Motivic Integration in Elementary Extensions of $\mathbb{Q}_p$

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Florian Severin
aus Herdecke

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Berichterstatter:

1. Prof. Dr. Immanuel Halupczok
   Heinrich-Heine-Universität Düsseldorf

2. Prof. Dr. Raf Cluckers
   Université de Lille, KU Leuven

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Meinem Vater

Kurzfassung

Anders als in der klassischen Maßtheorie sind die Werte eines *motivischen Maßes* im Allgemeinen keine reellen Zahlen, sondern von geometrischer Natur. Üblicherweise ist der Wertebereich von einem Grothendieck-Ring bestimmt, etwa dem Ring $K_0(\mathrm{Var})$ der Isomorphieklassen von Varietäten modulo einer Zerlegungsrelation. Eine *elementare Erweiterung* $N \succcurlyeq M$ einer Struktur M ist eine Oberstruktur, welche dieselben Formeln der Logik erster Stufe erfüllt. Ein Beispiel einer elementaren Erweiterung der ganzen Zahlen $\mathbb{Z}$ als angeordnete abelsche Gruppe erhält man durch dichtes Aneinanderreihen abzählbar unendlich vieler Kopien von $\mathbb{Z}$. Formal ausgedrückt ist dies die Menge $\mathbb{Q} \times \mathbb{Z}$ mit lexikographischer Ordnung und komponentenweiser Addition.

Ein typisches Phänomen in elementaren Erweiterungen angeordneter Strukturen ist die Existenz unendlich großer bzw. kleiner Elemente. Dies trifft auch auf echte elementare Erweiterungen $\mathbb{K} \succcurlyeq \mathbb{Q}_p$ zu – und bringt Schwierigkeiten für ein (motivisches) Maß mit sich. Einerseits verhindert es die Existenz eines Haarmaßes, da solch ein $\mathbb{K}$ nicht lokalkompakt ist: Jeder Ball kann leicht in unendlich viele disjunkte offene Bälle von infinitesimalem Radius zerlegt werden. Außerdem wollen wir unendliche kleine (und große) Mengen präzise messen; der Wertebereich des Maßes muss daher deutlich größer als $\mathbb{Q}$ sein (was für Maße definierbarer Teilmengen von $\mathbb{Q}_p$ ausreichend wäre).

Den Ring $R_{\mathrm{mot}}(Z)$ der Werte der motivischen Integration zu verstehen ist daher ein wichtiger Schritt dieser Arbeit. Dabei ist Z eine beliebige Parametermenge in der Wertegruppe Γ, welche notwendigerweise eine $\mathbb{Z}$-Gruppe, d. h. eine elementare Erweiterung von $\mathbb{Z}$ als angeordnete abelsche Gruppe, ist. Die detailierte Analyse des Grothendieck-Rings $K_b^{\Gamma}(Z)$ der beschränkten Z-definierbaren Mengen in Γ durch Raf Cluckers und Immanuel Halupczok erlaubt uns die konkrete Beschreibung $R_{\mathrm{mot}}(Z) \cong (K_b^{\Gamma}(Z) \otimes \mathbb{Q})[T^{Z}]/(T-p)$.

Um die Grundidee der Konstruktion unseres Maßes zu beschreiben, sei $X \subset \mathbb{K}$ eine beschränkte definierbare Teilmenge. Nach endlicher Zerlegung (und bis auf endlich viele Punkte) können wir annehmen, dass X eine Zelle, d.h. die disjunkte Vereinigung einer Familie von durch die Wertegruppe parametrisierten Bällen, ist. Anschaulich ist das Maß der Zelle X dann durch

$$\mu_{\mathrm{mot}}(X) = \sum_r r \cdot N_r$$

definiert, wobei r alle vorkommenden Radii durchläuft, während N_r die (ggf. unendliche) „Zahl“ der Bälle mit Radius r in der Zerlegung ist. Der Fall $X \subset \mathbb{K}^n$ für $n > 1$ ist komplizierter und erfordert einige technische Hilfsmittel, die wir entwickeln.

Der Nutzen unserer Konstruktion wird auch dadurch deutlich, dass sie das universelle motivische Maß liefert. Genauer handelt es sich um das allgemeinste (normalisierte) Maß, welches (1) additiv und multiplikativ ist, (2) „kleinen“ Mengen das Maß 0 zuordnet und (3) eine gewisse Variablensubstitutionsregel erfüllt. Der Beweis dieser Universalität lässt sich mit wenigen Anpassungen aus der vorausgehenden Arbeit von Raf Cluckers und Immanuel Halupczok im Fall von $\mathbb{Q}_p$ übertragen.

Synopsis

Unlike in classical measure theory, the values of a *motivic measure* are not necessarily real numbers, but they are of a more geometric nature. Usually, the set of values is related to some kind of Grothendieck ring, e.g., the ring $K_0(\mathrm{Var})$ of isomorphism classes of varieties modulo a scissor relation. An *elementary extension* of some given structure M is a superstructure $N \supset M$ that satisfies the same first-order formulas as M, and in that case, we write $N \succcurlyeq M$. One example of an elementary extension of the integers $\mathbb{Z}$ as an ordered abelian group can be obtained by densely stacking countably infinitely many copies of $\mathbb{Z}$ right next to each other. More formally, consider the set $\mathbb{Q} \times \mathbb{Z}$ with the lexicographic order and component-wise addition.

A typical phenomenon in elementary extensions of ordered structures is the existence of infinitely small and/or large elements. This also happens in proper elementary extensions $\mathbb{K} \succcurlyeq \mathbb{Q}_p$, and it is precisely what causes difficulties regarding a (motivic) measure. For one, it prevents the existence of a Haar measure, as such fields $\mathbb{K}$ are not locally compact. Indeed, any ball can easily be partitioned into infinitely many disjoint open balls of infinitesimal radius. Furthermore, to understand the motivic measure μ_{mot} in our case, we have to precisely measure infinitely small (and large) sets. In order to prevent the loss of information, the ring of values thus has to be much bigger than $\mathbb{Q}$ (which would suffice for measures of definable subsets of $\mathbb{Q}_p$).

Studying this ring $R_{\mathrm{mot}}(Z)$ of values of motivic integration is therefore an important step towards our actual goal. Here and in the following, Z is an arbitrary set of parameters in the value group Γ, which is necessarily a $\mathbb{Z}$-group, i.e., an elementary extension of $\mathbb{Z}$ as ordered abelian groups. Making use of the detailed analysis of the Grothendieck ring $K_b^\Gamma(Z)$ of bounded Z-definable sets in Γ by Raf Cluckers and Immanuel Halupczok, we obtain the concrete description $R_{\mathrm{mot}}(Z) \cong (K_b^\Gamma(Z) \otimes \mathbb{Q})[T^Z]/(T-p)$.

To get a basic idea of our construction of the measure, consider a bounded definable subset $X \subset \mathbb{K}$ of the valued field. After a finite decomposition (and up to finitely many points), we may assume that X is a cell, i.e., a disjoint union of a family of balls, parameterized by a subset of the value group. Intuitively, its measure is then defined as

$$\mu_{\mathrm{mot}}(X) = \sum_r r \cdot N_r,$$

where r runs over all possible radii, and where N_r is the (possibly infinite) “number” of balls of radius r in the decomposition. The case of $X \subset \mathbb{K}^n$ for $n > 1$ is more involved and requires some new technical tools that we develop.

Emphasizing the value of our construction, we show that the motivic measure obtained is universal, i.e., the most general (normalized) one that (1) is additive and multiplicative, (2) assigns measure 0 to “small” sets, and (3) satisfies a certain change of variables formula. The proof of this universality result can be adapted with few adjustments from the previous work of Raf Cluckers and Immanuel Halupczok in the case of $\mathbb{Q}_p$.

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Contents

1 Introduction

The origins of motivic integration lie in a lecture of Maxim Kontsevich at Orsay in 1995, [Kon95]. As a tool for proving that birationally equivalent Calabi-Yau manifolds have the same Hodge numbers, he introduced a theory of integration on arc spaces, and generalized Victor Batyrev's ideas from [Bat99] using p-adic integration.

In contrast to classical integration theories, motivic integrals do not generally evaluate to real numbers. Instead, and as the name suggests, the values are more geometric objects, e.g., classes of varieties in the Grothendieck ring $K_0(\mathrm{Var})$, which is obtained from the group of isomorphism classes by identifying $[X] + [Y]$ with $[X \dot{\cup} Y]$. Consequently, the associated motivic measure μ_{mot} (obtained by integrating characteristic functions) is not a measure in the usual sense, since it is not real-valued.

However, some important properties are still satisfied. Most notably, the motivic measure is additive and multiplicative, "small" sets are negligible, and there is a change of variables formula (i.e., a bijection with constant Jacobian determinant changes the measure by the norm of that determinant as a scaling factor).

The main outcome of this thesis is the construction of a motivic measure on elementary extensions $\mathbb{K} \succcurlyeq \mathbb{Q}_p$ of the p-adic numbers satisfying the above conditions. This is the content of Definition 5.1.9 and Theorem 5.1.10. While the trivial measure, assigning 0 to each set, obviously meets these requirements, the value of our construction lies in the fact that it yields the universal motivic measure, i.e., the most general one with the properties mentioned above. The proof of that key result, Theorem 5.3.4, is almost identical to the recent proof of the analogous statement for classical p-adic integration obtained by Raf Cluckers and Immanuel Halupczok in [CH21].

As an application of our results, we can deduce that the universal measure on an ultrapower $\mathcal{K}$ of $\mathbb{Q}_p$ is strictly finer than the one naturally obtained as the ultrapower of the p-adic measure, see Example 6.5.

Among others, Jan Denef, Raf Cluckers, François Loeser, Ehud Hrushovski and David Kazhdan developed motivic integration from a model-theoretic perspective (e.g., in [DL99], [CL08], [HK06], [DL02]). Their approaches apply to different classes of structures, and this thesis is a continuation along that line in the realm of valued fields.

Common p-adic integration, i.e., integration with respect to the Haar measure on $\mathbb{Q}_p$, is closely related to integration in the style of both Cluckers-Loeser as well as Hrushovski-Kazhdan, see for example [CL05], [CL15] and [CH21]. Despite being model-theoretic in nature, their work however does not cover arbitrary elementary extensions. In particular, [CL15] only applies to discretely valued fields, and the residue field characteristic is assumed to be 0 in [HK06].

By developing a theory of motivic integration in elementary extensions $\mathbb{K} \succcurlyeq \mathbb{Q}_p$ of the p-adic numbers, we take a first step towards closing this gap. In addition to explicitly constructing a motivic measure on definable subsets of any such $\mathbb{K}$, we provide some tools for computing the measure of a given subset and, similarly, the integral of a given function. Our methods rely on a good understanding of the value group of $\mathbb{K}$, which is necessarily a $\mathbb{Z}$-group, and we thus make extensive use of the results of [Clu03] and [CH18].

While the integral of a definable function in $\mathbb{Q}_p$ always evaluates to a rational number, passing to elementary extensions demands enlarging the ring of possible values of integrals. In the following, we write $R_{\mathrm{mot}}(Z)$ for this enlarged ring, where $Z \subset \Gamma$ is a set of parameters in the value group Γ of the valued field $\mathbb{K}$. Describing and analyzing $R_{\mathrm{mot}}(Z)$ is a substantial part of the work and a necessary step towards constructing and understanding the motivic integral.

A crucial result in this direction is Proposition 4.2.4, relating functions $\mathfrak{f} : \mathrm{RV}^*_* \to \mathrm{p}^\Gamma$ to piecewise polynomial functions on the value group. (Here and in the following $\mathrm{p}^\Gamma \cong \Gamma$ is the value group of $\mathbb{K}$ written in multiplicative notation.) In particular, together with a detailed analysis of the ring of those piecewise polynomial functions, it yields a rather explicit description of $R_{\mathrm{mot}}(Z)$: Corollary 4.3.11 states that there is an isomorphism

$$R_{\mathrm{mot}}(Z) \cong (K_b^\Gamma(Z) \otimes \mathbb{Q})[T^Z]/(T - p),$$

where $K_b^\Gamma(Z)$ is the Grothendieck ring of bounded Z-definable subsets of the value group, which has been exhaustively studied in [CH18].

When it comes to integration theories, a common approach – in the style of Lebesgue – is to start with some measure μ and then construct the integral $\int f \,\mathrm{d}\mu$ of a function f with respect to that measure. From this point of view, our main goal is to define a measure μ_{mot} on bounded definable subsets of $\mathbb{K}^n$, $n \in \mathbb{N}$, for an elementary extension $\mathbb{K} \succcurlyeq \mathbb{Q}_p$ of the p-adic numbers. However, we will first define an integral of (definable) functions from the RV-sorts to the value group p^Γ (in multiplicative notation).

Let us describe the intuition behind this approach in a bit more detail. Fix some set M of parameters and consider any bounded M-definable subset $X \subset \mathbb{K}$. Such a set can be decomposed into a family of balls and points, parameterized by a subset U of some RV^n_m, and both U and the family of sets it parameterizes can be chosen to be M-definable. (This is made precise in Section 5.1, namely in Definition 5.1.1 and Lemma 5.1.3, and is merely a slightly different formulation of the notion introduced in [Clu+21, Definitions 2.1.1].)

Consider, for this set U, the (M-definable) function $\mathfrak{f} : U \to \mathrm{p}^\Gamma$ sending an element $\boldsymbol{u} \in U$ to the radius of the ball corresponding to $\boldsymbol{u}$. Intuitively, the (hyper-)cardinality of the preimage under $\mathfrak{f}$ of some element $\alpha \in \mathrm{p}^{\Gamma}$ is then the "number of balls of radius α in the decomposition of X", so it makes sense to set

$$\mu_{\mathrm{mot}}(X) := \sum_{\alpha \in \mathrm{p}^{\Gamma}} \alpha \cdot \#\mathfrak{f}^{-1}(\alpha) := \int_{\mathrm{mot}} \mathfrak{f}, \tag{1.1}$$

where the right-hand side has yet to be defined. This defers a significant portion of the work to be done to the RV-sorts (and the value group).

The case $X \subset \mathrm{K}^n$ for $n > 1$ requires additional work, and we proceed a bit differently. Instead of constructing the measure for such sets, we will recursively define the integral of functions $\mathfrak{f} : \mathrm{K}^n \times \mathrm{RV}^*_* \to \mathrm{p}^{\Gamma}$ in general, where RV^*_* is a finite product of the sets $\mathrm{RV}^{n_i}_{m_i}$ for some $m_i, n_i \in \mathbb{N}_{>0}$. From there, one easily obtains the desired measure for (M-)definable bounded sets $X \subset \mathrm{K}^n$ by setting $\mu_{\mathrm{mot}}(X) := \int_{\mathrm{mot}} \mathrm{const}_X(1)$.

One final introductory remark: While the results presented in this thesis certainly depend on previous work in several areas of mathematics, particularly in recent model theory, no previous knowledge of motivic integration is required. We describe most constructions (more or less) explicitly and from the ground up. Albeit some proofs and definitions are quite technical in nature, this hopefully also enables readers without a strong background in model theory or motivic integration to grasp the essential concepts.

1.1 Outline

Chapter 1: Introduction

The present Chapter 1 serves as an introduction to the theme of this thesis and the overall ideas. Right here in Section 1.1, we give an outline of the structure, briefly saying what to expect from each chapter and section. In Section 1.2, we fix our notation and introduce some conventions we adhere to almost everywhere in this thesis (and the only exception is made clear).

Chapter 2: Preliminaries

The following Chapter 2 provides a few preliminaries of general nature for later reference.

Section 2.1 is a collection of specific algebraic tools for dealing with certain rings that will come up in the study of integrable functions. While the results there are quite basic (the whole section should be comprehensible to anyone who completed an introductory course on abstract Algebra), most are far from obvious and still demand proofs.

In contrast, Section 2.2 mostly reviews common knowledge from the model theory of valued fields, specifically the p-adic numbers. We give short proofs for some of the statements, and cite others.

The last preliminary section, Section 2.3, introduces the Grothendieck rings of definable subsets of the value group and of the RV-sorts. As the value group is elementary equivalent to $\mathbb{Z}$ in our setting, the former has been studied extensively in [CH18] and we just recall the most essential results we will later apply. We take advantage of the residue field being finite in order to show that the two Grothendieck rings are isomorphic.

Chapter 3: Presburger sets

In Chapter 3, we dive deeper into the study of $\mathbb{Z}$-groups, going beyond the structure of their Grothendieck ring.

Recalling important basic notions and results from [Clu03] and [CH18] makes up most of Section 3.1.

In Section 3.2, we introduce the more specific helpful notion of affine closure and we analyze some of its properties. In particular, we give a complete description of the affine closure of a Presburger cell, depending on its shape.

Building on these results, Section 3.3 provides a trichotomy result for linear functions on Presburger cells that serves as an important tool for the later study of families of integrable functions.

Chapter 4: Integrable functions on RV_*^*

The main part of the present thesis starts with Chapter 4, which is concerned with integrable functions from the RV-sorts to the value group. We introduce and study a Grothendieck ring $K_{\mathrm{int}}(Z)$ of those functions, as well as a variant $K_{\mathrm{int},S}(Z)$ for families of functions.

Section 4.1 lays the foundation by setting up the definitions and basic properties.

We then give an alternative and more explicit description of $K_{\mathrm{int},S}(Z)$ in terms of piecewise polynomial functions in Section 4.2. The results obtained in that section once more rely on [CH18], assuring that the hypercardinalities of families of sets in a $\mathbb{Z}$-group can be expressed as a polynomial in the parameters (up to finite partition of the parameter set).

The purpose of Section 4.3 is to get a better understanding of the ring $R_{\mathrm{mot}}(Z)$ of values of integrable functions, which is a quotient of $K_{\mathrm{int}}(Z)$. The main result is, intuitively speaking, that it suffices to consider integrable functions with finite images when dealing with $R_{\mathrm{mot}}(Z)$. As before, we in fact mostly work with a family version $R_{\mathrm{mot},S}(Z)$.

In Section 4.4, we show that an equality of the integrals of the corresponding functions in two families of integrable functions is already witnessed uniformly. Besides using our knowledge about $R_{\mathrm{mot},S}(Z)$ from the previous section, the tools developed in Section 3.2 and Section 3.3 come into play.

Chapter 5: Integrable functions on $\mathrm{K}^* \times \mathrm{RV}_*^*$

Our work culminates in Chapter 5, where we recursively define the integral for functions from $\mathrm{K}^e \times \mathrm{RV}_*^*$ to the value group, where RV_*^* is an arbitrary product of some of the RV_m. In particular, this leads to a motivic measure on definable subsets of K^e.

The actual construction is done in Section 5.1, using the notion of preparation as introduced in [Clu+21]. Most of the work is proving that the integral as constructed is well defined, by an induction on the ambient dimension e. A key ingredient in the induction step is Lemma 4.4.12 from Section 4.4, and a version of that statement for variables from K is obtained simultaneously in the proof.

In Section 5.2, we apply several results of [Clu+21] to obtain a change of variables formula for our integration. In particular, we define the Jacobian matrix of functions from K^e to K^ℓ, for $e, \ell \in \mathbb{N}_{>0}$.

We conclude with showing that the constructed motivic measure is the universal such. This is the content of Section 5.3, which is an adaptation of [CH21] (concerned with the same result for p-adic integration) to our setting.

Chapter 6: Outlook

Finally, we give a brief outlook and pose some open questions in Chapter 6.

1.2 Notation

Throughout this thesis, K is a fixed elementary extension of $\mathbb{Q}_p$, and we write

- Γ for its value group (in additive notation),
- $\mathrm{val} : \mathrm{K} \to \Gamma \cup \{\infty\}$ for the valuation map,
- $\mathcal{O} = \{x \in \mathrm{K} \mid \mathrm{val}(x) \geq 0\}$ for the valuation ring,
- $\mathfrak{m} = \{x \in \mathrm{K} \mid \mathrm{val}(x) > 0\}$ for the maximal ideal,
- $\mathrm{RV}_m = \mathrm{K}^\times/(1 + p^m\mathcal{O}) \cup \{0\}$ for the m-th RV-structure (combining information from the residue field and the value group), for $m \in \mathbb{N}_{>0}$,

- $\mathrm{rv}_m : \mathrm{K} \to \mathrm{RV}_m$ for the natural quotient map, extended to K by $\mathrm{rv}_m(0) = 0$,
- $\mathrm{AC}_m = \{\xi \in \mathrm{RV}_m \mid \mathrm{val}(\xi) = 0\} \cup \{0\}$ for the set of m-th angular components, where $\mathrm{AC}_0 = \{0\}$, and
- $\mathrm{ac}_m : \mathrm{K} \to \mathrm{AC}_m$ for a fixed m-th angular component map (for our precise definition, see Remark/Definition 2.2.3 below),

As already used above, $\mathrm{val} : \mathrm{K} \to \Gamma \cup \{\infty\}$ factors through RV_m for each m, and we denote the induced map from RV_m to $\Gamma \cup \{\infty\}$ by val as well.

Note also that the more common definition of ac_m as a map from K to $\mathcal{O}/\mathfrak{m}^m = \mathcal{O}/p^m\mathcal{O}$ coincides with ours, when viewing the former as a map onto its image $(\mathcal{O}/p^m\mathcal{O})^\times \cup \{0\}$. Indeed, for $\xi \in \mathrm{AC}_m \subset \mathrm{RV}_m$ with $\xi \neq 0$, we have

$$\xi = x \cdot (1 + p^m\mathcal{O}) = x + p^m \underbrace{x\mathcal{O}}_{=\mathcal{O}} = x + p^m\mathcal{O} \in \mathcal{O}/p^m\mathcal{O}$$

for an(y) appropriate choice of $x \in \mathcal{O}^\times$.

We work in the multi-sorted language $\mathcal{L}_{\mathrm{val}}$ with sorts for the valued field K, the value group Γ and the sorts RV_m for $m \in \mathbb{N}_{>0}$; with the ring language $\mathcal{L}_{\mathrm{ring}}$ on K, the language of ordered abelian groups $\mathcal{L}_{\mathrm{oag}}$ on Γ, and the maps $\mathrm{val} : \mathrm{K} \to \Gamma \cup \{\infty\}$, $\mathrm{val} : \mathrm{RV}_m \to \Gamma \cup \{\infty\}$ and $\mathrm{rv}_m : \mathrm{K} \to \mathrm{RV}_m$ between the sorts. However, as we are only interested in definability, the exact choice of language does not actually matter in most statements.

For a set X and an element c (regardless of the sorts), we write $\mathrm{const}_X(c)$ to denote the constant function on X with value c.

We want to be able to use both additive and multiplicative notation for the value group. In order to avoid confusion, we write Γ for the value group when using additive notation and we write p^Γ when using multiplicative notation. There is a canonical group isomorphism translating additive to multiplicative notation, given by

$$\begin{aligned} \Gamma &\to \mathrm{p}^\Gamma \\ a &\mapsto \mathrm{p}^{-a}. \end{aligned}$$

We denote its inverse by $\mathrm{val} : \mathrm{p}^\Gamma \to \Gamma$, using the same name val for now three different maps. However, it will always be clear from the context (i.e., the domain) which of those maps we are referring to.

We generally work with M-definable sets and functions, for some arbitrary parameter set $M \subset \mathrm{K} \cup \Gamma$, i.e., we allow parameters from both the valued field and the value group. Note that every element of the residue field (which is isomorphic to $\mathbb{F}_p$) is $\emptyset$-definable, that RV_m and Γ are interdefinable (that is, $\mathrm{RV}_m \subset \mathrm{dcl}(\Gamma)$ as well as $\Gamma \subset \mathrm{dcl}(\mathrm{RV}_m)$) for each $m \in \mathbb{N}_{>0}$, see Remark 2.2.4. Thus, there is no benefit in allowing additional parameters from the residue field or from any of the RV_m-sorts. Here and in the following, dcl denotes the definable closure operator, and we consider its image to lie in the (disjoint) union of all sorts by abuse of notation. More precisely, $\mathrm{dcl}(S)$, for an arbitrary set S, contains all S-definable elements of K, of Γ, and of each of the RV_m for $m \in \mathbb{N}$.

We often use $Z = \mathrm{dcl}(M) \cap \Gamma$ as the set of parameters for definable sets in Γ or RV, which simplifies notation without actually changing which sets are definable, see Lemma 2.2.6.

By a definable set or function, we mean "definable with parameters". However, we only use the term "definable" when describing results from a high-level perspective in prose (and these descriptions are often a bit imprecise in other aspects as well). In formal statements or proofs, we always use more precise terms like "M-definable", "Z-definable", or "$\emptyset$-definable".

We generally use the symbols

- x, y, z for elements of K, and X, Y, Z for subsets of K^n,
- u, v, w for elements of RV_m, and U, V, W for subsets of RV_m^n,
- a, b, c for elements of the value group Γ (written additively), and A, B, C for subsets of Γ^n,
- α, β, γ for elements of the value group p^Γ (written multiplicatively), and $\mathrm{A}, \mathrm{B}, \mathrm{C}$ for subsets of $(\mathrm{p}^\Gamma)^n$,
- ξ for elements of AC_m (we never need two distinct elements of AC_m, and it only appears briefly in Section 2.2)
- $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ for integrable (motivic) functions (see Chapter 4 and Chapter 5),
- D, E for M-definable subsets of $\mathrm{K}^* \times \mathrm{RV}_*^*$ (see below for the notation $\mathrm{K}^* \times \mathrm{RV}_*^*$).

An exception to this is Chapter 5, where we also use c to denote elements of K for simplicity of notation.

We use a boldface font for tuples, i.e., $\boldsymbol{x} = (x_1, \ldots, x_n) \in X \subset \mathrm{K}^n$, $\boldsymbol{a} = (a_1, \ldots, a_n) \in A \subset \Gamma^n$, etc. For a tuple $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathrm{K}^n$, we define

$$\begin{aligned} \mathbf{val}(\boldsymbol{x}) &:= (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \in \Gamma^n, \\ \mathbf{rv}_m(\boldsymbol{x}) &:= (\mathrm{rv}_m(x_1), \ldots, \mathrm{rv}_m(x_n)) \in \mathrm{RV}_m^n, \text{ and} \\ \mathbf{ac}_m(\boldsymbol{x}) &:= (\mathrm{ac}_m(x_1), \ldots, \mathrm{ac}_m(x_n)) \in \mathrm{AC}_m^n. \end{aligned}$$

Given $x \in \mathrm{K}$ and $a \in \Gamma$, we write $\mathcal{B}_{\geq a}(x)$ for the *ball around x of radius a*, i.e., for the set

$$\mathcal{B}_{\geq a}(x) = \{y \in \mathrm{K} \mid \mathrm{val}(y - x) \geq a\}.$$

Note that balls are never singletons, since we do not allow $a = \infty$.

Recall that, due to the ultrametric inequality, any two balls are either disjoint or one of them is contained in the other. Whenever we mention or implicitly use a topology on K, we mean the one generated by the family of all balls (this is also called the *valuation topology*). Note that any ball $B = \mathcal{B}_{\geq a}(x)$ can be written as the complement of the open set

$$\bigcup_{y \notin B} \mathcal{B}_{\geq \mathrm{val}(x-y)+1}(y),$$

and is hence clopen in the valuation topology.

Given $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^{\ell}$ for some $\ell \in \mathbb{N}$, we write $\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ as a shorthand for the product $\prod_{i=1}^{\ell} \mathrm{RV}_{m_i}^{n_i}$.

For convenience, we sometimes write "*M-definable subset of $\mathrm{K}^* \times \mathrm{RV}_*^*$*" instead of "$M$-definable subset of $\mathrm{K}^e \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ for some $e, \ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^{\ell}$".

Similarly, we use the term "*Z-definable subset of RV_*^**" as an abbreviation for "Z-definable subset of $\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ for some $\ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^{\ell}$". (Note that a subset of $\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ is M-definable if and only if it is Z-definable for $Z = \mathrm{dcl}(M) \cap \Gamma$, see Lemma 2.2.6.)

For $a \in \Gamma \cup \{-\infty\}$ and $b \in \Gamma \cup \{\infty\}$, we define

$$(a, b) := \{c \in \Gamma \mid a < c < b\}.$$

Additionally given some $d \in \mathbb{N}_{>0}$ for which we have $b - a \in d \cdot \Gamma \cup \{\infty\}$, we also define

$$\begin{aligned}
[a, b)_d &:= \{c \in \Gamma \mid a \leq c < b, c \equiv a \pmod{d}\} && \text{for } a \neq -\infty,\\
(a, b]_d &:= \{c \in \Gamma \mid a \leq c < b, c \equiv b \pmod{d}\} && \text{for } b \neq \infty, \text{ and}\\
[a, b]_d &:= \{c \in \Gamma \mid a \leq c \leq b, a \equiv c \equiv b \pmod{d}\} && \text{for } a \neq -\infty \text{ and } b \neq \infty
\end{aligned}$$

Let us emphasize again that, when using these notations, we always assume $b - a$ to be divisible by d (or infinite), i.e., the condition

$$a \equiv b \pmod{d} \quad \text{or} \quad a = -\infty \quad \text{or} \quad b = \infty$$

is implicit in the notations $[a, b)_d$, $(a, b]_d$, and $[a, b]_d$.

Moreover, we define

$$(a, b)_{\equiv_d r} := \{c \in \Gamma \mid a < c < b, c \equiv r \pmod{d}\}.$$

for $r \in \{0, \ldots, d-1\}$.

Given $n, k \in \mathbb{N}_{>0}$ with $k \leq n$ and arbitrary sets $X_1, \dots, X_n$ with $X = \prod_{i=1}^{n} X_i$, we write

$$\begin{aligned}
\mathrm{pr}_i &: X \to X_i, \\
\mathrm{pr}_{\leq k} &: X \to \prod_{i=1}^{k} X_i \text{ and} \\
\mathrm{pr}_{\neq k} &: X \to \prod_{i=1}^{k-1} X_i \times \prod_{i=k+1}^{n} X_i
\end{aligned}$$

for the canonical projection maps, and we analogously use $\mathrm{pr}_{\geq k}$, $\mathrm{pr}_{<k}$ and $\mathrm{pr}_{>k}$. Similarly, for a tuple $\boldsymbol{x} = (x_1, \dots, x_n) \in X$, we write $\boldsymbol{x}_{\leq k}$, $\boldsymbol{x}_{\neq k}$, etc. for its image under $\mathrm{pr}_{\leq k}$, $\mathrm{pr}_{\neq k}$, etc.

A ring by our definition is commutative and has a multiplicative identity, ring homomorphisms respect the latter. A subring of a ring R is a subset that is itself a ring with the induced (i.e., restricted) addition and multiplication and with the same multiplicative identity as R.

2 Preliminaries

We assume the reader to be familiar with the basic theory of valued fields and with the basics of model theory, including the compactness theorem of first order logic. For thorough introductions, containing much more than we need, see for example [EP05] and [Hod93].

In this chapter, we collect some general lemmata and remarks, laying a foundation for the main parts of the present thesis. While some of these statements are well-known and we refer to the literature, some others are more specific and deserve proofs.

In Section 2.1, we provide a few basic (but quite technical) algebraic tools for later use. In Section 2.2, we recall well-known results from the model theory of valued fields and establish some more specific auxiliary statements. Lastly, in Section 2.3, we introduce the Grothendieck rings $K_b^{\Gamma}(Z)$ of definable subsets of Γ and $K_b^{\mathrm{RV}}(Z)$ of definable subsets of RV_*^*. The former has been studied in detail in [CH18], and we cite the most important results. We also construct an isomorphism between those two Grothendieck rings, allowing us to only work with $K_b^{\Gamma}(Z)$ afterwards.

2.1 An algebraic toolkit

In this section, we state and prove some general algebraic statements that will be useful later on in the descriptions of certain rings. Let us start with a rather specific technical lemma which we later apply to prove the main result of Section 4.3.

Lemma 2.1.1. *Let R be a ring, $a, b, d \in R$ with $a \neq b$, and let $P \in R[X] \setminus \{0\}$ be a polynomial.*[a] *Then there is a polynomial $Q \in R[X]$ with $\deg(Q) \leq \deg(P)$ and some $m \in \{1, \ldots, \deg(P)+1\}$ such that*

$$(a-b)^m \cdot P(z) = a \cdot Q(z) - b \cdot Q(z+d) \tag{2.1}$$

holds for all $z \in R$.

Moreover, if R' is any ring containing R as a subring, we can choose $m \in \mathbb{N}_{>0}$ and $g \in R[X]$ as above such that (2.1) even holds for all $z \in R'$.

[a] In the case $P = 0$, the statement below trivially holds for all $m \in \mathbb{N}$ and $Q = P$.

Proof. We prove the statement by induction on $n := \deg(P)$.

Induction base, $n = 0$. Then P is a constant polynomial, and the choices $Q = P$ and $m = 1$ clearly satisfy (2.1).

Induction step. Fix $n \in \mathbb{N}$, assume that the claim holds for all polynomials of degree at most n, and let $P \in R[X]$ with $\deg(P) = n+1$. Write $P = \sum_{i=0}^{n+1} a_i X^i$ where $a_i \in R$ and consider the polynomial $P' \in R[X]$ given by

$$P'(X) := (a-b) \cdot \sum_{i=0}^{n} a_i X^i + b \cdot a_{n+1} \cdot \underbrace{\sum_{k=0}^{n} \binom{n+1}{k} \cdot d^{n+1-k} \cdot X^k}_{= (X+d)^{n+1} - X^{n+1}}$$

of degree at most n. By the induction hypothesis, we can find a polynomial $Q_0 \in R[X]$ and some $m \in \mathbb{N}$ with $\deg(Q_0) \leq \deg(P') \leq n$ and $1 \leq m \leq \deg(P') + 1$ such that

$$(a-b)^m \cdot P'(z) = a \cdot Q_0(z) - b \cdot Q_0(z+d)$$

for all $z \in R$. Consider $Q \in R[X]$ with $Q(X) := (a-b)^m \cdot a_{n+1} X^{n+1} + Q_0(X)$ and observe that we have

$$a \cdot Q(z) - b \cdot Q(z+d) = (a-b)^{m+1} \cdot P(z)$$

for all $z \in R$ by definition of P', Q_0 and Q. Since $\deg(Q) \leq n+1 = \deg(P)$ and $1 \leq m+1 \leq \deg(P') + 2 \leq n+2 = \deg(P) + 1$, this completes the inductive step.

The "moreover" part follows by replacing the two occurrences of "…for all $z \in R$." with "…for all $z \in R'$." in the above. ☐

The following statement says that a non-constant polynomial cannot coincide with an exponential function on an infinite set. While this is straight-forward when some basic concepts of calculus (e.g., derivatives or limits) are available, we will need it in a more general setting and provide a purely algebraic prove.

Proposition 2.1.2. *Let R be a ring with torsion-free additive group, let $a \in R$, $Q \in R[T]$, and $q \in \mathbb{Z} \subset R$ with $q \notin \{0, 1\}$. If $Q(t) = a \cdot q^t$ for all $t \in \mathbb{N}$, then $Q = 0$.*

Proof. Let $g : \mathbb{N} \to R, t \mapsto a \cdot q^t$ and suppose that $g(t) = Q(t)$ for all $t \in \mathbb{N}$, where $Q \in R[T]$ is some polynomial over R. Define $Q_0 := Q$ and $Q_{i+1}(T) := Q_i(T+1) - Q_i(T) \in R[T]$ for $i \in \mathbb{N}$.

Considering the binomial expansion of $(T+1)^{\deg(Q_i)}$ makes clear that $\deg(Q_{i+1}) \leq \deg(Q_i)$ for all i, where equality holds if and only if $Q_i = 0$, i.e., $\deg(Q_i) = -\infty$ (and in that case, we also have $Q_{i+1} = 0$). In particular, $Q_d = 0$ for $d > \deg(Q)$. On the other hand, recursively define maps $g_d : \mathbb{N} \to R$ for $d \in \mathbb{N}$ by

$$\begin{aligned} g_0(t) &:= g(t) = Q(t), \text{ and} \\ g_{d+1}(t) &:= g_d(t+1) - g_d(t) \end{aligned}$$

for all $t \in \mathbb{N}$. Then, by induction on d, we have $g_d(t) = a \cdot q^t \cdot (q-1)^d$ for all $t, d \in \mathbb{N}$. Put together, we get that $a \cdot q^t \cdot (q-1)^d = g_d(t) = Q_d(t) = 0$ for all $t \in \mathbb{N}$ and $d > \deg(Q)$. Since $q \in \mathbb{Z} \setminus \{0, 1\}$, we have $q^t \cdot (q-1)^d \in \mathbb{Z} \setminus \{0\}$. As $(R, +)$ is torsion-free, this implies $a = 0$, hence $Q(t) = 0$ for all $t \in \mathbb{N}$. This yields $Q = 0$ as claimed. □

The next lemma provides a useful criterion for checking whether the rationals embed into a given ring.

Lemma 2.1.3. *Let R be a ring and suppose there is a (necessarily unique) injective ring homomorphism $\varphi : \mathbb{Z} \hookrightarrow R$. We identify $k \in \mathbb{Z}$ with its image $\varphi(k) \in R$ and just write $k \in R$.*

Let $p \in \mathbb{Z}$ be some fixed prime and suppose there is an element $r \in R$ with $p \cdot r = 1$. Moreover, suppose that for each $d \in \mathbb{N}_{>0}$, there is an element $r_{p^d-1} \in R$ for which we have $(p^d - 1) \cdot r_{p^d-1} = 1$. Then φ uniquely extends to an injective homomorphism $\varphi : \mathbb{Q} \hookrightarrow R$. In particular, R is then torsion-free.

Proof. We first show that there is, for each $m \in \mathbb{N}_{>0}$, an element $r_m \in R$ with $m \cdot r_m = 1$. Firstly, for $m = p$, this is part of the assumptions, setting $r_p = r$.

Secondly, let m now be any other prime. Let $d \in \mathbb{N}_{>0}$ be the order of p in the cyclic group $(\mathbb{Z}/m\mathbb{Z})^\times$. Then we have $p^d \equiv 1 \pmod{m}$, i.e., $p^d - 1 = m \cdot k$ for some $k \in \mathbb{Z}$, and hence

$$m \cdot k \cdot r_{p^d-1} = (p^d - 1) \cdot r_{p^d-1} = 1,$$

so that $r_m = k \cdot r_{p^d-1}$ is as desired.

Finally, let us consider the general case, i.e., let $m \in \mathbb{N}_{>0}$ be an arbitrary positive integer. Let $m = \prod_{i \in I} q_i^{e_i}$ denote the prime decomposition of m, i.e., $e_i \in \mathbb{N}_{>0}$ and $q_i \in \mathbb{N}_{>0}$ prime for each $i \in I$ for some finite index set I. Then

$$r_m = \prod_{i \in I} r_{q_i}^{e_i}$$

is as desired, since we have

$$\begin{aligned} m \cdot r_m &= \prod_{i \in I} q_i^{e_i} \cdot \prod_{i \in I} r_{q_i}^{e_i} \\ &= \prod_{i \in I} (\underbrace{q_i \cdot r_{q_i}}_{=1})^{e_i} \\ &= 1. \end{aligned}$$

Note that the elements r_m are uniquely determined by m, since this is true in general for multiplicative inverses in a ring (if they exist). In particular, any ring homomorphism $\tilde{\varphi} : \mathbb{Q} \to R$ extending φ has to satisfy

$$\tilde{\varphi}(\frac{k}{m}) = k \cdot r_m.$$

We now claim that the above mapping rule indeed defines an injective ring homomorphism from $\mathbb{Q}$ to R.

$\tilde{\varphi}$ **is well-defined.** Let $\frac{k}{m} = \frac{k'}{m'}$, i.e., $k \cdot m' = k' \cdot m$. Then we have

$$\begin{aligned} k \cdot r_m &= k \cdot m' \cdot r_{m'} \cdot r_m \\ &= k' \cdot m \cdot r_{m'} \cdot r_m \\ &= k' \cdot r_{m'}. \end{aligned}$$

$\tilde{\varphi}$ **is injective.** Suppose that $k \cdot r_m = \tilde{\varphi}(\frac{k}{m}) = \tilde{\varphi}(\frac{k'}{m'}) = k' \cdot r_{m'}$. Then we have

$$\begin{aligned} k \cdot m' &= k \cdot r_m \cdot m \cdot m' \\ &= k' \cdot r_{m'} \cdot m \cdot m' \\ &= k' \cdot m \end{aligned}$$

in R, and hence in $\mathbb{Z} \subset R$, so $\frac{k}{m} = \frac{k'}{m'}$.

$\tilde{\varphi}$ **is a ring homomorphism.** Let $k, k' \in \mathbb{Z}$ and $m, m' \in \mathbb{N}_{>0}$. We have

$$\begin{aligned} \tilde{\varphi}(\frac{k}{m} \cdot \frac{k'}{m'}) &= k \cdot k' \cdot r_m \cdot r_{m'} \\ &= k \cdot r_m \cdot k' \cdot r_{m'} \\ &= \tilde{\varphi}(\frac{k}{m}) \cdot \tilde{\varphi}(\frac{k'}{m'}) \end{aligned}$$

and

$$\begin{aligned} \tilde{\varphi}(\frac{k}{m} + \frac{k'}{m'}) &= \tilde{\varphi}(\frac{km' + k'm}{mm'}) \\ &= (km' + k'm) \cdot r_{mm'} \\ &= km' \cdot r_{mm'} + k'm \cdot r_{mm'} \\ &= k \cdot r_m + k' \cdot r_{m'} \\ &= \tilde{\varphi}(\frac{k}{m}) + \tilde{\varphi}(\frac{k'}{m'}), \end{aligned}$$

where the second-to-last equality holds since $r_m = \tilde{\varphi}(\frac{1}{m}) = \tilde{\varphi}(\frac{m'}{mm'}) = m' \cdot r_{mm'}$ and, similarly, $r_{m'} = m \cdot r_{mm'}$. Lastly, we also have $\tilde{\varphi}(1) = 1$.

For the in particular part, let $k \in \mathbb{N}_{>0}$ and $r \in R$ and note that $k \cdot r = 0$ then already implies $r = \tilde{\varphi}(\frac{1}{k}) \cdot k \cdot r = 0$. □

When working with integrable functions later, we will need to make the additive groups of certain rings divisible by tensoring with $\mathbb{Q}$. The following statement is a useful tool which will help us to describe the resulting rings.

Lemma 2.1.4. *Let G be a torsion-free divisible abelian group, written additively, and let $H \subset G$ be a subgroup. Suppose that for each $g \in G$, there is an $n \in \mathbb{N}_{>0}$ with $n \cdot g \in H$.*

Then the map given by

$$\begin{aligned} \varphi : H \otimes \mathbb{Q} &\to G \\ h \otimes q &\mapsto h \cdot q \end{aligned}$$

is an isomorphism (of groups).

Proof. Because G is torsion-free, it is naturally embedded into $G \otimes \mathbb{Q}$ via $\iota : g \mapsto g \otimes 1$. Since G is divisible, this map is also surjective, hence an isomorphism. As $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, the natural $\mathbb{Z}$-module homomorphism $\eta : H \otimes \mathbb{Q} \hookrightarrow G \otimes \mathbb{Q}$ is injective.

Note that $\varphi = \iota^{-1} \circ \eta$, hence it is an injective $\mathbb{Z}$-module homomorphism. To show that φ is surjective, let $g \in G$ and fix $n \in \mathbb{N}_{>0}$ with $n \cdot g \in H$. Then $\varphi((n \cdot g) \otimes \frac{1}{n}) = g$, so φ is an isomorphism as claimed. □

2.2 Some model theory of valued fields

Recall from Section 1.2 that we work in a fixed elementary extension $\mathrm{K} \succcurlyeq \mathbb{Q}_p$ of the p-adic numbers in the multi-sorted language $\mathcal{L}_{\mathrm{val}}$ with sorts K, Γ and RV_m for $m \in \mathbb{N}_{>0}$. In this section, we moreover fix a set of parameters $M \subset \mathrm{K} \cup \Gamma$, and whenever we are concerned with definable sets (in any sort) we consider M-definable sets.

Albeit we do not have definable Skolem functions between arbitrary sorts, the order on the value group facilitates the following result.

Lemma 2.2.1 (Existence of definable Skolem functions from RV to Γ)**.** *Let $U \subset \mathrm{RV}_m^n$ be M-definable, $m, n \in \mathbb{N}_{>0}$, and let $\varphi(\boldsymbol{u}, a)$ be an $\mathcal{L}_{\mathrm{val}}$-formula satisfying*

$$\mathrm{K} \models \forall \boldsymbol{u} \in U \, \exists a \in \Gamma \; \varphi(\boldsymbol{u}, a).$$

Then there is an M-definable function $s : U \to \Gamma$ such that $\varphi(\boldsymbol{u}, s(\boldsymbol{u}))$ holds for all $\boldsymbol{u} \in U$.

Proof. As usual in model theory, we write $\varphi(\boldsymbol{u}, \Gamma)$ for the set of $a \in \Gamma$ with $\mathrm{K} \models \varphi(\boldsymbol{u}, a)$. Consider the M-definable sets

$$U_{-} = \{\boldsymbol{u} \in U \mid \varphi(\boldsymbol{u}, \Gamma) \text{ has a minimum}\} \text{ and}$$
$$U_{+} = \{\boldsymbol{u} \in U \mid \varphi(\boldsymbol{u}, \Gamma) \text{ has a maximum}\}.$$

Note that the minimum of a M-definable subset of Γ is again M-definable, and set

$$s(\boldsymbol{u}) := \min(\varphi(\boldsymbol{u}, \Gamma)) \text{ for } \boldsymbol{u} \in U_{-} \text{ and}$$
$$s(\boldsymbol{u}) := \max(\varphi(\boldsymbol{u}, \Gamma)) \text{ for } \boldsymbol{u} \in U_{+} \setminus U_{-}.$$

For the remaining case, let $\boldsymbol{u} \in U \setminus (U_{-} \cup U_{+})$. Then $\varphi(\boldsymbol{u}, \Gamma)$ has neither a minimum nor a maximum, and is hence unbounded (see Lemma 2.2.6 and Lemma 3.1.4), so we can set

$$s(\boldsymbol{u}) := \min(\varphi(\boldsymbol{u}, \Gamma) \cap \Gamma_{\geq 0}) \text{ for } \boldsymbol{u} \in U \setminus (U_{-} \cup U_{+}),$$

yielding the desired M-definable function $s : U \to \Gamma$. □

(Even though we use the later Lemma 2.2.6 and Lemma 3.1.4, note that the proofs of both do not depend on Lemma 2.2.1.)

Remark 2.2.2 (see also [CH18]). Note that the definable closure $\mathrm{dcl}(Z) \subset \Gamma$ of any subset $Z \subset \Gamma$ is an elementary substructure, hence we may assume $Z \preccurlyeq \Gamma$. In particular, Z is then itself a $\mathbb{Z}$-group.

Under this assumption, note that $\mathrm{acl}(Z) = \mathrm{dcl}(Z) = Z$, so any finite Z-definable set is already contained in Z.

Remark and Definition 2.2.3. Let $T = \mathrm{Th}(\mathbb{Q}_p)$ be the $\mathcal{L}_{\mathrm{val}}$-theory of $\mathbb{Q}_p$ and let $\mathrm{K} \models T$ be a model of T.

Then there is, for each $m \in \mathbb{N}_{>0}$, an $\mathcal{L}_{\mathrm{val}}$-definable *m-th angular component map*, i.e., a map

$$\mathrm{ac}_m : \mathrm{K} \to \mathrm{AC}_m = \{\xi \in \mathrm{RV}_m \mid \mathrm{val}(\xi) = 0\} \cup \{0\}$$

which extends $\mathrm{rv}_m \restriction (\mathcal{O}^\times \cup \{0\})$ and which restricts to a multiplicative group homomorphism from $\mathrm{K}^\times$ to $(\mathcal{O}/p^m\mathcal{O})^\times = \mathrm{AC}_m \setminus \{0\}$.

Proof. In the case $K = \mathbb{Q}_p$, the map

$$\begin{aligned} \mathrm{ac}_m : \mathrm{K} &\to \mathrm{AC}_m \\ x &\mapsto x \cdot p^{-\mathrm{val}(x)} \end{aligned}$$

is $\emptyset$-definable by [Den86, Lemma 2.1 (4)]. It is clear that this map satisfies the desired properties (i.e., it extends $\mathrm{rv}_m\restriction(\mathcal{O}^\times \cup \{0\})$ and restricts to a group homomorphism from $\mathrm{K}^\times$ to $\mathrm{AC}_m \setminus \{0\}$). Thus, the $\mathcal{L}_{\mathrm{val}}$-formula defining ac_m in $\mathrm{K} = \mathbb{Q}_p$ defines the desired map in any model of $\mathrm{Th}(\mathbb{Q}_p)$. $\square$

Remark 2.2.4. Note that, for $m \in \mathbb{N}_{>0}$, the set $\mathrm{AC}_m = \{\xi \in \mathrm{RV}_m \mid \mathrm{val}(\xi) = 0\} \cup \{0\}$ is contained in $\{0, 1, 2, \ldots, p^m - 1\} \subset \mathrm{dcl}(\emptyset)$.

Moreover, for each $u \in \mathrm{RV}_m$, we have $u \in \mathrm{dcl}(\mathrm{val}(u))$.

Proof (of the "moreover" part). Let $u \in \mathrm{RV}_m$. Consider the $\mathrm{val}(u)$-definable set $U := \{u' \in \mathrm{RV}_m \mid \mathrm{val}(u') = \mathrm{val}(u)\}$ and note that $\{u\} = \mathrm{ac}_m^{-1}(\underbrace{\mathrm{ac}_m(u)}_{\in \mathrm{dcl}(\emptyset)}) \cap U$ is the intersection of an $\emptyset$-definable set with a $\mathrm{val}(u)$-definable set, hence it is $\mathrm{val}(u)$-definable itself. $\square$

Lemma 2.2.5. *Let S be an arbitrary set of parameters (from arbitrary sorts). For each $m \in \mathbb{N}_{>0}$, we have $\mathrm{acl}(S) \cap \mathrm{RV}_m = \mathrm{dcl}(S) \cap \mathrm{RV}_m$.*

Proof. By Remark 2.2.4, it is enough to show that $\mathrm{val}(u) \in \mathrm{dcl}(S)$ for all $u \in \mathrm{acl}(S) \cap \mathrm{RV}_m$. So let $U \subset \mathrm{RV}_m$ be finite and S-definable with $u \in U$. Then $\mathrm{val}(U)$ is finite and S-definable, hence contained in $\mathrm{acl}(S) \cap \Gamma$. Since the order on Γ is $\emptyset$-definable (as it is part of our language), we have $\mathrm{acl}(S) \cap \Gamma = \mathrm{dcl}(S) \cap \Gamma$, and thus $\mathrm{val}(u) \in \mathrm{val}(U) \subset \mathrm{dcl}(S)$. By Remark 2.2.4, this yields $u \in \mathrm{dcl}(\mathrm{val}(u)) \subset \mathrm{dcl}(S)$ as claimed. $\square$

Lemma 2.2.6. *The sorts Γ and RV_m, for any $m \in \mathbb{N}_{>0}$ are stably embedded. More precisely, subsets $A \subset \Gamma^*$ and $U \subset \mathrm{RV}_m^*$ are M-definable if and only if they are Z-definable, where $Z = \mathrm{dcl}(M) \cap \Gamma$.*

Proof. Clearly any Z-definable set is M-definable. We now prove the other direction for subsets of $U \subset \mathrm{RV}_m^*$.

Consider the "leading term language" introduced in [Fle11], i.e., the multisorted language $\mathcal{L}_{\mathrm{rv}}$ with sorts for the valued field K and each of the RV_m, for $m \in \mathbb{N}_{>0}$, with the ring language on K, multiplication and partial addition on the RV_m and maps $\mathrm{rv}_m : \mathrm{K} \to \mathrm{RV}_m$ as well as $\mathrm{rv}_{\ell,m} : \mathrm{RV}_\ell \to \mathrm{RV}_m$ for $\ell \geq m$.

Note that multiplication and partial addition on RV_m are $\emptyset$-definable in $\mathcal{L}_{\mathrm{val}}$, and that Γ is interpretable in $\mathcal{L}_{\mathrm{rv}}$ in (each of the) RV_m over $\emptyset$, see [Fle11, Definition 2.1 and Proposition 2.8]. This allows us to replace a definition of U in $\mathcal{L}_{\mathrm{val}}$ by a definition using (almost) only $\mathcal{L}_{\mathrm{rv}}$ as follows: Let $\varphi(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{w}, x)$ be an $\mathcal{L}_{\mathrm{val}}$-formula defining U with parameters $\boldsymbol{a} \in \Gamma^*$, $\boldsymbol{w} \in \mathrm{RV}_*^*$, and $x \in \mathrm{K}^*$. By interpretability of Γ in RV_* over the empty set in the language $\mathcal{L}_{\mathrm{rv}}$, there is an $\mathcal{L}_{\mathrm{rv}}$-formula $\psi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, x)$ such that $\varphi(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{w}, x)$ holds in K – for any given tuple $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{w}, x)$ – if and only if we have the equivalence

$$\mathrm{K} \models_{\mathcal{L}_{\mathrm{rv}}} \psi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, x) \quad \Longleftrightarrow \quad \mathrm{val}(\boldsymbol{v}) = \boldsymbol{a}$$

for all $\boldsymbol{v}$. By relative quantifier elimination in the language $\mathcal{L}_{\mathrm{rv}}$, [Fle11, Proposition 4.3], we may assume that ψ does not contain any K-quantifiers. The only variables from the valued field that appear in ψ are then the coordinates of x, and they can only appear in the form $\mathrm{rv}_\ell(P(x))$ for some polynomial $P \in \mathbb{Z}[\boldsymbol{T}]$, where $\boldsymbol{T}$ is a tuple of the same length as x.

Hence we can already define U by an $\mathcal{L}_{\mathrm{rv}}$-formula just using parameters from the set $\{\mathrm{rv}_\ell(P(x)) \mid P \in \mathbb{Z}[\boldsymbol{T}], \ell \in \mathbb{N}_{>0}\}$, and $\boldsymbol{v}$ and $\boldsymbol{w}$. As the additional relations in $\mathcal{L}_{\mathrm{rv}}$ (i.e., multiplication and partial addition on the RV-sorts) are $\emptyset$-definable in $\mathcal{L}_{\mathrm{val}}$, the set U is then also definable over the same parameters in $\mathcal{L}_{\mathrm{val}}$. By (the moreover part of) Remark 2.2.4, all parameters are Z-definable for $Z = \mathrm{dcl}(M) \cap \Gamma$, yielding that U is indeed Z-definable in $\mathcal{L}_{\mathrm{val}}$.

Analogously showing that any M-definable subset $A \subset \Gamma^*$ is already Z-definable is left to the reader. □

The following observation is a handy tool to answer the question whether two elements of K have the same image in RV_d by solely comparing two valuations.

Remark 2.2.7 (cf. [CHR21, Remark 2.1.3], note that rv_λ for $\lambda = p^{-d}$ there is rv_{d+1} here). For $x, y \in \mathrm{K}$, we have

$$\mathrm{rv}_d(x) = \mathrm{rv}_d(y) \iff (x = y = 0) \text{ or } \mathrm{val}(x - y) \geq d + \mathrm{val}(x).$$

In Chapter 5, we use the equivalence in the following form: For $c, c', x \in \mathrm{K}$, we have

$$\mathrm{rv}_d(x - c) = \mathrm{rv}_d(x - c') \iff (c = c' = x) \text{ or } \mathrm{val}(c - c') \geq d + \mathrm{val}(x - c).$$

Let us give a first application of this observation that will be useful later on (in Chapter 5) to allow us to assume $\Gamma \neq \mathbb{Z}$.

Remark 2.2.8. If $\Gamma = \mathbb{Z}$, then we already have $\mathrm{K} = \mathbb{Q}_p$.

Proof. Let $x, y \in \mathrm{K}^\times$ be two distinct non-zero elements. By the assumption, the valuation of their difference is then an integer. In formulas, we have $\mathrm{val}(x-y) \in \Gamma = \mathbb{Z}$. Remark 2.2.7 thus implies

$$\mathrm{rv}_d(x) \neq \mathrm{rv}_d(y)$$

for all $d \in \mathbb{N}$ with $d > \mathrm{val}(x - y) - \mathrm{val}(x)$.

In other words, any element $x \in K^\times$ is already completely determined by the sequence $(\mathrm{rv}_d(x))_{d\in\mathbb{N}} \in \prod_{d\in\mathbb{N}} K^\times/(1 + p^d\mathcal{O}) \cong \prod_{d\in\mathbb{N}} \mathbb{Q}_p^\times/(1 + p^d\mathbb{Z}_p)$, so that the inclusion $\mathbb{Q}_p \hookrightarrow K$ must be surjective. □

There is no natural well-defined addition on RV_d, but the following lemma gives a helpful characterization of the sets of images under rv_d of certain sums in K. We will use it in Chapter 5 to prove that the integral (to be defined) on K is well-defined.

Lemma 2.2.9. *Let $d \in \mathbb{N}_{>0}$ and $u, v \in \mathrm{RV}_d$, and fix some $x_0, y_0 \in K$ with $\mathrm{rv}_d(x_0) = u$ and $\mathrm{rv}_d(y_0) = v$. Then the set*

$$\{\mathrm{rv}_d(x + y) \mid \mathrm{rv}_d(x) = u, \mathrm{rv}_d(y) = v\}$$

is equal to

(1) $\ldots\{\mathrm{rv}_d(x_0 + y_0)\}$, if $\mathrm{rv}(u) \neq \mathrm{rv}(-v)$,

(2) $\ldots\{w \in \mathrm{RV}_d \mid \mathrm{rv}_{d-\ell}(w) = \mathrm{rv}_{d-\ell}(x_0 + y_0)\}$, if $\mathrm{rv}_\ell(u) = \mathrm{rv}_\ell(-v)$ but $\mathrm{rv}_{\ell+1}(u) \neq \mathrm{rv}_{\ell+1}(-v)$ for some $1 \leq \ell < d$, and

(3) $\ldots\{w \in \mathrm{RV}_d \mid \mathrm{val}(w) \geq d + \mathrm{val}(u)\}$, if $u = -v$.

In particular, its cardinality is 1 in the first case, p^ℓ in the second case, and infinite in the third case.

Proof. Before we begin the case distinction, note that we have

$$\begin{aligned}\{x + y \mid \mathrm{rv}_d(x) = u, \mathrm{rv}_d(y) = v\} &= \mathrm{rv}_d^{-1}(u) + \mathrm{rv}_d^{-1}(v)\\ &= \mathcal{B}_{\geq d+\mathrm{val}(x_0)}(x_0) + \mathcal{B}_{\geq d+\mathrm{val}(y_0)}(y_0)\\ &= \mathcal{B}_{\geq d+\min\{\mathrm{val}(x_0),\mathrm{val}(y_0)\}}(x_0 + y_0)\end{aligned}$$

by Remark 2.2.7.

(1) If $\mathrm{rv}(x_0) = \mathrm{rv}(u) \neq \mathrm{rv}(-v) = \mathrm{rv}(-y_0)$, then we have the equality $\mathrm{val}(x_0 + y_0) = \min\{\mathrm{val}(x_0), \mathrm{val}(y_0)\}$ by Remark 2.2.7. Hence we have

$$\begin{aligned}\{x + y \mid \mathrm{rv}_d(x) = u, \mathrm{rv}_d(y) = v\} &= \mathcal{B}_{\geq d+\mathrm{val}(x_0+y_0)}(x_0 + y_0)\\ &= \mathrm{rv}_d^{-1}(\mathrm{rv}_d(x_0 + y_0)),\end{aligned}$$

which implies the claim.

(2) If $\mathrm{rv}_\ell(u) = \mathrm{rv}_\ell(-v)$ but $\mathrm{rv}_{\ell+1}(u) \neq \mathrm{rv}_{\ell+1}(-v)$, then we have $\mathrm{val}(x_0 + y_0) = \ell + \mathrm{val}(x_0) = \ell + \mathrm{val}(y_0)$ by Remark 2.2.7. Hence we have

$$\begin{aligned}\{x + y \mid \mathrm{rv}_d(x) = u, \mathrm{rv}_d(y) = v\} &= \mathcal{B}_{\geq d+\min\{\mathrm{val}(x_0),\mathrm{val}(y_0)\}}(x_0 + y_0)\\ &= \mathcal{B}_{\geq (d-\ell)+\mathrm{val}(x_0-y_0)}(x_0 + y_0)\\ &= \mathrm{rv}_{d-\ell}^{-1}(\mathrm{rv}_{d-\ell}(x_0 + y_0)),\end{aligned}$$

which implies the claim.

(3) If $u = -v$, we have $\mathrm{val}(x_0 - y_0) \geq d + \mathrm{val}(x_0) = d + \mathrm{val}(y_0)$ by Remark 2.2.7. Hence we have

$$\begin{aligned}\{x + y \mid \mathrm{rv}_d(x) = u, \mathrm{rv}_d(y) = v\} &= \mathcal{B}_{\geq d+\min\{\mathrm{val}(x_0),\mathrm{val}(y_0)\}}(x_0 + y_0) \\ &= \mathcal{B}_{\geq d+\mathrm{val}(x_0)}(x_0 + y_0) \\ &= \underbrace{\mathcal{B}_{\geq d+\mathrm{val}(x_0)}(x_0)}_{=\mathrm{rv}_d^{-1}(u)+y_0} + y_0 \\ &= \mathcal{B}_{\geq d+\mathrm{val}(x_0)}(0),\end{aligned}$$

as the latter two sets are balls of the same radius both containing 0, since $\mathrm{rv}_d(-y_0) = -v = u$. This implies the claim. $\square$

2.3 The Grothendieck ring of RV_*^*

The aim of this section is to get a basic understanding of Z-definable subsets of RV_*^* for some parameter set $Z \subset \Gamma$. Recall that we may assume that Z is an elementary substructure of Γ (with respect to the language $\mathcal{L}_{\mathrm{oag}}$), see Remark 2.2.2.

Let us first introduce the Grothendieck ring of bounded definable subsets of a $\mathbb{Z}$-group, as studied in [CH18]. We then analogously define the Grothendieck ring of the Z-definable bounded subsets of RV_*^*, and we will close this section by constructing an isomorphism between those two Grothendieck rings.

Definition 2.3.1 (cf. [CH18, Definition 2.3.1])**.** The ring $K_b^\Gamma(Z)$, denoting the *Grothendieck ring of Z-definable bounded subsets of Γ*, is defined as follows:

The additive group of $K_b^\Gamma(Z)$ is the free abelian group generated by symbols $[A]$ for each Z-definable bounded subset $A \subset \Gamma^n$, for some $n \in \mathbb{N}$, modulo the relations

(1) $[(A \cup B)] = [A] + [B]$, for disjoint sets $A, B \subset \Gamma^n$, and

(2) $[A] = [B]$, if there is a Z-definable bijection from $A \subset \Gamma^n$ to $B \subset \Gamma^m$.

We will write $\#A$ for the element in $K_b^\Gamma(Z)$ corresponding to A and call it the *hypercardinality of A*.

The multiplication on $K_b^\Gamma(Z)$ is given by $\#A \cdot \#B := \#(A \times B)$. (It is straightforward to check that this gives a ring structure on $K_b^\Gamma(Z)$.)

Lemma 2.3.2 ([CH18, Lemma 2.2.3 (2) and Lemma 2.2.6]). *The ring* $K_b^\Gamma(Z)$ *naturally embeds into* $K_b^\Gamma(Z)\otimes\mathbb{Q}$*. Moreover, the latter (and hence also the former) is an integral domain.*

Given a Z-definable bounded subset $U \subset \mathrm{RV}_m^n$, there are now two natural ways of defining the hypercardinality of U: Firstly, one could set $\#U := \sum_{\xi\in\mathrm{AC}_m^n} \#\,\mathbf{val}(U_\xi) \in K_b^\Gamma(Z)$, where the U_ξ are Z-definable sets on which the valuation map $\mathbf{val}$ is injective and which form a partition of U. (This is satisfied, for example, by the sets $U_\xi := U \cap \mathrm{ac}_m^{-1}(\xi) = \{\boldsymbol{u} \in U \mid \mathrm{ac}_m(\boldsymbol{u}) = \xi\}$ for $\xi \in \mathrm{ac}_m(U) \subset \mathrm{AC}_m^n$.)

On the other hand, one could define the hypercardinality of such a set U as the corresponding class in the Grothendieck ring $K_b^{\mathrm{RV}}(Z)$ of Z-definable subsets of RV, similar to Definition 2.3.1. We will show that both definitions agree, by following the latter approach and then constructing an isomorphism between $K_b^\Gamma(Z)$ and $K_b^{\mathrm{RV}}(Z)$ using partitions as above.

The following observation simplifies the definition of the Grothendieck ring, allowing us to restrict our attention to subsets of RV_m^n for $m, n \in \mathbb{N}_{>0}$ instead of arbitrary products $\prod_i \mathrm{RV}_{m_i}^{n_i}$.

Remark 2.3.3. Given $m, n, k, \ell \in \mathbb{N}$ with $k \geq m$, $\ell \geq n$, and a Z-definable set $U \subset \mathrm{RV}_m^n$, there is a Z-definable set $\widetilde{U} \subset \mathrm{RV}_k^\ell$ which is in Z-definable bijection with U. Indeed, fix some right-inverse $r : \mathrm{AC}_m^n \to \mathrm{AC}_k^n$ of $\mathrm{ac}_m\restriction\mathrm{AC}_k^n$ and define $s : \mathrm{RV}_m^n \to \mathrm{RV}_k^n$ by

$$s(\boldsymbol{u}) = \boldsymbol{v} \quad\iff\quad \begin{aligned} &\mathrm{ac}_k(\boldsymbol{v}) = r(\mathrm{ac}_m(\boldsymbol{u})) \text{ and} \\ &\mathbf{val}(\boldsymbol{v}) = \mathbf{val}(\boldsymbol{u}). \end{aligned}$$

Since AC_m^n and AC_k^n are contained in $\mathrm{dcl}(\emptyset)$, the map r is Z-definable, and hence so is s. Thus s induces a Z-definable bijection from U to $\widetilde{U} := s(U) \times \{0\}^{\ell-n} \subset \mathrm{RV}_k^\ell$.

Definition 2.3.4. The <u>*Grothendieck ring of bounded* Z*-definable subsets of* RV</u>, denoted by $K_b^{\mathrm{RV}}(Z)$, is defined as follows:

The additive group of $K_b^{\mathrm{RV}}(Z)$ is the free abelian group generated by symbols $[U]$ for each Z-definable bounded subset $U \subset \mathrm{RV}_m^n$ for some $m \in \mathbb{N}_{>0}, n \in \mathbb{N}$, modulo the relations

(1) $[U \cup V] = [U] + [V]$, if $U, V \subset \mathrm{RV}_m^n$ are disjoint, and

(2) $[U] = [V]$, if there is a Z-definable bijection from U to V.

As above, we will write $\#U$ for the element in $K_b^{\mathrm{RV}}(Z)$ corresponding to U, called its *hypercardinality*.

The multiplication on $K_b^{\mathrm{RV}}(Z)$ is given by $\#U \cdot \#V := \#(\widetilde{U} \times \widetilde{V})$ for any two Z-definable sets $\widetilde{U} \subset \mathrm{RV}_k^n$ and $\widetilde{V} \subset \mathrm{RV}_k^\ell$ being in Z-definable bijection with U and V respectively.[a] (And it is again straightforward to check that this definition makes $K_b^\Gamma(Z)$ a ring.)

[a] cf. Remark 2.3.3

Note that we could also have defined $K_b^{\mathrm{RV}}(Z)$ as being generated by $[U]$ for Z-definable sets $U \subset \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ for varying $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^\ell$, with multiplication then given by $\#U \cdot \#V := \#(U \times V)$. However, the natural inclusion map from the ring as we defined it to the one arising from this alternative definition induces an isomorphism. Thus we will stick to our definition to simplify the notation, and we will freely use Remark 2.3.3 to assume finitely many subsets to all lie in the same RV_m^n when beneficial. Similarly, we will also just use the notation $\#U$ for subsets of RV_*^*, implicitly making use of Remark 2.3.3.

Intuitively, it is no real surprise that $K_b^{\mathrm{RV}}(Z)$ and $K_b^\Gamma(Z)$ are isomorphic, since RV_m merely consists of finitely many copies of the value group. Instead of just proving this abstractly, we explicitly construct an isomorphism for later use, making the proof longer than it might have to be.

Lemma 2.3.5. *There is an isomorphism between the two Grothendieck rings $K_b^{\mathrm{RV}}(Z)$ and $K_b^\Gamma(Z)$ induced by*

$$\#U \mapsto \sum_{\xi \in \mathrm{AC}_m^n} \# \,\mathbf{val}(U_\xi)$$

for any Z-definable subset $U \subset \mathrm{RV}_m^n$, where

$$U_\xi := U \cap \mathrm{ac}_m^{-1}(\xi) = \{\boldsymbol{u} \in U \mid \mathrm{ac}_m(\boldsymbol{u}) = \xi\}$$

for each $\xi \in \mathrm{AC}_m^n$.

Proof. Let us write $\mathrm{FrAb}(\mathrm{RV})$ for the free abelian group generated by the symbols $[U]_{\mathrm{FrAb}}$ as in Definition 2.3.4. Consider the group homomorphism $\varphi : \mathrm{FrAb}(\mathrm{RV}) \to K_b^\Gamma(Z)$ induced by $\varphi([U]_{\mathrm{FrAb}}) := \sum_{\xi \in \mathrm{AC}_m^n} \# \,\mathbf{val}(U_\xi) \in K_b^\Gamma(Z)$ for all $m, n \in \mathbb{N}$ and all Z-definable $U \subset \mathrm{RV}_m^n$. We will now first show that φ induces a homomorphism $\bar{\varphi} : K_b^{\mathrm{RV}}(Z) \to K_b^\Gamma(Z)$, then note that $\bar{\varphi}$ respects the multiplication (and hence is a ring homomorphism) and finally prove that it is bijective.

To see that φ induces a homomorphism on the quotient $K_b^{\mathrm{RV}}(Z)$ of $\mathrm{FrAb}(\mathrm{RV})$, we check that it respects the two relations from Definition 2.3.4. Regarding the relation

(1), let $U, V \subset \mathrm{RV}_m^n$ be two bounded Z-definable sets with $U \cap V = \emptyset$. Then, for each $\xi \in \mathrm{AC}_m^n$, we have

$$\begin{aligned}(U \cup V)_\xi &= (U \cup V) \cap \mathrm{ac}_m^{-1}(\xi) \\ &= (U \cap \mathrm{ac}_m^{-1}(\xi)) \cup (V \cap \mathrm{ac}_m^{-1}(\xi)) \\ &= U_\xi \cup V_\xi,\end{aligned}$$

where the two bounded Z-definable sets $U_\xi \subset U$ and $V_\xi \subset V$ are disjoint. Moreover, note that $\mathbf{val}$ is injective on $(U \cup V)_\xi$ and hence we have $\mathbf{val}((U \cup V)_\xi) = \mathbf{val}(U_\xi) \cup \mathbf{val}(V_\xi)$, where $\mathbf{val}(U_\xi)$ and $\mathbf{val}(V_\xi)$ are also disjoint (and both are bounded and Z-definable). Thus

$$\begin{aligned}\varphi([U \cup V]_{\mathrm{FrAb}}) &= \sum_\xi \# \,\mathbf{val}((U \cup V)_\xi) \\ &= \sum_\xi (\# \,\mathbf{val}(U_\xi) + \# \,\mathbf{val}(V_\xi)) \\ &= \sum_\xi \# \,\mathbf{val}(U_\xi) + \sum_\xi \# \,\mathbf{val}(V_\xi) \\ &= \varphi([U]_{\mathrm{FrAb}}) + \varphi([V]_{\mathrm{FrAb}})\end{aligned}$$

Regarding the relation (2), let $U \subset \mathrm{RV}_m^n$ and $V \subset \mathrm{RV}_k^\ell$ be two bounded Z-definable sets which are in Z-definable bijection, say via $f : U \to V$. For each $\xi \in \mathrm{AC}_m^n$ and $\theta \in \mathrm{AC}_k^\ell$, let

$$\begin{aligned}U_{\xi,\theta} &:= U_\xi \cap f^{-1}(V_\theta) \\ &= \{\boldsymbol{u} \in U \mid \mathrm{ac}_m(\boldsymbol{u}) = \xi, \mathrm{ac}_k(f(\boldsymbol{u})) = \theta\}\end{aligned}$$

and

$$\begin{aligned}V_{\xi,\theta} &:= f(U_{\xi,\theta}) = f(U_\xi) \cap V_\theta \\ &= \{\boldsymbol{v} \in V \mid \mathrm{ac}_m(f^{-1}(\boldsymbol{v})) = \xi, \mathrm{ac}_k(\boldsymbol{v}) = \theta\}.\end{aligned}$$

Then f restricts to a Z-definable bijection between the bounded Z-definable sets $U_{\xi,\theta}$ and $V_{\xi,\theta}$. Moreover, $\mathbf{val}$ is injective on both of these sets (since it is already injective on their supersets U_ξ and V_θ). Hence there is a Z-definable bijection between $\mathbf{val}(U_{\xi,\theta})$ and $\mathbf{val}(V_{\xi,\theta})$. Thus we have

$$\begin{aligned}\varphi([U]_{\mathrm{FrAb}}) &= \varphi\left(\sum [U_{\xi,\theta}]_{\mathrm{FrAb}}\right) = \sum \# \,\mathbf{val}(U_{\xi,\theta}) \\ &= \sum \# \,\mathbf{val}(V_{\xi,\theta}) = \varphi\left(\sum [V_{\xi,\theta}]_{\mathrm{FrAb}}\right) = \varphi([V]_{\mathrm{FrAb}})\end{aligned}$$

as claimed.

This shows that φ induces a group homomorphism $\bar{\varphi} : K_b^{\mathrm{RV}}(Z) \to K_b^{\Gamma}(Z)$ given by

$$\bar{\varphi} : \#U \mapsto \sum_{\xi \in \mathrm{AC}_m^n} \# \operatorname{\mathbf{val}}(U_{\xi})$$

Now note that $\bar{\varphi}$ also respects the multiplicative structure of the rings $K_b^{\mathrm{RV}}(Z)$ and $K_b^{\Gamma}(Z)$. Indeed, for two bounded Z-definable subsets $U \subset \mathrm{RV}_m^n$ and $V \subset \mathrm{RV}_k^{\ell}$ we can first assume that $m = k$ by Remark 2.3.3, and then use the identity $\operatorname{\mathbf{val}}((U \times V)_{(\xi,\theta)}) = \operatorname{\mathbf{val}}(U_{\xi} \times V_{\theta}) = \operatorname{\mathbf{val}}(U_{\xi}) \times \operatorname{\mathbf{val}}(V_{\theta})$ to see that

$$\begin{aligned} \bar{\varphi}(\#(U \times V)) &= \sum_{(\xi,\theta)} \# \operatorname{\mathbf{val}}((U \times V)_{\xi,\theta}) = \sum_{(\xi,\theta)} (\# \operatorname{\mathbf{val}}(U_{\xi}) \cdot \# \operatorname{\mathbf{val}}(V_{\theta})) \\ &= (\sum_{\xi} \# \operatorname{\mathbf{val}}(U_{\xi})) \cdot (\sum_{\theta} \# \operatorname{\mathbf{val}}(V_{\theta})) = \bar{\varphi}(\#U) \cdot \bar{\varphi}(\#V). \end{aligned}$$

We now finish the proof by showing that $\bar{\varphi}$ is bijective. Firstly, given a bounded Z-definable subset $A \subset \Gamma^n$, the set $U := \{\boldsymbol{u} \in \mathrm{RV}_1^n \mid \operatorname{\mathbf{val}}(\boldsymbol{u}) \in A, \mathrm{ac}_1(\boldsymbol{u}) = 1\}$ is bounded and Z-definable and satisfies $\bar{\varphi}(\#U) = \#A$, so $\bar{\varphi}$ is surjective. Secondly, to see that it is also injective, first note that given a Z-definable bounded subset $U \subset \mathrm{RV}_m^n$, we can find a Z-definable bounded set U' which is in Z-definable bijection with U and on which $\operatorname{\mathbf{val}}$ is injective. E.g., $U' := \{(\boldsymbol{u}, j(\mathrm{ac}_m(\boldsymbol{u}))) \mid \boldsymbol{u} \in U\} \subset \mathrm{RV}_m^{n+1}$ meets these requirements whenever $j : \mathrm{AC}_m^n \to \mathrm{RV}_m$ is a map for which $\operatorname{\mathbf{val}} \circ j$ is injective. (Since AC_m is finite, it is straightforward to find such a map j, and each choice is automatically $\emptyset$-definable as $\mathrm{AC}_m \subset \operatorname{dcl}(\emptyset)$, see Remark 2.2.4.) It follows that

$$\bar{\varphi}(\#U) = \bar{\varphi}(\#U') = \# \operatorname{\mathbf{val}}(U').$$

For $U \subset \mathrm{RV}_m^n$ and $V \subset \mathrm{RV}_k^{\ell}$ bounded Z-definable with $\bar{\varphi}(\#U) = \bar{\varphi}(\#V)$, define U' as above and V' in the same way. Then we have $\# \operatorname{\mathbf{val}}(U') = \# \operatorname{\mathbf{val}}(V')$. This implies, by [CH18, Theorem 5.2.2], that there is a Z-definable bijection between $\operatorname{\mathbf{val}}(U')$ and $\operatorname{\mathbf{val}}(V')$. Since these two sets are in Z-definable bijection with U and V respectively, $\#U = \#V$ follows. Thus $\bar{\varphi} : K_b^{\mathrm{RV}}(Z) \to K_b^{\Gamma}(Z)$ is also injective and therefore a ring isomorphism. □

As mentioned before, the Grothendieck ring $K_b^{\Gamma}(Z)$ has been comprehensively studied in [CH18], and Lemma 2.3.5 now allows us to transfer the results to $K_b^{\mathrm{RV}}(Z)$. We close this section (and chapter) with one example of such a statement that will play a prominent role in Chapter 4.

Corollary 2.3.6. *Let $A \subset \Gamma^n$ be Z-definable and let $(U_{\boldsymbol{a}})_{\boldsymbol{a} \in A}$ be a Z-definable family of bounded sets $U_{\boldsymbol{a}} \subset \mathrm{RV}_*^*$.*

Then $\#U_{\boldsymbol{a}}$ is piecewise polynomial in $\boldsymbol{a}$ with coefficients in $K_b^{\mathrm{RV}}(Z) \otimes \mathbb{Q}$, i.e., there is a partition of A into finitely many Z-definable sets A_i such that we have, *for each* i*, a polynomial* $P_i \in (K_b^{\mathrm{RV}}(Z) \otimes \mathbb{Q})[T_1, \dots, T_n]$ *with*

$$\# U_{\boldsymbol{a}} = P_i(a_1, \dots, a_n)$$

for all $\boldsymbol{a} = (a_1, \dots, a_n) \in A_i$*.*

Proof. The same statement for definable families of subsets of Γ^* instead of RV_*^*, and $K_b^{\Gamma}(Z)$ instead of $K_b^{\mathrm{RV}}(Z)$, is proven in [CH18, Proposition 5.2.1]. The claim now follows from that statement by using Lemma 2.3.5. $\square$

3 Presburger sets

In this chapter, we study definable subsets of the value group Γ. Recall that a subset of the value group is M-definable for some set $M \subset \mathrm{K} \cup \Gamma$ if and only if it is Z-definable, where $Z = \mathrm{dcl}(M) \cap \Gamma$, see Lemma 2.2.6. Since the language on Γ is the pure language of ordered abelian groups, we can work solely within the value group, i.e., we can restrict our attention to Presburger sets. Using that the Grothendieck rings $K_b^{\mathrm{RV}}(Z)$ and $K_b^{\Gamma}(Z)$ are isomorphic (by Lemma 2.3.5), the results will still be useful when we later work with subsets of the RV-sorts.

Throughout this chapter, Γ is an arbitrary $\mathbb{Z}$-group, i.e., a group that is elementary equivalent to the integers $\mathbb{Z}$ in the language of ordered abelian groups $\mathcal{L}_{\mathrm{oag}} = (0, +, <)$. As in Section 2.3, we may work under the general assumption that Z is an elementary substructure of Γ (with respect to the language $\mathcal{L}_{\mathrm{oag}}$), see Remark 2.2.2.

3.1 Basic definitions and facts

Most of the material in this section is taken from or based on [Clu03] and [CH18].

Definition 3.1.1 ([CH18, Definition 3.1.1])**.** A map $f : A \to \Gamma$, for $A \subset \Gamma^n$, is called *linear* if there are $d \in \mathbb{N}_{>0}$, $m_1, \dots, m_n \in \mathbb{Z}$, and $c \in \Gamma$ for which we have

$$f(\boldsymbol{a}) = \frac{1}{d} \cdot (m_1 \cdot a_1 + \cdots + m_n \cdot a_n + c)$$

for all $\boldsymbol{a} = (a_1, \dots, a_n) \in A$.

A map $f : A \to \Gamma \cup \{\infty\}$ is called linear if it is either linear to Γ as defined above or if $f = \mathrm{const}_A(\infty)$. (And similarly for $f : A \to \Gamma \cup \{-\infty\}$.)

A map $f : A \to (\Gamma \cup \{\infty\})^m$, for $m \in \mathbb{N}_{>0}$, is called *linear*, if $\mathrm{pr}_i \circ f : A \to \Gamma \cup \{\infty\}$ is linear for each $i = 1, \dots, m$.

Remark 3.1.2. Note that, for any $A \subset \Gamma^n$ and any linear map $f : A \to \Gamma$, there is some $k \in \mathbb{N}_{>0}$ such that $k \cdot f$ extends to a linear map on all of Γ^n.

Definition 3.1.3 (Presburger Cell, adapted from [Clu03, Definition 2]). Given $i_1, \dots, i_n \in \{0,1\}$, we define the notion of a *$(i_1, \dots, i_n)$-cell* $C \subset \Gamma^n$ recursively as follows.

(1) A *(0)-cell* is a singleton $\{a\} \subset \Gamma$.

(2) A *(1)-cell* is an infinite set of the form $(a,b)_{\equiv_d r}$, where $d \in \mathbb{N}_{>0}$, $r \in \{0, \dots, d-1\}$, $a \in \Gamma \cup \{-\infty\}$ and $b \in \Gamma \cup \{\infty\}$.

(3) An *$(i_1, \dots, i_n, 0)$-cell* is the graph of a linear function from an $(i_1, \dots, i_n)$-cell to Γ, i.e., a set of the form

$$C = \{(\boldsymbol{a}, f(\boldsymbol{a})) \in \Gamma^{n+1} \mid \boldsymbol{a} \in A\}$$

for an $(i_1, \dots, i_n)$-cell $A \subset \Gamma^n$ (called the *base* of C) and a linear function $f : A \to \Gamma$.

(4) An *$(i_1, \dots, i_n, 1)$-cell* is a set of the form

$$C = \{(\boldsymbol{a}, b) \in \Gamma^{n+1} \mid \boldsymbol{a} \in A, b \in (f(\boldsymbol{a}), g(\boldsymbol{a}))_{\equiv_d r}\}$$

for an $(i_1, \dots, i_n)$-cell $A \subset \Gamma^n$ with $\mathrm{pr}_{\leq k}(C) = A$ (called the *base* of C), some $d \in \mathbb{N}_{>0}$, $r \in \{0, \dots, d-1\}$, and linear functions $f : C \to \Gamma \cup \{-\infty\}$ and $g : C \to \Gamma \cup \{\infty\}$, such that the cardinality of the fibers $C_{\boldsymbol{a}} = \{b \in \Gamma \mid (\boldsymbol{a}, b) \in C\}$ cannot be bounded uniformly in $\boldsymbol{a} \in A$ by an integer. (Note that the fibers of C over A are non-empty, as $A = \mathrm{pr}_{\leq k}(C)$.)

For an $(i_1, \dots, i_n)$-cell $C \subset \Gamma^n$, we also say that C is a cell of *shape* $(i_1, \dots, i_n) \in \{0,1\}^n$.

Lemma 3.1.4 (Presburger Cell Decomposition, [Clu03, Theorem 1]). *Let $A \subset \Gamma^n$ be Z-definable. Then there is a partition of A into finitely many Z-definable cells.*

Corollary 3.1.5. *If $\Gamma \neq \mathbb{Z}$, then any Z-definable map $f : \Gamma^k \to \Gamma$ with $\mathrm{im}(f) \subset \mathbb{N}$ has finite image.*

Proof. The set $\mathrm{im}(f) \subset \Gamma$ is Z-definable, so Lemma 3.1.4 implies that it is a finite union of sets of the form $[a,b)_d$. No infinite such interval is contained in $\mathbb{N}$, hence the assumption that $\mathrm{im}(f) \subset \mathbb{N}$ implies that all of them must be finite. Consequently, $\mathrm{im}(f)$ must be finite. □

Up to definable bijection and finite partition, one can avoid the technical notion of cells and instead work with the following notion, as in [CH18].

Definition 3.1.6 ([CH18, Definition 3.3.1]). A *cuboid* is a subset of Γ^n of the form $\prod_{i=1}^n [0, a_i)$ for some $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \Gamma \cup \{\infty\}$.

The following criterion for divisibility in $K_b^\Gamma(Z)$ will be helpful in Chapter 4. More precisely, we will apply it to prove surjectivity of the homomorphism χ_S in Proposition 4.2.4.

Lemma 3.1.7. *Let $C \subset \Gamma^n$ be a Z-definable cuboid, i.e., $C = \prod_{i=1}^n [0, c_i)$ for some $c_i \in Z$, and let $d \in \mathbb{N}_{>0}$. Then the following are equivalent:*

(1) The element $\#C$ is divisible by d, i.e., it lies in the ideal $(d) \subset K_b^\Gamma(Z)$.

(2) There are $d_1, \dots, d_n \in \mathbb{N}_{>0}$ such that $d = \prod_{i=1}^n d_i$ and $d_i | c_i$ for all i.

(3) There is a Z-definable subset $C' \subset \Gamma^m$ for which C is in Z-definable bijection to the disjoint union of d copies of C'.

Proof. First note that all conditions trivially hold if $C = \emptyset$, so let us restrict to $C \neq 0$, hence $c_i > 0$, in the following. The implication (3) $\Rightarrow$ (1) is clear by definition, and setting $C' := \prod_{i=1}^n [0, c_i)_{d_i}$ shows the implication (2) $\Rightarrow$ (3),

The proof of the remaining implication (1) $\Rightarrow$ (2) is more involved. Let $\#C$ be a multiple of $d \in \mathbb{N}_{>0}$ and assume, without loss of generality that $d \geq 2$.

First consider the case that C is finite, so $\#C = \prod_{i=1}^n c_i \in \mathbb{N}_{>0}$ where the product is well-defined in $\Gamma \supset \mathbb{N}_{>0}$ because all c_i are natural numbers. By [CH18, Lemma 2.2.4], there is a polynomial $P \in \mathbb{Z}[x_1, \dots, x_\ell]$ and a tuple $\boldsymbol{b} \in \Gamma^\ell$ such that $1, b_1, \dots, b_\ell$ are $\mathbb{Q}$-linearly independent as elements of $\Gamma \otimes \mathbb{Q}$ and $\#C = q \cdot P(\boldsymbol{b})$. Then $(d \cdot P - \#C)(\boldsymbol{b}) = 0$, hence applying [CH18, Lemma 2.2.3 (3)] to the polynomial $d \cdot P - \#C \in \mathbb{Z}[x_1, \dots, x_\ell]$ yields $d \cdot P = \#C$. Thus P is constant and $\#C = \prod_{i=1}^n c_i$ is a multiple of d, not only in $K_b^\Gamma(Z)$, but also in $\mathbb{N}$. Hence the existence of the desired decomposition $d = \prod_{i=1}^n d_i$ with $d_i | c_i$ follows by basic number theory.

To handle the case that C is infinite, write

$$c_i = d \cdot b_i + r_i \text{ for some } r_i \in \{0, \dots, d-1\}$$

for all i and let $C_I := \prod_{i \in I} [0, d \cdot b_i) \times \prod_{i \notin I} [0, r_i)$ for $I \subset \{1, \dots, n\}$. Then C is in Z-definable bijection to the disjoint union of the 2^n many cuboids C_I for all such I, so $\#C = \sum_I \#C_I$.

By the implication (2) $\Rightarrow$ (1) applied to the cuboids C_I, we have $\#C_I \in (d)$ for all $I \neq \emptyset$, and hence also $\#C_\emptyset = \#C - \sum_{I \neq \emptyset} \#C_I \in (d)$. Since $C_\emptyset = \prod_{i=1}^n [0, r_i)$ is finite (and we already handled the finite case above), there is a decomposition $d = \prod_{i=1}^n d_i$ such that $d_i | r_i$ for all i. As $c_i = d \cdot b_i + r_i$, this already implies $d_i | c_i$ for all i, finishing the proof. ∎

We close this section with a family version of (1-dimensional) Presburger cell decomposition, Lemma 3.1.4, over a parameter set in RV_*^*. Such families will appear in Chapter 4, and the following lemma allows us to reduce to simpler cases in some of the proofs.

Lemma 3.1.8. *Let $Z \preccurlyeq \Gamma$, let $\ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^\ell_{>0}$, let $S \subset \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ be Z-definable and let $X \subset S \times \Gamma$ be a Z-definable set whose fibers over S are bounded from below. Then there is a finite partition of X into (Z-definable) sets $Y \subset S \times \Gamma$ whose fibers over S are of the form*

$$Y_{\boldsymbol{s}} = [a(\boldsymbol{s}), b(\boldsymbol{s}))_d$$

for some $d \in \mathbb{N}$ and Z-definable functions $a : S \to \Gamma$ and $b : S \to \Gamma \cup \{\infty\}$.

Proof. For each $\boldsymbol{s} \in S$, since the fiber $X_{\boldsymbol{s}}$ is bounded from below, it can be partitioned into finitely many intervals of the form $[a, b)_d$ by cell decomposition in Γ, Lemma 3.1.4. By a standard compactness argument, similar to the proof of cell decomposition in higher dimensions in [Clu03], the claim follows: Let $\varphi_{r,\boldsymbol{d}}(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{b})$, for $r \in \mathbb{N}$ and $\boldsymbol{d} \in \mathbb{N}^r$, be the $\mathcal{L}_{\mathrm{val}}(Z)$-formula which holds in K if and only if

- $\boldsymbol{s} \in S$, and
- $X_{\boldsymbol{s}}$ is the disjoint union of $[a_j, b_j)_{d_j}$ for $j = 1, \ldots, r$,

(Note that we allow $b_j = \infty$ here for simplicity; this can easily be coded as a separate case in all formulas, e.g., by $b_j < a_j$.) Then, in every model K' of $\mathrm{Th}(\mathrm{K})$ in the language $\mathcal{L}_{\mathrm{val}}(Z)$, the union of the sets $S'_{r,\boldsymbol{d}}(K') := \{\boldsymbol{s} \in \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \mid K' \models \exists \boldsymbol{a}, \boldsymbol{b} \in \Gamma'^r : \varphi_{r,\boldsymbol{d}}(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{b})\}$ is all of $\boldsymbol{S}(K')$ by one-dimensional cell decomposition in the value group Γ'. In other words, consider the language $\mathcal{L} = \mathcal{L}_{\mathrm{val}}(Z) \cup \{\boldsymbol{s}\}$, with an additional constant symbol $\boldsymbol{s}$ of sort $\mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$, and let $\psi(\boldsymbol{s})$ denote the $\mathcal{L}_{\mathrm{val}}(Z)$-formula defining S. Then the $\mathcal{L}$-theory given by

$$T = \mathrm{Th}(\mathrm{K}) \cup \{\psi(\boldsymbol{s})\} \cup \{\neg \exists \boldsymbol{a}, \boldsymbol{b} \in \Gamma^r : \varphi_{r,\boldsymbol{d}}(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{b})\},$$

is inconsistent. By compactness, there is already a finite inconsistent subset of T. In other words, and working in K again, there is a finite set of tuples $(r_i, \boldsymbol{d}_i)$ with $r_i \in \mathbb{N}$ and $\boldsymbol{d}_i \in \mathbb{N}^{r_i}$ such that S is the union of the finitely many sets $S'_{r_i,\boldsymbol{d}_i} = S'_{r_i,\boldsymbol{d}_i}(\mathrm{K})$. Write $S_i := S'_{r_i,\boldsymbol{d}_i} \setminus \bigcup_{j<i} S'_{r_j,\boldsymbol{d}_j}$, so that S is the disjoint union of the sets S_i. By Lemma 2.2.1, there are Z-definable functions $a_{i,j} : S_i \to \Gamma$ and $b_{i,j} : S_i \to \Gamma \cup \{\infty\}$ for all i and $j = 1, \ldots, r_i$, such that $\varphi_{r_i,\boldsymbol{d}_i}(\boldsymbol{s}, \boldsymbol{a}_i(\boldsymbol{s}), \boldsymbol{b}_i(\boldsymbol{s}))$ holds for all i and $\boldsymbol{s} \in S_i$, where $\boldsymbol{a}_i(\boldsymbol{s}) = (a_{i,1}(\boldsymbol{s}), \ldots, a_{i,r_i}(\boldsymbol{s}))$ and $\boldsymbol{b}_i(\boldsymbol{s}) = (b_{i,1}(\boldsymbol{s}), \ldots, b_{i,r_i}(\boldsymbol{s}))$. Write $d_{i,j}$ for the j-th entry of $\boldsymbol{d}_i \in \mathbb{N}^{r_i}$. Then the finitely many sets

$$X_{i,j} := \{(\boldsymbol{s}, c) \in S \times \Gamma \mid \boldsymbol{s} \in S_i, c \in [a_{i,j}(\boldsymbol{s}), b_{i,j}(\boldsymbol{s}))_{d_{i,j}}\}$$

are pairwise disjoint and their union is X, proving the claim. $\square$

3.2 Relative affine hulls

In this section, we introduce the notion of "affine hull" in $\mathbb{Z}$-groups. In some aspects, it is quite similar to the same (well-studied) notion in vector spaces, see, for example, Remark 3.2.5 and Lemma 3.2.9. However, in a non-standard model $\Gamma \neq \mathbb{Z}$, there are also unintuitive phenomena, one of which we illustrate in Example 3.2.8.

The most important result for the remainder of this section is the classification of affine hulls of Presburger cells, Lemma 3.2.13. We will use it in Section 3.3 to study the behavior of linear functions on Presburger cells, and we will use its Corollary 3.2.14 in Section 4.4 to prove the crucial Lemma 4.4.10.

Definition 3.2.1. Let $n \in \mathbb{N}_{>0}$ and $A \subset B \subset \Gamma^n$. The *affine hull of A in B* is the set $\operatorname{aff}_B(A) \subset B$ of all points $\boldsymbol{b} \in B$ satisfying (for all $m \in \mathbb{N}_{>0}$) the condition

$$\text{For any linear map } f : B \to \Gamma^m \text{ with } f{\restriction}A = \operatorname{const}_A(\mathbf{0}), \text{ we have } f(\boldsymbol{b}) = \mathbf{0}. \tag{3.1}$$

In other words, $\operatorname{aff}_B(A)$ is the intersection of all those kernels of linear maps on B that contain A.

We say that A is *affinely closed in B* if $\operatorname{aff}_B(A) = A$.

Remark 3.2.2. Note that we can replace Γ^m with Γ in Definition 3.2.1 (3.1) without changing the meaning of the defined notion of "affine hull".

Indeed, if $f : B \to \Gamma^m$ is linear with $f{\restriction}A = \operatorname{const}_A(\mathbf{0})$, then each of the components f_i of f are constantly zero on A, hence the condition (3.1) for $m = 1$ already implies $f_i(\boldsymbol{a}) = 0$ for all i, and thus $f(\boldsymbol{a}) = \mathbf{0}$.

Remark 3.2.3. Note that, for $A \subset B \subset C$, we have $\operatorname{aff}_B(A) = \operatorname{aff}_C(A) \cap B$. We can (and will) therefore restrict our attention to $\operatorname{aff}_{\Gamma^n}$; and we will often just write aff instead of $\operatorname{aff}_{\Gamma^n}$, since one can always infer the exponent n from the context.

Proof. If $b \in \operatorname{aff}_C(A) \cap B$ and $f : B \to \Gamma^m$ is linear with $f{\restriction}A = \operatorname{const}_A(\mathbf{0})$, an appropriate multiple $g = d \cdot f$, for $d \in \mathbb{N}_{>0}$, extends to a linear map on C with $g{\restriction}A = \operatorname{const}_A(\mathbf{0})$. Hence $d \cdot f(b) = g(b) = \mathbf{0}$ and thus $f(b) = \mathbf{0}$, showing that $b \in \operatorname{aff}_B(A)$.

The (other) inclusion $\operatorname{aff}_B(A) \subset \operatorname{aff}_C(A) \cap B$ is clear by spelling out the definitions, which we leave to the reader. ☐

While the Definition 3.2.1 is only concerned with linear maps that are constantly equal to $\mathbf{0}$, translation by arbitrary elements of Γ^m does not change anything. More precisely, the definition immediately yields the following remark.

Remark 3.2.4. Let $n \in \mathbb{N}_{>0}$, let $A \subset \Gamma^n$ be an arbitrary subset, and let $f : \operatorname{aff}(A) \to \Gamma^m$ be a linear map. If f is constant on A, then it is already constant on $\operatorname{aff}(A)$.

Moreover, this property characterizes the affine hull, i.e., it serves as an alternative definition.

Another straight-forward observation is that aff is a closure operator.

Remark 3.2.5. For all $n \in \mathbb{N}_{>0}$, the map $\operatorname{aff} : \mathcal{P}(\Gamma^n) \to \mathcal{P}(\Gamma^n)$ is a closure operator, i.e., we have

- $A \subset \operatorname{aff}(A)$,
- $\operatorname{aff}(A) \subset \operatorname{aff}(B)$, and
- $\operatorname{aff}(\operatorname{aff}(A)) = \operatorname{aff}(A)$

for all $A \subset B \subset \Gamma^n$.

Let us now calculate some affine hulls to get used to the notion.

Example 3.2.6. Singletons are affinely closed: For $\boldsymbol{a} \in \Gamma^n$, we have $\operatorname{aff}(\{\boldsymbol{a}\}) = \{\boldsymbol{a}\}$.

Example 3.2.7. Two points span everything in Γ: For $a \neq b \in \Gamma$, we have $\operatorname{aff}(\{a, b\}) = \Gamma$. Indeed, if $f : \Gamma \to \Gamma$ is linear with

$$f(a) - f(b) = m \cdot \underbrace{(a - b)}_{\neq 0} = 0$$

then we must have $m = 0$, and thus f is constant on all of Γ.

In particular, $\operatorname{aff}([a, b)_d) = \Gamma$ unless $\#[a, b)_d \leq 1$.

Example 3.2.8. Two points can span more than a line: Let $\omega \in \Gamma_{>0} \setminus \mathbb{Z}$ and $A = \{(0, 0), (\omega, \omega + 1)\}$. Then $\operatorname{aff}(A) = \Gamma^2$.

Indeed, if $f : \Gamma^2 \to \Gamma$ is linear with $f(a, b) = n \cdot a + m \cdot b + c$ and satisfies $f{\restriction}A = \operatorname{const}_A(0)$, then $c = 0$ and hence $(n + m) \cdot \omega + m = f(\omega, \omega + 1) = 0$. As $\omega \in \Gamma_{>0} \setminus \mathbb{Z}$, we must have $n + m = 0 = m$, and thus $f = \operatorname{const}_{\Gamma^2}(0)$.

Lemma 3.2.9. *Let $n \in \mathbb{N}_{>0}$. The affinely closed subsets of Γ^n containing $\mathbf{0}$ are exactly the sets of the form $\ker(M)$ for some $\ell \in \mathbb{N}$ and some matrix $M \in \mathbb{Z}^{\ell \times n}$. (Without loss of generality, we can assume $\ell \leq n$.)*

In particular, each affinely closed subset containing $\mathbf{0}$ is $\emptyset$-definable.

Before we prove Lemma 3.2.9, note that it yields a classification of all affinely closed subsets, since "being affinely closed" is translation-invariant.

Corollary 3.2.10. *Let $n \in \mathbb{N}_{>0}$. The affinely closed subsets of Γ^n are exactly the sets of the form $\ker(M) + \boldsymbol{c}$ for some $\ell \in \mathbb{N}$, some matrix $M \in \mathbb{Z}^{\ell \times n}$, and some $\boldsymbol{c} \in \Gamma^n$. (Without loss of generality, we can assume $\ell \leq n$.)*

In particular, each affinely closed subset $A \subset \Gamma^n$ is $\boldsymbol{c}$-definable for any $\boldsymbol{c} \in A$.

Proof of Lemma 3.2.9. Fix any matrix $M \in \mathbb{Z}^{\ell \times n}$ and first note that $\ker(M)$ is $\emptyset$-definable, which implies the "in particular" part.

To see that $\ker(M)$ is affinely closed, it suffices to show $\operatorname{aff}(\ker(M)) \subset \ker(M)$. Towards this end, we fix $\boldsymbol{a} \in \Gamma^n \setminus \ker(M)$ and show $\boldsymbol{a} \notin \operatorname{aff}(\ker(M))$. Indeed, since $\ker(M) = \bigcap_{i=1}^{\ell} \ker(M_i)$, where M_i denotes the i-th row of M, we have $\boldsymbol{a} \notin \ker(M_i)$ for at least one i. As the map given by M_i is linear and constantly zero on $\ker(M)$, this yields $\boldsymbol{a} \notin \operatorname{aff}(\ker(M))$, showing that $\ker(M)$ is affinely closed.

Now let $A \subset \Gamma^n$ be an arbitrary affinely closed subset containing $\mathbf{0}$. Then $A = \bigcap_{f \in \mathcal{F}} \ker(f)$ for the family $\mathcal{F} = \{f : \Gamma^n \to \Gamma \text{ linear } \mid f{\restriction}A = \operatorname{const}_A(0)\}$ of linear maps from Γ^n to Γ which are constantly zero on A, see Remark 3.2.2. We will now show that there is a finite subset $\mathcal{F}_0 \subset \mathcal{F}$ for which $A = \bigcap_{f \in \mathcal{F}_0} \ker(f)$. For each $f \in \mathcal{F}$, pick $d_f \in \mathbb{N}_{>0}$, $m_{f,1}, \dots, m_{f,n} \in \mathbb{N}$, and $c_f \in \Gamma$ such that we have

$$f(\boldsymbol{a}) = \frac{1}{d_f} \cdot (m_{f,1} \cdot a_1 + \cdots + m_{f,n} \cdot a_n + c_f)$$

for all $\boldsymbol{a} = (a_1, \dots, a_n) \in \Gamma^n$. Since $\mathbf{0} \in A$, we have $f(\mathbf{0}) = 0$, hence $c_f = 0$, for all $f \in \mathcal{F}$. Moreover, as $\ker(d_f \cdot f) = \ker(f)$, we have $A = \bigcap_{f \in \mathcal{F}'} \ker(f)$ where $\mathcal{F}'$ is the set of those $f \in \mathcal{F}$ for which $d_f = 1$. Thus A is the solution set of the homogeneous system of the (possibly infinitely many) linear equations

$$m_{f,1} \cdot a_1 + \cdots + m_{f,n} \cdot a_n = 0$$

for $f \in \mathcal{F}'$, where all of the coefficients $m_{f,j}$ are integers. By (a variant of) Gaußian elimination, at most n equations suffice, i.e., there is a subset $\mathcal{F}_0 \subset \mathcal{F}'$ with $\#\mathcal{F}_0 \leq n$ and $A = \bigcap_{f \in \mathcal{F}_0} \ker(f)$. Let $M \in \mathbb{Z}^{(\#\mathcal{F}_0) \times n}$ be the matrix whose rows are given by the coefficients $m_{f,1}, \dots, m_{f,n}$ of the elements $f \in \mathcal{F}_0$. Then we have $A = \ker(M)$ as desired. $\square$

As a consequence of Lemma 3.2.9, we easily obtain that the affine hull of a definable set is again definable with the same parameters.

Corollary 3.2.11. *The affine hull of a Z-definable set is again Z-definable.*

Proof. Let $A \subset \Gamma^n$ be Z-definable. The existence of definable Skolem functions in Γ (see also Lemma 2.2.1) yields a point $\boldsymbol{a} \in A \cap \mathrm{dcl}(Z)$. Note that $\mathrm{aff}(A) = \mathrm{aff}(A - \boldsymbol{a}) + \boldsymbol{a}$, where the right-hand side is $\boldsymbol{a}$-definable (and hence Z-definable) since $\mathrm{aff}(A - \boldsymbol{a})$ contains $\mathbf{0}$ and is thus $\emptyset$-definable. □

Note that a linear map $f : A \to \Gamma$ for $A \subset \Gamma^n$ does not need to extend to Γ^n in general, an easy example is $A = [0, \infty)_2$ and $f : a \mapsto \frac{1}{2} \cdot a$.

However, the extension of a linear map to the affine hull of its domain is unique if it exists. Let us note this for later reference.

Remark 3.2.12. Let $f : A \to \Gamma^\ell$ be a linear function for some $A \subset \Gamma^n$ and let $A \subset A' \subset \mathrm{aff}(A)$. Then there is at most one linear map on A' extending f.

Indeed, the difference of any two linear functions on A' extending f is itself linear and moreover constantly zero on A. It is thus constantly zero on $\mathrm{aff}(A) \supset A'$, yielding the claim.

Lemma 3.2.13. *Let $C \subset \Gamma^{k+1}$ be a cell. Then*

(1) If $k = 0$ and C is a (0)-cell, then $\mathrm{aff}(C) = C$.

(2) If $k = 0$ and C is a (1)-cell, then $\mathrm{aff}(C) = \Gamma$.

(3) If $k \geq 1$ and C is an $(i_1, \ldots, i_k, 0)$-cell with base A, fix $d \in \mathbb{N}_{>0}$, $m_i \in \mathbb{N}$ and $c \in \Gamma$ such that $C = \mathrm{graph}(f)$ for the linear function $f : A \to \Gamma$ given by

$$f(\boldsymbol{a}) = \frac{1}{d} \cdot \left(\sum_i m_i \cdot a_i + c\right) \text{ for all } \boldsymbol{a} \in A$$

Then $\mathrm{aff}(C)$ is the graph of the (unique) linear extension $\tilde{f} : \tilde{A} \to \Gamma$ of f to the set

$$\tilde{A} = \{\boldsymbol{a} \in \mathrm{aff}(A) \mid \sum_i m_i \cdot a_i + c \text{ is divisible by } d\} \supset A,$$

i.e., we have

$$\operatorname{aff}(C) = \{(\boldsymbol{a}, b) \in \Gamma^{k+1} \mid \boldsymbol{a} \in \tilde{A}, b = \frac{1}{d} \cdot (\sum_i m_i \cdot a_i + c)\}$$

(4) If $k \geq 1$ *and* C *is an* $(i_1, \dots, i_k, 1)$*-cell with base* A*, then* $\operatorname{aff}(C) = \operatorname{aff}(A) \times \Gamma$.

Note that the statement (3) in particular shows that $\tilde{A}$ does not depend of the exact choices of d, m_i and c (although this can also be seen directly).

Proof. We already handled the statements (1) and (2) in Example 3.2.6 and Example 3.2.7 respectively.

(3) Let C be an $(i_1, \dots, i_k, 0)$-cell with base A, and let $f : A \to \Gamma$ be the linear function with $C = \operatorname{graph}(f)$, given by

$$f(\boldsymbol{a}) = \frac{1}{d} \cdot (\sum_i m_i \cdot a_i + c),$$

and let

$$\tilde{A} = \{\boldsymbol{a} \in \operatorname{aff}(A) \mid \sum_i m_i \cdot a_i + c \text{ is divisible by } d\}.$$

As in the statement of the lemma, let $\tilde{f} : \tilde{A} \to \Gamma$ be the unique linear extension of f to $\tilde{A}$. (Note that f extends to a linear map on $\tilde{A}$ by definition, and that this extension is unique by Remark 3.2.12.)

In order to show $\operatorname{aff}(C) \subset \operatorname{graph}(\tilde{f})$, consider the linear map on Γ^{k+1} given by

$$(\boldsymbol{a}, b) \mapsto \sum_i m_i \cdot a_i + c - d \cdot b,$$

which is constantly zero on $C = \operatorname{graph}(f)$, and hence constantly zero on $\operatorname{aff}(C)$. For $(\boldsymbol{a}, b) \in \operatorname{aff}(C)$, we thus have $\sum_i m_i \cdot a_i + c = d \cdot b$, and hence $\boldsymbol{a} \in \tilde{A}$ and $(\boldsymbol{a}, b) \in \operatorname{graph}(\tilde{f})$ as claimed.

To show $\operatorname{graph}(\tilde{f}) \subset \operatorname{aff}(C)$, let $g : \operatorname{graph}(\tilde{f}) \to \Gamma$ now be an arbitrary linear function that is constant on $C = \operatorname{graph}(f)$. Then the linear map $h : \tilde{A} \to \Gamma$ given by $h(\boldsymbol{a}) := g(\boldsymbol{a}, \tilde{f}(\boldsymbol{a}))$ is constant on A, and thus already constant on $\operatorname{aff}_{\tilde{A}}(A) = \tilde{A}$. By definition of h, this means that g is constant on $\operatorname{graph}(\tilde{f})$. As g was arbitrary, the claim $\operatorname{graph}(\tilde{f}) \subset \operatorname{aff}(C)$ follows.

(4) Let C be an $(i_1, \dots, i_k, 1)$-cell with base A, i.e.,

$$C = \{(\boldsymbol{a}, b) \in \Gamma^{k+1} \mid \boldsymbol{a} \in A, b \in (-f(\boldsymbol{a}), g(\boldsymbol{a}))_{\equiv_d r}\},$$

where A is an $(i_1, \dots, i_k)$-cell, $d \in \mathbb{N}_{>0}$, $r \in \{0, \dots, d-1\}$, and f and g are linear functions from A to $\Gamma \cup \{\infty\}$ (such that the cardinality of the fibers $C_{\boldsymbol{a}}$ cannot be bounded uniformly in $\boldsymbol{a} \in A$ by an integer).

To show that $\operatorname{aff}(C) \subset \operatorname{aff}(A) \times \Gamma$, let $\boldsymbol{a} \in \Gamma^k \setminus \operatorname{aff}(A)$. By definition, there is then some linear function $h : \Gamma^k \to \Gamma$ with $h \restriction A = \operatorname{const}_A(0)$ and $h(\boldsymbol{a}) \neq 0$. Consider the linear map $g : \Gamma^{k+1} \to \Gamma$ given by $g = h \circ \operatorname{pr}_{\leq k}$. Then we have $g \restriction C = \operatorname{const}_C(0)$, but $g(\boldsymbol{a}, b) = h(\boldsymbol{a}) \neq 0$ for all $b \in \Gamma$. Hence we have $(\boldsymbol{a}, b) \notin \operatorname{aff}(C)$ for all $b \in \Gamma$. Since this holds for all $\boldsymbol{a} \notin \operatorname{aff}(A)$, we have $\operatorname{aff}(C) \subset \operatorname{aff}(A) \times \Gamma$.

For the other direction, let $f : \Gamma^{k+1} \to \Gamma$ be linear with $f \restriction C = \operatorname{const}_C(0)$, say $f(\boldsymbol{a}, b) = \frac{1}{d} \cdot (\sum_{i=1}^k m_i \cdot a_i + n \cdot b + c)$ for $d \in \mathbb{N}_{>0}$, $m_1, \dots, m_k, n \in \mathbb{Z}$ and $c \in \Gamma$. Take some $\boldsymbol{a} \in A$ for which there are (at least) two distinct elements $b, b' \in \Gamma$ with $(\boldsymbol{a}, b), (\boldsymbol{a}, b') \in C$. Then we have $f(\boldsymbol{a}, b) - f(\boldsymbol{a}, b') = n \cdot (b - b')$, and the former vanishes since f is constant on C. Thus $n = 0$, meaning that f is constant on $A \times \Gamma$. Now consider the linear map $f(\bullet, b) : \Gamma^k \to \Gamma$ for any $b \in \Gamma$. It is constant on A, and hence constant on $\operatorname{aff}(A)$. Since the value of $f(\boldsymbol{a}, b)$ does not depend on b at all (as $n = 0$), this implies that f is constant on $\operatorname{aff}(A) \times \Gamma$, showing $\operatorname{aff}(C) \supset \operatorname{aff}(A) \times \Gamma$. □

Let us close this section with two Corollaries that both follow from Lemma 3.2.13 by induction on the ambient dimension n. (Recall that the shape of a Presburger cell is the tuple $(i_1, \dots, i_n) \in \{0, 1\}^n$ for which it is a $(i_1, \dots, i_n)$-cell, see Definition 3.1.3.)

Corollary 3.2.14. *If $C' \subset C \subset \Gamma^n$ are two cells of the same shape, then* $\operatorname{aff}(C') = \operatorname{aff}(C)$.

Corollary 3.2.15. *For any cell $C \subset \Gamma^n$, the affine hull $\operatorname{aff}(C)$ is a cell of the same shape (and hence also of the same dimension) as C.*

3.3 Linear functions on Presburger cells

The main purpose of this section is establishing Proposition 3.3.3, a trichotomy result saying that any linear function on a Presburger cell is either constant, or its value is infinitely small or infinitely big on a subcell of the same shape. This will be used later in the proof of the crucial Lemma 4.4.10.

Remark 3.3.1. Any Z-definable linear function $\ell : A \to \Gamma$ on a bounded (and necessarily Z-definable) subset $A \subset \Gamma^n$ is bounded. In particular, ℓ assumes a minimum and a maximum on A, both of which are Z-definable.

Indeed, if A is a Presburger cell, this statement can easily be expressed as a scheme of parameter-free Presburger formulas (parameterized by the coefficients of ℓ and of the linear functions describing the cell), each of which is trivially satisfied in $\mathbb{Z}$ since "bounded" just means "finite" in $\mathbb{Z}^n$. The statement for general A and the "in particular" part follow by cell decomposition applied to A and $\ell(A)$ respectively.

Recall that we use $<$ not only to compare elements of Γ, but also to compare subsets, in the following sense.

Notation 3.3.2. For $A, B \subset \Gamma$, we write $A < B$ if $a < b$ for all $a \in A$ and all $b \in B$. We also write $a < B$ for $\{a\} < B$ and $a > B$ for $\{a\} > B$.

Let us now state the trichotomy result mentioned above.

Proposition 3.3.3. *Let $C \subset \Gamma^n$ be a bounded cell and let $\ell : C \to \Gamma$ be a linear function. Then there is a bounded cell $C' \subset C$ of the same shape as C for which at least one of the following holds*

(1) $\ell(C') > \mathbb{Z}$, or

(2) $\ell(C') < \mathbb{Z}$, or

(3) ℓ is constant on C' (and hence on all of C).

Moreover, if C is Z-definable, then we can choose C' to be Z-definable.

We will prove it together with the following auxiliary lemma by mutual induction on n, using each of the two statements in the proof of the other only in the induction step and only for smaller n.

Lemma 3.3.4. *Let $C \subset \Gamma^n$ be a bounded cell, let $\ell : C \to \Gamma$ be a linear function and let*[a] *$\omega \in \Gamma_{>0} \setminus \mathbb{Z}$. Then there is a bounded cell $C' \subset C$ of the same shape as C for which we have*

$$\max \ell(C') - \min \ell(C') < \omega.$$

Moreover, if C is Z-definable and $\omega \in Z$, we can choose C' to be Z-definable.

[a] Note that we deviate from our convention to use latin letters for the elements of Γ here to emphasize that ω is required to be a non-standard integer.

Proof of Lemma 3.3.4. Without loss of generality, we can assume that $\ell(\boldsymbol{c}) = \frac{1}{k}(m_1 \cdot c_1 + \cdots + m_n \cdot c_n)$ for all $\boldsymbol{c} \in C$, since an absolute term does not influence the difference of maximum and minimum. Moreover, it suffices to consider the case $k = 1$. The general case then follows, as we always have $\max \ell(C') - \min \ell(C') \geq 0$.

Let $C \subset \Gamma^n$ be a bounded cell and proceed by induction on n.

Induction base, $n = 1$. If C is a (0)-cell or if $m_1 = 0$, then ℓ is constant, so choosing $C' = C$ finishes the proof. We now establish the claim in case C is (1)-cell and $m_1 \neq 0$. Write $C = (a, b)_{\equiv_d r}$ for some $a, b \in \Gamma$ with $b - a > \mathbb{Z}$. Let $\delta \in \Gamma$ be the maximal element still satisfying $|m_1| \cdot \delta < \omega$, set $b' = \min(b,\ a + \delta)$, and consider the bounded (1)-cell

$$C' = (a, b')_{\equiv_d r} \subset C.$$

Note that C' is Z-definable if C is Z-definable and $\omega \in Z$. Moreover, if $m_1 > 0$, we have

$$\begin{aligned} \min \ell(C') &> |m_1| \cdot a \text{ and} \\ \max \ell(C') &< |m_1| \cdot \min(b,\ a + \delta), \end{aligned}$$

and if $m_1 < 0$, we have

$$\begin{aligned} \min \ell(C') &> -|m_1| \cdot \min(b,\ a + \delta) \text{ and} \\ \max \ell(C') &< -|m_1| \cdot a. \end{aligned}$$

In both cases, this yields

$$\begin{aligned} \max \ell(C') - \min \ell(C') &< |m_1| \cdot (\min(b,\ a + \delta) - a) \\ &= |m_1| \cdot \min((b - a),\ \delta) \\ &\leq |m_1| \cdot \delta \\ &< \omega, \end{aligned}$$

as claimed.

Induction step. Let $A = \mathrm{pr}_{\leq n-1}(C) \subset \Gamma^{n-1}$ be the base of C. Consider the linear map $\ell' : A \to \Gamma$ given by $\ell'(\boldsymbol{a}) := m_1 \cdot a_1 + \cdots + m_{n-1} \cdot a_{n-1}$ for $\boldsymbol{a} = (a_1, \ldots, a_{n-1}) \in A$, so that $\ell(\boldsymbol{a}, b) = \ell'(\boldsymbol{a}) + m_n \cdot b$ for all $(\boldsymbol{a}, b) \in C$. We now proceed by a case distinction based on the shape of C.

Case 1: C is an $(i_1, \ldots, i_{n-1}, 0)$-cell. Then we have $C = \mathrm{graph}(f {\restriction} A)$ for some linear function $f : A \to \Gamma$. Let $\delta > \mathbb{Z}$ be maximal with $(|m_n| + 1) \cdot \delta < \omega$. By the induction hypothesis, there is then a bounded cell $A' \subset A$ with

$$\begin{aligned} \max \ell'(A') - \min \ell'(A') &< \delta, \text{ and} \\ \max f(A') - \min f(A') &< \delta, \end{aligned}$$

which can be chosen to be Z-definable if C is Z-definable. Consider the bounded $(i_1, \ldots, i_{n-1}, 0)$-cell

$$C' = \mathrm{graph}(f {\restriction} A') = C \cap (A' \times \Gamma),$$

which is Z-definable if C and ω are. We now claim that C' is as desired, i.e., that $\max \ell(C') - \min \ell(C') < \omega$. If $m_n \geq 0$, we have

$$\begin{aligned} \min \ell(C') &\geq \min \ell'(A') + |m_n| \cdot \min f(A'), \text{ and} \\ \max \ell(C') &\leq \max \ell'(A') + |m_n| \cdot \max f(A'), \end{aligned}$$

and if $m_n < 0$, we have

$$\begin{aligned}\min \ell(C') &\geq \min \ell'(A') - |m_n| \cdot \max f(A'), \text{ and} \\ \max \ell(C') &\leq \max \ell'(A') - |m_n| \cdot \min f(A').\end{aligned}$$

In both cases, this yields

$$\begin{aligned}&\max \ell(C') - \min \ell(C') \\ &\leq \underbrace{\max \ell'(A') - \min \ell'(A')}_{<\delta} + |m_n| \cdot (\underbrace{\max f(A') - \min f(A')}_{<\delta}) \\ &< (|m_n| + 1) \cdot \delta < \omega,\end{aligned}$$

as claimed.

Case 2: C is an $(i_1, \ldots, i_{n-1}, 1)$-cell. Then there are linear functions $f : A \to \Gamma$ and $g : A \to \Gamma$, whose difference $g - f$ is non-negative and cannot be bounded by an integer on A, with

$$C = \{(\boldsymbol{a}, b) \in A \times \Gamma \mid b \in (f(\boldsymbol{a}), g(\boldsymbol{a}))_{\equiv_d r}\}.$$

By Proposition 3.3.3 (for $A \subset \Gamma^{n-1}$), there is a bounded cell $A' \subset A$ of the same shape as A for which $(g - f)(A') > \mathbb{Z}$, and which can be chosen to be Z-definable if C is. Let $\delta > \mathbb{Z}$ be maximal with $\delta < \min(g - f)(A')$ and $(2 \cdot |m_n| + 1) \cdot \delta < \omega$. By the induction hypothesis, there is then a bounded cell $A'' \subset A'$ of the same shape as A', for which we have

$$\begin{aligned}\max \ell'(A'') - \min \ell'(A'') &< \delta, \text{ and} \\ \max f(A'') - \min f(A'') &< \delta,\end{aligned}$$

and which can be chosen to be Z-definable if C is Z-definable. Consider the bounded $(i_1, \ldots, i_{n-1}, 1)$-cell

$$C' = \{(\boldsymbol{a}, b) \in A'' \times \Gamma \mid b \in (f(\boldsymbol{a}), f(\boldsymbol{a}) + \delta)_{\equiv_d r}\} \subset C,$$

and note that it is Z-definable if both C and ω are. We now claim that C' is as desired, i.e., that $\max \ell(C') - \min \ell(C') < \omega$. If $m_n \geq 0$, we have

$$\begin{aligned}\min \ell(C') &\geq \min \ell'(A'') + |m_n| \cdot \min f(A''), \text{ and} \\ \max \ell(C') &\leq \max \ell'(A'') + |m_n| \cdot (\max f(A'') + \delta),\end{aligned}$$

and if $m_n < 0$, we have

$$\begin{aligned}\min \ell(C') &\geq \min \ell'(A'') - |m_n| \cdot (\max f(A'') + \delta), \text{ and} \\ \max \ell(C') &\leq \max \ell'(A'') - |m_n| \cdot \min f(A'').\end{aligned}$$

In both cases, this yields the claim, since

$$\begin{aligned}
&\max \ell(C') - \min \ell(C') \\
&\quad \le \underbrace{\max \ell'(A'') - \min \ell'(A'')}_{<\delta} + |m_n| \cdot \underbrace{(\max f(A'') + \delta - \min f(A''))}_{<2\delta} \\
&\quad < (2 \cdot |m_n| + 1) \cdot \delta \\
&\quad < \omega
\end{aligned}$$ □

Proof of Proposition 3.3.3. Let $C \subset \Gamma^n$. We establish the claim by induction on the ambient dimension $n \in \mathbb{N}_{>0}$.

Induction base, $n = 1$. If C is a (0)-cell, then it is a singleton, so ℓ is constant on C. Now let us assume that C is a (1)-cell. Write $C = (a, b)_{\equiv_d r}$ for some $a, b \in \Gamma$, $d \in \mathbb{N}_{>0}$, and $r \in \{0, \dots, d-1\}$ with $b - a > \mathbb{Z}$. We have $\ell(x) = \frac{1}{k} \cdot (m \cdot x + c)$ for some $k \in \mathbb{N}_{>0}$, some $m \in \mathbb{Z}$, and some $c \in \Gamma$. Moreover, we can assume that ℓ is non-constant, i.e., that $m \neq 0$ – otherwise, choosing $C' = C$ finishes the proof. Let δ be the unique element of Γ for which $3 \cdot \delta \in \{b - a - 1, b - a - 2, b - a - 3\}$. As $b - a > \mathbb{Z}$, we then have $\delta > \mathbb{Z}$. Consider the two bounded (1)-cells

$$C_0 := (a, a + \delta)_{\equiv_d r} \qquad \text{and} \qquad C_1 := (a + 2\delta, b)_{\equiv_d r},$$

and let $\sigma = \operatorname{sign}(m)$. Note that we have $a, b, \delta \in Z$ if C is Z-definable, hence both C_0 and C_1 are Z-definable if C is. We now claim that we must have at least one of $\sigma \cdot \ell(C_0) < \mathbb{Z}$ or $\sigma \cdot \ell(C_1) > \mathbb{Z}$. Suppose, towards a contradiction, that neither of these inequalities hold. Then there are $c_0 \in C_0$, $c_1 \in C_1$, and $k_0, k_1 \in \mathbb{Z}$, such that $\sigma \cdot \ell(c_0) \geq k_0$ and $\sigma \cdot \ell(c_1) \leq k_1$. We have

$$k_1 - k_0 \geq \sigma \cdot (\ell(c_1) - \ell(c_0)) = \frac{|m|}{k} \cdot \underbrace{(c_1 - c_0)}_{\geq \delta > \mathbb{Z}} > \mathbb{Z},$$

which is a contradiction to $k_0, k_1 \in \mathbb{Z}$. Hence, for $C' = C_0$ or $C' = C_1$, we have $\ell(C') < \mathbb{Z}$ or $\ell(C') > \mathbb{Z}$ as desired.

Induction step. Let $A = \operatorname{pr}_{\leq n-1}(C) \subset \Gamma^n$ be the base of C. As before, we proceed by a case distinction based on the shape of C.

Case 1: C is an $(i_1, \dots, i_{n-1}, 0)$-cell. Then $C = \operatorname{graph}(f)$ for some linear function $f : A \to \Gamma$, and the linear function

$$\begin{aligned}
\ell' : A &\to \Gamma \\
\boldsymbol{a} &\mapsto \ell(\boldsymbol{a}, f(\boldsymbol{a}))
\end{aligned}$$

satisfies $\ell(\boldsymbol{a}, b) = \ell'(\boldsymbol{a})$ for all $(\boldsymbol{a}, b) \in C$. Hence, if ℓ' is constant on A, then ℓ is constant on C and we are done. Otherwise, the induction hypothesis yields a bounded cell $A' \subset A$ of the same shape as A for which we have either $\ell'(A') < \mathbb{Z}$ or $\ell'(A') > \mathbb{Z}$, and which can be chosen to be Z-definable if C is. Consider the bounded $(i_1, \ldots, i_{n-1}, 0)$-cell $C' = \mathrm{graph}(f{\restriction}A') \subset C = C \cap (A' \times \Gamma)$. As it satisfies $\ell(C') = \ell'(A')$, the cell C', which is Z-definable if C is Z-definable, is as desired.

Case 2: C is an $(i_1, \ldots, i_{n-1}, 1)$-cell. Then there are linear functions $f : A \to \Gamma$ and $g : A \to \Gamma$, whose difference $g-f$ is non-negative and cannot be bounded by an integer on A, with

$$C = \{(\boldsymbol{a}, b) \in A \times \Gamma \mid b \in (f(\boldsymbol{a}), g(\boldsymbol{a}))_{\equiv_d r}\}.$$

Write $\ell(\boldsymbol{a}, b) = \frac{1}{k} \cdot (\ell'(\boldsymbol{a}) + m \cdot b)$ for some linear map $\ell' : A \to \Gamma$, some $k \in \mathbb{N}_{>0}$, and some $m \in \mathbb{Z}$. If $m = 0$, note that the induction hypothesis applied to ℓ' yields a bounded $(i_1, \ldots, i_{n-1})$-cell $A'' \subset A$ (which can be chosen to be Z-definable if C is) for which $C' = C \cap (A'' \times \Gamma)$ then is as desired. So let us from now on assume that $m \neq 0$.

Applying the induction hypothesis to the linear function $g - f$ on the bounded cell $A \subset \Gamma^{n-1}$ yields a bounded cell $A' \subset A$ of the same shape as A for which $(g - f)(A') > \mathbb{Z}$ (and which can again be chosen to be Z-definable if C is Z-definable). Note that $(g - f)(A') > \mathbb{Z}$ has a minimum in Γ by Remark 3.3.1 and let $\delta \in \Gamma$ be the maximal element still satisfying $5 \cdot \delta < \min(g - f)(A')$. Moreover, note that if C is Z-definable, then so is δ (i.e., we have $\delta \in Z$). Applying Lemma 3.3.4 twice (to $A' \subset \Gamma^{n-1}$ and a subset of A' obtained in the process), we can replace A' by a bounded subcell of the same shape for which we have

$$\begin{aligned} \max f(A') - \min f(A') &< \delta \text{ and} \\ \max \ell'(A') - \min \ell'(A') &< \delta. \end{aligned}$$

Similarly to the second case in the proof of the induction base above, consider the two bounded $(i_1, \ldots, i_{n-1}, 1)$-cells

$$\begin{aligned} C_0 &= \{(\boldsymbol{a}, b) \in A' \times \Gamma \mid b \in (f(\boldsymbol{a}), f(\boldsymbol{a}) + \delta)_{\equiv_d r}\} \text{ and} \\ C_1 &= \{(\boldsymbol{a}, b) \in A' \times \Gamma \mid b \in (f(\boldsymbol{a}) + 4\delta, g(\boldsymbol{a}))_{\equiv_d r}\}, \end{aligned}$$

both of which are subcells of C, and both of which are Z-definable if C is. For any choice of $(\boldsymbol{a}_0, b_0) \in C_0$ and $(\boldsymbol{a}_1, b_1) \in C_1$, we then have

$$\begin{aligned} b_1 - b_0 &> f(\boldsymbol{a}_1) + 4\delta - (f(\boldsymbol{a}_0) + \delta) \\ &= \underbrace{f(\boldsymbol{a}_1) - f(\boldsymbol{a}_0)}_{>-\delta} + 3\delta \\ &> 2\delta, \end{aligned}$$

and thus

$$\begin{aligned}
&\operatorname{sign}(m)\cdot(\ell(\boldsymbol{a}_1,b_1)-\ell(\boldsymbol{a}_0,b_0))\\
&\quad=\operatorname{sign}(m)\cdot\frac{1}{k}\cdot(\underbrace{\ell'(\boldsymbol{a}_1)-\ell'(\boldsymbol{a}_0)}_{\in(-\delta,\delta)}+m\cdot(b_1-b_0))\\
&\quad=\frac{1}{k}\cdot(\underbrace{\operatorname{sign}(m)\cdot(\ell'(\boldsymbol{a}_1)-\ell'(\boldsymbol{a}_0))}_{\geq-\delta}+|m|\cdot\underbrace{(b_1-b_0)}_{>2\delta})\\
&\quad>\frac{2\,|m|-1}{k}\cdot\delta\\
&\quad>\mathbb{Z}.
\end{aligned}$$

Just as in the proof of the induction base above, we now claim that at least one of the two inequalities $\operatorname{sign}(m)\cdot\ell(C_0)<\mathbb{Z}$ or $\operatorname{sign}(m)\cdot\ell(C_1)>\mathbb{Z}$ hold. Indeed, if both are false, we have $\operatorname{sign}(m)\cdot\ell(\boldsymbol{a}_0,b_0)\geq k_0$ and $\operatorname{sign}(m)\cdot\ell(\boldsymbol{a}_1,b_1)\leq k_1$ for some $(\boldsymbol{a}_0,b_0)\in C_0$, some $(\boldsymbol{a}_1,b_1)\in C_1$, and $k_0,k_1\in\mathbb{Z}$, leading to

$$k_1-k_0\geq\operatorname{sign}(m)\cdot(\ell(\boldsymbol{a}_1,b_1)-\ell(\boldsymbol{a}_0,b_0))>\mathbb{Z},$$

which is a contradiction. □

Lemma 3.3.5. *Suppose that $Z\neq\mathbb{Z}$. If C is a Z-definable cell, then we can find a bounded Z-definable cell $C'\subset C$ of the same shape as C.*

In particular, if $Z\neq\mathbb{Z}$, we can remove the assumption that C is bounded in the statements of Lemma 3.3.4 and Proposition 3.3.3.

Before we continue with the proof, let us point out that the assumption $Z\neq\mathbb{Z}$ is really necessary: Note that any $\mathbb{Z}$-definable subset of Γ is either finite or equals Γ. Thus any Z-definable cell $C\subset\Gamma$ is either a singleton or equals Γ. In particular, $C=\Gamma$ (which is an unbounded $\mathbb{Z}$-definable (1)-cell) does not contain a bounded $\mathbb{Z}$-definable (1)-cell.

Proof. Let $\omega\in Z$ with $\omega>\mathbb{Z}$. Let $C\subset\Gamma^n$ be a Z-definable cell and continue by induction on n.

Induction base, $n=1$. Write $C=(a,b)_{\equiv_d r}$ for some $d\in\mathbb{N}_{>0}$, $r\in\{0,\dots,d-1\}$, and $a\in\Gamma\cup\{-\infty\}$ and $b\in\Gamma\cup\{\infty\}$. In case $a,b\in\Gamma$, the cell C is already bounded, so there is nothing to show. In the remaining cases, C is a (1)-cell. Define

$$C'=\begin{cases}(-\omega,\omega)_{\equiv_d r}, & \text{if } a=-\infty \text{ and } b=\infty\\ (a,a+\omega)_{\equiv_d r}, & \text{if } a\in\Gamma \text{ and } b=\infty\\ (b-\omega,b)_{\equiv_d r}, & \text{if } a=-\infty \text{ and } b\in\Gamma\end{cases}.$$

Then, in all these cases, C' is a bounded Z-definable (1)-cell contained in C, as desired.

Induction step. Write $C = \{(\boldsymbol{a}, b) \in A \times \Gamma \mid b \in (f(\boldsymbol{a}), g(\boldsymbol{a}))_{\equiv_d r}\}$ for some $d \in \mathbb{N}_{>0}$, $r \in \{0, \dots, d-1\}$, and Z-definable linear functions $f : A \to \Gamma \cup \{-\infty\}$ and $g : A \to \Gamma \cup \{\infty\}$. By the induction hypothesis, there is a bounded Z-definable cell $A' \subset A$ of the same shape as A.

If $f(\boldsymbol{a}) \in \Gamma$ and $g(\boldsymbol{a}) \in \Gamma$ for all $\boldsymbol{a} \in A'$, consider the Z-definable cell $C' := C \cap (A' \times \Gamma)$. By Remark 3.3.1, f and g assume a minimum and maximum on A' respectively, so $C' \subset A' \times (\min f(A'), \max g(A'))_{\equiv_d r}$ is bounded. Moreover, the difference $g - f$ is a linear function on A and hence on A'. Thus, Proposition 3.3.3 yields a bounded cell $A'' \subset A'$ of the same shape as A, whose image under $g - f$ is either contained in $\Gamma_{>0} \setminus \mathbb{Z}$ or a singleton in $\mathbb{Z}$. In the first case, $g - f$ cannot be bounded by an integer on A', so C and C' are both of shape $(i_1, \dots, i_{n-1}, 1)$. In the second case, $g - f$ is constant on A with an integer value, since we have $\operatorname{aff}(A) = \operatorname{aff}(A'')$ by Corollary 3.2.14. Therefore, both C and C' are of shape $(i_1, \dots, i_{n-1}, 0)$. In both cases, $C' \subset C$ is a bounded Z-definable cell of the same shape as C, as desired. We have thus established the claim for $n > 1$ if f and g only take finite values.

In the remaining cases, note that C is always an $(i_1, \dots, i_{n-1}, 1)$-cell and consider the set $C' \subset A' \times \Gamma$ whose fibers $C'_{\boldsymbol{a}} = \{b \in \Gamma \mid (\boldsymbol{a}, b) \in C'\}$ over A' are given, for all $\boldsymbol{a} \in A'$, by

$$C'_{\boldsymbol{a}} = \begin{cases} [-\omega, \omega]_{\equiv_d r}, & \text{if } f = \mathrm{const}_{\Gamma^n}(-\infty) \text{ and } g = \mathrm{const}_{\Gamma^n}(\infty) \\ (f(\boldsymbol{a}), f(\boldsymbol{a}) + \omega)_{\equiv_d r}, & \text{if } f(A') \subset \Gamma \text{ and } g = \mathrm{const}_{\Gamma^n}(\infty) \\ (g(\boldsymbol{a}) - \omega, g(\boldsymbol{a}))_{\equiv_d r}, & \text{if } f = \mathrm{const}_{\Gamma^n}(-\infty) \text{ and } g(A') \subset \Gamma \end{cases}$$

Then, in all these cases, C' is a bounded Z-definable $(i_1, \dots, i_{n-1}, 1)$-cell contained in C, as desired. □

Recall that a linear function which is constantly equal to zero on some Presburger cell (or even some arbitrary subset of Γ^n, in general) is already constantly equal to zero on its affine hull. (This is just how we defined the affine hull, see Definition 3.2.1.) We close this section with proving the same result for polynomials instead of linear functions.

Lemma 3.3.6. *Let $C \subset \Gamma^n$ be a cell and let $P \in R[T_1, \dots, T_n]$ be a polynomial with $P(C) = 0$, where R is some integral domain whose additive group contains Γ as a subgroup. Then we already have $P(\operatorname{aff}(C)) = 0$.*

Proof. We establish the claim by induction on the ambient dimension $n \in \mathbb{N}_{>0}$, making use of Lemma 3.2.13 for the explicit description of $\operatorname{aff}(C)$.

Induction base, $n = 1$. If C is a (0)-cell, then $\operatorname{aff}(C) = C$, so there is nothing to show. If C is a (1)-cell, then it is infinite, so by the assumption $P(C) = 0$, the polynomial P has infinitely many zeros. Hence it must be the constant zero polynomial, showing that $P(\operatorname{aff}(C)) = P(\Gamma) = 0$.

Induction step. Let $A = \mathrm{pr}_{\leq n-1}(C)$ be the base of C. We proceed by a case distinction on the shape of C.

Case 1: C is an $(i_1, \dots, i_{n-1}, 0)$-cell. Let $f : A \to \Gamma$ be the linear function with $C = \mathrm{graph}(f)$, given by

$$f(\boldsymbol{a}) = \frac{1}{d} \cdot \left(\sum_{i=1}^{n-1} m_i \cdot a_i + c \right) \text{ for all } \boldsymbol{a} \in A$$

for some $d \in \mathbb{N}_{>0}$, $m_i \in \mathbb{N}$ and $c \in \Gamma$. Let $\tilde{A} \subset \mathrm{aff}(A)$ and $\tilde{f} : \tilde{A} \to \Gamma$ be as in Lemma 3.2.13 (3), i.e., so that $\mathrm{aff}(C) = \mathrm{graph}(\tilde{f})$.

Consider the polynomials $F, Q \in \mathrm{Frac}(R)[T_1, \dots, T_{n-1}]$ given by

$$F(T_1, \dots, T_{n-1}) = \frac{1}{d} \cdot (m_1 \cdot T_1 + \dots + m_{n-1} \cdot T_{n-1} + c) \text{ and}$$
$$Q(T_1, \dots, T_{n-1}) = P(T_1, \dots, T_{n-1}, F(T_1, \dots, T_{n-1})).$$

Fix some element $r \in R$ such that $r \cdot Q \in R[T_1, \dots, T_{n-1}]$ and note that we have $Q(A) = 0$, so the induction hypothesis applied to $r \cdot Q$ yields $Q(\mathrm{aff}(A)) = 0$. Since $\mathrm{aff}(C) = \mathrm{graph}(\tilde{f})$ by Lemma 3.2.13, we have $c_n = \tilde{f}(c_1, \dots, c_{n-1}) = F(c_1, \dots, c_{n-1})$ for all $\boldsymbol{c} = (c_1, \dots, c_n) \in \mathrm{aff}(C)$, and thus $P(\mathrm{aff}(C)) = Q(\tilde{A}) = 0$. As $\tilde{A}$ is a subset of $\mathrm{aff}(A)$ and $Q(\mathrm{aff}(A)) = 0$, the claim follows.

Case 2: C is an $(i_1, \dots, i_{n-1}, 1)$-cell. Let $f : A \to \Gamma \cup \{-\infty\}$ and $g : A \to \Gamma \cup \{\infty\}$ be linear functions with

$$C = \{(\boldsymbol{a}, b) \in A \times \Gamma \mid b \in (f(\boldsymbol{a}), g(\boldsymbol{a}))_{\equiv_d r}\}$$

for some $d \in \mathbb{N}$ and $r \in \{0, \dots, d-1\}$. As C is an $(i_1, \dots, i_{n-1}, 1)$-cell, $g - f$ then cannot be bounded by an integer on A. By Proposition 3.3.3, there is an $(i_1, \dots, i_{n-1})$-cell $A' \subset A$ for which $(g - f)(A') > \mathbb{Z}$. Fix some $\boldsymbol{a} = (a_1, \dots, a_{n-1}) \in A'$. Note that the set $(\{\boldsymbol{a}\} \times \Gamma) \cap C$ is infinite and consider the polynomial $Q_{\boldsymbol{a}} \in R[T_n]$ given by

$$Q_{\boldsymbol{a}}(T_n) = P(a_1, \dots, a_{n-1}, T_n).$$

Then we have $Q_{\boldsymbol{a}}(b) = 0$ for all infinitely many b with $(\boldsymbol{a}, b) \in C$. Hence $Q_{\boldsymbol{a}}$ has infinitely many zeros, so it must be the zero polynomial. By the definition of $Q_{\boldsymbol{a}}$ and since the argument above works for all $\boldsymbol{a} \in A'$, we thus have

$$P(a_1, \dots, a_{n-1}, b) = Q_{\boldsymbol{a}}(b) = 0$$

for all $\boldsymbol{a} \in A'$ and all $b \in \Gamma$. Now fix $b \in \Gamma$ and consider the polynomial $Q^b \in R[T_1, \dots, T_{n-1}]$ given by

$$Q^b(T_1, \dots, T_{n-1}) = P(T_1, \dots, T_{n-1}, b).$$

Then $Q^b(A') = 0$, so the induction hypothesis implies that $Q^b(\mathrm{aff}(A')) = 0$. Since A and A' are of the same shape, we have $\mathrm{aff}(A) = \mathrm{aff}(A')$, hence $Q^b(\mathrm{aff}(A)) = 0$. By the definition of Q^b and since the argument above works for all $b \in \Gamma$, we thus have

$$P(\boldsymbol{a}, b) = Q^b(\boldsymbol{a}) = 0$$

for all $(\boldsymbol{a}, b) \in \mathrm{aff}(A) \times \Gamma$. As $\mathrm{aff}(A) \times \Gamma = \mathrm{aff}(C)$ by Lemma 3.2.13, this finishes the proof. □

Corollary 3.3.7. *If $C' \subset C \subset \Gamma^n$ are two cells of the same shape and $P \in R[T_1, \dots, T_n]$ is a polynomial with $P(C') = 0$, where R is some integral domain whose additive group contains Γ as a subgroup, then $P(C) = 0$.*

Proof. By Lemma 3.3.6 we have $P(\mathrm{aff}(C')) = 0$, and by Corollary 3.2.14 we have $\mathrm{aff}(C') = \mathrm{aff}(C) \supset C$, implying the claim. □

Remark 3.3.8. Note that the assumption $\Gamma \subset R$ in Lemma 3.3.6 is only used twice in the proof, namely to ensure that F and Q (in Case 1) and $Q_{\boldsymbol{a}}$ and Q^b (in Case 2) are polynomials over $\mathrm{Frac}(R)$ and R, respectively.

It is straightforward to adapt the proof at these places to work under the weaker assumption that there is, for each $\boldsymbol{a} = (a_1, \dots, a_n) \in \Gamma^n$, an integral domain $R(\boldsymbol{a}) \supset R$ whose additive group contains the subgroup of Γ generated by the set $\{a_1, \dots, a_n\}$.

This is the situation in which we will apply Corollary 3.3.7 later on.

4 Integrable functions on RV_*^*

We now have all the necessary tools to start developing an integration theory for (definable and integrable) functions from subsets of RV_*^* to the (multiplicative) value group p^{Γ}. This is a major building block and first step towards developing integration of functions on K, as will be done in Chapter 5.

To analyze such functions, we introduce a specific Grothendieck ring $K_{\mathrm{int}}(Z)$, see Definition 4.1.2. The integral of a (Z-definable and integrable) function on RV_*^* will then be defined as its class in a certain quotient of $K_{\mathrm{int}}(Z)$.

Developing a good understanding of this Grothendieck ring (and some variants as mentioned below) is therefore a crucial ingredient of working with the integral. Intuitively speaking, the major challenge is handling infinte sums, corresponding to functions with infinite image in p^{Γ}. This becomes possible by describing the ring of interest in various ways, allowing us to switch descriptions to focus on certain aspects when suitable.

When later generalizing the aforementioned construction of the integral to functions defined on K^n for $n \geq 2$, we will also need a family version of $K_{\mathrm{int}}(Z)$, dubbed $K_{\mathrm{int},S}(Z)$. Since many results about $K_{\mathrm{int}}(Z)$ then just become special cases of the corresponding results about $K_{\mathrm{int},S}(Z)$, we mostly work with *families of integrable functions* right away. Nevertheless, we start with some basic definitions and results in the non-family setting in Section 4.1 to facilitate a first understanding of the concepts.

In Section 4.2 we will build on [CH18, Proposition 5.2.1], saying that the hypercardinality of a definable family of subsets of Γ^* is piecewise polynomial in the parameters. This yields a description of integrals by (piecewise) polynomial functions, which aids in getting a deeper understanding of the Grothendieck ring of integrable functions on RV_*^*. In particular, we obtain Corollary 4.2.12, allowing rather explicit calculation of integrals of functions with finite image.

The strength of this last-mentioned observation is highlighted by Section 4.3, in which we show that (and how) any integral (of a definable and integrable function on RV_*^*) can be expressed by solely considering integrals of functions with finite image.

Section 4.4 connects families of integrable functions back to $K_{\mathrm{int}}(Z)$. Intuitively, the result Lemma 4.4.12 (and similar ones) allows us to think about the integrals of (some) families of integrable functions as infinite sums of integrals. More precisely, given two families $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ of integrable functions satisfying $\int_{\mathrm{mot}} \mathfrak{f}_{\boldsymbol{s}} = \int_{\mathrm{mot}} \mathfrak{g}_{\boldsymbol{s}}$ for all $\boldsymbol{s} \in S$, the classes of

$$\sum_{\boldsymbol{s}\in S}[\mathfrak{f}_{\boldsymbol{s}}] \quad \text{and} \quad \sum_{\boldsymbol{s}\in S}[\mathfrak{g}_{\boldsymbol{s}}]$$

in the quotient $R_{\mathrm{mot}}(Z) = K_{\mathrm{int}}(Z)/(\mathrm{p} - p)$ are equal, whenever it makes sense to consider these sums (i.e., if they correspond to an integrable function on RV_*^*; for a precise statement of this condition, see Notation 4.4.6 and Remark 4.4.7).

Throughout the chapter, Z will always denote an elementary substructure of Γ (in the language of ordered abelian groups), as before. However, unlike in the previous Chapter 3, we will sometimes need to enlarge the parameter set Z a bit. We thus do not fix one particular such elementary substructure now. Instead let us fix the convention that each Definition, Remark, Theorem, etc. (invisibly) starts with the sentence “For all $Z \preccurlyeq \Gamma$, we have the following.”.

4.1 Integrable functions and families thereof

Recall that we write $\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ as a shorthand for the product $\prod_{i=1}^{\ell} \mathrm{RV}_{m_i}^{n_i}$ whenever $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^{\ell}$ for some $\ell \in \mathbb{N}$.

In this section, we want to define $\int_{\mathrm{mot}} \mathfrak{f}$ for (M-definable) integrable functions $\mathfrak{f} : U \to \mathrm{p}^{\Gamma}$ with $U \subset \mathrm{RV}_*^*$. The very first step is to say what “integrable” should mean, and this is a good moment to explain how (the two rightmost terms of) the guiding equation (1.1) motivates several definitions. With those two terms read from right to left, we have

$$\int_{\mathrm{mot}} \mathfrak{f} \overset{\text{want!}}{=\!=} \sum_{\alpha\in\mathrm{im}(\mathfrak{f})} \alpha \cdot \#\mathfrak{f}^{-1}(\alpha). \tag{4.1}$$

Even in the standard model $\mathbb{Q}_p$, the sum on the right-hand side does not always exist as an element of the reals, but it might diverge to infinity. While p^{Γ} contains infinite elements in an elementary extension, we still only want to integrate those (definable) functions $\mathfrak{f}$, for which the sum $\sum_{\alpha\in\mathrm{im}(\mathfrak{f})} \alpha \cdot \#\mathfrak{f}^{-1}(\alpha)$ is not “uncontrollably infinite”. In $\mathbb{Q}_p$, it converges if and only if $\mathrm{im}(\mathfrak{f})$ is bounded from above and the fibers of $\mathfrak{f}$ are finite. This motivates the following definition in our more general case $\mathrm{K} \succcurlyeq \mathbb{Q}_p$.

Definition 4.1.1. A Z-definable function $\mathfrak{f} : U \to \mathrm{p}^{\Gamma}$ with $U \subset \mathrm{RV}_*^*$ is called *integrable*, if

- $\mathrm{im}(\mathfrak{f}) \subset \mathrm{p}^{\Gamma}$ is bounded from above[a], and
- for each $\alpha \in \mathrm{p}^{\Gamma}$, the fiber $\mathfrak{f}^{-1}(\alpha) \subset U$ is bounded.

(Note that, by Lemma 2.2.6, $\mathfrak{f}$ is M-definable for some $M \subset \mathrm{K} \cup \Gamma$ if and only if it is Z-definable for $Z = \mathrm{dcl}(M) \cap \Gamma \preccurlyeq \Gamma$, hence we can restrict to parameters from the value group.)

[a]Equivalently, $\mathrm{val}(\mathrm{im}(\mathfrak{f})) \subset \Gamma$ is bounded from below, i.e., contained in some interval of the form $[a, \infty)$.

We will define the motivic integral of such an integrable function on RV^*_* as its residue class in (the quotient of) a Grothendieck ring. In other words, we specify some few (more precisely: three) conditions any integral should satisfy, and then consider the quotient of the free abelian group of integrable functions by those relations. We start with the two essential relations guaranteeing additivity and invariance under bijections between the fibers.

Definition 4.1.2. The *Grothendieck ring of Z-definable integrable functions on RV^*_**, denoted by $K_{\mathrm{int}}(Z)$, is defined as follows:

The additive group of $K_{\mathrm{int}}(Z)$ is the free abelian group generated by symbols $[\mathfrak{f}]_{\mathrm{FrAb}}$ for each integrable function $\mathfrak{f} : U \to \mathrm{p}^{\Gamma}$, where $U \subset \mathrm{RV}^*_*$, modulo the relations

1. $[\mathfrak{f} \cup \mathfrak{g}]_{\mathrm{FrAb}} = [\mathfrak{f}]_{\mathrm{FrAb}} + [\mathfrak{g}]_{\mathrm{FrAb}}$ if $\mathrm{dom}(\mathfrak{f})$ and $\mathrm{dom}(\mathfrak{g})$ are disjoint subsets of the same ambient set $\mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$, and
2. $[\mathfrak{f}]_{\mathrm{FrAb}} = [\mathfrak{g}]_{\mathrm{FrAb}}$ if there exists a Z-definable bijection $h : \mathrm{dom}(\mathfrak{f}) \to \mathrm{dom}(\mathfrak{g})$ with $\mathfrak{g} \circ h = \mathfrak{f}$.

We will write $[\mathfrak{f}]$ for the class of $\mathfrak{f}$ in $K_{\mathrm{int}}(Z)$.

The multiplication on $K_{\mathrm{int}}(Z)$ is given by $[\mathfrak{f}] \cdot [\mathfrak{g}] := [\mathfrak{f} \star \mathfrak{g}]$, where $\mathfrak{f} \star \mathfrak{g} : \mathrm{dom}(\mathfrak{f}) \times \mathrm{dom}(\mathfrak{g}) \to \mathrm{p}^{\Gamma}$ sends $(\boldsymbol{u}, \boldsymbol{v})$ to $\mathfrak{f}(\boldsymbol{u}) \cdot \mathfrak{g}(\boldsymbol{v})$.

Remark 4.1.3. Note that the condition "$\mathfrak{g} \circ h = \mathfrak{f}$" in (2) is equivalent to the demand that h restricts to bijections between the fibers $\mathfrak{f}^{-1}(\alpha)$ and $\mathfrak{g}^{-1}(\alpha)$ for each $\alpha \in \mathrm{p}^{\Gamma}$.

Note that the sum of two generators of $K_{\mathrm{int}}(Z)$ is a generator itself: Indeed, given two integrable functions $\mathfrak{f} : U \to \mathrm{p}^{\Gamma}$ and $\mathfrak{g} : V \to \mathrm{p}^{\Gamma}$, let $U' := \{(0, \boldsymbol{u}, \mathbf{0}) \mid \boldsymbol{u} \in U\}$ and $V' := \{(1, \mathbf{0}, \boldsymbol{v}) \mid \boldsymbol{v} \in V\}$, where the two occurrences of "$\mathbf{0}$" may mean different things, such that U' and V' are subsets of the same ambient set RV^*_*. The integrable functions $\mathfrak{f}' : U' \to \mathrm{p}^{\Gamma}$ and $\mathfrak{g}' : V' \to \mathrm{p}^{\Gamma}$ induced by $\mathfrak{f}$ and $\mathfrak{g}$ in the obvious way then satisfy $[\mathfrak{f}] = [\mathfrak{f}']$ as well as $[\mathfrak{g}] = [\mathfrak{g}']$, and their sum $[\mathfrak{f}'] + [\mathfrak{g}'] = [\mathfrak{f}' \cup \mathfrak{g}']$ is a generator as claimed.

Moreover, any element of $K_{\mathrm{int}}(Z)$ can, by definition, be written as the difference of (two) sums of generators. Hence, the above observation yields the following note.

Remark 4.1.4. Any element of $K_{\mathrm{int}}(Z)$ can be written as the difference of two generators, i.e., in the form $[\mathfrak{f}] - [\mathfrak{g}]$ for appropriate choices of integrable functions $\mathfrak{f}$ and $\mathfrak{g}$ on RV_*^*.

We have yet to justify calling $K_{\mathrm{int}}(Z)$ a Grothendieck *ring*:

Proposition 4.1.5. *With the multiplication defined above,* $K_{\mathrm{int}}(Z)$ *is a ring with unit* $[\mathrm{const}_{\{0\}}(1)]$.

Proof. Consider the free abelian group F generated by symbols $[\mathfrak{f}]_{\mathrm{FrAb}}$ for each integrable function $\mathfrak{f}$ on RV_*^*. Let $F_{\cong}$ denote the quotient of F by the relation (2) and write $[\mathfrak{f}]_{\cong}$ for the class of $\mathfrak{f}$ in $F_{\cong}$.

Now $[\mathfrak{f}]_{\cong} \cdot [\mathfrak{g}]_{\cong} := [\mathfrak{f} \star \mathfrak{g}]_{\cong}$ is a well-defined multiplication[1] on $F_{\cong}$: If $[\mathfrak{f}_1]_{\cong} = [\mathfrak{f}_2]_{\cong}$ and $[\mathfrak{g}_1]_{\cong} = [\mathfrak{g}_2]_{\cong}$, and $h_{\mathfrak{f}}$ and $h_{\mathfrak{g}}$ are Z-definable bijections with $\mathfrak{f}_2 \circ h_{\mathfrak{f}} = \mathfrak{f}_1$ and $\mathfrak{g}_2 \circ h_{\mathfrak{g}} = \mathfrak{g}_1$, then the map given by

$$\begin{aligned} h : \mathrm{dom}(\mathfrak{f}_1) \times \mathrm{dom}(\mathfrak{g}_1) &\to \mathrm{dom}(\mathfrak{f}_2) \times \mathrm{dom}(\mathfrak{g}_2) \\ (\boldsymbol{u}, \boldsymbol{v}) &\mapsto (h_{\mathfrak{f}}(\boldsymbol{u}), h_{\mathfrak{g}}(\boldsymbol{v})) \end{aligned}$$

is a Z-definable bijection satisfying $(\mathfrak{f}_2 \star \mathfrak{g}_2) \circ h = \mathfrak{f}_1 \star \mathfrak{g}_1$.

Note that this multiplication gives $F_{\cong}$ the structure of a ring, with unit $[\mathrm{const}_{\{0\}}(1)]_{\cong}$.

Thus it is left to show that the subgroup A of $F_{\cong}$ generated by the relation (1) is an ideal in the ring $F_{\cong}$. To this end, it suffices to note that

$$\begin{aligned} &[\mathfrak{f}]_{\cong} \cdot ([\mathfrak{g}]_{\cong} + [\mathfrak{h}]_{\cong} - [\mathfrak{g} \cup \mathfrak{h}]_{\cong}) \\ &= [\mathfrak{f} \star \mathfrak{g}]_{\cong} + [\mathfrak{f} \star \mathfrak{h}]_{\cong} - [\mathfrak{f} \star (\mathfrak{g} \cup \mathfrak{h})]_{\cong}) \\ &= [\mathfrak{f} \star \mathfrak{g}]_{\cong} + [\mathfrak{f} \star \mathfrak{h}]_{\cong} - [(\mathfrak{f} \star \mathfrak{g}) \cup (\mathfrak{f} \star \mathfrak{h})]_{\cong}) \end{aligned}$$

lies in A for all integrable functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$, where the domains of $\mathfrak{g}$ and $\mathfrak{h}$ are disjoint subsets of the same ambient set RV_*^*. ☐

We can now define $\int_{\mathrm{mot}} \mathfrak{f}$ for Z-definable integrable functions $\mathfrak{f} : U \to \mathrm{p}^{\Gamma}$ with $U \subset \mathrm{RV}_*^*$. Besides the relations in the Grothendieck ring $K_{\mathrm{int}}(Z)$, recall that we also want it to satisfy the equation (4.1) on p. 48. In the special case of the constant function $\mathfrak{f} = \mathrm{const}_{\{0\}}(\alpha)$ on the one-point set $\{0\}$ with value α, this equation simplifies to

$$\int_{\mathrm{mot}} \mathrm{const}_{\{0\}}(\alpha) \overset{\text{want!}}{=\!=} \alpha.$$

[1] More formally: the multiplication induced by the above, extending to sums in the natural way ensuring distributivity.

Looking at two different values of α, e.g., $\alpha = 1$ and $\alpha = \mathrm{p}$, this results in the relation

$$p \cdot \int_{\mathrm{mot}} \mathrm{const}_{\{0\}}(1) \overset{\text{want!}}{=\!=} p \cdot 1 \overset{\text{want!}}{=\!=} \mathrm{p} \overset{\text{want!}}{=\!=} \int_{\mathrm{mot}} \mathrm{const}_{\{0\}}(\mathrm{p}),$$

motivating the following definition.

Definition 4.1.6. Let $\mathfrak{f}$ be an integrable function on RV^{*}_{*}. The integral $\int_{\mathrm{mot}} \mathfrak{f}$ of $\mathfrak{f}$ is then defined as the residue class of $[\mathfrak{f}]$ in the quotient $R_{\mathrm{mot}}(Z) := K_{\mathrm{int}}(Z)/(\mathrm{p} - p)$, where $(\mathrm{p} - p)$ is the ideal generated by the element

$$\mathrm{p} - p := [\mathrm{const}_{\{0\}}(\mathrm{p})] - p \cdot \underbrace{[\mathrm{const}_{\{0\}}(1)]}_{= 1 \in K_{\mathrm{int}}(Z)}.$$

Moreover, $(\mathrm{p} - p)^{\mathbb{Q}}$ denotes the ideal generated by $\mathrm{p} - p$ in $K_{\mathrm{int}}(Z) \otimes \mathbb{Q}$.

Remark 4.1.7. Note that $p \cdot [\mathrm{const}_{\{0\}}(1)] = [\mathrm{const}_U(1)]$ for any Z-definable set $U \subset \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ which has exactly p elements. Indeed, we then have $U \subset \mathrm{dcl}(\mathrm{val}(U)) \subset \mathrm{dcl}(Z)$ by Remark 2.2.2 and Remark 2.2.4 and

$$p = 1 + \cdots + 1 = [\mathrm{const}_{\{0\}}(1)] + \cdots + [\mathrm{const}_{\{0\}}(1)] = [\mathrm{const}_U(1)].$$

in $K_{\mathrm{int}}(Z)$.

Remark 4.1.8. Note that the map from $\mathrm{p}^{\mathbb{Z}}$ to $K_{\mathrm{int}}(Z)$ given by $\alpha \mapsto [\mathrm{const}_{\{0\}}(\alpha)]$ is injective, as Corollary 4.2.10 will show, and it also respects multiplication, since

$$[\mathrm{const}_{\{0\}}(\alpha)] \cdot [\mathrm{const}_{\{0\}}(\beta)] = [\mathrm{const}_{\{0\}}(\alpha \cdot \beta)]$$

We will thus just write $\alpha \in K_{\mathrm{int}}(Z)$ for $[\mathrm{const}_{\{0\}}(\alpha)]$. Using this notation and observing that

$$[\mathrm{const}_U(1)] = \#U = \#\!\#U, \text{ for any finite } Z\text{-definable } U,$$

we see that $\int_{\mathrm{mot}}$ indeed satisfies the equation (4.1) on p. 48 whenever both $\mathrm{im}(\mathfrak{f})$ as well as all fibers of $\mathfrak{f}$ are finite: Given a Z-definable integrable function $\mathfrak{f}$ on

RV_*^* with finite image and finite fibers, we have

$$\begin{aligned}\int_{\mathrm{mot}} \mathfrak{f} &= [\mathfrak{f}] + (\mathrm{p} - p)\\ &= \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} [\mathrm{const}_{\mathfrak{f}^{-1}(\alpha)}(\alpha)] + (\mathrm{p} - p)\\ &= \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} [\mathrm{const}_{\{0\}}(\alpha)] \cdot [\mathrm{const}_{\mathfrak{f}^{-1}(\alpha)}(1)] + (\mathrm{p} - p)\\ &= \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} \alpha \cdot \#\mathfrak{f}^{-1}(\alpha) + (\mathrm{p} - p).\end{aligned}$$

(Note that $\mathrm{im}(\mathfrak{f})$ is a finite Z-definable set, so we have $\mathrm{im}(\mathfrak{f}) \subset \mathrm{p}^Z$, and hence $\mathfrak{f}^{-1}(\alpha) \subset \mathrm{dcl}(Z)$ for $\alpha \in \mathrm{im}(\mathfrak{f})$, by Remark 2.2.2 and Remark 2.2.4.)

Corollary 4.1.9. *Let $\mathfrak{f} : U \to \mathrm{p}^\Gamma$ be a Z-definable integrable function and let F be a finite Z-definable subset of RV_*^* for which $\#F$ is a power of p. Consider the projection $\mathrm{pr}_U : F \times U \to U$. Then*

$$\int_{\mathrm{mot}} \mathfrak{f} \circ \mathrm{pr}_U = \#F \cdot \int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \#F \cdot \mathfrak{f},$$

where $\#F \cdot \mathfrak{f}$ is the Z-definable integrable function from U to p^Γ sending $\boldsymbol{u} \in U$ to $\#F \cdot \mathfrak{f}(\boldsymbol{u}) \in \mathrm{p}^\Gamma$, i.e., $[\#F \cdot \mathfrak{f}] = [\mathrm{const}_{\{0\}}(\#F)] \cdot [\mathfrak{f}]$.

Proof. Note that we have

$$[\mathfrak{f} \circ \mathrm{pr}_U] = \sum_{\boldsymbol{f} \in F} \underbrace{[\mathfrak{f} \circ \mathrm{pr}_U \restriction (\{\boldsymbol{f}\} \times U)]}_{= [\mathfrak{f}]} = \#F \cdot [\mathfrak{f}],$$

proving the first equality. By Remark 4.1.8 applied to $\mathrm{const}_{\{0\}}(\#F)$, we moreover get

$$\begin{aligned}\int_{\mathrm{mot}} \mathfrak{f} \circ \mathrm{pr}_U = \#F \cdot \int_{\mathrm{mot}} \mathfrak{f} &= \int_{\mathrm{mot}} \mathrm{const}_{\{0\}}(\#F) \cdot \int_{\mathrm{mot}} \mathfrak{f}\\ &= \int_{\mathrm{mot}} (\mathrm{const}_{\{0\}}(\#F) \star \mathfrak{f})\\ &= \int_{\mathrm{mot}} \#F \cdot \mathfrak{f}\end{aligned}$$

as claimed. ∎

In the remainder of this section, we will work with the following family version of $K_{\mathrm{int}}(Z)$. Results about the non-family version then just follow as special cases, and we will state some important results explicitly as corollaries.

Definition 4.1.10. Let $S \subset \mathrm{RV}^*_*$ be Z-definable. The *Grothendieck ring of S-families of integrable functions on* RV^*_*, denoted by $K_{\mathrm{int},S}(Z)$, is defined as follows:

The additive group of $K_{\mathrm{int},S}(Z)$ is the free abelian group generated by symbols $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}}$ for each Z-definable family of integrable functions on RV^*_*, modulo the relations

(1) $[(\mathfrak{f}_{\boldsymbol{s}} \cup \mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}} = [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}} + [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}}$ if, for each $\boldsymbol{s} \in S$, the domains of the functions $\mathfrak{f}_{\boldsymbol{s}}$ and $\mathfrak{g}_{\boldsymbol{s}}$ are disjoint subsets of the same ambient set RV^*_*, and

(2) $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}} = [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}}$ if there is a Z-definable family of bijections $h_{\boldsymbol{s}} : \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) \to \mathrm{dom}(\mathfrak{g}_{\boldsymbol{s}})$ with $\mathfrak{g}_{\boldsymbol{s}} \circ h_{\boldsymbol{s}} = \mathfrak{f}_{\boldsymbol{s}}$ for all $\boldsymbol{s} \in S$.

We will write $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]$ for the class of $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ in $K_{\mathrm{int},S}(Z)$.

The multiplication on $K_{\mathrm{int},S}(Z)$ is given by

$$[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \cdot [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] := [(\mathfrak{f}_{\boldsymbol{s}} \star \mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}],$$

where $\star$ is defined just as in Definition 4.1.2. (Note that $(\mathfrak{f}_{\boldsymbol{s}} \star \mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ is a Z-definable family of integrable functions on RV^*_* whenever $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ are.)

Just as in the non-family version, the sum of two generators of $K_{\mathrm{int},S}(Z)$ is a generator itself, resulting in the following useful remark:

Remark 4.1.11. Any element of $K_{\mathrm{int},S}(Z)$ can be written as the difference of two generators, i.e., in the form $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]$ for appropriate choices of Z-definable families of integrable functions $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ on RV^*_*.

Proposition 4.1.12. *With the multiplication defined above, $K_{\mathrm{int},S}(Z)$ is a ring with unit* $[(\mathrm{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}]$.

Proof. The proof of Proposition 4.1.5 applies to this family version as well, mutatis mutandis. ☐

Remark 4.1.13. For any fixed $\boldsymbol{s}_0 \in S$, we have a natural specialization map

$$\begin{aligned} \mathrm{spz}_{\boldsymbol{s}_0} : K_{\mathrm{int},S}(Z) &\to K_{\mathrm{int}}(Z(\boldsymbol{s}_0)) \\ [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] &\mapsto [\mathfrak{f}_{\boldsymbol{s}_0}]. \end{aligned}$$

Note that $\mathrm{spz}_{\boldsymbol{s}_0}$ is a homomorphism of rings just by the definitions of $K_{\mathrm{int},S}(Z)$ and $K_{\mathrm{int}}(Z(\boldsymbol{s}_0))$.

We also need a family version of the ideal $(\mathrm{p} - p)$ and the quotient $R_{\mathrm{mot}}(Z)$ of $K_{\mathrm{int}}(Z)$ by that ideal, similar to Definition 4.1.6

Notation 4.1.14. We write $(\mathrm{p} - p)_S \subset K_{\mathrm{int},S}(Z)$ for the ideal generated by

$$[(\mathrm{const}_{\{0\}}(\mathrm{p}))_{\boldsymbol{s}\in S}] - p \cdot \underbrace{[(\mathrm{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}]}_{=\,1\,\in K_{\mathrm{int},S}(Z)} \in K_{\mathrm{int},S}(Z),$$

and we write $R_{\mathrm{mot},S}(Z)$ for the quotient $K_{\mathrm{int},S}(Z)/(\mathrm{p} - p)_S$. Moreover, $(\mathrm{p} - p)_S^{\mathbb{Q}}$ denotes the ideal generated by $(\mathrm{p} - p)_S^{\mathbb{Q}}$ in $K_{\mathrm{int},S}(Z) \otimes \mathbb{Q}$.

4.2 Understanding $K_{\mathrm{int},S}(Z)$ via polynomials

We now want to get some better understanding of Z-definable families of integrable functions on RV_*^* by analyzing the Grothendieck ring $K_{\mathrm{int},S}(Z)$ and its quotient $R_{\mathrm{mot},S}(Z) = K_{\mathrm{int},S}(Z)/(\mathrm{p} - p)_S$. Recall that, by Corollary 2.3.6, the hypercardinalities of subsets of RV_*^* in a Z-definable family over the value group are piecewise polynomial in the parameter. This will help us to describe the elements of $K_{\mathrm{int},S}(Z)$ in terms of polynomials, see Proposition 4.2.4.

Instead of $K_{\mathrm{int},S}(Z)$, we mostly work with the ring $K_{\mathrm{int},S}(Z)\otimes\mathbb{Q}$. The reason for doing that will become more apparent in Section 4.3, where we show that every element of $R_{\mathrm{mot},S}(Z)$ can be written as a rational multiple of the difference of the integrals of two integrable functions with finite images. While $\mathbb{Q}$ embeds into $R_{\mathrm{mot},S}(Z)$ (this is Lemma 4.3.2), certain fractions can only be written as integrals when allowing integrable functions with infinite images.

Moreover, we do not lose any information when tensoring with $\mathbb{Q}$, since $K_{\mathrm{int},S}(Z)$ embeds into $K_{\mathrm{int},S}(Z)\otimes\mathbb{Q}$ and $(K_{\mathrm{int},S}(Z)\otimes\mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}}$ is isomorphic to $K_{\mathrm{int},S}(Z)/(\mathrm{p}-p)_S = R_{\mathrm{mot},S}(Z)$, see Corollary 4.2.8 and Proposition 4.3.3.

Definition 4.2.1. Let $S \subset \mathrm{RV}_*^*$ be Z-definable. Consider the abelian group $\mathrm{Maps}(S \times \Gamma, K_b^{\Gamma}(\Gamma) \otimes \mathbb{Q})$ of maps from $S \times \Gamma$ to $K_b^{\Gamma}(\Gamma) \otimes \mathbb{Q}$ with pointwise addition, i.e., $(f+g)(\boldsymbol{s},a) = f(\boldsymbol{s},a) + g(\boldsymbol{s},a)$.

We will write $\mathbf{P}_S^{\mathbb{Q}}(Z)$ for the subgroup generated by the maps

$$f_{X,P} : S \times \Gamma \to K_b^{\Gamma}(\Gamma) \otimes \mathbb{Q}$$
$$(\boldsymbol{s},a) \mapsto \begin{cases} P(\mathrm{val}(s_1),\dots,\mathrm{val}(s_n),a) & \text{if } (\boldsymbol{s},a) \in X \\ 0 & \text{otherwise} \end{cases}$$

where $P \in (K_b^{\Gamma}(Z) \otimes \mathbb{Q})[T_1,\dots,T_{n+1}]$ is a polynomial (and where n is the number of coordinates of S) and $X \subset S \times \Gamma$ is a Z-definable subset whose fibers $X_{\boldsymbol{s}} = \{a \in \Gamma \mid (\boldsymbol{s},a) \in X\}$ are bounded from below for all $\boldsymbol{s} \in S$.

(Here, writing $a \in K_b^{\Gamma}(Z(a)) \subset K_b^{\Gamma}(\Gamma)$ for $a \in \Gamma$ is an abuse of notation using the natural injection $\Gamma \hookrightarrow K_b^{\Gamma}(\Gamma)$ given by $a \mapsto \#[0,a)$ for non-negative a and $a \mapsto -\#[0,-a)$ for negative a, see also [CH18, Lemma 2.2.6].)

Remark 4.2.2. Note that the group $\mathbf{P}_S^{\mathbb{Q}}(Z)$ as defined above is contained in the subgroup of $\mathrm{Maps}(S \times \Gamma, K_b^{\Gamma}(\Gamma) \otimes \mathbb{Q})$ consisting of those maps $f : S \times \Gamma \to K_b^{\Gamma}(\Gamma) \otimes \mathbb{Q}$ for which we have

$$f(\boldsymbol{s},a) \in K_b^{\Gamma}(Z(\boldsymbol{s},a)) \otimes \mathbb{Q} \text{ for all } (\boldsymbol{s},a) \in S \times \Gamma,$$

where $Z(\boldsymbol{s},a) = \mathrm{acl}(Z \cup \{\boldsymbol{s}\} \cup \{a\}) \cap \Gamma$.

Note that there is a natural ring structure on $\mathbf{P}_S^{\mathbb{Q}}(Z)$, given by the multiplication induced by $f_{X,P} \cdot f_{Y,Q} := f_{X \cap Y, P \cdot Q}$. However, any element is a zero divisor with respect to this multiplication, since $f_{X,P} \cdot f_{Y,Q} = 0$ whenever $X \cap Y = \emptyset$. As we want $\mathbf{P}_S^{\mathbb{Q}}(Z)$ to be isomorphic to $K_{\mathrm{int},S}(Z) \otimes \mathbb{Q}$ as rings, and the latter is a domain, we need another multiplication on the former. We will not give an explicit definition, but only define this multiplication implicitly using the group isomorphism between $\mathbf{P}_S^{\mathbb{Q}}(Z)$ and $K_{\mathrm{int},S}(Z) \otimes \mathbb{Q}$ from Proposition 4.2.4, so that this isomorphism automatically becomes an isomorphism of rings.

Intuitively (and in actual fact in some special cases, see Remark 4.2.6), the multiplication on $\mathbf{P}_S^{\mathbb{Q}}(Z)$ is given by a convolution in $a \in \Gamma$. Supporting this intuition, it can be helpful to think of an element $f \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ as a map sending $\boldsymbol{s} \in S$ to the formal sum $\sum_{a \in \Gamma} p^{-a} \cdot f(\boldsymbol{s},a)$. It is important here to note that these sums are formal, even $p^{-(-1)} \cdot 1$ is not equal to $p^0 \cdot p$. The quotient of $\mathbf{P}_S^{\mathbb{Q}}(Z)$ by the relation identifying these two formal summands is isomorphic to $R_{\mathrm{mot},S}(Z)$, as Proposition 4.2.4 shows. In other words, the only thing separating the map $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s} \in S} \mapsto [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s} \in S}]$ from the uniform integral is the (uniform) identification of the two different presentations of p described above.

The subgroup (and a posteriori, subring) $\mathbf{P}_S(Z)$ of $\mathbf{P}_S^{\mathbb{Q}}(Z)$ as defined below is isomorphic to $K_{\mathrm{int},S}(Z)$ as a subgroup (subring) of $K_{\mathrm{int},S}(Z) \otimes \mathbb{Q}$, see Proposition 4.2.4.

Definition and Lemma 4.2.3. Let $S \subset \mathrm{RV}_*^*$ be Z-definable. We define the subgroup

$$\mathbf{P}_S(Z) := \mathbf{P}_S^{\mathbb{Q}}(Z) \cap \mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma))$$

of $\mathbf{P}_S^{\mathbb{Q}}(Z)$, using Lemma 2.3.2 to construe $\mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma))$ as a subgroup of $\mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma) \otimes \mathbb{Q})$.

In other words, $\mathbf{P}_S(Z)$ is the subgroup of $\mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma))$ generated by those of the maps $f_{X,P} \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ as in Definition 4.2.1 whose image is contained in $K_b^\Gamma(\Gamma) \subset K_b^\Gamma(\Gamma) \otimes \mathbb{Q}$.

Proof. We have to prove that both definitions agree. Towards this end, let $M_1 := \mathbf{P}_S^{\mathbb{Q}}(Z) \cap \mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma))$ and let M_2 be the subgroup of $\mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma))$ generated by those of the $f_{X,P} \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ whose image is contained in $K_b^\Gamma(Z)$. First note that any such $f_{X,P}$ is an element of $\mathbf{P}_S^{\mathbb{Q}}(Z) \cap \mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma))$, so the inclusion $M_2 \subset M_1$ is clear. For the other direction, let $f \in M_1$ and write it as a (finite) sum of generators $f = \sum_i f_{X_i,P_i}$. Since the sum of two polynomials is a polynomial again, we can assume that the X_i are pairwise disjoint. As $f \in \mathrm{Maps}(S \times \Gamma, K_b^\Gamma(\Gamma))$, the image of each of the f_{X_i,P_i} must then be contained in $K_b^\Gamma(Z)$, so each of them is an element of M_1, and hence so is f. □

Note that, in Definition 4.2.3, we still allow coefficients in $K_b^\Gamma(Z) \otimes \mathbb{Q}$ for the polynomial, even though the image of a map in $\mathbf{P}_S(Z)$ has to be contained in $K_b^\Gamma(Z)$. The reason for defining $\mathbf{P}_S(Z)$ in exactly this way becomes more apparent with the following Proposition 4.2.4, which yields an isomorphism between $K_{\mathrm{int},S}(Z)$ and $\mathbf{P}_S(Z)$, requiring that maps like $f_{S\times[0,\infty)_2,\frac{1}{2}T_{n+1}}$ lie in $\mathbf{P}_S(Z)$.[2]

Moreover, note that $\mathrm{Maps}(S \times \Gamma, K_b^\Gamma(Z) \otimes \mathbb{Q})$ is torsion-free since $K_b^\Gamma(Z) \otimes \mathbb{Q}$ is (by Lemma 2.3.2), hence so is its subgroup $\mathbf{P}_S(Z)$. Also, for each $f \in \mathbf{P}_S^{\mathbb{Q}}(Z)$, there clearly is an $n \in \mathbb{N}_{>0}$ such that $n \cdot f \in \mathbf{P}_S(Z)$. Namely, we can chose n to be any common multiple of all denominators of the coefficients of all polynomials that appear when writing f as a sum of generators. Thus, Lemma 2.1.4 applied to $G = \mathbf{P}_S^{\mathbb{Q}}(Z)$ and $H = \mathbf{P}_S(Z)$ yields $\mathbf{P}_S(Z) \otimes \mathbb{Q} \cong \mathbf{P}_S^{\mathbb{Q}}(Z)$ via the canonical isomorphism $f \otimes q \mapsto q \cdot f$.

Proposition 4.2.4. *Let $S \subset \mathrm{RV}_*^*$ be Z-definable. Then there is an isomorphism of groups $\chi_S : K_{\mathrm{int},S}(Z) \to \mathbf{P}_S(Z)$ induced by*

$$[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \mapsto \big((\boldsymbol{s}, a) \mapsto \#\mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a})\big)$$

[2]Indeed, $\chi_S([(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]) = f_{S\times[0,\infty)_2,\frac{1}{2}T_{n+1}}$ for the family $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ of integrable functions given by $\mathfrak{f}_{\boldsymbol{s}} : \{(a,b) \mid 0 \le a < b\} \to \mathrm{p}^\Gamma, (a,b) \mapsto \mathrm{p}^{-2b}$ for all $\boldsymbol{s} \in S$.

for each Z-definable family of integrable functions $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ on RV_*^*.

In particular, since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, χ_S induces an isomorphism of groups $\chi_S^{\mathbb{Q}} : K_{\mathrm{int},S}(Z) \otimes \mathbb{Q} \to \mathbf{P}_S^{\mathbb{Q}}(Z)$.

Once we have proven the above, the isomorphism χ_S transfers the multiplication on $K_{\mathrm{int},S}(Z)$ to $\mathbf{P}_S(Z)$, hence inducing a ring structure on the latter. Similarly, $\chi_S^{\mathbb{Q}}$ induces a ring structure on $\mathbf{P}_S^{\mathbb{Q}}(Z)$ so that, a posteriori, all of the group isomorphisms χ_S, $\chi_S^{\mathbb{Q}}$, and $\mathbf{P}_S(Z)\otimes\mathbb{Q} \cong \mathbf{P}_S^{\mathbb{Q}}(Z)$, $f\otimes q \mapsto q\cdot f$ become in fact ring isomorphisms. Let us make note of the latter for later reference.

Remark 4.2.5. The canonical ring homomorphism

$$\begin{aligned} \mathbf{P}_S(Z)\otimes\mathbb{Q} &\to \mathbf{P}_S^{\mathbb{Q}}(Z) \\ f\otimes q &\mapsto q\cdot f \end{aligned}$$

is an isomorphism of rings.

Proof of Proposition 4.2.4. Let F_S denote the free abelian group generated by the symbols $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}}$ as in Definition 4.1.10. The mapping rule as given in the statement of the lemma clearly induces a homomorphism ψ from F_S to the group of maps $\mathrm{Maps}(S\times\Gamma, K_b^{\Gamma}(\Gamma))$.

We will now first show that the image of ψ is contained in $\mathbf{P}_S(Z)$. Given a Z-definable family $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ of integrable functions on RV_*^*, let $f := \psi([(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]_{\mathrm{FrAb}})$ be the image under ψ of its class in F_S, i.e.,

$$f(\boldsymbol{s}, a) := \#\mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a})$$

for all $\boldsymbol{s}\in S$ and all $a\in\Gamma$. Note that there is a finite partition of S into Z-definable sets S_i for $i\in I$ such that $\mathbf{val}$ is injective on each S_i. (E.g., we can let S_i be the set of those elements of S which have a specific combination of higher angular components specified by i.) Since $f = \sum_{i\in I} f{\restriction}(S_i\times\Gamma)$, it is thus enough to consider the case that $\mathbf{val}$ is injective on all of S. Consider the Z-definable family of subsets of RV_*^* given by $U_{(\mathbf{val}(\boldsymbol{s}),a)} := \mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a})$. Note that the hypercardinality of $U_{(\mathbf{val}(\boldsymbol{s}),a)}$ is piecewise polynomial in $(\mathbf{val}(\boldsymbol{s}), a)$ by Corollary 2.3.6, so $\mathrm{im}(\psi)$ is indeed contained in $\mathbf{P}_S(Z)$.

Moreover, ψ respects the relations (1) and (2) from Definition 4.1.10, so the subgroup of F_S generated by these two relations is contained in the kernel of ψ. Hence ψ induces a group homomorphism $\chi_S : K_{\mathrm{int},S}(Z) \to \mathbf{P}_S(Z)$.

It is left to show that χ_S is bijective. Injectivity follows from [CH18, Theorem 5.2.2]: Suppose that $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]$ lies in the kernel of χ_S. (Recall that any element of $K_{\mathrm{int},S}(Z)$ can be written as the difference of two generators, cf. Remark 4.1.11.) Then we have $\#\mathfrak{f}_{\boldsymbol{s}}^{-1}(\alpha) = \#\mathfrak{g}_{\boldsymbol{s}}^{-1}(\alpha)$ for all $\boldsymbol{s}\in S$ and all $\alpha\in\mathrm{p}^{\Gamma}$. We can, just as above, reduce to the case that $\mathbf{val}$ is injective on S so that the Z-definable families $\mathfrak{f}_{\boldsymbol{s}}^{-1}(\alpha)$ and

$\mathfrak{g}_{\boldsymbol{s}}^{-1}(\alpha)$ are parameterized by a Z-definable subset of (some power of) Γ. Hence, the implication (4) $\Rightarrow$ (5) of [CH18, Theorem 5.2.2], together with [CH18, Remark 2.3.2], yields a Z-definable family of bijections $h_{\boldsymbol{s},\alpha} : \mathfrak{f}_{\boldsymbol{s}}^{-1}(\alpha) \to \mathfrak{g}_{\boldsymbol{s}}^{-1}(\alpha)$. (Note that [CH18, Theorem 5.2.2] is formulated under the assumption that Γ is $|Z|^+$-saturated. However, applying it in a sufficiently saturated model $\Gamma^\star \succcurlyeq \Gamma$ shows that the cited theorem still holds if Γ is not saturated, as "being bounded" is a first-order property.)

Now, setting $h_{\boldsymbol{s}} := \bigcup_{\alpha \in \mathrm{im}(\mathfrak{f}_{\boldsymbol{s}})} h_{\boldsymbol{s},\alpha}$ gives us a Z-definable family of bijections as in Definition 4.1.10 (2), witnessing $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] = [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]$.

To show that χ_S is also surjective, let $f_{X,P}$ be a generator of $\mathbf{P}_S(Z)$, i.e., let $X \subset S \times \Gamma$ be Z-definable with fibers over S bounded from below, and $P \in (K_b^\Gamma(Z) \otimes \mathbb{Q})[T_1, \dots, T_{n+1}]$, and consider

$$
\begin{aligned}
f_{X,P} : S \times \Gamma &\to K_b^\Gamma(\Gamma) \\
(\boldsymbol{s}, a) &\mapsto \begin{cases} P(\mathrm{val}(s_1), \dots, \mathrm{val}(s_n), a) & \text{if } (\boldsymbol{s}, a) \in X \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Since we already proved that χ_S is a group homomorphism, we can write $f_{X,P}$ as a (finite) sum and then only have to find preimages for each summand. We will do this several times in the remainder of the proof to restrict to special cases that are easier to handle. Firstly, we can assume that

$$
P(\mathrm{val}(s_1), \dots, \mathrm{val}(s_n), a) = q \cdot \#A \cdot \prod_{i=1}^{n} \mathrm{val}(s_i)^{e_i} \cdot a^{e_{n+1}},
$$

where $q \in \mathbb{Q}$, $A \subset \Gamma^m$ is Z-definable, and $e_i \in \mathbb{N}$ for $i = 1, \dots, n+1$. (Here, read $0^0 = 1$.) Moreover, we can restrict to the case that $q = \frac{1}{M}$ for some $M \in \mathbb{N}_{>0}$.

Now consider the partition of $X \subset S \times \Gamma$ into the finitely many Z-definable sets $X_{\boldsymbol{\sigma}}$ for $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_{n+1}) \in \{\pm 1\}^{n+1}$ and the set $\widehat{X}$ with

$$
\begin{aligned}
X_{\boldsymbol{\sigma}} &= \{(\boldsymbol{s}, a) \in X \mid \sigma_i \cdot \mathrm{val}(s_i) > 0 \text{ for } i \le n, \text{ and } \sigma_{n+1} \cdot a > 0\}, \text{ and} \\
\widehat{X} &= \{(\boldsymbol{s}, a) \in X \mid \mathrm{val}(s_i) = 0 \text{ for at least one } i \le n, \text{ or } a = 0\} = X \setminus \bigcup_{\boldsymbol{\sigma}} X_{\boldsymbol{\sigma}}.
\end{aligned}
$$

Note that $f_{X,P} = \sum_{\boldsymbol{\sigma}} f_{X_{\boldsymbol{\sigma}},P}$, so we can fix one of the finitely many $\boldsymbol{\sigma}$ and assume that $X = X_{\boldsymbol{\sigma}}$. We will now construct a preimage of $f_{X,\sigma P}$ for $\sigma := \prod_{i=1}^{n+1} \sigma_i^{e_i} \in \{\pm 1\}$. This is enough to finish the proof – also in the case $\sigma = -1$, since $f_{X,-P} = -f_{X,P}$ and we already proved that χ_S is a group homomorphism.

Recall that

$$
P(\mathrm{val}(s_1), \dots, \mathrm{val}(s_n), a) = \frac{1}{M} \cdot \#A \cdot \prod_{i=1}^{n} \mathrm{val}(s_i)^{e_i} \cdot a^{e_{n+1}} \in K_b^\Gamma(\Gamma),
$$

with $\#A \in K_b^\Gamma(Z)$, where we can further assume that A is a Z-definable cuboid by [CH18, Proposition 3.3.2]. I.e., $A = \prod_{\ell=1}^m [0, a_\ell)$ for some $a_\ell \in Z$. For a given $(\boldsymbol{s}, a) \in X$, Lemma 3.1.7 implies the existence of $d_{i,\ell} \in \mathbb{N}$ for $0 \le i \le n+1$ with

$$\begin{aligned} d_{0,\ell} &| a_\ell && \text{for } 1 \le \ell \le m, \\ d_{i,\ell} &| \operatorname{val}(s_i) && \text{for } 1 \le i \le n \text{ and } 1 \le \ell \le e_i, \\ d_{n+1,\ell} &| a && \text{for } 1 \le \ell \le e_{n+1}, \\ \text{and } M = \prod d_{i,\ell}. \end{aligned} \tag{4.2}$$

Now consider the partition of X into pieces given, for each $\boldsymbol{d} \in \mathbb{N}_{>0}^{m+e_1+\dots+e_{n+1}}$, by

$$X_{\boldsymbol{d}} := \{(\boldsymbol{s}, a) \in X \mid \boldsymbol{d} \text{ is the (lexicographically) minimal tuple satisfying (4.2)}\}$$

Since the (Z-definable) sets $X_{\boldsymbol{d}}$ are empty for all but finitely many $\boldsymbol{d}$, this again yields a partition of X into finitely many pieces. Thus we can fix one such tuple $\boldsymbol{d}$ and assume that $X = X_{\boldsymbol{d}}$. Now consider the Z-definable (families of) sets

$$\begin{aligned} A' &:= \prod_{\ell=1}^m [0, a_\ell)_{d_{0,\ell}}, \\ B_{\boldsymbol{s}} &:= \{(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n) \mid \boldsymbol{b}_i \in \prod_{\ell=1}^{e_i} [0, \sigma_i \cdot \operatorname{val}(s_i))_{d_{i,\ell}}\} \subset \Gamma^{(e_1+\dots+e_n)}, \text{ and} \\ C_{\boldsymbol{s}} &:= \{(\boldsymbol{c}, a) \mid (\boldsymbol{s}, a) \in X, \boldsymbol{c} \in \prod_{\ell=1}^{e_{n+1}} [0, \sigma_{n+1} \cdot a)_{d_{n+1,\ell}}\} \subset \Gamma^{e_{n+1}} \times \Gamma, \end{aligned}$$

and their "counterparts" in RV_*^*,

$$\begin{aligned} U &:= ((\mathrm{ac}_1)^{-1}(1))^m \cap \mathbf{val}^{-1}(A') \subset \mathrm{RV}_1^m, \\ V_{\boldsymbol{s}} &:= ((\mathrm{ac}_1)^{-1}(1))^{e_1+\dots+e_n} \cap \mathbf{val}^{-1}(B_{\boldsymbol{s}}) \subset \mathrm{RV}_1^{e_1+\dots+e_n}, \text{ and} \\ W_{\boldsymbol{s}} &:= ((\mathrm{ac}_1)^{-1}(1))^{e_{n+1}+1} \cap \mathbf{val}^{-1}(C_{\boldsymbol{s}}) \subset \mathrm{RV}_1^{e_{n+1}+1}. \end{aligned}$$

Note that we then have $\#U = \#A'$ and $\#B_{\boldsymbol{s}} = \#V_{\boldsymbol{s}}$ as well as

$$\begin{aligned} &\#\{\boldsymbol{w'} \in \mathrm{RV}_1^{e_{n+1}} \mid \boldsymbol{w} = (\boldsymbol{w'}, w_{e_{n+1}+1}) \in W_{\boldsymbol{s}}\} \\ &= \#\{\boldsymbol{c} \in \Gamma^{e_{n+1}} \mid (\boldsymbol{c}, \operatorname{val}(w_{e_{n+1}+1})) \in C_{\boldsymbol{s}}\} \end{aligned}$$

for all $\boldsymbol{s} \in S$ by (the proof of) Lemma 2.3.5. Thus, the Z-definable family of integrable functions $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s} \in S}$ given by

$$\begin{aligned} \mathfrak{f}_{\boldsymbol{s}} : U \times V_{\boldsymbol{s}} \times W_{\boldsymbol{s}} &\to \mathrm{p}^\Gamma \\ (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &\mapsto p^{-\operatorname{val}(w_{e_{n+1}+1})}. \end{aligned}$$

satisfies

$$\begin{aligned}
\#\mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a}) &= \#\left(A' \times B_{\boldsymbol{s}} \times \prod_{\ell=1}^{e_{n+1}} [0, \sigma_{n+1} \cdot a)_{d_{n+1,\ell}}\right) \\
&= \frac{\# A}{\prod_{\ell=1}^{m} d_{0,\ell}} \cdot \left(\prod_{i=1}^{n} \prod_{\ell=1}^{e_i} \frac{\sigma_i \cdot \mathrm{val}(s_i)}{d_{i,\ell}}\right) \cdot \left(\prod_{\ell=1}^{e_{n+1}} \frac{\sigma_{n+1} \cdot a}{d_{n+1,\ell}}\right) \\
&= \frac{1}{\prod d_{i,\ell}} \cdot \# A \cdot \prod_{i=1}^{n} (\sigma_i \cdot \mathrm{val}(s_i))^{e_i} \cdot (\sigma_{n+1} \cdot a)^{e_{n+1}} \\
&= \underbrace{\prod_{i=1}^{n+1} \sigma_i^{e_i}}_{=\sigma} \cdot P(\mathbf{val}(\boldsymbol{s}), a),
\end{aligned}$$

for all $(\boldsymbol{s}, a) \in X$, as well as $\mathfrak{f}_{\boldsymbol{s}}^{-1}(p^{-a}) = \emptyset$ for $(\boldsymbol{s}, a) \notin X$. Hence we have $\chi_S([\mathfrak{f}]) = f_{X,\sigma \cdot P}$ as claimed. □

Remark 4.2.6. As already mentioned, the isomorphism χ_S from Proposition 4.2.4 induces a multiplication on the group $\mathbf{P}_S(Z)$ by transferring the multiplication from $K_{\mathrm{int},S}(Z)$. This gives $\mathbf{P}_S(Z)$ the structure of a ring with unit $f_{S \times \{0\},1}$.

For $f, g \in \mathbf{P}_S(Z)$, we have

$$(f \cdot g)(\boldsymbol{s}, a) = \sum_{a_1 + a_2 = a} f(\boldsymbol{s}, a_1) \cdot g(\boldsymbol{s}, a_2)$$

whenever the sum on the right-hand side is defined (in particular, for those $\boldsymbol{s}$, for which at least one of the maps $a \mapsto f(\boldsymbol{s}, a)$ or $a \mapsto g(\boldsymbol{s}, a)$ has finite support – note that the sum is then finite).

Corollary 4.2.7 (of Proposition 4.2.4)**.** *There is an isomorphism (a priori: of groups, a posteriori: of rings) $\chi : K_{\mathrm{int}}(Z) \to \mathbf{P}(Z)$, induced by*

$$[\mathfrak{f}] \mapsto \left(\boldsymbol{a} \mapsto \#\mathfrak{f}^{-1}(p^{-\boldsymbol{a}})\right),$$

and inducing a ring structure on $\mathbf{P}(Z)$ with unit $f_{\{0\},1} = \mathrm{const}_{\{0\}}(1)$.

Corollary 4.2.8 (of Proposition 4.2.4)**.** *The additive group of the ring $K_{\mathrm{int},S}(Z)$ is torsion-free. Equivalently, the canonical map $K_{\mathrm{int},S}(Z) \to K_{\mathrm{int},S}(Z) \otimes \mathbb{Q}$ is injective.*

Proof. This follows from the proposition since $\mathbf{P}_S(Z)$ is torsion-free (see the remark just before Proposition 4.2.4). $\square$

Lemma 4.2.9. *Let $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be Z-definable families of integrable functions on RV^*_* with $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] = [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \in K_{\mathrm{int},S}(Z)$.*

Then there exists a Z-definable family of bijections $h_{\boldsymbol{s}} : \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) \to \mathrm{dom}(\mathfrak{g}_{\boldsymbol{s}})$ satisfying $\mathfrak{g}_{\boldsymbol{s}} \circ h_{\boldsymbol{s}} = \mathfrak{f}_{\boldsymbol{s}}$.

Proof. This follows from (the proof of) Proposition 4.2.4: Given $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ with $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] = [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]$, we have

$$\#\mathfrak{f}_{\boldsymbol{s}}^{-1}(\alpha) = \#\mathfrak{g}_{\boldsymbol{s}}^{-1}(\alpha)$$

for all $\boldsymbol{s} \in S$ and all $\alpha \in \mathrm{p}^{\Gamma}$ by Proposition 4.2.4. By [CH18, Theorem 5.2.2] (and partitioning S so that $\mathbf{val}\restriction S$ is injective) this implies the existence of a Z-definable family of bijections $h_{\boldsymbol{s},\alpha} : \mathfrak{f}_{\boldsymbol{s}}^{-1}(\alpha) \to \mathfrak{g}_{\boldsymbol{s}}^{-1}(\alpha)$, just as in the proof of the injectivity of χ_S in Proposition 4.2.4. As there, fix $\boldsymbol{s} \in S$ and define $h_{\boldsymbol{s}} : \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) \to \mathrm{dom}(\mathfrak{g}_{\boldsymbol{s}})$ in the obvious way, by putting together the corresponding $h_{\boldsymbol{s},\alpha}$. Then $h_{\boldsymbol{s}}$ is as desired. $\square$

Corollary 4.2.10. *Let $\mathfrak{f}$ and $\mathfrak{g}$ be Z-definable integrable functions on RV^*_* with $[\mathfrak{f}] = [\mathfrak{g}]$. Then there is a Z-definable map $h : \mathrm{dom}(\mathfrak{f}) \to \mathrm{dom}(\mathfrak{g})$ with $\mathfrak{g} \circ h = \mathfrak{f}$.*

Corollary 4.2.11. *There is an injective ring homomorphism from $K_b^{\mathrm{RV}}(Z) \cong K_b^{\Gamma}(Z)$ to $K_{\mathrm{int}}(Z)$ induced by*

$$\#U \mapsto [\mathrm{const}_U(1)]$$

.

Proof. Note that, by Definition 2.3.4 and Definition 4.1.2, the given mapping rule respects the relations of $K_b^{\mathrm{RV}}(Z)$ – and hence yields a well-defined group homomorphism. Indeed, for any two disjoint Z-definable subsets $U, V \subset \mathrm{RV}^*_*$ we have

$$\begin{aligned}[\mathrm{const}_{U\cup V}(1)] &= [\mathrm{const}_U(1) \cup \mathrm{const}_V(1)] \\ &= [\mathrm{const}_U(1)] + [\mathrm{const}_V(1)],\end{aligned}$$

and for any two Z-definable sets $U' \subset \mathrm{RV}^*_*$ and $V' \subset \mathrm{RV}^*_*$ in Z-definable bijection, we have

$$[\mathrm{const}_{U'}(1)] = [\mathrm{const}_{V'}(1)].$$

Moreover, for arbitrary Z-definable sets $U \subset \mathrm{RV}_*^*$ and $V \subset \mathrm{RV}_*^*$, we have

$$\mathrm{const}_{U\times V}(1) = \mathrm{const}_U(1) \star \mathrm{const}_V(1)$$

by Definition 4.1.2, hence the group homomorphism from $K_b^{\mathrm{RV}}(Z)$ to $K_{\mathrm{int}}(Z)$ also respects the multiplication rules and therefore the ring structures.

Lastly, injectivity follows from Corollary 4.2.10, using the fact that any element of $K_b^{\mathrm{RV}}(Z)$ can be written as the difference of two generators: If such an element $\#U - \#V$ lies in the kernel of the homomorphism given by the mapping rule from the statement above, then we have

$$[\mathrm{const}_U(1)] = [\mathrm{const}_V(1)]$$

in $K_{\mathrm{int}}(Z)$, yielding a Z-definable bijection $h : U \to V$ and therefore $\#U = \#V$. $\square$

Using Corollary 4.2.11 to just write $\#U \in K_{\mathrm{int}}(Z)$ for Z-definable sets $U \subset \mathrm{RV}_*^*$, we obtain the following improvement of Remark 4.1.8.

Corollary 4.2.12. *The integral* $\int_{\mathrm{mot}}$ *satisfies the equality* (4.1) *on p. 48 whenever* $\mathrm{im}(\mathfrak{f})$ *is finite. More precisely, given an integrable function* $\mathfrak{f}$ *on* RV_*^* *with finite image, we have*

$$\begin{aligned}\int_{\mathrm{mot}} \mathfrak{f} &= [\mathfrak{f}] + (\mathrm{p} - p)\\ &= \sum_{\alpha\in\mathrm{im}(\mathfrak{f})} [\mathrm{const}_{\mathfrak{f}^{-1}(\alpha)}(\alpha)] + (\mathrm{p} - p)\\ &= \sum_{\alpha\in\mathrm{im}(\mathfrak{f})} [\mathrm{const}_{\{0\}}(\alpha)] \cdot [\mathrm{const}_{\mathfrak{f}^{-1}(\alpha)}(1)] + (\mathrm{p} - p)\\ &= \sum_{\alpha\in\mathrm{im}(\mathfrak{f})} \alpha \cdot \#\mathfrak{f}^{-1}(\alpha) + (\mathrm{p} - p).\end{aligned}$$

4.3 Integrable functions with finite images

We now want to get a better understanding of $R_{\mathrm{mot},S}(Z)$. In particular, note that we have not even seen that it is non-trivial yet.

The main purpose of this section is to prove Theorem 4.3.8. The non-family version of the same statement, Corollary 4.3.9, says that for any Z-definable integrable function $\mathfrak{f}$ on RV_*^*, there are two such functions with finite image such that the difference of their integrals equals the integral of $\mathfrak{f}$. This will simplify working with (families of) integrable functions on RV_*^*, since it allows us (for some purposes, at least) to assume their images to be finite. It also emphasizes the strength of Remark 4.1.8 and its generalization Corollary 4.2.12.

For some proofs in this section, it is more convenient to work with $(K_{\mathrm{int},S}(Z)\otimes\mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}}$ instead of $R_{\mathrm{mot},S}(Z) = K_{\mathrm{int},S}(Z)/(\mathrm{p}-p)_S$. In fact, Proposition 4.3.3 says that they are isomorphic, so we will not lose any information. Before we are able to prove this, we need the following two lemmata.

Lemma 4.3.1. *There is a (necessarily unique) injective homomorphism from $\mathbb{Z}$ into $R_{\mathrm{mot},S}(Z) = K_{\mathrm{int},S}(Z)/(\mathrm{p}-p)_S$.*

Proof. We will establish the claim for the ring $\mathbf{P}_S(Z)/(f_{\mathrm{p}-p,S})$, which is isomorphic to $R_{\mathrm{mot},S}(Z)$, where $(f_{\mathrm{p}-p,S})$ is the image of $(\mathrm{p}-p)_S$ under the isomorphism $\chi_S : K_{\mathrm{int},S}(Z) \to \mathbf{P}_S(Z)$ from Proposition 4.2.4, i.e., the ideal generated by

$$\begin{aligned} f_{\mathrm{p}-p,S} &:= f_{S\times\{-1\},1} - p\cdot f_{S\times\{0\},1} \\ &= \chi_S([(\mathrm{const}_{\{0\}}(\mathrm{p}))_{\boldsymbol{s}\in S}] - p\cdot[(\mathrm{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}]). \end{aligned}$$

Note that any homomorphism from $\mathbb{Z}$ to $\mathbf{P}_S(Z)/(f_{\mathrm{p}-p,S})$ has to send $1\in\mathbb{Z}$ to the multiplicative identity element $f_{S\times\{0\},1} + (f_{\mathrm{p}-p,S})$, and hence has to send $m\in\mathbb{Z}$ to $f_{S\times\{0\},m} + (f_{\mathrm{p}-p,S})$, where

$$\begin{aligned} f_{S\times\{0\},m} : S\times\Gamma &\to K_b^{\Gamma}(\Gamma) \\ (\boldsymbol{s},a) &\mapsto \begin{cases} m, & \text{if } a=0 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

We thus have to show that, for $m\in\mathbb{Z}$, the function $f_{S\times\{0\},m}$ only lies in the ideal $(f_{\mathrm{p}-p,S})$ if $m=0$. Towards this end, suppose that we have

$$\begin{aligned} f_{S\times\{0\},m} &= g\cdot f_{\mathrm{p}-p,S} \\ &= g\cdot(f_{S\times\{-1\},1} - p\cdot f_{S\times\{0\},1}) \end{aligned}$$

for some $m\in\mathbb{Z}$ and some $g\in\mathbf{P}_S(Z)$, and let us fix some $\boldsymbol{s}\in S$. We aim to show that $g(\boldsymbol{s},\bullet)=0$, which then implies $m=0$ and hence finishes the proof. For all $a\in\Gamma$, we have

$$\begin{aligned} f_{S\times\{0\},m}(\boldsymbol{s},a) &= (g\cdot(f_{S\times\{-1\},1} - p\cdot f_{S\times\{0\},1}))(\boldsymbol{s},a) \\ &= g(\boldsymbol{s},a+1) - p\cdot g(\boldsymbol{s},a). \end{aligned} \tag{4.3}$$

If $g(\boldsymbol{s},\bullet)\neq 0$, there is some minimal $a_0\in\Gamma$ with $g(\boldsymbol{s},a)=0$ for all $a\le a_0$, since the support of $g(\boldsymbol{s},\bullet)$ is bounded from below by definition of $\mathbf{P}_S(Z)$. We then have

$$\begin{aligned} f_{S\times\{0\},m}(\boldsymbol{s},a_0) &= g(\boldsymbol{s},a_0+1) - p\cdot g(\boldsymbol{s},a_0) \\ &= g(\boldsymbol{s},a_0+1)\neq 0, \end{aligned}$$

implying $a_0 = 0$ and $g(\boldsymbol{s}, 1) = f_{S\times\{0\},m}(\boldsymbol{s}, 0) = m$. For $a > 0$, the equation (4.3) yields

$$\begin{aligned} g(\boldsymbol{s}, a+1) &= \underbrace{f_{\{0\},m}(\boldsymbol{s}, a)}_{=0} + p \cdot g(\boldsymbol{s}, a) \\ &= p \cdot g(\boldsymbol{s}, a), \end{aligned}$$

and thus, by an induction on t, we have

$$g(\boldsymbol{s}, a+t) = p^t \cdot g(\boldsymbol{s}, a) \tag{4.4}$$

for all $a > 0$ and all $t \in \mathbb{N}$.

Now $g \in \mathbf{P}_S(Z)$ is piecewise polynomial, i.e., there is a partition of Γ into finitely many ($Z(\boldsymbol{s})$-definable) pieces such that $g(\boldsymbol{s}, \bullet)$ is polynomial on each piece, with coefficients in $K_b^\Gamma(Z(\boldsymbol{s}))$. We can assume that each of theses pieces is of the form $(a, b)_{\equiv_d r}$ for some $a \in \Gamma \cup \{-\infty\}$, $b \in \Gamma \cup \{\infty\}$ and $d \in \mathbb{N}_{>0}$, $r \in \{0, \dots, d-1\}$. Since there are only finitely many pieces covering all of Γ, there is, for each archimedean class $A \subset \Gamma$, at least one such piece for which the intersection $(a, b)_{\equiv_d r} \cap A$ is infinite and not bounded from above. Fix any element a' of that intersection and note that $Q(t) = g(\boldsymbol{s}, a' + t \cdot d)$ is polynomial in t, with coefficients in $K_b^\Gamma(Z(\boldsymbol{s}, a'))$, satisfying

$$\begin{aligned} Q(t) &= g(\boldsymbol{s}, a' + t \cdot d) \\ &= (p^d)^t \cdot g(\boldsymbol{s}, a') \\ &= (p^d)^t \cdot Q(0) \end{aligned}$$

for all $t \in \mathbb{N}$. By Proposition 2.1.2, we then have $Q = 0$, and hence $g(\boldsymbol{s}, a') = 0$, so that the equation (4.4) implies $g(\boldsymbol{s}, \bullet)\restriction A = 0$. Since this argument works for all archimedean classes $A \subset \Gamma$, we have $g(\boldsymbol{s}, \bullet) = 0$ and hence $m = 0$, as claimed. □

Lemma 4.3.2 (see also [CH21, Lemma 3.11.]). *There is a (unique) injective homomorphism from $\mathbb{Q}$ into $R_{\mathrm{mot},S}(Z) = K_{\mathrm{int},S}(Z)/(\mathrm{p} - p)_S$. In particular, $R_{\mathrm{mot},S}(Z)$ is torsion-free.*

Proof. By Lemma 4.3.1 and Lemma 2.1.3, we only have to show that $\frac{1}{p} \in R_{\mathrm{mot},S}(Z)$ and $\frac{1}{p^d - 1} \in R_{\mathrm{mot},S}(Z)$ for all $d \in \mathbb{N}$. Firstly, note that we have

$$p \cdot [(\mathrm{const}_{\{0\}}(\mathrm{p}^{-1}))_{\boldsymbol{s}\in S}] \equiv \underbrace{[(\mathrm{const}_{\{0\}}(\mathrm{p}))_{\boldsymbol{s}\in S}] \cdot [(\mathrm{const}_{\{0\}}(\mathrm{p}^{-1}))_{\boldsymbol{s}\in S}]}_{= [(\mathrm{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}] = 1} \pmod{(\mathrm{p} - p)_S},$$

and thus $[(\mathrm{const}_{\{0\}}(\mathrm{p}^{-1}))_{\boldsymbol{s}\in S}] + (\mathrm{p} - p)_S = \frac{1}{p} \in R_{\mathrm{mot},S}(Z)$.

Secondly, we will now show $\frac{1}{p^d-1} \in R_{\mathrm{mot},S}(Z)$. Towards this end, consider the Z-definable families $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ of integrable functions given by

$$\begin{aligned}
\mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) &= U = \{u \in \mathrm{RV}_1 \mid \mathrm{val}(u) \in [0,\infty)_d\,, \mathrm{ac}_1(u) = 1\} \text{ with} \\
&\mathfrak{f}_{\boldsymbol{s}}(u) = \mathrm{p}^{-d-\mathrm{val}(u)} \text{ for all } u \in U \text{ and} \\
\mathrm{dom}(\mathfrak{g}_{\boldsymbol{s}}) &= V = \{v \in \mathrm{RV}_1 \mid \mathrm{val}(v) \in [-d,\infty)_d\,, \mathrm{ac}_1(v) = 1\} = U \cup \{p^{-d}\} \text{ with} \\
&\mathfrak{g}_{\boldsymbol{s}}(v) = \mathrm{p}^{-d-\mathrm{val}(v)} \text{ for all } v \in V
\end{aligned}$$

for all $\boldsymbol{s} \in S$. Note that the Z-definable family of bijections $h_{\boldsymbol{s}} : U \to V, u \mapsto p^{-d} \cdot u$ satisfies

$$\begin{aligned}
(\mathfrak{g}_{\boldsymbol{s}} \circ h_{\boldsymbol{s}})(u) &= \mathfrak{g}_{\boldsymbol{s}}(p^{-d} \cdot u) \\
&= \mathrm{p}^{-d-\mathrm{val}(p^{-d}\cdot u)} \\
&= \mathrm{p}^{-d-(\mathrm{val}(u)-d)} \\
&= \mathrm{p}^{\mathrm{d}} \cdot \mathfrak{f}_{\boldsymbol{s}}(u) \\
&= \mathrm{const}_{\{0\}}(\mathrm{p}^{\mathrm{d}}) \star \mathfrak{f}_{\boldsymbol{s}}(u)
\end{aligned}$$

for all $\boldsymbol{s} \in S$ and all $u \in U$, so we have

$$\begin{aligned}
[(\mathrm{const}_{\{0\}}(\mathrm{p}^{\mathrm{d}}))_{\boldsymbol{s}\in S}] \cdot [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] &= [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \\
&= [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] + [(\mathrm{const}_{\{p^{-d}\}}(1))_{\boldsymbol{s}\in S}] \\
&= [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] + \underbrace{[(\mathrm{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}]}_{=1},
\end{aligned}$$

where the second equality is due to the fact that we have $V = U \cup \{p^{-d}\}$ with $\mathfrak{g}_{\boldsymbol{s}}\restriction U = \mathfrak{f}_{\boldsymbol{s}}$ and $\mathfrak{g}_{\boldsymbol{s}}(p^{-d}) = \mathrm{p}^{-d-\mathrm{val}(p^{-d})} = 1$ for all $\boldsymbol{s} \in S$. Since we have

$$(p^d - 1) \cdot [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \equiv \underbrace{([(\mathrm{const}_{\{0\}}(\mathrm{p}^{\mathrm{d}}))_{\boldsymbol{s}\in S}] - 1) \cdot [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]}_{=1} \pmod{(\mathrm{p}-p)_S},$$

this implies $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] + (\mathrm{p}-p)_S = \frac{1}{p^d-1}$ in $R_{\mathrm{mot},S}(Z) = K_{\mathrm{int},S}(Z)/(\mathrm{p}-p)_S$. □

Proposition 4.3.3. *There is a canonical isomorphism of rings*

$$\rho : \overbrace{K_{\mathrm{int},S}(Z)/(\mathrm{p}-p)_S}^{=R_{\mathrm{mot},S}(Z)} \to (K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}},$$
induced by $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] + (\mathrm{p}-p)_S \mapsto [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \otimes 1 + (\mathrm{p}-p)_S^{\mathbb{Q}}$.

In particular, $R_{\mathrm{mot}}(Z) = K_{\mathrm{int}}(Z)/(\mathrm{p}-p)$ *is isomorphic to* $(K_{\mathrm{int}}(Z)\otimes\mathbb{Q})/(\mathrm{p}-p)^{\mathbb{Q}}$.

Proof. Consider the ring homomorphism $\rho_0 : K_{\mathrm{int},S}(Z) \to (K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p} - p)_S^{\mathbb{Q}}$ with $\rho_0([(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]) = [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]\otimes 1+(\mathrm{p}-p)_S^{\mathbb{Q}}$, i.e., ρ_0 is the composition of the embedding $K_{\mathrm{int},S}(Z) \hookrightarrow K_{\mathrm{int},S}(Z)\otimes\mathbb{Q}$ and the quotient map $K_{\mathrm{int},S}(Z)\otimes\mathbb{Q} \twoheadrightarrow (K_{\mathrm{int},S}(Z)\otimes\mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}}$. We will show that ρ_0 is surjective with $\ker(\rho_0) = (\mathrm{p}-p)_S$, which yields the claim.

ρ_0 **is surjective.** Let $F\otimes q+(\mathrm{p}-p)_S^{\mathbb{Q}}$ be any element of $(K_{\mathrm{int},S}(Z)\otimes\mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}}$. Let $Q \in K_{\mathrm{int},S}(Z)$ with $Q + (\mathrm{p} - p)_S = q \in \mathbb{Q} \subset K_{\mathrm{int},S}(Z)/(\mathrm{p} - p)_S$ and note that

$$Q \otimes 1 + (\mathrm{p} - p)_S^{\mathbb{Q}} = 1 \otimes q + (\mathrm{p} - p)_S^{\mathbb{Q}}$$

in $(K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p} - p)_S^{\mathbb{Q}}$. Thus we have

$$\begin{aligned}\rho_0(F \cdot Q) &= \rho_0(F) \cdot \rho_0(Q)\\ &= (F \otimes 1) \cdot (Q \otimes 1) + (\mathrm{p} - p)_S^{\mathbb{Q}}\\ &= F \otimes q + (\mathrm{p} - p)_S^{\mathbb{Q}},\end{aligned}$$

proving that ρ_0 is surjective.

$\ker(\rho_0) = (\mathrm{p} - p)_S$**.** Clearly, $(\mathrm{p} - p)_S$ is contained in the kernel of ρ_0. For the other direction, let $F \in K_{\mathrm{int},S}(Z)$ with $\rho_0(F) = (\mathrm{p} - p)_S^{\mathbb{Q}}$, i.e., $F \otimes 1 \in (\mathrm{p} - p)_S^{\mathbb{Q}}$. Then there is some $G \in K_{\mathrm{int},S}(Z)$ and some $q \in \mathbb{Q}$ such that

$$F \otimes 1 = (G \otimes q) \cdot (P \otimes 1)$$

where $P = [(\mathrm{const}_{\{0\}}(\mathrm{p}))_{\boldsymbol{s}\in S}] - p \cdot [(\mathrm{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}]$ is the generator of the ideal $(\mathrm{p} - p)_S \subset K_{\mathrm{int},S}(Z)$. Write $q = \frac{q_1}{q_2}$ for some $q_1, q_2 \in \mathbb{Z}$ with $q_2 \neq 0$. Then we have

$$(q_2 \cdot F) \otimes 1 = ((q_1 \cdot G) \otimes 1) \cdot (P \otimes 1),$$

and hence

$$q_2 \cdot F = (q_1 \cdot G) \cdot P$$

since $K_{\mathrm{int},S}(Z)$ embeds into $K_{\mathrm{int},S}(Z) \otimes \mathbb{Q}$, by Corollary 4.2.8. Thus $q_2 \cdot F$ lies in $(\mathrm{p} - p)$, and by Lemma 4.3.2, we have $F \in (\mathrm{p} - p)$. □

Definition 4.3.4. The ring $K^{\mathrm{fin}}_{\mathrm{int},S}(Z)$ is the subring of $K_{\mathrm{int},S}(Z)$ generated by those symbols $[\mathfrak{f}] \in K_{\mathrm{int},S}(Z)$ for which $\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}})$ is finite for all $\boldsymbol{s} \in S$.

Definition 4.3.5. The ring $\mathbf{P}^{\mathrm{fin}}_S(Z)$ is the subring of $\mathbf{P}_S(Z)$ generated by those maps $f_{X,P} \in \mathbf{P}_S(Z)$ for which $X \cap \{\boldsymbol{s}\} \times \Gamma$ is finite for all $\boldsymbol{s} \in S$. (Equivalently, it is the subring generated by those $f_{X,P}$ for which $\mathrm{supp}\,(a \mapsto f_{X,P}(\boldsymbol{s}, a))$ is finite for all $\boldsymbol{s} \in S$.)

Finally, $\mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z)$ is the subring of $\mathbf{P}^{\mathbb{Q}}_S(Z)$ generated by those maps $f_{X,P} \in \mathbf{P}^{\mathbb{Q}}_S(Z)$

for which $X \cap \{\boldsymbol{s}\} \times \Gamma$ is finite for all $\boldsymbol{s} \in S$. (And again, equivalently, it is the subring generated by those $f_{X,P}$ for which $\operatorname{supp}(a \mapsto f_{X,P}(\boldsymbol{s}, a))$ is finite for all $\boldsymbol{s} \in S$.)

Note that χ_S restricts to an isomorphism from $K^{\mathrm{fin}}_{\mathrm{int},S}(Z)$ to $\mathbf{P}^{\mathrm{fin}}_S(Z)$: Consider any Z-definable family $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ of integrable functions on RV^*_* and let $f := \chi_S([(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}])$ be the image under χ_S of its class in $K_{\mathrm{int},S}(Z)$. Then, by definition of χ_S, we have $\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}}) = p^{-\operatorname{supp}(f(\boldsymbol{s},\bullet))}$ for all $\boldsymbol{s} \in S$, hence the image of $K^{\mathrm{fin}}_{\mathrm{int},S}(Z)$ under χ_S is contained in $\mathbf{P}^{\mathrm{fin}}_S(Z)$. Moreover, the preimage in $K_{\mathrm{int},S}(Z)$ of a given generator $f_{X,P}$ of $\mathbf{P}^{\mathrm{fin}}_S(Z)$ as constructed in the proof of surjectivity of χ_S (i.e., the second part of the proof of Proposition 4.2.4) already lies in $K^{\mathrm{fin}}_{\mathrm{int},S}(Z)$. Thus we have an isomorphism $\chi^{\mathrm{fin}}_S : K^{\mathrm{fin}}_{\mathrm{int},S}(Z) \to \mathbf{P}^{\mathrm{fin}}_S(Z)$, given by restricting χ_S.

Recall that there is an isomorphism from $\mathbf{P}_S(Z) \otimes \mathbb{Q}$ to $\mathbf{P}^{\mathbb{Q}}_S(Z)$ by Lemma 2.1.4 (see also the note just before Proposition 4.2.4), and note that it furthermore restricts to an isomorphism from $\mathbf{P}^{\mathrm{fin}}_S(Z) \otimes \mathbb{Q}$ to $\mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z)$.

Moreover, the following Lemma 4.3.6 provides an isomorphism between appropriate quotients of $\mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z)$ and $\mathbf{P}^{\mathbb{Q}}_S(Z)$ – a key ingredient for proving Theorem 4.3.8.

Lemma 4.3.6. *The canonical map* $\psi_0 : \mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z) \to \mathbf{P}^{\mathbb{Q}}_S(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$ *induces an isomorphism*

$$\psi : \mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}} \to \mathbf{P}^{\mathbb{Q}}_S(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$$

of rings, i.e., ψ_0 *is surjective and satisfies* $\ker(\psi_0) = (f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}}$.

Here and in the following, $(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$ *and* $(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}}$ *denote the ideals generated by* $f_{\mathrm{p}-p,S}$ *in* $\mathbf{P}^{\mathbb{Q}}_S(Z)$ *and in* $\mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z)$ *respectively, where*

$$f_{\mathrm{p}-p,S} = f_{S\times\{-1\},1} - \underbrace{p \cdot f_{S\times\{0\},1}}_{= f_{S\times\{0\},p}} \in \mathbf{P}^{\mathrm{fin}}_S(Z) \subset \mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z),$$

as introduced in the proof of Lemma 4.3.1.

(Note that $(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$ *is the image of* $(\mathrm{p}-p)^{\mathbb{Q}}_S$ *under the isomorphism* $\chi^{\mathbb{Q}}_S$ *from Proposition 4.2.4, as its generator* $f_{\mathrm{p}-p,S}$ *is the image of the generator of* $(\mathrm{p}-p)^{\mathbb{Q}}_S$*.)*

Proof. We first show that ψ_0 induces an injective homomorphism ψ and then prove surjectivity.

Step 1, $\ker(\psi_0) = (f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}}$**:** Since $\psi_0(f_{\mathrm{p}-p,S}) = 0$, we clearly have $\ker(\psi_0) \supset (f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}}$. For the other direction, let $f \in \ker(\psi_0)$, i.e., suppose that $f \in \mathbf{P}^{\mathbb{Q},\mathrm{fin}}_S(Z)$ and that there exists some $h \in \mathbf{P}^{\mathbb{Q}}_S(Z)$ for which $f = f_{\mathrm{p}-p,S} \cdot h$. We will show that h already lies in $\mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$, yielding that f lies in $(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}} \subset (f_{\mathrm{p}-p,S})^{\mathbb{Q}} \cap \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$.

By Remark 4.2.6, the condition $f = f_{\mathrm{p}-p,S} \cdot h$ means that

$$\begin{aligned} f(\boldsymbol{s}, a) &= f_{\mathrm{p}-p,S}(\boldsymbol{s}, -1) \cdot h(\boldsymbol{s}, a+1) + f_{\mathrm{p}-p,S}(\boldsymbol{s}, 0) \cdot h(\boldsymbol{s}, a) \\ &= h(\boldsymbol{s}, a+1) - p \cdot h(\boldsymbol{s}, a) \end{aligned} \tag{4.5}$$

for all $(\boldsymbol{s}, a) \in S \times \Gamma$.

To prove $h \in \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$, we fix some $\boldsymbol{s} \in S$ and show that the support of the map $a \mapsto h(\boldsymbol{s}, a)$ is finite. As $f \in \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$, the support of $a \mapsto f(\boldsymbol{s}, a)$ is finite (and Z-definable, i.e., already contained in $\mathrm{acl}(Z) = Z$), say $\mathrm{supp}(f(\boldsymbol{s}, \cdot)) = \{c_1, \ldots, c_\ell\}$ with $c_i < c_{i+1}$ for all i. Now fix a finite partition $\mathcal{P}$ of $\mathrm{supp}(h(\boldsymbol{s}, \cdot))$ consisting of Z-definable sets such that $h(\boldsymbol{s}, \cdot)$ is given by a polynomial on each piece of the partition. Refine the partition so that each piece

- is either equal to $\{c_i\}$ for some i or contained in one of the intervals $(-\infty, c_1)$, (c_i, c_{i+1}) for some i, or (c_ℓ, ∞), and
- is of the form $[a, b)_d$ for some $d \in \mathbb{N}$ and $a \in \Gamma$, $b \in \Gamma \cup \{\infty\}$.

Our claim is equivalent to the statement that each piece of the partition $\mathcal{P}$ is a finite set. So suppose, towards a contradiction, that $I = [a, b)_d \in \mathcal{P}$ is an infinite piece of the partition.

Let $b' = \infty$ if $b = \infty$ and $b' = b - d + 1$ otherwise, so that $[a, b')$ is the convex hull of I in both cases. Then we have $c \neq c_i$ for all i and all (supposedly infinitely many) $c \in [a, b')$. In particular, all $c \in [a, b')$ satisfy $f(\boldsymbol{s}, c) = 0$, so that the equality (4.5) yields

$$\begin{aligned} h(\boldsymbol{s}, c+1) &= f(\boldsymbol{s}, c) + p \cdot h(\boldsymbol{s}, c) \\ &= p \cdot h(\boldsymbol{s}, c) \end{aligned}$$

for all $c \in [a, b')$. By induction, and since $b' \ggg a$ by assumption, this implies

$$h(\boldsymbol{s}, a + d \cdot m) = (p^d)^m \cdot h(\boldsymbol{s}, a)$$

for all $m \in \mathbb{N}$.

As $h(\boldsymbol{s}, c)$ is polynomial in c on the supposedly infinite set $I = [a, b)_d$ and since $p^d \in \mathbb{Z} \setminus \{0, 1\}$, we can apply Proposition 2.1.2, yielding $h(\boldsymbol{s}, a) = 0$. But this is a contradiction to the assumption that $a \in I \subset \mathrm{supp}(h(\boldsymbol{s}, \cdot))$. Hence there can be no infinite set in the partition $\mathcal{P}$, i.e., it is a partition of $\mathrm{supp}(h(\boldsymbol{s}, \cdot))$ into finite sets. Consequently, the support of $h(\boldsymbol{s}, \cdot)$ is itself finite. Therefore h lies in $\mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$, showing that $f = f_{\mathrm{p}-p,S} \cdot h$ lies in $(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}}$, and we thus obtain the equality $\ker(\psi_0) = (f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}}$ as claimed.

Step 2, $\operatorname{im}(\psi_0) = \mathbf{P}_S^{\mathbb{Q}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$**:** To show that ψ_0 is surjective, fix a generator $f_{X,P}$ of $\mathbf{P}_S^{\mathbb{Q}}(Z)$, i.e., more explicitly: Let $X \subset S \times \Gamma$ be a Z-definable set whose fibers $X_{\boldsymbol{s}} \subset \Gamma$ are bounded from below for all $\boldsymbol{s} \in S$, let $P \in (K_b^\Gamma(Z) \otimes \mathbb{Q})[T_1, \dots, T_{n+1}]$ be a polynomial, and consider the map $f_{X,P} \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ given by

$$f_{X,P} : S \times \Gamma \to K_b^\Gamma(\Gamma) \otimes \mathbb{Q}$$
$$(\boldsymbol{s}, c) \mapsto \begin{cases} P(\operatorname{val}(s_1), \dots, \operatorname{val}(s_n), c) & \text{if } (\boldsymbol{s}, c) \in X \\ 0 & \text{otherwise} \end{cases}.$$

We now have to find a preimage of $f_{X,P} + (f_{\mathrm{p}-p,S})^{\mathbb{Q}} \in \mathbf{P}_S^{\mathbb{Q}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$ in the ring $\mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$, that is to say, we have to find an element $h \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ for which we have

$$f_{X,P} - h \cdot f_{\mathrm{p}-p,S} \in \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z).$$

While it suffices to find any such h for proving surjectivity of ψ_0, we will later (for Lemma 4.4.12) need one whose support does not deviate too much from the set X. More precisely, we will show that we can choose h in such a way that

$$\operatorname{supp}(h(\boldsymbol{s}, \bullet)) \subset [\min(X_{\boldsymbol{s}}) - k, \infty)$$

for some $k \in \mathbb{N}_{>0}$ (which only depends on X), i.e., such that $h(\boldsymbol{s}, a) = 0$ whenever $a < \min(X_{\boldsymbol{s}}) - k$.

By Lemma 3.1.8 (and since $f_{X,P} = \sum_i f_{X_i,P}$ if X is the disjoint union of finitely many (Z-definable) sets X_i), it is enough to handle the case that $X_{\boldsymbol{s}} = [a(\boldsymbol{s}), b(\boldsymbol{s}))_d$ for some $d \in \mathbb{N}_{>0}$ and some Z-definable maps $a : S \to \Gamma$ and $b : S \to \Gamma \cup \{\infty\}$.

Now apply Lemma 2.1.1 to the rings $R = (K_b^\Gamma(Z) \otimes \mathbb{Q})[T_1, \dots, T_n]$ and $R' = (K_b^\Gamma(\Gamma) \otimes \mathbb{Q})[T_1, \dots, T_n]$, the polynomial $Q \in R[T_{n+1}]$ given by

$$Q(T_{n+1}) = P(T_1, \dots, T_n, T_{n+1}),$$

and the elements $a = p^d$ and $b = 1$ of $\mathbb{Z} \subset R$. This yields the existence of a polynomial $Q' \in R[T_{n+1}]$ and an integer $m \in \mathbb{N}$ with

$$-(p^d - 1)^m \cdot Q(c) = Q'(c + d) - p^d \cdot Q'(c)$$

for all $c \in \Gamma \subset K_b^\Gamma(\Gamma) \otimes \mathbb{Q} \subset R'$, so that we have

$$Q(c) = Q_d(c + d) - p^d \cdot Q_d(c)$$

for all $c \in \Gamma$, where $Q_d := -(p^d - 1)^{-m} \cdot Q' \in R[T_{n+1}]$. Consider the polynomial $P_d \in (K_b^\Gamma(Z) \otimes \mathbb{Q})[T_1, \dots, T_n, T_{n+1}]$ with $Q_d(T_{n+1}) = P_d(T_1, \dots, T_n, T_{n+1})$. For $f_{X,P_d} \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ and $f_{\mathrm{p}^{\mathrm{d}}-p^d,S} := f_{S \times \{-d\},1} - f_{S \times \{0\},p^d} \in \mathbf{P}_S^{\mathbb{Q}}(Z)$, we have

$$\begin{aligned}(f_{\mathrm{p}^{\mathrm{d}}-p^d,S} \cdot f_{X,P_d})(\boldsymbol{s}, c) &= \left((f_{S \times \{-d\},1} - f_{S \times \{0\},p^d}) \cdot f_{X,P_d}\right)(\boldsymbol{s}, c) \\ &= f_{X,P_d}(\boldsymbol{s}, c + d) - p^d \cdot f_{X,P_d}(\boldsymbol{s}, c)\end{aligned}$$

for all $(\boldsymbol{s}, c) \in S \times \Gamma$. Thus, for fixed $\boldsymbol{s} \in S$, we have

$$\begin{aligned}
&(f_{X,P} - f_{\mathrm{p}^{\mathrm{d}}-p^d,S} \cdot f_{X,P_d})(\boldsymbol{s}, c) \\
&\quad = f_{X,P}(\boldsymbol{s}, c) - (f_{X,P_d}(\boldsymbol{s}, c+d) - p^d \cdot f_{X,P_d}(\boldsymbol{s}, c)) \\
&\quad = \begin{cases} P(\boldsymbol{s}, c) - \big(P_d(\boldsymbol{s}, c+d) - p^d \cdot P_d(\boldsymbol{s}, c)\big) = 0, \\ \qquad \text{for } c \in X_{\boldsymbol{s}} \cap (X_{\boldsymbol{s}} - d) = [a(\boldsymbol{s}, b(\boldsymbol{s}) - d)_d \\ 0 - 0 = 0, \\ \qquad \text{for } c \notin X_{\boldsymbol{s}} \cup (X_{\boldsymbol{s}} - d) = [a(\boldsymbol{s}) - d, b(\boldsymbol{s}))_d\,, \\ \star, \quad \text{otherwise, i.e., for } c \in X_{\boldsymbol{s}} \triangle (X_{\boldsymbol{s}} - d) \end{cases}
\end{aligned}$$

Hence the support of $(f_{X,P} - f_{\mathrm{p}-p,S} \cdot f_{X,P_d})(\boldsymbol{s}, \cdot)$ is, for fixed $\boldsymbol{s}$, contained in the finite Z-definable set $X_{\boldsymbol{s}} \triangle (X_{\boldsymbol{s}} - d) \subset \{a(\boldsymbol{s}) - d, b(\boldsymbol{s})\}$. Thus $f := f_{X,P} - f_{\mathrm{p}^{\mathrm{d}}-p^d,S} \cdot f_{X,P_d}$ lies in $\mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$.

It is left to show that $f_{\mathrm{p}^{\mathrm{d}}-p^d,S} \in (f_{\mathrm{p}-p,S})^{\mathbb{Q}}$, which will then yield $f_{X,P} - f = f_{\mathrm{p}^{\mathrm{d}}-p^d,S} \cdot f_{X,P_d} \in (f_{\mathrm{p}-p,S})^{\mathbb{Q}}$. And indeed, we have

$$\begin{aligned}
f_{\mathrm{p}^{\mathrm{d}}-p^d,S} &= f_{S\times\{-d\},1} - f_{S\times\{0\},p^d} \\
&= (f_{S\times\{-1\},1})^d - (f_{S\times\{0\},p})^d \\
&= (f_{\mathrm{p}-p,S} + f_{S\times\{0\},p})^d - (f_{S\times\{0\},p})^d \\
&= \left(\sum_{j=0}^{d} \binom{d}{j} \cdot (f_{\mathrm{p}-p,S})^j \cdot (f_{S\times\{0\},p})^{d-j}\right) - (f_{S\times\{0\},p})^d \\
&= \sum_{j=1}^{d} \binom{d}{j} \cdot (f_{\mathrm{p}-p,S})^j \cdot (f_{S\times\{0\},p})^{d-j} \\
&\in (f_{\mathrm{p}-p,S})^{\mathbb{Q}},
\end{aligned}$$

completing the proof that ψ_0 is surjective.

For our additional request on the choice of h, which we will need in the proof of Lemma 4.4.12, note that the above can be written as

$$\begin{aligned}
f &= f_{X,P} - f_{\mathrm{p}^{\mathrm{d}}-p^d,S} \cdot f_{X,P_d} \\
&= f_{X,P} - h \cdot f_{\mathrm{p}-p,S},
\end{aligned}$$

where

$$h = f_{X,P_d} \cdot \sum_{j=1}^{d} \binom{d}{j} \cdot (f_{\mathrm{p}-p,S})^{j-1} \cdot (f_{S\times\{0\},p})^{d-j}.$$

By Remark 4.2.6, we thus have $h(\boldsymbol{s}, a) = 0$ whenever

$$\begin{aligned}
a &< \min(X_{\boldsymbol{s}}) + \min_{j=1}^{d}(j-1)\cdot(-1) \\
&= \min(X_{\boldsymbol{s}}) - (d-1).
\end{aligned}$$

This completes the proof of Step 2. □

Putting the results of the two steps together, the map

$$\psi : \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}} \to \mathbf{P}_S^{\mathbb{Q}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}},$$

induced by the canonical map $\psi_0 : \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z) \to \mathbf{P}_S^{\mathbb{Q}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$ is an isomorphism.

Remark 4.3.7. Note that the additional claim in Step 2 of the proof generalizes to arbitrary elements of $\mathbf{P}_S^{\mathbb{Q}}(Z)$ with Z-definable support.

More precisely, let $f \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ and suppose that $X := \mathrm{supp}(f)$ is Z-definable. Fix a partition of X into finitely many Z-definable sets X_i and polynomials P_i such that $f = \sum_i f_{X_i,P_i}$. Then Step 2 of the proof of Lemma 4.3.6 yields elements $h_i \in \mathbf{P}_S^{\mathbb{Q}}(Z)$ and integers $k_i \in \mathbb{N}_{>0}$ with $f_{X_i,P_i} - h_i \cdot f_{\mathrm{p}-p,S} \in \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)$ and

$$\mathrm{supp}(h_i(\boldsymbol{s}, \bullet)) \subset [\min(X_{i\boldsymbol{s}}) - k_i, \infty)\,.$$

For $h = \sum_i h_i$, we thus have

$$\begin{aligned} f - h \cdot f_{\mathrm{p}-p,S} &= \sum_i f_{X_i,P_i} - (\sum_i h_i) \cdot f_{\mathrm{p}-p,S} \\ &= \sum_i (f_{X_i,P_i} - h_i \cdot f_{\mathrm{p}-p,S}) \in \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z) \end{aligned}$$

and

$$\begin{aligned} \mathrm{supp}(h(\boldsymbol{s}, \bullet)) &\subset \bigcup_i \mathrm{supp}(h_i(\boldsymbol{s}, \bullet)) \\ &= \bigcup_i [\min(X_{i\boldsymbol{s}}) - k_i, \infty) \\ &\subset \left[\min_i(\min(X_{i\boldsymbol{s}}) - k_i), \infty\right) \\ &\subset [\min(X_{\boldsymbol{s}}) - k, \infty) \\ &\subset [\min(\mathrm{supp}(f(\boldsymbol{s}, \bullet))) - k, \infty)\,, \end{aligned}$$

where $k = \max_i k_i \in \mathbb{N}_{>0}$.

We now have all ingredients to prove the desired main result of this chapter.

Theorem 4.3.8 (Finite images are enough)**.** *The natural inclusion* $K_{\mathrm{int},S}^{\mathrm{fin}}(Z) \otimes \mathbb{Q} \hookrightarrow K_{\mathrm{int},S}(Z) \otimes \mathbb{Q}$ *induces an isomorphism between the quotients* $(K_{\mathrm{int},S}^{\mathrm{fin}}(Z) \otimes \mathbb{Q})/(\mathrm{p} - p)_S^{\mathbb{Q},\mathrm{fin}}$ *and* $(K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p} - p)_S^{\mathbb{Q}} \cong R_{\mathrm{mot},S}(Z)$.

Corollary 4.3.9. *The natural inclusion* $K_{\mathrm{int}}^{\mathrm{fin}}(Z) \otimes \mathbb{Q} \hookrightarrow K_{\mathrm{int}}(Z) \otimes \mathbb{Q}$ *induces an isomorphism between the quotients* $(K_{\mathrm{int}}^{\mathrm{fin}}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)^{\mathbb{Q},\mathrm{fin}}$ *and* $(K_{\mathrm{int}}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)^{\mathbb{Q}} \cong R_{\mathrm{mot}}(Z)$.

Proof of Theorem 4.3.8. First note that the ideal $(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$ is the image of $(\mathrm{p}-p)_S^{\mathbb{Q}}$ under the isomorphism $\chi_S^{\mathbb{Q}}$ from Proposition 4.2.4. Similarly, $(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}}$ is the image of $(\mathrm{p}-p)_S^{\mathbb{Q},\mathrm{fin}}$ under the restriction of $\chi_S^{\mathbb{Q}}$. We thus obtain isomorphisms

$$(K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}} \xrightarrow{\cong} \mathbf{P}_S^{\mathbb{Q}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}}$$

and

$$(K_{\mathrm{int},S}^{\mathrm{fin}}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q},\mathrm{fin}} \xrightarrow{\cong} \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}},$$

both induced by $\chi_S^{\mathbb{Q}}$.

Together with Lemma 4.3.6, this leads to the following commutative diagram.

$$\begin{tikzcd}
(K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}} \arrow[drr, "\cong"] & & \\
K_{\mathrm{int},S}(Z) \otimes \mathbb{Q} \arrow[u, two heads] \arrow[r, "\cong", "4.2.4"'] & \mathbf{P}_S^{\mathbb{Q}}(Z) \arrow[r, two heads] \arrow[d, phantom, "\cup"] & \mathbf{P}_S^{\mathbb{Q}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q}} \\
K_{\mathrm{int},S}^{\mathrm{fin}}(Z) \otimes \mathbb{Q} \arrow[u, hook] \arrow[r, "\cong"] \arrow[d, two heads] & \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z) \arrow[r, two heads] \arrow[ur, two heads] & \mathbf{P}_S^{\mathbb{Q},\mathrm{fin}}(Z)/(f_{\mathrm{p}-p,S})^{\mathbb{Q},\mathrm{fin}} \arrow[u, "\cong", "4.3.6"'] \\
(K_{\mathrm{int},S}^{\mathrm{fin}}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q},\mathrm{fin}} \arrow[urr, "\cong"] & &
\end{tikzcd} \tag{4.6}$$

Composing the appropriate isomorphisms from the bottom to the top of this diagram now yields the claim. ☐

The relevance of Theorem 4.3.8 relies on the ring $K_{\mathrm{int}}^{\mathrm{fin}}(Z)$ (as well as its variants) being easier to understand and deal with than $K_{\mathrm{int}}(Z)$. Even though this is a natural expectation, we have not seen any formal arguments supporting it yet. The following lemma fulfills our hopes and gives a rather tangible description of $K_{\mathrm{int}}^{\mathrm{fin}}(Z)$. It can be viewed as a generalization of Corollary 4.2.12, lifting the equation (4.1) on p. 48 to the ring $K_{\mathrm{int}}^{\mathrm{fin}}(Z)$, and further supports the notation used there.

Lemma 4.3.10. *There is an isomorphism of rings* $\sigma : K^{\mathrm{fin}}_{\mathrm{int}}(Z) \to \left(K^{\Gamma}_{b}(Z)\right)[T^{Z}]$ *induced by*

$$\begin{aligned}[\mathfrak{f}] \mapsto & \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} \#\mathfrak{f}^{-1}(\alpha) \cdot T^{-\,\mathrm{val}(\alpha)} \\ &= \sum_{\mathrm{p}^{-a} \in \mathrm{im}(\mathfrak{f})} \#\mathfrak{f}^{-1}(\mathrm{p}^{-a}) \cdot T^{-a},\end{aligned}$$

and σ *further induces an isomorphism*

$$\hat{\sigma} : (K^{\mathrm{fin}}_{\mathrm{int}}(Z) \otimes \mathbb{Q})/(\mathrm{p}-p)^{\mathbb{Q},\mathrm{fin}} \to (K^{\Gamma}_{b}(Z) \otimes \mathbb{Q})[T^{Z}]/(T-p).$$

(Note that $\mathrm{im}(\mathfrak{f})$ *is contained in* $\mathrm{p}^{Z} \subset \mathrm{p}^{\Gamma}$*, since it is finite and* Z*-definable, see Remark 2.2.2.)*

Proof. We first have to show that the given mapping rule induces a well-defined map from $K^{\mathrm{fin}}_{\mathrm{int}}(Z)$ to $(K^{\Gamma}_{b}(Z))[T^{Z}]$. Towards that end, let $\mathfrak{f} : U \to \mathrm{p}^{\Gamma}$ and $\mathfrak{g} : V \to \mathrm{p}^{\Gamma}$ be two Z-definable integrable functions with finite images. If there is a Z-definable bijection $h : U \to V$ with $\mathfrak{g} \circ h = \mathfrak{f}$, then we have $\#\mathfrak{f}^{-1}(\alpha) = \#\mathfrak{g}^{-1}(\alpha)$ for all $\alpha \in \mathrm{p}^{\Gamma}$, and hence

$$\sum_{\alpha \in \mathrm{im}(\mathfrak{f})} \#\mathfrak{f}^{-1}(\alpha) \cdot T^{-\,\mathrm{val}(\alpha)} = \sum_{\alpha \in \mathrm{im}(\mathfrak{g})} \#\mathfrak{f}^{-1}(\alpha) \cdot T^{-\,\mathrm{val}(\alpha)}.$$

If U and V are disjoint subsets of the same ambient set RV^{*}_{*}, then we have $\#(\mathfrak{f}^{-1}(\alpha) \cup \mathfrak{g}^{-1}(\alpha)) = \#\mathfrak{f}^{-1}(\alpha) + \#\mathfrak{g}^{-1}(\alpha)$, and hence the mapping rule sends $[\mathfrak{f} \cup \mathfrak{g}]$ to

$$\begin{aligned}&\sum_{\alpha \in \mathrm{im}(\mathfrak{f}) \cup \mathrm{im}(\mathfrak{g})} (\#\mathfrak{f}^{-1}(\alpha) + \#\mathfrak{g}^{-1}(\alpha)) \cdot T^{-\,\mathrm{val}(\alpha)} \\ &= \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} \#\mathfrak{f}^{-1}(\alpha) \cdot T^{-\,\mathrm{val}(\alpha)} + \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} \#\mathfrak{g}^{-1}(\alpha)) \cdot T^{-\,\mathrm{val}(\alpha)}.\end{aligned}$$

It is thus compatible with the defining relations of $K^{\mathrm{fin}}_{\mathrm{int}}(Z)$, so σ indeed exists (as a group homomorphism, for now).

To see that it also respects multiplication, note that we have

$$\begin{aligned}(\mathfrak{f} \star \mathfrak{g})^{-1}(\gamma) &= \bigcup_{(\alpha,\beta)} \mathfrak{f}^{-1}(\alpha) \times \mathfrak{g}^{-1}(\beta) \\ &= \sum_{(\alpha,\beta)} \#\mathfrak{f}^{-1}(\alpha) \cdot \#\mathfrak{g}^{-1}(\beta),\end{aligned}$$

where the indices in the union and in the sum run over all $(\alpha, \beta) \in \operatorname{im}(\mathfrak{f})) \times \operatorname{im}(\mathfrak{g})$ with $\alpha \cdot \beta = \gamma$. Thus the image of $[\mathfrak{f}] \cdot [\mathfrak{g}] = [\mathfrak{f} \star \mathfrak{g}]$ under σ is

$$\sum_{\gamma} (\mathfrak{f} \star \mathfrak{g})^{-1}(\gamma) \cdot T^{-\operatorname{val}(\gamma)} = \sum_{\gamma} \left(\sum_{(\alpha,\beta)} \#\mathfrak{f}^{-1}(\alpha) \cdot \#\mathfrak{g}^{-1}(\beta) \right) \cdot T^{-\operatorname{val}(\gamma)},$$

which is just the product of $\sigma(\mathfrak{f})$ and $\sigma(\mathfrak{g})$ in $(K_b^\Gamma(Z))[T^Z]$.

To show injectivity, let $[\mathfrak{f}] - [\mathfrak{g}] \in \ker(\sigma)$. Then we have

$$\begin{aligned} 0 &= \sigma([\mathfrak{f}] - [\mathfrak{g}]) \\ &= \sum (\#\mathfrak{f}^{-1}(\alpha) - \#\mathfrak{g}^{-1}(\alpha)) \cdot T^{-\operatorname{val}(\alpha)}, \end{aligned}$$

i.e., $\#\mathfrak{f}^{-1}(\alpha) = \#\mathfrak{g}^{-1}(\alpha)$ for all $\alpha \in \mathrm{p}^\Gamma$. Hence Corollary 4.2.10 implies that $\operatorname{im}(\mathfrak{f}) = \operatorname{im}(\mathfrak{g})$ and that there is, for each $\alpha \in \operatorname{im}(\mathfrak{f}) = \operatorname{im}(\mathfrak{g})$, a Z-definable bijection $h_\alpha : \mathfrak{f}^{-1}(\alpha) \to \mathfrak{g}^{-1}(\alpha)$. Since $\operatorname{im}(\mathfrak{f}) = \operatorname{im}(\mathfrak{g})$ is finite, the map $h = \bigcup_\alpha h_\alpha$ is a Z-definable bijection from $\bigcup_\alpha \mathfrak{f}^{-1}(\alpha) = \operatorname{dom}(\mathfrak{f})$ to $\bigcup_\alpha \mathfrak{g}^{-1}(\alpha) = \operatorname{dom}(\mathfrak{g})$ with $\mathfrak{g} \circ h = \mathfrak{f}$, witnessing $[\mathfrak{f}] - [\mathfrak{g}] = 0$.

Finally, note that $(K_b^\Gamma(Z))[T^Z]$ is generated by the elements $\#U \cdot T^{-a}$, where U runs over all bounded Z-definable subsets of all $\mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ and a runs over all elements of Z. Thus, as we have $\#U \cdot T^{-a} = \sigma([\operatorname{const}_U(\mathrm{p}^{-a})])$ for each such U and a, the homomorphism σ is also surjective.

Furthermore σ now extends to an isomorphism between $K_{\mathrm{int}}^{\mathrm{fin}}(Z) \otimes \mathbb{Q}$ and $(K_b^\Gamma(Z) \otimes \mathbb{Q})[T^Z]$, and it sends the generator $([\operatorname{const}_0(\mathrm{p})] - p \cdot [\operatorname{const}_0(1)]) \otimes 1$ of the ideal $(\mathrm{p} - p)^{\mathbb{Q},\mathrm{fin}}$ to

$$\begin{aligned} \sigma([\operatorname{const}_0(\mathrm{p})]) - p \cdot \sigma([\operatorname{const}_0(1)]) \otimes 1 &= T^{-\operatorname{val}(\mathrm{p})} - p \cdot T^{-\operatorname{val}(1)} \\ &= T - p. \end{aligned}$$

Thus, σ induces an isomorphism $\hat{\sigma} : (K_{\mathrm{int}}^{\mathrm{fin}}(Z) \otimes \mathbb{Q})/(\mathrm{p} - p)^{\mathbb{Q},\mathrm{fin}} \to (K_b^\Gamma(Z) \otimes \mathbb{Q})[T^Z]/(T - p)$ as claimed. □

Combining Theorem 4.3.8 and Lemma 4.3.10 yields a rather explicit description and therefore leads to a better understanding of $R_{\mathrm{mot}}(Z)$.

Corollary 4.3.11. *There is an isomorphism*

$$R_{\mathrm{mot}}(Z) \xrightarrow{\cong} (K_b^\Gamma(Z) \otimes \mathbb{Q})[T^Z]/(T - p),$$

induced by

$$[\mathfrak{f}] + (\mathrm{p} - p) \longmapsto \sum_{\alpha \in \operatorname{im}(\mathfrak{f})} \#\mathfrak{f}^{-1}(\alpha) \cdot T^{-\operatorname{val}(\alpha)} + (T - p)$$

One first example is the following important observation.

Lemma 4.3.12. *For $Z \preccurlyeq Z' \preccurlyeq \Gamma$, the ring $R_{\mathrm{mot}}(Z)$ naturally embeds into $R_{\mathrm{mot}}(Z')$.*

Proof. This follows from Corollary 4.3.9 and Lemma 4.3.10: Note that $K_b^{\Gamma}(Z) \otimes \mathbb{Q}$ embeds into $K_b^{\Gamma}(Z') \otimes \mathbb{Q}$ by [CH18, Remark 2.2.8 (and Theorem 2.3.4)]. Therefore, there is a canonical injective homomorphism from $(K_b^{\Gamma}(Z) \otimes \mathbb{Q})[T^Z]$ to $(K_b^{\Gamma}(Z') \otimes \mathbb{Q})[T^{Z'}]$, which induces a homomorphism

$$\varphi : (K_b^{\Gamma}(Z) \otimes \mathbb{Q})[T^Z] \to (K_b^{\Gamma}(Z') \otimes \mathbb{Q})[T^{Z'}]/(T - p) \cong R_{\mathrm{mot}}(Z').$$

It is left to show that the kernel of φ is exactly the ideal generated by $T - p$. Indeed, it is clear that $\ker(\varphi)$ contains this ideal since it contains the generator $T - p$. For the other direction, let $f \in \ker(\varphi)$ with $f(T) = \sum_{a \in A} X_a \cdot T^a$ for some finite set $A \subset Z$ and coefficients $X_a \in K_b^{\Gamma}(Z) \otimes \mathbb{Q} \subset K_b^{\Gamma}(Z') \otimes \mathbb{Q}$. Then we have $f(T) = g(T) \cdot (T - p)$ for some $g \in (K_b^{\Gamma}(Z') \otimes \mathbb{Q})[T^{Z'}]$, say

$$g(T) = \sum_{a \in A'} Y_a \cdot T^a$$

for some finite set $A' \subset Z'$ and coefficients $Y_a \in K_b^{\Gamma}(Z') \otimes \mathbb{Q}$. I.e.,

$$\begin{aligned}
\sum_{a \in A} X_a \cdot T^a = f(T) &= g(T) \cdot (T - p) \\
&= \sum_{a \in A'} Y_a \cdot T^{a+1} - \sum_{a \in A'} p \cdot Y_a \cdot T^a \\
&= \sum_{a \in Z'} (Y_{a-1} - p \cdot Y_a) \cdot T^a,
\end{aligned}$$

where we just set $Y_a = 0$ for $a \notin A'$. Equating coefficients yields $Y_{a-1} - p \cdot Y_a = 0$ whenever $a \in Z' \setminus A$, so we have

$$f(T) = \sum_{a \in Z'} (Y_{a-1} - p \cdot Y_a) \cdot T^a = \sum_{a \in A} (Y_{a-1} - p \cdot Y_a) \cdot T^a.$$

Using that $Z = Z + 1$, we obtain $f(T) = h(T) \cdot (T - p)$ for

$$h(T) = \sum_{a \in A' \cap Z} Y_a \cdot T^a \in (K_b^{\Gamma}(Z) \otimes \mathbb{Q})[T^Z],$$

and hence f lies in the ideal generated by $(T - p)$ in $(K_b^{\Gamma}(Z) \otimes \mathbb{Q})[T^Z]$, as claimed. □

4.4 Uniform equality of integrals of families

Given two Z-definable families of integrable functions on RV_*^* over the same parameter set S, there are two natural notions of equality of their integrals. We certainly want that the integrals of the corresponding members of the families agree as elements of $(K_{\mathrm{int}}(Z \cup \{\boldsymbol{s}\}) \otimes \mathbb{Q})/(\mathrm{p} - p)^{\mathbb{Q}}$, but we can also demand that this equality is being witnessed uniformly in the parameter $\boldsymbol{s} \in S$. In this section, we will show that both notions are equivalent. One direction of this equivalence is straight-forward, but proving the other requires some work. Indeed, the purpose of most of Section 3.2 and Section 3.3 is to aid in this proof.

Definition 4.4.1. Let $S \subset \mathrm{RV}_*^*$ be Z-definable and let $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be two Z-definable families of integrable functions on RV_*^* over S. We say that $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have

(1) *pointwise equal integrals*, if we have $\int_{\mathrm{mot}} \mathfrak{f}_{\boldsymbol{s}} = \int_{\mathrm{mot}} \mathfrak{g}_{\boldsymbol{s}}$ in $R_{\mathrm{mot}}(Z(\boldsymbol{s}))$ for all $\boldsymbol{s} \in S$, and

(2) *uniformly equal integrals*, if we have $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \otimes 1 + (\mathrm{p} - p)_S^{\mathbb{Q}} = [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \otimes 1 + (\mathrm{p} - p)_S^{\mathbb{Q}}$ in $(K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p} - p)_S^{\mathbb{Q}}$.

Remark 4.4.2. Uniform equality of integrals implies pointwise equality of integrals.

Proof. Note that for each $\boldsymbol{t} \in S$, the specialization map $\mathrm{spz}_{\boldsymbol{t}} : K_{\mathrm{int},S}(Z) \to K_{\mathrm{int}}(Z(\boldsymbol{t}))$, introduced in Remark 4.1.13, induces a map $\mathrm{spz}_{\boldsymbol{t}} \otimes \mathrm{id} : K_{\mathrm{int},S}(Z) \otimes \mathbb{Q} \to K_{\mathrm{int}}(Z(\boldsymbol{t})) \otimes \mathbb{Q}$. Let φ denote the composition of $\mathrm{spz}_{\boldsymbol{t}} \otimes \mathrm{id}$ with the canonical projection $K_{\mathrm{int}}(Z(\boldsymbol{t})) \otimes \mathbb{Q} \twoheadrightarrow (K_{\mathrm{int}}(Z(\boldsymbol{t})) \otimes \mathbb{Q})/(\mathrm{p} - p)^{\mathbb{Q}}$. Since we have $\mathrm{spz}_{\boldsymbol{t}}((\mathrm{p} - p)_S) \subset (\mathrm{p} - p)$, and thus $(\mathrm{spz}_{\boldsymbol{t}} \otimes \mathrm{id})((\mathrm{p} - p)_S^{\mathbb{Q}}) \subset (\mathrm{p} - p)^{\mathbb{Q}}$, the kernel of φ contains the ideal $(\mathrm{p} - p)_S^{\mathbb{Q}}$.

Up to identifying the codomain $K_{\mathrm{int}}(Z(\boldsymbol{t}))$ of φ with $R_{\mathrm{mot}}(Z(\boldsymbol{t}))$ via the canonical isomorphism from Proposition 4.3.3, φ induces a homomorphism

$$
\begin{aligned}
(K_{\mathrm{int},S}(Z) \otimes \mathbb{Q})/(\mathrm{p} - p)_S^{\mathbb{Q}} &\to R_{\mathrm{mot}}(Z(\boldsymbol{t})) \text{ by} \\
[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \otimes 1 + (\mathrm{p} - p)_S^{\mathbb{Q}} &\mapsto \mathrm{spz}_{\boldsymbol{t}}([(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]) \otimes 1 + (\mathrm{p} - p)^{\mathbb{Q}} \\
&= [\mathfrak{f}_{\boldsymbol{t}}] \otimes 1 + (\mathrm{p} - p)^{\mathbb{Q}} \\
&= \int_{\mathrm{mot}} \mathfrak{f}_{\boldsymbol{t}}.
\end{aligned}
$$

The claim now follows by the definitions of uniform and pointwise equality of integrals respectively. □

The remainder of this section is dedicated to proving the other direction. Intuitively, the relation generating $(\mathrm{p} - p)$ allows us to modify an integrable function $\mathfrak{f}$ by shifting finitely many values and obtain an integrable function $\mathfrak{f}^\star$ with the same integral. If $\operatorname{im}(\mathfrak{f})$ is finite to begin with, we can guarantee that $\operatorname{val}(\operatorname{im}(\mathfrak{f}^\star)) \subset \Gamma$ contains at most one element of each archimedean class. We make this intuition precise in the following Definition 4.4.3 and Remark 4.4.4.

Definition 4.4.3. An integrable function $\mathfrak{f}$ with finite image is called *reduced*, if $\operatorname{val}(\operatorname{im}(\mathfrak{f})) \subset \Gamma$ contains at most one element of each archimedean class, i.e., if $a - a' \notin \mathbb{Z}$ for all $a \neq a'$ with $\mathrm{p}^{-a}, \mathrm{p}^{-a'} \in \operatorname{im}(\mathfrak{f})$.

We will not need the following remark as is, but it might still serve the purpose of conveying a better intuition regarding (reduced) integrable functions.

Remark 4.4.4. Let $\mathfrak{f}$ be an integrable function on RV^*_* with finite image. Then there is a reduced integrable function $\mathfrak{f}^\star$ on RV^*_* with $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{f}^\star$.

Indeed, it is straight-forward to establish this statement by induction on $\#\operatorname{im}(\mathfrak{f})$.

One reason why reduced integrable functions are especially nice is that equality of their integrals is quite easy to describe. More precisely, we have the following criterion for equality of the integrals of two reduced functions.

Lemma 4.4.5. *Let $\mathfrak{f}$ and $\mathfrak{g}$ be two reduced integrable functions on RV^*_*. Then the following are equivalent:*

(1) We have $\#\operatorname{im}(\mathfrak{f}) = \#\operatorname{im}(\mathfrak{g})$ and, for each $\alpha \in \operatorname{im}(\mathfrak{f})$, there is some $d \in \mathbb{Z}$ with

$$\#\mathfrak{f}^{-1}(\alpha) = p^{-d} \cdot \#\mathfrak{g}^{-1}(p^{-d} \cdot \alpha).$$

(2) We have $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$.

Proof. The direction (1) $\Rightarrow$ (2) is straight-forward using Corollary 4.2.12 and the assumption that $\mathfrak{f}$ and $\mathfrak{g}$ are reduced. For the other direction, we will make use of the description of $K_{\mathrm{int}}(Z)$ in terms of (piecewise) polynomial functions from Section 4.2.

(1) $\Longrightarrow$ (2)**:** Since $\mathfrak{g}$ is reduced, there is, for any given $\alpha \in \mathrm{p}^\Gamma$, at most one $d \in \mathbb{Z}$ for which $p^{-d} \cdot \alpha \in \operatorname{im}(\mathfrak{g})$. The premise thus implies that there is a unique map $\delta : \operatorname{im}(\mathfrak{f}) \to \mathbb{Z}$ for which we have

$$p^{-\delta(\alpha)} \cdot \alpha \in \operatorname{im}(\mathfrak{g}) \text{ and}$$
$$\#\mathfrak{f}^{-1}(\alpha) = p^{-\delta(\alpha)} \cdot \#\mathfrak{g}^{-1}(p^{-\delta(\alpha)} \cdot \alpha)$$

whenever $\alpha \in \mathrm{im}(\mathfrak{f})$. By the first of these two conditions, δ induces a map from $\mathrm{im}(\mathfrak{f})$ to $\mathrm{im}(\mathfrak{g})$ by sending $\alpha \in \mathrm{im}(\mathfrak{f})$ to $p^{-\delta(\alpha)} \cdot \alpha$. We now claim that this induced map is a bijection. Since $\#\,\mathrm{im}(\mathfrak{f}) = \#\,\mathrm{im}(\mathfrak{g})$, it suffices to prove injectivity, so let $\alpha, \beta \in \mathrm{im}(\mathfrak{f})$ with $p^{-\delta(\alpha)} \cdot \alpha = p^{-\delta(\beta)} \cdot \beta$. We then have $\alpha = p^{\delta(\alpha)-\delta(\beta)} \cdot \beta$, which already implies $\delta(\alpha) = \delta(\beta)$ and $\alpha = \beta$ since $\mathfrak{f}$ is reduced. Hence the map from $\mathrm{im}(\mathfrak{f})$ to $\mathrm{im}(\mathfrak{g})$ induced by δ is indeed bijective.

Corollary 4.2.12 now yields

$$\begin{aligned}
\int_{\mathrm{mot}} \mathfrak{f} &= \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} \alpha \cdot \#\mathfrak{f}^{-1}(\alpha) + (\mathrm{p} - p) \\
&= \sum_{\alpha \in \mathrm{im}(\mathfrak{f})} \alpha \cdot \overbrace{p^{-\delta(\alpha)} \cdot \#\mathfrak{g}^{-1}(p^{-\delta(\alpha)} \cdot \alpha)}^{\#\mathfrak{f}^{-1}(\alpha)=} + (\mathrm{p} - p) \\
&= \sum_{\beta \in \mathrm{im}(\mathfrak{g})} \beta \cdot \#\mathfrak{g}^{-1}(\beta) + (\mathrm{p} - p) \\
&= \int_{\mathrm{mot}} \mathfrak{g},
\end{aligned}$$

where the second-to-last equality holds, since $\alpha \mapsto \beta = p^{-\delta(\alpha)} \cdot \alpha$ is a bijection from $\mathrm{im}(\mathfrak{f})$ to $\mathrm{im}(\mathfrak{g})$.

(2) $\implies$ (1)**:** Let $\mathfrak{f}$ and $\mathfrak{g}$ be two reduced integrable functions on RV_*^* with $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$. Consider the map $\varphi : \Gamma \to K_b^{\Gamma}(\Gamma) \otimes \mathbb{Q}$ given by $a \mapsto \#\mathfrak{f}^{-1}(p^{-a}) - \#\mathfrak{g}^{-1}(p^{-a})$. In other words, φ is the image of $[\mathfrak{f}] \otimes 1 - [\mathfrak{g}] \otimes 1 \in K_{\mathrm{int}}(Z) \otimes \mathbb{Q}$ under the non-family version χ of the isomorphism χ_S from Proposition 4.2.4. Note that we have $[\mathfrak{f}] \otimes 1 - [\mathfrak{g}] \otimes 1 \in (\mathrm{p} - p)^{\mathbb{Q}}$ by the assumption $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$, and thus $\varphi = \psi \cdot \eta$ for some $\psi \in \mathbf{P}^{\mathbb{Q},\mathrm{fin}}(Z)$, where η is the image of the generator of $(\mathrm{p} - p)^{\mathbb{Q}}$ under χ, i.e.,

$$\begin{aligned}
\eta : \Gamma &\to K_b^{\Gamma}(\Gamma) \subset K_b^{\Gamma}(\Gamma) \otimes \mathbb{Q} \\
a &\mapsto \begin{cases} 1, & \text{if } a = -1 \\ -p, & \text{if } a = 0 \\ 0, & \text{otherwise} \end{cases} .
\end{aligned}$$

By Remark 4.2.6 (and just as in (4.5) in the proof of Lemma 4.3.6), this means that we have

$$\varphi(a) = (\psi \cdot \eta)(a) = \psi(a+1) - p \cdot \psi(a)$$

for all $a \in \Gamma$.

For each $\alpha \in \mathrm{im}(\mathfrak{f})$, we now have to find some $d \in \mathbb{Z}$ as in the claim. Towards this end, fix some $\alpha = p^{-a} \in \mathrm{im}(\mathfrak{f})$ and consider the set

$$A = (a + \mathbb{Z}) \cap \mathrm{supp}(\psi) = \{a' \in \mathrm{supp}(\psi) \mid a' - a \in \mathbb{Z}\}$$

If $A = \emptyset$, then we have $\psi(a+k) = 0$ for all $k \in \mathbb{Z}$, hence also $\varphi(a+k) = 0$ and thus $\#\mathfrak{f}^{-1}(a+k) - \#\mathfrak{g}^{-1}(a+k)$ for all $k \in \mathbb{Z}$. This implies that the choice $d = 0$ satisfies the claim. So let us, from now on, assume that A is non-empty. Set $a_0 := \min(A)$ and $a_1 := \max(A)$. Then we have $\psi(a_0 - 1) = 0$, hence $\varphi(a_0 - 1) = \psi(a_0) \neq 0$, and $\psi(a_1+1) = 0$, hence $\varphi(a_1) = -p \cdot \psi(a_1) \neq 0$. Since the set $p^{-\operatorname{supp}(\varphi)}$ is contained in $\operatorname{im}(\mathfrak{f}) \cup \operatorname{im}(\mathfrak{g})$, and $\mathfrak{f}$ and $\mathfrak{g}$ are reduced, the support of φ contains at most two elements of each archimedean class. Thus we must have $\operatorname{supp}(\varphi) = \{a_0 - 1, a_1\}$. In particular, this implies that $\psi(a_0+k) - p \cdot \psi(a_0+k-1) = \varphi(a_0 + k - 1) = 0$ for $k = 1, \ldots, a_1 - a_0$. Hence we have

$$\psi(a_0 + k) = p \cdot \psi(a_0 + k - 1)$$

for all $k = 1, \ldots, a_1 - a_0$. Applying this last equation repeatedly yields

$$\psi(a_1) = p^{a_1 - a_0} \cdot \psi(a_0).$$

For $d := a_1 - a_0 + 1$, we thus now have

$$\begin{aligned}
\#\mathfrak{f}^{-1}(p^{-a_1}) - \#\mathfrak{g}^{-1}(p^{-a_1}) &= \overbrace{\varphi(a_1)}^{\neq 0} \\
&= -p \cdot \psi(a_1) \\
&= -p^d \cdot \psi(a_0) \\
&= -p^d \cdot \varphi(\underbrace{a_0 - 1}_{= a_1 - d}) \\
&= -p^d \cdot \left(\#\mathfrak{f}^{-1}(p^{-a_1+d}) - \#\mathfrak{g}^{-1}(p^{-a_1+d}) \right).
\end{aligned}$$

Since $\mathfrak{f}$ and $\mathfrak{g}$ are both reduced, exactly one summand on each side of this equation is non-zero, in such a way that we have

$$\begin{aligned}
\#\mathfrak{f}^{-1}(p^{-a_1}) &= p^d \cdot \#\mathfrak{g}^{-1}(p^{-a_1+d}) \text{ and} \\
\#\mathfrak{g}^{-1}(p^{-a_1}) &= p^d \cdot \#\mathfrak{f}^{-1}(p^{-a_1+d}),
\end{aligned}$$

where exactly one of the two lines vanishes. In particular, we either have $\#\mathfrak{f}^{-1}(\alpha) = p^d \cdot \#\mathfrak{g}^{-1}(p^d \cdot \alpha)$ (in case $\alpha = p^{-a_1}$) or $\#\mathfrak{f}^{-1}(\alpha) = p^{-d} \cdot \#\mathfrak{g}^{-1}(p^{-d} \cdot \alpha)$ (in case $\alpha = p^{-a_0+d}$). Thus either $-d$ or d is as desired.

Moreover, we can do the same argument as above for all $\beta \in \operatorname{im}(\mathfrak{g})$ instead of $\alpha \in \operatorname{im}(\mathfrak{f})$. All put together, this implies that $\operatorname{im}(\mathfrak{f})$ contains an element of a given archimedean class if and only if $\operatorname{im}(\mathfrak{g})$ contains an element of the same archimedean class, hence we have $\# \operatorname{im}(\mathfrak{f}) = \# \operatorname{im}(\mathfrak{g})$. $\square$

We now have all the necessary prerequisites to show that pointwise equality implies uniform equality of integrals (see Lemma 4.4.10 and Lemma 4.4.11), but we will later also need the stronger statement Lemma 4.4.12. Towards the formulation (and proof) of the latter, we introduce the following handy notation for "merging" a Z-definable family of integrable functions into one single function.

Notation 4.4.6. Let $S \subset \mathrm{RV}_*^*$ be Z-definable and let $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be a Z-definable family of integrable functions on RV_*^* over S. Then we write $\bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}_{\boldsymbol{s}}$ for the Z-definable function given by

$$\bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}_{\boldsymbol{s}} : \bigsqcup_{\boldsymbol{s}\in S} \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) \to \mathrm{p}^{\Gamma}$$
$$(\boldsymbol{s}, \boldsymbol{u}) \mapsto \mathfrak{f}_{\boldsymbol{s}}(\boldsymbol{u}),$$

where $\bigsqcup_{\boldsymbol{s}\in S} \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) := \bigcup_{\boldsymbol{s}\in S}\{\boldsymbol{s}\} \times \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}})$.

Note that the operation $\bigsqcup$ does not induce a map from $K_{\mathrm{int},S}(Z)$ to $K_{\mathrm{int}}(Z)$ since the function $\mathfrak{f} = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}_{\boldsymbol{s}}$ does not need to be integrable if all of the $\mathfrak{f}_{\boldsymbol{s}}$ are. Two prototypical examples are the families $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ given by

$$\mathfrak{f}_{\boldsymbol{s}} = \mathrm{const}_{\{0\}}(1) \text{ and}$$
$$\mathfrak{g}_{\boldsymbol{s}} = \mathrm{const}_{\{0\}}(\mathrm{p}^{-\mathrm{val}(s)})$$

over some set $S \subset \mathrm{RV}_m$ which is not bounded (then $\mathfrak{f}^{-1}(1) = S$ is not bounded) or for which $\mathrm{val}(S)$ is not bounded from below (then $\mathrm{im}(\bigsqcup_{\boldsymbol{s}\in S} \mathfrak{g}_{\boldsymbol{s}})$ is not bounded from above).

However, when $\bigsqcup$ yields integrable functions, it does behave well. Most importantly, the following observation shows that it respects equality in the Grothendieck rings.

Remark 4.4.7. Let $S \subset \mathrm{RV}_*^*$ be Z-definable and let $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be two Z-definable families of integrable functions with

$$[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] = [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}].$$

Suppose moreover that $\mathfrak{f} = \bigsqcup \mathfrak{f}_{\boldsymbol{s}}$ and $\mathfrak{g} = \bigsqcup \mathfrak{g}_{\boldsymbol{s}}$ are integrable. Then we have

$$[\mathfrak{f}] = [\mathfrak{g}].$$

Proof. By Lemma 4.2.9, the assumption $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] = [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]$ yields the existence of a Z-definable family of bijections

$$b_{\boldsymbol{s}} : \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) \to \mathrm{dom}(\mathfrak{g}_{\boldsymbol{s}})$$

with $\mathfrak{f}_{\boldsymbol{s}} = \mathfrak{g}_{\boldsymbol{s}} \circ b_{\boldsymbol{s}}$ for all $\boldsymbol{s} \in S$. The Z-definable map $b = \bigsqcup b_{\boldsymbol{s}}$ then is a bijection from $\bigsqcup \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}) = \mathrm{dom}(\mathfrak{f})$ to $\bigsqcup \mathrm{dom}(\mathfrak{g}_{\boldsymbol{s}}) = \mathrm{dom}(\mathfrak{g})$ satisfying

$$\mathfrak{f} = b \circ \mathfrak{g},$$

so that integrability of $\mathfrak{f}$ and $\mathfrak{g}$ yields the claim. $\square$

When working with Notation 4.4.6, we want to guarantee that the resulting functions are integrable. A sufficient condition is that the parameter set $S \subset \mathrm{RV}_*^*$ is bounded. Indeed, the following Remark 4.4.8 implies that $\bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}_{\boldsymbol{s}}$ is then integrable for any Z-definable family $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ of integrable functions.

Remark 4.4.8. Let $S \subset \mathrm{RV}_*^*$ be a Z-definable bounded set (i.e., $\mathbf{val}(S)$ is bounded in Γ^*) and let $(U_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be a Z-definable family of bounded subsets of $\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ over S, where $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}_{>0}^{\ell}$.

Then $U = \bigcup_{\boldsymbol{s}\in S} U_{\boldsymbol{s}}$ is bounded.

Proof. Let $N = \sum_{i=1}^{\ell} n_i$, where $\boldsymbol{n} = (n_1, \dots, n_\ell)$, so that $\mathbf{val}(U_{\boldsymbol{s}}) \subset \Gamma^N$ for all $\boldsymbol{s} \in S$. Since $U_{\boldsymbol{s}}$ is bounded and Z-definable for each $\boldsymbol{s} \in S$, the functions

$$\begin{aligned} a_i : S &\to \Gamma \\ \boldsymbol{s} &\mapsto \min(\mathrm{pr}_i(\mathbf{val}(U_{\boldsymbol{s}}))) \text{ and} \\ b_i : S &\to \Gamma \\ \boldsymbol{s} &\mapsto \max(\mathrm{pr}_i(\mathbf{val}(U_{\boldsymbol{s}}))), \end{aligned}$$

for $i = 1, \dots, N$, are Z-definable and satisfy $U_{\boldsymbol{s}} \subset \prod_{i=1}^{N} [a_i(\boldsymbol{s}), b_i(\boldsymbol{s})]$.

By partitioning S into finitely many pieces so that $\mathbf{val}$ is injective on each piece and then applying Remark 3.3.1, we obtain lower and upper bounds $a_i^-, b_i^+ \in \Gamma$ for $a_i(S)$ and $b_i(S)$ respectively. This yields

$$U = \bigcup_{\boldsymbol{s}\in S} U_{\boldsymbol{s}} \subset \bigcup_{\boldsymbol{s}\in S} \prod_{i=1}^{n} [a_i(\boldsymbol{s}), b_i(\boldsymbol{s})] \subset \left[\min(a_i^-), \max(b_i^+)\right]^N,$$

showing that U is indeed bounded. $\square$

Note that Remark 4.4.8 does not generalize to sets merely being bounded from below (or above). E.g., let $S \subset \mathrm{RV}$ with $\mathrm{val}(S) = [0, \infty)$ and consider $U_s = \mathrm{val}^{-1}(\mathrm{val}(s)) \times \mathrm{val}^{-1}(-\mathrm{val}(s))$. For $U = \bigcup_{\boldsymbol{s}\in S} U_s$, the set

$$\mathbf{val}(U) = \{(a, -a) \mid a \in [0, \infty)\}$$

is then neither bounded from below, nor from above, even though $\mathrm{val}(S)$ is bounded from below and all of the sets $\mathbf{val}(U_{\boldsymbol{s}})$ are singletons.

On several occasions in the remainder of this section, the following criterion for integrability of a function obtained as in Notation 4.4.6 will be useful. Albeit a bit technical, it also works well for an unbounded parameter set $S \subset \mathrm{RV}_*^*$ and hence has a much broader scope than the criterion mentioned just before Remark 4.4.8.

Lemma 4.4.9. *Let $S \subset \mathrm{RV}_*^*$ be Z-definable, let $\mathfrak{f} : U \to \mathrm{p}^\Gamma$ be a Z-definable integrable function with $U \subset S \times \mathrm{RV}_*^*$ and let $(\mathfrak{h}_{\boldsymbol{s}})_{\boldsymbol{s} \in S}$ be a Z-definable family of integrable functions.*

Suppose that there is some element $\gamma \in \mathrm{p}^\Gamma$ with $\max(\mathrm{im}(\mathfrak{h}_{\boldsymbol{s}})) \leq \gamma \cdot \max(\mathrm{im}(\mathfrak{f}(\boldsymbol{s}, \bullet)))$ for all $\boldsymbol{s} \in S$. Then the Z-definable function $\mathfrak{h} = \bigsqcup_{\boldsymbol{s} \in S} \mathfrak{h}_{\boldsymbol{s}}$ given by

$$\begin{aligned} \mathfrak{h} : \bigcup_{\boldsymbol{s} \in S} \{\boldsymbol{s}\} \times \mathrm{dom}(\mathfrak{h}_{\boldsymbol{s}}) &\to \mathrm{p}^\Gamma \\ (\boldsymbol{s}, \boldsymbol{w}) &\mapsto \mathfrak{h}_{\boldsymbol{s}}(\boldsymbol{w}) \end{aligned}$$

is integrable.

Proof. We have to show that $\mathrm{im}(\mathfrak{h}) \subset \mathrm{p}^\Gamma$ is bounded from above and that the fiber $\mathfrak{h}^{-1}(\alpha)$ is bounded for each $\alpha \in \mathrm{im}(\mathfrak{h})$. Firstly, note that we have

$$\begin{aligned} \max(\mathrm{im}(\mathfrak{h}_{\boldsymbol{s}})) &\leq \gamma \cdot \max(\mathrm{im}(\mathfrak{f}(\boldsymbol{s}, \bullet)) \\ &\leq \gamma \cdot \max(\mathrm{im}(\mathfrak{f})) \end{aligned} \tag{4.7}$$

for all $\boldsymbol{s} \in S$, implying that $\mathrm{im}(\mathfrak{h})$ is indeed bounded from above since $\mathrm{im}(\mathfrak{f})$ is.

Now fix any $\alpha \in \mathrm{p}^\Gamma$ and consider the set

$$\begin{aligned} S_\alpha &= \{\boldsymbol{s} \in S \mid \gamma \cdot \max(\mathrm{im}(\mathfrak{f}(\boldsymbol{s}, \bullet)) \geq \alpha\} \\ &= \{\boldsymbol{s} \in S \mid \exists \beta \in \mathrm{im}(\mathfrak{f}(\boldsymbol{s}, \bullet)) : \gamma^{-1} \cdot \alpha \leq \beta\} \\ &= \mathrm{pr}_S \left(\bigcup_{\beta \in \mathrm{B}} \mathfrak{f}^{-1}(\beta) \right), \end{aligned}$$

where the union runs over the bounded $(Z \cup \{\alpha, \gamma\})$-definable set

$$\mathrm{B} = \{\beta \in \mathrm{p}^\Gamma \mid \gamma^{-1} \cdot \alpha \leq \beta \leq \max(\mathrm{im}(\mathfrak{f}))\}.$$

Since $\mathfrak{f}$ is integrable, the fiber $\mathfrak{f}^{-1}(\beta)$ is bounded for all β. By Remark 4.4.8, the set S_α is thus the projection of a bounded set, and hence itself bounded.

For $\boldsymbol{s} \in \mathrm{pr}_S(\mathfrak{h}^{-1}(\alpha))$, i.e., for $\alpha \in \mathrm{im}(\mathfrak{h}_{\boldsymbol{s}})$, we have $\alpha \leq \gamma \cdot \max(\mathrm{im}(\mathfrak{f}(\boldsymbol{s}, \bullet)))$ by the assumption, and hence $\boldsymbol{s} \in S_\alpha$. Thus $\mathrm{pr}_S(\mathfrak{h}^{-1}(\alpha)) \subset S_{\boldsymbol{\alpha}}$, so that the fiber

$$\begin{aligned} \mathfrak{h}^{-1}(\alpha) &= \{(\boldsymbol{s}, \boldsymbol{u}) \mid \boldsymbol{u} \in \mathfrak{h}_{\boldsymbol{s}}^{-1}(\alpha)\} \\ &= \bigcup_{\boldsymbol{s} \in \mathrm{pr}_S(\mathfrak{h}^{-1}(\alpha))} \{\boldsymbol{s}\} \times \mathfrak{h}_{\boldsymbol{s}}^{-1}(\alpha) \\ &= \bigcup_{\boldsymbol{s} \in S_\alpha} \{\boldsymbol{s}\} \times \mathfrak{h}_{\boldsymbol{s}}^{-1}(\alpha) \end{aligned}$$

is bounded for all $\alpha \in \mathrm{p}^\Gamma$ by Remark 4.4.8. $\square$

Let us now finally prove that pointwise equality of integrals implies uniform equality of integrals. We first handle the case of families of integrable functions with finite images. The general case then follows by using the results of Section 4.3.

Lemma 4.4.10 (Pointwise equality of integrals implies uniform equality of integrals if images are finite)**.** *Suppose that $\Gamma \neq \mathbb{Z}$. Let S be a Z-definable subset of RV_*^* and let $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be two Z-definable families of integrable functions on RV_*^* over S. If $\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})$ and $\operatorname{im}(\mathfrak{g}_{\boldsymbol{s}})$ are finite for all $\boldsymbol{s} \in S$ and $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have pointwise equal integrals, then they have uniformly equal integrals.*

Moreover, if both of the Z-definable functions $\mathfrak{f} = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}_{\boldsymbol{s}}$ and $\mathfrak{g} = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{g}_{\boldsymbol{s}}$ are integrable, then we also have $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$ in $R_{\mathrm{mot}}(Z)$.

Proof. First note that it suffices to partition S into finitely many Z-definable pieces and then show the claim on each piece individually. We will do this several times in the proof. (For the "moreover"-part, consider a partition of S into finitely many Z-definable sets S_i and note that $\mathfrak{f}$ is then integrable if and only if all of the Z-definable functions $\mathfrak{f}_i = \bigsqcup_{\boldsymbol{s}\in S_i} \mathfrak{f}_{\boldsymbol{s}}$ are integrable.)

For a start, this procedure allows us to restrict to the case that **val** is injective on S (just as in the proof of Proposition 4.2.4).

Now, we want to further restrict to the case that $\#\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})$ is constant on S. Note that $\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})$ is a Z-definable family of subsets of p^{Γ} over the parameter set S. By Lemma 3.1.8, applied to $X = \{(\boldsymbol{s}, a) \in S \times \Gamma \mid \mathrm{p}^{-a} \in \operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})\}$, there are finitely many Z-definable functions $a_i : S \to \Gamma$ and $b_i : S \to \Gamma$ and $d_i \in \mathbb{N}_{>0}$, for $i = 1, \dots, k$, such that we have

$$\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}}) = \left\{ \mathrm{p}^{-a} \mid a \in \bigcup_{i=1}^{k} [a_i(\boldsymbol{s}), b_i(\boldsymbol{s}))_{d_i} \right\}$$

for all $\boldsymbol{s} \in S$. Thus, the map $c : S \to \mathbb{N} \subset \Gamma$ given by $c(\boldsymbol{s}) = \#\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}}) = \sum_{i=1}^{k} \frac{1}{d_i} \cdot (b_i(\boldsymbol{s}) - a_i(\boldsymbol{s}))$ is Z-definable. Its image is hence finite by Corollary 3.1.5 (recall that we are assuming $\Gamma \neq \mathbb{Z}$), so we can indeed assume that $c(\boldsymbol{s}) = \#\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})$ is constant on S. Analogously, we can assume that $\#\operatorname{im}(\mathfrak{g}_{\boldsymbol{s}})$ is constant on S. Let $N_{\mathfrak{f}} = \#\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})$ and $N_{\mathfrak{g}} = \#\operatorname{im}(\mathfrak{g}_{\boldsymbol{s}})$ for any $\boldsymbol{s} \in S$. We now proceed by induction on $N := N_{\mathfrak{f}} + N_{\mathfrak{g}}$.

Induction base, $N \leq 1$. Note that the case $N = 1$ cannot occur. Indeed the integral of the empty function is 0, but the integral of any constant integrable function on a non-empty set $U \subset \mathrm{RV}_*^*$ is not. (The latter follows, for example, from Lemma 4.3.10.) In case $N = 0$, all of the $\mathfrak{f}_{\boldsymbol{s}}$ and hence all of the $\mathfrak{g}_{\boldsymbol{s}}$ are empty maps, so there is nothing to show.

Induction step. Consider the maps $a_i : \mathbf{val}(S) \to \Gamma$ for $i = 1, \dots, N_{\mathfrak{f}}$ and $b_j : \mathbf{val}(S) \to \Gamma$ for $j = 1, \dots, N_{\mathfrak{g}}$ with

$$\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}}) = \{\mathrm{p}^{-a_i(\mathbf{val}(\boldsymbol{s}))} \mid i = 1, \dots, N_{\mathfrak{f}}\} \text{ and}$$
$$\operatorname{im}(\mathfrak{g}_{\boldsymbol{s}}) = \{\mathrm{p}^{-b_j(\mathbf{val}(\boldsymbol{s}))} \mid j = 1, \dots, N_{\mathfrak{g}}\}$$

satisfying $a_i(\mathbf{val}(\boldsymbol{s})) < a_{i+1}(\mathbf{val}(\boldsymbol{s}))$ and $b_j(\mathbf{val}(\boldsymbol{s})) < b_{j+1}(\mathbf{val}(\boldsymbol{s}))$ for all appropriate i, j and all $\boldsymbol{s} \in S$. Note that these maps are well-defined, since $\mathbf{val}$ is injective on S, and Z-definable. Up to another partition of S into finitely many Z-definable pieces, we can thus assume that all of the maps a_i and b_j are linear. By Corollary 2.3.6, we can moreover assume that, for all i and all j,

$$\#\mathfrak{f}_{\boldsymbol{s}}^{-1}(p^{-a_i(\mathbf{val}(\boldsymbol{s}))}) \quad \text{and} \quad \#\mathfrak{g}_{\boldsymbol{s}}^{-1}(p^{-b_j(\mathbf{val}(\boldsymbol{s}))})$$

are polynomial in $\mathbf{val}(\boldsymbol{s})$ with coefficients in $K_b^{\Gamma}(Z) \otimes \mathbb{Q}$. Up to yet another partition, we can assume that $\mathbf{val}(S)$ is a Presburger cell.

We proceed by a case distinction on whether the differences $a_{i+1} - a_i$ and $b_{j+1} - b_j$ are constantly equal to integers.

Case 1: There is an i for which $a_{i+1} - a_i$ is constantly equal to an integer. Fix any such i and let $d \in \mathbb{Z}$ with $a_{i+1}(\mathbf{val}(\boldsymbol{s})) - a_i(\mathbf{val}(\boldsymbol{s})) = d$ for all $\boldsymbol{s} \in S$. Note that $d > 0$ and consider the Z-definable family of integrable functions $(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s} \in S}$ given by

$$\begin{aligned}
&\operatorname{dom}(\mathfrak{f}'_{\boldsymbol{s}}) = (U_{\boldsymbol{s}} \times F) \cup ((\operatorname{dom}(\mathfrak{f}_{\boldsymbol{s}}) \setminus U_{\boldsymbol{s}}) \times \{0\}) \text{ with}\\
&\mathfrak{f}'_{\boldsymbol{s}}(\boldsymbol{u}, \boldsymbol{f}) = p^{-a_{i+1}(\mathbf{val}(\boldsymbol{s}))} \text{ for } \boldsymbol{u} \in U_{\boldsymbol{s}}, \boldsymbol{f} \in F \text{ and}\\
&\mathfrak{f}'_{\boldsymbol{s}}(\boldsymbol{u}, \boldsymbol{f}) = \mathfrak{f}_{\boldsymbol{s}}(\boldsymbol{u}) \text{ for } \boldsymbol{u} \notin U_{\boldsymbol{s}}, \boldsymbol{f} = \mathbf{0}
\end{aligned}$$

where $U_{\boldsymbol{s}} = \mathfrak{f}_{\boldsymbol{s}}^{-1}(p^{-a_i(\mathbf{val}(\boldsymbol{s}))})$ and $F \subset \mathrm{RV}_*^*$ is a finite Z-definable set of cardinality $\#F = p^d$ with $\mathbf{0} \in F$. Then we have

$$\begin{aligned}
&[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s} \in S}] - [(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s} \in S}]\\
&= [(\operatorname{const}_{U_{\boldsymbol{s}}}(p^{-a_i(\mathbf{val}(\boldsymbol{s}))}))_{\boldsymbol{s} \in S}] - [(\operatorname{const}_{U_{\boldsymbol{s}} \times F}(p^{-a_{i+1}(\mathbf{val}(\boldsymbol{s}))}))_{\boldsymbol{s} \in S}],
\end{aligned} \tag{4.8}$$

which is the product of

$$[(\operatorname{const}_{U_{\boldsymbol{s}}}(p^{-a_i(\mathbf{val}(\boldsymbol{s}))}))_{\boldsymbol{s} \in S}]$$
and
$$[(\operatorname{const}_{\{0\}}(1))_{\boldsymbol{s} \in S}] - [(\operatorname{const}_F(p^{-d}))_{\boldsymbol{s} \in S}].$$

Note that the latter lies in $(\mathrm{p} - p)_S$, as the appearing functions do not actually depend on $\boldsymbol{s} \in S$ and we have

$$\int_{\mathrm{mot}} \operatorname{const}_{\{0\}}(1) = 1 \cdot \#\{0\} = 1 = p^{-d} \cdot \#F = \int_{\mathrm{mot}} \operatorname{const}_F(p^{-d})$$

by Remark 4.1.8. Put together, we have $[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]-[(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \in (\mathrm{p}-p)_S$, i.e., the families $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have uniformly equal integrals. In particular, by Remark 4.4.2, they have pointwise equal integrals and thus, so have $(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$. As $\#\operatorname{im}(\mathfrak{f}'_{\boldsymbol{s}}) = \#\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}}) - 1$, the induction hypothesis now implies that $(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have uniformly equal integrals, and hence, so have $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$.

For the "moreover"-part, now assume that $\mathfrak{f}$ and $\mathfrak{g}$ are integrable and consider the Z-definable functions

$$\begin{aligned} \mathfrak{f}' : \bigcup_{\boldsymbol{s}\in S} \{\boldsymbol{s}\} \times \operatorname{dom}(\mathfrak{f}'_{\boldsymbol{s}}) &\to \mathrm{p}^{\Gamma} \\ (\boldsymbol{s}, \boldsymbol{u}, \boldsymbol{f}) &\mapsto \mathfrak{f}'_{\boldsymbol{s}}(\boldsymbol{u}, \boldsymbol{f}) \end{aligned}$$

$$\begin{aligned} \text{and } \mathfrak{h} : \bigcup_{\boldsymbol{s}\in S} \{\boldsymbol{s}\} \times U_{\boldsymbol{s}} &\to \mathrm{p}^{\Gamma} \\ (\boldsymbol{s}, \boldsymbol{u}) &\mapsto p^{-a_i(\mathbf{val}(\boldsymbol{s}))}. \end{aligned}$$

Let us now show that both $\mathfrak{f}'$ and $\mathfrak{h}$ are integrable, so that the equality (4.8), rewritten as

$$\begin{aligned} &[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] + [(\mathfrak{h}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \cdot [(\operatorname{const}_F(p^{-d}))_{\boldsymbol{s}\in S}] \\ &= [(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] + [(\mathfrak{h}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \cdot [(\operatorname{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}], \end{aligned}$$

yields

$$[\mathfrak{f}] - [\mathfrak{f}'] = [\mathfrak{h}] \cdot (\underbrace{[\operatorname{const}_{\{0\}}(1)] - [\operatorname{const}_F(p^{-d})]}_{\in(\mathrm{p}-p)^{\mathbb{Q}} \text{ by Remark 4.1.8}}),$$

by Remark 4.4.7, showing that $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{f}'$.

Indeed, we have $\operatorname{im}(\mathfrak{f}') = \bigcup_{\boldsymbol{s}\in S} \operatorname{im}(\mathfrak{f}'_{\boldsymbol{s}}) \subset \bigcup_{\boldsymbol{s}\in S} \operatorname{im}(\mathfrak{f}_{\boldsymbol{s}}) = \operatorname{im}(\mathfrak{f})$, and similarly $\operatorname{im}(\mathfrak{h}) \subset \operatorname{im}(\mathfrak{f})$, so the images of $\mathfrak{f}'$ and $\mathfrak{h}$ are bounded from above. Further note that, for each $\alpha \in \mathrm{p}^{\Gamma}$, the fiber $(\mathfrak{f}')^{-1}(\alpha)$ is contained in the union of the two sets

$$\mathfrak{f}^{-1}(\alpha) \times \{\mathbf{0}\}$$

and

$$\begin{aligned} &\{(\boldsymbol{s}, \boldsymbol{u}) \mid \boldsymbol{s} \in S, \boldsymbol{u} \in U_{\boldsymbol{s}} = \mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a_i(\mathbf{val}(\boldsymbol{s}))}), \alpha = \mathrm{p}^{-a_{i+1}(\mathbf{val}(\boldsymbol{s}))}\} \times F \\ &\subset \mathfrak{f}^{-1}(\mathrm{p}^d \cdot \alpha) \times F, \end{aligned}$$

both of which are bounded because $\mathfrak{f}$ is integrable. Similarly, the fiber $\mathfrak{h}^{-1}(\alpha)$ is contained in

$$\begin{aligned} &\{(\boldsymbol{s}, \boldsymbol{u}) \mid \boldsymbol{s} \in S, \boldsymbol{u} \in U_{\boldsymbol{s}} = \mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a_i(\mathbf{val}(\boldsymbol{s}))}), \alpha = \mathrm{p}^{-a_i(\mathbf{val}(\boldsymbol{s}))}\} \\ &= \mathfrak{f}^{-1}(\alpha), \end{aligned}$$

and hence bounded. Thus both $\mathfrak{f}'$ and $\mathfrak{h}$ are integrable, implying $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{f}'$ by Remark 4.4.7.

Case 2: There is a j for which $b_{j+1} - b_j$ is constantly equal to an integer. Completely analogous to Case 1.

Case 3: None of $a_{i+1} - a_i$ or $b_{j+1} - b_j$ is constantly equal to any integer. (Note that this in particular includes the case $N_\mathfrak{f} = N_\mathfrak{g} = 1$.)

Recall that $\mathbf{val}(S)$ is a Presburger cell by our current assumptions (after partitioning S if necessary).

We now want to apply Proposition 3.3.3, but cannot directly assume $\mathrm{val}(S)$ to be bounded. Lemma 3.3.5 can be used to obtain a bounded definable subset, but this requires a parameter set larger than $\mathbb{Z}$. Thus, we temporarily work with the parameter set $Z' = \Gamma$ instead of Z, if needed. (In case $Z \neq \mathbb{Z}$, just set $Z' = Z$.) By Lemma 3.3.5 we can then indeed (repeatedly) apply Proposition 3.3.3 to obtain a Z'-definable subset $S' \subset S$ for which $\mathbf{val}(S')$ is a Presburger cell of the same shape as $\mathbf{val}(S)$, and for which we have

$$((a_{i+1} - a_i) \circ \mathbf{val})(S') > \mathbb{Z} \text{ as well as}$$
$$((b_{j+1} - b_j) \circ \mathbf{val})(S') > \mathbb{Z}$$

for all $i = 1, \dots, N_\mathfrak{f} - 1$ and all $j = 1, \dots, N_\mathfrak{g} - 1$. In other words, $\mathfrak{f}_{\boldsymbol{s}}$ and $\mathfrak{g}_{\boldsymbol{s}}$ are then reduced for all $\boldsymbol{s} \in S'$.

Now consider the difference $d = (b_1 - a_1) \circ \mathbf{val}$ as a linear Z-definable function on S. For each $\boldsymbol{s} \in S'$, as we have $\int_{\mathrm{mot}} \mathfrak{f}_{\boldsymbol{s}} = \int_{\mathrm{mot}} \mathfrak{g}_{\boldsymbol{s}}$, Lemma 4.4.5 (1) implies that $d(\boldsymbol{s})$ is an integer with

$$\#\mathfrak{f}_{\boldsymbol{s}}^{-1}(p^{-a_1(\mathbf{val}(\boldsymbol{s}))}) = p^{-d(\boldsymbol{s})} \cdot \#\mathfrak{g}_{\boldsymbol{s}}^{-1}(p^{-b_1(\mathbf{val}(\boldsymbol{s}))}).$$

Moreover, since $d(S') \subset \mathbb{Z}$, Proposition 3.3.3 ensures that d is constant on S'. Since we have $\mathrm{aff}(\mathbf{val}(S')) = \mathrm{aff}(\mathbf{val}(S))$ by Corollary 3.2.14, this already means that d is constant on all of S. In particular, we have

$$\#\mathfrak{f}_{\boldsymbol{s}}^{-1}(p^{-a_1(\mathbf{val}(\boldsymbol{s}))}) = p^{-d} \cdot \#\mathfrak{g}_{\boldsymbol{s}}^{-1}(p^{-b_1(\mathbf{val}(\boldsymbol{s}))}) \tag{4.9}$$

for all $\boldsymbol{s} \in S'$, where $d \in \mathbb{Z}$ is given by $d = (b_1 - a_1)(\mathbf{val}(\boldsymbol{s}))$ for any (all) $\boldsymbol{s} \in S$. Recall that the terms on both sides of (4.9) are polynomial in $\mathbf{val}(\boldsymbol{s})$ on all of S, with coefficients in $K_b^\Gamma(Z) \otimes \mathbb{Q}$. As $\mathbf{val}(S')$ is a Presburger cell of the same shape as $\mathbf{val}(S)$, Remark 3.3.8 therefore implies that (4.9) already holds for all $\boldsymbol{s} \in S$. (Note that the assumption $\Gamma \subset R$ is not actually satisfied for $R = K_b^\Gamma(Z) \otimes \mathbb{Q}$, hence we cannot apply Corollary 3.3.7 directly and resort to the variant described in the Remark.)

Hence we have $\int_{\mathrm{mot}} \mathfrak{f}_{0,s} = \int_{\mathrm{mot}} \mathfrak{g}_{0,s}$ and thus $\int_{\mathrm{mot}} \mathfrak{f}_{1,s} = \int_{\mathrm{mot}} \mathfrak{g}_{1,s}$ for

$$\begin{aligned}
\mathfrak{f}_{0,s} &:= \mathfrak{f}_s \restriction U_s = \mathrm{const}_{U_s}(p^{-a_1(\mathbf{val}(s))}),\\
\mathfrak{g}_{0,s} &:= \mathfrak{g}_s \restriction V_s = \mathrm{const}_{V_s}(p^{-b_1(\mathbf{val}(s))}),\\
\mathfrak{f}_{1,s} &:= \mathfrak{f}_s \restriction (\mathrm{dom}(\mathfrak{f}_s) \setminus U_s), \text{ and}\\
\mathfrak{g}_{1,s} &:= \mathfrak{g}_s \restriction (\mathrm{dom}(\mathfrak{g}_s) \setminus V_s),
\end{aligned}$$

where $U_s = \mathfrak{f}_s^{-1}(p^{-a_1(\mathbf{val}(s))}) \subset \mathrm{dom}(\mathfrak{f}_s)$ and $V_s = \mathfrak{g}_s^{-1}(p^{-b_1(\mathbf{val}(s))}) \subset \mathrm{dom}(\mathfrak{g}_s)$. Note that we have

$$\begin{aligned}
[(\mathfrak{f}_{0,s})_{s\in S}] &= [(\mathrm{const}_{U_s}(p^{-a_1(\mathbf{val}(s))}))_{s\in S}]\\
&= p^{-d} \cdot [(\mathrm{const}_{V_s}(p^{-a_1(\mathbf{val}(s))}))_{s\in S}]
\end{aligned}$$

as $\#V_s = p^d \cdot \#U_s$ for all $s \in S$ by (4.9). Moreover,

$$\begin{aligned}
[(\mathfrak{g}_{0,s})_{s\in S}] &= [(\mathrm{const}_{V_s}(p^{-b_1(\mathbf{val}(s))}))_{s\in S}]\\
&= [(\mathrm{const}_{V_s}(p^{-a_1(\mathbf{val}(s))-d}))_{s\in S}].
\end{aligned}$$

Thus, their difference $[(\mathfrak{f}_{0,s})_{s\in S}] - [(\mathfrak{g}_{0,s})_{s\in S}]$ is the product of

$$[(\mathrm{const}_{V_s}(p^{-a_1(\mathbf{val}(s))}))_{s\in S}]$$

and

$$p^{-d} \cdot [(\mathrm{const}_{\{0\}}(1))_{s\in S}] - [(\mathrm{const}_{\{0\}}(\mathrm{p}^{-\mathrm{d}}))_{s\in S}],$$

the latter of which lies in $(\mathrm{p} - p)_S^{\mathbb{Q}}$, as these families are constant and we have $p^{-d} \cdot \int_{\mathrm{mot}} \mathrm{const}_{\{0\}}(1) = p^{-d} = \int_{\mathrm{mot}} \mathrm{const}_{\{0\}}(\mathrm{p}^{-\mathrm{d}})$ by Remark 4.1.8. Therefore, the families $(\mathfrak{f}_{0,s})_{s\in S}$ and $(\mathfrak{g}_{0,s})_{s\in S}$ have uniformly equal integrals. Lastly, the induction hypothesis implies that the families $(\mathfrak{f}_{1,s})_{s\in S}$ and $(\mathfrak{g}_{1,s})_{s\in S}$ have uniformly equal integrals. Since $[(\mathfrak{f}_s)_{s\in S}] = [(\mathfrak{f}_{0,s})_{s\in S}] + [(\mathfrak{f}_{1,s})_{s\in S}]$ and $[(\mathfrak{g}_s)_{s\in S}] = [(\mathfrak{g}_{0,s})_{s\in S}] + [(\mathfrak{g}_{1,s})_{s\in S}]$, this finishes the proof.

The "moreover"-part follows just as in Case 1: Firstly, note that $\mathfrak{f}_i = \bigsqcup \mathfrak{f}_{i,s}$ and $\mathfrak{g}_i = \bigsqcup \mathfrak{g}_{i,s}$ are integrable for $i = 0, 1$, since they are merely restrictions of the integrable functions $\mathfrak{f} = \bigsqcup \mathfrak{f}_s$ and $\mathfrak{g} = \bigsqcup \mathfrak{g}_s$ respectively.

Secondly, consider the Z-definable function $\mathfrak{h} = \bigsqcup \mathfrak{h}_s$, where

$$\mathfrak{h}_s = \mathrm{const}_{V_s}(p^{-a_1(\mathbf{val}(s))}).$$

Similar to above, $\mathfrak{h}$ is then integrable. Indeed, its image is contained in $\mathrm{im}(\mathfrak{f})$, and hence bounded from above, and for each $\alpha \in \mathrm{p}^{\Gamma}$ we have

$$\mathfrak{h}^{-1}(\alpha) \subset \mathfrak{g}^{-1}(\mathrm{p}^{-d} \cdot \alpha),$$

using that $b_1(\mathbf{val}(\boldsymbol{s})) = d + a_1(\mathbf{val}(\boldsymbol{s}))$. As before, Remark 4.4.7 thus yields $\int_{\mathrm{mot}} \mathfrak{f}_0 = \int_{\mathrm{mot}} \mathfrak{g}_0$, and the induction hypothesis yields $\int_{\mathrm{mot}} \mathfrak{f}_1 = \int_{\mathrm{mot}} \mathfrak{g}_1$. Put together, we obtain

$$\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{f}_0 + \int_{\mathrm{mot}} \mathfrak{f}_1 = \int_{\mathrm{mot}} \mathfrak{g}_0 + \int_{\mathrm{mot}} \mathfrak{g}_1 = \int_{\mathrm{mot}} \mathfrak{g}$$

as claimed. □

Collecting the previous results, we can now show that two arbitrary families of integrable functions that have pointwise equal integrals already have uniformly equal integrals. More precisely, this is a consequence of Lemma 4.4.10, where we handled this same statement for families of integrable functions with finite images, and Theorem 4.3.8 ("Finite images are enough").

Lemma 4.4.11 (Pointwise equality of integrals implies uniform equality of integrals)**.** *Suppose that $\Gamma \neq \mathbb{Z}$. Let S be a Z-definable subset of RV_*^* and let $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be two Z-definable families of integrable functions on RV_*^* over S.*

If $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have pointwise equal integrals, then they have uniformly equal integrals.

Proof. The assumptions (in particular) mean that we have $\int_{\mathrm{mot}} \mathfrak{f}_{\boldsymbol{s}} = \int_{\mathrm{mot}} \mathfrak{g}_{\boldsymbol{s}}$ for all $\boldsymbol{s} \in S$. The isomorphism between $(K_{\mathrm{int},S}(Z)\otimes\mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q}}$ and $(K^{\mathrm{fin}}_{\mathrm{int},S}(Z)\otimes\mathbb{Q})/(\mathrm{p}-p)_S^{\mathbb{Q},\mathrm{fin}}$ from Theorem 4.3.8, together with Remark 4.1.11, yields Z-definable families $(\mathfrak{f}^\star_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}^\star_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ of integrable functions on RV_*^* for which we have

$$[(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - [(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \equiv [(\mathfrak{f}^\star_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - [(\mathfrak{g}^\star_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] \pmod{(\mathrm{p}-p)_S^{\mathbb{Q}}}, \tag{4.10}$$

where $\mathrm{im}(\mathfrak{f}^\star_{\boldsymbol{s}})$ and $\mathrm{im}(\mathfrak{g}^\star_{\boldsymbol{s}})$ are finite for all $\boldsymbol{s} \in S$. By Remark 4.4.2, this implies

$$\int_{\mathrm{mot}} \mathfrak{f}^\star_{\boldsymbol{s}} - \int_{\mathrm{mot}} \mathfrak{g}^\star_{\boldsymbol{s}} = \int_{\mathrm{mot}} \mathfrak{f}_{\boldsymbol{s}} - \int_{\mathrm{mot}} \mathfrak{g}_{\boldsymbol{s}} = 0 \quad \text{for all } \boldsymbol{s} \in S,$$

i.e., the families $(\mathfrak{f}^\star_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}^\star_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have pointwise equal integrals. By Lemma 4.4.10, they then already have uniformly equal integrals, and hence, equation (4.10) implies that $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have uniformly equal integrals, as claimed. □

As the final result of this chapter, we now also want to prove the "moreover"-part of Lemma 4.4.10 in the general case.

Lemma 4.4.12. *Suppose that $\Gamma \neq \mathbb{Z}$. Let $S \subset \mathrm{RV}_*^*$ be Z-definable and let $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ be two Z-definable families of integrable functions which have* *pointwise (and hence uniformly) equal integrals, i.e., for which we have*

$$\int_{\text{mot}} \mathfrak{f}_{\boldsymbol{s}} = \int_{\text{mot}} \mathfrak{g}_{\boldsymbol{s}} \text{ for all } \boldsymbol{s} \in S.$$

Suppose moreover that both of the Z-definable functions $\mathfrak{f} = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}_{\boldsymbol{s}}$ and $\mathfrak{g} = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{g}_{\boldsymbol{s}}$ are integrable.

Then we have $\int_{\text{mot}} \mathfrak{f} = \int_{\text{mot}} \mathfrak{g}$.

Proof. We will reduce the claim to the special case in which $\operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})$ and $\operatorname{im}(\mathfrak{g}_{\boldsymbol{s}})$ are finite for all $\boldsymbol{s} \in S$, and which we already handled in (the "moreover"-part of) Lemma 4.4.10.

Step 1: Rewriting the integrals of $\mathfrak{f}$ and $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$. The aim of this step is to find two Z-definable families of integrable functions $(\mathfrak{f}^+_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{f}^-_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ with finite images for which $\mathfrak{f}^+ = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}^+_{\boldsymbol{s}}$ and $\mathfrak{f}^- = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}^-_{\boldsymbol{s}}$ are integrable, and which moreover satisfy

$$m \cdot \int_{\text{mot}} \mathfrak{f} = \int_{\text{mot}} \mathfrak{f}^+ - \int_{\text{mot}} \mathfrak{f}^- \text{ and}$$
$$m \cdot \int_{\text{mot}} \mathfrak{f}_{\boldsymbol{s}} = \int_{\text{mot}} \mathfrak{f}^+_{\boldsymbol{s}} - \int_{\text{mot}} \mathfrak{f}^-_{\boldsymbol{s}}$$

for all $\boldsymbol{s} \in S$, where $m \in \mathbb{N}_{>0}$ is some (sufficiently large) integer.

Let $f = \chi_S([(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}]) \in \mathbf{P}_S(Z)$ be the image of (the class of) the family $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ under the isomorphism χ_S from Proposition 4.2.4. That is, $f : S\times\Gamma \to K_b^{\Gamma}(\Gamma)\otimes\mathbb{Q}$ is the map given by $f(\boldsymbol{s}, a) = \#\mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a})$ for $\boldsymbol{s} \in S$ and $a \in \Gamma$. Note that

$$\begin{aligned}\operatorname{supp}(f) &= \{(\boldsymbol{s}, a) \in S \times \Gamma \mid f(\boldsymbol{s}, a) \neq 0\}\\ &= \{(\boldsymbol{s}, a) \in S \times \Gamma \mid \#\mathfrak{f}_{\boldsymbol{s}}^{-1}(\mathrm{p}^{-a}) \neq 0\}\\ &= \{(\boldsymbol{s}, a) \in S \times \Gamma \mid \mathrm{p}^{-a} \in \operatorname{im}(\mathfrak{f}_{\boldsymbol{s}})\}\end{aligned}$$

is then Z-definable (since $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ is). By Remark 4.3.7 there is an element $h \in \mathbf{P}^{\mathbb{Q}}_S(Z)$ such that

$$f - h \cdot f_{\mathrm{p}-p,S} \in \mathbf{P}^{\mathbb{Q},\text{fin}}_S(Z), \tag{4.11}$$

where $f_{\mathrm{p}-p,S} = f_{S\times\{-1\},1} - p \cdot f_{S\times\{0\},1}$, and there is, moreover, an integer $k \in \mathbb{N}_{>0}$ for which we have

$$\operatorname{supp}(h(\boldsymbol{s}, \bullet)) \subset [\min(\operatorname{supp}(f(\boldsymbol{s}, \bullet))) - k, \infty)$$

for all $\boldsymbol{s} \in S$. By Remark 4.2.5, we have $m \cdot h \in \mathbf{P}_S(Z)$ for sufficiently large $m \in \mathbb{N}_{>0}$. Note that $\operatorname{supp}((m \cdot h)(\boldsymbol{s}, \bullet)) = \operatorname{supp}(h(\boldsymbol{s}, \bullet))$. Now consider the preimage of $m \cdot h$ under the isomorphism χ_S from Proposition 4.2.4. By Remark 4.1.11, it can be written as the difference of two generators of $K_{\mathrm{int},S}(Z)$, i.e., there are two Z-definable families of integrable functions $(\mathfrak{h}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}$ and $(\mathfrak{h}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}$ for which we have

$$\chi_S([(\mathfrak{h}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}] - [(\mathfrak{h}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}]) = m \cdot h \tag{4.12}$$

Let $a(\boldsymbol{s}) := \min(\mathrm{supp}(f(\boldsymbol{s}, \bullet)))$ and recall that we then have

$$\mathrm{supp}((m \cdot h)(\boldsymbol{s}, \bullet)) = \mathrm{supp}(h(\boldsymbol{s}, \bullet)) \subset [a(\boldsymbol{s}) - k, \infty)$$

for all $\boldsymbol{s} \in S$. We thus have $\#(\mathfrak{h}_{\boldsymbol{s}}^+)^{-1}(\mathrm{p}^{-a}) - \#(\mathfrak{h}_{\boldsymbol{s}}^-)^{-1}(\mathrm{p}^{-a}) = m \cdot h(\boldsymbol{s}, a) = 0$ for all $a < a(\boldsymbol{s}) - k$, and hence $[(\mathfrak{h}_{\boldsymbol{s}}^+\restriction U_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - [(\mathfrak{h}_{\boldsymbol{s}}^-\restriction V_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] = 0$ for the Z-definable families of sets

$$\begin{aligned} U_{\boldsymbol{s}} &= \{\boldsymbol{u} \in \mathrm{dom}(\mathfrak{h}_{\boldsymbol{s}}^+) \mid \mathfrak{h}_{\boldsymbol{s}}^+(\boldsymbol{u}) > \mathrm{p}^{-a(\boldsymbol{s})+k}\} \text{ and} \\ V_{\boldsymbol{s}} &= \{\boldsymbol{v} \in \mathrm{dom}(\mathfrak{h}_{\boldsymbol{s}}^-) \mid \mathfrak{h}_{\boldsymbol{s}}^-(\boldsymbol{v}) > \mathrm{p}^{-a(\boldsymbol{s})+k}\}. \end{aligned}$$

By replacing $\mathfrak{h}_{\boldsymbol{s}}^+$ and $\mathfrak{h}_{\boldsymbol{s}}^-$ with their restrictions to the complements of $U_{\boldsymbol{s}}$ and $V_{\boldsymbol{s}}$ respectively, we can therefore assume that $U_{\boldsymbol{s}}$ and $V_{\boldsymbol{s}}$ are empty for all $\boldsymbol{s} \in S$. In other words, we then have

$$\max(\mathrm{im}(\mathfrak{h}_{\boldsymbol{s}}^+) \cup \mathrm{im}(\mathfrak{h}_{\boldsymbol{s}}^-)) \le \mathrm{p}^{-a(\boldsymbol{s})+k} = \mathrm{p}^k \cdot \max(\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}})) \tag{4.13}$$

for all $\boldsymbol{s} \in S$, while still retaining (4.12). By Lemma 4.4.9, the two Z-definable functions $\mathfrak{h}^+ = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{h}_{\boldsymbol{s}}^+$ and $\mathfrak{h}^- = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{h}_{\boldsymbol{s}}^-$ are thus integrable.

Multiplying the equation (4.11) by m and applying χ_S^{-1} yields

$$m \cdot [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - \big(\underbrace{[(\mathfrak{h}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}] - [(\mathfrak{h}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}]}_{=\chi_S^{-1}(m\cdot h)}\big) \cdot \chi_S^{-1}(f_{\mathrm{p}-p,S}) \in K_{\mathrm{int},S}^{\mathrm{fin}}(Z),$$

where $\chi_S^{-1}(f_{\mathrm{p}-p,S}) = [(\mathrm{const}_{\{0\}}(\mathrm{p}))_{\boldsymbol{s}\in S}] - p \cdot [(\mathrm{const}_{\{0\}}(1))_{\boldsymbol{s}\in S}]$, i.e., we have

$$\begin{aligned} &m \cdot [(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - [(\mathrm{p} \cdot \mathfrak{h}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}] + [(\mathrm{p} \cdot \mathfrak{h}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}] + p \cdot [(\mathfrak{h}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}] - p \cdot [(\mathfrak{h}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}] \\ &= [(\mathfrak{f}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}] - [(\mathfrak{f}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}] \end{aligned} \tag{4.14}$$

for some Z-definable families $(\mathfrak{f}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}$ and $(\mathfrak{f}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}$ whose images $\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}}^+)$ and $\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}}^-)$ are finite for each $\boldsymbol{s} \in S$. Note that the set

$$\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}}) \cup \mathrm{im}(\mathrm{p} \cdot \mathfrak{h}_{\boldsymbol{s}}^+) \cup \mathrm{im}(\mathrm{p} \cdot \mathfrak{h}_{\boldsymbol{s}}^-) \cup \mathrm{im}(\mathfrak{h}_{\boldsymbol{s}}^+) \cup \mathrm{im}(\mathfrak{h}_{\boldsymbol{s}}^-)$$

is bounded from above by $\mathrm{p}^{k+1} \cdot \max(\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}}))$ by the inequality (4.13).

We continue just as above for $(\mathfrak{h}_{\boldsymbol{s}}^+)_{\boldsymbol{s}\in S}$ and $(\mathfrak{h}_{\boldsymbol{s}}^-)_{\boldsymbol{s}\in S}$: Similar to the argument there, the equality (4.14) implies $[(\mathfrak{f}^+\restriction U_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] - [(\mathfrak{f}^-\restriction V_{\boldsymbol{s}})_{\boldsymbol{s}\in S}] = 0$ for

$$\begin{aligned} U_{\boldsymbol{s}} &= \{\boldsymbol{u} \in \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}^+) \mid \mathfrak{f}_{\boldsymbol{s}}^+(\boldsymbol{u}) > \mathrm{p}^{k+1} \cdot \max(\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}}))\} \text{ and} \\ V_{\boldsymbol{s}} &= \{\boldsymbol{v} \in \mathrm{dom}(\mathfrak{f}_{\boldsymbol{s}}^-) \mid \mathfrak{f}_{\boldsymbol{s}}^-(\boldsymbol{v}) > \mathrm{p}^{k+1} \cdot \max(\mathrm{im}(\mathfrak{f}_{\boldsymbol{s}}))\}. \end{aligned}$$

We may therefore assume that U_s and V_s are empty, i.e., that we have

$$\max(\operatorname{im}(\mathfrak{f}_s^+) \cup \operatorname{im}(\mathfrak{f}_s^-)) \leq \mathrm{p}^{k+1} \cdot \operatorname{im}(\mathfrak{f}_s)$$

for all $s \in S$, so that Lemma 4.4.9 implies that the two Z-definable functions $\mathfrak{f}^+ = \bigsqcup_{s\in S} \mathfrak{f}_s^+$ and $\mathfrak{f}^- = \bigsqcup_{s\in S} \mathfrak{f}_s^-$ are integrable.

Applying Remark 4.4.7 to the equation (4.14) now yields

$$\begin{aligned} &m \cdot [\mathfrak{f}] - [\mathrm{p} \cdot \mathfrak{h}^+] + [\mathrm{p} \cdot \mathfrak{h}^-] + p \cdot [\mathfrak{h}^+] - p \cdot [\mathfrak{h}^-] \\ &= [\mathfrak{f}^+] - [\mathfrak{f}^-], \end{aligned}$$

which we can rewrite as

$$\begin{aligned} &m \cdot [\mathfrak{f}] - ([\mathfrak{f}^-] - [\mathfrak{f}^+]) \\ &= [\mathrm{p} \cdot \mathfrak{h}^+] - p \cdot [\mathfrak{h}^+] - [\mathrm{p} \cdot \mathfrak{h}^-] + p \cdot [\mathfrak{h}^+] \\ &= ([\mathfrak{h}^+] - [\mathfrak{h}^-]) \cdot (\mathrm{const}_0(\mathrm{p}) - p \cdot \mathrm{const}_0(1)) \in (\mathrm{p} - p). \end{aligned}$$

We thus have

$$\begin{aligned} m \cdot \int_{\mathrm{mot}} \mathfrak{f} &= \int_{\mathrm{mot}} \mathfrak{f}^+ - \int_{\mathrm{mot}} \mathfrak{f}^- \text{ and} \\ m \cdot \int_{\mathrm{mot}} \mathfrak{f}_s &= \int_{\mathrm{mot}} \mathfrak{f}_s^+ - \int_{\mathrm{mot}} \mathfrak{f}_s^- \end{aligned}$$

for all $s \in S$ by equation (4.14), as claimed.

Step 2: Rewriting the integrals of $\mathfrak{g}$ and $(\mathfrak{g}_s)_{s\in S}$. In the very same way as in Step 1, we can find an integer $m' \in \mathbb{N}_{>0}$ and two Z-definable families of integrable functions $(\mathfrak{g}_s^+)_{s\in S}$ and $(\mathfrak{g}_s^-)_{s\in S}$ with finite images for which we have

$$\begin{aligned} m' \cdot \int_{\mathrm{mot}} \mathfrak{g} &= \int_{\mathrm{mot}} \mathfrak{g}^+ - \int_{\mathrm{mot}} \mathfrak{g}^- \text{ and} \\ m' \cdot \int_{\mathrm{mot}} \mathfrak{g}_s &= \int_{\mathrm{mot}} \mathfrak{g}_s^+ - \int_{\mathrm{mot}} \mathfrak{g}_s^- \end{aligned}$$

for all $s \in S$. Moreover, by replacing both m and m' with their maximum, we can assume them to be equal. (Note that the only requirement for m in Step 1 was that it needs to be sufficiently big.)

Step 3: Reducing to "better" functions. By Step 1 and Step 2, we have

$$\int_{\mathrm{mot}} \mathfrak{f}_s^+ - \int_{\mathrm{mot}} \mathfrak{f}_s^- = m \cdot \int_{\mathrm{mot}} \mathfrak{f}_s = m \cdot \int_{\mathrm{mot}} \mathfrak{g}_s = \int_{\mathrm{mot}} \mathfrak{g}_s^+ - \int_{\mathrm{mot}} \mathfrak{g}_s^-$$

for all $s \in S$. Rewritten as

$$\int_{\mathrm{mot}} \mathfrak{f}_s^+ + \int_{\mathrm{mot}} \mathfrak{g}_s^- = \int_{\mathrm{mot}} \mathfrak{g}_s^+ + \int_{\mathrm{mot}} \mathfrak{f}_s^-,$$

this means that the two Z-definable families of integrable functions $(\mathfrak{f}'_{\boldsymbol{s}})_{\boldsymbol{s}\in S} = (\mathfrak{f}^+_{\boldsymbol{s}} \sqcup \mathfrak{g}^-_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}'_{\boldsymbol{s}})_{\boldsymbol{s}\in S} = (\mathfrak{g}^+_{\boldsymbol{s}} \sqcup \mathfrak{f}^-_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ have pointwise equal integrals. Note that the images of $\mathfrak{f}'_{\boldsymbol{s}}$ and $\mathfrak{g}'_{\boldsymbol{s}}$ are finite for all $\boldsymbol{s} \in S$ and the Z-definable functions

$$\mathfrak{f}' = \mathfrak{f}^+ \sqcup \mathfrak{g}^- = \bigsqcup_{\boldsymbol{s}\in S} (\mathfrak{f}^+_{\boldsymbol{s}} \sqcup \mathfrak{g}^-_{\boldsymbol{s}}) = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{f}'_{\boldsymbol{s}} \text{ and}$$
$$\mathfrak{g}' = \mathfrak{g}^+ \sqcup \mathfrak{f}^- = \bigsqcup_{\boldsymbol{s}\in S} (\mathfrak{g}^+_{\boldsymbol{s}} \sqcup \mathfrak{f}^-_{\boldsymbol{s}}) = \bigsqcup_{\boldsymbol{s}\in S} \mathfrak{g}'_{\boldsymbol{s}}$$

are integrable. We are therefore in the situation of the special case already handled in (the “moreover”-part of) Lemma 4.4.11, which implies $\int_{\mathrm{mot}} \mathfrak{f}' = \int_{\mathrm{mot}} \mathfrak{g}'$. We thus obtain

$$\begin{aligned} m \cdot \int_{\mathrm{mot}} \mathfrak{f} &= \int_{\mathrm{mot}} \mathfrak{f}^+ - \int_{\mathrm{mot}} \mathfrak{f}^- \\ &= \int_{\mathrm{mot}} \mathfrak{g}^+ - \int_{\mathrm{mot}} \mathfrak{g}^- = m \cdot \int_{\mathrm{mot}} \mathfrak{g}, \end{aligned}$$

and therefore $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$. This completes the last step and hence the proof. □

To end this section, let us state a slight reformulation of Lemma 4.4.12. While being an equivalent statement, it provides another (possibly more intuitive) perspective, starting from the functions $\mathfrak{f}$ and $\mathfrak{g}$ instead of the families $(\mathfrak{f}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(\mathfrak{g}_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$.

Remark 4.4.13. Suppose that $\Gamma \neq \mathbb{Z}$. Let $\mathfrak{f} : U \to \mathrm{p}^\Gamma$ and $\mathfrak{g} : V \to \mathrm{p}^\Gamma$ be two Z-definable integrable functions with $U \subset \mathrm{RV}_*^* \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ and $V \subset \mathrm{RV}_*^* \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ for some $\ell \in \mathbb{N}_{>0}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^\ell$.

Suppose that $\mathrm{pr}_{\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}}(U) = \mathrm{pr}_{\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}}(V) =: S$ and consider, for $\boldsymbol{s} \in S$, the $Z \cup \{\boldsymbol{s}\}$-definable functions

$$\begin{aligned} \mathfrak{f}(\bullet, \boldsymbol{s}) : U_{\boldsymbol{s}} &\to \mathrm{p}^\Gamma \\ \boldsymbol{u} &\mapsto \mathfrak{f}(\boldsymbol{u}, \boldsymbol{s}) \text{ and} \\ \mathfrak{g}(\bullet, \boldsymbol{s}) : V_{\boldsymbol{s}} &\to \mathrm{p}^\Gamma \\ \boldsymbol{v} &\mapsto \mathfrak{g}(\boldsymbol{v}, \boldsymbol{s}), \end{aligned}$$

where $U_{\boldsymbol{s}} = \{\boldsymbol{u} \in \mathrm{RV}_*^* \mid (\boldsymbol{u}, \boldsymbol{s}) \in U\}$ and $V_{\boldsymbol{s}} = \{\boldsymbol{v} \in \mathrm{RV}_*^* \mid (\boldsymbol{v}, \boldsymbol{s}) \in V\}$. (Note that these are integrable functions on RV_*^* since $\mathfrak{f}$ and $\mathfrak{g}$ are.)

Then by (and equivalent to) Lemma 4.4.12, the equality $\int_{\mathrm{mot}} \mathfrak{f}(\bullet, \boldsymbol{s}) = \int_{\mathrm{mot}} \mathfrak{g}(\bullet, \boldsymbol{s})$ for all $\boldsymbol{s} \in S$ already implies $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$.

5 Integrable functions on $\mathrm{K}^* \times \mathrm{RV}_*^*$

Let us briefly recall the most relevant aspects of our notation, as fixed in Section 1.2.

- K denotes a fixed elementary extension of $\mathbb{Q}_p$,
- $\mathcal{O}$ denotes its valuation ring,
- $\mathrm{RV}_m = \mathrm{K}^\times/(1+p^m\mathcal{O}) \cup \{0\}$ is the m-th RV-structure, for $m \in \mathbb{N}_{>0}$,
- Γ denotes the value group (in additive notation),
- p^Γ denotes the value group in multiplicative notation, canonicaly isomorphic to Γ via $a \mapsto \mathrm{p}^{-\mathrm{a}}$,
- $\mathrm{val} : \mathrm{K} \to \Gamma \cup \{\infty\}$ denotes the valuation map (and we also write $\mathrm{val} : \mathrm{RV}_m \to \Gamma \cup \{\infty\}$, and $\mathrm{val} : \mathrm{p}^\Gamma \to^G$, abusing notation),
- $\mathrm{rv}_m : \mathrm{K} \to \mathrm{RV}_m$ denotes the natural quotient map, extended to K by $\mathrm{rv}_m(0) = 0$,
- M is an arbitrary set $M \subset \mathrm{K} \cup \Gamma$ of parameters,
- parameters from $Z = \mathrm{dcl}(M) \cap \Gamma$ suffice when working with definable sets in Γ or RV, see Lemma 2.2.6,
- $\mathcal{B}_{\geq a}(x) = \{y \in \mathrm{K} \mid \mathrm{val}(y-x) \geq a\}$ denotes the ball around x of (additively valuative) radius $a \neq \infty$, which is never a singleton,
- "definable" means "definable with parameters" (but will mostly be avoided to prevent confusion),
- $\mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ is shorthand for $\prod_{i=1}^{\ell} \mathrm{RV}_{m_i}^{n_i}$, and
- "<u>*M-definable subset of $\mathrm{K}^* \times \mathrm{RV}_*^*$*</u>" means "$M$-definable subset of $\mathrm{K}^e \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ for some $e, \ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^{\ell}$" (and similarly for Z as parameter set).

We work in the multi-sorted language $\mathcal{L}_{\mathrm{val}}$ with sorts for the valued field K, the value group Γ and the sorts RV_m for $m \in \mathbb{N}_{>0}$; with the ring language $\mathcal{L}_{\mathrm{ring}}$ on K, the language of ordered abelian groups $\mathcal{L}_{\mathrm{oag}}$ on Γ, and the maps $\mathrm{val} : \mathrm{K} \to \Gamma \cup \{\infty\}$, $\mathrm{val} : \mathrm{RV}_m \to \Gamma \cup \{\infty\}$ and $\mathrm{rv}_m : \mathrm{K} \to \mathrm{RV}_m$ between the sorts. However, as we are only interested in definability, the exact choice of language does not actually matter in most statements.

We can now employ our knowledge about integrable functions on RV_*^* gained in Chapter 4 to define an integral for functions on K. Using the notion of preparation from [Clu+21], and since $\mathrm{K} \succcurlyeq \mathbb{Q}_p$ is 1-h-minimal, a definable subset of K can be partitioned into a (definable) family of particularly nice subsets parameterized by RV_*^*. A similar result holds for M-definable functions from (subsets of) K to p^Γ, see Definition 5.1.1 (3) and Lemma 5.1.3. Note that these partitions can also be obtained more or less directly from the cell decomposition results in [Pas90], predating [Clu+21]. We cite the latter for simplicity, since the formulation of the corresponding results fits our setting more conveniently.

Working with these partitions, we construct the integral $\int_{\mathrm{mot}} \mathfrak{f}$ for (integrable) functions $\mathfrak{f} : X \to \mathrm{p}^\Gamma$, with $X \subset \mathrm{K}$, by reducing to integrals of functions on RV_*^* as studied in Chapter 4. Family versions of the auxiliary preparation results, together with the powerful Lemma 4.4.12 allow us to extend this construction to functions on $\mathrm{K}^n \times \mathrm{RV}_*^*$ by induction on n.

We perform this construction of the integral on $\mathrm{K}^* \times \mathrm{RV}_*^*$ in Section 5.1, including some helpful statements for calculating it explicitly in some concrete cases.

In Section 5.2, we introduce the preliminaries to then establish a change-of-variables formula, see Proposition 5.2.15.

Finally, the results about p-adic integration obtained in [CH21] also apply in our situation, showing that the constructed motivic measure on K is the universal one. This is the content of Section 5.3. We sketch the approach of [CH21] and explain how to adapt it to the scope of this thesis.

In this part of the present thesis, it is often more intuitive to use multiplicative radii for balls. More precisely, we define the *ball of (multiplicative) radius* $\alpha \in \mathrm{p}^\Gamma$ *around* $x \in \mathrm{K}$ as the set

$$\mathcal{B}_{\leq \alpha}(x) = \{y \in \mathrm{K} \mid |y - x| \geq \alpha\}.$$

Note that the ball of multiplicative radius $\alpha = \mathrm{p}^{-a}$ is nothing but the ball of additive radius $a = \mathrm{val}(\alpha)$ around the same point. In formulas, we have $\mathcal{B}_{\leq \alpha}(x) = \mathcal{B}_{\geq a}(x)$ for $\alpha = \mathrm{p}^{-a}$ (i.e., $\mathrm{val}(\alpha) = a$). In particular, $\mathcal{B}_{\leq 1}(0) = \mathcal{B}_{\geq 0}(0) = \mathcal{O}$. Recall that we do not consider singletons to be balls, reflected by only allowing $\alpha \in \mathrm{p}^\Gamma$ in the above definition, with $0 \notin \mathrm{p}^\Gamma$.

From now on, in all of this chapter, we work under the assumption $\mathrm{K} \neq \mathbb{Q}_p$. Note that this allows us to freely use the results of Chapter 3 and Chapter 4 that require $\Gamma \neq \mathbb{Z}$, as Remark 2.2.8 the following remark shows.

5.1 Constructing the integral on $\mathrm{K}^* \times \mathrm{RV}_*^*$

In order to construct the integral for functions on (definable subsets of) $\mathrm{K}^* \times \mathrm{RV}_*^*$, we first need some technical preliminaries.

Definition 5.1.1. Let $d, r \in \mathbb{N}_{>0}$ and let $\boldsymbol{c} = (c_1, \dots, c_r) \in \mathrm{K}^r$. We define the following notions.

(1) A *$(\boldsymbol{c}, d)$-ball* (which is not necessarily a ball) is a non-empty subset of K of the form

$$(\mathrm{rv}_d^{\boldsymbol{c}})^{-1}(\boldsymbol{u}) = \{x \in \mathrm{K} \mid \mathrm{rv}_d^{\boldsymbol{c}}(x) = \boldsymbol{u}\},$$

for some $\boldsymbol{u} \in \mathrm{RV}_d^r$, where $\mathrm{rv}_d^{\boldsymbol{c}}(x)$ denotes the tuple

$$\mathrm{rv}_d^{\boldsymbol{c}}(x) = (\mathrm{rv}_d(x - c_1), \dots, \mathrm{rv}_d(x - c_r)) \in \mathrm{RV}_d^r.$$

We say that such a $(\boldsymbol{c}, d)$-ball is *proper* if $(\boldsymbol{u} \in \mathrm{RV}_d^{\times})^r$, and in that case, it is truly a ball in the sense of the definition on p. 7 in Section 1.2.

(2) A subset $X \subset \mathrm{K}$ is *$(\boldsymbol{c}, d)$-prepared* if it is a union of $(\boldsymbol{c}, d)$-balls.

(3) A function $\mathfrak{f} : X \to \mathrm{p}^{\Gamma}$ is *$(\boldsymbol{c}, d)$-prepared* if $X = \mathrm{dom}(\mathfrak{f})$ is $(\boldsymbol{c}, d)$-prepared and $\mathfrak{f}$ is constant on $(\boldsymbol{c}, d)$-balls, i.e., if the value of $\mathfrak{f}(x)$ only depends on $\mathrm{rv}_d^{\boldsymbol{c}}(x)$.

More generally, let $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ be a function and let $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^{\ell}$ for some $\ell \in \mathbb{N}$. Then we define the following notions.

(4) A subset $D \subset \mathrm{K}^e \times \mathrm{K} \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ is *$(\boldsymbol{c}, d)$-prepared* if the fiber

$$\{y \in \mathrm{K} \mid (\boldsymbol{x}, y, \boldsymbol{u}) \in D\}$$

is $(\boldsymbol{c}(\boldsymbol{x}), d)$-prepared for each $(\boldsymbol{x}, \boldsymbol{u}) \in \mathrm{K}^e \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$.

(5) A function $\mathfrak{f} : D \to \mathrm{p}^{\Gamma}$ is *$(\boldsymbol{c}, d)$-prepared* if $\mathfrak{f}(\boldsymbol{x}, \bullet, \boldsymbol{u})$ is $(\boldsymbol{c}(\boldsymbol{x}), d)$-prepared for each $\boldsymbol{x} \in \mathrm{K}^e$ and each $\boldsymbol{u} \in \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$, i.e., if its domain D is $(\boldsymbol{c}, d)$-prepared and the value of $\mathfrak{f}(\boldsymbol{x}, y, \boldsymbol{u})$ only depends on $(\boldsymbol{x}, \mathrm{rv}_d^{\boldsymbol{c}(\boldsymbol{x})}(y), \boldsymbol{u})$.

Note that $(\mathrm{rv}_d^{\boldsymbol{c}})^{-1}(\boldsymbol{u})$, for $\boldsymbol{u} = (u_1, \dots, u_r) \in (\mathrm{RV}_d^{\times})^r$, is the intersection of the (finitely many) balls

$$(\mathrm{rv}_d)^{-1}(u_i) + c_i = \mathcal{B}_{\geq \mathrm{val}(u_i) + d}(c_i + x_i)$$

for $i = 1, \dots, r$, where the $x_i \in \mathrm{K}$ are arbitrary with $\mathrm{rv}_d(x_i) = u_i$. Hence any proper $(\boldsymbol{c}, d)$-ball is indeed a ball (and the non-proper ones are one-point sets). Moreover, by the geometry of balls in K, we have

$$(\mathrm{rv}_d^{\boldsymbol{c}})^{-1}(\boldsymbol{u}) = \bigcap_{i=1}^{r} \mathrm{rv}_d^{-1}(u_j) + c_i = \mathrm{rv}_d^{-1}(u_j) + c_j$$

for any $j \in \{1, \dots, r\}$ satisfying $\mathrm{val}(u_j) = \max\{\mathrm{val}(u_1), \dots, \mathrm{val}(u_r)\}$. In particular, for fixed $\boldsymbol{c} \in \mathrm{K}^r$, the radius of the $(\boldsymbol{c}, d)$-ball $(\mathrm{rv}_d^{\boldsymbol{c}})^{-1}(\boldsymbol{u})$ only depends on d and $\boldsymbol{u}$

(given that $(\mathrm{rv}_d^c)^{-1}(\boldsymbol{u})$ is non-empty for that choice of d and $\boldsymbol{u}$ – otherwise it is not a (c, d)-ball). We will also denote it by $\mathrm{rad}_d(\boldsymbol{u}) \in \mathrm{p}^\Gamma$, construing rad_d as a function from RV_d^r to $\Gamma \cup \{\infty\}$ (for any $r \in \mathbb{N}$, abusing notation) given by

$$\mathrm{rad}_d(\boldsymbol{u}) = \mathrm{p}^{-d-\max\{\mathrm{val}(u_1),\ldots,\mathrm{val}(u_r)\}}.$$

(Note that $\mathrm{rad}_d(\boldsymbol{u}) = \mathrm{p}^{-\infty} = 0$ if we have $u_i = 0$ for at least one i, hence the above readily extends to non-proper (c, d)-balls, i.e., points.)

Given $\boldsymbol{c} \in \mathrm{K}^r$, $\boldsymbol{c}' \in \mathrm{K}^{r'}$ with $r' \geq r$ and $\mathrm{pr}_{\leq r}(\boldsymbol{c}') = \boldsymbol{c}$, as well as $d \in \mathbb{N}_{>0}$, note that any (c, d)-ball can be written as the disjoint union of (perhaps infinitely many) (c', d)-balls. How many (c', d)-balls one needs depends on the distance of the c_i' to the given (c, d)-ball; for a more detailed analysis in the case $r' = r + 1$, see also the case distinction in the proof of Lemma 5.1.11 (2).

As the wording suggests, our notion of “being prepared” is closely related to the one from [Clu+21]. The following Remark 5.1.2 makes this precise.

Remark 5.1.2. For $e = \ell = 0$, an M-definable function $\mathfrak{f} : X \to \mathrm{p}^\Gamma$ is $(\boldsymbol{c}, d)$-prepared in the sense of our Definition 5.1.1 (3) above if and only if $\mathfrak{f}$ is p^{-d+1}-prepared by the corresponding finite set $C = \{c_1, \ldots, c_r\} \subset \mathrm{K}$ in the sense of [Clu+21, Definition 2.1.1 (1)].

In particular, note that “preparation is first order” by [Clu+21, Lemma 2.3.1 (3)], i.e., whether a function is prepared by some tuple can be expressed by a first order formula.

Lemma 5.1.3 (Definable functions can be prepared.)**.** *Let* $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ *be an* M*-definable function with domain* $D \subset \mathrm{K}^e \times \mathrm{K} \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ *for some* $e, \ell \in \mathbb{N}$ *and* $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^\ell$*. Then the following hold.*

(1) There are $d, r \in \mathbb{N}_{>0}$ *and an* M*-definable function* $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ *such that* $\mathfrak{f}$ *is* (c, d)*-prepared.*

(2) Suppose that $\mathfrak{f}$ *is* (c, d)*-prepared for some* $d, r \in \mathbb{N}_{>0}$ *and some function* $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$*. Let* $d', r' \in \mathbb{N}_{>0}$ *with* $d' \geq d$ *and* $r' \geq r$ *and let* $\boldsymbol{c}' : \mathrm{K}^e \to \mathrm{K}^{r'}$ *be any function with* $\mathrm{pr}_{\leq r} \circ c' = c$*.*

Then $\mathfrak{f}$ *is also* (c', d')*-prepared. (In particular,* D *is also* (c', d')*-prepared.)*

Proof. Part (2) follows immediately from the definition of “being prepared”, i.e., Definition 5.1.1 (5). Part (1) follows from [Clu+21, Proposition 2.3.2] and a compactness argument:

Case 1, $e = 0$**.** By [Clu+21, Proposition 2.3.2] and Remark 5.1.2, $\mathfrak{f}$ is (c, d)-prepared for some $d, r \in \mathbb{N}$ and some finite M-definable tuple $\boldsymbol{c} \in \mathrm{K}^r$.

Case 2, $e > 0$. First note that we can assume that M is finite, since any formula defining $\mathfrak{f}$ can only use finitely many elements of M. For $d, r \in \mathbb{N}_{>0}$, let $\varphi_{d,r}(\boldsymbol{x}, \boldsymbol{c})$ be the $\mathcal{L}_{\mathrm{val}}(M)$-formula which holds in K if and only if $\boldsymbol{x} \in \mathrm{K}^e \setminus \mathrm{pr}_1(D)$ or $\mathfrak{f}(\boldsymbol{x}, \bullet, \bullet)$ is $(\boldsymbol{c}, d)$-prepared, where $\boldsymbol{c} = (c_1, \ldots, c_r)$.

Temporarily fix $\boldsymbol{x} \in \mathrm{K}^e$. By Case 1, there are $d, r \in \mathbb{N}_{>0}$ and an $(M \cup \{\boldsymbol{x}\})$-definable tuple $\boldsymbol{c}_{\boldsymbol{x}} \in \mathrm{K}^r$ such that $\mathfrak{f}(\boldsymbol{x}, \bullet, \bullet)$ is $(\boldsymbol{c}_{\boldsymbol{x}}, d)$-prepared. Let $\psi_{r,\boldsymbol{x}}(\boldsymbol{y}, \boldsymbol{c})$ be an $\mathcal{L}_{\mathrm{val}}(M)$-formula defining $\boldsymbol{c}_{\boldsymbol{x}}$, i.e., for which

$$\mathrm{K} \models \psi_{r,\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{c}) \text{ holds if and only if } \boldsymbol{c} = \boldsymbol{c}_{\boldsymbol{x}}$$

for all $\boldsymbol{c} \in \mathrm{K}^e$.

For each tuple (d, r, ψ) with $(d, r) \in \mathbb{N}^2_{>0}$ and $\psi \in \{\psi_{r,\boldsymbol{x}} \mid \boldsymbol{x} \in \mathrm{K}^e\}$, consider the set

$$P_{d,r,\psi} = \{\boldsymbol{y} \in \mathrm{K}^e \mid \mathrm{K} \models (\exists^{=1}\boldsymbol{c} \colon \psi(\boldsymbol{y}, \boldsymbol{c})) \wedge \forall \boldsymbol{c} \colon (\psi(\boldsymbol{y}, \boldsymbol{c}) \to \varphi_{d,r}(\boldsymbol{y}, \boldsymbol{c}))\}.$$

The (countable) union of the sets $P_{d,r,\psi}$ is then all of K^e by Case 1, and this argument works in all models of $\mathrm{Th}(\mathrm{K})$ in the language $\mathcal{L}_{\mathrm{val}}(M)$. By compactness, there are thus finitely many tuples (d_i, r_i, ψ_i) with $(d_i, r_i) \in \mathbb{N}^2_{>0}$ and $\psi_i \in \{\psi_{r_i,\boldsymbol{x}}\}$ such that $\bigcup_i P_{d_i,r_i,\psi_i} = \mathrm{K}^e$.

Using part (2), and by modifying the ψ_i to include more coordinates if needed, we can assume that there are $d, r \in \mathbb{N}_{>0}$ with $(d_i, r_i) = (d, r)$ for all i. To sum up, we now have finitely many formulas $\psi_i \in \{\psi_{r,\boldsymbol{x}} \mid \boldsymbol{x} \in \mathrm{K}^e\}$ defining maps $\boldsymbol{g}_i : \mathrm{K}^e \to \mathrm{K}^r$ such that for each $\boldsymbol{x} \in \mathrm{K}^e$, there is some i for which $\mathfrak{f}(\boldsymbol{x}, \bullet, \bullet)$ is $(\boldsymbol{g}_i(\boldsymbol{x}), d)$-prepared. Using [Clu+21, Lemma 2.3.1 (3)], this easily allows us to construct from the $\boldsymbol{g}_i$ a single function $\boldsymbol{c}$ for which $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared. ☐

Let us now describe in more detail how to define $\int_{\mathrm{mot}} \mathfrak{f}$ for suitable $\mathfrak{f} : X \to \mathrm{p}^\Gamma$ with $X \subset \mathrm{K}$. First, pick some $d, r \in \mathbb{N}_{>0}$ and $\boldsymbol{c} \in \mathrm{K}^r$ such that $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared. Consider then the function

$$\begin{aligned} \widetilde{\mathfrak{f}} \colon U &\to \mathrm{p}^\Gamma \\ \boldsymbol{u} &\mapsto \mathfrak{f}(x) \cdot \mathrm{rad}_d(\boldsymbol{u}), \end{aligned}$$

where the elements $\boldsymbol{u} \in U$ are in one-to-one correspondence to the $(\boldsymbol{c}, d)$-balls contained in X, with $\mathrm{rad}_d(u) \in \mathrm{p}^\Gamma$ being the (multiplicative) radius of the corresponding ball. Suppose that $\widetilde{\mathfrak{f}}$ is integrable as defined in Definition 4.1.1 (i.e., $\mathrm{im}(\widetilde{\mathfrak{f}}) \subset \mathrm{p}^\Gamma$ is bounded from above and each fiber of $\widetilde{\mathfrak{f}}$ over p^Γ is bounded in RV^*_*). Relying on Chapter 4, we can then set $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}}$. It remains to see that this definition does not depend on the exact choices of $d, r \in \mathbb{N}_{>0}$ and $\boldsymbol{c} \in \mathrm{K}^r$. For functions on $\mathrm{K}^* \times \mathrm{RV}^*_*$, the construction is essentially the same, using recursion and induction.

Before proving that this yields a well-defined notion of integral, let us fix some useful notations for the functions and sets involved.

Remark and Notation 5.1.4. Let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ be an M-definable function with domain $D \subset \mathrm{K}^e \times \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ for some $e, \ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^\ell_{>0}$. Moreover, let $d, r \in \mathbb{N}_{>0}$ and let $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ be a function such that $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared.

We then write $\mathfrak{f}_{(\boldsymbol{c},d)}$ for the (unique and well-defined) function with domain $D_{(\boldsymbol{c},d)}$ given by

$$\mathfrak{f}_{(\boldsymbol{c},d)}(x, \mathrm{rv}_d^{\boldsymbol{c}(x)}(y), \boldsymbol{v}) = \mathfrak{f}(x, y, \boldsymbol{v})$$

for all $(x, y, \boldsymbol{v}) \in D$, where

$$\begin{aligned} D_{(\boldsymbol{c},d)} &= \{(x, \boldsymbol{u}, \boldsymbol{v}) \mid \boldsymbol{u} = \mathrm{rv}_d^{\boldsymbol{c}(x)}(y) \text{ for some } (x, y, \boldsymbol{v}) \in D\} \\ &\subset \mathrm{K}^e \times \mathrm{RV}^r_d \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}. \end{aligned}$$

Moreover, we write $\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}$ for the function

$$\begin{aligned} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} : \widetilde{D_{(\boldsymbol{c},d)}} &\to \mathrm{p}^\Gamma \\ (x, \boldsymbol{u}, \boldsymbol{v}) &\mapsto \mathfrak{f}_{(\boldsymbol{c},d)}(x, \boldsymbol{u}, \boldsymbol{v}) \cdot \mathrm{rad}_d(\boldsymbol{u}) \\ &\qquad = \mathfrak{f}(x, y, \boldsymbol{v}) \cdot \mathrm{rad}_d(\boldsymbol{u}) \text{ for all } y \in \mathrm{K} \text{ with } \mathrm{rv}_d^{\boldsymbol{c}(x)}(y) = \boldsymbol{u}, \end{aligned}$$

where

$$\begin{aligned} \widetilde{D_{(\boldsymbol{c},d)}} &= D_{(\boldsymbol{c},d)} \cap (\mathrm{K}^e \times (\mathrm{RV}^\times_d)^r \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}) \\ &= \{(x, \boldsymbol{u}, \boldsymbol{v}) \in D_{(\boldsymbol{c},d)} \mid u_i \neq 0 \text{ for all } i = 1, \dots, r\}. \end{aligned}$$

Note that the $(\boldsymbol{c}(x), d)$-ball $(\mathrm{rv}_d^{\boldsymbol{c}(x)})^{-1}(\boldsymbol{u})$ is proper for $(x, \boldsymbol{u}, \boldsymbol{v}) \in \widetilde{D_{(\boldsymbol{c},d)}}$, and recall that $\mathrm{rad}_d(\boldsymbol{u}) = p^{-d-\max\{\mathrm{val}(u_1),\dots,\mathrm{val}(u_r)\}}$ is its (multiplicative) radius.

Note that both of the functions $\mathfrak{f}_{(\boldsymbol{c},d)}$ and $\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}$ are M-definable if $\mathfrak{f}$ and $\boldsymbol{c}$ are.

We have yet to define a criterion for integrability of functions from a subset $D \subset \mathrm{K}^* \times \mathrm{RV}^*_*$ to p^Γ.

Definition 5.1.5. Let $M \subset \mathrm{K} \cup \Gamma$ and let $e \in \mathbb{N}_{>0}$. We recursively define an M-definable function $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ with $D \subset \mathrm{K}^e \times \mathrm{RV}^*_*$ to be <u>*integrable*</u>, if there are $d, r \in \mathbb{N}_{>0}$ and an M-definable function $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ such that

(1) $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared, and

(2) the function $\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} : \mathrm{K}^{e-1} \times \mathrm{RV}^r_d \times \mathrm{RV}^*_* \to \mathrm{p}^\Gamma$ is integrable.

Remark 5.1.6. Let $\mathfrak{f} : D \to p^\Gamma$ be an M-definable function, with $D \subset K \times RV_*^*$ whose domain D is bounded and whose image $\mathrm{im}(\mathfrak{f}) \subset p^\Gamma$ is bounded from above. Then $\mathfrak{f}$ is integrable.

Proof. By Lemma 5.1.3 (1), there are $r, d \in \mathbb{N}$ and an M-definable tuple $\boldsymbol{c} \in K^r$ for which $\mathfrak{f}$ is (c, d)-prepared. By part (2) of the same lemma, we can assume that $c_r = 0$. Then we have

$$\begin{aligned}
\widetilde{\mathfrak{f}_{(c,d)}}(\mathrm{rv}_d^c(x), \boldsymbol{v}) &= \mathfrak{f}(x, \boldsymbol{v}) \cdot \mathrm{rad}_d(\mathrm{rv}_d^c(x)) \\
&= \mathfrak{f}(x, \boldsymbol{v}) \cdot p^{-d-\max\{\mathrm{val}(x-c_1),\ldots,\mathrm{val}(x-c_r)\}} \\
&\le \mathfrak{f}(x, \boldsymbol{v}) \cdot p^{-d-\mathrm{val}(x-c_r)} \\
&= \mathfrak{f}(x, \boldsymbol{v}) \cdot p^{-d-\mathrm{val}(x)},
\end{aligned}$$

yielding that $\mathrm{im}(\widetilde{\mathfrak{f}_{(c,d)}})$ is bounded from above because both of the sets $\mathrm{im}(\mathfrak{f}) \subset p^\Gamma$ and $\{p^{-d-\mathrm{val}(x)} \mid x \in \mathrm{pr}_1(D)\} \subset p^\Gamma$ are, the latter since D is bounded.

Now fix $\gamma \in \mathrm{im}(\widetilde{\mathfrak{f}_{(c,d)}})$ and note that the fiber

$$(\widetilde{\mathfrak{f}_{(c,d)}})^{-1}(\gamma) = \bigcup_{\alpha \cdot \beta = \gamma} \{(\mathrm{rv}_d^c(x), \boldsymbol{v}) \mid \mathfrak{f}(x, \boldsymbol{v}) = \alpha,\ \mathrm{rad}_d(\mathrm{rv}_d^c(x)) = \beta\}$$

is a bounded union of bounded sets, since both α and β in the union above can be bounded from above. By Remark 4.4.8, the fiber over γ is then bounded, for each $\gamma \in \mathrm{im}(\widetilde{\mathfrak{f}_{(c,d)}})$, yielding the claim. □

Example 5.1.7. Note that our Definition 5.1.5 covers a broader class of functions than those mentioned in Remark 5.1.6. For example, the function

$$\begin{aligned}
\mathfrak{f} : X &\to p^\Gamma \\
x &\mapsto p^{\frac{1}{2}\mathrm{val}(x)}
\end{aligned}$$

on the set $X = \{x \in \mathcal{O} \mid \mathrm{val}(x) \in 2 \cdot \Gamma_{\ge 0}\}$ is integrable, even though its image is not bounded from above. Indeed, note that $\mathfrak{f}$ is (c, d)-prepared for $c = 0$ and $d = 1$, with

$$\widetilde{X_{(c,d)}} = \{u \in RV_1 \mid \mathrm{val}(u) \in 2 \cdot \Gamma_{\ge 0}\}$$

and

$$\begin{aligned}\widetilde{\mathfrak{f}_{(c,d)}} : \widetilde{X_{(c,d)}} &\to \mathrm{p}^\Gamma\\ u &\mapsto \mathrm{p}^{\frac{1}{2}\operatorname{val}(u)} \cdot \operatorname{rad}_1(u)\\ &= \mathrm{p}^{\frac{1}{2}\operatorname{val}(u)} \cdot \mathrm{p}^{-1-\operatorname{val}(u)}\\ &= \mathrm{p}^{-\frac{1}{2}\operatorname{val}(u)-1}.\end{aligned}$$

Clearly, $\operatorname{im}(\widetilde{\mathfrak{f}_{(c,d)}}) = \{\mathrm{p}^{-a-1} \mid a \in \Gamma_{\geq 0}\}$ is bounded from above by p^{-1}, and for each $\alpha = \mathrm{p}^{-a-1} \in \operatorname{im}(\widetilde{\mathfrak{f}_{(c,d)}})$ the fiber

$$(\widetilde{\mathfrak{f}_{(c,d)}})^{-1}(\alpha) = \{u \in \mathrm{RV}_1 \mid \operatorname{val}(u) = 2 \cdot a\}$$

is even finite, and hence bounded. (And one easily calculates that $\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} = 1$, by using that $\mathrm{p} = p$ in $R_{\mathrm{mot}}(Z)$.)

Similarly, $\mathfrak{g} : \mathrm{K} \setminus \mathcal{O} \to \mathrm{p}^\Gamma, x \mapsto \mathrm{p}^{2\cdot\operatorname{val}(x)}$ is integrable even though its domain is not bounded. (And one similarly calculates that $\int_{\mathrm{mot}} \mathfrak{g} = \frac{1}{p}$.)

Given an M-definable integrable function $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ which is $(\boldsymbol{c}, d)$-prepared for some $(\boldsymbol{c}, d)$, we now want to (recursively) define

$$\int_{\mathrm{mot}} \mathfrak{f} := \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}.$$

To be able to do so, we have to prove that integrability of $\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}$ as well as the value of $\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}$ do not depend on the exact choices of $\boldsymbol{c}$ and d. The largest part of this section is concerned with establishing the latter by an induction on e, where $\operatorname{dom}(\mathfrak{f}) \subset \mathrm{K}^e \times \mathrm{RV}^*_*$.

Firstly, we need a result similar to Lemma 5.1.3 for integrability instead of preparation.

Lemma 5.1.8. *Let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ be an M-definable function with domain $D \subset \mathrm{K}^e \times \mathrm{K} \times \mathrm{RV}^{\boldsymbol{m}}_{\boldsymbol{m}}$ for some $e, \ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^\ell_{>0}$, let $d, r \in \mathbb{N}_{>0}$ and let $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ be an M-definable function. Suppose that $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared and that $\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}$ is integrable.*

Then for any $d' \geq d$, $r' \geq r$ and any M-definable function $\boldsymbol{c}' : \mathrm{K}^e \to \mathrm{K}^{r'}$ with $\operatorname{pr}_{\leq r} \circ \boldsymbol{c}' = \boldsymbol{c}$, the function $\widetilde{\mathfrak{f}_{(\boldsymbol{c}',d')}}$ is integrable. (Note that $\mathfrak{f}$ is $(\boldsymbol{c}', d')$-prepared by Lemma 5.1.3 (2).)

Proof. We proceed by induction on $e \in \mathbb{N}$.

Induction base, $e = 0$**.** Note that we have

$$\widetilde{\mathfrak{f}_{(c,d)}}(\boldsymbol{v}, \boldsymbol{w}) = \mathfrak{f}(y, \boldsymbol{w}) \cdot \mathrm{rad}_d(\boldsymbol{v}) \text{ and}$$
$$\widetilde{\mathfrak{f}_{(c',d')}}(\boldsymbol{v'}, \boldsymbol{w}) = \mathfrak{f}(y, \boldsymbol{w}) \cdot \mathrm{rad}_{d'}(\boldsymbol{v'})$$

for all $(y, \boldsymbol{w}) \in D$, where $\boldsymbol{v} = \mathrm{rv}^c_d(y)$ and $\boldsymbol{v'} = \mathrm{rv}^{c'}_{d'}(y)$.

For all y, we have the inequality

$$\begin{aligned}
\mathrm{rad}_{d'}(\boldsymbol{v'}) &= \mathrm{p}^{-d' - \max\{\mathrm{val}(\boldsymbol{v'}_1), \ldots, \mathrm{val}(\boldsymbol{v'}_{r'})\}} \\
&= \mathrm{p}^{-d' - \max\{\mathrm{val}(y - c'_1), \ldots, \mathrm{val}(y - c'_{r'})\}} \\
&\leq \mathrm{p}^{-d' - \max\{\mathrm{val}(y - c'_1), \ldots, \mathrm{val}(y - c'_{r})\}} \\
&= \mathrm{p}^{-d' - \max\{\mathrm{val}(y - c_1), \ldots, \mathrm{val}(y - c_{r})\}} \\
&= \mathrm{p}^{-d' - \max\{\mathrm{val}(\boldsymbol{v}_1), \ldots, \mathrm{val}(\boldsymbol{v}_{r})\}} \\
&= \mathrm{p}^{d - d'} \cdot \mathrm{rad}_d(\boldsymbol{v}),
\end{aligned}$$

yielding $\widetilde{\mathfrak{f}_{(c,d)}}(\boldsymbol{v}, \boldsymbol{w}) \leq \widetilde{\mathfrak{f}_{(c',d')}}(\boldsymbol{v'}, \boldsymbol{w})$ for all appropriate $\boldsymbol{v}$, $\boldsymbol{v'}$, and $\boldsymbol{w}$. Since $\widetilde{\mathfrak{f}_{(c,d)}}$ is integrable by assumption, Lemma 4.4.9 thus ensures that $\widetilde{\mathfrak{f}_{(c',d')}}$ is integrable.

Induction step, $e \geq 1$**.** For $i = e, \ldots, 1$, we recursively define $d_i, r_i \in \mathbb{N}_{>0}$, $c_i : \mathrm{K}^{i-1} \to \mathrm{K}^{r_i}$ and functions

$$\mathfrak{f}^{(i)} : \mathrm{K}^i \times \Big(\prod_{j=i+1}^{e} \mathrm{RV}^{r_j}_{d_j}\Big) \times \mathrm{RV}^{r}_{d} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \to \mathrm{p}^{\Gamma}$$
$$\mathfrak{g}^{(i)} : \mathrm{K}^i \times \Big(\prod_{j=i+1}^{e} \mathrm{RV}^{r_j}_{d_j}\Big) \times \mathrm{RV}^{r'}_{d'} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \to \mathrm{p}^{\Gamma}$$

as follows: Firstly, set $\mathfrak{f}^{(e)} = \widetilde{\mathfrak{f}_{(c,d)}}$ and $\mathfrak{g}^{(e)} = \widetilde{\mathfrak{f}_{(c',d')}}$. For the recursion on $i = e, \ldots, 1$, choose $d_i, r_i \in \mathbb{N}_{>0}$ and an M-definable function $c_i : \mathrm{K}^{i-1} \to \mathrm{K}^{r_i}$ such that both $\mathfrak{f}^{(i)}$ and $\mathfrak{g}^{(i)}$ are (c_i, d)-prepared and $\widetilde{(\mathfrak{f}^{(i)})_{(c_i,d_i)}}$ is moreover integrable. (This is possible by combining the induction hypothesis and Lemma 5.1.3 (2).) Further set

$$\mathfrak{f}^{(i-1)} = \widetilde{(\mathfrak{f}^{(i)})_{(c_i,d_i)}} \text{ and}$$
$$\mathfrak{g}^{(i-1)} = \widetilde{(\mathfrak{g}^{(i)})_{(c_i,d_i)}},$$

stopping with $\mathfrak{f}^{(0)}$ and $\mathfrak{g}^{(0)}$.

By definition, all of the $\mathfrak{f}^{(i)}$ are integrable, and we aim to show that the $\mathfrak{g}^{(i)}$ are integrable as well. For $i = e$, this implies the claim of the lemma, as $\mathfrak{g}^{(e)} = \widetilde{\mathfrak{f}_{(c',d')}}$.

We now proceed similarly as in the induction base. Note that we have

$$\begin{aligned}
&\mathfrak{f}^{(0)}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_e, \boldsymbol{v}, \boldsymbol{w}) \\
&\qquad = \mathfrak{f}(x, y, \boldsymbol{w}) \cdot (\prod_{i=1}^{e} \mathrm{rad}_{d_i}(\boldsymbol{u}_i)) \cdot \mathrm{rad}_d(\boldsymbol{v}) \text{ and} \\
&\mathfrak{g}^{(0)}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_e, \boldsymbol{v'}, \boldsymbol{w}) \\
&\qquad = \mathfrak{f}(x, y, \boldsymbol{w}) \cdot (\prod_{i=1}^{e} \mathrm{rad}_{d_i}(\boldsymbol{u}_i)) \cdot \mathrm{rad}_{d'}(\boldsymbol{v'})
\end{aligned}$$

for all $(x, y, \boldsymbol{w}) \in D$, where $\boldsymbol{u}_i = \mathrm{rv}_{d_i}^{c_i}(x_{<i})(x_i)$ for $i = 1, \dots, e$, and $\boldsymbol{v} = \mathrm{rv}_d^{c(x)}(y)$ and $\boldsymbol{v'} = \mathrm{rv}_{d'}^{c'(x)}(y)$.

Just as before, for all x and all y, we have the inequality

$$\begin{aligned}
\mathrm{rad}_{d'}(\boldsymbol{v'}) &= \mathrm{p}^{-d' - \max\{\mathrm{val}(\boldsymbol{v'}_1), \dots, \mathrm{val}(\boldsymbol{v'}_{r'})\}} \\
&= \mathrm{p}^{-d' - \max\{\mathrm{val}(y - (c'(x))_1), \dots, \mathrm{val}(y - (c'(x))_{r'})\}} \\
&\le \mathrm{p}^{-d' - \max\{\mathrm{val}(\boldsymbol{v}_1), \dots, \mathrm{val}(\boldsymbol{v}_r)\}} \\
&= \mathrm{p}^{d - d'} \cdot \mathrm{rad}_d(\boldsymbol{v}),
\end{aligned}$$

yielding

$$\mathfrak{f}^{(0)}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_e, \boldsymbol{v}, \boldsymbol{w}) \le \mathfrak{g}^{(0)}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_e, \boldsymbol{v'}, \boldsymbol{w})$$

for all appropriate $\boldsymbol{u}_1, \dots, \boldsymbol{u}_e$, $\boldsymbol{v}$, $\boldsymbol{v'}$, and $\boldsymbol{w}$. Since $\mathfrak{f}^{(0)}$ is integrable, Lemma 4.4.9 thus ensures that $\mathfrak{g}^{(0)}$ is integrable. By repeatedly applying the definition, we obtain that $\mathfrak{g}^{(i)}$ is integrable for all $i = 0, \dots, e$. In particular, $\mathfrak{g}^{(e)} = \widetilde{\mathfrak{f}_{(c', d')}}$ is then integrable, finishing the proof. □

Definition 5.1.9. We define the statements $\underline{\mathrm{intDef}}(e)$ and $\underline{\mathrm{intLem}}(e)$, for $e \in \mathbb{N}$, as follows:

$\underline{\mathrm{intDef}}(e)$**:** If $e \ge 1$, suppose that $\underline{\mathrm{intDef}}(e-1)$ holds. Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^{\Gamma}$ be an M-definable integrable function for some (necessarily M-definable) subset $D \subset \mathrm{K}^e \times \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$, where $\ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{\ell}_{>0}$ are arbitrary. Then the value of $\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}}$ is the same for all (c, d) for which $\mathfrak{f}$ is (c, d)-prepared. We can (and do) thus define

$$\int_{\mathrm{mot}} \mathfrak{f} := \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}}$$

for any (c, d) for which $\mathfrak{f}$ is (c, d)-prepared.

$\underline{\mathrm{intLem}}(e)$**:** Suppose that $\underline{\mathrm{intDef}}(e)$ holds. Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^{\Gamma}$ and

$\mathfrak{g} : E \to \mathrm{p}^{\Gamma}$ be two M-definable integrable functions for some (M-definable) subsets $D, E \subset \mathrm{K} \times \mathrm{K}^e \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ with $\mathrm{pr}_1(D) = \mathrm{pr}_1(E)$. Consider, for each $x \in \mathrm{pr}_1(D) = \mathrm{pr}_1(E)$, the $(M \cup \{x\})$-definable fibers

$$\begin{aligned} D_x &= \{(\boldsymbol{y}, \boldsymbol{v}) \in \mathrm{K}^e \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \mid (x, \boldsymbol{y}, \boldsymbol{v}) \in D\} \text{ and} \\ E_x &= \{(\boldsymbol{y}, \boldsymbol{v}) \in \mathrm{K}^e \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \mid (x, \boldsymbol{y}, \boldsymbol{v}) \in E\} \end{aligned}$$

and the $(M \cup \{x\})$-definable integrable functions $\mathfrak{f}_x : D_x \to \mathrm{p}^{\Gamma}, (\boldsymbol{y}, \boldsymbol{v}) \mapsto \mathfrak{f}(x, \boldsymbol{y}, \boldsymbol{v})$ and $\mathfrak{g}_x : E_x \to \mathrm{p}^{\Gamma}, (\boldsymbol{y}, \boldsymbol{v}) \mapsto \mathfrak{g}(x, \boldsymbol{y}, \boldsymbol{v})$. Suppose that

$$\int_{\mathrm{mot}} \mathfrak{f}_x = \int_{\mathrm{mot}} \mathfrak{g}_x \in R_{\mathrm{mot}}(Z(x)) \text{ for all } x \in \mathrm{K},$$

where $Z(x) = \mathrm{acl}(M \cup \{x\}) \cap \Gamma \succcurlyeq \mathrm{acl}(M) \cap \Gamma = Z$. Then we have $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$ in $R_{\mathrm{mot}}(Z)$.

Theorem 5.1.10. *For all $e \in \mathbb{N}$, the statements $\underline{\mathrm{intDef}}(e)$ and $\underline{\mathrm{intLem}}(e)$ hold. More precisely, we show*

(1) $\underline{\mathrm{intDef}}(0)$ holds.

(2) $\underline{\mathrm{intLem}}(0)$ holds.

(3) For $e \geq 1$, $\underline{\mathrm{intDef}}(e-1)$ and $\underline{\mathrm{intLem}}(e-1)$ imply $\underline{\mathrm{intDef}}(e)$.

(4) For $e \geq 1$, $\underline{\mathrm{intDef}}(e-1)$, $\underline{\mathrm{intDef}}(e)$, and $\underline{\mathrm{intLem}}(e-1)$ imply $\underline{\mathrm{intLem}}(e)$.

We will first establish a variant of (1) in a separate lemma (just below), and then prove the remaining statements of Theorem 5.1.10.

Lemma 5.1.11. *Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^{\Gamma}$ be an M-definable function, where $D \subset \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ for some $\ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{\ell}_{>0}$. Let $d, r, r' \in \mathbb{N}_{>0}$ with $r \leq r'$, and let $\boldsymbol{c} = (c_1, \ldots, c_r) \in \mathrm{K}^r$ and $\boldsymbol{c}' = (c_1, \ldots, c_r, c_{r+1}, \ldots, c_{r'}) \in \mathrm{K}^{r'}$ be M-definable. Suppose that $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared (and hence also $(\boldsymbol{c}', d)$-prepared and $(\boldsymbol{c}, d+1)$-prepared, by Lemma 5.1.3 (2)). Then we have*

(1) $\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d+1)}}$, and

(2) $\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}}$.

Proof. (1) Note that $\mathfrak{f}$ is $\underline{(\boldsymbol{c}, d+1)\text{-prepared}}$ by Lemma 5.1.3 (2). Consider the Z-definable map $h : \widetilde{D_{(\boldsymbol{c},d+1)}} \to \widetilde{D_{(\boldsymbol{c},d)}}$ given by $h(\mathrm{rv}^{\boldsymbol{c}}_{d+1}(x), \boldsymbol{v}) = (\mathrm{rv}^{\boldsymbol{c}}_{d}(x), \boldsymbol{v})$. Then h is a p-to-1-map (i.e., each fiber has cardinality p), since the value of $\mathrm{rv}^{\boldsymbol{c}}_{d+1}(x)$

is already determined by $\mathrm{rv}_{d+1}(x - c_i)$ for one of those i for which $\mathrm{val}(x - c_i)$ is maximal (see also the remarks just after Definition 5.1.1).

Moreover, we have

$$\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} \circ h = \mathrm{p} \cdot \widetilde{\mathfrak{f}_{(\boldsymbol{c},d+1)}},$$

i.e., the diagram

$$\begin{tikzcd}
\widetilde{D_{(\boldsymbol{c},d+1)}} \arrow[rr, "h"] \arrow[dr, "\mathrm{p}\cdot\widetilde{\mathfrak{f}_{(\boldsymbol{c},d+1)}}"'] & & \widetilde{D_{(\boldsymbol{c},d)}} \arrow[dl, "\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}"] \\
& \mathrm{p}^{\Gamma} &
\end{tikzcd}$$

commutes. Corollary 4.1.9 thus implies

$$\mathrm{p} \cdot \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} \circ h = \mathrm{p} \cdot \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d+1)}},$$

which yields the claim.

(2) We will only handle the case $r' = r+1$; the general case then follows immediately by induction on $r' - r$. Moreover, note that we have

$$\begin{aligned}
\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} &= \bigsqcup_{(\boldsymbol{u},\boldsymbol{v}) \in \widetilde{D_{(\boldsymbol{c},d)}}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} \restriction \{(\boldsymbol{u},\boldsymbol{v})\} \text{ and} \\
\widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}} &= \bigsqcup_{(\boldsymbol{u},\boldsymbol{v}) \in \widetilde{D_{(\boldsymbol{c},d)}}} \widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}} \restriction U(\boldsymbol{u},\boldsymbol{v}),
\end{aligned}$$

where

$$U(\boldsymbol{u},\boldsymbol{v}) = \{(\boldsymbol{u}',\boldsymbol{v}) \in \widetilde{D_{(\boldsymbol{c}',d)}} \mid \mathrm{pr}_{\leq r}(\boldsymbol{u}') = \boldsymbol{u}\}.$$

By Lemma 4.4.12, it is therefore enough to show that we have

$$\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} \restriction \{(\boldsymbol{u},\boldsymbol{v})\} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}} \restriction U(\boldsymbol{u},\boldsymbol{v})$$

for all $(\boldsymbol{u},\boldsymbol{v}) \in \widetilde{D_{(\boldsymbol{c},d)}}$. (Recall that we have $\mathrm{K} \neq \mathbb{Q}_p$, hence $\Gamma \neq \mathbb{Z}$, so we can indeed apply Lemma 4.4.12.) We can do this individually for each such $(\boldsymbol{u},\boldsymbol{v})$, i.e., we may from now on assume that $\#\widetilde{D_{(\boldsymbol{c},d)}} = 1$.

Say we have $\widetilde{D_{(\boldsymbol{c},d)}} = \{(\boldsymbol{u},\boldsymbol{v})\}$ for a fixed tuple $(\boldsymbol{u},\boldsymbol{v}) \in (\mathrm{RV}_d^{\times})^r \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$. Fix some $j \in \{1,\dots,r\}$ for which $\mathrm{val}(u_j)$ is maximal, i.e., for which $\mathrm{rad}_d(\boldsymbol{u}) = p^{-d-\mathrm{val}(u_j)}$.

Let $D' = D \setminus (\{c_1, \dots, c_r\} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}})$ and note that $\mathfrak{f}\restriction D'$ is both $(\boldsymbol{c}, d)$-prepared and $(\boldsymbol{c}', d)$-prepared, with

$$\widetilde{D'_{(\boldsymbol{c},d)}} = \widetilde{D_{(\boldsymbol{c},d)}} \text{ and } \widetilde{D'_{(\boldsymbol{c}',d)}} = \widetilde{D_{(\boldsymbol{c}',d)}}$$

and thus

$$(\widetilde{\mathfrak{f}\restriction D')_{(\boldsymbol{c},d)}} = \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} \text{ and } (\widetilde{\mathfrak{f}\restriction D')_{(\boldsymbol{c}',d)}} = \widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}}.$$

We can therefore assume that $D' = D$, i.e., that $\mathrm{pr}_1(D)$ does not contain any of the points $c_1, \dots, c_r \in \mathrm{K}$. (Note that this also implies $\widetilde{D_{(\boldsymbol{c},d)}} = D_{(\boldsymbol{c},d)}$, but not necessarily $\widetilde{D_{(\boldsymbol{c}',d)}} = D_{(\boldsymbol{c}',d)}$, since we might have $c_{r+1} \in \mathrm{pr}_1(D)$.)

Then, by the geometry of balls in K, we have

$$\begin{aligned}
D &= \{(x, \boldsymbol{v}) \in \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \mid \mathrm{rv}^{\boldsymbol{c}}_d(x) = \boldsymbol{u}\} \\
&= \{(x, \boldsymbol{v}) \in \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \mid \mathrm{rv}_d(x - c_1) = u_1, \ \dots \ , \mathrm{rv}_d(x - c_r) = u_r\} \\
&= \bigcap_{i=1}^{r} (\mathrm{rv}_d^{-1}(u_i) + c_i) \times \{\boldsymbol{v}\} \\
&= (\mathrm{rv}_d^{-1}(u_j) + c_j) \times \{\boldsymbol{v}\},
\end{aligned}$$

so that we can moreover assume $r = 1$. To summarize up to here, we are now in the situation that $\boldsymbol{c} = (c)$ and $\boldsymbol{c}' = (c, c')$ for some (M-definable) $c, c' \in \mathrm{K}$ with $c \neq c'$, and $\boldsymbol{u} = (u)$. Moreover, we can assume that $c = 0$ by translating $\mathfrak{f}$. We proceed by a case distinction based on the position of c' relative to the ball $\mathrm{pr}_{\mathrm{K}}(D) = \mathrm{rv}_d^{-1}(u)$.

Case 1: $\mathrm{rv}(c') \neq \mathrm{rv}(u)$. Then $c' \notin \mathrm{pr}_1(D)$, and for $x \in \mathrm{pr}_{\mathrm{K}}(D)$, the value of $\mathrm{rv}_d(x - c')$ is already determined by $\mathrm{rv}_d(x) = u$ and $\mathrm{rv}_d(c')$ by Lemma 2.2.9. Thus the set

$$\begin{aligned}
\widetilde{D_{(\boldsymbol{c}',d)}} = D_{(\boldsymbol{c}',d)} & \\
&= (\{u\} \times \{\mathrm{rv}_d(x - c') \mid x \in \mathrm{pr}_{\mathrm{K}}(D)\}) \times \{\boldsymbol{v}\} \\
&= (\{u\} \times \{\mathrm{rv}_d(x - c') \mid \mathrm{rv}_d(x) = u\}) \times \{\boldsymbol{v}\}
\end{aligned}$$

has exactly one element. The unique (M-definable) bijection $h : \widetilde{D_{(\boldsymbol{c},d)}} \to \widetilde{D_{(\boldsymbol{c}',d)}}$ then satisfies $\widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}} \circ h = \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}$, so we even have $[\widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}}] = [\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}]$. In particular, the claim $\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c}',d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}$ holds.

Case 2: $\mathrm{rv}_\ell(c') = \mathrm{rv}_\ell(u)$ and $\mathrm{rv}_{\ell+1}(c') \neq \mathrm{rv}_{\ell+1}(u)$ for some ℓ with $0 < \ell < d$. Then we also have $c' \notin \mathrm{pr}_1(D)$, and the set

$$\begin{aligned}
\widetilde{D_{(\boldsymbol{c}',d)}} = D_{(\boldsymbol{c}',d)} & \\
&= (\{u\} \times \{\mathrm{rv}_d(x - c') \mid x \in \mathrm{pr}_{\mathrm{K}}(D)\}) \times \{\boldsymbol{v}\} \\
&= (\{u\} \times \{\mathrm{rv}_d(x - c') \mid \mathrm{rv}_d(x) = u\}) \times \{\boldsymbol{v}\}
\end{aligned}$$

has exactly p^ℓ elements by Lemma 2.2.9. Therefore, the unique surjection $h : \widetilde{D_{(c',d)}} \twoheadrightarrow \widetilde{D_{(c,d)}}$ is a p^ℓ-to-1-map. Moreover, for all $x \in \mathrm{pr}_1(D)$, we have

$$\begin{aligned}\mathrm{rad}_d(u, \mathrm{rv}_d(x - c')) &= p^{-d-\max\{\mathrm{val}(u),\mathrm{val}(x-c')\}} \\ &= p^{-d-\ell-\mathrm{val}(u)} \\ &= p^{-\ell} \cdot \mathrm{rad}_d(u)\end{aligned}$$

by Remark 2.2.7 and the case assumption, and hence

$$\begin{aligned}\widetilde{\mathfrak{f}_{(c',d)}}((u, \mathrm{rv}_d(x - c')), \boldsymbol{v}) &= \mathrm{rad}_d(u, \mathrm{rv}_d(x - c')) \cdot \mathfrak{f}(x, \boldsymbol{v}) \\ &= p^{-\ell} \cdot \mathrm{rad}_d(u) \cdot \mathfrak{f}(x, \boldsymbol{v}) \\ &= p^{-\ell} \cdot \widetilde{\mathfrak{f}_{(c,d)}}(u, \boldsymbol{v}).\end{aligned}$$

Thus $\widetilde{\mathfrak{f}_{(c,d)}} \circ h = \mathrm{p}^\ell \cdot \widetilde{\mathfrak{f}_{(c',d)}}$. By Corollary 4.1.9, this implies

$$\mathrm{p}^\ell \cdot \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} \circ h = \mathrm{p}^\ell \cdot \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c',d)}},$$

which yields the claim.

Case 3: $\mathrm{rv}_d(c') = u$, i.e., $c' \in \mathrm{pr}_{\mathrm{K}}(D)$. This case is more involved, since $D_{(c',d)}$ is infinite (and now, $\widetilde{D_{(c',d)}} \neq D_{(c',d)}$). By Lemma 2.2.9, we have

$$\begin{aligned}D_{(c',d)} &= (\{u\} \times \{\mathrm{rv}_d(x - c') \mid x \in \mathrm{pr}_{\mathrm{K}}(D)\}) \times \{\boldsymbol{v}\} \\ &= (\{u\} \times \{w \in \mathrm{RV}_d \mid \mathrm{val}(w) \geq \mathrm{val}(u) + d\}) \times \{\boldsymbol{v}\},\end{aligned}$$

and thus

$$\widetilde{D_{(c',d)}} = (\{u\} \times \{w \in \mathrm{RV}_d^\times \mid \mathrm{val}(w) \geq \mathrm{val}(u) + d\}) \times \{\boldsymbol{v}\}.$$

Furthermore, we have $\widetilde{\mathfrak{f}_{(c',d)}}((u, w), \boldsymbol{v}) = \alpha \cdot \mathrm{p}^{-d-\mathrm{val}(w)}$ for all $w \in \mathrm{RV}_d^\times$ with $\mathrm{val}(w) \geq \mathrm{val}(u) + d$, where α is the unique value of $\mathfrak{f}$ on D. (Note that $\mathfrak{f}$ is constant on D by the assumption $\#D_{(c,d)} = 1$.) Consider the images $f, f' \in \mathbf{P}(Z)$ of $[\widetilde{\mathfrak{f}_{(c,d)}}]$ and $[\widetilde{\mathfrak{f}_{(c',d)}}]$, respectively, under the isomorphism χ from Corollary 4.2.7. That is,

$$f = \chi([\widetilde{\mathfrak{f}_{(c,d)}}]) \qquad \text{and} \qquad f' = \chi([\widetilde{\mathfrak{f}_{(c',d)}}]),$$

$$\begin{aligned}
\text{with } f(a) &= \#\widetilde{\mathfrak{f}_{(c,d)}}^{-1}(p^{-a}) \\
&= \begin{cases} 1, & \text{if } a = a_0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{and } f'(a) &= \#\widetilde{\mathfrak{f}_{(c',d)}}^{-1}(p^{-a}) \\
&= \begin{cases} (p-1)\cdot p^{d-1}, & \text{if } a \geq d + a_0 \\ 0, & \text{otherwise} \end{cases},
\end{aligned}$$

for all $a \in \Gamma$, where $a_0 = \mathrm{val}(\alpha) + d + \mathrm{val}(u)$. We thus have

$$\begin{aligned}
(f - f')(a) &= \#\widetilde{\mathfrak{f}_{(c,d)}}^{-1}(p^{-a}) - \#\widetilde{\mathfrak{f}_{(c',d)}}^{-1}(p^{-a}) \\
&= \begin{cases} 1, & \text{if } a = a_0 \\ -(p-1)\cdot p^{d-1}, & \text{if } a \geq d + a_0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

for all $a \in \Gamma$. Consider the map $h : \Gamma \to K_b^{\Gamma}(\Gamma)$ defined by

$$h(a) = \begin{cases} 0, & \text{if } a \leq a_0 \\ p^{k-1}, & \text{if } a = k + a_0 \text{ for } k \in \{1, \dots, d\} \\ p^{d-1}, & \text{if } a > d + a_0 \end{cases}.$$

Note that h is piecewise constant, so it is an element of $\mathbf{P}(Z)$. Moreover, we have

$$\begin{aligned}
&(h \cdot f_{\mathrm{p}-p})(a) \\
&= h(a+1) - p \cdot h(a) \\
&= \begin{cases} 0, & \text{if } a + 1 \leq a_0 \\ h(a+1) = p^{1-1} = 1, & \text{if } a = a_0 \\ p^k - p\cdot p^{k-1} = 0, & \text{if } a = k + a_0 \\ & \quad \text{(for some } 1 \leq k < d) \\ p^{d-1} - p \cdot p^{d-1} = -(p-1)\cdot p^{d-1} & \text{if } a = d + a_0 \\ p^{d-1} - p \cdot p^{d-1} = -(p-1)\cdot p^{d-1} & \text{if } a > d + a_0 \end{cases} \\
&= \begin{cases} 1, & \text{if } a = a_0 \\ -(p-1)\cdot p^{d-1}, & \text{if } a \geq d + a_0 \\ 0, & \text{otherwise} \end{cases} \\
&= (f - f')(a),
\end{aligned}$$

so $f - f' = h \cdot f_{\mathrm{p}-p}$ lies in the ideal $(f_{\mathrm{p}-p})$, which is the image of the ideal $(\mathrm{p} - p) \subset K_{\mathrm{int}}(Z)$ under χ. In other words,

$$\widetilde{\mathfrak{f}_{(c',d)}} = \chi^{-1}(f) \qquad \text{and} \qquad \widetilde{\mathfrak{f}_{(c,d)}} = \chi^{-1}(f')$$

are congruent modulo $(\mathrm{p}-p) \subset K_{\mathrm{int}}(Z)$, so we have $\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c',d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}}$, as claimed. □

Proof of Theorem 5.1.10. We mostly use the notations from the statements $\underline{\mathrm{intDef}}(e)$ and $\underline{\mathrm{intLem}}(e)$ as introduced in Definition 5.1.9 and refrain from repeating all details here.

(1) $\underline{\mathrm{intDef}}(0)$ holds.

This follows from Lemma 5.1.11: Suppose that $\mathfrak{f} : X \to \mathrm{p}^\Gamma$ is both (c, d)-prepared and (c', d')-prepared for some $d, d', r, r' \in \mathbb{N}_{>0}$ and $\boldsymbol{c} \in \mathrm{K}^r, \boldsymbol{c'} \in \mathrm{K}^{r'}$. Then it is $((c, c'), d'')$-prepared (and also $((c', c), d'')$-prepared) for $d'' = \max(d, d')$ by Lemma 5.1.3 (2). Hence Lemma 5.1.11 implies

$$\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{((c,c'),d'')}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{((c',c),d'')}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c',d')}}$$

(For the second equality, note that we even have $[\widetilde{\mathfrak{f}_{((c,c'),d'')}}] = [\widetilde{\mathfrak{f}_{((c',c),d'')}}]$ in $K_{\mathrm{int}}(Z)$.)

(2) $\underline{\mathrm{intLem}}(0)$ holds.

Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ and $\mathfrak{g} : E \to \mathrm{p}^\Gamma$, $D, E \subset \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ be as in the statement of $\underline{\mathrm{intLem}}(0)$, see Definition 5.1.9. By Lemma 5.1.3, we can choose some $d, r \in \mathbb{N}_{>0}$ and $\boldsymbol{c} \in \mathrm{K}^r$ such that both $\mathfrak{f}$ and $\mathfrak{g}$ are (c, d)-prepared, where $\boldsymbol{c}$ is M-definable. This means that whether $(x, \boldsymbol{v}) \in D$ only depends on $\mathrm{rv}^{\boldsymbol{c}}_d(x)$ and $\boldsymbol{v}$, and so does the value of $\mathfrak{f}(x, \boldsymbol{v})$ (and the same is true for E and $\mathfrak{g}$). For $x \in \mathrm{K}$ with $\boldsymbol{u} := \mathrm{rv}^{\boldsymbol{c}}_d(x) \in (\mathrm{RV}^\times_d)^r$ we can thus consider the $(M \cup \{x\})$-definable integrable functions

$$\begin{aligned} (\widetilde{\mathfrak{f}_{(c,d)}})(\boldsymbol{u}, \bullet) : (\widetilde{D_{(c,d)}})_{\boldsymbol{u}} &\to \mathrm{p}^\Gamma \\ \boldsymbol{v} &\mapsto \widetilde{\mathfrak{f}_{(c,d)}}(\boldsymbol{u}, \boldsymbol{v}) \\ &= \mathrm{rad}_d(\boldsymbol{u}) \cdot \mathfrak{f}(x, \boldsymbol{v}) \text{ and} \\ (\widetilde{\mathfrak{g}_{(c,d)}})(\boldsymbol{u}, \bullet) : (\widetilde{E_{(c,d)}})_{\boldsymbol{u}} &\to \mathrm{p}^\Gamma \\ \boldsymbol{v} &\mapsto \widetilde{\mathfrak{g}_{(c,d)}}(\boldsymbol{u}, \boldsymbol{v}) \\ &= \mathrm{rad}_d(\boldsymbol{u}) \cdot \mathfrak{g}(x, \boldsymbol{v}), \end{aligned}$$

where

$$\begin{aligned} (\widetilde{D_{(c,d)}})_{\boldsymbol{u}} &= \{\boldsymbol{v} \in \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \mid (x, \boldsymbol{v}) \in D\} \text{ and} \\ (\widetilde{E_{(c,d)}})_{\boldsymbol{u}} &= \{\boldsymbol{v} \in \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \mid (x, \boldsymbol{v}) \in E\}. \end{aligned}$$

In other words, for each $\boldsymbol{u} \in \mathrm{rv}_d^{\mathrm{c}}(\mathrm{K}) \cap (\mathrm{RV}_d^\times)^r$ we have $(\widetilde{\mathfrak{f}_{(c,d)}})(\boldsymbol{u}, \bullet) = \mathrm{rad}_d(\boldsymbol{u}) \cdot \mathfrak{f}_x$ and $(\widetilde{\mathfrak{g}_{(c,d)}})(\boldsymbol{u}, \bullet) = \mathrm{rad}_d(\boldsymbol{u}) \cdot \mathfrak{g}_x$ for any (and all) $x \in (\mathrm{rv}_d^{\mathrm{c}})^{-1}(\boldsymbol{u})$, and thus

$$\begin{aligned}\int_{\mathrm{mot}} (\widetilde{\mathfrak{f}_{(c,d)}})(\boldsymbol{u}, \bullet) &= \mathrm{rad}_d(\boldsymbol{u}) \cdot \int_{\mathrm{mot}} \mathfrak{f}_x \\ &= \mathrm{rad}_d(\boldsymbol{u}) \cdot \int_{\mathrm{mot}} \mathfrak{g}_x \\ &= \int_{\mathrm{mot}} (\widetilde{\mathfrak{g}_{(c,d)}})(\boldsymbol{u}, \bullet)\end{aligned}$$

in $R_{\mathrm{mot}}(Z(\boldsymbol{u})) \subset R_{\mathrm{mot}}(Z(x))$ for all $\boldsymbol{u} \in \mathrm{rv}_d^{\mathrm{c}}(\mathrm{K}) \cap (\mathrm{RV}_d^\times)^r$, by the assumption that $\int_{\mathrm{mot}} \mathfrak{f}_x = \int_{\mathrm{mot}} \mathfrak{g}_x$ for all $x \in \mathrm{K}$. (For the inclusion $R_{\mathrm{mot}}(Z(\boldsymbol{u})) \subset R_{\mathrm{mot}}(Z(x))$, see Lemma 4.3.12.)

Since we have $\widetilde{\mathfrak{f}_{(c,d)}} = \bigsqcup_{\boldsymbol{u}} \widetilde{\mathfrak{f}_{(c,d)}}(\boldsymbol{u}, \bullet)$ and $\widetilde{\mathfrak{g}_{(c,d)}} = \bigsqcup_{\boldsymbol{u}} \widetilde{\mathfrak{g}_{(c,d)}}(\boldsymbol{u}, \bullet)$ by definition, Lemma 4.4.12 thus implies

$$\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{g}_{(c,d)}} = \int_{\mathrm{mot}} \mathfrak{g},$$

as claimed.

(3) For $e \geq 1$, $\underline{\mathrm{intDef}}(e-1)$ and $\underline{\mathrm{intLem}}(e-1)$ imply $\underline{\mathrm{intDef}}(e)$.

Let $M \subset \mathrm{K} \cup \Gamma$ and $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ with $D \subset \mathrm{K} \times \mathrm{K}^{e-1} \times \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ be as in the statement of $\underline{\mathrm{intDef}}(e)$, and suppose that $\mathfrak{f}$ is both (c, d)-prepared and (c', d')-prepared, where $d, d' \in \mathbb{N}_{>0}$ and $c : \mathrm{K}^e \to \mathrm{K}^r$ and $c' : \mathrm{K}^e \to \mathrm{K}^{r'}$ are M-definable for some $r, r' \in \mathbb{N}_{>0}$. We then have to show the equality

$$\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c',d')}}.$$

For each $x \in \mathrm{pr}_1(D)$, consider the two $(M \cup \{x\})$-definable integrable functions

$$\begin{aligned}(\widetilde{\mathfrak{f}_{(c,d)}})(x, \bullet, \bullet, \bullet) : (\widetilde{D_{(c,d)}})_x &\to \mathrm{p}^\Gamma \\ (\boldsymbol{y}, \mathrm{rv}_d^{c(x,\boldsymbol{y})}(z), \boldsymbol{v}) &\mapsto \widetilde{\mathfrak{f}_{(c,d)}}(x, \boldsymbol{y}, \mathrm{rv}_d^{c(x,\boldsymbol{y})}(z), \boldsymbol{v}) \\ &= \mathrm{rad}_d(\mathrm{rv}_d^{c(x,\boldsymbol{y})}(z)) \cdot \mathfrak{f}(x, \boldsymbol{y}, z, \boldsymbol{v}), \\ \text{and } (\widetilde{\mathfrak{f}_{(c',d')}})(x, \bullet, \bullet, \bullet) : (\widetilde{D_{(c',d')}})_x &\to \mathrm{p}^\Gamma \\ (\boldsymbol{y}, \mathrm{rv}_{d'}^{c'(x,\boldsymbol{y})}(z), \boldsymbol{v}) &\mapsto \widetilde{\mathfrak{f}_{(c',d')}}(x, \boldsymbol{y}, \mathrm{rv}_{d'}^{c'(x,\boldsymbol{y})}(z), \boldsymbol{v}) \\ &= \mathrm{rad}_{d'}(\mathrm{rv}_{d'}^{c'(x,\boldsymbol{y})}(z)) \cdot \mathfrak{f}(x, \boldsymbol{y}, z, \boldsymbol{v}),\end{aligned}$$

where

$$\begin{aligned}
(\widetilde{D_{(\boldsymbol{c},d)}})_x &= \{(\boldsymbol{y}, \mathrm{rv}_d^{\boldsymbol{c}(x,\boldsymbol{y})}(z), \boldsymbol{v}) \mid (x, \boldsymbol{y}, z, \boldsymbol{v}) \in D, \mathrm{rv}_d^{\boldsymbol{c}(x,\boldsymbol{y})}(z) \in (\mathrm{RV}_d^\times)^r\} \\
&\subset \mathrm{K}^{e-1} \times (\mathrm{RV}_d^\times)^r \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}, \\
\text{and } (\widetilde{D_{(\boldsymbol{c}',d')}})_x &= \{(\boldsymbol{y}, \mathrm{rv}_{d'}^{\boldsymbol{c}'(x,\boldsymbol{y})}(z), \boldsymbol{v}) \mid (x, \boldsymbol{y}, z, \boldsymbol{v}) \in D, \mathrm{rv}_{d'}^{\boldsymbol{c}'(x,\boldsymbol{y})}(z) \in (\mathrm{RV}_{d'}^\times)^{r'}\} \\
&\subset \mathrm{K}^{e-1} \times (\mathrm{RV}_{d'}^\times)^{r'} \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}.
\end{aligned}$$

Recall that $\boldsymbol{c}$ and $\boldsymbol{c}'$ are M-definable functions from K^e to K^r and to $\mathrm{K}^{r'}$ respectively, and consider, for $x \in \mathrm{pr}_1(D)$, the $(M \cup \{x\})$-definable functions

$$\begin{aligned}
\boldsymbol{c}(x, \bullet) : \mathrm{K}^{e-1} &\to \mathrm{K}^r \\
\boldsymbol{y} &\mapsto \boldsymbol{c}(x, \boldsymbol{y}) \\
\text{and } \boldsymbol{c}'(x, \bullet) : \mathrm{K}^{e-1} &\to \mathrm{K}^{r'} \\
\boldsymbol{y} &\mapsto \boldsymbol{c}'(x, \boldsymbol{y}).
\end{aligned}$$

For any $x \in \mathrm{pr}_1(D)$, the function

$$\begin{aligned}
\mathfrak{f}_x = \mathfrak{f}(x, \bullet, \bullet, \bullet) : D_x &\to \mathrm{p}^\Gamma \\
(\boldsymbol{y}, z, \boldsymbol{v}) &\mapsto \mathfrak{f}(x, \boldsymbol{y}, z, \boldsymbol{v}),
\end{aligned}$$

where $D_x = \{(\boldsymbol{y}, z, \boldsymbol{v}) \in \mathrm{K}^{e-1} \times \mathrm{K} \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}} \mid (x, \boldsymbol{y}, z, \boldsymbol{v}) \in D\}$, is then both $(\boldsymbol{c}(x, \bullet), d)$-prepared and $(\boldsymbol{c}'(x, \bullet), d')$-prepared, and we moreover have

$$\mathrm{dom}((\widetilde{\mathfrak{f}_x)_{(\boldsymbol{c}(x,\bullet),d)}}) = (\widetilde{D_x)_{(\boldsymbol{c}(x,\bullet),d)}} = (\widetilde{D_{(\boldsymbol{c},d)}})_x = \mathrm{dom}((\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}})_x),$$

and furthermore

$$\begin{aligned}
(\widetilde{\mathfrak{f}_x)_{(\boldsymbol{c}(x,\bullet),d)}}(\boldsymbol{y}, \mathrm{rv}_d^{\boldsymbol{c}(x,\boldsymbol{y})}(z), \boldsymbol{v}) &= \mathrm{rad}_d(\mathrm{rv}_d^{\boldsymbol{c}(x,\boldsymbol{y})}(z)) \cdot \mathfrak{f}_x(\boldsymbol{y}, z, \boldsymbol{v}) \\
&= \mathrm{rad}_d(\mathrm{rv}_d^{\boldsymbol{c}(x,\boldsymbol{y})}(z)) \cdot \mathfrak{f}(x, \boldsymbol{y}, z, \boldsymbol{v}) \\
&= (\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}})_x(\boldsymbol{y}, \mathrm{rv}_d^{\boldsymbol{c}(x,\boldsymbol{y})}(z), \boldsymbol{v})
\end{aligned}$$

for all $(\boldsymbol{y}, \mathrm{rv}_d^{\boldsymbol{c}(x,\boldsymbol{y})}(z), \boldsymbol{v}) \in (\widetilde{D_{(\boldsymbol{c},d)}})_x$. Taken all together, we just argued that

$$(\widetilde{\mathfrak{f}_x)_{(\boldsymbol{c}(x,\bullet),d)}} = (\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}})_x \text{ for all } x \in \mathrm{pr}_1(D),$$

and we analogously obtain

$$(\widetilde{\mathfrak{f}_x)_{(\boldsymbol{c}'(x,\bullet),d')}} = (\widetilde{\mathfrak{f}_{(\boldsymbol{c}',d')}})_x \text{ for all } x \in \mathrm{pr}_1(D).$$

This implies

$$\int_{\mathrm{mot}} (\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}})_x = \int_{\mathrm{mot}} (\widetilde{\mathfrak{f}_x)_{(\boldsymbol{c}(x,\bullet),d)}} = \int_{\mathrm{mot}} (\widetilde{\mathfrak{f}_x)_{(\boldsymbol{c}'(x,\bullet),d')}} = \int_{\mathrm{mot}} (\widetilde{\mathfrak{f}_{(\boldsymbol{c}',d')}})_x$$

in $R_{\mathrm{mot}}(Z(x))$ for all $x \in \mathrm{K}$ by $\underline{\mathrm{intDef}}(e-1)$ applied to $\mathfrak{f}_x$ with respect to $(c(x, \bullet), d)$ and $(c'(x, \bullet), d')$. Applying $\underline{\mathrm{intLem}}(e-1)$ to $\widetilde{\mathfrak{f}_{(c,d)}}$ and $\widetilde{\mathfrak{f}_{(c',d')}}$ now yields

$$\int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c',d')}}$$

in $R_{\mathrm{mot}}(Z)$, as claimed.

(4) For $e \geq 1$, $\underline{\mathrm{intDef}}(e-1)$, $\underline{\mathrm{intDef}}(e)$, and $\underline{\mathrm{intLem}}(e-1)$ imply $\underline{\mathrm{intLem}}(e)$.

Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ and $\mathfrak{g} : E \to \mathrm{p}^\Gamma$ be two M-definable integrable functions with $D, E \subset \mathrm{K} \times \mathrm{K}^{e-1} \times \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$, as in the statement of $\underline{\mathrm{intLem}}(e)$. Suppose that $\int_{\mathrm{mot}} \mathfrak{f}_x = \int_{\mathrm{mot}} \mathfrak{g}_x$ in $R_{\mathrm{mot}}(Z(x))$ for all $x \in \mathrm{K}$.

Similarly to the proof of (2), Lemma 5.1.3 yields some tuple (c, d) such that both $\mathfrak{f}$ and $\mathfrak{g}$ are (c, d)-prepared. Similar to the proof of (3), and with the same notation, both of the functions $\mathfrak{f}_x$ and $\mathfrak{g}_x$ are then $(c(x, \bullet), d)$-prepared, and we have

$$\begin{aligned}
\int_{\mathrm{mot}} (\widetilde{\mathfrak{f}_{(c,d)}})_x &= \int_{\mathrm{mot}} \widetilde{(\mathfrak{f}_x)_{(c(x,\bullet),d)}} \\
&= \int_{\mathrm{mot}} \mathfrak{f}_x = \int_{\mathrm{mot}} \mathfrak{g}_x \\
&= \int_{\mathrm{mot}} \widetilde{(\mathfrak{g}_x)_{(c(x,\bullet),d)}} = \int_{\mathrm{mot}} (\widetilde{\mathfrak{g}_{(c,d)}})_x
\end{aligned}$$

for each $x \in \mathrm{pr}_1(D) = \mathrm{pr}_1(E)$ by $\underline{\mathrm{intDef}}(e-1)$. Applying $\underline{\mathrm{intDef}}(e)$ to $\mathfrak{f}$ and $\mathfrak{g}$ respectively, and applying $\underline{\mathrm{intLem}}(e-1)$ to $\widetilde{\mathfrak{f}_{(c,d)}}$ and $\widetilde{\mathfrak{g}_{(c,d)}}$ now yields

$$\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} = \int_{\mathrm{mot}} \widetilde{\mathfrak{g}_{(c,d)}} = \int_{\mathrm{mot}} \mathfrak{g}$$

in $R_{\mathrm{mot}}(Z)$, as claimed. □

As described before, the integral immediately leads to a notion of measure for subsets of K^* as follows.

Definition 5.1.12. The *measure* of a bounded M-definable subset $X \subset \mathrm{K}^*$, denoted by $\mu_{\mathrm{mot}}(X)$, is given as the integral of the constant function with value 1 on X, i.e.,

$$\mu_{\mathrm{mot}}(X) = \int_{\mathrm{mot}} \mathrm{const}_X(1).$$

The following observation gives a sufficient condition for the measure of set to be 0, which will be applied in Section 5.3.

Corollary 5.1.13 (of Theorem 5.1.10)**.** *Let $f : X \to \mathrm{K}$ be an M-definable function for some $X \subset \mathrm{K}^e$, $e \in \mathbb{N}$, and let $\mathfrak{f} : Y \to \mathrm{p}^\Gamma$ be an M-definable integrable function on $Y = \mathrm{graph}(f)$. Then we have $\int_{\mathrm{mot}} \mathfrak{f} = 0$.*

In particular, with $\mathfrak{f} = \mathrm{const}_Y(1)$, this yields $\mu_{\mathrm{mot}}(\mathrm{graph}(f)) = 0$ for any M-definable function $f : X \to \mathrm{K}$ with $X \subset \mathrm{K}^*$.

Proof. Let $f : X \to \mathrm{K}$ be M-definable, $X \subset \mathrm{K}^e$, and let $\mathfrak{f} : Y \to \mathrm{p}^\Gamma$ be an M-definable integrable function with domain $Y = \mathrm{graph}(f)$. Pick some $d, r \in \mathbb{N}_{>0}$ and some M-definable function $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ for which $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared. By Lemma 5.1.3 (2), we can assume that $\mathrm{pr}_r \circ \boldsymbol{c} = f$, and we then have

$$Y_{(\boldsymbol{c},d)} \subset \{(\boldsymbol{x}, \boldsymbol{u}) \in X \times \mathrm{RV}_d^r \mid \mathrm{pr}_r(\boldsymbol{u}) = 0\},$$

yielding $\widetilde{Y_{(\boldsymbol{c},d)}} = \emptyset$ and hence $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}} = 0$, as $[\underbrace{\widetilde{\mathfrak{f}_{(\boldsymbol{c},d)}}}_{=\emptyset}] = 0$. ☐

Remark 5.1.14. Note that Corollary 5.1.13 also implies that $\mu_{\mathrm{mot}}(Y') = 0$ for any M-definable subset $Y' \subset \mathrm{graph}(f)$, as we then have $Y' = \mathrm{graph}(f \restriction \mathrm{pr}_e(Y'))$, so Y' is itself the graph of an M-definable function.

Moreover, Corollary 5.1.13 also implies that subsets $X \subset \mathrm{K}^e$ with $\dim(X) < e$ have measure 0 in general, e.g., by applying cell decomposition (cf. [Den86]).

Spelling out the recursive definition of the integral yields a rather instructive way to compute $\int_{\mathrm{mot}} \mathfrak{f}$ for a given integrable function $\mathfrak{f} : D \to \mathrm{p}^\Gamma$. Let us do this once in a general fashion and then, just below, apply it to more specific cases.

Corollary 5.1.15 (of Theorem 5.1.10)**.** *Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ be an M-definable integrable function, where $D \subset \mathrm{K}^e \times \mathrm{RV}_{\boldsymbol{m}}^{\boldsymbol{n}}$ for some $e, \ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_{>0}^\ell$. Consider tuples $(\boldsymbol{c}_i, d_i)$ such that $\mathfrak{f}_i$ is $(\boldsymbol{c}_i, d_i)$-prepared for $i = 0, \ldots, e-1$, where $\mathfrak{f}_0 = \mathfrak{f}$ and $\mathfrak{f}_{i+1} = \widetilde{(\mathfrak{f}_i)_{(\boldsymbol{c}_i,d_i)}}$.*

Then we have $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{f}_i$ for all $i = 0, \ldots, e$. Moreover, writing

$$\boldsymbol{u}_j = \boldsymbol{u}_j(\boldsymbol{x}_{\le j}) = \mathrm{rv}_{d_{j-1}}^{\boldsymbol{c}_{j-1}(x_1,\ldots,x_{j-1})}(x_j) \in \mathrm{RV}_{d_{j-1}}^{r_{j-1}}$$

for $j = e-i+1, \ldots, e$ and $\boldsymbol{x} \in \mathrm{K}^e$, we have

$$\begin{aligned} &\mathfrak{f}_i(x_1, \ldots, x_{e-i}, \boldsymbol{u}_{e-i+1}, \ldots, \boldsymbol{u}_e, \boldsymbol{v}) \\ &= \mathfrak{f}(\boldsymbol{x}, \boldsymbol{v}) \cdot \mathrm{rad}_{d_{e-i}}(\boldsymbol{u}_{e-i+1}) \cdot \dots \cdot \mathrm{rad}_{d_{e-1}}(\boldsymbol{u}_e) \end{aligned}$$

for all $(x_1, \ldots, x_{e-i}, \boldsymbol{u}_{e-i+1}, \ldots, \boldsymbol{u}_e, \boldsymbol{v}) \in \mathrm{dom}(\mathfrak{f}_i) = \widetilde{D}_i$*, where*

$$\widetilde{D}_i = D_i \cap (\mathrm{K}^{e-i} \times \prod (\mathrm{RV}^{\times}_{d_{j-1}})^{r_{j-1}} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}) \text{ for}$$
$$D_i = \{(x_{\leq e-i}, \boldsymbol{u}_{e-i+1}(x_{\leq e-i+1}), \ldots, \boldsymbol{u}_e(x_{\leq e}), \boldsymbol{v}) \mid (x, \boldsymbol{v}) \in D\}.$$

(In the first line, the product runs over all j from $e-i+1$ to e.)

Let us consider two important applications of Corollary 5.1.15 which are much less technical and as such also more intuitive. Those two special cases will help us to explicitly calculate some integrals later on, and they will be particularly useful when proving the change of variables formula, Proposition 5.2.15.

Remark 5.1.16. Let $M \subset \mathrm{K} \cup \Gamma$ and let $X \subset \mathrm{K}^e \times \mathrm{K}$ be an M-definable set with

$$X = \{(x, y) \in \mathrm{pr}_{\leq e}(X) \times \mathrm{K} \mid \mathrm{rv}_d(y - c(x)) = v\}$$

for some M-definable function $c : \mathrm{K}^e \to \mathrm{K}$, some $d \in \mathbb{N}_{>0}$, and some (necessarily M-definable) $v \in \mathrm{RV}_d$.

Moreover, let $\mathfrak{f} : X \to \mathrm{p}^\Gamma$ be an M-definable integrable function for which the value of $\mathfrak{f}(x, y)$ only depends on x, i.e., there is a (necessarily M-definable) integrable function $\mathfrak{g} : \mathrm{pr}_{\leq e}(X) \to \mathrm{p}^\Gamma$ such that $\mathfrak{f}(x, y) = \mathfrak{g}(x)$ for all $(x, y) \in X$.

Then we have

$$\int_{\mathrm{mot}} \mathfrak{f} = \mathrm{rad}_d(v) \cdot \int_{\mathrm{mot}} \mathfrak{g}$$

Proof. Note that we have $\widetilde{\mathfrak{f}_{(c,d)}} = \mathfrak{g} \star \mathrm{const}_v(1)$, i.e., $\widetilde{\mathfrak{f}_{(c,d)}}(x, v) = \mathfrak{g}(x) \cdot 1$, for $\boldsymbol{c} = (c)$ by the assumptions. Corollary 5.1.15 hence yields

$$\begin{aligned}
\int_{\mathrm{mot}} \mathfrak{f} &= \int_{\mathrm{mot}} \widetilde{\mathfrak{f}_{(c,d)}} \\
&= \int_{\mathrm{mot}} \mathfrak{g} \star \mathrm{const}_v(1) \\
&= \int_{\mathrm{mot}} \tilde{\mathfrak{g}} \star \mathrm{const}_v(1) \\
&= \int_{\mathrm{mot}} \tilde{\mathfrak{g}} \cdot \int_{\mathrm{mot}} \mathrm{const}_v(1) \\
&= (\int_{\mathrm{mot}} \tilde{\mathfrak{g}}) \cdot \mathrm{rad}_d(v) \\
&= (\int_{\mathrm{mot}} \mathfrak{g}) \cdot \mathrm{rad}_d(v),
\end{aligned}$$

where $\tilde{\mathfrak{g}}$ is the integrable function on RV^*_* eventually obtained from $\mathfrak{g}$ through the recursive process described in Corollary 5.1.15. □

Example 5.1.17. Consider a "twisted box", i.e., a set of the form

$$X = \{\boldsymbol{x} = (x_1, \ldots, x_e) \in \mathrm{K}^e \mid \mathrm{rv}_{d_i}(x_i - c_i(\boldsymbol{x}_{<i})) = u_i\} \subset \mathrm{K}^e$$

for some M-definable functions $c_i : \mathrm{K}^i \to \mathrm{K}$, some $d_i \in \mathbb{N}_{>0}$ and some $u_i \in \mathrm{RV}_{d_i}$ for $i = 1, \ldots, e$.

Repeatedly applying Remark 5.1.16 to the constant function $\mathrm{const}_X(\gamma)$, for some $\gamma \in \mathrm{p}^\Gamma$, then yields the equality

$$\int_{\mathrm{mot}} \mathrm{const}_X(\gamma) = \gamma \cdot \prod_i \mathrm{rad}_{d_i}(u_i).$$

(Note that X is bounded and that any constant function on a bounded set is integrable, cf Definition 5.1.5.)

In particular, setting $\gamma = 1$, we have $\mu_{\mathrm{mot}}(X) = \prod_i \mathrm{rad}_{d_i}(u_i)$, i.e., the measure of a product of balls (or of a twisted box, more generally) is "what it should be", namely the product of the radii of the balls.

We close this section with another important consequence of Corollary 5.1.15, namely a generalization of Lemma 4.4.12 allowing K-coordinates.

Lemma 5.1.18. *Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ and $\mathfrak{g} : E \to \mathrm{p}^\Gamma$ be two M-definable integrable functions with $D \subset \mathrm{K}^* \times \mathrm{RV}^*_* \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ and $E \subset \mathrm{K}^* \times \mathrm{RV}^*_* \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ for some $\ell \in \mathbb{N}_{>0}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^\ell_{>0}$, and suppose that we have*

$$\int_{\mathrm{mot}} \mathfrak{f}(\bullet, \bullet, \boldsymbol{v}) = \int_{\mathrm{mot}} \mathfrak{g}(\bullet, \bullet, \boldsymbol{v})$$

for all $\boldsymbol{v} \in \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$. Then $\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \mathfrak{g}$.

Proof. By Corollary 5.1.15, there are (M-definable) integrable functions $\tilde{\mathfrak{f}} : \tilde{D} \to \mathrm{p}^\Gamma$ and $\tilde{\mathfrak{g}} : \tilde{E} \to \mathrm{p}^\Gamma$ for some $\tilde{D} \subset \mathrm{RV}^*_* \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ and $\tilde{E} \subset \mathrm{RV}^*_* \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ with

$$\int_{\mathrm{mot}} \tilde{\mathfrak{f}} = \int_{\mathrm{mot}} \mathfrak{f} \quad \text{and} \quad \int_{\mathrm{mot}} \tilde{\mathfrak{g}} = \int_{\mathrm{mot}} \mathfrak{g}.$$

By definition of $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ as in Corollary 5.1.15, this in particular implies

$$\int_{\mathrm{mot}} \tilde{\mathfrak{f}}(\bullet, \boldsymbol{v}) = \int_{\mathrm{mot}} \mathfrak{f}(\bullet, \bullet, \boldsymbol{v}) = \int_{\mathrm{mot}} \mathfrak{g}(\bullet, \bullet, \boldsymbol{v}) = \int_{\mathrm{mot}} \tilde{\mathfrak{g}}(\bullet, \boldsymbol{v})$$

for all $\boldsymbol{v} \in \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$, where the second equality is the assumption from the statement. Lemma 4.4.12 applied to $\tilde{\mathfrak{f}}(\bullet, \boldsymbol{v})$ and $\tilde{\mathfrak{g}}(\bullet, \boldsymbol{v})$ now yields the claim

$$\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} \tilde{\mathfrak{f}} = \int_{\mathrm{mot}} \tilde{\mathfrak{g}} = \int_{\mathrm{mot}} \mathfrak{g}. \qquad \square$$

5.2 Change of variables

An important aspect of integration is understanding how the integral of a function $\mathfrak{f} : X \to \mathrm{p}^{\Gamma}$ changes when composing it with some bijective map $\tau : X \to Y$. If that τ is "measure-preserving" (in some sense not defined yet), we expect the same value for the integral. More generally, in case $X, Y \subset \mathrm{K}$, the derivative (and the Jacobian matrix, as a higher dimensional analogon) of τ gets involved in the equation.

In this section, we make these notions precise in our setting, allowing us to then state and prove a change-of-variable formula, Proposition 5.2.15.

Definition 5.2.1. Let $B \subset \mathrm{K}$ be a ball of radius $\alpha \in \mathrm{p}^{\Gamma}$. We say that a function $f : \mathrm{K} \to \mathrm{K}$ has the *(valuative) Jacobian property on B* if it is either constant on B or there is some $\gamma \in \mathrm{p}^{\Gamma}$ such that

(1) the set $f(B)$ is a ball of radius $\gamma \cdot \alpha$ and

(2) for all $x, y \in B$ with $x \neq y$, we have $\left|\frac{f(x)-f(y)}{x-y}\right| = \gamma$.

For $d, r \in \mathbb{N}_{>0}$ and a tuple $\boldsymbol{c} \in \mathrm{K}^r$, we say that f has the *$(\boldsymbol{c}, d)$-Jacobian property*, if it has the Jacobian property on all proper $(\boldsymbol{c}, d)$-balls, i.e., on all sets of the form $(\mathrm{rv}^{\boldsymbol{c}}_d)^{-1}(\boldsymbol{u}) = \{x \in \mathrm{K} \mid \mathrm{rv}^{\boldsymbol{c}}_d(x) = \boldsymbol{u}\}$ for $\boldsymbol{u} \in (\mathrm{RV}^{\times}_d)^r$.

Remark 5.2.2. Note that a function which has the Jacobian property on a given ball is, in particular, either constant or injective on that ball. Indeed, the condition (2) from Definition 5.2.1 implies $|f(x) - f(y)| > 0$, for $x \neq y$, yielding injectivity on B.

Remark 5.2.3. Note that "having the $(\boldsymbol{c}, d)$-Jacobian property" can be expressed by a first-order formula. Indeed, (1) for $B = (\mathrm{rv}^{\boldsymbol{c}}_d)^{-1}(\boldsymbol{u})$ is equivalent to

$$\exists y_0 : \forall y \left(\underbrace{(\exists x : (\mathrm{rv}^{\boldsymbol{c}}_d(x) = \boldsymbol{u} \wedge y = f(x)))}_{y \in f(B)} \longleftrightarrow \underbrace{|y - y_0| \leq \gamma \cdot \mathrm{p}^{-d-\mathrm{val}(\boldsymbol{u})}}_{y \in B_{\leq \gamma \cdot \mathrm{p}^{-d-\mathrm{val}(\boldsymbol{u})}}(y_0)} \right),$$

and (2) is already (almost) a first-order formula as stated.

Lemma 5.2.4. *Let $B' \subset B \subset \mathrm{K}$ be two balls and suppose that $f : \mathrm{K} \to \mathrm{K}$ has the Jacobian property on B. Then f has the Jacobian property on B'.*

Proof. If f is constant on B, then it is constant on B'. Otherwise, fix $x \in B'$ and $\alpha, \beta \in \mathrm{p}^\Gamma$ (with $\beta \leq \alpha$) such that $B = \mathcal{B}_{\leq\alpha}(x)$ and $B' = \mathcal{B}_{\leq\beta}(x)$. Moreover, fix $\gamma \in \mathrm{p}^\Gamma$ as in Definition 5.2.1, i.e., such that $f(B)$ is a ball of radius $\gamma \cdot \alpha$. More explicitly, we then have $f(B) = \mathcal{B}_{\leq\gamma\cdot\alpha}(f(x))$.

Our claim now is that $f(B') = \mathcal{B}_{\leq\gamma\cdot\beta}(f(x))$. We prove the two inclusions separately. First, let $y \in B' \subset B$ with $y \neq x$. By Definition 5.2.1 (2), we then have

$$|f(y) - f(x)| = \gamma \cdot \underbrace{|y - x|}_{\leq\beta} \leq \gamma \cdot \beta,$$

i.e., $f(y) \in \mathcal{B}_{\leq\gamma\cdot\beta}(f(x))$, as claimed. For the other direction, consider

$$\begin{aligned} z &\in \mathcal{B}_{\leq\gamma\cdot\beta}(f(x)) \\ &\subset \mathcal{B}_{\leq\gamma\cdot\alpha}(f(x)) \\ &= f(B) \end{aligned}$$

and fix $y \in B$ with $z = f(y)$. Then Definition 5.2.1 (2) implies

$$|y - x| = \gamma^{-1} \cdot \underbrace{|z - f(x)|}_{\leq\gamma\cdot\beta} \leq \beta,$$

i.e., $y \in \mathcal{B}_{\leq\beta}(x) = B'$, and hence $z = f(y) \in f(B')$. This finishes the proof. $\square$

Let us recall the statement of Lemma 5.1.3: It says that (1) any M-definable integrable function is $(\boldsymbol{c}, d)$-prepared for some appropriate choice of $\boldsymbol{c}$ and d, and that (2) "being $(\boldsymbol{c}, d)$-prepared" is stable under enlarging $\boldsymbol{c}$ and under increasing d. We now establish an analogous statement for the $(\boldsymbol{c}, d)$-Jacobian property. Afterwards, we can also combine both statements to obtain preparation and the Jacobian property simultaneously, this is Remark 5.2.7.

Lemma 5.2.5. *Let $M \subset \mathrm{K} \cup \Gamma$ and let $f : \mathrm{K} \to \mathrm{K}$ be any M-definable function. Then the following hold.*

(1) There are $d, r \in \mathbb{N}_{>0}$ and some M-definable tuple $\boldsymbol{c} \in \mathrm{K}^r$ such that f has the $(\boldsymbol{c}, d)$-Jacobian property.

(2) Suppose that f has the $(\boldsymbol{c}, d)$-Jacobian property with $\boldsymbol{c} = (c_1, \ldots, c_r) \in \mathrm{K}^r$. If $\boldsymbol{c}' = (c_1, \ldots, c_r, c_{r+1}, \ldots, c_{r'})$ for some $r' \geq r$, where $c_{r+1}, \ldots, c_{r'} \in \mathrm{K}$, and $d' \in \mathbb{N}_{>0}$ with $d' \geq d$, then f has the $(\boldsymbol{c}', d')$-Jacobian property.

Proof. Part (1) follows from [Clu+21, Corollary 3.1.3]. Part (2) follows from the definition of the Jacobian property, Definition 5.2.1, and Lemma 5.2.4. □

While we only defined the Jacobian property for functions on unary sets, we can also deduce a family version of Lemma 5.2.5 (1) by compactness. We will make use of it in the proof of the change-of-variables formula, Proposition 5.2.15.

Lemma 5.2.6. *Let $M \subset \mathrm{K} \cup \Gamma$ and let $f : \mathrm{K}^e \times \mathrm{K} \to \mathrm{K}$ be an M-definable function for some $e \in \mathbb{N}_{>0}$. Then there are $d, r \in \mathbb{N}_{>0}$ and an M-definable function $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ such that $f(\boldsymbol{x}, \bullet)$ has the $(\boldsymbol{c}(\boldsymbol{x}), d)$-Jacobian property for each $\boldsymbol{x} \in \mathrm{K}^e$.*

Proof. Similarly as in the induction step in the proof of Lemma 5.1.3, this follows from Lemma 5.2.5 by compactness. Just as there, we can first assume that M is finite, and we will from now on work in the (countable) language $\mathcal{L}_{\mathrm{val}}(M)$.

For $d, r \in \mathbb{N}_{>0}$, let $\varphi_{d,r}(\boldsymbol{x}, \boldsymbol{c})$ be the $\mathcal{L}_{\mathrm{val}}(M)$-formula given in Remark 5.2.3 which holds in K if and only if $f(\boldsymbol{x}, \bullet)$ has the $(\boldsymbol{c}, d)$-Jacobian property, where $\boldsymbol{c} = (c_1, \dots, c_r)$.

Temporarily fix $\boldsymbol{x} \in \mathrm{K}^e$. By Lemma 5.2.5, there are $d, r \in \mathbb{N}_{>0}$ and an $(M \cup \{\boldsymbol{x}\})$-definable tuple $\boldsymbol{c}_{\boldsymbol{x}} \in \mathrm{K}^r$ such that $f(\boldsymbol{x}, \bullet)$ has the $(\boldsymbol{c}_{\boldsymbol{x}}, d)$-Jacobian property. Let $\psi_{r,\boldsymbol{x}}(\boldsymbol{y}, \boldsymbol{c})$ be an $\mathcal{L}_{\mathrm{val}}(M)$-formula defining $\boldsymbol{c}_{\boldsymbol{x}}$.

We can now continue (almost literally, with few exceptions) as in the induction step in the proof of Lemma 5.1.3 (1). Indeed, just replace the references to Case 1 and Lemma 5.1.3 (2) in the former by references to Lemma 5.2.5 (1) and (2), and replace the reference to [Clu+21, Lemma 2.3.1 (3)] by a reference to Remark 5.2.3. □

As mentioned above, we can combine Lemma 5.2.6 with Lemma 5.1.3 to obtain the following unified statement.

Remark 5.2.7. Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^{\Gamma}$ be any M-definable integrable function for some $D \subset \mathrm{K}^e \times \mathrm{K} \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ and let $f : \mathrm{K}^e \times \mathrm{K} \to \mathrm{K}$ be any (M-definable) function. Then there are $d, r \in \mathbb{N}_{>0}$ and an M-definable function $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ for which $\mathfrak{f}$ is $(\boldsymbol{c}, d)$-prepared and $f(\boldsymbol{x}, \bullet)$ has the $(\boldsymbol{c}(\boldsymbol{x}), d)$-Jacobian property for all $\boldsymbol{x} \in \mathrm{K}^e$.

We now want to show that the integral of an integrable function $\mathfrak{f} : D \to \mathrm{p}^{\Gamma}$ does not change when permuting the K-coordinates of D. (For the RV-coordinates, this is clear by, e.g., Corollary 5.1.15 and Definition 4.1.2 (2).)

The crucial (first) step is the special case where $D \subset \mathrm{K}^2$ is a twisted box. We will then deduce the general case in Lemma 5.2.9.

Lemma 5.2.8. *Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ be an M-definable integrable function, where $D \subset \mathrm{K}^2$. Suppose there are $d_1, d_2 \in \mathbb{N}$, an M-definable element $c_1 \in \mathrm{K}$ and an M-definable function $c_2 : \mathrm{K} \to \mathrm{K}$, such that*

- *there are $u \in \mathrm{RV}_{d_1}$ and $v \in \mathrm{RV}_{d_2}$ with*
$$D = \{(x, y) \in \mathrm{K}^2 \mid \mathrm{rv}_{d_1}(x - c_1) = u \text{ and } \mathrm{rv}_{d_2}(y - c_2(x)) = v\},$$
- *$\mathfrak{f}$ is (c_2, d_2)-prepared,*
- *$\widetilde{\mathfrak{f}}_{(c_2,d_2)}$ is (c_1, d_1)-prepared, and*
- *c_2 has the (c_1, d_1)-Jacobian property.*

(Note that the second and third condition together, under the assumption of the first, are equivalent to saying that $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ is a constant function.)

Then we have $\int_{\mathrm{mot}}(\mathfrak{f} \circ \mathrm{flip}) = \int_{\mathrm{mot}} \mathfrak{f}$, where $\mathrm{flip} : D \to \mathrm{flip}(D) \subset \mathrm{K}^2$ swaps the two coordinates, i.e., $\mathrm{flip}(x, y) = (y, x)$.

Proof. In the special case $u = 0$, we have $(\widetilde{\widetilde{D}_{(c_2,d_2)}})_{(c_1,d_1)} = \emptyset$, and $\mathfrak{f} \circ \mathrm{flip}$ is (c_1', d_1)-prepared for $c_1' = \mathrm{const}_{\mathrm{pr}_2(D)}(c_1) : \mathrm{pr}_2(D) \to \mathrm{K}$, with $\widetilde{\mathrm{flip}(D)}_{(c_1,d_1)} = \emptyset$. Thus both $\int_{\mathrm{mot}} \mathfrak{f}$ and $\int_{\mathrm{mot}}(\mathfrak{f} \circ \mathrm{flip})$ evaluate to 0, i.e., the claim holds. In the case $v = 0$, we have $\widetilde{D}_{(c_2,d_2)} = \emptyset$, so $\int_{\mathrm{mot}} \mathfrak{f} = 0$, and $\int_{\mathrm{mot}}(\mathfrak{f} \circ \mathrm{flip}) = 0$ follows just as in the Case 2b below.

From now on, let us assume that $u \in \mathrm{RV}_{d_1}^\times$ and $v \in \mathrm{RV}_{d_2}^\times$. Note that the assumptions in particular imply that $\mathfrak{f}$ is constant on D, say $\mathfrak{f} = \mathrm{const}_D(\alpha)$. Hence we have $\mathfrak{f} \circ \mathrm{flip} = \mathrm{const}_{\mathrm{flip}(D)}(\alpha)$ and $\int_{\mathrm{mot}} \mathfrak{f} = \alpha \cdot \mathrm{rad}_{d_1}(u) \cdot \mathrm{rad}_{d_2}(v)$, the latter by Example 5.1.17. We cannot, however, directly apply the same argument to $\mathfrak{f} \circ \mathrm{flip}$, as it requires us to know that $\mathrm{flip}(D)$ is a (twisted) box. Showing that it is – and calculating its size – is our concern for the remainder of the proof.

By Definition 5.2.1, the function c_2 is either constant on $B = \mathrm{pr}_1(D) = \mathrm{rv}_{d_1}^{-1}(u) + c_1$, or there is some $\gamma \in \mathrm{p}^\Gamma$ such that $c_2(B)$ is a ball of radius $\gamma \cdot \mathrm{rad}_{d_1}(u) = \gamma \cdot p^{-d_1 - \mathrm{val}(u)}$. We handle those two cases separately. For the remainder of the proof, we fix some element $(x_0, y_0) \in D$.

Case 1: c_2 is constant on $B = \mathrm{pr}_1(D) = \mathrm{rv}_{d_1}^{-1}(u) + c_1$. Then we have

$$\begin{aligned} D &= \{(x, y) \in \mathrm{K}^2 \mid \mathrm{rv}_{d_1}(x - c_1) = u, \mathrm{rv}_{d_2}(y - c_2(x_0)) = v\} \\ &= \mathrm{pr}_1(D) \times \mathrm{pr}_2(D), \end{aligned}$$

so that $\mathrm{flip}(D) = \mathrm{pr}_2(D) \times \mathrm{pr}_1(D)$, the integrable function $\mathfrak{f} \circ \mathrm{flip} : \mathrm{flip}(D) \to \mathrm{p}^\Gamma$ is (c_2', d_2')-prepared and $\widetilde{\mathfrak{f}}_{(c_2',d_2')}$ is (c_1', d_1')-prepared for $c_2' = \mathrm{const}_{\mathrm{pr}_1(D)}(c_1)$, $d_2' = d_1$, and $c'_1 = c_2(x_0)$, $d'_1 = d_2$, with $(\mathrm{flip}(D)_{(c'_2,d'_2)})_{(c'_1,d'_1)} = \{(v,u)\}$. We thus have

$$\begin{aligned}\int_{\mathrm{mot}} \mathfrak{f} \circ \mathrm{flip} &= \int_{\mathrm{mot}} \mathrm{const}_{\mathrm{flip}(D)}(\alpha) \\ &= \alpha \cdot \mathrm{rad}_{d'_1}(v) \cdot \mathrm{rad}_{d'_2}(u) \\ &= \alpha \cdot \mathrm{rad}_{d_2}(v) \cdot \mathrm{rad}_{d_1}(u) \\ &= \int_{\mathrm{mot}} \mathfrak{f}\end{aligned}$$

by Example 5.1.17.

Case 2: $c_2(B)$ is a ball of radius $\gamma \cdot \mathrm{rad}_{d_1}(u)$. Pick some $u' \in \mathrm{RV}_{d_1}$ with $\mathrm{ac}_{d_1}(u') = \mathrm{ac}_{d_1}(u)$ and $\mathrm{val}(u') = \mathrm{val}(\gamma) + \mathrm{val}(u)$. Moreover, pick some $y_1 \in K$ with $\mathrm{rv}_{d_1}(c_2(x_0) - y_1) = u'$. Then we have

$$c_2(B) = \mathrm{rv}_{d_1}^{-1}(u') + y_1,$$

since both are balls of radius $\mathrm{rad}_{d_1}(u') = \gamma \cdot \mathrm{rad}_{d_1}(u)$ containing $c_2(x_0)$. Thus

$$\begin{aligned}\mathrm{pr}_2(D) &= \{y \in K \mid \exists x \in B : \mathrm{rv}_{d_2}(y - c_2(x)) = v\} \\ &= \bigcup_{x \in B} \mathrm{rv}_{d_2}^{-1}(v) + c_2(x) \\ &= \mathrm{rv}_{d_2}^{-1}(v) + c_2(B) \\ &= \mathrm{rv}_{d_2}^{-1}(v) + \mathrm{rv}_{d_1}^{-1}(u') + y_1\end{aligned}$$

is a ball of radius $\max\{\mathrm{rad}_{d_2}(v), \mathrm{rad}_{d_1}(u')\}$ containing y_0, i.e., we have

$$\mathrm{pr}_2(D) = \begin{cases} \mathrm{rv}_{d_2}^{-1}(v) + c_2(x_0), & \text{if } d_1 + \mathrm{val}(\gamma) + \mathrm{val}(u) \geq d_2 + \mathrm{val}(v) \\ \mathrm{rv}_{d_1}^{-1}(u') + y_0 - c_2(x_0) + y_1, & \text{if } d_1 + \mathrm{val}(\gamma) + \mathrm{val}(u) \leq d_2 + \mathrm{val}(v) \end{cases}.$$

Recall that $v = \mathrm{rv}_{d_2}(y_0 - c_2(x_0))$, so for $x, y \in K$, we have the equivalence

$$\mathrm{rv}_{d_2}(y - c_2(x)) = v \iff \mathrm{val}(y - y_0 + c_2(x_0) - c_2(x)) \geq d_2 + \mathrm{val}(v),$$

by Remark 2.2.7.

Case 2a: $d_1 + \mathrm{val}(\gamma) + \mathrm{val}(u) \geq d_2 + \mathrm{val}(v)$. Let $x \in \mathrm{pr}_1(D)$ and $y \in \mathrm{pr}_2(D)$. We will show that $(x, y) \in D$, i.e., that $D = \mathrm{pr}_1(D) \times \mathrm{pr}_2(D)$. The claim then follows just as in Case 1. Towards that end, note that we have

$$\begin{aligned}\mathrm{val}(y - y_0) &= \mathrm{val}(y - c_2(x_0) - (y_0 - c_2(x_0))) \\ &\geq d_2 + \mathrm{val}(v)\end{aligned}$$

by Remark 2.2.7, since both y and y_0 lie in $\mathrm{pr}_2(D) = \mathrm{rv}_{d_2}^{-1}(v) + c_2(x_0)$. Similarly, we have

$$\begin{aligned}\mathrm{val}(c_2(x_0) - c_2(x)) &= \mathrm{val}(c_2(x_0) - y_1 - (c_2(x) - y_1)) \\ &\geq d_1 + \mathrm{val}(u') = d_1 + \mathrm{val}(\gamma) + \mathrm{val}(u) \\ &\geq d_2 + \mathrm{val}(v),\end{aligned}$$

since both $c_2(x_0)$ and $c_2(x)$ lie in $c_2(B) = \mathrm{rv}_{d_1}^{-1}(u') + y_1$.

Using both these inequalities, we get

$$\begin{aligned}\mathrm{val}(y - y_0 + c_2(x_0) - c_2(x)) &\geq \min\{y - y_0, c_2(x_0) - c_2(x)\}\\ &\geq d_2 + \mathrm{val}(v),\end{aligned}$$

and hence $\mathrm{rv}_{d_2}(y - c_2(x)) = v$ holds, i.e., $(x, y) \in D$ as claimed.

Case 2b: $d_1 + \mathrm{val}(\gamma) + \mathrm{val}(u) \leq d_2 + \mathrm{val}(v)$**.** Then $\mathrm{pr}_2(D) = c_2(B)+y_0-c_2(x_0)$, and since c_2 is injective on B (see Remark 5.2.2), we have

$$D = \{(x, y) \in B \times \mathrm{pr}_2(D) \mid \mathrm{val}(x - \tilde{c}_2(y)) \geq d_2 + \mathrm{val}(v) - \mathrm{val}(\gamma)\},$$

where the function $\tilde{c}_2 : \mathrm{pr}_2(D) \to B = \mathrm{pr}_1(D)$ given by $y \mapsto {c_2}^{-1}(y - y_0 + c_2(x_0))$ is $(M \cup \{x_0, y_0\})$-definable.

Indeed, using Remark 2.2.7 and the assumption that c_2 has the (c_1, d_1)-Jacobian property, we have

$$\begin{aligned}(x, y) \in D &\iff \mathrm{rv}_{d_2}(y - c_2(x)) = v\\ &\iff \mathrm{val}(y - y_0 + c_2(x_0) - c_2(x)) \geq d_2 + \mathrm{val}(v)\\ &\iff \mathrm{val}(\underbrace{{c_2}^{-1}(y - y_0 + c_2(x_0))}_{=:\, \tilde{c}_2(y)} - x) + \mathrm{val}(\gamma) \geq d_2 + \mathrm{val}(v)\\ &\iff \mathrm{val}(x - \tilde{c}_2(y)) \geq d_2 + \mathrm{val}(v) - \mathrm{val}(\gamma),\end{aligned}$$

for all $(x, y) \in B \times \mathrm{pr}_2(D)$.

Fixing some $z \in \mathbb{K}$ with $\mathrm{val}(z) = \mathrm{val}(v) - \mathrm{val}(\gamma)$, we then have

$$\begin{aligned}(x, y) \in D &\iff \mathrm{val}(x - \tilde{c}_2(y)) \geq d_2 + \mathrm{val}(v) - \mathrm{val}(\gamma)\\ &\iff \mathrm{val}(x - \tilde{c}_2(y) + z - z) \geq d_2 + \mathrm{val}(z)\\ &\iff \mathrm{rv}_{d_2}(x - \tilde{c}_2(y) + z) = \mathrm{rv}_{d_2}(z)\\ &\iff \mathrm{rv}_{d_2}(x - \tilde{c}_2(y) + z) = \mathrm{rv}_{d_2}(z)\end{aligned}$$

for all $(x, y) \in B \times \mathrm{pr}_2(D)$. Note that the set of x satisfying the equality $\mathrm{rv}_{d_2}(x - \tilde{c}_2(y) + z) = \mathrm{rv}_{d_2}(z)$ is a ball of radius $\mathrm{rad}_{d_2}(\mathrm{rv}_{d_2}(z))$ and B is a ball of radius $\mathrm{rad}_{d_1}(u)$ (which is larger than $\mathrm{rad}_{d_2}(\mathrm{rv}_{d_2}(z)) = p^{-d_2-\mathrm{val}(z)} = p^{-d_2-\mathrm{val}(v)+\mathrm{val}(\gamma)}$ by the case assumption), and both these balls contain x_0. Hence the condition $\mathrm{rv}_{d_2}(x - \tilde{c}_2(y) + z) = \mathrm{rv}_{d_2}(z)$ already implies $x \in B$.

Moreover, we have $\mathrm{pr}_2(D) = \mathrm{rv}_{d_1}^{-1}(u') + (y_0 - c_2(x_0) + y_1)$, and thus, all put together

$$\begin{aligned}D = \{(x, y) \in \mathbb{K}^2 \mid\ & x \in B,\ y \in \mathrm{pr}_2(D),\\ & \mathrm{rv}_{d_2}(x - \tilde{c}_2(y) + z) = \mathrm{rv}_{d_2}(z)\}\\ = \{(x, y) \in \mathbb{K}^2 \mid\ & \mathrm{rv}_{d_1}(y - (y_0 - c_2(x_0) + y_1)) = u',\\ & \mathrm{rv}_{d_2}(x - \tilde{c}_2(y) + z) = \mathrm{rv}_{d_2}(z)\},\end{aligned}$$

i.e.,

$$\begin{aligned}\mathrm{flip}(D) = \{(y, x) \in \mathrm{K}^2 \mid \mathrm{rv}_{d_1}(y - (y_0 - c_2(x_0) + y_1)) = u', \\ \mathrm{rv}_{d_2}(x - \tilde{c}_2(y) + z) = \mathrm{rv}_{d_2}(z)\}.\end{aligned}$$

Hence $(\mathfrak{f} \circ \mathrm{flip}) : \mathrm{flip}(D) \to \mathrm{p}^\Gamma$ is (c_2', d_2)-prepared and $\widetilde{(\mathfrak{f} \circ \mathrm{flip})}_{(c_2', d_2)}$ is (c_1', d_1)-prepared for the $(M \cup \{x_0, y_0\})$-definable element $c_1' = (y_0 - c_2(x_0) + y_1) \in \mathrm{K}$ and the $(M \cup \{x_0, y_0, z\})$-definable function $c_2' : \mathrm{K} \to \mathrm{K}$ given by $c_2'(x) = \tilde{c}_2(x) - z$. In $R_{\mathrm{mot}}(Z')$, for $Z' = \mathrm{dcl}(M \cup \{x_0, y_0, z\}) \cap \Gamma \succcurlyeq Z = \mathrm{dcl}(M) \cap \Gamma$, we thus have

$$\begin{aligned}\int_{\mathrm{mot}} \mathfrak{f} \circ \mathrm{flip} &= \int_{\mathrm{mot}} \mathrm{const}_{\mathrm{flip}(D)}(\alpha) \\ &= \alpha \cdot \mathrm{rad}_{d_1}(u') \cdot \mathrm{rad}_{d_2}(w) \\ &= \alpha \cdot p^{-d_1 - \mathrm{val}(u') - d_2 - \mathrm{val}(w)} \\ &= \alpha \cdot p^{-d_1 - \mathrm{val}(\gamma) - \mathrm{val}(u) - d_2 - \mathrm{val}(v) + \mathrm{val}(\gamma)} \\ &= \alpha \cdot p^{-d_1 - \mathrm{val}(u) - d_2 - \mathrm{val}(v)} \\ &= \alpha \cdot \mathrm{rad}_{d_1}(u) \cdot \mathrm{rad}_{d_2}(v) \\ &= \int_{\mathrm{mot}} \mathfrak{f},\end{aligned}$$

by Example 5.1.17. Since both $\mathfrak{f}$ and $\mathfrak{f} \circ \mathrm{flip}$ are M-definable integrable functions, the equality of their integrals also holds in $R_{\mathrm{mot}}(Z)$, as claimed.

This finishes the case distinction and thus the proof. □

Lemma 5.2.9. *Let $M \subset \mathrm{K} \cup \Gamma$ and let $\mathfrak{f} : D \to \mathrm{p}^\Gamma$ be an M-definable integrable function, where $D \subset \mathrm{K}^e \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}}$ for some $e \in \mathbb{N}_{>0}$, $\ell \in \mathbb{N}$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^\ell_{>0}$. Consider the permutation of coordinates*

$$\begin{aligned}\tau : \mathrm{K}^e \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} &\to \mathrm{K}^e \times \mathrm{RV}^{\boldsymbol{n}}_{\boldsymbol{m}} \\ (x_1, \ldots, x_e, \boldsymbol{u}) &\mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(e)}, \boldsymbol{u})\end{aligned}$$

for an arbitrary permutation $\sigma : \{1, \ldots, e\} \to \{1, \ldots, e\}$. Then $\int_{\mathrm{mot}} \mathfrak{f} \circ \tau = \int_{\mathrm{mot}} \mathfrak{f}$.

Proof. We will reduce the general situation to the special case handled in Lemma 5.2.8. First note that we can restrict to the case that σ is a transposition of two consecutive numbers, since any permutation of $\{1, \ldots, e\}$ can be written as a composition of such transpositions. By repeatedly applying <u>intLem</u>(•) (see Definition 5.1.9), we can restrict to the case that σ transposes 1 and 2. By repeatedly applying Corollary 5.1.15 to both $\mathfrak{f}$ and $\mathfrak{f} \circ \tau$, we can replace the last K-coordinate of D (i.e., the e-th coordinate) by an RV^*_*-coordinate, and hence we can restrict to the case that $e = 2$. By Lemma 5.1.18, we can furthermore restrict to the case that $\ell = 0$, i.e., $D = \mathrm{dom}(\mathfrak{f}) \subset \mathrm{K}^2$, and $\tau : \mathrm{K}^2 \to \mathrm{K}^2, (x, y) \mapsto (y, x)$.

Now, pick some $d_2, r_2 \in \mathbb{N}_{>0}$ and some M-definable function $\boldsymbol{c}_2 : \mathrm{K} \to \mathrm{K}^{r_2}$ for which $\mathfrak{f}$ is $(\boldsymbol{c}_2, d_2)$-prepared and consider the sets

$$Q_{\boldsymbol{v}} = \{(x, y) \in D \mid \mathrm{rv}_{d_2}^{\boldsymbol{c}_2(x)}(y) = \boldsymbol{v}\}$$

for $\boldsymbol{v} \in \mathrm{pr}_{\mathrm{RV}_{d_2}^{r_2}}(D_{(\boldsymbol{c}_2, d_2)})$. Note that D is the (disjoint) union of all the sets $Q_{\boldsymbol{v}}$, so Lemma 5.1.18 allows us to establish the claim separately for each $\mathfrak{f}{\restriction}Q_{\boldsymbol{v}}$. Hence we can assume that $D = Q_{\boldsymbol{v}}$ for some given $\boldsymbol{v} \in \mathrm{RV}_{d_2}^{r_2}$. Since the set of y with $\mathrm{rv}_{d_2}^{\boldsymbol{c}_2(x)}(y) = \boldsymbol{v}$ is an intersection of balls (possibly including non-proper ones) for each $x \in \mathrm{pr}_1(D)$, we can moreover assume that $r_2 = 1$ (and we will thus write c_2 instead of $\boldsymbol{c}_2$ from now on). By Remark 5.2.7, there are $d_1, r_1 \in \mathbb{N}_{>0}$ and an M-definable tuple $\boldsymbol{c}_1 \in \mathrm{K}^{r_1}$ for which $\widetilde{\mathfrak{f}_{(c_2,d_2)}}$ is $(\boldsymbol{c}_1, d_1)$-prepared and c_2 has the $(\boldsymbol{c}_1, d_1)$-Jacobian property. Similarly to above, Lemma 5.1.18 allows us to further restrict to the case that $r_1 = 1$, i.e.,

$$D = \{(x, y) \in \mathrm{K}^2 \mid \mathrm{rv}_{d_1}(x - c_1) = u, \mathrm{rv}_{d_2}(y - c_2(x)) = v\}$$

for some fixed $u \in \mathrm{RV}_{d_1}^{r_1}$. Note that c_2 still has the (c_1, d_1)-Jacobian property after this reduction, so that we are now in the situation already handled in Lemma 5.2.8. ☐

Definition 5.2.10 (adapted from [CHR21, Definition 3.1.1])**.** Let $f : X \to \mathrm{K}$ be an arbitrary function for some subset $X \subset \mathrm{K}$ and let $x \in X$. A (or a posteriori: the) (classical) *derivative of f at x* is an element $y \in \mathrm{K}$ such that for all $\beta \in \mathrm{p}^\Gamma$ there is some $\alpha \in \mathrm{p}^\Gamma$ for which we have $\mathcal{B}_{\leq\alpha}(x) \subset X$ and

$$\left| \frac{f(x) - f(x')}{x - x'} - y \right| < \beta.$$

for all $x' \in \mathcal{B}_{\leq\alpha}(x)$ with $x' \neq x$.

Note that such an element y is necessarily unique if it exists, i.e., there is at most one derivative of f at each $x \in X$.

We use the standard notation $f'(x)$ for the derivative of f at x, if it exists.

Lemma 5.2.11. *Let $M \subset \mathrm{K} \cup \Gamma$ and let $f : \mathrm{K} \to \mathrm{K}$ be an M-definable function. Then the set of points at which the derivative of f exists is cofinite and open.*

Proof. Cofiniteness follows from [CHR21, Theorem 3.1.4] and [CHR21, Theorem 5.1.5] together with [Clu+21, Proposition 3.1.1].

Openness follows from the definition of the derivative, Definition 5.2.10, using that a subset $X' \subset \mathrm{K}$ is open if and only if there is, for each $x \in X'$, an element $\alpha \in \mathrm{p}^\Gamma$ such that $\mathcal{B}_{\leq\alpha}(x) \subset X'$. ☐

We can now define the Jacobian matrix, show that it exists almost everywhere, and then finally state and prove the change-of-variables formula, Proposition 5.2.15.

Definition 5.2.12. Let $M \subset \mathrm{K} \cup \Gamma$ and let $\tau : X \to \mathrm{K}^\ell$ be an M-definable function for some (M-definable) $X \subset \mathrm{K}^e$. Fix some element $\boldsymbol{x} = (x_1, \ldots, x_e) \in X$ and consider the functions $\tau_{i,\boldsymbol{x}_{\neq j}} = (\mathrm{pr}_i \circ \tau)(x_1, \ldots, x_{j-1}, \bullet, x_{j+1}, \ldots x_e)$, for $i = 1, \ldots, \ell$ and $j = 1, \ldots, e$, given by

$$\begin{aligned} \tau_{i,\boldsymbol{x}_{\neq j}} : X_{\boldsymbol{x}_{\neq j}} &\to \mathrm{K} \\ y &\mapsto (\mathrm{pr}_i \circ \tau)(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots x_e) \end{aligned}$$

where $X_{\boldsymbol{x}_{\neq j}} = \{y \in \mathrm{K} \mid (x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_e) \in X\}$.

If all of the derivatives $\tau'_{i,\boldsymbol{x}_{\neq j}}(x_j)$ exist, we define the *Jacobian matrix of τ at $\boldsymbol{x}$* as the $\ell \times e$-matrix

$$\mathbf{Jac}_\tau(\boldsymbol{x}) = \begin{pmatrix} \tau'_{1,\boldsymbol{x}_{\neq 1}}(x_1) & \cdots & \tau'_{1,\boldsymbol{x}_{\neq e}}(x_e) \\ \vdots & \ddots & \vdots \\ \tau'_{\ell,\boldsymbol{x}_{\neq 1}}(x_1) & \cdots & \tau'_{\ell,\boldsymbol{x}_{\neq e}}(x_e) \end{pmatrix}$$

with entries in K.

Lemma 5.2.13. *The Jacobian matrix of a given M-definable function $\tau : X \to \mathrm{K}^\ell$ exists "almost everywhere". More formally, we have $\mu_{\mathrm{mot}}(X') = \mu_{\mathrm{mot}}(X)$ for the set X' of points $\boldsymbol{x} \in X$ at which $\mathbf{Jac}_\tau(\boldsymbol{x})$ exists.*

Moreover, the set X' is open in X.

Proof. Let $\tau : X \to \mathrm{K}^\ell$ be an M-definable function for some $X \subset \mathrm{K}^e$, $e \in \mathbb{N}_{>0}$. Note that $\mathbf{Jac}_\tau(\boldsymbol{x})$ exists if and only if $\mathbf{Jac}_{\tau_i}(\boldsymbol{x})$ exists for all $i \in \{1, \ldots, \ell\}$, so we can restrict to the case that $\ell = 1$. Consider the sets

$$\begin{aligned} Z_j &= \{\boldsymbol{x} \in X \mid \text{the derivative } \tau'_{\boldsymbol{x}_{\neq j}}(x_j) \text{ exists}\} \text{ and} \\ Y_j &= X \setminus Z_j \end{aligned}$$

for $j \in \{1, \ldots, e\}$, using the same notation as in Definition 5.2.12 above. Note that $X \setminus \bigcup_{j=1}^e Y_j = \bigcap_{j=1}^e Z_j \subset X$ is precisely the set X' of points $\boldsymbol{x} \in X$ at which the Jacobian matrix $\mathbf{Jac}_\tau(\boldsymbol{x})$ exists.

Now fix some $j \in \{1, \ldots, e\}$. By Lemma 5.1.3, there are $d, r \in \mathbb{N}_{>0}$ and an M-definable function $\boldsymbol{c} : \mathrm{K}^{e-1} \to \mathrm{K}^r$ such that $\sigma_j(Y_j)$ is $(\boldsymbol{c}, d)$-prepared, where

$$\begin{aligned} \sigma_j : \mathrm{K}^e &\to \mathrm{K}^e \\ (x_1, \ldots, x_e) &\mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_e, x_j) \end{aligned}$$

is a permutation of coordinates. Then each of the fibers $(\sigma_j(Y_j))_{\boldsymbol{x}_{\neq j}} = \{y \in \mathrm{K} \mid (x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_e) \in Y_j\}$ of $\sigma_j(Y_j)$, for $\boldsymbol{x}_{\neq j} \in \mathrm{pr}_{\neq j}(X) \subset \mathrm{K}^{e-1}$, is finite by Lemma 5.2.11. Since each such fiber is moreover $(\boldsymbol{c}(\boldsymbol{x}_{\neq j}), d)$-prepared, i.e., a union of $(\boldsymbol{c}(\boldsymbol{x}_{\neq j}), d)$-balls (which are either infinite or singletons, cf. Definition 5.1.1 (1)), we must already have

$$\sigma_j(Y_j) \subset \bigcup_{i=1}^{r} \mathrm{graph}(\mathrm{pr}_i \circ \boldsymbol{c}).$$

By Remark 5.1.14 and Lemma 5.2.9 we thus have $\mu_{\mathrm{mot}}(X \setminus X') = \mu_{\mathrm{mot}}(\bigcup_{j=1}^{e} Y_j) = 0$, showing that $\mu_{\mathrm{mot}}(X) = \mu_{\mathrm{mot}}(X')$, as claimed.

Moreover, note that $\bigcup_{j=1}^{e} Y_j$ is closed as it is the union of finitely many graphs up to coordinate permutations, and thus its relative complement X' is open in X. □

In the situation of Lemma 5.2.13, note that the set X' is not necessarily open in K^e, as the following example shows.

Example 5.2.14. Let $X = (\mathcal{O}^2 \setminus \Delta) \cup \{(0,0)\}$, where Δ denotes the diagonal set, i.e., $X = \{(x,y) \in \mathcal{O}^2 \mid x \neq y \text{ or } x = y = 0\}$, and consider the M-definable function

$$\begin{aligned} \tau : X &\to \mathrm{K} \\ (x,y) &\mapsto x + y. \end{aligned}$$

Then the Jacobian matrix of τ exists (and is equal to the matrix $\begin{pmatrix} 1 & 1 \end{pmatrix}$) at all points $(x,y) \in X$. However, X is not open in K^2, since it does not completely contain any open ball around $(0,0) \in X$.

An analogous statement is true for a constant function on X, with the Jacobian matrix then evaluating to zero everywhere.

We now have all the necessary ingredients to state and prove the main result of this section, establishing a change of variables formula.

Proposition 5.2.15 (Change of variables)**.** *Let $M \subset \mathrm{K} \cup \Gamma$, $e \in \mathbb{N}$, let $\tau : X \to \mathrm{K}^{e+1}$ be an M-definable injection for some (M-definable) open set $X \subset \mathrm{K}^{e+1}$ such that $\mathbf{Jac}_\tau(\boldsymbol{x})$ exists at all $\boldsymbol{x} \in X$. Further let $\mathfrak{f} : \tau(X) \to \mathrm{p}^\Gamma$ be an M-definable integrable function with domain $\tau(X)$. Then we have*

$$\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} |\det(\mathbf{Jac}_\tau)| \cdot (\mathfrak{f} \circ \tau)$$

where the integrand on the right-hand side denotes the M-definable integrable function with domain X which sends $x \in X$ to $|\det(\mathbf{Jac}_\tau(x))| \cdot \mathfrak{f}(\tau(x))$.

Proof. First note that Lemma 5.1.18 allows us to partition X into a family of sets parameterized by an M-definable subset of RV_*^* and to prove the claim for each piece of the partition individually.

We will now start the actual proof by induction on e.

Induction base, $e = 0$. By Remark 5.2.7, we can find $d, r \in \mathbb{N}_{>0}$ and a tuple $\boldsymbol{c} = (c_1, \ldots, c_r) \in \mathrm{K}^r$ such that τ has the $(\boldsymbol{c}, d)$-Jacobian property and $\mathfrak{f} \circ \tau$ is $(\boldsymbol{c}, d)$-prepared. Using Lemma 5.1.18 as noted above, we can assume that X is itself a $(\boldsymbol{c}, d)$-ball, i.e., there is some $\boldsymbol{u} \in \mathrm{RV}_d^r$ such that we have

$$X = \{x \in \mathrm{K} \mid \mathrm{rv}_d^{\mathbf{c}}(x) = \boldsymbol{u}\} = (\mathrm{rv}_d^{\mathbf{c}})^{-1}(\boldsymbol{u}).$$

In particular, τ then has the Jacobian property on X. By the definitions of the Jacobian property and of the derivative, Definition 5.2.1 and Definition 5.2.10, the image of X under τ is a ball of radius $\mathrm{rad}_d(\boldsymbol{u}) \cdot |\tau'(x_0))|$, for any $x_0 \in X$. (Note that τ is not constant on any ball, as it is injective on X.) For any $x_0 \in X$, we have $\det(\mathbf{Jac}_\tau(x_0)) = \tau'(x_0)$ by definition of the Jacobian matrix. Moreover, since $\mathfrak{f} \circ \tau$ is $(\boldsymbol{c}, d)$-prepared, we have $\mathfrak{f} \circ \tau = \mathrm{const}_X(\gamma)$ for some $\gamma \in \mathrm{p}^\Gamma$, and consequently $\mathfrak{f} = \mathrm{const}_{\tau(X)}(\gamma)$. Using Remark 5.1.16, the above yields the desired equality

$$\begin{aligned}
\int_{\mathrm{mot}} \mathfrak{f} &= \int_{\mathrm{mot}} \mathrm{const}_{\tau(X)}(\gamma) \\
&= \gamma \cdot \mathrm{rad}_d(\boldsymbol{u}) \cdot |\tau'(x_0)| \\
&= \left(\int_{\mathrm{mot}} \mathrm{const}_X(\gamma)\right) \cdot |\tau'(x_0)| \\
&= \left(\int_{\mathrm{mot}} \mathfrak{f} \circ \tau\right) \cdot |\tau'(x_0)| \\
&= \int_{\mathrm{mot}} |\tau'| \cdot (\mathfrak{f} \circ \tau X) \\
&= \int_{\mathrm{mot}} |\det(\mathbf{Jac}_\tau)| \cdot (\mathfrak{f} \circ \tau)
\end{aligned}$$

where $x_0 \in X$ is arbitrary. This completes the proof of the induction base.

Induction step, $e \geq 1$. By Remark 5.2.7, we can find $d, r \in \mathbb{N}_{>0}$ and an M-definable function $\boldsymbol{c} : \mathrm{K}^e \to \mathrm{K}^r$ such that $\mathfrak{f} \circ \tau$ is $(\boldsymbol{c}, d)$-prepared and all of the functions $(\mathrm{pr}_i \circ \tau)(\boldsymbol{x}, \bullet)$, for $i \in \{1, \ldots, e+1\}$, have the $(\boldsymbol{c}(\boldsymbol{x}), d)$-Jacobian property for each $\boldsymbol{x} \in \mathrm{K}^e$.

Just as above, Lemma 5.1.18 allows us to restrict our attention to the case that $X = \{(\boldsymbol{x}, y) \mid \mathrm{rv}_d^{c(\boldsymbol{x})}(y) = \boldsymbol{u}\}$ for some $\boldsymbol{u} \in \mathrm{RV}_d^r$. For each $\boldsymbol{x} \in \mathrm{pr}_{\leq e}(X)$, the fiber of X at $\boldsymbol{x}$ is then the $(\boldsymbol{c}(\boldsymbol{x}), d)$-ball

$$B(\boldsymbol{x}) := (\mathrm{rv}_d^{c(\boldsymbol{x})})^{-1}(\boldsymbol{u}) \subset \mathrm{K}.$$

Consider, for each $i \in \{1, \dots, e+1\}$, the set

$$X_{\mathrm{inj},i} = \{(\boldsymbol{x}, y) \in X \mid (\mathrm{pr}_i \circ \tau)(\boldsymbol{x}, \bullet) \text{ is injective on } B(\boldsymbol{x})\}.$$

We now show that we have $X = \bigcup_i X_{\mathrm{inj},i}$. If $\boldsymbol{u} = \boldsymbol{0}$, this is clear. Otherwise, the map $\tau(\boldsymbol{x}, \bullet)$ is not constant on the ball $B(\boldsymbol{x}) = (\mathrm{rv}_d^{c(\boldsymbol{x})})^{-1}(\boldsymbol{u})$ for any $\boldsymbol{x}$, since τ is injective. By the $(\boldsymbol{c}(\boldsymbol{x}), d)$-Jacobian property for $\tau(\boldsymbol{x}, \bullet)$, we thus have

$$\begin{aligned} X \setminus \bigcup_i X_{\mathrm{inj},i} &= \bigcap_i (X \setminus X_{\mathrm{inj},i}) \\ &\subset \{(\boldsymbol{x}, y) \in X \mid (\mathrm{pr}_i \circ \tau)(\boldsymbol{x}, \bullet) \text{ is constant on } B(\boldsymbol{x}) \text{ for all } i\} \\ &= \{(\boldsymbol{x}, y) \in X \mid \tau(\boldsymbol{x}, \bullet) \text{ is constant on } B(\boldsymbol{x})\} \\ &= \emptyset, \end{aligned}$$

as claimed.

Hence we have

$$\begin{aligned} \int_{\mathrm{mot}} \mathfrak{f} &= \sum_i \int_{\mathrm{mot}} \mathfrak{f}{\restriction}\tau(X'_{\mathrm{inj},i}) \text{ and} \\ \int_{\mathrm{mot}} |\det(\mathbf{Jac}_\tau)| \cdot (\mathfrak{f} \circ \tau) &= \sum_i \int_{\mathrm{mot}} |\det(\mathbf{Jac}_\tau)| \cdot (\mathfrak{f} \circ \tau{\restriction}X'_{\mathrm{inj},i}), \end{aligned}$$

where $X'_{\mathrm{inj},i} = X_{\mathrm{inj},i} \setminus \bigcup_{j<i} X_{\mathrm{inj},j}$. It thus suffices to show the equality

$$\int_{\mathrm{mot}} \mathfrak{f}{\restriction}\tau(X'_{\mathrm{inj},i}) = \int_{\mathrm{mot}} |\det(\mathbf{Jac}_\tau)| \cdot (\mathfrak{f} \circ \tau{\restriction}X'_{\mathrm{inj},i})$$

for each $i \in \{1, \dots, e+1\}$.

To do so, let us now fix one such i. For each $\boldsymbol{x} \in \mathrm{pr}_{\leq e}(X'_{\mathrm{inj},i})$, the map $(\tau{\restriction} X'_{\mathrm{inj},i})(\boldsymbol{x}, \bullet)$ is injective on the fiber of $X'_{\mathrm{inj},i} \subset X_{\mathrm{inj},i}$ at $\boldsymbol{x}$, since it is injective on $B(\boldsymbol{x})$. We can therefore write $\tau{\restriction}X'_{\mathrm{inj},i}$ as the composition of the maps

$$\begin{aligned} (\boldsymbol{x}, y) &\mapsto (\boldsymbol{x}, \tau_i(\boldsymbol{x}, y)) \\ &\mapsto (\boldsymbol{x}_{\leq i}, \tau_i(\boldsymbol{x}, y), \boldsymbol{x}_{>i}) \\ &\mapsto (\tau_1(\boldsymbol{x}, y), \dots, \tau_i(\boldsymbol{x}, y), \dots, \tau_{e+1}(\boldsymbol{x}, y)) = \tau(\boldsymbol{x}, y), \end{aligned}$$

each of which is either a permutation of coordinates or fixes at least one coordinate. By Lemma 5.2.9, we are thus left with proving the claim in the case that

τ fixes the last coordinate, i.e., $\tau(x, y) = (\tilde{\tau}(x, y), y)$ for some $\tilde{\tau} : X \to \mathrm{K}^e$. Note that we then have

$$\mathfrak{f}_y \circ (\tilde{\tau})_y = (\mathfrak{f} \circ \tau)_y,$$

where $X_y = \{x \in \mathrm{K}^e \mid (x, y) \in X\}$, and $\mathfrak{f}_y = \mathfrak{f}(\bullet, y)$ as well as $(\tilde{\tau})_y = \tilde{\tau}(\bullet, y)$. Moreover, we have

$$\mathbf{Jac}_\tau(x, y) = \begin{pmatrix} & & & \star \\ & \mathbf{Jac}_{(\tilde{\tau})_y}(x) & & \vdots \\ & & & \star \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

for each $x \in \mathrm{pr}_{\leq e}(X) \subset \mathrm{K}^e$, and thus $\det(\mathbf{Jac}_{(\tilde{\tau})_y}) = \det(\mathbf{Jac}_\tau(\bullet, y))$. Further note that $(\tilde{\tau})_y$ is injective for each $y \in \mathrm{K}$ since τ is injective. The induction hypothesis applied to $M \cup \{y\} \subset \mathrm{K}$, the $(M \cup \{y\})$-definable integrable function $\mathfrak{f}_y$, and the $(M \cup \{y\})$-definable injection $(\tilde{\tau})_y$ now yields

$$\begin{aligned} \int_{\mathrm{mot}} \mathfrak{f}_y &= \int_{\mathrm{mot}} \left|\det(\mathbf{Jac}_{(\tilde{\tau})_y})\right| \cdot (\mathfrak{f}_y \circ (\tilde{\tau})_y) \\ &= \int_{\mathrm{mot}} |\det(\mathbf{Jac}_\tau(\bullet, y))| \cdot (\mathfrak{f} \circ \tau)_y \\ &= \int_{\mathrm{mot}} \big(|\det(\mathbf{Jac}_\tau)| \cdot (\mathfrak{f} \circ \tau)\big)_y \in R_{\mathrm{mot}}(Z(y)) \end{aligned}$$

for all $y \in \mathrm{K}$, where $Z(y) = \mathrm{acl}(M \cup \{y\}) \cap \Gamma \succcurlyeq \mathrm{acl}(M) \cap \Gamma = Z$. Hence $\underline{\mathrm{intLem}}(e)$ (see Definition 5.1.9 and Theorem 5.1.10), together with Lemma 5.2.9 implies

$$\int_{\mathrm{mot}} \mathfrak{f} = \int_{\mathrm{mot}} |\det(\mathbf{Jac}_{\tilde{\tau}})| \cdot (\mathfrak{f} \circ \tau),$$

finishing the proof. $\square$

5.3 Universal motivic measure

In [CH21], Cluckers and Halupczok show that the Haar measure on semi-algebraic subsets of $\mathbb{Q}_p$ is the universal motivic measure, that is to say, it is the most general one that satisfies certain conditions one would expect from a measure. In this section, we will repeat the most important definitions and lemmata and adapt the notation to the scope of this thesis, i.e., with respect to the measure on a proper elementary extension $\mathrm{K} \succcurlyeq \mathbb{Q}_p$ as defined in Definition 5.1.12.

The work of Cluckers and Halupczok relies on model theory of $\mathbb{Q}_p$ and on their previous work in [CH18] (which is already formulated for arbitrary $\mathbb{Z}$-groups). Most of the results of [CH21] can thus easily be transferred to the situation handled in this thesis, essentially by replacing all occurrences of $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{Z}_p$ with $R_{\mathrm{mot}}(Z)$, K and $\mathcal{O}$ and most occurrences of $\mathbb{Z}$ with Γ. We will give some more details below.

Transferring [CH21, Proposition 4.3] requires a bit more work. We provide a proof that also applies in our setting, building on Section 3.3 and using arguments similar to some of those in the proof of Lemma 4.4.11.

Note that Cluckers and Halupczok in [CH21] work with the language $\mathcal{L} = (+, \cdot, \mathcal{O})$, considering the value group as an imaginary sort. However, as they note in [CH21, Convention-Remark 3.1], a subset of the value group is imaginary $\emptyset$-definable in the language $\mathcal{L}$ if and only if it is a $\emptyset$-definable Presburger set. As these are precisely the $\emptyset$-definable subsets of the value group in our multi-sorted language $\mathcal{L}_{\mathrm{val}}$, we may as well continue to work in this language. (See also the introductory remarks in Chapter 3.)

In this section we will need the existence of definable Skolem functions in K, [Dri84, Theorem 3.2], and thus only work with M-definable sets for some fixed set $M \subset \mathrm{K}$ of parameters. (Note that there are no definable Skolem functions if we allow arbitrary parameters from Γ, e.g., the set $\{x \in \mathrm{K} \mid \mathrm{val}(x) = a\}$ does not contain an $\{a\}$-definable element for $a \in \Gamma_{\geq 0} \setminus \mathbb{Z}$.) Similar to before, we set $Z = \mathrm{dcl}(M) \cap \Gamma$.

First, let us recall the most important notions of [CH21] that we will use in the following, starting with certain subsets of K^n of a simple form, parameterized by the value group.

Throughout the whole section, we fix a set $L \subset \mathrm{K}^*$, and we consider families of subsets of K^n paramterized by L, as subsets of $L \times \mathrm{K}^n$.

Notation 5.3.1 ([CH21, Definition 3.8]). For $n \in \mathbb{N}$, a M-definable set $\Lambda \subset L \times \Gamma^n$, and a M-definable map $\nu : \Lambda \to \Gamma$, consider the M-definable sets

$$P(\Lambda) = \{(\ell, x) \in L \times \mathrm{K}^n \mid (\ell, \mathbf{val}(x)) \in \Lambda, \mathrm{ac}(x_1) = \cdots = \mathrm{ac}(x_n) = 1\} \text{ and}$$
$$P(\Lambda, \nu) = (P(\Lambda) \times \mathrm{K}^{(1)}) \cap \{(\ell, x, y) \mid \mathrm{val}(y) = -n - 1 - \nu(\ell, \mathbf{val}(x)) - \sigma(x)\},$$

where $\mathrm{K}^{(1)} = \{y \in \mathrm{K} \mid \mathrm{ac}(y) = 1\}$ and $\sigma(x) = \sum_{i=1}^n \mathrm{val}(x_i)$.

Definition 5.3.2 ([CH21, Definition 2.1]). Let R_L be the quotient of the free abelian group generated by symbols $[X]$ for each M-definable set $X \subset L \times \mathrm{K}^n$, any $n \in \mathbb{N}_{>0}$, such that the fiber $X_\ell = \{x \in \mathrm{K}^n \mid (\ell, x) \in X\} \subset \mathrm{K}^n$ is bounded for each $\ell \in L$, modulo the relations

(R1) Additivity:

$$[X_1 \cup X_2] = [X_1] + [X_2]$$

for any two M-definable disjoint sets $X_1, X_2 \subset L \times \mathrm{K}^n$.

(R2) Negligible sets:

$$[X] = 0$$

for $X \subset L \times \mathrm{K}^n$ with $\dim(X_\ell) < n$ for each $\ell \in L$.

(R3) Change of variables:

$$[X] = [Y]$$

for $X, Y \subset L \times \mathrm{K}^n$ if X_ℓ and Y_ℓ are open for all $\ell \in L$ and there is a M-definable family of bijections $\phi_\ell : X_\ell \to Y_\ell$ such that $\mathbf{Jac}_x(\phi_\ell) = 1$ for all $\ell \in L$ and all $x \in X_\ell$.

(R4) Product with unit ball:

$$[X] = [X \times \mathcal{O}]$$

for any $X \subset L \times \mathrm{K}^n$.

The multiplication rule induced by $[X] \cdot [Y] = [X \times_L Y]$ yields a ring structure on R_L, where

$$X \times_L Y = \{(\ell, \boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \in X_\ell, \boldsymbol{y} \in Y_\ell\},$$

see also [CH21, Lemma 3.4].

Just as in the original work for the Haar measure on $\mathbb{Q}_p$, our measure μ_{mot} on K factors through this abstract ring in the following sense.

Remark 5.3.3 ([CH21, Definition 2.2]). Let $L \subset \mathrm{K}^*$ be M-definable and let $\ell \in L$. Then there is a (unique) canonical ring homomorphism $\mu_\ell : R_L \to R_{\mathrm{mot}}(Z(\ell))$ induced by

$$\mu_\ell([X]) = \mu_{\mathrm{mot}}(X_\ell)$$

for all M-definable sets $X \subset L \times \mathrm{K}^n$, any $n \in \mathbb{N}$, where μ_{mot} is the (motivic) measure from Definition 5.1.12 and where $X_\ell = \{\boldsymbol{x} \in \mathrm{K}^n \mid (\ell, \boldsymbol{x}) \in X\}$ denotes the fiber of X over ℓ.

Proof. It is clear that the measure we defined in Definition 5.1.12 satisfies (R1) and (R4). Moreover, (R3) follows from Proposition 5.2.15 and (R2) follows from cell decomposition in K together with Corollary 5.1.13. □

The main result of this section is the generalization of the main result of [CH21] to the scope of p-adically closed fields.

Theorem 5.3.4 ([CH21, Theorem 2.3]). *Let* $L \subset \mathrm{K}^*$ *be* $\emptyset$*-definable. The ring homomorphism sending* $\Xi \in R_L$ *to the function*

$$\begin{aligned} L &\to R_{\mathrm{mot}}(Z(L)) \\ \ell &\mapsto \mu_\ell(\Xi) \end{aligned}$$

is injective.

In the important special case $L = \{0\}$ one obtains the following non-family version, stating that our measure μ_{mot} agrees with the universal motivic measure $X \mapsto [X] \in R_{\{0\}}$.

Corollary 5.3.5 ([CH21, Corollary 2.4]). *The map*

$$\begin{aligned} \mu_0 : R_{\{0\}} &\to R_{\mathrm{mot}}(Z) \\ \Xi &\mapsto \mu_0(\Xi), \end{aligned}$$

induced by $[X] \mapsto \mu_{\mathrm{mot}}(X)$*, is an isomorphism of rings.*

Proof (using Theorem 5.3.4). By Theorem 5.3.4, μ_0 is an injective ring homomorphism, so it remains to show surjectivity. Note that Corollary 4.3.9 allows us to write any given element of $R_{\mathrm{mot}}(Z)$ as a sum of elements of the form

$$[\mathrm{const}_U(\alpha)] \otimes q + (\mathrm{p} - p)^{\mathbb{Q}} \tag{5.1}$$

for some Z-definable set $U \subset \mathrm{RV}_*^*$, some $\alpha \in \mathrm{p}^\Gamma$, and some $q \in \mathbb{Q}$. Since μ_0 is a homomorphism, it suffices to handle the case of only one summand, i.e., we now aim to find a preimage under μ_0 of the element given in (5.1). By (the adapted version of) [CH21, Lemma 3.11] together with Theorem 5.3.4, there is $X \subset \mathrm{K}^*$ with $\mu_{\mathrm{mot}}(X) = q$, so we can moreover assume that $q = 1$. Now Lemma 2.3.5 allows us to assume $\#U = \#A$ for some Z-definable subset $A \subset \Gamma^*$. Using (the generalization of) equation (1) of [CH21], one then calculates that we have

$$\mu_{\mathrm{mot}}(P(A, \mathrm{const}_A(0))) = \#A = \#U$$

in $R_{\mathrm{mot}}(Z) = K_{\mathrm{int}}(Z)/(\mathrm{p} - p)$. By Corollary 4.2.12 and under our assumption $q = 1$, the latter is equal to the element of $R_{\mathrm{mot}}(Z)$ given in (5.1). This shows that μ_0 is indeed surjective and hence an isomorphism. □

Using the following replacements, most of the statements and proofs of [CH21] immediately generalize to the situation of an elementary extension $\mathrm{K} \succcurlyeq \mathbb{Q}_p$ that we are concerned with.

Table 5.1: A dictionary for translating (most of) the results of [CH21] to our setting.

Notation in [CH21], for $\mathbb{Q}_p$	Notation here, for $\mathrm{K} \succcurlyeq \mathbb{Q}_p$	Remark
S and s	L and ℓ respectively	
$\mathbb{R}$	$R_{\mathrm{mot}}(Z(L))$	
$\mathbb{Q}_p$	K	
$\mathbb{Z}_p$	$\mathcal{O}$	
$\mathbb{Z}$	Γ	with exceptions[1]
$\mathbb{N}$	$\Gamma_{\geq 0}$	with exceptions[1]
$X \subset \mathbb{Q}_p^n$ has finite (Haar) measure	$X \subset \mathrm{K}^n$ is bounded	
finite subset of $\mathbb{Z}$	bounded subset of Γ	

[1] Exceptions: Do not replace $\mathbb{Z}$ in Lemma 3.11, Remark 3.12, Remark 3.14 (2), and in part (4) in the proof of Proposition 3.15. Do not replace $\mathbb{N}$ where obvious, i.e., when used for indices of a finite tuple or exponents such as in K^n.

More precisely, all of the following statements of [CH21] as well as their proofs can be translated using the dictionary given in Table 5.1, where one should not replace $\mathbb{Z}$ in those statements marked with (*): Definition 3.3, Lemma 3.4, Remark 3.5(*), Remark 3.6, Remark 3.7, Definition 3.8, Lemma 3.10, Lemma 3.11(*), Remark 3.12(*), Lemma 3.13, and Remark 3.14(*). A little more attention is required for Proposition 3.15: There, replace $\mathbb{Z}$ by Γ in the statement and in part (1) of the proof, but do not replace it in part (4) of the proof. Moreover, in the first part of the proof (before the case distinction), the equation

$$\phi_s(\lambda) = M\lambda + \mu_s$$

for some matrix $M \in \mathbb{Q}^{n\times n}$ and some vector $\mu_s \in \mathbb{Q}^n$ should in our case rather be written as

$$\phi_s(\lambda) = \frac{1}{d} \cdot (M\lambda + \mu_s)$$

for some $d \in \mathbb{N}_{>0}$, some matrix $M \in \mathbb{Q}^{n\times n}$ and some vector $\mu_s \in \Gamma^n$.

Further note that the equation (1) of [CH21] (right after Definition 3.8) also holds in our setting if Λ_ℓ is finite for all $\ell \in L$, see our equation (4.1) on p. 48 and Remark 4.1.8. (More generally, a variant of equation (1) of [CH21] holds if $\mathrm{im}(\nu_s)$ is finite for all s, see Corollary 4.2.12.)

A minor adaption (additional to the replacements mentioned above) is needed in the proof of Lemma 3.9: In the definition of $X_a = \prod_i \{p^\lambda x_i \mid \lambda \in \mathbb{N}, x_i \in \mathrm{res}^{-1}(a_i)\}$, read

$$\{p^\lambda x_i \mid x_i \in \mathrm{res}^{-1}(a_i)\}$$

as

$$\{y_i \mid \mathrm{ac}(y_i) = a_i, \mathrm{val}(y_i) = \lambda\}, \text{ if } a_i \neq 0$$
$$\text{and as } \{y_i \mid \mathrm{val}(y_i) \geq \lambda + 1\}, \text{ if } a_i = 0.$$

Furthermore, Theorem 3.16 and Proposition 3.17 are taken from [CH18], hence already formulated for Presburger groups in general. (Compared with [CH21], we need to replace $\mathbb{Z}$ by Γ in those statements, as well as $\mathbb{Q}$ by $K_b^{\Gamma}(\mathbb{Z}) \otimes \mathbb{Q}$ and $\#$ by $\#\!\!\#$ in Proposition 3.17.)

This only leaves Section 4 of [CH21] for closer investigation. There, especially in the proof of [CH21, Proposition 4.3], some further changes are required. Let us spell out the details, starting with an altered version of [CH21, Definition 4.1].

Definition 5.3.6 (adapted from [CH21, Definition 4.1])**.** Let $n \in \mathbb{N}$, and let $\Lambda \subset L \times \Gamma^n$ and $\nu : \Lambda \to \Gamma$ be M-definable. We then call $P(\Lambda, \nu)$ a *basic set* if each of the fibers

$$\Lambda_\ell = \{\boldsymbol{b} \in \Gamma^n \mid (\ell, \boldsymbol{b}) \in \Lambda\} \subset \Gamma^n$$

is bounded and the value of ν only depends on $\ell \in L$. Abusing notation, we then also just write $\nu(\ell)$ for the (constant) value of $\nu(\ell, \bullet)$ on Λ_ℓ.

Let R_L^{basic} denote the subgroup of R_L denoted by the basic sets. Note that the product of two basic sets is again basic by [CH21, Lemma 3.13], so that R_L^{basic} is a subring of R_L.

Just as in [CH21, Theorem 2.3], the proof of Theorem 5.3.4 is a combination of the following two propositions.

Proposition 5.3.7 ([CH21, Proposition 4.3])**.** *Let* $\Xi \in R_L^{\mathrm{basic}}$. *If we have* $\mu_\ell(\Xi) = 0$ *for all* $\ell \in L$, *then* $\Xi = 0$.

Proof. Let $\Xi \in R_L^{\mathrm{basic}}$ with $\mu_\ell(\Xi) = 0$ for all $\ell \in L$. Write

$$\Xi = \sum_{j=1}^{m} \delta_j \cdot [P(\Lambda_j, \nu_j)] \tag{5.2}$$

for some $m \in \mathbb{N}_{>0}$, $\delta_1, \ldots, \delta_m \in \{1, -1\}$, and some basic sets $P(\Lambda_j, \nu_j)$. Using cell decomposition in $\mathbb{K}$ as in the proof of [CH21, Proposition 4.3], and similar to the preparation arguments in Section 5.1, we may assume that the fibers $\Lambda_{j,\ell}$ and the value of $\nu_j(\ell)$ only depend on $\mathbf{val}(\ell)$.

Let us write $\widetilde{\nu_j}(\mathbf{val}(\ell)) = \nu_j(\ell)$ for each j. We now proceed similarly as in the proof of Lemma 4.4.10, refining the partition of L into finitely many sets several times. Firstly, we can assume

$$\widetilde{\nu_j}(\mathbf{val}(\ell)) \leq \widetilde{\nu_{j+1}}(\mathbf{val}(\ell))$$

for all $j < m$ and all $\ell \in L$. Moreover, Lemma 3.1.4, applied to the graphs of the maps $\widetilde{\nu_j} : \mathbf{val}(L) \to \Gamma$, allows us to assume that all of them are linear. Using [CH18, Proposition 5.2.1], we can further assume

$$\#\Lambda_{j,\ell} = g_j(\mathbf{val}(\ell))$$

for each j, where the g_j are polynomials with coefficients in $K_b^\Gamma(Z) \otimes \mathbb{Q}$. The equation (5.2) together with the assumption $\mu_\ell(\Xi) = 0$ then yields, for all $\ell \in L$,

$$\begin{aligned} \sum_{j=1}^{m} \delta_j \cdot g_j(\mathbf{val}(\ell)) \cdot \mathrm{p}^{\widetilde{\nu_j}(\mathbf{val}(\ell))} &= \sum_{j=1}^{m} \delta_j \cdot \mu_\ell(P(\Lambda_j, \nu_j)) \\ &= \mu_\ell(\Xi) = 0. \end{aligned} \tag{5.3}$$

We now complete the proof by induction on m. For $m = 0$, we have $\Xi = 0$ by (5.2). For $m > 0$, consider the following case distinction.

Case 1: There is a $j < m$ for which $\widetilde{\nu_{j+1}} - \widetilde{\nu_j}$ is constantly equal to an integer. Similar to the proof of [CH21, Proposition 4.3] and the strategy in the proof of Lemma 4.4.10, we can then “group” the sets Λ_j and Λ_{j+1} using [CH21, Remark 3.14 (2)] (or Remark 4.1.8 respectively). The claim then follows by the induction hypothesis.

Case 2: None of the differences $\widetilde{\nu_{j+1}} - \widetilde{\nu_j}$ is constantly equal to any integer. By Proposition 3.3.3, there is then a Presburger cell $A \subset \mathbf{val}(L)$ of the same shape as $\mathbf{val}(L)$ for which we have

$$((\widetilde{\nu_{j+1}} - \widetilde{\nu_j}) \circ \mathbf{val})(A) > \mathbb{Z}$$

for all $j < m$. The equation (5.3), together with Lemma 4.4.5 applied to appropriate integrable functions $\mathfrak{f}$ and $\mathfrak{g}$, thus yields

$$\delta_j \cdot g_j(\boldsymbol{a}) \cdot \mathrm{p}^{\widetilde{\nu_j}(\boldsymbol{a})} = 0$$

for all j and all $\boldsymbol{a} \in A$. Hence g_j must be constantly equal to 0 on A, and by Lemma 3.3.6, it is then constantly equal to 0 on all of $\mathbf{val}(L)$, using Corollary 3.2.14. We thus have

$$\#\Lambda_{j,\ell} = g_j(\mathbf{val}(\ell)) = 0$$

for all j and all $\ell \in L$. This implies $\Lambda_j = \emptyset$ and hence $[P(\Lambda_j, \nu_j)] = 0$ in $R_L^{\mathrm{basic}} \subset R_L$ for all j. By (5.2), we then obtain $\Xi = 0$ as claimed. □

Proposition 5.3.8 ([CH21, Proposition 4.4]). *Let $\Xi \in R_L$. Then there is some $d \in \mathbb{N}_{>0}$ with $d \cdot \Xi \in R_L^{\mathrm{basic}}$.*

Proof. We can translate the proof of the original [CH21, Proposition 4.4], using the dictionary given in Table 5.1 and replacing ℓ with d, up to the following adaptions.

- In the equation (5), $r \geq 1$ as well as the b_i are honest integers (i.e., we do not replace the hidden $\mathbb{Z}$ there with Γ), and so is b in Case 1.
- In Case 1, we obtain $b < 0$ not by calculating the measure of the set X, but by observing that its fibers

$$X_\ell = \{(x, y) \in \mathrm{K}^{(1)} \times \mathrm{K}^{(1)} \mid \mathrm{val}(x) \geq 0, \mathrm{val}(y) = -2 - (b+1) \cdot \mathrm{val}(x)\}$$

  would be unbounded if $b + 1 > 0$, but they are bounded by assumption.

The rest of the proof works without further changes (beyond the substitutions indicated in Table 5.1). □

Proof of Theorem 5.3.4. Exactly as in the proof of [CH21, Theorem 2.3], with only the accustomed notational changes. □

6 Outlook

In this final chapter, we explain how to obtain integrals for more general functions using the theory developed in Chapter 4 and Chapter 5. We also pose some questions regarding further generalizations, and we moreover explain how our construction relates to the natural measure on an ultrapower of $\mathbb{Q}_p$.

Considering functions to p^{Γ} as a foundational base case, our work readily admits integrating a much broader class of functions. More precisely, we can assign a value to the integral of any sufficiently nice function on $\mathrm{K}^* \times \mathrm{RV}_*^*$ with codomain $R_{\mathrm{mot}}(\Gamma)$ in the following way.

Definition 6.1. Let $X, Y \subset \mathrm{K}^*$ be M-definable bounded sets and suppose that we have (not necessarily finite) M-definable partitions of X into sets $X_{\boldsymbol{s}}$ and of Y into sets $Y_{\boldsymbol{s}}$, where $\boldsymbol{s}$ runs over an M-definable set $S \subset \mathrm{K}^* \times \mathrm{RV}_*^*$. Then the function

$$\begin{aligned} f : S &\to R_{\mathrm{mot}}(\Gamma) \\ \boldsymbol{s} &\mapsto \mu_{\mathrm{mot}}(X_{\boldsymbol{s}}) - \mu_{\mathrm{mot}}(Y_{\boldsymbol{s}}), \end{aligned}$$

is <u>*integrable over S*</u>, and the value of the integral is defined as

$$\int_S f := \mu_{\mathrm{mot}}(X) - \mu_{\mathrm{mot}}(Y) \in R_{\mathrm{mot}}(Z) \subset R_{\mathrm{mot}}(\Gamma).$$

(For the inclusion $R_{\mathrm{mot}}(Z) \subset R_{\mathrm{mot}}(\Gamma)$, see Lemma 4.3.12.)

Note that Theorem 5.1.10 (see also the statement <u>intLem</u>$(*)$ in Definition 5.1.9) and Lemma 5.1.18 guarantee that $\int_S f$ is well-defined, i.e., independent of the exact choices of X, Y, and the partitions $(X_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$ and $(Y_{\boldsymbol{s}})_{\boldsymbol{s}\in S}$.

Intuitively speaking, the only obstacle for a function $f : S \to R_{\mathrm{mot}}(\Gamma)$ to be integrable over S is, that $\sum_{\boldsymbol{s}\in S} f(\boldsymbol{s})$ has to still lie in $R_{\mathrm{mot}}(\Gamma)$, meaning that this sum does not become too large (i.e., infinite in a strong sense).

Furthermore, by Corollary 5.3.5, the ring $R_{\mathrm{mot}}(Z)$ is already generated by the measures of M-definable sets. Hence, stating Definition 6.1 for functions $f : S \to R_{\mathrm{mot}}(Z)$ whose values $f(\boldsymbol{s})$ are given by integrals of functions on $\mathrm{K}^* \times \mathrm{RV}_*^*$, instead of measures of subsets of K^*, would not make it more general.

The idea behind Definition 6.1 can be generalized further to permit integrating out individual variables of (sufficiently nice) functions defined on Cartesian products of K^* and RV^*_*. More precisely, we can define an integral for functions with codomain $R_{\mathrm{mot}}(Z)$ as follows.

Definition 6.2. Let $S \subset \mathrm{K}^* \times \mathrm{RV}^*_*$ and $T \subset \mathrm{K}^* \times \mathrm{RV}^*_*$ be M-definable sets and let $\Xi \subset S \times T$. Let $f : \Xi \to R_{\mathrm{mot}}(\Gamma)$ be a function with

$$f(\boldsymbol{s}, \boldsymbol{t}) = \mu_{\mathrm{mot}}(X_{\boldsymbol{s},\boldsymbol{t}}) - \mu_{\mathrm{mot}}(Y_{\boldsymbol{s},\boldsymbol{t}})$$

for some $(M \cup \{\boldsymbol{s}, \boldsymbol{t}\})$-definable bounded sets $X_{\boldsymbol{s},\boldsymbol{t}}, Y_{\boldsymbol{s},\boldsymbol{t}} \subset \mathrm{K}^{n_{\boldsymbol{t}}}$, where $n_{\boldsymbol{t}} \in \mathbb{N}_{>0}$. Assume moreover that, for fixed $\boldsymbol{t} \in T$, the sets $X_{\boldsymbol{s},\boldsymbol{t}}$ are pairwise disjoint and their union

$$X_{\boldsymbol{t}} := \bigcup_{\boldsymbol{s} \in \Xi_{\boldsymbol{t}}} X_{\boldsymbol{s},\boldsymbol{t}} \subset \mathrm{K}^{n_{\boldsymbol{t}}}$$

is bounded and M-definable, and assume the same holds for the sets $Y_{\boldsymbol{s},\boldsymbol{t}}$, analogously setting $Y_{\boldsymbol{t}} := \bigcup_{\boldsymbol{s} \in \Xi_{\boldsymbol{t}}} Y_{\boldsymbol{s},\boldsymbol{t}}$.

Then f is <u>*integrable over* $\mathrm{pr}_S(\Xi) \subset S$</u>, with $\int_{\mathrm{pr}_S(\Xi)} f(\boldsymbol{s}, \bullet)\, \mathrm{d}\boldsymbol{s} := F$ for the function

$$\begin{aligned} F : \Xi_{\boldsymbol{s}} &\to R_{\mathrm{mot}}(\Gamma) \\ \boldsymbol{t} &\mapsto \mu_{\mathrm{mot}}(X_{\boldsymbol{t}}) - \mu_{\mathrm{mot}}(Y_{\boldsymbol{t}}), \end{aligned}$$

where $\Xi_{\boldsymbol{s}} = \{\boldsymbol{t} \in T \mid (\boldsymbol{s}, \boldsymbol{t}) \in \Xi\} \subset T$ is the fiber of Ξ over $\boldsymbol{s} \in S$.

The remarks on Definition 6.1 above apply similarly to Definition 6.2. In particular, considering integrals instead of measures for the values of f does not further enlarge the class of integrable functions.

While Section 5.3 ensures that the motivic integral we constructed is the most general one possible in the case of elementary extensions $\mathrm{K} \succcurlyeq \mathbb{Q}_p$, some natural questions remain. Let us discuss some interesting ones posing problems that seem within reach.

Since most results in Chapter 5 do not make explicit use of K being a model of $\mathrm{Th}(\mathbb{Q}_p)$, but merely exploit the properties of h-minimality studied in [Clu+21], one could expect the developed methods to apply more generally.

Question 6.3. *Can the results be adapted to work in elementary extensions of finite (algebraic) field extensions of* $\mathbb{Q}_p$*?*

Note that the universality result from [CH21] treated in Section 5.3 makes use of definable Skolem functions, and we presumably have to expand the language to obtain the same result in the case of field extensions. It then seems reasonable to expect that our results admit a generalization to this setting, see also [CH21, Remark 4.5 (2)].

Question 6.4. *Can the results be adapted to work in h-minimal fields of mixed characteristic* $(0, p)$*? (See also [CHR21, Corollary 6.2.7] for examples of such fields.)*

Again, one obstacle for an immediate generalization of the universality result is the existence of definable Skolem functions. Moreover, our methods demand a good understanding of the (Grothendieck ring of the) value group, whereas our Chapter 3 merely handles $\mathbb{Z}$-groups.

While we constructed the universal motivic measure on arbitrary elementary extensions $\mathrm{K} \succcurlyeq \mathbb{Q}_p$, there is a model-theoretic construction to (more or less) explicitly obtain specific such K, bringing with it a version of p-adic integration: The ultraproduct. More precisely, given an ultrafilter $\mathcal{U}$ on an index set I, consider the ultrapower

$$\mathcal{K} := \mathbb{Q}_p^I/\mathcal{U} = (\prod_{i\in I} \mathbb{Q}_p)/\mathcal{U}.$$

This $\mathcal{K}$ can naturally be viewed as an $\mathcal{L}_{\mathrm{val}}$-structure, and is then moreover an elementary extension of $\mathbb{Q}_p$ by Łoś's Theorem (see, e.g., [Hod93], or any other sufficiently saturated introduction to model theory).

Write μ for the p-adic measure, i.e., the Haar measure on $\mathbb{Q}_p$, normalized by setting $\mu(\mathbb{Z}_p) = 1$, and define

$$\mu_{\mathcal{K}}(X) := \big(\mu(X_i)\big)_{i\in I} \in \mathbb{R}^{\mathcal{U}}$$

where $X \subset \mathcal{K}$ and $X_i \subset \mathbb{Q}_p$ are bounded definable[1] sets with $X = (\prod_I X_i)/\mathcal{U}$. Then $\mu_{\mathcal{K}}$ behaves like a measure, just by construction and the nature of ultrapowers. By Section 5.3, we thus have a map φ from $R_{\mathrm{mot}}(\mathcal{Z})$ to $\mathbb{R}^{\mathcal{U}}$ for $\mathcal{Z} = \mathrm{acl}(\mathcal{M}) \cap \Gamma$, where $\Gamma = \mathbb{Z}^{\mathcal{U}}$, such that the diagram

$$\begin{tikzcd} \left\{\begin{matrix}\mathcal{M}\text{-definable}\\ \text{subsets of } \mathrm{K}^*\end{matrix}\right\} \arrow[rr, "\mu_{\mathcal{K}}"] \arrow[dr, "\mu_{\mathrm{mot}}"'] & & \mathbb{R}^{\mathcal{U}} \\ & R_{\mathrm{mot}}(\mathcal{Z}) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes, where μ_{mot} denotes the measure constructed and defined in this thesis, see Definition 5.1.12.

In this case, the map φ is not injective, i.e., μ_{mot} preservers more information about definable subsets of $\mathcal{K}$ than the natural ultrapower measure $\mu_{\mathcal{K}}$.

[1] Definable here can be taken to mean $\mathcal{M}$-definable and M_i-definable, respectively, where $\mathcal{M} = (\prod_I M_i)/\mathcal{U}$.

Example 6.5. Let $a \in \Gamma_{\geq 0} \setminus \mathbb{Z}$ be an infinitely large element of the value group $\Gamma = \mathbb{Z}^{\mathcal{U}}$. By [CH18], we have

$$\#\,[0,a) \cdot \#\,[0,a) = \#\,([0,a) \times [0,a)) \neq \#\,[0,b)$$

in $K_b^{\Gamma}(\mathcal{Z})$, where $b = a^2 \in \Gamma$. More precisely, [CH18, Lemma 4.2.14] together with [CH18, Lemma 4.2.15 and Definition 4.2.9] ensures that there is no definable bijection between $[0,a) \times [0,a)$ and $[0,b)$, and [CH18, Theorem 5.2.2] then yields that their hypercardinalities cannot coincide.

Now choose subsets $X, Y \subset \mathrm{K}^*$ with $\mu_{\mathrm{mot}}(X) = \#\,[0,a)$ and $\mu_{\mathrm{mot}}(Y) = \#\,[0,b)$. (E.g., by using sets of the form $P(\Lambda, \nu)$ as defined in Notation 5.3.1 together with the equation (1) of [CH21].)

Then we have $\mu_{\mathrm{mot}}(X \times X) \neq \mu_{\mathrm{mot}}(Y)$ in $R_{\mathrm{mot}}(\mathcal{Z})$, but one easily computes $\mu_{\mathcal{K}}(X \times X) = \mu_{\mathcal{K}}(Y)$ in the ultrapower: Indeed, pick a sequence $(a_i)_{i \in I} \in \mathbb{Z}^I$ with $\lim_{\mathcal{U}}(a_i) = a$, and hence $\lim_{\mathcal{U}}({a_i}^2) = b$. Then we have

$$\begin{aligned}
\mu_{\mathcal{K}}(X \times X) &= \mu_{\mathcal{K}}(X) \cdot \mu_{\mathcal{K}}(X) \\
&= \big(\lim_{\mathcal{U}}\,(a_i)\big)^2 \\
&= \lim_{\mathcal{U}}\,({a_i}^2) \\
&= \mu_{\mathcal{K}}(Y),
\end{aligned}$$

as claimed.

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