Semistable Lévy Processes and Log-Periodically Disturbed Fractional Calculus

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Abstract

It is well-known that densities of stable Lévy processes solve particular fractional diffusion equations. On the other hand, fractional diffusion equations gain a stochastically meaningful interpretation by their connection to stable laws. That way, fractional calculus and the theory of stable laws are strongly connected, and both areas benefit noticeably from this connection. The present thesis investigates a similar relationship between semistable Lévy processes and generalized fractional derivatives with log-periodic perturbations, which we call semi-fractional derivatives. To develop a basic idea of these operators, we initially show essential characteristics like Fourier and Laplace transforms as well as different integral representations and their relations. Besides, a numerical approximation of Grünwald-Letnikov type is proven. A semigroup approach finally yields the desired connection between semistable densities and semi-fractional diffusion equations.

Due to this connection, the knowledge about semi-fractional derivatives is able to enrich our understanding of semistable laws and even offers the possibility to numerically approximate semistable densities. On that basis, we study different related issues to evaluate the potential of semi-fractional calculus. First, we consider general Cauchy problems with semi-fractional time and space derivatives and prove an integral representation of the solution. Also, we identify the stochastic processes governed by these equations. Namely, they appear as limiting processes of uncoupled Continuous Time Random Walks (CTRW limits). The CTRWs offer a microscopic description of the underlying system and are a valuable tool in applications. Thereby, the case of uncoupled CTRWs is quite a special one. Thus, we also study the far more general case of a possibly coupled CTRW and analyze its convergence as well as the resulting limiting distribution. Semi-fractional derivatives are non-local operators, and hence, semi-fractional space derivatives require the inclusion of the whole environment into their calculation. Similarly, semi-fractional time derivatives model long-time memory effects of the underlying system. Since the latter one is easier to handle for practical applications, we offer a space-time duality result. This states that a negatively-skewed space semi-fractional differential equation is equivalent to a particular inhomogeneous differential equation with semi-fractional time derivative.
To strengthen the theory of semi-fractional calculus, we finally study its potential to model real-world applications. Therefore, we explore different semi-fractional growth models and apply them to mobile use and cancer growth data. Additionally, we consider tempered semi-fractional diffusion and show how the therein included damping of huge events' probability yields good fits in stock data.

Zusammenfassung

Ein weithin bekanntes Resultat aus der Wahrscheinlichkeitstheorie besagt, dass die Dichten von stabilen Lévy-Prozessen bestimmte fraktionierte Diffusionsgleichungen lösen. Andererseits können fraktionierte Diffusionsgleichungen durch diese Verbindung stochastisch interpretiert werden. Somit besteht eine starke Verbindung zwischen diesen beiden Teilbereichen der Mathematik, von der beide Gebiete im Laufe der letzten Jahrzehte spürbar profitieren konnten. Die vorliegende Arbeit entwickelt eine analoge Beziehung zwischen semistabilen Lévy-Prozessen und fraktionierten Ableitungen mit log-periodischer Störung, welche wir semi-fraktionierte Ableitungen nennen. Um eine grundlegende Vorstellung dieser Operatoren zu gewinnen, analysieren wir nicht nur die Fourier- und Laplacetransformierten von semi-fraktionierten Ableitungen, sondern beweisen ebenfalls verschiedene Integraldarstellungen sowie deren Beziehungen untereinander. Zusätzlich bietet eine Approximation vom Grünwald-Letnikov Typ die Möglichkeit, semi-fraktionierte Ableitungen numerisch zu berechnen. Die gewünschte Darstellung semistabiler Dichten als Lösungen bestimmter semi-fraktionierter Diffusionsgleichungen wird schließlich mithilfe eines Halbgruppen-Ansatzes aufgezeigt.

Das gewonnene Wissen über semi-fraktionierte Ableitungen kann nun dafür verwendet werden, unser Verständnis von semistabilen Dichten zu vertiefen. Insbesondere bietet es die Möglichkeit, semistabile Dichten numerisch zu approximieren. Darauf aufbauend betrachten wir mehrere weiterführende Problemstellungen. Zunächst werden allgemeine Cauchy Probleme mit semi-fraktionierter Zeit- und Ortsableitung betrachtet und eine Integraldarstellung der Lösung bewiesen. Es stellt sich heraus, dass die Lösungen Dichten von Grenzwertprozessen von ungekoppelten Continuous Time Random Walks (CTRWs) sind. Diese bieten eine mikroskopische Beschreibung des zugrunde liegenden Systems und können daher leicht auf Anwendungsbeispiele übertragen werden. Neben dem ungekoppelten Fall betrachten wir auch die Konvergenz und die Grenzwertdichten allgemeiner CTRWs, in denen beliebige Abhängigkeiten zwischen den Sprüngen und Wartezeiten erlaubt sind. Semi-fraktionierte Ableitungen sind nicht-lokale Operatoren und daher muss für die Berechnung einer semi-fraktionierten Ortsableitung die gesamte Umgebung in die Berechnung einbezogen werden. Ebenso schließen semi-fraktionierte Zeitableitungen die gesamte Vergangenheit in ihre Berechnung mit ein. Da das Letztere als Langzeitgedächtnis des zugrunde liegenden Systems für praktische Anwendungen leichter zu interpretieren ist, stellen wir eine Ort-Zeit-Dualität vor. Dabei können bestimmte Diffusionsgleichungen mit semi-fraktionierter Ortsableitung in eine inhomogene Differentialgleichung mit semi-fraktionierter Zeitableitung überführt werden.

Gestärkt wird die Theorie semi-fraktionierter Differentialgleichungen schließlich durch die Betrachtungen von Anwendungsbeispielen. Dabei untersuchen wir zunächst verschiedene Wachstumsmodelle und wenden diese auf mobile Internetnutzung sowie Tumorwachstum an. Zusätzlich betrachten wir temperierte semi-fraktionierte Diffusionsgleichungen, welche zu guten Approximationen von Aktienmarktdaten führen.

Chapter 1

Introduction

The basic aim of this thesis is to deepen our understanding of semistable laws as well as to introduce and analyze semi-fractional derivatives and corresponding differential equations. To get an idea of how this theory is embedded in the mathematical context and to motivate the leading questions for this thesis, we briefly examine its historical development.

In 1827, the biologist Robert Brown discovered the trembling, irregular motion of pollen particles immersed in water [26], but the origin of this movement baffled the scientists at that time. Quickly known as Brownian motion, it took nearly eighty years to find a satisfying explanation for this phenomenon. Only in 1905, Einstein used thermodynamic results to prove that a particle suspended in a fluid performs an irregular movement due to random collisions with surrounding fluid molecules [39]. In his probabilistic model, the probability density for the particle's location at time $t > 0$ is given by the solution $x \mapsto p(x,t)$ to the diffusion equation

$$\frac{\partial}{\partial t} p(x,t) = D \frac{\partial^2}{\partial x^2} p(x,t), \tag{1.1}$$

where $D > 0$ is a particular diffusion coefficient. For a particle starting in the origin, the unique solution to (1.1) is given by centered normal densities with variance $2Dt$ for every $t > 0$. Hence a particle in this model travels an average distance proportional to $t^{\frac{1}{2}}$ until time $t > 0$. After his predictions were confirmed by experiments of Jean Baptiste Perrin only four years later [108], Einstein's diffusion model was fully accepted and celebrated, not only for clarifying the origin of the astonishing Brownian motion but also for being the final proof for the existence of atoms, which was highly controversial at that time.

From a mathematical perspective, the underlying stochastic process describing the diffusive behavior was of particular interest. Since the solution to (1.1) is given by a normal density at every time $t > 0$, a description of the movement as a Gaussian process seemed to be suitable. Besides, experiments indicated that the process has independent and identically distributed increments and that the paths, as irregular as they might seem, are continuous. However, the existence of such a process was uncertain until 1923, when Nor-

bert Wiener published a rigorous proof [149], which is why the corresponding stochastic process is sometimes called Wiener process instead of Brownian motion.
In the following years, Brownian motion was of considerable interest to mathematicians. Especially the fact that it is not only a connection between Gaussian and Lévy processes but also a martingale and a Markov process at the same time is an essential basis for further results and simplifies its handling. For more information about Brownian motion, we suggest the monographs [104] and [125]. Likewise, from the perspective of applied mathematics, the process is of particular interest. Even before Wiener proved its well-definedness, Louis Bachelier proposed an application of Brownian motion to value stock prices at the Paris stock exchange [9]. Thereby, he laid the foundation for modern mathematical finance where Brownian motion still plays an important role. Apart from finance, other scientists, especially physicists, benefit from the mathematical results, too. Thus, nowadays, there is comprehensive literature dealing with applications of Brownian motion and related processes. For a deeper insight into physical and financial applications, we refer to [86] and [129].

Although many applications were successfully modeled with Brownian motion, numerous experiments indicated a different behavior of the involved particles. Especially measurements in turbulent flow or diffusion on polymer chains display that the spreading rate is faster than Brownian motion predicts (see for example [99] or [54] and the references cited therein). To generalize the diffusion model to this so-called super-diffusive behavior, mathematicians thought about modifying (1.1) and helped themselves to an operator nearly as old as the classical derivative: the fractional derivative. First mentioned in a letter from Leibniz to L'Hôspital in 1695 [77], the idea of derivatives of arbitrary, positive real order emerged from time to time but did not attract much attention at first [118]. In 1823, Abel's work [2] advanced the idea of fractional derivatives and inspired other mathematicians like Liouville or Riemann to further work on this topic [118]. Over the years, many different suggestions were made to define a fractional derivative, and until today, there is an ongoing discussion whether there is a 'right' form in a given scenario. For this thesis, we will concentrate on the two perhaps most popular forms in modern fractional calculus: The Riemann-Liouville and the Caputo form. Thereby, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of a suitable function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{0+}^{\infty} f(x-y) y^{n-1-\alpha}\, dy,$$

where $n \in \mathbb{N}$ fulfills $n - 1 < \alpha < n$ [116]. In contrast, the Caputo form of the fractional derivative [27] arises from the Riemann-Liouville form by a formal change of integration and differentiation

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{0+}^{\infty} f^{(n)}(x-y) y^{n-1-\alpha}\, dy.$$

Note that for the Riemann-Liouville form, the function f is not necessarily assumed to be differentiable such that it may exist under weaker conditions than the Caputo form. However, even for functions f having both a Caputo and a Riemann-Liouville fractional derivative of order $\alpha > 0$, the difference between both definitions might be huge, which becomes evident by studying the Heavyside function $f(x) = \mathbb{1}_{(0,\infty)}(x)$. Under smoothness assumptions on f, both forms have Fourier transform $(-ik)^\alpha \widehat{f}(k)$ for every $k \in \mathbb{R}$, where $\widehat{f}$ is the Fourier transform of f. For that reason, a fractional derivative is often defined in the Fourier space in modern fractional calculus literature. Since we do not want to treat other forms of fractional derivatives here, we only refer to [56], [33], or [8] for an overview of different forms and their properties. Also, note that recent approaches tend to define a generalized form of fractional derivatives, including as many already known forms as possible (e.g., see [71]).

Now we return to the problem of modeling applications which show a spreading rate faster than the classical diffusion model predicts. If we replace the second-order space derivative in the classical diffusion equation (1.1) with a Caputo fractional derivative of order $\alpha \in (0, 2) \setminus \{1\}$, we obtain the fractional diffusion equation

$$\frac{\partial}{\partial t} p(x,t) = -D \frac{\partial^\alpha}{\partial x^\alpha} p(x,t) \tag{1.2}$$

under initial condition $p(x, 0) = \delta(x)$, where we assume that $D > 0$ for $\alpha \in (0, 1)$ and $D < 0$ for $\alpha \in (1, 2)$. The restriction of α to values in $(0, 2) \setminus \{1\}$ and of the algebraic sign of the constant D yields a probabilistic solution in the same way the Brownian motion offers a solution to the classical diffusion equation (1.1). Namely, the solutions $x \mapsto p(x, t)$ of (1.2) are the densities of an α-stable Lévy process [94].

The theory of stable distributions and stable Lévy processes grew independent of fractional calculus for a long time. Stable laws were probably first studied by Lévy (see for example [78]), who aimed to generalize the Central Limit Theorem and therefore studied sums of independent and identically distributed random variables. Together with the textbook of Khintchine [68] as well as further works of Feller [43], Gnedenko and Kolmogorov [48], or Sato [122], to name just a few, his results provide the basis of our current knowledge about stable laws. There are several equivalent ways to define stability (compare [94] and [152]), but the most suitable one for this thesis is the representation in the Fourier space. This is, a non-degenerate measure μ is α-stable for $\alpha \in (0, 2]$ if either $\alpha = 2$ and

$$\widehat{\mu}(k) := \int_{\mathbb{R}} e^{ikx} \, d\mu(x) = \exp\Big(iak - \frac{1}{2}\sigma^2 k^2\Big)$$

for some $a \in \mathbb{R}$ and $\sigma^2 > 0$ or $\alpha \in (0, 2)$ and

$$\widehat{\mu}(k) = \exp\left(iak - p\Gamma(1-\alpha)(-ik)^\alpha - q\Gamma(1-\alpha)(ik)^\alpha\right)$$

for some $a \in \mathbb{R}$ and $p, q \geq 0$ with $p + q > 0$ and $\Gamma(x)$ denotes the gamma function

[94, Proposition 3.10 and Proposition 3.12]. For $\alpha = 2$, the distribution is obviously Gaussian with mean a and variance σ^2. To obtain an impression of an α-stable law for $\alpha \in (0, 2)$, note that equivalently one may define stable measures by the assumption that μ is infinitely divisible and for every $c > 0$, there is $d(c) \in \mathbb{R}$ with

$$\mu^{*c} = (c^{\frac{1}{\alpha}}\mu) * \epsilon_{d(c)}, \tag{1.3}$$

where $(c^{\frac{1}{\alpha}}\mu)$ is the image measure under the dilatation $x \mapsto c^{\frac{1}{\alpha}}x$, $\epsilon_{d(c)}$ is the point measure in $d(c)$, and the c-fold convolution is well-defined through its Fourier transform. Therefore, the one-dimensional marginal distributions of the corresponding Lévy process $(X_t)_{t\geq 0}$ with $P_{X_t} = \mu^{*t}$ arise from μ only by scaling and a suitable shift. In the case $0 < \alpha < 2$, the resulting distribution is heavy-tailed, indicating that very large events occur with higher probabilities than in the case $\alpha = 2$. In detail, the left and right tail behave like $\mu(-\infty, -r) \sim C_1 r^{-\alpha}$ as well as $\mu(r, \infty) \sim C_2 r^{-\alpha}$ for every $r > 0$ and constants $C_1, C_2 \geq 0$ ([121, Proposition 1.2.15]). Due to this heavy-tailed behavior, for an α-stable random variable X, only the absolute moments $\mathbb{E}[|X|^p]$ for $p < \alpha$ exist [121, Property 1.2.16], making it more difficult to apply in finance or physics. To overcome these differences between the theoretical model and the claims of the applications, different approaches of tempered stable distributions were developed (see for example [117], [11] or [97]). The basic idea is to manipulate stable densities so that moments of every order exist, but the behavior on a finite domain around zero is quite similar to stable laws. Yielding physically meaningful models, tempered stable distributions are successfully applied to a steadily growing number of applications ([94], [112]).
Note that we can only name three closed-form expressions for stable densities, the Cauchy-density (belonging to $\alpha = 1$), the Lévy-density (belonging to $\alpha = \frac{1}{2}$), and the normal density corresponding to $\alpha = 2$. Hence, growing computational possibilities contributed to the increasing interest in stable laws and supported our imagination of this special class of distributions. We suggest the monographs [105], [152], and [121] to the interested reader for deeper insight and more properties of stable laws.

For the solution to the fractional diffusion equation (1.2), we consider α-stable Lévy processes, which are Lévy processes $(X_t)_{t\geq 0}$ such that P_{X_1} is α-stable for some $\alpha \in (0, 2]$. Note that for every $t > 0$, X_t has a $C^\infty(\mathbb{R})$-density [122, Example 28.2], and using inverse Fourier transform, the densities of the α-stable process with $a = 0$, $q = 0$, and $p = \frac{D}{\Gamma(1-\alpha)} > 0$ solve the diffusion equation (1.2) [94]. Based on this relation, the already known results about stable processes enrich the knowledge about solutions to fractional diffusion equations and vice versa. Note that particularly, the solution of (1.2) fulfills the scaling property

$$p(x, ct) = c^{-\frac{1}{\alpha}} p(xc^{-\frac{1}{\alpha}}, t)$$

for every $c, t > 0$ and $x \in \mathbb{R}$ [94]. Therefore, the scaling rate of a particle in this model equals $\frac{1}{\alpha}$ with $\alpha \in (0, 2)\setminus\{1\}$ and hence reflects the faster spreading we were searching for and for which the Brownian motion fails as a description. Being aware of this property, many different applications with heavy-tailed behavior have been successfully modeled using fractional diffusion equations (see for example [69], [94], or [120]).

Apart from (1.2), many different kinds of differential equations involving fractional derivatives have been analyzed (see [36] or [110] for an overview). We want to emphasize that fractional derivatives are defined for any choice of $\alpha > 0$, and hence the general theory of fractional differential equations is much wider than the theory of those equations offering probabilistic solutions or interpretations in which we are interested. However, even if we restrict our attention to the case $\alpha \in (0, 2]$, we are able to study various generalizations of fractional differential equations like the generalized Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta} u(x,t) = Lu(x,t) \tag{1.4}$$

under initial condition $u(x, 0) = f(x)$, where $\beta \in (0, 1)$, f is a suitable function, and L is the generator of a Feller semigroup. Problems of this kind were introduced and partly solved by Saichev and Zaslavsky [119] and later fully answered by Baeumer and Meerschaert [10]. Apart from different initial conditions, restrictions of fractional differential equations to bounded domains and the effects on the underlying process are also of interest for fractional calculus and its applications. However, this topic exceeds the scope of this thesis, and hence we suggest [12], [29], [35], and [90] for a first impression to the interested reader.
Similar to the theory of stable distributions, fractional calculus profited considerably from growing numerical possibilities. Beginning with the Grünwald-Letnikov formula, which offers a numerical approximation of fractional derivatives, the computational methods have improved ever since, and their enhancement is an ongoing challenge. For an introduction to established methods, we suggest the monograph [15].

However, the approach for this thesis comes from a probabilistic point of view as follows. As already mentioned by Lévy [78], the class of stable distributions is embedded into the larger class of semistable laws. Thereby, semistable laws fulfill the scaling property (1.3) only for a single $c > 1$ and by iteration for every integer power of c. Thus a semistable distribution depends not only on $\alpha \in (0, 2]$ but also on the parameter c, such that we refer to it as a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution. As a consequence, the one-dimensional marginal distributions of the corresponding Lévy process $(X_t)_{t\geq 0}$ are no longer shifted and stretched versions of the density $\mu = P_{X_1}$. This only holds on the discrete scale $t = c^k$ with $k \in \mathbb{Z}$. Again the only semistable distribution with index $\alpha = 2$ is Gaussian [122, Theorem 14.1] and hence even stable, such that we often exclude this well-known case. For $\alpha \in (0, 2)$, a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution μ is still heavy-tailed, but the discrete scaling property yields remarkable differences between stable and semistable laws: The weaker condition causes a log-periodic perturbation of the Lévy measure, and as we will see, this log-periodic disturbance is characteristic for semistable laws and appears on many different levels. Besides other effects, the additional log-periodic behavior of the Lévy measure can yield multimodal distributions [146, Proposition 1], which clearly

emphasizes the difference between stable and semistable laws.
In the past, semistable laws were mainly studied as limit distributions of i.i.d. sequences of random variables (see for example [91] or [98]) but have been of minor interest in probability theory. Therefore, our knowledge about this particular subclass of infinitely divisible distributions is relatively small compared to the results we have for stable laws. However, recent analyses, for example of earthquakes ([100], [132]), financial crises [131], or fractal systems [106], suggest exactly the characteristic power law behavior with additional log-periodic perturbations a semistable law would follow and hence motivated a closer analysis of these distributions.

Therefore the first purpose of this thesis is to extend our knowledge about semistable laws. Starting from already known results (see for example [91] and [122]), we aim to prove an explicit form of the Fourier transform and thereby derive further properties. Like the theory of stable distributions profited from the connection to fractional diffusion equations, we will thereby examine a similar relationship in the semistable case. More precisely, if we replace stable laws with the more general semistable ones, can we still find a connection to diffusion equations involving some kind of generalized fractional derivative? And if so, how do both sides - the semistable law and the differential equation - benefit from this connection? Additionally, the operator generalizing the fractional derivative is of special interest, and apart from its definition, we are curious about its properties. Due to their challenging handling, semistable distributions have not drawn much attention before. Thus, we finally try to evaluate whether their analysis can provide valuable approximations of real-world applications.

To answer these questions, the reader is guided through the following topics. First, we formally introduce semistable laws and afterward prove an integral representation for the log-characteristic function of a semistable distribution on $\mathbb{R}^d$ in Section 2.2, which will be the basis for following results. Additionally, to solve abstract Cauchy problems later on, some basic facts about convolution semigroups and their generators are recalled in Section 2.3.
The representation of the Fourier transform of semistable laws enables us to define an operator, which we call semi-fractional derivative, such that the semistable densities solve a corresponding diffusion equation. Partly, this was already published in [66], where we defined and studied one-dimensional fractional derivatives (see Appendix B for a detailed list of the individual contributions of the authors). From this basis, we successively generalize the definition first to directional semi-fractional derivatives and finally to the multidimensional case in Chapter 3. Regardless of whether one-dimensional or multidimensional versions are studied, we are able to define different forms of semi-fractional derivatives. Following the fractional case, we will mainly concentrate on Caputo and Riemann-Liouville type ones and their connection. Finally, we end the chapter about semi-fractional derivatives by proving an approximation formula of Grünwald-Letnikov type, enabling us to calculate semi-fractional derivatives and the solution to semi-fractional differential equations numerically. As an important characteristic valuable for solving semi-fractional differential equations, the Laplace transform of semi-fractional derivatives

is analyzed in Chapter 4.
These differential equations are afterward studied in Chapter 5. Due to our strategy, semi-fractional diffusion equations are solved by the densities of semistable Lévy processes, and thereby, the numerical approximation from the previous chapter enables us to plot semistable distributions. Note that pictures of semistable laws have been quite rare before, and hence this is an important step toward a better understanding of this class of distributions. To study semi-fractional differential equations more generally, we consider a semi-fractional version of (1.4), inspired by the work of Baeumer and Meerschaert [10] in the fractional case. By analyzing semistable subordinators, which are semistable Lévy processes with almost surely non-decreasing paths, we can eventually present solutions to this general kind of equation.

As it turns out, under particular assumptions, a solution to the semi-fractional Cauchy problem on a stochastic level is provided by the densities of the process $(A(E(t)))_{t\geq 0}$, where $(A(t))_{t\geq 0}$ is semistable and $(E(t))_{t\geq 0}$ is a hitting time process for a semistable subordinator, which is independent of $(A(t))_{t\geq 0}$. Such processes appear as limiting processes of uncoupled Continuous Time Random Walks (CTRWs), modeling random walks with additional random waiting times between the jumps. Based on this knowledge, we discuss which limit processes of CTRWs can appear in general by considering semistable processes under arbitrary dependencies between jumps and their corresponding waiting times. As outlined in Section 6.1, such coupled CTRWs yield various interesting limiting processes, and we even prove a representation of their densities in Section 6.2.
The last theoretical chapter treats the concept of space-time duality. This work was inspired by [64], where it was shown that the solution to a negatively skewed space-fractional derivative of order $\alpha \in (1,2)$ is equivalent to those of a particular time-fractional differential equation of order $\frac{1}{\alpha}$. Note that, due to the non-locality of the semi-fractional derivative, it is difficult to find a physical meaning of space-fractional differential equations, whereas time-fractional equations may be interpreted as systems with long-time memory. Hence the space-time duality result is an important tool for applications, and we aim to generalize it to the semi-fractional case.
To test the practical application of the findings, we end this thesis with a study of growth models applied to mobile data as well as cancer growth. Similar to stable laws, by an additional tempering, semistable laws can be transferred to distributions with finite moments. As we will see, they are then suitable for applications to stock prices.

Chapter 2

Semistable distributions

The following chapter is mainly devoted to familiarizing the reader with the definition of semistable laws and some of their properties, which are essential for this thesis. For a more general introduction, we refer to [91]. Especially, we consider log-characteristic functions of semistable distributions as these laws are best characterized through their Fourier transform. This representation will naturally yield the definition of semi-fractional derivatives in Chapter 3. To define Caputo or Riemann-Liouville type forms in Chapter 3, we finally introduce Feller semigroups and their generators. Again, we only name a few essential facts and suggest the monographs [62] or [135] for further information.

2.1 Definition and first properties

Semistable distributions are not as common as their stable subclass, and therefore we briefly summarize the definition and some crucial facts in this section. To clarify the general mathematical context we are working in, we start our consideration with the larger set of infinitely divisible distributions.

There are several equivalent ways to define infinite divisibility. Still, for our purpose, the most suitable approach is to define infinite divisibility by the Fourier transform of the probability measure, also called characteristic function. This is, a probability measure ν on $\mathbb{R}^d$ is infinitely divisible if the characteristic function

$$\widehat{\nu}(k) := \int\limits_{\mathbb{R}^d} e^{i\langle k,x\rangle}\, d\nu(x)$$

is given by $\exp(\Psi(k))$ for every $k \in \mathbb{R}^d$ with log-characteristic function

$$\Psi(k) = i\langle a, k\rangle - \frac{1}{2}\langle k, Qk\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle x,k\rangle} - 1 - \frac{i\langle x, k\rangle}{1 + ||x||^2}\right) d\Phi(x) \tag{2.1}$$

for some $a \in \mathbb{R}^d$, a non-negative definite matrix $Q \in \mathbb{R}^{d\times d}$, and a σ-finite Borel measure

Φ on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \min\{1, ||x||^2\} \, d\Phi(x) < \infty. \tag{2.2}$$

Every σ-finite Borel measure Φ on $\mathbb{R}^d \setminus \{0\}$ fulfilling (2.2) is called a Lévy measure. Furthermore, the triple $[a, Q, \Phi]$, called Lévy-Khintchine triple of ν, is uniquely determined [91, Theorem 3.1.11]. Note that by (2.1), we can uniquely split every infinitely divisible distribution into the following three components: The first term in (2.1) is simply a drift, whereas the second one is the log-characteristic function of a centered normal density with covariance matrix Q and therefore called Gaussian component. The last term is called Poisson component since, at least for finite measures Φ, it is the log-characteristic function of a shifted compound Poisson distribution [91, Definition 3.1.12].

Even if not denoted with this term, Bruno de Finetti was the first to study infinitely divisible distributions, followed by Kolmogorov, Lévy, and Khintchine, to name just a few (see [84] for a historical survey and the associated references). Nowadays, there is comprehensive literature concerning these distributions, but since we concentrate on the smaller class of semistable distributions, we only refer to [91] and [122] for more information about infinitely divisible laws in general.

For a precise definition of semistable laws, we need the following generalization of non-degenerate measures on $\mathbb{R}$ to full measures on $\mathbb{R}^d$.

Definition 2.1.1. (Full measure)
A probability measure ν on $\mathbb{R}^d$ is full if ν is not supported on any $(d-1)$-dimensional hyperplane. Similarly, a Lévy measure Φ on $\mathbb{R}^d \setminus \{0\}$ is full if Φ is not supported on any $(d-1)$-dimensional hyperplane.

Note that an infinitely divisible distribution ν with Lévy-Khintchine triple $[a, 0, \Phi]$ is full if and only if the Lévy measure Φ is full [91, Proposition 3.1.20]. Under the assumption of fullness, semistable laws now arise from infinitely divisible ones under the additional requirement of a discrete scaling property.

Definition 2.1.2. (Semistable distribution)
For fixed $\alpha \in (0, 2]$ and $c > 1$, a full, infinitely divisible distribution ν on $\mathbb{R}^d$ is called $(c^{\frac{1}{\alpha}}, c)$-semistable if there is $y \in \mathbb{R}^d$ such that

$$\nu^{*c} = (c^{\frac{1}{\alpha}} \nu) * \epsilon_y, \tag{2.3}$$

where ϵ_y is the Dirac measure in y and $(c^{\frac{1}{\alpha}} \nu)$ is the image measure of ν under the dilation $x \mapsto c^{\frac{1}{\alpha}} x$. Note that the c-fold convolution ν^{*c} is well-defined through its Fourier transform. If $y = 0$ in (2.3), then the distribution is called strictly semistable.

By iteration, we obtain (2.3) for every integer power of c. Hence, for every $t = c^k$ with

$k \in \mathbb{Z}$, the distribution ν^{*t} is no more than a shifted and scaled version of ν itself. To become more familiar with this definition, we consider some special cases.

Example 2.1.3. (Special cases of semistable distributions)

(i) If (2.3) holds for any $c > 1$ and some $y = y(c) \in \mathbb{R}^d$, then ν is α-stable. In this sense, stability strengthens semistability by demanding the scaling property on a continuous and not only on a discrete scale. Note that in this case, the distribution ν^{*t} is indeed a stretched and shifted version of ν for every $t > 0$. Due to this more restrictive assumption, stable laws are easier to handle, and hence, many properties like the explicit shape of the log-characteristic function, different representations, and their ability to model various applications have been investigated (see for example [94], [105], or [152]). Nevertheless, there are only a few cases where a stable density is explicitly known (these cases are the Cauchy distribution ($\alpha = 1$), the Lévy distribution ($\alpha = \frac{1}{2}$), and the normal distribution ($\alpha = 2$)).

(ii) If $\alpha = 2$, then every semistable distribution is Gaussian (compare [122, Theorem 14.1]). However, the normal distribution is well-known, and this is why we often exclude this special case.

(iii) As a concrete example, consider the limit distribution of the total gain in successive St. Petersburg games. In a St. Petersburg game, a fair coin is tossed until it shows head for the first time. If this happens in the n-th toss, the player wins 2^n Euro. The expectation of the gain is infinite such that a fair entrée fee cannot be constructed with naive methods, and hence, the problem became a popular paradox ([60], [37]). Now let S_N be the total gain of the player in N independent St. Petersburg games. It was shown by Feller [43, Chapter X:4] that

$$\frac{S_N}{N \log_2(N)} \to 1$$

in probability, such that a fair entrée fee for N independent games can by approximated by $N \log_2(N)$ for large values of N. Later on, Anders Martin-Löf [85, Theorem 1] offered a limit distribution for the total gain, proving that

$$\frac{S_N - N \log_2(N)}{N}$$

has a $(c^{\frac{1}{\alpha}}, c)$-semistable limit distribution with $c = 2$ and $\alpha = 1$ along the sequence $N = 2^n$ with $n \in \mathbb{N}$. In this case, the Lévy measure Φ is concentrated on $2^{\mathbb{Z}}$ with

$$\Phi(2^k) = 2^{-k} \quad \forall\, k \in \mathbb{Z},$$

and (2.3) is fulfilled with $y = 2$.

Historically, semistable distributions were not defined by (2.3) but appeared as limit distributions of sums of i.i.d. random variables first [78]. Similar to Example 2.1.3 (iii), the convergence of the partial sums has to be studied along special subsequences in the following way.

Theorem 2.1.4. *[91, Proposition 8.3.16] Let A be a random variable with full distribution ν on $\mathbb{R}^d$. Then ν is semistable if and only if there is an i.i.d. sequence $X, X_1, X_2, \dots$ of random variables on $\mathbb{R}^d$ with common distribution μ, an increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ with $k_n \to \infty$ and*

$$\frac{k_{n+1}}{k_n} \to c$$

as $n \to \infty$ as well as real numbers $a_n > 0$ such that

$$a_n(X_1 + \ldots + X_{k_n}) + b_n \xrightarrow{d} A$$

for non-random vectors $b_n \in \mathbb{R}^d$. Thereby, $\xrightarrow{d}$ indicates convergence in distribution. On the level of measures, this reads as the weak convergence

$$(a_n\mu)^{*k_n} * \epsilon_{b_n} \xrightarrow{w} \nu.$$

In this case, we say that X (or μ) belongs to the domain of semistable attraction of A (or ν). If $b_n = 0$ for every $n \in \mathbb{N}$, then X (or μ) is in the strict domain of semistable attraction of A (or ν).

Remark 2.1.5. As already seen in Example 2.1.3 (i), stable laws are embedded in the set of semistable distributions and hence can be described as limit distributions of sums of random variables as in Theorem 2.1.4. In this case, the sequence $(k_n)_{n\in\mathbb{N}}$ can be chosen such that $k_n = n$ for every $n \in \mathbb{N}$ (compare [91, Definition 7.3.1]).

For every infinitely divisible distribution ν, there is a Lévy process $(A(t))_{t\geq 0}$, unique in distribution, with $P_{A(1)} = \nu$ [62, Theorem 13.12]. Note that throughout this thesis, a stochastic process $(A(t))_{t\geq 0}$ is defined to be a Lévy process if $A(0) = 0$ almost surely, the increments are independent and identically distributed, and the process is continuous in probability. If the distribution ν is additionally $(c^{\frac{1}{\alpha}}, c)$-semistable, then the process is called $(c^{\frac{1}{\alpha}}, c)$-semistable Lévy process. Due to the discrete scaling property of ν, this process fulfills

$$A(ct) \overset{f.d.}{=} c^{\frac{1}{\alpha}} A(t) + y(t)$$

for every $t > 0$ and $y(t) \in \mathbb{R}^d$, where $\overset{f.d.}{=}$ denotes equality of the finite-dimensional marginal distributions. This property is called wide-sense (or broad-sense) semi-selfsimilarity in the literature (see for example [122] or [122]) and offers an equivalent definition for a Lévy process to be semistable. If ν is strictly semistable, then the corresponding Lévy process is semi-selfsimilar, meaning that

$$A(ct) \overset{f.d.}{=} c^{\frac{1}{\alpha}} A(t)$$

for every $t > 0$. Note that every semistable Lévy process has $C^\infty(\mathbb{R}^d)$-densities $x \mapsto p(x,t)$ for every $t > 0$ [122, Example 28.2], and the wide-sense semi-selfsimilarity carries over to these densities. More precisely, the densities of a strict semistable process fulfill

$$p(x, ct) = c^{-\frac{1}{\alpha}} p(c^{-\frac{1}{\alpha}} x, t) \tag{2.4}$$

for every $t > 0$. If $(A(t))_{t \geq 0}$ is a stable Lévy process, then it is wide-sense selfsimilar (with index $\frac{1}{\alpha}$), meaning that

$$A(ct) \overset{f.d.}{=} c^{\frac{1}{\alpha}} A(t) + y(t)$$

for every $t > 0$, $c > 0$, and $y(t) \in \mathbb{R}^d$. If in addition $y(t) = 0$ for every $t > 0$, then the process is selfsimilar (with index $\frac{1}{\alpha}$), corresponding to a strictly stable Lévy process [122, Propositon 13.5], and (2.4) is fulfilled for every $c > 0$ and $t > 0$.

2.2 Log-characteristic function

Even not possible for most stable laws, we do not hope to find an explicit representation of semistable densities. Alternatively, we characterize these distributions by their Fourier transforms in terms of the Lévy-Khintchine triple. The explicit formula we prove in this section will provide the basis for the following calculations and is especially needed for the definition of semi-fractional derivatives in Chapter 3.
Since every 2-semistable distribution is Gaussian (Example 2.1.3 (ii)), we exclude this case from our consideration. For $\alpha < 2$, we can use a spectral representation of the Lévy measure given in [91] to characterize the Lévy-Khintchine triple of a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution. Thus, in the following, let $S \coloneqq \{x \in \mathbb{R}^d : ||x||^2 = 1\}$ denote the d-dimensional unit sphere with respect to a fixed norm $||\cdot||$ on $\mathbb{R}^d$.

Theorem 2.2.1. *(Spectral representation, [91, Theorem 7.4.3])*
Let ν be a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution on $\mathbb{R}^d$ for some $c > 1$ and $\alpha \in (0,2)$. Then the Lévy-Khintchine triple of ν is given by $[a, 0, \Phi]$ for some $a \in \mathbb{R}^d$ and a Lévy measure Φ fulfilling

$$\Phi\{t\theta : t > r, \theta \in D\} = \int_D r^{-\alpha} K_\theta(\log(r))\, dM(\theta) \tag{2.5}$$

for every $r > 0$ and $D \subseteq \mathcal{B}(S)$, where

$$M(D) = \Phi\{t\theta : t > 1, \theta \in D\}$$

is a finite Borel measure on the unit sphere S and $(K_\theta)_{\theta \in S}$ is a family of θ-measurable, non-negative and $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic functions with

$$K_\theta(x + \delta) e^{-\alpha\delta} \leq K_\theta(x - \delta) e^{\alpha\delta} \tag{2.6}$$

for every $x \in \mathbb{R}$*,* $\delta > 0$*, and every* $\theta \in S$*.*

Note that the growth condition (2.6) ensures that the functions $r \mapsto r^{-\alpha} K_\theta(\log(r))$ in the integral (2.5) are non-increasing in $r > 0$ for every $\theta \in S$ and is hence a necessary assumption for Φ to be a well-defined measure. Additionally, it follows from (2.6) that for every $\theta \in S$, the function K_θ is either strictly positive or identically zero. However, due to the fullness of ν, the Lévy measure Φ is full, and there is at least one $\theta \in S$ such that K_θ is non-vanishing. For a simpler handling, we will quantify the properties of the set $(K_\theta)_{\theta \in S}$ in the term 'admissable' later on (compare Definition 3.1.1 or Definition 3.3.1 respectively).
The spectral representation above allows us to simplify the log-characteristic function (2.1) of a semistable distribution using spherical coordinates.

Lemma 2.2.2. *(Semistable log-characteristic function)*
Let ν *be a* $(c^{\frac{1}{\alpha}}, c)$*-semistable distribution on* $\mathbb{R}^d$ *for some* $c > 1$ *and* $\alpha \in (0,2) \setminus \{1\}$*. Then the log-characteristic function* Ψ *of* ν *is given by*

$$\Psi(k) = i\langle b, k\rangle - \int\limits_S h_\theta(\langle k, \theta\rangle)\, dM(\theta) \tag{2.7}$$

for every $k \in \mathbb{R}^d$*, some* $b \in \mathbb{R}^d$*, and the finite Borel measure* M *on* S *from Theorem 2.2.1, where the function* $h_\theta : \mathbb{R} \to \mathbb{C}$ *is given as the Riemann-Stieltjes integral*

$$h_\theta(x) = \int\limits_{0+}^{\infty} \left(e^{irx} - \sum_{p=0}^{\lfloor \alpha \rfloor} (irx)^p \right) dG_{K_\theta}(r) \tag{2.8}$$

for every $\theta \in S$*,* $\lfloor \alpha \rfloor$ *is the integer part of* α*, and* $G_{K_\theta} : (0,\infty) \to [0,\infty)$ *is the non-increasing function* $G_{K_\theta}(r) \coloneqq r^{-\alpha} K_\theta(\log(r))$*.*

Proof. Consider the case $\alpha \in (0,1)$ first. According to [91, Lemma 6.3.11]

$$\int\limits_{\{0<||x||<R\}} ||y||\, d\Phi(y) < \infty$$

for every $R > 0$ such that with [91, Theorem 3.1.14], we can simplify the log-characteristic function to

$$\Psi(k) = i\langle b, k\rangle + \int\limits_{\mathbb{R}^d \setminus \{0\}} \left(e^{i\langle x, k\rangle} - 1 \right) d\Phi(x)$$

for every $k \in \mathbb{R}^d$, where

$$b \coloneqq a - \int\limits_{\mathbb{R}^d \setminus \{0\}} \frac{x}{1 + ||x||^2}\, d\Phi(x). \tag{2.9}$$

For every $\theta \in S$, let $G_{K_\theta} : (0,\infty) \to [0,\infty)$ be the function $G_{K_\theta}(r) = r^{-\alpha} K_\theta(\log(r))$ and note that G_{K_θ} is non-increasing. Using spherical coordinates $x = r\theta$ with $r > 0$ and $\theta \in S$, we show that the Lévy measure Φ decomposes as

$$d\Phi(x) = -dG_{K_\theta}(r) dM(\theta). \tag{2.10}$$

Therefore apply a similar technique as in the proof of Theorem 7.3.3 in [91] and consider the sets $A_{s,D} \coloneqq \{t\theta : t > s, \theta \in D\}$ for arbitrary $s > 0$ and a Borel set $D \subseteq S$. Since these sets are $\cap$-stable and generate the Borel sets on $\mathbb{R}^d \setminus \{0\}$, it is sufficient to show (2.10) for sets of this type. According to (2.5), we have

$$\begin{aligned}
\Phi(A_{s,D}) &= \int\limits_D s^{-\alpha} K_\theta(\log(s))\, dM(\theta) = -\int\limits_D \int\limits_s^\infty dG_{K_\theta}(r)\, dM(\theta) \\
&= -\int\limits_{A_{s,D}} dG_{K_\theta}(r)\, dM(\theta)
\end{aligned}$$

such that (2.10) holds. Then it follows that

$$\begin{aligned}
\Psi(k) &= i\langle b,k\rangle - \int\limits_S \int\limits_{0+}^\infty \left(e^{ir\langle \theta,k\rangle} - 1\right) dG_{K_\theta}(r)\, dM(\theta) \\
&= i\langle b,k\rangle - \int\limits_S h_\theta(\langle \theta,k\rangle)\, dM(\theta)
\end{aligned}$$

for every $k \in \mathbb{R}^d$. If on the contrary $\alpha \in (1,2)$, we find

$$\int\limits_{\{||x||\geq R\}} ||y||\, d\Phi(y) < \infty$$

for every $R > 0$ such that

$$\Psi(k) = i\langle b,k\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle x,k\rangle} - 1 - i\langle x,k\rangle\right) d\Phi(x)$$

for every $k \in \mathbb{R}^d$ (compare [91, Lemma 6.3.17 and Theorem 3.1.14]) with

$$b \coloneqq a - \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(\frac{x}{1+||x||^2} - x\right) d\Phi(x). \tag{2.11}$$

Use (2.10) to obtain

$$\begin{aligned}\Psi(k) &= i\langle b,k\rangle - \int_S \int_{0+}^{\infty} \left(e^{ir\langle\theta,k\rangle} - 1 - ir\langle\theta,k\rangle\right) dG_{K_\theta}(r)\, dM(\theta)\\ &= i\langle b,k\rangle - \int_S h_\theta(\langle\theta,k\rangle)\, dM(\theta).\end{aligned}$$

□

Remark 2.2.3. (Interpretation of the measure M)
Note that the functions h_θ in (2.8) are log-characteristic functions of one-dimensional $(c^{\frac{1}{\alpha}}, c)$-semistable distributions with Lévy-Khintchine triple $[a_\theta, 0, \Phi_\theta]$, where

$$a_\theta := \begin{cases} \displaystyle\int_{0+}^{\infty} \frac{x}{1+x^2}\, d\Phi_\theta(x) & \text{if } \alpha \in (0,1) \\ \displaystyle\int_{0+}^{\infty} \left(\frac{x}{1+x^2} - x\right) d\Phi_\theta(x) & \text{if } \alpha \in (1,2) \end{cases}$$

and the Lévy measure Φ_θ is supported on the positive real line with

$$\Phi_\theta(r,\infty) = G_{K_\theta}(r) = r^{-\alpha} K_\theta(\log(r))$$

for every $r > 0$. Then the Lévy measure M on the d-dimensional unit sphere can be interpreted as a mixing measure, weighting each radial direction.

Remark 2.2.4. For $\alpha = 1$, the spectral representation in Theorem 2.2.1 is valid likewise, but note that in this case neither the integral in (2.9) nor the integral in (2.11) are finite. Alternatively, with ([91, Theorem 3.1.14]), we obtain

$$\Psi(k) = i\langle b,k\rangle - \int_S \int_{0+}^{\infty} \left(e^{ir\langle\theta,k\rangle} - 1 - ir\langle\theta,k\rangle \mathbb{1}_{\{r<1\}}\right) dG_{K_\theta}(r)\, dM(\theta)$$

for every $k \in \mathbb{R}^d$, where

$$b := a - \int_{\mathbb{R}^d\setminus\{0\}} \left(\frac{x}{1+||x||^2} - x\mathbb{1}_{\{||x||<1\}}\right) d\Phi(x).$$

Example 2.2.5. (Stable log-characteristic function)
If $K_\theta \equiv 1$, then ν is stable and for $\alpha \in (0,1)$,

$$h_\theta(x) = -\alpha \int_{0+}^{\infty} \left(e^{irx} - 1\right) r^{-\alpha-1}\, dr = \Gamma(1-\alpha)(-ix)^\alpha$$

for every $x \in \mathbb{R}$ (compare [94, Proposition 3.10]). Following Lemma 2.2.2, the log-characteristic function equals

$$\Psi(k) = i\langle b, k\rangle - \Gamma(1-\alpha)\int_S (-i\langle k,\theta\rangle)^\alpha \, dM(\theta) \tag{2.12}$$

for every $k \in \mathbb{R}^d$. In the same way, for $\alpha \in (1,2)$, we obtain

$$\begin{aligned}\Psi(k) &= i\langle b, k\rangle + \frac{\Gamma(2-\alpha)}{\alpha-1}\int_S (-i\langle k,\theta\rangle)^\alpha \, dM(\theta)\\ &= i\langle b, k\rangle - \Gamma(1-\alpha)\int_S (-i\langle k,\theta\rangle)^\alpha \, dM(\theta)\end{aligned}$$

for every $k \in \mathbb{R}^d$ (compare [94, Proposition 3.12]). Especially in the one-dimensional case this yields the well-known formula

$$\Psi(k) = ibk - p\Gamma(1-\alpha)(-ik)^\alpha - q\Gamma(1-\alpha)(ik)^\alpha$$

for every $k \in \mathbb{R}$, where $p := M\{1\} \geq 0$ and $q := M\{-1\} \geq 0$ with $p+q > 0$ (e.g., see [94, Proposition 3.10 and Proposition 3.12]).

Example 2.2.6. Assume that for every $\theta \in S$, the function K_θ is smooth; this is K_θ is continuous and piecewise continuously differentiable. Due to its smoothness, the Fourier series agrees with the function K_θ in every point $x \in \mathbb{R}$ [46, Theorem 2.1] such that we are able to write

$$K_\theta(x) = \sum_{n\in\mathbb{Z}} c_{n,\theta} e^{in\tilde{c}x}$$

for Fourier coefficients $(c_{n,\theta})_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ and $\tilde{c} := \frac{2\pi\alpha}{\log(c)}$ guaranteeing the right period. In addition, the Fourier coefficients decay like $|c_n| \leq C|n|^{-\frac{3}{2}}$ for a constant $C > 0$ and every $n \in \mathbb{Z}$ [46, Theorem 2.6]. In [66], we already considered this special case in one dimension, and since the multidimensional case works similarly, we only name the results here. Note that in [66], we assumed that the Fourier coefficients decay like n^{-2}, but all results can be proven similarly under the weaker assumption above.
As in the stable case, it was shown in [66, Theorem 3.1] that for $\alpha \in (0,2) \setminus \{1\}$ the integral defining h_θ can be solved explicitly using this Fourier series approach, and we obtain

$$h_\theta(x) = \sum_{n\in\mathbb{Z}} c_{n,\theta}\Gamma(in\tilde{c} - \alpha + 1)(-ix)^{\alpha - in\tilde{c}}$$

for every $x \in \mathbb{R}$. Note that this series converges absolutely due to the exponential decay of the gamma function [3, Corollary 1.4.4]. Hence, the log-characteristic function is given by

$$\Psi(k) = i\langle b,k\rangle - \int_S \sum_{n\in\mathbb{Z}} c_{n,\theta}\Gamma(in\tilde{c}-\alpha+1)(-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}}\,dM(\theta)$$

for every $k \in \mathbb{R}^d$. If we define functions $\eta_\theta : \{z \in \mathbb{C} : |\operatorname{Im}(z)| = \frac{\pi}{2}\} \to \mathbb{C}$ by

$$\eta_\theta(x) := \sum_{n\in\mathbb{Z}} c_{n,\theta}\Gamma(in\tilde{c}-\alpha+1)e^{-in\tilde{c}x} \tag{2.13}$$

for every $\theta \in S$, it follows that

$$\Psi(k) = i\langle b,k\rangle - \int_S (-i\langle k,\theta\rangle)^{\alpha}\eta_\theta(\log(-i\langle k,\theta\rangle))\,dM(\theta).$$

In comparison to the log-characteristic function of a stable distribution (2.12), we see that the log-periodic perturbation of the Lévy measure yields a similar log-periodic disturbance in this integral representation of the log-characteristic function.

To close this section, we want to give an example of how semistable distributions arise as limit laws of i.i.d. sums of random variables and therefore introduce disturbed Pareto laws. Similar to ordinary Pareto laws, these distributions are heavy-tailed, but additionally, the tail is log-periodic disturbed. Since Pareto distributions have been successfully applied to various heavy-tailed phenomena like the distribution of income [136] or daily changes in stock prices [94], we furthermore hope to provide even better models by this generalized class of distributions (compare Chapter 8).

Example 2.2.7. (Disturbed Pareto distribution)
We say that a real-valued random variable X has a one-sided disturbed Pareto distribution if

$$P(X \geq t) = \begin{cases} \frac{c}{K(0)}t^{-\alpha}K(\log(t)), & \text{if } t \geq c^{\frac{1}{\alpha}} \\ 1 & \text{if } t < c^{\frac{1}{\alpha}} \end{cases}$$

for some $\alpha \in (0,2)\setminus\{1\}$, $c > 1$, and a positive, $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic function K such that (2.6) holds. If we assume that K is continuously differentiable, then X has the Lebesgue density

$$\begin{aligned} f(t) &= \frac{c}{K(0)}t^{-\alpha-1}(\alpha K(\log(t)) - K'(\log(t)))\mathbb{1}_{(c^{\frac{1}{\alpha}},\infty)}(t) \\ &=: g(t)\mathbb{1}_{(c^{\frac{1}{\alpha}},\infty)}(t). \end{aligned}$$

Note that disturbed Pareto laws differ from semi-Pareto distributions as treated in [109], and we therefore consciously relinquish to include the prefix 'semi' in the definition. To evaluate the connection between disturbed Pareto laws and semistable distributions, we calculate the characteristic function of X. Concentrating on the case $\alpha \in (0,1)$ first, we obtain

$$\begin{aligned}\mathbb{E}[e^{ikX}] &= \int_{c^{\frac{1}{\alpha}}}^{\infty} e^{ikt} f(t)\,dt = \int_{c^{\frac{1}{\alpha}}}^{\infty} (1+e^{ikt}-1) f(t)\,dt \\ &= 1 + \int_{0_+}^{\infty} (e^{ikt}-1) g(t)\,dt - \int_{0+}^{c^{\frac{1}{\alpha}}} (e^{ikt}-1) g(t)\,dt\end{aligned}$$

for every $k \in \mathbb{R}$. Since K is smooth, we are able to express K with its Fourier series

$$K(x) = \sum_{n\in\mathbb{Z}} c_{n,1} e^{in\tilde{c}x}$$

for Fourier coefficients $(c_{n,1})_{n\in\mathbb{Z}}$ and $\tilde{c} = \frac{2\pi\alpha}{\log(c)}$. Following Example 2.2.6, the first integral is given by

$$\int_{0_+}^{\infty} (e^{ikt}-1) g(t)\,dt = -\frac{c}{K(0)} (-ik)^{\alpha} \eta_1(\log(k))$$

with η_1 from (2.13), whereas for the second integral, using a Taylor approximation, we find

$$\left| \int_{0+}^{c^{\frac{1}{\alpha}}} (e^{ikt}-1) g(t)\,dt \right| \le \int_{0+}^{c^{\frac{1}{\alpha}}} |e^{ikt}-1| g(t)\,dt \le \int_{0+}^{c^{\frac{1}{\alpha}}} |kt| g(t)\,dt = O(k).$$

Now if $X_1, X_2, \ldots$ are i.i.d. copies of X, then the sum $c^{-\frac{n}{\alpha}} \sum_{j=1}^{\lfloor c^n \rfloor} X_j$ has Fourier transform

$$\begin{aligned}\left(\mathbb{E}[e^{ikc^{-\frac{n}{\alpha}}X}]\right)^{\lfloor c^n \rfloor} &= \left(1 - \frac{c}{K(0)} (-ic^{-\frac{n}{\alpha}}k)^{\alpha} \eta_1\left(\log\left(c^{-\frac{n}{\alpha}}k\right)\right) + O\left(kc^{-\frac{n}{\alpha}}\right)\right)^{\lfloor c^n \rfloor} \\ &= \left(1 - \frac{c^{1-n}}{K(0)} (-ik)^{\alpha} \eta_1(\log(k)) + O(c^{-\frac{n}{\alpha}})\right)^{\lfloor c^n \rfloor} \\ &\to \exp\left(-\frac{c}{K(0)} (-ik)^{\alpha} \eta_1(\log(k))\right)\end{aligned}$$

for every $k \in \mathbb{R}$ as $n \to \infty$. However, this is the Fourier transform of a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution ν on $\mathbb{R}$ with $\nu \sim [a, 0, \Phi]$, where

$$a = \int\limits_{\mathbb{R}\setminus\{0\}} \frac{x}{1+x^2}\, d\Phi(x)$$

and

$$\Phi(-\infty, -r) = 0 \quad \text{ and } \quad \Phi(r, \infty) = r^{-\alpha}\frac{c}{K(0)}K(\log(r))$$

for every $r > 0$. Hence, we obtain convergence of the normalized sum toward a semistable random variable with distribution ν. Similarly, for $\alpha \in (1, 2)$, we have

$$\begin{aligned}
\mathbb{E}[e^{ikX}] &= \int\limits_{c^{\frac{1}{\alpha}}}^{\infty} (1 + ikt + e^{ikt} - 1 - ikt) f(t)\, dt \\
&= 1 + ik\mathbb{E}[X] + \int\limits_{0_+}^{\infty} (e^{ikt} - 1 - ikt) g(t)\, dt - \int\limits_{0+}^{c^{\frac{1}{\alpha}}} (e^{ikt} - 1 - ikt) g(t)\, dt \\
&= 1 + ik\mathbb{E}[X] - \frac{c}{K(0)}(-ik)^{\alpha}\eta_1(\log(t)) - \int\limits_{0+}^{c^{\frac{1}{\alpha}}} (e^{ikt} - 1 - ikt) g(t)\, dt
\end{aligned}$$

for every $k \in \mathbb{R}$, where the last integral is bounded with

$$\begin{aligned}
\left| \int\limits_{0+}^{c^{\frac{1}{\alpha}}} (e^{ikt} - 1 - ikt) g(t)\, dt \right| &\le \int\limits_{0+}^{c^{\frac{1}{\alpha}}} |e^{ikt} - 1 - ikt| g(t)\, dt \\
&\le \int\limits_{0+}^{c^{\frac{1}{\alpha}}} k^2 t^2 g(t)\, dt \\
&= O(k^2).
\end{aligned}$$

Then the shifted sum $c^{-\frac{n}{\alpha}} \sum\limits_{j=1}^{\lfloor c^n \rfloor} (X_j - \mathbb{E}[X])$ has Fourier transform

$$\begin{aligned}
\left(E[e^{ikc^{-\frac{n}{\alpha}}(X - \mathbb{E}[X])}]\right)^{\lfloor c^n \rfloor} &= \left(1 - \frac{c}{K(0)}(-ic^{-\frac{n}{\alpha}}k)^{\alpha}\eta_1\left(\log\left(c^{-\frac{n}{\alpha}}k\right)\right) + O(c^{-\frac{2n}{\alpha}})\right)^{\lfloor c^n \rfloor} \\
&\to \exp\left(-\frac{c}{K(0)}(-ik)^{\alpha}\eta_1(\log(k))\right)
\end{aligned}$$

for every $k \in \mathbb{R}$ as $n \to \infty$. Again the scaled and shifted sum converges to a semistable distribution. Hence, in a similar way Pareto distributions are used to approximate stable distributions, we are able to calculate semistable laws as limit of normalized sums of i.i.d. disturbed Pareto distributions.

2.3 Convolution Semigroups

As a last introductory section, we recall some basic facts about convolution semigroups and their generators. Since every infinitely divisible law generates a Feller semigroup, this theory delivers an additional description of Lévy processes. Especially for this thesis, not only the definition of semi-fractional derivatives depends on this connection (compare Lemma 3.1.10), but it also allows us to solve semi-fractional Cauchy problems in Chapter 5.

Definition 2.3.1. Let $(T(t))_{t\geq 0}$ be a family of linear operators on a Banach space $(\mathcal{X}, ||\cdot||)$ with $T(0)$ being the identity operator. We use the following definitions:

- If $T(s+t) = T(s)T(t)$ for every $s, t \geq 0$, then $(T(t))_{t\geq 0}$ is a semigroup.
- If for every $t \geq 0$ there is $M_t > 0$ with $||T(t)f|| \leq M_t||f||$ for all $f \in X$, then we call $(T(t))_{t\geq 0}$ bounded.
- $(T(t))_{t\geq 0}$ is uniformly bounded if there is a constant $M > 0$ with $||T(t)f|| \leq M||f||$ for all $f \in X$ and all $t \geq 0$.
- We say that $(T(t))_{t\geq 0}$ is strongly continuous if
$$||T(t)f - f|| \to 0 \quad \text{as } t \downarrow 0.$$

A strongly continuous, uniformly bounded semigroup with constant $M = 1$ is called a Feller semigroup.

In what follows, let ν be an infinitely divisible law on $\mathbb{R}^d$ with Lévy-Khintchine representation $[a, Q, \Phi]$, where $a \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ is a non-negative definite matrix, and Φ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$. By $(A(t))_{t\geq 0}$, we denote the Lévy process on $\mathbb{R}^d$ with $P_{A(1)} = \nu$.
In addition, consider the Banach space $L^1(\mathbb{R}^d)$ of integrable functions equipped with the $||\cdot||_1$-norm and define the family of linear operators $(T(t))_{t\geq 0}$ on $L^1(\mathbb{R}^d)$ by

$$T(t)f(x) := \int_{\mathbb{R}^d} f(x-y)\, d\nu^{*t}(y) = \mathbb{E}[f(x - A(t))]. \tag{2.14}$$

According to [57, Theorem 23.13.1] and [10, Proposition 2.1], these operators form a Feller semigroup on $L^1(\mathbb{R}^d)$. For such a semigroup, we define the generator L as

$$Lf := \lim_{t\downarrow 0} \frac{T(t)f - f}{t},$$

and denote with $\mathrm{Dom}(L)$ the domain of L containing all functions $f \in L^1(\mathbb{R}^d)$ for which the above limit exists. For further calculations, we need the following additional properties of L and $\mathrm{Dom}(L)$ taken from [107, Theorem 2.4 and Corollary 2.5].

Lemma 2.3.2. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator L.*

(a) $\mathrm{Dom}(L)$ *is dense in* $L^1(\mathbb{R}^d)$ *and* L *is a closed linear operator.*

(b) For every $f \in L^1(\mathbb{R}^d)$*, we have* $\int_0^t T(s)f\,ds \in \mathrm{Dom}(L)$ *and*

$$T(t)f = L\left(\int_0^t T(s)f\,ds\right) + f.$$

For suitable functions f, the generator Lf can be written down explicitly using the Lévy-Khintchine representation of ν. As shown below, a sufficient condition on f for this representation is to assume that $f \in W^2(\mathbb{R}^d)$, where for $n \in \mathbb{N}$ and an open subset $\mathcal{A} \subseteq \mathbb{R}^d$, we denote by $W^n(\mathcal{A})$ the space of all $L^1(\mathcal{A})$-functions whose partial derivatives up to order n exist and are again $L^1(\mathcal{A})$-functions.

Lemma 2.3.3. *([10, Theorem 2.2]) Let ν be an infinitely divisible distribution on $\mathbb{R}^d$ with Lévy-Khintchine representation $[a, Q, \Phi]$. Then the generator L of the semigroup $(T(t))_{t\geq 0}$ defined in (2.14) is given by*

$$Lf(x) = -\langle a, \nabla f(x)\rangle + \frac{1}{2}\langle \nabla, Q\nabla f(x)\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(f(x-y) - f(x) + \frac{\langle \nabla f(x), y\rangle}{1+||y||^2}\right) d\Phi(y)$$

for all $f \in W^2(\mathbb{R}^d)$. Furthermore it holds that $\mathcal{F}(Lf)(k) = \Psi(k)\mathcal{F}(f)(k)$, where Ψ is the log-characteristic function of ν and $\mathcal{F}$ denotes the Fourier transform.

In the context of semistable laws, we are mainly interested in the following special case of Lemma 2.3.3.

Example 2.3.4. For $\alpha \in (0,2)$ and $c > 1$, let ν be $(c^{\frac{1}{\alpha}}, c)$-semistable with Lévy-Khintchine representation $[a_1 + a_2, 0, \Phi]$, where $a_1 \in \mathbb{R}^d$ and

$$a_2 = \begin{cases} \int\limits_{\mathbb{R}^d\setminus\{0\}} \frac{x}{1+||x||^2}\,d\Phi(x) & \text{if } \alpha \in (0,1) \\ \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(\frac{x}{1+||x||^2} - x\mathbb{1}_{\{||x||<1\}}\right) d\Phi(x) & \text{if } \alpha = 1 \\ \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(\frac{x}{1+||x||^2} - x\right) d\Phi(x) & \text{if } \alpha \in (1,2) \end{cases}. \tag{2.15}$$

Then together with (2.10), the generator L of the semigroup $(T(t))_{t\geq 0}$ for $\alpha \in (0,1)$ is given by

$$\begin{aligned} Lf(x) &= -\langle a_1, \nabla f(x)\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} (f(x-y) - f(x))\, d\Phi(y) \\ &= -\langle a_1, \nabla f(x)\rangle + \int\limits_S \int\limits_{0+}^{\infty} (f(x) - f(x-r\theta))\, dG_\theta(r)\, dM(\theta) \end{aligned}$$

for every $x \in \mathbb{R}^d$, whereas for $\alpha \in (1,2)$

$$\begin{aligned} Lf(x) &= -\langle a_1, \nabla f(x)\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} (f(x-y) - f(x) + \langle y, \nabla f(x)\rangle)\, d\Phi(y) \\ &= -\langle a_1, \nabla f(x)\rangle + \int\limits_S \int\limits_{0+}^{\infty} (f(x) - f(x-r\theta) - r\langle \theta, \nabla f(x)\rangle)\, dG_\theta(r)\, dM(\theta). \end{aligned}$$

In the case $\alpha = 1$, together with the spherical representation in Remark 2.2.4, we obtain

$$Lf(x) = -\langle a_1, \nabla f(x)\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} (f(x-y) - f(x) + \langle y, \nabla f(x)\rangle \mathbb{1}_{\{||y||<1\}})\, d\Phi(y)$$

for every $x \in \mathbb{R}^d$.

Example 2.3.5. If ν is a 2-semistable distribution, then it is Gaussian, and if we choose $a = 0$ in Lemma 2.3.3, the corresponding generator is given by

$$Lf(x) = \frac{1}{2}\langle \nabla, Q\nabla f(x)\rangle.$$

Especially, if $Q = \sigma^2\,\mathrm{Id}$ for some $\sigma^2 > 0$ and the identity matrix $\mathrm{Id} \in \mathbb{R}^{d\times d}$, this simplifies to

$$Lf(x) = \frac{1}{2}\sigma^2 \Delta f(x),$$

which in the one-dimensional case yields

$$Lf(x) = \frac{1}{2}\sigma^2 \frac{d^2}{dx^2} f(x).$$

A reason why studying semigroups and its generators is so important for this thesis is their connection to Cauchy problems. Since $(T(t))_{t\geq 0}$ in (2.14) is a strongly continuous semigroup with generator L, for any initial condition $p_0 \in \mathrm{Dom}(L)$ the function

$$q(x,t) \coloneqq T(t)p_0(x) = \mathbb{E}[p_0(x - A(t))]$$

is the unique solution to the abstract Cauchy problem

$$\frac{\partial}{\partial t} q(x,t) = Lq(x,t), \quad q(x,0) = p_0(x) \tag{2.16}$$

for every $x \in \mathbb{R}^d$, $t > 0$ ([150, Theorem 1.14]). If ν^{*t} has a Lebesgue-density $x \mapsto p(x,t)$ for every $t > 0$, we obtain the representation

$$q(x,t) = \int_{\mathbb{R}^d} p_0(x-y)p(y,t)\,dy,$$

and the set $\{p(\cdot,t) : t > 0\}$ is called Green's function solution to (2.16). On the other hand, the process $(A(t))_{t\geq 0}$ is called stochastic solution to the abstract Cauchy problem (2.16).

Chapter 3

Semi-fractional derivatives

Although the previous chapter was mainly of introductory character, the more explicit form of the log-characteristic function proven in Lemma 2.2.2 is a crucial step toward a better understanding of semistable laws. For further insights, we orientate ourselves by the results known for the special case of stable laws. As mentioned before, the theory of stable distributions profited noticeably from its connection to fractional differential equations, and for this reason, the following chapter is devoted to the presentation of a generalized operator creating a similar connection in the semistable case.

In their origin, fractional derivatives did not appear as an instrument to investigate stable laws but arose naturally as a generalization of ordinary derivatives in a letter from Leibniz to L'Hôspital in 1695 [77]. However, the connection to stable distributions was only discovered many years later when both - the theory of fractional calculus and the theory of stable laws - were well-established research areas.
On the contrary, we now particularly define a so-called semi-fractional derivative, which creates an equal connection between semistable laws and associated diffusion equations. Like fractional derivatives, the theory of semi-fractional ones can be studied outside of a probability context and, in this sense, may also enrich the general theory of fractional calculus. However, this task is outside the scope of this thesis.

The following chapter starts with introducing semi-fractional derivatives in the one-dimensional case yielding the desired connection between semistable laws and corresponding diffusion equations. Note that under more restrictive assumptions, the results in Section 3.1 have partly been published in [66]. Afterward, we stepwise generalize the idea of semi-fractional derivatives by first considering directional and then multidimensional versions. Particularly, we analyze different forms of these operators like Caputo and Riemann-Liouville forms and their reciprocal connection. Finally, we provide a numerical approach to both directional and multidimensional semi-fractional derivatives in section 3.4.

3.1 Introduction to one-dimensional semi-fractional derivatives

To familiarize the reader with semi-fractional derivatives, we concentrate on a one-dimensional setting in this very first section. Apart from different characterizations - either by Fourier transforms or Caputo and Riemann-Liouville type forms - studying examples improves our comprehension of this generalization of fractional derivatives.

As outlined in the introduction, there are several different approaches to define a fractional derivative. However, as a starting point for our generalization, we affiliate with the most common one in probability theory and use the Fourier space definition. This is, the fractional derivative of order $\alpha > 0$ of a suitable function $f : \mathbb{R} \to \mathbb{R}$ is the function with Fourier transform $(-ik)^{\alpha}\widehat{f}(k)$. Note that in the case $\alpha \in \mathbb{N}$, the fractional derivative simplifies to an ordinary derivative of order n, and hence this definition is a natural extension of the classical one. Although the fractional derivative is defined for every $\alpha > 0$, we restrict our attention to derivatives of order $\alpha \in (0, 2]$ since these cases are connected to stable distributions [94].

In contrast to fractional derivatives, a semi-fractional one not only depends on the order α but also on an additional perturbation modeled by a periodic function. Instead of allowing an arbitrary periodic function in the definition, we demand some qualities, which we consolidate in the term 'admissable'. By doing so, we retain the connection between semi-fractional derivatives and semistable laws (compare Example 5.1.3). For this and subsequent definitions alike, we exclude the integer cases $\alpha = 1$ and $\alpha = 2$, which will be justified below (see Remark 3.1.11).

Definition 3.1.1. (Admissable function)
Given $\alpha \in (0, 2) \setminus \{1\}$ and $c > 1$, we call a function $K : \mathbb{R} \to \mathbb{R}$ admissable with respect to α and c if the following three conditions are satisfied:

(i) K is $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic,

(ii) K is strictly positive,

(iii) the function K fulfills the growth restriction, meaning that $G_K : (0, \infty) \to (0, \infty)$ with $G_K(r) = r^{-\alpha}K(\log(r))$ is non-increasing.

Hereafter, for an admissable function K , let G_K always denote the associated function $G_K : (0, \infty) \to (0, \infty)$ with $G_K(r) = r^{-\alpha}K(\log(r))$.

Most of the time, conditions (i) and (ii) in Definition 3.1.1 are easily shown, but it may be more challenging to verify the growth restriction (iii). Hence, we first prove equivalent criteria for this last requirement.

Lemma 3.1.2. *Let $K : \mathbb{R} \to (0, \infty)$ be a periodic function. The following statements are equivalent:*

(i) The function G_K is non-increasing.

(ii) The function K fulfills

$$K(x+\delta) \leq e^{\alpha\delta} K(x) \quad \textit{and} \quad K(x-\delta) \geq e^{-\alpha\delta} K(x)$$

for every $x > 0$ and $\delta > 0$.

(iii) The function K fulfills $K(x+\delta) \leq e^{\alpha\delta} K(x)$ for every $x > 0$ and $\delta > 0$.

(iv) The function K fulfills $K(x+\delta) \leq e^{\alpha\delta} K(x)$ for every $x \in \mathbb{R}$ and $\delta > 0$.

(v) The function K fulfills $K(x+\delta)e^{-\alpha\delta} \leq K(x-\delta)e^{\alpha\delta}$ for every $x \in \mathbb{R}$ and $\delta > 0$.

If K is differentiable, then each statement is furthermore equivalent to

(vi) The derivative K' of K is bounded from above with $K'(x) \leq \alpha K(x)$ for every $x \in \mathbb{R}$.

Formulation (v) is equal to the assumption (2.6) required for the functions K_1 and K_{-1} in the one-dimensional Lévy measure (2.5) of a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution and thereby already adverts to a connection to semistable laws. In a non-probability context, one could think about much weaker assumptions to define an admissable perturbation for semi-fractional derivatives.

Proof. The equivalence of (i), (ii), and (v) has already been proven in [91, compare p. 275 and 276]. Note that (ii) trivially implies (iii). If (iii) holds true, then using the periodicity of K we obtain (iv). Now (iv) implies (ii) since for $\delta > 0$ and every $x \in \mathbb{R}$

$$K(x) = K(x - \delta + \delta) \leq e^{\alpha\delta} K(x - \delta).$$

It remains to prove each of these statements' equivalence to (vi), and therefore we assume that K is differentiable. If $K'(x) \leq \alpha K(x)$ for every $x \in \mathbb{R}$, then the function G_K is differentiable with derivative

$$\frac{d}{dr} G_K(r) = r^{-\alpha-1}(-\alpha K(\log(r)) + K'(\log(r))) \leq 0$$

for every $r > 0$ and consequently non-increasing. On the other hand, if K is differentiable and fulfills (iv), then

$$\begin{aligned} & K(x+\delta) \leq e^{\alpha\delta} K(x) \\ \Leftrightarrow \quad & \frac{K(x+\delta) - K(x)}{\delta} \leq \frac{e^{\alpha\delta} - 1}{\delta} K(x) \end{aligned}$$

for every $x \in \mathbb{R}$ and $\delta > 0$. Take limits on both sides to obtain

$$K'(x) = \lim_{\delta \downarrow 0} \frac{K(x+\delta) - K(x)}{\delta} \leq \lim_{\delta \downarrow 0} \frac{e^{\alpha\delta} - 1}{\delta} K(x) = \alpha K(x)$$

for every $x \in \mathbb{R}$, completing the proof. $\square$

Using the definition of admissable functions, we can now define semi-fractional derivatives in terms of Fourier transforms.

Definition 3.1.3. (One-dimensional semi-fractional derivative)
Choose $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and a function $K : \mathbb{R} \to (0,\infty)$ being admissable with respect to these parameters. The (positive) semi-fractional derivative of $f \in L^1(\mathbb{R})$ is given by the function with Fourier transform $Dh(k)\widehat{f}(k)$ if this function exists. Thereby h is defined by

$$h(k) := \int_{0+}^{\infty} \left(e^{irk} - \sum_{p=0}^{\lfloor \alpha \rfloor} (irk)^p \right) dG_K(r) \tag{3.1}$$

and $D = D(\alpha) := (-1)^{\lfloor \alpha \rfloor}$. Analogously, the negative semi-fractional derivative can be defined as the function with Fourier transform $Dh(-k)\widehat{f}(k)$ if this function exists.

Before we formulate sufficient conditions for the existence of semi-fractional derivatives, we record some notations and introductory remarks.

Remark 3.1.4. (Notations and introductory remarks)

(i) Since a semi-fractional derivative depends on α and c as well as on the admissable function K, we denote the semi-fractional derivative by the symbol $\dfrac{\partial^\alpha}{\partial_{c,K} x^\alpha}$, whereas we write $\dfrac{\partial^\alpha}{\partial_{c,K} (-x)^\alpha}$ for the negative semi-fractional derivative.

(ii) In many of the following calculations, the cases $\alpha \in (0,1)$ and $\alpha \in (1,2)$ have to be treated separately, and often the results distinguish in their algebraic sign. For this reason, we fix the letter $D = D(\alpha)$ in the above sense for the whole thesis to obtain closed-form expressions.

(iii) The semi-fractional derivative of a suitable function f is defined such that

$$\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f \right)(k) = -D\Psi(k)\widehat{f}(k),$$

where $\Psi(k)$ is the log-characteristic function of the $(c^{\frac{1}{\alpha}}, c)$-semistable law ν on $\mathbb{R}$ from Lemma 2.2.2 with $b = 0$ and $M = \epsilon_1$. Then the semigroup theory outlined in Section 2.3 easily delivers that this semistable density solves a corresponding diffusion equation, showing that this definition accomplishes our preceded goal. A detailed proof is presented in Section 5.1 below.

(iv) Note that in [66], semi-fractional derivatives were introduced by the generator form (see Lemma 3.1.10 below), and it was shown afterward that their Fourier transform is given by $Dh(k)\widehat{f}(k)$ for every $k \in \mathbb{R}$. In this thesis, we decided to reverse this procedure to create comparability with the fractional and the subsequent multidimensional case. Besides, we often used a Fourier series approach in [66], assuming that the function K is smooth, to gain explicit results. As far as possible, we avoid this additional assumption here and only consider this case as an example.

Merely from Definition 3.1.3, it may be unclear how the Fourier transform of a semi-fractional derivative is structured, i.e., how the function h given by the integral (3.1) behaves. The following remark shows that the Fourier transform is indeed not more than a log-periodically disturbed version of the Fourier transform of a fractional derivative.

Remark 3.1.5. (Shape of the Fourier transform of semi-fractional derivatives)
To interpret the Fourier transform of a semi-fractional derivative, define the function $u_1 : \mathbb{R} \to \mathbb{C}$ by $u_1(y) := (-ie^y)^{-\alpha} h(e^y)$. We first show that similar to the perturbation K the so-defined function u_1 is $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic. Note that for every $y \in \mathbb{R}$

$$\begin{aligned} u_1\left(y + \log\left(c^{\frac{1}{\alpha}}\right)\right) &= (-ie^y c^{\frac{1}{\alpha}})^{-\alpha} h(e^y c^{\frac{1}{\alpha}}) \\ &= (-ie^y)^{-\alpha} c^{-1} \int\limits_{0+}^{\infty} \left(e^{ire^y c^{\frac{1}{\alpha}}} - \sum_{p=0}^{\lfloor \alpha \rfloor} (ire^y c^{\frac{1}{\alpha}})^p \right) dG_K(r), \end{aligned}$$

and using the substitution $z := rc^{\frac{1}{\alpha}}$ for Riemann-Stieltjes integrals ([4, Theorem 7.7]) we obtain

$$u_1\left(y + \log\left(c^{\frac{1}{\alpha}}\right)\right) = (-ie^y)^{-\alpha} c^{-1} \int\limits_{0+}^{\infty} \left(e^{ie^y z} - \sum_{p=0}^{\lfloor \alpha \rfloor} (ie^y z)^p \right) dG_K(zc^{-\frac{1}{\alpha}}).$$

Since K is $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic itself, for every $z \in (0, \infty)$, we find

$$\begin{aligned} G_K(zc^{-\frac{1}{\alpha}}) &= (zc^{-\frac{1}{\alpha}})^{-\alpha} K\left(\log\left(zc^{-\frac{1}{\alpha}}\right)\right) \\ &= cz^{-\alpha} K\left(\log(z) - \log\left(c^{\frac{1}{\alpha}}\right)\right) \\ &= cz^{-\alpha} K(\log(z)) \\ &= cG_K(z) \end{aligned}$$

and thereby

$$\begin{aligned} u_1\left(y + \log\left(c^{\frac{1}{\alpha}}\right)\right) &= (-ie^y)^{-\alpha} c^{-1} \int\limits_{0+}^{\infty} \left(e^{ie^y z} - \sum_{p=0}^{\lfloor \alpha \rfloor} (ie^y z)^p \right) c\, dG_K(z) \\ &= u_1(y). \end{aligned}$$

Hence u_1 is a $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic function, and if we rearrange the defining equation for u_1, it follows that $h(k) = (-ik)^{\alpha} u_1(\log(k))$ for every $k > 0$. Analogously with $u_2 : \mathbb{R} \to \mathbb{C}$ defined by $u_2(y) = (ie^y)^{-\alpha}h(-e^y)$, we obtain $h(k) = (-ik)^\alpha u_2(\log(-k))$ for every $k < 0$, and the function u_2 is likewise $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic. Consequently, the Fourier transform behaves like a Fourier transform of a fractional derivative log-periodically disturbed by u_1 or u_2 on the positive and negative real line. Note that the periodic functions u_1, u_2 are continuous as compositions of continuous functions, and thus $C_1 := \max_{x\in\mathbb{R}}\{|u_1(x)|, |u_2(x)|\}$ exists. Then for every $k \in \mathbb{R}$, the Fourier transform of the semi-fractional derivative is bounded with

$$\left|\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}f\right)(k)\right| = \left|Dh(k)\widehat{f}(k)\right| \leq C_1|k|^\alpha|\widehat{f}(k)|. \tag{3.2}$$

Example 3.1.6. For smooth functions K, the integral defining h in (3.1) can be computed explicitly as in Example 2.2.6. This is, if we write

$$K(x) = \sum_{n\in\mathbb{Z}} c_n e^{in\tilde{c}x}$$

for every $x \in \mathbb{R}$ with Fourier coefficients $(c_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ and $\tilde{c} = \frac{2\pi\alpha}{\log(c)}$, then the semi-fractional derivative of a suitable function f is given by the function with Fourier transform $Dh(k)\widehat{f}(k)$, where

$$h(k) = \sum_{n\in\mathbb{Z}} \omega_n(-ik)^{\alpha-in\tilde{c}}$$

and $\omega_n := c_n\Gamma(in\tilde{c} - \alpha + 1)$ for every $n \in \mathbb{Z}$ (compare [66, Remark 3.3]).

Example 3.1.7. (Fractional derivatives)
Searching for a generalization of fractional derivatives, we want to ensure that these operators are a subset of all semi-fractional derivatives. For this purpose, choose $K(x) = \frac{1}{|\Gamma(1-\alpha)|} = \frac{D}{\Gamma(1-\alpha)}$ for every $x \in \mathbb{R}$, which is a smooth admissable function. According to Example 3.1.6, the semi-fractional derivative of a suitable function f is given by the function with Fourier transform $(-ik)^\alpha\widehat{f}(k)$ for every $k \in \mathbb{R}$, which, per definition, is the fractional derivative of order α. In addition, the negative semi-fractional derivative equals the negatively skewed fractional derivative under this choice of K.

To ensure the existence of the Fourier transform $\widehat{f}$ in the definition of semi-fractional derivatives, we necessarily need $f \in L^1(\mathbb{R}^d)$. However, the subsequent lemma shows that we have to demand even more quality of the function to secure the existence of a function with Fourier transform $k \mapsto Dh(k)\widehat{f}(k)$.

Lemma 3.1.8. *Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let K be an admissable function with respect to these parameters. For every $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R})$, the semi-fractional derivative*

$$\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}f(x) = \frac{D}{2\pi i}\int_{\mathbb{R}} e^{-ikx}h(k)\widehat{f}(k)\,dk$$

exists with Fourier transform $Dh(k)\widehat{f}(k)$.

Proof. According to the Lemma of Riemann-Lebesgue (see for example [47, Theorem 8.22]), for $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R})$, we have

$$|\widehat{f}(k)| \leq \frac{C_2}{(1+|k|)^{\lfloor\alpha\rfloor+2}}$$

for a constant $C_2 > 0$. Then with (3.2)

$$\left|\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}f\right)(k)\right| = |h(k)\widehat{f}(k)| \leq C_1C_2\frac{|k|^\alpha}{(1+|k|)^{\lfloor\alpha\rfloor+2}}.$$

Since $\lfloor\alpha\rfloor + 2 > \alpha + 1$, the function $k \mapsto \frac{|k|^\alpha}{(1+|k|)^{\lfloor\alpha\rfloor+2}}$ is integrable for large $|k|$, and the Fourier inversion theorem yields the result. □

Remark 3.1.9. It follows immediately from the proof of Lemma 3.1.8 that the negative semi-fractional derivative exists under the same assumptions.

For many applications and a better intuition of semi-fractional derivatives, we aim to characterize this operator not only in the Fourier space but by its concrete action on a function f alike. In [66, Section 2.1], two different integral representations of semi-fractional derivatives were introduced, which we shortly recall here. The first one is based on the semigroup theory introduced in Section 2.3.

Lemma 3.1.10. *(Generator form of the one-dimensional semi-fractional derivative) Let $\alpha \in (0,2)\setminus\{1\}$, $c > 1$, and let ν be a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution with Lévy-Khintchine triple $[a, 0, \Phi]$, where a is defined as in (2.15) and the Lévy measure Φ is given by*

$$\Phi(-\infty, -r) = 0 \quad \textit{and} \quad \Phi(r, \infty) = G_K(r)$$

for every $r > 0$ and an admissable function K. If L denotes the corresponding generator given in Example 2.3.4, then for every $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R})$, the generator form of the semi-fractional derivative exists with

$$\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}f(x) = -DLf(x) \tag{3.3}$$

for every $x \in \mathbb{R}$. Analogously we obtain the negative semi-fractional derivative by a reflection of the Lévy measure and (3.3).

Note that this integral representation arises immediately from the fact that the Fourier transform of the generator L is given by $\widehat{Lf}(k) = \Psi(k)\widehat{f}(k) = -h(k)\widehat{f}(k)$ with the log-characteristic function Ψ of the underlying law ν. Then $-DLf(x)$ has Fourier transform $Dh(k)\widehat{f}(k)$, and the result is verified with the uniqueness of the Fourier transform.

Utilizing the representation of the generator L in Example 2.3.4 we formulate (3.3) explicitly as follows. Let $\alpha \in (0,2)\setminus\{1\}$, $c > 1$, and let K be an admissable function. For every $f \in W^{\lfloor \alpha \rfloor + 2}(\mathbb{R})$, the generator form of the semi-fractional derivative

$$\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) = D \int_{0+}^{\infty} \left(f(x-y) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p f^{(p)}(x) \right) dG_K(y) \tag{3.4}$$

exists with Fourier transform $Dh(k)\widehat{f}(k)$. The generator form of the negative semi-fractional derivative given by

$$\frac{\partial^\alpha}{\partial_{c,K} (-x)^\alpha} f(x) = D \int_{0+}^{\infty} \left(f(x+y) - \sum_{p=0}^{\lfloor \alpha \rfloor} y^p f^{(p)}(x) \right) dG_K(y)$$

with Fourier transform $Dh(-k)\widehat{f}(k)$ exists under the same assumptions.

Remark 3.1.11. Note that in Definition 3.1.3, the cases $\alpha = 1$ and $\alpha = 2$ were excluded for different reasons. For $\alpha = 1$, we proposed a semi-fractional derivative in [66, Section 2.2]. However, since this case is only a side issue and calculations become rather lengthy, we do not treat it here and refer to the paper only. For $\alpha = 2$, using (3.3), the generator form coincides with an ordinary second-order derivative (see Example 2.3.5), which corresponds to the fact that every 2-semistable distribution is Gaussian (compare Example 2.1.3 (ii)). For this reason, considering $\alpha = 2$ will not yield any new results, and hence we draw no further attention to this particular choice of α.

Remark 3.1.12. As one may expect, positive and negative semi-fractional derivatives are not completely different operators but connected in a certain way. Namely for every function $f \in W^{\lfloor \alpha \rfloor + 2}(\mathbb{R})$ and $x \in \mathbb{R}$, it follows that

$$\frac{\partial^\alpha}{\partial_{c,K} (-x)^\alpha} f(x) = \frac{\partial^\alpha}{\partial_{c,K} x^\alpha} g(-x) \tag{3.5}$$

where $g(x) \coloneqq f(-x)$. This equality is directly obtained from studying the right-hand side of (3.5) given by

$$\begin{aligned} \frac{\partial^\alpha}{\partial_{c,K} x^\alpha} g(-x) &= D \int_{0+}^{\infty} \left(g(-x-y) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p g^{(p)}(-x) \right) dG_K(y) \\ &= D \int_{0+}^{\infty} \left(f(x+y) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p (-1)^p f^{(p)}(x) \right) dG_K(y) \\ &= \frac{\partial^\alpha}{\partial_{c,K} (-x)^\alpha} f(x) \end{aligned}$$

for every $x \in \mathbb{R}$.

Under additional assumptions on f, the generator form yields related forms, specifically the Caputo and the Riemann-Liouville form. For their definition, let $C_0(\mathbb{R}^d)$ be the space

of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with $\lim_{|x|\to\infty} f(x) = 0$. Furthermore, for fixed $n \in \mathbb{N}$, define the space $C_0^n(\mathbb{R}^d)$ of n-times partially differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that f and all partial derivatives up to order n belong to $C_0(\mathbb{R}^d)$.
Then the Caputo form of the semi-fractional derivative is defined as follows. Let $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R})$ and consider the case $\alpha \in (0,1)$ first. Integration by parts for Riemann-Stieltjes integrals (compare [76, Chapter X, Proposition 1.4]) of the generator form (3.4) yields the Caputo form of the semi-fractional derivative

$$\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) = \int_{0+}^{\infty} f'(x-y) G_K(y)\, dy \tag{3.6}$$

(compare [66, Section 2.1]). If instead $\alpha \in (1,2)$, we obtain

$$\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) = \int_{0+}^{\infty} (f'(x) - f'(x-y)) G_K(y)\, dy. \tag{3.7}$$

Remark that in this case with repeated integration by parts,

$$\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) = \int_{0+}^{\infty} f''(x-y) H_K(y)\, dy, \tag{3.8}$$

where similar to G_K, the function $H_K : (0,\infty) \to (0,\infty)$ can be written as $H_K(y) = y^{1-\alpha}\gamma(\log(y))$ and $\gamma : \mathbb{R} \to (0,\infty)$ is a continuous, admissable function with respect to the same parameters $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$ such that

$$\int_{y}^{\infty} G_K(x)\, dx = \int_{y}^{\infty} x^{-\alpha} K(\log(x))\, dx = y^{1-\alpha}\gamma(\log(y)) = H_K(y) \tag{3.9}$$

(see [66, Lemma 2.2]). In addition, if K is smooth with Fourier series representation $K(x) = \sum_{n\in\mathbb{Z}} c_n e^{in\tilde{c}x}$ with $\tilde{c} = \frac{2\pi\alpha}{\log(c)}$ and Fourier coefficients $(c_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$, then γ has the Fourier series representation

$$\gamma(x) = \sum_{n\in\mathbb{Z}} \frac{c_n}{\alpha - 1 - in\tilde{c}} e^{in\tilde{c}x}$$

for every $x \in \mathbb{R}$ [66, Lemma 3.2]. Since we always assume $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R})$, we refer to both (3.7) and (3.8) as Caputo forms for $\alpha \in (1,2)$. Note that using Remark 3.1.12, also negatively skewed forms of the Caputo semi-fractional derivative are defined.

Finally, we present a Riemann-Liouville type form of the semi-fractional derivative not mentioned in [66]. It arises from the Caputo form by a formal change of integration and differentiation; this is for $\alpha \in (0,1)$, we define

$$\left(\frac{\partial}{\partial_{c,K} x}\right)^{\alpha} f(x) := \frac{d}{dx} \int\limits_{0+}^{\infty} f(x-y) G_K(y)\, dy \tag{3.10}$$

to be the Riemann-Liouville form of semi-fractional derivative, whereas for $\alpha \in (1,2)$, we set

$$\left(\frac{\partial}{\partial_{c,K} x}\right)^{\alpha} f(x) := \frac{d^2}{dx^2} \int\limits_{0+}^{\infty} f(x-y) H_K(y)\, dy, \tag{3.11}$$

whenever these integrals exist. In accordance with the fractional case, we define the negative semi-fractional derivative of Riemann-Liouville type by

$$\left(\frac{\partial}{\partial_{c,K} (-x)}\right)^{\alpha} f(x) := -\frac{d}{dx} \int\limits_{0+}^{\infty} f(x+y) G_K(y)\, dy$$

for $\alpha \in (0,1)$ and

$$\left(\frac{\partial}{\partial_{c,K} (-x)}\right)^{\alpha} f(x) := \frac{d^2}{dx^2} \int\limits_{0+}^{\infty} f(x+y) H_K(y)\, dy$$

if $\alpha \in (1,2)$. In general, the Caputo and Riemann-Liouville form do not agree and hence are denoted with different symbols. Their difference will be analyzed in a more general setting in the next section (see Lemma 3.2.12 and Lemma 3.2.15).

Remark 3.1.13. We aim to show that the relationship between positive and negative Caputo forms in Remark 3.1.12 still holds for the Riemann-Liouville semi-fractional derivative.

Consider the case $\alpha \in (0,1)$ first and define $J(x) := \int\limits_{0+}^{\infty} f(x+y) G_K(y)\, dy$. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) := f(-x)$ for every $x \in \mathbb{R}$, then

$$\begin{aligned}\left(\frac{\partial}{\partial_{c,K} x}\right)^{\alpha} g(x) &= \frac{d}{dx} \int\limits_{0+}^{\infty} g(x-y) G_K(y)\, dy \\ &= \frac{d}{dx} \int\limits_{0+}^{\infty} f(-x+y) G_K(y)\, dy = \frac{d}{dx}(J(-x)),\end{aligned}$$

and the chain rule yields

$$\left(\frac{\partial}{\partial_{c,K} x}\right)^{\alpha} g(x) = -\frac{dJ}{dx}(-x).$$

Now insert the point $-x$ to get

$$\left(\frac{\partial}{\partial_{c,K} x}\right)^{\alpha} g(-x) = -\frac{dJ}{dx}(x) = \left(\frac{\partial}{\partial_{c,K}(-x)}\right)^{\alpha} f(x)$$

for every $x \in \mathbb{R}$. Applying the chain rule twice, we obtain the same result for $\alpha \in (1, 2)$.

In analogy to stable and semistable laws, semi-fractional derivatives are more challenging than fractional ones since the log-periodic perturbation complicates the calculations. Nevertheless, the following example shows that considering the more complicated case yields noticeable different results and therefore justifies its studying with all associated troubles.

Example 3.1.14. Let $f : \mathbb{R} \to (0, \infty)$ be defined by $f(x) = e^{-ax^2}$ for some $a > 0$. Then $f \in W^3(\mathbb{R}) \cap C_0^2(\mathbb{R})$, such that the assumptions of Lemma 3.1.8 are fulfilled for any choice of $\alpha \in (0, 2) \setminus \{1\}$. Hence for fixed $\alpha \in (0, 2) \setminus \{1\}$, $c > 1$, and K admissable with respect to these parameters, the Caputo form of the semi-fractional derivative of f exists with Fourier transform $Dh(k)\widehat{f}(k)$. Additionally, the Riemann-Liouville form exists and equals the Caputo form since in (3.10), we can differentiate under the integral due to the exponential decay and smoothness of f.

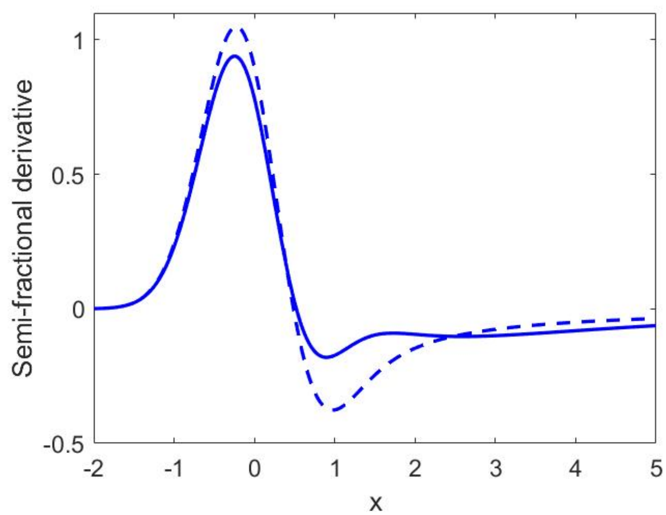

Figure 3.1: Caputo semi-fractional derivative of $f(x) = e^{-2x^2}$ with respect to $\alpha = 0.4$, $c = e^{2\pi\alpha}$, and K as in (3.12) (solid line) in comparison to the fractional case with $K(x) = \frac{1}{\Gamma(1-\alpha)}$ (dashed line) in Example 3.1.14.

Even in this straightforward case, it is impossible to obtain a closed-form expression for the semi-fractional derivative. However, we can calculate the result numerically, as shown in Appendix C for the following choice of parameters. Set $a = 2$, $\alpha = 0.4$, $c = e^{2\pi\alpha}$ as well as

$$K(x) = \frac{1}{\Gamma(1-\alpha)} + \frac{1}{10}\sin(x) + \frac{2}{25}\sin(2x) \tag{3.12}$$

admissable with respect to these parameters. Figure 3.1 shows the resulting semi-fractional derivative of f. For comparison, we furthermore display the associated fractional derivative corresponding to the constant function $K(x) = \frac{1}{\Gamma(1-\alpha)}$. Note that for large values of $|x|$, the fractional and the semi-fractional derivative are comparable due to the fast decay of f. However, for small values of $|x|$, the semi-fractional derivative shows significantly different behavior from the fractional one, and the additional log-periodic disturbance even changes the monotonicity of the resulting derivative. Due to this remarkable distinction, studying semi-fractional derivatives instead of the easier fractional ones may be worth the effort.

As the previous example implies, for most functions $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R})$, we are not able to calculate a closed-form expression of semi-fractional derivatives. However, the Caputo form of semi-fractional derivatives exists under much weaker assumptions on f, and some of these cases are explicitly solvable. The integral defining the Caputo form can even be finite for functions $f \notin L^1(\mathbb{R}^d)$. In this case, f does not have af Fourier transform, and we do not obtain a semi-fractional derivative in the sense of Definition 3.1.3. The following example displays such a case and yet justifies in what sense the calculated object is still a semi-fractional derivative.

Example 3.1.15. Let $\alpha \in (0, 1)$, $c > 1$, and let K be a smooth admissable function with respect to these parameters. Furthermore, denote by $(c_n)_{n\in\mathbb{N}}$ the Fourier coefficients of K and note that due to the smoothness of K, the coefficients are absolutely summable. We study the semi-fractional derivative of $f(x) = x^p \mathbb{1}_{(0,\infty)}(x)$ for some $p > 0$. Clearly, f is not integrable and hence does not satisfy the assumptions in Lemma 3.1.8. Nevertheless, the Caputo form (3.6) exists with

$$\begin{aligned}\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) &= \int_{0+}^{x} p(x-y)^{p-1} y^{-\alpha} K(\log(y))\, dy \\ &= p \int_{0+}^{x} \sum_{n\in\mathbb{Z}} c_n (x-y)^{p-1} y^{-\alpha+in\tilde{c}}\, dy\end{aligned}$$

for every $x > 0$ using the Fourier series representation of K. By dominated convergence, we change integration and summation such that

$$\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) = p \sum_{n\in\mathbb{Z}} c_n \int_{0+}^{x} (x-y)^{p-1} y^{-\alpha+in\tilde{c}}\, dy.$$

Now substitute $z := \frac{y}{x}$ to obtain

$$\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) = px \sum_{n\in\mathbb{Z}} c_n \int_{0+}^{1} (x-xz)^{p-1} (xz)^{-\alpha+in\tilde{c}}\, dz$$

$$
\begin{aligned}
&= p \sum_{n\in\mathbb{Z}} c_n x^{p-\alpha+in\tilde{c}} \int_{0+}^{1} (1-z)^{p-1} z^{-\alpha+in\tilde{c}}\, dz \\
&= p x^{p-\alpha} \sum_{n\in\mathbb{Z}} c_n x^{in\tilde{c}} B(p, 1-\alpha+in\tilde{c}),
\end{aligned}
$$

where $B(x,y)$ denotes the beta function

$$
B(x,y) = \int_0^1 (1-t)^{x-1} t^{y-1}\, dt
$$

for every $x, y \in \mathbb{C}$ with $\mathrm{Re}(x), \mathrm{Re}(y) > 0$. Define $\zeta : (0,\infty) \to \mathbb{R}$ by

$$
\zeta(x) = \sum_{n\in\mathbb{Z}} c_n B(p, 1-\alpha+in\tilde{c}) e^{in\tilde{c}x}
$$

and note that ζ is a well-defined, real-valued, and $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic function. Finally, ζ is strictly positive since, with dominated convergence, it follows that

$$
\begin{aligned}
\zeta(x) &= \int_0^1 \sum_{n\in\mathbb{Z}} c_n t^{p-1} (1-t)^{-\alpha+in\tilde{c}} e^{in\tilde{c}x}\, dt \\
&= \int_0^1 t^{p-1} (1-t)^{-\alpha} K(\log(1-t)+x)\, dt > 0
\end{aligned}
$$

for every $x \in \mathbb{R}^d$. Altogether, the Caputo form of the semi-fractional derivative is given by

$$
\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) = p x^{p-\alpha} \zeta(\log(x)),
$$

and hence the Caputo semi-fractional derivative of f oscillates around a multiple of $x^{p-\alpha}$. Especially, we obtain the fractional derivative of order α corresponding to $K_1(x) = \frac{1}{\Gamma(1-\alpha)}$ as

$$
\frac{\partial^\alpha}{\partial x^\alpha} f(x) = p x^{p-\alpha} \frac{\Gamma(p)}{\Gamma(p+1-\alpha)} = x^{p-\alpha} \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}.
$$

For $\alpha = \frac{1}{2}$ and two different choices of p ($p = 0.3$ and $p = 0.6$), Figure 3.2 displays the semi-fractional derivative with perturbation

$$
\begin{aligned}
K_2(x) &= \frac{1}{10}(\sin(2x) + \cos(x)) + \frac{1}{\Gamma(1-\alpha)} \\
&= \frac{1}{10}(\sin(2x) + \cos(x)) + K_1(x)
\end{aligned}
$$

in comparison to the fractional derivative corresponding to the constant function K_1. Using Lemma 3.1.2, the function K_2 is admissable and oscillates around K_1. Both plots are shown on a double logarithmic scale, such that the fractional derivative is a straight line with slope $p - \alpha$. The Matlab code for the calculation is given in Appendix C.

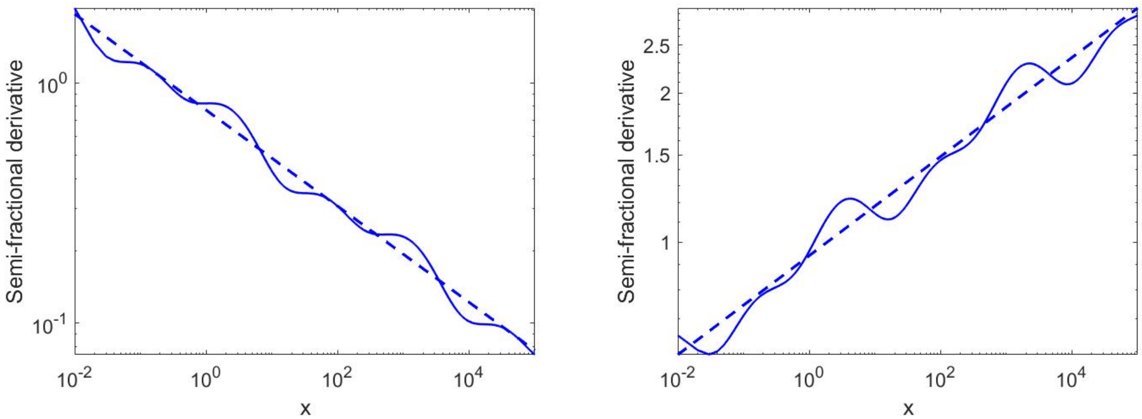

Figure 3.2: Semi-fractional derivative with respect to $\alpha = \frac{1}{2}$, $c = e^{2\pi\alpha}$, and K_2 (solid line) for $p = 0.3$ (left) and $p = 0.6$ (right) on a double logarithmic scale in comparison to the fractional derivative corresponding to K_1 (dashed line) in Example 3.1.15.

Similar to the Caputo form, the Riemann-Liouville form of the semi-fractional derivative exists and equals the Caputo form, which can be derived either by direct calculation or by the general result in Lemma 3.2.12.
Note that with $f \notin L^1(\mathbb{R})$, the semi-fractional derivative of f does not exist in the sense of Definition 3.1.3, and the above results only show the existence and form of the integral (3.6). However, this statement can be strengthened in the following way. If we consider f and the Fourier transform in the distributional sense, then $\widehat{f}(k) = \Gamma(p+1)(-ik)^{-p-1}$ [58, Example 7.1.17]. On the other hand, we can calculate the Fourier transform of the Caputo form in the distributional sense. Therefore, let ϕ be an element of the Schwartz space $\mathcal{S}(\mathbb{R})$ consisting of all rapidly decreasing functions. Then we obtain

$$
\begin{aligned}
\int\limits_{\mathbb{R}} \mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f\right)(x)\phi(x)\,dx &= \int\limits_{\mathbb{R}} \left(\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f\right)(x)\widehat{\phi}(x)\,dx \\
&= \int\limits_0^\infty p x^{p-\alpha}\zeta(\log(x))\widehat{\phi}(x)\,dx \\
&= p\int\limits_0^\infty \sum_{n\in\mathbb{Z}} c_n B(p, 1-\alpha+in\tilde{c}) x^{p-\alpha+in\tilde{c}}\widehat{\phi}(x)\,dx.
\end{aligned}
$$

Using the absolute convergence of the Fourier coefficients of ζ and the fact that with ϕ, the Fourier transform $\widehat{\phi}$ is likewise rapidly decreasing, we change the order of integration and summation yielding

$$\begin{aligned}\int\limits_{\mathbb{R}} \mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K}x^\alpha} f\right)(x)\phi(x)\,dx &= p\sum_{n\in\mathbb{Z}} c_n B(p, 1-\alpha+in\tilde{c}) \int\limits_0^\infty x^{p-\alpha+in\tilde{c}}\widehat{\phi}(x)\,dx \\ &= p\sum_{n\in\mathbb{Z}} c_n B(p, 1-\alpha+in\tilde{c}) \int\limits_{\mathbb{R}} \mathcal{F}(x^{p-\alpha+in\tilde{c}}\mathbb{1}_{(0,\infty)}(x))\phi(x)\,dx.\end{aligned}$$

According to [58, Example 7.1.17], the Fourier transform of $g(x) = x^{p-\alpha+in\tilde{c}}\mathbb{1}_{(0,\infty)}(x)$ in the distributional sense is given by $\Gamma(p-\alpha+in\tilde{c}+1)(-ix)^{-p+\alpha-in\tilde{c}-1}$ and hence

$$\int\limits_{\mathbb{R}} \mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K}x^\alpha} f\right)(x)\phi(x)\,dx = \sum_{n\in\mathbb{Z}} c_n\Gamma(p+1)\Gamma(1-\alpha+in\tilde{c}) \int\limits_{\mathbb{R}} (-ix)^{\alpha-p-in\tilde{c}-1}\phi(x)\,dx.$$

Changing integration and summation once again, we finally get

$$\begin{aligned}\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K}x^\alpha} f\right)(k) &= \Gamma(p+1)(-ik)^{-p-1}\sum_{n\in\mathbb{Z}} c_n\Gamma(in\tilde{c}-\alpha+1)(-ik)^{\alpha-in\tilde{c}} \\ &= \widehat{f}(k)h(k)\end{aligned}$$

as stated in Example 3.1.6. At least, the definition of semi-fractional derivatives is fulfilled in this weaker, distributional sense, and one may thereby still talk about a semi-fractional derivative.

3.2 Directional semi-fractional derivatives

So far, we only considered one-dimensional functions $f : \mathbb{R} \to \mathbb{R}$ and their semi-fractional derivatives. However, real-world applications can rarely be reduced to such a one-dimensional setting, and therefore we have to think about multidimensional generalizations of fractional derivatives. This is, we need to extend the definition of fractional derivatives to fractional equivalents of multidimensional differential operators such as the gradient or the Laplace operator.

Even in the special case of fractional derivatives, whose one-dimensional forms have been known for many decades, this is a relatively new and unexplored research area. The first approach toward a fractional gradient was given by Ben Adda [19], who replaced the ordinary derivative in the gradient's definition with a Sonine-Liouville fractional one (sometimes also called Nishimoto derivative; see for example [120, Section 22]). Based on the variety of one-dimensional fractional derivatives, several different suggestions for multidimensional differential operators appeared in the following years (see for example [41], [31], [137], or the overview given in [138]). Recently, Meerschaert et al. proposed a definition of fractional gradients and related operators based on directional fractional derivatives in [87] (also compare [89]). Due to its generality, their definition includes several known forms, and therefore we apply a similar technique in this thesis to obtain

a multidimensional semi-fractional differential operator.
The basic idea in [89] is to display the multivariable fractional derivative as a mixture of directional fractional ones, so this section is devoted to define and analyze the semi-fractional counterpart. Guided by [89], this will yield the multidimensional semi-fractional derivative in Section 3.3.

Recall that for a function $f : \mathbb{R}^d \to \mathbb{R}$, the ordinary directional derivative along a given vector $\theta \in S$, where again S is the d-dimensional unit sphere, is given by

$$\partial_\theta f(x) := \langle \theta, \nabla f(x) \rangle = \sum_{i=1}^{d} \theta_i \frac{\partial}{\partial x_i} f(x) = \frac{\partial b}{\partial s}(x,s)\Big|_{s=0},$$

where $b : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is defined by $b(x,s) := f(x+s\theta)$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner dot product on $\mathbb{R}^d$. Similarly, we define directional semi-fractional derivatives.

Definition 3.2.1. (Directional semi-fractional derivative)
Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let K be an admissable function with respect to these parameters. In addition, fix a unit vector $\theta \in S$. If this function exists, the directional semi-fractional derivative along θ is given by

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) := \frac{\partial^\alpha}{\partial_{c,K} s^\alpha} b(x,s)\Big|_{s=0}, \tag{3.13}$$

where $b : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is defined by $b(x,s) := f(x+s\theta)$ and the one-dimensional derivative is the generator form of the semi-fractional derivative (3.4). We will refer to this derivative as the generator form of the directional semi-fractional derivative in the following.

Using that

$$\frac{\partial}{\partial s} b(x,s) = \langle \theta, \nabla f(x+s\theta) \rangle = \partial_\theta f(x+s\theta),$$

the generator form of the semi-fractional derivative in (3.4) reads as

$$\begin{aligned}
\frac{\partial^\alpha}{\partial_{c,K} s^\alpha} b(x,s) &= D \int_{0+}^{\infty} \left(b(x,s-y) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p b^{(p)}(x,s) \right) dG_K(y) \\
&= D \int_{0+}^{\infty} \left(f(x+(s-y)\theta) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \partial_\theta^{(p)} f(x+s\theta) \right) dG_K(y)
\end{aligned}$$

with $\partial_\theta^{(0)} f := f$ and $\partial_\theta^{(1)} f := \partial_\theta f$ for every $\theta \in S$ and $x \in \mathbb{R}^d$. Now evaluate this expression in $s = 0$ to gain the following explicit representation of the directional semi-fractional derivative.

Lemma 3.2.2. *(Explicit representation of the generator form)*
Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let K be an admissable function with respect to these

parameters. In addition, let $\theta \in S$ be a fixed unit vector. An explicit representation of the directional semi-fractional derivative along θ is given by

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = D \int_{0+}^{\infty} \left(f(x - y\theta) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right) dG_K(y) \tag{3.14}$$

for every $x \in \mathbb{R}^d$.

Remark 3.2.3. Our considerations are inspired by [89] and [88], in which the authors studied a directional fractional derivative defined by

$$\frac{\partial_\theta^\alpha}{\partial x^\alpha} f(x) = \frac{\partial^\alpha}{\partial x^\alpha} b(x,s)\Big|_{s=0}$$

with b as in Definition 3.2.1. By choosing $K(x) = \frac{1}{|\Gamma(1-\alpha)|}$, their operator is included in our more general definition.

Example 3.2.4. (Connection to one-dimensional semi-fractional derivatives)
Consider the one-dimensional case $S = \{-1, 1\}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that semi-fractional derivatives of every order exist. For $\theta = 1$, it follows that

$$\frac{\partial_1^\alpha}{\partial_{c,K} x^\alpha} f(x) = D \int_{0+}^{\infty} \left(f(x - y) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \frac{d^p}{dx^p} f(x) \right) dG_K(y) = \frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x)$$

such that the directional semi-fractional derivative equals the generator form of the positive semi-fractional derivative. In the same way, we obtain equality of the directional semi-fractional derivative along $\theta = -1$ with the negative form of the semi-fractional derivative.

As in the one-dimensional case, the directional semi-fractional derivative exists under smoothness assumptions on f, but additionally, we need some qualities of the resulting derivative. Using the generator form, we can prove the following result similar to [88, Lemma 2.4].

Lemma 3.2.5. *Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let K be an admissable function with respect to these parameters. The directional semi-fractional derivative along $\theta \in S$ of $f \in W^{\lfloor \alpha \rfloor + 2}(\mathbb{R}^d) \cap C_0^{\lfloor \alpha \rfloor + 1}(\mathbb{R}^d)$ exists and*

$$\int_{\mathbb{R}^d} \left| \frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) \right| dx \leq C$$

for a constant $C > 0$ independent of θ.

Proof. First, note that for $f \in W^{\lfloor \alpha \rfloor + 2}(\mathbb{R}^d) \cap C_0^{\lfloor \alpha \rfloor + 1}(\mathbb{R}^d)$, the semi-fractional derivative of $b(x,s) = f(x+s\theta)$ in (3.13) exists according to Lemma 3.1.8. To prove that the directional semi-fractional derivative is integrable, use Lemma 3.2.2 yielding

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \left| \frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) \right| dx &= \int\limits_{\mathbb{R}^d} \left| D \int\limits_{0+}^{\infty} \left(f(x-y\theta) - \sum_{p=0}^{\lfloor\alpha\rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right) dG_K(y) \right| dx \\ &\leq I_1(x) + I_2(x)\end{aligned}$$

for every $x \in \mathbb{R}^d$, where I_1 and I_2 are defined by

$$I_1(x) := \int\limits_{\mathbb{R}^d} \left| D \int\limits_{0+}^{1} \left(f(x-y\theta) - \sum_{p=0}^{\lfloor\alpha\rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right) dG_K(y) \right| dx$$

$$I_2(x) := \int\limits_{\mathbb{R}^d} \left| D \int\limits_{1}^{\infty} \left(f(x-y\theta) - \sum_{p=0}^{\lfloor\alpha\rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right) dG_K(y) \right| dx.$$

Considering the integral $I_2(x)$ first, we use Tonelli's theorem and find

$$\begin{aligned}I_2(x) &\leq \int\limits_{1}^{\infty} \int\limits_{\mathbb{R}^d} \left| f(x-y\theta) - \sum_{p=0}^{\lfloor\alpha\rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right| dx\, dG_K(y) \\ &\leq \int\limits_{1}^{\infty} \left(||f||_1 + \sum_{p=0}^{\lfloor\alpha\rfloor} y^p ||\partial_\theta^{(p)} f||_1 \right) dG_K(y) \\ &= 2||f||_1 \int\limits_{1}^{\infty} dG_K(y) + ||\partial_\theta f||_1 \left(\int\limits_{1}^{\infty} y\, dG_K(y) \right) \mathbb{1}_{(1,2)}(\alpha) \\ &= 2||f||_1 \int\limits_{1}^{\infty} dG_K(y) + ||\nabla f||_1 \left(\int\limits_{1}^{\infty} y\, dG_K(y) \right) \mathbb{1}_{(1,2)}(\alpha).\end{aligned}$$

Since $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d)$, it follows that $||f||_1, ||\partial_\theta f||_1 < \infty$. Furthermore, G_K behaves like $G_K(y) \sim y^{-\alpha}$ for large values of y, and hence the first integral is finite. With the same argument, the second one is finite for $\alpha \in (1,2)$. Summarizing, we get $I_2(x) < \infty$ for every $x \in \mathbb{R}^d$.
It remains to show that I_1 is likewise finite. With $\alpha < 2$, we can rewrite $I_1(x)$ as

$$I_1(x) = \int\limits_{\mathbb{R}^d} \left| D \int\limits_{0+}^{1} \left(f(x-y\theta) - \sum_{p=0}^{\lfloor\alpha\rfloor} \frac{(-y)^p}{p!} \partial_\theta^{(p)} f(x) \right) dG_K(y) \right| dx.$$

As f is $(\lfloor\alpha\rfloor + 1)$-times partially differentiable with continuous derivatives, use a Taylor series expansion with integral representation of the remainder to obtain

$$f(x-y\theta)-\sum_{p=0}^{\lfloor\alpha\rfloor}\frac{(-y)^p}{p!}\partial_\theta^{(p)}f(x)=\frac{(-1)^{\lfloor\alpha\rfloor+1}}{\lfloor\alpha\rfloor!}\int\limits_0^y\partial_\theta^{(\lfloor\alpha\rfloor+1)}f(x-s\theta)(y-s)^{\lfloor\alpha\rfloor}\,ds.$$

Inserting this expression in $I_1(x)$ yields

$$\begin{aligned}I_1(x)&=\int\limits_{\mathbb{R}^d}\int\limits_{0+}^1\left|\frac{(-1)^{\lfloor\alpha\rfloor+1}}{\lfloor\alpha\rfloor!}\int\limits_0^y\partial_\theta^{(\lfloor\alpha\rfloor+1)}f(x-s\theta)(y-s)^{\lfloor\alpha\rfloor}\,ds\right|\,dG_K(y)\,dx\\&\leq\frac{1}{\lfloor\alpha\rfloor!}\int\limits_{\mathbb{R}^d}\int\limits_{0+}^1\int\limits_0^y\left|\partial_\theta^{(\lfloor\alpha\rfloor+1)}f(x-s\theta)(y-s)^{\lfloor\alpha\rfloor}\right|\,ds\,dG_K(y)\,dx.\end{aligned}$$

Finally, apply Tonelli's theorem to obtain

$$\begin{aligned}I_1&\leq\frac{1}{\lfloor\alpha\rfloor!}\int\limits_{0+}^1\int\limits_0^y\int\limits_{\mathbb{R}^d}\left|\partial_\theta^{(\lfloor\alpha\rfloor+1)}f(x-s\theta)\right|\,dx\,(y-s)^{\lfloor\alpha\rfloor}\,ds\,dG_K(y)\\&\leq\frac{1}{\lfloor\alpha\rfloor!}||\partial_\theta^{(\lfloor\alpha\rfloor+1)}f||_1\int\limits_{0+}^1\int\limits_0^y(y-s)^{\lfloor\alpha\rfloor}\,ds\,dG_K(y)\\&=\frac{1}{\lfloor\alpha\rfloor!}||\partial_\theta^{(\lfloor\alpha\rfloor+1)}f||_1\int\limits_{0+}^1\left[-\frac{1}{\lfloor\alpha\rfloor+1}(y-s)^{\lfloor\alpha\rfloor+1}\right]_0^y\,dG_K(y)\\&=\frac{1}{(\lfloor\alpha\rfloor+1)!}||\partial_\theta^{(\lfloor\alpha\rfloor+1)}f||_1\int\limits_{0+}^1y^{\lfloor\alpha\rfloor+1}\,dG_K(y).\end{aligned}$$

Now $||\partial_\theta^{(\lfloor\alpha\rfloor+1)}f||_1<\infty$ due to our assumption on f and the norm is bounded independent of θ by the norm of the gradient or Hessian matrix respectively. Additionally, $y^{\lfloor\alpha\rfloor+1}$ is integrable around zero with respect to G_K. Hence $I_1(x)<\infty$, which finishes the proof. □

When considering the d-dimensional sphere with respect to the Euclidean norm, we can connect directional semi-fractional derivatives for different choices of θ similar to Remark 3.1.12 as the following lemma shows.

Lemma 3.2.6. *Let S be the d-dimensional sphere with respect to the Euclidean norm. Furthermore, choose $\alpha\in(0,2)\setminus\{1\}$, $c>1$, and let K be an admissable function with respect to theses parameters. In addition, let $r:\mathbb{R}^d\to\mathbb{R}^d$ be a reflection, this is $r(x)=Ax$ for an orthogonal, symmetric matrix $A\in\mathbb{R}^{d\times d}$ and every $x\in\mathbb{R}^d$. Then for a fixed unit* *vector* $\theta \in S$

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = \frac{\partial_{r(\theta)}^\alpha}{\partial_{c,K} x^\alpha} g(r(x)), \tag{3.15}$$

where $g(x) := f(r(x))$ *for every* $x \in \mathbb{R}^d$.

Proof. First note that with $\theta \in S$ and the fact that r is orthogonal, we have $r(\theta) \in S$, and the directional derivative is well-defined. We evaluate the right-hand side of (3.15). In view of (3.14), for every $x \in \mathbb{R}^d$

$$\begin{aligned}
\frac{\partial_{r(\theta)}^\alpha}{\partial_{c,K} x^\alpha} g(x) &= D \int_{0+}^{\infty} \left(g(x - yr(\theta)) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \partial_{r(\theta)}^{(p)} g(x) \right) dG_K(y) \\
&= D \int_{0+}^{\infty} \Big(g(x - yr(\theta)) - g(x) + y \partial_{r(\theta)} g(x) \mathbb{1}_{(1,2)}(\alpha) \Big) \, dG_K(y) \\
&= D \int_{0+}^{\infty} \Big(f(r(x) - y\theta) - f(r(x)) + y \langle r(\theta), \nabla g(x) \rangle \mathbb{1}_{(1,2)}(\alpha) \Big) \, dG_K(y)
\end{aligned}$$

since r is linear and self-inverse. For $\alpha \in (1,2)$, we analyze the gradient of g. Utilizing the chain rule, ∇g is given by

$$\nabla g(x) = D_r(x)^T \nabla f(r(x))$$

for every $x \in \mathbb{R}^d$, where D_r is the Jacobian matrix of $x \mapsto r(x)$. However, r is given by $r(x) = Ax$ with A symmetric such that

$$\nabla g(x) = A \nabla f(r(x)).$$

This yields

$$\begin{aligned}
\langle r(\theta), \nabla g(x) \rangle &= \langle A\theta, A \nabla f(r(x)) \rangle \\
&= \langle \theta, \nabla f(r(x)) \rangle \\
&= \partial_\theta f(r(x))
\end{aligned}$$

using the orthogonality and symmetry of A once again. Altogether, we find

$$\frac{\partial_{r(\theta)}^\alpha}{\partial_{c,K} x^\alpha} g(x) = D \int_{0+}^{\infty} \Big(f(r(x) - y\theta) - f(r(x)) + y \partial_\theta f(r(x)) \mathbb{1}_{(1,2)}(\alpha) \Big) \, dG_K(y)$$

for every $x \in \mathbb{R}^d$. Finally evaluate this expression in $r(x)$ to obtain

$$\frac{\partial^\alpha_{r(\theta)}}{\partial_{c,K} x^\alpha} g(r(x)) = D \int\limits_{0+}^{\infty} \Big(f(x - y\theta) - f(x) + y\partial_\theta f(x) \mathbb{1}_{(1,2)}(\alpha) \Big) \, dG_K(y)$$
$$= \frac{\partial^\alpha_\theta}{\partial_{c,K} x^\alpha} f(x)$$

for every $x \in \mathbb{R}^d$ with (3.14). □

Remark 3.2.7. In one dimension, choose $r(x) = -x$ for every $x \in \mathbb{R}$. Then r is a reflection as defined in Lemma 3.2.6, and according to Example 3.2.4 and Lemma 3.2.6, it follows that

$$\frac{\partial^\alpha}{\partial_{c,K}(-x)^\alpha} f(x) = \frac{\partial^\alpha_{-1}}{\partial_{c,K} x^\alpha} f(x) = \frac{\partial^\alpha_{1}}{\partial_{c,K} x^\alpha} g(-x) = \frac{\partial^\alpha}{\partial_{c,K} x^\alpha} g(-x)$$

for every $x \in \mathbb{R}$, where $g(x) = f(-x)$. Hence we regain the result of Remark 3.1.12.

Apart from the explicit representation of the directional semi-fractional derivative, we need to evaluate its Fourier transform.

Lemma 3.2.8. *(Fourier transform of directional semi-fractional derivatives)*
Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let K be an admissable function with respect to these parameters. In addition, let $f \in W^{\lfloor \alpha \rfloor + 2}(\mathbb{R}^d) \cap C_0^{\lfloor \alpha \rfloor + 1}(\mathbb{R}^d)$. Then for every fixed $\theta \in S$,

$$\mathcal{F}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K} x^\alpha} f \right)(k) = Dh(\langle k, \theta \rangle) \widehat{f}(k)$$

for every $k \in \mathbb{R}^d$, where h is given by (3.1).

Proof. According to (3.4), the generator form of the semi-fractional derivative of $b(x,s) = f(x + s\theta)$ is given by

$$\frac{\partial^\alpha}{\partial_{c,K} s^\alpha} b(x,s) = D \int\limits_{0+}^{\infty} \left(b(x, s-y) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p b^{(p)}(x,s) \right) \, dG_K(y).$$

Applying a d-dimensional Fourier transform in $k \in \mathbb{R}^d$ yields

$$\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K} s^\alpha} b \right)(k,s) = D \int\limits_{\mathbb{R}^d} e^{i\langle k, x \rangle} \int\limits_{0+}^{\infty} \left(b(x, s-y) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p b^{(p)}(x,s) \right) \, dG_K(y) \, dx$$
$$= D \int\limits_{\mathbb{R}^d} e^{i\langle k, x \rangle} \int\limits_{0+}^{\infty} \left(f(x + (s-y)\theta) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \frac{\partial^p}{\partial s^p} f(x + s\theta) \right) dG_K(y) \, dx.$$

In view of Lemma 3.2.5, we can switch the order of integration with Fubini's theorem such that

$$\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K} s^\alpha} b\right)(k,s) = D\int_{0+}^{\infty}\int_{\mathbb{R}^d} e^{i\langle k,x\rangle}\left(f(x+(s-y)\theta) - \sum_{p=0}^{\lfloor\alpha\rfloor}(-y)^p \frac{\partial^p}{\partial s^p} f(x+s\theta)\right) dx\, dG_K(y).$$

With the linearity of the Fourier transform, we evaluate the inner integral term by term. First notice that with the component-wise substitution $z = x + u$, it results that

$$\int_{\mathbb{R}^d} e^{i\langle k,x\rangle} f(x+u)\, dx = e^{-i\langle k,u\rangle}\int_{\mathbb{R}^d} e^{i\langle k,z\rangle} f(z)\, dz = e^{-i\langle k,u\rangle}\widehat{f}(k)$$

for every $u \in \mathbb{R}^d$. Additionally, for $\alpha \in (1,2)$, we have

$$\int_{\mathbb{R}^d} e^{i\langle k,x\rangle}\frac{\partial}{\partial s} f(x+s\theta)\, dx = \frac{\partial}{\partial s}\widehat{f}(k)e^{-i\langle k,s\theta\rangle} = -i\langle k,\theta\rangle\widehat{f}(k)e^{-i\langle k,s\theta\rangle}$$

using dominated convergence. Then the Fourier transform is given by

$$\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K} s^\alpha} b\right)(k,s) = D\int_{0+}^{\infty}\left(\widehat{f}(k)e^{-i\langle k,(s-y)\theta\rangle} - \sum_{p=0}^{\lfloor\alpha\rfloor}(iy\langle k,\theta\rangle)^p\widehat{f}(k)e^{-i\langle k,s\theta\rangle}\right) dG_K(y).$$

Evaluate this expression in $s = 0$ to obtain

$$\begin{aligned}\mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,\theta} s^\alpha} b\right)(k,0) &= D\widehat{f}(k)\int_{0+}^{\infty}\left(e^{iy\langle k,\theta\rangle} - \sum_{p=0}^{\lfloor\alpha\rfloor}(iy\langle k,\theta\rangle)^p\right) dG_K(y)\\ &= Dh(\langle k,\theta\rangle)\widehat{f}(k)\end{aligned}$$

with h as in (3.1). By definition, it follows that

$$\mathcal{F}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K} x^\alpha} f\right)(k) = \mathcal{F}\left(\frac{\partial^\alpha}{\partial_{c,K} s^\alpha} b\right)(k,0) = Dh(\langle k,\theta\rangle)\widehat{f}(k)$$

for every $k \in \mathbb{R}^d$. $\square$

As for the one-dimensional derivative, the generator form is the basis for other semi-fractional derivatives, namely the Caputo and Riemann-Liouville form.

Lemma 3.2.9. *(Caputo form of directional semi-fractional derivatives)*
Let $\alpha \in (0,2)\setminus\{1\}$, $c > 1$, and let K be an admissable function with respect to these parameters. In addition, fix a unit vector $\theta \in S$. For every $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d)\cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R}^d)$, the generator form of the directional semi-fractional derivative exists and coincides with

the Caputo form

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = D \int_{0+}^{\infty} \left(\partial_\theta f(x-y\theta) - \partial_\theta f(x) \mathbb{1}_{(1,2)}(\alpha)\right) G_K(y)\, dy \tag{3.16}$$

for every $x \in \mathbb{R}^d$*.*

Proof. Using integration by parts for Riemann-Stieltjes integrals [76, Chapter X, Proposition 1.4], we get

$$\begin{aligned}\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) =& D \int_{0+}^{\infty} \left(f(x-y\theta) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right) \, dG_K(y) \\ =& D \left[\left(f(x-y\theta) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right) G_K(y) \right]_{r=0+}^{\infty} \\ &- D \int_{0+}^{\infty} \left(\langle \nabla f(x-y\theta), -\theta \rangle + \sum_{p=1}^{\lfloor \alpha \rfloor} p(-y)^{p-1} \partial_\theta^{(p)} f(x) \right) G_K(y)\, dy.\end{aligned}$$

With $f \in C_0^{\lfloor \alpha \rfloor + 1}(\mathbb{R}^d)$, use a Taylor approximation yielding

$$\begin{aligned} f(x-y\theta) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \partial_\theta^{(p)} f(x) &= f(x-y\theta) - f(x) + y \partial_\theta f(x) \mathbb{1}_{(1,2)}(\alpha) \\ &\leq C_3 y^{\lfloor \alpha \rfloor + 1} \end{aligned}$$

for a constant $C_3 > 0$ such that with the boundedness of K, it follows that

$$\lim_{y \to 0} \left| \left(f(x-y\theta) - \sum_{p=0}^{\lfloor \alpha \rfloor} (-y)^p \partial_\theta^{(p)} f(x) \right) G_k(y) \right| \leq C_3 \lim_{y \to 0} y^{\lfloor \alpha \rfloor + 1} y^{-\alpha} K(\log(y)) = 0.$$

Note that with $f \in C_0^{\lfloor \alpha \rfloor + 1}(\mathbb{R}^d)$, the evaluation at ∞ vanishes as well and hence

$$\begin{aligned}\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) &= -D \int_{0+}^{\infty} \left(\langle \nabla f(x-y\theta), -\theta \rangle + \sum_{p=1}^{\lfloor \alpha \rfloor} p(-y)^{p-1} \partial_\theta^{(p)} f(x) \right) G_K(y)\, dy \\ &= -D \int_{0+}^{\infty} \left(\langle \nabla f(x-y\theta), -\theta \rangle + \partial_\theta f(x) \mathbb{1}_{(1,2)}(\alpha) \right) G_K(y)\, dy \\ &= D \int_{0+}^{\infty} \left(\partial_\theta f(x-y\theta) - \partial_\theta f(x) \mathbb{1}_{(1,2)}(\alpha) \right) G_K(y)\, dy.\end{aligned}$$

□

As in the one-dimensional semi-fractional case, repeated integration by parts yields the equivalent form of the directional semi-fractional derivative for $\alpha \in (1,2)$

$$\begin{aligned}\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) &= \int\limits_{0+}^{\infty} \Big(\partial_\theta f(x - y\theta) - \partial_\theta f(x) \mathbb{1}_{(1,2)}(\alpha) \Big) \, dH_K(y) \\ &= \int\limits_{0+}^{\infty} \langle \theta, \mathcal{H}_f(x - y\theta)\theta \rangle H_K(y) \, dy \end{aligned} \tag{3.17}$$

for every $x \in \mathbb{R}^d$, where $\mathcal{H}_f$ is the Hessian matrix of f and H_K is given by (3.9). Again, we will refer to both formulas as Caputo form. A formal change of integration and differentiation finally yields the Riemann-Liouville form of the directional semi-fractional derivative.

Definition 3.2.10. (Riemann-Liouville form of directional semi-fractional derivatives) Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let K be an admissable function with respect to these parameters. For $\alpha \in (0,1)$ and $\theta \in S$, we define

$$\left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^\alpha f(x) := \left\langle \nabla \int\limits_{0+}^{\infty} f(x - y\theta) G_K(y) \, dy, \theta \right\rangle \tag{3.18}$$

for every $x \in \mathbb{R}^d$ to be the Riemann-Liouville form of the directional semi-fractional derivative, if this function exists. Whenever this function exists, let the Riemann-Liouville form of the semi-fractional derivative of order $\alpha \in (1,2)$ be defined by

$$\left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^\alpha f(x) := \langle \theta, \mathcal{H}_{I(f)}(x)\theta \rangle, \tag{3.19}$$

where $I(f) : \mathbb{R}^d \to \mathbb{R}$ is the function $I(f)(x) := \int\limits_{0+}^{\infty} f(x - y\theta) H_K(y) \, dy$ with H_K as introduced in (3.9).

Remark 3.2.11. First note that the definition of Riemann-Liouville directional derivatives in one dimension coincides with the definitions (3.10) and (3.11) if we choose $\theta = 1$. Besides, the relation between different directional derivatives obtained in Lemma 3.2.6 still holds for the Riemann-Liouville form of the directional semi-fractional derivative, which is proven in the following.

Consider the case $\alpha \in (0,1)$ first and define $J(x) := \int\limits_{0+}^{\infty} f(x - y\theta) G_K(y) \, dy$ for every $x \in \mathbb{R}^d$. In addition, let $r : \mathbb{R}^d \to \mathbb{R}^d$ be a reflection as given in Lemma 3.2.6. We want to verify that

$$\left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^\alpha f(x) = \left(\frac{\partial_{r(\theta)}}{\partial_{c,K} x} \right)^\alpha g(r(x)) \tag{3.20}$$

for every $x \in \mathbb{R}^d$, where $g(x) := f(r(x))$. According to (3.18), the right-hand side of (3.20) equals

$$\begin{aligned}\left(\frac{\partial_{r(\theta)}}{\partial_{c,K}x}\right)^{\alpha} g(x) &= \left\langle \nabla \int\limits_{0+}^{\infty} g(x - yr(\theta))G_K(y)\,dy, r(\theta)\right\rangle \\ &= \left\langle \nabla \int\limits_{0+}^{\infty} f(r(x) - y\theta)G_K(y)\,dy, r(\theta)\right\rangle \\ &= \langle \nabla(J(r(x))), r(\theta)\rangle\end{aligned}$$

for every $x \in \mathbb{R}^d$. Applying the chain rule to the gradient yields

$$\nabla(J(r(x))) = D_r^T(x)\nabla J(r(x)) = A\nabla J(r(x)),$$

where $D_r(z) = A$ is the Jacobian of $x \mapsto r(x) = Ax$ with A symmetric. Using the orthogonality of A, we furthermore get

$$\left(\frac{\partial_{r(\theta)}}{\partial_{c,K}x}\right)^{\alpha} g(x) = \left\langle A\nabla J(r(x)), A\theta\right\rangle = \left\langle \nabla J(r(x)), \theta\right\rangle$$

for every $x \in \mathbb{R}^d$. Finally evaluate this expression in $r(x)$ to obtain

$$\left(\frac{\partial_{r(\theta)}}{\partial_{c,K}x}\right)^{\alpha} g(r(x)) = \left\langle \nabla J(x), \theta\right\rangle = \left(\frac{\partial_{\theta}}{\partial_{c,K}x}\right)^{\alpha} f(x).$$

Now let $\alpha \in (1, 2)$. Again we aim to show (3.20). First note that with (3.19),

$$\left(\frac{\partial_{r(\theta)}}{\partial_{c,K}x}\right)^{\alpha} g(x) = \langle A\theta, \mathcal{H}_{I(g)}(x)A\theta\rangle = \langle \theta, A\mathcal{H}_{I(g)}(x)A\theta\rangle$$

for every $x \in \mathbb{R}^d$ since A is symmetric. To analyze the Hessian matrix, consider the term

$$\begin{aligned}I(g)(x) &= \int\limits_{0+}^{\infty} g(x - yr(\theta))H_K(y)\,dy \\ &= \int\limits_{0+}^{\infty} f(r(x) - y\theta)H_K(y)\,dy = I(f)(r(x)) = I(f)(Ax).\end{aligned}$$

Then using the chain rule and the symmetry of A, the Hessian matrix of $I(f)$ is given by

$$H_{I(g)}(x) = AH_{I(f)}(r(x))A.$$

Finally, using the orthogonality of A once again we have

$$\left(\frac{\partial_{r(\theta)}}{\partial_{c,K} x}\right)^{\alpha} g(x) = \langle \theta, A\mathcal{H}_{I(g)}(x)A\theta\rangle = \langle \theta, H_{I(f)}(r(x))\rangle$$

for every $x \in \mathbb{R}^d$. Evaluating this expression in $r(x)$ yields the result.

To close this section, we discuss the differences between Caputo and Riemann-Liouville forms of directional semi-fractional derivatives. As a special case, we gain the difference in the one-dimensional setting, and as promised, we thereby close the gap in Section 3.1.

In the above theorems, we deal with functions $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R}^d)$. Under these restrictive assumptions, differentiation under the integral in the Riemann-Liouville form shows that both forms yield identical results. However, for many physical or financial applications, such a model is unsuitable. For instance, a one-dimensional time-dependent experiment is typically modeled by a function $f : [0, \infty) \to \mathbb{R}$. We can extend f to a function on the whole real line by setting $f(t) = 0$ for $t < 0$, but this extension might not be differentiable or even continuous in $t = 0$. Nevertheless, the one-dimensional semi-fractional Caputo derivative

$$\frac{\partial^{\alpha}}{\partial_{c,K} t^{\alpha}} f(t) = \int\limits_{0+}^{t} f'(x-y) G_K(y)\, dy$$

for $\alpha \in (0, 1)$ or

$$\frac{\partial^{\alpha}}{\partial_{c,K} t^{\alpha}} f(t) = \int\limits_{0+}^{t} f''(x-y) H_K(y)\, dy$$

for $\alpha \in (1, 2)$ exists under certain assumptions on f. Similarly, the Riemann-Liouville form exists for sufficiently smooth functions f having support on the positive half-line $[0, \infty)$. Since functions of this type are of great interest, we aim to analyze how a discontinuity in $t = 0$ changes the difference between both forms.
To obtain a result as general as possible, return to the multidimensional setting and consider the space $\mathbb{R}^d_+ = (0, \infty)^d$. In the following, we denote by $\overline{\mathbb{R}^d_+}$ the closure of $\mathbb{R}^d_+$. Additionally, let $C_0(\overline{\mathbb{R}^d_+})$ be the space of functions $f : \mathbb{R}^d_+ \to \mathbb{R}$ which are continuous on $\mathbb{R}^d_+$, can be extended continuously to $\overline{\mathbb{R}^d_+}$, and for which $\lim_{|x|\to\infty} f(x) = 0$. For $n \in \mathbb{N}$, denote by $C_0^n(\overline{\mathbb{R}^d_+})$ the set of all functions which are n-times partially differentiable on $\mathbb{R}^d_+$ such that all partial derivatives lie in $C_0(\overline{\mathbb{R}^d_+})$.
We first analyze the difference between both forms in the case $\alpha \in (0, 1)$.

Lemma 3.2.12. (*Difference between Caputo and Riemann-Liouville form, $\alpha \in (0, 1)$*) *Let $\alpha \in (0, 1)$, $c > 1$, and let K be an admissable function with respect to these parameters. In addition, choose $\theta \in S$ with $\theta_i > 0$ for at least one $i \in \{1, \dots, d\}$ and let*

$f \in W^{\lfloor \alpha \rfloor + 2}(\mathbb{R}_+^d) \cap C_0^{\lfloor \alpha \rfloor + 1}(\overline{\mathbb{R}_+^d})$ be such that the Riemann-Liouville form of the directional semi-fractional derivative exists. Denote with $a = a(x) \in \mathbb{R}^d$ the boundary point of $\mathbb{R}_+^d$ we reach first when moving from x in direction $-\theta$. Then for every $x \in \mathbb{R}_+^d$, we have

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = \left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^\alpha f(x) - f(a) \, G_K(||x - a||).$$

Proof. Consider the Riemann-Liouville form of the directional semi-fractional derivative

$$\left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^\alpha f(x) = \sum_{i=1}^d \theta_i \frac{\partial}{\partial x_i} \int\limits_{0+}^{\infty} f(x - y\theta) G_K(y) \, dy$$

for every $x \in \mathbb{R}_+^d$. In the one-dimensional case and for $\theta = 1$, this integral is indeed finite as f is supported on $[0, \infty)$. We aim to show that a similar result holds for the multidimensional case. Note that we integrate $f(x - y\theta) G_K(y)$ along $y \in (0, \infty)$. However, f is supported on $\overline{\mathbb{R}_+^d}$, and studying $f(x - y\theta)$, where θ has at least one positive component, there is a minimal $r_{\min} > 0$ with $a := x - r_{\min}\theta \in \partial \mathbb{R}_+^d$. Here and in the following, $\partial \mathbb{R}_+^d$ denotes the boundary of $\mathbb{R}_+^d$. For every $y > r_{\min}$, we have $x - y\theta \in \mathbb{R}^d \setminus \overline{\mathbb{R}_+^d}$, and hence the integral is finite with

$$\left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^\alpha f(x) = \sum_{i=1}^d \theta_i \frac{\partial}{\partial x_i} \int\limits_{0+}^{r_{\min}} f(x - y\theta) G_K(y) \, dy.$$

Furthermore, the boundary of $\mathbb{R}_+^d$ is given by

$$\{x \in \mathbb{R}^d : x_i = 0 \text{ for at least one } i \in \{1, \ldots, d\}\}$$

such that we obtain

$$r_{\min} = \min_{\substack{i=1,\ldots,d, \\ \theta_i > 0}} \frac{x_i}{\theta_i} =: \frac{x_l}{\theta_l},$$

which exists due to our assumptions on θ. Since f and all partial derivatives of f are continuous on $\mathbb{R}_+^d$, we can apply Leibniz's rule to obtain

$$\begin{aligned}
\left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^\alpha f(x) &= \sum_{i=1}^d \theta_i \left(\int\limits_{0+}^{\frac{x_l}{\theta_l}} \frac{\partial}{\partial x_i} f(x - y\theta) G_K(y) \, dy + f\left(x - \frac{x_l}{\theta_l}\theta \right) G_K\left(\frac{x_l}{\theta_l} \right) \theta_l^{-1} \frac{dx_l}{dx_i} \right) \\
&= \left(\sum_{i=1}^d \theta_i \int\limits_{0+}^{\frac{x_l}{\theta_l}} \frac{\partial}{\partial x_i} f(x - y\theta) G_K(y) \, dy \right) + f\left(x - \frac{x_l}{\theta_l}\theta \right) G_K\left(\frac{x_l}{\theta_l} \right)
\end{aligned}$$

$$= \frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) + f\left(x - \frac{x_l}{\theta_l}\theta\right) G_K\left(\frac{x_l}{\theta_l}\right)$$

for every $x \in \mathbb{R}_+^d$ using (3.16). Recall that $a = x - r_{\min}\theta = x - \frac{x_l}{\theta_l}\theta$ and note that

$$\frac{x_l}{\theta_l} = \left|\frac{x_l}{\theta_l}\right| = \left|\left|\frac{x_l}{\theta_l}\theta\right|\right| = ||x - a|| \tag{3.21}$$

to finally obtain

$$\left(\frac{\partial_\theta}{\partial_{c,K} x}\right)^\alpha f(x) = \frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) + f\left(a\right) G_K\left(||x - a||\right)$$

for every $x \in \mathbb{R}_+^d$. □

Example 3.2.13. In the fractional case, $K(x) = \frac{1}{\Gamma(1-\alpha)}$ such that under the assumptions of Lemma 3.2.12, the difference between the fractional Riemann-Liouville and Caputo form is given by

$$\frac{\partial_\theta^\alpha}{\partial x^\alpha} f(x) = \left(\frac{\partial_\theta}{\partial x}\right)^\alpha f(x) - \frac{1}{\Gamma(1-\alpha)} f\left(a\right) ||x - a||^{-\alpha}$$

for every $x \in \mathbb{R}_+^d$. Especially in one dimension and for $\theta = 1$, this yields a relation between one-dimensional positive fractional derivatives

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \left(\frac{\partial}{\partial x}\right)^\alpha f(x) - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} f\left(0\right).$$

Note that this formula for the one-dimensional case is already known by [139, (17.37)].

Example 3.2.14. (Difference in the one-dimensional case)
In the one-dimensional case, the semi-fractional derivative coincides with the directional derivative for $\theta = 1$ (compare Example 3.2.4) and we obtain

$$\begin{aligned}\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f(x) &= \left(\frac{\partial}{\partial_{c,K} x}\right)^\alpha f(x) - f\left(0\right) G_K\left(x\right) \\ &= \left(\frac{\partial}{\partial_{c,K} x}\right)^\alpha f(x) - f\left(0\right) x^{-\alpha} K(\log(x))\end{aligned}$$

for every $x > 0$.

In the case $\alpha \in (1, 2)$, the argumentation works similarly, but now the difference consists of two terms.

Lemma 3.2.15. (*Difference between Caputo and Riemann-Liouville form, $\alpha \in (1, 2)$*)
Let $\alpha \in (1, 2)$, $c > 1$, and let K be a smooth and admissable function with respect to these parameters. In addition, choose $\theta \in S$ with $\theta_i > 0$ for at least one $i \in \{1, \ldots, d\}$ and let

$f \in W^{\lfloor\alpha\lfloor+2}(\mathbb{R}_+^d) \cap C_0^{\lfloor\alpha\lfloor+1}(\overline{\mathbb{R}_+^d})$ be such that the Riemann-Liouville form of the directional semi-fractional derivative exists. Denote with $a = a(x) \in \mathbb{R}^d$ the boundary point of $\mathbb{R}_+^d$ which we reach first when moving from x in direction $-\theta$. Then

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = \left(\frac{\partial_\theta}{\partial_{c,K} x}\right)^\alpha f(x) + f\left(a\right) G_K\left(||x-a||\right) - \langle \theta, \nabla f(a)\rangle H_K\left(||x-a||\right)$$

for every $x \in \mathbb{R}_+^d$.

Proof. We show the statement for $x \in (\delta, \infty)^d$. The result then follows for every $x \in \mathbb{R}_+^d$ by considering the limit $\delta \to 0$. Recall the construction in the proof of Lemma 3.2.12; this is, let $a = x - r_{\min}\theta$ be the boundary point we reach first when moving from x in direction $-\theta$. Thereby, $r_{\min}$ is given by

$$r_{\min} \coloneqq \min_{\substack{i=1,\dots,d,\\ \theta_i>0}} \frac{x_i}{\theta_i} = \frac{x_l}{\theta_l}.$$

This time, we start with the Caputo form of the directional semi-fractional derivative, which now reads as the finite integral

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = \sum_{i,j=1}^d \theta_i\theta_j \int_{0+}^{\frac{x_l}{\theta_l}} \frac{\partial^2}{\partial x_i \partial x_j} f(x - r\theta) H_K(r)\, dr.$$

For further calculations, we need the argument of f to be bounded away from the boundary of $\mathbb{R}_+^d$. Hence we write

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = \lim_{\epsilon\downarrow 0} \sum_{i,j=1}^d \theta_i\theta_j \int_{0+}^{\frac{x_l}{\theta_l}-\epsilon} \frac{\partial^2}{\partial x_i \partial x_j} f(x - r\theta) H_K(r)\, dr. \tag{3.22}$$

To simplify the notation, define $\Upsilon : \mathbb{R}_+^d \times (0,\infty) \to \mathbb{R}$ by $\Upsilon(x,s) = f(x - s\theta)H_K(s)$ and note that Υ is continuously differentiable. We evaluate the integral on the right-hand side of (3.22) first. For every fixed $\epsilon > 0$ and $i, j \in \{1, \dots, d\}$ apply Leibniz's rule to the integral yielding

$$\int_{0+}^{\frac{x_l}{\theta_l}-\epsilon} \frac{\partial^2}{\partial x_i \partial x_j} \Upsilon(x,r)\, dr = \frac{\partial}{\partial x_i} \int_{0+}^{\frac{x_l}{\theta_l}-\epsilon} \frac{\partial}{\partial x_j} \Upsilon(x,r)\, dr - D_{i,j}^{1,\epsilon}(x)$$

with

$$D_{i,j}^{1,\epsilon}(x) = \frac{\partial \Upsilon}{\partial x_j}\left(x, \frac{x_l}{\theta_l} - \epsilon\right) \theta_l^{-1} \frac{dx_l}{dx_i} = \frac{\partial \Upsilon}{\partial x_j}\left(x, \frac{x_l}{\theta_l} - \epsilon\right) \theta_l^{-1} \delta_{l,i}$$

for every $x \in \mathbb{R}^d_+$. Note that

$$\begin{aligned}\sum_{i,j=1}^{d} \theta_i \theta_j D^{1,\epsilon}_{i,j}(x) &= \sum_{j=1}^{d} \theta_j \frac{\partial \Upsilon}{\partial x_j} \left(x, \frac{x_l}{\theta_l} - \epsilon \right) \\ &= \sum_{j=1}^{d} \theta_j \frac{\partial}{\partial x_j} f \left(x - \left(\frac{x_l}{\theta_l} - \epsilon \right) \theta \right) H_K \left(\frac{x_l}{\theta_l} - \epsilon \right) \\ &\to \sum_{j=1}^{d} \theta_j \frac{\partial f}{\partial x_j} \left(x - \frac{x_l}{\theta_l} \theta \right) H_K \left(\frac{x_l}{\theta_l} \right),\end{aligned}$$

as $\epsilon \downarrow 0$ since the partial derivatives of f and H_K are continuous. Inserting the definition of the boundary point a and (3.21), we obtain

$$\begin{aligned}\sum_{i,j=1}^{d} \theta_i \theta_j D^{1,\epsilon}_{i,j}(x) &\to \sum_{j=1}^{d} \theta_j \frac{\partial f}{\partial x_j} (a) \, H_K \left(||x - a|| \right) \\ &= \langle \theta, \nabla f(a) \rangle H_K \left(||x - a|| \right)\end{aligned}$$

for every $x \in \mathbb{R}^d_+$. Repeat this procedure to obtain

$$\begin{aligned}\int_{0+}^{\frac{x_l}{\theta_l} - \epsilon} \frac{\partial^2}{\partial x_i \partial x_j} \Upsilon(x, r) \, dr &= \frac{\partial}{\partial x_i} \int_{0+}^{\frac{x_l}{\theta_l} - \epsilon} \frac{\partial}{\partial x_j} \Upsilon(x, r) \, dr - D^{1,\epsilon}_{i,j} \\ &= \frac{\partial^2}{\partial x_i \partial x_j} \int_{0+}^{\frac{x_l}{\theta_l} - \epsilon} \Upsilon(x, r) \, dr - D^{1,\epsilon}_{i,j}(x) - D^{2,\epsilon}_{i,j}(x)\end{aligned}$$

with

$$D^{2,\epsilon}_{i,j}(x) = \frac{\partial}{\partial x_i} \left(\Upsilon \left(x, \frac{x_l}{\theta_l} - \epsilon \right) \theta_l^{-1} \delta_{j,l} \right).$$

For $i \neq l$, it follows that

$$\begin{aligned}D^{2,\epsilon}_{i,l}(x) &= \frac{\partial}{\partial x_i} \left(f \left(x - \left(\frac{x_l}{\theta_l} - \epsilon \right) \theta \right) H_K \left(\frac{x_l}{\theta_l} - \epsilon \right) \theta_l^{-1} \right) \\ &= \frac{\partial f}{\partial x_i} \left(x - \left(\frac{x_l}{\theta_l} - \epsilon \right) \theta \right) H_K \left(\frac{x_l}{\theta_l} - \epsilon \right) \theta_l^{-1},\end{aligned} \tag{3.23}$$

whereas for $i = l$, we obtain

$$\begin{aligned}D^{2,\epsilon}_{l,l}(x) &= \frac{\partial}{\partial x_l} \left(\Upsilon \left(x, \frac{x_l}{\theta_l} - \epsilon \right) \theta_l^{-1} \right) \\ &= \theta_l^{-1} \frac{\partial}{\partial x_l} \left(f \left(x - \left(\frac{x_l}{\theta_l} - \epsilon \right) \theta \right) H_K \left(\frac{x_l}{\theta_l} - \epsilon \right) \right).\end{aligned}$$

Using (3.23) yields

$$\frac{\partial}{\partial x_l}\left(f\left(x-\left(\frac{x_l}{\theta_l}-\epsilon\right)\theta\right)\right)=\sum_{\substack{m=1,\\ m\neq l}}^{d}\frac{\partial f}{\partial x_m}\left(x-\left(\frac{x_l}{\theta_l}-\epsilon\right)\theta\right)\left(-\theta_m\theta_l^{-1}\right)$$
$$=-H_K\left(\frac{x_l}{\theta_l}-\epsilon\right)^{-1}\sum_{\substack{m=1,\\ m\neq l}}^{d}\theta_m D_{m,l}^{2,\epsilon},$$

and since

$$\frac{\partial}{\partial x_l}H_K\left(\frac{x_l}{\theta_l}-\epsilon\right)=-G_K\left(\frac{x_l}{\theta_l}-\epsilon\right)\theta_l^{-1},$$

the derivative is given by

$$D_{l,l}^{2,\epsilon}(x)=-\theta_l^{-1}\sum_{\substack{m=1,\\ m\neq l}}^{d}\theta_m D_{m,l}^{2,\epsilon}-\theta_l^{-2}f\left(x-\left(\frac{x_l}{\theta_l}-\epsilon\right)\theta\right)G_K\left(\frac{x_l}{\theta_l}-\epsilon\right).$$

Summing all $D_{i,j}^{2,\epsilon}$, we find

$$\sum_{i,j=1}^{d}\theta_i\theta_j D_{i,j}^{2,\epsilon}=\sum_{i=1}^{d}\theta_i\theta_l D_{i,l}^{2,\epsilon}$$
$$=-f\left(x-\left(\frac{x_l}{\theta_l}-\epsilon\right)\theta\right)G_K\left(\frac{x_l}{\theta_l}-\epsilon\right).$$

Since G_K is assumed to be continuous and f is likewise continuous, this yields

$$\sum_{i,j=1}^{d}\theta_i\theta_j D_{i,j}^{2,\epsilon}\to -f\left(a\right)G_K\left(||x-a||\right)$$

as $\epsilon\downarrow 0$ for every $x\in\mathbb{R}_+^d$. To finish the proof, it remains to show that

$$\lim_{\epsilon\downarrow 0}\frac{\partial^2}{\partial x_i\partial x_j}\int_{0+}^{\frac{x_l}{\theta_l}-\epsilon}\Upsilon(x,r)\,dr=\frac{\partial^2}{\partial x_i\partial x_j}\int_{0+}^{\frac{x_l}{\theta_l}}\Upsilon(x,r)\,dr$$

for every $x\in\mathbb{R}_+^d$. Starting with the right-hand side, we have

$$\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^{\alpha}f(x)=\frac{\partial^2}{\partial x_i\partial x_j}\int_{0+}^{\frac{x_l}{\theta_l}}\Upsilon(x,r)\,dr=\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\lim_{\epsilon\downarrow 0}\int_{0+}^{\frac{x_l}{\theta_l}-\epsilon}\Upsilon(x,r)\,dr\right).$$

The functions $g_{1,\epsilon}(x) = \int\limits_{0+}^{\frac{x_l}{\theta_l}-\epsilon} \Upsilon(x,r)\,dr$ are continuously partially differentiable with

$$\frac{\partial}{\partial x_j} g_{1,\epsilon}(x) = \int\limits_{0}^{\frac{x_l}{\theta_l}-\epsilon} \frac{\partial}{\partial x_j}\Upsilon(x,r)\,dr$$

for every $j \neq l$ and

$$\frac{\partial}{\partial x_l} g_{1,\epsilon}(x) = \int\limits_{0}^{\frac{x_l}{\theta_l}-\epsilon} \frac{\partial}{\partial x_l}\Upsilon(x,r)\,dr + \theta_l^{-1}\Upsilon\left(x, \frac{x_l}{\theta_l} - \epsilon\right).$$

In addition, we have $g_{1,\epsilon}(x) \to g_1(x)$ for every $x \in \mathbb{R}^d_+$, where $g_1(x) = \int\limits_{0+}^{\frac{x_l}{\theta_l}} \Upsilon(x,r)\,dr$. We furthermore need to prove that the partial derivatives of $g_{1,\epsilon}$ converge uniformly to those of g_1. For $j \neq l$

$$\left|\frac{\partial}{\partial x_j} g_{1,\epsilon}(x) - \frac{\partial}{\partial x_j} g_1(x)\right| = \left|\int\limits_{\frac{x_l}{\theta_l}-\epsilon}^{\frac{x_l}{\theta_l}} \frac{\partial}{\partial x_j} f(x - r\theta) H_K(r)\,dr\right| \leq C_4\epsilon$$

for a constant $C_4 > 0$ since the partial derivatives of f are bounded and H_K is bounded on every closed interval $[\delta - \epsilon, \infty)$. Similarly, for $i = l$, we have

$$\begin{aligned}\left|\frac{\partial}{\partial x_l} g_{1,\epsilon}(x) - \frac{\partial}{\partial x_l} g(x)\right| &= \left|\int\limits_{\frac{x_l}{\theta_l}-\epsilon}^{\frac{x_l}{\theta_l}} \frac{\partial}{\partial x_l} f(x - r\theta) H_K(r)\,dr + \theta_l^{-1}\Upsilon\left(x, \frac{x_l}{\theta_l} - \epsilon\right) - \theta_l^{-1}\Upsilon\left(x, \frac{x_l}{\theta_l}\right)\right| \\ &\leq C_4\epsilon + |\theta_l^{-1}| \left|\Upsilon\left(x, \frac{x_l}{\theta_l} - \epsilon\right) - \Upsilon\left(x, \frac{x_l}{\theta_l}\right)\right|.\end{aligned}$$

Note that $\Upsilon(x,r)$ is differentiable with respect to r, and according to the mean value theorem, there is $x_0 \in \left[\frac{x_l}{\theta_l} - \epsilon, \frac{x_l}{\theta_l}\right]$ such that

$$\Upsilon\left(x, \frac{x_l}{\theta_l} - \epsilon\right) - \Upsilon\left(x, \frac{x_l}{\theta_l}\right) = \epsilon \frac{\partial \Upsilon}{\partial r}(x, x_0).$$

The derivative of Υ is given by

$$\frac{\partial \Upsilon}{\partial r}(x,r) = \langle -\theta, \nabla f(x - r\theta)\rangle - f(x - r\theta) G_K(r),$$

which is bounded for every $x \in \mathbb{R}^d_+$ and $r \in (\frac{\delta}{\theta_l} - \epsilon, \infty)$. Hence the derivatives of $g_{1,\epsilon}$

converge uniformly to those of g_1 and thereby

$$\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f(x) = \frac{\partial}{\partial x_i}\left(\lim_{\epsilon\downarrow 0}\frac{\partial}{\partial x_j}\int_{0+}^{\frac{x_l}{\theta_l}-\epsilon}\Upsilon(x,r)\,dr\right).$$

Now use the fact that with $f \in C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}_+^d})$, the second order partial derivatives of f are bounded likewise and due to our assumptions, G_K is differentiable. Then we can repeat the above procedure such that

$$\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f(x) = \lim_{\epsilon\downarrow 0}\left(\frac{\partial^2}{\partial x_i\partial x_j}\int_{0+}^{\frac{x_l}{\theta_l}-\epsilon}\Upsilon(x,r)\,dr\right),$$

and the result follows.

□

Example 3.2.16. (Difference in the one-dimensional case)
Consider the one-dimensional case with $\theta = 1$. Then under the assumptions of Lemma 3.2.15, the (positive) Caputo and Riemann-Liouville forms of the semi-fractional derivative are connected by

$$\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}f(x) = \left(\frac{\partial}{\partial_{c,K}x}\right)^\alpha f(x) + f(0)\,G_K(x) - f'(0)\,H_K(x)$$

for every $x > 0$. Especially for $K = -\frac{1}{\Gamma(1-\alpha)}$, corresponding to the fractional case, this yields

$$\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}f(x) = \left(\frac{\partial}{\partial_{c,K}x}\right)^\alpha f(x) - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}f(0) - \frac{x^{-\alpha+1}}{\Gamma(2-\alpha)}f'(0)$$

coinciding with [139, (17.37)].

To close this section, we display an example illustrating the difference between Caputo and Riemann-Liouville forms.

Example 3.2.17. Consider a two-dimensional setting ($d = 2$) and let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be defined by $f(x,y) = (x^2+y)\mathbb{1}_{\mathbb{R}_+^2}(x,y)$. In addition, let $\alpha = \frac{2}{3}$, $c = e^{2\pi\alpha}$, and

$$K(x) = \sin(x) + \cos(2x) + 5$$

admissable with respect to these parameters. If we choose $\theta = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then the directional semi-fractional derivative of f along θ exists in the Caputo as well as in the Riemann-Liouville sense. We calculate both forms using (3.16) and (3.18) as well as numerical integration and differentiation (see Appendix C for the Matlab code). The result is shown in Figure 3.3. The forms differ clearly, especially when the point of calculation is close to the boundary of $\mathbb{R}^2_+$. Note that this coincides with Lemma 3.2.12 since the term

$$G_K(||x-a||) = ||x-a||^{-\alpha} K(\log(||x-a||))$$

in the difference between both forms diverges for $x \to a$. On the contrary, the forms coincide on the diagonal line $y = x$ because when moving from (x, x) in direction $-\theta = -\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, we end up in $a = (0, 0)$, and the difference vanishes with $f(0, 0) = 0$. In Figure 3.4, we additionally plot both forms restricted to the diagonal line (x, x) to show their equality.

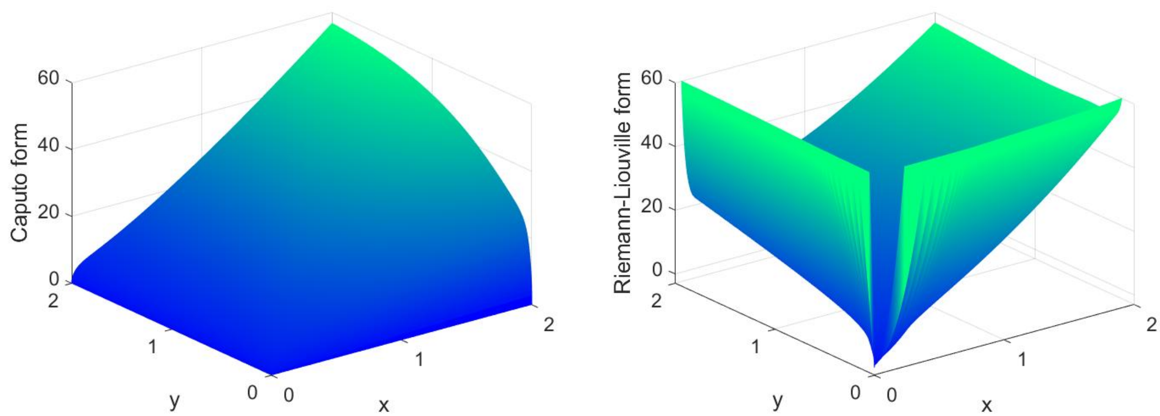

Figure 3.3: Comparison between Caputo (left) and Riemann-Liouville form (right) of the directional semi-fractional derivative in Example 3.2.17.

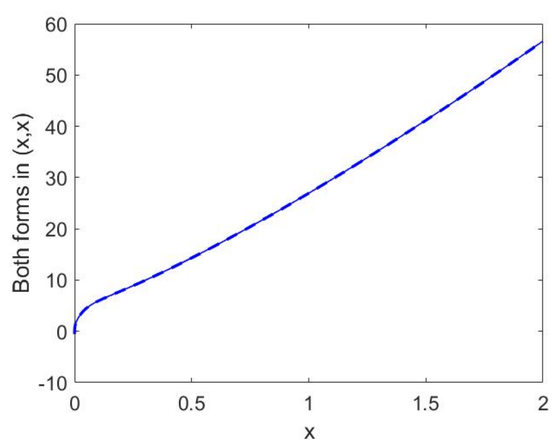

Figure 3.4: Comparison between Caputo (dashed line) and Riemann-Liouville form (solid line) on the diagonal line (x, x) in Example 3.2.17.

3.3 Multidimensional semi-fractional derivatives

The main reason for studying directional semi-fractional derivatives in the previous chapter was the aspiration toward semi-fractional differential operators for multivariable functions. Since we follow the idea of [87], we aim to define a multidimensional semi-fractional derivative as a mixture of directional ones in this chapter.
In detail, the concept in the fractional case is to define the multidimensional fractional derivative of order $\alpha \in (0,2] \setminus \{1\}$ of a suitable function $f : \mathbb{R}^d \to \mathbb{R}$ as the function with Fourier transform

$$\widehat{\mathbb{D}^{\alpha,M} f}(k) := \left(\int_S (-i\langle k, \theta \rangle)^\alpha \, dM(\theta) \right) \widehat{f}(k), \tag{3.24}$$

where M is a probability measure on the unit sphere S (compare [94],[89], or [87]). However, (3.24) is an integral over Fourier transforms of directional fractional derivatives taken in each radial direction and thus represents a mixture of directional fractional derivatives. Consequently, the measure M is called a mixing measure ([87] and [24]). Here we derive a similar formula for semi-fractional derivatives.

The first step toward such a generalization is to extend the concept of admissable functions to a set of functions $(K_\theta)_{\theta \in S}$.

Definition 3.3.1. (Admissable set of functions)
Let $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$ be fixed. A set of functions $(K_\theta)_{\theta \in S}$ is called admissable with respect to α and c if for every $\theta \in S$, the function K_θ is either admissable in the sense of Definition 3.1.1 or identically zero, but there is at least one $\theta \in S$ such that K_θ is non-zero.

As in the fractional case, we define the semi-fractional derivative using the Fourier space representation.

Definition 3.3.2. (Multidimensional semi-fractional derivative)
Choose $\alpha \in (0,2) \setminus \{1\}$, $c > 1$ as well as a set of admissable functions $(K_\theta)_{\theta \in S}$ and a probability measure M on the unit sphere S. The semi-fractional derivative of $f \in L^1(\mathbb{R}^d)$ with respect to these parameters is given by the function with Fourier transform

$$D \int_S h_\theta(\langle k, \theta \rangle) \, dM(\theta) \, \widehat{f}(k) \tag{3.25}$$

with h_θ as in (2.8) if this function exists. In accordance with directional semi-fractional derivatives, we denote the multidimensional operator with $\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta \in S}}$. However, note that the parameters depend on each other, and all of them uniquely describe the Lévy measure Φ of a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution and vice versa.

Remark 3.3.3. In Remark 3.1.5, it was shown that for every $\theta \in S$

$$h_\theta(\langle k, \theta \rangle) \leq C_\theta |\langle k, \theta \rangle|^\alpha \leq C_\theta |k|^\alpha$$

for a constant $C_\theta > 0$. If $\int\limits_S C_\theta \, dM(\theta) < \infty$ and $f \in W^{\lfloor \alpha \rfloor + d + 1}(\mathbb{R}^d)$, then the multidimensional semi-fractional derivative exists with Fourier transform (3.25) by the Fourier inversion theorem.

Example 3.3.4. If $K_\theta(x) = \frac{1}{|\Gamma(1-\alpha)|}$ for every $x \in \mathbb{R}$ and $\theta \in S$, then the set $(K_\theta)_{\theta \in S}$ is admissable, and according to Example 2.2.5, we have

$$\begin{aligned}
\mathcal{F}(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta \in S}} f)(k) &= D \int\limits_S h_\theta(\langle k, \theta \rangle) \, dM(\theta) \, \widehat{f}(k) \\
&= \int\limits_S (-i\langle k, \theta \rangle)^\alpha \, dM(\theta) \, \widehat{f}(k) \\
&= \mathcal{F}(\mathbb{D}^{\alpha,M} f)(k).
\end{aligned}$$

Hence the semi-fractional derivative coincides with the fractional derivative in this case.

Example 3.3.5. In one dimension, the unit sphere consists of only two points such that

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta \in S}} f(x) = M\{1\} \frac{\partial^\alpha}{\partial_{c,K_1} x^\alpha} f(x) + M\{-1\} \frac{\partial^\alpha}{\partial_{c,K_{-1}} (-x)^\alpha} f(x),$$

and the semi-fractional derivative is a weighted mixture of one-dimensional semi-fractional derivatives.

Remark 3.3.6. (Connection to classical multidimensional differential operators)
In classical analysis, partial and the related directional derivatives of a multivariable function $f : \mathbb{R}^d \to \mathbb{R}$ are well-known, but there is no such thing as a multidimensional derivative. Instead, the gradient or the Laplace operator are suitable tools to consider multidimensional settings. We shortly want to classify our result into the set of those typical multidimensional operators.
If M is a discrete measure concentrated on finitely many points $\theta_1, \ldots, \theta_n \in S$, then

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta \in S}} f(x) = \sum_{j=1}^n M\{\theta_j\} \frac{\partial^\alpha_{\theta_j}}{\partial_{c,K_{\theta_j}} x^\alpha} f(x),$$

and hence the multidimensional semi-fractional derivative is a weighted sum of directional derivatives. Especially for $n = d$ and M being uniformly distributed on $\theta_i = e_i$, where for every $i \in \{1, \ldots, d\}$, e_i is the standard coordinate vector, we obtain

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta \in S}} f(x) = \frac{1}{d} \sum_{j=1}^n \frac{\partial^\alpha}{\partial_{c,K_{\theta_j}} x_i^\alpha} f(x).$$

In this sense, we can interpret the multidimensional semi-fractional derivative as a general version of the Laplace operator. As done in [89] for the fractional case, it is possible to define semi-fractional generalized versions of operators like the gradient or the curl operator. However, this topic is outside the scope of this thesis and will be investigated elsewhere.

Remark 3.3.7. (Relation to the fractional Laplacian)
A well-established differential operator for multivariable functions in fractional calculus is the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$ defined in the way, that for a suitable function $f : \mathbb{R}^d \to \mathbb{R}$, the Fourier transform of $\Delta^{\frac{\alpha}{2}} f$ is given by $-||k||^\alpha \widehat{f}(k)$. To analyze its connection to the multidimensional semi-fractional derivative, assume that M is uniform over S and that $K_\theta = K$ for every $\theta \in S$, where $K : \mathbb{R} \to (0, \infty)$ is a smooth admissable function with Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$. Then due to the symmetry of M, the Fourier transform of the multidimensional semi-fractional derivative of a suitable function $f : \mathbb{R}^d \to \mathbb{R}$ is given by

$$\mathcal{F}\left(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}} f\right)(k) = D \int\limits_S \frac{1}{2}\left(h_\theta(\langle k, \theta\rangle) + h_{-\theta}(\langle k, -\theta\rangle)\right) dM(\theta) \widehat{f}(k)$$

for every $k \in \mathbb{R}^d$, and using Example 2.2.6, h_θ has the representation

$$h_\theta(x) = \sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1)(-ix)^{\alpha - in\tilde{c}}$$

for every $x \in \mathbb{R}$. Note that for every $u \in \mathbb{R}$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
(-iu)^{\alpha-in\tilde{c}} + (iu)^{\alpha-in\tilde{c}} &= (-i|u|)^{\alpha-in\tilde{c}} + (i|u|)^{\alpha-in\tilde{c}} \\
&= |u|^{\alpha-in\tilde{c}}\left(e^{-i\frac{\pi}{2}(\alpha-in\tilde{c})} + e^{i\frac{\pi}{2}(\alpha-in\tilde{c})}\right) \\
&= 2|u|^{\alpha-in\tilde{c}} \cos\left(\frac{\pi}{2}(\alpha - in\tilde{c})\right),
\end{aligned}$$

and hence we obtain

$$\begin{aligned}
\frac{1}{2}\left(h_\theta(u) + h_{-\theta}(-u)\right) &= \sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1)\frac{1}{2}\left((-iu)^{\alpha-in\tilde{c}} + (iu)^{\alpha-in\tilde{c}}\right) \\
&= \sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1)|u|^{\alpha-in\tilde{c}} \cos\left(\frac{\pi}{2}(\alpha - in\tilde{c})\right)
\end{aligned}$$

for every $u \in \mathbb{R}$. Write $k = r\omega$ for some $r > 0$ and $||\omega|| = 1$ to see that

$$\mathcal{F}\left(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}} f\right)(k) = \sum_{n\in\mathbb{Z}} u_n ||k||^{\alpha-in\tilde{c}} \widehat{f}(k), \tag{3.26}$$

where the Fourier coefficients $(u_n)_{n\in\mathbb{Z}}$ are given by

$$\begin{aligned} u_n :&= Dc_n\Gamma(in\tilde{c}-\alpha+1)\cos\Big(\frac{\pi}{2}(\alpha-in\tilde{c})\Big)\int\limits_S |\langle\omega,\theta\rangle|^{\alpha-in\tilde{c}}\,dM(\theta) \\ &= Dc_n\Gamma(in\tilde{c}-\alpha+1)\cos\Big(\frac{\pi}{2}(\alpha-in\tilde{c})\Big)\int\limits_S |\theta_1|^{\alpha-in\tilde{c}}\,dM(\theta). \end{aligned}$$

Thereby, the second equality follows from the fact that the integral only depends on the angle between ω and θ. However, since θ varies over the whole sphere and M is uniform, any choice of ω yields the same result, and we choose $\omega = e_1$ without loss of generality (compare [94, Example 6.24]). Especially in the case $K(x) = \frac{D}{\Gamma(1-\alpha)}$ corresponding to the multidimensional fractional derivative, we obtain

$$\mathcal{F}\left(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}f\right)(k) = u_0||k||^\alpha\widehat{f}(k),$$

which is a multiple of the fractional Laplacian. Comparing this result with (3.26), the multidimensional semi-fractional derivative can likewise be interpreted as a log-periodically disturbed version of the fractional Laplacian.

Due to our leading motivation, we relate Definition 3.3.2 to directional semi-fractional derivatives as introduced in the previous chapter in the following way.

Lemma 3.3.8. *Fix $\alpha\in(0,2)\backslash\{1\}$, $c>1$ as well as an admissable set of functions $(K_\theta)_{\theta\in S}$ and a probability measure M on the unit sphere. For every $f\in W^{\lfloor\alpha\rfloor+d+1}(\mathbb{R}^d)\cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R}^d)$, the multidimensional semi-fractional derivative is given by*

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}f(x) = \int\limits_S \frac{\partial^\alpha_\theta}{\partial_{c,K_\theta}x^\alpha}f(x)\,dM(\theta) \tag{3.27}$$

for almost every $x\in\mathbb{R}^d$.

Remark 3.3.9. One can just as well define the multidimensional semi-fractional derivative by (3.27) for every $x\in\mathbb{R}$ and then calculate the Fourier transform.

Proof. Evaluating the Fourier transform of the right-hand side of (3.27) yields

$$\int\limits_{\mathbb{R}^d}\int\limits_S e^{i\langle k,x\rangle}\frac{\partial^\alpha_\theta}{\partial_{c,K_\theta}x^\alpha}f(x)\,dM(\theta)\,dx = \int\limits_S\int\limits_{\mathbb{R}^d} e^{i\langle k,x\rangle}\frac{\partial^\alpha_\theta}{\partial_{c,K_\theta}x^\alpha}f(x)\,dx\,dM(\theta),$$

where we can change the order of integration with Lemma 3.2.5 and the fact that M is a probability measure on the unit sphere. Then the result holds in view of Lemma 3.2.8. □

Using the explicit representation of the directional semi-fractional derivative in Lemma 3.2.2, we immediately obtain a generator form for the multidimensional semi-fractional derivative. Namely, for every $f \in W^{\lfloor\alpha\rfloor+d+1}(\mathbb{R}^d) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R}^d)$, the generator form is given by

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}} f(x) = D \int_S \int_{0+}^{\infty} \left(f(x - r\theta) - \sum_{p=0}^{\lfloor\alpha\rfloor} (-r)^p \partial_\theta^{(p)} f(x) \right) dG_{K_\theta}(r)\, dM(\theta),$$

whereas the Caputo form reads as

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}} f(x) = D \int_S \int_{0+}^{\infty} \left(\partial_\theta f(x - r\theta) - \sum_{p=1}^{\lfloor\alpha\rfloor} \partial_\theta^{(p)} f(x) \right) G_{K_\theta}(r)\, dr\, dM(\theta).$$

Similar to the one-dimensional case, the generator form is closely connected to semistable distributions, which follows directly from Example 2.3.4.

Lemma 3.3.10. *Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let ν be a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution on $\mathbb{R}^d$ with Lévy-Khintchine Triple $[a, 0, \Phi]$, where a is defined as in (2.15) and the Lévy measure Φ is given by (2.5) for an admissable set of functions $(K_\theta)_{\theta\in S}$ and a probability measure M on the unit sphere. If L denotes the corresponding generator given in Example 2.3.4, then for every $f \in W^{\lfloor\alpha\rfloor+d+1}(\mathbb{R}^d) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R}^d)$, the generator form of the semi-fractional derivative exists with*

$$\mathbb{D}^{\alpha,M}_{\alpha,(K_\theta)_{\theta\in S}} f(x) = -DLf(x)$$

for every $x \in \mathbb{R}^d$.

3.4 A numerical approach

Now that we are familiar with semi-fractional derivatives in both the one- and multi-dimensional setting, we end this chapter by naming a few numerical results. As seen above, a semi-fractional derivative can rarely be calculated analytically. However, using numerical integration, the Caputo or Riemann-Liouville form can be approximated with negligible error. The primary motivation for these operators' definition was to connect semistable densities and solutions to semi-fractional diffusion equations. Since they are even more complicated to calculate analytically, a numerical approximation in these cases is at least of equal importance. Eventually, the possibility to compute numerical solutions to semi-fractional differential equations will increase our knowledge about semistable laws and thereby contributes to the satisfaction of our underlying motivation.

The most natural numerical approach toward approximate solutions of differential equations is the usage of finite difference methods, which demands approximating all operators involved by finitely many point evaluations. Even if we are not able to find such a finite approximation, a generalized form with countably many point evaluations can be found in analogy to the fractional case.

Recall that the one-dimensional fractional derivative of order $\alpha \in (0,2) \setminus \{1\}$ can be approximated by the Grünwald-Letnikov formula

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \lim_{\delta \downarrow 0} \delta^{-\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - j\delta) \tag{3.28}$$

([94, Proposition 2.1]) for suitable functions $f : \mathbb{R} \to \mathbb{R}$. As outlined in [88], the result can be extended to the multidimensional setting by considering directional fractional derivatives first. The non-locality of the fractional derivative prohibits an approximation with finitely many point evaluations. However, (3.28) can be seen as a generalized finite difference approximation, which is still suitable for numerical calculations.

During the last decades, growing computational power yielded an increasing interest in numerical methods of fractional calculus, and nowadays, there is comprehensive literature about different methods and their properties (e.g., see [15] or [63]). Specifically, the numerical progress allows the application of fractional differential equations to real-world problems and thereby remarkably contributes to the overall growing interest in the general theory of fractional calculus.

We now turn toward directional semi-fractional derivatives and define a Grünwald-Letnikov type formula as a generalization of (3.28). Note that for the one-dimensional case, the results have partly been published in [66, Section 4]. In contrast to the last sections, we need to demand some qualities of admissable functions. Namely, if $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and K is an admissable function with respect to these parameters, throughout this section we assume that $K \in C^2_{pw}(\mathbb{R})$, where $C^2_{pw}(\mathbb{R})$ is the space of continuously differentiable functions f such that f' is piecewise smooth and f has the Fourier series representation

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{in\tilde{c}x} \tag{3.29}$$

for every $x \in \mathbb{R}$ and $\tilde{c} = \frac{2\pi\alpha}{\log(c)}$. The following definition of directional Grünwald-Letnikov differences yields an approximation of directional semi-fractional derivatives.

Definition 3.4.1. (Grünwald-Letnikov differences)
Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. For a fixed unit vector $\theta \in S$ and $f : \mathbb{R}^d \to \mathbb{R}$ bounded, define directional Grünwald-Letnikov differences by

$${}^K\Delta_\delta^{\alpha,\theta} f(x) := D \sum_{n \in \mathbb{Z}} \sum_{j=0}^{\infty} \omega_n \delta^{in\tilde{c}-\alpha} \binom{\alpha - in\tilde{c}}{j} (-1)^j f(x - j\delta\theta) \tag{3.30}$$

for every $x \in \mathbb{R}^d$ and every $\delta > 0$, where $\omega_n := c_n \Gamma(in\tilde{c} - \alpha + 1)$ for every $n \in \mathbb{Z}$. Thereby, for every $z, w \in \mathbb{C} \setminus \{-1, -2, \ldots\}$ with $z - w \notin \{-1, -2, \ldots\}$, the general binomial coefficient is given by

$$\binom{z}{w} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}.$$

Since we aim to approximate a real-valued function, we first ensure that the approximation is both well-defined and real-valued.

Lemma 3.4.2. *Let $\alpha \in (0,2)\setminus\{1\}$, $c > 1$, and let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. For a fixed unit vector $\theta \in S$ and $f : \mathbb{R}^d \to \mathbb{R}$ bounded, it follows that ${}^K\Delta^{\alpha,\theta}_\delta f(x) \in \mathbb{R}$ for every $x \in \mathbb{R}^d$ and $\delta > 0$. If in addition $f \in L^1(\mathbb{R}^d)$, then also ${}^K\Delta^{\alpha,\theta}_\delta f \in L^1(\mathbb{R}^d)$.*

Proof. First note that the double series in (3.30) converges absolutely since

$$\begin{aligned} |{}^K\Delta^{\alpha,\theta}_\delta f(x)| &\le \sum_{n\in\mathbb{Z}}\sum_{j=0}^{\infty}\left|\omega_n \delta^{in\tilde{c}-\alpha}\binom{\alpha - in\tilde{c}}{j}(-1)^j f(x - j\delta\theta)\right| \\ &= \delta^{-\alpha}\sum_{n\in\mathbb{Z}}\sum_{j=0}^{\infty}\left|\omega_n \binom{\alpha - in\tilde{c}}{j} f(x - j\delta\theta)\right|. \end{aligned}$$

According to [45, Theorem VI.1], the binomial coefficient is bounded by

$$\left|\binom{z}{j}\right| \le C_5 \frac{j^{-\operatorname{Re}(z)-1}}{|\Gamma(-z)|} \tag{3.31}$$

for every $j \in \mathbb{N}$, $z \in \mathbb{C}\setminus\mathbb{N}_0$, and a constant $C_5 > 0$ such that

$$|{}^K\Delta^{\alpha,\theta}_\delta f(x)| \le \delta^{-\alpha}\sum_{n\in\mathbb{Z}}|\omega_n|\left(|f(x)| + \frac{C_5}{|\Gamma(in\tilde{c}-\alpha)|}\sum_{j=1}^{\infty} j^{-\alpha-1}|f(x-j\delta\theta)|\right).$$

Using the boundedness of f, we find a constant $C_6 > 0$ with

$$\begin{aligned} |{}^K\Delta^{\alpha,\theta}_\delta f(x)| &\le C_6\delta^{-\alpha}\sum_{n\in\mathbb{Z}}\frac{|\omega_n|}{|\Gamma(in\tilde{c}-\alpha)|} \\ &= C_6\delta^{-\alpha}\sum_{n\in\mathbb{Z}}\frac{|c_n\Gamma(in\tilde{c}-\alpha+1)|}{|\Gamma(in\tilde{c}-\alpha)|} \\ &= C_6\delta^{-\alpha}\sum_{n\in\mathbb{Z}}|c_n||in\tilde{c}-\alpha|, \end{aligned}$$

and since $K \in C^2_{pw}(\mathbb{R})$, the Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$ decay like $|c_n| \sim |n|^{-\frac{5}{2}}$ [46, Theorem 2.6] such that the series is finite. To see that the Grünwald-Letnikov differences deliver a real-valued approximation, note that we have

$${}^K\Delta^{\alpha,\theta}_\delta f(x) = D\delta^{-\alpha}\sum_{n\in\mathbb{Z}} a_n e^{in\tilde{c}\log(\delta)},$$

where

$$a_n := \sum_{j=0}^{\infty} \omega_n \binom{\alpha - in\tilde{c}}{j} (-1)^j f(x - j\delta\theta)$$

for every $n \in \mathbb{Z}$. Since K is real-valued, the Fourier coefficients fulfill $\overline{c_{-n}} = c_n$ for every $n \in \mathbb{Z}$ and hence

$$\begin{aligned}
\overline{a_{-n}} &= \sum_{j=0}^{\infty} \overline{\omega_{-n}} \overline{\binom{\alpha + in\tilde{c}}{j}} (-1)^j f(x - j\delta\theta) \\
&= \sum_{j=0}^{\infty} \overline{c_{-n}} \overline{\Gamma(-in\tilde{c} - \alpha + 1)} \overline{\frac{\Gamma(\alpha + in\tilde{c} + 1)}{\Gamma(j+1)\Gamma(\alpha + in\tilde{c} - j + 1)}} (-1)^j f(x - j\delta\theta) \\
&= \sum_{j=0}^{\infty} c_n \Gamma(in\tilde{c} - \alpha + 1) \frac{\Gamma(\alpha - in\tilde{c} + 1)}{\Gamma(j+1)\Gamma(\alpha - in\tilde{c} - j + 1)} (-1)^j f(x - j\delta\theta) \\
&= a_n.
\end{aligned}$$

Therefore, ${}^K\Delta_\delta^{\alpha,\theta} f(x) \in \mathbb{R}$ for every $x \in \mathbb{R}^d$. Finally note that for $f \in L^1(\mathbb{R}^d)$, using the absolute convergence of the double series, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} \left| {}^K\Delta_\delta^{\alpha,\theta} f(x) \right| dx &= \int_{\mathbb{R}^d} \left| \sum_{n\in\mathbb{Z}} \sum_{j=0}^{\infty} \omega_n \delta^{in\tilde{c}-\alpha} \binom{\alpha - in\tilde{c}}{j} (-1)^j f(x - j\delta\theta) \right| dx \\
&\leq \sum_{n\in\mathbb{Z}} \sum_{j=0}^{\infty} \left| \omega_n \delta^{in\tilde{c}-\alpha} \binom{\alpha - in\tilde{c}}{j} (-1)^j \right| \int_{\mathbb{R}^d} |f(x - j\delta\theta)| \, dx \\
&= ||f||_1 \sum_{n\in\mathbb{Z}} \sum_{j=0}^{\infty} \left| \omega_n \delta^{in\tilde{c}-\alpha} \binom{\alpha - in\tilde{c}}{j} (-1)^j \right| \\
&< \infty,
\end{aligned}$$

which yields the result. □

To verify that Grünwald-Letnikov differences indeed approximate directional semi-fractional derivatives, note that the following result holds.

Lemma 3.4.3. *(Fourier transform of Grünwald-Letnikov differences)*
Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. For $f \in L^1(\mathbb{R}^d)$ bounded and every $\theta \in S$, the Fourier transform of ${}^K\Delta_\delta^{\alpha,\theta} f$ converges

$$\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) \to Dh(\langle k, \theta \rangle)\widehat{f}(k)$$

as $\delta \downarrow 0$ for every $k \in \mathbb{R}^d$.

Proof. For every $\delta > 0$ and $k \in \mathbb{R}^d$, the Fourier transform of the Grünwald-Letnikov differences is given by

$$\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) = \int\limits_{\mathbb{R}^d} e^{i\langle k,x\rangle}\ {}^K\Delta_\delta^{\alpha,\theta} f(x)\,dx.$$

According to Lemma 3.4.2, ${}^K\Delta_\delta^{\alpha,\theta} f \in L^1(\mathbb{R}^d)$ such that

$$\begin{aligned}\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) &= D\sum_{n\in\mathbb{Z}}\sum_{j=0}^{\infty}\omega_n\delta^{in\tilde{c}-\alpha}\binom{\alpha-in\tilde{c}}{j}(-1)^j\int\limits_{\mathbb{R}^d} e^{i\langle k,x\rangle} f(x-j\delta\theta)\,dx\\ &= D\sum_{n\in\mathbb{Z}}\sum_{j=0}^{\infty}\omega_n\delta^{in\tilde{c}-\alpha}\binom{\alpha-in\tilde{c}}{j}(-1)^j\widehat{f}(k)e^{i\langle k,j\delta\theta\rangle}.\end{aligned}$$

Note that with $\mathrm{Re}(\alpha-in\tilde{c})>0$ and the general binomial theorem [70, Satz 247],

$$\sum_{j=0}^{\infty}\binom{\alpha-in\tilde{c}}{j}(-1)^j e^{ij\delta\langle k,\theta\rangle} = (1-e^{i\delta\langle k,\theta\rangle})^{\alpha-in\tilde{c}}$$

such that

$$\begin{aligned}\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) &= D\sum_{n\in\mathbb{Z}}\omega_n\delta^{in\tilde{c}-\alpha}(1-e^{i\delta\langle k,\theta\rangle})^{\alpha-in\tilde{c}}\widehat{f}(k)\\ &= D\sum_{n\in\mathbb{Z}}\omega_n\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}}\widehat{f}(k).\end{aligned} \tag{3.32}$$

Furthermore, the fraction can be displayed as in terms of the negative differential quotient of $x\mapsto e^{ix\langle k,\theta\rangle}$ in $x=0$; this is

$$\lim_{\delta\downarrow 0}\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right) = -\frac{d}{dx}e^{ix\langle k,\theta\rangle}\Big|_{x=0} = -i\langle k,\theta\rangle$$

for every $k\in\mathbb{R}^d$. Then by dominated convergence, we obtain

$$\begin{aligned}\lim_{\delta\downarrow 0}\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) &= \lim_{\delta\downarrow 0} D\sum_{n\in\mathbb{Z}}\omega_n\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}}\widehat{f}(k)\\ &= D\sum_{n\in\mathbb{Z}}\omega_n\left(-i\langle k,\theta\rangle\right)^{\alpha-in\tilde{c}}\widehat{f}(k).\end{aligned}$$

Using Example 3.1.6, the right-hand side finally reads as

$$\lim_{\delta\downarrow 0}\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) = Dh(\langle k,\theta\rangle)\widehat{f}(k)$$

for every $k\in\mathbb{R}^d$. □

In view of Lemma 3.4.3, the Grünwald-Letnikov differences converge to the corresponding semi-fractional derivative in the Fourier space. However, the following theorem even shows how the approximation error behaves dependent on the choice of the step size $\delta > 0$. The proof of this theorem was developed in cooperation with Matthias Häußler and can also be found in his unpublished master thesis [53].

Theorem 3.4.4. *(Convergence of Grünwald-Letnikov differences)*
Let $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. In addition, fix a unit vector $\theta \in S$. For every bounded function $f \in W^{\lfloor \alpha \rfloor + d + 2}(\mathbb{R}^d) \cap C_0^{\lfloor \alpha \rfloor + 1}(\mathbb{R}^d)$, we have

$$\frac{\partial^\alpha_\theta}{\partial_{c,K} x^\alpha} f(x) = \lim_{\delta \downarrow 0} {}^K\Delta^{\alpha,\theta}_\delta f(x)$$

for almost every $x \in \mathbb{R}^d$, and the convergence is of order $O(\delta)$.

To prove the rate of convergence in Theorem 3.4.4, we first show the following auxiliary result.

Lemma 3.4.5. *For $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$ let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. Then for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ and every $L \in \mathbb{N}$,*

$$\left| \sum_{n \in \mathbb{Z}} \omega_n z^{\alpha - in\tilde{c}} - \sum_{|n| \leq L} \omega_n z^{\alpha - in\tilde{c}} \right| = \left| \sum_{|n| > L} \omega_n z^{\alpha - in\tilde{c}} \right| \leq C_9 L^{-1} |z|^\alpha \tag{3.33}$$

for a constant $C_9 > 0$.

Proof. To obtain (3.33), write $z = re^{i\phi}$ for some $r > 0$ and $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that

$$\begin{aligned}
\left| \sum_{n \in \mathbb{Z}} \omega_n z^{\alpha - in\tilde{c}} - \sum_{|n| \leq L} \omega_n z^{\alpha - in\tilde{c}} \right| &\leq \sum_{|n| > L} |\omega_n (re^{i\phi})^{\alpha - in\tilde{c}}| \\
&\leq \sum_{|n| > L} |\omega_n| r^\alpha e^{n\tilde{c}\phi} \\
&= r^\alpha \sum_{|n| > L} |c_n \Gamma(in\tilde{c} - \alpha + 1)| e^{n\tilde{c}\phi},
\end{aligned}$$

inserting the definition of ω_n. Using the asymptotic behavior of the gamma function [3, Corollary 1.4.4], we obtain

$$\begin{aligned}
\left| \sum_{n \in \mathbb{Z}} \omega_n z^{\alpha - in\tilde{c}} - \sum_{|n| \leq L} \omega_n z^{\alpha - in\tilde{c}} \right| &\leq C_7 r^\alpha \sum_{|n| > L} |c_n| |n|^{\frac{1}{2} - \alpha} e^{-\frac{\pi}{2}|n|\tilde{c}} e^{n\tilde{c}\phi} \\
&\leq C_7 r^\alpha \sum_{|n| > L} |c_n| |n|^{\frac{1}{2} - \alpha}
\end{aligned}$$

for a constant $C_7 > 0$. Due to the smoothness of K, the Fourier coefficients decay like $|c_n| \leq C_8 |n|^{-\frac{5}{2}}$ for a constant $C_8 > 0$ and every $n \in \mathbb{Z}$ (compare [46, Theorem 2.6]) such that

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}}\omega_n z^{\alpha-in\tilde{c}} - \sum_{|n|\le L}\omega_n z^{\alpha-in\tilde{c}}\right| &\le C_7C_8r^\alpha \sum_{|n|>L}|n|^{-\alpha-2}\\ &= 2C_7C_8r^\alpha\sum_{n>L}|n|^{-1-\alpha}|n|^{-1}\\ &\le 2C_7C_8r^\alpha L^{-1}\sum_{n=1}^{\infty}|n|^{-1-\alpha}\\ &= C_9L^{-1}r^\alpha,\end{aligned}$$

where $C_9 := 2C_7C_8\sum\limits_{n=1}^{\infty}|n|^{-1-\alpha} < \infty$. □

Proof of Theorem 3.4.4. To show convergence of the claimed order, first use (3.32) for fixed $\delta > 0$ to obtain

$$\begin{aligned}\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta}f\right)(k) &= D\sum_{n\in\mathbb{Z}}\omega_n\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}}\widehat{f}(k)\\ &= D\sum_{n\in\mathbb{Z}}\omega_n\widehat{f}(k)\left[(-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}} + \left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}} - (-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}}\right]\\ &= \mathcal{F}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K}x^\alpha}f\right)(k) + D\widehat{f}(k)\sum_{n\in\mathbb{Z}}\omega_n\left[\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}} - (-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}}\right],\end{aligned}$$

according to Lemma 3.2.8 and Example 3.1.6. Let $L \ge \delta^{-1}$. With $\operatorname{Re}(-i\langle k,\theta\rangle) = 0$ apply (3.33) to the series

$$\sum_{n\in\mathbb{Z}}\omega_n(-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}} = \sum_{|n|\le L}\omega_n(-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}} + O(\delta)|\langle k,\theta\rangle|^\alpha.$$

Since

$$\operatorname{Re}\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right) \ge 0,$$

we apply (3.33) to the first series as well, yielding

$$\sum_{n\in\mathbb{Z}}\omega_n\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}} = \sum_{|n|\le L}\omega_n\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}} + O(\delta)\left|\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)\right|^\alpha.$$

Note that using a Taylor expansion,

$$e^{i\delta\langle k,\theta\rangle} - 1 = i\langle k,\theta\rangle\delta + O(\delta^2)$$

such that

$$\sum_{n\in\mathbb{Z}}\omega_n\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}}=\sum_{|n|\le L}\omega_n\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}}+O(\delta)\left|\langle k,\theta\rangle\right|^{\alpha}.$$

Hence we obtain

$$\begin{aligned}&\left|\mathcal{F}\left({}^{K}\Delta_{\delta}^{\alpha,\theta}f\right)(k)-\mathcal{F}\left(\frac{\partial_{\theta}^{\alpha}}{\partial_{c,K}x^{\alpha}}f\right)(k)\right|\\ =&\left|D\widehat{f}(k)\sum_{n\in\mathbb{Z}}\omega_n\left[\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}}-(-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}}\right]\right|\\ \le&\left|D\widehat{f}(k)\sum_{|n|\le L}\omega_n\left[\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}}-(-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}}\right]\right|+O(\delta)|\langle k,\theta\rangle|^{\alpha}|\widehat{f}(k)|\end{aligned}$$

for every $k\in\mathbb{R}^d$. To calculate an upper bound for the first term, consider the functions $d_n:\mathbb{R}\to\mathbb{C}$ with

$$d_n(x):=\left(\frac{1-e^{i\delta x}}{\delta}\right)^{\alpha-in\tilde{c}}-(-ix)^{\alpha-in\tilde{c}} \tag{3.34}$$

for every $n\in\mathbb{N}$. Then

$$\begin{aligned}|d_n(x)|&=\left|\left(\frac{1-e^{i\delta x}}{-ix\delta}\right)^{\alpha-in\tilde{c}}-1\right|\left|(-ix)^{\alpha-in\tilde{c}}\right|\\ &\le|x|^{\alpha}e^{-\frac{\pi}{2}n\tilde{c}}\left|\left(\frac{1-e^{i\delta x}}{-ix\delta}\right)^{\alpha-in\tilde{c}}-1\right|\end{aligned}$$

for every $n\in\mathbb{Z}$. The Taylor expansion of

$$u\mapsto\left(\frac{1-e^{iu}}{-iu}\right)^{\alpha-in\tilde{c}}$$

in $u=0$ is given by

$$1+\frac{i(\alpha-in\tilde{c})}{2}u+O(u^2)$$

such that

$$\left|\left(\frac{1-e^{i\delta x}}{-ix\delta}\right)^{\alpha-in\tilde{c}}-1\right|\le|\alpha-in\tilde{c}||x|\delta$$

for δ sufficient small and every $n \leq L$. Summarizing, we obtain

$$|d_n(x)| \leq |\alpha - in\tilde{c}||x|^{\alpha+1} e^{-\frac{\pi}{2}n\tilde{c}}\delta$$

for every $x \in \mathbb{R}$. Then the difference between the Fourier transforms is bounded by

$$\begin{aligned}
&\left|\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) - \mathcal{F}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f\right)(k)\right| \\
\leq &\left| D\widehat{f}(k) \sum_{|n|\leq L} \omega_n d_n(\langle k, \theta\rangle)\right| + O(\delta)|\langle k,\theta\rangle|^\alpha |\widehat{f}(k)| \\
\leq &\delta |\widehat{f}(k)||\langle k,\theta\rangle|^{\alpha+1} \sum_{|n|\leq L} |\alpha - in\tilde{c}||\omega_n| e^{-\frac{\pi}{2}n\tilde{c}} + O(\delta)|\langle k,\theta\rangle|^\alpha |\widehat{f}(k)| \\
\leq &\delta |\widehat{f}(k)||\langle k,\theta\rangle|^{\alpha+1} \sum_{n\in\mathbb{Z}} |\alpha - in\tilde{c}||\omega_n| e^{-\frac{\pi}{2}n\tilde{c}} + O(\delta)|\langle k,\theta\rangle|^\alpha |\widehat{f}(k)| \\
= &O(\delta)|k|^\alpha (1+|k|)|\widehat{f}(k)|.
\end{aligned}$$

Thereby, the series $\sum_{n\in\mathbb{Z}} |\alpha - in\tilde{c}||\omega_n| e^{-\frac{\pi}{2}n\tilde{c}}$ converges as shown in the proof of Lemma 3.4.5. According to the Lemma of Riemann-Lebesgue [47, Theorem 8.22],

$$|\widehat{f}(k)| \leq \frac{1}{(1+|k|)^{\lfloor\alpha\rfloor + d + 2}}$$

and by Fourier inversion, we obtain

$$\begin{aligned}
\left|{}^K\Delta_\delta^{\alpha,\theta} f(x) - \frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x)\right| &= \left|\int_{\mathbb{R}^d} e^{i\langle x,k\rangle}\left(\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) - \mathcal{F}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f\right)(k)\right) dk\right| \\
&\leq \int_{\mathbb{R}^d} \left|\mathcal{F}\left({}^K\Delta_\delta^{\alpha,\theta} f\right)(k) - \mathcal{F}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f\right)(k)\right| dk \\
&= O(\delta) \int_{\mathbb{R}^d} \frac{|k|^\alpha}{(1+|k|)^{\lfloor\alpha\rfloor + d + 1}}\, dk.
\end{aligned}$$

Since $\lfloor\alpha\rfloor + d + 1 > \alpha + d$, the integral is finite, and hence the convergence is of order $O(\delta)$. □

Remark 3.4.6. Due to the representation of the multidimensional semi-fractional derivatives as a mixture of directional semi-fractional ones (compare Lemma 3.3.8), we also gain an approximation of the multidimensional derivative. For $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$, let $(K_\theta)_{\theta\in S} \subset C^2_{pw}(\mathbb{R})$ be an admissable set of functions such that for the Fourier coefficients, it holds that $|c_{n,\theta}| \leq Cn^{-\frac{5}{2}}$ for every $\theta \in S$, $n \in \mathbb{Z}$, and a constant $C > 0$ independent of θ. Note that in this case, the convergence of order $O(h)$ in Theorem 3.4.4 is independent of θ and hence

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}} f(x) = \int\limits_S {}^K\Delta^{\alpha,\theta}_{\delta} f(x)\, dM(\theta) + O(\delta)$$

for almost every $x \in \mathbb{R}^d$. Dependent on the mixing measure M, the integral over S has to be discretized appropriately. As an example, we refer to [88], where the multidimensional fractional derivative is discretized for the special cases of a discrete measure M as well as for M having a Lipschitz-continuous Lebesgue density.

For fractional diffusion, using the standard Grünwald-Letnikov formula (3.28) often causes instabilities of the chosen numerical method, especially if $\alpha \in (1,2)$. To overcome these difficulties, a shifted form of Grünwald-Letnikov differences was proposed in [95], yielding at least conditionally stable methods (e.g., see [95] or [96]). For the semi-fractional case and $\alpha \in (0,1)$, a particular stability result can be found in [53], but since it is a generalization of fractional diffusion, we are convinced that instabilities occur similarly for $\alpha \in (1,2)$. However, analyzing the stability of particular methods for semi-fractional differential equations exceeds the scope of this thesis. Nevertheless, we introduce a shifted form for our Grünwald-Letnikov formula and provide it for the following numerical calculations in Appendix C.

Definition 3.4.7. (Shifted Grünwald-Letnikov differences)
Let $\alpha \in (0,2)\setminus\{1\}$, $c > 1$, and let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. For a fixed unit vector $\theta \in S$ and $f : \mathbb{R}^d \to \mathbb{R}$ bounded, define shifted directional Grünwald-Letnikov differences by

$$ {}^K\Delta^{\alpha,\theta}_{\delta,p} f(x) := D \sum_{n\in\mathbb{Z}} \sum_{j=0}^{\infty} \omega_n \delta^{in\tilde{c}-\alpha} \binom{\alpha - in\tilde{c}}{j} (-1)^j f(x-(j-p)\delta\theta) \tag{3.35}$$

for every $x \in \mathbb{R}^d$ and every $\delta > 0$, where $p \in \mathbb{N}_0$ is a shift parameter.

Note that Lemma 3.4.2 transfers directly to the shifted form of Grünwald-Letnikov differences. This is, the shifted differences in (3.35) are real-valued and ${}^K\Delta^{\alpha,\theta}_{\delta,p} f \in L^1(\mathbb{R}^d)$ if $f \in L^1(\mathbb{R}^d)$. In addition, we obtain the following convergence result.

Theorem 3.4.8. *(Convergence of shifted Grünwald-Letnikov differences)*
Let $\alpha \in (0,2)\setminus\{1\}$, $c > 1$, and let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. In addition, fix a unit vector $\theta \in S$ and $p \in \mathbb{N}_0$. For every bounded function $f \in W^{\lfloor\alpha\rfloor+d+2}(\mathbb{R}^d) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R}^d)$, we have

$$\frac{\partial^\alpha_\theta}{\partial_{c,K} x^\alpha} f(x) = \lim_{\delta\downarrow 0} {}^K\Delta^{\alpha,\theta}_{\delta,p} f(x)$$

for almost every $x \in \mathbb{R}^d$, and the convergence is of order $O(\delta)$.

Proof. Similar to the proof of Lemma 3.4.3, the Fourier transform of the shifted Grünwald-Letnikov differences is given by

$$\mathcal{F}\left({}^K\Delta_{\delta,p}^{\alpha,\theta} f\right)(k) = De^{-ip\delta\langle k,\theta\rangle} \sum_{n\in\mathbb{Z}} \omega_n \left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}} \widehat{f}(k)$$

for every $k \in \mathbb{R}^d$ and $\delta > 0$. Then according to the proof of Lemma 3.4.4, we obtain

$$\begin{aligned}
&\left|\mathcal{F}\left({}^K\Delta_{\delta,p}^{\alpha,\theta} f\right)(k) - \mathcal{F}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K}x^\alpha} f\right)(k)\right| \\
=&\left|D\widehat{f}(k) \sum_{|n|\le L} \omega_n \left[e^{-ip\delta\langle k,\theta\rangle}\left(\frac{1-e^{i\delta\langle k,\theta\rangle}}{\delta}\right)^{\alpha-in\tilde{c}} - (-i\langle k,\theta\rangle)^{\alpha-in\tilde{c}}\right]\right| + O(\delta)|k|^\alpha \widehat{f}(k)
\end{aligned}$$

for $L \ge \delta^{-1}$. Repeating the procedure in the proof of Lemma 3.4.4 with

$$D_n(x) \coloneqq e^{-ip\delta x}\left(\frac{1-e^{i\delta x}}{\delta}\right)^{\alpha-in\tilde{c}} - (-ix)^{\alpha-in\tilde{c}}$$

instead of d_n in (3.34), we see that

$$\begin{aligned}
|D_n(x)| &\le \left|e^{-ip\delta x}\left(\frac{1-e^{i\delta x}}{-ix\delta}\right)^{\alpha-in\tilde{c}} - 1\right| \left|(-ix)^{\alpha-in\tilde{c}}\right| \\
&\le |x|^\alpha e^{-\frac{\pi}{2}n\tilde{c}} \left|e^{-ip\delta x}\left(\frac{1-e^{i\delta x}}{-ix\delta}\right)^{\alpha-in\tilde{c}} - 1\right| \\
&= |x|^\alpha e^{-\frac{\pi}{2}n\tilde{c}} \left|\left(\frac{1-e^{i\delta x}}{-ix\delta}\right)^{\alpha-in\tilde{c}} - 1 + 1 - e^{ip\delta x}\right| \\
&\le |x|^\alpha e^{-\frac{\pi}{2}n\tilde{c}} \left(|x|\delta\cdot|\alpha-in\tilde{c}| + 2p\delta|x|\right), \qquad (3.36)
\end{aligned}$$

and the result follows as in the proof of Theorem 3.4.4. □

Remark 3.4.9. Note that (3.36) implies choosing p as the smallest number, which leads to a stable method.

In order to ensure the existence of the directional semi-fractional derivatives, we assumed that $f \in W^{\lfloor\alpha\rfloor+d+2}(\mathbb{R}^d) \cap C_0^{\lfloor\alpha\rfloor+1}(\mathbb{R}^d)$ in the above theorems, in which case the Caputo and Riemann-Liouville forms are equal, and both are approximated by the Grünwald-Letnikov formula. However, we also want to analyze situations where the function f is supported on $[0,\infty)^d$, yielding a possible difference between Caputo and Riemann-Liouville forms (compare Lemma 3.2.12 and Lemma 3.2.15). Recall from Section 3.2 that this phenomenon especially appears when considering a time dependent system but may also occur in multivariable settings. Asking for the accordance of the Grünwald-Letnikov formula in this particular case, we find the following.

Lemma 3.4.10. *Let $\alpha \in (0,2)\backslash\{1\}$, $c > 1$, and let $K \in C^2_{pw}(\mathbb{R})$ be an admissable function with respect to these parameters. In addition, let $f : \mathbb{R}^d_+ \to \mathbb{R}$ fulfill the assumptions of Lemma 3.2.12 for $\alpha \in (0,1)$ or Lemma 3.2.15 for $\alpha \in (1,2)$. For every unit vector $\theta \in S$ with $\theta_i > 0$ for at least one $i = 1, \ldots, d$, the Grünwald-Letnikov scheme coincides with the Riemann-Liouville form of the directional semi-fractional derivative.*

Proof. The subsequent proof is based on the calculations in [110, p. 52-55] for the case of fractional derivatives. We first use Pascal's identity [111, (1.27)]

$$\binom{z}{j} = \binom{z-1}{j} + \binom{z-1}{j-1},$$

which holds for every $z \in \mathbb{C}$ and $j \in \mathbb{N}$, to rearrange ${}^K\Delta^{\alpha,\theta}_\delta f$. This is, we write

$${}^K\Delta^{\alpha,\theta}_\delta f(x) = \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha}(S(x) + f(x))$$

with

$$\begin{aligned} S(x) :&= \sum_{j=1}^{\infty} \binom{\alpha - in\tilde{c}}{j} (-1)^j f(x - j\delta\theta) \\ &= \sum_{j=1}^{\infty} \left(\binom{\alpha - in\tilde{c} - 1}{j} + \binom{\alpha - in\tilde{c} - 1}{j-1} \right) (-1)^j f(x - j\delta\theta). \end{aligned} \tag{3.37}$$

Note that f is supported on $\overline{\mathbb{R}^d_+}$, and since θ has at least one positive component, for every $\delta > 0$ there is $N(x) \in \mathbb{N}$ such that

$$x - N(x)\delta\theta \in \overline{\mathbb{R}^d_+}$$

and

$$x - (N(x)+1)\delta\theta \notin \overline{\mathbb{R}^d_+}.$$

Then $S(x)$ is indeed a finite series given by

$$\begin{aligned} S(x) &= \sum_{j=1}^{N(x)} \left(\binom{\alpha - in\tilde{c} - 1}{j} + \binom{\alpha - in\tilde{c} - 1}{j-1} \right) (-1)^j f(x - j\delta\theta) \\ &= \sum_{j=1}^{N(x)} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j f(x - j\delta\theta) + \sum_{j=1}^{N(x)} \binom{\alpha - in\tilde{c} - 1}{j-1} (-1)^j f(x - j\delta\theta) \\ &= \sum_{j=1}^{N(x)} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j f(x - j\delta\theta) + \sum_{k=0}^{N(x)-1} \binom{\alpha - in\tilde{c} - 1}{k} (-1)^{k+1} f(x - (k+1)\delta\theta) \end{aligned}$$

using the shift $k = j - 1$ in the second sum. Now combine both series to obtain

$$S(x) = \sum_{j=1}^{N(x)-1} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j (f(x - j\delta\theta) - f(x - (j+1)\delta\theta)) + R_\delta^1(x) - f(x - \delta\theta)$$

with

$$R_\delta^1(x) \coloneqq \binom{\alpha - in\tilde{c} - 1}{N(x)} (-1)^{N(x)} f(x - N(x)j\delta)$$

belonging to the index $N(x)$ in the first sum, whereas the term $-f(x - \delta\theta)$ belongs to the first term in the second sum associated with $k = 0$.
First consider the case $\alpha \in (0, 1)$. Then

$$\frac{f(x - j\delta\theta) - f(x - (j+1)\delta\theta)}{\delta} = \partial_\theta f(x - j\delta\theta) + O(\delta)$$

is a first-order approximation to the directional derivative of f in $x - j\delta\theta$ such that

$$\begin{aligned}
\delta^{-1}(S(x) + f(x)) &= \sum_{j=1}^{N(x)-1} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j \left(\partial_\theta f(x - j\delta\theta) + O(\delta)\right) \\
&\quad + \frac{f(x) - f(x - \delta\theta)}{\delta} + \delta^{-1} R_\delta^1(x) \\
&= \sum_{j=0}^{N(x)-1} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j \partial_\theta f(x - j\delta\theta) + \delta^{-1} R_\delta^1(x) + O(\delta).
\end{aligned}$$

For the Grünwald-Letnikov differences, it follows that

$$\begin{aligned}
{}^K\Delta_\delta^{\alpha,\theta} f(x) &= \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha+1} \delta^{-1}(S(x) + f(x)) \\
&= \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha+1} \left(\sum_{j=0}^{N(x)-1} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j \partial_\theta f(x - j\delta\theta) + \delta^{-1} R_\delta^1(x) + O(\delta) \right) \\
&= I_\delta^1(x) + \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^1(x) + O(\delta^{2-\alpha}),
\end{aligned}$$

where

$$I_\delta^1(x) \coloneqq \sum_{n\in\mathbb{Z}} \sum_{j=0}^{N(x)-1} \omega_n \delta^{in\tilde{c}-\alpha+1} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j \partial_\theta f(x - j\delta\theta).$$

We first analyze the convergence of I_δ^1. Let $a \in \mathbb{R}^d$ be the boundary point of $\mathbb{R}_+^d$ we reach first when moving from x in direction $-\theta$. Then we aim to show that

$$\lim_{\substack{\delta\downarrow 0,\\ x-N(x)\delta\theta=a}} I_\delta^1(x) = \int_{0+}^{x} \partial_\theta f(x-r\theta) G_K(r)\,dr, \tag{3.38}$$

which equals the Caputo form of semi-fractional derivatives. To do so, let

$$a_j^n := \Gamma(in\tilde{c}-\alpha+1)\binom{\alpha-in\tilde{c}-1}{j} j^{\alpha-in\tilde{c}}(-1)^j$$

for $j\in\mathbb{N}_0$ and $n\in\mathbb{Z}$. Then with the definition of $\omega_n = c_n\Gamma(in\tilde{c}-\alpha+1)$ for every $n\in\mathbb{Z}$, we can write

$$\begin{aligned}
I_\delta^1(x) &= \sum_{n\in\mathbb{Z}}\sum_{j=0}^{N(x)-1} c_n\Gamma(in\tilde{c}-\alpha+1)\delta^{in\tilde{c}-\alpha+1}\binom{\alpha-in\tilde{c}-1}{j}(-1)^j\partial_\theta f(x-j\delta\theta)\\
&= \sum_{n\in\mathbb{Z}}\sum_{j=0}^{N(x)-1} c_n a_j^n j^{in\tilde{c}-\alpha}\delta^{in\tilde{c}-\alpha+1}\partial_\theta f(x-j\delta\theta)\\
&= \delta\sum_{n\in\mathbb{Z}}\sum_{j=0}^{N(x)-1} c_n a_j^n (j\delta)^{in\tilde{c}-\alpha}\partial_\theta f(x-j\delta\theta)\\
&= \delta\sum_{j=0}^{N(x)-1}\partial_\theta f(x-j\delta\theta)(j\delta)^{-\alpha}\sum_{n\in\mathbb{Z}} c_n a_j^n (j\delta)^{in\tilde{c}}.
\end{aligned}$$

Note that with

$$\binom{z}{j}(-1)^j = \binom{j-z-1}{j}$$

for every $z\in\mathbb{C}$ and $j\in\mathbb{N}_0$, we obtain

$$\begin{aligned}
a_j^n &= \Gamma(in\tilde{c}-\alpha+1)\binom{in\tilde{c}-\alpha+1+j-1}{j} j^{\alpha-in\tilde{c}}\\
&= \Gamma(in\tilde{c}-\alpha+1)\frac{\Gamma(in\tilde{c}-\alpha+j+1)}{j!\Gamma(in\tilde{c}-\alpha+1)} j^{\alpha-in\tilde{c}}\\
&= \frac{\Gamma(in\tilde{c}-\alpha+j+1)}{j!} j^{\alpha-in\tilde{c}}\\
&= \Gamma(in\tilde{c}-\alpha+1)\frac{(in\tilde{c}-\alpha+j)\dots(in\tilde{c}-\alpha+1)}{j!j^{in\tilde{c}-\alpha}},
\end{aligned}$$

and with

$$\Gamma(in\tilde{c}-\alpha+1) = \lim_{j\to\infty}\frac{j!j^{in\tilde{c}-\alpha+1}}{(in\tilde{c}-\alpha+1)\dots(in\tilde{c}-\alpha+1+j)}$$

(see for example [113, p. 39]), it holds that $a_j^n \to 1$ as $j \to \infty$. Then we can think of $I_\delta^1(x)$ as a Riemann sum approximation of the Caputo semi-fractional directional derivative as follows. Choose a grid over $[0, x]$ with step size δ, such that the grid points are given by $j\delta$ with $j = 0, \ldots, \frac{x}{\delta}$. For simplicity we assume that δ is chosen such that $\frac{x}{\delta} \in \mathbb{N}$. Then a Riemann sum approximation of the Caputo form is given by

$$\begin{aligned}
\int_{0+}^{x} \partial_\theta f(x - r\theta) G_K(r)\, dr &= \int_{0+}^{x} \partial_\theta f(x - r\theta) r^{-\alpha} K(\log(r))\, dr \\
&= \int_{0+}^{x} \sum_{n\in\mathbb{Z}} c_n \partial_\theta f(x - r\theta) r^{-\alpha + in\tilde{c}}\, dr \\
&= \lim_{\substack{\delta\downarrow 0,\\ x - N(x)\delta\theta = a}} \delta \sum_{j=0}^{N(x)-1} \sum_{n\in\mathbb{Z}} c_n \partial_\theta f(x - j\delta\theta)(j\delta)^{-\alpha + in\tilde{c}}
\end{aligned}$$

by using the series representation of K in (3.29) and evaluating the function in the left endpoints $j\delta$ of the interval $[j\delta, (j+1)\delta]$. This sum equals $I_\delta^1(x)$ up to the multiplication with a_j^n, but since this coefficients converge to 1 as $j \to \infty$, we obtain (3.38). It remains to prove convergence of $\sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^1(x)$. Therefore, note that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^1(x) &= \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} \binom{\alpha - in\tilde{c} - 1}{N(x)} (-1)^{N(x)} f(a) \\
&= \sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1) \delta^{in\tilde{c}-\alpha} \binom{\alpha - in\tilde{c} - 1}{N(x)} (-1)^{N(x)} f(a) \\
&= \sum_{n\in\mathbb{Z}} c_n a_{N(x)}^n N(x)^{in\tilde{c}-\alpha} \delta^{in\tilde{c}-\alpha} f(a)
\end{aligned}$$

with $a_{N(x)}^n$ as defined above. Then

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^1(x) &= \sum_{n\in\mathbb{Z}} c_n a_{N(x)}^n (N(x)\delta)^{in\tilde{c}-\alpha} f(a) \\
&= \sum_{n\in\mathbb{Z}} c_n a_{N(x)}^n ||N(x)\delta\theta||^{in\tilde{c}-\alpha} f(a) \\
&= \sum_{n\in\mathbb{Z}} c_n a_{N(x)}^n ||x - a||^{in\tilde{c}-\alpha} f(a),
\end{aligned}$$

and with the convergence $a_{N(x)}^n \to 1$, we conclude

$$\begin{aligned}
\lim_{\substack{\delta\downarrow 0,\\ x - N(x)\delta\theta = a}} \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^1(x) &= \sum_{n\in\mathbb{Z}} c_n ||x - a||^{in\tilde{c}-\alpha} f(a) \\
&= G_K(||x - a||) f(a).
\end{aligned}$$

Altogether, we have

$$ {}^K\Delta_\delta^{\alpha,\theta} f(x) \to \int_{0+}^{x} \partial_\theta f(x-r\theta) G_K(r)\,dr + G_K(||x-a||) f(a), $$

yielding the result for the case $\alpha \in (0,1)$ together with Lemma 3.2.12.
For $\alpha \in (1,2)$, repeat the procedure in (3.37) and obtain

$$ \begin{aligned} S(x) &= \sum_{j=1}^{N(x)-1} \binom{\alpha - in\tilde{c} - 1}{j} (-1)^j (f(x-j\delta\theta) - f(x-(j+1)\delta\theta)) + R_\delta^1(x) - f(x-\delta\theta) \\ &= \sum_{j=1}^{N(x)-1} \left(\binom{\alpha - in\tilde{c} - 2}{j} + \binom{\alpha - in\tilde{c} - 2}{j-1} \right) (-1)^j (f(x-j\delta\theta) - f(x-(j+1)\delta\theta)) \\ &\quad + R_\delta^1(x) - f(x-\delta\theta) \\ &= \sum_{j=1}^{N(x)-2} \binom{\alpha - in\tilde{c} - 2}{j} (-1)^j (f(x-j\delta\theta) - 2f(x-(j+1)\delta\theta) + f(x-(j+2)\delta\theta)) \\ &\quad - 2f(x-\delta\theta) + f(x-2\delta\theta) + R_\delta^2(x) + R_\delta^1(x) \end{aligned} $$

with

$$ R_\delta^2(x) := \binom{\alpha - in\tilde{c} - 2}{N(x)-1} (-1)^{N(x)-1} (f(x-(N(x)-1)\delta\theta) - f(x-N(x))\delta\theta). $$

Similar to the case $\alpha \in (0,1)$,

$$ \frac{f(x-j\delta\theta) - 2f(x-(j+1)\delta\theta) + f(x-(j+2)\delta\theta)}{\delta^2} = \langle \theta, H_f(x-j\delta\theta)\theta \rangle + O(\delta) $$

such that for the Grünwald-Letnikov differences, we obtain

$$ \begin{aligned} {}^K\Delta_\delta^{\alpha,\theta} f(x) = &- \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha+2} \sum_{j=0}^{N(x)-2} \binom{\alpha - in\tilde{c} - 2}{j} (-1)^j \langle \theta, H_f(x-j\delta\theta)\theta \rangle \\ &- \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} (R_\delta^1(x) + R_\delta^2(x)) + O(\delta^{3-\alpha}). \end{aligned} $$

Again we show that the limit of

$$ I_\delta^2(x) := - \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha+2} \sum_{j=0}^{N(x)-2} \binom{\alpha - in\tilde{c} - 2}{j} (-1)^j \langle \theta, H_f(x-j\delta\theta)\theta \rangle $$

is given by the Caputo form of the semi-fractional derivative, this is

$$\lim_{\substack{\delta\downarrow 0,\\ N(x)\delta\theta=a}} I_\delta^2(x) = \int_{0+}^{x} \langle \theta, H_f(x-r\theta)\theta\rangle H_K(r)\,dr$$

with H_K from (3.9). Keep the series representation of γ in the definition of H_K in mind to write I_δ^2 as

$$\begin{aligned} I_\delta^2(x) &= \sum_{j=0}^{N(x)-2} \langle \theta, H_f(x-j\delta\theta)\theta\rangle \sum_{n\in\mathbb{Z}} \frac{c_n}{\alpha-1-in\tilde{c}} b_j^n j^{in\tilde{c}-\alpha+1}\delta^{in\tilde{c}-\alpha+2} \\ &= \delta \sum_{j=0}^{N(x)-2} \langle \theta, H_f(x-j\delta\theta)\theta\rangle (j\delta)^{-\alpha+1} \sum_{n\in\mathbb{Z}} \frac{c_n}{\alpha-1-in\tilde{c}} b_j^n (j\delta)^{in\tilde{c}} \end{aligned}$$

with

$$b_j^n := \Gamma(in\tilde{c}-\alpha+2)\binom{\alpha-in\tilde{c}-2}{j}(-1)^j j^{-in\tilde{c}+\alpha-1}$$

for every $j\in\mathbb{N}_0$ and $n\in\mathbb{Z}$. Note that b_j^n coincides with a_j^n when considering $\tilde{\alpha}=\alpha-1\in(0,1)$ such that $b_j^n\to 1$ as $n\to\infty$. Then using a similar construction as in the case $\alpha\in(0,1)$, I_δ^2 can be seen as a Riemann sum approximation of the Caputo directional semi-fractional derivative. In addition, as shown above

$$\lim_{\substack{\delta\downarrow 0,\\ N(x)\delta\theta=a}} \sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^1(x) = G_K(||x-a||)f(a),$$

where $a\in\mathbb{R}^d$ is the boundary point we reach first when moving from x in direction $-\theta$ such that we only have to investigate the convergence of

$$\sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^2(x) = -\sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha}\binom{\alpha-in\tilde{c}-2}{N(x)-1}(-1)^{N(x)-1}(f(a+\delta\theta)-f(a)).$$

To do so, set

$$\begin{aligned} B^n :&= \Gamma(in\tilde{c}-\alpha+2)\binom{\alpha-in\tilde{c}-2}{N(x)-1}(-1)^{N(x)-1}N(x)^{\alpha-in\tilde{c}-1} \\ &= \left(\frac{N(x)}{N(x)-1}\right)^{\alpha-in\tilde{c}-1} b_{N(x)-1}^n, \end{aligned}$$

and note that with the convergence of b_j^n, the factor B^n converges to 1 as well. Then

$$\sum_{n\in\mathbb{Z}} \omega_n \delta^{in\tilde{c}-\alpha} R_\delta^2(x) = \sum_{n\in\mathbb{Z}} \frac{c_n}{\alpha-in\tilde{c}-1} B^n (N(x)\delta)^{in\tilde{c}-\alpha+1}\frac{f(a+\delta\theta)-f(a)}{\delta}$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \frac{c_n}{\alpha - in\tilde{c} - 1} B^n ||x-a||^{in\tilde{c}-\alpha+1} \frac{f(a+\delta\theta)-f(a)}{\delta} \\
&\to H_K(||x-a||)\partial_\theta f(a).
\end{aligned}
$$

Altogether we obtain

$$
{}^K\Delta_\delta^{\alpha,\theta} f(x) \to \int\limits_{0+}^{x} \langle \theta, H_f(x-r\theta)\theta\rangle H_K(r)\,dr - G_K(||x-a||)f(a) + H_K(||x-a||)\partial_\theta f(a),
$$

which coincides with the Riemann-Liouville form of the semi-fractional directional derivative according to Lemma 3.2.15. □

Chapter 4

Laplace transform of semi-fractional derivatives

In the following chapter, we study semi-fractional differential equations and their connection to semistable laws. However, a last important tool for this task is still missing. When solving ordinary or fractional differential equations, using the Laplace transform is a powerful method since it converts the differential equation into an algebraic statement in the Laplace domain. The arising equation can then be solved more easily, and the solution of the initial differential equation is obtained by inverse Laplace transform (e.g., see [124], [110], or [101]). This chapter is devoted to analyzing the Laplace transform of semi-fractional derivatives to extend this valuable method to the case of semi-fractional differential equations. Throughout this thesis, we denote the Laplace transform of a suitable function $f : \mathbb{R}_+^d \to \mathbb{R}$ with $\mathcal{L}(f)$ or $\widetilde{f}$, this is

$$\mathcal{L}(f)(s) = \widetilde{f}(s) := \int\limits_{\mathbb{R}_+^d} e^{-\langle s,x\rangle} f(x)\, dx.$$

Typically, the Laplace transform is applied to the time variable but similarly may be utilized to transform arbitrary positive variables. Hence, to obtain a general result we study functions $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}_+^d) \cap C_0(\overline{\mathbb{R}_+^d})$, which are supported on $\overline{\mathbb{R}_+^d}$. Recall from the previous chapter that in this case, the Caputo and Riemann-Liouville form differ dependent on the boundary values of f on $\partial\mathbb{R}_+^d$ (see Lemma 3.2.12 and Lemma 3.2.15), and this difference is reflected in their Laplace transforms. For every $i = 1, \ldots, d$ and $x \in \mathbb{R}_+^d$, we denote by

$$f_i(x) := f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d)$$

the value of the continuous extension of f in the boundary point

$$(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) \in \partial\mathbb{R}_+^d.$$

4.1 Laplace transform of the Caputo form

We first concentrate on the directional semi-fractional Caputo form and analyze the Riemann-Liouville form afterward. However, to ensure the Laplace transform's existence, we restrict our attention to directions $\theta \in S \cap \overline{\mathbb{R}^d_+}$ with non-negative components.

Lemma 4.1.1. *Let $\alpha \in (0,1)$, $c > 1$, let K be an admissable function with respect to these parameters, and let $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d_+) \cap C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}^d_+})$. In addition, choose a fixed unit vector $\theta \in S \cap \overline{\mathbb{R}^d_+}$ and assume that the function*

$$(x,r) \mapsto e^{-\langle s,x\rangle}\partial_\theta f(x-r\theta)\mathbb{1}_{\mathbb{R}^d_+}(x-r\theta)G_K(r)$$

is integrable for every $s \in \mathbb{R}^d_+$ with respect to the Lebesgue measure on $\mathbb{R}^d_+ \times \mathbb{R}_+$. Then for every $s \in \mathbb{R}^d_+$, the Laplace transform of the Caputo form of the directional semi-fractional derivative is given by

$$\mathcal{L}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K}x^\alpha}f\right)(s) = \widetilde{G_K}(\langle s,\theta\rangle)\left(\langle s,\theta\rangle\widetilde{f}(s) - \sum_{i=1}^{d}\theta_i\widetilde{f}_i(s)\right).$$

Remark 4.1.2. Note that in Lemma 4.1.1, for every $i \in \{1,\dots,d\}$, the function $\widetilde{f}_i$ is the $(d-1)$-dimensional Laplace transform of f in the variables $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)$. Especially for $d=1$, it coincides with $f(0)$.

Proof. Since the function f is supported on $\overline{\mathbb{R}^d_+}$, according to (3.16), we have

$$\mathcal{L}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K}x^\alpha}f\right)(s) = \int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle}\int\limits_{0+}^{\infty}\partial_\theta f(x-r\theta)\mathbb{1}_{\mathbb{R}^d_+}(x-r\theta)G_K(r)\,dr\,dx$$

for every $s \in \mathbb{R}^d_+$. Using the assumption of integrability, we change the order of integration with Fubini's theorem and obtain

$$\begin{aligned}\mathcal{L}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K}x^\alpha}f\right)(s) &= \int\limits_{0+}^{\infty}\int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle}\partial_\theta f(x-r\theta)\mathbb{1}_{\mathbb{R}^d_+}(x-r\theta)G_K(r)\,dx\,dr\\ &= \int\limits_{0+}^{\infty}\int\limits_{\mathbb{R}^d_+} e^{-\langle s,z+r\theta\rangle}\partial_\theta f(z)G_K(r)\,dz\,dr\end{aligned}$$

with the component-wise substitution $z \coloneqq x - r\theta$ and $\theta \in S \cap \overline{\mathbb{R}^d_+}$. Now we can separate the integrals

$$\mathcal{L}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K}x^\alpha}f\right)(s) = \int\limits_{0+}^{\infty} e^{-r\langle s,\theta\rangle}G_K(r)\,dr\int\limits_{\mathbb{R}^d_+} e^{-\langle s,z\rangle}\partial_\theta f(z)\,dz$$

$$= \widetilde{G_K}(\langle s, \theta \rangle) \int\limits_{\mathbb{R}_+^d} e^{-\langle s,z \rangle} \partial_\theta f(z)\, dz.$$

Note that with $s \in \mathbb{R}_+^d$ and $\theta \in S \cap \overline{\mathbb{R}_+^d}$, we have $\langle s, \theta \rangle > 0$ and the Laplace transform is well-defined. For the second integral, using the rule for Laplace transforms of partial derivatives (compare [34, Equations (44) and (45)]), we find

$$\begin{aligned}
\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z \rangle} \partial_\theta f(z)\, dz &= \sum_{i=1}^d \theta_i \int\limits_{\mathbb{R}_+^d} e^{-\langle s,z \rangle} \frac{\partial}{\partial x_i} f(z)\, dz \\
&= \sum_{i=1}^d \theta_i \left(s_i \widetilde{f}(s) - \widetilde{f}_i(s) \right) \\
&= \langle s, \theta \rangle \widetilde{f}(s) - \sum_{i=1}^d \theta_i \widetilde{f}_i(s)
\end{aligned}$$

such that the statement follows. □

To obtain a more explicit form of the Laplace transform, let us analyze $\widetilde{G_K}$ more closely.

Lemma 4.1.3. *Let $\alpha \in (0,1)$, $c > 1$, and let K be an admissable function with respect to these parameters. Then for every $s > 0$,*

$$\widetilde{G_K}(s) = s^{\alpha-1} \eta_1(\log(s)),$$

where $\eta_1 : \mathbb{R} \to (0, \infty)$ given by

$$\eta_1(x) := e^{x(1-\alpha)} \int\limits_0^\infty e^{-e^x t} t^{-\alpha} K(\log(t))\, dt \tag{4.1}$$

is a positive and $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic $C^\infty(\mathbb{R})$-function. Furthermore, the function $x \mapsto \eta_1(-x)$ is admissable with respect to α and c. If additionally K is smooth with Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$, then η_1 has the Fourier series representation

$$\eta_1(x) = \sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1) e^{-in\tilde{c}x}$$

for every $x \in \mathbb{R}$.

Proof. To prove the statement above, we conduct a direct calculation. However, note that equally, one may use the theory of Bernstein functions, which we introduce later on in Section 7.1. Although this theory probably yields an easier proof, its structure is not suitable for the case $\alpha \in (1,2)$, and for consistency of this chapter, we decided to postpone the introduction of Bernstein functions at this point.

For every $s > 0$, we find

$$\begin{aligned}
\widetilde{G_K}(s) &= \int_0^\infty e^{-st} G_K(t)\, dt \\
&= s^{\alpha-1} s^{1-\alpha} \int_0^\infty e^{-st} t^{-\alpha} K(\log(t))\, dt \\
&= s^{\alpha-1} \eta_1(\log(s))
\end{aligned}$$

if $\eta_1 : \mathbb{R} \to \mathbb{R}$ is defined as above. Note that it follows directly from the definition that η_1 is strictly positive. In addition, η_1 is $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic since

$$\begin{aligned}
\eta_1\left(x + \log\left(c^{\frac{1}{\alpha}}\right)\right) &= e^{x(1-\alpha)} c^{\frac{1}{\alpha}(1-\alpha)} \int_0^\infty e^{-e^x t c^{\frac{1}{\alpha}}} t^{-\alpha} K(\log(t))\, dt \\
&= e^{x(1-\alpha)} c^{\frac{1}{\alpha}-1} c^{-\frac{1}{\alpha}} \int_0^\infty e^{-e^x y} (y c^{-\frac{1}{\alpha}})^{-\alpha} K\left(\log\left(y c^{-\frac{1}{\alpha}}\right)\right)\, dy \\
&= e^{x(1-\alpha)} \int_0^\infty e^{-e^x y} y^{-\alpha} K(\log(y))\, dy \\
&= \eta_1(x)
\end{aligned}$$

with the substitution $y := tc^{\frac{1}{\alpha}}$ and the periodicity of K. Besides, $\eta_1 \in C^\infty(\mathbb{R})$ as composition of $C^\infty(\mathbb{R})$-functions. It remains to show that $x \mapsto \eta_1(-x)$ is admissable. As argued above, the function is strictly positive and $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic. In addition, η_1 fulfills the growth restriction since

$$\begin{aligned}
\eta_1(-(x+\delta)) &= e^{-(x+\delta)(1-\alpha)} \int_0^\infty e^{-e^{-(x+\delta)} t} t^{-\alpha} K(\log(t))\, dt \\
&= e^{-x(1-\alpha)} e^{\delta\alpha-\delta} e^{\delta} \int_0^\infty e^{-e^{-x} y} (y e^{\delta})^{-\alpha} K\left(\log\left(y e^{\delta}\right)\right)\, dy \\
&= e^{-x(1-\alpha)} \int_0^\infty e^{-e^{-x} y} y^{-\alpha} K\left(\log(y) + \delta\right)\, dy
\end{aligned}$$

with $y := e^{-\delta} t$ for every $x \in \mathbb{R}$ and $\delta > 0$. Now K is admissable such that

$$\eta_1(-(x+\delta)) \leq e^{\alpha\delta} e^{-x(1-\alpha)} \int_0^\infty e^{-e^{-x} y} y^{-\alpha} K\left(\log(y)\right)\, dy = e^{\alpha\delta} \eta_1(-x)$$

for every $x \in \mathbb{R}$ and $\delta > 0$. Hence $x \mapsto \eta_1(-x)$ is admissable in view of Lemma 3.1.2. Finally assume that K is smooth with Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$. Then

$$\begin{aligned}\eta_1(x) &= e^{x(1-\alpha)} \int\limits_0^\infty \sum_{n\in\mathbb{Z}} c_n e^{-e^x t} t^{-\alpha+in\tilde{c}}\, dt \\ &= e^{x(1-\alpha)} \sum_{n\in\mathbb{Z}} c_n \int\limits_0^\infty e^{-e^x t} t^{-\alpha+in\tilde{c}}\, dt,\end{aligned}$$

where we can interchange the order of integration and summation since

$$\int\limits_0^\infty \sum_{n\in\mathbb{Z}} \left| c_n e^{-e^x t} t^{-\alpha+in\tilde{c}} \right| dt = \left(\sum_{n\in\mathbb{Z}} |c_n| \right) \int\limits_0^\infty e^{-e^x t} t^{-\alpha}\, dt < \infty.$$

With the substitution $y \coloneqq e^x t$, we get

$$\begin{aligned}\eta_1(x) &= e^{-\alpha x} \sum_{n\in\mathbb{Z}} c_n \int\limits_0^\infty e^{-y} (e^{-x} y)^{-\alpha+in\tilde{c}}\, dy \\ &= \sum_{n\in\mathbb{Z}} c_n e^{-in\tilde{c}x} \int\limits_0^\infty e^{-y} y^{-\alpha+in\tilde{c}}\, dy \\ &= \sum_{n\in\mathbb{Z}} c_n \Gamma(1-\alpha+in\tilde{c}) e^{-in\tilde{c}x},\end{aligned}$$

which finishes the proof. □

Combining Lemma 4.1.1 and Lemma 4.1.3, we obtain the following.

Theorem 4.1.4. (*Laplace transform of Caputo derivative, $\alpha \in (0,1)$*)
Let $\alpha \in (0,1)$, $c > 1$, let K be an admissable function with respect to these parameters, and let $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d_+) \cap C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}^d_+})$. In addition, choose a unit vector $\theta \in S \cap \overline{\mathbb{R}^d_+}$ and assume that the function

$$(x,r) \mapsto e^{-\langle s,x\rangle} \partial_\theta f(x - r\theta) \mathbb{1}_{\mathbb{R}^d_+}(x - r\theta) G_K(r)$$

is integrable for every $s \in \mathbb{R}^d_+$ with respect to the Lebesgue measure on $\mathbb{R}^d_+ \times \mathbb{R}_+$. Then for every $s \in \mathbb{R}^d_+$, the Laplace transform of the Caputo form of the directional semi-fractional derivative is given by

$$\mathcal{L}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K} x^\alpha} f \right)(s) = \langle s,\theta\rangle^{\alpha-1} \eta_1(\log(\langle s,\theta\rangle)) \left(\langle s,\theta\rangle \widetilde{f}(s) - \sum_{i=1}^d \theta_i \widetilde{f}_i(s) \right)$$

with η_1 as in (4.1).

Using this explicit representation, we can study some important examples.

Example 4.1.5. (Laplace transform of the one-dimensional Caputo form, $\alpha \in (0,1)$)
In the one-dimensional case and for $\theta = 1$, we obtain the Laplace transform of the semi-fractional Caputo form with

$$\mathcal{L}\left(\frac{\partial^\alpha}{\partial_{c,K} x^\alpha} f\right)(s) = s^{\alpha-1}\eta_1(\log(s))\left(s\widetilde{f}(s) - f(0)\right)$$

for every $s > 0$.

Example 4.1.6. (Laplace transform of the fractional Caputo form, $\alpha \in (0,1)$)
Let $K(x) = \frac{1}{\Gamma(1-\alpha)}$ be the constant function corresponding to the fractional derivative. Since K is smooth, it follows immediately from Lemma 4.1.3 that $\eta_1(x) = 1$ for every $x \in \mathbb{R}$, and hence the Laplace transform of the directional fractional derivative equals

$$\mathcal{L}\left(\frac{\partial_\theta^\alpha}{\partial x^\alpha} f\right)(s) = \langle s, \theta\rangle^{\alpha-1}\left(\langle s, \theta\rangle \widetilde{f}(s) - \sum_{i=1}^{d} \theta_i \widetilde{f}_i(s)\right)$$

for every $s \in \mathbb{R}_+^d$. Especially in one dimension and for $\theta = 1$, this yields

$$\mathcal{L}\left(\frac{\partial^\alpha}{\partial x^\alpha} f\right)(s) = s^{\alpha-1}\left(s\widetilde{f}(s) - f(0)\right)$$

for every $s > 0$, which is already known by [94, p. 39].

In the case $\alpha \in (1,2)$, the argumentation works similarly.

Lemma 4.1.7. *Let $\alpha \in (1,2)$, $c > 1$, let K be an admissable function with respect to these parameters, and let $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}_+^d) \cap C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}_+^d})$. In addition, let $\theta \in S \cap \overline{\mathbb{R}_+^d}$ be a fixed unit vector and assume that the function*

$$(x, r) \mapsto e^{-\langle s, x\rangle}\langle \theta, \mathcal{H}_f(x - r\theta)\theta\rangle H_K(r) \mathbb{1}_{\mathbb{R}_+^d}(x - r\theta)$$

with H_K from (3.9) is integrable for every $s \in \mathbb{R}_+^d$ with respect to the Lebesgue measure on $\mathbb{R}_+^d \times \mathbb{R}_+$. Then for every $s \in \mathbb{R}_+^d$, the Laplace transform of the Caputo form of the directional semi-fractional derivative is given by

$$\mathcal{L}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f\right)(s) = \widetilde{H_K}(\langle s, \theta\rangle) \int_{\mathbb{R}_+^d} e^{-\langle s, z\rangle}\langle \theta, \mathcal{H}_f(z)\theta\rangle \, dz.$$

Proof. According to (3.17), for a function $f : \mathbb{R}^d \to \mathbb{R}$ supported on $\overline{\mathbb{R}_+^d}$, the Laplace transform of the Caputo form is given by

$$\mathcal{L}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f\right)(s) = \int_{\mathbb{R}_+^d} e^{-\langle s, x\rangle} \int_{0+}^{\infty} \langle \theta, \mathcal{H}_f(x - r\theta)\theta\rangle H_K(r) \mathbb{1}_{\mathbb{R}_+^d}(x - r\theta)\, dr\, dx.$$

Using the assumption of integrability, we change the order of integration with Fubini's theorem and substitute $z := x - r\theta$ to obtain

$$\begin{aligned}\mathcal{L}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f\right)(s) &= \int_{0+}^{\infty} \int_{\mathbb{R}_+^d} e^{-\langle s,x\rangle} \langle \theta, \mathcal{H}_f(x - r\theta)\theta\rangle H_K(r) \mathbb{1}_{\mathbb{R}_+^d}(x - r\theta)\, dx\, dr \\ &= \int_{0+}^{\infty} \int_{\mathbb{R}_+^d} e^{-\langle s,z+r\theta\rangle} \langle \theta, \mathcal{H}_f(z)\theta\rangle H_K(r)\, dz\, dr\end{aligned}$$

since $\theta \in S \cap \overline{\mathbb{R}_+^d}$. Then separating the integrals as

$$\mathcal{L}\left(\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f\right)(s) = \widetilde{H_K}(\langle s,\theta\rangle) \int_{\mathbb{R}_+^d} e^{-\langle s,z\rangle} \langle \theta, \mathcal{H}_f(z)\theta\rangle\, dz$$

for every $s \in \mathbb{R}_+^d$ yields the result. □

Similar to the case $\alpha \in (0,1)$, the Laplace transform of H_K can be represented using a periodic function.

Lemma 4.1.8. *Let $\alpha \in (1,2)$, $c > 1$, and let K be an admissable function with respect to these parameters. Then for every $s > 0$,*

$$\widetilde{H_K}(s) = s^{\alpha-2} \eta_2(\log(s)),$$

where $\eta_2 : \mathbb{R} \to (0,\infty)$ given by

$$\eta_2(x) = e^{x(2-\alpha)} \int_0^\infty e^{-e^x t} \int_t^\infty z^{-\alpha} K(\log(z))\, dz\, dt \tag{4.2}$$

is a positive and $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic $C^\infty(\mathbb{R})$-function. In addition, the function $x \mapsto \eta_2(-x)$ is admissable with respect to α and c. If additionally K is smooth with Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$, then η_2 has the Fourier series representation

$$\eta_2(x) = -\sum_{n\in\mathbb{Z}} c_n \Gamma(1 - \alpha + in\tilde{c}) e^{-in\tilde{c}x}$$

for every $x \in \mathbb{R}$.

Proof. For every $s > 0$, we have

$$\widetilde{H_K}(s) = \int_0^\infty e^{-st} H_K(t)\, dt$$

$$
\begin{aligned}
&= \int_0^\infty e^{-st} \int_t^\infty z^{-\alpha} K(\log(z))\, dz\, dt \\
&= s^{\alpha-2} s^{2-\alpha} \int_0^\infty e^{-st} \int_t^\infty z^{-\alpha} K(\log(z))\, dz\, dt \\
&= s^{\alpha-2} \eta_2(\log(s))
\end{aligned}
$$

by (3.9) if η_2 is defined as in (4.2). Again it follows directly from the integral representation that η_2 is strictly positive. Besides, for every $x \in \mathbb{R}$, we get

$$
\eta_2\left(x + \log\left(c^{\frac{1}{\alpha}}\right)\right) = e^{x(2-\alpha)} c^{\frac{1}{\alpha}(2-\alpha)} \int_0^\infty e^{-e^x t c^{\frac{1}{\alpha}}} \int_t^\infty z^{-\alpha} K(\log(z))\, dz\, dt.
$$

Substitute $y := tc^{\frac{1}{\alpha}}$ to obtain

$$
\eta_2\left(x + \log\left(c^{\frac{1}{\alpha}}\right)\right) = e^{x(2-\alpha)} c^{\frac{1}{\alpha}(1-\alpha)} \int_0^\infty e^{-e^x y} \int_{yc^{-\frac{1}{\alpha}}}^\infty z^{-\alpha} K(\log(z))\, dz\, dy,
$$

and with $u := zc^{\frac{1}{\alpha}}$, we see that

$$
\begin{aligned}
\eta_2\left(x + \log\left(c^{\frac{1}{\alpha}}\right)\right) &= e^{x(2-\alpha)} c^{-1} \int_0^\infty e^{-e^x y} \int_y^\infty (uc^{-\frac{1}{\alpha}})^{-\alpha} K\left(\log\left(uc^{-\frac{1}{\alpha}}\right)\right)\, du\, dy \\
&= e^{x(2-\alpha)} \int_0^\infty e^{-e^x y} \int_y^\infty u^{-\alpha} K\left(\log(u)\right)\, du\, dy \\
&= \eta_2(x)
\end{aligned}
$$

using the periodicity of K. Hence η_2 is $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic. Additionally, note that $u_2 \in C^\infty(\mathbb{R})$ as a composition of infinitely differentiable functions. To show that $x \mapsto \eta(-x)$ is admissable, let $\delta > 0$ and study

$$
\begin{aligned}
\eta_2(-(x+\delta)) &= e^{-(x+\delta)(2-\alpha)} \int_0^\infty e^{-e^{-(x+\delta)} t} \int_t^\infty z^{-\alpha} K(\log(z))\, dz\, dt \\
&= e^{-x(2-\alpha)} e^{-\delta(1-\alpha)} \int_0^\infty e^{-e^{-x} y} \int_{ye^\delta}^\infty z^{-\alpha} K(\log(z))\, dz\, dy \\
&= e^{-x(2-\alpha)} e^{\delta\alpha} \int_0^\infty e^{-e^{-x} y} \int_y^\infty (ue^\delta)^{-\alpha} K\left(\log\left(ue^\delta\right)\right)\, du\, dy
\end{aligned}
$$

$$= e^{-x(2-\alpha)} \int_0^\infty e^{-e^{-x}y} \int_y^\infty u^{-\alpha} K(\log(u)+\delta)\, du\, dy$$

with the substitutions $y \coloneqq e^{-\delta}t$ and $u \coloneqq ze^{-\delta}$. Since K is admissable, the integral is bounded by

$$\begin{aligned}\eta_2(-(x+\delta)) &\leq e^{\alpha\delta} e^{-x(2-\alpha)} \int_0^\infty e^{-e^{-x}y} \int_y^\infty u^{-\alpha} K(\log(u))\, du\, dt \\ &= e^{\alpha\delta}\eta_2(-x),\end{aligned}$$

and therefore $x \mapsto \eta_2(-x)$ is admissable as well. Finally, assume that K is smooth with Fourier coefficients $(c_n)_{n\in\mathbb{N}}$. Then

$$\eta_2(x) = e^{x(2-\alpha)} \int_0^\infty e^{-e^x t} \int_t^\infty \sum_{n\in\mathbb{Z}} c_n z^{-\alpha+in\tilde{c}}\, dz\, dt.$$

Due to the fact that

$$\begin{aligned}\int_0^\infty \int_t^\infty \sum_{n\in\mathbb{Z}} \left| e^{-e^x t} c_n z^{-\alpha+in\tilde{c}} \right| dz\, dt &= \int_0^\infty \int_t^\infty \sum_{n\in\mathbb{Z}} e^{-e^x t} |c_n| z^{-\alpha}\, dz\, dt \\ &= \sum_{n\in\mathbb{Z}} |c_n| \int_0^\infty \frac{1}{\alpha-1} t^{-\alpha+1} e^{-e^x t}\, dt \\ &= \sum_{n\in\mathbb{Z}} \frac{|c_n|}{\alpha-1} e^{-x} \int_0^\infty (e^{-x}z)^{-\alpha+1} e^{-z}\, dz \\ &= \sum_{n\in\mathbb{Z}} \frac{|c_n|}{\alpha-1} e^{(\alpha-2)x} \Gamma(2-\alpha) \\ &< \infty\end{aligned}$$

with $z \coloneqq e^x t$, we can change the order of integration and summation such that with the same substitution, we get

$$\begin{aligned}\eta_2(x) &= e^{x(2-\alpha)} \sum_{n\in\mathbb{Z}} c_n \int_0^\infty e^{-e^x t} \int_t^\infty z^{-\alpha+in\tilde{c}}\, dz\, dt \\ &= e^{x(2-\alpha)} \sum_{n\in\mathbb{Z}} c_n \int_0^\infty e^{-e^x t} \frac{1}{\alpha-1-in\tilde{c}} t^{1-\alpha+in\tilde{c}}\, dt\end{aligned}$$

$$
\begin{aligned}
&= e^{x(2-\alpha)} \sum_{n\in\mathbb{Z}} \frac{c_n}{\alpha - 1 - in\tilde{c}} \int_0^\infty e^{-z}(ze^{-x})^{1-\alpha+in\tilde{c}} e^{-x}\, dt \\
&= \sum_{n\in\mathbb{Z}} \frac{c_n}{\alpha - 1 - in\tilde{c}} e^{x(\alpha-2-in\tilde{c}+2-\alpha)} \Gamma(2-\alpha+in\tilde{c}) \\
&= -\sum_{n\in\mathbb{Z}} c_n \Gamma(1-\alpha+in\tilde{c}) e^{-in\tilde{c}x},
\end{aligned}
$$

which finishes the proof. □

Combining Lemma 4.1.7 and Lemma 4.1.8, we obtain the following.

Theorem 4.1.9. (*Laplace transform of Caputo derivative, $\alpha \in (1,2)$*)
Let $\alpha \in (1,2)$, $c > 1$, let K be an admissable function with respect to these parameters, and let $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d_+) \cap C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}^d_+})$. In addition, let $\theta \in S \cap \overline{\mathbb{R}^d_+}$ be a fixed unit vector and assume that the function

$$(x,r) \mapsto e^{-\langle s,x\rangle} \langle \theta, \mathcal{H}_f(x - r\theta)\theta\rangle H_K(r) \mathbb{1}_{\mathbb{R}^d_+}(x - r\theta)$$

is integrable for every $s \in \mathbb{R}^d_+$ with respect to the Lebesgue measure on $\mathbb{R}^d_+ \times \mathbb{R}_+$. Then for every $s \in \mathbb{R}^d_+$, the Laplace transform of the Caputo form of the directional semi-fractional derivative is given by

$$\mathcal{L}\left(\frac{\partial^\alpha_\theta}{\partial_{c,K} x^\alpha} f\right)(s) = \langle s,\theta\rangle^{\alpha-2} \eta_2(\log(\langle s,\theta\rangle)) \int_{\mathbb{R}^d_+} e^{-\langle s,z\rangle} \langle \theta, \mathcal{H}_f(z)\theta\rangle\, dz, \tag{4.3}$$

where η_2 is defined as in (4.2).

Remark 4.1.10. As in the case $\alpha \in (0,1)$, it is possible to evaluate the integral in (4.3) using boundary values of f and its partial derivatives. For every $\alpha \in (1,2)$ and $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d_+) \cap C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}^d_+})$, we can extend f as well as its partial derivatives continuously to the boundary of $\mathbb{R}^d_+$. Similar to the definition of f_i for every $i = 1, \ldots, d$, we denote by

$$f_{i,j}(x) = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d)$$

for $i, j \in \{1, \ldots, d\}$ and $x \in \mathbb{R}^d_+$ the value of f in this boundary point and by

$$\left(\frac{\partial}{\partial x_i} f\right)_j (x) = \lim_{z\downarrow 0} \left(\frac{\partial}{\partial x_i} f\right)(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d)$$

the value of the partial derivative of f in $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d)$. Additionally, we assume that for every $i \neq j$

$$\frac{\partial}{\partial x_i} f_j(x) = \left(\frac{\partial}{\partial x_i} f\right)_j (x). \tag{4.4}$$

Then under the assumptions of Theorem 4.1.9,

$$\begin{aligned}\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\langle\theta,\mathcal{H}_f(z)\theta\rangle\,dz &= \sum_{i,j=1}^{d}\theta_i\theta_j\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\frac{\partial^2}{\partial x_i\partial x_j}f(z)\,dz\\ &= \sum_{i=1}^{d}\theta_i^2\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\frac{\partial^2}{\partial x_i^2}f(z)\,dz + \sum_{\substack{i,j=1,\\ i\neq j}}^{d}\theta_i\theta_j\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\frac{\partial^2}{\partial x_i\partial x_j}f(z)\,dz.\end{aligned}$$

Note that with [34, Equations (46)-(48)], for every $i\in\{1,\dots,d\}$

$$\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\frac{\partial^2}{\partial x_i^2}f(z)\,dz = s_i^2\widetilde{f}(s) - s_i\widetilde{f}_i(s) - \mathcal{L}\left(\left(\frac{d}{dx_i}f\right)_i\right)(s),$$

whereas for $i,j\in\{1,\dots,d\}$ with $i\neq j$

$$\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\frac{\partial^2}{\partial x_i\partial x_j}f(z)\,dz = s_is_j\widetilde{f}(s) - s_j\widetilde{f}_i(s) - s_i\widetilde{f_j}(s) + \widetilde{f_{i,j}}(s)$$

using (4.4). Similar to the case $\alpha\in(0,1)$, the Laplace transform $\widetilde{f}_i$ is $(d-1)$-dimensional, whereas that of $f_{i,j}$ is $(d-2)$-dimensional. Altogether we get

$$\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\langle\theta,\mathcal{H}_f(z)\theta\rangle\,dz = \langle s,\theta\rangle^2\ \widetilde{f}(s) - R(s),$$

where

$$R(s) \coloneqq \sum_{i=1}^{d}\theta_i^2\left(s_i\widetilde{f}_i(s) + \mathcal{L}\left(\left(\frac{d}{dx_i}f\right)_i\right)(s)\right) + \sum_{\substack{i,j=1,\\ i\neq j}}^{d}\theta_i\theta_j(2s_i\widetilde{f_j}(s) - \widetilde{f_{i,j}}(s))$$

for every $s\in\mathbb{R}_+^d$. Especially if $\theta=e_i$ for $i=1,\dots,d$ is a coordinate vector, then the formula simplifies to

$$\int\limits_{\mathbb{R}_+^d} e^{-\langle s,z\rangle}\langle e_i,\mathcal{H}_f(z)e_i\rangle\,dz = s_i^2\ \widetilde{f}(s) - s_i\widetilde{f}_i(s) - \mathcal{L}\left(\left(\frac{d}{dx_i}f\right)_i\right)(s)$$

for every $s\in\mathbb{R}_+^d$.

Again, we study the two examples most valuable for the following chapters.

Example 4.1.11. (Laplace transform of the one-dimensional Caputo form, $\alpha\in(1,2)$) First consider the one-dimensional case with $\theta=1$. Then the directional semi-fractional derivative coincides with the positive one-dimensional semi-fractional derivative, and for

f as in Theorem 4.1.9, the semi-fractional derivative has Laplace transform

$$\begin{aligned}\mathcal{L}\left(\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}f\right)(s) &= s^{\alpha-2}\eta_2(\log(s))\int\limits_{\mathbb{R}_+} e^{-\langle s,z\rangle}\langle\theta,\mathcal{H}_f(z)\theta\rangle\,dz \\ &= s^{\alpha-2}\eta_2(\log(s))(s^2\widetilde{f}(s)-sf(0)-f'(0))\end{aligned}$$

for every $s>0$.

Example 4.1.12. (Laplace transform of the one-dimensional fractional form, $\alpha\in(1,2)$) Let $K(x)=-\frac{1}{\Gamma(1-\alpha)}$ be the constant function corresponding to the fractional derivative. Again, K is smooth such that according to Lemma 4.1.8 we have $\eta_2(x)=1$ for every $x\in\mathbb{R}$. Then we obtain the Laplace transform of the fractional derivative by

$$\begin{aligned}\mathcal{L}\left(\frac{\partial^\alpha}{\partial x^\alpha}f\right)(s) &= s^{\alpha-2}(s^2\widetilde{f}(s)-sf(0)-f'(0)) \\ &= s^\alpha\widetilde{f}(s)-s^{\alpha-1}f(0)-s^{\alpha-2}f'(0),\end{aligned}$$

coinciding with [94, p. 34].

4.2 Laplace transform of the Riemann-Liouville form

After studying semi-fractional Caputo forms, we now turn to the Riemann-Liouville form of directional semi-fractional derivatives. Again, we start our consideration by analyzing the case $\alpha\in(0,1)$.

Theorem 4.2.1. (*Laplace transform of Riemann-Liouville derivative,* $\alpha\in(0,1)$) *Let* $\alpha\in(0,1)$, $c>1$, *let* K *be an admissable function with respect to these parameters, and let* $\theta\in S\cap\overline{\mathbb{R}^d_+}$ *be a fixed unit vector. For* $f\in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}^d_+)\cap C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}^d_+})$ *such that the Riemann-Liouville form of the semi-fractional derivative exists and for which*

$$(x,r)\mapsto e^{-\langle s,x\rangle}f(x-r\theta)\mathbb{1}_{\mathbb{R}^d_+}(x-r\theta)G_K(r)$$

is integrable for every $s\in\mathbb{R}^d_+$ *with respect to the Lebesgue measure on* $\mathbb{R}^d_+\times\mathbb{R}_+$, *the Laplace transform of the Riemann-Liouville form of the directional semi-fractional derivative is given by*

$$\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f\right)(s)=\langle s,\theta\rangle^\alpha\eta_1(\log(\langle s,\theta\rangle))\widetilde{f}(s)$$

for every $s\in\mathbb{R}^d_+$ *and* η_1 *as in (4.1).*

To prove Theorem 4.2.1, we will directly compute the Laplace transform. Similarly, one can obtain the result by utilizing the Laplace transform of the Caputo form in Theorem 4.1.4 and the difference between both forms in Lemma 3.2.12. However, for $\alpha\in(1,2)$, this method is not valid since the Laplace transform of the difference does not exist, and hence we use direct calculation in both proofs to obtain consistency for all $\alpha \in (0,2)\setminus\{1\}$.

Proof. For every function $f : \mathbb{R}_+^d \to \mathbb{R}$, the Laplace transform of the directional semi-fractional derivative is according to (3.18) given by

$$\begin{aligned}\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f\right)(s) &= \int_{\mathbb{R}_+^d} e^{-\langle s,x\rangle}\left\langle \nabla \int_{0+}^{\infty} f(x-r\theta)\mathbb{1}_{\mathbb{R}_+^d}(x-r\theta)G_K(r)\,dr, \theta\right\rangle dx \\ &= \sum_{i=1}^{d}\theta_i \int_{\mathbb{R}_+^d} e^{-\langle s,x\rangle}\frac{\partial}{\partial x_i}\int_{0+}^{\infty} f(x-r\theta)G_K(r)\mathbb{1}_{\mathbb{R}_+^d}(x-r\theta)\,dr\,dx.\end{aligned}$$

For fixed $\theta \in S\cap\overline{\mathbb{R}_+^d}$, define

$$y(x) := \min_{\substack{j=1,\dots,d,\\ \theta_j\neq 0}} \frac{x_j}{\theta_j}.$$

For every $i \in \{1,\dots,d\}$ with $\theta_i \neq 0$, we analyze the integral

$$\begin{aligned}A(s_i) &:= \int_{\mathbb{R}_+} e^{-s_i x_i}\frac{\partial}{\partial x_i}\int_{0+}^{\infty} f(x-r\theta)G_K(r)\mathbb{1}_{\mathbb{R}_+^d}(x-r\theta)\,dr\,dx_i \\ &= \int_{\mathbb{R}_+} e^{-s_i x_i}\frac{\partial}{\partial x_i}\int_{0+}^{y(x)} f(x-r\theta)G_K(r)\,dr\,dx_i \\ &= s_i\int_{\mathbb{R}_+} e^{-s_i x_i}\int_{0+}^{y(x)} f(x-r\theta)G_K(r)\,dr\,dx_i - \lim_{x_i\downarrow 0}\int_{0+}^{y(x)} f(x-r\theta)G_K(r)\,dr \\ &= s_i\int_{\mathbb{R}_+} e^{-s_i x_i}\int_{0+}^{y(x)} f(x-r\theta)G_K(r)\,dr\,dx_i.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f\right)(s) &= \sum_{i=1}^{d}\theta_i s_i \int_{\mathbb{R}_+^d} e^{-\langle s,x\rangle}\int_{0+}^{\infty} f(x-r\theta)G_K(r)\mathbb{1}_{\mathbb{R}_+^d}(x-r\theta)\,dr\,dx \\ &= \langle s,\theta\rangle \int_{0+}^{\infty}\int_{\mathbb{R}_+^d} e^{-\langle s,x\rangle} f(x-r\theta)G_K(r)\mathbb{1}_{\mathbb{R}_+^d}(x-r\theta)\,dx\,dr\end{aligned}$$

with Fubini's theorem using the assumption of integrability. With the substitution $z \coloneqq x - r\theta$, we obtain

$$\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f\right)(s) = \langle s, \theta\rangle \int\limits_{0+}^{\infty} \int\limits_{\mathbb{R}_+^d} e^{-\langle s, z+r\theta\rangle} f(z) G_K(r)\, dz\, dr$$

since $\theta \in S \cap \overline{\mathbb{R}_+^d}$. The result follows from the separation of the integrals

$$\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f\right)(s) = \langle s, \theta\rangle \widetilde{G_K}(\langle s, \theta\rangle)\widetilde{f}(s)$$

together with Lemma 4.1.3. □

Example 4.2.2. (Laplace transform of the Riemann-Liouville form, $d = 1$ and $\alpha \in (0, 1)$) Note that in one dimension and for $\theta = 1$, we obtain the Laplace transform of the semi-fractional Riemann-Liouville form with

$$\mathcal{L}\left(\left(\frac{\partial}{\partial_{c,K}x}\right)^\alpha f\right)(s) = s^\alpha \eta_1(\log(s))\widetilde{f}(s)$$

for every $s > 0$.

Example 4.2.3. (Laplace transform of the fractional Riemann-Liouville form, $\alpha \in (0, 1)$) If we choose $K(x) = \frac{1}{\Gamma(1-\alpha)}$, the semi-fractional derivative equals the fractional one, and from Lemma 4.1.3, it follows that $\eta_1(x) = 1$ for every $x \in \mathbb{R}$. Then

$$\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial x}\right)^\alpha f\right)(s) = \langle s, \theta\rangle^\alpha \widetilde{f}(s)$$

for every $s \in \mathbb{R}_+^d$, which especially yields

$$\mathcal{L}\left(\left(\frac{\partial}{\partial x}\right)^\alpha f\right)(s) = s^\alpha \widetilde{f}(s)$$

in the one-dimensional case with $\theta = 1$ (compare [94, p. 39]).

Finally, we consider the Riemann-Liouville form in the case $\alpha \in (1, 2)$.

Theorem 4.2.4. (*Laplace transform of Riemann-Liouville derivative, $\alpha \in (1, 2)$*) *Let $\alpha \in (1, 2)$, $c > 1$, let K be an admissable function with respect to these parameters, and let $\theta \in S \cap \overline{\mathbb{R}_+^d}$ be a fixed unit vector. For $f \in W^{\lfloor\alpha\rfloor+2}(\mathbb{R}_+^d) \cap C_0^{\lfloor\alpha\rfloor+1}(\overline{\mathbb{R}_+^d})$ such that the Riemann-Liouville form of the semi-fractional derivative exists and*

$$(x, r) \mapsto e^{-\langle s, x\rangle} f(x - r\theta)\mathbb{1}_{\mathbb{R}_+^d}(x - r\theta) H_K(r)$$

is integrable for every $s \in \mathbb{R}_+^d$ with respect to the Lebesgue measure on $\mathbb{R}_+^d \times \mathbb{R}_+$, the Laplace transform of the Riemann-Liouville form of the directional semi-fractional derivative is *given by*

$$\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f\right)(s) = \langle s,\theta\rangle^\alpha \eta_2(\log(\langle s,\theta\rangle))\widetilde{f}(s)$$

for every $s \in \mathbb{R}^d_+$ *and* η_2 *as in (4.2).*

Proof. According to (3.19),

$$\begin{aligned}\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha f\right)(s) &= \int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle}\langle\theta, \mathcal{H}_{I(f)}(x)\theta\rangle\, dx \\ &= \sum_{i,j=1}^d \theta_i\theta_j \int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle}\frac{\partial^2}{\partial x_i \partial x_j} I(f)(x)\, dx\end{aligned}$$

for every $s \in \mathbb{R}^d_+$, where

$$I(f)(x) = \int\limits_{0+}^{\infty} f(x - r\theta)\mathbb{1}_{\mathbb{R}^d_+}(x - r\theta)H_K(r)\, dr.$$

For fixed $i, j \in \{1, \ldots, d\}$, consider the integral

$$B(s) := \int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle}\frac{\partial^2}{\partial x_i \partial x_j} I(f)(x)\, dx.$$

Integration by parts with respect to x_i yields

$$B(s) = \left[\int\limits_{\mathbb{R}^{d-1}_+} e^{-\langle s,x\rangle}\frac{\partial}{\partial x_j} I(f)(x)\, dx^{(i)}\right]_{x_i=0}^{\infty} + s_i \int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle}\frac{\partial}{\partial x_j} I(f)(x)\, dx,$$

where $x^{(i)} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$ is the projection of x onto all but the i-th component. To evaluate the first term, note that

$$\int\limits_{\mathbb{R}^{d-1}_+} e^{-\langle s,x\rangle}\frac{\partial}{\partial x_j} I(f)(x)\, dx^{(i)} = \left[\int\limits_{\mathbb{R}^{d-2}_+} e^{-\langle s,x\rangle} I(f)(x)\, dx^{(i,j)}\right]_{x_j=0}^{\infty} + s_j \int\limits_{\mathbb{R}^{d-1}_+} e^{-\langle s,x\rangle} I(f)(x)\, dx^{(i)},$$

where $x^{(i,j)} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)$ is the projection of x onto all but the i-th and j-th components. Similar to the case $\alpha \in (0,1)$, with

$$y(x) := \min_{\substack{j=1,\dots,d,\\ \theta_j \neq 0}} \frac{x_j}{\theta_j}$$

for $\theta \in S \cap \overline{\mathbb{R}^d_+}$, we find

$$\lim_{x_j \to 0} I(f)(x) = \lim_{x_j \to 0} \int_{0+}^{y(x)} f(x - r\theta) H_K(r)\, dr = 0,$$

and since $f \in C_0(\overline{\mathbb{R}^d_+})$ and $H_K(y) \sim y^{1-\alpha}$, we additionally obtain

$$\lim_{x_j \to \infty} e^{-s_j x_j} I(f)(x) = \lim_{x_j \to \infty} e^{-s_j x_j} \int_{0+}^{y(x)} f(x - r\theta) H_K(r)\, dr = 0$$

such that

$$\int_{\mathbb{R}^{d-1}_+} e^{-\langle s,x\rangle} \frac{\partial}{\partial x_j} I(f)(x)\, dx^{(i)} = s_j \int_{\mathbb{R}^{d-1}_+} e^{-\langle s,x\rangle} I(f)(x)\, dx^{(i)}.$$

Following these arguments, the integral B is given by

$$B(s) = s_i \int_{\mathbb{R}^d_+} e^{-\langle s,x\rangle} \frac{\partial}{\partial x_j} I(f)(x)\, dx$$

for every $s \in \mathbb{R}^d_+$. Repeated integration by parts delivers

$$\begin{aligned} B(s) &= \left[s_i \int_{\mathbb{R}^{d-1}_+} e^{-\langle s,x\rangle} I(f)(x)\, dx^{(j)} \right]_{x_j=0}^{\infty} + s_i s_j \int_{\mathbb{R}^d_+} e^{-\langle s,x\rangle} I(f)(x)\, dx \\ &= s_i s_j \int_{\mathbb{R}^d_+} e^{-\langle s,x\rangle} I(f)(x)\, dx \end{aligned}$$

for every $s \in \mathbb{R}^d_+$. Then the Laplace transform of the semi-fractional derivative is given by

$$\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K} x} \right)^{\alpha} f \right)(s) = \sum_{i,j=1}^{d} \theta_i \theta_j s_i s_j \int_{\mathbb{R}^d_+} e^{-\langle s,x\rangle} I(f)(x)\, dx = \langle s, \theta\rangle^2 \widetilde{I}(f)(s).$$

Now using the assumption of integrability, with Fubini's theorem, it follows that

$$
\begin{aligned}
\widetilde{I}(f)(s) &= \int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle} \int\limits_{0+}^{\infty} f(x-r\theta)\mathbb{1}_{\mathbb{R}^d_+}(x-r\theta)H_K(r)\,dr\,dx \\
&= \int\limits_{0+}^{\infty} \int\limits_{\mathbb{R}^d_+} e^{-\langle s,x\rangle} f(x-r\theta)\mathbb{1}_{\mathbb{R}^d_+}(x-r\theta)H_K(r)\,dx\,dr \\
&= \int\limits_{0+}^{\infty} \int\limits_{\mathbb{R}^d_+} e^{-\langle s,z+r\theta\rangle} f(z)H_K(r)\,dz\,dr \\
&= \widetilde{f}(s)\widetilde{H_K}(\langle s,\theta\rangle)
\end{aligned}
$$

for every $s \in \mathbb{R}^d_+$, completing the proof together with Lemma 4.1.8. □

Example 4.2.5. (Laplace transform of the Riemann-Liouville form, $d = 1$ and $\alpha \in (1,2)$) In the one-dimensional case with $\theta = 1$, the Laplace transform of the semi-fractional derivative is given by

$$
\mathcal{L}\left(\left(\frac{\partial}{\partial_{c,K} x}\right)^{\alpha} f\right)(s) = s^{\alpha}\eta_2(\log(s))\widetilde{f}(s)
$$

for every $s > 0$.

Example 4.2.6. (Laplace transform of the fractional Riemann-Liouville form, $\alpha \in (1,2)$) If we choose $K(x) = -\frac{1}{\Gamma(1-\alpha)}$, then the semi-fractional derivative equals the fractional one. In addition, according to Lemma 4.1.8, we obtain $\eta_2(x) = 1$ for every $x \in \mathbb{R}$ and hence

$$
\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K} x}\right)^{\alpha} f\right)(s) = \langle s,\theta\rangle^{\alpha}\widetilde{f}(s)
$$

for every $s \in \mathbb{R}^d_+$. Especially in one dimension with $\theta = 1$, this yields the well-known formula [94, p. 34]

$$
\mathcal{L}\left(\left(\frac{\partial_\theta}{\partial_{c,K} x}\right)^{\alpha} f\right)(s) = s^{\alpha}\widetilde{f}(s)
$$

for every $s > 0$.

Chapter 5

Semi-fractional Cauchy problems

The overall motivation for the definition of semi-fractional derivatives and the analysis of their properties in the last chapters was to construct a connection between semistable densities and solutions to semi-fractional differential equations. In Remark 3.1.4 (iii), we already touched on this connection in order to justify the definition of semi-fractional derivatives. The first aim of this chapter is to formulate and prove this connection rigorously. Consequently, both theories can profit from each other: Semistable distributions can now be characterized in a new way and benefit from obtained and following results about semi-fractional diffusion. Especially, we can use the Grünwald-Letnikov formula to approximate semistable laws numerically. On the other hand, the connection to semistable laws may arouse interest in the quite new theory of semi-fractional differential equations and inspire new work on this topic.

However, we do not restrict our attention to those semi-fractional differential equations directly connected to semistable laws. Instead, we consider a wide class of semi-fractional Cauchy problems. In our context, such an equation may include a semi-fractional derivative in the time as well as in the space variable and is studied under a given initial condition. Since such an initial state of the underlying system is known for many real-world applications, Cauchy problems are widely used in mathematical modeling and are thereby of particular importance.

Specifically, the theory of fractional Cauchy problems gained influence during the last decades, and there is a rapidly growing number of publications dealing with existence and uniqueness results for different kinds of Cauchy equations (e.g., see [36] or [69] and the references cited therein). Note that the whole theory of fractional differential equation spreads much wider than the analysis of Cauchy problems and, for example, includes several non-linear equations under different initial or boundary conditions (see for example [36], [120], and [143]) as well as fractional differential problems on bounded domains (compare [13], [29], or [12]). Partly, these results can surely be extended to the case of semi-fractional diffusion and would be fascinating to investigate. However, as a start, we concentrate on semi-fractional Cauchy problems here. Nevertheless, we hope to offer a glimpse into the potential of semi-fractional differential equations in general.

The chapter is structured as follows. In Section 5.1, we use semigroup theory to study semi-fractional diffusion equations. These are differential equations with a semi-fractional space but an ordinary time derivative under given initial conditions, which finally deliver the connection between semistable laws and semi-fractional derivatives. Afterward, we study semistable subordinators in Section 5.2, enabling us to solve more general Cauchy problems involving a time and space semi-fractional derivative in Section 5.3.

5.1 Semi-fractional diffusion and semistable densities

To finally establish a connection between semistable laws and semi-fractional diffusion, consider the following setting. Fix $\alpha \in (0,2) \setminus \{1\}$, $c > 1$, and an admissable set of functions $(K_\theta)_{\theta \in S}$ as well as a probability measure M on the unit sphere. We study the semi-fractional diffusion equation

$$\frac{\partial}{\partial t} u(x,t) = -D \mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}} u(x,t) \tag{5.1}$$

under the initial condition $u(x,0) = u_0(x)$ for every $x \in \mathbb{R}^d$ and $t > 0$. Thereby, the multidimensional semi-fractional derivative is the generator form, which coincides with the Caputo form under smoothness assumptions on u (compare Section 3.3). Besides, recall that $D = D(\alpha) = (-1)^{\lfloor \alpha \rfloor}$, and hence, the sign on the right-hand side of (5.1) differs for the two cases $\alpha \in (0,1)$ and $\alpha \in (1,2)$. Using the semigroup theory introduced in Section 2.3, we obtain the following result.

Lemma 5.1.1. *For fixed $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$, let $(K_\theta)_{\theta \in S}$ be an admissable set of functions and let M be a probability measure on the unit sphere. Furthermore, let ν be the $(c^{\frac{1}{\alpha}}, c)$-semistable distribution with Lévy-Khintchine triple $[a, 0, \Phi]$, where*

$$a = \begin{cases} \displaystyle\int_{\mathbb{R}^d \setminus \{0\}} \frac{x}{1+||x||^2}\, d\Phi(x) & \text{if } \alpha \in (0,1) \\ \displaystyle\int_{\mathbb{R}^d \setminus \{0\}} \left(\frac{x}{1+||x||^2} - x \right) d\Phi(x) & \text{if } \alpha \in (1,2) \end{cases} \tag{5.2}$$

and the Lévy measure Φ is given by (2.5), and denote by $(A(t))_{t\geq 0}$ the corresponding semistable Lévy process. If $u_0 \in \mathrm{Dom}\left(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}\right)$, then the unique solution to (5.1) is given by $u(x,t) = \mathbb{E}[u_0(x - A(t))]$.

Proof. Let L be the generator of the semigroup $(T(t))_{t\geq 0}$ defined in (2.14). According to Lemma 3.3.10, the multidimensional semi-fractional derivative and the generator are connected by

$$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}} u(x,t) = -DLu(x,t)$$

for every $x \in \mathbb{R}^d$ and $t > 0$. Then from $u_0 \in \operatorname{Dom}\left(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}\right)$, it follows that $u_0 \in \operatorname{Dom}(L)$, and according to Section 2.3, the unique solution to the Cauchy problem

$$\frac{\partial}{\partial t}u(x,t) = Lu(x,t) = -D\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}u(x,t)$$

is given by $u(x,t) = \mathbb{E}[u_0(x - A(t))]$. □

Example 5.1.2. Consider the one-dimensional diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = -\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}u(x,t) \tag{5.3}$$

for $\alpha = 0.6$, $c = e^{\pi\alpha}$, and

$$K(x) = \frac{3}{15}\cos(2x) + \frac{3}{20}\sin(4x) + \frac{3}{\Gamma(1-\alpha)}$$

admissable with respect to these parameters. In addition, choose the initial condition $u_0(x) = e^{-x^2}$ and note that $u_0 \in \operatorname{Dom}(\frac{\partial^\alpha}{\partial_{c,K}x^\alpha})$. Figure 5.1 displays the solution $x \mapsto u(x,t)$ for two different times ($t = 0.5$ and $t = 1$). For comparison, we additionally show the solution to the corresponding fractional equation

$$\frac{\partial}{\partial t}u(x,t) = -3\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) \tag{5.4}$$

with $u(x,0) = u_0(x)$. The code for the calculation is attached in Appendix C. We observe that the periodic perturbation in the semi-fractional derivative causes an oscillation of the solution around the fractional one. Hence, the overall behavior is preserved, whereas, on a small scale, we find remarkable differences between both curves.

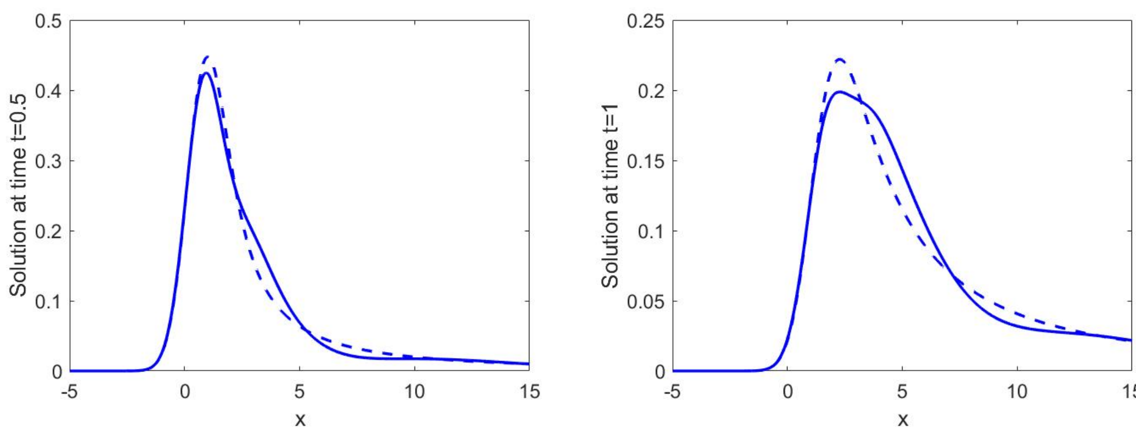

Figure 5.1: Solution to the semi-fractional diffusion equation (5.3) in Example 5.1.2 at time $t = 0.5$ (left) and $t = 1$ (right) shown as a solid line in comparison to the solution to the corresponding fractional equation (5.4) displayed as a dashed line.

In the situation of Lemma 5.1.1, the solution of the semi-fractional Cauchy problem is a classical one. This is, u is continuous in $t \geq 0$, continuously differentiable in $t > 0$, $u(\cdot, t) \in \mathrm{Dom}\left(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}\right)$ for every $t > 0$, and (5.1) is fulfilled point-wise. However, to construct a connection between semistable laws and semi-fractional diffusion, we have to weaken the assumptions on u_0, yielding a slightly weaker solution.

Theorem 5.1.3. *(Connection between semistable densities and semi-fractional diffusion) For fixed $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$, let ν be the $(c^{\frac{1}{\alpha}}, c)$-semistable distribution in Lemma 5.1.1 with corresponding Lévy process $(A(t))_{t\geq 0}$. Then the densities $x \mapsto p(x,t)$ of $A(t)$ solve the semi-fractional Cauchy problem (5.1) under the initial condition $u(x,0) = \delta(x)$ for almost every $x \in \mathbb{R}^d$.*

Proof. Note that a proof of this theorem in one dimension has already been published in [66, Theorem 5.1], and the multidimensional case works similarly. Apply a Fourier transform to (5.1) to obtain

$$\frac{\partial}{\partial t}\widehat{u}(k,t) = \mathcal{F}\left(\frac{\partial}{\partial t}u\right)(k,t) = -D\mathcal{F}\left(\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}u\right)(k,t) = \Psi(k)\widehat{u}(k,t)$$

using (3.25) and (2.7), where Ψ is the log-characteristic function of ν. If we especially consider the semistable density p, then

$$\frac{\partial}{\partial t}\widehat{p}(k,t) = \frac{\partial}{\partial t}e^{t\Psi(k)} = \Psi(k)\widehat{p}(k,t),$$

and hence p solves (5.1) in the Fourier domain. Finally, Fourier inversion yields the result. □

Together with the numerical approximation of the semi-fractional derivative in Section 3.4, this result enables us to plot semistable densities as a solution to the Cauchy problem. Since no explicit representations of purely semistable densities are known, pictures of such distributions were quite rare before, and hence this result contributes to a better understanding of semistable laws.

Example 5.1.4. Let $\alpha \in (0,1)$, $c > 0$, and let K be an admissable function with respect to these parameters. In addition, let ν be the semistable distribution on $\mathbb{R}$ with Lévy-Khintchine triple $[a, 0, \Phi]$, where a is as in (5.2) and

$$\Phi(-\infty, -r) = 0, \quad \Phi(r,\infty) = r^{-\alpha}K(\log(r)) \tag{5.5}$$

for every $r > 0$. According to Theorem 5.1.3, the densities $x \mapsto p(x,t)$ of the corresponding Lévy process solve the diffusion equation

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial^\alpha}{\partial_{c,K}x^\alpha}p(x,t).$$

For $\alpha = \frac{1}{2}$, $c = 2$, and

$$K(x) = \frac{1}{80}\left(\cos\left(\frac{4\pi}{\log(4)}x\right) + 2\cos\left(\frac{2\pi}{\log(4)}x\right)\right) + \frac{1}{\Gamma(1-\alpha)} \tag{5.6}$$

admissable with respect to these parameters, the semistable density $p(x,1)$ of ν is shown in Figure 5.2. The approximation indicates that this semistable density is not unimodal. Concerning this property, semistable laws differ crucially from stable ones, which are known to be unimodal (see [152, Theorem 2.7.6]). Thereby, we see that the class of semistable laws is much wider than this of stable ones, which underlines the importance of studying both. Note that in general semistable distributions can be either uni- or multimodal. For more general information, we refer to [122], [145], or [146].

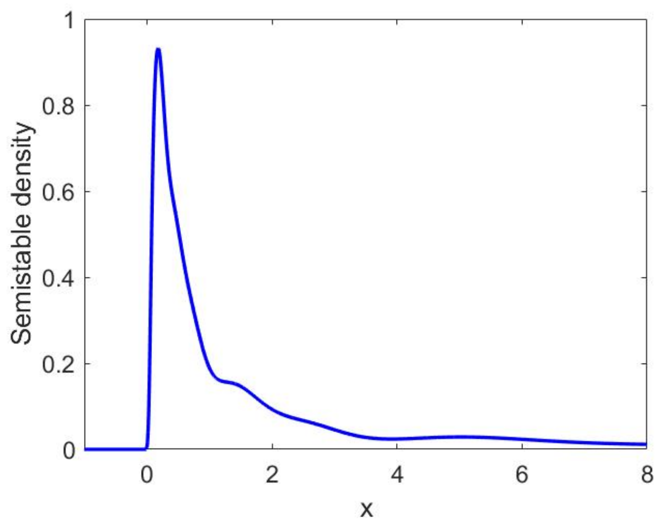

Figure 5.2: Density of the semistable distribution $\nu = p(x,1)$ in Example 5.1.4.

Since no explicit expressions of semistable densities are known, we cannot compare our approximation with the real density. However, in Section 2.1, we observed that semistable densities fulfill a particular scaling, and we can at least check whether this property is satisfied by our approximation. First note that for $\alpha \in (0,1)$, Lemma 2.14 in [74] ensures that ν is even strictly semistable. Then according to (2.4),

$$p(x,2) = \frac{1}{4}p\left(\frac{x}{4},1\right)$$

for every $x \in \mathbb{R}$. In Figure 5.3, we plotted both sides of this equation, finding that they are in good agreement, and in this sense, our approach is strengthened. Again, the Matlab code for the calculation is given in Appendix C.

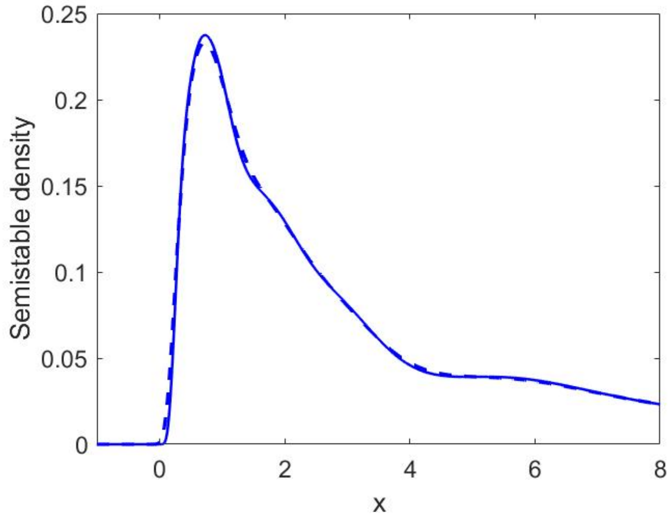

Figure 5.3: Density $x \mapsto p(x,2)$ (solid line) and $x \mapsto \frac{1}{4}p\left(\frac{x}{4},1\right)$ (dashed line) in Example 5.1.4.

Remark 5.1.5. To the best of our knowledge, the only alternative approach to plot semistable densities is given in [28], where one-dimensional semistable densities were approximated using inverse Laplace transforms. We want to compare both methods in the situation of Example 5.1.4.
Instead of interpreting the semistable density as a solution to a semi-fractional differential equation, the author of [28] used the inverse Laplace method of Abate and Whitt [1], stating the following. If the density $p = p(x,1)$ of ν is supported on $(0,\infty)$, then

$$p(x) \approx \frac{e^{\frac{A}{2}}}{2x}\operatorname{Re}\left(\widetilde{p}\left(\frac{A}{2x}\right)\right) + \frac{e^{\frac{A}{2}}}{x}\sum_{k=1}^{\infty}(-1)^k \operatorname{Re}\left(\widetilde{p}\left(\frac{A+2ki\pi}{2x}\right)\right) \tag{5.7}$$

for every $x > 0$, where $A > 0$ is a tuning parameter. Using an additional shift, the author of [28] also approximated semistable densities on the whole real line. However, since this procedure may cause further errors, we decided to stick to the case of densities supported on $\mathbb{R}_+$ here. So as in Example 5.1.4, we study a strictly semistable density p, which according to [74, Lemma 2.14] is supported on $\mathbb{R}_+$. Remark that according to Example 2.2.6, the Laplace transform of the density p is given by

$$\widetilde{p}(s) = \exp\left(-\sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1)s^{\alpha - in\tilde{c}}\right)$$

for every $s > 0$. Inserting this expression in (5.7) yields a numerical approximation of the semistable density.

Independent from the choice of A, the approximated density preserves the scaling property

(2.4). To see this, note that according to (5.7)

$$p(c^{-\frac{1}{\alpha}}x,t) \approx \frac{e^{\frac{A}{2}}}{2x} c^{\frac{1}{\alpha}} \widetilde{p}\left(\frac{A}{2x}c^{\frac{1}{\alpha}},t\right) + \frac{e^{\frac{A}{2}}}{x} c^{\frac{1}{\alpha}} \sum_{k=1}^{\infty} (-1)^k \operatorname{Re}\left(\widetilde{p}\left(\frac{A+2ki\pi}{2x}c^{\frac{1}{\alpha}},t\right)\right)$$

for every $x > 0$ and $t > 0$. Since

$$\begin{aligned}\widetilde{p}(yc^{\frac{1}{\alpha}},t) &= \exp\left(-t\sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c}-\alpha+1)(yc^{\frac{1}{\alpha}})^{\alpha-in\tilde{c}}\right)\\ &= \exp\left(-tc\sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c}-\alpha+1)y^{\alpha-in\tilde{c}}\right)\\ &= \widetilde{p}(y,ct)\end{aligned}$$

for every $y > 0$, we obtain

$$p(c^{-\frac{1}{\alpha}}x,t) \approx c^{\frac{1}{\alpha}} \left(\frac{e^{\frac{A}{2}}}{2x}\widetilde{p}\left(\frac{A}{2x},ct\right) + \frac{e^{\frac{A}{2}}}{x}\sum_{k=1}^{\infty}(-1)^k \operatorname{Re}\left(\widetilde{p}\left(\frac{A+2ki\pi}{2x},ct\right)\right)\right),$$

which is the approximation formula of $c^{\frac{1}{\alpha}}p(x,ct)$ according to the method in [28].

To compare both methods, we use the parameters from Example 5.1.4. This is, let $\alpha = \frac{1}{2}$, $c = 2$ as well as K as in (5.6) and Φ as in (5.5). As shown in Figure 5.4, the approximation using the inverse Laplace transform method yields slightly different results depending on the tuning parameter A.

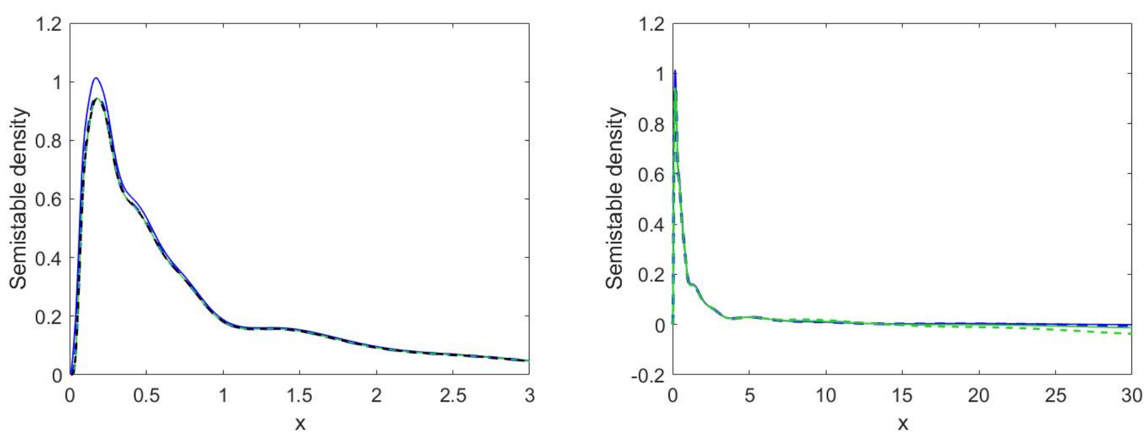

Figure 5.4: Approximation of the semistable density $p(x,1)$ using the method of [28] for different values of A ($A = 2$ blue solid line), $A = 4$ (blue dashed line), $A = 6$ (green solid line) and $A = 8$ (black dashed line) for $x \in [0,3]$ (left) and on a large scale $x \in [0,30]$ (right) in Remark 5.1.5.

Note that the approximation for $A = 2$ (blue solid line) differs from all other curves in the height of its first maximum and also exhibits a greater oscillation for values of x near 0.5. After behaving similarly for $x \in [1, 15]$, the approximations corresponding to larger values of A ($A = 6$ and $A = 8$) take negative values, which contradicts the fact that we approximate a semistable density. Hence we stick to the smaller values $A = 2$ and $A = 4$ for the following comparison.

We now compare the remaining curves corresponding to $A = 2$ and $A = 4$ with the Grünwald-Letnikov approximation. As seen in Figure 5.5, the overall behavior is similar, but there is a slight difference for small values of x. Nevertheless, the general agreement of both approaches strengthens the validity of both approximations.

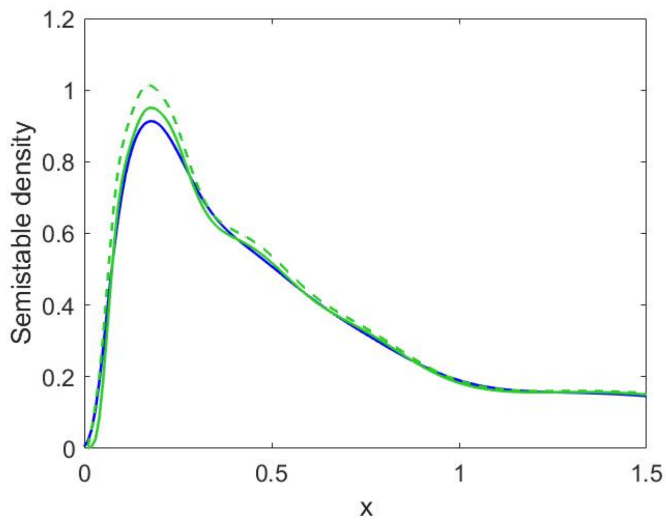

Figure 5.5: Comparison of the approximation using Grünwald-Letnikov differences (blue solid line) and the approximation with the method of Abate and Whitt for $A = 2$ (green dashed line) and $A = 4$ (green solid line) in Remark 5.1.5.

Note that by calculating an approximation of the semistable density with the inverse Laplace transform method for every $A \in \{1.8, 1.81, \ldots, 4.19, 4.20\}$, we find the best agreement with our result for $A = 3.89$. Thereby, the difference between both approximations was measured using the least squared error on $x \in [0, 8]$. Both approximations are in good agreement, as shown in Figure 5.6. All Matlab scripts used in this remark can be found in Appendix C.

So how can we obtain a reliable approximation of semistable densities? A weakness of the inverse Laplace transform method is undoubtedly the right choice of the tuning parameter A. In [28], the author applied his method to stable densities likewise and found the best agreement with the actual density for larger values of A (for example $A = 9$). However, for our particular example, these values failed as an approximation of the semistable density. In contrast, the approximation with Grünwald-Letnikov differences is more expensive to calculate numerically but does not depend on an additional tuning parameter. Additionally, our method directly transfers to semistable densities on the whole real line. For these reasons, the approximation with Grünwald-Letnikov difference might be preferred to the inverse Laplace transform method or at least may be used as additional validation.

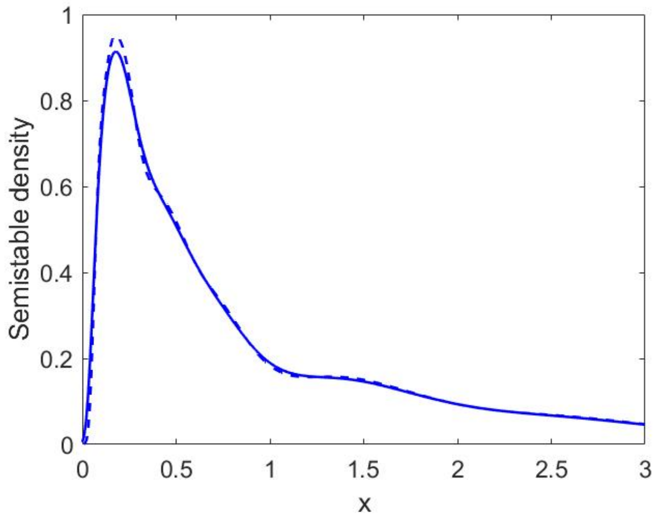

Figure 5.6: Comparison of the approximation using Grünwald-Letnikov differences (blue solid line) and the approximation with the method of Abate and Whitt for $A = 3.89$ (blue dashed line) in Remark 5.1.5.

5.2 Semistable subordinators

In contrast to ordinary derivatives, the fractional and the semi-fractional derivative are non-local operators. For this reason, the solution of the semi-fractional diffusion equation (5.1) at a given point is influenced by the whole surrounding environment. A similar behavior is observed in many physical applications when considering the time variable. More precisely, the current status of a physical system often depends on all of its past states. In this case, the corresponding system is subject to long-term memory effects. Modeling exactly such an outcome, differential equations with time-fractional derivatives have been considered extensively (see for example [71], [10], or the monograph [72] for a comprehensive overview).

In the following, we investigate semi-fractional Cauchy problems with an additional semi-fractional time derivative. Thereby, we model a long-time memory effect with log-periodic perturbations and hence extend the possible models for such situations. As a preparation, this section analyzes semistable subordinators, which are semistable processes with almost surely non-decreasing paths. These processes and their properties deliver the key ideas to solve general Cauchy problems in Section 5.3.

Subordinators are of particular interest not only in the case of stable or semistable processes. This section only proves a few properties for semistable subordinators, which are necessary to solve abstract Cauchy problems later on. However, we refer to [22], [122], and [74] for information about subordinators in general. First note that the property of non-decreasing paths can equally be obtained from the Lévy-Khintchine representation.

Theorem 5.2.1. *(Characterization of subordinators, [74, Lemma 2.14])*
A Lévy process $(X(t))_{t\geq 0}$ *on* $\mathbb{R}$ *with* $X_1 \sim [a, Q, \Phi]$ *is a subordinator if and only if* $Q = 0$, $\Phi(-\infty, 0) = 0$ *as well as*

$$\int_0^\infty \min\{1, x\}\, d\Phi(x) < \infty$$

and

$$a - \int_{0+}^\infty \frac{x}{1+x^2}\, d\Phi(x) \geq 0.$$

To construct a semistable subordinator, consider the following situation. Let μ be a $(d^{\frac{1}{\beta}}, d)$-semistable distribution for some $\beta \in (0, 1)$ and $d > 1$ with Lévy-Khintchine triple $[a, 0, \Phi]$. We assume that the Lévy measure Φ is concentrated on the positive real axis, this is

$$\Phi(-\infty, -r) = 0 \quad \text{ and } \quad \Phi(r, \infty) = G_V(r) = r^{-\beta} V(\log(r))$$

for every $r > 0$ and an admissable, $\log\left(d^{\frac{1}{\beta}}\right)$-periodic function V. Besides, we choose the drift coefficient $a \in \mathbb{R}$ as

$$a := \int_{0+}^\infty \frac{x}{1+x^2}\, d\Phi(x).$$

Then the log-characteristic function Ψ of μ can be written as

$$\Psi(x) = \int_{0+}^\infty \left(e^{ixy} - 1\right)\, d\Phi(y), \tag{5.8}$$

and according to Theorem 5.2.1, the Lévy process $(D(t))_{t\geq 0}$ with $P_{D(1)} = \mu$ is a semistable subordinator. In addition, $\Phi(0, \infty) = \infty$, which implies that the sample paths are even strictly increasing almost surely ([122, Theorem 21.3]). According to [122, Example 28.2], the process has $C^\infty(\mathbb{R})$-densities $x \mapsto g(x, t)$ for every $t > 0$, and for further calculations, we need an explicit form of their Laplace transform.

Lemma 5.2.2. *(Laplace transform of semistable subordinator densities)*
Let $(D(t))_{t\geq 0}$ *be the semistable subordinator with density* $x \mapsto g(x, t)$ *for every* $t > 0$.

Then

$$\widetilde{g}(s,t) = \int_0^\infty e^{-xs} g(x,t)\, dx = e^{-t\Gamma_{D(1)}(s)} \tag{5.9}$$

for every $s > 0$ with

$$\Gamma_{D(1)}(s) = s^\beta \eta_1(\log(s)). \tag{5.10}$$

Thereby, $\eta_1 : \mathbb{R} \to (0,\infty)$ is the $\log\left(d^{\frac{1}{\beta}}\right)$-periodic, positive $C^\infty(\mathbb{R})$-function defined in (4.1).

Proof. The existence of the Laplace exponent in (5.9) follows immediately from the Lévy-Khintchine triple, so we only have to prove the explicit representation and the properties of $\Gamma_{D(1)}$. Using integration by parts for Riemann-Stieltjes integrals [76, Chapter X, Proposition 1.4], we obtain

$$\begin{aligned}
\Gamma_{D(1)}(s) &= \int_{0+}^\infty (1 - e^{-sx})\, d\Phi(x) \\
&= -\int_{0+}^\infty (1 - e^{-sx})\, dG_V(x) \\
&= -[(1 - e^{-sx})G_V(x)]_{0+}^\infty + \int_{0+}^\infty s e^{-sx} G_V(x)\, dx
\end{aligned}$$

for every $s > 0$. Note that the first part vanishes since $G_V(x) \to 0$ as $x \to \infty$, and using the boundedness of V, we get

$$\begin{aligned}
\lim_{x\downarrow 0} |(1 - e^{-sx})G_V(x)| &= \lim_{x\downarrow 0} |(1 - e^{-sx})x^{-\beta} V(\log(x))| \\
&\le C_{10} \lim_{x\downarrow 0} |s x^{1-\beta}| \\
&= 0
\end{aligned}$$

for a constant $C_{10} > 0$ using a Taylor expansion. Then

$$\Gamma_{D(1)}(s) = s \int_{0+}^\infty e^{-sx} G_V(x)\, dx = s\widetilde{G_V}(s)$$

for every $s > 0$ and according to Lemma 4.1.3

$$\widetilde{G_V}(s) = s^{\beta-1}\eta_1(\log(s)),$$

where $\eta_1 : \mathbb{R} \to \mathbb{R}_+$ has the claimed properties. $\square$

Using this representation, we can prove an additional property of the Laplace exponent $\Gamma_{D(1)}$.

Lemma 5.2.3. *The function* $\Gamma_{D(1)} : (0, \infty) \to (0, \infty)$ *is bijective.*

Proof. According to (5.10), the function $\Gamma_{D(1)}$ is continuous with $\lim_{s \downarrow 0} \Gamma_{D(1)}(s) = 0$ and $\lim_{s \to \infty} \Gamma_{D(1)}(s) = \infty$. We show that $\Gamma_{D(1)}$ is strictly increasing, which yields the result. For every $s > 0$ and $h > 0$, we obtain

$$\Gamma_{D(1)}(s+h) - \Gamma_{D(1)}(s) = \int_{0+}^{\infty} \left(1 - e^{-(s+h)y} - 1 + e^{-sy}\right) d\Phi(y)$$
$$= \int_{0+}^{\infty} e^{-sy} \underbrace{\left(1 - e^{-hy}\right)}_{>0} d\Phi(y) > 0$$

since Φ is non-degenerate. Hence $\Gamma_{D(1)}$ is strictly increasing. $\square$

The paths of $(D(t))_{t \geq 0}$ are strictly increasing almost surely such that the inverse semistable subordinator given by

$$E(t) \coloneqq \inf\{u > 0 : D(u) > t\} \qquad \forall t \geq 0,$$

is well-defined, and from the definition it follows immediately that

$$\{E(t_i) \leq s_i \text{ for } i = 1, \ldots, m\} = \{D(s_i) \geq t_i \text{ for } i = 1, \ldots, m\} \tag{5.11}$$

for any $s_1, \ldots, s_m, t_1, \ldots, t_m > 0$ with $m \in \mathbb{N}$. Due to the fact that $(D(t))_{t \geq 0}$ is a strictly semistable Lévy process, it is semi-selfsimilar according to [122, Proposition 13.5], and this property carries over to the inverse process in the subsequent sense.

Lemma 5.2.4. *The inverse semistable subordinator* $(E(t))_{t \geq 0}$ *is semi-selfsimilar with*

$$\{E(d^{\frac{1}{\beta}} t)\}_{t \geq 0} \stackrel{f.d.}{=} \{dE(t)\}_{t \geq 0}.$$

Proof. First note that since $(D(t))_{t \geq 0}$ is a strictly $(d^{\frac{1}{\beta}}, d)$-semistable Lévy process, it is semi-selfsimilar [122, Proposition 13.5] with

$$\{D(d^k t)\}_{t \geq 0} \stackrel{f.d.}{=} \{d^{\frac{k}{\beta}} D(t)\}_{t \geq 0}$$

for every $k \in \mathbb{Z}$. Then using (5.11), for $0 \leq t_1 < t_2 < \ldots < t_m$ and $s_1, \ldots, s_m \geq 0$ we obtain

$$P(E(d^{\frac{1}{\beta}} t_i) \leq s_i \text{ for } i = 1, \ldots, m) = P(D(s_i) \geq d^{\frac{1}{\beta}} t_i \text{ for } i = 1, \ldots, m)$$

$$\begin{aligned}&= P(D(d^{-1}s_i) \geq t_i \text{ for } i = 1, \dots, m)\\&= P(E(t_i) \leq d^{-1}s_i \text{ for } i = 1, \dots, m)\\&= P(dE(t_i) \leq s_i \text{ for } i = 1, \dots, m).\end{aligned}$$

□

In the special case of a β-stable subordinator $(D(t))_{t\geq 0}$, the inverse stable subordinator $(E(t))_{t\geq 0}$ is even selfsimilar with index β (compare [92, Proposition 3.1]). The connection between subordinators and their inverse counterpart now enables us to prove a representation of the Lebesgue density of $E(t)$ for every $t > 0$.

Lemma 5.2.5. *(Density of the inverse semistable subordinator)*
For every $t > 0$, the distribution $P_{E(t)}$ has a Lebesgue density $x \mapsto h(x,t)$ on $(0,\infty)$ with

$$h(x,t) = -\frac{\partial}{\partial x} \int_0^t g(y,x)\, dy, \quad \forall x > 0, \tag{5.12}$$

where $x \mapsto g(x,t)$ is the density of $D(t)$.

Proof. From (5.11), we obtain

$$P(E(t) \leq x) = P(D(x) \geq t) = \int_t^\infty g(y,x)\, dy = 1 - \int_0^t g(y,x)\, dy$$

for every $x, t > 0$ since g is a density on $(0,\infty)$. Additionally, the function $x \mapsto P(E(t) \leq x)$ is monotonically increasing. Hence it is differentiable almost everywhere with

$$h(x,t) = \frac{\partial}{\partial x} P(E(t) \leq x) = -\frac{\partial}{\partial x} \int_0^t g(y,x)\, dy.$$

Note that due to the monotonicity, h is a non-negative function with

$$\begin{aligned}\int_0^\infty h(x,t)\, dx &= -\left[\int_0^t g(y,x)\, dy\right]_0^\infty\\&= \lim_{x\to 0} P(D(x) \leq t) - \lim_{x\to\infty} P(D(x) \leq t)\\&= 1\end{aligned}$$

since $D(t) \xrightarrow{p} D(0) = 0$ and using the fact that the paths of $(D(t))_{t\geq 0}$ are unbounded almost surely [30, Theorem 3.2]. □

Remark 5.2.6. In view of Lemma 5.2.5, we are able to calculate the density of an inverse semistable subordinator numerically. Let $\alpha = 0.6$, $c = e^{2\pi\alpha}$, and

$$V(x) = \frac{1}{10}\left(\sin(2x) + \cos(x)\right) + \frac{1}{\Gamma(1-\alpha)}$$

admissable with respect to these parameters. We consider the thereby defined semistable subordinator $(D(t))_{t\geq 0}$, where $D(1)$ has log-characteristic function (5.8), as well as the corresponding inverse semistable subordinator $(E(t))_{t\geq 0}$. Figure 5.7 displays the density of $E(1)$ in comparison to those of the corresponding inverse stable subordinator. As expected, the density of $E(1)$ oscillates around the density of the inverse stable subordinator. The code for the calculation is attached in Appendix C.

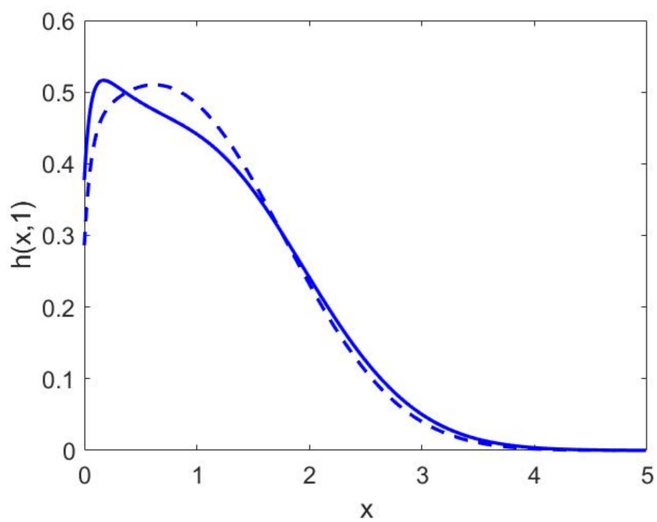

Figure 5.7: Density $h(x, 1)$ of the inverse semistable subordinator at time $t = 1$ (solid line) in comparison to the density of the corresponding inverse stable subordinator (dashed line) in Remark 5.2.6.

In [93, Theorem 3.1], the authors proved the following equivalent representation of the density h of the inverse semistable subordinator.

Lemma 5.2.7. *For every $t > 0$, the density $x \mapsto h(x, t)$ of $E(t)$ is representable as*

$$h(x,t) = \int_0^t \Phi(t-y, \infty)\, dP_{D(x)}(y) \tag{5.13}$$

for every $x > 0$, where Φ is the Lévy measure of $D(1)$.

Since the semistable subordinator $(D(t))_{t\geq 0}$ has Lebesgue densities $x \mapsto g(x, t)$, we can simplify (5.13) to

$$h(x,t) = \int_0^t \Phi(t-y,\infty) g(y,x)\, dy.$$

Both representations have their advantages and will be used for further calculations.

Remark 5.2.8. At first glance, it is not obvious that the representations of the density $x \mapsto h(x,t)$ of the inverse semistable subordinator $E(t)$ in Lemma 5.2.5 and Lemma 5.2.7 are equivalent. Hence, we shortly evaluate their connection. First note that the density h with representation (5.12) has Laplace transform

$$\widetilde{h}(x,s) = -\frac{\partial}{\partial x} s^{-1} \widetilde{g}(s,x) = -\frac{\partial}{\partial x} s^{-1} e^{-x\Gamma_{D(1)}(s)} = \frac{\Gamma_{D(1)}(s)}{s} e^{-x\Gamma_{D(1)}(s)}$$

for every $s > 0$ using (5.9). On the other hand, the representation (5.13) has Laplace transform

$$\begin{aligned}
\widetilde{h}(x,s) &= \int_0^\infty e^{-st} \int_0^t \Phi(t-y,\infty) g(y,x)\, dy\, dt \\
&= \int_0^\infty e^{-st} \int_0^t \Phi(z,\infty) g(t-z,x)\, dz\, dt,
\end{aligned}$$

where we substituted $z \coloneqq t-y$. Now with Tonelli's theorem and the substitution $u \coloneqq t-z$

$$\begin{aligned}
\widetilde{h}(x,s) &= \int_0^\infty \int_z^\infty e^{-st} g(t-z,x)\, dt\, \Phi(z,\infty)\, dz \\
&= \int_0^\infty \int_0^\infty e^{-s(u+z)} g(u,x)\, du\, \Phi(z,\infty)\, dz \\
&= \int_0^\infty e^{-sz} \widetilde{g}(s,x)\, \Phi(z,\infty)\, dz \\
&= \widetilde{g}(s,x) \widetilde{G_V}(s) \\
&= e^{-x\Gamma_{D(1)}(s)} \frac{\Gamma_{D(1)}(s)}{s}
\end{aligned}$$

according to Lemma 4.1.3. With the uniqueness theorem for the Laplace transform, the two representations are equivalent.

As a consequence of the previous remark, we state the following lemma.

Lemma 5.2.9. *(Laplace transform of inverse semistable subordinator densities)*
The density $x \mapsto h(x,t)$ of the inverse semistable subordinator $E(t)$ has Laplace transform

$$\widetilde{h}(x,s) = \frac{\Gamma_{D(1)}(s)}{s} e^{-x\Gamma_{D(1)}(s)}$$

for every $x > 0$ and $s > 0$.

5.3 Semi-fractional Cauchy problems

Keeping the results about semistable subordinators in mind, we now study abstract semi-fractional Cauchy problems. These are Cauchy problems involving an additional semi-fractional time derivative. By doing so, we model log-periodically disturbed long-time memory effects of the underlying system and provide another opportunity to model real-world applications.

For the whole section, fix $\beta \in (0,1)$, $d > 1$, and a smooth, admissable function V with respect to these parameters. In addition, let L be the generator of a semigroup $(T(t))_{t\geq 0}$ driven by an infinitely divisible law μ on $\mathbb{R}^d$ as defined in (2.14). Then we study the semi-fractional Cauchy problem

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} u(x,t) = Lu(x,t) + u_0(x) G_V(t) \tag{5.14}$$

for $x \in \mathbb{R}^d$, $t > 0$, and a suitable function u_0. In the following, we evaluate a solution to (5.14). Thereby, our considerations are inspired by the handling of the corresponding fractional case in [10].

Remark 5.3.1. Note that on the left-hand side of (5.14), we have the Riemann-Liouville form of the semi-fractional derivative. However, if $t \mapsto u(x,t)$ fulfills the assumptions of Lemma 3.2.12 for every $x \in \mathbb{R}^d$, then the difference between Riemann-Liouville and Caputo form is given by $u_0(x)G_V(t)$, and hence the equation simplifies to

$$\frac{\partial^{\beta}}{\partial_{d,V} t^{\beta}} u(x,t) = Lu(x,t)$$

under the initial condition $u(x,0) = u_0(x)$ for every $x \in \mathbb{R}^d$.

We define the subsequent class of operators.

Definition 5.3.2. For every $t \geq 0$, define the linear operator

$$S(t)f := \int_0^{\infty} T(u)f \; h(u,t)\, du \tag{5.15}$$

on $L^1(\mathbb{R}^d)$, where $x \mapsto h(x,t)$ is the density of the inverse semistable subordinator $E(t)$ from section 5.2 and $(T(t))_{t\geq 0}$ is the semigroup (2.14) driven by the infinitely divisible distribution μ on $\mathbb{R}^d$.

As the following lemma displays, the operators $(S(t))_{t\geq 0}$ exhibit some useful properties.

Lemma 5.3.3. *The family of operators $(S(t))_{t\geq 0}$ is uniformly bounded and strongly continuous with $S(0) = Id$.*

Proof. For every $f \in L^1(\mathbb{R}^d)$, we obtain

$$S(0)f = \int_0^\infty T(u)f\; h(u,0)\,du = T(0)f = f$$

since $h(x,0) = \delta(x)$ and $T(0)$ is the identity mapping. Besides, the semigroup $(T(t))_{t\geq 0}$ is uniformly bounded with $||T(t)f||_1 \leq ||f||_1$ for every $f \in L^1(\mathbb{R}^d)$ (compare Section 2.3) such that an application of Bochner's theorem (see for example [5, Theorem 1.1.4]) yields

$$\begin{aligned} ||S(t)f||_1 &= \left\| \int_0^\infty T(u)f\; h(u,t)\,du \right\|_1 \\ &\leq \int_0^\infty ||T(u)f\; h(u,t)||_1\,du \\ &\leq ||f||_1 \int_0^\infty h(u,t)\,du \\ &= ||f||_1. \end{aligned}$$

Thereby, the last equality holds since for every $t > 0$, $x \mapsto h(x,t)$ is a density on $(0,\infty)$. Hence $S(t)$ is well-defined for every $t \geq 0$ and uniformly bounded. It remains to show that $(S(t))_{t\geq 0}$ is strongly continuous. In view of Section 2.3, $(T(t))_{t\geq 0}$ is strongly continuous and thus $||T(t)f - f||_1 \to 0$ as $t \downarrow 0$. Then for every $\epsilon > 0$, there is $s_0 > 0$ with $||T(t)f - f||_1 < \frac{\epsilon}{2}$ for all $t \leq s_0$. It follows that

$$\begin{aligned} ||S(t)f - f||_1 &= \left\| \int_0^\infty T(u)f\; h(u,t)\,du - f \right\|_1 \\ &= \left\| \int_0^\infty (T(u)f - f)\; h(u,t)\,du \right\|_1 \\ &\leq \int_0^\infty ||T(u)f - f||_1\; h(u,t)\,du \end{aligned}$$

$$= \int_0^{s_0} ||T(u)f - f||_1 \, h(u,t)\, du + \int_{s_0}^{\infty} ||T(u)f - f||_1 \, h(u,t)\, du$$
$$\leq \frac{\epsilon}{2} + \int_{s_0}^{\infty} ||T(u)f - f||_1 \, h(u,t)\, du$$

for every $t > 0$ using Bochner's theorem [1, Theorem 1.1.4]. The process $(E(t))_{t\geq 0}$ is continuous in probability with $E(0) = 0$ almost surely such that for fixed $f \in L^1(\mathbb{R}^d)$, we can choose $t_0 > 0$ with $\int_{s_0}^{\infty} h(u,t)\, du < \frac{\epsilon}{4||f||_1}$ for all $t \leq t_0$. Then with $||T(u)f||_1 \leq ||f||_1$ for every $u \geq 0$, $||S(t)f - f||$ is bounded by

$$||S(t)f - f||_1 \leq \frac{\epsilon}{2} + 2||f||_1 \int_{s_0}^{\infty} h(u,t)\, du < \epsilon$$

for t sufficient small. Consequently, $||S(t)f - f||_1 \to 0$ as $t \downarrow 0$, showing that $(S(t))_{t\geq 0}$ is strongly continuous. □

Since $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are families of bounded operators, the Laplace transform of $t \mapsto T(t)f$ as well as the Laplace transform of $t \mapsto S(t)f$ exist for every $f \in L^1(\mathbb{R}^d)$, and the following equality connect both forms.

Lemma 5.3.4. *Let $f \in L^1(\mathbb{R}^d)$. The Laplace transform of $S(t)f$ is given by*

$$\int_0^{\infty} e^{-st} S(t)f\, dt = \frac{\Gamma_{D(1)}(s)}{s} \int_0^{\infty} e^{-u\Gamma_{D(1)}(s)} T(u)f\, du$$

for every $s > 0$.

Proof. Use the definition (5.15) of $S(t)$ to obtain

$$\int_0^{\infty} e^{-st} S(t)f\, dt = \int_0^{\infty} e^{-st} \int_0^{\infty} T(u)f\, h(u,t)\, du\, dt$$

for every $s > 0$. Note that with

$$\int_0^{\infty}\int_0^{\infty} ||e^{-st} T(u)f\, h(u,t)||_1\, du\, dt = \int_0^{\infty}\int_0^{\infty} e^{-st} ||T(u)f||_1 h(u,t)\, du\, dt$$
$$\leq ||f||_1 \int_0^{\infty}\int_0^{\infty} e^{-st} h(u,t)\, du\, dt$$
$$= ||f||_1 \int_0^{\infty} e^{-st}\, dt$$

$$= \frac{||f||_1}{s} < \infty,$$

by Fubini's theorem, we have

$$\int_0^\infty e^{-st} S(t) f \, dt = \int_0^\infty \int_0^\infty e^{-st} h(u,t) \, dt \, T(u) f \, du = \int_0^\infty \widetilde{h}(u,s) \, T(u) f \, du.$$

Now Lemma 5.2.9 yields

$$\int_0^\infty e^{-st} S(t) f \, dt = \frac{\Gamma_{D(1)}(s)}{s} \int_0^\infty e^{-u\Gamma_{D(1)}(s)} T(u) f \, du.$$

□

Turning back to the semi-fractional Cauchy problem (5.14), we want to describe the solution in terms of the operators $(S(t))_{t\geq 0}$. Since we prove our results using Laplace transforms, we will not obtain classical solutions but strong solutions defined as follows.

Definition 5.3.5. (Strong solution)
A function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is called strong solution of the semi-fractional Cauchy problem (5.14), if $u \in \operatorname{Dom}\left(\left(\frac{\partial}{\partial_{d,P} t}\right)^\beta\right) \cap \operatorname{Dom}(L)$ and (5.14) is fulfilled for almost every $t > 0$.

Since V is smooth, it is representable by its Fourier series

$$V(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\tilde{d}x} \tag{5.16}$$

for every $x \in \mathbb{R}$, where $(d_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ are the Fourier coefficients and $\tilde{d} = \frac{2\pi\beta}{\log(d)}$ determines the period of V. In this section, we demand that the Fourier coefficients possess an arbitrary small exponential decay, this is

$$|d_n| \leq C_{11} e^{-|n|\tilde{d}\epsilon} \tag{5.17}$$

for a constant $C_{11} > 0$ and some $\epsilon > 0$. This necessarily implies that $V \in C^\infty(\mathbb{R}^d)$ [46, Theorem 2.6]. Besides, we need a further assumption on V. Recall from Lemma 3.1.2 that the admissable and differentiable function V fulfills $V'(x) \leq \beta V(x)$ for every $x \in \mathbb{R}$. In what follows, we assume that this inequality is even strictly fulfilled, this is

$$V'(x) < \beta V(x) \tag{5.18}$$

for every $x \in \mathbb{R}$. Then we can characterize solutions to the semi-fractional Cauchy problems as follows.

Theorem 5.3.6. *(Strong solutions to semi-fractional Cauchy problems)*
Let V be an admissable function with respect to $\beta \in (0,1)$ and $d > 1$ given by (5.16) such that (5.17) and (5.18) are satisfied. In addition, let $u_0 \in L^1(\mathbb{R}^d)$. Then the functions $u(x,t) = S(t)u_0(x)$ solve the semi-fractional Cauchy problem (5.14) in the strong sense.

We will split the proof of Theorem 5.3.6 into several steps. We start by extending the Laplace exponent $\Gamma_{D(1)}$ to a sectorial region. Therefore let

$$\mathbb{C}(\vartheta) := \{re^{i\varphi} \in \mathbb{C} : r > 0, |\varphi| < \vartheta\}$$

be the sectorial region of angle ϑ for every $\vartheta \in (0, \pi]$. If $\vartheta = \frac{\pi}{2}$, then the sectorial region equals the open right half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, whereas for $\vartheta = \pi$, we obtain the whole complex plane except for the negative real axis.

Lemma 5.3.7. *Let V be an admissable function with respect to $\beta \in (0,1)$ and $d > 1$ given by (5.16) such that (5.17) and (5.18) are satisfied and choose $0 < \epsilon' < \epsilon$. The function $s \mapsto \Gamma_{D(1)}(s)$ has an analytic extension to the sectorial region $\mathbb{C}\left(\frac{\pi}{2} + \epsilon'\right)$ with*

$$|\Gamma_{D(1)}(z)| \leq C_{13}|z|^\beta \tag{5.19}$$

for a constant $C_{13} > 0$ and every $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon'\right)$. In addition, there is $\epsilon'' \in (0, \epsilon']$ such that

$$\Gamma_{D(1)} : \mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right) \to \mathbb{C}(\eta)$$

for some $\eta \in \left(0, \frac{\pi}{2}\right)$.

We want to emphasize that since the sectorial regions $\mathbb{C}(\vartheta)$ with $\vartheta \in (0, \pi]$ are open subsets of the complex plane, by extending $\Gamma_{D(1)}$ analytically to $\mathbb{C}\left(\frac{\pi}{2} + \epsilon'\right)$ for every $\epsilon' \in (0, \epsilon)$, we even obtain the result for the whole region $\mathbb{C}\left(\frac{\pi}{2} + \epsilon\right)$. However, the proof below shows that the constant C_{13} in (5.19) depends on ϵ' and diverges for $\epsilon' \to \epsilon$ such that we restrict our considerations to a smaller sectorial region in order to preserve this bound.

Proof. According to Lemma 5.2.2 and Lemma 4.1.3, for a smooth function V with Fourier series (5.16), the function $\Gamma_{D(1)}$ is given by

$$\Gamma_{D(1)}(s) = \sum_{n\in\mathbb{Z}} d_n \Gamma(1 - \beta + in\tilde{d}) s^{\beta - in\tilde{d}}$$

for every $s > 0$. Hence we define an extension of $\Gamma_{D(1)}$ by

$$\Gamma_{D(1)}(z) = \sum_{n\in\mathbb{Z}} d_n \Gamma(1 - \beta + in\tilde{d}) z^{\beta - in\tilde{d}} \tag{5.20}$$

for every $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon'\right)$. First note that the series is absolutely convergent since for every

$z = re^{i\phi}$ with $r > 0$ and $|\phi| < \frac{\pi}{2} + \epsilon'$, we find

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left| d_n \Gamma(1-\beta+in\tilde{d}) z^{\beta-in\tilde{d}} \right| &\leq r^\beta \sum_{n\in\mathbb{Z}} \left| d_n \Gamma(1-\beta+in\tilde{d}) \right| \left| e^{i\phi(\beta-in\tilde{d})} \right| \\ &\leq r^\beta \sum_{n\in\mathbb{Z}} \left| d_n \Gamma(1-\beta+in\tilde{d}) \right| e^{\phi n\tilde{d}} \\ &\leq C_{11}\, r^\beta \sum_{n\in\mathbb{Z}} e^{-|n|\tilde{d}\epsilon} \left| \Gamma(1-\beta+in\tilde{d}) \right| e^{\phi n\tilde{d}}\end{aligned}$$

due to (5.17). Recall from ([3, Corollary 1.4.4]) that the gamma function decays exponentially such that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left| d_n \Gamma(1-\beta+in\tilde{d}) z^{\beta-in\tilde{d}} \right| &\leq C_{12}\, r^\beta \left(\Gamma(1-\beta) + \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-|n|\tilde{d}\epsilon} e^{-\frac{\pi}{2}|n|\tilde{d}} |n|^{\frac{1}{2}-\beta} e^{\phi n\tilde{d}} \right) \\ &\leq C_{12}\, r^\beta \left(\Gamma(1-\beta) + \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{\frac{1}{2}-\beta} e^{-|n|\tilde{d}(\frac{\pi}{2}+\epsilon-\frac{\pi}{2}-\epsilon')} \right) \\ &= C_{12}\, r^\beta \left(\Gamma(1-\beta) + \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{\frac{1}{2}-\beta} e^{-|n|\tilde{d}(\epsilon-\epsilon')} \right)\end{aligned}$$

for a constant $C_{12} > 0$. Due to our assumptions, $\epsilon > \epsilon'$ and the series is finite. Especially it follows that

$$|\Gamma_{D(1)}(z)| \leq C_{13} |z|^\beta$$

with $C_{13} > 0$ for every $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon'\right)$. Next we prove that (5.20) defines an analytic function. Therefore note that we can differentiate $\Gamma_{D(1)}$ piecewise such that

$$\begin{aligned}\Gamma_{D(1)}^{(l)}(z) &= \sum_{n\in\mathbb{Z}} (\beta - in\tilde{d}) \cdots (\beta - in\tilde{d} - l + 1) d_n \Gamma(1-\beta+in\tilde{d}) z^{\beta-in\tilde{d}-l} \\ &= l! \sum_{n\in\mathbb{Z}} \binom{\beta - in\tilde{d}}{l} d_n \Gamma(1-\beta+in\tilde{d}) z^{\beta-in\tilde{d}-l}\end{aligned}$$

for every $l \in \mathbb{N}$. Then for every $z_0 \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon'\right)$, the Taylor series Q in z is given by

$$\begin{aligned}Q(z) &= \sum_{l=0}^{\infty} \frac{\Gamma_{D(1)}^{(l)}(z_0)}{l!} (z - z_0)^l \\ &= \sum_{l=0}^{\infty} \sum_{n\in\mathbb{Z}} \binom{\beta - in\tilde{d}}{l} d_n \Gamma(1-\beta+in\tilde{d}) z_0^{\beta-in\tilde{d}-l} (z - z_0)^l.\end{aligned}$$

To change the order of summation, note that

$$
\begin{aligned}
|Q(z)| &\leq \sum_{l=0}^{\infty}\sum_{n\in\mathbb{Z}}\left|\binom{\beta - in\tilde{d}}{l} d_n\Gamma(1-\beta+in\tilde{d})z_0^{\beta-in\tilde{d}-l}(z-z_0)^l\right| \\
&= \sum_{n\in\mathbb{Z}}\left|d_n\Gamma(1-\beta+in\tilde{d})z_0^{\beta-in\tilde{d}}\right| + \sum_{l=1}^{\infty}\sum_{n\in\mathbb{Z}}\left|\binom{\beta - in\tilde{d}}{l}\frac{d_n\Gamma(1-\beta+in\tilde{d})}{z_0^{-\beta+in\tilde{d}}}\left(\frac{z-z_0}{z_0}\right)^l\right|,
\end{aligned}
$$

where the first series is finite as shown above. For the second series, use (3.31) for an upper bound of the binomial coefficients to obtain

$$
\begin{aligned}
&\sum_{l=1}^{\infty}\sum_{n\in\mathbb{Z}}\left|\binom{\beta - in\tilde{d}}{l} d_n\Gamma(1-\beta+in\tilde{d})z_0^{\beta-in\tilde{d}}\left(\frac{z-z_0}{z_0}\right)^l\right| \\
\leq & C_5\sum_{l=1}^{\infty}\sum_{n\in\mathbb{Z}}\left|\frac{l^{-1-\beta}}{\Gamma(in\tilde{d}-\beta)}d_n\Gamma(1-\beta+in\tilde{d})z_0^{\beta-in\tilde{d}}\left(\frac{z-z_0}{z_0}\right)^l\right| \\
= & C_5\sum_{n\in\mathbb{Z}}|d_n(in\tilde{c}-\beta)|\sum_{l=1}^{\infty}\left|l^{-1-\beta}z_0^{\beta-in\tilde{d}}\left(\frac{z-z_0}{z_0}\right)^l\right| \\
\leq & C_{14}|z_0|^\beta\sum_{l=1}^{\infty}l^{-1-\beta}\left|\frac{z}{z_0}-1\right|^l
\end{aligned}
$$

for a constant $C_{14} > 0$ with the exponential decay of the Fourier coefficients $(d_n)_{n\in\mathbb{Z}}$ in (5.17). For z in a sufficiently small neighborhood of z_0, we have $\left|\frac{z}{z_0}-1\right| < 1$, and the series converges due to the direct comparison test with a generalized harmonic series as convergent majorant. Then we can interchange the order of the series in $Q(z)$, and for $\left|\frac{z}{z_0}-1\right| < 1$ using the generalized binomial theorem [70, Satz 247], we obtain

$$
\begin{aligned}
Q(z) &= \sum_{n\in\mathbb{Z}} d_n\Gamma(1-\beta+in\tilde{d})z_0^{\beta-in\tilde{d}}\sum_{l=0}^{\infty}\binom{\beta - in\tilde{d}}{l}\left(\frac{z}{z_0}-1\right)^l \\
&= \sum_{n\in\mathbb{Z}} d_n\Gamma(1-\beta+in\tilde{d})z_0^{\beta-in\tilde{d}}\left(\frac{z}{z_0}\right)^{\beta-in\tilde{d}} \\
&= \Gamma_{D(1)}(z)
\end{aligned}
$$

for every z in a sufficiently small neighborhood of z_0. Hence the function $\Gamma_{D(1)}$ is analytic in $\mathbb{C}\left(\frac{\pi}{2}+\epsilon'\right)$.

Finally, we prove the existence of $\epsilon'' \in (0, \epsilon']$ such that $\Gamma_{D(1)}(z) \in \mathbb{C}(\eta)$ for every $z \in \mathbb{C}\left(\frac{\pi}{2}+\epsilon''\right)$ and some $\eta \in \left(0, \frac{\pi}{2}\right)$. We show the equivalent statement that $\cos\Big(\arg(\Gamma_{D(1)}(z))\Big)$ is bounded away from zero, where $\arg(z)$ is the argument of the complex number z. In view of (5.19), we get

$$
\cos\Big(\arg(\Gamma_{D(1)}(z))\Big) = \frac{\mathrm{Re}(\Gamma_{D(1)}(z))}{|\Gamma_{D(1)}(z)|} \geq \frac{\mathrm{Re}(\Gamma_{D(1)}(z))}{C_{13}|z|^\beta}
$$

for every $z \in \mathbb{C}\left(\frac{\pi}{2}+\epsilon'\right)$. Now we analyze the real part of the Laplace exponent. Therefore note that for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, the function $\Gamma_{D(1)}$ has the integral representation

$$\Gamma_{D(1)}(z) = \int_0^\infty (1 - e^{-zt})\, d\Phi(t).$$

This follows immediately with the same calculation as in [66] for the real-valued case. Now fix $\vartheta \in \left(0, \frac{\pi}{2}\right)$ and consider the subsequent three cases.

First case: $z \in \mathbb{C}(\vartheta)$. Write $z = re^{i\varphi}$ with $r > 0$ and $|\varphi| < \vartheta$, yielding

$$\begin{aligned}
\mathrm{Re}(\Gamma_{D(1)}(z)) &= \int_0^\infty \mathrm{Re}(1 - e^{-zt})\, d\Phi(t) \\
&= \int_0^\infty (1 - e^{-r\cos(\varphi)t} \cos(r\sin(\varphi)t))\, d\Phi(t) \\
&\geq \int_0^\infty (1 - e^{-r\cos(\varphi)t})\, d\Phi(t) \\
&\geq \int_0^\infty (1 - e^{-r\cos(\vartheta)t})\, d\Phi(t)
\end{aligned}$$

since $\cos(\vartheta) < \cos(\varphi)$. This integral has already been solved (compare for example Lemma 4.1.3) with

$$\int_0^\infty (1 - e^{-yt})\, d\Phi(t) = \sum_{n\in\mathbb{Z}} d_n \Gamma(in\tilde{d} - \beta + 1) y^{\beta - in\tilde{d}} = y^\beta \eta_1(\log(y))$$

for every $y > 0$ with $\eta_1 : \mathbb{R} \to (0, \infty)$ as in (4.1). Since η_1 is continuous and periodic, it is bounded, and with the positivity of the function, there is a constant $C_{15} > 0$ with

$$\mathrm{Re}(\Gamma_{D(1)}(z)) \geq (r\cos(\vartheta))^\beta \eta_1(\log(r\cos(\vartheta))) \geq C_{15} r^\beta.$$

Second Case: $z \in \mathbb{C}_+ \setminus \mathbb{C}(\vartheta)$. Again we write $z = re^{i\varphi}$ with $r > 0$ and $\vartheta \leq |\varphi| < \frac{\pi}{2}$. Using the integral representation of $\Gamma_{D(1)}$ as before, the real part is given by

$$\begin{aligned}
\mathrm{Re}(\Gamma_{D(1)}(z)) &= \int_0^\infty (1 - e^{-r\cos(\varphi)t} \cos(r\sin(\varphi)t))\, d\Phi(t) \\
&= -\int_0^\infty (1 - e^{-r\cos(\varphi)t} \cos(r\sin(\varphi)t))\, dG_V(t).
\end{aligned}$$

The function $G_V(t) = t^{-\beta}V(\log(t))$ is continuously differentiable due to (5.17) such that the Riemann-Stieltjes integral equals

$$\operatorname{Re}(\Gamma_{D(1)}(z)) = -\int_0^\infty (1 - e^{-r\cos(\varphi)t}\cos(r\sin(\varphi)t))G_V'(t)\,dt$$

(compare [134, Example 1.2.2 (i)]). For the derivative, we obtain

$$G_V'(t) = t^{-\beta-1}(-\beta V(\log(t)) + V'(\log(t)))$$

for every $t > 0$. Besides, the function $x \mapsto \beta V(x) - V'(x)$ is periodic, continuous, and according to (5.18) strictly positive. Then there is a constant $C_{16} > 0$ with

$$\beta V(x) - V'(x) \geq C_{16} > 0$$

for every $x \in \mathbb{R}$. It follows that

$$\begin{aligned}\operatorname{Re}(\Gamma_{D(1)}(z)) &\geq C_{16}\int_0^\infty (1 - e^{-r\cos(\varphi)t}\cos(r\sin(\varphi)t))t^{-\beta-1}\,dt \\ &= C_{16}\int_0^\infty (1 - e^{-r\cos(\varphi)t}\cos(r|\sin(\varphi)|t))t^{-\beta-1}\,dt\end{aligned}$$

using the symmetry of the cosine function. With the substitution $y \coloneqq r|\sin(\varphi)|t$, we gain

$$\begin{aligned}\operatorname{Re}(\Gamma_{D(1)}(z)) &\geq \frac{C_{16}}{r|\sin(\varphi)|}\int_0^\infty (1 - e^{-|\cot(\varphi)|y}\cos(y))(yr^{-1}|\sin(\varphi)|^{-1})^{-\beta-1}\,dy \\ &= C_{16}r^\beta|\sin(\varphi)|^\beta\int_0^\infty (1 - e^{-|\cot(\varphi)|y}\cos(y))y^{-\beta-1}\,dy.\end{aligned}$$

Now for every $\vartheta \leq |\varphi| < \frac{\pi}{2}$,

$$\operatorname{Re}(\Gamma_{D(1)}(z)) \geq C_{16}r^\beta\sin^\beta(\vartheta)\int_0^\infty (1 - |\cos(y)|)y^{-\beta-1}\,dy.$$

Since $1 - |\cos(y)| \sim \frac{y^2}{2}$ as $y \to 0$, the integral is finite such that

$$\operatorname{Re}(\Gamma_{D(1)}(z)) \geq C_{17}r^\beta$$

for a constant $C_{17} > 0$.

Third Case: $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon\right) \setminus \mathbb{C}_+$. Write $z = re^{i\varphi}$ for some $r > 0$ and $|\varphi| \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \epsilon\right)$.

Then

$$\Gamma_{D(1)}(z) = r^{\beta}\eta_{\varphi}(\log(r)),$$

where for every φ, the function $\eta_{\varphi} : \mathbb{R} \to \mathbb{C}$ is given by

$$\eta_{\varphi}(x) = e^{i\varphi\beta} \sum_{n\in\mathbb{Z}} d_n \Gamma(in\tilde{d} - \beta + 1) e^{n\tilde{d}\varphi} e^{-in\tilde{d}x}.$$

Since $\Gamma_{D(1)}(z)$ is analytic in $\mathbb{C}\left(\frac{\pi}{2} + \epsilon\right)$, η_{φ} is analytic in the same region. Hence, the real part of η_{φ}

$$\mathrm{Re}(\eta_{\varphi}(\log(r))) = \mathrm{Re}(r^{-\beta}\Gamma_{D(1)}(re^{i\varphi}))$$

is continuous in φ. From the previous two cases, it follows that

$$\mathrm{Re}(r^{-\beta}\Gamma_{D(1)}(re^{i\varphi})) \geq \min\{C_{15}, C_{17}\}$$

for every $r > 0$ and $|\varphi| < \frac{\pi}{2}$. Then using the continuity, for every $r > 0$, there is $\epsilon'_r \leq \epsilon'$ such that

$$\mathrm{Re}(\eta_{\varphi}(\log(r))) \geq \frac{1}{2}\min\{C_{15}, C_{17}\}$$

for every $|\varphi| < \frac{\pi}{2} + \epsilon'_r$. However, $\eta_{\varphi}(x)$ is periodic and continuous in x such that $\epsilon'' = \min_{r>0} \epsilon'_r \in (0, \epsilon']$ exists. Then

$$\mathrm{Re}(\eta_{\varphi}(\log(r))) \geq \frac{1}{2}\min\{C_{15}, C_{17}\}$$

for every $r > 0$ and $|\varphi| < \frac{\pi}{2} + \epsilon''$ and we obtain

$$\mathrm{Re}(\Gamma_{D(1)}(re^{i\varphi})) \geq \frac{1}{2} r^{\beta}\min\{C_{15}, C_{17}\}$$

for every $r > 0$ and $\frac{\pi}{2} \leq |\varphi| < \epsilon''$.
Summarizing all three cases,

$$\cos\Big(\arg(\Gamma_{D(1)}(z))\Big) \geq \frac{\mathrm{Re}(\Gamma_{D(1)}(z))}{C_{13}|z|^{\beta}} \geq \frac{1}{2}\frac{\min\{C_{15}, C_{17}\}}{C_{13}} > 0$$

for every $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right)$, and hence there is $\eta \in \left(0, \frac{\pi}{2}\right)$ with $\Gamma_{D(1)} : \mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right) \to \mathbb{C}(\eta)$. □

Lemma 5.3.8. *The function* $q : (0, \infty) \to L^1(\mathbb{R}^d)$ *with*

$$q(s) \coloneqq \int_0^\infty e^{-st} T(t) f \, dt$$

has an analytic extension to $\mathbb{C}_+ = \mathbb{C}\left(\frac{\pi}{2}\right)$.

Proof. For every fixed $z \in \mathbb{C}_+$, let $F(t) \coloneqq e^{-zt} T(t) f$. By Bochner's theorem, F is integrable over $(0, \infty)$ with

$$\begin{aligned}
||q(z)||_1 &\leq \int_0^\infty ||F(t)||_1 \, dt = \int_0^\infty ||e^{-zt} T(t) f||_1 \, dt \\
&= \int_0^\infty e^{-\operatorname{Re}(z)t} ||T(t) f||_1 \, dt \\
&\leq \int_0^\infty e^{-\operatorname{Re}(z)t} ||f||_1 \, dt \\
&\leq \frac{||f||_1}{\operatorname{Re}(z)} \qquad (5.21)
\end{aligned}$$

for every $f \in L^1(\mathbb{R}^d)$ and $z \in \mathbb{C}_+$. Then the abscissa of convergence $abs(f)$ as defined in [5, Section 1.4] is given by

$$abs(T(t)f) \coloneqq \inf\{\operatorname{Re}(\lambda) : \mathcal{L}(T(t)f)(\lambda) = q(\lambda) \text{ exists}\} \leq 0,$$

and according to Theorem 1.5.1 in [5], the function q is analytic in $\mathbb{C}_+$. □

Lemma 5.3.9. *The function* $\widetilde{r} : (0, \infty) \to L^1(\mathbb{R}^d)$ *with* $\widetilde{r}(s) \coloneqq \int_0^\infty e^{-st} S(t) f \, dt$ *is analytic in* $\mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right)$ *with* $\epsilon'' > 0$ *from Lemma 5.3.7. Additionally,* $\widetilde{r}$ *is the Laplace transform of some function* r *analytic in* $\mathbb{C}(\epsilon'')$.

To prove this result, we use the following theorem taken from [5].

Theorem 5.3.10. *[5, Theorem 2.6.1] Let* $0 < \zeta \leq \frac{\pi}{2}$, $x \in \mathbb{R}$, *and* $f : (x, \infty) \to L^1(\mathbb{R}^d)$. *The following two statements are equivalent:*

- *There is an analytic function* $g : \mathbb{C}(\zeta) \to L^1(\mathbb{R}^d)$ *such that* $\sup_{z \in \mathbb{C}(\zeta')} ||e^{-xz} g(z)||_1 < \infty$ *for every* $0 < \zeta' < \zeta$ *and* $\widetilde{g}(\lambda) = f(\lambda)$ *for all* $\lambda > x$.
- *The function* f *has an analytic extension* $f : x + \mathbb{C}\left(\frac{\pi}{2} + \zeta\right) \to L^1(\mathbb{R}^d)$ *such that*

$$\sup_{z \in x + \mathbb{C}\left(\frac{\pi}{2} + \zeta'\right)} ||(z - x) f(z)||_1 < \infty$$

for all $0 < \zeta' < \zeta$.

Proof of Lemma 5.3.9. According to Lemma 5.3.4,

$$\widetilde{r}(s) = \frac{\Gamma_{D(1)}(s)}{s} q(\Gamma_{D(1)}(s)) \tag{5.22}$$

for every $s > 0$. However, by Lemma 5.3.7, $\Gamma_{D(1)}$ is analytic in $\mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right)$ and fulfills $\mathrm{Re}(\Gamma_{D(1)}(z)) > 0$ for every $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right)$. In addition, according to Lemma 5.3.8, the function q also has an analytic extension to $\mathbb{C}_+$ such that the right-hand side of (5.22) is well-defined for every $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right)$ and analytic in this region. Hence, $\widetilde{r}$ is analytic in $\mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right)$. In order to apply Theorem 5.3.10, we prove that for every $0 < \epsilon''' < \epsilon''$, we have

$$\sup_{z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon'''\right)} ||z\widetilde{r}(z)||_1 < \infty. \tag{5.23}$$

Use (5.22) and (5.21) to obtain

$$||z\widetilde{r}(z)||_1 = \left|\left|\Gamma_{D(1)}(z) q(\Gamma_{D(1)}(z))\right|\right|_1 \leq |\Gamma_{D(1)}(z)| \frac{||f||_1}{\mathrm{Re}(\Gamma_{D(1)}(z))}$$

for every $z \in \mathbb{C}\left(\frac{\pi}{2} + \epsilon'''\right)$. According to Lemma 5.3.7, we can write $\Gamma_{D(1)}(z) = re^{i\varphi}$ for some $r > 0$ and $|\varphi| < \eta$ with $\eta \in \left(0, \frac{\pi}{2}\right)$. Then

$$||z\widetilde{r}(z)||_1 \leq |re^{i\varphi}| \frac{||f||_1}{r\cos(\varphi)} = \frac{||f||_1}{\cos(\varphi)} \leq \frac{||f||_1}{\cos(\eta)}, \tag{5.24}$$

and the supremum in (5.23) is finite. According to Theorem 5.3.10, $\widetilde{r}$ is the Laplace transform of a function r analytic in $\mathbb{C}(\epsilon'')$. ☐

Lemma 5.3.11. *For every $f \in L^1(\mathbb{R}^d)$, the function*

$$t \mapsto \int_0^t G_V(t-u) S(u) f \, du$$

has an analytic extension to the sectorial region $\mathbb{C}(\epsilon'')$ with ϵ'' from Lemma 5.3.7.

Proof. The Laplace transform of the function above is given by

$$\begin{aligned}\int_0^\infty e^{-st} \int_0^t G_V(t-u) S(u) f \, du \, dt &= \left(\int_0^\infty e^{-st} G_V(t) \, dt\right) \left(\int_0^\infty e^{-st} S(t) f \, dt\right) \\ &= \widetilde{G_V}(s) \widetilde{r}(s)\end{aligned}$$

using the convolution rule for Laplace transforms, where $\tilde{r}$ is given as in Lemma 5.3.9. Besides, as calculated in Lemma 4.1.3,

$$\widetilde{G_V}(s) = s^{\beta-1}\sum_{n\in\mathbb{Z}} d_n\Gamma(in\tilde{d}-\beta+1)s^{-in\tilde{d}} = \frac{\Gamma_{D(1)}(s)}{s} \tag{5.25}$$

for every $s>0$. Hence the Laplace transform is given by

$$\int_0^\infty e^{-st}\int_0^t G_V(t-u)S(u)f\,du\,dt = \frac{\Gamma_{D(1)}(s)}{s}\tilde{r}(s), \tag{5.26}$$

which is an analytic function on $\mathbb{C}\left(\frac{\pi}{2}+\epsilon''\right)$ according to Lemma 5.3.7 and 5.3.9. Again we want to apply Theorem 5.3.10 to obtain the claimed result. Therefore, let $x>0$. Then the considered function has an analytic extension to $x+\mathbb{C}\left(\frac{\pi}{2}+\epsilon''\right)$, and we have to show that in addition

$$\sup_{z\in x+\mathbb{C}\left(\frac{\pi}{2}+\epsilon'''\right)}\left|\left|(z-x)\frac{\Gamma_{D(1)}(z)}{z}\tilde{r}(z)\right|\right|_1 < \infty \tag{5.27}$$

holds for every $0<\epsilon'''<\epsilon''$. First note that

$$\begin{aligned}\left|\left|(z-x)\frac{\Gamma_{D(1)}(z)}{z}\tilde{r}(z)\right|\right|_1 &= |z-x|\cdot|z|^{-2}|\Gamma_{D(1)}(z)|\cdot||z\tilde{r}(z)||_1\\ &\le |z-x|\cdot|z|^{-2}|\Gamma_{D(1)}(z)|\cdot\frac{||f||_1}{\cos(\eta)}\end{aligned}$$

with (5.24). In addition, according to Lemma 5.3.7,

$$|\Gamma_{D(1)}(z)| \le C_{13}|z|^\beta$$

such that

$$\begin{aligned}\left|\left|(z-x)\frac{\Gamma_{D(1)}(z)}{z}r(z)\right|\right|_1 &\le C_{13}|z-x|\cdot|z|^{\beta-2}\cdot\frac{||f||_1}{\cos(\eta)}\\ &= C_{13}\frac{|z-x|}{|z|}\cdot|z|^{\beta-1}\cdot\frac{||f||_1}{\cos(\eta)}\\ &\le C_{13}\frac{|z|+|x|}{|z|}\cdot|z|^{\beta-1}\cdot\frac{||f||_1}{\cos(\eta)}.\end{aligned}$$

Note that $|z|$ is bounded away from zero for every $z\in x+\mathbb{C}\left(\frac{\pi}{2}+\epsilon''\right)$, and hence there is a constant $C_{18}>0$ with

$$\left|\left|(z-x)\frac{\Gamma_{D(1)}(z)}{z}r(z)\right|\right|_1 \le C_{18}\frac{||f||_1}{\cos(\eta)}$$

for every $z \in x + \mathbb{C}\left(\frac{\pi}{2} + \epsilon''\right)$. Then the supremum in (5.27) is finite, and according to Theorem 5.3.10, the stated function is analytic in $\mathbb{C}(\epsilon'')$. □

Having all this auxiliary results in mind, we can finally establish Theorem 5.3.6.

Proof of Theorem 5.3.6. We have to prove that the functions $u(x,t) = S(t)u_0(x)$ solve the semi-fractional Cauchy problem (5.14) in the strong sense. For every $u_0 \in L^1(\mathbb{R}^d)$, according to Lemma 2.3.2,

$$T(t)u_0 = L\left(\int_0^t T(x)u_0\,dx\right) + u_0,$$

where L is again the generator of the semigroup $(T(t))_{t\geq 0}$. Applying the Laplace transform to this equation yields

$$\begin{aligned}
q(s) &= \int_0^\infty e^{-st}T(t)u_0\,dt \\
&= \int_0^\infty e^{-st}\left(L\left(\int_0^t T(x)u_0\,dx\right) + u_0\right)dt \\
&= \int_0^\infty e^{-st}L\left(\int_0^t T(x)u_0\,dx\right)dt + s^{-1}u_0
\end{aligned}$$

for every $s > 0$. Using the fact that L is closed (see Lemma 2.3.2), it was shown in [10, Proof of Theorem 3.1] that

$$\int_0^\infty e^{-st}L\left(\int_0^t T(x)u_0\,dx\right)dt = s^{-1}L(q(s)) = s^{-1}L\left(\int_0^\infty e^{-st}T(t)u_0\,dt\right)$$

for every $s > 0$ such that

$$q(s) = s^{-1}L\left(\int_0^\infty e^{-st}T(t)u_0\,dt\right) + s^{-1}u_0. \tag{5.28}$$

Besides, in view of Lemma 5.2.3, the function $\Gamma_{D(1)} : (0,\infty) \to (0,\infty)$ is bijective such that for every $s > 0$, there is a unique $u > 0$ with $s = \Gamma_{D(1)}(u)$. Inserting this in (5.28) yields

$$q(\Gamma_{D(1)}(u)) = \Gamma_{D(1)}(u)^{-1}L\left(\int_0^\infty e^{-\Gamma_{D(1)}(u)t}T(t)u_0\,dt\right) + \Gamma_{D(1)}(u)^{-1}u_0$$

$$= \Gamma_{D(1)}(u)^{-1} L\left(q(\Gamma_{D(1)}(u))\right) + \Gamma_{D(1)}(u)^{-1} u_0.$$

Using (5.22) on both sides, we receive

$$\frac{u}{\Gamma_{D(1)}(u)}\widetilde{r}(u) = \Gamma_{D(1)}(u)^{-1} L\left(\frac{u}{\Gamma_{D(1)}(u)}\widetilde{r}(u)\right) + \Gamma_{D(1)}(u)^{-1} u_0,$$

and since L is a closed linear operator, we have

$$\frac{u}{\Gamma_{D(1)}(u)}\widetilde{r}(u) = \frac{u}{\Gamma_{D(1)}(u)^2} L\left(\widetilde{r}(u)\right) + \Gamma_{D(1)}(u)^{-1} u_0$$

for every $u > 0$. Now multiply the equation with $u^{-2}\Gamma_{D(1)}(u)^2$ to obtain

$$u^{-1}\Gamma_{D(1)}(u)\widetilde{r}(u) = u^{-1} L\left(\widetilde{r}(u)\right) + u^{-2}\Gamma_{D(1)}(u) u_0. \tag{5.29}$$

We want to apply an inverse Laplace transform to (5.29). Therefore first note that according to (5.26), the Laplace inversion of the left-hand side is given by

$$\int_0^t G_V(t-y) S(y) u_0 \, dy.$$

On the other hand, for every $u > 0$, the Laplace transform of $t \mapsto \int_0^t G_V(y)\, dy$ in u is given by $u^{-1}\widetilde{G_V}(u) = u^{-2}\Gamma_{D(1)}(u)$ according to (5.25). Finally, in view of Lemma 5.3.9, the function $t \mapsto L\left(\int_0^t S(y) u_0 \, dy\right)$ has Laplace transform $u^{-1}L(\widetilde{r}(u))$ such that (5.29) equals

$$\int_0^t G_V(t-y) S(y) u_0 \, dy = L\left(\int_0^t S(y) u_0 \, dy\right) + \int_0^t G_V(y)\, dy \, u_0$$

for almost every $t > 0$. The function on the left-hand side is analytic in the sectorial region $\mathbb{C}(\epsilon'')$ and thereby differentiable for every $t > 0$. The derivative

$$\frac{d}{dt}\int_0^t G_V(t-u) S(u) u_0 \, du$$

is the Riemann-Liouville form of the semi-fractional derivative of order β applied to $t \mapsto S(t)u_0$, and in particular, it follows that $S(t)u_0 \in \mathrm{Dom}\left(\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta}\right)$. Then differentiation of the whole equation yields

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} S(t)u_0 = \frac{d}{dt} L \left(\int_0^t S(u)u_0 \, du \right) + G_V(t)u_0.$$

Finally note that using the fact that L is closed, we can take the derivative inside the generator to obtain

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} S(t)u_0 = L\left(S(t)u_0\right) + G_V(t)u_0$$

for almost every $t > 0$, where $S(t)u_0 \in \mathrm{Dom}(L)$. □

According to Theorem 5.3.6, we can solve quite general Cauchy problems involving a semi-fractional time derivative. In the following, we study some special choices of the infinitely divisible law μ and the corresponding semigroup $(T(t))_{t\geq 0}$ with generator L. First, choose μ as a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution on $\mathbb{R}$ for some $\alpha \in (0,2)\setminus\{1\}$ and $c > 0$. Then the corresponding generator is a mixture of positive and negative semi-fractional derivatives, and hence the corresponding semi-fractional Cauchy problem (5.14) includes semi-fractional derivatives in time and space.

Lemma 5.3.12. *(Time and space semi-fractional Cauchy problems)*
Let V be an admissable function with respect to $\beta \in (0,1)$ and $d > 1$ given by (5.16) such that (5.17) and (5.18) are satisfied. In addition, let ν be a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution for some $\alpha \in (0,2)\setminus\{1\}$ and $c > 1$ with Lévy-Khintchine triple $[a, 0, \Phi]$. Thereby, choose a as in (5.2) and let

$$\Phi(-\infty, -r) = r^{-\alpha} K_1(\log(r)) \quad \text{and} \quad \Phi(r, \infty) = r^{-\alpha} K_2(\log(r))$$

for every $r > 0$, where K_1, K_2 are admissable functions with respect to α and c. Denote by $x \mapsto p(x,t)$ the densities of the corresponding Lévy process for every $t > 0$. Then the functions

$$u(x,t) = \int_0^\infty p(x,s)h(s,t)\, ds \tag{5.30}$$

are strong solutions to

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} u(x,t) = -D \frac{\partial^\alpha}{\partial_{c,K_2} x^\alpha} u(x,t) - D \frac{\partial^\alpha}{\partial_{c,K_1} (-x)^\alpha} u(x,t) + G_p(t)\delta(x). \tag{5.31}$$

Proof. The representation of the generator L is given in Lemma 3.1.10, and with $u_0(x) = \delta(x)$, we get

$$S(t)u_0(x) = \int_0^\infty T(s)\delta(x)h(s,t)\,ds = \int_0^\infty p(x,s)h(s,t)\,ds.$$

The result now follows immediately from Theorem 5.3.6. □

In the situation of Lemma 5.3.12, we directly identify the stochastic process governed by equation (5.31). Therefore, denote by $(A(t))_{t\geq 0}$ the $(c^{\frac{1}{\alpha}}, c)$-semistable Lévy process with $P_{A(1)} = \nu$, and as before, let $(E(t))_{t\geq 0}$ be the inverse semistable subordinator belonging to the $(d^{\frac{1}{\beta}}, d)$-semistable subordinator $(D(t))_{t\geq 0}$. If we assume that $A(1)$ and $D(1)$ are independent, then for the subordinated process $(A(E(t)))_{t\geq 0}$, we have

$$P(A(E(t)) \leq x) = \int_0^\infty P(A(s) \leq x)\,dP_{E(t)}(s) = \int_0^\infty \int_{-\infty}^x p(y,s)\,dy\,h(s,t)\,ds,$$

and with Tonelli's theorem, we receive

$$P(A(E(t)) \leq x) = \int_{-\infty}^x \int_0^\infty p(y,s)h(s,t)\,ds\,dy$$

for every $x \in \mathbb{R}$. Differentiate this expression to obtain the density

$$u(x,t) = \int_0^\infty p(x,s)h(s,t)\,ds. \tag{5.32}$$

Hence, a solution u to (5.31) is the density of $(A(E(t)))_{t\geq 0}$, where $(A(t))_{t\geq 0}$ and the subordinator $(D(t))_{t\geq 0}$ are independent. Such processes appear as limiting processes of so-called Continuous Time Random Walks (CTRWs), which we study more generally in the next chapter.

Example 5.3.13. We consider a concrete example of Lemma 5.3.12. Let $\beta = 0.8$, $d = e^{2\pi\beta}$, and

$$V(x) = \frac{1}{20}\sin(2x) + \frac{1}{15}\cos(x) + \frac{1}{\Gamma(1-\beta)}$$

admissable with respect to these parameters. Note that by this choice of V, we ensure that (5.17) and (2.6) are satisfied. Denote by $x \mapsto h(x,t)$ the density of the inverse semistable subordinator corresponding to these parameters. Besides, for $\alpha = 0.5$ and $c = e^{\pi\alpha}$, let ν be the $(c^{\frac{1}{\alpha}}, c)$-semistable distribution with Lévy-Khintchine triple $[a, 0, \Phi]$ with a from

(5.2) and

$$\Phi(-\infty, -r) = 0 \quad \text{ and } \quad \Phi(r, \infty) = r^{-\alpha} K(\log(r))$$

for every $r > 0$ and a function K admissable with respect to α and c. If $x \mapsto p(x,t)$ is the density of the corresponding Lévy process at time $t > 0$, then the functions u in (5.30) solve

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} u(x,t) = -\frac{\partial^{\alpha}}{\partial_{c,K} x^{\alpha}} u(x,t) + G_p(t)\delta(x). \tag{5.33}$$

Choosing

$$K(x) = \frac{1}{10}\left(\frac{1}{2}\sin(4x) + \cos(2x)\right) + \frac{1}{\Gamma(1-\alpha)},$$

we obtain an admissable function with respect to α and c. Then the corresponding solution at time $t = 1$, calculated with the Matlab script given in Appendix C, is displayed in Figure 5.8.

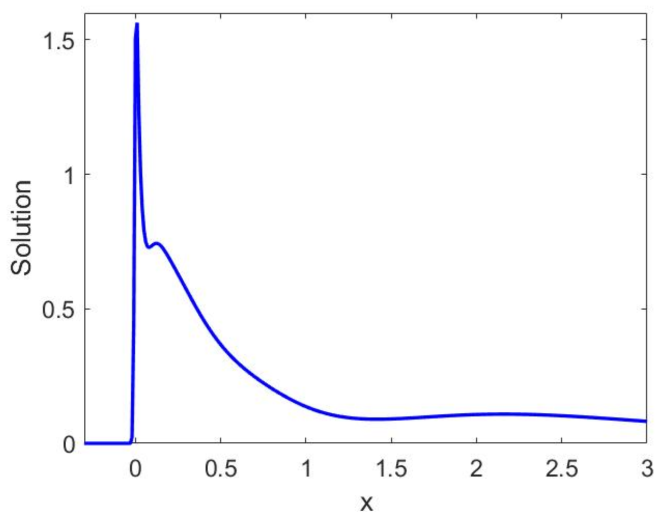

Figure 5.8: Solution $x \mapsto u(x, 1)$ of (5.33) at time $t = 1$ in Example 5.3.13.

Another important consequence of Theorem 5.3.6 is the following equation, involving an ordinary space derivative.

Example 5.3.14. Let ν be a degenerated distribution on $\mathbb{R}$ with log-characteristic function $\Psi(k) = ik$ for every $k \in \mathbb{R}$. Then the generator of the corresponding semigroup is given by

$$Lf(x) = -\frac{d}{dx} f(x),$$

and hence the semi-fractional Cauchy problem with $u_0(x) = \delta(x)$ reads as

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} u(x,t) = -\frac{d}{dx} u(x,t) + G_V(t)\delta(x). \tag{5.34}$$

Under the assumptions of Theorem 5.3.6, a solution to (5.34) is

$$u(x,t) = \int_0^{\infty} T(w)\delta(x)\, h(w,t)\, dw = \int_0^{\infty} \delta(x-w)\, h(w,t)\, du = h(x,t)$$

and therefore given by the densities of the inverse semistable subordinator.

Example 5.3.15. Let ν be a multivariate centered normal distribution with covariance matrix $Q = \mathrm{Id}$. The corresponding Lévy process $(B(t))_{t\geq 0}$ is a Brownian motion having densities

$$x \mapsto p(x,t) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{2t}||x||^2}$$

for every $t > 0$. Additionally, in view of Example 2.3.5, the generator L of the semigroup $(T(t))_{t\geq 0}$ is given by

$$Lf(x) = \frac{1}{2}\Delta f(x)$$

for every $x \in \mathbb{R}^d$. Under the assumptions of Theorem 5.3.6, the semi-fractional Cauchy problem

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} u(x,t) = \frac{1}{2}\Delta u(x,t) + G_V(t)\delta(x)$$

is solved by

$$u(x,t) = \int_0^{\infty} p(x,w) h(w,t)\, dw.$$

Similar to (5.32), $x \mapsto u(x,t)$ is the density of $B(E(t))$ for every $t > 0$. Thereby, $(E(t))_{t\geq 0}$ is the inverse semistable subordinator belonging to the $(d^{\frac{1}{\beta}}, d)$-semistable subordinator with Lévy measure

$$\Phi(-\infty, -r) = 0 \quad \text{and} \quad \Phi(r, \infty) = G_V(r)$$

for every $r > 0$ and $(B(t))_{t\geq 0}$ is an independent Brownian motion.

Chapter 6

Continuous Time Random Walks

The semi-fractional Cauchy problem (5.14) analyzed and solved in the previous section offers the opportunity to model log-periodic disturbed anomalous diffusion with an additional long-time memory effect. However, for a better understanding as well as for applications, a description of the underlying process on a microscopic scale is undoubtedly an advantage. Such a representation is offered by Continuous Time Random Walks (CTRWs), initially introduced by Montroll and Weiss ([102] and [123]). These models generalize a classical random walk by additionally considering random waiting times between two consecutive jumps. CTRWs have not only been studied extensively in different settings but also have been successfully applied to many applications in physics, biology, or finance (see [73] or [99] and the references cited therein for an overview).

In the simplest setting of a so-called uncoupled CTRW, we consider waiting times modeled by i.i.d. random variables $(J_n)_{n\in\mathbb{N}}$ and jumps described by i.i.d. random variables $(X_n)_{n\in\mathbb{N}}$ independent of $(J_n)_{n\in\mathbb{N}}$. If

$$N(t) := \max\{n \geq 0 : \sum_{j=1}^{n} J_j \leq t\}$$

denotes the number of jumps until time $t > 0$, then we study the CTRW $\sum_{j=1}^{N(t)} X_j$. This process models the position of a particle at time $t > 0$, which starts its movement in the origin at time $t = 0$. Dependent on the choice of $(J_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$, the CTRW converges to a limit process under an appropriate scaling. In the previous chapter, we already met such a limiting process. Therein, the process $(A(E(t)))_{t\geq 0}$, whose densities solve the time and space semi-fractional Cauchy problem (5.31), is the limit process of an uncoupled CTRW. In this particular case, X_1 lies in the domain of semistable attraction of $A(1)$ and J_1 lies in the domain of semistable attraction of the subordinator $D(1)$, which generates the inverse semistable subordinator $(E(t))_{t\geq 0}$. Note that characteristically the microscopic behavior is connected with the semi-fractional differential equation in the way that the densities of the limiting process provide solutions to this very equation.

However, in general, many different types of limiting processes can appear, and hence this chapter is devoted to studying general CTRW limits for random variables in the domain of semistable attraction. Thereby, we not only investigate the simpler case of uncoupled CTRW but also allow arbitrary dependencies between jumps and waiting times. Afterward, we analyze the densities of the limit process in Section 6.2. That way, we extend the possibilities to model anomalous diffusion on a microscopic scale to the situation of dependent jumps and waiting times.

6.1 Limit theorems

Primarily, to talk about process limits, we need to introduce a proper framework, which we define in terms of sample paths. Note that we are dealing with stochastic processes having càdlàg paths; these are right-continuous paths such that left-hand limits exist. Hence, we consider the space $\mathcal{D}([0,\infty),\mathbb{R}^d)$ of all càdlàg functions $f:[0,\infty)\to\mathbb{R}^d$ and endow this space with the J_1-topology, originally introduced by Skorokhod [130]. For the construction of the J_1-topology, first consider a finite domain $[0,T]$. Then the J_1-topology on $\mathcal{D}([0,T],\mathbb{R}^d)$ is given as follows: Let Λ be the set of all strictly increasing functions $\lambda:[0,T]\to[0,T]$ such that λ and its inverse λ^{-1} are continuous. If Id is the identity mapping, then for $f,g\in\mathcal{D}([0,T],\mathbb{R}^d)$

$$d_{J_1}(f,g)=\inf_{\lambda\in\Lambda}\{||f(\lambda)-g||_\infty\vee||\lambda-\operatorname{Id}||_\infty\},$$

where $a\vee b=\max\{a,b\}$. We extend the topology to $\mathcal{D}([0,\infty),\mathbb{R}^d)$ by assuming that a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}([0,\infty),\mathbb{R}^d)$ converges to $f\in\mathcal{D}([0,\infty),\mathbb{R}^d)$ if the corresponding restrictions to $[0,T]$ converge in $\mathcal{D}([0,T],\mathbb{R}^d)$ for every continuity point T of f. Note that this topology seems more suitable for càdlàg functions than, for example, the topology introduced by the maximum norm since it allows functions to be close even if they jump at slightly different times. For details on Skorokhod spaces and topologies, we refer to the monograph [147].

In this framework, we study a CTRW constructed in the subsequent way. A particle is placed at the origin of $\mathbb{R}^d$ at time $t=0$. The particle jumps randomly, where the jumps are modeled by i.i.d. random variables $(X_j)_{j\in\mathbb{N}}$. Then the position after n jumps is given by $S(n):=\sum_{j=1}^n X_j$. For simplicity, let $S(t):=S(\lfloor t\rfloor)$ for any $t>0$. Additionally, the jumps appear not constantly but only after random waiting times described by i.i.d. random variables $J_1,J_2,\ldots>0$. With $T(0):=0$ and $T(n):=\sum_{j=1}^n J_j$ for $n\in\mathbb{N}$, we denote the time of the n-th jump. Again, let $T(t):=T(\lfloor t\rfloor)$ for every $t>0$. Besides, let

$$N(t):=\max\{n\geq 0:T(n)\leq t\}$$

describe the number of jumps until time $t>0$. It follows immediately that

$$\{N(t_i)\geq s_i,\, i=1,\ldots,m\}=\{T(s_i)\leq t_i,\, i=1,\ldots,m\}\tag{6.1}$$

for every $0<t_1<\ldots<t_m$ and $s_1,\ldots,s_m>0$ and $m\in\mathbb{N}$. Now the process

$S(N(t)) = \sum_{j=1}^{N(t)} X_j$ is a CTRW, which models the position of the particle at time $t > 0$. For this thesis, we assume that the random pairs $(X_j, J_j)_{j\in\mathbb{N}}$ are i.i.d. random variables. However, we explicitly allow arbitrary dependence between X_j and J_j. If X_j and J_j are independent, the random walk is called uncoupled. In all other cases, we obtain a so-called coupled CTRW.

Based on our motivation to model log-periodically disturbed anomalous diffusion, we study limit theorems for particular choices of $(X_n)_{n\in\mathbb{N}}$ and $(J_n)_{n\in\mathbb{N}}$. More precisely, we assume that $J = J_1$ lies in the domain of semistable attraction of some $(d^{\frac{1}{\beta}}, d)$-semistable distribution μ. Note that here, d is the parameter of the semistable distribution and not the space dimension. However, the respective context clarifies the meaning of the parameter. For the random variable X_1, we go a step further and only assume that $X = X_1$ lies in the strict domain of operator semistable attraction, which is defined as follows. Let $L(\mathbb{R}^d)$ denote the set of all linear operators $A : \mathbb{R}^d \to \mathbb{R}^d$. If there are linear operators $A_n \in L(\mathbb{R}^d)$ and an increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ with $k_n \to \infty$ and $\frac{k_{n+1}}{k_n} \to c > 1$ such that

$$A_n \sum_{j=1}^{k_n} X_j \xrightarrow{d} Y \tag{6.2}$$

for a full random variable Y, then X_1 lies in the strict domain of operator semistable attraction of Y ([91, Definition 3.3.20]). Besides, at least for large values of n, we have $A_n \in GL(\mathbb{R}^n)$ due to the fullness assumption of Y, where $GL(\mathbb{R}^d)$ is the set of invertible linear operators ([91, Lemma 3.3.21]). If (6.2) holds, then Y has a strictly operator semistable distribution, meaning that for $\nu = P_Y$, it holds that

$$\nu^{*c} = (c^E \nu)$$

for some $E \in L(\mathbb{R}^d)$ ([91, Theorem 7.1.10]). In analogy to the semistable case, we say that Y is strictly (c^E, c)-operator semistable. Necessarily, the real parts of the eigenvalues of E are contained in the set $\left[\frac{1}{2}, \infty\right)$ (compare [91, Theorem 7.1.10]) and thus $E \in GL(\mathbb{R}^d)$. Note that in the special case $E = \frac{1}{\alpha}\,\mathrm{Id}$, the random variable X_1 lies in the strict domain of semistable attraction of some $(c^{\frac{1}{\alpha}}, c)$-semistable distribution. However, to prove limit theorems for the coupled case, it is not sufficient to assume a particular limiting behavior for both processes separately, but we need to impose some conditions on the joint distribution. Involving the assumptions above, we study the following setting.

We assume that there are sequences of linear operators $(A_n)_{n\in\mathbb{N}} \subset GL(\mathbb{R}^d)$ and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ as well as sequences $(k_n)_{n\in\mathbb{N}}$, $(h_n)_{n\in\mathbb{N}}$ of natural numbers with $k_n \uparrow \infty$, $h_n \uparrow \infty$ and

$$\frac{k_{n+1}}{k_n} \to c > 1 \quad \text{and} \quad \frac{h_{n+1}}{h_n} \to d > 1$$

for some $c > 1$ and $d > 1$, such that

$$\left(A_n \sum_{j=1}^{\lfloor k_n t \rfloor} X_j, \frac{1}{a_n} \sum_{j=1}^{\lfloor h_n t \rfloor} J_j \right) \xrightarrow{d} (A(t), D(t)), \tag{6.3}$$

for every $t > 0$ as $n \to \infty$ in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$, where $(A(t), D(t))_{t \geq 0}$ is a stochastic process such that $P_{(A(1),D(1))}$ is full.

Remark 6.1.1. Note that by projection onto the first components, $X = X_1$ lies in the domain of the strictly (c^E, c)-operator semistable distribution $A(1)$. Similarly, by projection onto the second component, $J = J_1$ lies in the domain of semistable attraction of $D(1)$. Since $J_1 > 0$, the process $(D(t))_{t \geq 0}$ has non-decreasing paths almost surely and hence is a semistable subordinator. Then $D(1)$ necessarily has a $(d^{\frac{1}{\beta}}, d)$-semistable distribution for some $\beta \in (0,1)$ and $d > 1$ (compare Theorem 5.2.1). Besides, it follows from Section 5.2 that $(D(t))_{t \geq 0}$ has even strictly increasing paths almost surely, corresponding to a non-finite Lévy measure on the positive real axis in the Lévy-Khintchine representation ([122, Theorem 21.3]).

In general, assumption (6.3) is hard to verify and is challenging to handle due to the different number of terms in the individual components. Furthermore, we cannot ensure that the resulting process is a Lévy process, which additionally complicates the calculation. For this reason, we restrict our attention to two particular cases.

First, consider the uncoupled case, in which X and J are independent. Then the joint convergence in (6.3) is equal to the convergence of the individual components

$$A_n \sum_{j=1}^{\lfloor k_n t \rfloor} X_j \xrightarrow{d} A(t) \quad \text{and} \quad a_n^{-1} \sum_{j=1}^{\lfloor h_n t \rfloor} J_j \xrightarrow{d} D(t)$$

in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d)$ and $\mathcal{D}([0,\infty), [0,\infty))$ respectively. Furthermore, $(A(t))_{t \geq 0}$ and $(D(t))_{t \geq 0}$ are (operator) semistable Lévy processes, and with the independence of the components, the limiting process $(A(t), D(t))_{t \geq 0}$ is a Lévy process likewise. We aim to show that we obtain joint convergence with an equal number of terms in both components in (6.3) by proceeding to particular subsequences. Therefore we use the theory of regular variation, which proved to be a powerful tool in the theory of stochastic process limits. The subsequent definition of regularly varying sequences and functions is taken from [91, Definition 4.1.1, Definition 4.2.8].

Definition 6.1.2. (Regularly varying sequences and functions)
A sequence $(x_n)_{n \in \mathbb{N}}$ of positive real numbers is regularly varying with index $\lambda \in \mathbb{R}$ if

$$\frac{x_{\lfloor tn \rfloor}}{x_n} \to t^{\lambda}$$

for every $t > 0$ as $n \to \infty$. Similarly, a sequence of linear operators $(A_n)_{n \in \mathbb{N}} \subset GL(\mathbb{R}^d)$

varies regularly with index $E \in L(\mathbb{R}^d)$ if

$$A_{\lfloor tn \rfloor} A_n^{-1} \to t^E$$

for every $t > 0$ as $n \to \infty$. Finally, a function $f : [0, \infty) \to GL(\mathbb{R}^d)$ varies regularly with index $E \in L(\mathbb{R}^d)$ if

$$f(tx)f(x)^{-1} = t^E$$

for every $t > 0$ as $x \to \infty$.

According to [91, Corollary 4.2.15] in combination with [16, III, Semistable Distributions, Theorem 2.1], there is a sequence of positive real numbers $(b_n)_{n\in\mathbb{N}}$ regularly varying with index $\frac{1}{\beta}$ such that $a_n = b_{h_n}$ for every $n \in \mathbb{N}$. Besides, by defining $b(x) = b_{\lfloor x \rfloor}$, we extend the sequence $(b_n)_{n\in\mathbb{N}}$ to a function $b : [1, \infty) \to (0, \infty)$ regularly varying with the same index. Hence (6.3) reads as

$$\left(A_n \sum_{j=1}^{\lfloor k_n t \rfloor} X_j, b(h_n)^{-1} \sum_{j=1}^{\lfloor h_n t \rfloor} J_j \right) \xrightarrow{d} (A(t), D(t)) \tag{6.4}$$

for every $t > 0$ in the J_1-topology on $\mathcal{D}([0, \infty), \mathbb{R}^d \times [0, \infty))$.

Remark 6.1.3. (Asymptotic inverse function)
According to [127, p. 21], there is an asymptotic inverse function $b^{\leftarrow}$ of b; this is

$$\lim_{x\to\infty} \frac{b(b^{\leftarrow}(x))}{x} = \lim_{x\to\infty} \frac{b^{\leftarrow}(b(x))}{x} = 1.$$

In the following, for functions $f, g : [A, \infty) \to \mathbb{R}$ with $A \geq 0$, we write

$$f \sim g \quad \Leftrightarrow \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

to denote asymptotic equivalence. Additionally, the function $b^{\leftarrow}$ is regularly varying with index β. One possible choice of $b^{\leftarrow}$ is to take the generalized inverse function

$$b^{-1}(x) \coloneqq \inf\{y \in [1, \infty) : b(y) > x\}.$$

We define $\gamma : (0, \infty) \to (0, 1]$ by

$$\gamma(x) \coloneqq \frac{x}{h_n}$$

for $h_{n-1} < x \leq h_n$, $n \in \mathbb{N}$, with $h_0 \coloneqq 0$. The following lemma is an essential tool to obtain the same number of terms in the assumption of joint convergence.

Lemma 6.1.4. *There is a sequence $(x_n)_{n\in\mathbb{N}} \subset [1, \infty)$ and a version of the asymptotic inverse function $b^{\leftarrow}$ of b such that* *(i)* $x_n \uparrow \infty$ *as* $n \to \infty$,

(ii) $\gamma\left(b^{\leftarrow}(x_n)\right) \to r$ *as* $n \to \infty$ *for some* $r \in [d^{-1}, 1]$, *and*

(iii) there is another sequence $(u_n)_{n\in\mathbb{N}}$ *of natural numbers with* $b^{\leftarrow}(x_n) = k_{u_n}$ *for every* $n \in \mathbb{N}$.

Proof. Define $y_n := b(k_n)$ for every $n \in \mathbb{N}$. Since $k_n \uparrow \infty$ and $b(n) \uparrow \infty$ as $n \to \infty$, it follows that $y_n \uparrow \infty$. In addition,

$$b^{\leftarrow}(y_n) = b^{\leftarrow}(b(k_n)) \sim k_n$$

as $n \to \infty$. Now the asymptotic inverse function $b^{\leftarrow}$ is unique only up to asymptotic equivalence such that we can choose $b^{\leftarrow}$ fulfilling $b(y_n) = k_n$ for every $n \in \mathbb{N}$. Then

$$\gamma\left(b^{\leftarrow}(y_n)\right) = \gamma(k_n).$$

The sequence $(\gamma(k_n))_{n\in\mathbb{N}}$ is bounded, and according to the Bolzano-Weierstrass theorem, there is a convergent subsequence $(\gamma(k_{u_n}))_{n\in\mathbb{N}}$. The sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n := y_{u_n}$ is the sequence we were searching for. $\square$

Lemma 6.1.5. *Let* $(x_n)_{n\in\mathbb{N}}$ *and* $(u_n)_{n\in\mathbb{N}}$ *be the sequences in Lemma 6.1.4. In the uncoupled case, the convergence in (6.3) yields*

$$\sum_{j=1}^{\lfloor k_{u_n} t\rfloor} \left(A_{u_n} X_j, x_n^{-1} J_j\right) \xrightarrow{d} (A(t), r^{-\frac{1}{\beta}} D(rt)) \tag{6.5}$$

for every $t > 0$ *as* $n \to \infty$ *in the* J_1*-topology on* $\mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$, *where* $r \in [d^{-1}, 1]$ *is the limit of* $\gamma\left(b^{\leftarrow}(x_n)\right)$ *in Lemma 6.1.4.*

Proof. Since X and J are independent, it is sufficient to study the convergence of the individual components. The convergence of the first component follows immediately from (6.3). For the second component, note that for fixed $t > 0$,

$$x_n^{-1} \sum_{j=1}^{\lfloor k_{u_n} t\rfloor} J_j = x_n^{-1} T(k_{u_n} t) = x_n^{-1} T(b^{\leftarrow}(x_n) t)$$

using the properties of the sequence $(u_n)_{n\in\mathbb{N}}$ from Lemma 6.1.4. Now for every $n \in \mathbb{N}$, there is $p(x_n) \in \mathbb{R}$ such that

$$b^{\leftarrow}(x_n) = h_{p(x_n)} \gamma(b^{\leftarrow}(x_n)). \tag{6.6}$$

Since $b^{\leftarrow}(x_n) \uparrow \infty$ as $n \to \infty$ we also have $p(x_n) \uparrow \infty$ as $n \to \infty$. Then we obtain

$$x_n^{-1} \sum_{j=1}^{\lfloor k_{u_n} t\rfloor} J_j = x_n^{-1} T(h_{p(x_n)} \gamma(b^{\leftarrow}(x_n)) t) = \frac{b(h_{p(x_n)})}{x_n b(h_{p(x_n)})} T(h_{p(x_n)} \gamma(b^{\leftarrow}(x_n)) t).$$

In view of (6.6), we receive the convergence

$$
\begin{aligned}
\frac{b(h_{p(x_n)})}{x_n} &\sim \frac{b(h_{p(x_n)})}{b(b^{\leftarrow}(x_n))} \\
&= \frac{b(h_{p(x_n)})}{b(h_{p(x_n)}\gamma(b^{\leftarrow}(x_n)))} \\
&\to r^{-\frac{1}{\beta}}
\end{aligned}
$$

as $n \to \infty$ using the regular variation of b and the convergence $\gamma(b^{\leftarrow}(x_n)) \to r$. Then for every $t > 0$, we find

$$
x_n^{-1} \sum_{j=1}^{\lfloor k_{u_n} t \rfloor} J_j = \frac{b(h_{p(x_n)})}{x_n b(h_{p(x_n)})} \sum_{j=1}^{\lfloor h_{p(x_n)}\gamma(b^{\leftarrow}(x_n))t \rfloor} J_j \xrightarrow{d} r^{-\frac{1}{\beta}} D(rt)
$$

in the J_1-topology on $\mathcal{D}([0,\infty),[0,\infty))$ in view of (6.4). □

Before we formulate a CTRW limit theorem, we present a limit theorem for the number of jumps $(N(t))_{t\geq 0}$.

Lemma 6.1.6. *(Limit theorem for the number of jumps - Uncoupled case)*
Let $(x_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ be the sequences from Lemma 6.1.4. In the uncoupled case, we obtain

$$
b^{\leftarrow}(x_n)^{-1} N(x_n t) \xrightarrow{d} r^{-1} E(r^{\frac{1}{\beta}} t)
$$

for every $t > 0$ as $n \to \infty$ in the J_1-topology on $\mathcal{D}([0,\infty),[0,\infty))$.

Proof. First, we show that the above convergence holds for the finite-dimensional marginal distributions. So let $0 < t_1 < t_2 < \ldots < t_m$ and $s_1, s_2, \ldots, s_m > 0$. Then

$$
\begin{aligned}
P(b^{\leftarrow}(x_n)^{-1} N(x_n t_k) \leq s_k\,,\, k = 1,\ldots,m) &= P(N(x_n t_k) \leq b^{\leftarrow}(x_n) s_k\,,\, k = 1,\ldots,m) \\
&= P(N(x_n t_k) < \lfloor b^{\leftarrow}(x_n) s_k \rfloor + 1\,,\, k = 1,\ldots,m)
\end{aligned}
$$

since N only takes integer values. Using (6.1), we have

$$
\begin{aligned}
P(b^{\leftarrow}(x_n)^{-1} N(x_n t_k) \leq s_k\,,\, k = 1,\ldots,m) &= P(T(\lfloor b^{\leftarrow}(x_n) s_k \rfloor + 1) > x_n t_k\,,\, k = 1,\ldots,m) \\
&= P\left(\frac{T(\lfloor b^{\leftarrow}(x_n) s_k \rfloor + 1)}{x_n} > t_k\,,\, k = 1,\ldots,m\right).
\end{aligned}
$$

Note that for every $k = 1,\ldots,m$

$$
x_n^{-1} T(\lfloor b^{\leftarrow}(x_n) s_k \rfloor + 1) = x_n^{-1} T(\lfloor k_{u_n} s_k \rfloor) + x_n^{-1} J_{\lfloor k_{u_n} s_k \rfloor + 1}
$$

for the sequence $(u_n)_{n\in\mathbb{N}}$ from Lemma 6.1.4. In addition, $x_n \to \infty$ as $n \to \infty$ such that

$x_n^{-1} J_{\lfloor k_{u_n} s_k \rfloor + 1} \xrightarrow{p} 0$, and from Lemma 6.1.5, we obtain

$$P(x_n^{-1} T(\lfloor k_{u_n} s_k \rfloor) > t_k\,,\ k = 1, \ldots, m) \to P(r^{-\frac{1}{\beta}} D(r s_k) > t_k\,,\ k = 1, \ldots, m).$$

Hence

$$\begin{aligned} P(b^{\leftarrow}(x_n)^{-1} N(x_n t_k) \leq s_k\,,\ k = 1, \ldots, m) &\to P(D(r s_k) > r^{\frac{1}{\beta}} t_k\,,\ k = 1, \ldots, m) \\ &= P(D(r s_k) \geq r^{\frac{1}{\beta}} t_k\,,\ k = 1, \ldots, m) \\ &= P(E(r^{\frac{1}{\beta}} t_k) \leq r s_k\,,\ k = 1, \ldots, m) \\ &= P(r^{-1} E(r^{\frac{1}{\beta}} t_k) \leq s_k\,,\ k = 1, \ldots, m) \end{aligned}$$

using (5.11) since $D(t)$ has a continuous density for every $t > 0$. In addition, $D(t)$ is strictly increasing almost surely such that the limit process $r^{-1} E(r^{\frac{1}{\beta}} t)$ has continuous sample paths almost surely, and the process $b^{\leftarrow}(x_n)^{-1} N(x_n t_k)$ has monotonically increasing paths. Then Theorem 3 of [23] directly yields the convergence in distribution with respect to the J_1-topology. □

Now we can prove one of the main theorems of this chapter, the limit of uncoupled CTRWs in the semistable framework. Indeed, we study two different constructions. First, consider the case where a particle waits a random time J_1 before it jumps for the first time with height X_1 and then waits for the second jump and so forth. This model belongs to the classical CTRW described by $S(N(t))$. On the contrary, the particle can jump initially at time $t = 0$ and then wait a random time J_1 for the second jump to appear. The latter is called Overshoot Continuous Time Random Walk (OCTRW) and is described by $S(N(t) + 1)$. As the following theorem shows, for the uncoupled case, both constructions yield the same limiting process. However, in the coupled case, which we analyze afterward, both models can result in completely different limiting processes (see Example 6.2.7 below). For this reason, we already mention both constructions here.

Apart from càdlàg functions, we additionally deal with càglàd functions in the following. In our setting, these are left-continuous functions $f : [0, \infty) \to \mathbb{R}^d$ such that the right-hand limits exist. For a càdlàg function $f : [0, \infty) \to \mathbb{R}^d$, denote by $t \mapsto f(t-)$ the càglàd version of f. On the other hand, for a càglàd function $g : [0, \infty) \to \mathbb{R}^d$, denote by $t \mapsto (g(t))_+$ the càdlàg version of g. Note that the discrepancy between both notations seems unusual but will simplify the arguments and notation in further calculations.

Theorem 6.1.7. *(CTRW and OCTRW limit theorem - Uncoupled case)*
Let $(x_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$ be the sequences in Lemma 6.1.4. In the uncoupled case, the CTRW limit is given by

$$A_{u_n} S(N(x_n t)) \xrightarrow{d} M_r(t) := A(r^{-1} E(r^{\frac{1}{\beta}} t))$$

for every $t > 0$ in the J_1-topology on $\mathcal{D}([0, \infty), \mathbb{R}^d)$. In addition, the limit of the OCTRW $A_{u_n} S(N(x_n t) + 1)$ coincides with the CTRW limit almost surely.

Proof. For the proof of Theorem 6.1.7, we apply a method more general than necessary. However, the main part of the proof does not require independent components. Hence, we can transfer the method to the coupled case we study later on.
Consider the partial sum process

$$(S_n(t), T_n(t)) := \sum_{j=1}^{\lfloor k_{u_n} t \rfloor} (A_{u_n} X_j, x_n^{-1} J_j).$$

According to Lemma 6.1.5, we have

$$(S_n(t), T_n(t)) \xrightarrow{d} (A(t), r^{-\frac{1}{\beta}} D(rt))$$

as $n \to \infty$ in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$. For the CTRW and OCTRW limit processes, we apply a method of Straka and Henry in [133] based on continuous mapping arguments. Therefore, let $\mathcal{D}_{u,\uparrow} \subset \mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$ be the space of all functions (u, v) in $\mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$ such that v is unbounded and non-decreasing. In addition, let $\mathcal{D}_{u,\uparrow\uparrow}$ be the space of all functions $(u, v) \in \mathcal{D}_{u,\uparrow}$, which have a strictly increasing second component. Denote by v^{-1} the generalized inverse of v, and consider the functions $\Phi^{\uparrow} : \mathcal{D}_{u,\uparrow} \to \mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$ with

$$\Phi^{\uparrow}(u(t), v(t)) = ((u(v^{-1}(t-)-))_+, (v(v^{-1}(t-)-))_+)$$

and $\Psi^{\uparrow} : \mathcal{D}_{u,\uparrow} \to \mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$ with

$$\Psi^{\uparrow}(u(t), v(t)) = (u(v^{-1}(t)), v(v^{-1}(t))).$$

According to Proposition 2.3 in [133], the functions $\Phi^{\uparrow}$ and $\Psi^{\uparrow}$ are continuous on $\mathcal{D}_{u,\uparrow\uparrow}$ with respect to the J_1-topology and $P(\mathcal{D}_{u,\uparrow\uparrow}) = 1$. Now since $(D(t))_{t\geq 0}$ has strictly increasing and unbounded paths almost surely, Theorem 3.6 in [133] yields

$$\Phi^{\uparrow}(S_n(t), T_n(t)) \to \Phi^{\uparrow}(A(t), r^{-\frac{1}{\beta}} D(rt)) \tag{6.7}$$

and

$$\Psi^{\uparrow}(S_n(t), T_n(t)) \to \Psi^{\uparrow}(A(t), r^{-\frac{1}{\beta}} D(rt)). \tag{6.8}$$

We analyze the components of $\Phi^{\uparrow}$ and $\Psi^{\uparrow}$ in (6.7) and (6.8) separately. Note that in [44, Theorem 1], it was shown that

$$\begin{aligned} T_n^{-1}(t) &= \inf\{s \geq 0 : T_n(s) > t\} \\ &= \sup\{s \geq 0 : T_n(s) \leq t\} \end{aligned}$$

such that

$$\begin{aligned}
T_n^{-1}(t) &= \sup\left\{s \geq 0 : x_n^{-1}\sum_{j=1}^{\lfloor k_{u_n}s\rfloor} J_j \leq t\right\} \\
&= \sup\left\{s \geq 0 : \sum_{j=1}^{\lfloor k_{u_n}s\rfloor} J_j \leq x_n t\right\} \\
&= k_{u_n}^{-1}\sup\left\{s \geq 0 : \sum_{j=1}^{\lfloor s\rfloor} J_j \leq x_n t\right\} \\
&= k_{u_n}^{-1}\left(\max\left\{n \in \mathbb{N}_0 : \sum_{j=1}^{n} J_j \leq x_n t\right\} + 1\right) \\
&= k_{u_n}^{-1}(N(x_n t) + 1).
\end{aligned}$$

Then the first component of $\Psi^{\uparrow}(S_n(t), T_n(t))$ is given by

$$\begin{aligned}
S_n(T_n^{-1}(t)) &= A_{u_n}\sum_{j=1}^{\lfloor k_{u_n}k_{u_n}^{-1}(N(x_n t)+1)\rfloor} X_j \\
&= A_{u_n}S(N(x_n t) + 1).
\end{aligned}$$

To analyze the first component of the limit, note that

$$\begin{aligned}
\left(r^{-\frac{1}{\beta}}D(rt)\right)^{-1} &= \inf\{s \geq 0 : r^{-\frac{1}{\beta}}D(rs) > t\} \\
&= \inf\{s \geq 0 : D(rs) > r^{\frac{1}{\beta}}t\} \\
&= r^{-1}\inf\{s \geq 0 : D(s) > r^{\frac{1}{\beta}}t\} \\
&= r^{-1}E(tr^{\frac{1}{\beta}})
\end{aligned} \tag{6.9}$$

such that by (6.8), we obtain

$$A_{u_n}S(N(x_n t) + 1) \xrightarrow{d} O_r(t) := A(r^{-1}E(tr^{\frac{1}{\beta}})).$$

Here we only use the convergence of the first component since it already yields the claimed result. We deal with $\Phi^{\uparrow}$ similarly. The càglàd version of the generalized inverse function T_n^{-1} is given by

$$T_n^{-1}(t-) = k_{u_n}^{-1}(N(x_n t-) + 1)$$

such that the first component of $\Phi^{\uparrow}(S_n(t), T_n(t))$ equals

$$(S_n(T_n^{-1}(t-)-))_+ = (S_n(k_{u_n}^{-1}(N(x_n t-) + 1)-))_+.$$

Now for every $t > 0$,

$$S_n(t-) = \lim_{s\uparrow t} A_{u_n} \sum_{j=1}^{\lfloor k_{u_n} s\rfloor} X_j = \lim_{s\uparrow t} A_{u_n} \sum_{j=1}^{\lceil k_{u_n} s\rceil - 1} X_j = A_{u_n} \sum_{j=1}^{\lceil k_{u_n} t\rceil - 1} X_j$$

such that

$$\begin{aligned}
(S_n(T_n^{-1}(t-)-))_+ &= \left(A_{u_n} \sum_{j=1}^{\lceil N(x_n t-)+1\rceil - 1} X_j\right)_+ \\
&= \left(A_{u_n} \sum_{j=1}^{\lceil N(x_n t-)\rceil} X_j\right)_+ \\
&= A_{u_n} \sum_{j=1}^{N(x_n t)} X_j \\
&= A_{u_n} S(N(x_n t)).
\end{aligned}$$

With (6.9), the first component of the limit in (6.7) is given by

$$(A(r^{-1}E(tr^{\frac{1}{\beta}}-)-))_+ = (A(r^{-1}E(tr^{\frac{1}{\beta}})-))_+$$

since $(E(t))_{t\geq 0}$ has continuous sample paths almost surely. Then

$$A_{u_n} S(N(x_n t)) \xrightarrow{d} M_r(t) := (A(r^{-1}E(tr^{\frac{1}{\beta}})-))_+$$

in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d)$.

Up to this point, the proof works without the additional assumption of independence. Next, we show that the OCTRW and CTRW limit processes above are equal almost surely if X and J are independent. In this case, $(A(t))_{t\geq 0}$ and $(r^{-\frac{1}{\beta}} D(rt))_{t\geq 0}$ are independent as well, and hence the processes have no simultaneous jumps almost surely. To show equality, we evaluate the càglàd version of $(O_r(t))_{t\geq 0}$ and argue that this version coincides with $A(r^{-1}E(tr^{\frac{1}{\beta}})-)$ almost surely. The càglàd version of $(O_r(t))_{t\geq 0}$ is given by $\lim_{s\uparrow t} A(r^{-1}E(sr^{\frac{1}{\beta}}))$. Now $(E(t))_{t\geq 0}$ is non-decreasing almost surely such that

$$\lim_{s\uparrow t} A(r^{-1}E(sr^{\frac{1}{\beta}})) = \begin{cases} A(r^{-1}E(tr^{\frac{1}{\beta}})-) & \text{if } E(tr^{\frac{1}{\beta}}) > E(ur^{\frac{1}{\beta}}) \quad \text{for all } u < t \\ A(r^{-1}E(tr^{\frac{1}{\beta}})) & \text{otherwise.} \end{cases}$$

In the second case, $r^{\frac{1}{\beta}} D(rt)$ has a jump in $r^{-1}E(tr^{\frac{1}{\beta}})$. Then due to our assumption, with probability one $A(t)$ has no jump in $r^{-1}E(tr^{\frac{1}{\beta}})$ such that we can write

$$\lim_{s\uparrow t} A(r^{-1}E(sr^{\frac{1}{\beta}})) = A(r^{-1}E(tr^{\frac{1}{\beta}})-)\text{almost surely}$$

in both cases. Hence, the limiting processes coincide almost surely. □

We want to name some special case of the above theorem.

Example 6.1.8. (Special cases of the uncoupled CTRW limit)

1. If $r \in \{d^{-1}, 1\}$, then due to the semi-selfsimilarity of $(E(t))_{t\geq 0}$ (see Lemma 5.2.4), we have

$$A_{u_n} S(N(x_n t)) \xrightarrow{d} M_r(t) = A(E(t)),$$

in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d)$.

2. If $r \in [d^{-1}, 1]$ can be written as $r = c^{-k}$ for some $k \in \mathbb{N}_0$, then

$$M_r(t) = A(c^k E(tc^{-\frac{k}{\beta}})) = c^{\frac{k}{\alpha}} A(E(tc^{-\frac{k}{\beta}}))$$

using the semi-selfsimilarity of $(A(t))_{t\geq 0}$.

3. If the random waiting times $J_1, J_2, \ldots$ belong to the strict domain of attraction of some stable law, then the inverse stable subordinator $(E(t))_{t\geq 0}$ is selfsimilar with index β (see Lemma 5.2.4). Consequently, the limiting process in Theorem 6.1.7 is given by

$$M_r(t) = A(r^{-1} E(tr^{\frac{1}{\beta}})) = A(E(t))$$

for every $t > 0$.

4. If the random variables $X_1, X_2, \ldots$ modeling the jumps are chosen in the strict domain of attraction of some stable law, then $(A(t))_{t\geq 0}$ is selfsimilar with index $\frac{1}{\alpha}$, and we obtain

$$A_{u_n} S(N(x_n t)) \xrightarrow{d} M_r(t) = r^{-\frac{1}{\alpha}} A(E(tr^{\frac{1}{\beta}}))$$

in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d)$.

After studying uncoupled CTRWs, we extend our considerations to coupled cases, where we allow arbitrary dependencies between X and J in (6.3). As already mentioned, for general sequences $(k_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$, the different number of terms in the individual components in (6.3) causes difficulties, and we therefore assume that $k_n = h_n$ for every $n \in \mathbb{N}$, which yields

$$\sum_{j=1}^{\lfloor k_n t\rfloor} (A_n X_j, a_n^{-1} J_j) \xrightarrow{d} (A(t), D(t)) \tag{6.10}$$

for every $t > 0$ in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d \times [0,\infty))$. Note that in this case, the limiting process $(A(t), D(t))_{t\geq 0}$ is operator semistable and hence a Lévy process. Consequently, (X, J) is in the strict domain of attraction of the semistable Lévy process. Using the previous case, we obtain the following limit theorem for the number of jumps.

Lemma 6.1.9. *(Limit theorem for the number of jumps - Coupled case)*
Assume that $h_n = k_n$ for every $n \in \mathbb{N}$. Then

$$k_n^{-1} N(a_n t) \xrightarrow{d} E(t)$$

for every $t > 0$ as $n \to \infty$ in the J_1-topology on $\mathcal{D}([0,\infty),[0,\infty))$.

Proof. We want to adapt the proof of the corresponding limit theorem in the uncoupled case. First, note that the assumption of uncoupled components was not used in the proof of Lemma 6.1.6. Instead, we only needed the representation (6.5), where the sequences $(u_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ as well as $r \in [d^{-1}, 1]$ are given by Lemma 6.1.4. However, in the coupled case with $h_n = k_n$, we can equally write

$$\sum_{j=1}^{\lfloor k_n t \rfloor} (A_n X_j, b(k_n)^{-1} J_j) \xrightarrow{d} (A(t), D(t)),$$

where b is a regularly varying function with index $\frac{1}{\beta}$ such that $a_n = b(k_n)$ for every $n \in \mathbb{N}$. Choosing $u_n = n$ and $x_n = b(k_n)$ for every $n \in \mathbb{N}$, we have $x_n \uparrow \infty$ as $n \to \infty$, $b^{\leftarrow}(x_n) = k_n = k_{u_n}$ for a particular choice of the asymptotic inverse function $b^{\leftarrow}$, and

$$\gamma(b^{\leftarrow}(x_n)) = \gamma(k_n) = \gamma(h_n) = 1$$

for every $n \in \mathbb{N}$. Thus, we can display the joint convergence (6.10) in the form (6.5) and the proof of Lemma 6.1.6 yields

$$k_n^{-1} N(a_n t) = k_n^{-1} N(b(k_n)t) = b^{\leftarrow}(x_n)^{-1} N(x_n t) \xrightarrow{d} E(t)$$

for every $t > 0$ as $n \to \infty$ in the J_1-topology. □

Using a similar construction, we likewise gain a limit theorem for the coupled CTRW and OCTRW.

Theorem 6.1.10. *(CTRW and OCTRW limit theorem - Coupled case)*
Assume that $h_n = k_n$ for every $n \in \mathbb{N}$. For every $t > 0$, the limit of the coupled CTRW is given by

$$A_n S(N(a_n t)) \xrightarrow{d} M(t) := (A(E(t)-))_+,$$

whereas for the OCTRW, we obtain

$$A_n S(N(a_n t) + 1) \xrightarrow{d} O(t) := A(E(t))$$

in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d)$.

Proof. Using the construction in the proof of Lemma 6.1.9, the proof of Theorem 6.1.7

directly yields the OCTRW limit

$$A_{u_n} S(N(x_n t) + 1) \xrightarrow{d} A(r^{-1} E(tr^{\frac{1}{\beta}}))$$

for every $t > 0$ in the J_1-topology on $\mathcal{D}([0,\infty), \mathbb{R}^d)$. Inserting $u_n = n$, $x_n = b(k_n) = a_n$ as well as $r = 1$, we obtain the claimed result. Similarly, for the CTRW, the limit is given by

$$A_n S(N(a_n t)) = A_{u_n} S(N(x_n t)) \xrightarrow{d} (A(r^{-1} E(tr^{\frac{1}{\beta}}) -))_+ = (A(E(t)-))_+$$

for every $t > 0$ in the J_1-topology. □

Example 6.1.11. To close this section, we study a concrete example of CTRWs and OCTRWs. Let Y be a disturbed Pareto distribution as introduced in Example 2.2.7 with index $\alpha = \frac{3}{2}$, $c = e^\alpha$, and perturbation

$$K(x) = 5 + \sin(2\pi x)$$

admissable with respect to these parameters. Now model the jumps with the centered variable $X = Y - \mathbb{E}[Y]$, where with $\alpha > 1$, the expected value exists. According to Example 2.2.7, the sum $c^{-\frac{n}{\alpha}} \sum_{j=1}^{\lfloor c^n \rfloor} X_j$ converges to the semistable distribution with log-characteristic function

$$\Psi(k) = -\frac{c}{5}(-ik)^\alpha \eta_1(\log(k)),$$

where η_1 arises from the Fourier coefficients $(c_n)_{n\in\mathbb{N}}$ of K as

$$\eta_1(x) = \sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1) e^{-in\tilde{c}x}.$$

Furthermore, we model the waiting times J_j, $j \in \mathbb{N}$, independent of the jumps, to be disturbed Pareto random variables with index $\beta = 0.75$, $d = e^{2\beta} = c$, and perturbation

$$V(x) = 6 + \cos(\pi x)$$

admissable with respect to β and d. In Figure 6.1, a path simulation of

$$S(t) = \sum_{j=1}^{\lfloor t \rfloor} X_j \quad \text{and} \quad T(t) = \sum_{j=1}^{\lfloor t \rfloor} J_j \tag{6.11}$$

for $t \in (0, 1000]$ is shown. Since we study the uncoupled scenario here, the limit distribution of the OCTRW equals those of the CTRW (compare Theorem 6.1.7). For $t \in (0, 1000]$, a typical path of both processes is shown in Figure 6.2. The zoomed cut on the right-hand side displays that the OCTRW is by definition always one jump ahead. Nevertheless, in view of the picture on the large scale on the left-hand side of Figure 6.1, it seems reasonable that both models yield the same limiting process. The Matlab code for this simulation is inspired by the fractional case displayed in [91, Section 5.2] and is attached in Appendix C.

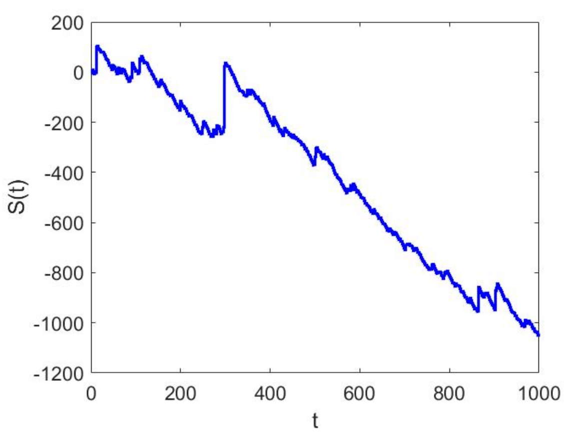

Figure 6.1: Simulation of the cumulative sum $S(t)$ (left) and $T(t)$ (right) in (6.11) for $t \in (0, 1000]$ in Example 6.1.11.

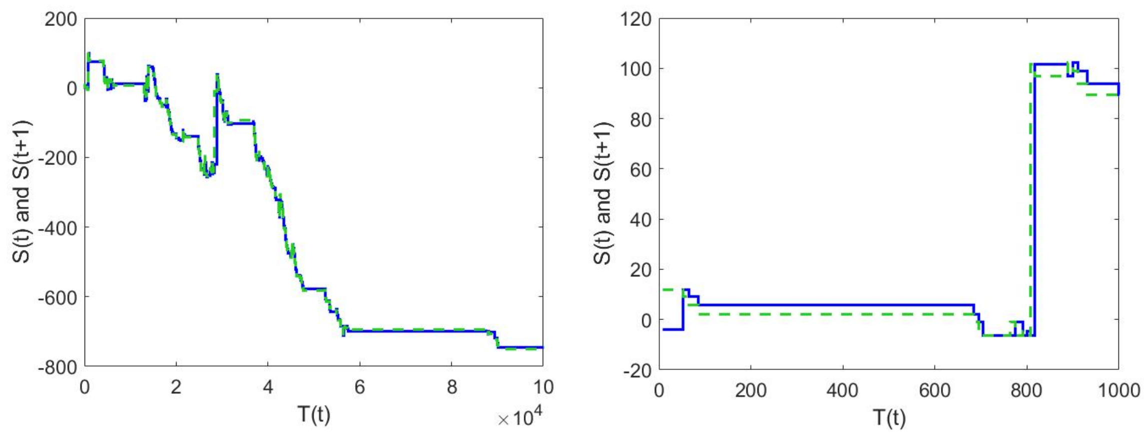

Figure 6.2: Path simulation of the CTRW (blue solid line) and the OCTRW (green dashed line) in Example 6.1.11 on a large scale (left) and zoomed to the first jumps (right).

Similarly, one can consider the OCTRW and CTRW limit, where the waiting times $J, J_1, \ldots$ are constructed as above and the jumps are chosen as $X = J$. This case models a totally coupled scenario, and here, the limiting processes show a completely different path behavior. We analyze their difference in detail in the following section (compare Example 6.2.7). However, studying a path simulation as displayed in Figure 6.3, one may already conjecture a discrepancy between both limits.

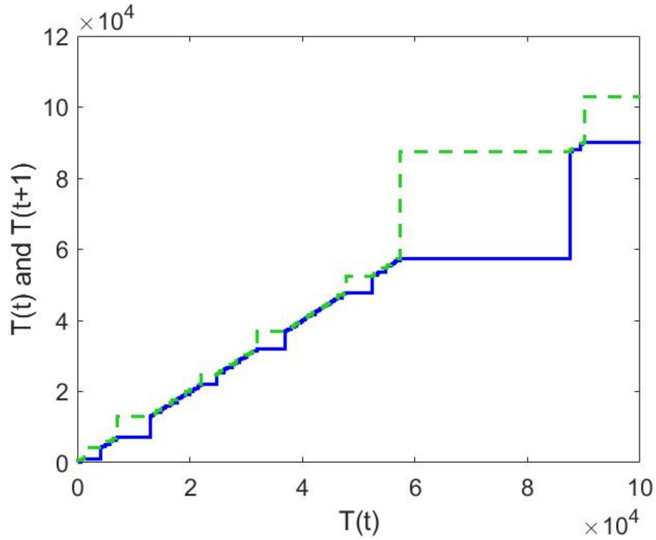

Figure 6.3: Path simulation of the coupled CTRW (blue solid line) and the OCTRW (green dashed line) in Example 6.1.11.

6.2 Limit distributions

To further analyze and describe the limit processes of the CTRW and the OCTRW, we study the corresponding distributions and the thereby deduced densities. As a special case, we regain the connection between uncoupled CTRWs and semi-fractional Cauchy problems displayed in Section 5.3. However, we are able to consider the far more complex situation of coupled CTRWs/OCTRWs likewise. Especially the case of totally coupled waiting times and jumps delivers interesting results. The proofs in this section are inspired by the works of Julewicz, Kern, Meerschaert and Scheffler (e.g., see [18], [17], or [61]) dealing with densities of CTRW and OCTRW limits in the operator stable case.

In the previous section, we distinguished the two cases of independent random variables X and J and the coupled case with $h_n = k_n$ for every $n \in \mathbb{N}$. To avoid repetitions, we summarize both cases as suggested in the proof of Lemma 6.1.9. This is, we have

$$A_{u_n} S(N(x_n t)) \to M_r(t) = (A(r^{-1} E(tr^{\frac{1}{\beta}})-))_+$$

and

$$A_{u_n} S(N(x_n t) + 1) \to O_r(t) = A(r^{-1} E(tr^{\frac{1}{\beta}}))$$

for every $t > 0$ in the J_1-topology on $\mathcal{D}([0, \infty), \mathbb{R}^d)$, where the sequences $(x_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are given as in Lemma 6.1.4.
The aim of this section is to develop an integral representation for the densities of the limit-

ing processes in terms of the joint density and its Lévy measure using Fourier and Laplace transforms. In [18, Lemma 2.1], it was shown that for the Lévy process $(A(u), D(ru))_{u\geq 0}$, there exists a uniquely determined continuous function $\Lambda : \mathbb{R}^d \times [0,\infty) \to \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ such that the Fourier-Laplace transform of $P_{(A(u),D(ru))}$ is given by

$$\overline{P_{(A(u),D(ru))}}(k,s) := \int\limits_{\mathbb{R}^d} \int\limits_0^\infty e^{i\langle k,x\rangle} e^{-st} \, P_{(A(u),D(ru))}(dx,dt) = e^{-u\Lambda(k,s)}$$

for every $k \in \mathbb{R}^d$ and $s \geq 0$. We call Λ the log-Fourier-Laplace transform (log-FLT) of $(A(1), D(r))$. Thereby, Λ has the representation

$$\Lambda(k,s) = i\langle a,k\rangle + bs + \langle k, Qk\rangle + \int\limits_{\mathbb{R}^d\times[0,\infty)\setminus\{(0,0)\}} \left(1 - e^{i\langle k,x\rangle} e^{-st} + \frac{i\langle k,x\rangle}{1+||x||^2}\right) \Phi_{(A(1),D(r))}(dx,dt) \tag{6.12}$$

for some $a \in \mathbb{R}^d$, $b \geq 0$, a non-negative definite matrix $Q \in \mathbb{R}^{d\times d}$, and a Lévy measure $\Phi_{(A(1),D(r))}$ on $\mathbb{R}^d \times [0,\infty) \setminus \{(0,0)\}$. Note that more precisely, in [18], the authors studied operator stable processes. However, for the proof, the authors used arguments in [115], which similarly hold for this more general case. Due to

$$\begin{aligned}
\mathrm{Re}(\Lambda(k,s)) &= bs + \langle k, Qk\rangle + \int\limits_{\mathbb{R}^d\times[0,\infty)\setminus\{(0,0)\}} \left(1 - \mathrm{Re}\left(e^{i\langle k,x\rangle} e^{-st}\right)\right) \Phi_{(A(1),D(r))}(dx,dt) \\
&= bs + \langle k, Qk\rangle + \int\limits_{\mathbb{R}^d\times[0,\infty)\setminus\{(0,0)\}} (1 - \cos(\langle k,x\rangle) e^{-st}) \, \Phi_{(A(1),D(r))}(dx,dt),
\end{aligned}$$

the log-FLT transform has even strictly positive real part for every $s > 0$. The first two terms are non-negative since $b \geq 0$ and Q is non-negative definite. Besides, due to our fullness assumption on $P_{(A(1),D(r))}$ in the previous section, the Lévy measure is full, and hence the integral is strictly positive for every $s > 0$.

Consequently, the log-FLT directly yields the already known Fourier and Laplace exponents of $P_{A(1)}$ and $P_{D(r)}$ in the following sense. By considering $k = 0$, we obtain the Laplace transform of $D(r)$ with Laplace exponent

$$\begin{aligned}
\Lambda(0,s) &= bs + \int\limits_{(0,\infty)} \left(1 - e^{-st}\right) \Phi_{(A(1),D(r))}(\mathbb{R}^d, dt) \\
&= bs + \int\limits_{(0,\infty)} \left(1 - e^{-st}\right) d\Phi_{D(r)}(t) \\
&= \Gamma_{D(r)}(s)
\end{aligned} \tag{6.13}$$

for every $s > 0$, where $\Phi_{D(r)}$ is the Lévy measure of $D(r)$. Due to our construction in the previous section, $(D(t))_{t\geq 0}$ is the semistable subordinator without drift, and hence we have $b = 0$. Similarly, we receive the log-characteristic function of $A(1)$ by considering $s = 0$, this is

$$\begin{aligned}
\Lambda(k,0) &= i\langle a,k\rangle + \langle k,Qk\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{i\langle k,x\rangle} + \frac{i\langle k,x\rangle}{1+||x||^2}\right) \Phi_{(A(1),D(r))}(dx,(0,\infty)) \\
&= i\langle a,k\rangle + \langle k,Qk\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{i\langle k,x\rangle} + \frac{i\langle k,x\rangle}{1+||x||^2}\right) d\Phi_{A(1)}(x) \\
&= \Psi_{A(1)}(k) \qquad (6.14)
\end{aligned}$$

for every $k \in \mathbb{R}^d$, where $\Psi_{A(1)}$ is the Lévy measure in the Lévy-Khintchine triple of $A(1)$. In contrast to the previous chapters, we subscript the log-characteristic function with A to clarify the affiliation of the function to the process.
Recall that we assume $(X_n, J_n)_{n\in\mathbb{N}}$ to be i.i.d. random variables distributed as (X, J). However, we allow arbitrary dependencies between X and J. Then a multidimensional generalization of [61, Lemma 4.3] shows that the Fourier-Laplace transform of the CTRW or OCTRW is given by the subsequent formula.

Lemma 6.2.1. *The Fourier-Laplace transform of the distribution of $S(N(t)+1)$ is given by*

$$\int\limits_0^\infty e^{-st}\widehat{P}_{S(N(t)+1)}(k)\,dt = \frac{1}{s}\frac{\widehat{P}_X(k) - \overline{P}_{(X,J)}(k,s)}{1-\overline{P}_{(X,J)}(k,s)},$$

whereas for the CTRW, we obtain

$$\int\limits_0^\infty e^{-st}\widehat{P}_{S(N(t))}(k)\,dt = \frac{1}{s}\frac{1-\tilde{P}_J(s)}{1-\overline{P}_{(X,J)}(k,s)}$$

for every $k \in \mathbb{R}^d$ and $s > 0$.

Using this result and the convergence of $S(N(t)+1)$ and $S(N(t))$ in Theorem 6.1.7 and 6.1.10, we gain the following result for the Fourier-Laplace transforms of the distribution of the limiting processes.

Lemma 6.2.2. *(Fourier-Laplace transform of the OCTRW and CTRW limit)*
The Fourier-Laplace transform of the distribution $P_{O_r(t)}$ of the OCTRW limit is given by

$$\int\limits_0^\infty e^{-st}\widehat{P}_{O_r(t)}(k)\,dt = \frac{1}{s}\frac{\Lambda(k,r^{-\frac{1}{\beta}}s) - \Psi_{A(1)}(k)}{\Lambda(k,r^{-\frac{1}{\beta}}s)}, \qquad (6.15)$$

whereas for the distribution $P_{M_r(t)}$ of the CTRW limit, we obtain

$$\int_0^\infty e^{-st}\widehat{P}_{M_r(t)}(k)\,dt = \frac{1}{s}\frac{\Gamma_{D(r)}(r^{-\frac{1}{\beta}}s)}{\Lambda(k, r^{-\frac{1}{\beta}}s)} \tag{6.16}$$

for every $k \in \mathbb{R}^d$ and $s > 0$.

Proof. Recall from the previous section that

$$(A_{u_n}S(k_{u_n}), x_n^{-1}T(k_{u_n})) = \sum_{j=1}^{k_{u_n}} \left(A_{u_n}X_j, x_n^{-1}J_j\right) \xrightarrow{d} (A(1), r^{-\frac{1}{\beta}}D(r))$$

as $n \to \infty$, and the convergence in distribution implies convergence of the log-Fourier-Laplace transforms. Then

$$k_{u_n}\log\left(\overline{P}_{(X,J)}(A_{u_n}y, x_n^{-1}s)\right) \to -\Lambda(y, r^{-\frac{1}{\beta}}s)$$

for every $y \in \mathbb{R}^d$ and $s > 0$. A Taylor expansion of the logarithm yields

$$k_{u_n}(1 - \overline{P}_{(X,J)}(A_{u_n}y, x_n^{-1}s)) \to \Lambda(y, r^{-\frac{1}{\beta}}s). \tag{6.17}$$

Especially if $s = 0$, (6.17) reads as

$$\begin{aligned} k_{u_n}(1 - \overline{P}_{(X,J)}(A_{u_n}y, 0)) &= k_{u_n}(1 - \widehat{P}_X(A_{u_n}y)) \\ &\to \Lambda(y, 0) \\ &= \Psi_{A(1)}(y) \end{aligned} \tag{6.18}$$

for every $y \in \mathbb{R}^d$, whereas $y = 0$ results in

$$\begin{aligned} k_{u_n}(1 - \overline{P}_{(X,J)}(0, x_n^{-1}s)) &= k_{u_n}(1 - \widetilde{P}_J(x_n^{-1}s)) \\ &\to \Lambda(0, r^{-\frac{1}{\beta}}s) \\ &= \Gamma_{D(1)}(r^{-\frac{1}{\beta}}s) \end{aligned} \tag{6.19}$$

for every $s > 0$. First, we analyze the Fourier-Laplace transform of the OCTRW, and the claimed result for this case follows from considering the corresponding limits. For every $s > 0$ and $y \in \mathbb{R}^d$, we find

$$\int_0^\infty e^{-st}\widehat{P}_{A_{u_n}S(N(x_nt)+1)}(y)\,dt = \int_0^\infty e^{-st}\widehat{P}_{S(N(x_nt)+1)}(A_{u_n}y)\,dt,$$

and the substitution $z \coloneqq x_n t$ yields

$$\int_0^\infty e^{-st}\widehat{P}_{A_{u_n}S(N(x_n t)+1)}(y)\,dt = x_n^{-1}\int_0^\infty e^{-sx_n^{-1}z}\widehat{P}_{S(N(z)+1)}(A_{u_n}y)\,dz.$$

However, according to Lemma 6.2.1, this integral is given by

$$\begin{aligned}\int_0^\infty e^{-st}\widehat{P}_{A_{u_n}S(N(x_n t)+1)}(y)\,dt &= \frac{1}{s}\frac{\widehat{P}_X(A_{u_n}y)-\overline{P}_{(X,J)}(A_{u_n}y,x_n^{-1}s)}{1-\overline{P}_{(X,J)}(A_{u_n}y,x_n^{-1}s)}\\ &= \frac{k_{u_n}(\widehat{P}_X(A_{u_n}y)-1)-k_{u_n}(\overline{P}_{(X,J)}(A_{u_n}y,x_n^{-1}s)-1)}{s\,k_{u_n}(1-\overline{P}_{(X,J)}(A_{u_n}y,x_n^{-1}s))}.\end{aligned}$$

Using the convergence in (6.17) and (6.18), we receive

$$\int_0^\infty e^{-st}\widehat{P}_{A_{u_n}S(N(x_n t)+1)}(y)\,dt \to \frac{1}{s}\frac{\Lambda(y,r^{-\frac{1}{\beta}},s)-\Psi_{A(1)}(y)}{\Lambda(y,r^{-\frac{1}{\beta}}s)}$$

for every $y \in \mathbb{R}^d$ and $s > 0$. On the other hand, the OCTRW converges to the limit process $(O_r(t))_{t\geq 0}$ such that

$$\int_0^\infty e^{-st}\widehat{P}_{A_{u_n}S(N(x_n t)+1)}(y)\,dt \to \int_0^\infty e^{-st}\widehat{P}_{O_r(t)}(y)\,dt,$$

which proves (6.15). We handle the CTRW limit similarly. Consider the Fourier-Laplace transform of the CTRW, this is

$$\begin{aligned}\int_0^\infty e^{-st}\widehat{P}_{A_{u_n}S(N(x_n t))}(y)\,dt &= \int_0^\infty e^{-st}\widehat{P}_{S(N(x_n t))}(A_{u_n}y)\,dt\\ &= x_n^{-1}\int_0^\infty e^{-sx_n^{-1}z}\widehat{P}_{S(N(z))}(A_{u_n}y)\,dz\end{aligned}$$

for every $y \in \mathbb{R}^d$ and $s > 0$, where again $z \coloneqq x_n t$. In view of Lemma 6.2.1, the Fourier-Laplace transform is given by

$$\begin{aligned}\int_0^\infty e^{-st}\widehat{P}_{A_{u_n}S(N(x_n t))}(y)\,dt &= \frac{1}{s}\frac{1-\widetilde{P}_J(x_n^{-1}s)}{1-\overline{P}_{(X,J)}(A_{u_n}y,x_n^{-1}s)}\\ &= \frac{1}{s}\frac{k_{u_n}(1-\widetilde{P}_J(x_n^{-1}s))}{k_{u_n}(1-\overline{P}_{(X,J)}(A_{u_n}y,x_n^{-1}s))}\end{aligned}$$

$$\to \frac{1}{s}\frac{\Gamma_{D(1)}(r^{-\frac{1}{\beta}}s)}{\Lambda(y, r^{-\frac{1}{\beta}}s)}$$

using (6.17) and (6.19). Applying the same arguments as for the OCTRW proves the claimed result (6.16). $\square$

Using the knowledge about the log-Fourier-Laplace transforms, we characterize the distribution of the limiting processes more precisely. Therefore, let $\mathcal{B}(\mathbb{R}^d)$ denote the Borel sets of $\mathbb{R}^d$ and define the sets

$$B - x \coloneqq \{y - x : y \in B\}$$

for any $B \in \mathcal{B}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Then the distribution of the limit process $(O_r(t))_{t\geq 0}$ of the OCTRW can be represented as follows.

Lemma 6.2.3. *(Distribution of the OCTRW limit)*
For every $B \in \mathcal{B}(\mathbb{R}^d)$ and $t > 0$, the limit process $(O_r(t))_{t\geq 0}$ of the OCTRW fulfills

$$P(O_r(t) \in B) = \int_{u=0}^{\infty} \int_{x\in\mathbb{R}^d} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{(A(1),D(r))}(B - x, (tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(dx, d\tau)\, du,$$

where Φ is the Lévy measure in the log-FLT of $P_{(A(u),D(ru))}$ in (6.12).

Proof. First, we show that the mappings $\rho(\cdot, t) : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$ with

$$\rho(B,t) \coloneqq \int_{u=0}^{\infty} \int_{x\in\mathbb{R}^d} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{(A(1),D(r))}(B - x, (tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(dx, d\tau)\, du$$

are well-defined probability measures on $\mathbb{R}^d$ for every fixed $t > 0$. Obviously, $\rho(B,t) \geq 0$ for every $B \in \mathcal{B}(\mathbb{R}^d)$ and every $t > 0$. In addition, $\Phi_{(A(1),D(r))}$ is a measure such that ρ is σ-additive. With Tonelli's theorem, we obtain

$$\begin{aligned}
\rho(\mathbb{R}^d, t) &= \int_{u=0}^{\infty} \int_{x\in\mathbb{R}^d} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{(A(1),D(r))}(\mathbb{R}^d, (tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(dx, d\tau)\, du \\
&= \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{(A(1),D(r))}(\mathbb{R}^d, (tr^{\frac{1}{\beta}} - \tau, \infty))\, dP_{D(ru)}(\tau)\, du \\
&= \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, dP_{D(ru)}(\tau)\, du
\end{aligned}$$

for every $t > 0$. Note that $\Phi_{D(r)}(B) = r\Phi_{D(1)}(B)$ for every $B \in \mathcal{B}((0,\infty))$ such that

$$\rho(\mathbb{R}^d, t) = r \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(1)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, dP_{D(ru)}(\tau)\, du.$$

Combining this result with Lemma 5.2.7, we find

$$\rho(\mathbb{R}^d, t) = r \int_0^{\infty} h(ru, tr^{\frac{1}{\beta}})\, du,$$

where $x \mapsto h(x,t)$ is the density of the inverse semistable subordinator $E(t)$. Substituting $y \coloneqq ru$, we receive

$$\rho(\mathbb{R}^d, t) = \int_0^{\infty} h(y, tr^{\frac{1}{\beta}})\, dy = 1$$

since h is a density on $(0,\infty)$. Hence ρ is a probability measure. Now calculate the Fourier-Laplace transform of ρ, this is

$$\overline{\rho}(k,s) = \int_{t=0}^{\infty} \int_{y\in\mathbb{R}^d} e^{-st} e^{i\langle k,y\rangle}\, \rho(dy, dt)$$

for every $k \in \mathbb{R}^d$ and $s > 0$. In order to apply Fubini's theorem, consider

$$\begin{aligned} I_1(k,s) :&= |\overline{\rho}(k,s)| \\ &\le \int_{t=0}^{\infty} \int_{y\in\mathbb{R}^d} \int_{u=0}^{\infty} \int_{x\in\mathbb{R}^d} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{-st}\, \Phi_{(A(1),D(r))}(dy, (tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(dx, d\tau)\, du\, dt. \end{aligned}$$

Using Tonelli's theorem, we obtain

$$\begin{aligned} I_1(k,s) &\le \int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \int_{x\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} e^{-st}\, \Phi_{(A(1),D(r))}(dy, (tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(dx, d\tau)\, du\, dt \\ &= \int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \int_{x\in\mathbb{R}^d} e^{-st}\, \Phi_{(A(1),D(r))}(\mathbb{R}^d, (tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(dx, d\tau)\, du\, dt \end{aligned}$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \int_{x \in \mathbb{R}^d} e^{-st} \Phi_{D(r)}((tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(dx, d\tau)\, du\, dt$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{-st} \Phi_{D(r)}((tr^{\frac{1}{\beta}} - \tau, \infty))\, P_{(A(u),D(ru))}(\mathbb{R}^d, d\tau)\, du\, dt$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{-st} \Phi_{D(r)}((tr^{\frac{1}{\beta}} - \tau, \infty))\, dP_{D(ru)}(\tau)\, du\, dt$$

for every $k \in \mathbb{R}^d$ and $s > 0$. Note that again we can express the integrals in terms of the density $x \mapsto h(x,t)$ of the inverse semistable subordinator $E(t)$ in the following way

$$I_1(k,s) \le \int_{t=0}^{\infty} e^{-st} r \int_{u=0}^{\infty} h(ru, tr^{\frac{1}{\beta}})\, du\, dt = \int_{t=0}^{\infty} e^{-st}\, dt = \frac{1}{s} < \infty$$

for every $s > 0$. Hence, we can apply Fubini's theorem to the Fourier-Laplace transform of ρ yielding

$$\overline{\rho}(k,s) = \int_{u=0}^{\infty} \int_{x \in \mathbb{R}^d} \int_{\tau=0}^{\infty} e^{i\langle k,x\rangle} I_2(k,s)\, P_{(A(u),D(ru))}(dx, d\tau)\, du,$$

where

$$I_2(k,s) \coloneqq \int_{t=r^{-\frac{1}{\beta}}\tau}^{\infty} \int_{y \in \mathbb{R}^d} e^{-st} e^{i\langle k,y\rangle}\, \Phi_{(A(1),D(r))}(dy, (tr^{\frac{1}{\beta}} - \tau, \infty))\, dt.$$

First substitute $v \coloneqq tr^{\frac{1}{\beta}} - \tau$ to obtain

$$I_2(k,s) = r^{-\frac{1}{\beta}} \int_{v=0}^{\infty} \int_{y \in \mathbb{R}^d} e^{-sr^{-\frac{1}{\beta}}(v+\tau)} e^{i\langle k,y\rangle}\, \Phi_{(A(1),D(r))}(dy, (v, \infty))\, dv$$

and write

$$I_2(k,s) = r^{-\frac{1}{\beta}} \int_{v=0}^{\infty} \int_{y \in \mathbb{R}^d} \int_{m=0}^{\infty} e^{-sr^{-\frac{1}{\beta}}(v+\tau)} e^{i\langle k,y\rangle} \mathbb{1}_{(v,\infty)}(m)\, \Phi_{(A(1),D(r))}(dy, dm)\, dv.$$

Using Fubini's theorem again, I_2 is given by

$$\begin{aligned}
I_2(k,s) =& r^{-\frac{1}{\beta}} \int\limits_{y\in\mathbb{R}^d} \int\limits_{m=0}^{\infty} \int\limits_{v=0}^{\infty} e^{-sr^{-\frac{1}{\beta}}(v+\tau)} e^{i\langle k,y\rangle} \mathbb{1}_{(v,\infty)}(m)\, dv\, \Phi_{(A(1),D(r))}(dy,dm) \\
=& r^{-\frac{1}{\beta}} \int\limits_{y\in\mathbb{R}^d} \int\limits_{m=0}^{\infty} e^{-sr^{-\frac{1}{\beta}}\tau} \left[-\frac{r^{\frac{1}{\beta}}}{s} e^{-sr^{-\frac{1}{\beta}}v} \right]_{v=0}^{m} e^{i\langle k,y\rangle}\, \Phi_{(A(1),D(r))}(dy,dm) \\
=& \frac{1}{s} \int\limits_{y\in\mathbb{R}^d} \int\limits_{m=0}^{\infty} e^{-sr^{-\frac{1}{\beta}}\tau} \left(1 - e^{-sr^{-\frac{1}{\beta}}m} \right) e^{i\langle k,y\rangle}\, \Phi_{(A(1),D(r))}(dy,dm) \\
=& \frac{1}{s} e^{-sr^{-\frac{1}{\beta}}\tau} \int\limits_{y\in\mathbb{R}^d} \int\limits_{m=0}^{\infty} \left(e^{i\langle k,y\rangle} - 1 - \frac{i\langle k,y\rangle}{1+||y||^2} \right) \Phi_{(A(1),D(r))}(dy,dm) \\
&+ \frac{1}{s} e^{-sr^{-\frac{1}{\beta}}\tau} \int\limits_{y\in\mathbb{R}^d} \int\limits_{m=0}^{\infty} \left(1 - e^{-sr^{-\frac{1}{\beta}}m} e^{i\langle k,y\rangle} + \frac{i\langle k,y\rangle}{1+||y||^2} \right) \Phi_{(A(1),D(r))}(dy,dm) \\
=& \frac{1}{s} e^{-sr^{-\frac{1}{\beta}}\tau} (-\Psi_{A(1)}(k) + \Lambda(k, sr^{-\frac{1}{\beta}}))
\end{aligned}$$

in view of (6.14) and (6.12). Then

$$\begin{aligned}
\overline{\rho}(k,s) &= \frac{1}{s}(\Lambda(k,sr^{-\frac{1}{\beta}}) - \Psi_{A(1)}(k)) \int\limits_{u=0}^{\infty} \int\limits_{x\in\mathbb{R}^d} \int\limits_{\tau=0}^{\infty} e^{i\langle k,x\rangle} e^{-sr^{-\frac{1}{\beta}}\tau}\, P_{(A(u),D(ru))}(dx,d\tau)\, du \\
&= \frac{1}{s}(\Lambda(k,sr^{-\frac{1}{\beta}}) - \Psi_{A(1)}(k)) \int\limits_{u=0}^{\infty} e^{-u\Lambda(k,sr^{-\frac{1}{\beta}})}\, du \\
&= \frac{\Lambda(k,sr^{-\frac{1}{\beta}}) - \Psi_{A(1)}(k)}{s\Lambda(k,sr^{-\frac{1}{\beta}})}
\end{aligned}$$

for every $k \in \mathbb{R}^d$ and $s > 0$, where we used that $\mathrm{Re}(\Lambda(k,s)) > 0$. Hence, according to Lemma 6.2.2, the Fourier-Laplace transform coincides with that of the limit process $(O_r(t))_{t\geq 0}$. Note that the process $(O_r(t))_{t\geq 0}$ has càdlàg paths, and thus $t \mapsto P_{O_r(t)}$ is weakly right-continuous. To show equality of the probability measures, it is sufficient to show that $t \mapsto \rho(\cdot,t)$ is weakly right-continuous as well (compare [61, Lemma 4.6]). Using the continuity theorem for Fourier transform, we prove this by showing right-continuity of the Fourier transform. So for $h > 0$, consider the difference

$$\widehat{\rho}(k,t) - \widehat{\rho}(k,t+h) = \int\limits_{\mathbb{R}^d} e^{i\langle k,y\rangle}\, \rho(dy,t) - \int\limits_{\mathbb{R}^d} e^{i\langle k,y\rangle}\, \rho(dy,t+h).$$

By defining

$$a_1(dy,\tau,h) = \Phi_{(A(1),D(r))}(dy,(tr^{\frac{1}{\beta}}-\tau,\infty)) - \Phi_{(A(1),D(r))}(dy,((t+h)r^{\frac{1}{\beta}}-\tau,\infty)) \quad (6.20)$$

and

$$a_2(dy,\tau,h) := \Phi_{(A(1),D(r))}(dy,((t+h)r^{\frac{1}{\beta}}-\tau,\infty)), \quad (6.21)$$

we can express the difference as

$$\widehat{\rho}(k,t) - \widehat{\rho}(k,t+h) = I_3(k,t,h) + I_4(k,t,h)$$

with

$$I_3(k,t,h) = \int\limits_{y\in\mathbb{R}^d}\int\limits_{u=0}^{\infty}\int\limits_{x\in\mathbb{R}^d}\int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{i\langle k,y+x\rangle} a_1(dy,\tau,h)\, P_{(A(u),D(ru))}(dx,d\tau)\, du$$

and

$$I_4(k,t,h) = -\int\limits_{y\in\mathbb{R}^d}\int\limits_{u=0}^{\infty}\int\limits_{x\in\mathbb{R}^d}\int\limits_{\tau=tr^{\frac{1}{\beta}}}^{(t+h)r^{\frac{1}{\beta}}} e^{i\langle k,y+x\rangle} a_2(dy,\tau,h)\, P_{(A(u),D(ru))}(dx,d\tau)\, du.$$

Note that $a_1 \geq 0$ such that with Tonelli's theorem

$$\begin{aligned}
|I_3(k,t,h)| &\leq \int\limits_{u=0}^{\infty}\int\limits_{x\in\mathbb{R}^d}\int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}}\int\limits_{y\in\mathbb{R}^d} a_1(dy,\tau,h)\, P_{(A(u),D(ru))}(dx,d\tau)\, du \\
&= \int\limits_{u=0}^{\infty}\int\limits_{x\in\mathbb{R}^d}\int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} a_1(\mathbb{R}^d,\tau,h)\, P_{(A(u),D(ru))}(dx,d\tau)\, du \\
&= \int\limits_{u=0}^{\infty}\int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} a_1(\mathbb{R}^d,\tau,h)\, P_{(A(u),D(ru))}(\mathbb{R}^d,d\tau)\, du \\
&= \int\limits_{u=0}^{\infty}\int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} a_1(\mathbb{R}^d,\tau,h) dP_{D(ru)}(\tau)\, du.
\end{aligned}$$

We want to consider the limit $\lim\limits_{h\downarrow 0} I_3(k,t,h)$. First, we have

$$|a_1(\mathbb{R}^d,\tau,h)| = \Phi_{D(r)}(tr^{\frac{1}{\beta}}-\tau,\infty) - \Phi_{D(r)}((t+h)r^{\frac{1}{\beta}}-\tau,\infty) \leq \Phi_{D(r)}(tr^{\frac{1}{\beta}}-\tau,\infty)$$

and

$$\int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, dP_{D(ru)}\tau\, du = r \int_{u=0}^{\infty} h(ru, tr^{\frac{1}{\beta}})\, du = 1$$

with the density $x \mapsto h(x,t)$ of the inverse semistable subordinator $(E(t))_{t\geq 0}$. Then using dominated convergence, we have

$$\lim_{h\downarrow 0} |I_3(k,t,h)| \leq \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \lim_{h\downarrow 0} a_1(\mathbb{R}^d, \tau, h)\, dP_{D(ru)}(\tau)\, du.$$

Recall from the previous section that the tail of $\Phi_{D(r)}$ is given by

$$\Phi_{D(r)}(t,\infty) = r\Phi_{D(1)}(t,\infty) = rt^{-\beta}V(\log(t))$$

for every $t > 0$. Since V is a periodic, admissable function, it is continuous almost everywhere. Then

$$0 \leq \lim_{h\downarrow 0} a_1(\mathbb{R}^d, \tau, h) = \lim_{h\downarrow 0} \Big(\Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty) - \Phi_{D(r)}((t+h)r^{\frac{1}{\beta}} - \tau, \infty) \Big) = 0$$

for almost every $\tau \in [0, tr^{\frac{1}{\beta}})$ such that the integral vanishes. Next we show that $I_4(k,t,h)$ vanishes likewise. Using Tonelli's theorem, we obtain

$$\begin{aligned}
|I_4(k,t,h)| &\leq \int_{u=0}^{\infty} \int_{\tau=tr^{\frac{1}{\beta}}}^{(t+h)r^{\frac{1}{\beta}}} \int_{x\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} a_2(dy,\tau,h)\, P_{(A(u),D(ru))}(dx, d\tau)\, du \\
&= \int_{u=0}^{\infty} \int_{\tau=tr^{\frac{1}{\beta}}}^{(t+h)r^{\frac{1}{\beta}}} a_2(\mathbb{R}^d,\tau,h)\, P_{(A(u),D(ru))}(\mathbb{R}^d, d\tau)\, du \\
&= \int_{u=0}^{\infty} \int_{\tau=tr^{\frac{1}{\beta}}}^{(t+h)r^{\frac{1}{\beta}}} a_2(\mathbb{R}^d,\tau,h)\, dP_{D(ru)}(\tau)\, du.
\end{aligned}$$

As in [67], define the occupation measure W_r by

$$W_r(B) = \int_{u=0}^{\infty} P\{D(ru) \in B\}\, du$$

for Borel sets $B \in \mathcal{B}((0,\infty))$. Then

$$|I_4(k,t,h)| \leq \int\limits_{\tau=tr^{\frac{1}{\beta}}}^{(t+h)r^{\frac{1}{\beta}}} a_2(\mathbb{R}^d,\tau,h)\,dW_r(\tau).$$

If $(E_r(t))_{t\geq 0}$ denotes the inverse semistable subordinator belonging to $(D_r(t) = D(rt))_{t\geq 0}$, then according to [67, Lemma 6.1]

$$\int\limits_0^a \Phi_{D(r)}(a+x-\tau,\infty)\,dW_r(\tau) = P\{D_r(E_r(a)) > a+x\}$$

such that

$$\begin{aligned}|I_4(k,t,h)| &\leq \int\limits_{\tau=tr^{\frac{1}{\beta}}}^{(t+h)r^{\frac{1}{\beta}}} \Phi_{D(r)}((t+h)r^{\frac{1}{\beta}}-\tau,\infty)\,dW_r(\tau)\\ &= P\{D_r(E_r((t+h)r^{\frac{1}{\beta}})) > (t+h)r^{\frac{1}{\beta}}\} - P\{D_r(E_r(tr^{\frac{1}{\beta}})) > (t+h)r^{\frac{1}{\beta}}\}.\end{aligned}$$

In [67, Corollary 6.2], it was shown that $P\{D_r(E_r(t)) > t\} = 1$ for every $t > 0$ such that

$$\begin{aligned}|I_4(k,t,h)| &\leq 1 - P\{D_r(E_r(tr^{\frac{1}{\beta}})) > (t+h)r^{\frac{1}{\beta}}\}\\ &= P\{D_r(E_r(tr^{\frac{1}{\beta}})) \leq (t+h)r^{\frac{1}{\beta}}\}.\end{aligned}$$

As shown in Proposition 5 of [67], this is a continuous function in $h \geq 0$ such that with Corollary 6.2 in [67], we receive

$$\lim_{h\downarrow 0} |I_4(k,t,h)| \leq P\{D_r(E_r(tr^{\frac{1}{\beta}})) \leq tr^{\frac{1}{\beta}}\} = 0.$$

Hence the mapping $t \mapsto \widehat{\rho}(k,t)$ is right-continuous for every $k \in \mathbb{R}^d$ and the claimed result follows. □

Lemma 6.2.4. *(Distribution of the CTRW limit)*
For every $B \in \mathcal{B}(\mathbb{R}^d)$ and $t > 0$, the limit process $(M_r(t))_{t\geq 0}$ of the CTRW fulfills

$$P(M_r(t) \in B) = \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(r)}(tr^{\frac{1}{\beta}}-\tau,\infty)\,P_{(A(u),D(ru))}(B,d\tau)\,du.$$

Proof. Again define

$$\eta(B,t) := \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, P_{(A(u),D(ru))}(B, d\tau)\, du$$

for Borel sets $B \subseteq \mathbb{R}^d$ and every $t > 0$. Then η is a probability measure for every fixed $t > 0$ since with Tonelli's theorem

$$\begin{aligned}
\eta(\mathbb{R}^d,t) &= \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, P_{(A(u),D(ru))}(\mathbb{R}^d, d\tau)\, du \\
&= \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, dP_{D(ru)}(\tau)\, du \\
&= r \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(1)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, P_{D(ru)}(d\tau)\, du \\
&= r \int_{u=0}^{\infty} h(ru, tr^{\frac{1}{\beta}})\, du \\
&= 1
\end{aligned}$$

as in the proof of Lemma 6.2.3, where $x \mapsto h(x,t)$ is the density of the inverse semistable subordinator $E(t)$. For the Fourier-Laplace transform of η, we obtain

$$\begin{aligned}
\overline{\eta}(k,s) &= \int_{t=0}^{\infty} \int_{y\in\mathbb{R}^d} e^{-st} e^{i\langle k,y\rangle}\, \eta(dy,dt) \\
&= \int_{t=0}^{\infty} \int_{y\in\mathbb{R}^d} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{-st} e^{i\langle k,y\rangle} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, P_{(A(u),D(ru))}(dy, d\tau)\, du\, dt
\end{aligned}$$

for every $k \in \mathbb{R}^d$ and $s > 0$. Again, we are allowed to use Fubini's theorem since

$$\begin{aligned}
&\int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \int_{y\in\mathbb{R}^d} \left| e^{-st} e^{i\langle k,y\rangle} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty) \right| P_{(A(u),D(ru))}(dy, d\tau)\, du\, dt \\
&= \int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \int_{y\in\mathbb{R}^d} e^{-st} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty)\, P_{(A(u),D(ru))}(dy, d\tau)\, du\, dt
\end{aligned}$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{-st} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty) \, dP_{D(ru)}(\tau) \, du \, dt$$

applying Tonelli's theorem. Use the representation of h in (5.13) to obtain

$$\begin{aligned}
&\int_{t=0}^{\infty} \int_{u=0}^{\infty} \int_{\tau=0}^{tr^{\frac{1}{\beta}}} \int_{y\in\mathbb{R}^d} \left| e^{-st} e^{i\langle k,y\rangle} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty) \right| P_{(A(u),D(ru))}(dy, d\tau) \, du \, dt \\
=& r \int_{t=0}^{\infty} \int_{u=0}^{\infty} e^{-st} h(ru, tr^{\frac{1}{\beta}}) \, du \, dt \\
=& \int_{t=0}^{\infty} e^{-st} \, dt \\
=& \frac{1}{s} < \infty.
\end{aligned}$$

Then with Fubini's theorem, it follows that

$$\overline{\eta}(k, s) = \int_{u=0}^{\infty} \int_{y\in\mathbb{R}^d} \int_{\tau=0}^{\infty} \int_{t=r^{-\frac{1}{\beta}}\tau}^{\infty} e^{-st} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty) \, dt \, e^{i\langle k,y\rangle} \, P_{(A(u),D(ru))}(dy, d\tau) \, du.$$

Note that with the substitution $v := tr^{\frac{1}{\beta}} - \tau$ and Tonelli's theorem, we have

$$\begin{aligned}
\int_{t=r^{-\frac{1}{\beta}}\tau}^{\infty} e^{-st} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty) \, dt &= r^{-\frac{1}{\beta}} \int_{v=0}^{\infty} e^{-s(v+\tau)r^{-\frac{1}{\beta}}} \Phi_{D(r)}(v, \infty) \, dv \\
&= r^{-\frac{1}{\beta}} \int_{v=0}^{\infty} \int_{m=0}^{\infty} e^{-s(v+\tau)r^{-\frac{1}{\beta}}} \mathbb{1}_{(v,\infty)}(m) \, d\Phi_{D(r)}(m) \, dv \\
&= r^{-\frac{1}{\beta}} e^{-\tau s r^{-\frac{1}{\beta}}} \int_{m=0}^{\infty} \left[-\frac{r^{\frac{1}{\beta}}}{s} e^{-svr^{-\frac{1}{\beta}}} \right]_{v=0}^{m} d\Phi_{D(r)}(m) \\
&= \frac{e^{-\tau s r^{-\frac{1}{\beta}}}}{s} \int_{m=0}^{\infty} \left(1 - e^{-smr^{-\frac{1}{\beta}}} \right) d\Phi_{D(r)}(m) \\
&= \frac{e^{-\tau s r^{-\frac{1}{\beta}}}}{s} \Gamma_{D(r)}(sr^{-\frac{1}{\beta}})
\end{aligned}$$

together with (6.13). Hence, the Fourier-Laplace transform of η is given by

$$\begin{aligned}\overline{\eta}(k,s) &= \frac{\Gamma_{D(r)}(r^{-\frac{1}{\beta}}s)}{s}\int\limits_{u=0}^{\infty}\int\limits_{y\in\mathbb{R}^d}\int\limits_{\tau=0}^{\infty} e^{-\tau s r^{-\frac{1}{\beta}}}e^{i\langle k,y\rangle}\,P_{(A(u),D(ru))}(dy,d\tau)\,du \\ &= \frac{\Gamma_{D(r)}(r^{-\frac{1}{\beta}}s)}{s}\int\limits_{u=0}^{\infty} e^{-u\Lambda(k,r^{-\frac{1}{\beta}}s)}\,du \\ &= \frac{\Gamma_{D(r)}(r^{-\frac{1}{\beta}}s)}{s\Lambda(k,r^{-\frac{1}{\beta}}s)}\end{aligned}$$

for every $k\in\mathbb{R}^d$ and $s>0$, using that for these parameters $\mathrm{Re}(\Lambda(k,s))>0$. Again, the process $M_r(t)=(A(r^{-1}E(r^{\frac{1}{\beta}}t)-))_+$ is a càdlàg process such that as in the proof of Lemma 6.2.3, it is sufficient to show that $t\mapsto\eta(\cdot,t)$ is weakly right-continuous. For every $k\in\mathbb{R}^d$ and $t>0$

$$\begin{aligned}\widehat{\eta}(k,t)-\widehat{\eta}(k,t+h) &= \int\limits_{x\in\mathbb{R}^d}\int\limits_{u=0}^{\infty}\int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{i\langle k,x\rangle}\Phi_{D(r)}(tr^{\frac{1}{\beta}}-\tau,\infty)\,P_{(A(u),D(ru))}(dx,d\tau)\,du \\ &\quad-\int\limits_{x\in\mathbb{R}^d}\int\limits_{u=0}^{\infty}\int\limits_{\tau=0}^{(t+h)r^{\frac{1}{\beta}}} e^{i\langle k,x\rangle}\Phi_{D(r)}((t+h)r^{\frac{1}{\beta}}-\tau,\infty)\,P_{(A(u),D(ru))}(dx,d\tau)\,du \\ &= \int\limits_{x\in\mathbb{R}^d}\int\limits_{u=0}^{\infty}\int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} e^{i\langle k,x\rangle}a_1(\mathbb{R}^d,\tau,h)\,P_{(A(u),D(ru))}(dx,d\tau)\,du \\ &\quad-\int\limits_{x\in\mathbb{R}^d}\int\limits_{u=0}^{\infty}\int\limits_{\tau=tr^{\frac{1}{\beta}}}^{(t+h)r^{\frac{1}{\beta}}} e^{i\langle k,x\rangle}a_2(\mathbb{R}^d,\tau,h)\,P_{(A(u),D(ru))}(dx,d\tau)\,du,\end{aligned}$$

where a_1 and a_2 are defined as in (6.20) and (6.21). As in the proof of Lemma 6.2.3, this expression converges to 0 as $h\downarrow 0$ such that the Fourier transform $t\mapsto\widehat{\eta}(k,t)$ is right-continuous for every $k\in\mathbb{R}^d$. □

For both of our cases, the uncoupled as well as the coupled case with $h_n=k_n$ for every $n\in\mathbb{N}$, the limit process $(A(t),r^{\frac{1}{\beta}}D(rt))_{t\geq 0}$ is a Lévy process. In the uncoupled case, the process has a product Lebesgue density whereas in the coupled case, the limiting process in operator semistable and has a Lebesgue density as well [80, Theorem 2.2]. From the previous result, we obtain the following direct consequence from Lemma 6.2.4.

Lemma 6.2.5. *(Density of the CTRW limit)*
Denote by $(x,y)\mapsto w_r(x,y,t)$ the density of $(A(t),D(rt))$ for every $t>0$. Then the

CTRW limit process $(M_r(t))_{t\geq 0}$ *has a Lebesgue density* $x \mapsto m_r(x,t)$ *with*

$$m_r(x,t) = \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{tr^{\frac{1}{\beta}}} \Phi_{D(r)}(tr^{\frac{1}{\beta}} - \tau, \infty) w_r(x,\tau,u)\, d\tau\, du$$

for every $t > 0$.

To close this section, we study some examples of CTRW and OCTRW limits. First, we consider uncoupled jumps and waiting times. This particular case links the microscopic CTRW/OCTRW model with the macroscopic description of anomalous diffusion delivered by the semi-fractional Cauchy problems studied in Chapter 5.

Example 6.2.6. We assume that the jumps X_j, $j \in \mathbb{N}$, and waiting times J_j, $j \in \mathbb{N}$, are independent and for simplicity, we furthermore assume that $r = 1$. In Section 5.3, we already argued that the semi-fractional Cauchy problem is solved by an uncoupled CTRW limit. However, Lemma 6.2.5 now yields the same result.In this uncoupled case, the Fourier-Laplace exponent Λ decomposes as

$$\Lambda(k,s) = \Psi_{A(1)}(k) + \Gamma_{D(1)}(s)$$

for every $k \in \mathbb{R}^d$ and $s > 0$. Note that according to Theorem 6.1.7, the limiting processes $(M_1(t))_{t\geq 0}$ of the CTRW and $(O_1(t))_{t\geq 0}$ of the OCTRW coincide almost surely. According to Lemma 6.2.5, the density $x \mapsto m_1(x,t)$ of $M_1(t)$ is given by

$$m_1(x,t) = \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \Phi_{D(1)}(t-\tau, \infty) w_1(x,\tau,u)\, d\tau\, du$$

for every $(x,t) \in \mathbb{R}^d \times (0,\infty)$. Thereby, $x \mapsto w_1(x,\tau,u)$ is the density of $(A(u), D(u))$. Similar to the log-FLT, the density of $(A(u), D(u))$ decomposes due to the independence of the components such that

$$w_1(x,y,t) = p(x,t)g(y,t)$$

for every $x \in \mathbb{R}^d$, $y > 0$, and $t > 0$, where $x \mapsto p(x,t)$ is the density of $A(t)$ and $y \mapsto g(y,t)$ is the density of $D(t)$. Hence using Lemma 5.2.7, we obtain

$$\begin{aligned} m_1(x,t) &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \Phi_{D(1)}(t-\tau, \infty) p(x,u) g(\tau,u)\, d\tau\, du \\ &= \int\limits_{u=0}^{\infty} h(u,t) p(x,u)\, du. \end{aligned}$$

Under the assumptions of Theorem 5.3.6 and together with Lemma 5.3.12, this density is a strong solution to the semi-fractional Cauchy problem

$$\left(\frac{\partial}{\partial_{d,V} t}\right)^{\beta} u(x,t) = Lu(x,t) + \delta(x) G_V(t)$$

with the generator L corresponding to $(A(t))_{t\geq 0}$, and the process $(M_1(t))_{t\geq 0}$ is a stochastic solution to this very equation.

As a contrary example, we study the case of totally coupled processes. Furthermore, this example impressively displays the possible difference between the OCTRW and the CTRW limit in the coupled case and thereby justifies both cases' treatment.

Example 6.2.7. Consider the totally coupled case, where the waiting time before or after each jump equals the jump's height ($X = J$). First, we analyze the CTRW. In view of Lemma 6.2.4, the cumulative distribution function is given by

$$\begin{aligned} P(M_1(t) \leq x) &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \Phi_{D(1)}(t-\tau,\infty)\, P_{(D(u),D(u))}([0,x], d\tau)\, du \\ &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \int\limits_{y=0}^{x} \Phi_{D(1)}(t-\tau,\infty)\, P_{(D(u),D(u))}(dy, d\tau)\, du. \end{aligned}$$

For every $u > 0$, the distribution of $(D(u), D(u))$ is concentrated on the diagonal line $\{(t,t) : t \geq 0\}$ such that $P_{(D(u),D(u))}(dy, d\tau) = d\epsilon_\tau(dy) dP_{D(u)}(\tau)$ and hence

$$\begin{aligned} P(M_1(t) \leq x) &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \int\limits_{y=0}^{x} \Phi_{D(1)}(t-\tau,\infty)\, d\epsilon_\tau(dy)\, dP_{D(u)}(\tau)\, du \\ &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \Phi_{D(1)}(t-\tau,\infty) \mathbb{1}_{(0,x)}(\tau)\, dP_{D(u)}(\tau)\, du. \end{aligned}$$

Note that for $x \geq t$, it follows that

$$\begin{aligned} P(M_1(t) \leq x) &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \Phi_{D(1)}(t-\tau,\infty)\, dP_{D(u)}(\tau)\, du \\ &= \int\limits_{u=0}^{\infty} h(u,t)\, du \\ &= 1 \end{aligned}$$

using the density $x \mapsto h(x,t)$ of the inverse semistable subordinator $E(t)$ in Lemma 5.2.7.

On the other hand, for $x < t$, we obtain

$$
\begin{aligned}
P(M_1(t) \leq x) &= \int_{u=0}^{\infty} \int_{\tau=0}^{x} \Phi_{D(1)}(t-\tau, \infty)\, dP_{D(u)}(\tau)\, du \\
&= \int_{\tau=0}^{x} \int_{u=0}^{\infty} \Phi_{D(1)}(t-\tau, \infty) g(\tau, u)\, du\, d(\tau)
\end{aligned}
$$

using Tonelli's theorem and the density $x \mapsto g(x,t)$ of $D(t)$. Differentiating with respect to x yields the density of the CTRW limit process

$$
\begin{aligned}
m_1(x,t) &= \frac{d}{dx} P(M_1(t) \leq x) \\
&= \left(\int_0^{\infty} \Phi_{D(1)}(t-x, \infty) g(x,u)\, du \right) \mathbb{1}_{(0,t)}(x) \\
&= \Phi_{D(1)}(t-x, \infty) \left(\int_0^{\infty} g(x,u)\, du \right) \mathbb{1}_{(0,t)}(x) \\
&= (t-x)^{-\beta} V(\log(t-x)) \left(\int_0^{\infty} g(x,u)\, du \right) \mathbb{1}_{(0,t)}(x).
\end{aligned}
$$

Define $\zeta : \mathbb{R} \to (0, \infty)$ by

$$
\zeta(x) := e^{x(1-\beta)} \int_0^{\infty} g(e^x, u)\, du. \tag{6.22}
$$

Then ζ is $\log\left(d^{\frac{1}{\beta}}\right)$-periodic since

$$
\zeta\left(x + \log\left(d^{\frac{1}{\beta}}\right)\right) = e^{x(1-\beta)} d^{\frac{1-\beta}{\beta}} \int_0^{\infty} g(e^x d^{\frac{1}{\beta}}, u)\, du.
$$

Besides, $(D(t))_{t \geq 0}$ is a strictly semistable subordinator such that we can use the scaling property (2.4) to obtain

$$
\begin{aligned}
\zeta\left(x + \log\left(d^{\frac{1}{\beta}}\right)\right) &= e^{x(1-\beta)} d^{\frac{1}{\beta}-1} \int_0^{\infty} d^{-\frac{1}{\beta}} g(e^x, d^{-1}u)\, du \\
&= e^{x(1-\beta)} \int_0^{\infty} g(e^x, z)\, dz
\end{aligned}
$$

$$= \zeta(x)$$

with $z \coloneqq d^{-1}u$. In conclusion, the density m_1 of the CTRW limit is given by

$$m_1(x,t) = (t-x)^{-\beta} V(\log(t-x)) x^{\beta-1} \zeta(\log(x)) \mathbb{1}_{(0,t)}(x),$$

which can be interpreted as a disturbed Beta density on $(0,t)$. For comparison, we also compute the density of the OCTRW limit process $(O_1(t))_{t\geq 0}$. According to Lemma 6.2.3,

$$\begin{aligned} P(O_1(t) \leq x) &= \int\limits_{u=0}^{\infty} \int\limits_{y=0}^{x} \int\limits_{\tau=0}^{t} \Phi_{(D(1),D(1))}((0,x-y),(t-\tau,\infty))\, P_{(D(u),D(u))}(dy,d\tau)\, du \\ &= \int\limits_{u=0}^{\infty} \int\limits_{y=0}^{x} \int\limits_{\tau=0}^{t} \Phi_{(D(1),D(1))}((0,x-y),(t-\tau,\infty))\, d\epsilon_\tau(dy)\, dP_{D(u)}(\tau)\, du \end{aligned}$$

since the distribution of $(D(t),D(t))_{t\geq 0}$ is supported on $\{(t,t) : t \geq 0\}$. For the same reason

$$\Phi_{(D(1),D(1))}((0,x-y),(t-\tau,\infty)) = \Phi(t-\tau,x-y)\mathbb{1}_{(t-\tau,\infty)}(x-y)$$

such that

$$\begin{aligned} P(O_1(t) \leq x) &= \int\limits_{u=0}^{\infty} \int\limits_{y=0}^{x} \int\limits_{\tau=0}^{t} \Phi(t-\tau,x-y)\mathbb{1}_{(t-\tau,\infty)}(x-y)\, d\epsilon_\tau(dy)\, dP_{D(u)}(\tau)\, du \\ &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \Phi(t-\tau,x-\tau)\mathbb{1}_{(t,\infty)}(x)\mathbb{1}_{(0,x)}(\tau)\, dP_{D(u)}(\tau)\, du \\ &= \int\limits_{u=0}^{\infty} \int\limits_{\tau=0}^{t} \Phi(t-\tau,x-\tau)\mathbb{1}_{(t,\infty)}(x) g(\tau,u)\, d\tau\, du \end{aligned}$$

with the density $x \mapsto g(x,t)$ of $D(t)$. Use (6.22) and Tonelli's theorem to obtain

$$P(O_1(t) \leq x) = \int\limits_{0}^{t} \Phi(t-\tau,x-\tau)\tau^{\beta-1}\zeta(\log(\tau))\mathbb{1}_{(t,\infty)}(x)\, d\tau.$$

Now

$$\Phi(t-\tau,x-\tau) = (t-\tau)^{-\beta}V(\log(t-\tau)) - (x-\tau)^{-\beta}V(\log(x-\tau))$$

such that

$$P(O_1(t) \leq x) = C_{19} \mathbb{1}_{(t,\infty)}(x) - \int\limits_0^t (x-\tau)^{-\beta} V(\log(x-\tau)) \tau^{\beta-1} \zeta(\log(\tau))\, d\tau \, \mathbb{1}_{(t,\infty)}(x)$$

with

$$C_{19} \coloneqq \int\limits_0^t (t-\tau)^{-\beta} V(\log(t-\tau)) \tau^{\beta-1} \zeta(\log(\tau))\, d\tau.$$

Aiming to find an explicit representation of the density, we assume that the periodic functions V and ζ are smooth and thus representable by their Fourier series

$$V(x) = \sum_{n\in\mathbb{Z}} d_n e^{in\tilde{d}x} \text{ and } \zeta(x) = \sum_{n\in\mathbb{Z}} u_n e^{in\tilde{d}x}$$

with $(d_n)_{z\in\mathbb{Z}}, (u_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ and $\tilde{d} = \frac{2\pi\beta}{\log(d)}$. Then for every $x > t$, the distribution function of $O_1(t)$ is given by

$$\begin{aligned} P(O_1(t) \leq x) &= -\int\limits_0^t (x-\tau)^{-\beta} \tau^{\beta-1} \left(\sum_{n\in\mathbb{Z}} d_n (x-\tau)^{in\tilde{d}} \right) \left(\sum_{m\in\mathbb{Z}} u_m \tau^{im\tilde{d}} \right) d\tau \\ &= C_{19} - \int\limits_0^t (x-\tau)^{-\beta} \tau^{\beta-1} \left(\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} d_n (x-\tau)^{in\tilde{d}} u_{m-n} \tau^{i(m-n)\tilde{d}} \right) d\tau. \end{aligned}$$

Note that the double series is bounded as a product of two periodic, continuous functions, and hence we can change the order of integration and summation as

$$\begin{aligned} P(O_1(t) \leq x) &= C_{19} - \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} d_n u_{m-n} \int\limits_0^t (x-\tau)^{-\beta+in\tilde{d}} \tau^{\beta-1+i(m-n)\tilde{d}} \, d\tau \\ &= C_{19} - \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} d_n u_{m-n} x^{-\beta+in\tilde{d}} \int\limits_0^t \left(1 - \frac{\tau}{x}\right)^{-\beta+in\tilde{d}} \tau^{\beta-1+i(m-n)\tilde{d}} \, d\tau. \end{aligned}$$

Substituting $y \coloneqq \frac{\tau}{x}$, we get

$$\begin{aligned} P(O_1(t) \leq x) &= C_{19} - \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} d_n u_{m-n} x^{im\tilde{d}} \int\limits_0^{\frac{t}{x}} (1-y)^{-\beta+in\tilde{d}} y^{\beta-1+i(m-n)\tilde{d}} \, dy \\ &= C_{19} - \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} d_n u_{m-n} x^{im\tilde{d}} B\left(\frac{t}{x}, \beta + i(m-n)\tilde{d}, 1 - \beta + in\tilde{d}\right) \end{aligned}$$

for every $x > t$, where $B(x, a, b)$ is the incomplete Beta-function

$$B(x,a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

for every $a, b \in \mathbb{C}$ with $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ and $x > 0$. Finally, differentiation yields the Lebesgue density of the OCTRW limit

$$\begin{aligned}
o_1(x,t) &= \frac{d}{dx} P(O_1(t) \le x) \\
&= -\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} d_n u_{m-n} x^{im\tilde{d}} \left(im\tilde{d}x^{-1} - \frac{t}{x^2}\left(1-\frac{t}{x}\right)^{-\beta+in\tilde{d}} \left(\frac{t}{x}\right)^{\beta-1+i(m-n)\tilde{d}} \right) \mathbb{1}_{(t,\infty)}(x) \\
&= -\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} d_n u_{m-n} x^{im\tilde{d}-1} \left(im\tilde{d} - t^{\beta+i(m-n)\tilde{d}} x^{-im\tilde{d}} (x-t)^{-\beta+in\tilde{d}} \right) \mathbb{1}_{(t,\infty)}(x) \\
&= -\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} d_n u_{m-n} x^{im\tilde{d}-1} \left(im\tilde{d} - \left(\frac{t}{x}\right)^{im\tilde{d}} \left(\frac{t}{x-t}\right)^{\beta-in\tilde{d}} \right) \mathbb{1}_{(t,\infty)}(x).
\end{aligned}$$

We want to emphasize that, in general, the Fourier coefficients $(d_n)_{n\in\mathbb{Z}}$ are known but the Fourier coefficients $(u_n)_{n\in\mathbb{N}}$ of ζ have to be calculated using (6.22). Nevertheless, the fundamental difference of both limiting densities directly follows from their disjoint supports.

In the special case of a β-stable subordinator $(D(t))_{t\ge 0}$, the calculation can be done explicitly. In this case, $V(x) = \frac{1}{\Gamma(1-\beta)}$ is a constant function. Additionally, we obtain ζ using the equality

$$x^{\beta-1}\zeta(\log(x)) = \int_0^\infty g(x,u)\,du.$$

The right-hand side has the Laplace transform

$$\int_0^\infty e^{-sx} \int_0^\infty g(x,u)\,du\,dx = \int_0^\infty e^{-us^\beta}\,du = s^{-\beta},$$

which is also the Laplace transform of

$$x \mapsto \frac{x^{\beta-1}}{\Gamma(\beta)}.$$

With the uniqueness of the Laplace transform, we find $\zeta(x) = \frac{1}{\Gamma(\beta)}$ for every $x \in \mathbb{R}$. Hence the density of the CTRW limit process is given by

$$m_1(x,t) = \frac{(t-x)^{-\beta}x^{\beta-1}}{\Gamma(1-\beta)\Gamma(\beta)} \mathbb{1}_{(0,t)}(x).$$

Besides, with $V(x) = \frac{1}{\Gamma(1-\beta)}$ and $\zeta(x) = \frac{1}{\Gamma(\beta)}$ for every $x \in \mathbb{R}$, the density of the OCTRW limit is given by

$$o_1(x,t) = \frac{1}{\Gamma(1-\beta)} \frac{1}{\Gamma(\beta)} x^{-1} \left(\frac{t}{x-t} \right)^{\beta} \mathbb{1}_{(t,\infty)}(x).$$

These results for the stable case are already known by [61, Example 5.2].

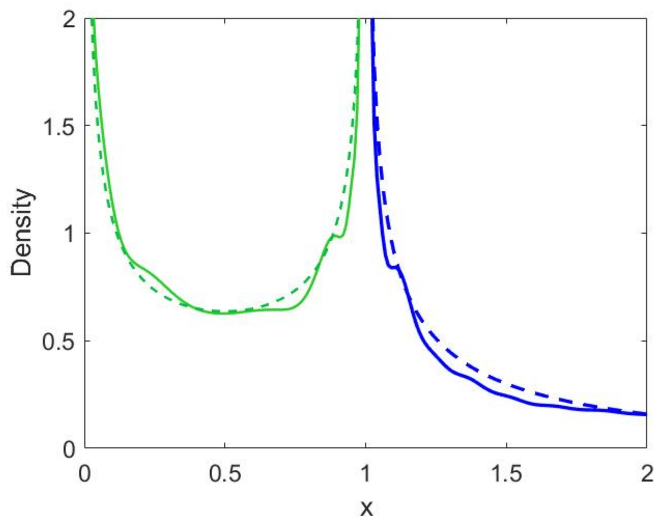

Figure 6.4: Distribution $x \mapsto m_1(x,1)$ of the CTRW limit (green solid line) and $x \mapsto o_1(x,1)$ of the OCTRW limit (blue solid line) at time $t = 1$ in comparison to the corresponding limit distributions in the stable case (dashed lines) in Example 6.2.7.

Finally, consider a concrete example of semistable subordinators. Therefore, let $D(1)$ have a $(d^{\frac{1}{\beta}}, d)$-semistable distribution for $\beta = 0.5$, $d = e^1$, and

$$V(x) = \frac{1}{40}\cos(2\pi x) + \frac{1}{20}\cos(\pi x) + \frac{1}{\Gamma(1-\beta)}$$

admissable with respect to these parameters. Figure 6.4 displays the density $x \mapsto m_1(x,1)$ of the CTRW limit and the density $x \mapsto o_1(x,1)$ of the OCTRW limit at time $t = 1$. As already seen, the densities are supported on $[0,t]$ and $[t,\infty)$ respectively, which underlines the difference between both processes. For comparison, we also plotted the densities of both limits in the case of a β-stable subordinator. As expected, the densities corresponding to the semistable subordinator oscillate around those of the stable case. The Matlab code for the calculation is attached in Appendix C.

Chapter 7

Space-time duality

From a physical point of view, a semi-fractional derivative with respect to the space variable can hardly be interpreted since it necessarily requires to include the whole environment into the calculations. In contrast, most physicists consider closed and bounded systems. However, a semi-fractional time derivative is easier to handle as the non-locality of the operator can be understood as a long-time memory effect, which is inherent in many physical systems. This chapter shows how a semi-fractional derivative in space can be shifted to a semi-fractional time derivative and thereby offers an interpretation at least for a class of space semi-fractional diffusion equations.
It was Zolotarev who in 1961 first connected stable densities of different indices [151]. To follow his considerations, note that the Fourier transform of a non-degenerate stable density with index $\alpha \in (0,2) \setminus \{1\}$ can not only be characterized by its Lévy-Khintchine triple but is also uniquely given by the representation

$$\widehat{p}(k) = \exp\left(ikb - c|k|^\alpha \exp\left(-\frac{i\pi\gamma}{2}\operatorname{sign}(k)\right)\right), \tag{7.1}$$

where $b \in \mathbb{R}$, $c > 0$, and $|\gamma| \leq \alpha$ for $\alpha \in (0,1)$, whereas $|\gamma| \leq 2 - \alpha$ for $\alpha \in (1,2)$. Zolotarev himself worked with a slightly different parametrization. However, since his result can be formulated easier this way, we decided to use (7.1). For the equivalence of parametrizations and the connection of different forms, we refer to [14]. Since every α-stable distribution is uniquely determined by b, c, and γ, we denote the corresponding density with $x \mapsto p_\alpha(x, b, c, \gamma)$. Using complex contour integrals, Zolotarev showed that

$$p_\alpha(x, 0, 1, \gamma) = x^{-1-\alpha} p_{\frac{1}{\alpha}}(x^{-\alpha}, 0, 1, \gamma^*)$$

holds for every $\alpha \in (1,2)$ and $x > 0$, where $\gamma^* = \frac{\gamma-1}{\alpha} + 1$ (compare [14, Theorem 2.1]). The result was proven differently by Lukacs [81, Theorem 3.3] using a series representation of stable densities, which has been independently obtained by Bergström [20] and Feller [42]. In 2009 Boris Baeumer et al. used Zolotarev's duality result to link space fractional and time fractional differential equations [14]. In detail, they proved that the solution of

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t)$$

with $u(x,0) = \delta(x)$ for $\alpha \in (1,2)$ is equivalent to the solution of

$$\frac{\partial^{\frac{1}{\alpha}}}{\partial t^{\frac{1}{\alpha}}}h(x,t) = -\frac{\partial}{\partial x}h(x,t)$$

for every $x > 0$ under the similar initial condition $h(x,0) = \delta(x)$. Thereby, equivalence of the solutions means that $u(x,t) = \alpha h(x,t)$ for every $(x,t) \in \mathbb{R}^2_+$. Note that the space fractional diffusion equation is solved by the densities of an α-stable Lévy process supported on $\mathbb{R}$, whereas the densities of the inverse $\frac{1}{\alpha}$-stable subordinator $(E(t))_{t\geq 0}$ supported on $\mathbb{R}_+$ solve the time fractional equation (see Example 5.3.14). Then for every $t > 0$, both densities coincide on the positive real line up to a constant.
Using Fourier-Laplace transforms instead of Zolotarev's duality result, Kelly and Meerschaert gave a different proof of space-time duality for fractional diffusion and applied it to open problems in hydrology [64]. As a side effect, they regain Zolotarev's law for a special choice of γ.

Naturally, the question arises if such an equality similarly holds for the semi-fractional setting. Since we are also missing a physical interpretation of space semi-fractional diffusion, such a result would strengthen our approach in applications. Generalizing Zolotarev's idea might cause difficulties due to the unknown log-periodic perturbation in the semi-fractional derivative. However, we are already familiar with Fourier and Laplace transforms of semi-fractional derivatives, such that a generalization of the proof given in [64] seems more promising. Parts of the following chapter have already been published in [65] (see Appendix B for a detailed list of the individual contributions of the authors), but we are now able to answer some open questions in [65] using the theory of Bernstein functions.

To introduce the reader to the theory of Bernstein functions, we shortly name the most important definitions in Section 7.1 and establish the subclass of selfsimilar Bernstein functions. Afterward, we prove a space-time duality result for the semi-fractional case in Section 7.2. In contrast to the fractional case, the negatively skewed semi-fractional equation will lead to an inhomogeneous time semi-fractional differential equation, which is no longer the density of the inverse semistable subordinator. Nevertheless, the result offers a physically meaningful interpretation of space semi-fractional differential equations.

7.1 Bernstein functions

Already in 1921, Hausdorff studied functions, which he called totally monotone. Under this definition, he understood functions $f : [0,\infty) \to [0,\infty)$ such that $f \in C^\infty(\mathbb{R}_+)$ and the derivatives have an alternating algebraic sign [52]. These functions are nowadays commonly known under the name completely monotone functions, and due to their appearance in many different mathematical areas, there is comprehensive literature concerning this class of functions (see for example [148] or [126]). Formally they are defined as follows.

Definition 7.1.1. (Completely monotone functions)
A function $f : \mathbb{R}_+ \to \mathbb{R}$ is completely monotone if $f \in C^\infty(\mathbb{R}_+)$ and

$$(-1)^n f^{(n)}(s) \geq 0$$

for all $n \in \mathbb{N}_0$ and $s > 0$.

In probability theory, completely monotone functions are also known as Laplace transforms of measures on the positive real line due to the following equivalence theorem by Bernstein based on [21].

Theorem 7.1.2. *(Bernstein's theorem, [126, Theorem 1.4])*
Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a completely monotone function. Then f is the Laplace transform of a unique measure μ on $[0, \infty)$, i.e. for all $s > 0$,

$$f(s) = \widetilde{\mu}(s) = \int_0^\infty e^{-st}\, d\mu(t).$$

Conversely, whenever the Laplace transform of a measure $\widetilde{\mu}(s)$ is finite for every $s > 0$, the function $s \mapsto \widetilde{\mu}(s)$ is completely monotone.

In our considerations, the closely related class of Bernstein functions appears. Bernstein functions are named after the Russian mathematician Sergei Bernstein. However, they can be found under different names in many mathematical areas, and lots of interesting properties have been investigated and published. For an overview and background information, we refer to the monograph [126].

Definition 7.1.3. (Bernstein function)
A function $f : \mathbb{R}_+ \to \mathbb{R}$ is a Bernstein function if $f \in C^\infty(\mathbb{R}_+)$, $f(s) \geq 0$ for all $s > 0$ and

$$(-1)^{n-1} f^{(n)}(s) \geq 0$$

for all $n \in \mathbb{N}$ and $s > 0$.

Note that from this definition, it follows immediately that a non-negative function $f \in C^\infty(\mathbb{R}_+)$ is a Bernstein function if its first derivative is a completely monotone function. Conversely, the primitive of a completely monotone function is a Bernstein function, if it is positive. Using this observation, an integral representation of Bernstein functions can be shown.

Lemma 7.1.4. *([126, Theorem 3.2]) A function $f : \mathbb{R}_+ \to \mathbb{R}$ is a Bernstein function if and only if it admits the representation*

$$f(s) = a + bs + \int_0^\infty (1 - e^{-sy})\, d\Phi(y), \tag{7.2}$$

where $a, b \geq 0$, and Φ is a measure on $(0, \infty)$ satisfying

$$\int_0^\infty \min\{1, t\}\, d\Phi(t) < \infty. \tag{7.3}$$

In particular, the triple $[a, b, \Phi]$ determines f uniquely.

Remark 7.1.5. Comparing the above representation of a Bernstein function f with Theorem 5.2.1, we see that the class of Bernstein functions with $a = 0$ equals those of Laplace exponents of infinitely divisible subordinators with drift, and we can uniquely determine the parameters $[a, 0, \Phi]$ of the Lévy-Khintchine triple form (7.2). Therefore, one refers to the representation (7.2) as the Lévy-Khintchine representation of f.

Since our interest in Bernstein functions was driven by semistable laws with discrete scale invariance, we consider Bernstein functions with a similar property.

Definition 7.1.6. (Selfsimilar Bernstein functions)
A Bernstein function $f : \mathbb{R}_+ \to \mathbb{R}$ is selfsimilar with respect to $\alpha \in (0, 1)$ and $c > 1$ if

$$f(c^{\frac{1}{\alpha}} s) = cf(s)$$

for every $s > 0$.

Similar to Lemma 7.1.4, we obtain a Lévy-Khintchine representation for selfsimilar Bernstein functions. Note that the additional property of selfsimilarity allows us to simplify the representation (7.2) noticeably.

Lemma 7.1.7. *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is a selfsimilar Bernstein function with respect to $\alpha \in (0, 1)$ and $c > 1$ if and only if it admits the representation*

$$f(s) = \int_0^\infty (1 - e^{-sy})\, d\Phi(y),$$

where Φ is a Lévy measure given by

$$\Phi(-\infty, -r) = 0 \quad \text{and} \quad \Phi(r, \infty) = r^{-\alpha} K(\log(r)) \tag{7.4}$$

for every $r > 0$ and a function $K : \mathbb{R} \to \mathbb{R}_+$ admissable with respect to α and c.

Proof. Let f be a selfsimilar Bernstein function with respect to $\alpha \in (0,1)$ and $c > 1$. According to Lemma 7.1.4, f can be written as

$$f(s) = a + bs + \int_0^\infty (1 - e^{-sy})\, d\Phi(y),$$

where $a, b \geq 0$, and Φ is a measure on $(0,\infty)$ fulfilling (7.3). Since f is selfsimilar, by iteration, we receive

$$f(c^{\frac{m}{\alpha}} s) = c^m f(s)$$

for every $m \in \mathbb{Z}$ and $s > 0$. As $m \downarrow -\infty$, this implies $\lim\limits_{s\downarrow 0} f(s) = 0$ and consequently $a = 0$. Then the selfsimilarity reads as

$$\begin{aligned} f(c^{\frac{1}{\alpha}} s) &= bc^{\frac{1}{\alpha}} s + \int_0^\infty (1 - e^{-c^{\frac{1}{\alpha}} sy})\, d\Phi(y) \\ &= bc^{\frac{1}{\alpha}} s + \int_0^\infty (1 - e^{-sy})\, d(c^{\frac{1}{\alpha}} \Phi)(y) \\ &= bcs + c \int_0^\infty (1 - e^{-sy})\, d\Phi(y) \\ &= cf(s). \end{aligned} \tag{7.5}$$

However, according to Lemma 7.1.4, the triple $[a, b, \Phi]$ is unique such that we have $b = 0$ as well as $(c^{\frac{1}{\alpha}} \Phi) = c \cdot \Phi$. Using Lemma 7.1.6 and Corollary 7.4.4 in [91], Φ is the Lévy measure of a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution supported on the positive real line. Thus it is given by (7.4) in view of Theorem 2.2.1.
Conversely, if f admits the above representation, it is obviously a Bernstein function and the selfsimilarity follows from the calculations in (7.5). □

Hence, selfsimilar Bernstein functions correspond to the Laplace exponents of semistable subordinators (compare Theorem 5.2.1). We can thereby use the calculations in Lemma 5.2.2 to obtain an explicit representation of these functions.

Lemma 7.1.8. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a selfsimilar Bernstein function with Lévy measure (7.4). Then*

$$f(s) = s^\alpha \eta_1(\log(s))$$

for every $s > 0$, where η_1 defined in Lemma 4.1.3 is a positive, $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic, $C^\infty(\mathbb{R})$-function.

Finally, we provide a sufficient condition for an inverse function to be a Bernstein function.

Lemma 7.1.9. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a $C^\infty(\mathbb{R}_+)$-function such that f' is a Bernstein function and $f^{(n)}(s) \neq 0$ for all $s > 0$ and $n \in \mathbb{N}$. Then its inverse f^{-1} is a Bernstein function with $(f^{-1})^{(n)}(s) \neq 0$ for all $s > 0$ and $n \in \mathbb{N}$.*

Proof. First note that f^{-1} is obviously strictly positive and $f^{-1} \in \mathbb{C}^\infty(\mathbb{R}_+)$ as the inverse function of f. For f^{-1} to be a Bernstein function, we need to calculate the algebraic sign of all its derivatives. By differentiation of the equation

$$f(f^{-1})(s) = s,$$

we derive the well-known formula

$$(f^{-1})'(s) = \frac{1}{f'(f^{-1}(s))}$$

for every $s > 0$. Repeating the differentiation n-times with $n \in \mathbb{N}$ yields Faà di Bruno's formula

$$\begin{aligned} 0 &= \frac{d^n}{ds^n}(f(f^{-1}(s))) \\ &= \sum_{\substack{k_1,\dots,k_n \in \mathbb{N}_0 \\ k_1+2k_2+\dots+nk_n=n}} \frac{n!}{k_1!\dots k_n!} f^{(k_1+\dots+k_n)}(f^{-1}(s)) \prod_{j=1}^{n} \left(\frac{(f^{-1})^{(j)}(s)}{j!} \right)^{k_j}. \end{aligned}$$

Since $k_1, \dots, k_n$ are chosen such that $k_1 + 2k_2 + \dots + nk_n = n$, necessarily $k_n \in \{0, 1\}$. Split these two cases to obtain

$$0 = \sum_{\substack{k_1,\dots,k_{n-1} \in \mathbb{N}_0 \\ k_1+2k_2+\dots+(n-1)k_{n-1}=n}} \frac{n!}{k_1!\dots k_{n-1}!} f^{(k_1+\dots+k_{n-1})}(f^{-1}(s)) \prod_{j=1}^{n-1} \left(\frac{(f^{-1})^{(j)}(s)}{j!} \right)^{k_j} + f'(f^{-1}(s))(f^{-1})^{(n)}(s)$$

or equally

$$(f^{-1})^{(n)}(s) = -\frac{1}{f'(f^{-1}(s))} \sum_{\substack{k_1,\dots,k_{n-1} \in \mathbb{N}_0 \\ k_1+2k_2+\dots+(n-1)k_{n-1}=n}} \frac{n!}{k_1!\dots k_{n-1}!} f^{(k_1+\dots+k_{n-1})}(f^{-1}(s)) \prod_{j=1}^{n-1} \left(\frac{(f^{-1})^{(j)}(s)}{j!} \right)^{k_j} \tag{7.6}$$

for every $s > 0$. Using this formula, we prove that

$$(-1)^{n-1}(f^{-1})^{(n)}(s) > 0 \tag{7.7}$$

by induction over $n \in \mathbb{N}$. As mentioned before, for $n = 1$, we have

$$(f^{-1})'(s) = \frac{1}{f'(f^{-1}(s))}$$

for every $s > 0$, and since f' is strictly positive, so is $(f^{-1})'$. For the second derivative, we receive

$$(f^{-1})''(s) = -\frac{1}{f'(f^{-1}(s))} f''(f^{-1}(s))((f^{-1})'(s))^2 = -\frac{f''(f^{-1}(s))}{(f'(f^{-1}(s)))^3}$$

for every $s > 0$. Since f' is a Bernstein function, f' and f'' are strictly positive. Then

$$(-1)(f^{-1})''(s) > 0$$

for every $s > 0$. Now let the assumption (7.7) be fulfilled for all derivatives up to order $n - 1 \in \mathbb{N}$ with $n \geq 2$. Then the n-th derivative of f^{-1} is given by (7.6). To calculate the algebraic sign of this derivative, we analyze the sign of the individual summands. First note that since f' is a Bernstein function

$$\operatorname{sign}(f^{(k_1+\ldots+k_{n-1})}(f^{-1}(s))) = (-1)^{k_1+\ldots+k_{n-1}},$$

whereas by induction

$$\begin{aligned}
\operatorname{sign}\left(\prod_{j=1}^{n-1}\left(\frac{(f^{-1})^{(j)}(s)}{j!}\right)^{k_j}\right) &= \prod_{j=1}^{n-1} \operatorname{sign}((f^{-1})^{(j)}(s))^{k_j} \\
&= \prod_{j=1}^{n-1} (-1)^{(j-1)k_j} \\
&= (-1)^{\sum_{j=1}^{n-1}(j-1)k_j}.
\end{aligned}$$

Then every single summand in (7.6) has algebraic sign $(-1)^{k_1+2k_1+\ldots+(n-1)k_{n-1}} = (-1)^n$. Note that since all summands have the same sign and all derivatives of f fulfill $f^{(n)}(s) \neq 0$ for all $s \in \mathbb{R}_+$, by induction, the n-th derivative of f^{-1} is non-vanishing in every point $s \in \mathbb{R}_+$. Together with the additional minus sign in (7.6), the result follows. □

7.2 Space-time duality for semi-fractional diffusion

Using the theory of Bernstein functions, we are now able to consider space-time duality in the semi-fractional setting, which offers a physical interpretation for systems underlying negatively skewed log-periodically disturbed diffusive behavior. As in the fractional case, we start with the negatively skewed diffusion equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^\alpha}{\partial_{c,K}(-x)^\alpha} u(x,t) \tag{7.8}$$

under the initial condition $u(x, 0) = \delta(x)$ for $\alpha \in (1, 2)$, $c > 1$, and K being admissable with respect to these parameters. According to Lemma 5.1.1, a solution to (7.8) is given by the densities of the $(c^{\frac{1}{\alpha}}, c)$-semistable Lévy process $(X(t))_{t\geq 0}$, where $P_{X(1)}$ has Lévy-

Khintchine representation $[a, 0, \Phi]$ with

$$a = \int_{\mathbb{R}} \left(\frac{1}{1+y^2} - y \right) d\Phi(y)$$

and

$$\Phi(-\infty, -r) = r^{-\alpha} K(\log(r)) = G_K(r) \quad \text{and} \quad \Phi(r, \infty) = 0$$

for every $r > 0$. Additionally, we assume that K is smooth with Fourier series

$$K(x) = \sum_{n \in \mathbb{Z}} c_n e^{in\tilde{c}x}$$

for Fourier coefficients $(c_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ and $\tilde{c} = \frac{2\pi\alpha}{\log(c)}$. Following Lemma 2.2.2 and Example 2.2.6, the corresponding log-characteristic function is given by

$$\Psi(k) = \int_{0+}^{\infty} \left(1 - e^{-irk} - irk\right) dG_K(r) = -\sum_{n\in\mathbb{Z}} \omega_n (ik)^{\alpha - in\tilde{c}} \tag{7.9}$$

for every $k \in \mathbb{R}$, where $\omega_n \coloneqq c_n \Gamma(in\tilde{c} - \alpha + 1)$ for every $n \in \mathbb{Z}$. Again define $\eta_2 : \mathbb{R} \to \mathbb{R}_+$ by

$$\eta_2(x) = -\sum_{n\in\mathbb{Z}} \omega_n e^{-in\tilde{c}x},$$

and note that according to Lemma 4.1.8, η_2 is a positive, $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic, $C^\infty(\mathbb{R})$-function with

$$\Psi(-ik) = k^\alpha \eta_2(\log(k)) \tag{7.10}$$

for every $k > 0$. As mentioned before, we want to adapt the method of proof in [64], which uses Fourier-Laplace transform. Similar to the already defined Fourier-Laplace transform of distributions, for a suitable function $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, the Fourier-Laplace transform of f is given by the integral

$$\overline{f}(k, s) = \int_{\mathbb{R}} \int_0^\infty e^{ikx} e^{-st} f(x, t)\, dt\, dx.$$

Throughout this thesis, we denote the Fourier-Laplace transform of f with $\overline{f}$ or $\mathcal{FL}(f)$ respectively. Applying the Fourier-Laplace transform to (7.8) yields

$$s\overline{u}(k, s) - \mathcal{F}(u)(k, 0) = \Psi(k)\overline{u}(k, s),$$

for every $k \in \mathbb{R}$ and $s > 0$, where the right-hand side with the log-characteristic function Ψ in (7.9) follows from Lemma 3.1.10. Note that $u(x,0) = \delta(x)$ has constant Fourier transform $\mathcal{F}(u)(k,0) = 1$ such that equally, we obtain

$$\overline{u}(k,s) = \frac{1}{s - \Psi(k)}. \tag{7.11}$$

From this expression, we derive the Laplace transform of u by applying an inverse Fourier transform to $\overline{u}$. Since this will be done by a closed contour integral in the lower half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) \leq 0\}$, we need some knowledge about the behavior of $z \mapsto \frac{1}{s-\Psi(z)}$ in this region. Similar to the proof of Lemma 5.3.7, the log-characteristic function Ψ can be extended analytically to the lower half-plane, and both the series and the integral representation in (7.9) are valid for every $z \in \mathbb{C}$ with $\mathrm{Im}(z) \leq 0$. Additionally, $\overline{u}(\cdot, s)$ in (7.11) has only one pole in the lower half-plane, and we can even locate this pole on the negative imaginary axis, which is proven in the following Lemma.

Lemma 7.2.1. *For every $s > 0$, there is a unique $z = z(s)$ in the lower half-plane such that $s = \Psi(z(s))$ and $z(s) = -i\zeta(s)$ with $\zeta(s) > 0$ lies on the negative imaginary axis.*

Proof. Fix $s > 0$. We first show that every possible $z = z(s)$ with $\Psi(z(s)) = s$ lies on the negative imaginary axis. Therefore consider $z = k_1 + ik_2$ with $k_1 > 0$ and $k_2 \leq 0$. Then

$$\begin{aligned}
\mathrm{Im}(\Psi(z)) &= \mathrm{Im}\left(\int_{0+}^{\infty} (1 - e^{-irz} - irz)\, dG_K(r)\right) \\
&= \int_{0+}^{\infty} \mathrm{Im}\left(1 - e^{-irz} - irz\right)\, dG_K(r) \\
&= \int_{0+}^{\infty} \mathrm{Im}\left(1 - e^{-irk_1 + rk_2} - irk_1 + rk_2\right)\, dG_K(r) \\
&= \int_{0+}^{\infty} \left(e^{rk_2}\sin(rk_1) - rk_1\right)\, dG_K(r).
\end{aligned}$$

Since $k_1 r > \sin(k_1 r)$ for every $r > 0$ and $e^{k_2 r} \leq 1$, the function $r \mapsto e^{rk_2}\sin(rk_1) - rk_1$ is negative for every choice of $(k_1, k_2) \in (0,\infty) \times (-\infty, 0]$. Additionally, G_K is monotonically decreasing such that $\mathrm{Im}(\Psi(z)) > 0$ yielding $\Psi(z) \neq s \in \mathbb{R}$. Similarly, for $z = k_1 + ik_2$ with $k_1 < 0$ and $k_2 \leq 0$, we find

$$\mathrm{Im}(\Psi(z)) = \int_{0+}^{\infty} \left(r(-k_1) - e^{rk_2}\sin(-rk_1)\right)\, dG_K(r) < 0$$

and again $\Psi(z) \neq s \in \mathbb{R}$. Thus, if there is any $z(s)$ with $\Psi(z(s)) = s$ in the lower half-plane, then $\mathrm{Re}(z(s)) = 0$ and hence $z(s) = -i\zeta(s)$ for $\zeta(s) > 0$. It remains to show that there is exactly one such $\zeta(s)$ for every $s > 0$. To do so, consider the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(k) := \Psi(-ik)$. Then according to (7.9) and (7.10), ψ is given by

$$\psi(k) = \int_{0+}^{\infty} (1 - e^{-rk} - rk)\, dG_K(r) = k^\alpha \eta_2(\log(k))$$

showing that ψ is continuously differentiable. It follows directly from the fact that $\eta_2(-x)$ is admissable (see Lemma 4.1.8) that the function ψ is non-decreasing. However, to obtain uniqueness of $\zeta(s)$ such that $\psi(\zeta(s)) = s$, we need ψ to be even strictly increasing, which we show by calculating the derivative of ψ. To differentiate under the integral sign, consider the partial derivative

$$\frac{d}{dk}(1 - e^{-rk} - rk) = r(e^{-rk} - 1).$$

Restrict the consideration to $k \in (0,1)$ first and set $g(r) := r^2 \mathbb{1}_{(0,1)}(r) + r\mathbb{1}_{[1,\infty)}(r)$. Since

$$\left| r(e^{-rk} - 1) \right| \le r$$

for every $r \ge 1$ and

$$\left| r(e^{-rk} - 1) \right| \le r^2 k \le r^2$$

for every $r < 1$ using a Taylor approximation, we have

$$\left| \frac{d}{dk}(1 - e^{-rk} - rk) \right| = \left| r(e^{-rk} - 1) \right| \le g(r)$$

for every $r \in (0, \infty)$ and $k \in (0,1)$. Besides, g is integrable with respect to G_K such that the assumptions for differentiation under the integral sign are fulfilled. Then

$$\frac{d}{dk}\psi(k) = \int_{0+}^{\infty} \underbrace{r(e^{-rk} - 1)}_{<0}\, dG_K(r) > 0$$

for every $k \in (0,1)$ such that ψ is strictly increasing for every $k \in (0,1)$. Using the alternative representation of ψ, this yields

$$\psi'(k) = k^{\alpha-1}(\alpha\eta_2(\log(k)) + \eta_2'(\log(k))) > 0$$

for every $k \in (0,1)$ such that

$$\alpha\eta_2(y) > -\eta_2'(y)$$

for every $y \in (-\infty, 0)$. However, η_2 is periodic on the whole real line yielding

$$\alpha\eta_2(y) > -\eta_2'(y)$$

for every $y \in \mathbb{R}$. Going backwards, this implies that ψ is strictly increasing for every $k \in (0,\infty)$. Together with $\psi(0) = 0$ and $\lim_{k\to\infty} \psi(k) = \infty$, for every $s > 0$, there is a unique $z(s)$ in the lower half-plane with $\Psi(z(s)) = s$, and $z(s)$ lies on the negative imaginary axis. □

The function $s \mapsto \zeta(s)$ from Lemma 7.2.1 describing the position of the pole of $z \mapsto \frac{1}{s-\Psi(z)}$ in the lower half-plane has some nice properties.

Lemma 7.2.2. *The function $s \mapsto \zeta(s)$ is a selfsimilar Bernstein function with respect to $\frac{1}{\alpha} \in (\frac{1}{2}, 1)$ and $d := c^{\frac{1}{\alpha}}$ such that $\zeta^{(n)}(s) \neq 0$ for every $s > 0$ and $n \in \mathbb{N}$. Additionally, for $s > 0$, we have $\zeta(s) = s^{\frac{1}{\alpha}} g(\log(s))$ for some $\log(c)$-periodic, strictly positive function $g \in C^\infty(\mathbb{R})$.*

Proof. Recall that $\psi(k) = \Psi(-ik)$ from the proof of Lemma 7.2.1 is the inverse function of ζ, and ψ is differentiable with

$$\psi'(k) = \int_{0+}^{\infty} r(e^{-rk} - 1)\, dG_K(r).$$

Define the measure $d\Phi^*(y) = -y\, dG_K(y)$, and note that Φ^* integrates $\min\{1, y\}$ over $(0,\infty)$. Then according to Lemma 7.1.4, ψ' is a Bernstein function. The derivatives of ψ are given by

$$\psi^{(n)}(k) = \int_{0+}^{\infty} (-1)^{n-1} r^n e^{-rk}\, dG_K(r),$$

which reflects the Bernstein property but also shows that $\psi^{(n)}(k) \neq 0$ for every $k > 0$. Then it follows from Lemma 7.1.9 that ζ is a Bernstein function with non-vanishing derivatives.

For the selfsimilarity, we observe that

$$\begin{aligned}
\psi(c^{\frac{1}{\alpha}}\zeta(s)) &= c\zeta(s)^\alpha \eta_2\left(\log\left(c^{\frac{1}{\alpha}}\zeta(s)\right)\right) \\
&= c\zeta(s)^\alpha \eta_2\left(\log(\zeta(s)) + \log\left(c^{\frac{1}{\alpha}}\right)\right) \\
&= c\zeta(s)^\alpha \eta_2\left(\log(\zeta(s))\right) \\
&= c\psi(\zeta(s)) \\
&= cs \\
&= \psi(\zeta(cs))
\end{aligned}$$

for every $s > 0$ using the periodicity of η_2. Since ψ is strictly increasing, it follows that

$$d\zeta(s) = c^{\frac{1}{\alpha}}\zeta(s) = \zeta(cs) = \zeta(d^\alpha s)$$

for every $s > 0$, and hence, ζ is selfsimilar with respect to $\frac{1}{\alpha}$ and $d = c^{\frac{1}{\alpha}}$. The claimed representation of ζ as the product of a power function and a $\log(c) = \log(d^\alpha)$-periodic function follows immediately from Lemma 7.1.8. □

We are now able to calculate the Laplace transform of the solution $x \mapsto u(x,t)$ of (7.8).

Lemma 7.2.3. *For $\alpha \in (1,2)$, the Laplace transform with respect to time of the semistable densities corresponding to the semi-fractional diffusion equation (7.8) takes the form*

$$\widetilde{u}(x,s) = \frac{\zeta(s)}{\alpha}\frac{e^{-x\zeta(s)}}{s+f(s)}$$

for every $x > 0$ and $s > 0$, where $f(s) \coloneqq \frac{1}{\alpha}\zeta(s)^\alpha \eta_2'(\log(\zeta(s)))$.

Proof. We use our knowledge about the Fourier-Laplace transform to gain a closed-form expression of the Laplace transform by inverse Fourier transform. Using equation (4.8.18) in [103], an inversion of the Fourier transform of $\overline{u}(k,s) = \frac{1}{s-\Psi(k)}$ for fixed $s > 0$ gives

$$\widetilde{u}(x,s) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T-i\zeta_0}^{T-i\zeta_0} \frac{e^{-ikx}}{s-\Psi(k)}\,dk, \tag{7.12}$$

where we choose $\zeta_0 \in (0,\zeta(s))$. We compute the integral using the closed contour displayed in Figure 7.1 consisting of the straight line L_T and the cut semicircle C_T. Note that in the plot, we changed the variable T in (7.12) to a slightly smaller value. However, we take the limit $T \to \infty$ such that this procedure is valid.
By Lemma 7.2.1, there is only one pole $-i\zeta(s)$ inside the semicircle such that with Cauchy's residue theorem, we obtain

$$\int_{C_T+L_T} \frac{e^{-ikx}}{s-\Psi(k)}\,dk = -2\pi i \cdot \mathrm{Res}(-i\zeta(s)), \tag{7.13}$$

where $\mathrm{Res}(z)$ is the residue of the function $z \mapsto \frac{e^{-ixz}}{s-\Psi(z)}$ in $z \in \mathbb{C}$ and the negative algebraic sign on the right-hand side indicates a negatively orientated path.

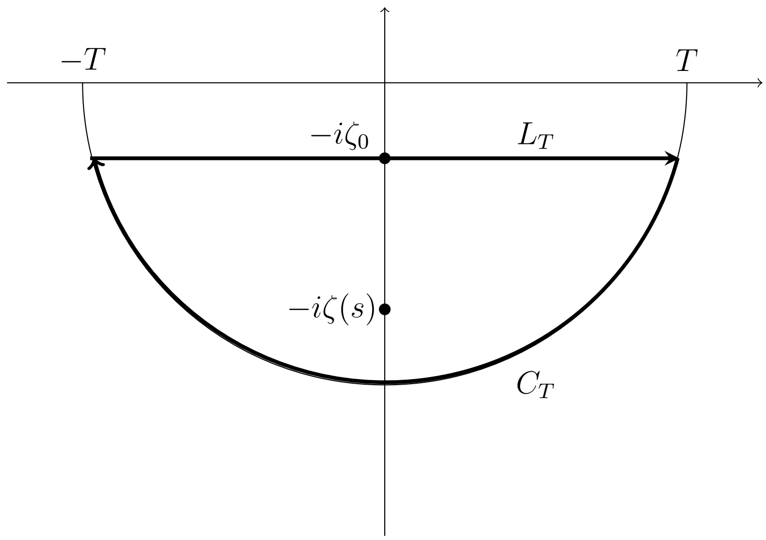

Figure 7.1: Plot of the closed contour for the evaluation of the inverse Fourier transform in the proof of Lemma 7.2.3.

First, consider the integral around C_T and describe the path as $\theta \mapsto Te^{-i\theta}$ for $\theta \in [\epsilon, \pi - \epsilon]$ and some $\epsilon > 0$. Note that ϵ depends on T with $\epsilon(T) \to 0$ as $T \to \infty$. Then the integral around the cut semicircle C_T is given by

$$\begin{aligned}\left| \int\limits_{C_T} \frac{e^{-ikx}}{s - \Psi(k)}\, dk \right| &\leq \int\limits_0^\pi T \left| \frac{e^{-iTe^{-i\theta}x}}{s - \Psi(Te^{-i\theta})} \right| d\theta \\ &= T \int\limits_0^\pi \frac{e^{-T\sin(\theta)x}}{|s - \Psi(Te^{-i\theta})|}\, d\theta.\end{aligned}$$

To simplify the denominator, remark that

$$\begin{aligned}\operatorname{Re}(\Psi(Te^{-i\theta})) &= \int\limits_{0+}^\infty \operatorname{Re}\left(1 - e^{-irTe^{-i\theta}} - irTe^{-i\theta}\right) dG_K(r) \\ &= \int\limits_{0+}^\infty \left(1 - e^{-rT\sin(\theta)} \operatorname{Re}(e^{-irT\cos(\theta)}) - rT\sin(\theta)\right) dG_K(r) \\ &= \int\limits_{0+}^\infty \left(1 - e^{-rT\sin(\theta)} \cos(rT\cos(\theta)) - rT\sin(\theta)\right) dG_K(r).\end{aligned}$$

With $\sin(\theta) \geq 0$ for $\theta \in [0, \pi]$, the first two terms in the integral are bounded for all T but the last growths linear with T. Together with the monotonicity of G_K, we obtain

$\mathrm{Re}(\Psi(Te^{-i\theta})) \to \infty$ as $T \to \infty$. Then for T sufficient large

$$|s - \Psi(Te^{-i\theta})| \geq 1$$

such that

$$\begin{aligned}\left| \int_{C_T} \frac{e^{-ikx}}{s - \Psi(k)}\, dk \right| &\leq T \int_0^{\pi} e^{-T\sin(\theta)x}\, d\theta \\ &= T \int_0^{\frac{\pi}{2}} e^{-T\sin(\theta)x}\, d\theta + T \int_{\frac{\pi}{2}}^{\pi} e^{-T\sin(\pi-\theta)x}\, d\theta \\ &= 2T \int_0^{\frac{\pi}{2}} e^{-T\sin(\theta)x}\, d\theta.\end{aligned}$$

For every $\theta \in (0, \frac{\pi}{2})$,

$$e^{-T\sin(\theta)x} \leq e^{-\frac{2T\theta x}{\pi}}$$

such that for every $x > 0$, we find

$$\left| \int_{C_T} \frac{e^{-ikx}}{s - \Psi(k)}\, dk \right| \leq 2T \int_0^{\frac{\pi}{2}} e^{-\frac{2T\theta x}{\pi}}\, d\theta \to 0$$

as $T \to \infty$ (compare [64, Equation (A2)]). Hence, the integral over the cut semicircle vanishes, and it remains to calculate the residue in (7.13). Note that by differentiating (7.9), we have

$$\begin{aligned}\Psi'(z)\,\big|_{z=-i\zeta(s)} &= -i \sum_{n\in\mathbb{Z}} \omega_n(\alpha - in\tilde{c})(iz)^{\alpha-in\tilde{c}-1}\Big|_{z=-i\zeta(s)} \\ &= i\zeta(s)^{\alpha-1}(\alpha\eta_2(\log(\zeta(s))) + \eta_2'(\log(\zeta(s)))),\end{aligned}$$

which is non-vanishing since $\zeta(s) > 0$ and $x \mapsto \eta_2(-x)$ is admissable according to Lemma 4.1.8. Then the pole $-i\zeta(s)$ is of order one with

$$\begin{aligned}\mathrm{Res}(-i\zeta(s)) &= \frac{e^{-x\zeta(s)}}{-\Psi'(-i\zeta(s))} \\ &= \frac{ie^{-x\zeta(s)}}{\zeta(s)^{\alpha-1}(\alpha\eta_2(\log(\zeta(s))) + \eta_2'(\log(\zeta(s))))} \\ &= \zeta(s)\frac{ie^{-x\zeta(s)}}{\alpha\psi(\zeta(s)) + \zeta(s)^{\alpha}\eta_2'(\log(\zeta(s)))}\end{aligned}$$

$$= \frac{i\zeta(s)}{\alpha} \frac{e^{-x\zeta(s)}}{s + \frac{1}{\alpha}\zeta(s)^\alpha \eta_2'(\log(\zeta(s)))}$$

since ζ is the inverse function of ψ. Finally, we obtain

$$\widetilde{u}(x,s) = -i\,\mathrm{Res}(-i\zeta(s)) = \frac{\zeta(s)}{\alpha} \frac{e^{-x\zeta(s)}}{s + \frac{1}{\alpha}\zeta(s)^\alpha \eta_2'(\log(\zeta(s)))}$$

for every $x > 0$ and $s > 0$. □

Recall from Lemma 7.2.3 that the function $f : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$f(s) = \frac{1}{\alpha}\zeta(s)^\alpha \eta_2'(\log(\zeta(s)))$$

and set

$$\widetilde{h}(x,s) := \frac{\zeta(s)e^{-x\zeta(s)}}{s + f(s)} = \alpha\widetilde{u}(x,s) \tag{7.14}$$

for every $x > 0$, $s > 0$. Below we show that $\widetilde{h}$ is the Laplace transform of a function $h : (0,\infty) \to (0,\infty)$, which retrospectively justifies the notation $\widetilde{h}$ in this definition. Applying a Fourier transform to (7.14) yields

$$\begin{aligned}
\overline{h}(k,s) &= \frac{\zeta(s)}{s + f(s)} \int_0^\infty e^{ikx} e^{-x\zeta(s)}\, dx \\
&= \frac{\zeta(s)}{s + f(s)} \left[\frac{1}{ik - \zeta(s)} e^{x(ik - \zeta(s))} \right]_{x=0}^\infty \\
&= \frac{\zeta(s)}{s + f(s)} \frac{1}{\zeta(s) - ik} \\
&= \left(\frac{1}{s} - \frac{1}{s} \frac{f(s)}{s + f(s)} \right) \frac{\zeta(s)}{\zeta(s) - ik}
\end{aligned}$$

or equally

$$\zeta(s)\overline{h}(k,s) - ik\overline{h}(k,s) - \frac{1}{s}\zeta(s) = -\frac{1}{s}\frac{f(s)}{s + f(s)}\zeta(s) \tag{7.15}$$

for every $k \in \mathbb{R}, s > 0$. We want to apply the inverse Fourier-Laplace transform to this equation. Starting with the left-hand side of (7.15), first note that by Lemma 7.2.2,

$$\zeta(s)\overline{h}(k,s) - s^{-1}\zeta(s) = s^{\frac{1}{\alpha}} g(\log(s))(\overline{h}(k,s) - s^{-1}). \tag{7.16}$$

According to Lemma 4.1.4, we can interpret (7.16) as the Fourier-Laplace transform of a Caputo semi-fractional derivative in time of order $\frac{1}{\alpha}$ with respect to $d = c^{\frac{1}{\alpha}} > 1$ and the

$\log(c)$-periodic function $K^*(x)$ implicitly defined by (4.1) as

$$\begin{aligned} g(x) &= e^{x(1-\frac{1}{\alpha})} \int_0^\infty e^{-e^x t} t^{-\frac{1}{\alpha}} K^*(\log(t))\, dt \\ &= \int_0^\infty e^{-u} u^{-\frac{1}{\alpha}} K^*(\log(u) - x)\, du \end{aligned}$$

if the thereby given function K^* is admissable and $h(x, 0) = \delta(x)$.

To show admissability of K^*, we need an explicit representation of this periodic function, which can be obtained by the Fourier series approach as follows. Since $g \in C^\infty(\mathbb{R})$, express it by its Fourier series

$$g(x) = \sum_{n \in \mathbb{Z}} d_n e^{in\tilde{d}x} \tag{7.17}$$

for Fourier coefficients $(d_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ and $\tilde{d} = \frac{2\pi}{\log(c)} = \frac{2\pi}{\alpha \log(d)}$. Due to the infinitely differentiability of g, the Fourier coefficients decay exponentially. We evaluate their behavior more precisely using already known results from the literature.
In view of Lemma 1 and Lemma 2 in [7, §12], exponential decay of the Fourier coefficients corresponds to analytic extensions of the function to horizontal strips in the complex plane and vice versa. Recall that ζ is a Bernstein function, and hence, it extends analytically to the right half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and is continuous on $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ [126, Proposition 3.6]. Then $g(x) = e^{\frac{x}{\alpha}}\zeta(e^x)$ extends analytically to the strip $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \frac{\pi}{2}\}$ and is continuous on its closure. Note that additionally, if K^* is bounded, $g(x + iy)$ is bounded for every fixed $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ since

$$\begin{aligned} |g(x+iy)| &= \left| e^{(x+iy)(1-\frac{1}{\alpha})} \int_0^\infty e^{-e^{(x+iy)}t} t^{-\frac{1}{\alpha}} K^*(\log(t))\, dt \right| \\ &\leq C_{20} e^{x(1-\frac{1}{\alpha})} \int_0^\infty |e^{-e^{(x+iy)}t}| t^{-\frac{1}{\alpha}}\, dt \\ &= C_{20} e^{x(1-\frac{1}{\alpha})} \int_0^\infty e^{-e^x \cos(y) t} t^{-\frac{1}{\alpha}}\, dt \\ &= C_{20} e^{x(1-\frac{1}{\alpha})} \int_0^\infty e^{-y} (y e^{-x} \cos(y)^{-1})^{-\frac{1}{\alpha}} e^{-x} \cos(y)^{-1}\, dt \\ &= C_{20} \cos(y)^{\frac{1}{\alpha}-1} \Gamma\left(1 - \frac{1}{\alpha}\right) \end{aligned}$$

for a constant $C_{20} > 0$ and using the substitution $y = e^x \cos(y)t$. According to [7, §12, Lemma 1], this indicates that the rate of decay for the Fourier coefficients $(d_n)_{n\in\mathbb{Z}}$ is given by $|d_n| \sim e^{-\frac{\pi}{2}|n|\tilde{d}}$. However, for smoothness and admissability of the function K^*, we need a little more quality, and therefore we assume that the Fourier coefficients even decay like

$$|d_n| \leq C_{21} e^{-\frac{\pi}{2}|n|\tilde{d}} |n|^{-\frac{3}{2}-\frac{1}{\alpha}-\epsilon} \tag{7.18}$$

for a constant $C_{21} > 0$ and some $\epsilon > 0$. In view of Lemma 4.1.3, the function K^* fulfilling (7.18) is given by

$$K^*(x) = \sum_{n\in\mathbb{Z}} \frac{d_n}{\Gamma(in\tilde{d} - \frac{1}{\alpha} + 1)} e^{-in\tilde{d}x} \tag{7.19}$$

and has the following properties.

Lemma 7.2.4. *If the Fourier coefficients $(d_n)_{n\in\mathbb{Z}}$ of g in (7.17) fulfill (7.18), then the function $K^* : \mathbb{R} \to \mathbb{R}_+$ defined in (7.19) is well-defined for every $x \in \mathbb{R}$, smooth, and admissable.*

Proof. Using the asymptotic behavior of the gamma function [3, Corollary 1.4.4], the Fourier coefficients of K^* fulfill

$$\left|\frac{d_n}{\Gamma(in\tilde{d} - \frac{1}{\alpha} + 1)}\right| \leq C_{22} |d_n| |n|^{-\frac{1}{2}+\frac{1}{\alpha}} e^{\frac{\pi}{2}|n|\tilde{d}}$$

for a constant $C_{22} > 0$. Now according to the assumption (7.18), we obtain

$$\begin{aligned}\left|\frac{d_n}{\Gamma(in\tilde{d} - \frac{1}{\alpha} + 1)}\right| &\leq C_{21} C_{22} e^{-\frac{\pi}{2}|n|\tilde{d}} |n|^{-\frac{3}{2}-\frac{1}{\alpha}-\epsilon} |n|^{-\frac{1}{2}+\frac{1}{\alpha}} e^{\frac{\pi}{2}|n|\tilde{d}} \\ &= C_{21} C_{22} |n|^{-2-\epsilon}\end{aligned}$$

for some $\epsilon > 0$ showing that the series in (7.19) converges absolutely and that the resulting function is continuously differentiable [46, Theorem 2.6]. It remains to show that K^* is admissable. By Lemma 7.2.2, ζ is a selfsimilar Bernstein function with respect to $\frac{1}{\alpha} \in (\frac{1}{2}, 1)$ and $d = c^{\frac{1}{\alpha}}$. Following Lemma 7.1.7, ζ is given by

$$\zeta(s) = \int_0^\infty (1 - e^{-sy})\, d\Phi^*(y), \tag{7.20}$$

where the Lévy measure Φ^* is supported on the positive real line with

$$\Phi^*(r, \infty) = r^{-\frac{1}{\alpha}} L(\log(r))$$

for every $r > 0$ and an admissable function $L : \mathbb{R} \to \mathbb{R}_+$ with respect to $\frac{1}{\alpha}$ and d. We aim to show that the function L coincides with K^* and thereby conclude that K^* is admissable. According to Lemma 4.1.3,

$$\zeta(s) = -\int_0^\infty (1 - e^{-sy})\, d(y^{-\alpha} L(\log(y))) = s^{\frac{1}{\alpha}} \eta_1(\log(s))$$

for every $s > 0$, where $\eta_1 \in C^\infty(\mathbb{R})$ is given by (4.1). On the other hand, in view of Lemma 7.2.2 and (7.17), ζ is given by

$$\zeta(s) = s^{\frac{1}{\alpha}} g(\log(s)) = s^{\frac{1}{\alpha}} \sum_{n\in\mathbb{Z}} d_n s^{in\tilde{d}}.$$

With the uniqueness of Fourier coefficients, the Fourier coefficients $(a_n)_{n\in\mathbb{Z}}$ of η_1 coincide with those of g. However, due to the decay of $(d_n)_{n\in\mathbb{Z}}$ in (7.18) and in view of Lemma 4.1.3, L is smooth with Fourier series

$$L(x) = \sum_{n\in\mathbb{Z}} \frac{a_n}{\Gamma(in\tilde{c} - \frac{1}{\alpha} + 1)} e^{in\tilde{d}x} = \sum_{n\in\mathbb{Z}} \frac{d_n}{\Gamma(in\tilde{c} - \frac{1}{\alpha} + 1)} e^{in\tilde{d}x}$$

and hence coincides with K^*. Thus, K^* is admissable.

□

Since K^* is admissable, Fourier-Laplace inversion of the left-hand side of (7.15) yields a Caputo semi-fractional derivative in time of order $\frac{1}{\alpha}$ with respect to d and K^*. To calculate the inverse Fourier-Laplace transform of the right-hand side of (7.15), note that

$$\begin{aligned}\psi'(s) &= s^{\alpha-1}(\alpha\eta_2(\log(s)) + \eta_2'(\log(s))) \\ &= s^{-1}\alpha\psi(s) + s^{\alpha-1}\eta_2'(\log(s)),\end{aligned}$$

and use that $\psi(\zeta(s)) = s$ for every $s > 0$, to obtain

$$\begin{aligned}\psi'(\zeta(s)) &= \zeta(s)^{-1}\alpha s + \zeta(s)^{\alpha-1}\eta_2'(\log(\zeta(s))) \\ &= \zeta(s)^{-1}\alpha s + \alpha\zeta(s)^{-1} f(s)\end{aligned}$$

with $f(s)$ as in Lemma 7.2.3. Then insert

$$f(s) = \frac{\zeta(s)}{\alpha}\psi'(\zeta(s)) - s$$

into the right-hand side of (7.15), which yields

$$-\frac{1}{s}\frac{f(s)}{s + f(s)}\zeta(s) = -\frac{1}{s}\left(\zeta(s) - \frac{\alpha s}{\psi'(\zeta(s))}\right).$$

According to Lemma (4.1.3), the inverse Fourier-Laplace transform of the first term is given by $-t^{-\frac{1}{\alpha}}K^*(\log(t))\delta(x)$. For the second term, differentiate $\psi(\zeta(s)) = s$ to obtain

$$\psi'(\zeta(s))\zeta'(s) = 1$$

such that

$$\frac{\alpha}{\psi'(\zeta(s))} = \alpha\zeta'(s).$$

Using the integral representation of ζ in (7.20), we find

$$\begin{aligned}
\frac{\alpha}{\psi'(\zeta(s))} &= \alpha \int_0^\infty y e^{-sy}\, d\Phi^*(y) \\
&= -\alpha \int_0^\infty y e^{-sy}\, d(y^{-\frac{1}{\alpha}}K^*(\log(y))) \\
&= \left[-\alpha y e^{-sy} y^{-\frac{1}{\alpha}} K^*(\log(y))\right]_0^\infty + \alpha \int_0^\infty (1 - sy) e^{-sy} y^{-\frac{1}{\alpha}} K^*(\log(y))\, dy \\
&= \alpha \int_0^\infty e^{-sy} y^{-\frac{1}{\alpha}} K^*(\log(y))\, dy - s\alpha \int_0^\infty e^{-xy} y^{1-\frac{1}{\alpha}} K^*(\log(y))\, dy,
\end{aligned}$$

which is the Laplace transform of

$$\alpha t^{-\frac{1}{\alpha}}K^*(\log(t)) - \alpha\frac{d}{dt}t^{1-\frac{1}{\alpha}}K^*(\log(t)) = t^{-\frac{1}{\alpha}}K^*(\log(t)) - \alpha t^{-\frac{1}{\alpha}}(K^*)'(\log(t)).$$

Altogether, the right-hand side of (7.15) is the Fourier-Laplace transform of

$$-\alpha t^{-\frac{1}{\alpha}}(K^*)'(\log(t))\delta(x),$$

which finally yields the following space-time duality result.

Theorem 7.2.5. *(Space-time duality)*
Let $\alpha \in (1, 2)$, $c > 1$, and K a smooth admissable function with respect to these parameters. In addition, let $x \mapsto u(x, t)$ be the solution to the negatively skewed space semi-fractional diffusion equation (7.8) at time $t > 0$. If K^* denotes the periodic function (7.19) and if the assumption (7.18) is fulfilled, then u is equivalent to the solution of the inhomogeneous time semi-fractional equation

$$\left(\frac{\partial}{\partial_{d,K^*}t}\right)^{\frac{1}{\alpha}} h(x, t) + \frac{\partial}{\partial x}h(x, t) = -\alpha t^{-\frac{1}{\alpha}}(K^*)'(\log(t))\delta(x) \qquad (7.21)$$

with $h(x, 0) = \delta(x)$ on the positive real line $x > 0$ for every $t > 0$. Here, equivalence means that $h(x, t) = \alpha u(x, t)$ for every $(x, t) \in \mathbb{R}^2_+$.

Remark 7.2.6. The fractional space-time duality in [64] is included in our result and can be reobtained as follows. Choosing $K(x) = -\frac{1}{\Gamma(1-\alpha)}$ for every $x \in \mathbb{R}$, we start with the fractional diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t) \tag{7.22}$$

with $u(x,0) = \delta(x)$ and $\alpha \in (1,2)$. Using the series representation, we immediately obtain $\eta_2 \equiv 1$ and $\psi(k) = k^\alpha$ in this case. Hence, the inverse function ζ can be calculated explicitly in this case and is given by $\zeta(s) = s^{\frac{1}{\alpha}}$ such that $g \equiv 1$ likewise. Then according to (7.19), $K^*(x) = \frac{1}{\Gamma(1-\frac{1}{\alpha})}$, and by Theorem 7.2.5, the solution u to (7.22) is equivalent to the solution of the time fractional equation

$$\left(\frac{\partial}{\partial t}\right)^{\frac{1}{\alpha}} h(x,t) + \frac{\partial}{\partial x}h(x,t) = 0 \tag{7.23}$$

for every $x > 0$ under the initial condition $h(x,0) = \delta(x)$. Note that by Example 5.3.14, the time fractional equation (7.22) is solved by the densities of an inverse stable subordinator $(E(t))_{t\geq 0}$. On the other hand, the time semi-fractional equation (7.21) is homogeneous if and only if the derivatives are fractional ones, which can be seen by going backwards through the above arguments.

Remark 7.2.7. By considering semi-fractional instead of fractional derivatives, we lose the connection to the inverse semistable subordinator, whose densities no longer solve the inhomogeneous time semi-fractional equation. Nevertheless, the space semi-fractional diffusion equation gains a physical interpretation as a system with a long-time memory effect and additional perturbation.

Chapter 8

Applications

Throughout the previous chapters, we developed a basic theory of semi-fractional derivatives and semi-fractional differential equations. To strengthen this theoretical approach, we finally consider some real data problems and compare fractional with semi-fractional methods.

Due to growing computational power, fractional differential equations have been applied to various diverse and widespread applications during the last decades. Among other fields, fractional models are used in physics ([142], [139], or [55]), biology ([59], [83]), and finance ([82], [94], [140], and [51]) yielding good agreements with actual data. Additionally, the number of applications is steadily growing and hence contribute to the overall interest in fractional calculus. Adding a periodic perturbation, we now widen the class of possible models. We have already seen that semi-fractional derivatives and solutions to semi-fractional differential equations differ from their fractional counterparts. Therefore, studying these more complicated operators is worth the trouble. So, this very last chapter aims to evaluate whether the difference caused by the periodic disturbance yields noticeable improvements in real-world applications.

As already mentioned, there is an ongoing discussion in fractional calculus about the correct usage of different forms of fractional derivatives. For initial value problems, which we discuss in this section, the Caputo form is often preferred to the Riemann-Liouville form since initial values can be chosen similar to the integer cases. In contrast, a fractional differential equation including Riemann-Liouville type derivatives often requires initial values involving a fractional derivative of the unknown solution in a given point (see for example [110] or [120]). We do not treat this topic in detail here. However, for this reason we concentrate on semi-fractional Caputo derivatives in the following only.

In section 8.1, we study two different growth models and define their semi-fractional versions. As a concrete example, we apply the growth models to mobile use date as well as to tumor growth. To close this thesis, we define tempered semistable distributions in Section 8.2, which exponentially dampen the probability of large events, and test this approach in modeling daily price changes of stock prices.

8.1 Semi-fractional growth models

The term growth model is widely used for a large class of differential equations, modeling various situations like population growth [38], tumor growth [75], the reproduction of bacteria [153], and many others. This section studies two different approaches, namely the exponential and the Gompertz model, and generalizes them to semi-fractional models.

The easiest growth model is the exponential one, given as the solution to

$$\frac{d}{dt}V(t) = aV(t) \quad t > 0$$

with $V(0) = V_0 \neq 0$ and $a > 0$. Uniquely solved by $V(t) = V_0 e^{at}$, the population in this model increases steadily with an unbounded limit. Therefore it is not suitable for many applications when applied over a long time period. Nevertheless, considering only a finite period $t \in [0, T]$, the model attains good fits in applications (e.g., see [114] or [32]). In [6], the corresponding fractional model

$$\frac{\partial^\alpha}{\partial t^\alpha}V(t) = aV(t) \tag{8.1}$$

for $\alpha \in (0, 1)$ under the similar initial condition $V(0) = V_0 \neq 0$ was studied. The unique solution of (8.1) is given by

$$V(t) = V_0 E_\alpha(at^\alpha), \tag{8.2}$$

where E_α is the one-parameter Mittag-Leffler function

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}$$

for every $x \in \mathbb{R}$ [36, Theorem 6.11]. In the special case $\alpha = 1$, the Mittag-Leffler function coincides with the exponential function

$$E_1(at) = e^{at},$$

and we regain the classical model. For every $\alpha \in (0, 1)$, the Mittag-Leffler function asymptotically still grows exponentially, since for every fixed $a > 0$,

$$E_\alpha(at^\alpha) = \frac{1}{\alpha}\exp\left(a^{\frac{1}{\alpha}}t\right) + O(t^{-\frac{1}{\alpha}}) \tag{8.3}$$

as $t \to \infty$ (see [50, Proposition 3.6]). Hence, the fractional model has the same weakness as the classical one when considering long time periods. In Figure 8.1, the solutions for $V_0 = 1$ and different values of a and α are shown. For the calculation of Mittag-Leffler functions here and in the following, we used the Matlab function provided by Podlubny [156], which is also able to handle complex arguments.

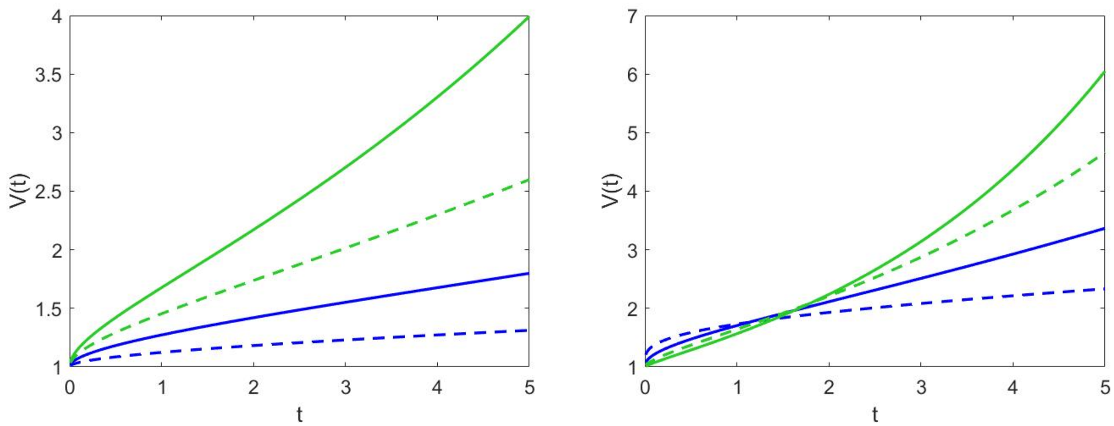

Figure 8.1: Left: Solution (8.2) of (8.1) for $\alpha = \frac{1}{2}$ and $a = 0.1$ (blue dashed line), $a = 0.2$ (blue solid line), $a = 0.3$ (green dashed line), and $a = 0.4$ (green solid line). Right: Solution (8.2) of (8.1) for $a = 0.4$ and $\alpha = 0.2$ (blue dashed line), $\alpha = 0.4$ (blue solid line), $\alpha = 0.6$ (green dashed line), and $\alpha = 0.8$ green solid line.

We want to go a step further and study a corresponding semi-fractional exponential model. As a first approach, consider the equation

$$\frac{\partial^\alpha}{\partial_{c,K} t^\alpha} V(t) = aV(t) \tag{8.4}$$

under the initial condition $V(0) = V_0 \neq 0$ with $\alpha \in (0,1)$, $c > 1$, and K admissable with respect to these parameters. Solving this semi-fractional equation is more complicated than the fractional one and we did not find an analytical formula for the solution. Nevertheless, we can compute a solution as the inverse Laplace transform of

$$\widetilde{V}(s) = \frac{s^{\alpha-1}\eta_1(\log(s))}{s^\alpha \eta_1(\log(s)) - a} V_0 \tag{8.5}$$

for every $s > 0$ and η_1 as in (4.1). This representation allows us to analyze the asymptotic behavior of the solution to the semi-fractional equation (8.4).

Lemma 8.1.1. *Let $V : [0,\infty) \to \mathbb{R}$ be a solution to the semi-factional exponential equation (8.4) and define $g(x) = x^\alpha \eta_1(\log(x))$ for every $x > 0$ and η_1 from (4.1). Then there is a unique $u \in \mathbb{R}_+$ with $g(u) = a$ and*

$$\lim_{t\to\infty} V(t)e^{-ut} = C < \infty, \tag{8.6}$$

where $C := \dfrac{aV_0}{\alpha a + u^\alpha \eta_1'(\log(u))} > 0.$

Proof. First note that according to Lemma 4.1.3 and Lemma 5.2.3, the function $g(x)$ is a strictly increasing, $C^\infty(\mathbb{R})$-function with $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to\infty} g(x) = \infty$ such that for every $a > 0$, there is a unique $u > 0$ with $g(u) = a$. Define the function $f : [0, \infty) \to \mathbb{R}$ by $f(t) = e^{-ut} V(t)$. Then using (8.5), f has Laplace transform

$$\begin{aligned}
\widetilde{f}(s) &= \int_0^\infty e^{-st} e^{-ut} V(t)\, dt \\
&= \widetilde{V}(s+u) \\
&= \frac{(s+u)^{\alpha-1} \eta_1(\log(s+u))}{(s+u)^\alpha \eta_1(\log(s+u)) - a} V_0 \\
&= \frac{(s+u)^{-1} g(s+u)}{g(s+u) - a} V_0
\end{aligned}$$

for every $s > 0$. The only pole of the Laplace transform $\widetilde{f}$ lies in the origin such that we are able to apply the final value theorem (see for example [124, Theorem 2.36]). Then we obtain

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \widetilde{f}(s) = \lim_{s\to 0} \frac{s(s+u)^{-1} g(s+u)}{g(s+u) - a} V_0.$$

With L'Hôspital's rule and the differentiability of g, we find

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{(s+u)^{-1} g(s+u) - s(s+u)^{-2} g(s+u) + s(s+u)^{-1} g'(s+u)}{g'(s+u)} V_0.$$

The derivative g' is given by

$$g'(x) = (x^\alpha \eta_1(\log(x)))' = x^{\alpha-1}(\alpha \eta_1(\log(x)) + \eta_1'(\log(x))) > 0$$

for every $x > 0$ since g is strictly increasing. Additionally, g' is continuous such that

$$\begin{aligned}
\lim_{s\to 0} g'(s+u) &= u^{\alpha-1}(\alpha \eta_1(\log(u)) + \eta_1'(\log(u))) \\
&= \alpha u^{-1} g(u) + u^{\alpha-1} \eta_1'(\log(u)) \\
&= \alpha a u^{-1} + u^{\alpha-1} \eta_1'(\log(u))
\end{aligned}$$

due to the choice of u. Finally, the limiting behavior is given by

$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \frac{u^{-1} g(u)}{g'(u)} V_0 \\
&= \frac{u^{-1} a}{\alpha a u^{-1} + u^{\alpha-1} \eta_1'(\log(u))} V_0 \\
&= \frac{a}{\alpha a + u^\alpha \eta_1'(\log(u))} V_0.
\end{aligned}$$

□

Remark 8.1.2. In the fractional case, $K(x) = \frac{1}{\Gamma(1-\alpha)}$ yields $\eta_1 = 1$ such that $g(x) = x^\alpha$, and we reobtain the asymptotic behavior $V(t) \sim \frac{V_0}{\alpha} \exp\left(a^{\frac{1}{\alpha}} t\right)$ as stated before.

The assumption on K to be admissable restricts our perturbation choices and has to be checked separately every time. To widen the class of possible models, first note that using (3.6), for every $b \geq 0$, we have

$$\begin{aligned}
\frac{\partial^\alpha}{\partial_{c,K} t^\alpha} V(t) &= \int\limits_{0+}^{\infty} V'(t-y) y^{-\alpha} K(\log(y))\, dy \\
&= \int\limits_{0+}^{\infty} V'(t-y) y^{-\alpha} \left(K(\log(y)) + \frac{b}{\Gamma(1-\alpha)} - \frac{b}{\Gamma(1-\alpha)} \right) dy \\
&= \frac{\partial^\alpha}{\partial_{c,K_{new}} t^\alpha} V(t) - b \frac{\partial^\alpha}{\partial t^\alpha} V(t),
\end{aligned}$$

where

$$K_{new}(x) \coloneqq K(x) + \frac{b}{\Gamma(1-\alpha)}$$

is again admissable. Hence (8.4) is equivalent to

$$\frac{\partial^\alpha}{\partial_{c,K_{new}} t^\alpha} V(t) - b \frac{\partial^\alpha}{\partial t^\alpha} V(t) = a V(t) \tag{8.7}$$

for every $b \geq 0$.

On the other hand, starting from (8.7), the function K_{new} has to be admissable with respect to α and $c > 1$, whereas K itself is not necessarily admissable. For b sufficient large, K can be chosen as an arbitrary $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic function. Therefore b is non-unique but bounded from below. If K is not admissable itself, we are not allowed to write equation (8.7) in the form (8.4) since we defined semi-fractional derivatives for admissable perturbations only. However, in the Laplace transform, the terms including b cancel themselves, and we can still calculate the function as the inverse Laplace transform of (8.5). Figure 8.2 shows the solution of (8.7) for $\alpha = \frac{1}{2}$, $a = 0.2$, $V_0 = 1$, $c = e^{3\alpha}$, and

$$K(x) = \frac{1}{2} \cos\left(\frac{2\pi}{3} x \right) + \frac{1}{\Gamma(1-\alpha)}$$

on a double logarithmic scale. The function K itself is not admissable but

$$K_{new}(x) = K(x) + \frac{b}{\Gamma(1-\alpha)} \tag{8.8}$$

is admissable for every $b \geq 3.12$, where we calculated the lower bound for b numerically. Both solutions were calculated using inverse Laplace methods and (8.5). The Matlab code for the calculation is attached in Appendix C.

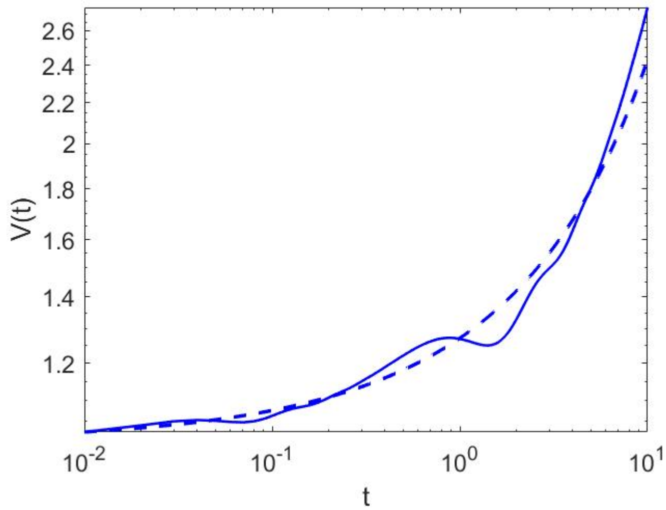

Figure 8.2: Solution of the exponential model (8.7) for $\alpha = \frac{1}{2}$, $a = 0.2$, $V_0 = 1$, and K_{new} in (8.8) (solid line) in comparison to the corresponding fractional case (dashed line) on a double logarithmic scale.

Remark 8.1.3. Note that studying the more general form (8.7) of the semi-fractional exponential equation might change the limiting behavior of the solution. As in Lemma 8.1.1, let $g(x) = x^\alpha \eta_1(\log(x))$, where η_1 is calculated using K as in (4.1). If K is not admissable itself, g is not necessarily increasing, and the arguments in Lemma 8.1.1 fail to describe the limit behavior. However, heuristically, we expect the semi-fractional derivative to oscillate around a fractional one. Thus, for small perturbations, we conjecture that the overall behavior is still dominated by exponential growth.

Example 8.1.4. (Web use on mobiles, part one)
In [6], the authors studied the percentage of mobile web use worldwide between December 2009 and August 2014 and fitted this data using the fractional exponential model (8.1). To reproduce their results, we downloaded the monthly reported percentage of mobile web use from [157], yielding 68 data points. Applying a best-fit approach in Matlab, we obtain $\alpha = 0.3465$ and $a = 0.3293$ for the fractional model, which coincides with the values calculated in [6] using Bayesian analysis. We want to check whether the semi-fractional model (8.7) offers a notable improvement in the fitting process. Note that the fractional model is included in the semi-fractional one by a particular choice of the perturbation such that we expect a small improvement in any case. To value the fit of the models, we use the mean squared error (MSE) given by

$$\mathrm{MSE} = \sum_{i=1}^{N} (V(t_i) - v_i)^2,$$

comparing the real data $V(t_i)$ with the estimated values v_i of the model at reporting time

t_i for $i = 1, \ldots, N$. For the semi-fractional model, we consider perturbations of the form

$$K(x) = d_1 \cos\left(\frac{2\pi}{p}x\right) + d_2 \sin\left(\frac{2\pi}{p}x\right) + \frac{1}{\Gamma(1-\alpha)}$$

for coefficients $d_1, d_2 \in \mathbb{R}$ and a period $p > 0$. Note that here, as well as in the following examples, we always assume that K has such a simple form. Indeed, one can consider perturbations with any number of terms. However, this may yield over-fitted models such that we stick to this simple case here. Additionally, the minimization of the MSE becomes quite difficult with a growing number of unknown parameters.

	Fractional model	Semi-fractional model
α	0.3465	0.9798
a	0.3293	0.0383
p	–	6.7047
d_1	–	-0.7981
d_2	–	0.0532
MSE	33.5127	18.5797

Table 8.1: Evaluated parameters for the fractional and semi-fractional exponential model in Example 8.1.4 describing the percentage of mobile web use worldwide between January 2009 and August 2014.

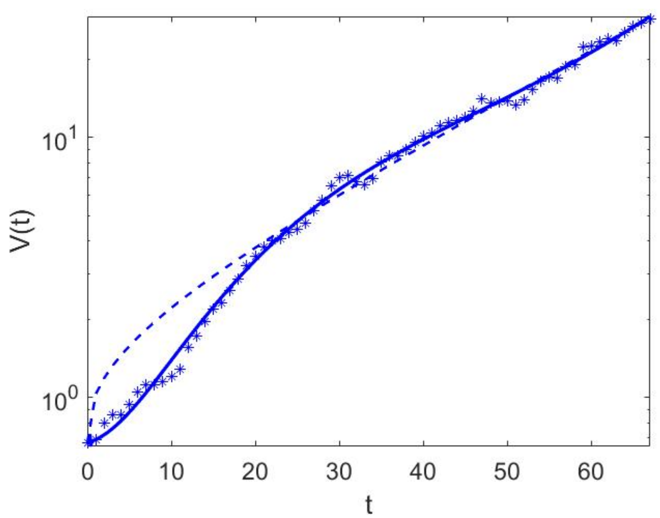

Figure 8.3: Monthly reported percentage of mobile web use between January 2009 and August 2014 (stars), fractional fit (dashed line), and semi-fractional fit (solid line) in Example 8.1.4 on a semi-logarithmic scale.

Using the Matlab function *'fminsearchbnd.m'* [154], we calculate parameters $\alpha \in (0, 1)$,

$d_1 \in \mathbb{R}, d_2 \in \mathbb{R}$, $p > 0$, and $a > 0$ minimizing the MSE. We want to emphasize that the minimum is only a local one depending on the starting values, and there might be even better fits. The calculated values are displayed in Table 8.1, and the fractional as well as the semi-fractional solution are shown in Figure 8.3. Since we reduced the MSE to over 44 percent compared to the original one, the semi-fractional model can be seen as a notable improvement.
In both models, the fractional model (8.1) as well as the semi-fractional exponential model (8.7), the calculated solutions for this application grow exponentially. However, for this particular situation in which we study percentages, any unbounded long-time behavior is impossible. Therefore, none of the above models is suitable for a long-time prediction but might be used for a short-time analysis.

For a long-time prediction, many applications require bounded models. One possibility for such a growth model is to consider Gompertz equations. Initially introduced by Gompertz in [49], the growth is thereby described as the function $V : [0, \infty) \to (0, \infty)$ solving

$$\frac{d}{dt} V(t) = a_1 V(t) - a_2 V(t) \log\left(\frac{V(t)}{V_0}\right) \tag{8.9}$$

for constants $a_1, a_2 > 0$ and an initial condition $V(0) = V_0 > 0$. The solution to this classical equation is given by

$$V(t) = V_0 \exp\left(\frac{a_1}{a_2}\left(1 - e^{-a_2 t}\right)\right)$$

for every $t \geq 0$. Note that in contrast to the exponential model, this equation demands V to converge to a boundary V_∞ as $t \to \infty$, which is given by

$$V_\infty := \lim_{t \to \infty} V(t) = V_0 \exp\left(\frac{a_1}{a_2}\right).$$

To derive a fractional version of the Gompertz model (8.9), we follow the steps in [25]. This is, we define $y(t) := \log\left(\frac{V(t)}{V_0}\right)$ such that

$$\frac{d}{dt} y(t) = \frac{1}{V(t)} \frac{d}{dt} V(t).$$

Then (8.9) equals

$$\frac{d}{dt} y(t) = a_1 - a_2 y(t)$$

with $y(0) = 0$. Based on this equation, the fractional model was defined as the function $V : [0, \infty) \to (0, \infty)$ such that $y(t) = \log\left(\frac{V(t)}{V_0}\right)$ solves the fractional equation

$$\frac{\partial^\alpha}{\partial t^\alpha} y(t) = a_1 - a_2 y(t)$$

with $y(0) = 0$ and $\alpha \in (0,1)$. The solution can again be formulated in terms of the Mittag-Leffler function as

$$y(t) = \frac{a_1}{a_2}\left(1 - E_\alpha\left(-a_2 t^\alpha\right)\right)$$

(see for example [25, Equation (2.4)]), and hence by recalling the substitution, we obtain

$$V(t) = V_0 \exp\left(\frac{a_1}{a_2}\left(1 - E_\alpha\left(-a_2 t^\alpha\right)\right)\right). \tag{8.10}$$

For $V_0 = 1$, $a_1 = 1$, and $a_2 = 0.5$, the solution for different values of α is shown in Figure 8.4.

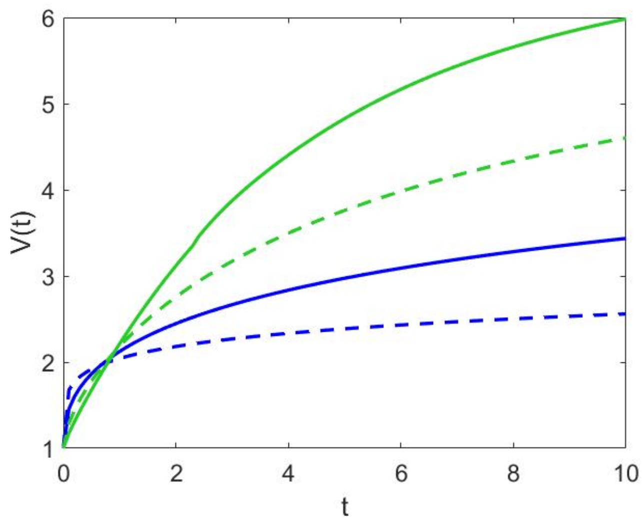

Figure 8.4: Solution $V(t)$ in (8.10) to the fractional Gompertz equation for $\alpha = 0.2$ (blue dashed line), $\alpha = 0.4$ (blue solid line), $\alpha = 0.6$ (green dashed line), and $\alpha = 0.8$ (green solid line).

Note that both the ordinary and the fractional solution approach the same limit as $t \to \infty$ since for the fractional model

$$\lim_{t\to\infty} V_0 \exp\left(\frac{a_1}{a_2}(1 - E_\alpha(-a_2 t^\alpha))\right) = V_0 \exp\left(\frac{a_1}{a_2}\right)$$

using (8.3), which is also the limit of the classical Gompertz equation. We extend this model to a semi-fractional Gompertz equation, this is we want to find $V : [0,\infty) \to (0,\infty)$ such that $y(t) = \log\left(\frac{V(t)}{V_0}\right)$ solves

$$\frac{\partial^\alpha}{\partial_{c,K} t^\alpha} y(t) = a_1 - a_2 y(t) \tag{8.11}$$

with $y(0) = 0$ for $\alpha \in (0,1)$, $c > 1$, and a $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic function K admissable with respect to α and c. An analytical form of the solution is difficult to find, so we characterize the solution as the function with Laplace transform

$$\widetilde{y}(s) = \frac{1}{s}\frac{a_1 + s^\alpha \eta_1(\log(s))y(0)}{s^\alpha \eta_1(\log(s)) + a_2} = \frac{1}{s}\frac{a_1}{s^\alpha \eta_1(\log(s)) + a_2}$$

for every $s > 0$, where again η_1 is defined as in (4.1). Note that the solution has the same limiting behavior as the fractional model, which is justified by the subsequent lemma.

Lemma 8.1.5. *Let $y : [0, \infty) \to \mathbb{R}$ be a solution of (8.11). Then the solution $V : [0, \infty) \to (0, \infty)$ of the semi-fractional Gompertz model with $V(t) = \log\left(\frac{V(t)}{V_0}\right)$ fulfills*

$$\lim_{t\to\infty} V(t) = V_0 \exp\left(\frac{a_1}{a_2}\right).$$

Proof. We analyze the asymptotic behavior of y using the final value theorem for the Laplace transform [124, Theorem 2.36]. The Laplace transform of y has $s = 0$ as its only pole and hence

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\widetilde{y}(s) = \lim_{s\to 0} \frac{a_1}{s^\alpha \eta_1(\log(s)) + a_2} = \frac{a_1}{a_2}.$$

Then the result follows from the fact that $V(t) = V_0 \exp(y(t))$. □

Similar to the exponential model, we weaken the assumptions on K by considering the modified equation

$$\frac{\partial^\alpha}{\partial_{c,K_{new}} t^\alpha} y(t) - b\frac{\partial^\alpha}{\partial t^\alpha} y(t) = a_1 - a_2 y(t)$$

for $b \geq 0$ such that

$$K_{new} = K(x) + \frac{b}{\Gamma(1-\alpha)}$$

is admissable with respect to α and $c > 1$. If b is sufficient large, K can be chosen as an arbitrary $\log\left(c^{\frac{1}{\alpha}}\right)$-periodic function. However, similar to the considerations in Remark 8.1.3, the limit in Lemma 8.1.5 may change if K is not admissable itself.

Example 8.1.6. (Web use on mobiles, part two)
In Example 8.1.4, we analyzed the percentage of mobile use between January 2009 and August 2014 and fitted a fractional and a semi-fractional exponential model in good agreement with the data. However, both models indicate a steadily increasing percentage, even exceeding the 100 percent bound in finite time. For this reason, we extend the analysis in [6] by considering fractional as well as semi-fractional Gompertz equations. For the semi-fractional one, we again assume that the periodic function K is given by

$$K(x) = d_1 \cos\left(\frac{2\pi}{p}x\right) + d_2 \sin\left(\frac{2\pi}{p}x\right) + \frac{1}{\Gamma(1-\alpha)}$$

for coefficients $d_1, d_2 \in \mathbb{R}$ and a period $p > 0$. As a data basis, we take the monthly reported percentages of mobile web use between January 2009 and January 2021. To evaluate our models' prediction capability, we only take the data up to August 2018 for the calculation of the parameters, which correlates with 80 percentage of all data points. Afterward, we calculate the MSE between the remaining points and the real data to compare the prediction capability. All calculated values are given in Table 8.2.

	Fractional Gompertz model	Semi-fractional Gompertz model
α	0.9691	0.6538
a_1	0.1171	0.3334
a_2	0.0235	0.0804
p	–	5.7980
d_1	–	0.6132
d_2	–	0.4647
Prediction MSE	7752.3937	324.2828

Table 8.2: Calculated values for the fractional and semi-fractional Gompertz equation in Example 8.1.6 based on the data from January 2009 to August 2018 as well as the prediction MSE of the data between September 2018 and January 2021.

In this scenario, the semi-fractional Gompertz model offers a better prediction than the fractional one, which is also displayed by the fits shown in Figure 8.5. The Matlab code for this calculation can be found in Appendix C.

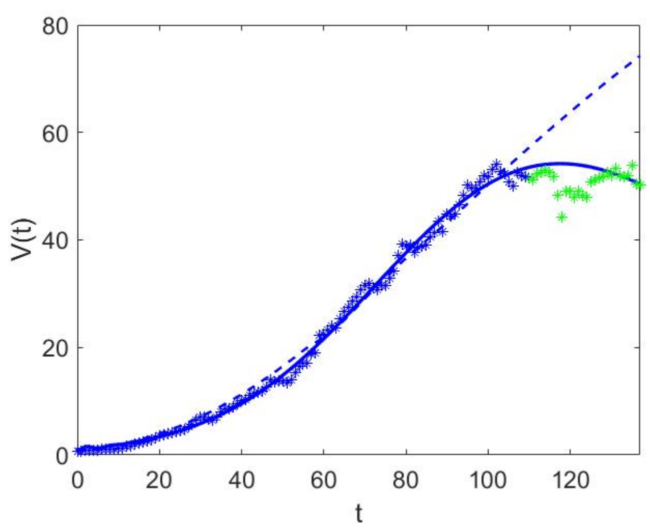

Figure 8.5: Monthly reported percentage of mobile web use between January 2009 and August 2018 (blue stars) as well as between September 2018 and January 2021 (green stars). In addition, the fractional Gompertz fit (dashed line) and the semi-fractional Gompertz fit (solid line) in Example 8.1.6 based on the first data set are shown.

Remark 8.1.7. In some applications, especially in medicine, another parametrization of the Gompertz equation appears. Setting $a_1 = a_2 \log\left(\frac{V_\infty}{V_0}\right)$ for $V_\infty > V_0$, equation (8.9) reads as

$$\frac{d}{dt}V(t) = a_2 V(t) \log\left(\frac{V_\infty}{V(t)}\right) \tag{8.12}$$

with $V(0) = V_0$. Note that the variable V_∞ indeed characterizes the limit of $V(t)$ as $t \to \infty$, which justifies the notation and allows an easier interpretation of the variables. Using the results above, the solution is given by

$$V(t) = V_0 \exp\Big(\log\left(\frac{V_\infty}{V_0}\right)(1 - e^{-a_2 t})\Big).$$

To derive a semi-fractional equation, substitute $y(t) \coloneqq \log\left(\frac{V_\infty}{V(t)}\right)$. Then (8.12) reads as

$$\frac{d}{dt}y(t) = -a_2 y(t)$$

with $y(0) = \log\left(\frac{V_\infty}{V_0}\right)$. The corresponding fractional equation

$$\frac{\partial^\alpha}{\partial t^\alpha}y(t) = -a_2 y(t)$$

with $y(0) = \log\left(\frac{V_\infty}{V_0}\right)$ is solved by

$$y(t) = \log\Big(\frac{V_\infty}{V_0}\Big)E_\alpha(-a_2 t^\alpha)$$

such that

$$\begin{aligned} V(t) = V_\infty \exp(-y(t)) &= V_\infty \exp\Big(-\log\Big(\frac{V_\infty}{V_0}\Big)E_\alpha(-a_2 t^\alpha)\Big) \\ &= V_0 \exp\Big(\log\Big(\frac{V_\infty}{V_0}\Big)(1 - E_\alpha(-a_2 t^\alpha))\Big). \end{aligned}$$

Again we can replace the time derivative with a semi-fractional one yielding the problem to find $V : [0,\infty) \to (0,\infty)$ such that $y(t) = \log\left(\frac{V_\infty}{V(t)}\right)$ solves

$$\frac{\partial^\alpha}{\partial_{c,K_{new}} t^\alpha}y(t) - b\frac{\partial^\alpha}{\partial t^\alpha}y(t) = -a_2 y(t)$$

with $y(0) = \log\left(\frac{V_\infty}{V_0}\right)$ and $K_{new}(x) = K(x) + \frac{b}{\Gamma(1-\alpha)}$ admissable with respect to α and c. In this parametrization, the solution y can be described as the inverse Laplace transform of

$$\widetilde{y}(s) = \frac{1}{s}\frac{s^\alpha \eta_1(\log(s))}{s^\alpha \eta_1(\log(s)) + a_2} y(0).$$

Besides, using Lemma 8.1.5, V_∞ describes the asymptotic behavior $V_\infty = \lim_{t\to\infty} V(t)$ whenever K is admissable.

Example 8.1.8. (Modeling tumor growth)
Predicting tumor growth is an important challenge in oncology to improve individualized therapy options in the clinic. Recently, fractional Gompertz models have been applied to this kind of data set, yielding better fits than classical models (compare for example [25] or [144]). We want to check whether the semi-fractional Gompertz model offers an even better fit. Again we assume that the periodic perturbation is given by

$$K(x) = d_1 \cos\left(\frac{2\pi}{p}x\right) + d_2 \sin\left(\frac{2\pi}{p}x\right) + \frac{1}{\Gamma(1-\alpha)}$$

for $d_1, d_2 \in \mathbb{R}$ and $p > 0$. We use the data reported in [79] of tumor growth in mice as a data basis. In this clinical study, tumors arose from chemical mutagenesis on the skin, and therefore the length $L(t)$ and height $H(t)$ were measured with a measuring caliper twice a week. Every measurement was repeated three times, and the median value was taken to reduce measurement errors. The tumor was assumed to have an ellipsoid shape such that the volume can be approximated by

$$V(t) = \frac{\pi}{6}(L(t)H(t))^{\frac{3}{2}}.$$

Tumor 1	Fractional Gompertz model	Semi-fractional Gompertz model
α	0.9820	0.999999996
V_∞	783.4445	269.4751
a_2	0.0255	0.0290
p	–	1.8589
d_1	–	−0.8245
d_2	–	0.9867
MSE	15722.7858	6553.4488

Table 8.3: Calculated values for the fractional and semi-fractional Gompertz equation for tumor 1 in Example 8.1.8.

Exemplarily, we analyze tumors of two non-medicated mice in detail. These tumors are tumor 2 of Mouse CM37 and tumor 3 of Mouse CM78 in the study, which we simply call tumor 1 and tumor 2 here. In Figure 8.6, the data as well as the fractional and semi-fractional fits for both tumors are shown. The calculated parameters using the alternative parametrization in Remark 8.1.7 are displayed in Table 8.3 and Table 8.4. In both cases, the semi-fractional Gompertz approach models the growth more efficiently. However, we want to emphasize that due to the small number of data points, the error in the calculation of the variables may be high. Nevertheless, the good fit inside this small data set may justify further studies on the improvement of semi-fractional derivatives in tumor growth models.

Tumor 2	Fractional Gompertz model	Semi-fractional Gompertz model
α	0.9804	0.9999
V_∞	473.2977	1841.6511
a_2	0.0221	0.0019
p	–	19.0063
d_1	–	−0.2221
d_2	–	0.1969
MSE	1686.0212	530.8454

Table 8.4: Calculated values for the fractional and semi-fractional Gompertz equation for tumor 2 in Example 8.1.8.

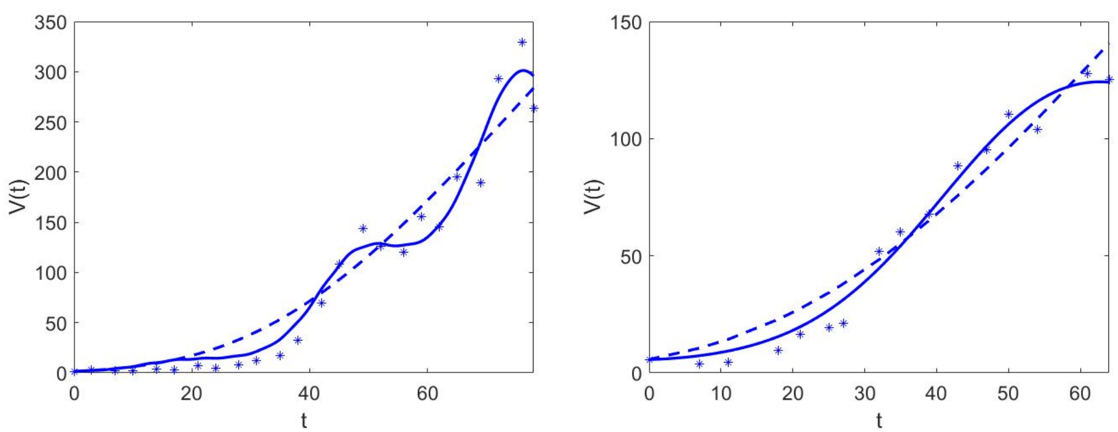

Figure 8.6: Tumor growth data of tumor 1 (left) and tumor 2 (right) displayed as stars as well as the fractional (dashed line) and the semi-fractional fit (solid line) of the Gompertz model in Example 8.1.8.

8.2 Tempered semistable distributions

For many physical applications, the moments of a distribution have an inherent physical meaning. However, for semistable and for stable densities of order $\alpha \in (1, 2)$, only the first integer moment is finite. Even more difficult for applications, none of the integer moments exist for (semi-)stable distributions of order $\alpha \in (0,1)$. To overcome these difficulties, the idea of tempered stable distributions emerged. Thereby, the probability of large observations is exponentially reduced, yielding finite moments of every order. For an introduction to the theory of tempered stable laws, we refer to [117], [94], and [11] and the references therein. In this section, we apply the same method to obtain tempered semistable distributions and afterward consider an explicit example in finance.

Let ν be a $(c^{\frac{1}{\alpha}}, c)$-semistable distribution on $\mathbb{R}$ for some $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$ with Lévy-Khintchine triple $[a, 0, \Phi]$, where a is given by (5.2) and

$$\Phi(-\infty, -r) = 0 \quad \text{ and } \quad \Phi(r, \infty) = r^{-\alpha} K(\log(r))$$

for every $r > 0$ and an admissable function K. We assume that K is continuously differentiable with Fourier coefficients $(c_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that according to Example 2.2.6, the log-characteristic function Ψ of ν is given by

$$\Psi(k) = -\sum_{n\in\mathbb{Z}} c_n \Gamma(in\tilde{c} - \alpha + 1)(-ik)^{\alpha - in\tilde{c}} \tag{8.13}$$

for every $k \in \mathbb{R}$. Similar to the proof of Lemma 5.3.7, the function $k \mapsto \Psi(k)$ can be extended to the complex half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$, and hence we obtain

$$\int_{-\infty}^{\infty} e^{ikx} e^{-sx} \, d\nu(x) = \exp(\Psi(k + is)) \tag{8.14}$$

for every $s > 0$ and $k \in \mathbb{R}$. By $x \mapsto p(x,t)$, we denote the Lebesgue density of the corresponding Lévy process $(X(t))_{t\geq 0}$ with $P_{X(1)} = \nu$. Note that according to Lemma 5.1.1, the densities solve the diffusion equation

$$\frac{\partial}{\partial t} p(x,t) = -D \frac{\partial^\alpha}{\partial_{c,K} x^\alpha} p(x,t),$$

where again $D = (-1)^{\lfloor \alpha \rfloor}$. Now we define the tempered semistable density in the following way.

Lemma 8.2.1. *(Tempered semistable density)*
For $\lambda > 0$, the tempered semistable densities

$$x \mapsto p_\lambda(x,t) := e^{-\lambda x} e^{-t\Psi(i\lambda)} p(x,t)$$

are well-defined with Fourier transform

$$\widehat{p_\lambda}(k,t) = e^{-t\Psi(i\lambda)} e^{t\Psi(k+i\lambda)}$$

for every $k \in \mathbb{R}$, $t > 0$.

Proof. First note that in view of (8.13), $\Psi(i\lambda) \in \mathbb{R}$ such that $p_\lambda \geq 0$. Besides, using

(8.14), we obtain

$$\int_{-\infty}^{\infty} e^{-\lambda x} e^{-t\Psi(i\lambda)} p(x,t)\,dx = e^{-t\Psi(i\lambda)} \int_{-\infty}^{\infty} e^{ik\cdot 0} e^{-\lambda x} p(x,t)\,dx$$
$$= e^{-t\Psi(i\lambda)} e^{t\Psi(i\lambda)}$$
$$= 1.$$

Hence, p_λ is a well-defined probability density. To calculate the Fourier transform, consider the integral

$$\widehat{p_\lambda}(k,t) = \int_{-\infty}^{\infty} e^{ikx} e^{-\lambda x} e^{-t\Psi(i\lambda)} p(x,t)\,dx$$
$$= e^{-t\Psi(i\lambda)} \exp(t\Psi(k+i\lambda))$$
$$= \exp(t(\Psi(k+i\lambda) - \Psi(i\lambda)))$$

for every $k \in \mathbb{R}$ and $t > 0$. $\square$

The tempered semistable distribution can also be obtained from exponentially tempering the Lévy measure Φ of ν as follows. For $\lambda > 0$, consider the measure

$$d\Phi_\lambda(x) = e^{-\lambda x} d\Phi(x).$$

Then Φ_λ is a Lévy measure since it integrates $\min\{1, ||x||^2\}$. Thus, there exists an infinitely divisible distribution with log-characteristic function

$$\Psi_\lambda(k) = \int_{0+}^{\infty} \left(e^{ixk} - \sum_{p=0}^{\lfloor\alpha\rfloor} (ikx)^p \right) d\Phi_\lambda(x)$$
$$= \int_{0+}^{\infty} \left(e^{(ik-\lambda)x} - \sum_{p=0}^{\lfloor\alpha\rfloor} (ikx)^p e^{-\lambda x} \right) d\Phi(x)$$

for every $k \in \mathbb{R}$. In the case $\alpha \in (0,1)$, this yields

$$\Psi_\lambda(k) = \int_{0+}^{\infty} \left(e^{(ik-\lambda)x} - e^{-\lambda x} \right) d\Phi(x)$$
$$= \int_{0+}^{\infty} \left(e^{i(k+i\lambda)x} - 1 \right) d\Phi(x) + \int_{0+}^{\infty} \left(1 - e^{-\lambda x} \right) d\Phi(x)$$
$$= \Psi(k+i\lambda) - \Psi(i\lambda),$$

which is the log-characteristic function of the tempered semistable density p_λ at time

$t = 1$. If $\alpha \in (1, 2)$, we receive

$$\begin{aligned}
\Psi_\lambda(k) &= \int\limits_{0+}^{\infty} \left(e^{(ik-\lambda)x} - e^{-\lambda x} - ikxe^{-\lambda x}\right) d\Phi(x) \\
&= \int\limits_{0+}^{\infty} \left(e^{(ik-\lambda)x} - 1 - (ik - \lambda)x\right) d\Phi(x) + \int\limits_{0+}^{\infty} (1 - e^{-\lambda x} - \lambda x)\, d\Phi(x) \\
&\quad + ik \int\limits_{0+}^{\infty} x(1 - e^{-\lambda x})\, d\Phi(x) \\
&= \Psi(k + i\lambda) - \Psi(i\lambda) + ik \int\limits_{0+}^{\infty} x(1 - e^{-\lambda x})\, d\Phi(x)
\end{aligned}$$

for every $k \in \mathbb{R}$. Hence, we have proven the following Lemma.

Lemma 8.2.2. *(Tempered semistable Lévy process)*
For $\lambda > 0$, let μ_λ be the infinitely divisible distribution with Lévy-Khintchine representation $[a_\lambda, 0, \Phi_\lambda]$ with

$$a_\lambda := \begin{cases} \int\limits_{0+}^{\infty} \dfrac{x}{1+x^2}\, d\Phi_\lambda(x) & \text{if } \alpha \in (0,1) \\ \int\limits_{0+}^{\infty} \left(\dfrac{x}{1+x^2} - x\right) d\Phi_\lambda(x) & \text{if } \alpha \in (1,2). \end{cases}$$

Then the corresponding Lévy process $(X_\lambda(t))_{t\geq 0}$ has densities $x \mapsto p_\lambda(x - q_\lambda)$ with p_λ as defined in Lemma 8.2.1 and

$$q_\lambda := \begin{cases} 0 & \text{if } \alpha \in (0,1) \\ \int\limits_{0+}^{\infty} x(1 - e^{-\lambda x})\, d\Phi(x) & \text{if } \alpha \in (1,2). \end{cases} \tag{8.15}$$

Remark 8.2.3. Just as well, one may add the shift q_λ to the constant a_λ and then study the densities of the corresponding Lévy process without any shifts. However, in order to obtain accordance with the already existing tempered stable laws and tempered fractional derivatives, we decided to stick to the notation in Lemma 8.2.2.

An essential advantage of the tempered distribution is the existence of moments of arbitrary order.

Lemma 8.2.4. *(Moments of tempered semistable laws)*
For $\lambda > 0$, let $x \mapsto p_\lambda(x, t)$ be a tempered semistable density for some $t > 0$. Then for every $n \in \mathbb{N}$, the n-th moment of p_λ exist.

Proof. According to [122, Theorem 25.3], the n-th moment

$$\int_{-\infty}^{\infty} x^n p_\lambda(x,t)\,dx$$

exists for every $t > 0$ if and only if

$$\int_{|x|>1} x^n\,d\Phi_\lambda(x) < \infty.$$

However, in our case the Lévy measure is concentrated on $(0,\infty)$, and hence we obtain

$$\int_{|x|>1} x^n\,d\Phi_\lambda(x) = \int_{1}^{\infty} x^n e^{-\lambda x}\,d\Phi(x) < \infty$$

for every $n \in \mathbb{N}$. □

As seen by the previous proof, the Lévy measure is strongly related to the asymptotic behavior of the (tempered) semistable random variable. In detail, if X is a semistable random variable, then due to the continuity of K, the distribution is subexponential ([128, Theorem 1.3]) such that

$$P(X > x) \sim \Phi(x,\infty) \tag{8.16}$$

for large values of x. If now X_λ has a tempered semistable distribution with $\lambda > 0$, the following Lemma provides a similar statement showing that the tail of the tempered semistable distribution asymptotically behaves like the tempered Lévy measure.

Lemma 8.2.5. *Let $(X(t))_{t\geq 0}$ be a semistable Lévy process, $(X_\lambda(t))_{t\geq 0}$ the corresponding tempered Lévy process as defined in Lemma 8.2.2 and $Y = X_\lambda(t) - q_\lambda$ for some fixed $t > 0$ and q_λ as in (8.15). Then*

$$P(Y > r) \sim C(t)t\Phi_\lambda(r,\infty)$$

as $r \to \infty$, where

$$C(t) = e^{-t\Psi(i\lambda)} = \begin{cases} \exp\left(t\displaystyle\int_{0+}^{\infty}(1-e^{-\lambda x})\,d\Phi(x)\right) & \text{if } \alpha \in (0,1) \\ \exp\left(t\displaystyle\int_{0+}^{\infty}(1-\lambda x-e^{-\lambda x})\,d\Phi(x)\right) & \text{if } \alpha \in (1,2). \end{cases} \tag{8.17}$$

Proof. According to (8.16), we have

$$1 = \lim_{r\to\infty} \frac{P(X(t) > r)}{t\Phi(r,\infty)} = \lim_{r\to\infty} \frac{\int_r^\infty p(x,t)\,dx}{t\int_r^\infty d\Phi(x)}$$

$$= -\lim_{r\to\infty} \frac{\int_r^\infty p(x,t)\,dx}{t\int_r^\infty dG_K(x)}$$

using the density $x \mapsto p(x,t)$ of $X(t)$. Since G_K is differentiable, using l'Hospital's rule we find

$$1 = -\lim_{r\to\infty} \frac{\int_r^\infty p(x,t)\,dx}{t\int_r^\infty G_K'(x)dx} = -\lim_{r\to\infty} \frac{p(r,t)}{tG_K'(r)}. \tag{8.18}$$

For the asymptotic behavior of Y, we study the limit

$$\begin{aligned}
\lim_{r\to\infty} \frac{P(Y > r)}{C(t)t\Phi_\lambda(r,\infty)} &= \lim_{r\to\infty} \frac{P(X_\lambda(t) - q_\lambda > r)}{C(t)t\Phi_\lambda(r,\infty)} \\
&= \lim_{r\to\infty} \frac{\int_{r+q_\lambda}^\infty p_\lambda(x,t)\,dx}{C(t)t\int_r^\infty e^{-\lambda x}\,d\Phi(x)} \\
&= -\lim_{r\to\infty} \frac{\int_{r+q_\lambda}^\infty e^{-\lambda(x-q_\lambda)}e^{-t\Psi(i\lambda)}p(x-q_\lambda,t)\,dx}{C(t)t\int_r^\infty e^{-\lambda x}G_K'(x)\,dx} \\
&= -\lim_{r\to\infty} \frac{\int_r^\infty e^{-\lambda x}e^{-t\Psi(i\lambda)}p(x,t)\,dx}{C(t)t\int_r^\infty e^{-\lambda x}G_K'(x)\,dx}.
\end{aligned}$$

Using l'Hospital's rule once again yields

$$\lim_{r\to\infty} \frac{P(Y > r)}{C(t)t\Phi_\lambda(r,\infty)} = -\lim_{r\to\infty} \frac{e^{-\lambda r}e^{-t\Psi(i\lambda)}p(r,t)}{C(t)te^{-\lambda r}G_K'(r)} = -\lim_{r\to\infty} \frac{p(r,t)}{tG_K'(r)} = 1$$

according to the definition of $C(t)$ in (8.17) and (8.18).

□

Similar to the semistable case, we gain a governing equation for the tempered semistable process. Namely, the densities $x \mapsto p_\lambda(x - q_\lambda, t)$ solve the equation

$$\frac{\partial}{\partial t} u(x,t) = L_\lambda u(x,t)$$

with $u(x,0) = \delta(x)$, where L_λ is the generator of the semigroup corresponding to μ_λ. If we formally define a tempered semi-fractional derivative as

$$\frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f(x) = -D L_\lambda f(x),$$

then the tempered semi-fractional derivative of a suitable function f is the function with Fourier transform

$$\widehat{\frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f}(k) = -D(\Psi(k + i\lambda) - \Psi(i\lambda)).$$

From the explicit representation of the generator in Lemma 2.3.3, we also receive an integral form of the tempered semi-fractional derivative as

$$\begin{aligned}
\frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f(x) &= -D \int_{0+}^{\infty} \Big(f(x-y) - f(x) + y f'(x) \mathbb{1}_{(1,2)}(\alpha) \Big)\, d\Phi_\lambda(y) \\
&= D \int_{0+}^{\infty} \Big(f(x) - f(x-y) - y f'(x) \mathbb{1}_{(1,2)}(\alpha) \Big)\, e^{-\lambda y}\, d\Phi(y).
\end{aligned}$$

For $\lambda \to 0$, the tempered semi-fractional derivative coincides with the semi-fractional one. However, even for $\lambda > 0$, we can describe the relation between both operators.

Lemma 8.2.6. *(Connection between tempered and ordinary semi-fractional derivatives) For $\alpha \in (0,2) \setminus \{1\}$ and $c > 1$, let K be an admissable function with respect to these parameters. For every $\lambda > 0$, we have*

$$\frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f(x) = e^{-\lambda x} \frac{\partial^{\alpha}}{\partial_{c,K} x^\alpha} (e^{\lambda x} f(x)) + D\Psi(i\lambda) f(x) - f'(x) q_\lambda$$

for every $x \in \mathbb{R}$.

Proof. First consider $\alpha \in (0,1)$, and note that $q_\lambda = 0$ in this case. Then with (3.4), for every $\lambda > 0$ and $\alpha \in (0,1)$, we obtain

$$\begin{aligned}
e^{-\lambda x} \frac{\partial^{\alpha}}{\partial_{c,K} x^\alpha} (e^{\lambda x} f(x)) &= e^{-\lambda x} \int_{0+}^{\infty} \Big(e^{\lambda(x-y)} f(x-y) - e^{\lambda x} f(x) \Big)\, dG_K(y) \\
&= \int_{0+}^{\infty} \Big(e^{-\lambda y} f(x-y) - f(x) \Big)\, dG_K(y)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{0+}^{\infty} (f(x-y) - f(x))\, e^{-\lambda y}\, dG_K(y) - f(x) \int_{0+}^{\infty} (1 - e^{-\lambda y})\, dG_K(y) \\
&= \int_{0+}^{\infty} (f(x) - f(x-y))\ d\Phi_\lambda(y) - f(x) \int_{0+}^{\infty} (1 - e^{-\lambda y})\, dG_K(y) \\
&= \frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f(x) - \Psi(i\lambda) f(x),
\end{aligned}
$$

which proves the statement for $\alpha \in (0,1)$. Similarly, for $\alpha \in (1,2)$ it follows with (3.4) that

$$
\begin{aligned}
e^{-\lambda x} \frac{\partial^\alpha}{\partial_{c,K} x^\alpha}(e^{\lambda x} f(x)) =& e^{-\lambda x} \int_{0+}^{\infty} \Big(e^{\lambda x} f(x) - e^{\lambda(x-y)} f(x-y) - y e^{\lambda x}(f'(x) + \lambda f(x)) \Big)\ dG_K(y) \\
=& \int_{0+}^{\infty} \Big(f(x) - e^{-\lambda y} f(x-y) - y(f'(x) + \lambda f(x))) \Big)\ dG_K(y) \\
=& \int_{0+}^{\infty} (f(x) - f(x-y) - y f'(x)) e^{-\lambda y}\, dG_K(y) \\
& + f(x) \int_{0+}^{\infty} (1 - e^{-\lambda y} - \lambda y)\, dG_K(y) + f'(x) q_\lambda \\
=& \frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f(x) + \Psi(i\lambda) f(x) + f'(x) q_\lambda
\end{aligned}
$$

for every $x \in \mathbb{R}$. □

Example 8.2.7. In the fractional case,

$$
\Psi(i\lambda) = \int_{0+}^{\infty} (e^{-\lambda y} - 1)\, d\Phi(y) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} (e^{-\lambda y} - 1) y^{-\alpha-1}\, dy = -\lambda^\alpha
$$

for $\alpha \in (0,1)$ (compare [94, Proposition 3.10]), whereas for $\alpha \in (1,2)$ we obtain

$$
\begin{aligned}
\Psi(i\lambda) &= \int_{0+}^{\infty} (e^{-\lambda y} - 1 + \lambda y)\, d\Phi(x) \\
&= -\frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} (e^{-\lambda y} - 1 + \lambda y) y^{-\alpha-1}\, dy \\
&= -\frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1-\alpha)} \lambda^\alpha
\end{aligned}
$$

$$= \lambda^\alpha$$

(see [94, Proposition 3.12]). Additionally, the shift q_λ is given by

$$q_\lambda = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} x^{-\alpha}(1 - e^{-\lambda x})\, dx = \alpha\lambda^{\alpha-1}$$

for $\alpha \in (1, 2)$. Hence, we get the relation

$$\frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f(x) = e^{-\lambda x} \frac{\partial^\alpha}{\partial_{c,K} x^\alpha}(e^{\lambda x} f(x)) - \lambda^\alpha f(x)$$

for $\alpha \in (0, 1)$ and

$$\frac{\partial^{\alpha,\lambda}}{\partial_{c,K} x^\alpha} f(x) = e^{-\lambda x} \frac{\partial^\alpha}{\partial_{c,K} x^\alpha}(e^{\lambda x} f(x)) - \lambda^\alpha f(x) - \alpha\lambda^{\alpha-1} f'(x)$$

for $\alpha \in (1, 2)$ respectively, which coincides with known results ([94, Equation (7.11) and (7.16)]).

Example 8.2.8. As an application of tempered semi-fractional laws, we want to study the percentage of absolute daily price changes (in US dollar) for Amazon Inc. stock. Data of this kind has been modeled with fractional approaches before (see for example [97]), and we hope to improve these results. Therefore, the historical data between 02.01.1998 and 31.07.2020 was taken from [158], yielding 5682 trading days. We assume that we can model the absolute daily price change percentage with a random variable $X > 0$ having a tempered $(d^{\frac{1}{\beta}}, d)$-semistable distribution for some $\beta \in (0, 1)$ and $d > 1$. Besides, we assume that the admissable function V in the tail of the Lévy measure Φ_λ is continuously differentiable with Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$ such that $x \mapsto \Phi_\lambda(x, \infty)$ is continuous in $x > 0$. According to Lemma 8.2.5, the asymptotic behavior can be characterized by

$$P(X > x) \sim C\Phi_\lambda(x, \infty)$$

for a constant $C > 0$ as $x \to \infty$. For the tail of the tempered Lévy measure, use integration by parts to obtain

$$\begin{aligned}\Phi_\lambda(x, \infty) &= \int_x^\infty e^{-\lambda y}\, d\Phi(y) = \Big[-e^{-\lambda y} G_V(y)\Big]_x^\infty - \lambda \int_x^\infty e^{-\lambda y} G_V(y)\, dy \\ &= e^{-\lambda x} G_V(x) - \lambda \int_x^\infty e^{-\lambda y} G_V(y)\, dy\end{aligned}$$

and the series representation of the log-periodic perturbation V in $G_V(x) = x^{-\beta} V(\log(x))$

yields

$$\begin{aligned}\Phi_\lambda(x,\infty) &= e^{-\lambda x} G_V(x) - \lambda \sum_{n\in\mathbb{Z}} c_n \int_x^\infty e^{-\lambda y} y^{in\tilde{d}-\beta}\, dy \\ &= e^{-\lambda x} G_V(x) - \sum_{n\in\mathbb{Z}} c_n \lambda^{\beta - in\tilde{d}} \Gamma(1-\beta+in\tilde{d}, \lambda x)\end{aligned}$$

with $\Gamma(a,b) = \int_b^\infty e^{-x} x^{a-1}\, dx$ being the incomplete gamma function. By equation (11.12) in [141], we have

$$\Gamma(a,b) = b^{a-1} e^{-b} \sum_{k=0}^\infty \frac{(-1)^k (1-a)\cdots(k-a)}{b^k} = b^{a-1} e^{-b} \sum_{k=0}^\infty \binom{a-1}{k} \frac{k!}{b^k}$$

such that

$$\begin{aligned}\Phi_\lambda(x,\infty) &= e^{-\lambda x} G_V(x) - \sum_{n\in\mathbb{Z}} c_n \lambda^{\beta-in\tilde{d}} (\lambda x)^{-\beta+in\tilde{d}} e^{-\lambda x} \sum_{k=0}^\infty \binom{in\tilde{d}-\beta}{k} \frac{k!}{(\lambda x)^k} \\ &= e^{-\lambda x} G_V(x) - \sum_{n\in\mathbb{Z}} c_n x^{-\beta+in\tilde{d}} e^{-\lambda x} - \sum_{n\in\mathbb{Z}} c_n x^{-\beta+in\tilde{d}} e^{-\lambda x} \sum_{k=1}^\infty \binom{in\tilde{d}-\beta}{k} \frac{k!}{(\lambda x)^k} \\ &= -\sum_{n\in\mathbb{Z}} c_n x^{-\beta+in\tilde{d}} e^{-\lambda x} \sum_{k=1}^\infty \binom{in\tilde{d}-\beta}{k} \frac{k!}{(\lambda x)^k} \\ &\sim -\frac{1}{\lambda} \sum_{n\in\mathbb{Z}} c_n x^{-\beta+in\tilde{d}-1} e^{-\lambda x} (in\tilde{d}-\beta).\end{aligned}$$

Using the differentiability of V, we have

$$\begin{aligned}\Phi_\lambda(x,\infty) &\sim -\frac{e^{-\lambda x}}{\lambda} \left(\frac{\partial}{\partial x} G_V(x) \right) \\ &= \frac{e^{-\lambda x}}{\lambda} x^{-\beta-1} (\beta V(\log(x)) - V'(\log(x))).\end{aligned}$$

Remark that the function $\beta V - V'$ is positive and $\log\left(d^{\frac{1}{\beta}}\right)$-periodic due to the admissability of V but not necessarily admissable itself. In this sense, the tail of the semistable random variable equals the tail of a tempered disturbed Pareto distribution with index $\alpha = \beta + 1 \in (1,2)$ and perturbation $K = C\lambda^{-1}(\alpha V - V')$.

To describe the percentage of daily price changes for Amazon Inc. stock, we took the 50 largest observations in the period between 02.01.1998 and 31.07.2020 and fit a Pareto, a tempered Pareto, and a tempered disturbed Pareto model to the data. For the disturbed Pareto model, we assume that

$$K(x) = d_1 + d_2 \cos\left(x \frac{2\pi}{p}\right) + d_3 \sin\left(x \frac{2\pi}{p}\right).$$

The result is shown in Figure 8.7, whereas the calculated parameters are displayed in Table 8.5. The Matlab code for the calculation can be found in Appendix C. The largest data points occur with such a low probability, that even though the tempered Pareto model approximates these points more accurately than the ordinary Pareto approach, with our accuracy, there is no difference in the MSE. However, the tempered disturbed Pareto model is an even better fit to the data and also captures the highest data points more precisely.

	Pareto model	Tempered Pareto	Tempered disturbed Pareto
α	3.6175	2.1578	1.0468
λ	–	0.0798	0.2090
p	–	–	0.8226
d_1	138.2396	8.7096	4.3405
d_2	–	–	0.8881
d_3	–	–	-1.0691
MSE $\cdot 10^6$	5.1195	5.1195	1.6780

Table 8.5: Calculated parameters for the different models in Example 8.2.8 for the daily price changes in Amazon Inc. stock.

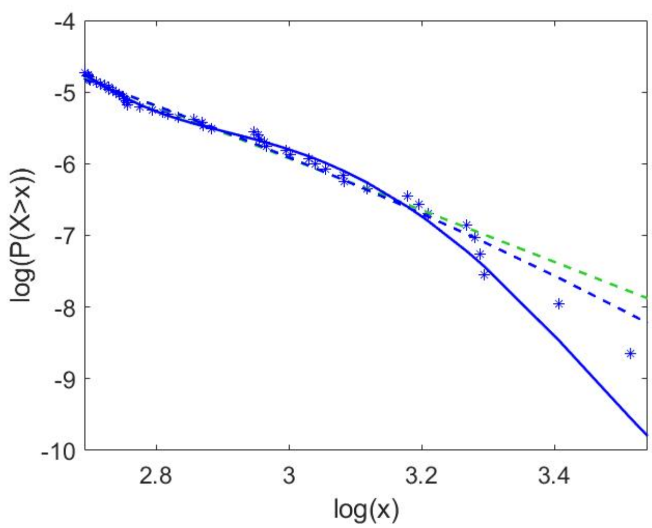

Figure 8.7: Tail of the best Pareto (green dashed line), tempered Pareto (blue dashed line), and tempered disturbed Pareto model (blue solid line) in comparison to the empirical tail (stars) consisting of the 50 largest observations of the percentage of absolute daily price change in Amazon Inc. stock between 02.01.1998 and 31.07.2020 in Example 8.2.8.

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Appendix A: Index of notation

Sets and spaces:

$\overline{A}, A^\circ$	Closure and interior of $A \subseteq \mathbb{R}^d$
∂A	Boundary of $A \subseteq \mathbb{R}^d$
$\mathbb{R}^d_+$	$\mathbb{R}^d_+ := (0,\infty)^d$
$\mathbb{C}(\vartheta)$	$\{re^{i\varphi} \in \mathbb{C} : r > 0, \lvert\varphi\rvert < \vartheta\}$
$\mathbb{C}_+$	$\mathbb{C}\left(\frac{\pi}{2}\right)$
$\mathcal{B}(A)$	Borel sets of $A \subseteq \mathbb{R}^d$
S	Unit sphere $S := \{x \in \mathbb{R}^d : \lvert\lvert x\rvert\rvert^2 = 1\}$
$L^1(\mathcal{A})$	Space of integrable functions $f : \mathcal{A} \to \mathbb{R}$ with $\subseteq \mathbb{R}^d$
$W^n(\mathcal{A})$	$W^n(\mathcal{A}) := \{f : \mathcal{A} \to \mathbb{R} : f$ is n-times partially differentiable on $\mathcal{A}^\circ$ and all partial derivatives up to order n belong to $L^1(\mathcal{A})\}$, $\mathcal{A} \subseteq \mathbb{R}^d$
$C_0(\mathbb{R}^d)$	Space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with $\lim\limits_{\lvert x\rvert \to \infty} f(x) = 0$
$C_0^n(\mathbb{R}^d)$	Space of n-times differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that f and all partial derivatives up to order n belong to $C_0(\mathbb{R}^d)$
$C_0^n(\overline{\mathbb{R}^d_+})$	Space of functions $f : \mathbb{R}^d_+ \to \mathbb{R}$ such that $f \in C_0^n(\mathbb{R}^d_+)$ and f and all partial derivatives up to order n can be extended continuously to the boundary of $\mathbb{R}^d_+$
$C^2_{pw}(\mathbb{R})$	Space of continuously differentiable functions such that f' is piecewise smooth

$\mathcal{S}(\mathbb{R})$	Schwartz space of all rapidly decreasing functions
$\mathcal{D}([0,\infty),\mathbb{R}^d)$	Space of all càdlàg functions $f:[0,\infty)\to\mathbb{R}^d$
$L(\mathbb{R}^d)$	Set of linear operators $A:\mathbb{R}^d\to\mathbb{R}^d$
$GL(\mathbb{R}^d)$	Set of invertable linear operators $A:\mathbb{R}^d\to\mathbb{R}^d$
$\mathrm{Dom}(L)$	Domain of a generator L

Measures and special functions:

ϵ_x	Point measure in $x\in\mathbb{R}^d$
$(A\nu)$	Image measure of ν on $\mathbb{R}^d$ under a linear transform $A:\mathbb{R}^d\to\mathbb{R}^d$
$\arg(z)$	Complex argument of $z\in\mathbb{C}$
$\Gamma(z)$	Gamma function $\Gamma(z)=\int\limits_0^\infty t^{z-1}e^{-t}\,dt$ for $\mathrm{Re}(z)>0$
$\Gamma(z,x)$	Incomplete gamma function $\Gamma(z,x)=\int\limits_x^\infty t^{z-1}e^{-t}\,dt$ for $\mathrm{Re}(z)>0$
$B(x,y)$	Beta function $B(x,y)=\int\limits_0^1 (1-t)^{x-1}t^{y-1}\,dt$ for $\mathrm{Re}(x),\mathrm{Re}(y)>0$
$B(x,a,b)$	Incomplete Beta function $B(x,a,b)=\int\limits_0^x t^{a-1}(1-t)^{b-1}\,dt$ for every $a,b\in\mathbb{C}$ with $\mathrm{Re}(a),\mathrm{Re}(b)>0$ and $x\in(0,1]$
$E_\alpha(z)$	Mittag-Leffler function $E_\alpha(x)=\sum\limits_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k+1)}$

Integral transforms:

$\widehat{f}$ or $\mathcal{F}(f)$	Fourier transform of a suitable function $f : \mathbb{R}^d \to \mathbb{R}$ $\widehat{f}(k) = \int\limits_{\mathbb{R}^d} e^{i\langle k,x\rangle} f(x)\,dx$
$\widehat{\nu}$	Fourier transform of a measure ν on $\mathbb{R}^d$ $\widehat{\nu}(k) = \int\limits_{\mathbb{R}^d} e^{i\langle k,x\rangle}\,d\nu(x)$
$\widetilde{f}$ or $\mathcal{L}(f)$	Laplace transform of a suitable function $f : \mathbb{R}^d_+ \to \mathbb{R}$ $\tilde{f}(s) = \int\limits_{\mathbb{R}^d_+} e^{-\langle s,t\rangle} f(t)\,dt$
$\widetilde{\nu}$	Laplace transform of a measure ν on $\mathbb{R}^d_+$ $\tilde{\nu}(s) = \int\limits_{\mathbb{R}^d_+} e^{-\langle s,t\rangle}\,d\nu(t)$
$\overline{f}$ or $\mathcal{FL}(f)$	Fourier-Laplace transform of a suitable function $f : \mathbb{R}^d \times \mathbb{R}^d_+ \to \mathbb{R}$ $\overline{f}(k,s) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d_+} e^{i\langle k,x\rangle} e^{-\langle s,t\rangle} f(x,t)\,dt\,dx$
$\overline{\nu}$	Fourier-Laplace transform of a measure ν on $\mathbb{R}^d \times \mathbb{R}^d_+$ $\overline{\nu}(k,s) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d_+} e^{i\langle k,x\rangle} e^{-\langle s,t\rangle} d\nu(x,t)$

Derivatives:

D_f	Jacobian matrix of f
H_f	Hessian matrix of f
$\Delta^{\frac{\alpha}{2}}$	Fractional Laplacian
δ_θ	Ordinary directional derivative in direction $\theta \in S$
$\dfrac{\partial^\alpha}{\partial x^\alpha}$	Caputo form of the positive fractional derivative of order α

$\dfrac{\partial^\alpha}{\partial(-x)^\alpha}$	Caputo form of the negative fractional derivative of order α
$\left(\dfrac{\partial}{\partial x}\right)^\alpha$	Riemann-Liouville form of the positive fractional derivative of order α
$\left(\dfrac{\partial}{\partial(-x)}\right)^\alpha$	Riemann-Liouville form of the negative fractional derivative of order α
$\mathbb{D}^{\alpha,M}$	Multidimensional fractional derivative of order α with respect to the finite Borel measure M on the unit sphere
$\dfrac{\partial^\alpha}{\partial_{c,K}x^\alpha}$	Caputo form of the positive semi-fractional derivative of order α with respect to $c > 1$ and $K : \mathbb{R} \to (0,\infty)$ admissable
$\dfrac{\partial^\alpha}{\partial_{c,K}(-x)^\alpha}$	Caputo form of the negative semi-fractional derivative of order α with respect to $c > 1$ and $K : \mathbb{R} \to (0,\infty)$ admissable
$\left(\dfrac{\partial}{\partial_{c,K}x}\right)^\alpha$	Riemann-Liouville form of the positive semi-fractional derivative of order α with respect to $c > 1$ and $K : \mathbb{R} \to (0,\infty)$
$\left(\dfrac{\partial}{\partial_{c,K}(-x)}\right)^\alpha$	Riemann-Liouville form of the negative semi-fractional derivative of order α with respect to $c > 1$ and $K : \mathbb{R} \to (0,\infty)$
$\dfrac{\partial_\theta^\alpha}{\partial_{c,K}x^\alpha}$	Caputo form of the directional semi-fractional derivative of order α with respect to $c > 1$ and $K : \mathbb{R} \to (0,\infty)$ admissable in direction $\theta \in S$
$\left(\dfrac{\partial_\theta}{\partial_{c,K}x}\right)^\alpha$	Riemann-Liouville form of the directional semi-fractional derivative of order α with respect to $c > 1$ and $K : \mathbb{R} \to (0,\infty)$ in direction $\theta \in S$
$\mathbb{D}^{\alpha,M}_{c,(K_\theta)_{\theta\in S}}$	Multidimensional semi-fractional derivative of order α with respect to $c > 1$, an admissable set of functions $(K_\theta)_{\theta\in S}$, and a finite Borel measure M on S
$\dfrac{\partial^{\alpha,\lambda}}{\partial_{c,K}x^\alpha}$	Tempered semi-fractional derivative of order α with respect to $c > 1$, K admissable, and $\lambda > 0$

Appendix B: Author contribution statement

In the following, I list the individual contributions to the two already published papers, which are cited in this thesis.

Kern, P., Lage, S., and Meerschaert, M. M. (2019). Semi-fractional diffusion equations. Fractional Calculus and Applied Analysis 22(2), pp. 326-357.
This article emerged in collaboration with Mark M. Meerschaert and Peter Kern. Mark M. Meerschaert developed the initial idea of a semi-fractional derivative as well as the idea of a Zolotarev semi-fractional derivative of order $\alpha = 1$, which is not treated in this thesis. The concrete elaboration and most of the results in this article arose from my master thesis under the supervision of Peter Kern. His support and ideas finalized the results.

Kern, P., and Lage, S. (2021). Space-time duality for semi-fractional diffusions. In: Freiberg, U., Hambly, B., Hinz, M., and Winter, S. (eds.). Fractal Geometry and Stochastics VI. Progress in Probability, Birkhäuser, Basel.
This article emerged in collaboration with Peter Kern. The first two parts of this article are a survey of existing results and were written by Peter Kern. The following results for the semi-fractional case were developed by me, where Peter Kern supported and supplemented their development.

Appendix C: Code

For completeness and reproducibility, we provide all Matlab scripts used in this thesis. For many calculations, the complex gamma function is needed, which is not implemented in Matlab. Hence, we additionally used a script provided by Paul Godfrey [155], which we renamed *'ComplexGamma.m'* to avoid a conflict with the implemented real gamma function.

Test a function on admissability: The function *'Testadmissability.m'* enables the user to test whether a p-periodic function K given as a finite Fourier series

$$K(x) = \sum_{n=-k}^{k} c_n e^{in\tilde{c}x}$$

for $k \in \mathbb{N}$ and $(c_n)_{n\in\{-k,\dots,k\}} \subset \mathbb{C}$ is admissable with respect to $\alpha \in (0,2) \setminus \{1\}$ and $c = e^{\alpha p}$ or not. As input parameters, the function demands the parameter $\alpha \in (0,2) \setminus \{1\}$ and a vector cn containing the $2k+1$ Fourier coefficients $cn = (c_{-k}, c_{-k+1}, \dots, c_0, c_1, \dots, c_{k-1}, c_k)$. Besides, the user has to deliver the period p of the periodic function. The script first tests the positivity of the function and then the growth restriction in Definition 3.1.1 using the derivative of K as in Lemma 3.1.2 (vi). As output, the script generates a statement of whether the function is admissable or not if $num = 0$. For all other values of the input parameter num, the script generates a binary output of this statement.

```
% Test a periodic function K on admissability

% Input parameters:
% alpha     =    Parameter of admissability
% cn        =    Vector of Fourier coefficients [c_{-k},...c_0,...c_k] of K
% p         =    Period of K
% num       =    Create numerical or non-numerical output

% Output
% Statement whether K is admissable with respect to alpha and c=e^(alpha*p) or not

function f=Testadmissability(alpha,cn,p,num)

% Fixed constants
  n=numel(cn);              % Total number of Fourier coefficients
  k=(n-1)/2;                % Number of one-sided Fourier coefficients c_1,..c_k
  tildec=2*pi/p;
  x=-2*p:0.05:2*p;

% Calculate the function K
  K=zeros(1,numel(x));
  for m=-k:k
      K=K+cn(m+k+1)*exp(i*m*tildec.*x);
  end

% Calculate the derivative of K
  DeriK=zeros(1,numel(x));
  for m=-k:k
```

```
        DeriK=DeriK+cn(m+k+1)*(i*m*tildec)*exp(i*m*tildec.*x);
    end
    y=alpha*K-DeriK;

% Create the output
if (K>0)
        if (y>=0)
            if (num==0)
             disp('Function is admissable')
            end
             f=0;
        else
            if (num==0)
                disp('Growth restriction not fulfilled')
            end
            f=1;
        end
     else
         if (num==0)
             disp('Function takes non-positive values')
         end
         f=1;
end
end
```

Code for Example 3.1.14: The following code calculates the semi-fractional derivative of $f(x) = e^{-2x^2}$ in Example 3.1.14.

```
% Calculate the semi-fractional derivative of f(x)=exp(-ax^2) in Example 3.1.14
clear all
close all

% Chosen parameters
a=2;                                                          % Parameter of f
alpha=0.4;                                                    % Order of semi-fractional derivative
cn=[-1/(25*i),-1/(20*i),1/gamma(1-alpha),1/(20*i),1/(25*i)];  % Fourier coefficients of K
Testadmissability(alpha,cn,2*pi,0);                           % Test K on admissability

% Initialize the (semi-)fractional derivative
h=0.01;                                                        % Grid size
points=-2:h:5;                                                 % Points of calculation
n=numel(points);                                               % Number of points
Semi_fractional_Caputo=zeros(1,n);
Fractional_Caputo=zeros(1,n);

% Calculate the (semi-)fractional derivative
counter=0;
for x=points(1):h:points(n)
    counter=counter+1
    z=@(y) (x-y).*exp(-a*(x-y).^2).*y.^(-alpha).*(1/gamma(1-alpha)+1/10*sin(log(y))+2/25*sin(2*log(y)));
    g=@(y) (x-y).*exp(-a*(x-y).^2).*y.^(-alpha).*1/gamma(1-alpha);
    Semi_fractional_Caputo(counter)=-2*a*integral(z,0,inf);
    Fractional_Caputo(counter)=-2*a*integral(g,0,inf);
end
```

Code for Example 3.1.15: The following code calculates the semi-fractional derivatives of the power function $f(x) = x^p$ in Example 3.1.15.

```
% Calculate the semi-fractional derivative of the power function in Example
% 3.1.15
clear all
close all

% Fixed constants
p=0.3; % p=0.6                                    % Parameter of f
alpha=1/2;                                        % Order of semi-fractional derivative
cn=[-1/(20*i),1/20,1/gamma(1-alpha),1/20,1/(20*i)]; % Fourier coefficients of K_2
n=(numel(cn)-1)/2;                                % Number of one-sided Fourier coefficients
period=2*pi;                                      % Period of K_2
Testadmissability(alpha,cn,period,0)              % Check K_2 on admissability
tildec=2*pi/period;
x=0:0.01:100000;                                  % Points of calculation
```

```
% Fractional derivative corresponding to K_1
Frac_Deriv=p*gamma(p)/gamma(p+1-alpha)*x.^(p-alpha);

% Semi-fractional derivative with respect to K_2
sum=zeros(1,numel(x));
for k=-n:n
    sum=sum+cn(k+n+1)*ComplexGamma(1-alpha+i*k*tildec)...
        /ComplexGamma(p+1-alpha+i*k*tildec)*x.^(p-alpha+i*k*tildec);
end
Semi_frac_Deriv=p*gamma(p)*real(sum);
```

Code for Example 3.2.17: The following code calculates the directional semi-fractional derivatives in the Caputo and Riemann-Liouville sense in Example 3.2.17. To plot both forms, note that they are given by

$$\frac{\partial_\theta^\alpha}{\partial_{c,K} x^\alpha} f(x) = \int_{0+}^{\infty} \langle \theta, \nabla f(x - y\theta) \rangle G_K(y)\, dy$$
$$= \theta_1 \int_{0+}^{\min\{\frac{x_1}{\theta_1}, \frac{x_2}{\theta_2}\}} 2(x_1 - y\theta_1) G_K(y)\, dy + \theta_2 \int_{0+}^{\min\{\frac{x_1}{\theta_1}, \frac{x_2}{\theta_2}\}} G_K(y)\, dy$$

and

$$\left(\frac{\partial_\theta}{\partial_{c,K} x}\right)^\alpha f(x) = \left\langle \nabla \int_{0+}^{\infty} f(x - y\theta) G_K(y)\, dy, \theta \right\rangle$$
$$= \left\langle \nabla \int_{0+}^{\min\{\frac{x_1}{\theta_1}, \frac{x_2}{\theta_2}\}} ((x_1 - y\theta_1)^2 + (x_2 - y\theta_2)) G_K(y)\, dy, \theta \right\rangle$$

for every $x \in \mathbb{R}_+^2$.

```
% Calculate the directional semi-fractional derivative in Example 3.2.17
clear all
close all

% Chosen parameters
theta=[1/sqrt(2),1/sqrt(2)];          % Direction of directional derivative
a=2/3;                                % Order of semi-fractional derivative
cn=[1/2,-1/(2*i),5,1/(2*i),1/2];      % Fourier coefficients of K
p=2*pi;                               % Period of K

% Test the function on admissability
Testadmissability(a,cn,p,0)

% Initialize the grid
h=0.001;
x=0:h:2;
y=0:h:2;
n=numel(x);
m=numel(y);
Caputo=zeros(n,m);
RL=zeros(n,m);

% Calulate the Caputo semi-fractional derivative
g1=@(r) r.^(-a).*(sin(log(r))+cos(2*log(r))+5);
g2=@(r) r.^(1-a).*(sin(log(r))+cos(2*log(r))+5);
for k=1:n
    for l=1:m
        a1=integral(g1,0.0001,min(x(k)/theta(1),y(l)/theta(2)));
```

```
        a2=integral(g2,0.0001,min(x(k)/theta(1),y(l)/theta(2)));
        Caputo(k,l)=(2*theta(1)*x(k)+theta(2))*a1-2*theta(1)^2*a2;
    end
end

% Calculate the Riemann-Liouville semi-fractional derivative
A=zeros(n,m);
for k=1:n
    for l=1:m
        g=@(r) ((x(k)-r*theta(1)).^2+(y(l)-r*theta(2))).*r.^(-a).*(sin(log(r))+cos(2*log(r))+5);
        A(k,l)=integral(g,0.0001,min(x(k)/theta(1),y(l)/theta(2)));
    end
end
for k=1:n-1
    for l=1:m-1
        RL(k,l)=1/h*(theta(1)*(A(k+1,l)-A(k,l))+theta(2)*(A(k,l+1)-A(k,l)));
    end
end
```

Calculation of solutions to semi-fractional diffusion equations using Grünwald-Letnikov differences: The function *'Semi_fractional_Differential_Eq.m'* calculates the solution to the semi-fractional diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = -D\frac{\partial^\alpha}{\partial_{c,K_1}x^\alpha}u(x,t) - D\frac{\partial^\alpha}{\partial_{c,K_2}(-x)^\alpha}u(x,t) \tag{C.1}$$

under the initial condition $u(x,0) = u_0(x)$ for p-periodic, admissable functions $K_1, K_2 : \mathbb{R} \to \mathbb{R}_+$ with finite Fourier series

$$K_1(x) = \sum_{n=-k_1}^{k_1} c_n e^{in\tilde{c}x} \quad \text{and} \quad K_2(x) = \sum_{n=-k_2}^{k_2} d_n e^{in\tilde{c}x},$$

where $k_1, k_2 \in \mathbb{N}_0$ and $(c_n)_{n=-k_1,\dots,k_1}, (d_n)_{n=-k_2,\dots,k_2} \subset \mathbb{C}$. If $u_0(x) = \delta(x)$, then the solution coincides with the density of a $(c^{\frac{1}{\alpha}}, c)$-semistable density.
The algorithm needs the order $\alpha \in (0,2) \setminus \{1\}$, vectors $cn = [c_{-k_1}, \dots, c_0, \dots, c_{k_1}]$ and $dn = [d_{-k_2}, \dots, d_0, \dots, d_{k_2}]$ containing the Fourier coefficients of K_1 and K_2 as well as their period p as input parameters. Furthermore, the vector x contains the points of calculation, whereas the vector *start* describes the initial values in x. If *start* $= 0$ or if *start* is a zero vector of any dimension, then the script takes an approximation of the delta function as initial condition. Apart from the step sizes h and τ in space and time, the user can choose a method for the discretization of the time derivative and a shift in the Grünwald-Letnikov formula. Possible methods are implicit and explicit Euler methods, whereas the shift should be chosen to be zero for $\alpha \in (0,1)$ and 1 for $\alpha \in (1,2)$.
As output, the script calculates a matrix, which contains the solution at time $i \cdot \tau$ in x in column i for $i = 1, \dots, \frac{T}{\tau}$.
Note that dependent on the choices of h and τ, the method might be unstable. In the fractional case, [95] and [96] give criteria for convergence at least for $\alpha \in (1,2)$. We are currently working on similar stability results for the semi-fractional case.

```
% Calculate the solution to the semi-fractional diffusion equation (B.1)

% Input parameters:
% a         =   Order of semi-fractional derivative
% cn        =   Vector of Fourier coefficients of K1
```

```
% dn        =   Vector of Fourier coefficients of K2
% p         =   Period of K1 and K2
% x         =   Points of calculation
% start     =   Initial values (if start=0, the script calulates initial
%               values as an approximation to the delta function)
% h         =   Step size in space
% tau       =   Step size in time
% T         =   Endpoint of calculation in time
% Method    =   Decide between explicit and implicit Euler method
%               (method=1 -> explicit Euler, method=2 -> implicit Euler)
% Shift     =   Shift in Gruenwald-Letnikov formula
%               (Shift=0 or Shift=1 possible)

% Output
% Matrix containig the density in x at all time steps

function f=Semi_fractional_Differential_Eq(a,cn,dn,p,x,start,h,tau,T,Method,Shift)

% Fix constants
M=numel(x);            % Number of stpes in space
N=T/tau;               % Number of stpes in time

% If no initial values are delivered, calculate initial values as an
% approximation to the delta-function
if all (start==0)
    epsilon=h;
    start=1/(sqrt(pi)*epsilon)*exp(-x.^2/epsilon^2);
end

% Calculate the iteration matrix
A=Iterationmatrix(a,cn,dn,p,x,h,tau,Method,Shift);

% Initialize the solution
Sol=zeros(M,N+1);
Sol(:,1)=start;

% Caluculate the solution
if Method==1
    for k=1:N
        Sol(:,k+1)=A*Sol(:,k);
    end
    f=Sol;
elseif Method==2
    for k=1:N
        Sol(:,k+1)=A\Sol(:,k);
    end
    f=Sol;
end
end
```

The function *'Iterationmatrix.m'* calculates the iteration matrix for the computation of semistable densities. The method of computation in this script was developed by Matthias Häußler in his unpublished Master thesis [53].

```
% Calculate the iteration matrix for the solution to the semi-fractional
% diffusion equation (B.1)

% Input parameters:
% a         =   Order of semi-fractional derivative
% cn        =   Vector of Fourier coefficients of K1
% dn        =   Vector of Fourier coefficients of K2
% p         =   Period of K1 and K2
% x         =   Points of calculation
% h         =   Step size in space
% tau       =   Step size in time
% Method    =   Decide between explicit and implicit Euler method
%               (method=1 -> explicit Euler, method=2 -> implicit Euler)
% Shift     =   Shift in Gruenwald-Letnikov formula
%               (Shift=0 or Shift=1 possible)

% Output
% Iteration matrix A


function A=Iterationmatrix(a,cn,dn,p,x,h,tau,Method,Shift)

% Fix constants
n1=numel(cn);          % Number of Fourier coefficients for positive s.-f. derivative
n2=numel(dn);          % Number of Fourier coefficients for negative s.-f. derivative
k1=(n1-1)/2;
k2=(n2-1)/2;
```

```
tildec=2*pi/p;
M=numel(x);          % Number of steps in space

% 1.Step: Calculate Gruenwald-Letnikow weights for the positive s.-f. derivative
if all (cn==0)
        pos_weights=zeros(1,M+1);
else
    syms J;
    for m=-k1:k1
        v=exp(gammaln(sym(a-i*m*tildec+1))+gammaln(sym(i*m*tildec-a+1))...
            -gammaln(sym(a-i*m*tildec-J+1))-gammaln(J+1));
        S1(:,m+k1+1)=cn(m+k1+1)*h^(i*m*tildec)*(-1).^J.*v;
    end
    s1=h^(-a)*sum(S1,2);
    s2=real(s1);
    f(J)=s2;

    J=0:1:M;
    a1=f(J);
    pos_weights=vpa(a1);
end

% 2.Step: Calculate Gruenwald-Letnikow weights for the negative s.-f. derivative
if all (dn==0)
        neg_weights=zeros(1,M+1);
elseif all (dn-cn==0)
    neg_weights=pos_weights;
else
    syms L;
    for m=-k2:k2
        v=exp(gammaln(sym(a-i*m*tildec+1))+gammaln(sym(i*m*tildec-a+1))...
            -gammaln(sym(a-i*m*tildec-L+1))-gammaln(L+1));
        D2(:,m+k2+1)=dn(m+k2+1)*h^(i*m*tildec)*(-1).^L.*v;
    end
    d1=h^(-a)*sum(D2,2);
    d2=real(d1);
    f(L)=d2;

    L=0:1:M;
    a2=f(L);
    neg_weights=vpa(a2);
end

% Calculate the iteration matrix
if (Shift==0)
    A1=tril(toeplitz(pos_weights(1:M)));
    A2=triu(toeplitz(neg_weights(1:M)));
    U=diag(ones(1,M));
    if (Method==1)
        A=U-tau*A1-tau*A2;
    elseif (Method==2)
        A=U+tau*A1+tau*A2;
    end

elseif (Shift==1)
    pos_weights1=pos_weights(2:M+1);
    neg_weights1=neg_weights(2:M+1);
    B1=tril(toeplitz(pos_weights1))+diag(pos_weights(1)*ones(1,M-1),1);
    B2=triu(toeplitz(neg_weights1))+diag(neg_weights(1)*ones(1,M-1),-1);
    U=diag(ones(1,M));
    if (Method==1)
        A=U-tau*B1-tau*B2;
    elseif (Method==2)
        A=U+tau*B1+tau*B2;
    end
end
end
```

Code for Example 5.1.2: The following code calculates the solution to the semi-fractional diffusion equation (5.3) in Example 5.1.2.

```
% Calculate a solution to the semi-fractional diffusion in Example 5_1_2
clear all
close all

% Fixed constants
a=0.6;                                    % Order of semi-fractional derivative
p=pi;                                     % Period of K
dn=3/gamma(1-a);
cn=[-3/(40*i),3/30,dn,3/30,3/(40*i)];     % Fourier coefficients of K
Testadmissability(a,cn,p,0);              % Test K on admissability
```

```
h=0.01;                                       % Step size in space
tau=0.01;                                     % Step size in time
x=-5:h:15;                                    % Points of calculation
T=1;                                          % Endpoint of calculation (in time)
N=T/tau;                                      % Number of steps in time

% Initial values
start=exp(-x.^2);

% Calculate the solution for different initial values
solution_semi_frac=Semi_fractional_Differential_Eq(a,cn,0,p,x,start,h,tau,T,1,0);
solution_frac=Semi_fractional_Differential_Eq(a,dn,0,p,x,start,h,tau,T,1,0);
```

Code for Example 5.1.4: The following code calculates the semistable density ν in Example 5.1.4 as well as the functions $x \mapsto p(x,2)$ and $x \mapsto \frac{1}{4}p\left(\frac{x}{4},1\right)$ to check whether the scaling property (2.4) holds.

```
% Test the scaling property of the semistable distribution in Example 5.1.4
clear all
close all

% Fixed constants
a=1/2;                                    % Order of semi-fractional derivative
p=log(4);                                 % Period of K
cn=[1/160,1/80,1/gamma(1-a),1/80,1/160];  % Fourier coefficients of K
Testadmissability(a,cn,p,0);

% Parameters for the calculation
h=0.01;          % Step size in space
tau=0.02;        % Step size in time
x=-4:h:16;       % Points of calculation

%Calculate semistable density up to time T=2
p1=Semi_fractional_Differential_Eq(a,cn,0,p,x,0,h,tau,2,1,0);
[a1,b1]=size(p1);
p1end=p1(:,b1-1);        % Solution at time T=2
p2=p1(:,(b1-1)/2);       % Solution at time T=1
```

Code for Remark 5.1.5: The following code calculates the semistable density in Remark 5.1.5 as a solution to the semi-fractional diffusion equation. In addition, using the function *'Laplace_Inv_Abate_Whitt.m'*, the density was calculated using the inverse Laplace method of Abate and Whitt [1] as in [28].

```
% Comparison of numerical calculations of semistable densities in Remark
% 5.1.5
close all
clear all

% Fixed constants
a=1/2;                                    % Order of semi-fractional derivative
p=log(4);                                 % Period of K
cn=[1/160,1/80,1/gamma(1-a),1/80,1/160];  % Fourier coefficients of K
Testadmissability(a,cn,p,0);

% Initialize the solution of diffusion equation
h=0.01;          % Step size in space
tau=0.01;        % Step size in time
T=1;             % Endpoint of calculation in time
x=-1:h:8;        % Points of calculation

% Solution with GL-differences
Diffusion=Semi_fractional_Differential_Eq(a,cn,0,p,x,0,h,tau,T,1,0);
SolDiffusion=Diffusion(:,end-1);

% Initialize the solution with the method of Abate and Whitt for different
% values of A
y=h:h:30;          % Points of calculation
n=numel(y);        % Number of points
```

```
M=1000;
Sol=zeros(n,4);
for A=2:2:8
    Sol(:,A/2)=Laplace_Inv_Abate_Whitt(a,cn,p,y,A,M);
end

% Calculate solution for A betwenn 1.8 and 5.2 and find nearest
% approximation
Index=0;
error=10000;
m=round((5.2-1.8)/h);
z=h:h:8;
b=numel(z);
Sol1=zeros(b,m+1);
for k=0:1:m
     Sol1(:,k+1)=Laplace_Inv_Abate_Whitt(a,cn,p,z,1.8+k*h,M);
     if norm(Sol1(:,k+1)-SolDiffusion(end-b+1:end))<error
         error=norm(Sol1(:,k+1)-SolDiffusion(end-b+1:end));
         Index=k;
     end
end
```

To compare the numerical approximations of semistable densities in Remark 5.1.5, we use the following implementation for the inverse Laplace transform method

```
% Calculation of semistable density with inverse Laplace transform by Abate and Whitt

% Input parameter
% a    =    Order of semi-fractional derivative
% cn   =    Vector of Fourier coefficients of K
% p    =    Period of the periodic function K
% x    =    Points of caluclation
% A    =    Tuning parameter
% M    =    Number of considered points

% Output
% Approximation of the semistable density in x

function f=Laplace_Inv_Abate_Whitt(a,cn,p,x,A,M)

% Calculate the first summand
m1=Laplacetransform(a,cn,p,A./(2*x));
p1=exp(A/2)./(2*x).*real(m1);

% Calulate the remaining sum
sum=zeros(1,numel(x));
for k=1:M
  m2=Laplacetransform(a,cn,p,(A+2*k*i*pi)./(2*x));
  sum=sum+(-1)^k.*real(m2);
end
p2=exp(A/2)./x.*sum;
f=p1+p2;
end
```

The following script calculates the Laplace transform of the semistable density in Remark 5.1.5 needed to calculate the approximation of the semistable density with the method of Abate and Whitt.

```
% Function to calculate the Laplace transform of a semistable density

% Input parameter
% a    =    Order of semi-fractional derivative
% cn   =    Vector of Fourier coefficients of K
% p    =    Period of the periodic function K
% y    =    Points of caluclation

% Output
% Laplace transform of the semistable density in y

function f=Laplacetransform(a,cn,p,y)

% Fixed constants
 n=(numel(cn)-1)/2;          % Number of one-sided Fourier coefficients
 m=numel(y);                 % Number of points of calculations
 tildec=2*pi/p;
```

```
 sum=zeros(1,m);
   for k=-n:n
      sum=sum+cn(k+n+1)*ComplexGamma(i*tildec*k-a+1)*y.^(a-i*tildec*k);
   end
f=exp(-sum);

end
```

Code for Remark 5.2.6: The following code calculates the density of the inverse semistable and inverse stable subordinator in Remark 5.2.6.

```
% Calculate the density of an inverse semistable subordinator in Remark
% 5.2.6
close all
clear all

% Useful constants
T=1;                                                  % Endpoint of calculation in time
a=0.6;                                                % Order of semistable law
p=2*pi;                                               % Period of K
cn=[-1/(20*i),1/20,1/gamma(1-a),1/20,1/(20*i)];       % Fourier coefficients of K
dn=1/gamma(1-a);
Testadmissability(a,cn,p,0)

h=0.01;                                               % Step size in space
tau=0.01;                                             % Step size in time
y=-1:h:2;                                             % Points of calculation
L=20;

% Calculate the (semi-)stable density g
A1=Semi_fractional_Differential_Eq(a,cn,0,p,y,0,h,tau,L,1,0);
A2=Semi_fractional_Differential_Eq(a,dn,0,p,y,0,h,tau,L,1,0);

% Restrict the density to the positive real line
n1=numel(-1:h:0);
n2=numel(-1:h:T);
Subordinator1=A1(n1-1:n2,:);
Subordinator2=A2(n1-1:n2,:);
[a1,b]=size(Subordinator1);

% Calculate the density of the inverse semistable subordinator
density=zeros(1,b-1);
for k=1:b-1
    for l=1:a1
      density(k)=density(k)+h/tau*(Subordinator1(l,k)-Subordinator1(l,k+1));
    end
end

% Calculate the density of the inverse stable subordinator
densityStable=zeros(1,b-1);
for k=1:b-1
    for l=1:a1
      densityStable(k)=densityStable(k)+h/tau*(Subordinator2(l,k)-Subordinator2(l,k+1));
    end
end
```

Code for Example 5.3.13: The following code calculates the solution $x \mapsto u(x,1)$ of (5.33) at time $t = 1$ in Example 5.3.13.

```
% Calculate the solution of the semi-fractional Cauchy problem in Example
% 5.3.13
clear all
close all

% Fixed constants
beta=0.8;                                             % Order of semi-fractional time derivative
pV=2*pi;                                              % Period of V
dn=[-1/(40*i),1/30,1/gamma(1-beta),1/30,1/(40*i)];    % Fourier coefficients of V
Testadmissability(beta,dn,pV,0);                      % Test V on admissability
alpha=0.5;                                            % Order of semi-fractional space derivative
pK=pi;                                                % Period of K
cn=[-1/(40*i),1/20,1/gamma(1-alpha),1/20,1/(40*i)];   % Fourier coefficients of K
Testadmissability(alpha,cn,pK,0);                     % Test K on admissability
```

```
% Calculate the density h(x,t) of the inverse semistable subordinator
T=1;          % Point of calculation
h=0.01;       % Step size in space
tau=h;        % Step size in time
y=-1:h:2;
L=20;

% Calculate the density of the semistable subordinator at time t=1
A=Semi_fractional_Differential_Eq(beta,dn,0,pV,y,0,h,tau,L,1,0);
n1=numel(-1:h:0);
n2=numel(-1:h:T);
Subordinator=real(A(n1-1:n2,:));
[a1,b]=size(Subordinator);

% Calculate the density h(x,1) of the inverse semistable subordinator
density=zeros(1,b-1);
for k=1:b-1
    for l=1:a1
      density(k)=density(k)+h/tau*(Subordinator(l,k)-Subordinator(l,k+1));
    end
end

% Calculate the semistable density p
z=-2:h:5;
p=Semi_fractional_Differential_Eq(alpha,cn,0,pK,z,0,h,tau,L,1,0);

% Calculate the solution u(x,1)
u=zeros(numel(z),1);
for m=1:b-1
    u=u+p(:,m)*density(m)*h;
end
```

Code for Example 6.1.11: The following code simulates the sample paths of $S(t)$ and $T(t)$ in (6.11) and the sample paths of the CTRW in Example 6.1.11. For the calculation, we use the fact that for a disturbed Pareto distribution with cumulative distribution function $F(y) = P(X \leq y)$, we have $X \stackrel{d}{=} F^{-1}(U)$, where U is uniformly distributed on $[0,1]$. Note that in our case $F(y) = 1 - \frac{c}{K(0)} y^{-\alpha} K(\log(y))$.

```
% Path simulation in Example 6.1.11
clear all
close all

% Initialize the distribution of X
a=3/2;
c=exp(a);
p1=log(c^(1/a));                   % Period of K
cn=[-1/(2*i),5,1/(2*i)];           % Fourier coefficients of K
Testadmissability(a,cn,p1,0);

% Create a sample of X_1,X_2,...,X_1000
U1=rand(1000,1);
n1=numel(U1);
x=0.01:0.01:200000;
m=numel(x);
y=zeros(1,n1);

 % Calculate the expected value of X
 E1=integral(@(s) s.^(-a).*(a*(5+sin(2*pi*log(s))-2*pi*cos(2*pi*log(s)))),c^(1/a),inf);
 E=E1*c/5;

 % Find the numerical inverse F^(-1)
for k=1:n1
    w=1-c/5*x.^(-a).*(5+sin(2*pi*log(x)))-U1(k);
    i=find(w>=0,1,'first');
    y(k)=x(i)-E;
end
S=cumsum(y);

% Initialize the distribution of J
beta=0.75;
d=exp(2*beta);
p2=log(d^(1/beta));              % Period of V
dn=[1/2,6,1/2];                  % Fourier coefficients of V
Testadmissability(beta,dn,p2,0);

% Create a sample of J_1,J_2,...,J_1000
U2=rand(1000,1);
```

```
n2=numel(U2);
z=zeros(1,n2);

% Find the numerical inverse F^(-1)
for k=1:n2
    w=1-d/7*x.^(-beta).*(6+cos(pi*log(x)))-U2(k);
    i=find(w>=0,1,'first');
    z(k)=x(i);
end
T=cumsum(z);
```

Code for Example 6.2.7: The following code calculates the density of the CTRW limit $x \mapsto m_1(x,1)$ and of the OCTRW limit $x \mapsto o_1(x,1)$ at time $t = 1$ in the totally coupled case described in Example 6.2.7.

```
% Calculate the densities of the OCTRW and CTRW limit in Example 6.2.7.
close all
clear all

% Fixed constants
beta=0.5;
p=2;                                            % Period of V
cn=[1/80,1/40,1/gamma(1-beta),1/40,1/80];       % Fourier coefficients of V
Testadmissability(beta,cn,p,0);
tilded=2*pi/p;

T=1;                                             % Time of calculation
L=30;
h=0.01;                                          % Step size in space
tau=0.01;                                        % Step size in time
x=-1:h:round(exp(p)/h)*h;                        % Points of calculation

% Density g of the semistable subordinator
g=Semi_fractional_Differential_Eq(beta,cn,0,p,x,0,h,tau,L,1,0);
[a,b]=size(g);

% Calculate zeta
x1=-1:h:0;
x2=-1:h:round(exp(p)/h)*h;
n1=numel(x1);
n2=numel(x2);
u=zeros(n2-n1+1,1);
for k=1:b-1
    u=u+tau*1/2*(g(n1:n2,k)+g(n1:n2,k+1));
end

y=0:h:T;
n3=numel(y);

% Density of the CTRW limit
m1=(T-y).^(-beta).*(1/20*cos(2*pi/p*log(T-y))+1/40*cos(4*pi/p*log(T-y))+1/gamma(1-beta)).*u(1:n3)';

% Density of the CTRW limit in the stable case
m1_stabel=1/(gamma(1-beta)*gamma(beta))*(T-y).^(-beta).*y.^(beta-1);

% Calculate the Fourier coefficients of zeta
z=1:h:round(exp(p)/h)*h;
n4=numel(z);
zeta=z.^(1-beta).*u(end-n4+1:end)';
% Initialize the Fourier coefficients of zeta
M=20;
dn=zeros(1,2*M+1);
for l=-M:M
    for k=1:n4-1
        dn(l+M+1)=dn(l+M+1)+1/p*h/2*+(zeta(k)*z(k).^(-1-2*pi*i*l/p)+zeta(k+1)*z(k+1)^(-1-2*pi*i*l/p));
    end
end

% Density of the OCTRW
o1=zeros(1,n4);
k=(numel(cn)-1)/2;
for l=-k:k
    for m=l-M:l+M
        o1=o1-cn(l+k+1)*dn(m-l+M+1)*z.^(i*m*tilded-1).*(i*m*tilded-(T./z).^(i*m*tilded)...
            .*(T./(z-T)).^(beta-i*l*tilded));
    end
end

% Density of the CTRW limit in the stable case
o1_stable=1/(gamma(1-beta)*gamma(beta))*z.^(-1).*(T./(z-T)).^(beta);
```

Code for Figure 8.2: The following code calculates the solution to the semi-fractional exponential model (8.7) and the corresponding fractional solution. For the semi-fractional case, the solution is calculated by (8.5) using the inverse Laplace method of Abate and Whitt in [1]. The function *'LaplaceTransformExp.m'* calculates the Laplace transform of the semi-fractional solution in (8.5).

```
% Calculate solution to the semi-fractional equation

% Fixed constants
alpha=1/2;                              % Order of semi-fractional derivative
cn=[1/4,1/gamma(1-alpha),1/4];          % Periodic perturbation
p=3;                                    % Period of K
V0=1;                                   % Initial value
a=0.2;

% Calculate the inverse Laplace transform
M=1000;                                              % Number of summands
A=6;                                                 % Tuning parameter
s=0.01:0.01:10;                                      % Points of calculation
n1=numel(s);
V=zeros(1,n1);
    for k=1:M
        V=V+(-1)^k*(LaplaceTransformExp((A+2*pi*i*k)./(2*s),alpha,cn,p,a,V0));
    end
Sol_sf=real(V).*exp(A/2)./s+exp(A/2)./(2*s).*real(LaplaceTransformExp(A./(2*s),alpha,cn,p,a,V0));
Sol_f=V0*mlf(alpha,1,a*s.^(alpha));
```

```
% Input parameters
% x          =   Points of calculation
% alpha      =   Order of semi-fractional derivative
% cn         =   Fourier coefficients of K
% p          =   Period of K
% a,V_0      =   Parameters of the model

% Output
% Laplace transform in (8.5) in x

function f=LaplaceTransformExp(x,alpha,cn,p,a,V0)

% Fixed constants
k=(numel(cn)-1)/2;
tildec=2*pi/p;

% Calculate g(x)=x^(alpha)eta_1(log(x))
g=zeros(1,numel(x));
for l=-k:k
    g=g+cn(l+k+1)*ComplexGamma(i*l*tildec-alpha+1)*x.^(alpha-i*l*tildec);
end

f=1./x.*g./(g-a)*V0;
```

Code for Example 8.1.4: The following code fits a fractional as well as a semi-fractional exponential model to the data of mobile web use in Example 8.1.4 using the function *'fminsearchbnd.m'* from [154].

```
% Analyze the percentage of mobile usage in Example 8.1.4
close all
clear all

% Include the data
x=0:1:144;
y=csvread('DataMobileUse2021.csv');

% Analyse the time 01.09-08.14
x1=0:1:67;
y1=y(1:68);

% Fit the fractional parameters
fit_exp_frac=@(a) sum((y1(1)*mlf(a(1),1,a(2)*x1.^(a(1)),10)-y1').^2);
b1_exp=fminsearchbnd(fit_exp_frac,[0.35,0.33],[0,0],[1,100]);
```

```
% Fit the semi-fractional parameters
Start=[-0.8,0.04,0.98,6.68,0.04];
options = optimset('Display','iter');
[b2_exp,fval,exitflag,output]=fminsearchbnd(@MSE_MobileUse_Exp_Model,Start,...
    [-10,-10,0,0,0],[10,10,1,20,2],options);
```

```
% Function to calculate MSE between approximative solution and real data in mobile use data

% Input parameter:
% c     =   Vector containing a version of the Fourier coefficient of the
%           considered perturbation as well as the period as its last entry

% Output
% MSE between mobile use data and calculated solution

function f=MSE_MobileUse_Exp_Model(c)

cn=[c(1)/2-c(2)/(2*i),1/gamma(1-c(3)),c(1)/2+c(2)/(2*i)];
p=c(4);
x1=1:1:67;
y=csvread('DataMobileUse2021.csv');
y1=y(2:68);

A=10;
M=1000;
V=zeros(1,67);
for k=1:M
    V=V+(-1)^k*real(LaplaceTransformExp((A+2*pi*i*k)./(2*x1),c(3),cn,p,c(5),y(1)));
end
 V=real(V).*exp(A/2)./x1+exp(A/2)./(2*x1).*real(LaplaceTransformExp(A./(2*x1),c(3),cn,p,c(5),y(1)));

f=sum((V'-y1).^2);
```

Code for Example 8.1.6: The following code calculates parameters for the fractional and semi-fractional Gompertz model applied to the percentage of mobile web use in Example 8.1.6. As auxiliary functions, the scripts needs the two scripts *'MSE_MobileUse_Gomp_Model.m'* and *'LaplaceTransformGomp.m'*, which calculate the MSE between the semi-fractional Gompertz model and the Laplace transform of the solution to the semi-fractional Gompertz equation.

```
% Analyze the percentage of mobile usage in Example 8.1.6
close all
clear all

% Include the data
x1=0:1:144;
y1=csvread('DataMobileUse2021.csv');
m=116;
x=x1(1:m);
y=y1(1:m);

% Fit the fractional parameters for Gompertz model
fit_gomp_frac=@(a) sum((y(1)*exp(a(1)/a(2)*(1-mlf(a(3),1,-a(2)*x.^(a(3)),10)))-y').^2);
start=[0.14,0.03,0.92];
b1_gomp=fminsearchbnd(fit_gomp_frac,start,[0,0,0],[10,10,1]);

% Fit the parameters for the semi-fractional Gompertz model
options = optimset('Display','iter');
Start=[-0.0155,0.7434,0.5884,4.6994,0.3897,0.2];
[b2_gomp,fval,exitflag,output]=fminsearchbnd(@MSE_MobileUse_Gomp_Model,Start,...
    [-10,-10,0,0,0,0],[10,10,1,10,10,10],options)
```

```
% Function to calculate MSE in mobile use data

% Input parameter:
% c     =   Vector containing a version of the Fourier coefficient of the
%           considered perturbation as well as the period as its last entry

% Output
% MSE between mobile use data and calculated solution
```

```
function f=MSE_MobileUse_Gomp_Model(c)
cn=[c(1)/2-c(2)/(2*i),1/gamma(1-c(3)),c(1)/2+c(2)/(2*i)];
p=c(4);

% Include the data
m=116;
x1=1:1:m-1;
y=csvread('DataMobileUse2021.csv');
y1=y(2:m);

A=10;
M=1000;
V=zeros(1,m-1);
for k=1:M
     V=V+(-1)^k*real(LaplaceTransformGomp((A+2*pi*i*k)./(2*x1),c(3),cn,p,c(5),c(6)));
end
 V=real(V).*exp(A/2)./x1+exp(A/2)./(2*x1).*real(LaplaceTransformGomp(A./(2*x1),c(3),cn,p,c(5),c(6)));

 u=y(1)*exp(V);
f=sum((u'-y1).^2);
```

Code for Example 8.1.8: The following code calculates parameters for the fractional and semi-fractional Gompertz model applied to tumor growth in Example 8.1.8.

```
% Analyze the tumor growth data in Example 8.1.8
close all
clear all

% Include the data and calculate the volume
z=csvread('MouseCM37T2.csv');
x=z(1,:);
H=1/3*(z(2,:)+z(3,:)+z(4,:));
L=1/3*(z(5,:)+z(6,:)+z(7,:));
Volume=pi/6*(H.*L).^(3/2);

% Fit the fractional parameters
fit_gomp_frac=@(a) sum((Volume(1)*exp(log(a(1)/Volume(1))*(1-mlf(a(3),1,-a(2)*x.^(a(3)),10)))...
    -Volume).^2);
b1_gomp=fminsearchbnd(fit_gomp_frac,[307,0.02,0.98],[0,0,0],[2000,10,1])
alpha_gomp=b1_gomp(3)
Dif=b1_gomp(2)
Vinfty=b1_gomp(1)

% Calulate the semi-fractional solution
Start=[-0.4123    0.4934    0.9999    1.8589    0.0290   269.4657];
[b2_gomp,fval,exitflag,output]=fminsearchbnd(@MSE_TumorGrowth_Gomp_Model,Start,...
    [-20,-20,0,0,0,0],[10,10,1,40,10,2000],options);
```

```
% Function to calculate MSE in tumor growth data

% Input parameter:
% c    =  Vector containing a version of the Fourier coefficient of the
%         considered perturbation as well as the period as its last entry

% Output
% MSE between tumor growth data and calculated solution

function f=MSE_TumorGrowth_Gomp_Model(c)

cn=[c(1)-c(2)/i,1/gamma(1-c(3)),c(1)+c(2)/i,c(4)];

% Include the data and calculate the volume
z=csvread('MouseCM37T2.csv');
x=z(1,2:end);
H=1/3*(z(2,:)+z(3,:)+z(4,:));
L=1/3*(z(5,:)+z(6,:)+z(7,:));
Vol=pi/6*(H.*L).^(3/2);
Volume=Vol(2:end);

A=10;
M=1000;
x2=x;
V=zeros(1,numel(x2));
for k=1:M
        V=V+(-1)^k*real(LaplaceTransformGomp_AltRep((A+2*pi*i*k)./(2*x2),c(3),cn,c(5),c(6),Vol(1)));
end
 V=real(V).*exp(A/2)./x2+exp(A/2)./(2*x2).*real(LaplaceTransformGomp_AltRep(A./(2*x2),c(3),...
    cn,c(5),c(6),Vol(1)));

 u=c(6)*exp(-V);
```

```
f=sum((u-Volume).^2);
```

```
% Function to calculate the Laplace transform of the solution to the
% semi-fractional Gompertz model in the alternative representation

% Input parameters
% x             =   Points of calculation
% alpha         =   Order of semi-fractional derivative
% cn            =   Vector of Fourier coefficients of K
% Dif, Vinfty   =   Parameters of the model
% V0            =   Initial value

% Output
% Laplace transform in x


function f=LaplaceTransformGomp_AltRep(x,alpha,cn,Dif,Vinfty,V0)

% Fixed constants
k=(numel(cn)-2)/2;
p=cn(end);
tildec=2*pi/p;

L=zeros(1,numel(x));
for l=-k:k
    L=L+cn(l+k+1)*ComplexGamma(i*l*tildec-alpha+1)*x.^(alpha-i*l*tildec);
end

f=1./x.*L./(L+Dif)*log(Vinfty/V0);
```

Code for Example 8.2.8: The following code calculates parameters for the (tempered) Pareto tail as well as the tempered disturbed Pareto tail in Example 8.2.8.

```
% Code for Example 8.2.7
close all
clear all

% Include the data
y=csvread('AMZN1.csv');
n=5682;

% Calculate the largest observations
Change=abs(y(2:n,4)-y(1:n-1,4))./(y(1:n-1,4))*100;      % Daily price changes
z=sort(Change);                                         % Sorted daily price changes
z(diff(z)==0)=[] ;
m=numel(z);
Prob=zeros(1,m);
for k=1:m
    u=find(Change>z(k));
    Prob(k)=numel(u)/(n-1);
end

% Split the tail
Tail=z(m-50:end);
ProbTail=Prob(m-50:end);

% Fit the best Pareto
fit_pareto=@(a) sum((a(1)*Tail.^(-a(2))-ProbTail').^2);
b1_pareto=fminsearch(fit_pareto,[12,3.7]);

% Fit the tempered Pareto
fit_pareto_temp=@(a) sum((a(1)*Tail.^(-a(2)).*exp(-a(3)*Tail)-ProbTail').^2);
Start1=[1.1     1.99     0.2];
b1_pareto_temp=fminsearch(fit_pareto_temp,Start1);

% Fit the tempered disturbed Pareto
Start2=[1.5     4    0     0     1     0.1];
b2_pareto_temp=fminsearchbnd(@MSE_Tempered,Start2,[],[2,10,10,10,10,10]);
```

```
% Function to calculate MSE for the tempered semistable density

% Input parameter:
% c     =   Vector containing alpha, a version of the Fourier coefficient of the
%           considered perturbation as well as the period as its last entry
```

```
% Output
% MSE between stock data and calculated solution


function f=MSE_Tempered(c)
alpha=c(1);
cn=[c(4)/2-c(3)/(2*i),c(2),c(4)/2+c(3)/(2*i)];
p=c(5);

y=csvread('AMZN1.csv');
n=5682;
Change=abs(y(2:n,4)-y(1:n-1,4))./(y(1:n-1,4))*100;

z=sort(Change);
z(diff(z)==0)=[] ;
m=numel(z);

Prob=zeros(1,m);
for k=1:m
    u=find(Change>z(k));
    Prob(k)=numel(u)/(n-1);
end

% Split the tail
Tail=z(m-50:end);
ProbTail=Prob(m-50:end);

f=sum((exp(-c(6)*Tail).*Tail.^(-c(1)).*(c(2)+c(3)*sin(log(Tail)*2*pi/c(5))...
        +c(4)*cos(log(Tail)*2*pi/c(5)))-ProbTail').^2);
```

Eidesstattliche Erklärung

Ich versichere an Eides statt, dass die Dissertation von mir selbstständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist. Die Dissertation wurde in der vorgelegten oder ähnlicher Form noch bei keiner anderen Institution eingereicht. Ich habe bisher keine erfolglosen Promotionsversuche unternommen.

Düsseldorf, den 30.03.2021

Svenja Lage