilled if the characteristic equation $\sinh^2(\lambda\theta_0) = \lambda^2 \sin^2(\theta_0)$ has no solution $\lambda \in \mathbb{C}$ on the line $\mathrm{Im}\lambda = -(2 - \frac{2+\gamma}{p})$. For the case $\gamma = 0$, i.e. Kondrat'ev weight $\rho^\gamma = 1$, Corollary C.3 implies that there is a finite set $N \subset (1,\infty)$ such that condition (4.4) is satisfied for all $p \in (1,\infty) \setminus N$.

Theorem 4.6, Lemma 5.5 and the last remark imply the following result

Proposition 5.7. *Let* $\theta_0 \in (0,\pi)$ *and* $\rho = |(x_1,x_2)|$. *There is a finite set* $N \subset (1,\infty)$ *such that for every* $p \in (1,\infty) \setminus N$ *we have the following: Equation* (3.1) *is for each* $f \in \widehat{W}^{-1,p}(G)$ *uniquely solvable with a solution* ϕ *satisfying*

$$\rho^{|\alpha|-3}\partial^\alpha\phi \in L^p(G) \quad (|\alpha| \le 3).$$

Now, we can show strong optimal regularity of the stationary Stokes equations (1.3) for the same range of p.

Theorem 5.8. *Let* $\theta_0 \in (0,\pi)$. *There is a finite set* $N \subset (1,\infty)$ *such that for every* $p \in (1,\infty) \setminus N$ *we have the following: For each* $f \in L^p(G,\mathbb{R}^2)$ *there exists a unique solution*

$$(u,\nabla p) \in \left\{ v \in K^2_p(G,\mathbb{R}^2) : \operatorname{div} v = 0 \text{ in } G,\ v = 0 \text{ on } \partial G \right\} \times L^p(G,\mathbb{R}^2)$$

of (1.3).

Proof. The proof goes along the lines of the proof of Theorem 5.4. □

6. Sectoriality in a neighborhood of $p = 2$

Here we consider the Stokes resolvent problem

$$\left.\begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } G, \\ \operatorname{div} u &= 0 \quad \text{in } G, \\ u &= 0 \quad \text{on } \partial G. \end{aligned}\right\} \tag{6.1}$$

In the sequel we always consider $1 < p < \infty$. Its weak formulation reads

$$\lambda(u,v) + (\nabla u, \nabla v) = \langle f, v\rangle_{W^{-1,p}_\sigma, W^{1,p'}_{0,\sigma}} \quad (v \in W^{1,p'}_{0,\sigma}).$$

Since in this section the domain is always a wedge G, we drop it in the notation of the space, i.e., we write L^p, $W^{k,p}$, etc.

To recover the pressure once a solution of the weak formulation is given, the following lemma will be helpful.

Lemma 6.1. *We have* $(H^1_{0,\sigma})^\perp = \nabla L^2$.

Proof. We prove $H^1_{0,\sigma} = (\nabla L^2)^\perp$. Then the assertion follows by reflexivity. It is obvious that $H^1_{0,\sigma} \subset (\nabla L^2)^\perp$. Conversely, for $w \in H^1_0$ such that

$$(w, \nabla p) = 0 \quad (p \in L^2),$$

we obtain $w \in L^2_\sigma$. The fact that $H^1_{0,\sigma} = H^1_0 \cap L^2_\sigma$ implies the result. □

In the lemma below, we will consider (6.1) also with inhomogeneous divergence condition $\operatorname{div} u = g$. To this end, we define the Banach space

$$R_p := \operatorname{div}(W^{1,p}_0), \ \|g\|_{R_p} := \inf\{\|w\|_{W^{1,p}_0};\ g = \operatorname{div} w\}.$$

Furthermore, we will utilize the following observation: since the embedding operator $J = J_{p'} : W^{1,p'}_{0,\sigma} \to W^{1,p'}_0$ is bounded, injective, and has closed range, its dual operator

$$J' = J'_p : W^{-1,p} \to W^{-1,p}_\sigma \tag{6.2}$$

is bounded and surjective, where $W^{-1,p}_\sigma := (W^{1,p'}_{0,\sigma})'$. It is also clear that $(J_p)_{1<p<\infty}$ and $(J'_p)_{1<p<\infty}$ are consistent scales.

Lemma 6.2. *Let $\psi \in (0,\pi)$. Let $\lambda \in \Sigma_{\pi-\psi}$. There is a $\kappa = \kappa(\lambda) > 0$ such that for $p \in I_\kappa := ((2+\kappa)', 2+\kappa)$ (6.1) is well-posed. Indeed, for every $f \in W^{-1,p}_\sigma$ there is a unique solution $(u,p) \in W^{1,p}_{0,\sigma} \times L^p$ to (6.1). Furthermore, if $S_p : f \mapsto (u,p)$ denotes the solution operator to (6.1), then $(S_p)_{p\in I_\kappa}$ is consistent on $(W^{-1,p})_{p\in I_\kappa}$ and uniformly bounded w.r.t. $p \in I_\kappa$.*

Proof. Our aim is to apply the Sneiberg type extrapolation result in the form given in [11, Theorem 2.7]. First, pick $r \in (4,\infty)$ and set

$$F_p := [R_{r'}, R_r]_s\,, \quad \frac{1}{p} = (1-s)\frac{1}{r'} + s\frac{1}{r},\ \ s \in [0,1].$$

By the reiteration theorem for the complex interpolation functor, see e.g. [15, Remark 1.9.3/1], $(F_p)_{p\in(r',r)}$ is a complex interpolation scale. Thus, $\left(W^{1,p}_0 \times L^p\right)_{p\in(r',r)}$ and $\left(W^{-1,p} \times F_p\right)_{p\in(r',r)}$ are complex interpolation scales, too. It is also obvious, that

$$\mathcal{L}_p : W^{1,p}_0 \times L^p \to W^{-1,p} \times F_p, \quad \mathcal{L}_p(u,p) := \begin{pmatrix} (\lambda-\Delta)u + \nabla p \\ \operatorname{div} u \end{pmatrix}$$

is bounded.

Next, we show that for $\lambda \in \Sigma_{\pi-\psi}$ the map $\mathcal{L}_2$ is isomorphic. For $(f,g) \in H^{-1} \times F_2$ first choose $w \in H^1_0$ such that $\operatorname{div} w = g$. Note that

$$J'h := J'(f - (\lambda-\Delta)w) \in H^{-1}_\sigma = (H^1_{0,\sigma})'.$$

By standard Hilbert space arguments we hence obtain a unique $v \in H^1_{0,\sigma}$ such that

$$\lambda(v,\varphi) + (\nabla v, \nabla\varphi) = \langle J'h, \varphi\rangle_{H^{-1}_\sigma, H^1_{0,\sigma}} = \langle h, J\varphi\rangle_{H^{-1}, H^1_0} \quad (\varphi \in H^1_{0,\sigma}).$$

This implies

$$\langle (\lambda-\Delta)Jv - h, J\varphi\rangle_{H^{-1},H^1_0} = 0 \quad (\varphi \in H^1_{0,\sigma}).$$

Thanks to Lemma 6.1 there is a $p \in L^2$ such that

$$(\lambda-\Delta)Jv - h = -\nabla p \quad \text{in } H^{-1}.$$

Thus, setting $u := Jv + w$, we conclude $\mathcal{L}_2(u,p) = (f,g)$. It is obvious that $\mathcal{L}_2(u,p) = 0$ implies $(u,p) = (0,0)$. Consequently,

$$\mathcal{L}_2 \in \mathscr{L}_{is}\left(H^1_0 \times L^2,\ H^{-1} \times F_2\right).$$

Due to [11, Theorem 2.7] there is a $\kappa(\lambda) > 0$ such that

$$\mathcal{L}_p \in \mathscr{L}_{is}\Big(W_0^{1,p} \times L^p,\, W^{-1,p} \times F_p\Big) \quad (p \in I_\kappa).$$

Note that consistency and uniform boundedness w.r.t. $p \in I_\kappa$ of $(\mathcal{L}_p^{-1})_{p\in I_\kappa}$ on $\big(W^{-1,p} \times F_p\big)_{p\in I_\kappa}$ also follow from the results in [11]. Thus, $(S_p)_{p\in I_\kappa}$ is consistent and uniformly bounded, too, as claimed. □

From Lemma 6.2 we have

Corollary 6.3. *Let $\psi \in (0, \pi)$. Then there is a $\kappa = \kappa(\psi) > 0$ such that for $p \in I_\kappa = ((2+\kappa)', (2+\kappa))$ and all $\lambda \in \overline{\Sigma}_{\pi-\psi}$ with $|\lambda| = 1$ (6.1) is well-posed. Indeed, for every $f \in W_\sigma^{-1,p}$ and every $\lambda \in \overline{\Sigma}_{\pi-\psi}$ with $|\lambda| = 1$ there is a unique solution $(u,p) \in W_{0,\sigma}^{1,p} \times L^p$ to (6.1). Furthermore, if $S_p : f \mapsto (u,p)$ denotes the solution operator to (6.1), then $(S_p)_{p\in I_\kappa}$ is consistent on $(W^{-1,p})_{p\in I_\kappa}$.*

Proof. Let $\psi' := \psi/2$. For $\lambda \in \Sigma_{\pi-\psi'}$ choose $\kappa(\lambda) > 0$ according to Lemma 6.2. Let A_p be the operator defined by the left-hand side of (6.1) in $W_\sigma^{-1,p}$ with domain $D(A_p) = W_{0,\sigma}^{1,p} \times W_\sigma^{-1,p}$. Then we have that $\lambda \in \rho(-A_p)$ for all $p \in I_{\kappa(\lambda)}$. Since $\rho(-A_p)$ is open and since $(\lambda + A_p)^{-1}$ is uniformly bounded w.r.t. $p \in I_{\kappa(\lambda)}$, there exists an $\varepsilon(\lambda) > 0$ such that $B_{\varepsilon(\lambda)}(\lambda) \subset \rho(-A_p)$ for all $p \in I_{\kappa(\lambda)}$. The set $M := \{\lambda \in \overline{\Sigma}_{\pi-\psi} :\ |\lambda| = 1\} \subset \Sigma_{\pi-\psi'}$ is compact with $M \subset \bigcup_{\lambda\in\Sigma_{\pi-\psi'}} B_{\varepsilon(\lambda)}(\lambda)$. Now choose $\lambda_1, \dots, \lambda_m \in \Sigma_{\pi-\psi'}$ such that $M \subset \bigcup_{k=1}^m B_{\varepsilon(\lambda_k)}(\lambda_k)$. We set $I := \bigcap_{k=1}^m I_\kappa(\lambda_k) = I_\kappa$ with $\kappa = \min\{\kappa(\lambda_k)\} > 0$ independent of $\lambda \in M$. Then $\lambda \in \rho(-A_p)$ for all $\lambda \in M$ and all $p \in I_\kappa$. □

We define the Stokes operator in $W_\sigma^{-1,p} = (W_{0,\sigma}^{1,p'})'$ by

$$A_p : D_{-1}(A_p) := W_{0,\sigma}^{1,p} \to W_\sigma^{-1,p}, \quad u \mapsto A_p u := (\nabla u, \nabla\cdot).$$

From Corollary 6.3 we derive

Corollary 6.4. *Let $\psi \in (0, \pi)$ and $p \in I_\kappa$. Then $K_\psi := \big\{\lambda \in \overline{\Sigma}_{\pi-\psi} :\ |\lambda| = 1\big\} \subset \rho(-A_p)$ and for every $\lambda \in K_\psi$ the scale $\big((\lambda + A_p)^{-1}\big)_{p\in I_\kappa}$ is consistent on $(W_\sigma^{-1,p})_{p\in I_\kappa}$.*

Proof. Note that by Corollary 6.3 we can choose a uniform $\kappa > 0$ for all $\lambda \in K_\psi$. For $f \in W_\sigma^{-1,p}$, by surjectivity of J' we find a $h \in W^{-1,p}$ such that $f = J'h$. Setting $(u,p) := S_p h \in W_{0,\sigma}^{1,p} \times L^p$, from this we infer

$$\begin{aligned}\lambda(u,\varphi) + (\nabla u, \nabla\varphi) &= \langle h, J\varphi\rangle_{W^{-1,p},W_0^{1,p'}} = \langle J'h, \varphi\rangle_{W_\sigma^{-1,p},W_{0,\sigma}^{1,p'}} \\ &= \langle f, \varphi\rangle_{W_\sigma^{-1,p},W_{0,\sigma}^{1,p'}} \qquad (\varphi \in W_{0,\sigma}^{1,p'}).\end{aligned}$$

This shows that the Stokes resolvent $(\lambda + A_p)^{-1} \in \mathscr{L}(W_\sigma^{-1,p})$ in $-\lambda \in K_\psi$ exists. In particular, we have

$$(\lambda + A_p)^{-1} J'h = (\lambda + A_p)^{-1} f = u = (S_p h)^1.$$

By the properties of J and J' it is clear that for $f \in W_\sigma^{-1,p} \cap W_\sigma^{-1,r}$ we find $h \in W^{-1,p} \cap W^{-1,r}$ such that $f = J'h$. Thus, the consistency of the Stokes resolvent follows from the consistency of $(S_p)_{p\in I_\kappa}$ given by Corollary 6.3. □

For fixed $\lambda_0 \in \rho(-A_p)$ we set

$$D(A_p) := (\lambda_0 + A_p)^{-1}(L^p_\sigma).$$

The restriction of A_p on $D_{-1}(A_p)$ to $D(A_p)$ we again denote by A_p. For later purposes also note that

$$W^{1,p}_{0,\sigma} \overset{d}{\hookrightarrow} \widehat{W}^{1,p}_{0,\sigma} \quad \Rightarrow \quad \widehat{W}^{-1,p}_\sigma := (\widehat{W}^{1,p'}_{0,\sigma})' \overset{d}{\hookrightarrow} W^{-1,p}_\sigma. \tag{6.3}$$

Furthermore, we define the Stokes operator on homogeneous spaces as

$$\widehat{A}_p : \widehat{W}^{1,p}_{0,\sigma} \to \widehat{W}^{-1,p}_\sigma, \quad u \mapsto \widehat{A}_p u := (\nabla u, \nabla \cdot). \tag{6.4}$$

Then obviously $\widehat{A}_p \in \mathscr{L}\big(\widehat{W}^{1,p}_{0,\sigma}, \widehat{W}^{-1,p}_\sigma\big)$ and for $u \in W^{1,p}_{0,\sigma}$ we obtain

$$\langle \widehat{A}_p u, \varphi\rangle_{\widehat{W}^{-1,p}_\sigma, \widehat{W}^{1,p'}_{0,\sigma}} = (\nabla u, \nabla\varphi) = \langle A_p u, \varphi\rangle_{W^{-1,p}_\sigma, W^{1,p'}_{0,\sigma}} \quad (\varphi \in W^{1,p'}_{0,\sigma}),$$

hence $A_p \subset \widehat{A}_p$. Corollary 6.4 and a scaling argument result in

Proposition 6.5. *The Stokes operator $A_p : D(A_p) \to L^p_\sigma$ is sectorial on L^p_σ with spectral angle $\phi_{A_p} = 0$ for $p \in I_\kappa$. In addition, we have*

$$\sup_{\lambda \in \Sigma_{\pi-\psi}} \Big(\|\lambda(\lambda + A_p)^{-1}\|_{\mathscr{L}(\widehat{W}^{-1,p}_\sigma)} + \|A_p(\lambda + A_p)^{-1}\|_{\mathscr{L}(\widehat{W}^{-1,p}_\sigma)} \Big) < \infty$$

for $\psi \in (0, \pi)$.

Proof. Thanks to Corollary 6.3 K_ψ lies in the resolvent set of $-A_p$ in $W^{-1,p}_\sigma$ and we obtain

$$\begin{aligned}
\|(\lambda + A_p)^{-1} f\|_p \le \|(\lambda + A_p)^{-1} f\|_{W^{1,p}_{0,\sigma}} &\le C(\lambda)\|f\|_{W^{-1,p}_\sigma} \\
&\le C(\lambda)\|f\|_p \quad (f \in L^p_\sigma,\ \lambda \in K_\psi).
\end{aligned}$$

Thus, the resolvent of A_p on L^p_σ in $\lambda \in K_\psi$ exists. The fact that

$$K_\psi \subset \rho(-A_p) \ni \lambda \mapsto (\lambda + A_p)^{-1} \in \mathscr{L}(L^p_\sigma)$$

is a holomorphic map yields

$$\sup_{\lambda \in \overline{\Sigma}_{\pi-\psi}, |\lambda|=1} \|\lambda(\lambda + A_p)^{-1}\|_{\mathscr{L}(L^p_\sigma)} < \infty.$$

Utilizing the fact that a wedge is scaling invariant, a scaling argument gives the claimed sectoriality on L^p_σ.

Taking into account (6.3) and the outcome of the lines after this fact, for $\lambda \in \Sigma_{\pi-\psi}$ we further calculate

$$\begin{aligned}
\|A_p(\lambda + A_p)^{-1} f\|_{\widehat{W}^{-1,p}_\sigma} &\le \|(\lambda + A_p)^{-1} f\|_{\widehat{W}^{1,p}_{0,\sigma}} \\
&\le \|(\lambda + A_p)^{-1} f\|_{W^{1,p}_{0,\sigma}} \\
&\le C(\lambda)\|f\|_{W^{-1,p}_\sigma} \\
&\le C(\lambda)\|f\|_{\widehat{W}^{-1,p}_\sigma} \quad (f \in L^p_\sigma).
\end{aligned}$$

Since $\Sigma_{\pi-\psi} \ni \lambda \mapsto (\lambda + A_p)^{-1} \in \mathscr{L}(\widehat{W}^{-1,p}_\sigma)$ still is holomorphic and by the fact that $\widehat{W}^{-1,p}_\sigma$ is homogeneous the same scaling argument as for L^p_σ yields the second assertion. The consistency of the family $((\lambda + A_p)^{-1})_{p \in I_\kappa}$ for $\lambda \in K_\psi$ given by Corollary 6.4 obviously implies its consistency for every $\lambda \in \Sigma_{\pi-\psi}$. □

7. Stationary consistency

Consider the stationary Stokes equations (1.3) on the wedge G. Its weak formulation reads

$$(\nabla u, \nabla v) = \langle f, v\rangle_{\widehat{W}_\sigma^{-1,p},\widehat{W}_{0,\sigma}^{1,p'}} \quad (v \in \widehat{W}_{0,\sigma}^{1,p'}). \tag{7.1}$$

Note that this can be expressed in terms of the Stokes operator on homogeneous spaces $\widehat{A}_q : \widehat{W}_{0,\sigma}^{1,p} \to \widehat{W}_\sigma^{-1,p}$, $u \mapsto f$ as defined in (6.4). From Theorem 5.4 we deduce

Proposition 7.1. *For $p \in (1,\infty)$ we have*

$$\widehat{A}_p \in \mathscr{L}_{is}\left(\widehat{W}_{0,\sigma}^{1,p}, \widehat{W}_\sigma^{-1,p}\right).$$

Proof. The case $p = 2$ follows from standard Hilbert space arguments. For $p \neq 2$ we set $F_p := \widehat{W}_{0,\sigma}^{1,p} \cap K_p^1$. In accordance with (6.2) note that the dual operator

$$\widehat{J}' = \widehat{J}_p' : \widehat{W}^{-1,p} \to \widehat{W}_\sigma^{-1,p}$$

of the embedding operator $\widehat{J} = \widehat{J}_{p'} : \widehat{W}_{0,\sigma}^{1,p'} \to \widehat{W}_0^{1,p'}$ is bounded and surjective. Now, pick $f \in \widehat{W}_\sigma^{-1,p}$ and choose $h \in \widehat{W}^{-1,p}$ such that $f = \widehat{J}'h$. From Theorem 5.4 we obtain unique $(u, \nabla p) \in F_p \times \widehat{W}^{-1,p}$ satisfying (1.3) with right-hand side $\widehat{J}'h$, hence

$$\begin{aligned}(\nabla u, \nabla v) &= \langle h, Jv\rangle_{\widehat{W}^{-1,p},\widehat{W}_0^{1,p'}} = \langle J'h, v\rangle_{\widehat{W}_\sigma^{-1,p},\widehat{W}_{0,\sigma}^{1,p'}} \\ &= \langle f, v\rangle_{\widehat{W}_\sigma^{-1,p},\widehat{W}_{0,\sigma}^{1,p'}} \qquad (v \in \widehat{W}_{0,\sigma}^{1,p'}).\end{aligned}$$

So, we have proved

$$\widehat{A}_{p'} \in \mathscr{L}_{is}\left(F_{p'}, \widehat{W}_\sigma^{-1,p'}\right).$$

Dualizing this implies

$$(\widehat{A}_{p'})' \in \mathscr{L}_{is}\left(\widehat{W}_{0,\sigma}^{1,p}, F_{p'}'\right).$$

It is not difficult to see that $\widehat{A}_p \subset (\widehat{A}_{p'})'$. On the other hand, we also have

$$\widehat{A}_p \in \mathscr{L}\left(\widehat{W}_{0,\sigma}^{1,p}, \widehat{W}_\sigma^{-1,p}\right).$$

The fact that $F_{p'} \overset{d}{\hookrightarrow} \widehat{W}_{0,\sigma}^{1,p'}$, hence $\widehat{W}_\sigma^{-1,p} \overset{d}{\hookrightarrow} F_{p'}'$, then implies that $F_p = \widehat{W}_{0,\sigma}^{1,p}$ with equivalent norms. Thus, the assertion follows also for $p \neq 2$. $\square$

Remark 7.2. The fact that the norms of F_p and $\widehat{W}_{0,\sigma}^{1,p}$ are equivalent also follows as a consequence of the Hardy inequality, which is applicable for $p \neq 2$.

Lemma 7.3. *Let $\psi \in (0,\pi)$ and $p \in I_\kappa$ with I_κ from Corollary 6.3. Then*

$$\lim_{k\to\infty} (1/k + A_p)^{-1} = \widehat{A}_p^{-1} \quad \text{in} \quad \mathscr{L}\left(\widehat{W}_\sigma^{-1,p}, \widehat{W}_{0,\sigma}^{1,p}\right).$$

Proof. Pick $f \in \widehat{W}_\sigma^{-1,p}$. Thanks to Propositions 6.5 and 7.1 for $\ell \in \mathbb{N}$ by the resolvent identity we obtain

$$\begin{aligned}&\|(1/(k+\ell) + A_p)^{-1} f - (1/k + A_p)^{-1} f\|_{\widehat{W}_{0,\sigma}^{1,p}} \\ &\leq C\big\|\big(1/(k+\ell) - 1/k\big)(1/k + A_p)^{-1}\widehat{A}_p(1/(k+\ell) + A_p)^{-1} f\big\|_{\widehat{W}_\sigma^{-1,p}} \\ &\leq C\big\|\big(k/(k+\ell) - 1\big)A_p(1/(k+\ell) + A_p)^{-1} f\big\|_{\widehat{W}_\sigma^{-1,p}} \\ &\leq C\big\|\big(k/(k+\ell) - 1\big) f\big\|_{\widehat{W}_\sigma^{-1,p}} \to 0 \quad (k \to \infty).\end{aligned}$$

Thus $(1/k + A_p)^{-1} f \to v$ in $\widehat{W}^{1,p}_{0,\sigma}$. The fact that $\widehat{A}_p \in \mathscr{L}\big(\widehat{W}^{1,p}_{0,\sigma}, \widehat{W}^{-1,p}_{\sigma}\big)$ yields

$$\widehat{A}_p v = \lim_{k\to\infty} \widehat{A}_p (1/k + A_p)^{-1} f = \lim_{k\to\infty} A_p (1/k + A_p)^{-1} f = f,$$

hence $v = \widehat{A}_p^{-1} f$. □

We come to the main result of this section.

Proposition 7.4. *The scale* $(\widehat{A}_p^{-1})_{p\in(1,\infty)}$ *of the operators* $\widehat{A}_p^{-1} : \widehat{W}^{-1,p}_{\sigma} \to \widehat{W}^{1,p}_{0,\sigma}$ *is consistent on* $(\widehat{W}^{-1,p}_{\sigma})_{p\in(1,\infty)}$.

Proof. Thanks to Proposition 7.1 $\widehat{A}_p^{-1}$ exists for every $p \in (1,\infty)$. Let $B_{p,2} : D(B_{p,2}) \to \widehat{W}^{-2,p}$ be the operator corresponding to the bi-harmonic equation as defined in the beginning of Section 5.1 for $\gamma = 0$. According to Proposition 5.3 we have

$$B_{p,2} \in \mathscr{L}_{is}\big(D(B_{p,2}), \widehat{W}^{-2,p}\big).$$

The proof of Theorem 5.4 shows that

$$u = \widehat{A}_p^{-1} f = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} B_{p,2}^{-1} \operatorname{curl} f \tag{7.2}$$

for $f \in \widehat{W}^{-1,p}$. We will show that

$$(B_{p,2}^{-1})_{p\in I} \text{ is consistent on } (\widehat{W}^{-2,p})_{p\in I} \tag{7.3}$$

for $I = (1,2)$ and for $I = (2,\infty)$.

Given (7.3), by (7.2) $(\widehat{A}_p^{-1})_{p\in I}$ is consistent on $(\widehat{W}^{-1,p}_{\sigma})_{p\in I}$ for $I = (1,2)$ and for $I = (2,\infty)$ as well. Then consistency on $(1,\infty)$ for the scale of operators $\widehat{A}_p^{-1} : \widehat{W}^{-1,p}_{\sigma} \to \widehat{W}^{1,p}_{0,\sigma}$ results from the following argument: Let $p, r \in I_\kappa$ and $f \in \widehat{W}^{-1,p}_{\sigma} \cap \widehat{W}^{-1,r}_{\sigma}$. Thanks to Corollary 6.4 we have

$$(1/k + A_p)^{-1} f = (1/k + A_r)^{-1} f \quad (k \in \mathbb{N}).$$

Letting $k \to \infty$, Lemma 7.3 yields $\widehat{A}_p^{-1} f = \widehat{A}_r^{-1} f$. By the fact that $(1,2) \cap I_\kappa \neq \emptyset$ and $(2,\infty) \cap I_\kappa \neq \emptyset$ the assertion follows.

Hence, it remains to prove (7.3). This is very similar to the proof of [7, Lemma 4, Proposition 2]. Indeed, it is even easier, since we can work with compactly supported functions all along. For the readers convenience we sketch the proof: Let $T_{p,-2} : W^{2,p}_0(\Omega) \to W^{-2,p}(\Omega)$ denote the operator corresponding to the transformed problem (3.9) as defined in the beginning of Section 4. Thanks to Theorem 4.5 we have

$$T_{p,-2} \in \mathscr{L}_{is}\left(W^{2,p}_0(\Omega), W^{-2,p}(\Omega)\right).$$

Further, let Θ^p_*, Θ^*_p and $\widetilde{\Theta}^p_*, \widetilde{\Theta}^*_p$ be the push-forwards and pull-backs as introduced in Section 3. Then we have

$$B_{p,2}^{-1} = \Theta^{p,2}_* T_{p,-2}^{-1} \widetilde{\Theta}^*_{p,2}, \tag{7.4}$$

Note that the single operators on the right-hand side by definition cannot be consistent, just their combination.

Now, let $I = (1,2)$ or $I = (2,\infty)$ pick $p, r \in I$ and set $\beta_p = 2 - 2/p$. It is straight forward to show that

$$e^{(\beta_r - \beta_p)x} T_{r,-2} e^{-(\beta_r - \beta_p)x} v = T_{p,-2} v \quad (v \in C_c^\infty(\Omega)).$$

Suitable manipulations (similar to the ones given in the proof of [7, Lemma 4]) lead to

$$T_{p,-2}^{-1} e^{-(\beta_r - \beta_p)x} g = e^{-(\beta_r - \beta_p)x} T_{r,-2}^{-1} g \quad (g \in C_c^\infty(\Omega)).$$

Then, based on representation (7.4) very similar to the proof of [7, Proposition 2] it follows

$$B_{p,2}^{-1} f = B_{r,2}^{-1} f \quad (f \in \widehat{W}^{-2,p} \cap \widehat{W}^{-2,r}).$$

Hence, we arrive at (7.3) and the proof is complete. □

Remark 7.5. Note that (7.3) can not be expected on $I = (1, \infty)$, neither for $(\widehat{A}_p^{-1})_{p\in(1,\infty)}$ regarded as scale of operators from $\widehat{W}_\sigma^{-1,p}$ to $\{v \in K_p^1(G, \mathbb{R}^2) : \operatorname{div} v = 0 \text{ in } G,\ v = 0 \text{ on } \partial G\}$, see also Remark 7.2.

8. Strong sectoriality in a neighborhood of $p = 2$

Now, we are in position to prove higher regularity for p close to 2. We start with a lemma on weak-strong consistency.

Lemma 8.1. *Assume that $1 < p < \infty$ such that (1.3) is uniquely solvable in the weak and the strong setting and let $f \in L^p_\sigma \cap \widehat{W}_\sigma^{-1,p}$. If*

1. *$v \in \{w \in K_p^2 :\ \operatorname{div} w = 0,\ w = 0 \text{ on } \partial G\}$ is the velocity of the unique solution of (1.3) to the right-hand side $f \in L^p_\sigma$ and*
2. *$u \in \widehat{W}_{0,\sigma}^{1,p}$ is the unique solution of (7.1), i.e. the weak form of (1.3), to the right-hand side $f \in \widehat{W}_\sigma^{-1,p}$,*

then $u = v$ (in L^1_{loc}).

Proof. This follows along the lines of the proof of Theorem 7.4. In fact, the two a priori different solutions are represented by

$$u = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} B_{p,2}^{-1} \operatorname{curl} f,$$

$$v = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} B_{p,1}^{-1} \operatorname{curl} f.$$

The two operators related to the bi-harmonic equation, in turn, are given as

$$B_{p,k}^{-1} = \Theta_*^{p,k}\, T_{p,-k}^{-1}\, \widetilde{\Theta}_{p,k}^*$$

for $k = 1, 2$. Note that the push-forwards and pull-backs depend on $\beta_p = \beta_{p,-k}$, hence in particular on k, see Definition 3.1 and what follows. To indicate this dependence, here we also put a sub- or superscript k.

By interchanging the roles of p and k from this point on we can follow the last lines of the proof of Theorem 7.4, in order to obtain in a similar way that

$$B_{p,2}^{-1} f = B_{p,1}^{-1} f \quad (f \in \widehat{W}^{-1,p} \cap \widehat{W}^{-2,r})$$

in the L^1_{loc}-sense. The above representations for u and v then give the result. □

As before, if A_p is the Stokes operator associated to (6.1), then $u = (\lambda + A_p)^{-1} f$ gives the velocity to (6.1).

Theorem 8.2. *Let $\psi \in (0, \pi)$. There is a $\kappa > 0$ such that for $p \in I_\kappa = \big((2+\kappa)', 2+\kappa\big)$ the following holds: For $f \in L^p_\sigma$ let $u = (\lambda + A_p)^{-1} f \in D(A_p)$. Then we have $u \in W^{2,p}$ and*

$$\|\lambda u\|_p + \|\nabla^2 u\|_p + \|\nabla p\|_p \le C \|f\|_p \quad (f \in L^p_\sigma,\ \lambda \in \Sigma_{\pi-\psi}).$$

Furthermore, the scale $\big((\lambda + A_p)^{-1}\big)_{p \in I_\kappa}$ is consistent on $(L^p_\sigma)_{p \in I_\kappa}$.

Proof. According to Theorem 5.8 the set N of possibly singular $p \in (1,\infty)$ concerning strong solvability of (1.3) is discrete in $(1,\infty)$. It might happen that $2 \in N$ or $2 \notin N$. Either way there is a $\kappa > 0$ such that for I_κ as defined we have $N \cap (I_\kappa \setminus \{2\}) = \emptyset$. We also may assume that the κ chosen here is smaller as or equal to the κ in Proposition 6.5.

We fix $p \in I_\kappa$, $p \neq 2$, and pick $f \in C^\infty_{c,\sigma}$. As a consequence of Corollary 6.4 (or Proposition 6.5) we have

$$u = (\lambda + A_p)^{-1} f \in W^{1,p}_{0,\sigma}.$$

Since $W^{1,p}_{0,\sigma} \hookrightarrow \widehat{W}^{1,p}_{0,\sigma}$, this yields

$$\widehat{A}_p u = A_p u = f - \lambda u =: g \in L^p_\sigma \cap \widehat{W}^{-1,p}_\sigma.$$

Due to Theorem 5.8 there exists a unique solution

$$v \in \left\{ w \in K^2_p : \ \operatorname{div} w = 0, \ w = 0 \text{ on } \partial G \right\} \subset \widehat{W}^{2,p}$$

of (1.3) with right-hand side $g \in L^p_\sigma$. Lemma 8.1 implies $u = v$ and we deduce

$$\|\nabla^2 u\|_p \leq C\|g\|_p \leq C\left(\|f\|_p + \|\lambda u\|_p\right) \leq C\|f\|_p$$

with $C > 0$ independent of f and λ and where we applied Proposition 6.5 for the last estimate. A density argument yields

$$\|\lambda u\|_p + \|\nabla^2 u\|_p \leq C\|f\|_p \quad (f \in L^p_\sigma, \ \lambda \in \Sigma_{\pi-\psi}).$$

The estimate for the pressure gradient then follows from equations (6.1) and the assertion is proved for $p \neq 2$.

By standard arguments it can now be proved that $A'_p = A_{p'}$ for $p \in I_\kappa \setminus \{2\}$. Completely analogously to [7, Proposition 3] it also follows that $(\lambda + A_p)^{-1}$ and $(\lambda + A'_p)^{-1}$ are consistent on $L^p_\sigma \cap L^{p'}_\sigma$ for $\lambda \in \Sigma_{\pi-\psi}$. Combining these two facts, we see that $\left((\lambda + A_p)^{-1}\right)_{p \in I_\kappa \setminus \{2\}}$ is consistent on $(L^p_\sigma)_{p \in I_\kappa \setminus \{2\}}$. But then the case $p = 2$ follows by interpolation. □

Appendix A. Elements from functional analysis

In the following let $G \subset \mathbb{R}^2$ be the wedge defined as in (1.2). The next lemma is already known by [3, Remark II.6.3]. Here, we give a more detailed version of its proof.

Lemma A.1. *Let $1 < p < \infty$. Let $\Omega_1 := \mathbb{R} \times (0, \theta_0)$ be a layer domain, and $\Omega_2 := G \cap (B_{2r}(0) \setminus \overline{B}_r(0))$ with $0 < r < \infty$. Moreover, let $k \in \mathbb{N}$. Then $\widehat{W}^{k,p}_0(\Omega_i)$ and $W^{k,p}_0(\Omega_i)$ are isomorphic for $i \in \{1, 2\}$.*

Proof. Let $i \in \{1, 2\}$. Since $\phi \in C^\infty_c(\Omega_i) \overset{d}{\hookrightarrow} W^{k,p}_0(\Omega_i)$, the Poincaré inequality implies that

$$\|\phi\|_{L^p(\Omega_i)} \leq C\|\nabla\phi\|_{L^p(\Omega_i)} \quad (\phi \in C^\infty_c(\Omega_i)),$$

for a constant $C > 0$. We again have by the Poincaré inequality that

$$\|\nabla\phi\|_{L^p(\Omega_i)} \leq C\|\nabla^2\phi\|_{L^p(\Omega_i)} \quad (\nabla\phi \in C^\infty_c(\Omega_i)),$$

and hence

$$\|\phi\|_{L^p(\Omega_i)} \leq C_2\|\nabla^2\phi\|_{L^p(\Omega_i)},$$

with $C_2 = C^2 > 0$. Induction implies for all $k \in \mathbb{N}$ the estimate

$$\|\phi\|_{L^p(\Omega_i)} \leq C_k\|\nabla^k\phi\|_{L^p(\Omega_i)} = C_k\|\phi\|_{\widehat{W}^{k,p}(\Omega_i)}. \tag{A.1}$$

Now let $u \in \widehat{W}^{k,p}_0(\Omega_i)$ and let $(\phi_l)_{l\in\mathbb{N}} \subset C^\infty_c(\Omega_i)$ such that $\phi_l \to u$ in $\widehat{W}^{k,p}_0(\Omega_i)$ as $l \to \infty$. This and (A.1) imply that

$$\|\phi_l - \phi_m\|_{L^p(\Omega_i)} \leq C_k\|\phi_l - \phi_m\|_{\widehat{W}^{k,p}(\Omega_i)} \underset{l,m\to\infty}{\longrightarrow} 0.$$

Hence $(\phi_l)_{l\in\mathbb{N}} \subset L^p(\Omega_i)$ is a Cauchy sequence, thus $\phi_l \to v$ in $L^p(\Omega_i)$ as $l \to \infty$ for some $v \in L^p(\Omega_i)$.

It remains to show that $u = v \in L^p(\Omega_i)$. From [3, Lemma II.6.1] it follows that $\widehat{W}^{k,p}(\Omega_i) \hookrightarrow L^p_{loc}(\Omega_i)$, and since $\widehat{W}^{k,p}_0(\Omega_i) \hookrightarrow \widehat{W}^{k,p}(\Omega_i)$, it follows that $\phi_l \to u$ in $L^p_{loc}(\Omega_i)$ as $l \to \infty$. On the other hand since $L^p(\Omega) \hookrightarrow L^p_{loc}(\Omega_i)$ we also have that $\phi_l \to v$ in $L^p_{loc}(\Omega_i)$ as $l \to \infty$. Hence, $u = v \in L^p(\Omega_i)$ since $L^p_{loc}(\Omega_i)$ is a Hausdorff space.

Let $u \in \widehat{W}^{k,p}_0(\Omega_i)$. Then (A.1) implies

$$\|u\|_{L^p(\Omega_i)} = \lim_{l\to\infty} \|\phi_l\|_{L^p(\Omega_i)} \le \lim_{l\to\infty} C_k \|\phi_l\|_{\widehat{W}^{k,p}(\Omega_i)} = C_k \|u\|_{\widehat{W}^{k,p}(\Omega_i)},$$

and hence

$$\|u\|_{W^{k,p}(\Omega_i)} \le C\|u\|_{\widehat{W}^{k,p}(\Omega_i)} \quad (u \in \widehat{W}^{k,p}_0(\Omega_i)).$$

Now let $u \in W^{k,p}_0(\Omega_i)$. Then it follows directly that

$$\|u\|_{\widehat{W}^{k,p}(\Omega_i)} \le C\|u\|_{W^{k,p}(\Omega_i)}$$

for a constant $C > 0$. □

Lemma A.2. *Let $1 < p < \infty$ with $1 = 1/p + 1/p'$, $\gamma \in \mathbb{R}$, $k \in \mathbb{N}$ and $\rho = |(x_1, x_2)|$. Then for any $p \in (1,\infty)$, functionals of the form*

$$\mathcal{F}(u) = (f, u), \quad f \in C_c^\infty(G),\ u \in \widehat{W}^{k,p}_{0,\gamma}(G) \tag{A.2}$$

are dense in $(\widehat{W}^{k,p}_{0,\gamma}(G))'$.

Proof. Let $f \in C_c^\infty(G)$ with $\operatorname{supp}(f) \subset\subset G$, such that $\operatorname{supp}(f) \subset G \cap \big(B_{2r}(0) \setminus \overline{B}_r(0)\big) =: G_r$ for $r > 0$. Then we have that $r < \rho < 2r$ in G_r and it follows that

$$\widehat{W}^{k,p}_{0,\gamma}(G_r) \widehat{=} \widehat{W}^{k,p}_0(G_r).$$

Lemma A.1 implies that

$$\widehat{W}^{k,p}_0(G_r) \widehat{=} W^{k,p}_0(G_r),$$

and since $W^{k,p}_0(G_r) \hookrightarrow L^p(G_r)$, we have

$$\widehat{W}^{k,p}_{0,\gamma}(G_r) \hookrightarrow L^p(G_r). \tag{A.3}$$

Now consider

$$\mathcal{F}(u) = (f, u), \quad f \in C_c^\infty(G),\ u \in \widehat{W}^{k,p}_{0,\gamma}(G). \tag{A.4}$$

Applying the Hölder inequality and (A.3) in (A.4) we get

$$\begin{aligned} |(f,u)_G| = \left| \int_{G_r} f(x)u(x)dx \right| &\le \|f\|_{L^{p'}(G_r)} \|u\|_{L^p(G_r)} \le C\|f\|_{L^{p'}(G_r)} \|u\|_{\widehat{W}^{k,p}_{0,\gamma}(G_r)} \\ &\le C\|f\|_{L^{p'}(G)} \|u\|_{\widehat{W}^{k,p}_{0,\gamma}(G)} \end{aligned}$$

for $u \in \widehat{W}^{k,p}_{0,\gamma}(G)$ and some constant $C > 0$. Hence, we can characterize the normed dual space $\left(\widehat{W}^{k,p}_{0,\gamma}(G)\right)'$ by

$$\|\mathcal{F}\|_{\left(\widehat{W}^{k,p}_{0,\gamma}(G)\right)'} = \sup_{u \in \widehat{W}^{k,p}_{0,\gamma}(G),\ \|u\|_{\widehat{W}^{k,p}_{0,\gamma}(G)} = 1} |(f,u)_G| < \infty.$$

The proof that the functionals $\mathcal{F}$ of the form (A.2) are dense in $\left(\widehat{W}^{k,p}_{0,\gamma}(G)\right)'$ is analogous to the proof of [3, Lemma II.8.1]. □

Appendix B. Transformations from the wedge onto the layer domain

In this section we give a detailed calculation about the transformed functions between the wedge and layer domain and the relation about the transformed $W_0^{k,p}, \widehat{W}_0^{k,p}$-spaces on the layer and wedge domain for all $k \in \mathbb{N}_0$. Let in the following $G \subset \mathbb{R}^2$ be the wedge domain defined as in (1.2) with opening angle θ_0, $\Omega := \mathbb{R} \times (0, \theta_0)$ be the layer domain. Moreover, let $\psi := \psi_P \circ \psi_E : \Omega \to G$ with $\tilde{v} := \Psi\tilde{u}$ with inverse Ψ^{-1} and let $\Theta_*^p : G \to \Omega$ with $u := \Theta_*^p v$ with inverse Ψ, Θ_p^* be defined as in Section 3. The corresponding pull-back and push-forward operators on $W^{-k,p}$-spaces are depending on k and p, hence, weighted functions appear in the transformed setting. Choosing the right transformation, roughly speaking the right k and p included in the pull-back and push-forward respectively, we can then work in unweighted $W^{-k,p}$-spaces on the layer for $k \in \mathbb{N}_0$. Now, set

$$\begin{aligned} g &:= \widetilde{\Theta}_p^* f = e^{lx}\Theta_p^* f = e^{(l-\beta_p)x}\Psi f, \\ f &:= \widetilde{\Theta}_*^p g = \Theta_*^p e^{-lx} g = \Psi^{-1} e^{(\beta_p - l)x} g, \end{aligned} \tag{B.1}$$

with $l \in \mathbb{N}$ and $\beta_p \in \mathbb{R}$. Next, we give the detailed proof about the calculation of the transformed functions between the domains.

Lemma B.1. *Let $l \in \mathbb{N}$ and $\beta_p \in \mathbb{R}$. Then we have*

(1) $\Psi(\nabla\tilde{u}) = e^{-x}\begin{pmatrix} h^1_{0,0}(\partial_x,\partial_\theta) \\ h^2_{0,0}(\partial_x,\partial_\theta)\end{pmatrix}\tilde{v}$ *for* $\tilde{u} = \Psi^{-1}\tilde{v}$,

(2) $\Theta_p^*(\nabla u) = e^{-x}\begin{pmatrix} h^1_{\beta_p,0}(\partial_x,\partial_\theta) \\ h^2_{\beta_p,0}(\partial_x,\partial_\theta)\end{pmatrix} v$ *for* $u = \Theta_*^p v$,

(3) $\widetilde{\Theta}_p^*(\nabla f) = e^{-x}\begin{pmatrix} h^1_{\beta_p-l,0}(\partial_x,\partial_\theta) \\ h^2_{\beta_p-l,0}(\partial_x,\partial_\theta)\end{pmatrix} g$ *for* $f = \widetilde{\Theta}_*^p g$,

with

$$\begin{aligned} h^1_{r,j}(\partial_x,\partial_\theta) &:= \cos\theta\ (r+j+\partial_x) - \sin\theta\ \partial_\theta \\ h^2_{r,j}(\partial_x,\partial_\theta) &:= \sin\theta\ (r+j+\partial_x) - \cos\theta\ \partial_\theta \end{aligned} \tag{B.2}$$

for $r, j \in \mathbb{R}$.

Proof. (1) The gradient $\Psi\nabla\tilde{u}$ is given as

$$\Psi\nabla\tilde{u} = \Psi\nabla(\Psi^{-1}\tilde{v}) = e^{-x}\begin{pmatrix}\cos\theta\ \partial_x - \sin\theta\ \partial_\theta \\ \sin\theta\ \partial_x - \cos\theta\ \partial_\theta\end{pmatrix}\tilde{v}.$$

(2) Employing $u = \Theta_*^p v$, (1) yields that

$$\nabla u = \nabla(\Theta_*^p v) = \nabla(\Psi^{-1}\mathcal{M}_{\beta_p} v) = \Psi^{-1}e^{(\beta_p-1)x}\begin{pmatrix}\cos\theta\ (\beta_p+\partial_x) - \sin\theta\ \partial_\theta \\ \sin\theta\ (\beta_p+\partial_x) - \cos\theta\ \partial_\theta\end{pmatrix} v,$$

and hence

$$\Theta_p^*(\nabla u) = \mathcal{M}_{-\beta_p}\Psi\nabla u = e^{-x}\begin{pmatrix}\cos\theta\ (\beta_p+\partial_x) - \sin\theta\ \partial_\theta \\ \sin\theta\ (\beta_p+\partial_x) - \cos\theta\ \partial_\theta\end{pmatrix} v.$$

(3) In the same way we compute for $f = \widetilde{\Theta}_p^* g$ that

$$\nabla f = \nabla(\widetilde{\Theta}_*^p g) = \nabla(\Psi^{-1}\mathcal{M}_{\beta_p - l}\ g) = \Psi^{-1}e^{(\beta_p-(l+1))x}\begin{pmatrix}\cos\theta\ (\beta_p - l+\partial_x) - \sin\theta\ \partial_\theta \\ \sin\theta\ (\beta_p - l+\partial_x) - \cos\theta\ \partial_\theta\end{pmatrix} g,$$

hence

$$\widetilde{\Theta}_p^*(\nabla f) = \mathcal{M}_{l-\beta_p}\Psi\nabla f = e^{-x}\begin{pmatrix}\cos\theta\ (\beta_p - l+\partial_x) - \sin\theta\ \partial_\theta \\ \sin\theta\ (\beta_p - l+\partial_x) - \cos\theta\ \partial_\theta\end{pmatrix} g.$$

□

Lemma B.2. *Let $k \in \mathbb{N}_0$, $\beta_p \in \mathbb{R}$ and $\rho := |(x_1, x_2)|$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and $m = (m_1, m_2) \in \mathbb{N}^2$ with $m \leq \alpha$. Let $h^i_{r,j}(\partial_x, \partial_\theta)$ for $i \in \{1,2\}$, $r, j \in \mathbb{R}$ be defined as in* (B.2)*. Then we have*

1. $\widetilde{\Theta}_p^*(D^\alpha f) = e^{-|\alpha|x} \prod_{n=1}^{\alpha_1} h^1_{r_1,-|\alpha|+n}(\partial_x, \partial_\theta) \prod_{n=1}^{\alpha_2} h^2_{r_1,-\alpha_2+n}(\partial_x, \partial_\theta) g$ *with* $r_1 := \beta_p - l$ *and for* $f = \widetilde{\Theta}_*^p g$,
2. $\Theta_p^*(D^\alpha u) = e^{-|\alpha|x} \prod_{n=1}^{\alpha_1} h^1_{\beta_p,-|\alpha|+n}(\partial_x, \partial_\theta) \prod_{n=1}^{\alpha_2} h^2_{\beta_p,-\alpha_2+n}(\partial_x, \partial_\theta) v$ *for* $u = \Theta_*^p v$,
3. $\Psi(D^\alpha \tilde{u}) = e^{-|\alpha|x} \prod_{n=1}^{\alpha_1} h^1_{0,-|\alpha|+n}(\partial_x, \partial_\theta) \prod_{n=1}^{\alpha_2} h^2_{0,-\alpha_2+n}(\partial_x, \partial_\theta) \tilde{v}$ *for* $\tilde{u} = \Psi^{-1}\tilde{v}$,
4. $\Psi(D^\alpha(\rho^{2k} f)) = \sum_{m \leq \alpha} \left(\binom{\alpha}{m} e^{(\beta_p - l)x} e^{-|\alpha|x} e^{2kx} \prod_{n=1}^{m_1} h^1_{r_1,-|\alpha|+2k+n}(\partial_x, \partial_\theta) \cdot \right.$
$\prod_{n=1}^{m_2} h^2_{r_1,-|\alpha-m|-m_2+2k+n}(\partial_x, \partial_\theta) \prod_{n=1}^{\alpha_1 - m_1} h^1_{r_1,-|\alpha-m|+n}(\partial_x, \partial_\theta) \cdot$
$\left. \prod_{n=1}^{\alpha_2 - m_2} h^2_{r_1,-(\alpha_2-m_2)+n}(\partial_x, \partial_\theta) g \right)$ *with* $r_1 := \beta_p - l$ *and for* $f = \widetilde{\Theta}_*^p g$.

Proof. (1) We calculate $D^\alpha f$ for any multiindex $\alpha \in \mathbb{N}_0^2$ satisfying $|\alpha| \leq k$.

First we compute $D_i^j f$ for a fixed $j \in \mathbb{N}_0$ and $i \in \{1,2\}$. If $j = 0$ then it follows by the definition of the push-forward that $f = \Psi^{-1} e^{(\beta_p - l)x} g$. Now let $j \in \mathbb{N} \backslash \{0\}$. We will use in the following the relation

$$h^i_{r,j}(\partial_x, \partial_\theta) e^{sx} \tilde{g} = e^{sx} h^i_{r,s+j}(\partial_x, \partial_\theta) \tilde{g} \quad (r, s \in \mathbb{R},\ \tilde{g} := \Psi \tilde{f},\ i \in \{1,2\}). \tag{B.3}$$

Now define

$$g_{i,j} := \widetilde{\Theta}_p^* \left(\frac{\partial^j}{\partial x_i^j} f \right) \quad (i \in \{1,2\}),$$

then using Lemma B.1 (3) j-times for each first partial derivative yields

$$\begin{aligned}
D_i^j f &= \frac{\partial^j}{\partial x_i^j} f = \frac{\partial}{\partial x_i} \frac{\partial^{j-1}}{\partial x_i^{j-1}} f = \frac{\partial}{\partial x_i} \widetilde{\Theta}_*^p g_{i,j-1} \\
&= \Psi^{-1} e^{(\beta_p - (l+1))x} h^i_{\beta_p - l, 0}(\partial_x, \partial_\theta) g_{i,j-1} \\
&= \Psi^{-1} e^{(\beta_p - (l+1))x} h^i_{\beta_p - l, 0}(\partial_x, \partial_\theta) e^{-x} h^i_{\beta_p - l, 0}(\partial_x, \partial_\theta) g_{i,j-2} \\
&\dots \\
&= \Psi^{-1} e^{(\beta_p - l)x} \underbrace{e^{-x} h^i_{\beta_p - l, 0}(\partial_x, \partial_\theta) \cdot \ldots \cdot e^{-x} h^i_{\beta_p - l, 0}(\partial_x, \partial_\theta)}_{j \times e^{-x} h^i_{\beta_p - l, 0}(\partial_x, \partial_\theta)} g \\
&\overset{\text{(B.3)}}{=} \Psi^{-1} e^{(\beta_p - l)x} e^{-jx} h^i_{\beta_p - l, -j+1}(\partial_x, \partial_\theta) \cdot \ldots \cdot h^i_{\beta_p - l, -j+j}(\partial_x, \partial_\theta) g \\
&= \Psi^{-1} e^{(\beta_p - l)x} e^{-jx} \prod_{n=1}^{j} h^i_{\beta_p - l, -j+n}(\partial_x, \partial_\theta) g \quad (i \in \{1,2\}).
\end{aligned} \tag{B.4}$$

Induction implies (B.4) for arbitrary $j \in \mathbb{N}_0$.

Now let $\alpha \in \mathbb{N}_0^2$ satisfying $|\alpha| \leq k$ with $\alpha = (\alpha_1, \alpha_2)$. Then the last calculation yields that

$$\begin{aligned}
D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} f &= \Psi^{-1} e^{(\beta_p - l)x} e^{-\alpha_1 x} \prod_{n=1}^{\alpha_1} h^1_{\beta_p - l, -\alpha_1 + n}(\partial_x, \partial_\theta) g_{2,\alpha_2} \\
&= \Psi^{-1} e^{(\beta_p - l)x} e^{-\alpha_1 x} \prod_{n=1}^{\alpha_1} h^1_{\beta_p - l, -\alpha_1 + n}(\partial_x, \partial_\theta)\ e^{-\alpha_2 x} \prod_{n=1}^{\alpha_2} h^2_{\beta_p - l, -\alpha_2 + n}(\partial_x, \partial_\theta) g
\end{aligned}$$

$$= \Psi^{-1} e^{(\beta_p - l)x} e^{-(\alpha_1+\alpha_2)x} \prod_{n=1}^{\alpha_1} h^1_{\beta_p - l, -(\alpha_1+\alpha_2)+n}(\partial_x, \partial_\theta) \cdot$$
$$\prod_{n=1}^{\alpha_2} h^2_{\beta_p - l, -\alpha_2+n}(\partial_x, \partial_\theta) g.$$

(2) and (3) follow analogously to (1). We compute (4) by using the Leibniz rule for partial derivatives. Here by $\rho^{2k} = \Psi^{-1} e^{2kx}$, $f = \widetilde{\Theta}^p_* g$ and by (1), (3) and (B.3) we have

$$D^\alpha(\rho^{2k} f) = \sum_{m \leq \alpha} \binom{\alpha}{m} D^m \rho^{2k} D^{\alpha-m} f$$
$$= \sum_{m \leq \alpha} \binom{\alpha}{m} \Psi^{-1} e^{-|m|x} \prod_{n=1}^{m_1} h^1_{0,-|m|+n}(\partial_x, \partial_\theta) \prod_{n=1}^{m_2} h^2_{0,-m_2+n}(\partial_x, \partial_\theta) e^{2kx} \cdot$$
$$e^{(\beta_p - l)x} e^{-|\alpha-m|x} \prod_{n=1}^{\alpha_1 - m_1} h^1_{r_1, -|\alpha-m|+n}(\partial_x, \partial_\theta) \prod_{n=1}^{\alpha_2 - m_2} h^2_{r_1, -(\alpha_2-m_2)+n}(\partial_x, \partial_\theta) g$$
$$= \Psi^{-1} \sum_{m \leq \alpha} \binom{\alpha}{m} e^{-|\alpha|x} e^{2kx} e^{(\beta_p - l)x} \prod_{n=1}^{m_1} h^1_{r_1, -|\alpha|+2k+n}(\partial_x, \partial_\theta) \cdot$$
$$\prod_{n=1}^{m_2} h^2_{r_1, -|\alpha-m|-m_2+2k+n}(\partial_x, \partial_\theta) \prod_{n=1}^{\alpha_1-m_1} h^1_{r_1, -|\alpha-m|+n}(\partial_x, \partial_\theta) \cdot$$
$$\prod_{n=1}^{\alpha_2-m_2} h^2_{r_1, -(\alpha_2-m_2)+n}(\partial_x, \partial_\theta) g,$$

with $r_1 := \beta_p - l$. □

For the next lemma we recall that $K^{l-k}_{p,\gamma}(G)$ is equipped with the norm

$$\|u\|_{K^{l-k}_{p,\gamma}(G)} = \left(\sum_{|\alpha| \leq l-k} \|\rho^{|\alpha|-(l-k)} \partial^\alpha u\|^p_{L^p_\gamma(G)} \right)^{1/p} \tag{B.5}$$

with $L^p_\gamma(G) = L^p(G, \rho^\gamma d(x_1, x_2))$, $\rho := |(x_1, x_2)|$ and $\gamma \in \mathbb{R}$.

Lemma B.3. *Let $1 < p < \infty$ and $1 = \frac{1}{p} + \frac{1}{p'}$. Let $\gamma \in \mathbb{R}$, $l \in \mathbb{N}$, $k \in \mathbb{N}_0$ such that $l \geq k$, $\beta_p := l - k - \frac{2+\gamma}{p}$ and $\rho := |(x_1, x_2)|$. Then*

(1) $\Theta^p_* \in \mathscr{L}_{is}\left(W^{l-k,p}(\Omega), K^{l-k}_{\gamma,p}(G)\right)$,
(2) $\rho^{2k} \widetilde{\Theta}^p_* \in \mathscr{L}_{is}\left(\widehat{W}^{k,p}_0(\Omega), \widehat{W}^{k,p}_{0,\gamma}(G)\right)$,
(3) $(\rho^{2k} \widetilde{\Theta}^{p'}_*)' = \widetilde{\Theta}^*_p$ *and* $(\widetilde{\Theta}^*_{p'} \rho^{-2k})' = \widetilde{\Theta}^p_*$,
(4) $\widetilde{\Theta}^*_p \in \mathscr{L}_{is}\left(\widehat{W}^{-k,p}_\gamma(G), \widehat{W}^{-k,p}(\Omega)\right)$,
(5) $\widetilde{\Theta}^*_p \in \mathscr{L}_{is}\left(\widehat{W}^{-k,p}_\gamma(G), W^{-k,p}(\Omega)\right)$.

Proof. (1) Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$. Let $v \in W^{l-k,p}(\Omega)$. Then Lemma B.2 (2) implies that

$$\sum_{|\alpha| \leq l-k} \|\rho^{|\alpha|-(l-k)} D^\alpha \Theta^p_* v\|^p_{L^p_\gamma(G)}$$

$$= \sum_{|\alpha|\le l-k} \int_\Omega \Bigg| e^{(|\alpha|-(l-k))x} e^{(l-k-\frac{2+\gamma}{p})x} e^{-|\alpha|x} \prod_{n=1}^{\alpha_1} h^1_{\beta_p,-|\alpha|+n}(\partial_x,\partial_\theta)\cdot$$
$$\prod_{n=1}^{\alpha_2} h^2_{\beta_p,-\alpha_2+n}(\partial_x,\partial_\theta)v(x,\theta)\Bigg|^p e^{(\gamma+2)x}dxd\theta$$
$$\le C \sum_{|\alpha|\le l-k} \|v\|^p_{W^{|\alpha|,p}(\Omega)} \le C\|v\|^p_{W^{l-k,p}(\Omega)}. \tag{B.6}$$

Since Θ^p_* is linear (B.6) also implies that

$$\Theta^p_* : W^{l-k,p}(\Omega) \to K^{l-k}_{\gamma,p}(G)$$

is continuous. The open mapping principle implies then that its inverse $\Theta^*_p : K^{l-k}_{\gamma,p}(G) \to W^{l-k,p}(\Omega)$ is continuous, too.

(2) Let $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2$. Let $g \in \widehat{W}^{k,p}_0(\Omega)$, then Lemma B.2 (4) and Lemma A.1 imply

$$\|\rho^{2k}\widetilde{\Theta}^p_* g\|^p_{\widehat{W}^{k,p}_\gamma(G)} = \sum_{|\alpha|=k} \|D^\alpha(\rho^{2k}\widetilde{\Theta}^p_* g)\|^p_{L^p_\gamma(G)}$$
$$= \sum_{|\alpha|=k}\int_\Omega \Bigg|\sum_{m\le\alpha}\binom{\alpha}{m} e^{(\beta_p-l)x}e^{-|\alpha|x}e^{2kx}\prod_{n=1}^{m_1}h^1_{r_1,j_1+n}(\partial_x,\partial_\theta)\prod_{n=1}^{m_2}h^2_{r_1,j_2+n}(\partial_x,\partial_\theta)\cdot$$
$$\prod_{n=1}^{\alpha_1-m_1}h^1_{r_1,-|\alpha-m|+n}(\partial_x,\partial_\theta)\prod_{n=1}^{\alpha_2-m_2}h^2_{r_1,-(\alpha_2-m_2)+n}(\partial_x,\partial_\theta)g(x,\theta)\Bigg|^p e^{(2+\gamma)x}dxd\theta$$
$$\le C\sum_{|\alpha|=k}\int_\Omega \Bigg| e^{2kx}e^{((l-k-\frac{2+\gamma}{p}-l)x}e^{-kx}\sum_{m\le\alpha}\prod_{n=1}^{m_1}h^1_{r_1,j_1+n}(\partial_x,\partial_\theta)\prod_{n=1}^{m_2}h^2_{r_1,j_2+n}(\partial_x,\partial_\theta)$$
$$\prod_{n=1}^{\alpha_1-m_1}h^1_{r_1,-|\alpha-m|+n}(\partial_x,\partial_\theta)\prod_{n=1}^{\alpha_2-m_2}h^2_{r_1,-(\alpha_2-m_2)+n}(\partial_x,\partial_\theta)g(x,\theta)\Bigg|^p e^{(2+\gamma)x}dxd\theta$$
$$\le C\|g\|^p_{W^{k,p}(\Omega)} \le C\|g\|^p_{\widehat{W}^{k,p}(\Omega)},$$

with $j_1 := -|\alpha| + 2k$ and $j_2 := -|\alpha - m| - m_2 + 2k$.

Since $\rho^{2k}\widetilde{\Theta}^p_*$ is linear, the last estimates yield that

$$\rho^{2k}\widetilde{\Theta}^p_* : \widehat{W}^{k,p}_0(\Omega) \to \widehat{W}^{k,p}_{0,\gamma}(G)$$

is continuous. The open mapping principle implies that its inverse $\widetilde{\Theta}^*_p\rho^{-2k}$ is also continuous from $\widehat{W}^{k,p}_{0,\gamma}(G)$ to $\widehat{W}^{k,p}_0(\Omega)$.

(3) We recall that by (B.1) we have $\widetilde{\Theta}^p_* u = \Psi^{-1}\mathcal{M}_{\beta_p-l}u$. Lemma A.2 implies that $\overline{C^\infty_c(G)} = (\widehat{W}^{k,p}_{0,\gamma}(G))'$. Hence, for a $\varphi \in C^\infty_c(G)$ and by

$$\beta_{p'} = l - k - \frac{2+\gamma}{p'} = -\beta_p + 2(l-k-1) - \gamma$$

we compute

$$\left(\rho^{2k}\widetilde{\Theta}^{p'}_* g,\, \varphi\right)_G = \int_G \rho^{2k}\varphi(y)(\mathcal{M}_{\beta_{p'}-l}\, g)(\psi^{-1}(y))\rho^\gamma dy$$

$$
\begin{aligned}
&= \int_\Omega e^{2kx}\varphi(\psi(x,\theta))(\mathcal{M}_{\beta_{p'}-l}\, g)(x,\theta)\; e^{(\gamma+2)x} dx d\theta \\
&= \int_\Omega \varphi(\psi(x,\theta)) e^{2kx} e^{(-\beta_p+2(l-k-1)-\gamma-l)x} g(x,\theta) e^{(\gamma+2)x} dx d\theta \\
&= \int_\Omega (\mathcal{M}_{l-\beta_p}\Psi\varphi)(x,\theta) g(x,\theta) dx d\theta \\
&= \left(g,\ \widetilde{\Theta}^*_p\varphi\right)_\Omega \quad (g \in \widehat{W}^{k,p}_0(\Omega), \varphi \in C^\infty_c(G)),
\end{aligned}
$$

and hence $(\rho^{2k}\widetilde{\Theta}^{p'}_*)' = \widetilde{\Theta}^*_p$.

The next relation follows since

$$(\widetilde{\Theta}^*_{p'}\rho^{-2k})' = ((\rho^{2k}\widetilde{\Theta}^{p'}_*)^{-1})' = ((\rho^{2k}\widetilde{\Theta}^{p'}_*)')^{-1} = (\widetilde{\Theta}^*_p)^{-1} = \widetilde{\Theta}^p_*.$$

(4) is a consequence from relations (2) and (3).

(5) We have that $\left(W^{k,p'}_0(\Omega)\right)' = W^{-k,p}(\Omega)$ and $\left(\widehat{W}^{k,p'}_0(\Omega)\right)' = \widehat{W}^{-k,p}(\Omega)$. Lemma A.1 implies for $\Omega = \mathbb{R}\times(0,\theta_0)$ that $W^{k,p'}_0(\Omega)$ and $\widehat{W}^{k,p'}_0(\Omega)$ are isomorphic. This yields that

$$W^{-k,p}(\Omega) \widehat{=} \widehat{W}^{-k,p}(\Omega),$$

and the assumption follows by relation (4). □

In the following we give the transformation of an elliptic operator from the wedge domain onto a layer domain. We consider

$$\left.\begin{aligned} \Delta^i u = f &\quad \text{in} \quad G, \\ B(u) = 0 &\quad \text{on} \quad \partial G, \end{aligned}\right\} \tag{B.7}$$

$B(u)$ defines the boundary conditions, $u = u(x_1,x_2)$, $f = f(x_1,x_2)$ and $i \in \mathbb{N}$. Then we have:

Lemma B.4. *Let $i \in \mathbb{N}$, u be the solution of problem* (B.7), $\beta_p \in \mathbb{R}$. *Then we have*

$$\Theta^*_p(\Delta^i u) = e^{-2ix}\prod_{j=1}^{i}\left(r_{\beta_p-2(i-j)}(\partial_x) + \partial^2_\theta\right) v \tag{B.8}$$

with the polynomial

$$r_a(\partial_x) := (\partial_x + a)^2 \quad (a \in \mathbb{R}).$$

Proof. Let $i \in \mathbb{N}$ be fixed. By definition of the pull-back and the calculations of [4] we know that for $u = \Theta^p_* v$ we have

$$\Delta u = \Delta(\Theta^p_* v) = \Delta(\Psi^{-1}\mathcal{M}_{\beta_p} v) = \Psi^{-1}\mathcal{M}_{\beta_p-2}(r_{\beta_p}(\partial_x) + \partial^2_\theta)v.$$

Hence $\Theta^*_p(\Delta u) = \mathcal{M}_{-2}(r_{\beta_p}(\partial_x) + \partial^2_\theta)v$. This yields that

$$
\begin{aligned}
\Delta^i u &= \Psi^{-1}\mathcal{M}_{\beta_p-2}\left(r_{\beta_p}(\partial_x) + \partial^2_\theta\right)\Theta^*_p(\Delta^{i-1}u) \\
&= \Psi^{-1}\mathcal{M}_{\beta_p-2}\left(r_{\beta_p}(\partial_x) + \partial^2_\theta\right)\mathcal{M}_{-\beta_p}\Psi\Delta^{i-1}u \\
&= \Psi^{-1}\mathcal{M}_{\beta_p-2}\left(r_{\beta_p}(\partial_x) + \partial^2_\theta\right)\mathcal{M}_{-2}\left(r_{\beta_p}(\partial_x) + \partial^2_\theta\right)\mathcal{M}_{-\beta_p}\Psi\Delta^{i-2}u \\
&\dots \\
&= \Psi^{-1}\mathcal{M}_{\beta_p}\underbrace{\mathcal{M}_{-2}\left(r_{\beta_p}(\partial_x) + \partial^2_\theta\right)\mathcal{M}_{-2}\left(r_{\beta_p}(\partial_x) + \partial^2_\theta\right)\cdot\dots\cdot\mathcal{M}_{-2}\left(r_{\beta_p}(\partial_x) + \partial^2_\theta\right)}_{i\times\mathcal{M}_{-2}(r_{\beta_p}(\partial_x)+\partial^2_\theta)} v.
\end{aligned}
$$

Since

$$\partial^2_\theta\mathcal{M}_{-\alpha}v = \mathcal{M}_{-\alpha}\partial^2_\theta v \quad \text{and} \quad r_{\beta_p}(\partial_x)\mathcal{M}_{-\alpha}v = \mathcal{M}_{-\alpha}r_{\beta_p-\alpha}(\partial_x)v \quad (\alpha \in \mathbb{R}),$$

we then have

$$\begin{aligned}\Theta_p^*(\Delta^i u) &= \mathcal{M}_{-\beta_p}\Psi(\Delta^i u)\\ &= \mathcal{M}_{-\beta_p}\mathcal{M}_{\beta_p-2i}\left(r_{\beta_p-2(i-1)}(\partial_x)+\partial_\theta^2\right)\cdot\dots\cdot\left(r_{\beta_p-2(i-i)}(\partial_x)+\partial_\theta^2\right)v\\ &= \mathcal{M}_{-2i}\prod_{j=1}^{i}\left(r_{\beta_p-2(i-j)}(\partial_x)+\partial_\theta^2\right)v.\end{aligned}$$

This proves the assertion for all $i\in\mathbb{N}$. □

Appendix C. Critical values

In this section we consider for $0<\theta_0<\pi$ the set

$$N := \Big\{\,\lambda\in\mathbb{C} : \sinh^2(\lambda\theta_0)=\lambda^2\sin^2(\theta_0)\,\Big\},$$

which represents the set of zeros of an entire analytic function. In particular, N is locally finite. We are interested in the intersection $N\cap S(a,\,b)$ of N and a strip $S(a,\,b) := \{\,\lambda\in\mathbb{C} : a\le \operatorname{Im}\lambda\le b\,\}$, where $a,\,b\in\mathbb{R}$ with $a<b$. With the aid of the following lemma we will show that $N\cap S(a,\,b)$ is finite.

Lemma C.1. *Let $\phi>0$ and let $\sigma\in\mathbb{R}$.*

(1) *For every solution $\tau\in\mathbb{R}$ of the equation $\cosh(\phi\tau)=\sigma$ it holds that*
$$|\tau|\le \tfrac{1}{\phi}\log(1+2|\sigma|).$$
(2) *For every solution $\tau\in\mathbb{R}$ of the equation $\sinh(\phi\tau)=\sigma\tau$ it holds that*
$$|\tau|\le \tfrac{2}{\phi}\log(1+\tfrac{2}{\phi}|\sigma|).$$

Proof. (1) If $\sigma<1$, then the equation $\cosh(\phi\tau)=\sigma$ has no solution $\tau\in\mathbb{R}$ and there is nothing that needs to be proved. So assume that $\sigma\ge 1$ in the following. Then the equation $\cosh(\phi\tau)=\sigma$ has precisely one solution $\tau\ge 0$ and precisely one solution $\tau'\le 0$ and it holds that $\tau'=-\tau$. Since we have $\cosh(\phi t)\ge \frac12\exp(\phi t)-\frac12$ for all $t\in\mathbb{R}$, we infer that $\tau\le\tau^*$, where $\tau^*>0$ is defined by the equation $\frac12\exp(\phi\tau^*)-\frac12=\sigma$. Since $\tau^*=\frac{1}{\phi}\log(1+2\sigma)$ we obtain the asserted estimate.

(2) For $\sigma\le 0$ the function $t\mapsto\sigma t:\mathbb{R}\longrightarrow\mathbb{R}$ represents a straight line with non-positive slope. In this case, $\tau=0$ is the only solution of the equation $\sinh(\phi\tau)=\sigma\tau$ and there is nothing that needs to be proved. So assume that $\sigma>0$ in the following. Then the equation $\sinh(\phi\tau)=\sigma\tau$ has precisely three solutions: The solution $\tau_0=0$, a positive solution $\tau>0$ and a negative solution $\tau'<0$. Since $\tau'=-\tau$ we only need to estimate the positive solution τ. To this end we observe that

$$\frac{\sigma}{\phi}=\frac{1}{\phi\tau}\sinh(\phi\tau)=\sum_{k\ge0}\frac{(\phi\tau)^{2k}}{(2k+1)!}\ge\sum_{k\ge0}\frac{(\phi\tau)^{2k}}{(2k)!\cdot 2^{2k}}=\sum_{k\ge0}\frac{(\frac{\phi}{2}\tau)^{2k}}{(2k)!}=\cosh(\tfrac{\phi}{2}\tau)=:\sigma'$$

and (1) yields

$$\tau\le\tfrac{2}{\phi}\log(1+2\sigma')\le\tfrac{2}{\phi}\log(1+\tfrac{2}{\phi}\sigma)$$

due to the monotonicity of the logarithm. This proves the estimate (2). □

Corollary C.2. *We have:*

(1) *For all $k\in\mathbb{Z}$ the set $N\cap S(k\frac{\pi}{\theta_0}-\frac13\frac{\pi}{\theta_0},\,k\frac{\pi}{\theta_0}+\frac13\frac{\pi}{\theta_0})$ is finite.*
(2) *For all $\ell\in\mathbb{Z}$ the set $N\cap S((\ell+\frac12)\frac{\pi}{\theta_0}-\frac13\frac{\pi}{\theta_0},\,(\ell+\frac12)\frac{\pi}{\theta_0}+\frac13\frac{\pi}{\theta_0})$ is finite.*
(3) *For all $a,\,b\in\mathbb{R}$ with $a<b$ the set $N\cap S(a,\,b)$ is finite.*

Proof. Let $\lambda \in N$. Then, by definition of N there exists $\varepsilon \in \{\pm 1\}$ such that $\sinh(\lambda\theta_0) = \varepsilon\lambda\sin(\theta_0)$. Taking the real and the imaginary part of this equation, respectively, we arrive at

$$\text{(i)}\ \sinh(\tau\theta_0)\cos(\alpha\theta_0) = \varepsilon\tau\sin(\theta_0), \quad \text{(ii)}\ \cosh(\tau\theta_0)\sin(\alpha\theta_0) = \varepsilon\alpha\sin(\theta_0),$$

where $\tau,\ \alpha \in \mathbb{R}$ such that $\lambda = \tau + i\alpha$.

To prove (1) assume that $k\frac{\pi}{\theta_0} - \frac{1}{3}\frac{\pi}{\theta_0} \le \alpha \le k\frac{\pi}{\theta_0} + \frac{1}{3}\frac{\pi}{\theta_0}$ for some $k \in \mathbb{Z}$. Then we have $|\cos(\alpha\theta_0)| \ge \frac{1}{2}$ and (i) implies that

$$\sinh(\tau\theta_0) = \varepsilon\tau\frac{\sin(\theta_0)}{\cos(\alpha\theta_0)} =: \sigma\tau$$

with $|\sigma| \le 2\sin(\theta_0)$. Hence, Lemma C.1 (2) yields

$$|\tau| \le \tfrac{2}{\theta_0}\log(1 + \tfrac{2}{\theta_0}|\sigma|) \le \tfrac{2}{\theta_0}\log(1 + \tfrac{4}{\theta_0}\sin(\theta_0)).$$

Thus, all points of the set $N \cap S(k\frac{\pi}{\theta_0} - \frac{1}{3}\frac{\pi}{\theta_0},\ k\frac{\pi}{\theta_0} + \frac{1}{3}\frac{\pi}{\theta_0})$ are contained in the compact set

$$A := \Big\{ z \in \mathbb{C} : |\operatorname{Re} z| \le \tfrac{2}{\theta_0}\log(1 + \tfrac{4}{\theta_0}\sin(\theta_0)),\ k\tfrac{\pi}{\theta_0} - \tfrac{1}{3}\tfrac{\pi}{\theta_0} \le \operatorname{Im} z \le k\tfrac{\pi}{\theta_0} + \tfrac{1}{3}\tfrac{\pi}{\theta_0} \Big\}.$$

Since N is locally finite, $N \cap A$ is finite. Therefore, we infer that the set $N \cap S(k\frac{\pi}{\theta_0} - \frac{1}{3}\frac{\pi}{\theta_0},\ k\frac{\pi}{\theta_0} + \frac{1}{3}\frac{\pi}{\theta_0})$ is also finite.

To prove (2) assume that $(\ell + \frac{1}{2})\frac{\pi}{\theta_0} - \frac{1}{3}\frac{\pi}{\theta_0} \le \alpha \le (\ell + \frac{1}{2})\frac{\pi}{\theta_0} + \frac{1}{3}\frac{\pi}{\theta_0}$ for some $\ell \in \mathbb{Z}$. Then we have $|\sin(\alpha\theta_0)| \ge \frac{1}{2}$ and (ii) implies that

$$\cosh(\tau\theta_0) = \varepsilon\alpha\frac{\sin(\theta_0)}{\sin(\alpha\theta_0)} =: \sigma$$

with $|\sigma| \le 2(|\ell| + \frac{5}{6})\frac{\pi}{\theta_0}\sin(\theta_0)$. Hence, Lemma C.1 (1) yields

$$|\tau| \le \tfrac{1}{\theta_0}\log(1 + 2|\sigma|) \le \tfrac{1}{\theta_0}\log(1 + (|\ell| + \tfrac{5}{6})\tfrac{4\pi}{\theta_0}\sin(\theta_0)).$$

Thus, all points of the set $N \cap S((\ell + \frac{1}{2})\frac{\pi}{\theta_0} - \frac{1}{3}\frac{\pi}{\theta_0},\ (\ell + \frac{1}{2})\frac{\pi}{\theta_0} + \frac{1}{3}\frac{\pi}{\theta_0})$ are contained in the compact set

$$A := \Big\{ z \in \mathbb{C} : |\operatorname{Re} z| \le \tfrac{1}{\theta_0}\log(1 + (|\ell| + \tfrac{5}{6})\tfrac{4\pi}{\theta_0}\sin(\theta_0)),$$
$$(\ell + \tfrac{1}{2})\tfrac{\pi}{\theta_0} - \tfrac{1}{3}\tfrac{\pi}{\theta_0} \le \operatorname{Im} z \le (\ell + \tfrac{1}{2})\tfrac{\pi}{\theta_0} + \tfrac{1}{3}\tfrac{\pi}{\theta_0} \Big\}.$$

Since N is locally finite, $N \cap A$ is finite. Therefore, we infer that the set $N \cap S((\ell + \frac{1}{2})\frac{\pi}{\theta_0} - \frac{1}{3}\frac{\pi}{\theta_0},\ (\ell + \frac{1}{2})\frac{\pi}{\theta_0} + \frac{1}{3}\frac{\pi}{\theta_0})$ is also finite.

To prove (3) let $a,\ b \in \mathbb{R}$ with $a < b$. Since there exist finite sets $K \subset \mathbb{Z}$ and $L \subset \mathbb{Z}$ such that

$$S(a,\ b) \subset \bigcup_{k\in K} S(k\tfrac{\pi}{\theta_0} - \tfrac{1}{3}\tfrac{\pi}{\theta_0},\ k\tfrac{\pi}{\theta_0} + \tfrac{1}{3}\tfrac{\pi}{\theta_0}) \cup \bigcup_{\ell\in L} S((\ell + \tfrac{1}{2})\tfrac{\pi}{\theta_0} - \tfrac{1}{3}\tfrac{\pi}{\theta_0},\ (\ell + \tfrac{1}{2})\tfrac{\pi}{\theta_0} + \tfrac{1}{3}\tfrac{\pi}{\theta_0}),$$

the assertion is a consequence of (1) and (2). □

Corollary C.3. *For every $k \in \mathbb{Z}$ there exists at most finitely many values $p \in [1,\ \infty]$ such that the equation*

$$\sinh^2(\lambda\theta_0) = \lambda^2\sin^2(\theta_0)$$

has a solution $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda = -(k + 3 - \frac{2}{p}) = -(k + 1 + \frac{2}{q})$, where $q \in [1,\ \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. According to Corollary C.2 the set $N \cap S(-(k + 3),\ -(k + 2))$ is finite. To each point in this set corresponds one "critical" value of $p \in [1,\ \infty]$. □

Of particular interest is the case $k = -2$ in Corollary C.3. Since $-1 \le -(1-\frac{2}{p}) \le 1$, we employ the following result.

Lemma C.4. *We have $N \cap S(-1, 1) = \{0, \pm\tau_1, \pm i\}$, where $\tau_1 > 0$ such that $\pm\tau_1$ are the two non-trivial real solutions of the equation $\sinh(\tau_1\theta_0) = \tau_1 \sin(\theta_0)$.*

Proof. As in the proof of Corollary C.2 we assume that $\lambda \in N \cap S(-1, 1)$ and infer that there exists $\varepsilon \in \{\pm 1\}$ such that

$$\text{(i)}\ \sinh(\tau\theta_0)\cos(\alpha\theta_0) = \varepsilon\tau\sin(\theta_0), \quad \text{(ii)}\ \cosh(\tau\theta_0)\sin(\alpha\theta_0) = \varepsilon\alpha\sin(\theta_0),$$

where $\tau \in \mathbb{R}$ and $\alpha \in [-1, 1]$ such that $\lambda = \tau + i\alpha$. Since we have that $-\pi < -\theta_0 \le \alpha\theta_0 \le \theta_0 < \pi$, we observe that $\sin(\alpha\theta_0) = 0$, if and only if $\alpha = 0$. In this case (i) yields $\tau \in \{0, \pm\tau_1\}$. For $\alpha \in (0, 1]$ we have $\sin(\alpha\theta_0) > 0$ and (ii) yields $\varepsilon = 1$ and

$$\cosh(\tau\theta_0) = \frac{\alpha\sin(\theta_0)}{\sin(\alpha\theta_0)}. \tag{$*$}$$

Now, we consider the functions

$$u,\, v : [0, \tfrac{\pi}{\theta_0}] \longrightarrow \mathbb{R}, \qquad u(s) := s\sin(\theta_0), \quad v(s) := \sin(s\theta_0), \quad 0 \le s \le \tfrac{\pi}{\theta_0}.$$

We have $u(0) = 0 = v(0)$ and $u'(0) = \sin(\theta_0) < \theta_0 = v'(0)$. Hence, if we assume that $u(s_0) = v(s_0)$ for some $0 < s_0 \le \frac{\pi}{\theta_0}$ such that $u(s) < v(s)$ for all $0 < s < s_0$, then we necessarily have $u'(s_0) = \sin(\theta_0) \ge \theta_0\cos(s_0\theta_0)$. Thus, since $\cos(\,\cdot\,\theta_0) : [0, \frac{\pi}{\theta_0}] \longrightarrow \mathbb{R}$ is strictly decreasing, we infer that $u'(s) > v'(s)$ for all $s_0 < s \le \frac{\pi}{\theta_0}$, which implies that $u(s) > v(s)$ for all $s_0 < s \le \frac{\pi}{\theta_0}$. This shows that the graphs of u and v have at most two intersection points in $[0, \frac{\pi}{\theta_0}]$ and one is given by $s = 0$. However, since $u(1) = v(1)$ and $0 < 1 < \frac{\pi}{\theta_0}$, the other one is given by $s = 1$. As a consequence, $\alpha\sin(\theta_0) < \sin(\alpha\theta_0)$ for all $\alpha \in (0, 1)$, in which case the equation $(*)$ has no solution $\tau \in \mathbb{R}$. Therefore, $\alpha \in (0, 1]$ implies $\alpha = 1$. In this case (ii) yields $\tau = 0$. Analogously, or, alternatively, by symmetry, we infer that $\alpha \in [-1, 0)$ implies $\alpha = -1$. In this case (ii) yields $\tau = 0$ again. ☐

Corollary C.5. *The equation*

$$\sinh^2(\lambda\theta_0) = \lambda^2\sin^2(\theta_0)$$

has no solution $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda = -(1-\frac{2}{p}) = 1 - \frac{2}{q}$ for all $p \in (1, 2) \cup (2, \infty)$. Here $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Appendix D. Interpolation results

Proposition D.1. *Let $\theta_0 > 0$ and let $\Omega = \mathbb{R} \times (0, \theta_0)$. Let $k, \ell, m \in \mathbb{N}$ and let $0 < \eta < 1$ such that $k = \eta m \ge \ell$. Let $1 < p < \infty$. Then we have*

$$\big[L^p(\Omega),\ W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega)\big]_\eta = W^{k,p}(\Omega) \cap W_0^{\ell,p}(\Omega),$$

where $[\,\cdot\,,\,\cdot\,]_\eta$ denotes the complex interpolation functor. In particular, we have $[L^p(\Omega),\ W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)]_{\frac{1}{2}} = W_0^{2,p}(\Omega)$.

Proof. (1) Let $\mathcal{E}_0^* : C_c^\infty(\Omega) \longrightarrow C_c^\infty(\mathbb{R}^2)$ be given as $\mathcal{E}_0^*\phi(x) := \phi(x)$, if $x \in \Omega$, and $\mathcal{E}_0^*\phi(x) := 0$, if $x \in \mathbb{R}^2 \backslash \Omega$, for $\phi \in C_c^\infty(\Omega)$. Then $\|\mathcal{E}_0^*\phi\|_{L^p(\mathbb{R}^2)} = \|\phi\|_{L^p(\Omega)}$ for all $\phi \in C_c^\infty(\Omega)$ and, hence, $\mathcal{E}_0^*$ extends to a continuous linear isometry $\mathcal{E}_0 : L^p(\Omega) \longrightarrow L^p(\mathbb{R}^2)$. Moreover, $\|\mathcal{E}_0\phi\|_{W^{m,p}(\mathbb{R}^2)} = \|\mathcal{E}_0^*\phi\|_{W^{m,p}(\mathbb{R}^2)} = \|\phi\|_{W^{m,p}(\Omega)}$ for all $\phi \in C_c^\infty(\Omega)$, which implies that $\mathcal{E}_0 u \in W^{m,p}(\mathbb{R}^2)$ with $\|\mathcal{E}_0 u\|_{W^{m,p}(\mathbb{R}^2)} = \|u\|_{W_0^{m,p}(\Omega)}$ for all $u \in W_0^{m,p}(\Omega)$. Indeed, let $u \in W_0^{m,p}(\Omega)$ and let $(\phi_j)_{j\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\phi_j \to u$ in $W_0^{m,p}(\Omega)$ as $j \to \infty$. Then, on the one hand, $\phi_j \to u$ in $L^p(\Omega)$ as $j \to \infty$, which implies that $\mathcal{E}_0\phi_j = \mathcal{E}_0^*\phi_j \to \mathcal{E}_0 u$ in

$L^p(\mathbb{R}^2)$ as $j \to \infty$. On the other hand, $(\mathcal{E}_0\phi_j)_{j\in\mathbb{N}} \subset W^{m,p}(\mathbb{R}^2)$ is a Cauchy sequence and, thus, $\mathcal{E}_0\phi_j \to v$ in $W^{m,p}(\mathbb{R}^2)$ as $j \to \infty$ for some $v \in W^{m,p}(\mathbb{R}^2)$. It follows that $\mathcal{E}_0 u = v$ and $\|v\|_{W^{m,p}(\mathbb{R}^2)} = \lim_{j\to\infty} \|\mathcal{E}_0\phi_j\|_{W^{m,p}(\mathbb{R}^2)} = \lim_{j\to\infty} \|\phi_j\|_{W_0^{m,p}(\Omega)} = \|u\|_{W_0^{m,p}(\Omega)}$.

(2) Let $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\chi|_{(-\infty, \theta_0/3]} \equiv 0$ and $\chi|_{[2\theta_0/3, \infty)} \equiv 1$. Let $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{R}$ be the solution of the linear system of equations

$$\sum_{j=1}^{m+1} \left(-\frac{j}{m+2}\right)^i \alpha_j = 1, \qquad i = 0, \ldots, m.$$

We define $\mathcal{R}_m^* : C_c^\infty(\mathbb{R}^2) \longrightarrow C_c^\infty(\bar{\Omega})$ as

$$\begin{aligned}\mathcal{R}_m^*\varphi(x, \theta) :=& \varphi(x, \theta) - \sum_{j=1}^{m+1} \alpha_j (1-\chi)(\theta)\varphi(x, -\tfrac{j}{m+2}\theta) \\ &- \sum_{j=1}^{m+1} \alpha_j \chi(\theta)\varphi(x, (1+\tfrac{j}{m+2})\theta_0 - \tfrac{j}{m+2}\theta)\end{aligned}$$

for $(x, \theta) \in \Omega$ and $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then $\|\mathcal{R}_m^*\varphi\|_{L^p(\Omega)} \le C\|\varphi\|_{L^p(\mathbb{R}^2)}$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ and, hence, $\mathcal{R}_m^*$ extends to a continuous linear operator $\mathcal{R}_m : L^p(\mathbb{R}^2) \longrightarrow L^p(\Omega)$. For $\phi \in C_c^\infty(\Omega)$ we have $\mathcal{R}_m\mathcal{E}_0\phi = \phi$, which implies that $\mathcal{R}_m\mathcal{E}_0 u = u$ for all $u \in L^p(\Omega)$, i. e. $\mathcal{R}_m$ is a continuous linear retraction and $\mathcal{E}_0$ is a corresponding continuous linear coretraction. Moreover, $\|\mathcal{R}_m\varphi\|_{W^{m,p}(\Omega)} = \|\mathcal{R}_m^*\varphi\|_{W^{m,p}(\Omega)} \le C\|\varphi\|_{W^{m,p}(\mathbb{R}^2)}$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$, which implies that $\mathcal{R}_m v \in W^{m,p}(\Omega)$ with $\|\mathcal{R}_m v\|_{W^{m,p}(\Omega)} \le C\|v\|_{W^{m,p}(\mathbb{R}^2)}$ for all $v \in W^{m,p}(\mathbb{R}^2)$. This follows with the same argument as used in Step (1) above. Finally, by construction we have $(\partial_\theta^i \mathcal{R}_m\varphi)|_{\partial\Omega} \equiv 0$ for $i = 0, \ldots, m$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$, which implies that $\mathcal{R}_m v \in W_0^{m,p}(\Omega)$ for all $v \in W^{m,p}(\mathbb{R}^2)$. Thus, $\mathcal{R}_m|_{W^{m,p}(\mathbb{R}^2)} : W^{m,p}(\mathbb{R}^2) \longrightarrow W_0^{m,p}(\Omega)$ is a continuous linear retraction and $\mathcal{E}_0|_{W_0^{m,p}(\Omega)} : W_0^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^2)$ is a corresponding continuous linear coretraction. Now, [15, Thm. 2.4.2/1 (7)] shows that

$$\left[L^p(\mathbb{R}^2),\ W^{m,p}(\mathbb{R}^2)\right]_\eta \cong W^{k,p}(\mathbb{R}^2) \tag{D.1}$$

and the retraction principle, [15, Thm. 1.2.4], yields

$$\left[L^p(\Omega),\ W_0^{m,p}(\Omega)\right]_\eta \cong W_0^{k,p}(\Omega). \tag{D.2}$$

(3) Let $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ and $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{R}$ as in Step (2). We define $\mathcal{E}_m^* : C_c^\infty(\bar{\Omega}) \longrightarrow C^m(\mathbb{R}^2)$ as

$$\mathcal{E}_m^*\phi(x, \theta) := \begin{cases} \phi(x, \theta), & 0 < \theta < \theta_0, \\ \sum_{j=1}^{m+1} \alpha_j (1-\chi)(-\tfrac{j}{m+2}\theta)\phi(x, -\tfrac{j}{m+2}\theta), & \theta \le 0, \\ \sum_{j=1}^{m+1} \alpha_j \chi((1+\tfrac{j}{m+2})\theta_0 - \tfrac{j}{m+2}\theta)\phi(x, (1+\tfrac{j}{m+2})\theta_0 - \tfrac{j}{m+2}\theta), & \theta \ge \theta_0, \end{cases}$$

for $(x, \theta) \in \mathbb{R}^2$ and $\phi \in C_c^\infty(\bar{\Omega})$. As in the proof of [1, Thm. 4.26] one verifies that $\mathcal{E}_m^*$ is well-defined and satisfies $\|\mathcal{E}_m^*\phi\|_{L^p(\mathbb{R}^2)} \le C\|\phi\|_{L^p(\Omega)}$ as well as $\|\mathcal{E}_m^*\phi\|_{W^{m,p}(\mathbb{R}^2)} \le C\|\phi\|_{W^{m,p}(\Omega)}$ for all $\phi \in C_c^\infty(\bar{\Omega})$. Therefore, $\mathcal{E}_m^*$ extends to a continuous linear operator $\mathcal{E}_m : L^p(\Omega) \longrightarrow L^p(\mathbb{R}^2)$ that satisfies $\mathcal{E}_m u \in W^{m,p}(\mathbb{R}^2)$ with $\|\mathcal{E}_m u\|_{W^{m,p}(\mathbb{R}^2)} \le C\|u\|_{W^{m,p}(\Omega)}$ for all $u \in W^{m,p}(\Omega)$. This follows with the same argument as used in Step (1) above.

(4) We define $\mathcal{R}_0^* : C^m(\mathbb{R}^2) \longrightarrow C^m(\bar{\Omega})$ as $\mathcal{R}_0^*\varphi := \varphi|_\Omega$ for $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then, clearly, $\mathcal{R}_0^*$ extends to a continuous linear operator $\mathcal{R}_0 : L^p(\mathbb{R}^2) \longrightarrow L^p(\Omega)$ that satisfies $\mathcal{R}_0\mathcal{E}_m\phi =$

ϕ for all $\phi \in C_c^\infty(\bar{\Omega})$. Hence, $\mathcal{R}_0\mathcal{E}_m u = u$ for all $u \in L^p(\Omega)$, i. e. $\mathcal{R}_0$ is a continuous linear retraction and $\mathcal{E}_m$ is a corresponding continuous linear coretraction. Clearly, we also have $\mathcal{R}_0 v \in W^{m,p}(\Omega)$ with $\|\mathcal{R}_0 v\|_{W^{m,p}(\Omega)} \leq C\|v\|_{W^{m,p}(\mathbb{R}^2)}$ for all $v \in W^{m,p}(\mathbb{R}^2)$. Thus, $\mathcal{R}_0|_{W^{m,p}(\mathbb{R}^2)} : W^{m,p}(\mathbb{R}^2) \longrightarrow W^{m,p}(\Omega)$ is a continuous linear retraction and $\mathcal{E}_m|_{W^{m,p}(\Omega)} : W^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^2)$ is a corresponding continuous linear coretraction. Therefore, (D.1) and the retraction principle yield

$$\big[L^p(\Omega),\ W^{m,p}(\Omega)\big]_\eta \cong W^{k,p}(\Omega). \tag{D.3}$$

(5) Since the inclusions

$$W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega) \subset W^{m,p}(\Omega)$$

are continuous, the same is true for the inclusions

$$\begin{aligned} W_0^{k,p}(\Omega) &\cong \big[L^p(\Omega),\ W_0^{m,p}(\Omega)\big]_\eta \subset \big[L^p(\Omega),\ W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega)\big]_\eta \\ &\subset \big[L^p(\Omega),\ W^{m,p}(\Omega)\big]_\eta \cong W^{k,p}(\Omega), \end{aligned}$$

where we used (D.2) and (D.3), respectively. Hence, $[L^p(\Omega),\ W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega)]_\eta$ is a linear subspace of $W^{k,p}(\Omega)$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ constitutes an equivalent norm on $[L^p(\Omega),\ W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega)]_\eta$. According to [10] the trace operator

$$\mathcal{T} : W^{\ell,p}(\Omega) \longrightarrow \prod_{j=0}^{\ell-1} W^{\ell-j-1/p,p}(\partial\Omega)$$

is a continuous linear retraction. Due to [15, Thm. 1.9.3 (c)] the space

$$L^p(\Omega) \cap \big(W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega)\big) = \Big\{ u \in W^{m,p}(\Omega)\ :\ \mathcal{T}u = 0 \Big\} =: E$$

is a dense linear subspace of $[L^p(\Omega),\ W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega)]_\eta$ and, thus, $[L^p(\Omega),\ W^{m,p}(\Omega) \cap W_0^{\ell,p}(\Omega)]_\eta$ coincides with the completion of E in $W^{k,p}(\Omega)$, which is given as

$$\Big\{ u \in W^{k,p}(\Omega)\ :\ \mathcal{T}u = 0 \Big\} = W^{k,p}(\Omega) \cap W_0^{\ell,p}(\Omega).$$

This completes the proof. □

Corollary D.2. *Let $\theta_0 > 0$ and let $\Omega = \mathbb{R} \times (0, \theta_0)$. Let $k,\ \ell,\ m \in \mathbb{N}$ and let $0 < \eta < 1$ such that $k = \eta m \geq \ell$. Let $1 < p,\ p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have*

$$\big[L^p(\Omega),\ (W^{m,p'}(\Omega) \cap W_0^{\ell,p'}(\Omega))'\big]_\eta = (W^{k,p'}(\Omega) \cap W_0^{\ell,p'}(\Omega))',$$

where $[\,\cdot\,,\,\cdot\,]_\eta$ denotes the complex interpolation functor. In particular, we have $[L^p(\Omega),\ (W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega))']_{\frac{1}{2}} = W_0^{2,p'}(\Omega)' = W^{-2,p}(\Omega)$.

Proof. This is a direct consequence of Proposition D.1, the identification $(L^{p'}(\Omega))' \cong L^p(\Omega)$ and [15, Thm. 1.11.3]. □

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Mathematisches Institut, Angewandte Analysis, Heinrich-Heine-Universität Düsseldorf, 40204 Düsseldorf, Germany

E-mail address: matthias.koehne@hhu.de

E-mail address: juergen.saal@hhu.de

E-mail address: laura.westermann@hhu.de

Chapter 4

Manuscript 3
Optimal Regularity of the Stokes Equations on a 2D Wedge Domain Subject to Navier Boundary Conditions

Joint Work with Matthias Köhne and Jürgen Saal.

OPTIMAL REGULARITY FOR THE STOKES EQUATIONS ON A 2D WEDGE DOMAIN SUBJECT TO NAVIER BOUNDARY CONDITIONS

MATTHIAS KÖHNE, JÜRGEN SAAL, AND LAURA WESTERMANN

Abstract. We consider the Stokes equations subject to Navier boundary conditions on a two-dimensional wedge domain with opening angle $\theta_0 \in (0, \pi)$. We prove existence and uniqueness of solutions with optimal regularity in an L^p-setting. The results are based on optimal regularity results for the Stokes equations subject to perfect slip boundary conditions on a two-dimensional wedge domain that have been obtained by the authors in [7]. Based on a detailed study of the corresponding trace operator on anisotropic Sobolev-Slobodeckij type function spaces on a two-dimensional wedge domain we are able to generalize the results proved in [7] to the case of inhomogeneous boundary conditions. Existence and uniqueness of solutions to the Stokes equations subject to (inhomogeneous) Navier boundary conditions are then obtained using a perturbation argument.

Contents

1. Introduction and Main Result

The main objective of this note is to study the (instationary) Stokes equations subject to (inhomogeneous) Navier boundary conditions

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla p &= f \quad \text{in} \quad J \times G, \\ \operatorname{div} u &= g \quad \text{in} \quad J \times G, \\ \alpha u \cdot \tau - \tau^T D_\pm(u)\nu &= h_1 \quad \text{on} \quad J \times \Gamma, \\ u \cdot \nu &= h_0 \quad \text{on} \quad J \times \Gamma, \\ u(0) &= u_0 \quad \text{in} \quad G, \end{aligned}\right\} \tag{1.1}$$

on a two-dimensional wedge domain G. We aim at existence and uniqueness of solutions with optimal regularity in an L^p-setting for $p \in (1, \infty)$. The wedge domain is defined as

$$G := \{(x_1, x_2) \in \mathbb{R}^2 : \; 0 < x_2 < x_1 \tan\theta_0\} \tag{1.2}$$

with opening angle $\theta_0 \in (0, \pi)$ and $J = (0, T)$ with $T > 0$. Here α is a given (variable) parameter, ν and τ denote the unit outer normal vector and a unit tangential vector on $\Gamma := \partial G \setminus \{0\}$ respectively. We have $\nu_1 = -e_2$ and set $\tau_1 = -e_1$ as the unit outer normal vector and a unit tangential vector on $\Gamma_1 := (-\infty, 0) \cdot \tau_1$. Furthermore, we have $\nu_2 = (-\sin\theta_0, \cos\theta_0)^T$ and set $\tau_2 = (\cos\theta_0, \sin\theta_0)^T$ as the unit normal vector and a unit tangential vector on $\Gamma_2 := (0, \infty) \cdot \tau_2$. Thus, the boundary of G is decomposed as

$$\Gamma = \Gamma_1 \cup \Gamma_2 = \partial G \setminus \{0\} \tag{1.3}$$

and we have $(\tau, \nu) = (\tau_j, \nu_j)$ on Γ_j for $j = 1, 2$. Note that (τ_j, ν_j) is positively oriented for $j = 1, 2$. The boundary conditions in the third and fourth equation of system (1.1) have to be understood as:

$$\begin{aligned} \alpha u \cdot \tau_1 - \tau_1^T D_\pm(u)\nu_1 &= h_1^{(1)} \text{ on } J \times \Gamma_1, \\ \alpha u \cdot \tau_2 - \tau_2^T D_\pm(u)\nu_2 &= h_1^{(2)} \text{ on } J \times \Gamma_2, \\ u \cdot \nu_1 &= h_0^{(1)} \text{ on } J \times \Gamma_1, \\ u \cdot \nu_2 &= h_0^{(2)} \text{ on } J \times \Gamma_2, \end{aligned}$$

where $h_\ell^{(j)} := h_\ell|_{\Gamma_j}$ for $\ell = 0, 1$ and $j = 1, 2$. Moreover, $D_\pm(u) := \frac{1}{2}(\nabla u \pm \nabla u^T)$ denote the rate of deformation tensor and the rate of rotation tensor, respectively.

If $\psi : G \longrightarrow \mathbb{R}$ or $\psi : \Gamma \longrightarrow \mathbb{R}$ is a function, we denote by $\langle \psi \rangle_j := \lim_{x\to 0} \psi|_{\Gamma_j}(x)$ its trace at the corner $x = 0$ of the wedge G taken w. r. t. its values on Γ_j for $j = 1, 2$, whenever it exists. By $\langle\!\langle \psi \rangle\!\rangle_\bullet := \langle \psi \rangle_2 - \langle \psi \rangle_1$ we denote its *jump* across the corner, whenever the two traces exist. Finally, we denote by $\langle \psi \rangle_\bullet := \langle \psi \rangle_1 = \langle \psi \rangle_2$ its unique trace at the corner, provided that $\langle\!\langle \psi \rangle\!\rangle_\bullet = 0$. Thus, a condition like $\langle \psi \rangle_\bullet = 0$ implicitly requires $\langle\!\langle \psi \rangle\!\rangle_\bullet = 0$.

We aim at solutions

$$(u, p) \in \mathbb{E} := \mathbb{E}_u \times \mathbb{E}_p, \tag{1.4}$$

where

$$\begin{aligned} \mathbb{E}_u &:= W^{1,p}(J, L^p(G, \mathbb{R}^2)) \cap L^p(J, W^{2,p}(G, \mathbb{R}^2)) \\ \mathbb{E}_p &:= L^p(J, \widehat{W}^{1,p}(G)) \end{aligned}$$

are given as anisotropic (homogeneous) Sobolev spaces; see Section 2. Of course, in this setting uniqueness of the pressure p has to be understood as uniqueness up to an additive constant. Then, necessarily, the given data in (1.1) have to satisfy the regularity conditions

$$\begin{aligned} f \in \mathbb{F}_f \quad &:= L^p(J, L^p(G, \mathbb{R}^2)), \\ g \in \mathbb{F}_g \quad &:= W_p^{1/2}(J, L^p(G)) \cap L^p(J, W^{1,p}(G)), \\ h_1 \in \mathbb{F}_\tau \quad &:= \{\, h : \Gamma \longrightarrow \mathbb{R} \,:\, h|_{\Gamma_j} \in \mathbb{F}_\tau^{(j)} \text{ for } j = 1, 2 \,\}, \text{ where} \\ \mathbb{F}_\tau^{(j)} \quad &:= W_p^{1/2-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{1-1/p}(\Gamma_j)), \;\; j = 1, 2, \\ h_0 \in \mathbb{F}_\nu \quad &:= \{\, h : \Gamma \longrightarrow \mathbb{R} \,:\, h|_{\Gamma_j} \in \mathbb{F}_\nu^{(j)} \text{ for } j = 1, 2 \,\}, \text{ where} \\ \mathbb{F}_\nu^{(j)} \quad &:= W_p^{1-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{2-1/p}(\Gamma_j)), \;\; j = 1, 2, \\ u_0 \in \mathbb{F}_0 \quad &:= W_p^{2-2/p}(G, \mathbb{R}^2), \end{aligned}$$

i. e. we have to work with anisotropic Sobolev-Slobodeckij spaces; see Section 2. For convenience we abbreviate

$$\mathbb{F} := \mathbb{F}_f \times \mathbb{F}_g \times \mathbb{F}_\tau \times \mathbb{F}_\nu \times \mathbb{F}_0. \tag{1.5}$$

We employ the space $BUC^1(\Gamma) := \{\, \alpha : \Gamma \longrightarrow \mathbb{R} \,:\, \alpha|_{\Gamma_j} \in BUC^1(\Gamma_j), \; j = 1, 2 \,\}$ for the coefficients. Besides the obvious necessary compatibility conditions between the right-hand side g in the divergence equation and the initial datum u_0 and between the boundary datum h_j and the initial datum u_0, respectively, there is a somewhat hidden but well-known necessary compatibility condition between g and the normal boundary datum h_0. To formulate this compatibility condition we denote by $p' \in (1,\infty)$ the dual exponent of $p \in (1,\infty)$ and define the functional $F(\gamma,\eta) : W^{1,p'}(G) \longrightarrow \mathbb{R}$ for $\gamma \in \mathbb{F}_g$ and $\eta \in \mathbb{F}_\nu$ as

$$[F(\gamma,\eta)](\phi) := (\eta,\,\phi)_\Gamma - (\gamma,\,\phi)_G, \qquad \phi \in W^{1,p'}(G). \tag{1.6}$$

Since

$$\begin{aligned}[F(g,h_0)](\phi) &= (h_0,\,\phi)_\Gamma - (g,\,\phi)_G = (u\cdot\nu,\,\phi)_\Gamma - (\operatorname{div} u,\,\phi)_G \\ &= (u,\,\nabla\phi)_G \in W^{1,p}(J), \qquad \phi \in W^{1,p'}(G),\end{aligned}$$

we infer that

$$F(g,h_0) \in W^{1,p}\big(J,(W^{1,p'}(G), \|\nabla\cdot\|_{L^{p'}(G,\mathbb{R}^2)})'\big).$$

By the fact that $C_c^\infty(\bar G)$ is dense in $\widehat{W}^{1,p'}(G)$ it follows that $F(g,h_0) \in W^{1,p}(J,\widehat{W}^{-1,p}(G))$.

Remark 1.1. For $g \in \mathbb{F}_g$ the requirement $F(g,0) \in W^{1,p}(J,\widehat{W}^{-1,p}(G))$ is equivalent to $g \in W^{1,p}(J,\widehat{W}^{-1,p}(G))$, while for $h_0 \in \mathbb{F}_\nu$ the requirement $F(0,h_0) \in W^{1,p}(J,\widehat{W}^{-1,p}(G))$ is equivalent to $h_0|_{\Gamma_j} \in W^{1,p}(J,\widehat{W}_p^{-1/p}(\Gamma_j))$ for $j = 1, 2$.

Now, our main result reads as follows.

Theorem 1.2. *Let $J = (0,T)$ with $0 < T < \infty$ and let $G \subset \mathbb{R}^2$ be defined as in (1.2) with $\theta_0 \in (0,\,\pi)$. Let $p \in (1,\infty) \setminus \{\frac{2\theta_0}{3\theta_0-\pi},\ \frac{2\theta_0}{3\theta_0-2\pi},\ \frac{3}{2},\ 2,\ 3\}$. Let $\alpha \in BUC^1(\Gamma)$ with $\langle\alpha\rangle_\bullet = 0$. Suppose the data satisfy the regularity condition*

$$(f,\ g,\ h_1,\ h_0,\ u_0) \in \mathbb{F}$$

and the compatibility conditions

$$\begin{aligned} \operatorname{div} u_0 &= g|_{t=0}, &&\text{if } p>2,\\ u_0\cdot\nu &= h_0|_{t=0}, &&\text{if } p>\tfrac{3}{2},\\ \alpha u_0\cdot\tau - \tau^T D_\pm(u_0)\nu &= h_1|_{t=0}, &&\text{if } p>3,\end{aligned}$$

as well as

$$F(g,h_0) \in W^{1,p}(J,\widehat{W}^{-1,p}(G)).$$

If the boundary condition is posed based on D_+, then assume the compatibility conditions $\langle\!\langle \partial_\tau h_0 + h_1\rangle\!\rangle_\bullet = 0$ in J, if $p > 2$, and

$$\tfrac{1}{2}\langle\partial_{\tau_1} h_0\rangle_1 + \tfrac{1}{2}\langle\partial_{\tau_2} h_0\rangle_2 = \langle\partial_\tau h_0 + h_1\rangle_\bullet \quad \text{in } J, \qquad \text{if } \theta_0 = \tfrac{\pi}{2} \text{ and } p>2.$$

If the boundary condition is posed based on D_-, then assume the compatibility conditions $\langle\!\langle h_1\rangle\!\rangle_\bullet = 0$ in J, if $p > 2$, and

$$-\tfrac{1}{2}\langle\partial_{\tau_1} h_0\rangle_1 - \tfrac{1}{2}\langle\partial_{\tau_2} h_0\rangle_2 = \langle h_1\rangle_\bullet \quad \text{in } J, \qquad \text{if } \theta_0 = \tfrac{\pi}{2} \text{ and } p>2.$$

Then there exists a unique solution $(u,p) \in \mathbb{E}$ to (1.1).

Remark 1.3. The values $p = 2$, $p = \frac{2\theta_0}{3\theta_0-\pi}$ and $p = \frac{2\theta_0}{3\theta_0-2\pi}$ with $\theta_0 \in (0,\pi)$ are excluded in Theorem 1.2 due to technical reasons. In Section 3 we solve the Laplace equation subject to Neumann boundary conditions on the wedge domain by transforming this problem into a problem on a layer domain. The latter is then solved using the operator sum method, which is based on the Kalton-Weis theorem. Due to this method a condition on the spectrum of the operators appears, which excludes $p = \frac{2\theta_0}{3\theta_0-\pi}$ and $p = \frac{2\theta_0}{3\theta_0-2\pi}$. Moreover, the transformation from the layer back to the wedge introduces weights. The norms in the corresponding weighted function spaces can be estimated thanks to Hardy's inequality for all $p \in (1,\infty)$ except for $p = 2$. See Lemma A.2 for Hardy's inequality on the wedge.

Thanks to the solvability of the Laplace equation we can then prove the solvability of equation (1.7) below, which is a crucial step for the proof of Theorem 1.2.

To provide an outline for the following sections we summarize the strategy of the proof of Theorem 1.2. At the end, problem (1.1) is a perturbed variant of the problem

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla p &= f && \text{in } J \times G, \\ \operatorname{div} u &= g && \text{in } J \times G, \\ -\tau^T D_\pm(u)\nu &= h_1 && \text{on } J \times \Gamma, \\ u \cdot \nu &= h_0 && \text{on } J \times \Gamma, \\ u(0) &= u_0 && \text{in } G, \end{aligned}\right\} \tag{1.7}$$

with fully inhomogeneous right-hand sides $(f, g, h_1, h_0, u_0) \in \mathbb{F}$. Therefore, it is sufficient to show existence and uniqueness of solutions $(u, p) \in \mathbb{E}$ to problem (1.7), provided the data satisfy appropriate compatibility conditions. This is achieved by Corollary 4.7.

This result, in turn, relies on the unique solvability of the Stokes equations subject to inhomogeneous perfect slip boundary conditions

$$u \cdot \nu = h_0, \qquad \operatorname{curl} u = h_1 \qquad \text{on } J \times \Gamma.$$

The latter problem is dealt with in Theorem 4.6. On the one hand, the proof of Theorem 4.6 relies on the result [7, Corollary 1], which provides optimal regularity for the Stokes equations subject to homogeneous perfect slip boundary conditions in the L^p-setting for all $p \in (1, \infty)$. On the other hand, to cope with the inhomogeneous boundary conditions, for the proof of Theorem 4.6 we also need to show optimal regularity for the Laplace equation subject to Neumann boundary conditions in the space $\widehat{W}^{1,p}(G)$ for all $p \in (1, \infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi}, \frac{2\theta_0}{3\theta_0 - 2\pi}, 2\}$. This is accomplished by Corollary 3.8, where we show the invertibility of the operator $A_{L,T}\phi := \Delta\phi$ associated to the problem

$$\left.\begin{aligned} \Delta\phi &= f && \text{in } J \times G, \\ \partial_\nu \phi &= 0 && \text{on } J \times \Gamma, \end{aligned}\right\} \tag{1.8}$$

to obtain $\phi \in L^p(J, K^3_p(G))$ for $f \in L^p(J, \widehat{W}^{1,p}(G))$. For a definition of the weighted Sobolev space K^3_p see (2.1) below.

Now, this note is organized as follows. In Section 2 we introduce the notation. Section 3 is devoted to the proof of Corollary 3.8, i. e. , to the treatment of the Laplace equation subject to Neumann boundary conditions in a wedge within the above function spaces. Finally, in Section 4 we prove the unique solvability of problem (1.7) and we provide a complete proof of Theorem 1.2. As auxiliary results, we provide several generic trace theorems for the wedge domain G for anisotropic Sobolev-Slobodeckij spaces, which may be of independent interest. For convenience this note is complemented by an appendix, where we discuss Hardy's inequality for the wedge domain G.

2. Notation

Let X be a Banach space, let $1 \leq p \leq \infty$ and let $\Omega \subset \mathbb{R}^2$ a domain. We set $C_c^\infty(\Omega) := \{u \in C_c^\infty(\Omega) : \operatorname{supp}(u) \subset\subset \Omega\}$ where $\operatorname{supp}(u)$ is the support of u. We denote by $L^p(\Omega, X)$ the X-valued Bochner-Lebesgue space. For $n \in \{1, 2\}$ we define $W^{k,p}(\Omega, \mathbb{R}^n)$ to be the Sobolev space of order $k \in \mathbb{N}$ and we set $W^{0,p} := L^p$. We denote by $W_0^{k,p}(\Omega, \mathbb{R}^n)$ the closure of $C_c^\infty(\Omega, \mathbb{R}^n)$ in the space $W^{k,p}(\Omega, \mathbb{R}^n)$. Furthermore, for $s = k + \lambda$ with $k \in \mathbb{N}_0$

and $0 < \lambda < 1$ we define $W_p^s(\Omega)$ to be the Sobolev-Slobodeckij space that consists of all functions $u \in W^{k,p}(\Omega)$ satisfying

$$\|u\|_{W_p^s(\Omega)} := \|u\|_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \left(\int_\Omega \int_\Omega \frac{|\partial^\alpha u(y) - \partial^\alpha u(x)|^p}{|y-x|^{n+\lambda p}} dy dx \right)^{1/p} < \infty.$$

For $k \in \mathbb{N}_0$ the homogeneous Sobolev space of scalar valued functions is defined as

$$\widehat{W}^{k,p}(\Omega) := \{ u \in L^1_{loc}(\Omega) : \ \partial^\alpha u \in L^p(\Omega), \ |\alpha| = k \},$$

and equipped with the seminorm

$$\|u\|_{\widehat{W}^{k,p}(\Omega)} := \|u\|_{\widehat{W}^{k,p}} := \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p(\Omega)}.$$

Now, let $G \subset \mathbb{R}^2$ be the wedge domain defined in (1.2) with opening angle $\theta_0 \in (0, \pi)$. We define the Kondrat'ev spaces as

$$L^p_\gamma(G) := L^p(G, \rho^\gamma d(x_1, x_2)), \quad \rho = |(x_1, x_2)|, \ \gamma \in \mathbb{R},$$

and for $m \in \mathbb{N}_0$ as

$$K^m_{p,\gamma}(G) := \{ u \in L^1_{loc}(G) : \ \rho^{|\alpha|-m} \partial^\alpha u \in L^p_\gamma(G), \ |\alpha| \le m \}, \quad \gamma \in \mathbb{R}. \tag{2.1}$$

The space $K^m_{p,\gamma}(G)$ equipped with the norm

$$\|u\|_{K^m_{p,\gamma}} := \|u\|_{K^m_{p,\gamma}(G)} := \left(\sum_{|\alpha| \le m} \|\rho^{|\alpha|-m} \partial^\alpha u\|^p_{L^p_\gamma(G)} \right)^{1/p}$$

is a Banach space for all $m \in \mathbb{N}_0$ and all $\gamma \in \mathbb{R}$ and we abbreviate $K^m_p(G) := K^m_{p,0}(G)$. For $k \in \mathbb{N}$ the weighted homogeneous Sobolev space is defined as

$$\widehat{W}^{k,p}_\gamma(G) := \{ u \in L^1_{loc}(G) : \ \partial^\alpha u \in L^p_\gamma(G), \ |\alpha| = k \}, \quad \gamma \in \mathbb{R},$$

and equipped with the seminorm

$$\|u\|_{\widehat{W}^{k,p}_\gamma(G)} := \|u\|_{\widehat{W}^{k,p}_\gamma} := \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p_\gamma(G)}$$

for $k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$.

The norm on a generic Banach space X is usually denoted by $\| \cdot \|_X$. If Y is another Banach space, then $\mathscr{L}(X, Y)$ denotes the space of all bounded linear operators from X to Y and $\mathscr{L}_{is}(X, Y)$ denotes the subspace of all linear isomorphisms from X onto Y. For $Y = X$ we employ the abbreviations $\mathscr{L}(X)$ and $\mathscr{L}_{is}(X)$, respectively.

If $\psi : \Gamma \longrightarrow \mathbb{R}$ is a function, we occasionally denote by $\psi^{(j)} = \psi|_{\Gamma_j}$ the restriction to Γ_j for $j = 1, 2$. The same notation is also occasionally used for vector fields $\psi : \Gamma \longrightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ and should not be confused with the components of ψ in this case. Moreover, if $\psi : \Gamma_j \longrightarrow \mathbb{R}$ with $j \in \{ 1, 2 \}$ is a function that is defined on one of the smooth parts of the boundary of G only, we also employ the notation $\langle \psi \rangle_j := \lim_{x \to 0} \psi(x)$ its trace at the corner $x = 0$ of the wedge G.

3. The Laplace Equation subject to Neumann Boundary Conditions

Let $G \subset \mathbb{R}^2$ be the wedge domain defined as in (1.2) and $J = (0,T)$ with $0 < T < \infty$. The objective of this section is to consider the problem

$$\begin{aligned} \Delta\phi &= f \quad \text{in } J \times G, \\ \partial_\nu \phi &= 0 \quad \text{on } J \times \Gamma, \end{aligned}$$

and to show its optimal regularity. Here ν denotes the unit outer normal vector at Γ with Γ defined as in (1.3). Recall that $\tau_1 = -e_1$ and $\nu_1 = -e_2$ on $\Gamma_1 := (-\infty, 0) \cdot \tau_1$ and $\tau_2 = (\cos\theta_0, \sin\theta_0)^T$ and $\nu_2 = (-\sin\theta_0, \cos\theta_0)^T$ on $\Gamma_2 := (0,\infty) \cdot \tau_2$, respectively. The boundary condition in the above system is to be understood as:

$$\begin{aligned} \partial_{\nu_1}\phi &= 0 \quad \text{on } \Gamma_1, \\ \partial_{\nu_2}\phi &= 0 \quad \text{on } \Gamma_2. \end{aligned}$$

Here, optimal regularity of the Neumann-Laplace equation means to show the invertibility of the operator $A_{L,T}\phi := \Delta\phi$, where

$$A_{L,T} : \ L^p(J, K_p^3(G)) \to L^p(J, \widehat{W}^{1,p}(G))$$

for all $p \in (1,\infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi}, \ \frac{2\theta_0}{3\theta_0 - 2\pi}, 2\}$.

The strategy will be to start considering the time independent Neumann problem for the Laplace operator

$$\left.\begin{aligned} \Delta\phi &= f \ \text{ in } \ G, \\ \partial_\nu \phi &= 0 \ \text{ on } \ \Gamma, \end{aligned}\right\} \tag{3.1}$$

and to transform it onto a layer domain $\Omega := \mathbb{R} \times (0, \theta_0)$ in a first step. Using the operator sum method we can then show the well-posedness of the transformed problem in the unweighted L^p-setting. In a second step we will show higher regularity of the transformed problem and then transform it back onto the wedge domain.

Remark 3.1. In [5, Chapter 4] the Laplace equation subject to general boundary conditions, where the Neumann boundary conditions are included, is studied on polygonal domains. There, localizing the vertices and transforming the Laplace equation to a layer domain yields the same form of the Laplace equation on the layer as in our setting. Hence, alternatively to the operator sum method, by modifying a step in the proof of [5, Theorem 4.3.2.3] we could also prove the invertibility of the transformed Neumann-Laplace operator on the layer. For this approach a suitable variant of the condition [5, (4.3.2.10)] has to be satisfied, which leads to a constraint on the parameter p of the L^p-space. Now, inserting into that equation $\beta_p = 3 - \frac{2+\gamma}{p}$ instead of $\frac{2}{p'} = 2 - \frac{2}{p}$, we get a condition that is equivalent to our spectral condition (3.9); see also Remark 3.6. Thus, this approach would lead to optimal regularity for the Neumann problem for the Laplace operator for the same values of p. However, we prefer to provide a self-contained proof based on the operator sum method.

Let's start with the transformation of problem (3.1) onto the layer domain. We set $\Omega := \mathbb{R} \times I$, with $I := (0, \theta_0)$ where θ_0 is the angle of the wedge G. We write the inverse of the transform to polar coordinates as

$$\psi_P : \ \mathbb{R}_+ \times I \to G, \quad (r,\theta) \mapsto (r\cos\theta, r\sin\theta) = (x_1, x_2).$$

We use the Euler transformation $r = e^x$ in radial direction and write by an abuse of notation $x \in \mathbb{R}$ for the new variable. We set

$$\psi_E : \ \Omega \to \mathbb{R}_+ \times I, \quad (x,\theta) \mapsto (e^x, \theta) =: (r,\theta).$$

It is not difficult to see that

$$\psi := \psi_P \circ \psi_E : \Omega \to G$$

is a diffeomorphism. We set

$$\Psi\phi := \phi \circ \psi \quad \text{and} \quad \Psi^{-1}\varphi := \varphi \circ \psi^{-1}.$$

Analogously to [9] we define pull-back and push-forward by

$$\varphi := \Theta_p^* \phi := e^{-\beta_p x}\Psi\phi \quad \text{and} \quad \phi := \Theta_*^p \varphi := \Psi^{-1} e^{\beta_p x}\varphi \tag{3.2}$$

with $\beta_p \in \mathbb{R}$. Let ϕ be the solution of (3.1), then by [5, Chapter 4] we have that

$$\Theta_p^*(\Delta\phi) = e^{-2x}(r_{\beta_p}(\partial_x) + \partial_\theta^2)\varphi, \tag{3.3}$$

where

$$r_{\beta_p}(\partial_x)\varphi := (\partial_x + \beta_p)^2\varphi. \tag{3.4}$$

To absorb the factor e^{-2x} in (3.3), we put

$$g = \widetilde{\Theta}_p^* f = e^{2x}\Theta_p^* f \tag{3.5}$$

with inverse $(\widetilde{\Theta}_p^*)^{-1} = \widetilde{\Theta}_*^p$. By the choice of

$$\beta_p = 3 - \frac{2+\gamma}{p} \tag{3.6}$$

Lemma 3.5 implies that

$$\widetilde{\Theta}_p^* \in \mathscr{L}_{is}\left(\widehat{W}_\gamma^{1,p}(G), W^{1,p}(\Omega)\right).$$

We notice that β_p, the pull-back and the push-forward depend on p. That means that the corresponding operator families are not consistent in p.

After transforming the boundary conditions of (3.1) to the layer domain we obtain

$$\partial_\theta \varphi = 0 \quad \text{on } \partial\Omega = \mathbb{R} \times \{0, \theta_0\}.$$

Hence, (3.1) is equivalent to

$$\left.\begin{aligned} -(r_{\beta_p}(\partial_x) + \partial_\theta^2)\varphi &= g \text{ in } \Omega \\ \partial_\theta\varphi &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{3.7}$$

The proof of the well-posedness of problem (3.7) needs some preparation. We start to describe the operators associated to the single parts of (3.7):

(1) Let r_{β_p} be the polynomial given in (3.4) with β_p given as in (3.6). We define $\mathcal{T}_x$ in $L^p(\mathbb{R})$ by

$$\mathcal{T}_x\varphi := -r_{\beta_p}(\partial_x)\varphi, \quad \varphi \in D(\mathcal{T}_x) := W^{2,p}(\mathbb{R}).$$

The spectrum of $\mathcal{T}_x$ is given by the parabola $-r_{\beta_p}(i\mathbb{R})$, which is symmetric w.r.t. the real axis, open to the right and has its intersection point with the real axis at $-\beta_p^2$. It is known that $\mathcal{T}_x + d \in \mathcal{H}^\infty(L^p(\mathbb{R}))$ for $d > \beta_p^2$ with $\phi^\infty_{\mathcal{T}_x+d} < \frac{\pi}{2}$, see [11, 9]. These properties are also true for the canonical extension of $\mathcal{T}_x$ to $L^p(\mathbb{R}, L^p(I))$, that is

$$T_x\varphi := \mathcal{T}_x\varphi, \quad \varphi \in D(T_x) := W^{2,p}(\mathbb{R}, L^p(I)),$$

see, for instance, [12, 3, 8] for operator-valued Fourier multiplier results.

(2) We define $\mathcal{T}_\theta$ in $L^p(I)$ by

$$\mathcal{T}_\theta\varphi := -\partial_\theta^2\varphi, \quad \varphi \in D(\mathcal{T}_\theta) := \left\{\phi \in W^{2,p}(I):\ \partial_\theta\phi = 0 \text{ on } \partial I\right\}.$$

It is straight forward to calculate its spectrum, which is given as

$$\sigma(\mathcal{T}_\theta) = \{0\} \cup \left\{ \left(\frac{\pi k}{\theta_0}\right)^2,\ k \in \mathbb{N} \right\} \tag{3.8}$$

with corresponding eigenfunctions

$$\tilde{e}_k(\theta) := \cos\left(\frac{\pi k}{\theta_0}\theta\right), \quad k \in \mathbb{N}_0,\ \theta \in I,$$

see also [9]. Since $\mathcal{T}_\theta$ is self-adjoint in $L^2(I)$, the eigenfunctions form a basis of $L^2(I)$. We denote by $(\lambda_i)_{i\in\mathbb{N}_0}$ the set of eigenvalues of $\mathcal{T}_\theta$, i.e., $(\lambda_i)_{i\in\mathbb{N}_0} = \sigma(\mathcal{T}_\theta)$ such that $\lambda_0 < \lambda_1 < \dots$. Setting $e_0 := \frac{\tilde{e}_0}{\sqrt{\theta_0}}$, where $\tilde{e}_0$ is the eigenfunction to the eigenvalue $\lambda_0 = 0$, and setting $e_i := \frac{\tilde{e}_i\sqrt{2}}{\sqrt{\theta_0}}$, where $\tilde{e}_i$ is the eigenfunction to the eigenvalue λ_i for all $i \in \mathbb{N}$, we have

$$\langle e_i, e_j\rangle = \frac{2}{\theta_0}\int_0^{\theta_0} \tilde{e}_i \cdot \tilde{e}_j\, d\theta = \delta_{ij} \qquad (i, j \in \mathbb{N}),$$

and

$$\langle e_0,\ e_j\rangle = \frac{\sqrt{2}}{\theta_0}\int_0^{\theta_0} \tilde{e}_0 \cdot \tilde{e}_j\, d\theta = 0 \quad (j \in \mathbb{N}),$$
$$\langle e_0,\ e_0\rangle = \frac{1}{\theta_0}\int_0^{\theta_0} \tilde{e}_0 \cdot \tilde{e}_0\, d\theta = 1.$$

By Fourier series techniques it is straight forward to see that $\mathcal{T}_\theta$ admits an $\mathcal{H}^\infty$-calculus on $L^q(I)$ with $\phi^\infty_{\mathcal{T}_\theta} = 0$; see [4] for more details. Again these facts remain valid for the canonical extension of $\mathcal{T}_\theta$ to $L^p(\mathbb{R}, L^p(I))$, which is defined by

$$T_\theta\varphi := \mathcal{T}_\theta\varphi, \quad D(T_\theta) := L^p(\mathbb{R}, D(\mathcal{T}_\theta)).$$

Optimal regularity for (3.7) is reduced to invertibility of the operator

$$T_p := T_x + T_\theta : D(T_p) \to L^p(\Omega)$$

if we can show that

$$D(T_p) = \left\{\varphi \in W^{2,p}(\Omega):\ \partial_\theta\varphi = 0 \text{ on } \partial\Omega\right\} = D(T_x) \cap D(T_\theta).$$

To this end, for $m \in \mathbb{N}$ let

$$P^c_{m,p}\varphi = \sum_{i=0}^m \langle\varphi, e_i\rangle\, e_i$$

be the projection of $\varphi \in L^p(I)$ onto $\langle e_0, \dots, e_m\rangle$ and put $P_{m,p} := 1 - P^c_{m,p}$. We also set $E^p_m := P_{m,p}\left(L^p(I)\right)$. It is obvious that $\left(P_{m,p}\right)_{1<p<\infty}$ is a consistent family on $(L^p(I))_{1<p<\infty}$, so we omit the index p and write P_m. If $\mathbb{P}_m$ denotes the canonical extension of P_m to $L^p(\mathbb{R}, L^p(I))$, then $\mathbb{P}_m \in \mathscr{L}(L^p(\Omega))$ is a projector onto $L^p(\mathbb{R}, E^p_m)$. Consequently, we have the decomposition

$$L^p(\Omega) = L^p(\mathbb{R}, \langle e_0, \dots, e_m\rangle) \oplus L^p(\mathbb{R}, E^p_m).$$

The proof of the following properties is straight forward.

Lemma 3.2. *Let $1 < p < \infty$. Let $d > \beta_p^2$ with β_p as given in* (3.6)*, $m \in \mathbb{N}$ and T_x, T_θ be given as above. Then we have*

(1) $\mathbb{P}_m\varphi \in D(T_x)$ and $\mathbb{P}_m T_x\varphi = T_x\mathbb{P}_m\varphi$ for $\varphi \in D(T_x)$,
(2) $\mathbb{P}_m\varphi \in D(T_\theta)$ and $\mathbb{P}_m T_\theta\varphi = T_\theta\mathbb{P}_m\varphi$ for $\varphi \in D(T_\theta)$,
(3) $T_x + d, T_\theta \in \mathcal{H}^\infty\left(L^p(\mathbb{R}, E_m^p)\right) \cap \mathcal{H}^\infty\left(L^p(\mathbb{R}, \langle e_0, \dots, e_m\rangle)\right)$ with the corresponding angles $\phi^\infty_{T_x+d} < \frac{\pi}{2}$, $\phi^\infty_{T_\theta} = 0$,
(4) $\mathbb{P}_m$, $(\lambda - T_x)^{-1}$ and $(\mu - T_\theta)^{-1}$ commute pairwise for $\lambda \in \rho(T_x)$ and $\mu \in \rho(T_\theta)$.

The invertibility of $T_p = T_x + T_\theta$ essentially follows by the operator sum method. For instance one can apply [10, Proposition 3.5], which is a consequence of the Kalton-Weis theorem [6, Cororallary 5.4].

Proposition 3.3. *Let $1 < p < \infty$ and β_p be defined as in* (3.6)*. Then*

$$T_p \in \mathscr{L}_{is}\big(D(T_p),\, L^p(\Omega)\big)$$

if and only if

$$\beta_p{}^2 \notin \sigma(T_\theta) = \{(\pi k/\theta_0)^2,\, k \in \mathbb{N}_0\}. \tag{3.9}$$

Proof. Relying on Lemma 3.2, the fact that

$$T_p \in \mathscr{L}_{is}\big(D(T_x) \cap D(T_\theta),\, L^p(\Omega)\big)$$

follows by copying almost verbatim the lines of the proof of [7, Theorem 2.3]. The proof of [7, Lemma 2.5] in addition shows that

$$W^{2,p}(\Omega) = W^{2,p}(\mathbb{R}, L^p(I)) \cap L^p(\mathbb{R}, W^{2,p}(I)).$$

The definition of the Sobolev space then yields that

$$D(T_x) \cap D(T_\theta) = \left\{\varphi \in W^{2,p}(\Omega) : \partial_\theta\varphi = 0 \text{ on } \partial\Omega\right\}.$$

This completes the proof. □

Next, we show higher regularity of the transformed problem (3.7).

Corollary 3.4. *Let $1 < p < \infty$, β_p be defined as in* (3.6) *and condition* (3.9) *be fulfilled. Then for every $g \in W^{1,p}(\Omega)$ the solution φ of* (3.7) *satisfies the estimate*

$$\|\varphi\|_{W^{3,p}(\Omega)} \le C\|g\|_{W^{1,p}(\Omega)}$$

for some constant $C > 0$ that is independent of φ and g.

Proof. Denote by $D_1^h\varphi$ the difference quotient

$$D_1^h\varphi(x,\theta) := \frac{\varphi((x,\theta) + he_1) - \varphi(x,\theta)}{h} \quad (h \in \mathbb{R},\ h \neq 0),$$

where $e_1 := (1, 0)$. Let $\varphi \in D(T_p)$ be the solution of (3.7). Applying D_1^h to (3.7) and using the fact that D_1^h commutes with T_p, we obtain

$$\begin{aligned} &D_1^h T_p\varphi = D_1^h g \text{ in } \mathcal{D}'(\Omega) \\ \Leftrightarrow\ &T_p D_1^h\varphi = D_1^h g \text{ in } \mathcal{D}'(\Omega). \end{aligned} \tag{3.10}$$

Now, let $g \in W^{1,p}(\Omega)$. The above calculation and Proposition 3.3 imply

$$\|D_1^h\varphi - D_1^{h'}\varphi\|_{W^{2,p}(\Omega)} \le C\|(D_1^h - D_1^{h'})g\|_{L^p(\Omega)} \tag{3.11}$$

for a constant $C > 0$.

For the right-hand side of (3.11) it is straight forward to see that

$$\|(D_1^h - D_1^{h'})g\|_{L^p(\Omega)} \underset{h,h'\to 0}{\longrightarrow} 0,$$

which implies that $D_1^h g$ is a Cauchy sequence in $L^p(\Omega)$ converging to $\partial_x g \in L^p(\Omega)$.

It follows by the estimate (3.11) that $D_1^h\varphi$ is a Cauchy sequence in $D(T_p)$ converging to $\partial_x\varphi \in D(T_p)$. The last calculations imply

$$\|\partial_x\varphi\|_{W^{2,p}(\Omega)} \le C\|\partial_x g\|_{L^p(\Omega)} \le C\|g\|_{W^{1,p}(\Omega)}, \qquad g \in W^{1,p}(\Omega)$$

for a constant $C > 0$. This yields that $\partial_x\varphi \in W^{2,p}(\Omega)$, i.e.

$$\varphi, \nabla\varphi, \nabla^2\varphi, \nabla^2\partial_x\varphi \in L^p(\Omega).$$

We still have to prove that $\partial_\theta^3\varphi \in L^p(\Omega)$. This, however, follows by $T_p\varphi = g$. Since

$$T_p\varphi = (\partial_x^2 + 2\beta_p\partial_x + \beta_p^2 + \partial_\theta^2)\varphi$$

we have that

$$\partial_\theta^2\varphi = -\left(\partial_x^2 + 2\beta_p\partial_x + \beta_p^2\right)\varphi + g \in W^{1,p}(\Omega),$$

and, hence, $\partial_\theta^3\varphi \in L^p(\Omega)$. □

Now, we consider the equivalence of problem (3.1) and (3.7). We define $A_L\phi := \Delta\phi$ on the wedge domain as

$$A_L\phi := \Delta\phi, \qquad \phi \in D(A_L) = \left\{\eta \in K^3_{p,\gamma}(G) : \ \partial_\nu\eta = 0 \text{ on } \Gamma\right\}. \tag{3.12}$$

Lemma 3.5. *Let $p \in (1,\infty)$, $\gamma \in \mathbb{R}$ such that $\gamma \neq p-2$ and $\beta_p = 3 - \frac{2+\gamma}{p}$. Let Θ_*^p, Θ_p^* be defined as in* (3.2) *and $\widetilde{\Theta}_*^p, \widetilde{\Theta}_p^*$ be defined as in* (3.5)*. Then we have*

$$\widetilde{\Theta}_p^* \in \mathscr{L}_{is}\left(\widehat{W}_\gamma^{1,p}(G), W^{1,p}(\Omega)\right), \quad \Theta_p^* \in \mathscr{L}_{is}\left(D(A_L), D(T_p)\right)$$

where $\|\cdot\|_{D(A_L)} = \|\cdot\|_{K^3_{p,\gamma}(G)}$ and $\|\cdot\|_{D(T_p)} = \|\cdot\|_{W^{3,p}(\Omega)}$.
In particular, $\phi \in D(A_L)$ is the unique solution of (3.1) *to the right-hand side $f \in \widehat{W}_\gamma^{1,p}(G)$ if and only if $\varphi = \Theta_p^*\phi \in D(T_p)$ is the unique solution of* (3.7) *to the right-hand side $g = \widetilde{\Theta}_p^* f \in W^{1,p}(\Omega)$.*

Proof. The proof of $\Theta_p^* \in \mathscr{L}_{is}\left(K^3_{p,\gamma}(G), W^{3,p}(\Omega)\right)$ follows from Chapter 3, Lemma B.3 (1) of this thesis with $l - k := 3$. In combination with the boundary conditions transformed at the beginning of this section, we obtain

$$\Theta_p^* \in \mathscr{L}\left(D(A_L), D(T_p)\right) \quad \text{and} \quad \Theta_*^p \in \mathscr{L}\left(D(T_p), D(A_L)\right),$$

and since Θ_p^* is the inverse of Θ_*^p the second assertion is proved.

Now, set

$$h^1_{r_1,j}(\partial_x, \partial_\theta) := \cos\theta(r_1 + j + \partial_x) - \sin\theta,$$
$$h^2_{r_1,j}(\partial_x, \partial_\theta) := \sin\theta(r_1 + j + \partial_x) - \cos\theta$$

with $r_1 := \beta_p - 2,\ j \in \mathbb{R}$. Let $g \in W^{1,p}(\Omega)$. Then by Chapter 3, Lemma B.1 (1) of this thesis with $l := 2$ we have

$$\|\widetilde{\Theta}^p_* g\|^p_{\widehat{W}^{1,p}_\gamma(G)} = \sum_{|\alpha|=1} \|D^\alpha(\widetilde{\Theta}^p_* g)\|^p_{L^p_\gamma(G)}$$
$$= \sum_{|\alpha|=1} \int_\Omega \left| e^{(\beta_p-2)x} e^{-|\alpha|x} \prod_{n=1}^{\alpha_1} h^1_{r_1,-|\alpha|+n}(\partial_x,\partial_\theta) \prod_{n=1}^{\alpha_2} h^2_{r_1,-\alpha_2+n}(\partial_x,\partial_\theta) g(x,\theta) \right|^p e^{(2+\gamma)x} dx d\theta$$
$$= \sum_{|\alpha|=1} \int_\Omega \left| e^{(3-\frac{2+\gamma}{p}-2)x} e^{-x} \prod_{n=1}^{\alpha_1} h^1_{r_1,-|\alpha|+n}(\partial_x,\partial_\theta) \prod_{n=1}^{\alpha_2} h^2_{r_1,-\alpha_2+n}(\partial_x,\partial_\theta) g(x,\theta) \right|^p e^{(2+\gamma)x} dx d\theta$$
$$\le C\|g\|^p_{W^{1,p}(\Omega)}$$

for some constant $C > 0$.

Next, we show the converse estimate. Let $f \in \widehat{W}^{1,p}_\gamma(G)$ such that $f(0) = 0$ if $\gamma < p-2$ and $f(\infty) = 0$ if $\gamma > p-2$. Then Hardy's inequality, see Lemma A.2, implies

$$\begin{aligned}\|\widetilde{\Theta}^*_p f\|^p_{L^p(\Omega)} &= \int_\Omega |e^{2x} e^{-\beta_p x} \Psi f(x,\theta)|^p d(x,\theta) \\ &= \int_G |\rho^2 \rho^{-(3-\frac{2+\gamma}{p})} f(x_1,x_2)|^p \rho^{-2} d(x_1,x_2) \\ &= \|\rho^{-1} f\|^p_{L^p_\gamma(G)} \le C\|\nabla f\|^p_{L^p_\gamma(G)},\end{aligned}$$

for some constant $C := C(p,\gamma) > 0$. Moreover, we have

$$\|\widetilde{\Theta}^*_p f\|^p_{\widehat{W}^{1,p}(\Omega)} = \int_\Omega |\nabla e^{(2-\beta_p)x} \Psi f(x,\theta)|^p d(x,\theta)$$
$$= \int_G \left| \rho^{2-\beta_p} \left(\begin{pmatrix} 2-\beta_p \\ 0 \end{pmatrix} f(x_1,x_2) + \rho \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) \nabla f(x_1,x_2) \right|^p \rho^{-2} d(x_1,x_2)$$
$$\le \int_G |\rho^{2-(3-\frac{2+\gamma}{p})} \begin{pmatrix} 2-\beta_p \\ 0 \end{pmatrix} f(x_1,x_2)|^p \rho^{-2} d(x_1,x_2)$$
$$+ \int_G |\rho^{3-(3-\frac{2+\gamma}{p})} \nabla f(x_1,x_2)|^p \rho^{-2} d(x_1,x_2)$$
$$\le C(\|\rho^{-1} f\|^p_{L^p_\gamma(G)} + \|f\|^p_{L^p_\gamma(G)}) \le C\|\nabla f\|^p_{L^p_\gamma(G)},$$

for a constant $C := C(p,\gamma) > 0$. Hence, the first assertion $\widetilde{\Theta}^*_p \in \mathscr{L}_{is}(\widehat{W}^{1,p}_\gamma(G), W^{1,p}(\Omega))$ follows. □

Remark 3.6. (a) For $\beta_p = 3 - \frac{2+\gamma}{p}$ the condition (3.9) is fulfilled, if every eigenvalue λ_i of $T_{p,\theta}$ satisfies

$$\lambda_i \ne \beta_p^2 = \left(3 - \frac{2+\gamma}{p}\right)^2. \tag{3.13}$$

For the case $\gamma = 0$, i.e. for the Kondrat'ev weight $\rho^\gamma \equiv 1$, we then have

$$\lambda_i \ne \beta_p^2 \;\Leftrightarrow\; \left(3 - \frac{2}{p}\right)^2 \ne \left(\frac{i\pi}{\theta_0}\right)^2, \qquad i \in \mathbb{N}_0.$$

This is equivalent to

$$p \ne \frac{2\theta_0}{3\theta_0 - i\pi}, \qquad i \in \mathbb{N}_0.$$

Since $\theta_0 \in (0,\pi)$, the above relation is always fulfilled for $p \in (1,\infty) \setminus \{\frac{2\theta_0}{3\theta_0-\pi}, \frac{2\theta_0}{3\theta_0-2\pi}\}$.

(b) Lemma 3.5 is fulfilled for all $p \in (1,\infty)$ such that $\gamma \neq p-2$ with $\gamma \in \mathbb{R}$. For $\gamma = 0$ this is equivalent to $p \neq 2$.

Proposition 3.3, Corollary 3.4, Lemma 3.5 and the last remark yield the following result.

Corollary 3.7. *Let* $p \in (1,\infty) \setminus \{\frac{2\theta_0}{3\theta_0-\pi},\ \frac{2\theta_0}{3\theta_0-2\pi},\ 2\}$, $\theta_0 \in (0,\pi)$ *and* $\rho = |(x_1,x_2)|$. *Then equation* (3.1) *is for each* $f \in \widehat{W}^{1,p}(G)$ *uniquely solvable with a solution* ϕ *satisfying*

$$\rho^{|\alpha|-3}\partial^\alpha \phi \in L^p(G), \qquad |\alpha| \leq 3.$$

The next corollary generalizes the above result to time dependent data:

Corollary 3.8. *Let* $p \in (1,\infty) \setminus \{\frac{2\theta_0}{3\theta_0-\pi},\ \frac{2\theta_0}{3\theta_0-2\pi},\ 2\}$, $\theta_0 \in (0,\pi)$ *and* $\rho = |(x_1,x_2)|$. *Let* $J = (0,T)$ *with* $0 < T < \infty$. *Then for every* $f \in L^p(J,\widehat{W}^{1,p}(G))$ *the equation*

$$\left.\begin{aligned} \Delta\phi &= f \ \text{ in } \ J\times G,\\ \partial_\nu \phi &= 0 \ \text{ on } \ J\times\Gamma, \end{aligned}\right\} \tag{3.14}$$

has a unique solution ϕ *satisfying*

$$\rho^{|\alpha|-3}\partial^\alpha \phi \in L^p(J, L^p(G)), \qquad |\alpha| \leq 3.$$

Proof. Assume that $f \in C^\infty(\overline{J\times G}) \cap L^p(J,\widehat{W}^{1,p}(G))$. For every $t \in \mathbb{R}$ choose $\phi(t,\,\cdot\,) \in K^3_p(G)$ to be the unique solution to the problem

$$\Delta\phi(t,\cdot) = f(t,\cdot) \quad \text{in } G, \qquad \partial_\nu\phi(t,\cdot) = 0 \quad \text{on } \Gamma,$$

which exists due to the Corollary 3.7. Now, we have

$$\|\phi\|^p_{L^p(J,K^3_p(G))} = \int_0^T \|\phi(t,\cdot)\|^p_{K^3_p(G)} dt \leq C^p \int_0^T \|f(t,\cdot)\|^p_{\widehat{W}^{1,p}(G)} dt = C^p \|f\|^p_{L^p(J,\widehat{W}^{1,p}(G))}$$

for a constant $C > 0$ which is independent of u, f and $t \in \mathbb{R}$. This shows unique solvability of (3.14) for a right-hand side $f \in C^\infty(\overline{J\times G}) \cap L^p(J,\widehat{W}^{1,p}(G))$. Now, since the latter space is dense in $L^p(J,\widehat{W}^{1,p}(G))$, using an approximation argument yields the assertion for every right-hand side $f \in L^p(J,\widehat{W}^{1,p}(G))$. □

4. The Stokes Equations subject to Navier Boundary Conditions

Let $J = (0,T)$ with $0 < T < \infty$ and let $G \subset \mathbb{R}^2$ be the wedge defined as in (1.2) with opening angle $\theta_0 \in (0,\pi)$. The aim of this section is to prove Theorem 1.2, that is the unique solvability of problem (1.1) in the L^p-setting for all $p \in (1,\infty) \setminus \{\frac{2\theta_0}{3\theta_0-\pi},\ \frac{2\theta_0}{3\theta_0-2\pi},\ \frac{3}{2},\ 2,\ 3\}$. We start with a proof of the well-posedness of the Stokes equations subject to inhomogeneous perfect slip boundary conditions.

4.1. **Inhomogeneous Perfect Slip Boundary Conditions.** Let $\mathbb{E}$ and $\mathbb{F}$ be defined as in (1.4) and (1.5), respectively. Here we consider the system

$$\left.\begin{aligned} \partial_t u - \Delta u + \nabla p &= f \ \text{ in } \ J\times G,\\ \operatorname{div} u &= g \ \text{ in } \ J\times G,\\ \operatorname{curl} u &= h_1 \text{ on } \ J\times\Gamma,\\ u\cdot\nu &= h_0 \text{ on } \ J\times\Gamma,\\ u(0) &= u_0 \ \text{ in } \ G, \end{aligned}\right\} \tag{4.1}$$

where the boundary of G is decomposed as in (1.3) as $\partial G = \Gamma \cup \{0\}$ with its smooth part given as $\Gamma = \Gamma_1 \cup \Gamma_2$. Recall that $(\tau,\,\nu) = (\tau_j,\,\nu_j)$ for $j = 1,\,2$ denotes the positively oriented pair of unit tangential and unit outer normal vector on Γ_j as introduced in Section 1. Of course, the boundary conditions in (4.1) have to be understood as

$$\begin{aligned} \operatorname{curl} u &= h_1^{(1)} \quad \text{on } J \times \Gamma_1, \\ \operatorname{curl} u &= h_1^{(2)} \quad \text{on } J \times \Gamma_2, \\ u \cdot \nu_1 &= h_0^{(1)} \quad \text{on } J \times \Gamma_1, \\ u \cdot \nu_2 &= h_0^{(2)} \quad \text{on } J \times \Gamma_2, \end{aligned}$$

where $h_\ell^{(j)} = h_\ell|_{\Gamma_j}$ for $\ell = 0$, 1 and $j = 1$, 2. We aim at solutions

$$(u, p) \in \mathbb{E}$$

and, hence, the given data in (4.1) have to satisfy the regularity conditions

$$(f,\ g,\ h_1,\ h_0,\ u_0) \in \mathbb{F}.$$

In order to treat problem (4.1) we first need the following result concerning traces on the wedge domain G.

Proposition 4.1. *Let $J = (0, T)$ with $0 < T < \infty$ and let $G \subset \mathbb{R}^2$ be the wedge domain defined as in* (1.2) *with opening angle $\theta_0 \in (0, \pi)$. Let $1 < p < \infty$ with $p \neq 2$. Furthermore, let $\Gamma_1 = (-\infty, 0) \cdot \tau_1$ and $\Gamma_2 = (0, \infty) \cdot \tau_2$ with*

$$\tau_1 = -e_1, \quad \nu_1 = -e_2, \quad \tau_2 = (\cos\theta_0, \sin\theta_0)^T, \quad \nu_2 = (-\sin\theta_0, \cos\theta_0)^T$$

such that $\partial G = \Gamma_1 \,\dot\cup\, \Gamma_2 \,\dot\cup\, \{0\}$. Now, suppose that

$$\begin{aligned} g_j &\in W_p^{1-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{2-1/p}(\Gamma_j)), \qquad j = 1, 2, \\ h_j &\in W_p^{1/2-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{1-1/p}(\Gamma_j)), \qquad j = 1, 2, \end{aligned}$$

such that

$$\begin{aligned} \langle g_1 \rangle_1 &= \langle g_2 \rangle_2 && \text{in } J, \\ \langle \partial_{\tau_1} g_1 \rangle_1 + \cos\theta_0 \cdot \langle \partial_{\tau_2} g_2 \rangle_2 &= \sin\theta_0 \cdot \langle h_2 \rangle_2 && \text{in } J, \quad \text{if } p > 2, \\ -\langle \partial_{\tau_2} g_2 \rangle_2 - \cos\theta_0 \cdot \langle \partial_{\tau_1} g_1 \rangle_1 &= \sin\theta_0 \cdot \langle h_1 \rangle_1 && \text{in } J, \quad \text{if } p > 2. \end{aligned}$$

Then there exists a function $u \in W^{1,p}(J, L^p(G)) \cap L^p(J, W^{2,p}(G))$ that satisfies

$$\begin{aligned} u &= g_1 \quad \text{and} \quad \partial_{\nu_1} u = h_1 \quad \text{on } J \times \Gamma_1, \\ u &= g_2 \quad \text{and} \quad \partial_{\nu_2} u = h_2 \quad \text{on } J \times \Gamma_2. \end{aligned}$$

Proof. Step 1. We first show that we can w. l. o. g. assume that $\langle g_j \rangle_j = 0$ as well as $\langle \partial_{\tau_j} g_j \rangle_j = \langle h_j \rangle_j = 0$, if $p > 2$, for $j = 1$, 2. Indeed, there exist extensions

$$\begin{aligned} \hat g_1 &\in W_p^{1-1/2p}(J, L^p(\Sigma_1)) \cap L^p(J, W_p^{2-1/p}(\Sigma_1)), \\ \hat h_1 &\in W_p^{1/2-1/2p}(J, L^p(\Sigma_1)) \cap L^p(J, W_p^{1-1/p}(\Sigma_1)) \end{aligned}$$

of g_1 and h_1, respectively, to the hyperplane $\Sigma_1 := \mathbb{R} \cdot \tau_1$; cf. [1, Thm. 4.26]. Now, the trace theory for anisotropic function spaces on the halfspace implies that there exists

$$v \in W^{1,p}(J, L^p(\mathbb{R} \times (0, \infty))) \cap L^p(J, W^{2,p}(\mathbb{R} \times (0, \infty)))$$

such that $v = \hat g_1$ and $\partial_{\nu_1} v = \hat h_1$ on $J \times \Sigma_1$. Then we set $u = v + \hat u$ and infer that $\hat u \in W^{1,p}(J, L^p(G)) \cap L^p(J, W^{2,p}(G))$ has to satisfy the boundary conditions

$$\begin{aligned} \hat u &= 0 \quad \text{and} \quad \partial_{\nu_1} \hat u = 0 \quad \text{on } J \times \Gamma_1, \\ \hat u &= \hat g_2 \quad \text{and} \quad \partial_{\nu_2} \hat u = \hat h_2 \quad \text{on } J \times \Gamma_2 \end{aligned}$$

for $\hat{g}_2 := g_2 - v|_{\Gamma_2}$ and $\hat{h}_2 := h_2 - \partial_{\nu_2} v$. Due to the choice of v and the compatibility conditions for the boundary data we have $\langle \hat{g}_2 \rangle_2 = \langle g_2 \rangle_2 - \langle g_1 \rangle_1 = 0$ and

$$\begin{aligned}\langle \partial_{\tau_2} \hat{g}_2 \rangle_2 &= \langle \partial_{\tau_2} g_2 \rangle_2 - \langle \partial_{\tau_2} v \rangle_2 \\ &= -\cos\theta_0 \cdot \langle \partial_{\tau_1} g_1 \rangle_1 - \sin\theta_0 \cdot \langle h_1 \rangle_1 - \langle \partial_{\tau_2} v \rangle_2 \\ &= \cos\theta_0 \cdot \langle \partial_{x_1} v \rangle_\bullet + \sin\theta_0 \cdot \langle \partial_{x_2} v \rangle_\bullet - \langle \partial_{\tau_2} v \rangle_2 = 0, \qquad \text{if } p > 2,\end{aligned}$$

as well as

$$\begin{aligned}\langle \hat{h}_2 \rangle_2 &= \langle h_2 \rangle_2 - \langle \partial_{\nu_2} v \rangle_2 \\ &= \tfrac{1}{\sin\theta_0} \big(\langle \partial_{\tau_1} g_1 \rangle_1 + \cos\theta_0 \cdot \langle \partial_{\tau_2} g_2 \rangle_2 \big) - \langle \partial_{\nu_2} v \rangle_\bullet \\ &= \tfrac{1}{\sin\theta_0} \big(\langle \partial_{\tau_1} g_1 \rangle_1 - \cos^2\theta_0 \cdot \langle \partial_{\tau_1} g_1 \rangle_1 - \sin\theta_0 \cdot \cos\theta_2 \cdot \langle h_1 \rangle_1 \big) - \langle \partial_{\nu_2} v \rangle_2 \\ &= \sin\theta_0 \cdot \langle \partial_{\tau_1} g_1 \rangle_1 - \cos\theta_0 \cdot \langle h_1 \rangle_1 - \langle \partial_{\nu_2} v \rangle_2 \\ &= -\sin\theta_0 \cdot \langle \partial_{x_1} v \rangle_\bullet + \cos\theta_0 \cdot \langle \partial_{x_2} v \rangle_\bullet - \langle \partial_{\nu_2} v \rangle_2 = 0, \qquad \text{if } p > 2.\end{aligned}$$

Hence, $\langle \partial_{\tau_2} \hat{g}_2 \rangle_2 = \langle \hat{h}_2 \rangle_2 = 0$, if $p > 2$.

Step 2. Now, assume that $\langle g_j \rangle_j = 0$ as well as $\langle \partial_{\tau_j} g_j \rangle_j = \langle h_j \rangle_j = 0$, if $p > 2$, for $j = 1, 2$. Let $\widetilde{G} := (0,\infty)^2$ be the wedge domain with opening angle $\frac{\pi}{2}$. Here we set $\widetilde{\Gamma}_1 := \Gamma_1$ and $\widetilde{\Gamma}_2 := \{ 0 \} \times (0,\infty)$ to obtain the decomposition $\partial\widetilde{G} = \widetilde{\Gamma}_1 \,\dot\cup\, \widetilde{\Gamma}_2 \,\dot\cup\, \{ 0 \}$ of the boundary of $\widetilde{G}$. We abbreviate $\rho := |x| = |(x_1, x_2)|$ for $x \in \mathbb{R}^2$ and define a transformation

$$\Phi : G \to \widetilde{G}, \qquad \Phi(x_1, x_2) = \left(\rho \cos\left(\frac{\pi}{2\theta_0} \arccos\left(\frac{x_1}{\rho} \right) \right),\ \rho \sin\left(\frac{\pi}{2\theta_0} \arccos\left(\frac{x_1}{\rho} \right) \right) \right).$$

It is not difficult to see that $\Phi : G \to \widetilde{G}$ is a C^∞-diffeomorphism. We set $\widetilde{g}_1 := g_1$, $\widetilde{h}_1 := h_1$ as well as

$$\widetilde{g}_2(t, se_2) := g_2(t, s\tau_2), \qquad \widetilde{h}_2(t, se_2) := h_2(t, s\tau_2), \qquad t \in J,\ s > 0.$$

Then we have

$$\begin{aligned}\widetilde{g}_j &\in W_p^{1-1/2p}(J, L^p(\widetilde{\Gamma}_j)) \cap L^p(J, W_p^{2-1/p}(\widetilde{\Gamma}_j)), \qquad j = 1, 2, \\ \widetilde{h}_j &\in W_p^{1/2-1/2p}(J, L^p(\widetilde{\Gamma}_j)) \cap L^p(J, W_p^{1-1/p}(\widetilde{\Gamma}_j)), \qquad j = 1, 2,\end{aligned}$$

and $\lim_{s\to 0} \widetilde{g}_j(t, se_j) = 0$ as well as $\lim_{s\to 0} \partial_{x_j} \widetilde{g}_j(t, se_j) = \lim_{s\to 0} \widetilde{h}_j(t, se_j) = 0$, if $p > 2$, for $t \in J$ and $j = 1, 2$. Now, we apply [2, Theorem VIII.1.8.5], which shows that there exists $\widetilde{u} \in W^{1,p}(J, L^p(\widetilde{G})) \cap L^p(J, W^{2,p}(\widetilde{G}))$ satisfying

$$\begin{aligned}\widetilde{u} &= \widetilde{g}_1 \quad \text{and} \quad \partial_{\nu_1} \widetilde{u} = \widetilde{h}_1 \quad \text{on } J \times \widetilde{\Gamma}_1, \\ \widetilde{u} &= \widetilde{g}_2 \quad \text{and} \quad \partial_{\nu_2} \widetilde{u} = \widetilde{h}_2 \quad \text{on } J \times \widetilde{\Gamma}_2.\end{aligned}$$

Finally, we set $u = \widetilde{u} \circ \Phi \in W^{1,p}(J, L^p(G)) \cap L^p(J, W^{2,p}(G))$. By construction, u satisfies all desired boundary conditions. Note that we indeed have $u \in L^p(J, W^{2,p}(G))$, which can be seen as follows: We have $\partial_j \Phi \sim \rho^0$ as $\rho \to 0$ and $\rho \to \infty$ for $j = 1, 2$ for the first derivatives of Φ and $\partial_j \partial_k \Phi \sim \rho^{-1}$ as $\rho \to 0$ and $\rho \to \infty$ for $j, k = 1, 2$ for the second derivatives of Φ, i. e. $\partial_j \Phi_n,\ \rho \partial_j \partial_k \Phi_n \in L^\infty(G)$ for $j, k, n = 1, 2$. Moreover, $\det \nabla\Phi \equiv \frac{\pi}{2\theta_0}$. However, the chain rule shows that

$$\partial_j \partial_k (\widetilde{u} \circ \Phi) = \sum_{m,n=1}^{2} \big((\partial_m \partial_n \widetilde{u}) \circ \Phi \big) \partial_j \Phi_m \partial_k \Phi_n + \sum_{n=1}^{2} \big((\partial_n \widetilde{u}) \circ \Phi \big) \partial_j \partial_k \Phi_n, \quad j, k = 1, 2$$

and we have $\rho^{-1}\partial_j\widetilde{u} \in L^p(J, L^p(\widetilde{G}))$ for $j = 1, 2$ due to Hardy's inequality; cf. Lemma A.2. Note that by construction we have $\partial_j\widetilde{u}(\,\cdot\,, 0) = 0$ in J for $j = 1, 2$, if $p > 2$, since $\lim_{s\to 0} \widetilde{h}_j(t, se_j) = 0$, for $t \in J$ and $j = 1, 2$, if $p > 2$. □

Remark 4.2. For $\theta_0 = \frac{\pi}{2}$ we have $\cos\theta_0 = 0$ and $\sin\theta_0 = 1$ as well as $\tau_1 = -e_1$, $\nu_1 = -e_2$, $\tau_2 = e_2$ and $\nu_2 = -e_1$. In this case the compatibility conditions in Proposition 4.1 read

$$\begin{aligned} \langle g_1\rangle_1 &= \langle g_2\rangle_2 && \text{in } J, \\ -\langle \partial_{x_1} g_1\rangle_1 &= \langle h_2\rangle_2 && \text{in } J, \quad \text{if } p > 2, \\ -\langle \partial_{x_2} g_2\rangle_2 &= \langle h_1\rangle_1 && \text{in } J, \quad \text{if } p > 2. \end{aligned}$$

These are precisely the compatibility conditions [2, (VIII.1.8.7) & (VIII.1.8.8)]. This is not surprising, since for $\theta_0 = \frac{\pi}{2}$ Proposition 4.1 is a special case of [2, Thm. VIII.1.8.5].

Corollary 4.3. *Let $J = (0, T)$ with $0 < T < \infty$ and let $G \subset \mathbb{R}^2$ be the wedge domain defined as in (1.2) with opening angle $\theta_0 \in (0, \pi)$. Let $1 < p < \infty$ with $p \neq 2$. Furthermore, let $\Gamma_1 = (-\infty, 0)\cdot\tau_1$ and $\Gamma_2 = (0, \infty)\cdot\tau_2$ with*

$$\tau_1 = -e_1, \quad \nu_1 = -e_2, \quad \tau_2 = (\cos\theta_0, \sin\theta_0)^T, \quad \nu_2 = (-\sin\theta_0, \cos\theta_0)^T$$

such that $\partial G = \Gamma_1 \,\dot\cup\, \Gamma_2 \,\dot\cup\, \{0\}$ and set $\Gamma = \Gamma_1 \cup \Gamma_2$. Now, suppose that

$$\begin{aligned} h_0^{(j)} &\in W_p^{1-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{2-1/p}(\Gamma_j)), && j = 1, 2, \\ h_1^{(j)} &\in W_p^{1/2-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{1-1/p}(\Gamma_j)), && j = 1, 2, \end{aligned}$$

such that $\langle\!\langle h_1\rangle\!\rangle_\bullet = 0$ in J, if $p > 2$, and

$$\langle \partial_{\tau_1} h_0\rangle_1 + \langle \partial_{\tau_2} h_0\rangle_2 = \langle h_1\rangle_\bullet \quad \text{in } J, \qquad \text{if } \theta_0 = \tfrac{\pi}{2} \text{ and } p > 2.$$

Then there exists a function $u \in W^{1,p}(J, L^p(G, \mathbb{R}^2)) \cap L^p(J, W^{2,p}(G, \mathbb{R}^2))$ that satisfies

$$u \cdot \nu = h_0 \quad \text{and} \quad \operatorname{curl} u = h_1 \quad \text{on } J \times \Gamma. \tag{4.2}$$

Proof. First note that for $v \in W^{1,p}(J, L^p(G, \mathbb{R}^2)) \cap L^p(J, W^{2,p}(G, \mathbb{R}^2))$ we have

$$\operatorname{curl} v = \partial_1 v_2 - \partial_2 v_1 = \partial_\tau(v\cdot\nu) - \partial_\nu(v\cdot\tau) \quad \text{on } J \times \Gamma.$$

Hence, if $v \cdot \nu = h_0$ and $\operatorname{curl} v = h_1$ on $J \times \Gamma$, then $\partial_\nu(v\cdot\tau) = \partial_\tau h_0 - h_1$ on $J \times \Gamma$.

Now, we choose $g_j \in W_p^{1-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{2-1/p}(\Gamma_j))$ for $j = 1, 2$ such that

$$\begin{aligned} \cos\theta_0 \cdot \partial_{\tau_1} g_1 &= -(1 - \cos\theta_0)\cdot\partial_{\tau_1} h_0^{(1)} + \tfrac{1}{2}\sin^2\theta_0 \cdot h_1^{(1)} && \text{at } J \times \{0\}, \\ \cos\theta_0 \cdot \partial_{\tau_2} g_2 &= (1 - \cos\theta_0)\cdot\partial_{\tau_2} h_0^{(2)} - \tfrac{1}{2}\sin^2\theta_0 \cdot h_1^{(2)} && \text{at } J \times \{0\} \end{aligned}$$

and $g_j(\,\cdot\,, 0) = 0$ in J for $j = 1, 2$, if $\theta_0 \neq \frac{\pi}{2}$ and $p > 2$, and $g_j := 0$ for $j = 1, 2$, if $\theta_0 = \frac{\pi}{2}$.

Next, we define $\widetilde{h}_0^{(j)} \in W_p^{1-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{2-1/p}(\Gamma_j))$ for $j = 1, 2$ as

$$\begin{aligned} \sin\theta_0 \cdot \widetilde{h}_0^{(1)}(t, -s\tau_1) &:= h_0^{(2)}(t, s\tau_2) + \cos\theta_0 \cdot h_0^{(1)}(t, -s\tau_1) + g_1(t, -s\tau_1), && t \in J,\ s > 0, \\ \sin\theta_0 \cdot \widetilde{h}_0^{(2)}(t, s\tau_2) &:= -h_0^{(1)}(t, -s\tau_1) - \cos\theta_0 \cdot h_0^{(2)}(t, s\tau_2) + g_2(t, s\tau_2), && t \in J,\ s > 0, \end{aligned}$$

and $H_0 \in W_p^{1-1/2p}(J, L^p(\Gamma, \mathbb{R}^2)) \cap L^p(J, W_p^{2-1/p}(\Gamma, \mathbb{R}^2))$ as $H_0 := \widetilde{h}_0 \cdot \tau + h_0 \cdot \nu$. By construction we then have $H_0 \cdot \nu = h_0$ on $J \times \Gamma$.

Finally, we define $\widetilde{h}_1^{(j)} \in W_p^{1/2-1/2p}(J, L^p(\Gamma_j)) \cap L^p(J, W_p^{1-1/p}(\Gamma_j))$ for $j = 1, 2$ as

$$\begin{aligned} \sin\theta_0 \cdot \widetilde{h}_1^{(1)}(t, -s\tau_1) &:= (\partial_{\tau_2} g_2)(t, s\tau_2) + (1 - \cos\theta_0)\cdot(\partial_{\tau_1} h_0^{(1)})(t, -s\tau_1), && t \in J,\ s > 0, \\ \sin\theta_0 \cdot \widetilde{h}_1^{(2)}(t, s\tau_2) &:= (\partial_{\tau_1} g_1)(t, -s\tau_1) - (1 - \cos\theta_0)\cdot(\partial_{\tau_2} h_0^{(2)})(t, s\tau_2), && t \in J,\ s > 0, \end{aligned}$$

and $H_1 \in W_p^{1/2-1/2p}(J, L^p(\Gamma, \mathbb{R}^2)) \cap L^p(J, W_p^{1-1/p}(\Gamma, \mathbb{R}^2))$ as $H_1 := (\partial_\tau h_0 - h_1) \cdot \tau + \widetilde{h}_1 \cdot \nu$. By construction we then have $H_1 \cdot \tau = \partial_\tau h_0 - h_1$ on $J \times \Gamma$.

Now, it is readily checked that

$$\begin{aligned} \langle H_0 \rangle_1 &= \langle H_0 \rangle_2 && \text{in } J, \\ \langle \partial_{\tau_1} H_0 \rangle_1 + \cos\theta_0 \cdot \langle \partial_{\tau_2} H_0 \rangle_2 &= \sin\theta_0 \cdot \langle H_1 \rangle_2 && \text{in } J, \quad \text{if } p > 2, \\ -\langle \partial_{\tau_2} H_0 \rangle_2 - \cos\theta_0 \cdot \langle \partial_{\tau_1} H_0 \rangle_1 &= \sin\theta_0 \cdot \langle H_1 \rangle_1 && \text{in } J, \quad \text{if } p > 2, \end{aligned}$$

Therefore, due to Proposition 4.1 there exists $u \in W^{1,p}(J, L^p(G, \mathbb{R}^2)) \cap L^p(J, W^{2,p}(G, \mathbb{R}^2))$ that satisfies

$$u = H_0 \quad \text{and} \quad \partial_\nu u = H_1 \quad \text{on } J \times \Gamma.$$

By construction this function satisfies the desired boundary conditions. □

Remark 4.4. For $\theta_0 = \frac{\pi}{2}$ we have $\cos\theta_0 = 0$ and $\sin\theta_0 = 1$ as well as $\tau_1 = -e_1$, $\nu_1 = -e_2$, $\tau_2 = e_2$ and $\nu_2 = -e_1$. In this case the compatibility conditions in Corollary 4.3 read

$$\begin{aligned} \langle\!\langle h_1 \rangle\!\rangle_\bullet &= 0 && \text{in } J, \quad \text{if } p > 2, \\ -\langle \partial_{x_1} h_0 \rangle_1 + \langle \partial_{x_2} h_0 \rangle_2 &= \langle h_1 \rangle_\bullet && \text{in } J, \quad \text{if } p > 2, \end{aligned}$$

which explains the additional compatibility condition between h_0 and h_1 that is necessary in this case:

$$-\langle \partial_{x_1} h_0 \rangle_1 + \langle \partial_{x_2} h_0 \rangle_2 = \langle \partial_{x_1} u_2 \rangle_1 - \langle \partial_{x_2} u_1 \rangle_2 = \langle \operatorname{curl} u \rangle_\bullet = \langle h_1 \rangle_\bullet, \quad \text{if } p > 2,$$

for every $u \in W^{1,p}(J, L^p(G, \mathbb{R}^2)) \cap L^p(J, W^{2,p}(G, \mathbb{R}^2))$ that satisfies $u \cdot \nu = h_0$ as well as $\operatorname{curl} u = h_1$ on $J \times \Gamma$.

The next auxiliary result is important, since it allows for the inhomogeneous divergence constraint in problem (4.1).

Proposition 4.5. *Let $J = (0, T)$ with $0 < T < \infty$ and let $G \subset \mathbb{R}^2$ be the wedge domain defined as in* (1.2) *with opening angle $\theta_0 \in (0, \pi)$ and let $\Gamma = \partial G \setminus \{0\}$. Assume that $p \in (1, \infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi}, \frac{2\theta_0}{3\theta_0 - 2\pi}, 2\}$. Then for each*

$$g \in W^{1,p}(J, \widehat{W}^{-1,p}(G)) \cap L^p(J, \widehat{W}^{1,p}(G))$$

there exists a function $u \in W^{1,p}(J, L^p(G, \mathbb{R}^2)) \cap L^p(J, W^{2,p}(G, \mathbb{R}^2))$ such that

$$\left.\begin{aligned} \operatorname{div} u = g &\quad \text{in } J \times G, \\ \text{curl } u = 0, \ u \cdot \nu = 0 &\quad \text{on } J \times \Gamma. \end{aligned}\right\} \tag{4.3}$$

Proof. Let $\phi \in L^p(J, K_p^3(G))$ be the unique solution of the problem

$$\begin{aligned} \Delta\phi = g &\text{ in } J \times G, \\ \partial_\nu \phi = 0 &\text{ on } J \times \Gamma, \end{aligned}$$

which exists according to Corollary 3.8, since $g \in L^p(J, \widehat{W}^{1,p}(G))$.

By the fact that we also have $g \in W^{1,p}(J, \widehat{W}^{-1,p}(G))$ it follows that ϕ is also a weak solution to the above problem, i. e. $\nabla\phi \in W^{1,p}(J, L^p(G, \mathbb{R}^2))$. Note that we have $\partial^\alpha \phi \in L^p(J, L^p(G))$ for $|\alpha| = 3$, since $\phi \in L^p(J, K_p^3(G))$. Now, let $u := \nabla\phi$. We have then $u, \partial^\alpha u \in L^p(J, L^p(G, \mathbb{R}^2))$ for $|\alpha| = 2$. Interpolation (e. g. using the Gagliardo-Nirenberg inequality) yields that $\partial^\alpha u \in L^p(J, L^p(G, \mathbb{R}^2))$ also for $|\alpha| = 1$. Summarizing we have $u \in W^{1,p}(J, L^p(G, \mathbb{R}^2)) \cap L^p(J, W^{2,p}(G, \mathbb{R}^2))$. Moreover, $\operatorname{div} u = \Delta\phi = g$ in $J \times G$ and $\operatorname{curl} u = \operatorname{curl} \nabla\phi = 0$ on $J \times \Gamma$. Finally, $u \cdot \nu = \partial_\nu \phi = 0$ on $J \times \Gamma$. □

Now, we are in position to prove the main result of this subsection.

Theorem 4.6. *Let* $J = (0, T)$ *with* $0 < T < \infty$ *and let* $G \subset \mathbb{R}^2$ *be the wedge domain defined as in* (1.2) *with opening angle* $\theta_0 \in (0, \pi)$ *and let* $\Gamma = \partial G \setminus \{0\}$*. Assume that* $p \in (1, \infty) \setminus \{\frac{2\theta_0}{3\theta_0 - \pi}, \frac{2\theta_0}{3\theta_0 - 2\pi}, \frac{3}{2}, 2, 3\}$*. Suppose the data satisfy the regularity condition*

$$(f, g, h_1, h_0, u_0) \in \mathbb{F}$$

and the compatibility conditions

$$\begin{aligned} \operatorname{div} u_0 &= g|_{t=0}, && \text{if } p > 2, \\ u_0 \cdot \nu &= h_0|_{t=0}, && \text{if } p > \tfrac{3}{2}, \\ \operatorname{curl} u_0 &= h_1|_{t=0}, && \text{if } p > 3, \end{aligned}$$

as well as

$$F(g, h_0) \in W^{1,p}(J, \widehat{W}^{-1,p}(G))$$

and $\langle\!\langle h_1 \rangle\!\rangle_\bullet = 0$ *in* J*, if* $p > 2$*, and*

$$\langle \partial_{\tau_1} h_0 \rangle_1 + \langle \partial_{\tau_2} h_0 \rangle_2 = \langle h_1 \rangle_\bullet \quad \text{in } J, \qquad \text{if } \theta_0 = \tfrac{\pi}{2} \text{ and } p > 2.$$

Then there exists a unique solution $(u, p) \in \mathbb{E}$ *to problem* (4.1)*.*

Proof. The uniqueness of the solution $(u, p) \in \mathbb{E}$ follows directly from [7, Corollary 1].

To show the existence of the solution to (4.1) we proceed in three steps: First, we employ Corollary 4.3 and choose $u_1 \in \mathbb{E}_u$ such that

$$\begin{aligned} \operatorname{curl} u_1 &= h_1 && \text{on } J \times \Gamma, \\ u_1 \cdot \nu &= h_0 && \text{on } J \times \Gamma. \end{aligned}$$

Next, we employ Proposition 4.5 and choose $u_2 \in \mathbb{E}_u$ such that

$$\begin{aligned} \operatorname{div} u_2 = g - \operatorname{div} u_1 \quad && \text{in } J \times G, \\ \operatorname{curl} u_2 = 0, \ u_2 \cdot \nu = 0 \quad && \text{on } J \times \Gamma. \end{aligned}$$

Note that the compatibility conditions and the fact that $u_1 \cdot \nu = h_0$ on $J \times \Gamma$ ensure that $g - \operatorname{div} u_1 \in W^{1,p}(J, \widehat{W}^{-1,p}(G)) \cap L^p(J, W^{1,p}(G))$; cf. Remark 1.1. Finally, we employ [7, Corollary 1] and choose $(u_3, p_3) \in \mathbb{E}$ such that

$$\begin{aligned} \partial_t u_3 - \Delta u_3 + \nabla p = f - \partial_t u_1 + \Delta u_1 - \partial_t u_2 + \Delta u_2 \quad && \text{in } J \times G, \\ \operatorname{div} u_3 = 0 \quad && \text{in } J \times G, \\ \operatorname{curl} u_3 = 0, \ u_3 \cdot \nu = 0 \quad && \text{on } J \times \Gamma, \\ u_3(0) = u_0 - u_1(0) - u_2(0) \quad && \text{in } G. \end{aligned}$$

By construction $(u, p) := (u_1 + u_2 + u_3, p) \in \mathbb{E}$ is a solution to (4.1). □

4.2. **Inhomogeneous Free and Perfect Slip Boundary Conditions.** Let $\mathbb{E}$ and $\mathbb{F}$ be defined as in (1.4) and (1.5), respectively. Here we consider the system (1.7) and show that it is uniquely solvable within the maximal regularity class $\mathbb{E}$. Recall that the boundary of G is decomposed as in (1.3) as $\partial G = \Gamma \cup \{0\}$ with its smooth part given as $\Gamma = \Gamma_1 \cup \Gamma_2$. Also recall that $(\tau, \nu) = (\tau_j, \nu_j)$ for $j = 1, 2$ denotes the positively oriented pair of unit tangential and unit outer normal vector on Γ_j as introduced in Section 1.

For the boundary conditions in problem (1.7) we observe that

$$\begin{aligned} \tau^T D_\pm(u)\nu &= \frac{1}{2} \begin{pmatrix} \partial_{x_1} u_1 \pm \partial_{x_1} u_1 & \partial_{x_1} u_2 \pm \partial_{x_2} u_1 \\ \partial_{x_2} u_1 \pm \partial_{x_1} u_2 & \partial_{x_2} u_2 \pm \partial_{x_2} u_2 \end{pmatrix} \nu \cdot \tau \\ &= \frac{1}{2} \begin{pmatrix} \partial_{x_1}(u \cdot \nu) \\ \partial_{x_2}(u \cdot \nu) \end{pmatrix} \cdot \tau \pm \frac{1}{2} \begin{pmatrix} \partial_\nu u_1 \\ \partial_\nu u_2 \end{pmatrix} \cdot \tau \end{aligned}$$

$$= \tfrac{1}{2}\partial_\tau(u\cdot\nu) \pm \tfrac{1}{2}\partial_\nu(u\cdot\tau) \qquad \text{on } J\times\Gamma,$$

which implies that

$$\begin{aligned}\tau^T D_+(u)\nu &= \tfrac{1}{2}\partial_\tau(u\cdot\nu) + \tfrac{1}{2}\partial_\nu(u\cdot\tau)\\ &= \partial_\tau(u\cdot\nu) - \tfrac{1}{2}\partial_\tau(u\cdot\nu) + \tfrac{1}{2}\partial_\nu(u\cdot\tau) = \partial_\tau(u\cdot\nu) - \tfrac{1}{2}\operatorname{curl} u \qquad \text{on } J\times\Gamma\end{aligned}$$

as well as

$$\tau^T D_-(u)\nu = \tfrac{1}{2}\partial_\tau(u\cdot\nu) - \tfrac{1}{2}\partial_\nu(u\cdot\tau) = \tfrac{1}{2}\operatorname{curl} u \qquad \text{on } J\times\Gamma.$$

Therefore, if the tangential boundary condition in (1.7) is posed based on D_+, then (1.7) is equivalent to

$$\begin{aligned}\partial_t u - \Delta u + \nabla p &= f && \text{in } J\times G,\\ \operatorname{div} u &= g && \text{in } J\times G,\\ u\cdot\nu &= h_0 && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } G\end{aligned} \tag{4.4}$$

together with the boundary condition

$$\operatorname{curl} u = 2(\partial_\tau h_0 + h_1) \quad \text{on } J\times\Gamma. \tag{4.5}$$

Analogously, if the tangential boundary condition in problem (1.7) is posed based on D_-, then (1.7) is equivalent to (4.4) together with the boundary condition

$$\operatorname{curl} u = -2h_1 \quad \text{on } J\times\Gamma. \tag{4.6}$$

Both systems (4.4, 4.5) and (4.4, 4.6) are uniquely solvable using Theorem 4.6 and, hence, we obtain the following result.

Corollary 4.7. *Let $J = (0,T)$ with $0 < T < \infty$ and let $G \subset \mathbb{R}^2$ be the wedge domain defined as in* (1.2) *with opening angle $\theta_0 \in (0,\pi)$ and let $\Gamma = \partial G \setminus \{0\}$. Assume that $p \in (1,\infty) \setminus \{\frac{2\theta_0}{3\theta_0-\pi},\ \frac{2\theta_0}{3\theta_0-2\pi},\ \frac{3}{2},\ 2,\ 3\}$. Suppose the data satisfy the regularity condition*

$$(f,\ g,\ h_1,\ h_0,\ u_0) \in \mathbb{F}$$

and the compatibility conditions

$$\begin{aligned}\operatorname{div} u_0 &= g|_{t=0}, && \text{if } p > 2,\\ u_0\cdot\nu &= h_0|_{t=0}, && \text{if } p > \tfrac{3}{2},\\ -\tau^T D_\pm(u_0)\nu &= h_1|_{t=0}, && \text{if } p > 3,\end{aligned}$$

as well as

$$F(g,h_0) \in W^{1,p}(J, \widehat{W}^{-1,p}(G)).$$

If the boundary condition is posed based on D_+, then assume the compatibility conditions $\langle\!\langle \partial_\tau h_0 + h_1 \rangle\!\rangle_\bullet = 0$ in J, if $p > 2$, and

$$\tfrac{1}{2}\langle \partial_{\tau_1} h_0\rangle_1 + \tfrac{1}{2}\langle \partial_{\tau_2} h_0\rangle_2 = \langle \partial_\tau h_0 + h_1\rangle_\bullet \quad \text{in } J, \qquad \text{if } \theta_0 = \tfrac{\pi}{2} \text{ and } p > 2.$$

If the boundary condition is posed based on D_-, then assume the compatibility conditions $\langle\!\langle h_1 \rangle\!\rangle_\bullet = 0$ in J, if $p > 2$, and

$$-\tfrac{1}{2}\langle \partial_{\tau_1} h_0\rangle_1 - \tfrac{1}{2}\langle \partial_{\tau_2} h_0\rangle_2 = \langle h_1\rangle_\bullet \quad \text{in } J, \qquad \text{if } \theta_0 = \tfrac{\pi}{2} \text{ and } p > 2.$$

Then there exists a unique solution $(u,p) \in \mathbb{E}$ to problem (1.7).

4.3. **Proof of Theorem 1.2.** A unique solution to (1.1) can now be obtained with the aid of the usual perturbation argument. To this end, we denote by $L : \mathbb{E} \longrightarrow \mathbb{F}$ the linear operator induced by the left-hand side of problem (1.7). Now, if $(f, g, h_1, h_0, u_0) \in \mathbb{F}$ satisfy all compatibility conditions stated in Theorem 1.2, then (1.1) is equivalent to

$$L(u, p) = (f, g, h_1 - \alpha u \cdot \tau, h_0, u_0).$$

Now, we choose $\widetilde{h}_1 \in \mathbb{F}_\tau$ such that $\langle \widetilde{h}_1 \rangle_\bullet = 0$, if $p > 2$ and $\widetilde{h}_1|_{t=0} = \alpha u_0|_\Gamma \cdot \tau$. This is possible, since $\langle \alpha u_0 \cdot \tau \rangle_\bullet = 0$, if $p > 2$, due to the requirement $\langle \alpha \rangle_\bullet = 0$. By construction, the data $(f, g, h_1 - \widetilde{h}_1, h_0, u_0) \in \mathbb{F}$ satisfy all compatibility conditions stated in Corollary 4.7.

Hence, Corollary 4.7 shows that there exists a unique solution (u_*, p_*) to the problem $L(u_*, p_*) = (f, g, h_1 - \widetilde{h}_1, h_0, u_0)$. Thus, the ansatz $(u, p) = (u_*, p_*) + (v, q)$ leads to the problem

$$\mathbb{L}(v, q) = \widetilde{h}_1 - \alpha u_*|_\Gamma \cdot \tau - \alpha v|_\Gamma \cdot \tau, \qquad (v, q) \in {}_0\mathbb{E},$$

where the linear operator $\mathbb{L} : {}_0\mathbb{E} \longrightarrow {}_0\mathbb{F}_\tau$ between the spaces

$${}_0\mathbb{E} := \left\{ (w, r) \in \mathbb{E} \; : \; \begin{array}{c} \partial_t w - \Delta w + \nabla r = 0 \text{ in } J \times G, \; \operatorname{div} w = 0 \text{ in } J \times G \\ w \cdot \nu = 0 \text{ on } J \times \Gamma, \; w|_{t=0} = 0 \text{ in } G \end{array} \right\}$$

and

$${}_0\mathbb{F}_\tau := \Big\{ h \in \mathbb{F}_\tau \; : \; \langle h \rangle_\bullet = 0, \text{ if } p > 2, \; h|_{t=0} = 0, \text{ if } p > 3 \Big\}$$

is given as $\mathbb{L}(w, r) := (\operatorname{curl} w)|_\Gamma$ for $(w, r) \in {}_0\mathbb{E}$.

Thanks to the homogeneous initial conditions the operator $\mathbb{L}$ is a linear isomorphism by Corollary 4.7, where the operator norm of $\mathbb{L}^{-1}$ does not depend on the length $T > 0$ of the time interval $J = (0, T)$ under consideration. Moreover, we have

$$\langle \widetilde{h}_1 - \alpha u_* \cdot \tau \rangle_\bullet = \langle \alpha v \cdot \tau \rangle_\bullet = 0, \quad \text{if } p > 2,$$

since $\langle \alpha \rangle_\bullet = \langle \widetilde{h}_1 \rangle_\bullet = 0$, as well as

$$(\widetilde{h}_1 - \alpha u_*|_\Gamma \cdot \tau)|_{t=0} = \widetilde{h}_1|_{t=0} - \alpha u_0|_\Gamma \cdot \tau = 0, \quad \text{if } p > 3,$$

which shows that $\widetilde{h}_1 - \alpha u_*|_\Gamma \cdot \tau \in {}_0\mathbb{F}_\tau$. Clearly, we also have $\alpha v|_\Gamma \cdot \tau \in {}_0\mathbb{F}_\tau$ for all $(v, q) \in {}_0\mathbb{E}$ and we are left with the task to solve the problem

$$(1 - \mathbb{L}^{-1} R)(v, q) = \mathbb{L}^{-1}(\widetilde{h}_1 - \alpha u_*|_\Gamma \cdot \tau), \qquad (v, q) \in {}_0\mathbb{E},$$

where the linear operator $R : {}_0\mathbb{E} \longrightarrow {}_0\mathbb{F}_\tau$ is given as $R(v, q) := -\alpha v|_\Gamma \cdot \tau$. However, this operator is of lower order and the usual estimates employed within perturbation arguments for parabolic problems show that $1 - \mathbb{L}^{-1} R$ is invertible by a Neumann series argument, at least for small values $T < T^*$. Here $T^* > 0$ is independent of the data. Consequently, problem (1.1) may be solved this way successively on small time intervals, which cover any given time interval $J = (0, T)$ after finitely many steps. This completes the proof of Theorem 1.2.

Appendix A. Hardy's inequality on the 2D wedge domain

The famous Hardy's inequality is well known and many proofs exist within the literature. However, a proper formulation for the wedge requires boundary conditions at the corner point $x = 0$ or at infinity, if one wants to have a version of Hardy's inequality at hand, that is not only valid for equivalence classes of functions that differ by additive constants. A version in the latter sense is easily deduced as a consequence of, for instance, [2, Corollary VIII.1.5.3].

However, we note that $L^p_\gamma(0,\infty) \hookrightarrow L^1_{\mathrm{loc}}(0,\infty)$ for $1 < p < \infty$ by Hölder's inequality, provided that $\gamma \in \mathbb{R}$ with $\gamma < p-1$. Hence, if $u \in L^p_{\mathrm{loc}}(0,\infty)$ and $u' \in L^p_\gamma(0,\infty)$, then $u \in W^{1,1}_{\mathrm{loc}}(0,\infty)$, if $\gamma < p-1$, which shows that the trace $u(0)$ is well-defined in this case. Analogously, the value $u(\infty) = \lim_{x\to\infty} u(x)$ is well-defined, if $\gamma > p-1$. Now, with the same proof as given in [2] we obtain the following version of [2, Corollary VIII.1.5.3], which is Hardy's inequality on the halfline $(0,\infty)$.

Corollary A.1. *Suppose $1 < p < \infty$, $\gamma \in \mathbb{R}$ and $\gamma \neq p-1$. Let $u \in L^p_{\mathrm{loc}}(0,\infty)$ with $u' \in L^p_\gamma(0,\infty)$ such that $u(0) = 0$, if $\gamma < p-1$, and $u(\infty) = 0$, if $\gamma > p-1$, respectively. Then we have*

$$\left\| \frac{u}{x} \right\|_{L^p_\gamma(0,\infty)} \leq C(p,\gamma)\, \|u'\|_{L^p_\gamma((0,\infty))}$$

with a constant $C(p,\gamma) > 0$ that is independent of u.

In the following let $\psi := \psi_p \circ \psi_E : \Omega \to G$ be the transformation from the wedge onto the layer domain defined at the beginning of Section 3. As consequences of Corollary A.1 we obtain:

Lemma A.2. *Let $1 < p < \infty$, $\gamma \in \mathbb{R}$ such that $\gamma \neq p-2$ and $\rho := |(x_1,x_2)|$. Let $u \in L^p_{\mathrm{loc}}(G)$ with $\nabla u \in L^p_\gamma(G)$ such that $u(0) = 0$, if $\gamma < p-2$, and $u(\infty) = 0$, if $\gamma > p-2$, respectively. Then we have*

$$\|\rho^{-1} u\|_{L^p_\gamma(G)} \leq C(p,\gamma)\, \|\nabla u\|_{L^p_\gamma(G)}$$

with a constant $C(p,\gamma) > 0$ that is independent of u.

Proof. Let $\tilde\gamma \in \mathbb{R}$ such that $\tilde\gamma \neq p-1$. Let $v \in L^p_{\mathrm{loc}}(0,\infty)$ with $v' \in L^p_{\tilde\gamma}(0,\infty)$ such that $v(0) = 0$, if $\tilde\gamma < p-1$, and $v(\infty) = 0$, if $\tilde\gamma > p-1$, respectively. Then by Lemma A.1 we have that

$$\begin{aligned}\int_{\mathbb{R}} e^{(\tilde\gamma - p)x} |v(e^x)|^p e^x dx &= \int_0^\infty \left| \frac{v(y)}{y} \right|^p y^{\tilde\gamma} dy \leq C(p,\tilde\gamma) \int_0^\infty |v'(y)|^p y^{\tilde\gamma} dy \\ &= C(p,\tilde\gamma) \int_{\mathbb{R}} e^{\tilde\gamma x} |v'(e^x)| e^x dx. \end{aligned} \tag{A.1}$$

Now, let $u \in L^p_{\mathrm{loc}}(G)$ with $\nabla u \in L^p_\gamma(G)$ such that $u(0) = 0$, if $\gamma < p-2$, and $u(\infty) = 0$, if $\gamma > p-2$, respectively, and set $\tilde\gamma := \gamma + 1$. Then the above calculation implies

$$\begin{aligned}\|\rho^{-1}u\|^p_{L^p_\gamma(G)} &= \int_G \left| \frac{u(x_1,x_2)}{\rho(x_1,x_2)} \right|^p \rho^\gamma d(x_1,x_2) = \int_0^{\theta_0} \int_{\mathbb{R}} e^{(\gamma-p)x} |u(\psi(x,\theta))|^p e^{2x} dx d\theta \\ &= \int_0^{\theta_0} \int_{\mathbb{R}} e^{(\gamma-p+1)x} |u(\psi(x,\theta))|^p e^{x} dx d\theta \\ &\leq C(p,\gamma+1) \int_0^{\theta_0} \int_{\mathbb{R}} e^{(\gamma+1)x} |\nabla u(\psi(x,\theta))|^p e^{x} dx d\theta \\ &= C(p,\gamma+1) \int_0^{\theta_0} \int_{\mathbb{R}} e^{\gamma x} |\nabla u(\psi(x,\theta))|^p e^{2x} dx d\theta \\ &= C(p,\gamma+1) \int_G \rho^\gamma |\nabla u(x_1,x_2)|^p d(x_1,x_2) \\ &= C(p,\gamma+1)\, \|\nabla u\|^p_{L^p_\gamma(G)}.\end{aligned}$$

Note that $\tilde\gamma \gtrless p-1$, if and only if $\gamma \gtrless p-2$. $\square$

Lemma A.3. *Let* $1 < p < \infty$, $\gamma \in \mathbb{R}$ *such that* $\gamma \neq -2$ *and* $\rho := |(x_1, x_2)|$. *Let* $u \in L^p_{\text{loc}}(G)$ *with* $\rho \nabla u \in L^p_\gamma(G)$ *such that* $u(0) = 0$, *if* $\gamma < -2$, *and* $u(\infty) = 0$, *if* $\gamma > -2$, *respectively. Then we have*

$$\|u\|_{L^p_\gamma(G)} \leq C(p,\gamma)\, \|\rho \nabla u\|_{L^p_\gamma(G)}$$

with a constant $C(p,\gamma) > 0$ *that is independent of* u.

Proof. Let $\tilde{\gamma} \in \mathbb{R}$ such that $\tilde{\gamma} \neq p-1$. Let $v \in L^p_{\text{loc}}(0,\infty)$ with $v' \in L^p_{\tilde{\gamma}}(0,\infty)$ such that $v(0) = 0$, if $\tilde{\gamma} < p-1$, and $v(\infty) = 0$, if $\tilde{\gamma} > p-1$, respectively. Then as above by Lemma A.1 we obtain (A.1).

Now, let $u \in L^p_{\text{loc}}(G)$ with $\rho \nabla u \in L^p_\gamma(G)$ i. e. $\rho^{1+\frac{\gamma}{p}} \nabla u \in L^p(G)$, such that $u(0) = 0$, if $\gamma < -2$, and $u(\infty) = 0$, if $\gamma > -2$, respectively, and set $\tilde{\gamma} := \gamma + p + 1$. Then the above calculation implies

$$\begin{aligned}
\|u\|^p_{L^p_\gamma(G)} &= \int_G |u(x_1,x_2)|^p \rho^\gamma d(x_1,x_2) = \int_0^{\theta_0} \int_{\mathbb{R}} e^{\gamma x} |u(\psi(x,\theta))|^p e^{2x} dx d\theta \\
&= \int_0^{\theta_0} \int_{\mathbb{R}} e^{(\gamma+1)x} |u(\psi(x,\theta))|^p e^{x} dx d\theta \\
&\leq C(p,\gamma+p+1) \int_0^{\theta_0} \int_{\mathbb{R}} e^{(\gamma+p+1)x} |\nabla u(\psi(x,\theta))|^p e^{x} dx d\theta \\
&= C(p,\gamma+p+1) \int_0^{\theta_0} \int_{\mathbb{R}} e^{(\gamma+p)x} |\nabla u(\psi(x,\theta))|^p e^{2x} dx d\theta \\
&= C(p,\gamma+p+1) \int_G \rho^{\gamma+p} |\nabla u(x_1,x_2)|^p d(x_1,x_2) \\
&= C(p,\gamma+p+1)\, \|\rho \nabla u\|^p_{L^p_\gamma(G)}.
\end{aligned}$$

Note that $\tilde{\gamma} \gtrless p-1$, if and only if $\gamma \gtrless -2$. □

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Mathematisches Institut, Angewandte Analysis, Heinrich-Heine-Universität Düsseldorf, 40204 Düsseldorf, Germany

E-mail address: `matthias.koehne@hhu.de`

E-mail address: `juergen.saal@hhu.de`

E-mail address: `laura.westermann@hhu.de`