On the Analysis of a Model on the Mechanisms of Tropical Storms Coupled to Moisture Dynamics

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Zusammenfassung

Mathematische Modelle spielen im Bereich der Wettervorhersage eine immer größere Rolle. Um diese Modelle bei der Berechnung von Vorhersagen anwenden zu können muss zuvor verifiziert werden, dass sie eindeutig lösbar sind. Aus diesem Grund beweisen wir in dieser Arbeit die Existenz und Eindeutigkeit von zeitlich lokalen, starken Lösungen eines Modells zur Beschreibung von Luftströmungen, die in tropischen Stürmen beobachtet werden.

Dieses System berücksichtigt Geschwindigkeit, Temperatur, Druck und Feuchtigkeitsentwicklung tropischer Stürme indem es Impuls-, Masse- und Energieerhaltung nutzt und mit nichtlinearen Feuchtigkeitsdynamiken koppelt. Es ist auf einem Zeitintervall und einem beschränkten, zylindrischen Gebiet definiert. Um die Lösbarkeit des Modells zu zeigen, linearisieren wir es geeignet und beweisen maximale L_p-Regularität für das linearisierte Modell.

Das linearisierte Modell besteht aus Stokes-Gleichungen mit Free-Slip-Randbedingungen und variablen Koeffizienten und parabolischen Gleichungssystemen mit Robin-Randbedingungen. Eine weitere Schwierigkeit ergibt sich aus der Tatsache, dass alle diese Systeme variable Koeffizienten enthalten, d. h. sie hängen von den räumlichen Komponenten ab. In dieser Arbeit untersuchen wir die Stokes-Gleichungen mit Perfect-Slip- und Free-Slip-Randbedingungen, sowie parabolische Problemen mit Robin-Randbedingungen, Neumann-Dirichlet Randbedingungen, Perfect-Slip- und Free-Slip-Randbedingungen. Wir zeigen die maximale L_p-Regularität all dieser Systeme mit variablen Koeffizienten in zylindrischen Gebieten. Dafür benötigen wir die Retraktionseigenschaft der beteiligten Spuroperatoren für Sobolev- und Bessel-Potentialräume in zylindrischen Gebieten, die wir detailliert in dieser Arbeit beweisen. Außerdem verwenden wir ein Lokalisierungsargument um die maximale L_p-Regularität von Stokes-Gleichungen mit konstanten Koeffizienten in zylindrischen Gebieten auf solche mit variablen Koeffizienten zu übertragen.

Wir nutzen die Theorie der anisotropen Sobolev- und Bessel-Potentialräume um optimale Abschätzungen für die nichtlinearen Terme unseres Modells zu erhalten. Dies führt in Kombination mit dem Fixpunktsatz von Banach und der maximalen L_p-Regularität des linearisierten Modells zur Existenz und Eindeutigkeit von zeitlich lokalen, starken Lösungen des vollständigen Modells mit optimalen Grenzen für den Integrationsparameter p.

Summary

Mathematical models play an increasingly important role in the field of weather forecasting. In order to use these models for the calculation of predictions, their solvability has to be verified first. For this reason, in this thesis we show the existence and uniqueness of local-in-time, strong solutions to a model describing the air flow observed in tropical storms.

This model takes velocity, temperature, pressure and moisture ratios into account by using the conservation of momentum, mass and energy, and coupling them to nonlinear moisture dynamics. It is posed on a time interval and a bounded, cylindrical domain. In order to show solvability of the model, we linearise it suitably and prove maximal L_p-regularity for the linearised model.

The linearised model is composed of the Stokes equations with free slip boundary conditions, and parabolic systems with Robin boundary conditions. Another difficulty is given by the fact that all these systems contain variable coefficients, i. e., they depend on the spatial components. In this thesis we study the Stokes equations with perfect slip and free slip boundary conditions, as well as parabolic problems with Robin boundary conditions, Neumann-Dirichlet boundary conditions, perfect slip and free slip boundary conditions. We show maximal L_p-regularity of all these systems with variable coefficients in cylindrical domains. In order to do so, we need the retraction property of the involved trace operators for Sobolev and Bessel potential spaces in cylindrical domains, which we prove in detail in this thesis. Moreover, we use a localisation argument to translate the maximal L_p-regularity of the Stokes equations in cylindrical domains with constant coefficients to those with variable coefficients.

We use the theory of anisotropic Sobolev and Bessel potential spaces to obtain optimal estimates for the nonlinear terms of our model. In combination with the Fixed-Point Theorem of Banach and the maximal L_p-regularity of the linearised model this allows us to show existence and uniqueness of local-in-time, strong solutions to the full model in an L_p-setting with optimal restrictions on the integrability parameter p.

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Contents

Introduction

We cannot direct the wind.
But we can adjust the sails.

Aristotle

Idai, Haiyan and Katrina. These names are associated with the most devastating tropical storm of recent years. The cyclone Idai was causing over 700 casualties in Mozambique and Zimbabwe in 2019 [9], the typhoon Haiyan in the Philippines in 2013 over 6400 casualties [14], and hurricane Katrina in the Caribbean and the East of the United States in 2005 about 1800 casualties [22]. All of these events have one thing in common: they all demonstrate the destructive power of winds. But how can such catastrophes be prevented in the future?

The observation of natural phenomena dates back to the beginning of human history. From Aristotle, who studied the dynamic of winds as early as 300 B.C., via Kepler, who in the early 17th century used observation to deduce laws governing the motion of planets around the sun, to the present day. In order to be able to describe natural phenomena, particularly Kepler's laws of planetary motion, Newton developed infinitesimal calculus in the middle of the 17th century. Independently of Newton, Leibniz, using a geometric approach instead, also developed this calculus at about the same time. From a today's perspective, infinitesimal calculus can be regarded as a precursor to differential calculus and modern analysis. Almost two-hundred years later, scientists such as Navier and Stokes have, independently of each other, been able to describe the inner friction of fluids using differential equations. The Navier-Stokes equations are a mathematical model to describe the dynamics of viscous Newtonian fluids and gases. Mathematical models are nowadays a popular tool for the description of phenomena in physics. Through the use of methods from calculus such models can be analysed and statements about their solvability can be made. This makes it possible to decide whether it is worthwhile to use numerical methods to approximate precise solutions of the model, in order to obtain predictions for real world events.

As Aristotle once said, we cannot direct the wind, but with the help of more precise predictions we may be able to better judge where and with which impact tropical storms may occur, in order to soften the consequences of catastrophes such as those caused by hurricane Katrina, that is, "to adjust the sails". In order to make better predictions concerning tropical storms, it is worthwhile to investigate mathematical models in more depth. Nolan and Montgomery [44] utilised the fact that the behaviour of air and water can be described using Navier-Stokes equations, and developed the following mathematical model in order to describe the dynamics of tropical storms:

$$\begin{aligned}
\rho\partial_t u + \rho(u\cdot\nabla)u - \epsilon\rho\Delta u - \epsilon\rho\nabla\mathrm{div}u + \nabla q &= \rho\tfrac{\bar\theta-\theta}{\bar\theta}\nabla F - \omega e_3\times\rho u && \text{in } J\times\Omega,\\
\mathrm{div}(\rho u) &= 0 && \text{in } J\times\Omega,\\
\rho\partial_t\theta + \rho(u\cdot\nabla)\theta - \epsilon\rho\Delta\theta &= \rho(u\cdot\nabla)\bar\theta - \kappa\Delta\bar\theta && \text{in } J\times\Omega,\\
u\cdot\nu = h_u\cdot\nu,\quad \beta^u P_\Gamma D_+(u)\nu &= P_\Gamma h_u && \text{on } J\times\Gamma,\\
\beta^\theta\partial_\nu\theta + \sigma^\theta\theta &= h_\theta && \text{on } J\times\Gamma,\\
u(0) &= u_0 && \text{in } \Omega,\\
\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned}$$

The model introduced by them mainly consists of extended Navier-Stokes equations (first two equations), which are coupled to the heat equation (third equation). The density ρ is assumed to be a given, time-independent positive function, ϵ denotes a constant eddy viscosity, and $\bar\theta$ denotes a given mean value. The model describes velocity u, pressure q and temperature θ of a tropical storm using conservation of momentum

$$\rho\left(\partial_t u + (u\cdot\nabla)u\right) = \mathrm{div}(T - \mathrm{Id}\, q) + G,$$

conservation of mass

$$\mathrm{div}(\rho u) = \partial_t\rho = 0,$$

and conservation of energy

$$\rho\left(\partial_t(\theta-\bar\theta) + (u\cdot\nabla)(\theta-\bar\theta)\right) = \epsilon\rho\Delta(\theta-\bar\theta).$$

Within the equation for the conservation of momentum, the term for the acceleration $\partial_t u + (u\cdot\nabla)u$ is contrasted with the inner friction $\mathrm{div}(T - \mathrm{Id}\, q)$, where T is the stress tensor. In the model by Nolan and Montgomery, the inner friction is represented by the term $\epsilon\rho\Delta u - \epsilon\rho\nabla\mathrm{div}\, u + \nabla q$. The function G represents the external forces, which in the model are given by the Coriolis force $-\omega e_3\times\rho u$ and the buoyancy $\rho\frac{\bar\theta-\theta}{\bar\theta}\nabla F$. In [49] Saal has shown existence and uniqueness of solutions to this model in the Hilbert-space setting. This leads to the question whether it is also possible to prove existence and uniqueness of solutions to this model in the general L_p-setting and whether improvements to the model are possible. In a next step, this could motivate the numerical investigation of this model in order to find approximate solutions which may in turn improve predictions for tropical storms. However, it is not easy to see whether the model is thermodynamically consistent. Furthermore, the model does not take moisture dynamics into account, which however are known to be a major influence on tropical storms in terms of size and intensity, as shown for instance by the works of Hill and Lackmann [26], as well as Wu, Su, Fovell, Dunkerton, Wang and Kahn [56]. This is why we decided to modify the model of Nolan and Montgomery. We extensively touch upon the modifications of the resulting model in Section 4.1. We slightly adapt the coefficients of the model of Nolan and Montgomery to the setting of Novotný, Růžička and Thäter [45], as well as add the term

$$\frac{p_0}{\bar\theta}\mathrm{div}(\rho u F),$$

which models the creation of heat through volume work. This ensures thermodynamic consistency within the model. Part of these modifications are the substitution of constant coefficients by variable ones, i. e., all relevant coefficients such as viscosity, density, etc. are replaced by given, positive functions. In order to appropriately incorporate moisture, we couple the modified system of Nolan and Montgomery to the following nonlinear moisture dynamics

$$
\begin{aligned}
\partial_t m_v + (u\cdot\nabla)m_v - \eta_v\Delta m_v - S_{ev} + S_{cd} &= 0 && \text{in } J\times\Omega,\\
\partial_t m_c + (u\cdot\nabla)m_c - \eta_c\Delta m_c - S_{cd} + S_{ac} + S_{cr} &= 0 && \text{in } J\times\Omega,\\
\partial_t m_r + (u\cdot\nabla)m_r - \eta_r\Delta m_r - S_{ac} - S_{cr} + S_{ev} &= \tfrac{V}{g\rho_m}\mathrm{e}_3\cdot\nabla(\rho_m m_r) && \text{in } J\times\Omega,\\
\beta^{m_v}\partial_\nu m_v + \sigma^{m_v} m_v &= h_v && \text{on } J\times\Gamma,\\
\beta^{m_c}\partial_\nu m_c + \sigma^{m_c} m_c &= h_c && \text{on } J\times\Gamma,\\
\beta^{m_r}\partial_\nu m_r + \sigma^{m_r} m_r &= h_r && \text{on } J\times\Gamma,\\
m_v(0) = m_{v,0},\quad m_c(0) = m_{c,0},\quad m_r(0) &= m_{r,0} && \text{in } \Omega,
\end{aligned}
$$

which were introduce by Hittmeir, Klein, Li and Titi in [27]. The model of nonlinear moisture dynamics by Hittmeir, Klein, Li and Titi does not only describe the moisture dynamics with respect to rain water (m_r), but also the moisture dynamics with respect to vapour (m_v) and cloud water (m_c). Through the coupling of these models we obtain a system that unifies the temporal and spatial description of motion, pressure, temperature and moisture.

The goal of this thesis is to provide a proof of existence and uniqueness of local-in-time, strong solutions to the above model on a cylinder (Theorem 4.2). That is, we want to show solvability of a system of partial differential equations consisting of the Navier-Stokes equations, the heat equation, and nonlinear moisture dynamics, with variable coefficients on a cylindrical domain. Here, a cylindrical domain Ω refers to a Cartesian product consisting of a bounded $\mathcal{C}^3$-domain A and an interval $(-a, a)$ with $a > 0$, i. e.

$$\Omega := A \times (-a, a).$$

We study the model on a time interval $(0, T)$ with $T > 0$, and on a cylindrical domain Ω, because we are interested in both the temporal, as well as the spatial dynamics of tropical storms. We have decided to use a cylinder as our model domain, because its geometry suits the shape of tropical storms, such as tornadoes, very well. One could also model tropical storms on the upper half plane, which might serve as a simplified version of the surface of the earth. With equal right, one could also model tropical storms on a sphere, serving as an approximation to earth as a whole. However, these two domains go beyond the scope of the present thesis, and are left for future research. For the purpose of proving solvability of our model, we reduce it to a linear system using linearisation techniques, such as the ones presented in [6] by Amann. This linearised system contains all linear terms of highest order and takes on the following form:

$$\begin{aligned} \rho\partial_t u - \epsilon\rho\Delta u + \nabla q &= f_u && \text{in } J\times\Omega,\\ \operatorname{div}(\rho u) &= 0 && \text{in } J\times\Omega,\\ u\cdot\nu &= h_u\cdot\nu && \text{on } J\times\Gamma,\\ \beta^u P_\Gamma D_+(u)\nu &= P_\Gamma h_u && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

$$\begin{aligned} \rho\partial_t\theta - \epsilon\rho\Delta\theta &= f_\theta && \text{in } J\times\Omega,\\ \beta^\theta\partial_\nu\theta + \sigma^\theta\theta &= h_\theta && \text{on } J\times\Gamma,\\ \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned}$$

$$\begin{aligned} \partial_t m_j - \eta_j\Delta m_j &= f_j && \text{in } J\times\Omega,\\ \beta^{m_j}\partial_\nu m_j + \sigma^{m_j} m_j &= h_j && \text{on } J\times\Gamma,\\ m_j(0) &= m_{j,0} && \text{in } \Omega, \end{aligned}$$

where the last three equations are repeated independently for every $j \in \{v, c, r\}$. The linearised model is thus composed of the Stokes equations with variable coefficients, as well as of various parabolic problems with Robin boundary conditions and variable coefficients, each being defined on a cylindrical domain Ω.

Boundary value problems on cylindrical domains have for instance been investigated by Nau, Saal and Denk [39, 40, 43, 42, 17]. However, all of these works only studied boundary value problems with constant not variable coefficients. First investigations of the Stokes equations with variable coefficients in L_p go back to Abels and Terasawa [1, 2]. Moreover, Abels and Weber [3] analysed the inhomogeneous Navier-Stokes equations with variable density. On the other hand, the Stokes equations with constant coefficients have been the subject of many studies. Miyakawa [38], Giga [24], Shibata and Shimizu [51] have been the first to approach the Stokes equations with first-order boundary conditions in L_p in a rigorous mathematical way. Some investigations of the Stokes equations with Robin boundary conditions may be found in Saal [48, 47], Shibata and Shimada [50, 52]. For further investigations about the Stokes equations see [11, 19, 23], and for a detailed overview of the Stokes equations in the L_p-setting we refer to [25].

Because both the Stokes equations, as well as parabolic systems with variable coefficients in cylindrical domains, have attracted little attention in the literature, we dedicate the first part of this thesis (Chapters 1–3) to the development of an L_p-theory of such boundary value problems on cylindrical domain. In order to make such a theory valid not only for constant but also variable coefficients, we adapt a similar strategy as Denk, Hieber and Prüss in [15, Therorem 5.7]. They used a localisation argument to transfer maximal regularity of elliptic operators in Banach spaces of class $\mathcal{HT}$ with constant coefficients to the same operators with variable coefficients. In the second part of this thesis (Chapter 4), we describe the model on tropical storms given above more comprehensively, and rigorously show its solvability. This is done by showing that the aforementioned linearised model, i. e. the resulting Stokes equations and parabolic problems, admit a unique solution. To this end, we make use of results obtained in Sections 3.2 and 2.2, respectively. With the help of a perturbation argument, we are able to add the nonlinear terms, as well as the linear terms of lower order as perturbations to the linearised model. The existence of a unique solution to our model can then be obtained using the Fixed-Point Theorem of Banach. In this thesis we study our model on cylindrical domains, and show solvability for arbitrary data and small time intervals. The investigation of the model's behaviour on different domains, and its solvability for arbitrary time intervals and small data are left for future research.

This thesis is organised as follows.

Chapter 1 starts by introducing general notation and basic function spaces. There, we establish most of the mathematical notions that are used in later chapters, including Banach spaces of class $\mathcal{HT}$, property (α), $\mathcal{R}$-boundedness and maximal regularity. Moreover, we give a brief introduction to cylindrical domains and the Helmholtz projection. This chapter includes a first overview of parabolic problems and the Stokes equations, as well as of different boundary conditions, which are later discussed in Chapters 2 and 3. We also present a proof of the retraction property of trace maps with respect to the aforementioned boundary conditions.

Chapter 2 studies elliptic and parabolic problems with variable coefficients on cylindrical domains. More precisely, here, we investigate elliptic problems with Neumann boundary conditions, both in the case of time-dependent, as well as time-independent data. Furthermore, we prove maximal regularity for parabolic problems with Robin boundary conditions, as well as Neumann-Dirichlet boundary conditions, perfect slip boundary conditions and free slip boundary conditions. Maximal regularity of parabolic problems with Robin boundary conditions and variable coefficients prove to be particularly useful in Chapter 4 to show solvability of our model describing the dynamics of tropical storms. The results in this chapter allow us to extend the L_p-theory for cylindrical boundary value problems with constant coefficients to such problems with variable coefficients.

Chapter 3 extents the investigation of the previous chapter by studying the Stokes equations with variable coefficients on cylindrical domains. Here, we prove maximal regularity for the Stokes equations with perfect slip and free slip boundary conditions, thereby complementing the L_p-theory of cylindrical boundary value problems with variable coefficients as developed in Chapter 2.

Chapter 4 studies in detail the model on the mechanisms of tropical storms that was already briefly described in this introduction. Moreover, we prove existence and uniqueness of local-in-time, strong solutions to this model by using the results of Chapters 2 and 3. Put in another way, we make use of the maximal regularity of parabolic problems with Robin boundary conditions and of the maximal regularity of the Stokes equations with free slip boundary conditions. Finally, we use a perturbation argument in order to prove solvability for this model for arbitrary data and small time intervals.

Cylindrical Domains

1 Preliminaries

The purpose of this chapter is to recall and define terms and concepts that are frequently used throughout this thesis. This is also to ensure that during the study of this thesis these concepts are able to be found in one place such that the flow of reading may not be disturbed. We also give a first introduction to the problems discussed in the first part of the thesis.

In Section 1.1 we give an overview of the general notation used throughout this thesis. We also introduce function spaces, reminding the reader of their important properties such as property α and the class $\mathcal{HT}$. In Section 1.2 we introduce important concepts of operators such as sectoriality, $\mathcal{R}$-boundedness and maximal regularity. Especially the concept of maximal regularity plays an important role throughout this thesis. Since we exclusively focus on problems on cylindrical domains, we devote Section 1.3 entirely to their introduction. This includes a discussion of their boundary as well as the behaviour of the Helmholtz projection on cylindrical domains. In Section 1.4 we give a first overview of problems later on discussed in Chapters 2 and 3, i. e. boundary conditions used in the context of these problems, as well as necessary regularity and compatibility conditions. Finally, in Section 1.5 the presentation gets somewhat more formal as we show that trace maps with respect to the boundary conditions, introduced in Section 1.4, are retractions.

1.1 Essentials

In the first part of this section we collect some basic definitions and notations which are used throughout this thesis. The second part of this section is meant to serve as a brief reminder of some function spaces, such as L_p-spaces, and of some of their important properties such as property α and the class $\mathcal{HT}$. The remainder of this section is arranged into two paragraphs: an introduction to isotropic function spaces followed by an introduction to anisotropic function spaces.

General Notation

This paragraph is devoted to an explanation of notations used throughout this thesis. Among other things, we introduce the divergence, the gradient and the Laplacian of matrices and vector fields. In addition, we also review the definition of Fréchet differentiability. Furthermore, normalized vector spaces and their dual spaces are also recalled.

As usual, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural, real and complex numbers, respectively. For any natural number $n \in \mathbb{N}$, we then denote by $\mathbb{N}^n$, $\mathbb{R}^n$ and $\mathbb{C}^n$ the corresponding n-dimensional natural, real and complex space, respectively. As we adapt the standpoint that the natural numbers do not contain zero, we introduce the additional set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for distinguishability. For any vector $x \in \mathbb{R}^n$ or matrix $S \in \mathbb{R}^{n\times n}$ their transposed versions are denoted by x^T and S^T, respectively. Likewise x_j, for $j \in \{1, \ldots, n\}$, and $S_{j,k}$, for $j,\ k \in \{1, \ldots, n\}$, denote the components of x and S, respectively. Furthermore, for any two vectors $x,\ y \in \mathbb{R}^n$, $x \cdot y := \sum_{j=1}^{n} x_j y_j$ indicates their inner product. We also abbreviate the element-wise product of two matrices $S,\ T \in \mathbb{R}^{n\times n}$ by $S : T := \sum_{j,k=1}^{n} S_{j,k}T_{j,k}$. Moreover, $x \perp y$ indicates that vector x is orthogonal to vector y. Additionally, for any real function α, by $(\alpha)^+ := \max\{0, \alpha\}$ we denote the positive part.

If it is clear from the context, $e_j \in \mathbb{R}^n$ denotes for $j \in \{1, \ldots, n\}$ the entire j-th real unit vector, whose components are zero, except for the one at j-th position, which is one. Then, $\mathrm{Id} = (e_1, \ldots, e_n)$ denotes the identity matrix in $\mathbb{R}^{n\times n}$. We also denote the identity mapping between normed vector spaces by Id, since this is unlikely to lead to misconception.

For a normed, real vector space $(X, \|\cdot\|_X)$ we set X' to be the related dual space and $\langle \cdot \mid \cdot \rangle$ to be the dual pairing. For two Banach spaces X and Y we write $X \doteq Y$, if X and Y are identical up to equivalence of norms. By $\mathcal{L}(X,Y)$ we denote the space of all linear continuous operators from the normed space X to the normed space Y and by $\mathcal{L}_{\mathrm{is}}(X,Y)$ the space of all isomorphisms. For $X = Y$, we use the abbreviations $\mathcal{L}(X)$ and $\mathcal{L}_{\mathrm{is}}(X)$. We use $\mathfrak{R}(T)$ for the range and $\mathfrak{N}(T)$ for the kernel of an operator $T \in \mathcal{L}(X,Y)$. Furthermore, by $\|T\|_{X\to Y}$ we denote the operator norm in $\mathcal{L}(X,Y)$, which we sometimes also may write as $\|T\|_{\mathcal{L}(X,Y)}$. We use $\rho(T)$ for the resolvent set of an operator T. In general, $D(T)$ we denote the domain of an operator T.

The divergence of a continuously differentiable vector field $u\colon D \longrightarrow \mathbb{R}^n$, with $D \subseteq \mathbb{R}^n$ an open subset, is denoted by $\mathrm{div}(u)$. For a matrix $S\colon D \longrightarrow \mathbb{R}^{n\times n}$ with columns $s_1, \ldots, s_n\colon D \longrightarrow \mathbb{R}^n$, such that for $x \in D$ the identity $S(x) = (s_1(x), \ldots, s_n(x))^T$ holds true, we write

$$\mathrm{div}(S) := \mathrm{div}(s_1, \ldots, s_n) = (\mathrm{div}(s_1), \ldots, \mathrm{div}(s_n))^T$$

for the divergence. Throughout this thesis, the gradient of a differentiable function $f\colon D \longrightarrow \mathbb{R}$ is denoted by

$$\nabla f := (\partial_1 f, \ldots, \partial_n f)^T$$

and the gradient of a continuously differentiable vector field $u\colon D \longrightarrow \mathbb{R}^n$ by

$$\nabla u := \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_1}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}.$$

Then, the directional derivative of the vector field u is written as $(\nabla u)^T h =: \partial_h u$ for $h \in \mathbb{R}^n$, and the Laplacian as $\Delta u := \mathrm{div}(\nabla u)$. We use the notation $u_{|A}$ to restrict the domain of a vector field $u\colon D \longrightarrow \mathbb{R}^n$ to an open subset $A \subseteq D$. In addition, by $\mathrm{supp}(u)$ we denote the support of the vector field u, which is the closure in D of the set of points in D where u is non-zero.

Let X and Y be normed vector spaces and $U \subseteq X$ an open subset. We call a function $A\colon U \longrightarrow Y$ Fréchet differentiable at $x \in U$, if there exists a continuous linear operator $B \in \mathcal{L}(X,Y)$, such that

$$\lim_{\|h\|\to 0} \frac{\|A(x+h) - A(x) - Bh\|_Y}{\|h\|_X} = 0.$$

We denote the Fréchet derivative of A at point $x \in U$ by $DA(x) := B$. If the derivative exists at all points $x \in U$, we call the resulting function $DA\colon U \longrightarrow \mathcal{L}(X,Y)$ the Fréchet derivative of A, which maps $x \longmapsto DA(x)$.

The closure, the interior and the boundary of a set G are denoted by $\bar{G}$, G° and ∂G, respectively. Moreover, $G \cup H$, $G \dot{\cup} H$ and $G \cap H$ stand for the union, the disjoint union and the intersection of two sets G and H, respectively. An example of a set, which is often used in this thesis, is the open ball $B_r(z)$. It is constructed with respect to the Euclidean norm with radius $r > 0$ and center point z. The Euclidean norm on $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ are be denoted by $|\cdot|$. In accordance with the notation above, $\bar{B}_r(z)$ denotes the closure of the ball $B_r(z)$.

We also make frequent use of positive constants in estimations or equations. For better readability when used in chains of inequalities, we denote all positive constants in each estimate of the chain by C, whenever the actual value of the constant is not important. If the actual values are important or should be emphasised, we indicate them with their values or primes, e. g. C', C'', ... and so on.

Function Spaces

Throughout this thesis, we work with many different function spaces, whose definition we briefly recall here.
Let $D \subseteq \mathbb{R}^n$ be an open subset with dimension $n \in \mathbb{N}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $l \in (0,1]$ and $m \in \mathbb{N}_0$. Moreover, assume $x \in \mathbb{R}^n$ and $j \in \mathbb{N}_0^n$ to be a multi-index with $|j| := \sum_{i=1}^n j_i$ and $\partial_x^j := \partial_{x_1}^{j_1} \dots \partial_{x_n}^{j_n}$. Then we have

$$\begin{aligned}
\mathcal{C}^k(D) &:= \{f \colon D \longrightarrow \mathbb{R} : f \text{ is k-times continuously differentiable}\},\\
\mathcal{C}^{k,\lambda}(D) &:= \left\{f \in \mathcal{C}^k(D) : \partial_x^j f \text{ is Hölder continuous with exponent } \lambda \text{ for all } |j| = k\right\},\\
\mathcal{BC}^k(D) &:= \left\{f \in \mathcal{C}^k(D) : \sup_{x\in D}|\partial_x^j f(x)| < \infty \text{ for all } |j| \leqslant k\right\},\\
\mathcal{BUC}(D) &:= \left\{f \in \mathcal{BC}^0(D) : f \text{ is uniformly continuous on } D\right\},\\
\mathcal{BUC}^m(D) &:= \left\{f \colon D \longrightarrow \mathbb{R} : \partial_x^j f \in \mathcal{BUC}(D),\ |j \leqslant m\right\}.
\end{aligned}$$

Each of these function spaces can also be restricted to include functions with compact support only, and we denote them by

$$\begin{aligned}
Y_c(D) &:= \{f \in Y(D) : \mathrm{supp}(f) \text{ is a compact subset of } D\},\\
Y_c(\bar{D}) &:= \{f_{|D} : f \in Y_c(\mathbb{R}^n)\},
\end{aligned}$$

for $Y \in \{\mathcal{C}^k, \mathcal{C}^{k,\lambda}, \mathcal{BUC}, \mathcal{BUC}^m\}$. We set

$$\mathcal{C}^\infty_{c,\sigma}(D) := \{u \in \mathcal{C}_c^\infty(D)^n : \mathrm{div}(u) = 0\},$$

and define the space of solenoidal functions as

$$L_{p,\sigma}(D) := \overline{\mathcal{C}^\infty_{c,\sigma}(D)},$$

where the closure is taken in $L_p(D, \mathbb{R}^n)$, for $1 < p < \infty$.
Throughout this thesis, vector-valued L_p-spaces are frequently used. In order to define them, let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let E be a Banach space. Then, for $1 \leqslant p < \infty$,

we define

$$\begin{aligned}
\mathfrak{L}_p(\mu;E) &:= \{u\colon \Omega \longrightarrow E \text{ measurable} : \|u\|_{L_p(\mu;E)} := \big(\textstyle\int_\Omega |u|^p \,\mathrm{d}\mu(x)\big)^{1/p} < \infty\},\\
\mathfrak{N}(\mu;E) &:= \{u \in \mathfrak{L}_1(\mu;E) : \|u\|_{L_1(\mu;\,E)} = 0\},\\
L_p(\mu;E) &:= \mathfrak{L}_p(\mu;E)/\mathfrak{N}(\mu;E),\\
L_\infty(\mu;E) &:= \{u\colon \Omega \longrightarrow E \text{ measurable} : \|u\|_{L_\infty}(\mu;E) := \operatorname{ess\,sup}_{x\in\Omega} |u(x)| < \infty\},\\
L_{p,\mathrm{loc}}(\mu;E) &:= \{u\colon \Omega \longrightarrow E \text{ measurable} : \textstyle\int_K |u|^p \,\mathrm{d}\mu(x) < \infty, \forall K \subseteq \Omega \text{ compact}\}.
\end{aligned}$$

If $\mathcal{A}$ is a Borel-Lebesgue σ-algebra and μ is the Lebesgue measure, we use standard notation by writing $L_p(\Omega,E) := L_p(\mu;E)$. Furthermore, we write $L_p(\Omega)$ if $E = \mathbb{R}$ and $L_p(\Omega)^n$ if $E = \mathbb{R}^n$, $n \in \mathbb{N}$. In case the underlying domain Ω is understood from context, and μ is the Lebesgue measure, we write $\|\cdot\|_p$ for the L_p-norm, $1 \leqslant p \leqslant \infty$, and $\int u \,\mathrm{d}x = \int u \,\mathrm{d}\mu(x)$. For a domain Ω or real manifold Γ, we denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ the inner product of $L_2(\Omega)^n$ and $L_2(\Gamma)^n$, respectively. Then

$$(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x$$

stands for the mean value of a function $u \in L_1(\Omega)^n$. With the help of vector-valued L_p-spaces we are able to define the class $\mathcal{HT}$.

Definition 1.1. cf. [16, Definition 1.11] Let E be a Banach space.

(i) The *Hilbert transform* $\mathfrak{H}\colon \mathcal{S}(\mathbb{R},E) \longrightarrow \mathcal{S}'(\mathbb{R},E)$ of a function $f \in \mathcal{S}(\mathbb{R},E)$ is defined through

$$(\mathfrak{H}\, f)(x) := \tfrac{1}{\pi} \lim_{\varepsilon\to 0} \int_{|x-y|\geqslant\varepsilon} \frac{f(y)}{x-y}\,\mathrm{d}y,\ \ x \in \mathbb{R}.$$

(ii) The Banach space E is of *class* $\mathcal{HT}$ if the Hilbert transformation $\mathfrak{H}$ can be extended for one (and thus for all; see [6]) $p \in (1,\infty)$ to a continuous and linear operator $\mathfrak{H} \in \mathcal{L}(L_p(\mathbb{R},E))$.

We denote by $\mathcal{S}(\mathbb{R}^n,E)$ the E-valued Schwartz space on $\mathbb{R}^n$, by $\mathfrak{F}\colon \mathcal{S}(\mathbb{R}^n,E) \longrightarrow \mathcal{S}(\mathbb{R}^n,E)$ the Fourier transformation on this space and by $\mathcal{S}'(\mathbb{R}^n,E)$ the space of E-valued tempered distributions. For a comprehensive approach to vector-valued distribution spaces and Fourier multipliers, see [7].

Remark 1.2. There exists alternative descriptions of the class $\mathcal{HT}$. In particular, E is of class $\mathcal{HT}$ if and only if the property "E is a UMD-space" holds, where UMD stands for *unconditional martingale differences*, cf. [30, Theorem 2.1.19].

Besides the class $\mathcal{HT}$, the property (α) is an important property of Banach spaces.

Definition 1.3. [46, Definition 4.2.7] Let $(\Omega,\mathcal{A},\mu)$ and $(\Omega',\mathcal{A}',\mu')$ be probability spaces. A Banach space E has *property* (α), if a constant $\alpha > 0$ exists such that

$$\Big\|\sum_{i,j=1}^{N} \alpha_{ij}\varepsilon_i\varepsilon'_j x_{ij}\Big\|_{L_2(\Omega\times\Omega',E)} \leqslant \alpha \Big\|\sum_{i,j=1}^{N} \varepsilon_i\varepsilon'_j x_{ij}\Big\|_{L_2(\Omega\times\Omega',E)}$$

for all $\alpha_{ij} \in \{-1,1\}$, $x_{ij} \in E$, $N \in \mathbb{N}$ and all $\{-1,1\}$-valued random variables ε_i on $(\Omega,\mathcal{A},\mu)$ and ε'_j on $(\Omega',\mathcal{A}',\mu')$.

Isotropic Function Spaces

We would like to remind the reader that for a Banach space E and $-\infty < s < \infty$ the *Bessel potentials* are defined by

$$\begin{aligned} \mathcal{B}^s u &:= \mathfrak{F}^{-1}(\xi \longmapsto B^s(\xi)\mathfrak{F}u(\xi)), && u \in \mathcal{S}(\mathbb{R}^n, E), \\ B^s(\xi) &:= (1 + |\xi|^2)^{s/2}, && \xi \in \mathbb{R}^n. \end{aligned}$$

Thus, we can define the *vector-valued Bessel potential spaces* as

$$H^s_p(\mathbb{R}^n, E) := \{u \in \mathcal{S}'(\mathbb{R}^n, E) : u = \mathcal{B}^{-s} f,\ f \in L_p(\mathbb{R}^n, E)\}, \quad -\infty < s < \infty,\ 1 < p < \infty,$$

with norm

$$\|u\|_{H^s_p(\mathbb{R}^n, E)} := \|f\|_{L_p(\mathbb{R}^n, E)}, \quad u \in H^s_p(\mathbb{R}^n, E),\ f \in L_p(\mathbb{R}^n, E),\ u = \mathcal{B}^{-s} f.$$

Moreover, the *Sobolev spaces* for $s \in \mathbb{N}_0$ and $1 \leqslant p < \infty$ are defined as follows,

$$W^s_p(\mathbb{R}^n, E) := \{u \in L_p(\mathbb{R}^n, E) : \partial^\alpha u \in L_p(\mathbb{R}^n, E),\ |\alpha| \leqslant s\},$$

with norm

$$\|u\|_{W^s_p(\mathbb{R}^n, E)} := \left(\sum_{|\alpha| \leqslant s} \|\partial^\alpha u\|^p_{L_p(\mathbb{R}^n, E)} \right), \quad u \in W^s_p(\mathbb{R}^n, E).$$

There is also the following relationship between Sobolev spaces and Bessel potential spaces:

Remark 1.4. If E is a Banach space of class $\mathcal{HT}$, then

$$H^s_p(\mathbb{R}^n, E) \doteq W^s_p(\mathbb{R}^n, E),$$

for every $s \in \mathbb{N}_0$, and every $1 < p < \infty$. This can be seen e. g. in [58, Proposition 3].

The Sobolev spaces are defined for $s \in \mathbb{N}_0$ only. We can extend this scale to $0 < s < \infty$ by the construction of the Sobolev-Slobodeckij spaces. Let $\lfloor \cdot \rfloor : (0, \infty) \longrightarrow \mathbb{N}_0$ with $\lfloor s \rfloor := \max\{m \in \mathbb{N}_0 : m \leqslant s\}$ for $s \in (0, \infty)$ be the floor function and

$$|u|_{\dot{W}^s_p(\mathbb{R}^n, E)} := \left(\sum_{|\alpha| = \lfloor s \rfloor} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|^p_E}{|x - y|^{n + (s - \lfloor s \rfloor)p}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p},$$

for all $0 < s < \infty$ with $s \notin \mathbb{N}$ and $1 \leqslant p < \infty$. Then the *Sobolev-Slobodeckij spaces* are for $s \in (0, \infty) \backslash \mathbb{N}$ and $1 \leqslant p < \infty$ defined by

$$\begin{aligned} W^s_p(\mathbb{R}^n, E) &:= \left\{ u \in W^{\lfloor s \rfloor}_p(\mathbb{R}^n, E) : |u|_{\dot{W}^s_p(\mathbb{R}^n, E)} < \infty \right\}, \\ \|u\|_{W^s_p(\mathbb{R}^n, E)} &:= \left(\|u\|^p_{W^{\lfloor s \rfloor}_p(\mathbb{R}^n, E)} + |u|^p_{\dot{W}^s_p(\mathbb{R}^n, E)} \right)^{1/p}, \quad u \in W^s_p(\mathbb{R}^n, E). \end{aligned}$$

By assuming E to be a Banach space, we can define the *E-valued Besov spaces* for $-\infty < s < \infty$, $\delta > 0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ by

$$B^s_{p,q}(\mathbb{R}^n, E) := \left(H^{s-\delta}_p(\mathbb{R}^n, E), H^{s+\delta}_p(\mathbb{R}^n, E) \right)_{1/2, q},$$

using real interpolation. The family of real interpolation functors is denoted by $(\cdot,\cdot)_{\theta,q}$ on the category of all interpolation couples. Later on, we also apply complex interpolation. For an introduction to real and complex interpolation we would like to refer the reader to [54] and [10]. We use the abbreviation $B^s_p := B^s_{p,p}$ and get the relation

$$B^s_p(\mathbb{R}^n, E) \doteq W^s_p(\mathbb{R}^n, E), \quad s \in (0,\infty)\backslash\mathbb{N},\ 1 < p < \infty,$$

see e. g. [7, Equation (5.8)]. For vector-valued function spaces on arbitrary domains $\Omega \subseteq \mathbb{R}^n$ we set

$$\begin{aligned} H^s_p(\Omega, E) &:= \{u \in \mathcal{D}'(\Omega, E) :\ \text{it exists a } g \in H^s_p(\mathbb{R}^n, E) \text{ with } g_{|\Omega} = u\}, \\ &\quad\ s \in (-\infty,\infty),\ 1 < p < \infty, \\ W^s_p(\Omega, E) &:= \{u \in \mathcal{D}'(\Omega, E) :\ \text{it exists a } g \in W^s_p(\mathbb{R}^n, E) \text{ with } g_{|\Omega} = u\}, \\ &\quad\ s \in [0,\infty),\ 1 \leqslant p < \infty, \\ B^s_{p,q}(\Omega, E) &:= \{u \in \mathcal{D}'(\Omega, E) :\ \text{it exists a } g \in B^s_{p,q}(\mathbb{R}^n, E) \text{ with } g_{|\Omega} = u\}, \\ &\quad\ s \in (-\infty,\infty),\ 1 < p < \infty,\ 1 \leqslant q \leqslant \infty, \end{aligned}$$

where $\mathcal{D}'(\Omega, E)$ is the space of E-valued distributions on Ω and $g_{|\Omega}$ is the restriction of g on Ω in a distributional sense. The corresponding norms are defined by

$$\|u\|_{Y(\Omega,E)} := \inf_{\substack{g \in Y(\mathbb{R}^n,E) \\ g_{|\Omega}=u}} \|g\|_{Y(\mathbb{R}^n,E)},$$

for $Y \in \{H^s_p,\ W^s_p,\ B^s_{p,q}\}$. Furthermore, we set

$${}_0Y(J, E) := \{u \in Y(J, E) : u(0) = 0\},$$

on an interval $J = (0, T)$, $T > 0$, for $Y \in \{H^s_p,\ W^s_p,\ B^s_{p,q}\}$ and corresponding s, p and q as above. These spaces are essentially identical to Bessel potential, Sobolev, Sobolev-Slobodeckij an Besov spaces. They only differ in so far that every function has additionally an initial value of zero. The function space

$$\dot{H}^1_p(\Omega) := \{\phi \in L_{p,\mathrm{loc}}(\Omega) : \nabla\phi \in L_p(\Omega)\}.$$

defines the *homogeneous Sobolev space of order one*, which becomes a semi-normed space via

$$|\phi|_{\dot{H}^1_p(\Omega)} = \|\nabla\phi\|_{L_p(\Omega,\mathbb{R}^n)}, \quad \phi \in \dot{H}^1_p(\Omega).$$

For a more detailed treatment of vector-valued Bessel potential, Sobolev, Sobolev-Slobodeckij or Besov spaces, cf. [7] and [30].

Anisotropic Function Spaces

Let again E be a Banach space. By $\gamma \in \mathbb{N}$ we denote the number of slices, in which we divide the Euclidean space in order to allow different regularities in space. By $n = (n_1, \ldots, n_\gamma) \in \mathbb{N}^\gamma$ we denote the dimensions of the slices and use the abbreviation $\mathbb{R}^n := \mathbb{R}^{n_1} \times \ldots \mathbb{R}^{n_\gamma}$. Moreover, by $\omega = (\omega_1, \ldots, \omega_\gamma) \in \mathbb{N}^\gamma$ we denote an arbitrary weight. We use the abbreviation $\xi = (\xi_1, \ldots, \xi_\gamma) \in \mathbb{R}^n$, and $\dot{\omega} := \mathrm{lcm}\{\omega_1, \ldots, \omega_\gamma\}$ to denote the least common multiple of the weight entries $\omega_1, \ldots, \omega_\gamma$. Then, for $-\infty < s < \infty$ the *weighted Bessel potentials* are defined by

$$\begin{aligned} B^{s,\omega} &:= \mathfrak{F}^{-1}(\xi \longmapsto B^{s,\omega}(\xi)\mathfrak{F}u(\xi)), \qquad u \in \mathcal{S}(\mathbb{R}^n, E), \\ B^{s,\omega}(\xi) &:= \left(1 + \textstyle\sum_{k=1}^{\gamma} |\xi_k|^{2\dot{\omega}/\omega_k}\right)^{s/2\dot{\omega}}, \qquad \xi \in \mathbb{R}^n. \end{aligned}$$

Thus, we can define the *E-valued anisotropic Bessel potential spaces* as

$$H_p^{s,\omega}(\mathbb{R}^n, E) := \{u \in \mathcal{S}'(\mathbb{R}^n, E) : u = \mathcal{B}^{-s,\omega} f,\ f \in L_p(\mathbb{R}^n, E)\},$$

for $s \in (-\infty, \infty)$ and $1 < p < \infty$, with norm

$$\|u\|_{H_p^{s,\omega}(\mathbb{R}^n,E)} := \|f\|_{L_p(\mathbb{R}^n,E)}, \quad u \in H_p^{s,\omega}(\mathbb{R}^n, E),\ f \in L_p(\mathbb{R}^n, E),\ u = \mathcal{B}^{-s,\omega} f.$$

Moreover, the anisotropic Sobolev spaces for $s \in \dot{\omega} \cdot \mathbb{N}_0$ and $1 \leqslant p < \infty$ can be defined in a similar manner as follows

$$W_p^{s,\omega}(\mathbb{R}^n, E) := \left\{ u \in L_p(\mathbb{R}^n, X) : \begin{array}{c} \partial_k^\alpha u \in L_p(\mathbb{R}^n, E),\ \alpha \in \mathbb{N}_0^{n_k} \\ |\alpha| \leqslant s/\omega_k,\ k \in \{1, \dots, \gamma\} \end{array} \right\},$$

with norm

$$\|u\|_{W_p^{s,\omega}(\mathbb{R}^n,\, E)} := \left(\sum_{k=1}^{\gamma} \sum_{|\alpha| \leqslant s/\omega_k} \|\partial_k^\alpha u\|_{L_p(\mathbb{R}^n,\, E)}^p \right)^{1/p}, \quad u \in W_p^{s,\omega}(\mathbb{R}^n, E).$$

Here, $\partial_k^\alpha = \partial^{|\alpha|}/\partial_{x_k^\alpha}$ denotes the partial derivative with respect to the k-th component $x_k \in \mathbb{R}^{n_k}$ of $x = (x_1, \dots, x_\gamma) \in \mathbb{R}^n$. According to Amann [8], anisotropic Bessel potential and Sobolev spaces can also be characterized as follows:

Proposition 1.5. *cf. [34, Theorems 3.7.2 & 3.7.3] Let $\gamma \in \mathbb{N}$ and $n,\ \omega \in \mathbb{N}^\gamma$. In addition $n_k' := n \backslash \{n_k\}$, $\omega' := \omega \backslash \{\omega_k\}$ for $n,\ \omega \in \mathbb{N}^\gamma$ and $k \in \{1, \dots, \gamma\}$. Let E be an* UMD*-space, that has property (α) if $\omega \neq (\dot{\omega}, \dots, \dot{\omega})$. The spaces $H_p^{s/\omega_k}(\mathbb{R}^{n_k}, \dots)$ and $W_p^{s/\omega}(\mathbb{R}^{n_k}, \dots)$ stands for the isotropic vector-valued Bessel potential and Sobolev spaces on the slice $\mathbb{R}^{n_k}$, respectively. Then the equations*

$$\begin{aligned}
H_p^{s,\omega}(\mathbb{R}^n, E) &= H_p^{s/\omega_1}(\mathbb{R}^{n_1}, L_p(\mathbb{R}^{n_1'}, E)) \cap L_p(\mathbb{R}^{n_1}, H_p^{s,\omega_1'}(\mathbb{R}^{n_1'}, E)) \\
&= \textstyle\bigcap_{k=1}^{\gamma} H_p^{s/\omega_k}(\mathbb{R}^{n_k}, L_p(\mathbb{R}^{n_k'}, E)), \qquad 0 < s < \infty,\ 1 < p < \infty, \\
W_p^{s,\omega}(\mathbb{R}^n, E) &= W_p^{s/\omega_1}(\mathbb{R}^{n_1}, L_p(\mathbb{R}^{n_1'}, E)) \cap L_p(\mathbb{R}^{n_1}, W_p^{s,\omega_1'}(\mathbb{R}^{n_1'}, E)) \\
&= \textstyle\bigcap_{k=1}^{\gamma} W_p^{s/\omega_k}(\mathbb{R}^{n_k}, L_p(\mathbb{R}^{n_k'}, E)), \qquad s \in \dot{\omega} \cdot \mathbb{N},\ 1 \leqslant p < \infty,
\end{aligned}$$

are valid.

From now on we assume E to be a Banach space of class $\mathcal{HT}$ that has property (α). Then, we can introduce the anisotropic Sobolev-Slobodeckij spaces by

$$\begin{aligned}
W_p^{s,\omega}(\mathbb{R}^n, E) &= W_p^{s/\omega_1}(\mathbb{R}^{n_1}, L_p(\mathbb{R}^{n_1'}, E)) \cap L_p(\mathbb{R}^{n_1}, W_p^{s,\omega_1'}(\mathbb{R}^{n_1'}, E)) \\
&= \textstyle\bigcap_{k=1}^{\gamma} W_p^{s/\omega_k}(\mathbb{R}^{n_k}, L_p(\mathbb{R}^{n_k'}, E)), \qquad 0 < s < \infty,\ 1 \leqslant p < \infty,
\end{aligned}$$

and thus extend the Sobolev scale. By using real interpolation, we can finally define the anisotropic Besov spaces by

$$B_{p,q}^{s,\omega}(\mathbb{R}^n, E) := (H_p^{s-\delta,\omega}(\mathbb{R}^n, E), H_p^{s+\delta,\omega}(\mathbb{R}^n, E))_{1/2,q},$$

for $-\infty < s < \infty$, $\delta > 0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Using the abbreviation $B_p^{s,\omega} := B_{p,p}^{s,\omega}$ we can again employ another characterisation of anisotropic Besov spaces by

$$\begin{aligned}
B_p^{s,\omega}(\mathbb{R}^n, E) &= B_p^{s/\omega_1}(\mathbb{R}^{n_1}, L_p(\mathbb{R}^{n_1'}, E)) \cap L_p(\mathbb{R}^{n_1}, B_p^{s,\omega_1'}(\mathbb{R}^{n_1'}, E)) \\
&= \textstyle\bigcap_{k=1}^{\gamma} B_p^{s/\omega_k}(\mathbb{R}^{n_k}, L_p(\mathbb{R}^{n_k'}, E)), \qquad 0 < s < \infty,\ 1 < p < \infty,
\end{aligned}$$

using to [34, Proposition 1.4].
For a comprehensive approach of anisotropic function spaces we refer e. g. to [55], [8] and [34]. Using the same construction as for isotropic function spaces, we can define the anisotropic function spaces on a domain $\Omega \subseteq \mathbb{R}^n$.

1.2 $\mathcal{R}$-boundedness and Maximal Regularity

This section is devoted to the explanation of the concept of maximal regularity. Maximal regularity is of particular importance in this thesis, since every linear system of equations, such as parabolic problems and the Stokes equations, that has the property of maximal regularity has a unique solution. Moreover, the solution of a linear system of equations that has the property of maximal regularity does not “loose” regularity with respect to the regularity of the given data. This in turn enables us to solve nonlinear partial differential equation systems. In Chapter 4 we introduce a model, describing the dynamics of tropical storms, and use the approach of maximal regularity to solve this model. This is done by linearising the model and by proving an “optimal” regularity of the linearised equations. The crucial point, therefore, is to prevent any loss of regularity for the linearised system, i. e. to prove maximal regularity. The concept of maximal regularity is closely related to sectoriality, $\mathcal{R}$-sectoriality and $\mathcal{R}$-boundedness of operator families. For this reason, we would like to recall the definitions of these concepts, following references [15] and [36].

The class of sectorial operators is one of the most important classes of closed but unbounded linear operators. It is defined as follows:

Definition 1.6. cf. [15, Definition 1.1] Let E be a complex Banach space and A a closed linear operator in E. A is called *sectorial* if the following two conditions are satisfied,

(S1) $\overline{D(A)} = E$, $\overline{\Re(A)} = E$, $(-\infty, 0) \subseteq \rho(A)$;

(S2) $\|t(t+A)^{-1}\|_{\mathcal{L}(E)} \leqslant M$ for all $t > 0$, and some $M < \infty$.

The class of sectorial operators in E is denoted by $\mathcal{S}(E)$. The *spectral angle* ϕ_A of $A \in \mathcal{S}(X)$ is defined by

$$\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(E)} < \infty\}.$$

The sector Σ_θ in the complex plane is defined by $\Sigma_\theta := \{z \in \mathbb{C}\backslash\{0\};\ |\arg z| < \theta\}$ for $0 < \theta \leqslant \pi$. Another important concept for linear operators is $\mathcal{R}$-boundedness. The latter also provides an important connection to maximal regularity, which is defined later on.

Definition 1.7. [15, Definition 3.1] Let E and F be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(E, F)$ is called $\mathcal{R}$-*bounded*, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_1, \dots T_N \in \mathcal{T}$, $x_1, \dots, x_n \in E$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{A}, \mu)$ the inequality

$$\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_{L_p(\Omega, F)} \leqslant C \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_{L_p(\Omega, E)}$$

is valid. The smallest such C is called $\mathcal{R}$-*bound* of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$.

Analogously to the definition of sectoriality we define $\mathcal{R}$-sectoriality.

Definition 1.8. cf. [15, Definition 4.1] Let E be a Banach space. A sectorial operator $A \in \mathcal{S}(E)$ is called *$\mathcal{R}$-sectorial* if

$$\mathcal{R}_A(0) := \mathcal{R}\{t(t+A)^{-1} : t > 0\} < \infty.$$

The *$\mathcal{R}$-angle* ϕ_A^R of A is defined by means of

$$\phi_A^R := \inf\{\phi \in (0,\pi) : \rho(-A) \supseteq \Sigma_{\pi-\phi}, \mathcal{R}_A(\pi-\phi) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda+A)^{-1} : \lambda \neq 0, |\arg\lambda| \leqslant \theta\}.$$

Maximal Regularity

In this paragraph we explain the concept of maximal regularity (cf. [36, Section 1.3]) and its relation to $\mathcal{R}$-boundedness. Let A be a sectorial operator in a Banach space E with spectral angle $\phi_A < \frac{\pi}{2}$. The Cauchy problem

$$\dot{u}(t) + Au(t) = f(t), \quad t \geqslant 0, \quad u(0) = 0$$

has a solution for any given $f \in L_p([0,T), E)$, $[0,T) \subset [0,\infty)$, which is, according to the Variation of Constants Formula, formally given by

$$u(t) = \int_0^t \mathrm{e}^{-At} f(t-s)\,\mathrm{d}s, \quad t \in [0,T).$$

The operator A has the property of *maximal regularity of type L_p* for $1 < p < \infty$ on $J = [0,\infty)$, if the solution u of the Cauchy problem is (Fréchet) differentiable almost everywhere, the solution u takes its values in $D(A)$ almost everywhere and $\dot{u}$ and Au satisfy the estimate

$$\|\dot{u}\|_{L_p(J,X)} + \|Au\|_{L_p(J,X)} \leqslant C\left(\|u_0\|_{D_A(1-1/p,p)} + \|f\|_{L_p(J,X)}\right),$$

for a constant $C > 0$, cf. [46, Section 3.5]. The trace space $D_A(1-1/p,p)$ is given by

$$D_A(1-1/p,p) = \left\{x \in E : [x]_{1-1/p,p} := \left(\int_0^\infty |t^{1/p} A\mathrm{e}^{-At}x|^p\,\mathrm{d}t/t\right)^{1/p} < \infty\right\},$$

with norm

$$\|x\|_{1-1/p,p} := |x| + [x]_{1-1/p,p}, \quad x \in D_A(1-1/p,p),$$

cf. [46, Section 3.4]. That is, the notion of "maximal regularity" refers to the fact that the regularity of $\dot{u}$ and Au is not worse than the one of the given function f, i. e. no regularity is lost.

Finally, let us consider a theorem describing the relation between maximal regularity and $\mathcal{R}$-boundedness, as well as $\mathcal{R}$-sectoriality.

Theorem 1.9. *[15, Theorem 4.4] Let E be a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, and let A be a sectorial operator in E with spectral angle $\phi_A < \frac{\pi}{2}$. Then the Cauchy problem*

$$\dot{u}(t) + Au(t) = f(t), \quad t \geqslant 0, \quad u(0) = 0,$$

with given function $f \in L_p(\mathbb{R}_+, E)$ has maximal regularity of type L_p on $[0,\infty)$, if and only if A is $\mathcal{R}$-sectorial with $\phi_A^R < \frac{\pi}{2}$. More precisely, the following statements are equivalent

(i) *The Cauchy problem has maximal regularity of type* L_p *on* $[0,\infty)$*;*

(ii) *the set* $\{A(i\rho + A)^{-1} : \rho \in \mathbb{R}\}$ *is* $\mathcal{R}$*-bounded;*

(iii) *the set* $\{A(\lambda + A)^{-1} : \lambda \in \Sigma_\theta\}$ *is* $\mathcal{R}$*-bounded, for some* $\theta > \frac{\pi}{2}$*;*

(iv) *the set* $\{\mathrm{e}^{-Az} : z \in \Sigma_\vartheta\}$ *is* $\mathcal{R}$*-bounded for some* $\vartheta > 0$*;*

(v) *the sets* $\{\mathrm{e}^{-At} : t > 0\}$ *and* $\{tA\mathrm{e}^{-At} : t > 0\}$ *are* $\mathcal{R}$*-bounded.*

It can also be shown that statement (*iii*) is equivalent to A being $\mathcal{R}$-sectorial with angel $\phi_A^R < \frac{\pi}{2}$. This establishes the aforementioned connection between maximal regularity and $\mathcal{R}$-sectoriality.

1.3 Cylindrical Domains and the Helmholtz Projection

In this thesis we mainly consider systems of equations on cylindrical domains $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}\backslash\{1\}$. These cylindrical domains are given as the cartesian product of a bounded $\mathcal{C}^3$-domain A and an interval $(-a, a)$ with $a > 0$, i. e.

$$\Omega := A \times (-a, a).$$

The topological boundary of Ω consists of five different party: the boundary of the top part $\Gamma_{\text{top}} := A \times \{+a\}$ and the bottom part $\Gamma_{\text{bot}} := A \times \{-a\}$ of the cylindrical domain, the lateral boundary $\Sigma := \partial A \times (-a, a)$, as well as the upper edge $\mathcal{R}^{\text{top}}$ and the lower edge $\mathcal{R}^{\text{bot}}$ of the cylindrical domain. By

$$\Gamma := \Gamma_{\text{top}} \,\dot\cup\, \Gamma_{\text{bot}} \,\dot\cup\, \Sigma$$

we denote the smooth part of the boundary, by

$$\mathcal{R} := \mathcal{R}^{\text{top}} \,\dot\cup\, \mathcal{R}^{\text{bot}}$$

the edges, and by

$$\partial\Omega = \Gamma \,\dot\cup\, \mathcal{R}$$

the entire boundary of Ω. However, we mainly use the smooth part Γ of the boundary when we study problems on the boundary of Ω, e. g. when studying boundary conditions.

Boundary Operators

For a cylindrical domain $\Omega = A \times (-a, a) \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$,

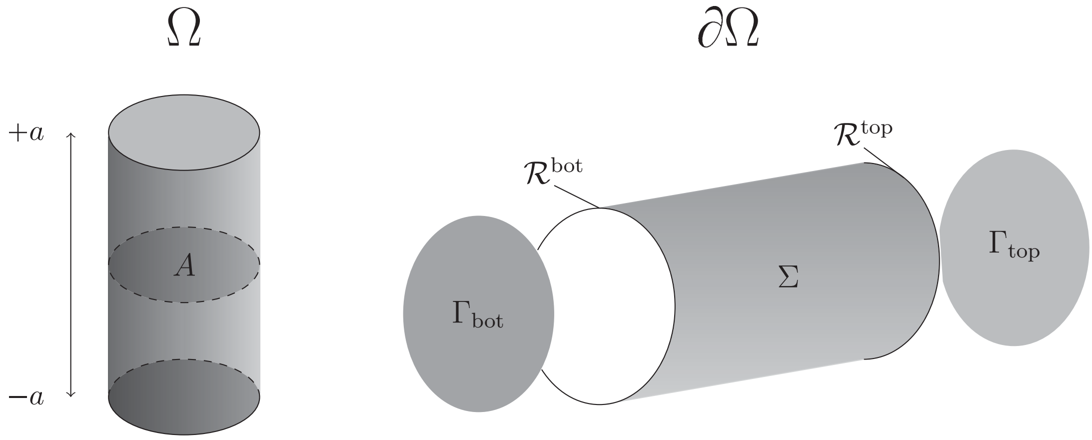
$$\nu_\Gamma : \Gamma \longrightarrow \mathbb{R}^n$$

denotes the outward pointing vector, normal to the boundary Γ and

$$P_\Gamma(x) := \mathrm{Id}(x) - \nu_\Gamma(x) \otimes \nu_\Gamma(x) : \mathbb{R}^n \longrightarrow T_x\Gamma,\ x \in \Gamma,$$

the projection on the tangent bundle $T\Gamma$ of Γ. Note that

$$P_\Gamma u = u - (\nu_\Gamma \otimes \nu_\Gamma)u = u - (\nu_\Gamma \nu_\Gamma^T)u = u - (\nu_\Gamma \cdot u)\nu_\Gamma \quad \text{on } \Gamma,$$



Cylindrical domain Ω and its boundary $\partial\Omega$ for $n = 3$ and circular cross-section A

for a vector field u on Ω. Since Γ is the disjoint union of different boundary parts, also the outer normal vector ν must be understood accordingly. Thus, P_Γ can be considered a system of projections onto the respective boundaries. For example, the notation

$$P_\Gamma(\nabla u)\nu = 0 \text{ on } \Gamma,$$

is supposed to be interpreted as the system

$$\begin{aligned} P_{\Gamma_{\text{top}}}(\nabla u)\nu_{\Gamma_{\text{top}}} &= 0 \quad \text{on } \Gamma_{\text{top}}, \\ P_\Sigma(\nabla u)\nu_\Sigma &= 0 \quad \text{on } \Sigma, \\ P_{\Gamma_{\text{bot}}}(\nabla u)\nu_{\Gamma_{\text{bot}}} &= 0 \quad \text{on } \Gamma_{\text{bot}}, \end{aligned}$$

for $u \in H_p^3(\Omega)$. It is

$$\begin{aligned} \nu_{\Gamma_{\text{top}}}(x) &= +\mathrm{e}_n & &\text{for } x \in \Gamma_{\text{top}}, \\ \nu_{\Gamma_{\text{bot}}}(x) &= -\mathrm{e}_n & &\text{for } x \in \Gamma_{\text{bot}}, \\ \nu_\Sigma(x) &= (\nu_A(x_1, \dots, x_{n-1}), 0) \perp \pm\mathrm{e}_n & &\text{for } x \in \Sigma, \end{aligned}$$

where $\nu_A(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ is the outer normal vector on ∂A. In case the underlying boundary is understood from the context, we just write ν for the outer normal vector on the respective boundaries. Moreover, $P_{\Gamma_{\text{top}}}$, P_Σ, $P_{\Gamma_{\text{bot}}}$ are projections on the tangent bundles $T\Gamma_{\text{top}}$, $T\Sigma$ and $T\Gamma_{\text{bot}}$, respectively. The projection onto the normal bundle $N\Gamma$ of Γ is denoted by

$$Q_\Gamma(x) := \mathrm{Id}(x) - P_\Gamma(x) : \mathbb{R}^n \longrightarrow N_x\Gamma, \ x \in \Gamma.$$

As for P_Γ, we can define different parts for Q_Γ. For a detailed discussion of outer unit vectors, tangential and normal projections, as well as tangential and normal bundles we refer the reader to [5].

The deformation (+) and the rotation tensor (−) of a vector field u on Ω are denoted by

$$D_\pm(u) := \tfrac{1}{2}(\nabla u \pm (\nabla u)^T).$$

For dimension $n = 3$ and u being a vector field, a simple computation reveals the identity

$$D_-(u)\nu = -\tfrac{1}{2}\nu \times \operatorname{curl} u \quad \text{on } \Gamma.$$

For any dimension $n \in \mathbb{N}$, the deformation tensor also satisfies the relation

$$\begin{aligned} P_\Gamma D_-(u)\nu &= \tfrac{1}{2}\big((\nabla u)\nu - (\nabla u)^T\nu - \nu\nu^T(\nabla u)\nu + \nu\nu^T(\nabla u)^T\nu\big) \\ &= \tfrac{1}{2}\big((\nabla u)\nu - (\nabla u)^T\nu - \nu\nu^T(\nabla u)\nu + \nu\nu^T(\nabla u)\nu\big) \\ &= \tfrac{1}{2}\big(\nabla u - (\nabla u)^T\big)\,\nu \\ &= D_-(u)\nu \qquad \text{on } \Gamma. \end{aligned}$$

on Γ. Furthermore,

$$D_-(\nabla p)_{ij} = \tfrac{1}{2}\left(\partial_i(\nabla p)_j - \partial_j(\nabla p)_i\right) = \tfrac{1}{2}\left(\partial_i\partial_j p - \partial_j\partial_i p\right) = 0,$$

is valid in the distributional sense for any $p \in \dot{H}^1_p(\Omega)$ due to the symmetry of second derivatives.

The Helmholtz Projection

In this paragraph we study the Helmholtz projection in $L_p(\Omega)^n$, $n \in \mathbb{N}$, on a cylindrical domain $\Omega = A \times (-a, a)$ where $A \subseteq \mathbb{R}^{n-1}$ is a bounded $\mathcal{C}^3$-domain and $a > 0$. According to [41], we have the decomposition

$$L_p(\Omega)^n = L_{p,\sigma}(\Omega) \oplus \nabla\dot{H}^1_p(\Omega), \quad 1 < p < \infty,$$

on Ω. In this case, we can write the space of solenoidal functions also as

$$L_{p,\sigma}(\Omega) = \{u \in L_p(\Omega)^n : \operatorname{div}(u) = 0,\ u \cdot \nu = 0 \text{ on } \Gamma\},$$

which is equivalent to the definition given on page 11 in Section 1.1. By $H\colon L_p(\Omega)^n \longrightarrow L_p(\Omega)^n$ we denote the *Helmholtz projection*, that projects $L_p(\Omega)^n$ onto $L_{p,\sigma}(\Omega)$ along $\nabla\dot{H}^1_p(\Omega)$. The construction of H relies on the existence of a unique solution $q \in \dot{H}^1_p(\Omega)$ to the weak Neumann problem

$$(\nabla q, \nabla\phi)_\Omega = \langle f \mid \phi\rangle, \quad \phi \in \dot{H}^1_{p'}(\Omega),$$

for $f \in {}_0\dot{H}^{-1}_p(\Omega) := \dot{H}^1_{p'}(\Omega)'$, $\frac{1}{p} + \frac{1}{p'} = 1$. Indeed, given $u \in L_p(\Omega)^n$ we solve the weak Neumann problem for $q \in \dot{H}^1_p(\Omega)$ and $f \in {}_0\dot{H}^{-1}_p(\Omega)$ given as $\langle f \mid \phi\rangle := (u, \nabla\phi)_\Omega$ for $\phi \in \dot{H}^1_{p'}(\Omega)$ and then we obtain $Hu = u - \nabla q$.

There are two results concerning the Helmholtz decomposition which we are going to prove in the following paragraph for later use. Firstly, that the Helmholtz projection has even higher regularity than shown in [41]. In order to prove that we make use of some results from elliptic problems that are later discussed in Section 2.1. In addition, in this proof we use a result about Neumann traces which goes back to Bothe, Köhne, Maier and Saal in [12, Lemma 3.4] and can be seen in Lemma 1.16 of this thesis. This leads us to the propositions:

Proposition 1.10. *Let $T > 0$ and let $J = (0, T)$ or $J = \mathbb{R}$. Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain and $a > 0$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain and $1 < p < \infty$. Let $H\colon L_p(\Omega)^n \longrightarrow L_p(\Omega)^n$ be the Helmholtz projection. Then we have:*

(i) if $u \in H^m_p(\Omega)^n$ for $m \in \{1,\ 2\}$, then $Hu \in H^m_p(\Omega)^n$,

(ii) if $u \in H^1_p(J, L_p(\Omega))^n \cap L_p(J, H^2_p(\Omega))^n$, then $Hu \in H^1_p(J, L_{p,\sigma}(\Omega)) \cap L_p(J, H^2_p(\Omega))^n$.

Proof. *(i)* Assume that $u \in H^1_p(\Omega)^n$. The fact that the trace operator $\partial_\nu \colon H^2_p(\Omega) \longrightarrow W^{1-1/p}_p(\Gamma)$ is a continuous linear retraction, Lemma 1.16, combined with Theorem 2.1 imply the existence of a unique solution $q \in H^2_p(\Omega)$ to the problem

$$-\Delta q = -\operatorname{div} u \qquad \text{in } \Omega, \qquad\qquad \partial_\nu q = u \cdot \nu \qquad \text{on } \Gamma.$$

For $\phi \in C^\infty_c(\bar{\Omega})$, we then have

$$(\nabla q - u,\, \nabla\phi)_\Omega = (\partial_\nu q - u \cdot \nu,\, \phi)_\Gamma - (\Delta q - \operatorname{div} u,\, \phi)_\Omega = 0,$$

which shows that q is a solution to the weak Neumann problem, because $C^\infty_c(\bar{\Omega})$ is dense in $\dot{H}^1_{p'}(\Omega)$. Hence, we have $Hu = u - \nabla q \in H^1_p(\Omega)^n$. Now, if we assume that $u \in H^2_p(\Omega)^n$, then the fact that the trace operator $\partial_\nu \colon H^3_p(\Omega) \longrightarrow W^{2-1/p}_p(\Gamma)$ is a continuous linear retraction, Remark 1.17, and using Lemma 2.3 imply that we even have $q \in H^3_p(\Omega)$. Thus, $Hu \in H^2_p(\Omega)^n$.

(ii) **Step 1.** Let $u(t,\cdot) \in L_p(\Omega)^n$ for $t \in \mathbb{R}$. Then, for every $t \in \mathbb{R}$, $Hu(t,\cdot) \in L_{p,\sigma}(\Omega)$ is the image of $u(t,\cdot)$ via the Helmholtz projection, due to [41]. Now, we have

$$\|Hu\|^p_{L_p(\mathbb{R},\, L_{p,\sigma}(\Omega))} = \int_{\mathbb{R}} \|Hu(t,\cdot)\|^p_{L_{p,\sigma}(\Omega)}\, \mathrm{d}t \leqslant C^p \int_{\mathbb{R}} \|u(t,\cdot)\|^p_{L_p(\Omega)^n}\, \mathrm{d}t = \|u\|^p_{L_p(J,\, L_p(\Omega))^n}$$

for some constant $C > 0$ that is independent of u and $t \in \mathbb{R}$. Using an approximation argument, we obtain $Hu \in L_p(\mathbb{R}, L_{p,\sigma}(\Omega))$ to be the image of $u \in L_p(J, L_p(\Omega))^n$ via the Helmholtz projection for $J = \mathbb{R}$. For $J = \mathbb{R}$ the time derivative ∂_t can be approximated in $L_p(\mathbb{R})$ by difference quotients. This way we obtain $Hu \in H^1_p(\mathbb{R}, L_{p,\sigma}(\Omega))$ for the solution constructed above, if we additionally have that $u \in H^1_p(\mathbb{R}, L_{p,\sigma}(\Omega))$. Now, using extension and restriction operators between $H^1_p(J)$ and $H^1_p(\mathbb{R})$ we are able to obtain the same result also for $J = (0,T)$.

Step 2. Now, let $u(t,\cdot) \in H^2_p(\Omega)^n$. Then, for every $t \in \mathbb{R}$, $Hu(t,\cdot) \in H^2_p(\Omega)^n$ is the image of $u(t,\cdot)$ via the Helmholtz projection of, due to *(i)*. By using the same arguments as in step 1, we obtain $Hu \in L_p(J, H^2_p(\Omega))^n$ to be the image of $u \in L_p(J, H^2_p(\Omega))^n$ via the Helmholtz projection for $J = (0,T)$ or $J = \mathbb{R}$.

Step 3. By combining step 1 and 2, we obtain $Hu \in H^1_p(J, L_{p,\sigma}(\Omega)) \cap L_p(J, H^2_p(\Omega))^n$ to be the image of $u \in H^1_p(J, L_p(\Omega))^n \cap L_p(J, H^2_p(\Omega))^n$ via the Helmholtz projection for $J = (0,T)$ or $J = \mathbb{R}$. □

The second important result of the theory of the Helmholtz decomposition is the fact that the divergence of a particular antisymmetric matrix is contained in the solenoidal space $L_{p,\sigma}(\Omega)$.

Proposition 1.11. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$ with $T > 0$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, ν to be the outer normal vector on the boundary Γ of Ω, $C \in L_p(J, W^1_p(\Omega))^{n\times n}$ to be a skew-symmetric matrix with $C\nu = 0$ on $J \times \Gamma$ and $1 < p < \infty$. Then we have* $\operatorname{div}(C) \in L_p(J, L_{p,\sigma}(\Omega))$.

Proof. For an arbitrary $\psi \in W^1_{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, we can choose a sequence $(\psi_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}^\infty_c(\bar{\Omega})$ with $\psi_k \longrightarrow \psi$ for $k \longrightarrow \infty$ in $W^1_{p'}(\Omega)$, [4, Theorem 3.18]. Since Ω is bounded and $\partial\Omega$ is continuous, the function spaces $\dot{W}^1_{p'}(\Omega)$ and $W^1_{p'}(\Omega)$ are algebraically the same. Note that

$$\operatorname{div}(A\nabla\psi_k) = \operatorname{div}(A) \cdot \nabla\psi_k + A : \nabla^2\psi_k$$

for an $A\colon \Omega \longrightarrow \mathbb{R}^{n\times n}$. Hence, we deduce

$$\operatorname{div}(C\nabla\psi_k) = \operatorname{div}(C)\cdot\nabla\psi_k + C : \nabla^2\psi_k = \operatorname{div}(C)\cdot\nabla\psi_k,$$

due to $C : \nabla^2\psi_k = 0$ by the skew-symmetry of C and the symmetry of $\nabla^2\psi_k$. By taking the limit $k \longrightarrow \infty$, we obtain

$$\operatorname{div}(C\nabla\psi) = \operatorname{div}(C)\cdot\nabla\psi. \tag{1.3.1}$$

By using the Gaussian integral Theorem [35, Chapter 2.5], which is also valid for L_p-spaces due to [29, Chapter II.2.5], equation (1.3.1) and $C\nu = 0$ on the boundary, we have

$$\int_\Omega \nabla\psi\cdot\operatorname{div} C \,\mathrm{d}x = \int_\Omega \operatorname{div}(C\nabla\psi)\,\mathrm{d}x = \int_{\partial\Omega} C\nabla\psi\cdot\nu \,\mathrm{d}\sigma = -\int_{\partial\Omega}\nabla\psi\cdot C\nu\,\mathrm{d}\sigma = 0.$$

Since the equation is valid for any $\psi \in W^1_{p'}(\Omega)$, we conclude $\operatorname{div}(C) \in L_p(J, L_{p,\sigma}(\Omega))$. □

1.4 Regularity and Compatibility Conditions

The Chapters 2 and 3 deal with the development of an L_p-theory on cylindrical domains. More precisely, in the next two chapters we study the solvability of parabolic systems of equations and Stokes equations with different boundary conditions on cylinders. The special feature of our consideration is that we allow, not only constant, but variable coefficients in these systems.

This section deals with the introduction of parabolic problems

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= f && \text{in } J\times\Omega,\\ \mathcal{B}^V(u) &= h && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{$(P|J)_V$}$$

and the Stokes equations

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J\times\Omega,\\ \operatorname{div}(\rho u) &= g && \text{in } J\times\Omega,\\ \mathcal{B}^V(u,q) &= h && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{$(S|J)_V$}$$

Moreover, we discuss necessary regularity and compatibility conditions of these systems. As always, by $\Omega := A\times(-a,a) \subseteq \mathbb{R}^n$ we denote a cylindrical domain being a cartesian product of a bounded $\mathcal{C}^3$-domain A and an interval $(-a,a)$ for some $a > 0$. In addition, $J = (0,T)$, $T > 0$, denotes a time interval. Moreover, we consider the boundary conditions exclusively on the smooth part $\Gamma = \Gamma_{\text{top}} \,\dot\cup\, \Gamma_{\text{bot}} \,\dot\cup\, \Sigma$ of the boundary of Ω. Here, Γ_{top} denotes the boundary of the top and Γ_{bot} the boundary of the bottom of Ω and Σ the lateral boundary. For a more comprehensive discussion of cylindrical domains and their boundary we refer the reader to Section 1.3. The density ρ, the coefficient α and the viscosity μ are assumed to be given. They may be constant or variable, both of which cases are studied in subsequent chapters. In the constant case, we have $\rho,\ \alpha,\ \mu > 0$. In the variable case, we assume $\rho \in W^2_\infty(\Omega)$ to be a time independent positive function with positive inverse $\frac{1}{\rho} \in W^2_\infty(\Omega)$ and $\alpha \in \mathcal{BUC}^1(\Omega)$, $\mu \in \mathcal{BUC}(\Omega)$ with $\inf_\Omega \alpha$, $\inf_\Omega \mu > 0$. Of course, assuming the coefficients constant is a special case of the coefficients being variable. But we see later in the Chapters 2 and 3 that for proving maximal regularity of these systems with variable coefficients we need the maximal regularity of these systems with constant coefficients, which is why we distinguish the constant and variable case.

Remark 1.12. (i) Regardless of whether the coefficients are assumed to be constant or variable, it is sufficient to view the first equation of $(P|J)_V$ as

$$\partial_t u - \mu \Delta u = f \quad \text{in } J \times \Omega,$$

since we can replace μ by $\tilde{\mu} = \frac{1}{\rho}\mu$ and f by $\tilde{f} = \frac{1}{\rho} f$.

(ii) Assuming $\rho > 0$, $\mu > 0$ and $\alpha > 0$ to be constant, the analogous consideration as in (i) is sufficient to simplify the momentum equation of $(S|J)_V$ as follows

$$\partial_t u - \mu \Delta u + \alpha \nabla q = f \quad \text{in } J \times \Omega.$$

Additionally, it is sufficient to rewrite the divergence equation of $(S|J)_V$ as

$$\operatorname{div}(u) = g \quad \text{in } J \times \Omega,$$

since we can replace g by $\tilde{g} = \frac{1}{\rho} g$.

Now, we introduce the boundary operators $\mathcal{B}^V$ with parameter $V \in \{R,\ ND,\ S\pm\}$. After that, we deal with the necessary regularity condition of the systems $(P|J)_V$ and $(S|J)_V$, and finally we study their compatibility conditions.

Boundary Conditions

In the study of parabolic problems $(P|J)_V$ and the Stokes equations $(S|J)_V$, the parabolic problem with Robin boundary condition and the Stokes equations with free slip boundary conditions are of special interest to us. They are particularly useful when solving problems of physics. We see this later when examining a model which describes the dynamics of tropical storms in Chapter 4. We study the parabolic system with Neumann-Dirichlet, with perfect slip and with free slip boundary conditions in Chapter 2 and we study the Stokes equations with perfect slip and with free slip boundary conditions in Chapter 3. We do not study the Stokes equations with Robin and Neumann-Dirichlet boundary conditions in this thesis. Therefore, the Robin and the Neumann-Dirichlet boundary operator, which are defined in the following paragraph, depend on the velocity u only. Moreover, the Neumann-Dirichlet, the perfect slip and the free slip boundary operators contain a boundary condition in normal as well as in tangential direction. The *Robin boundary operator* is denoted by

$$\mathcal{B}^R(u) := \beta^u \partial_\nu u + \sigma^u u \quad \text{on } J \times \Gamma,$$

the *Neumann-Dirichlet boundary operator* is denoted by

$$\mathcal{B}^{ND}(u) = h \quad \text{on } J \times \Gamma \quad :\Longleftrightarrow \quad \begin{array}{rcll} u \cdot \nu &=& h \cdot \nu & \text{on } J \times \Gamma, \\ \delta \partial_\nu P_\Gamma u &=& P_\Gamma h & \text{on } J \times \Gamma, \end{array}$$

with $\delta > 0$, the *perfect slip boundary operator* is denoted by

$$\mathcal{B}^{S-}(u) := \mathcal{B}^{S-}(u,q) = h \quad \text{on } J \times \Gamma \quad :\Longleftrightarrow \quad \begin{array}{rcll} u \cdot \nu &=& h \cdot \nu & \text{on } J \times \Gamma, \\ -\beta^u P_\Gamma D_-(u)\nu &=& P_\Gamma \nu & \text{on } J \times \Gamma, \end{array}$$

and the *free slip boundary operator* is denoted by

$$\mathcal{B}^{S+}(u) := \mathcal{B}^{S+}(u,q) = h \quad \text{on } J \times \Gamma \quad :\Longleftrightarrow \quad \begin{array}{rcll} u \cdot \nu &=& h \cdot \nu & \text{on } J \times \Gamma, \\ \beta^u P_\Gamma D_+(u)\nu &=& P_\Gamma h & \text{on } J \times \Gamma. \end{array}$$

The coefficients β^u and σ^u of the boundary are also assumed to be either constant with $\beta^u > 0$, $\sigma^u \geqslant 0$ or variable, more precisely

$$\beta^u \in \mathcal{BC}^1(J \times \Gamma, (0, \infty)) \text{ with } \inf_\Gamma \beta^u > 0,$$

and

$$\sigma^u \in \mathcal{BC}^2(J \times \Gamma, [0, \infty)).$$

Since $\Gamma = \Gamma_{\text{top}} \dot\cup \Gamma_{\text{bot}} \dot\cup \Sigma$, the boundary operators on Γ are considered separately on each part $\Gamma_{\text{top}},\ \Sigma,\ \Gamma_{\text{bot}}$ of the boundary Γ. For example the perfect slip boundary operator

$$\mathcal{B}^{S-}(u) = h \quad \text{on } J \times \Gamma,$$

applied to a vector field u is a shorthand notation for the system

$$\begin{aligned} u \cdot \nu_{\Gamma_{\text{top}}} &= h \cdot \nu_{\Gamma_{\text{top}}}, & \beta^u P_{\Gamma_{\text{top}}} D_+(u)\nu_{\Gamma_{\text{top}}} &= P_{\Gamma_{\text{top}}} h && \text{on } J \times \Gamma_{\text{top}}, \\ u \cdot \nu_\Sigma &= h \cdot \nu_\Sigma, & \beta^u P_{\Gamma_\Sigma} D_+(u)\nu_\Sigma &= P_\Sigma h && \text{on } J \times \Sigma, \\ u \cdot \nu_{\Gamma_{\text{bot}}} &= h \cdot \nu_{\Gamma_{\text{bot}}}, & \beta^u P_{\Gamma_{\text{bot}}} D_+(u)\nu_{\Gamma_{\text{bot}}} &= P_{\Gamma_{\text{bot}}} h && \text{on } J \times \Gamma_{\text{bot}}. \end{aligned}$$

Remark 1.13. When applying $\mathcal{B}^{S\pm}$ to $(P|J)_{S\pm}$ or $(S|J)_{S\pm}$ it is sufficient to consider the tangential boundary condition as

$$\pm P_\Gamma D_\pm(u)\nu = P_\Gamma h \quad \text{on } J \times \Gamma,$$

since we can replace h by $\tilde{h} = \frac{1}{\beta^u} h$, regardless of whether β^u is assumed to be constant or variable.

Necessary Regularity Conditions

To establish an L_p-theory for $(P|J)_V$ and $(S|J)_V$, $V \in \{R,\ ND,\ S\pm\}$ it is particularly important to show maximal regularity of these systems of equations. The concept of maximal regularity, which was introduced in Section 1.2, essentially depends on the function spaces in which the equations are considered. Therefore, it is essential to detect the necessary and sufficient regularity conditions for our problems in order to find unique solutions to them. In order to find them, we proceed similarly to Köhne in [32, Chapter 3.1]. That is, we consider the first equation of $(P|J)_V$ and the momentum equation of $(S|J)_V$ in the base space $L_p(J \times \Omega)^n$. Thus, we require

$$f \in \mathbb{F}_p^f(J) := L_p(J \times \Omega)^n.$$

Later, we see that the Stokes equations $(S|J)_V$ can be reduced to the parabolic problems $(P|J)_V$. A unique solution of $(P|J)_V$ and $(S|J)_V$ should satisfy

$$u \in \mathbb{E}_p^u(J) := H_p^1(J, L_p(\Omega))^n \cap L_p(J, H_p^2(\Omega))^n,$$

for $V \in \{ND,\ S\pm\}$, and

$$u \in \mathbb{E}_p^z(J) := H_p^1(J, L_p(\Omega)) \cap L_p(J, H_p^2(\Omega)),$$

for $V = R$. In order to obtain a solution to $(S|J)_V$ we not only need the velocity u, but also the pressure q. For a unique solution of the Stokes equations the latter should satisfy

$$\nabla q \in L_p(J \times \Omega)^n.$$

Since the momentum equation of $(S|J)_V$ only implies regularity for the gradient of the pressure, we additionally need to require

$$q \in L_p(J, \dot{H}^1_p(\Omega)).$$

By considering the boundary operators $\mathcal{B}^V$, $V \in \{ND,\ S\pm\}$, which do not contain the pressure, we conclude that a solution of the pressure for the Stokes equations is only unique up to a constant. Hence, we may fix a particular pressure q by requiring $(q)_\Omega = 0$ for its mean value. Since Ω is a bounded domain consisting of a bounded $\mathcal{C}^3$-domain A and an interval $(-a, a)$, we can use the Poincaré inequality and obtain

$$\|q - (q)_\Omega\|_{L_p(\Omega)} \leqslant C\|\nabla q\|_{L_p(\Omega)} = |q|_{\dot{H}^1_p(\Omega)}, \quad p \in \dot{H}^1_p(\Omega),$$

for some constant $C > 0$ that is independent of $q \in L_p(\Omega)$. Thus, we require

$$q \in \mathbb{E}^q_p(J) := \{p \in L_p(J, H^1_p(\Omega)) : (p)_\Omega = 0\}$$

for the pressure in $(S|J)_V$.

In the next step, we would like to investigate the divergence equation of $(S|J)_V$, in particular the regularity of the function g. To derive the necessary regularity conditions for g we use the *Mixed Derivative Theorem*, which goes back to the work of Sobolevskii [53]. Following the proof of e. g. Köhne [32, Proposition 3.9], we obtain the following result, since the Laplacian fulfils the necessary properties of the proof also on cylindrical domains.

Proposition 1.14. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3-$domain, $a > 0$ and $\Omega = A \times (-a, a)$ be a cylindrical domain with $\Gamma = \Gamma_{top} \mathbin{\dot\cup} \Gamma_{bot} \mathbin{\dot\cup} \Sigma$ as the smooth part of the boundary of Ω. Let $T > 0$, $1 < p < \infty$, $\tau \in (0, 1]$ and $\sigma \in (0, 2]$. Then the embeddings*

$$H^\tau_p((0, T), L_p(\Omega)) \cap L_p((0, T, H^\sigma_p(\Omega)) \hookrightarrow H^{(1-\theta)\tau}_p((0, T), H^{\theta\sigma}_p(\Omega)), \quad \theta \in [0, 1]$$

are valid.

We obtain

$$\mathbb{E}^u_p(J) \hookrightarrow H^{1/2}_p(J, H^1_p(\Omega))^n,$$

by applying Proposition 1.14 to $\mathbb{E}^u_p(J)$ and using the regularity of the cylindrical domain Ω. This implies

$$g \in \mathbb{F}^g_p(J) := H^{1/2}_p(J, L_p(\Omega)) \cap L_p(J, H^1_p(\Omega)).$$

To analyse the boundary operators $\mathcal{B}^V$ for $V \in \{R,\ ND,\ S\pm\}$ in more detail, we need a result about traces, which can be proven using the work of Denk, Hieber and Prüss [15].

Proposition 1.15. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3-$domain, $a > 0$ and $\Omega = A \times (-a, a)$ be a cylindrical domain with $\Gamma = \Gamma_{top} \mathbin{\dot\cup} \Gamma_{bot} \mathbin{\dot\cup} \Sigma$ as the smooth part of the boundary of Ω. Let $1 < p < \infty$. Then for $\Lambda \in \{\Gamma_{top}, \Sigma, \Gamma_{bot}\}$ the following assertions are valid.*

(i) The trace operator

$$\gamma_\Lambda \colon H^1_p(J, L_p(\Omega)) \cap L_p(J, H^2_p(\Omega)) \longrightarrow W^{1-1/2p}_p(J, L_p(\Lambda)) \cap L_p(J, W^{2-1/p}_p(\Lambda))$$

is bounded.

(ii) The trace operator

$$\gamma_\Lambda \colon H_p^{1/2}(J, L_p(\Omega)) \cap L_p(J, H_p^1(\Omega)) \longrightarrow W_p^{1/2-1/2p}(J, L_p(\Lambda)) \cap L_p(J, W_p^{1-1/p}(\Lambda))$$

is bounded.

Proof. (i) We prove this assertion for $\Lambda = \Gamma_{\text{top}}$. According to [4, Theorem 5.28] we can extend $u \in H_p^1(J, L_p(\Omega)) \cap L_p(J, H_p^2(\Omega))$ to $\tilde{u} \in H_p^1(J, L_p(\mathbb{R}^n)) \cap L_p(J, H_p^2(\mathbb{R}^n))$. The trace operator

$$\gamma \colon H_p^1(J, L_p(\mathbb{R}^n)) \cap L_p(J, H_p^2(\mathbb{R}^n)) \longrightarrow W_p^{1-1/2p}(J, L_p(H)) \cap L_p(J, W_p^{2-1/p}(H)),$$

with $H = \mathbb{R}^{n-1} \times \{a\}$ is bounded according to [15]. Then, we have that

$$\gamma_{\Gamma_{\text{top}}} \colon H_p^1(J, L_p(\Omega)) \cap L_p(J, H_p^2(\Omega)) \longrightarrow W_p^{1-1/2p}(J, L_p(\Gamma_{\text{top}})) \cap L_p(J, W_p^{2-1/p}(\Gamma_{\text{top}})),$$

with

$$\gamma_{\Gamma_{\text{top}}}(u) = \gamma(\tilde{u})_{|\Gamma_{\text{top}}}$$

is bounded. The cases $\Lambda \in \{\Sigma, \Gamma_{\text{bot}}\}$ can be proven analogously by setting $H = A \times \{-\infty, \infty\}$ for $\Lambda = \Sigma$ and $H = \mathbb{R}^{n-1} \times \{-a\}$ for $\Lambda = \Gamma_{\text{bot}}$.
(ii) This assertion can be obtained similar to (i), since

$$\gamma \colon H_p^{1/2}(J, L_p(\mathbb{R}^n)) \cap L_p(J, H_p^1(\mathbb{R}^n)) \longrightarrow W_p^{1/2-1/2p}(J, L_p(H)) \cap L_p(J, W_p^{1-1/p}(H))$$

is bounded according to [15]. □

Now, combining Proposition 1.14 and 1.15 we obtain for each part Γ_{top}, Γ_Σ, Γ_{bot} of the boundary Γ that

$$\begin{aligned}
\gamma_\Lambda(v) &\in W_p^{1-1/2p}(J, L_p(\Lambda))^n \cap L_p(J, W_p^{2-1/p}(\Lambda))^n, \\
\partial_\nu v,\ \gamma_\Lambda(\partial_k v) &\in W_p^{1/2-1/2p}(J, L_p(\Lambda))^n \cap L_p(J, W_p^{1-1/p}(\Lambda))^n, \quad \Lambda \in \{\Gamma_{\text{top}}, \Sigma, \Gamma_{\text{bot}}\},
\end{aligned}$$

for all $v \in \mathbb{E}_p^u(J)$ and all $j \in \{1, \dots, n\}$, where γ_Λ is the corresponding trace on the boundary Λ. Thus, if $u \in \mathbb{E}_p^u(J)$, we have

$$\begin{aligned}
&\mathcal{B}^R(u) \in \mathbb{F}_p^{R,h}(J) \text{ with} \\
&\mathbb{F}_p^{R,h}(J) := \{h \colon \Gamma \longrightarrow \mathbb{R}^n : h_{|\Gamma_{\text{top}}} =: h^{\text{top}} \in \mathbb{F}_p^{R,\Gamma_{\text{top}}}(J), \\
&\qquad h_{|\Sigma} =: h^\Sigma \in \mathbb{F}_p^{R,\Sigma}(J),\ h_{|\Gamma_{\text{bot}}} =: h^{\text{bot}} \in \mathbb{F}_p^{R,\Gamma_{\text{bot}}}\} \text{ and} \\
&\mathbb{F}_p^{R,\Lambda}(J) := W_p^{1/2-1/2p}(J, L_p(\Lambda))^n \cap L_p(J, W_p^{1-1/p}(\Lambda))^n, \quad \Lambda \in \{\Gamma_{\text{top}}, \Sigma, \Gamma_{\text{bot}}\},
\end{aligned}$$

for the Robin boundary operator,

$$\begin{aligned}
&P_\Gamma \mathcal{B}^{ND}(u) \in \mathbb{T}_p^h(J) \text{ with} \\
&\quad \mathbb{T}_p^h(J) := \{h \colon \Gamma \longrightarrow \mathbb{R}^n : h_{|\Gamma_{\text{top}}} =: h^{\text{top}} \in \mathbb{T}_p^{\Gamma_{\text{top}}}(J), \\
&\qquad h_{|\Sigma} =: h^\Sigma \in \mathbb{T}_p^\Sigma(J),\ h_{|\Gamma_{\text{bot}}} =: h^{\text{bot}} \in \mathbb{T}_p^{\Gamma_{\text{bot}}}(J)\} \text{ and} \\
&\quad \mathbb{T}_p^\Lambda(J) := W_p^{1/2-1/2p}(J, L_p(\Lambda))^n \cap L_p(J, W_p^{1-1/p}(\Lambda))^n, \quad \Lambda \in \{\Gamma_{\text{top}}, \Sigma, \Gamma_{\text{bot}}\} \\
&Q_\Gamma B^{ND} \in \mathbb{N}_p^h(J) \text{ with} \\
&\quad \mathbb{N}_p^h(J) := \{h \colon \Gamma \longrightarrow \mathbb{R}^n : h_{|\Gamma_{\text{top}}} =: h^{\text{top}} \in \mathbb{N}_p^{\Gamma_{\text{top}}}(J), \\
&\qquad h_{|\Sigma} =: h^\Sigma \in \mathbb{N}_p^\Sigma(J),\ h_{|\Gamma_{\text{bot}}} =: h^{\text{bot}} \in \mathbb{N}_p^{\Gamma_{\text{bot}}}(J)\} \text{ and} \\
&\quad \mathbb{N}_p^\Lambda(J) := W_p^{1-1/2p}(J, L_p(\Lambda))^n \cap L_p(J, W_p^{2-1/p}(\Lambda))^n,
\end{aligned}$$

for the Neumann-Dirichlet boundary operator and

$$P_\Gamma \mathcal{B}^{S\pm}(u) \in \mathbb{T}_p^h(J),$$
$$Q_\Gamma \mathcal{B}^{S\pm}(u) \in \mathbb{N}_p^h(J),$$

for the perfect slip and free slip boundary operators. The norm of $\mathbb{F}_p^\Lambda(J)$ is defined as

$$\|h\|_{\mathbb{F}_p^\Lambda(J)} = \big(\|P_\Lambda h\|^p_{W_p^{1/2-1/2p}(J,L_p(\Lambda))} + \|P_\Lambda h\|^p_{L_p(J,W_p^{1-1/p}(\Lambda))} + \|h\cdot\nu_\Lambda\|^p_{W_p^{1-1/2p}(J,L_p(\Lambda))} + \|h\cdot\nu_\Lambda\|^p_{L_p(J,W_p^{2-1/p}(\Lambda))}\big)^{1/p}, \quad h \in \mathbb{F}_p^\Lambda(J),$$

and the norm of $\mathbb{F}_p^h(J)$ is defined as

$$\|h\|_{\mathbb{F}_p^h(J)} := \left(\|h_{|\Gamma_{\mathrm{top}}}\|^p_{\mathbb{F}_p^{\Gamma_{\mathrm{top}}}} + \|h_{|\Sigma}\|^p_{\mathbb{F}_p^\Sigma} + \|h_{|\Gamma_{\mathrm{bot}}}\|^p_{\mathbb{F}_p^{\Gamma_{\mathrm{bot}}}}\right)^{1/p}, \quad h \in \mathbb{F}_p^h(J).$$

Then the regularity class for the data h on the boundary of system $(P|J)_R$ is given by

$$h \in \mathbb{F}_p^{R,h}(J),$$

and the regularity class for the data h on the boundary of the systems $(P|J)_{NV}$, $(P|J)_{S\pm}$, $(S|J)_{NV}$ and $(S|J)_{S\pm}$ is given by

$$h \in \mathbb{F}_p^h(J) := \{h \in \mathbb{T}_p^h(J) : h\cdot\nu \in \mathbb{N}_p^h(J)\}.$$

Finally, we consider the initial equation of $(P|J)_V$ and $(S|J)_V$, $V \in \{\ R,\ ND,\ S\pm\}$, with initial data u_0. Thanks to Proposition 1.14 and Sobolev's embedding theorem, the embedding

$$\mathbb{E}_p^u(J) \hookrightarrow \mathcal{BUC}(J, W_p^{2-2/p}(\Omega))^n$$

is valid. Thus we obtain

$$u_0 \in \mathbb{F}_p^0 := W_p^{2-2/p}(\Omega)^n.$$

Necessary Compatibility Conditions

Additionally to the regularity conditions, the data with respect to the systems $(P|J)_V$ and $(S|J)_V$, $V \in \{R,\ ND,\ S\pm\}$, has to fulfil certain compatibility conditions. With the aid of compatibility conditions, we may assume that the different pairs of data, (f,g,h,u_0) for $(S|J)_V$ and (f,h,u_0) for $(P|J)_V$, are compatible with each other. We assume the solutions and data that follows to be in the function spaces that were identified in the paragraph above. First of all, the condition

$$\mathrm{div}(u_0) = g(0) \quad \text{if } p \geqslant 2 \tag{C1}$$

is necessary, i. e. the data g of the divergence equation has to be compatible with the initial data u_0. Furthermore, there is a hidden compatibility condition, which arises from the divergence condition and the normal boundary condition. We set ${}_0H_p^{-1}(\Omega) := H^1_{p'}(\Omega)'$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and define the linear functional

$$\mathcal{F}\colon \mathbb{F}_p^g \times \mathbb{N}_p^h \longrightarrow L_p(J, {}_0H_p^{-1}(\Omega))$$

for $\psi \in \mathbb{F}_p^g$ and $\eta \in \mathbb{N}_p^h$ through

$$\langle \phi \mid \mathcal{F}(\psi, \eta)\rangle = \int_\Gamma \phi\eta \,\mathrm{d}\sigma - \int_\Omega \phi\psi \,\mathrm{d}x, \quad \phi \in H^1_{p'}(\Omega).$$

Integration by parts leads to

$$\langle \phi \mid \mathcal{F}(\mathrm{div}(u), u\cdot\nu)\rangle = \int_\Omega \nabla\phi \cdot u \,\mathrm{d}x, \quad \phi \in H^1_{p'}(\Omega),$$

from which we can infer

$$\|\langle \phi \mid \mathcal{F}(\mathrm{div}(u), u\cdot\nu)\rangle\|_{L_p(J)} \leqslant \|u\|_{\mathbb{E}_p^u(J)} |\phi|_{\dot{H}^1_{p'}(\Omega)}, \quad \phi \in H^1_{p'}(\Omega),$$

and

$$\|\langle \phi \mid \partial_t\mathcal{F}\,(\mathrm{div}(u), u\cdot\nu)\rangle\|_{L_p(J)} \leqslant \|u\|_{\mathbb{E}_p^u(J)} |\phi|_{\dot{H}^1_{p'}(\Omega)}, \quad \phi \in H^1_{p'}(\Omega).$$

Since Ω is bounded and $\partial\Omega$ is continuous, the function spaces $H^1_{p'}(\Omega)$ and $\dot{H}^1_{p'}(\Omega)$ are algebraically the same. Therefore, the relation ${}_0\dot{H}_p^{-1}(\Omega) = \dot{H}^1_{p'}(\Omega)'$ implies the compatibility condition

$$\mathcal{F}(g, h\cdot\nu) \in H^1_p(J, {}_0\dot{H}_p^{-1}(\Omega)). \tag{$C2$}$$

Moreover, the data h on the boundary has to be compatible with the initial data u_0. Since the boundary operators $\mathcal{B}^V$ have different boundary conditions for every $V \in \{R,\, ND,\, S\pm\}$, different compatibility conditions arise for h and u_0 with respect to each boundary operator. However, the perfect slip and the free slip boundary operators only differ up to a sign, which is why we consider the compatibility conditions of these two operators at the same time. Because of Remark 1.13, we consider the tangential boundary condition of $\mathcal{B}^{S\pm}$ without the coefficient β^u. Likewise, we also consider the compatibility condition with respect to the boundary operators $\mathcal{B}^{S\pm}$ without the coefficient β^u. Then, we have the compatibility condition

$$\beta^u\partial_\nu u_0 + \sigma^u u_0 = h(0) \text{ if } p > 3 \tag{$C3)_R$}$$

with respect to the Robin boundary operator. We require

$$\begin{aligned} u_0\cdot\nu &= h(0)\cdot\nu && \text{if } p > \tfrac{3}{2}, \\ \delta\partial_\nu P_\Gamma(u_0) &= P_\Gamma h(0) && \text{if } p > 3, \end{aligned} \tag{$C3)_{ND}$}$$

with respect to the Neumann-Dirichlet boundary operator, and

$$\begin{aligned} u_0\cdot\nu &= h(0)\cdot\nu && \text{if } p > \tfrac{3}{2}, \\ \pm P_\Gamma D_\pm(u_0)\nu &= P_\Gamma h(0) && \text{in } p > 3, \end{aligned} \tag{$C3)_{S\pm}$}$$

with respect to the perfect slip and free slip boundary operator.
Lastly, there are compatibility conditions for h which arise from the boundary conditions on the edges of Ω. We would like to remind the reader that we denote the data h on Γ_{top} by h^{top}, on Σ by h^Σ, and on Γ_{bot} by h^{bot}. Since the boundary Γ is composed out of mutually disjoint parts Γ_{top}, Σ and Γ_{bot}, we have to put special emphasis to make sure that the continuations of h on these distinct boundary parts are compatible on the respective connecting edges. That means, we have to show that h^{top} is compatible with

h^Σ on the connecting edge of the top and the lateral boundary of the cylindrical domain, and that h^{bot} is compatible with h^Σ on the connecting edge of the bottom and the lateral boundary of the cylindrical domain. These kind of compatibility conditions only arise with respect to the Neumann-Dirichlet boundary operator and with respect to the perfect slip and free slip boundary operators, because they have different conditions for the tangential and normal part of the boundary. First, we consider the Neumann-Dirichlet boundary conditions, followed by the perfect slip/free slip boundary conditions.

Neumann-Dirichlet: We proceed in two steps. In the first step we study the compatibility of the data h^{top} with h^Σ on the connecting edge of the top and the lateral boundary of the cylindrical domain Ω. In the second step we consider the compatibility of the data h^{bot} with h^Σ on the connecting edge of the bottom and the lateral boundary of Ω, which we obtain analogously to the one of the top edge. For the top of Ω and the lateral boundary we have

$$[h^{\text{top}} \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}} = [u \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}}, \tag{1.4.1}$$

$$[h^\Sigma \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} = [u \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}}, \tag{1.4.2}$$

$$[\delta \partial_{\nu_{\Gamma_{\text{top}}}} P_{\Gamma_{\text{top}}} u \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} = [P_{\Gamma_{\text{top}}} h^{\text{top}} \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}}, \tag{1.4.3}$$

$$[\delta \partial_{\nu_\Sigma} P_\Sigma u \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}} = [P_\Sigma h^\Sigma \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}}, \tag{1.4.4}$$

if $p > 2$. Here, $[\cdot]_{\mathcal{R}^{\text{top}}}$ denotes the trace on the edge $\mathcal{R}^{\text{top}}$ of the cylinder Ω. Considering the left-hand side of (1.4.3), a straightforward calculation shows

$$\begin{aligned}
[\delta \partial_{\nu_{\Gamma_{\text{top}}}} P_{\Gamma_{\text{top}}} u \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} &= [\delta P_{\Gamma_{\text{top}}} \partial_{\nu_{\Gamma_{\text{top}}}} u \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= [\delta P_{\Gamma_{\text{top}}} (\nabla u)^T \nu_{\Gamma_{\text{top}}} \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= \delta[\nu_{\Gamma_{\text{top}}} \cdot (\nabla u P_{\Gamma_{\text{top}}}) \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= \delta[\nu_{\Gamma_{\text{top}}} \cdot (\nabla u) \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= \delta[(\nabla u)^T \nu_{\Gamma_{\text{top}}} \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= \delta[(\partial_{\nu_{\Gamma_{\text{top}}}} u) \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= \delta[\partial_{\nu_{\Gamma_{\text{top}}}} (u \cdot \nu_\Sigma) - u \cdot \partial_{\nu_{\Gamma_{\text{top}}}} \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= \delta[\partial_{\nu_{\Gamma_{\text{top}}}} (h^\Sigma \cdot \nu_\Sigma)]_{\mathcal{R}^{\text{top}}} \\
&= \delta[(\partial_{\nu_{\Gamma_{\text{top}}}} h^\Sigma) \cdot \nu_\Sigma + h^\Sigma \cdot \partial_{\nu_{\Gamma_{\text{top}}}} \nu_\Sigma]_{\mathcal{R}^{\text{top}}} \\
&= \delta[(\partial_{\nu_{\Gamma_{\text{top}}}} h^\Sigma) \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}}
\end{aligned}$$

where we used (1.4.2) and the fact that $\partial_{\nu_{\Gamma_{\text{top}}}} P_{\Gamma_{\text{top}}} = P_{\Gamma_{\text{top}}} \partial_{\nu_{\Gamma_{\text{top}}}}$, $P_{\Gamma_{\text{top}}} \nu_\Sigma = \nu_\Sigma$, $\partial_{\nu_{\Gamma_{\text{top}}}} \nu_\Sigma = 0$, $\partial_{\nu_\Sigma} \nu_{\Gamma_{\text{top}}} = 0$ on $\mathcal{R}^{\text{top}}$. Considering the left-hand side of (1.4.4), we similarly deduce

$$[\delta \partial_{\nu_\Sigma} P_\Sigma u \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}} = \delta[(\partial_{\nu_\Sigma} h^{\text{top}}) \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}}$$

again by using (1.4.1) and the fact that $\partial_{\nu_\Sigma} P_\Sigma = P_\Sigma \partial_{\nu_\Sigma}$, $P_\Sigma \nu_{\Gamma_{\text{top}}} = \nu_{\Gamma_{\text{top}}}$, $\partial_{\nu_{\Gamma_{\text{top}}}} \nu_\Sigma = 0$, $\partial_{\nu_\Sigma} \nu_{\Gamma_{\text{top}}} = 0$ on $\mathcal{R}^{\text{top}}$. From the right-hand side of (1.4.3) we obtain

$$[P_{\Gamma_{\text{top}}} h^{\text{top}} \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}} = [h^{\text{top}} \cdot P_{\Gamma_{\text{top}}} \nu_\Sigma]_{\mathcal{R}^{\text{top}}} = [h^{\text{top}} \cdot \nu_\Sigma]_{\mathcal{R}^{\text{top}}}$$

and analogously from the right-hand side of (1.4.4) we get

$$[P_\Sigma h^\Sigma \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}} = [h^\Sigma \cdot \nu_{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}}.$$

Combining these results, we derive the compatibility conditions

$$\begin{aligned} \delta[(\partial_{\nu_{\Gamma_{\rm top}}} h^{\Sigma})]_{\mathcal{R}^{\rm top}} \cdot \nu_{\Sigma} &= [h^{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}} \cdot \nu_{\Sigma} \quad \text{if } p > 2, \\ \delta[(\partial_{\nu_{\Sigma}} h^{\rm top})]_{\mathcal{R}^{\rm top}} \cdot \nu_{\Gamma_{\rm top}} &= [h^{\Sigma}]_{\mathcal{R}^{\rm top}} \cdot \nu_{\Gamma_{\rm top}} \quad \text{if } p > 2, \end{aligned} \tag{$(C4)_{ND}$}$$

for the data h on the connecting edge of top and lateral boundary of Ω. Here, $\nu_\Lambda = [\nu_\Lambda]_{\mathcal{R}^{\rm top}}$ and $\nu_\Lambda = [\nu_\Lambda]_{\mathcal{R}^{\rm bot}}$, $\Lambda \in \{\Gamma_{\rm top}, \Gamma_{\rm bot}, \Sigma\}$, represent the extension of ν_Λ to the boundary of the edges $\mathcal{R}^{\rm top}$ and $\mathcal{R}^{\rm bot}$, respectively. Analogously we obtain

$$\begin{aligned} \delta[(\partial_{\nu_{\Gamma_{\rm bot}}} h^{\Sigma})]_{\mathcal{R}^{\rm bot}} \cdot \nu_{\Sigma} &= [h^{\Gamma_{\rm bot}}]_{\mathcal{R}^{\rm bot}} \cdot \nu_{\Sigma} \quad \text{if } p > 2, \\ \delta[(\partial_{\nu_{\Sigma}} h^{\rm bot})]_{\mathcal{R}^{\rm bot}} \cdot \nu_{\Gamma_{\rm bot}} &= [h^{\Sigma}]_{\mathcal{R}^{\rm bot}} \cdot \nu_{\Gamma_{\rm bot}} \quad \text{if } p > 2, \end{aligned} \tag{$(C5)_{ND}$}$$

for the compatibility of the data h on the connecting edge of bottom and lateral boundary of Ω.

Perfect slip/free slip: As above, we proceed in two steps. First, we study the compatibility of the data $h^{\rm top}$ with h^Σ on the connecting edge of the top and the lateral boundary of the cylindrical domain Ω, and then infer the compatibility of the data $h^{\rm bot}$ with h^Σ on the connecting edge of bottom and lateral boundary. For the top of Ω and the lateral boundary we have

$$[h^{\rm top} \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}} = [u \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}}, \tag{1.4.5}$$

$$[h^{\Sigma} \cdot \nu_{\Sigma}]_{\mathcal{R}^{\rm top}} = [u \cdot \nu_{\Sigma}]_{\mathcal{R}^{\rm top}}, \tag{1.4.6}$$

$$[\pm P_{\Gamma_{\rm top}} D_\pm(u) \nu_{\Gamma_{\rm top}} \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}} = [P_{\Gamma_{\rm top}} h^{\rm top} \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}}, \tag{1.4.7}$$

$$[\pm P_{\Sigma} D_\pm(u) \nu_{\Sigma} \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}} = [P_{\Sigma} h^{\Sigma} \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}}, \tag{1.4.8}$$

if $p > 2$. Again, $[\cdot]_{\mathcal{R}^{\rm top}}$ denotes the trace on the edge $\mathcal{R}^{\rm top}$ of the cylinder Ω. A straightforward calculation shows that we may rewrite the left-hand side of (1.4.7) like

$$\begin{aligned} [\pm P_{\Gamma_{\rm top}} D_\pm(u) \nu_{\Gamma_{\rm top}} \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}} &= \tfrac{1}{2}[(\partial_{\nu_{\Gamma_{\rm top}}} u) \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}} \pm \tfrac{1}{2}[(\partial_{\nu_\Sigma} u) \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}} \\ &= \tfrac{1}{2}[(\partial_{\nu_{\Gamma_{\rm top}}} h^\Sigma) \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}} \pm \tfrac{1}{2}[(\partial_{\nu_\Sigma} h^{\rm top}) \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}} \end{aligned}$$

where we used (1.4.5), (1.4.6) and the fact that $P_{\Gamma_{\rm top}} \nu_\Sigma = \nu_\Sigma$, $\partial_{\nu_{\Gamma_{\rm top}}} \nu_\Sigma = 0$, $\partial_{\nu_\Sigma} \nu_{\Gamma_{\rm top}} = 0$ on $\mathcal{R}^{\rm top}$. Considering the left-hand side of (1.4.8), we similarly arrive at

$$[\pm P_{\Sigma} D_\pm(u) \nu_{\Sigma} \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}} = \pm\tfrac{1}{2}[(\partial_{\nu_{\Gamma_{\rm top}}} h^\Sigma) \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}} + \tfrac{1}{2}[(\partial_{\nu_\Sigma} h^{\rm top}) \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}},$$

where again we used (1.4.5), (1.4.6) and the fact that $P_\Sigma \nu_{\Gamma_{\rm top}} = \nu_{\Gamma_{\rm top}}$, $\partial_{\nu_{\Gamma_{\rm top}}} \nu_\Sigma = 0$, $\partial_{\nu_\Sigma} \nu_{\Gamma_{\rm top}} = 0$ on $\mathcal{R}^{\rm top}$. Moreover, from the right-hand side of (1.4.7) we see that

$$[P_{\Gamma_{\rm top}} h^{\rm top} \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}} = [h^{\rm top} \cdot P_{\Gamma_{\rm top}} \nu_\Sigma]_{\mathcal{R}^{\rm top}} = [h^{\rm top} \cdot \nu_\Sigma]_{\mathcal{R}^{\rm top}}$$

and analogously from the right-hand side of (1.4.8) that

$$[P_\Sigma h^\Sigma \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}} = [h^\Sigma \cdot \nu_{\Gamma_{\rm top}}]_{\mathcal{R}^{\rm top}}.$$

Combining these results, we finally arrive at the compatibility conditions

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\rm top}} \cdot \nu_{\Gamma_{\rm top}} &= \pm[h^{\rm top}]_{\mathcal{R}^{\rm top}} \cdot \nu_\Sigma \quad \text{if } p > 2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\rm top}}} h^\Sigma]_{\mathcal{R}^{\rm top}} \cdot \nu_\Sigma \pm \tfrac{1}{2}[\partial_{\nu_\Sigma} h^{\rm top}]_{\mathcal{R}^{\rm top}} \cdot \nu_{\Gamma_{\rm top}} &= [h^{\rm top}]_{\mathcal{R}^{\rm top}} \cdot \nu_\Sigma \quad \text{if } p > 2, \end{aligned} \tag{$(C4)_{S\pm}$}$$

for the data h on the connecting edge of top and lateral boundary of Ω. Again $\nu_\Lambda = [\nu_\Lambda]_{\mathcal{R}^{\mathrm{top}}}$ and $\nu_\Lambda = [\nu_\Lambda]_{\mathcal{R}^{\mathrm{bot}}}$, $\Lambda \in \{\Gamma_{\mathrm{top}}, \Gamma_{\mathrm{bot}}, \Sigma\}$, represent the extension of ν_Λ to the boundary of the edges $\mathcal{R}^{\mathrm{top}}$ and $\mathcal{R}^{\mathrm{bot}}$, respectively. Analogously we obtain

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\mathrm{bot}}} \cdot \nu_{\Gamma_{\mathrm{bot}}} &= \pm [h^{\mathrm{bot}}]_{\mathcal{R}^{\mathrm{bot}}} \cdot \nu_\Sigma \quad \text{if } p > 2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\mathrm{bot}}}} h^\Sigma]_{\mathcal{R}^{\mathrm{bot}}} \cdot \nu_\Sigma \pm \tfrac{1}{2}[\partial_{\nu_\Sigma} h^{\mathrm{bot}}]_{\mathcal{R}^{\mathrm{bot}}} \cdot \nu_{\Gamma_{\mathrm{bot}}} &= [h^{\mathrm{bot}}]_{\mathcal{R}^{\mathrm{bot}}} \cdot \nu_\Sigma \quad \text{if } p > 2, \end{aligned} \qquad (C5)_{S\pm}$$

for the compatibility of the data h on the connecting edge of bottom and lateral boundary of Ω.

1.5 Trace Maps

This section is dedicated to the trace operators with respect to the boundary conditions defined in Section 1.4. That is, such a trace operator map a vector field to one of the boundary conditions defined in Section 1.4 with respect to all parts Γ_{top}, Σ, Γ_{bot} of the boundary Γ. For example,

$$\begin{aligned} \gamma\colon \mathbb{E}_p^u(J) &\longrightarrow \tilde{\mathbb{F}}_p^h(J) \\ c &\longmapsto \big((c_{|\Gamma_{\mathrm{top}}} \cdot \nu)\nu - P_\Gamma D_-(c)_{|\Gamma_{\mathrm{top}}}\nu, \\ &\qquad (c_{|\Sigma} \cdot \nu)\nu - P_\Gamma D_-(c)_{|\Sigma}\nu, \\ &\qquad (c_{|\Gamma_{\mathrm{bot}}} \cdot \nu)\nu - P_\Gamma D_-(c)_{|\Gamma_{\mathrm{bot}}}\nu\big) \end{aligned}$$

is the trace operator with respect to the perfect slip boundary operator $\mathcal{B}^{S-}$. Here the data space is defined as

$$\begin{aligned} \tilde{\mathbb{F}}_p^h(J) := \{&(h^{\mathrm{top}}, h^\Sigma, h^{\mathrm{bot}}) \in \mathbb{F}_p^{\Gamma_{\mathrm{top}}}(J) \times \mathbb{F}_p^\Sigma(J) \times \mathbb{F}_p^{\Gamma_{\mathrm{bot}}}(J) : \\ &h^{\mathrm{top}} \text{ and } h^\Sigma \text{ fulfil } (C4)_{S-};\ h^{\mathrm{bot}} \text{ and } h^\Sigma \text{ fulfil } (C5)_{S-}\}. \end{aligned}$$

In particular, we would like to prove that the trace operators with respect to the boundary conditions are retractions. However, for the trace operator with respect to the Robin boundary operator $\mathcal{B}^R$ it is not necessary to prove that it is a retraction. This trace is used exclusively in the context of the parabolic problem with Robin boundary conditions $(P|J)_R$ in Section 2.2, and there we are able to rely on a result about the trace operator with respect to Neumann boundary conditions and a perturbation argument. That the Neumann trace operator is a retraction on a three-dimensional cylindrical domain $\Omega := A \times (-a, a) \subseteq \mathbb{R}^3$ has already been proven by Bothe, Köhne, Maier and Saal in [12, Lemma 3.4]. In Proposition 1.21 we proceed similarly to proof of this lemma in order to prove that the trace operator with respect to perfect slip boundary conditions is a linear retraction on n-dimensional cylindrical domains $\Omega \subseteq \mathbb{R}^n$. Accordingly, the result of Bothe, Köhne, Maier and Saal can also be proved for n-dimensional cylindrical domains $\Omega \subseteq \mathbb{R}^n$ and we obtain the following lemma.

Lemma 1.16. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $\Omega = A \times (-a, a)$ be a cylindrical domain with $\Gamma = \Gamma_{top} \,\dot\cup\, \Gamma_{bot} \,\dot\cup\, \Sigma$ as the smooth part of the boundary of Ω. Let $1 < p < \infty$ with $p \neq 3$ and let $J = (0, T)$ with $T > 0$. Let also*

$$\tilde{\mathbb{K}}_p^h(J) := \{(h^{top}, h^\Sigma, h^{bot}) \in \mathbb{F}_p^{R,\Gamma_{top}}(J) \times \mathbb{F}_p^{R,\Sigma}(J) \times \mathbb{F}_p^{R,\Gamma_{bot}}(J)\}.$$

Then the Neumann trace operator

$$\begin{aligned} \gamma\colon \mathbb{E}_p^z(J) &\longrightarrow \tilde{\mathbb{K}}_p^h(J) \\ c &\longmapsto \big(\partial_\nu c_{|\Gamma_{top}}, \partial_\nu c_{|\Sigma}, \partial_\nu c_{|\Gamma_{bot}}\big) \end{aligned}$$

is a linear retraction.

Remark 1.17. With the same assumptions as in Lemma 1.16 and the same argument as in [12, Lemma 3.4], one can also show that the trace operator

$$\partial_\nu \colon W_p^3(\Omega) \longrightarrow W_p^{2-1/p}(\Gamma)$$

is a continuous linear retraction.

The trace operators with respect to the Neumann-Dirichlet, the perfect slip and the free slip boundary conditions are more complicated to work with, since their boundary conditions differ in the tangential and in the normal direction. For these trace operators, we must pay particular attention to the geometry of the cylindrical domain Ω and its boundary Γ. We show exemplarily that the trace operator with respect to the perfect slip boundary operator is a retraction. The proof of the retraction property of the trace operators with respect to the Neumann-Dirichlet and the free slip boundary operators are simpler versions of this proof and we omit them in this thesis. To show that the trace operator with respect to the perfect slip boundary operator is a retraction, we must first prove two lemmas. The first lemma gives us the boundedness of a trace map on a hyper-surface.

Lemma 1.18. *Let $c > 0$ and let $D := \mathbb{R}^{n-1} \times \{c\}$ be a affine hyper-surface. Assume $J = (0,T)$ and $1 < p < \infty$. Then the following assertions are valid.*

(i) The trace map

$$\gamma \colon W_p^{1-1/2p}(J, L_p(D)) \cap L_p(J, W_p^{2-1/p}(D)) \longrightarrow W_p^{2-3/p}(D) \tag{1.5.1}$$

is bounded for $p > \frac{3}{2}$.

(ii) The trace map

$$\gamma \colon W_p^{1/2-1/2p}(J, L_p(D)) \cap L_p(J, W_p^{1-1/p}(D)) \longrightarrow W_p^{1-3/p}(D) \tag{1.5.2}$$

is bounded for $p > 3$.

Proof. Due to [8, Theorem 3.8.1], we can identify

$$W_p^{1-1/2p}(J, L_p(D)) \cap L_p(J, W_p^{2-1/p}(D)) = B_{p,p}^{2-1/p,(2,1)}(J \times D)$$

and

$$W_p^{1/2-1/2p}(J, L_p(D)) \cap L_p(J, W_p^{1-1/p}(D)) = B_{p,p}^{1-1/p,(2,1)}(J \times D),$$

as $1 - \frac{1}{2p}, 2 - \frac{1}{p}, \frac{1}{2} - \frac{1}{2p}, 1 - \frac{1}{p} \notin \mathbb{N}_0$ are non-integer numbers for $1 < p < \infty$. Here, $B_{p,p}^{s,(\omega_1,\omega_2)}$ denotes an anisotropic Besov space, see page 14 in Section 1.1. From [8, Theorem 4.5.2] we obtain that the trace maps

$$\gamma \colon B_{p,p}^{2-1/p,(2,1)}(J \times D) \longrightarrow B_{p,p}^{2-1/p-2/p,(1)}(D) = B_{p,p}^{2-3/p}(D)$$

and

$$\gamma \colon B_{p,p}^{1-1/p,(2,1)}(J \times D) \longrightarrow B_{p,p}^{1-1/p-2/p,(1)}(D) = B_{p,p}^{1-3/p}(D).$$

are bounded. Together with the identities

$$B_{p,p}^{2-3/p}(D) = W_p^{2-3/p}(D), \quad B_{p,p}^{1-3/p}(D) = W_p^{1-3/p}(D)$$

for $p > \frac{3}{2}$ and $p > 3$, respectively, the assertion follows. □

Remark 1.19. Lemma 1.18 is also applicable for D as the boundary of a bounded $\mathcal{C}^{3-}$-domain. This case can be proven by localisation.

The second lemma gives us the surjectivity of the trace operator with respect to the perfect slip and free slip boundary operators on a $\mathcal{C}^{k,\lambda}$-domain with compact boundary, i. e. not yet on a cylindrical domain.

Lemma 1.20. *Let $k \geqslant 2$, $\lambda \geqslant 0$ and let A be a $\mathcal{C}^{k,\lambda}$- domain with compact boundary ∂A. Suppose that $1 < p < \infty$ and $1+\frac{1}{p} < s < k+\lambda+\frac{1}{p}$. Moreover, let $D_{\pm}(c) := \frac{1}{2}(\nabla c \pm (\nabla c)^T)$ and*

$$\mathbb{F}^c := \{z \in W_p^{s-1-1/p}(\partial A)^n : z \cdot \nu \in W_p^{s-1/p}(\partial A)\}.$$

Then the trace operator

$$\begin{aligned}\gamma\colon W_p^s(A) &\longrightarrow \mathbb{F}^c\\ c &\longmapsto -P_{\partial A} D_{\pm}(c)\nu + (c\cdot\nu)\nu\end{aligned}$$

is a surjective. In case $s - \frac{1}{p}$ is not an integer there exists a bounded, linear right inverse of γ.

Proof. Let $(g, h) \in W_p^{s-1-1/p}(\partial A)^n \times W_p^{s-1/p}(\partial A)^n$. We prove the existence of a $c \in W_p^s(A)^n$, which satisfies the following system

$$\begin{aligned}-P_{\partial A} D_{\pm}(c)\nu &= P_{\partial A} g && \text{on } \partial A,\\ P_{\partial A} c &= P_{\partial A} h && \text{on } \partial A,\\ c\cdot\nu &= h\cdot\nu && \text{on } \partial A.\end{aligned}\tag{1.5.3}$$

Consider

$$\begin{aligned}-P_{\partial A} D_{\pm}(c)\nu &= -\tfrac{1}{2} P_{\partial A}(\nabla c \pm (\nabla c)^T)\nu\\ &= \mp\tfrac{1}{2} P_{\partial A}\partial_\nu c - \tfrac{1}{2} P_{\partial A}(\nabla c)\nu\\ &= \mp\tfrac{1}{2} P_{\partial A}\partial_\nu c + \tfrac{1}{2} P_{\partial A}\left((\nabla\nu)c - \nabla(c\cdot\nu)\right)\\ &= \mp\tfrac{1}{2} P_{\partial A}\partial_\nu c + \tfrac{1}{2} P_{\partial A}(\nabla\nu)c - \tfrac{1}{2}\nabla_{\partial A}(c\cdot\nu)\\ &= \mp\tfrac{1}{2} P_{\partial A}\partial_\nu c + \tfrac{1}{2}(\nabla_{\partial A}\nu)c - \tfrac{1}{2}\nabla_{\partial A}(h\cdot\nu),\end{aligned}$$

with $\nabla_{\partial A}$ being the gradient on the boundary. The last equation is valid if and only if $c\cdot\nu = h\cdot\nu$ on the boundary. Then (1.5.3) is equivalent to

$$\begin{aligned}P_{\partial A}\partial_\nu c &= \mp(2P_{\partial A} g - (\nabla_{\partial A}\nu)c + \nabla_{\partial A}(h\cdot\nu)) && \text{on } \partial A,\\ P_{\partial A} c &= P_{\partial A} h && \text{on } \partial A,\\ c\cdot\nu &= h\cdot\nu && \text{on } \partial A.\end{aligned}$$

The relation $-(\nabla_{\partial A}\nu)c = -(\nabla_{\partial A}\nu)h$ is valid, since $c = h$ on ∂A. The vector $-(\nabla_{\partial A}\nu)c$ is tangential on the boundary. Note that $\nabla_{\partial A}(h\cdot\nu)$ is tangential on ∂A, too. As a consequence, $\tilde{g}\cdot\nu = 0$ for

$$W_p^{1-3/p}(\partial A) \ni \tilde{g} := \mp\left(2P_{\partial A} g - (\nabla_{\partial A}\nu)c + \nabla_{\partial A}(h\cdot\nu)\right).$$

We can then rewrite (1.5.3) as

$$\begin{aligned}P_{\partial A}\partial_\nu c &= P_{\partial A}\tilde{g} && \text{on } \partial A,\\ P_{\partial A} c &= P_{\partial A} h && \text{on } \partial A,\\ c\cdot\nu &= h\cdot\nu && \text{on } \partial A.\end{aligned}\tag{1.5.4}$$

According to [37, Theorem 2] the trace

$$\begin{aligned} \partial_\nu c &= \tilde{g} \qquad \text{on } \partial A, \\ c &= h \qquad \text{on } \partial A, \end{aligned}$$

is surjective and has, in case $s - \frac{1}{p}$ is not an integer, a bounded linear right inverse. Therefore (1.5.4) and thus (1.5.3) are also surjective and have a bounded linear right inverse, for the above case. By choosing $g := P_{\partial A}\left(-P_{\partial A} D_\pm(c)\nu + (c \cdot \nu)\nu\right) = P_{\partial A} z$ and $h := \big((-P_{\partial A} D_\pm(c)\nu + (c \cdot \nu)\nu) \cdot \nu\big)\nu = (z \cdot \nu)\nu$ for a given $z \in \mathbb{F}^c$ we have shown the assertion. □

With the help of these two lemmas we can now go on to prove the retraction property of the trace operator with respect to the perfect slip boundary operator on a cylindrical domain.

Proposition 1.21. *Let $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$, $D_-(u) = \frac{1}{2}((\nabla u - (\nabla u)^T)$ and $J = (0,T)$. Assume $A \subseteq \mathbb{R}^{n-1}$ to be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $\Omega := A \times (-a, a)$ a cylindrical domain with boundary Γ. Let also*

$$\begin{aligned} \tilde{\mathbb{F}}_p^h(J) := \{&(h^{top}, h^\Sigma, h^{bot}) \in \mathbb{F}_p^{\Gamma_{top}}(J) \times \mathbb{F}_p^\Sigma(J) \times \mathbb{F}_p^{\Gamma_{bot}}(J) : \\ &h^{top} \text{ and } h^\Sigma \text{ fulfil } (C4)_{S-};\ h^{bot} \text{ and } h^\Sigma \text{ fulfil } (C5)_{S-}\}. \end{aligned}$$

Then the trace operator

$$\begin{aligned} \gamma\colon \mathbb{E}_p^u(J) &\longrightarrow \tilde{\mathbb{F}}_p^h(J) \\ c &\longmapsto \big((c_{|\Gamma_{top}} \cdot \nu)\nu - P_\Gamma D_-(c)_{|\Gamma_{top}}\nu, \\ &\qquad (c_{|\Sigma} \cdot \nu)\nu - P_\Gamma D_-(c)_{|\Sigma}\nu, \\ &\qquad (c_{|\Gamma_{bot}} \cdot \nu)\nu - P_\Gamma D_-(c)_{|\Gamma_{bot}}\nu\big) \end{aligned}$$

is a bounded linear retraction.

Proof. This proof is to some extend similar to the proof of [12, Lemma 3.4], where a different trace map was considered. Let $(h^{\mathrm{top}}, h^\Sigma, h^{\mathrm{bot}}) \in \tilde{\mathbb{F}}_p^h(J)$. We prove the existence of a vector field $c \in \mathbb{E}_p^u(J)$, which satisfies the system

$$\begin{aligned} c \cdot \nu_{\Gamma_{\mathrm{top}}} &= h^{\mathrm{top}} \cdot \nu_{\Gamma_{\mathrm{top}}} && \text{on } J \times \Gamma_{\mathrm{top}}, \\ -P_{\Gamma_{\mathrm{top}}} D_-(c)\nu_{\Gamma_{\mathrm{top}}} &= P_{\Gamma_{\mathrm{top}}} h^{\mathrm{top}} && \text{on } J \times \Gamma_{\mathrm{top}}, \\ c \cdot \nu_\Sigma &= h^\Sigma \cdot \nu_\Sigma && \text{on } J \times \Sigma, \\ -P_\Sigma D_-(c)\nu_\Sigma &= P_\Sigma h^\Sigma && \text{on } J \times \Sigma, \\ c \cdot \nu_{\Gamma_{\mathrm{bot}}} &= h^{\mathrm{bot}} \cdot \nu_{\Gamma_{\mathrm{bot}}} && \text{on } J \times \Gamma_{\mathrm{bot}}, \\ -P_{\Gamma_{\mathrm{bot}}} D_-(c)\nu_{\Gamma_{\mathrm{bot}}} &= P_{\Gamma_{\mathrm{bot}}} h^{\mathrm{bot}} && \text{on } J \times \Gamma_{\mathrm{bot}}. \end{aligned} \tag{1.5.5}$$

For that purpose we split (1.5.5) into the two systems

$$\begin{aligned} v \cdot \nu_{\Gamma_{\mathrm{top}}} &= h^{\mathrm{top}} \cdot \nu_{\Gamma_{\mathrm{top}}} && \text{on } J \times \Gamma_{\mathrm{top}}, \\ -P_{\Gamma_{\mathrm{top}}} D_-(v)\nu_{\Gamma_{\mathrm{top}}} &= P_{\Gamma_{\mathrm{top}}} h^{\mathrm{top}} && \text{on } J \times \Gamma_{\mathrm{top}}, \\ v \cdot \nu_{\Gamma_{\mathrm{bot}}} &= h^{\mathrm{bot}} \cdot \nu_{\Gamma_{\mathrm{bot}}} && \text{on } J \times \Gamma_{\mathrm{bot}}, \\ -P_{\Gamma_{\mathrm{bot}}} D_-(v)\nu_{\Gamma_{\mathrm{bot}}} &= P_{\Gamma_{\mathrm{bot}}} h^{\mathrm{bot}} && \text{on } J \times \Gamma_{\mathrm{bot}}, \end{aligned} \tag{1.5.6}$$

and

$$
\begin{aligned}
z \cdot \nu_{\Gamma_{\text{top}}} &= 0 && \text{on } J \times \Gamma_{\text{top}}, \\
-P_{\Gamma_{\text{top}}} D_-(z)\nu_{\Gamma_{\text{top}}} &= 0 && \text{on } J \times \Gamma_{\text{top}}, \\
z \cdot \nu_\Sigma &= (h^\Sigma - v)\cdot \nu_\Sigma && \text{on } J \times \Sigma, \\
-P_\Sigma D_-(z)\nu_\Sigma &= P_\Sigma(h^\Sigma + D_-(v)\nu_\Sigma) && \text{on } J \times \Sigma, \\
z \cdot \nu_{\Gamma_{\text{bot}}} &= 0 && \text{on } J \times \Gamma_{\text{bot}}, \\
-P_{\Gamma_{\text{bot}}} D_-(z)\nu_{\Gamma_{\text{bot}}} &= 0 && \text{on } J \times \Gamma_{\text{bot}},
\end{aligned}
\tag{1.5.7}
$$

with $c = v + z$. We proceed in two steps.

Step 1. To prove the existence of a solution to system (1.5.6), we are using maximal regularity of the Stokes equations with boundary conditions, which correspond to the first two lines of (1.5.6), on a lower half-space and maximal regularity of the Stokes equations with boundary conditions, which correspond to the last two lines of (1.5.6), on an upper half-space. By adding the two resulting solutions and multiplying them with appropriate cut-off functions, we obtain a solution of (1.5.6). The main issue here is to extend h^{top} and h^{bot} to a lower and upper half-space, respectively. Moreover we have to define data that accomplishes the appropriate compatibility conditions of such Stokes equations.

We work with the boundaries $\Gamma_{\text{top}} = A \times \{a\}$, $\Gamma_{\text{bot}} = A \times \{-a\}$ and the outer normals $\nu_{\Gamma_{\text{top}}} = \mathrm{e}_n$ on Γ_{top}, $\nu_{\Gamma_{\text{bot}}} = -\mathrm{e}_n$ on Γ_{bot}. Define the half-spaces

$$H_a := \mathbb{R}^{n-1} \times (-\infty, a), \quad H_{-a} := \mathbb{R}^{n-1} \times (-a, \infty).$$

Then, we have $\Gamma_{\text{top}} \subseteq \partial H_a$ and $\Gamma_{\text{bot}} \subseteq \partial H_{-a}$. There is an extension of h^{top} to

$$
\begin{aligned}
\tilde{h}^{\text{top}} \in \mathbb{F}_p^{\partial H_a}(J) := \{h \in W_p^{1/2-1/2p}(J, L_p(\partial H_a))^n \cap L_p(J, W_p^{1-1/p}(\partial H_a))^n :& \\
h \cdot \nu \in W_p^{1-1/2p}(J, L_p(\partial H_a)) \cap L_p(J, W_p^{2-1/p}(\partial H_a))\}&
\end{aligned}
$$

and of h^{bot} to

$$
\begin{aligned}
\tilde{h}^{\text{bot}} \in \mathbb{F}_p^{\partial H_{-a}}(J) := \{h \in W_p^{1/2-1/2p}(J, L_p(\partial H_{-a}))^n \cap L_p(J, W_p^{1-1/p}(\partial H_{-a}))^n :& \\
h \cdot \nu \in W_p^{1-1/2p}(J, L_p(\partial H_{-a})) \cap L_p(J, W_p^{2-1/p}(\partial H_{-a}))\},&
\end{aligned}
$$

using [4, Theorem 4.26].

Next, we have to define appropriate data that accomplish the compatibility conditions of the Stokes equations with perfect slip boundary conditions on H_a and H_{-a}, respectively. We do this exemplarily for $\tilde{h}^{\text{top}}$ and the Stokes equations on H_a. The case of $\tilde{h}^{\text{bot}}$ and H_{-a} is analogous. From Lemma 1.18, we obtain

$$\tilde{h}^{\text{top}}(0) \cdot \nu \in W_p^{2-3/p}(\partial H_a),$$

and

$$P_\Gamma \tilde{h}^{\text{top}}(0) \in W_p^{1-3/p}(\partial H_a).$$

To get an appropriate initial value $\tilde{v}_0^{\text{top}}$ for the Stokes equations, we choose $\tilde{v}_0^{\text{top}} = (\bar{u}_0^{\text{top}}, \bar{w}_0^{\text{top}}) = (0,0) \in W_p^{2-2/p}(H_a)^n$ in the case of $1 < p < \frac{3}{2}$. In case of $\frac{3}{2} < p < 3$ we again choose $\bar{u}_0^{\text{top}} = 0 \in W_p^{2-2/p}(H_a)^{n-1}$, but we choose $\bar{w}_0^{\text{top}} \in W_p^{2-2/p}(H_a)^1$, such that $[\bar{w}_0^{\text{top}}]_{\partial H_a} = \tilde{h}_w^{\text{top}}(0)$. Since $[\cdot]_{\partial H_a} \colon W_p^{2-2/p}(H_a) \longrightarrow W_p^{2-3/p}(\partial H_a)$ is surjective, cf. [37, Theorem 2], such a $\bar{w}_0^{\mathrm{top}}$ exists. Finally, in case $p > 3$ we choose the variable $\bar{w}_0^{\mathrm{top}}$ as in the case before and $\bar{u}_0^{\mathrm{top}} \in W_p^{2-2/p}(H_a)^{n-1}$, such that

$$-\frac{1}{2}\partial_\nu \bar{u}_0^{\mathrm{top}} = \tilde{h}_u^{\mathrm{top}}(0) - \frac{1}{2}\nabla_\Gamma \tilde{h}_w^{\mathrm{top}}(0).$$

Such a $\bar{u}_0^{\mathrm{top}}$ exists, because $\partial_\nu \colon W_p^{2-2/p}(\partial H_a) \longrightarrow W_p^{1-3/p}(H_a)$ is surjective, due to [37, Theorem 2]. We use the notation $\nabla_\Gamma := (\partial_1, \partial_2, \cdots, \partial_{n-1})^{\mathrm{T}}$ and $\tilde{h}^{\mathrm{top}} = (\tilde{h}_u^{\mathrm{top}}, \tilde{h}_w^{\mathrm{top}})$, where $\tilde{h}_u^{\mathrm{top}}$ is the tangential and $\tilde{h}_w^{\mathrm{top}}$ is the normal part of $\tilde{h}^{\mathrm{top}}$. By choosing $\tilde{v}_0^{\mathrm{top}}$ as described above, it satisfies

$$\begin{aligned} \tilde{v}_0^{\mathrm{top}} \cdot \nu_{\Gamma_{\mathrm{top}}} &= \tilde{h}^{\mathrm{top}}(0) \cdot \nu_{\Gamma_{\mathrm{top}}} && \text{on } \partial H_a \text{ if } p > \tfrac{3}{2}, \\ -P_{\Gamma_{\mathrm{top}}} D_-(\tilde{v}_0^{\mathrm{top}}) \nu_{\Gamma_{\mathrm{top}}} &= P_{\Gamma_{\mathrm{top}}} \tilde{h}^{\mathrm{top}}(0) && \text{on } \partial H_a \text{ if } p > 3. \end{aligned}$$

Now, it is left to define a $\tilde{g}^{\mathrm{top}}$ that satisfies

$$\begin{gathered} \operatorname{div}(\tilde{v}_0^{\mathrm{top}}) = \tilde{g}^{\mathrm{top}}(0) \\ \mathcal{F}(\tilde{g}^{\mathrm{top}}, \tilde{h}^{\mathrm{top}} \cdot \nu_{\Gamma_{\mathrm{top}}}) \in H_{p'}^1(J, (H_{p'}^1(H_a), \|\nabla \cdot\|_{p'})'). \end{gathered}$$

For this, consider the system

$$\begin{aligned} \partial_t \tilde{\varphi} - \mu \Delta \tilde{\varphi} &= 0 && \text{in } J \times H_a, \\ [\varphi_u]_{\partial H_a} &= \psi && \text{on } J \times \partial H_a, \\ [\varphi_w]_{\partial H_a} &= \tilde{h}_w^{\mathrm{top}} && \text{on } J \times \partial H_a, \\ \partial_t \psi - \mu \Delta_\Gamma \psi &= 0 && \text{in } J \times \partial H_a, \\ \psi(0) &= [\bar{u}_0^{\mathrm{top}}]_{\partial H_a} && \text{in } \partial H_a, \\ \tilde{\varphi}(0) &= \tilde{v}_0^{\mathrm{top}} && \text{in } H_a, \end{aligned}$$

with $\Delta_\Gamma := \sum_{k=1}^{n-1} \partial_k^2$ and $\tilde{\varphi} = (\tilde{\varphi}_u, \tilde{\varphi}_w)$, where $\tilde{\varphi}_u$ is the tangential and $\tilde{\varphi}_w$ is the normal part of $\tilde{\varphi}$. The system has a unique solution $\tilde{\varphi} \in W_p^1(J, L_p(H_a))^n \cap L_p(J, W_p^2(H_a))^n$, cf. [18, Theorem 2.1]. Define $\tilde{g}^{\mathrm{top}} := \operatorname{div}(\tilde{\varphi})$, then the equations

$$\begin{aligned} \partial_t v^{\mathrm{top}} - \mu \Delta v^{\mathrm{top}} + \nabla p^{\mathrm{top}} &= 0 && \text{in } J \times H_a, \\ \operatorname{div}(v^{\mathrm{top}}) &= \tilde{g}^{\mathrm{top}} && \text{in } J \times H_a, \\ v^{\mathrm{top}} \cdot \nu_{\Gamma_{\mathrm{top}}} &= \tilde{h}^{\mathrm{top}} \cdot \nu_{\Gamma_{\mathrm{top}}} && \text{on } J \times \partial H_a, \\ -P_{\Gamma_{\mathrm{top}}} D_-(v^{\mathrm{top}}) &= P_{\Gamma_{\mathrm{top}}} \tilde{h}^{\mathrm{top}} && \text{on } J \times \partial H_a, \\ v^{\mathrm{top}}(0) &= \tilde{v}_0^{\mathrm{top}} && \text{in } H_a, \end{aligned}$$

are well-posed and the compatibility conditions $(C1)$, $(C2)$, $(C3)_{S-}$ are satisfied by construction. These are the Stokes equations we were looking for. Due to [32, Corollary 5.6], we obtain a unique solution

$$v^{\mathrm{top}} \in H_p^1(J, L_p(H_a))^n \cap L_p(J, H_p^2(H_a))^n, \quad p^{\mathrm{top}} \in L_p(J, H_p^1(H_a)).$$

In the case of the bottom of our cylindrical domain we proceed analogously and obtain a unique solution

$$v^{\mathrm{bot}} \in H_p^1(J, L_p(H_{-a}))^n \cap L_p(J, H_p^2(H_{-a}))^n, \quad p^{\mathrm{bot}} \in L_p(J, H_p^1(H_{-a}))$$

for the system

$$\begin{aligned}
\partial_t v^{\text{bot}} - \mu\Delta v^{\text{bot}} + \nabla p^{\text{bot}} &= 0 && \text{in } J \times H_{-a},\\
\operatorname{div}(v^{\text{bot}}) &= \tilde{g}^{\text{bot}} && \text{in } J \times H_{-a},\\
v^{\text{bot}} \cdot \nu_{\Gamma_{\text{bot}}} &= \tilde{h}^{\text{bot}} \cdot \nu_{\Gamma_{\text{bot}}} && \text{on } J \times \partial H_{-a},\\
-P_{\Gamma_{\text{bot}}} D_-(v^{\text{bot}}) &= P_{\Gamma_{\text{bot}}} \tilde{h}^{\text{bot}} && \text{on } J \times \partial H_{-a},\\
v^{\text{bot}}(0) &= \tilde{v}_0^{\text{bot}} && \text{in } H_{-a}.
\end{aligned}$$

Let the cut-off function $\zeta \in \mathcal{C}^\infty(\mathbb{R}^n, [0,1])$ be defined like

$$\zeta(x) = \begin{cases} 0, & \text{if } x_n \in (-\infty, -a/3),\\ 1, & \text{if } x_n \in (a/3, \infty). \end{cases} \tag{1.5.8}$$

Then the convex-combination

$$v = \zeta v^{\text{top}}_{|\Omega} + (1-\zeta) v^{\text{bot}}_{|\Omega} \in \mathbb{E}^u_p(J)$$

fulfils the boundary conditions on bottom and top of Ω by construction.

Step 2. It remains to be shown that a vector field $z \in \mathbb{E}^u_p(J)$ exists in such a way that system (1.5.7) is valid. For that we extend and reflect the cylindrical domain Ω to obtain smooth $\mathcal{C}^3$ domains G_+ and G_-. As in the previous step we try to find solutions $\hat{z}^+$ and $\hat{z}^-$ for Stokes equations on G_+ or G_- with boundary conditions

$$\begin{aligned}
\hat{z}_\pm \cdot \nu_\Sigma &= \tilde{h}^\Sigma \cdot \nu_\Sigma,\\
-P_\Sigma D_-(\hat{z}_\pm)\nu_\Sigma &= P_\Sigma \tilde{h}^\Sigma
\end{aligned}$$

that also fulfil $\hat{z}_+ \cdot \nu_{\Gamma_{\text{top}}}$ and $-P_{\Gamma_{\text{top}}} D_-(\hat{z}_+)\nu_{\Gamma_{\text{top}}} = 0$ for $\hat{z}_+$ on Γ_{top} and for $\hat{z}_-$ on Γ_{bot}, respectively. Then we add up these solutions with appropriate cut-off functions to obtain the solution of (1.5.7). The crucial step is the compatibility between lines (1)–(4) and lines (3)–(6) of system (1.5.7). That means that $\tilde{h}^{\text{top}} = 0$ and $\tilde{h}^\Sigma$ with $\tilde{h}^\Sigma \cdot \nu_\Sigma = (h^\Sigma - v)\cdot \nu_\Sigma$ on Σ and $P_\Sigma \tilde{h}^\Sigma = P_\Sigma(h^\Sigma + D_-(v)\nu_\Sigma)$ on Σ have to satisfy $(C4)_{S-}$ and $\tilde{h}^{\text{bot}} = 0$ and $\tilde{h}^\Sigma$ have to satisfy $(C5)_{S-}$. Moreover, we need an appropriate extension and reflection for the domain Ω to G_+ and G_- and also of $\tilde{h}^\Sigma$ from Ω to G_+ and G_-. Additionally we need appropriate data, such that the compatibility conditions of Stokes equations on G_+ and G_+ are satisfied, respectively. We start to extend the equations (1.5.7) to a bounded $\mathcal{C}^3$-domain. In order to do so, we define Ω_{-a} as the domain that results from extending Ω in a bounded and smooth (at least in the $\mathcal{C}^3$-sense) way on the top. We set $\Sigma_{-a} := \partial\Omega_{-a} \backslash \bar{\Gamma}^{\text{bot}}$. In the same manner we define Ω_{+a} and Σ_{+a} as the appropriate extension of Ω on the bottom. Next, let $G_\pm$ denote the domains resulting from reflecting $\Omega_{\pm a}$ at $\pm a$ and set $\Gamma_\pm := \partial G_\pm$. For example, if $n = 3$ and the cross-section A of Ω is a circle, we connect a smooth cap to Ω at Γ_{top} to obtain Ω_{-a}, and we connect a smooth cap to Ω at Γ_{bot} to obtain Ω_{+a}. Therefore, the domains $G_\pm$ have both the form of a "pill", as shown in Figure 1.1. Following this strategy, we can always find a suitable extension, such that $G_\pm$ is of class $\mathcal{C}^3$.

Next, we show that $\tilde{h}^{\text{top}}$ and $\tilde{h}^\Sigma$ satisfy the compatibility conditions $(C4)_{S-}$ and extend $\tilde{h}^\Sigma$ to G_+. The case of $\tilde{h}^{\text{bot}}$, $\tilde{h}^\Sigma$ and G_- is analogous. Let $\zeta \in \mathcal{C}^\infty(\mathbb{R}^n, [0,1])$ be a cut-off function satisfying (1.5.8). We have that h^{top} and h^Σ fulfil $(C4)_{S-}$, the compatibility conditions are linear and ζv satisfies

$$\begin{aligned}
\zeta v \cdot \nu_{\Gamma_{\text{top}}} &= h^{\text{top}}(0) \cdot \nu_{\Gamma_{\text{top}}} && \text{on } \Gamma_{\text{top}},\\
-P_{\Gamma_{\text{top}}} D_-(\zeta v)\nu_{\Gamma_{\text{top}}} &= P_{\Gamma_{\text{top}}} h^{\text{top}}(0) && \text{on } \Gamma_{\text{top}}, .
\end{aligned}$$

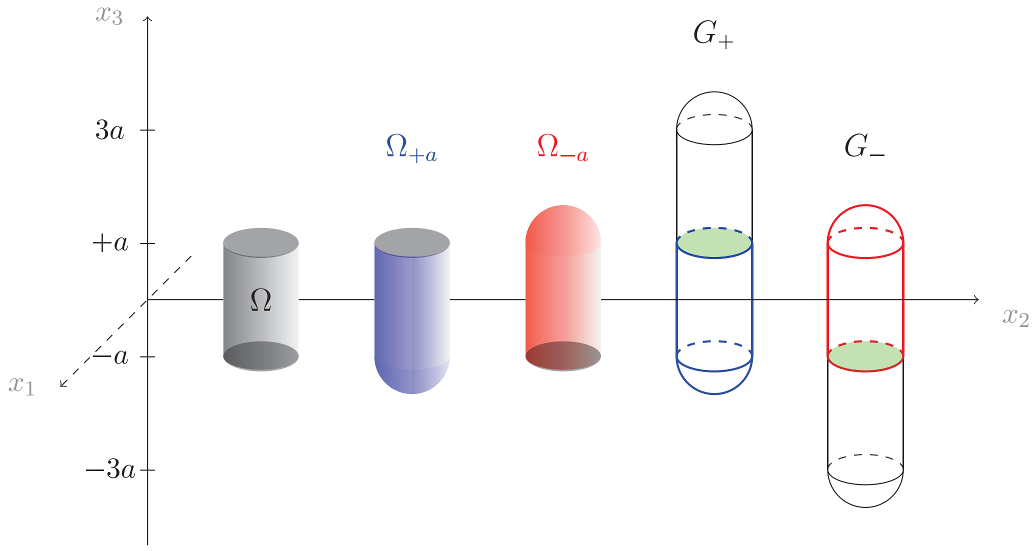


Figure 1.1: Extension of Ω for $n = 3$ and a circular cross-section A of Ω

From this, we obtain that $\tilde{h}^{\text{top}}$ and $\zeta\tilde{h}^{\Sigma}$ satisfy $(C4)_{S-}$. Now, we extend $\zeta\tilde{h}^{\Sigma}$ by zero to a function

$$\begin{aligned}\tilde{h}^{\Sigma}_{+} \in \mathbb{F}^{\Sigma_{+a}}_{p}(J) :=\{&h \in W^{1/2-1/2p}_{p}(J, L_p(\Sigma_{+a}))^n \cap L_p(J, W^{1-1/p}(\Sigma_{+a}))^n : \\ &h\cdot\nu \in W^{1-1/2p}_{p}(J, L_p(\Sigma_{+a})) \cap L_p(J, W^{2-1/p}_{p}(\Sigma_{+a})); \\ &h \text{ and } \tilde{h}^{\text{top}} \text{ fulfil } (C4)_{S-}\}.\end{aligned}$$

Then, we extend $\tilde{h}^{\Sigma}_{+}$ to Γ_+ by an even reflection of $(\tilde{h}^{\Sigma}_{+})_{1,\dots,n-1}$ at point $+a$ to

$$(\hat{h}^{\Sigma}_{+})_k(t, x', x_n) := \begin{cases} (\tilde{h}^{\Sigma}_{+})_k(t, x', x_n), & \text{if } x_n < a, \\ (\tilde{h}^{\Sigma}_{+})_k(t, x', 2a - x_n), & \text{if } x_n \geqslant a, \end{cases}$$

for $k = 1, \dots, n-1$, $(x', x_n) \in \Gamma_+$ and $t \in J$, and by an odd reflection of $(\tilde{h}^{\Sigma}_{+})_n$ at point $+a$ to

$$(\hat{h}^{\Sigma}_{+})_n(t, x', x_n) := \begin{cases} (\tilde{h}^{\Sigma}_{+})_n(t, x', x_n), & \text{if } x_n < a, \\ -(\tilde{h}^{\Sigma}_{+})_n(t, x', 2a - x_n), & \text{if } x_n \geqslant a, \end{cases}$$

for $(x', x_n) \in \Gamma_+$ and $t \in J$. We obtain $\hat{h}^{\Sigma}_{+}\colon \Gamma_+ \longrightarrow \mathbb{R}^n$ and have to show that

$$\begin{aligned}\hat{h}^{\Sigma}_{+} \in \mathbb{F}^{\Gamma_+}_{p}(J) := \{&h \in W^{1/2-1/2p}_{p}(J, L_p(\Gamma_+))^n \cap L_p(J, W^{1-1/p}(\Gamma_+))^n : \\ &h\cdot\nu \in W^{1-1/2p}_{p}(J, L_p(\Gamma_+)) \cap L_p(J, W^{2-1/p}_{p}(\Gamma_+)); \\ &h \text{ and } \tilde{h}^{\text{top}} \text{ fulfil } (C4)_{S-}\}.\end{aligned}$$

The outer normal vector $\tilde{\nu}_{\Sigma}\colon \Gamma_+ \longrightarrow \mathbb{R}^n$ on Γ_+ is also defined by a reflection

$$\tilde{\nu}_{\Sigma}(x', x_n) := \begin{cases} \nu_{\Sigma}(x', x_n) \quad , & \text{if } x_n < a, \\ \nu_{\Sigma}(x', 2a - x_n), & \text{if } x_n \geqslant a, \end{cases}$$

for $(x', x_n) \in \Gamma_+$. Note that $(\hat{h}^{\Sigma}_{+})_k \in W^{1/2-1/2p}_{p}(J, L_p(\Gamma_+)) \cap L_p(J, W^{1-1/p}_{p}(\Gamma_+))$ holds true for $k = 1, \dots, n-1$, because it was extended evenly. Moreover, we have that

$(\hat{h}_+^\Sigma)_n \in W_p^{1/2-1/2p}(J, L_p(\Gamma_+)) \cap L_p(J, W_p^{1-1/p}(\Gamma_+))$, since

$$\tilde{h}_+^\Sigma \cdot \nu_{\Gamma_{\text{top}}} = \tilde{h}^{\text{top}} \cdot \nu_\Sigma = 0 \quad \text{if } p > 2,$$

holds true. This can be seen from the first equation of the compatibility condition $(C4)_{S-}$. We also obtain that

$$\hat{h}_+^\Sigma \cdot \tilde{\nu}_\Sigma \in W_p^{1-1/2p}(J, L_p(\Gamma_+)) \cap L_p(J, W_p^{2-1/p}(\Gamma_+)),$$

because of

$$\partial_{x_n}(\tilde{h}_+^\Sigma \cdot \nu_\Sigma)_{|\mathcal{R}^{\text{top}}} = \partial_{\nu_{\Gamma_{\text{top}}}}(\tilde{h}_+^\Sigma \cdot \nu_\Sigma)_{|\mathcal{R}^{\text{top}}} = 0 \quad \text{if } p > 2,$$

which holds true because of the second equation of the compatibility condition $(C4)_{S-}$. Consequently $\hat{h}_+^\Sigma \in \mathbb{F}_p^{\Gamma_+}(J)$.
In the forthcoming step, we have to find appropriate data that satisfy the necessary compatibility conditions $(C1)$, $(C2)$, $(C3)_{S-}$ with respect to the Stokes equations on G_+ with perfect slip boundary conditions. From Remark 1.19 we obtain

$$\hat{h}_+^\Sigma(0) \cdot \tilde{\nu}_\Sigma \in W_p^{2-3/p}(\Gamma_+),\ \hat{h}_+^\Sigma(0) \in W_p^{1-3/p}(\Gamma_+).$$

For the initial data we choose $\hat{z}_0^+ \in W_p^{2-2/p}(G_+)$, such that $\hat{z}_0^+ = 0$ for $1 < p < \frac{3}{2}$. For the case $\frac{3}{2} < p < 3$ the initial data $\hat{z}_0^+$ has to satisfy

$$\begin{aligned} \hat{z}_0^+ \cdot \tilde{\nu}_\Sigma &= \hat{h}_+^\Sigma(0) \cdot \tilde{\nu}_\Sigma && \text{on } \Gamma_+, \\ -P_\Sigma D_-(\hat{z}_0^+)\tilde{\nu}_\Sigma &= 0 && \text{on } \Gamma_+. \end{aligned} \tag{1.5.9}$$

Finally, for the case $p > 3$, the initial data has to satisfy

$$\begin{aligned} \hat{z}_0^+ \cdot \tilde{\nu}_\Sigma &= \hat{h}_+^\Sigma(0) \cdot \tilde{\nu}_\Sigma && \text{on } \Gamma_+, \\ -P_\Sigma D_-(\hat{z}_0^+)\tilde{\nu}_\Sigma &= P_\Sigma \hat{h}_+^\Sigma(0) && \text{on } \Gamma_+. \end{aligned} \tag{1.5.10}$$

We can find a unique $\hat{z}_0^+$ solving (1.5.9) on account of [37, Theorem 2] and solving (1.5.10) on account of Lemma 1.20. Note that in both cases $\hat{z}_0^+$ is constructed using a bounded linear right inverse to the trace map. Hence, $(\hat{z}_0^+)_k$ is even for $k = 1, \ldots, n-1$ and $(\hat{z}_0^+)_n$ is odd. By choosing $\hat{z}_0^+$ like this, it satisfies

$$\begin{aligned} \hat{z}_0^+ \cdot \tilde{\nu}_\Sigma &= \tilde{h}_+^\Sigma(0) \cdot \tilde{\nu}_\Sigma && \text{on } \Gamma_+ \text{ if } p > \tfrac{3}{2}, \\ -P_\Sigma D_-(\hat{z}_0^+)\tilde{\nu}_\Sigma &= P_\Sigma \tilde{h}_+^\Sigma(0) && \text{on } \Gamma_+ \text{ if } p > 3. \end{aligned}$$

Now, it is left to define a $\tilde{g}^+$ that accomplishes

$$\begin{gathered} \operatorname{div}(\hat{z}_0^+) = \tilde{g}^+(0) \\ \mathcal{F}(\tilde{g}^+, \tilde{h}_+^\Sigma \cdot \tilde{\nu}_\Sigma) \in H_{p'}^1(J, (H_{p'}^1(G_+), \|\nabla \cdot\|_{p'})'). \end{gathered}$$

For that, consider the system

$$\begin{aligned} \partial_t\tilde{\varphi} - \mu\Delta\tilde{\varphi} &= 0 && \text{in } J \times G_+, \\ [P_\Gamma\tilde{\varphi}]_{\Gamma_+} &= \psi && \text{on } J \times \Gamma_+, \\ [\tilde{\varphi} \cdot \nu]_{\Gamma_+} &= \hat{h}_+^\Sigma \cdot \tilde{\nu}_\Sigma && \text{on } J \times \Gamma_+, \\ \partial_t\psi - \mu\Delta_\Gamma\psi &= 0 && \text{on } J \times \Gamma_+, \\ \psi(0) &= [P_\Gamma\hat{z}_0^+]_{\Gamma_+} && \text{on } \Gamma_+, \\ \tilde{\varphi}(0) &= \hat{z}_0^+ && \text{in } G_+, \end{aligned} \tag{1.5.11}$$

which has a unique solution $\tilde{\varphi} \in W_p^1(J, L_p(G_+))^n \cap L_p(J, W_p^2(G_+))^n$, due to [18, Theorem 2.1]. Defining $\tilde{g}^+ := \operatorname{div}(\tilde{\varphi})$, the equations

$$\begin{aligned}
\partial_t \hat{z}^+ - \mu\Delta\hat{z}^+ + \nabla p^+ &= 0 && \text{in } J \times G_+,\\
\operatorname{div}(\hat{z}^+) &= \tilde{g}^+ && \text{in } J \times G_+,\\
\hat{z}^+ \cdot \tilde{\nu}_\Sigma &= \hat{h}_+^\Sigma \cdot \tilde{\nu}_\Sigma && \text{on } J \times \Gamma_+,\\
-P_\Sigma D_-(\hat{z}^+)\tilde{\nu}_\Sigma &= P_\Sigma \hat{h}_+^\Sigma && \text{on } J \times \Gamma_+,\\
\hat{z}^+(0) &= \hat{z}_0^+ && \text{in } G_+,
\end{aligned}$$

become well-posed and satisfy the compatibility conditions $(C1)$, $(C2)$, $(C3)_{S-}$ by construction. These are the Stokes equations we were looking for. Due to [32, Theorem 3.30] we obtain a unique solution

$$\hat{z}^+ \in H_p^1(J, L_p(G_+))^n \cap L_p(J, H_p^2(G_+))^n, \quad p^+ \in L_p(J, H_p^1(G_+)).$$

Defining

$$\tilde{z}^+ := \hat{z}^+_{|\Omega_{+a}} \in \mathbb{E}_p^{\Omega_{+a}}(J),$$

we have $\tilde{z}^+ \cdot \nu_\Sigma = \zeta\tilde{h}_+^\Sigma \cdot \nu_\Sigma$ on Σ and $-P_\Sigma D_-(\tilde{z}^+)\cdot\nu = P_\Sigma\zeta\tilde{h}_+^\Sigma$ on Σ. Since $\hat{h}_+^\Sigma \cdot \nu_{\Gamma_{\text{top}}}$, $\hat{z}_0^+ \cdot \nu_{\Gamma_{\text{top}}}$ are odd and $P_{\Gamma_{\text{top}}}\hat{h}_+^\Sigma$, $P_{\Gamma_{\text{top}}}\hat{z}_0^+$ are even with respect to Γ_{top}, we infer that $(\tilde{z}_+)_k$ is even for $k = 1, \dots, n-1$ and $(\tilde{z}_+)_n$ is odd. Therefore, we get $\tilde{z}_+ \cdot \nu_{\Gamma_{\text{top}}} = 0$ and $-P_{\Gamma_{\text{top}}}D_-(\tilde{z}_+)\nu_{\Gamma_{\text{top}}} = 0$ on Γ_{top}. Analogously we proceed with Γ_{bot}. In this case we extend $(1-\zeta)\tilde{h}^\Sigma$ to obtain

$$\tilde{z}^- := \hat{z}^-_{|\Omega_{-a}} \in \mathbb{E}_p^{\Omega_{-a}}(J)$$

with $\tilde{z}^-\cdot\nu_\Sigma = (1-\zeta)\tilde{h}^\Sigma\cdot\nu_\Sigma$ on Σ, $-P_\Sigma D_-(\tilde{z}^-)\nu_\Sigma = P_\Gamma(1-\zeta)\tilde{h}^\Sigma$ on Σ, $\tilde{z}^-\cdot\nu_{\Gamma_{\text{bot}}} = 0$ on Γ_{bot} and $-P_{\Gamma_{\text{bot}}}D_-(\tilde{z}^-)\nu_{\Gamma_{\text{bot}}} = 0$ on Γ_{bot}. Let some cut-off functions $\zeta_1, \zeta_2 \in \mathcal{C}^\infty(\mathbb{R}^n, [0,1])$ satisfy

$$\zeta_1(x) = \begin{cases} 0, & \text{if } x_n \in (-\infty, -2a/3) \\ 1, & \text{if } x_n \in (-a/2, \infty) \end{cases}, \quad \zeta_2(x) = \begin{cases} 1, & \text{if } x_n \in (-\infty, a/2) \\ 0, & \text{if } x_n \in (2a/3, \infty) \end{cases}.$$

Then the sum

$$z = \zeta_1\tilde{z}^+_{|\Omega} + \zeta_2\tilde{z}^-_{|\Omega} \in \mathbb{E}_p^u(J)$$

satisfies the equations of (1.5.7) on Γ_{top} and Γ_{bot} by construction. It is only left to show that it also satisfies the equations on Σ. To this end, we consider

$$\begin{aligned}
z \cdot \nu_\Sigma &= \zeta_1\tilde{z}^+ \cdot \nu_\Sigma + \zeta_2\tilde{z}^- \cdot \nu_\Sigma\\
&= \zeta_1\zeta\tilde{h}^\Sigma \cdot \nu_\Sigma + \zeta_2(1-\zeta)\tilde{h}^\Sigma \cdot \nu_\Sigma\\
&= \zeta\tilde{h}^\Sigma \cdot \nu_\Sigma + (1-\zeta)\tilde{h}^\Sigma \cdot \nu_\Sigma\\
&= \tilde{h}^\Sigma \cdot \nu_\Sigma = (h^\Sigma - v)\cdot\nu_\Sigma \qquad \text{on } J \times \Sigma
\end{aligned}$$

and

$$\begin{aligned}
-P_\Sigma D_-(z)\nu &= -P_\Sigma D_-(\zeta_1\tilde{z}^+)\nu - P_\Sigma D_-(\zeta_2\tilde{z}^-)\nu \\
&= \zeta_1(-P_\Sigma D_-(\tilde{z}^+)\nu_\Sigma) + \zeta_2(-P_\Sigma D_-(\tilde{z}^-)\nu_\Sigma) \\
&\quad - \zeta_1'(\tilde{z}^+\cdot\nu_\Sigma)\mathrm{e}_n - \zeta_2'(\tilde{z}^-\cdot\nu_\Sigma)\mathrm{e}_n \\
&= \zeta_1(P_\Sigma\zeta\tilde{h}^\Sigma) + \zeta_2(P_\Sigma(1-\zeta)\tilde{h}^\Sigma) \\
&\quad - \zeta_1'(\zeta\tilde{h}^\Sigma\cdot\nu_\Sigma)\mathrm{e}_n - \zeta_2'((1-\zeta)\tilde{h}^\Sigma\cdot\nu_\Sigma)\mathrm{e}_n \\
&= P_\Sigma\zeta_1\zeta\tilde{h}^\Sigma + P_\Sigma\zeta_2(1-\zeta)\tilde{h}^\Sigma \\
&= P_\Sigma\zeta\tilde{h}^\Sigma + P_\Sigma(1-\zeta)\tilde{h}^\Sigma \\
&= P_\Sigma\tilde{h}^\Sigma = P_\Sigma(h^\Sigma + D_-(v)) \qquad \text{on } J\times\Sigma.
\end{aligned}$$

Defining $c := v + w \in \mathbb{E}_p^u$ by combining step 1 and 2 of this proof, we obtain a solution of system (1.5.5). Therefore, we have proven that there exists a bounded linear right-inverse to γ for $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$, which implies that the trace operator $\gamma\colon \mathbb{E}_p^u(J) \longrightarrow \tilde{\mathbb{F}}_p^h(J)$ is a linear retraction. □

Remark 1.22. The system

$$\begin{aligned}
\partial_t\tilde{\varphi} - \mu\Delta\tilde{\varphi} &= 0 && \text{in } J\times G_+, \\
\tilde{\varphi}\cdot\nu &= \hat{h}_+^\Sigma\cdot\nu && \text{on } J\times\Gamma_+, \\
-P_\Gamma D_-(\tilde{\varphi})\cdot\nu &= P_\Gamma\hat{h}_+^\Sigma && \text{on } J\times\Gamma_+, \\
\tilde{\varphi}(0) &= \hat{z}_0^+ && \text{in } G_+,
\end{aligned}$$

does not satisfy the Lopatinskii-Shapiro conditions, see [46, p. 253] for a definition of these conditions. In the proof of Proposition 1.21 we were therefore using the Stokes equations instead of using a parabolic problem. Hence, we had to use the dynamic system (1.5.11) to fulfil all necessary compatibility conditions.

Remark 1.23. Proposition 1.21 is also applicable to bounded $\mathcal{C}^3$-domains and to bent half spaces. This cases may be proven analogously to the proof of Proposition 1.21 or by use of [54, Theorem 2.9.1].

The proof of the retraction property of the trace operators with respect to the Neumann-Dirichlet and the free slip boundary operators are simpler versions of the proof of Proposition 1.21 and we omit them, here. Therefore, we have established the two important results:

Proposition 1.24. *Let $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$, $\delta > 0$ and $J = (0, T)$. Assume $A \subseteq \mathbb{R}^{n-1}$ to be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $\Omega := A \times (-a, a)$ a cylindrical domain with boundary Γ. Let also*

$$\begin{aligned}
\tilde{\mathbb{G}}_p^h(J) := \{&(h^{top}, h^\Sigma, h^{bot}) \in \mathbb{F}_p^{\Gamma_{top}}(J)\times\mathbb{F}_p^\Sigma(J)\times\mathbb{F}_p^{\Gamma_{bot}}(J) : \\
&h^{top} \text{ and } h^\Sigma \text{ fulfil } (C4)_{ND};\ h^{bot} \text{ and } h^\Sigma \text{ fulfil } (C5)_{ND}\}.
\end{aligned}$$

Then the trace operator

$$\begin{aligned}
\gamma\colon \mathbb{E}_p^u(J) &\longrightarrow \tilde{\mathbb{G}}_p^h(J) \\
c &\longmapsto \big((c_{|\Gamma_{top}}\cdot\nu)\nu + \delta\partial_\nu P_\Gamma c_{|\Gamma_{top}}, \\
&\qquad (c_{|\Sigma}\cdot\nu)\nu + \delta\partial_\nu P_\Gamma c_{|\Sigma}, \\
&\qquad (c_{|\Gamma_{bot}}\cdot\nu)\nu + \delta\partial_\nu P_\Gamma c_{|\Gamma_{bot}}\big)
\end{aligned}$$

is a bounded linear retraction.

Proposition 1.25. *Let $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$, $D_+(u) := \frac{1}{2}((\nabla u + (\nabla u)^T)$ and $J = (0,T)$. Assume $A \subseteq \mathbb{R}^{n-1}$ to be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $\Omega := A \times (-a, a)$ a cylindrical domain with boundary Γ. Let also*

$$\begin{aligned}\tilde{\mathbb{H}}_p^h(J) := \{(h^{top}, h^{\Sigma}, h^{bot}) \in \mathbb{F}_p^{\Gamma_{top}} \times \mathbb{F}_p^{\Sigma} \times \mathbb{F}_p^{\Gamma_{bot}} :\\ h^{top} \text{ and } h^{\Sigma} \text{ fulfil } (C4)_{S+};\ h^{bot} \text{ and } h^{\Sigma} \text{ fulfil } (C5)_{S+}\}.\end{aligned}$$

Then the trace operator

$$\begin{aligned}\gamma\colon \mathbb{E}_p^u(J) &\longrightarrow \tilde{\mathbb{H}}_p^h(J)\\ c &\longmapsto \big((c_{|\Gamma_{top}} \cdot \nu)\nu + P_\Gamma D_+(c)_{|\Gamma_{top}}\nu,\\ &\qquad (c_{|\Sigma} \cdot \nu)\nu + P_\Gamma D_+(c)_{|\Sigma}\nu,\\ &\qquad (c_{|\Gamma_{bot}} \cdot \nu)\nu + P_\Gamma D_+(c)_{|\Gamma_{bot}}\nu\big)\end{aligned}$$

is a bounded linear retraction.

2 Maximal L_p-Regularity for Elliptic and Parabolic Problems

In this chapter we extensively study the L_p-theory of elliptic and parabolic problems on cylindrical domains. Laplace operators on cylindrical domains were investigated by Nau already [39], but for different boundary conditions than those we are interested in. Also we would like to investigate systems with constant and variable coefficients. On the one hand, this enables us to construct auxiliary solutions for the pressure using elliptic problems, see Proposition 3.4. On the other hand, with the help of solutions to parabolic problems and the Helmholtz decomposition, we gain a better understanding of the Stokes equations. Of particular interest here are parabolic problems with Robin boundary conditions as they provide a deeper insight into physical problems, e. g. for proving the solvability of a model describing the dynamics of tropical storms in Chapter 4.

We start with an analysis of elliptic problems with Neumann boundary condition

$$\begin{aligned} -\mathrm{div}(\alpha\nabla w) &= d && \text{in } \Omega, \\ \partial_\nu w &= 0 && \text{on } \Gamma, \end{aligned} \tag{E}$$

and then consider parabolic problems of the form

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= f && \text{in } J\times\Omega, \\ \mathcal{B}^V(u) &= h && \text{on } J\times\Gamma, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{$(P|J)_V$}$$

The boundary operators $\mathcal{B}^V$, $V \in \{R,\ ND,\ S\pm\}$ are assumed to be as on page 23 in Section 1.4. By $\Omega := A \times (-a,a) \subseteq \mathbb{R}^n$ we denote a cylindrical domain consisting of a bounded $\mathcal{C}^3$-domain A and an interval $(-a,a)$ with $a > 0$. In addition, $J = (0,T)$, $T > 0$, denotes a time interval. Moreover, we consider the above systems exclusively on the smooth part $\Gamma = \Gamma_{\text{top}} \mathbin{\dot\cup} \Gamma_{\text{bot}} \mathbin{\dot\cup} \Sigma$ of the boundary of Ω. Here, Γ_{top} is the boundary of the top and Γ_{bot} of the bottom of Ω, and Σ the lateral boundary. For a more comprehensive investigation of cylindrical domains and their boundaries, cf. Section 1.3.

For parabolic problems we use the following data spaces

$$\begin{aligned}
\mathbb{F}_p^f(J) &= L_p(J\times\Omega)^n, \\
\mathbb{F}_p^\Lambda(J) &= \{h \in W_p^{1/2-1/2p}(J, L_p(\Lambda))^n \cap L_p(J, W_p^{1-1/p}(\Lambda))^n : \\
&\qquad h\cdot\nu \in W_p^{1-1/2p}(J, L_p(\Lambda)) \cap L_p(J, W_p^{2-1/p}(\Lambda))\},\ \Lambda \in \{\Gamma_{\text{top}}, \Gamma_{\text{bot}}, \Sigma\} \\
\mathbb{F}_p^h(J) &= \{h\colon \Gamma \longrightarrow \mathbb{R}^n : h_{|\Gamma_{\text{top}}} =: h^{\text{top}} \in \mathbb{F}_p^{\Gamma_{\text{top}}}(J), \\
&\qquad h_{|\Sigma} =: h^\Sigma \in \mathbb{F}_p^\Sigma(J),\ h_{|\Gamma_{\text{bot}}} =: h^{\text{bot}} \in \mathbb{F}_p^{\Gamma_{\text{bot}}}(J)\}, \\
\mathbb{F}_p^0 &= W_p^{2-2/p}(\Omega)^n, \\
\mathbb{F}_p^{R,\Lambda}(J) &= W_p^{1/2-1/2p}(J, L_p(\Lambda)) \cap L_p(J, W_p^{1-1/p}(\Lambda)), \\
\mathbb{F}_p^{R,h}(J) &= \{h\colon \Gamma \longrightarrow \mathbb{R}^n : h_{|\Gamma_{\text{top}}} =: h^{\text{top}} \in \mathbb{F}_p^{R,\Gamma_{\text{top}}}(J), \\
&\qquad h_{|\Sigma} =: h^\Sigma \in \mathbb{F}_p^{R,\Sigma}(J),\ h_{|\Gamma_{\text{bot}}} =: h^{\text{bot}} \in \mathbb{F}_p^{R,\Gamma_{\text{bot}}}\}.
\end{aligned}$$

We use the solution space

$$\mathbb{E}_p^z(J) = H_p^1(J, L_p(\Omega)) \cap L_p(J, H_p^2(\Omega)),$$

for parabolic problems with Robin boundary conditions, and

$$\mathbb{E}_p^u(J) = H_p^1(J, L_p(\Omega))^n \cap L_p(J, H_p^2(\Omega))^n,$$

for parabolic problems with Neumann-Dirichlet, perfect slip or free slip boundary conditions. These spaces are defined on page 24 in Section 1.4. From now on we consider the first equation of the parabolic problems $(P|J)_V$ to be of the form

$$\partial_t u - \mu \Delta u = f \quad \text{in } J \times \Omega,$$

regardless of whether the coefficients are constant or variable, cf. Remark 1.12.

2.1 Elliptic Problems: Neumann Boundary Condition

The Laplacian with constant coefficients has been studied already on cylindrical domains, see [39]. For us, however, the study of elliptic problems on cylindrical domains with Neumann boundary condition of the form

$$\begin{aligned} -\operatorname{div}(\alpha \nabla w) &= d \quad && \text{in } \Omega, \\ \partial_\nu w &= 0 && \text{on } \Gamma, \end{aligned} \tag{E}$$

is of importance. Here, $\Omega \subseteq \mathbb{R}^n$ is a cylindrical domain and every data $d \in L_p(\Omega, \mathbb{R})$ has to satisfy the compatibility condition $\int_\Omega d \, \mathrm{d}x = 0$.

In this section we are interested in systems of the form (E) with variable coefficient $\alpha \in \mathcal{BC}^1(\Omega, \mathbb{R})$. In addition, we would like to study the elliptic problem with constant coefficient $\alpha \equiv 1$ for time-dependent data d. Thereby we are interested in even higher regularity for elliptic problems. We have a special interest in this kind of elliptic problems, since we encounter them over and over again throughout this thesis as auxiliary problems, e. g. to prove maximal regularity of the Stokes equations with variable coefficients (see Proposition 3.8).

Variable Coefficients

Throughout this paragraph we always assume

$$\alpha \in \mathcal{BC}^1(\Omega, \mathbb{R}) \text{ with } \inf_\Omega \alpha > 0$$

to be a (time-independent) positive function. In order to prove that the elliptic problem (E) has a unique solution $w \in H_p^2(\Omega)$, we first show that the associated resolvent problem induces an isomorphism for any data $d \in L_p(\Omega)$. In the following step we use the maximal regularity of the resolvent problem to infer the solvability of elliptic problems with Neumann boundary condition.

Theorem 2.1. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain and let $a > 0$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $\alpha \in \mathcal{BUC}^1(\Omega)$ with $\inf_\Omega \alpha > 0$ and $1 < p < \infty$. Then there exists a solution $w \in H_p^2(\Omega)$ to the elliptic problem (E) for every $d \in L_p(\Omega)$ with $\int_\Omega d \, dx = 0$. Under the additional constraint $\int_\Omega w \, \mathrm{d}x = 0$ this solution is unique.*

Proof. **Step 1.** First we consider the operator $A_p := -\operatorname{div}(\alpha \nabla w)$ in $L_p(\Omega)$ with domain $D(A_p) = \{w \in H_p^2(\Omega) : \partial_\nu w = 0\}$. For $\lambda > 0$ the resolvent problem for A_p is given by

$$\begin{aligned} \lambda w - \nabla\alpha \cdot \nabla w - \alpha \Delta w &= d && \text{in } \Omega, \\ \partial_\nu w &= 0 && \text{on } \Gamma. \end{aligned} \tag{2.1.1}$$

Now, the Laplacian with constant coefficients on a cylindrical domain is $\mathcal{R}$-sectorial; see e. g. [39, Theorem 8.22]. Therefore, in case $\alpha \equiv 1$ there exists $\lambda_0 \geqslant 0$ such that for every $\lambda > \lambda_0$ the equations (2.1.1) have the property of maximal regularity, i. e. for every $d \in L_p(\Omega)$ there exists a unique solution $w \in D(A_p)$ to (2.1.1). A similar assertion is valid for the general case $\alpha \in \mathcal{BC}^1(\Omega)$ with $\inf_\Omega \alpha > 0$ as can be seen by using a suitable localisation argument as in [15, Theorem 5.7] or as demonstrated in the proof of Proposition 3.8. As a consequence, A_p is a closed operator for all $1 < p < \infty$.

Step 2. Now, we consider the operator $B_p w := -\operatorname{div}(\alpha \nabla w)$ in $E_p := \{q \in L_p(\Omega) : \int_\Omega q \,\mathrm{d}x = 0\}$ with domain $D(B_p) = \{w \in H_p^2(\Omega) : \partial_\nu w = 0,\ \int_\Omega w \,\mathrm{d}x = 0\}$. Since E_p is a closed subspace of $L_p(\Omega)$ and $D(B_p) = D(A_p) \cap E_p$ is the intersection of $D(A_p)$ with a closed subspace of $L_p(\Omega)$, we infer that B_p is a closed operator for all $1 < p < \infty$. Because the embedding $D(B_p) \hookrightarrow L_p(\Omega)$ is compact due to [4, Theorem 6.2], B_p has a compact resolvent. Hence, the spectrum $\sigma(B_p)$ consists of eigenvalues only. Now, assume that $p = 2$, $w \in D(B_2)$ and $B_2 w = 0$. Then, by using partial integration we obtain

$$\begin{aligned} 0 &= \int_\Omega -\operatorname{div}(\alpha \nabla w) \cdot \bar{w} \,\mathrm{d}x \\ &= \int_\Omega \alpha \nabla w \cdot \bar{w} \,\mathrm{d}x - \int_\Gamma \alpha \partial_\nu w \cdot \bar{w} \,\mathrm{d}\sigma \\ &= \int_\Omega \alpha |\nabla w|^2 \,\mathrm{d}x. \end{aligned}$$

Since $\inf_\Omega \alpha > 0$, we have $\nabla w = 0$ almost everywhere in Ω. Hence, w is constant. From $\int_\Omega w \,\mathrm{d}x = 0$ we conclude $w = 0$. Thus, for the kernel of B_2 we have $\mathfrak{N}(B_2) = \{0\}$, i. e. B_2 is injective. Since Ω is bounded, we have the linear embedding

$$\mathcal{G} \colon D(B_p) \hookrightarrow D(B_2), \qquad 2 \leqslant p < \infty.$$

Thus, if $2 \leqslant p < \infty$ and $w \in D(B_p)$ with $B_p w = 0$, then $\mathcal{G}(w) \in D(B_2)$ with $B_2(\mathcal{G}(w)) = 0$, which yields $\mathcal{G}(w) = 0$, since $\mathfrak{N}(B_2) = \{0\}$ holds true. Therefore, B_p is injective for all $2 \leqslant p < \infty$. We can then conclude $B_p \in \mathcal{L}_{is}(D(B_p), E_p)$ for all $2 \leqslant p < \infty$. Since $B_{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ is the dual operator of B_p, the Closed Range Theorem [57, Ch. 5] yields that $B_p \in \mathcal{L}_{is}(D(B_p), E_p)$ for all $1 < p \leqslant 2$. □

Elliptic Problems with Time Dependent Data

In this paragraph we generalise the result above to time dependent data, at least for the case of constant coefficients, more precisely for $\alpha \equiv 1$.

Corollary 2.2. *Let $T > 0$ and let $J = (0, T)$ or $J = \mathbb{R}$. Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain and $a > 0$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain and $1 < p < \infty$. Then for every $d \in L_p(J \times \Omega)$ with $\int_\Omega d(t, \cdot\,) \,dx = 0$ for almost all $t \in J$ the equation*

$$\begin{aligned} -\Delta w &= d && \text{in } J \times \Omega, \\ \partial_\nu w &= 0 && \text{on } J \times \Gamma, \end{aligned} \tag{2.1.2}$$

has a unique solution $w \in L_p(J, H_p^2(\Omega))$ with $\int_\Omega w(t, \cdot\,) \,dx = 0$ for almost all $t \in J$. If, in addition, $d \in H_p^\tau(J, L_p(\Omega))$ for some $\tau \in (0, 1]$, then $w \in H_p^\tau(J, H_p^2(\Omega))$.

Proof. **Step 1.** We first assume $d \in C^\infty(\mathbb{R}\times\bar{\Omega})$. For every $t \in \mathbb{R}$ we choose $w(t, \cdot) \in H^2_p(\Omega)$ to be the unique solution to the problem

$$-\Delta w(t, \cdot) = d(t, \cdot) \qquad \text{in } \Omega, \qquad\qquad \partial_\nu w = 0 \qquad \text{on } \Gamma, \qquad\qquad \int_\Omega w(t, \cdot)\,\mathrm{d}x = 0,$$

which exists based on Theorem 2.1 applied for $\alpha \equiv 1$. Now, we have

$$\|w\|^p_{L_p(\mathbb{R},\, H^2_p(\Omega))} = \int_\mathbb{R} \|w(t, \cdot)\|^p_{H^2_p(\Omega)}\,\mathrm{d}t \leqslant C^p \int_\mathbb{R} \|d(t, \cdot)\|^p_{L_p(\Omega)}\,\mathrm{d}t = C^p \|d\|^p_{L_p(\mathbb{R}\times\Omega)}$$

for some constant $C > 0$ that is independent of w, d and t. Using an approximation argument we obtain a unique solution $w \in L_p(\mathbb{R}, H^2_p(\Omega))$ to (2.1.2) for $J = \mathbb{R}$ that satisfies $\int_\Omega w(t, \cdot)\,\mathrm{d}x = 0$ for almost all $t \in \mathbb{R}$ for every $d \in L_p(\mathbb{R} \times \Omega)$ with $\int_\Omega d(t, \cdot)\,\mathrm{d}x = 0$ for almost all $t \in \mathbb{R}$. Using extension and restriction operators between $L_p(J)$ and $L_p(\mathbb{R})$ we obtain the same result also for $J = (0, T)$.

Step 2. For $J = \mathbb{R}$ the time derivative ∂_t can be approximated in $L_p(\mathbb{R})$ by difference quotients. This way we obtain $w \in H^1_p(\mathbb{R}, H^2_p(\Omega))$ for the solution constructed in the first step, if we additionally have that $d \in H^1_p(\mathbb{R}, L_p(\Omega))$. Now, using extension and restriction operators between $H^1_p(J)$ and $H^1_p(\mathbb{R})$ we obtain the same result also for $J = (0, T)$. Finally, an interpolation argument yields the additional assertion for $0 < \tau < 1$. □

Higher Regularity of Elliptic Problems

In this paragraph, we study higher regularity of elliptic problems for the case $\alpha \equiv 1$. Moreover, we would like to apply this higher regularity to this kind of elliptic problems with time-dependent data.

Lemma 2.3. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain and $a > 0$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain and $1 < p < \infty$. Then for every $d \in L_p(\Omega)$ with $\int_\Omega d\,\mathrm{d}x = 0$ the equation*

$$\begin{aligned} -\Delta w &= d && \textit{in } \Omega, \\ \partial_\nu w &= 0 && \textit{on } \Gamma, \end{aligned} \tag{2.1.3}$$

has a unique solution $w \in H^2_p(\Omega)$ with $\int_\Omega w\,\mathrm{d}x = 0$. If, in addition, $d \in H^1_p(\Omega)$, then $w \in H^3_p(\Omega)$.

Proof. We start with the unique solution $w \in H^2_p(\Omega)$ of system (2.1.3) with data $d \in \mathcal{C}^\infty_c(\bar{\Omega}) \subseteq H^1_p(\Omega)$ subject to the constraints $\int_\Omega w\,\mathrm{d}x = \int_\Omega d\,\mathrm{d}x = 0$ which exists due to Theorem 2.1. Since $\partial_n d \in L_p(\Omega)$, there exists a unique solution $v \in H^2_p(\Omega)$ to

$$\begin{aligned} -\Delta v &= f && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma_{\text{top}} \cup \Gamma_{\text{bot}}, \\ \partial_\nu v &= 0 && \text{on } \Sigma, \end{aligned} \tag{2.1.4}$$

for $f = \partial_n d$. Existence and uniqueness of a solution $v \in H^2_p(\Omega)$ can be proved with the same arguments as used in the proof of Theorem 2.1; one just has to consider the operator $B_p := -\Delta v$ with domain $D(B_p) := \{v \in H^2_p(\Omega, X) : v = 0 \text{ on } \Gamma_{\text{top}} \cup \Gamma_{\text{bot}},\ \partial_\nu v = 0 \text{ on } \Sigma\}$ in $L_p(\Omega)$ using [39, Theorem 8.10].

Next, we show that $\partial_n w = v \in H^2_p(\Omega)$. To this end, we consider the weak problem

$$(\nabla z, \nabla \zeta)_\Omega = \langle f, \zeta \rangle, \qquad\qquad \zeta \in X_2 := \{\, \psi \in H^1_2(\Omega) \,:\, \psi = 0 \text{ on } \Gamma_{\text{top}} \cup \Gamma_{\text{bot}} \,\}, \tag{2.1.5}$$

which has a unique solution $z \in X_2$ for every right-hand side $f \in X_2'$. Indeed, if $z \in X_2$ is a solution for $f = 0$, then we obtain $|\nabla z|^2_{L_2(\Omega)} = (\nabla z, \nabla z)_\Omega = 0$. This implies that z is constant almost everywhere in Ω, i. e. $z = 0$ almost everywhere in Ω due to the boundary conditions. Moreover, if $f \in L_2(\Omega)$, we choose $z \in H^2_2(\Omega)$ to be the unique solution to (2.1.4) and obtain

$$(\nabla z, \nabla \zeta)_\Omega = (\partial_\nu z, \zeta)_\Gamma - (\Delta z, \zeta)_\Omega = (f, \zeta)_\Omega, \qquad \zeta \in X_2,$$

where we used that $\partial_\nu z = 0$ on Σ and $\zeta = 0$ on $\Gamma_{\text{top}} \cup \Gamma_{\text{bot}}$. This shows that z is a solution to (2.1.5) with

$$\begin{aligned}\|\nabla z\|^2_{L_2(\Omega)} = (\nabla z, \nabla z)_\Omega = (f, z)_\Omega &\leqslant \tfrac{1}{4\varepsilon}\|f\|^2_{X_2'} + \varepsilon \|z\|^2_{X_2} \\ &\leqslant \tfrac{1}{4\varepsilon}\|f\|^2_{X_2'} + \varepsilon C_p \|\nabla z\|^2_{L_2(\Omega)},\end{aligned}$$

for all $\varepsilon > 0$. Here, $C_p > 0$ denotes the constant in Poincaré's inequality and we used the inequality of Young. Therefore, we have $\|\nabla z\|_{L_2(\Omega)} \leqslant C\|f\|_{X_2'}$. Now, for $\phi \in X_0 := \{\, \psi \in C^\infty_c(\bar\Omega) \,:\, \psi = 0 \text{ on } \Gamma_{\text{top}} \cup \Gamma_{\text{bot}} \,\}$ we have

$$\begin{aligned}&(\nabla v - \nabla \partial_n w, \nabla \phi)_\Omega \\ &\quad = \underbrace{(\partial_\nu v, \phi)_\Gamma}_{=0} - (\Delta v, \phi)_\Omega - \underbrace{((\nu \cdot e_n)\,\nabla w, \nabla \phi)_\Gamma}_{=0} + (\nabla w, \nabla \partial_n \phi)_\Omega \\ &\quad = (\partial_n d, \phi)_\Omega + \underbrace{(\partial_\nu w, \partial_n \phi)_\Gamma}_{=0} - (\Delta w, \partial_n \phi)_\Omega \\ &\quad = \underbrace{((\nu \cdot e_n) d, \phi)_\Gamma}_{=0} - (d, \partial_n \phi) + (d, \partial_n \phi)_\Omega = 0.\end{aligned}$$

Here, we used that $\partial_\nu v = 0$ on Σ, $\phi = 0$ and $\nabla \phi = (\partial_n \phi) e_n$ on $\Gamma_{\text{top}} \cup \Gamma_{\text{bot}}$, $\nu \cdot e_n = 0$ on Σ, $\partial_n w = \pm \partial_\nu w = 0$ on $\Gamma_{\text{top}} \cup \Gamma_{\text{bot}}$ and that $\partial_\nu w = 0$ on Σ. We have $v \in H^2_p(\Omega)$ for all $1 < p < \infty$, due to the fact that $\partial_n d \in \mathcal{C}^\infty_c(\bar\Omega)$. Since X_0 is dense in X_2 we infer that

$$(\nabla v - \nabla \partial_n w, \nabla \zeta)_\Omega = 0, \qquad \zeta \in X_2.$$

This, in term implies that $X_2 \ni \partial_n w = v \in H^2_2(\Omega)$ due to the unique solvability of (2.1.5). It follows that $\partial_n w = v \in H^2_p(\Omega)$ for all $1 < p < \infty$ with

$$\|\partial_n^2 w\|_{H^1_p(\Omega)} = \|\partial_n v\|_{H^1_p(\Omega)} \leqslant \|v\|_{H^2_p(\Omega)} \leqslant C\|\partial_n d\|_{L_p(\Omega)} \leqslant C\|d\|_{H^1_p(\Omega)}, \tag{2.1.6}$$

for $d \in \mathcal{C}^\infty_c(\bar\Omega)$. Due to the fact that $\mathcal{C}^\infty_c(\bar\Omega) \subseteq H^1_p(\Omega)$ is dense, we infer that the solution $w \in H^2_p(\Omega)$ to (2.1.3) for a right-hand side $d \in H^1_p(\Omega)$ subject to the constraints $\int_\Omega w \,\mathrm{d}x = \int_\Omega d \,\mathrm{d}x = 0$ satisfies $\partial_n w \in H^2_p(\Omega)$ and the estimate (2.1.6). Hence, we obtain

$$\begin{aligned}-\Delta_A w &= -\Delta w + \partial_n^2 w \\ &= d + \partial_n^2 w \in H^1_p((-a,a), L_p(A)) \cap L_p((-a,a), H^1_p(A)) && \text{in } A, \\ \partial_{\nu_A} w &= 0 && \text{on } \partial A,\end{aligned}$$

with $-\Delta_A$ the Dirichlet-Laplace operator on the bounded $\mathcal{C}^3$-domain A. We thus have

$$w \in H^1_p((-a,a), H^2_p(A)) \cap L_p((-a,a), H^3_p(A))$$

and deduce that $\partial_j \partial_k w \in H^1_p(\Omega)$ for all $j, k \in \{1, \ldots, n-1\}$. In combination with $\partial_n w \in H^2_p(\Omega)$ we therefore obtain $w \in H^3_p(\Omega)$ for the solution to (2.1.3). □

Now, with the same arguments as used in the proof of Corollary 2.2 we obtain the following result based on Lemma 2.3 instead of Theorem 2.1.

Corollary 2.4. *Let $T > 0$ and let $J = (0, T)$ or $J = \mathbb{R}$. Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain and $a > 0$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain and $1 < p < \infty$. Then for every $d \in L_p(J, H_p^1(\Omega))$ with $\int_\Omega d(t, \cdot)\,\mathrm{d}x = 0$ for almost all $t \in J$ the equations*

$$\begin{aligned} -\Delta w &= d && \text{in } J \times \Omega, \\ \partial_\nu w &= 0 && \text{on } J \times \Gamma, \end{aligned}$$

have a unique solution $w \in L_p(J, H_p^3(\Omega))$ with $\int_\Omega w(t, \cdot)\,\mathrm{d}x = 0$ for almost all $t \in J$. If, in addition, $d \in H_p^\tau(J, H_p^1(\Omega))$ for some $\tau \in (0, 1]$ then $w \in H_p^\tau(J, H_p^3(\Omega))$.

2.2 Parabolic Problems: Robin Boundary Condition

This section is devoted to the study of parabolic problems on cylindrical domains with Robin boundary conditions, i. e. to the study of systems of equations of the form

$$\begin{aligned} \partial_t u - \mu\Delta u &= f && \text{in } J \times \Omega, \\ \beta^u \partial_\nu u + \sigma^u u &= h && \text{on } J \times \Gamma, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \qquad (P|J)_R$$

As mentioned before, Nau studied this kind of problems with constant coefficients μ, β^u and σ^u, see [39]. Considering $(P|J)_R$ is particularly important for solving physical problems. We say more about this later when examining a model on the mechanisms of tropical storms in Chapter 4.

Again $\Omega \subseteq \mathbb{R}^n$ is assumed to be a cylindrical domain and $J = (0, T)$, $T > 0$ a time interval. We aim to prove the existence of a unique solution

$$u = u(t, x) \in \mathbb{E}_p^z(J)$$

to system $(P|J)_R$ for every data

$$(f, h, u_0) \in \mathbb{F}_p^{P,R}(J)$$

which meets the necessary regularity and compatibility conditions. On this account we introduce the data space $\mathbb{F}_p^{P,R}(J)$, which is defined to consist of all

$$(f, h, u_0) \in L_p(J \times \Omega) \times \mathbb{F}_p^{R,h}(J) \times W_p^{2-2/p}(\Omega)$$

that satisfy the compatibility condition

$$\beta^u \partial_\nu u_0 + \sigma^u u_0 = h(0) \quad \text{on } \Gamma \text{ if } p > 3, \qquad (C3)_R$$

which stems from the boundary condition of $(P|J)_R$. Its necessity was shown in Section 1.4.

Variable Coefficients

In this paragraph we assume

$$\mu \in \mathcal{BUC}(\Omega;\mathbb{R}) \text{ with } \inf_{\Omega} \mu > 0,$$

$$\beta^u \in \mathcal{BC}^1(J \times \Gamma, (0,\infty)) \text{ with } \inf_{\Gamma} \beta^u > 0,$$

and

$$\sigma^u \in \mathcal{BC}^2(J \times \Gamma, [0,\infty)).$$

To prove maximal regularity for $(P|J)_R$ with variable coefficients we first need a result, which shows the maximal regularity for perturbed parabolic problems, if the respective parabolic system has the property of maximal regularity. In the following lemma we prove maximal regularity for parabolic problems subject to all boundary operators defined on page 23 in Section 1.4. The reason is that want to use the lemma not for parabolic problems with Robin boundary conditions only, but also for parabolic problems with Neumann-Dirichlet boundary conditions, perfect slip boundary conditions and free slip boundary conditions.

Lemma 2.5. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $\mu \in \mathcal{BUC}(\Omega;\mathbb{R})$ with $\inf_\Omega \mu > 0$ and $\mathcal{B}^V$ with $V \in \{R,\, ND,\, S\pm\}$ to be one of the boundary operators defined in Section 1.4. Assume additionally*

- *for $V = R$ that $\mathbb{X}(J) := \mathbb{E}_p^z(J)$. Let $1 < p < \infty$ with $p \neq 3$, $\hat{f} \in L_p(J \times \Omega)$, $R\colon {}_0\mathbb{X}(J) \longrightarrow \mathbb{F}_p^{R,h}(J)$ to be a linear function with $R(w)(0) = 0$ for $w \in {}_0\mathbb{X}(J)$ with $w(0) = 0$ in Ω. Let $\hat{h} \in \mathbb{F}_p^{R,h}(J)$, such that $(\hat{f}, \hat{h}, 0)$ satisfy the necessary compatibility condition $(C3)_R$.*
- *for $V \in \{ND,\, S\pm\}$ that $\mathbb{X}(J) := \mathbb{E}_p^u(J)$. Let Let $1 < p < \infty$ with $p \notin \{2, \frac{3}{2}, 3\}$, $\hat{f} \in \mathbb{F}_p^f(J)$, $R\colon {}_0\mathbb{X}(J) \longrightarrow \mathbb{F}_p^h(J)$ to be a linear function with $R(w)(0) = 0$ for $w \in {}_0\mathbb{X}(J)$ with $w(0) = 0$ in Ω. Let $\hat{h} \in \mathbb{F}_p^{R,h}(J)$, such that $(\hat{f}, \hat{h}, 0)$ satisfy the necessary compatibility condition $(C3)_{ND}$ – $(C5)_{ND}$, $(C3)_{S\pm}$ – $(C5)_{S\pm}$, respectively.*

Let the system

$$\begin{aligned} \partial_t v - \mu \Delta v &= f && \text{in } J \times \Omega, \\ \mathcal{B}^V(v) &= h && \text{on } J \times \Gamma, \\ v(0) &= v_0 && \text{in } \Omega. \end{aligned} \tag{2.2.1}$$

have a unique solution $v \in \mathbb{X}(J)$ for every data $(f, h, v_0) \in \mathbb{F}_p^{P,V}$ with $V \in \{R,\, ND,\, S\pm\}$. If

$$\|R(z)\|_{{}_0\mathbb{F}_p^{R,h}(J)} \leqslant C|J|^\tau \|z\|_{{}_0\mathbb{X}(J)}, \text{ for } V = R,$$

and

$$\|R(z)\|_{{}_0\mathbb{F}_p^{h}(J)} \leqslant C|J|^\tau \|z\|_{{}_0\mathbb{X}(J)}, \text{ for } V \in \{ND,\, S\pm\},$$

for every $z \in \mathbb{X}(J)$ with $z(0) = 0$ in Ω and some constants τ, $C > 0$, which are independent of J, then system

$$\begin{aligned} \partial_t w - \mu \Delta w &= \hat{f} && \text{in } J \times \Omega, \\ \mathcal{B}^V(w) &= \hat{h} - R(w) && \text{on } J \times \Gamma, \\ w(0) &= 0 && \text{in } \Omega \end{aligned} \tag{2.2.2}$$

has a unique solution $w \in \mathbb{X}(J)$.

Proof. We show that (2.2.2) has the property of maximal regularity for the given time interval J. To this end, we first show that (2.2.2) has the property of maximal regularity for a small time interval $\bar{J} = (0, \bar{T})$, $\bar{T} > 0$. Let us establish $L\colon \mathbb{X}(\bar{J}) \longrightarrow \mathbb{F}_p^V(\bar{J})$ with $V \in \{R, ND, S\pm\}$ as the the operator defined through the left-hand side of (2.2.1). Since (2.2.1) has the property of maximal regularity, the operator L is invertible. We use the notation ${}_0L := L_{|_0\mathbb{E}_p(\bar{J})}\colon {}_0\mathbb{X}(\bar{J}) \longrightarrow {}_0\mathbb{F}_p^{P,V}(\bar{J})$. The operator ${}_0L^{-1}$ is bounded independently of $\bar{J}$, due to the fact that all $u \in {}_0\mathbb{X}(\bar{J})$ satisfy $u(0) = 0$ in Ω. Since $(\hat{f}, \hat{h}, 0)$ satisfy the necessary compatibility condition according to our assumptions and $R(w)(0) = 0$, if $w(0) = 0$, we can transform (2.2.2) into the operator equation

$${}_0L(w) = (\hat{f}, \hat{h} - R(w)) = (\hat{f}, \hat{h}) + (0, -R(w))$$

which is equivalent to

$$w = {}_0L^{-1}(\hat{f}, \hat{h}) + {}_0L^{-1}(0, -R(w)),$$

since the operator L is invertible. Subtraction of ${}_0L^{-1}(0, -R)(w)$ yields

$$(\mathrm{Id} - {}_0L^{-1}(0, -R))(w) = {}_0L^{-1}(\hat{f}, \hat{h}).$$

It is now left to show that $(\mathrm{Id} - {}_0L^{-1}(0, -R))^{-1}$ exists, since this would imply that

$$w = (\mathrm{Id} - {}_0L^{-1}(0, -R))^{-1}{}_0L^{-1}(\hat{f}, \hat{h})$$

is the unique solution to (2.2.2). The Neumann series argument provides the existence of $(\mathrm{Id} - {}_0L^{-1}(0, -R))^{-1}$, if $\|{}_0L^{-1}(0, -R)\| < 1$. Note that

$$\|{}_0L^{-1}(0, -R)\|_{{}_0\mathbb{X}(\bar{J}) \to {}_0\mathbb{X}(\bar{J})} \leqslant \|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{P,V}(\bar{J}) \to {}_0\mathbb{X}(\bar{J})} \cdot \|(0, -R)\|_{{}_0\mathbb{X}(\bar{J}) \to {}_0\mathbb{F}_p^{P,V}(\bar{J})}.$$

According to our assumptions we can estimate

$$\|R(w)\|_{{}_0\mathbb{F}_p^{R,h}(\bar{J})} \leqslant C|\bar{J}|^\tau \|w\|_{{}_0\mathbb{X}(\bar{J})}, \quad \text{for } V = R,$$

and

$$\|R(w)\|_{{}_0\mathbb{F}_p^{h}(\bar{J})} \leqslant C|\bar{J}|^\tau \|w\|_{{}_0\mathbb{X}(\bar{J})}, \quad \text{for } V \in \{ND, S\pm\},$$

for some constants τ, $C > 0$, which are independent of $\bar{J}$. Thus, we make $\|R(w)\|$ small by choosing the interval $\bar{J}$ sufficiently small, such that $\|{}_0L^{-1}(0, -R)\|_{{}_0\mathbb{X}(\bar{J}) \to {}_0\mathbb{X}(\bar{J})}$ is less than one. This is possible, because $\|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{P,R}(\bar{J}) \to {}_0\mathbb{X}(\bar{J})}$ is bounded independently of $\bar{J}$.

Now, we can conclude that (2.2.2) has the property of maximal regularity for a small time interval $\bar{J}$. To get maximal regularity for (2.2.2) for the time interval J, we choose $\bar{T}$ sufficiently small and such that

$$k\bar{T} = T \quad \text{for some } k \in \mathbb{N},$$

and successively solve the parabolic system (2.2.2) in a cylindrical domain on the time intervals

$$(0, \bar{T}),\ (\bar{T}, 2\bar{T}),\ \ldots,\ ((k-1)\bar{T}, k\bar{T}).$$

With this strategy we obtain a unique solution to (2.2.2) on the time interval $J = (0, T)$. □

Using some results from Nau [39, Theorems 8.10 & 8.22] about parabolic problems on cylindrical domains with constant coefficients and applying a similar argument as Denk, Hieber and Prüss in [15, Theorem 5.7], where they proved maximal regularity for elliptic operators with variable coefficients in a Banach space of class $\mathcal{HT}$ we can prove maximal regularity for parabolic problems with variable coefficients and Neumann boundary conditions. With Lemma 2.5 we can then infer maximal regularity for $(P|J)_R$.

Theorem 2.6. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0, T)$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $(f, h, u_0) \in \mathbb{F}_p^{P,R}(J)$, $\mu \in \mathcal{BUC}(\Omega; \mathbb{R})$ with $\inf_\Omega \mu > 0$, $\beta^u \in \mathcal{BC}^1(J \times \Gamma)$, $\sigma^u \in \mathcal{BC}^2(J \times \Gamma)$, $\inf_\Gamma \beta^u > 0$, and $1 < p < \infty$ with $p \neq 3$. Then system $(P|J)_R$ has a unique solution $u \in \mathbb{E}_p^z(J)$.*

Proof. To prove Theorem 2.6 for a time interval J, we split system $(P|J)_R$ into a parabolic system with Neumann boundary conditions and a perturbed parabolic problem with initial value zero. For this we set $u := v + w$ and choose $\tilde{h} \in \mathbb{F}_p^{R,h}(J)$ with

$$\tilde{h}(0) = \beta^u \partial_\nu u_0 \quad \text{on } \Gamma,$$

if $p > 3$. Thus, $(P|J)_R$ can be rewritten as the two systems

$$\begin{aligned} \partial_t v - \mu\Delta v &= f && \text{in } J \times \Omega, \\ \beta^u \partial_\nu v &= h && \text{on } J \times \Gamma, \\ v(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{2.2.3}$$

and

$$\begin{aligned} \partial_t w - \mu\Delta w &= 0 && \text{in } J \times \Omega, \\ \beta^u \partial_\nu w &= h - \tilde{h} - \sigma^u v - \sigma^u w && \text{on } J \times \Gamma, \\ w(0) &= 0 && \text{in } \Omega. \end{aligned} \tag{2.2.4}$$

Now, to obtain maximal regularity for $(P|J)_R$, we prove maximal regularity for both systems independently of each other. We proceed in two steps.

Step 1: Let $\hat{v} \in \mathbb{E}_p^z(J)$ be a solution to

$$\beta^u \partial_\nu \hat{v} = \tilde{h} \quad \text{on } J \times \Gamma,$$

which exists due to Lemma 1.16. Furthermore, system

$$\begin{aligned} \partial_t \tilde{v} - \mu\Delta\tilde{v} &= f - (\partial_t \hat{v} - \mu\Delta\hat{v}) && \text{in } J \times \Omega, \\ \beta^u \partial_\nu \tilde{v} &= 0 && \text{on } J \times \Gamma, \\ \tilde{v}(0) &= u_0 - \hat{v}(0) && \text{in } \Omega, \end{aligned}$$

has the unique solution $\tilde{v} \in \mathbb{E}_p^z(J)$. Since the Laplace operator is $\mathcal{R}$-sectorial on a cylindrical domain, [39, Theorem 8.22], the proof of maximal regularity for the above system can be proven by the same methods as the proof of maximal regularity for elliptic operators with variable coefficients in a Banach space of class $\mathcal{HT}$ [15, Theorem 5.7]. We discuss this approach in detail during the proof of Proposition 3.8. Then $v := \hat{v} + \tilde{v} \in \mathbb{E}_p^z(J)$ is the unique solution of (2.2.3).

Step 2: We define the perturbation $R\colon {}_0\mathbb{E}_p^z(J) \longrightarrow {}_0\mathbb{F}_p^{R,h}(J)$ through $R(w) := \sigma^u w$, which is a linear function with $R(w)(0) = 0$, if $w(0) = 0$ in Ω. We set

$$\hat{h} := h - \tilde{h} - \sigma^u v \in \mathbb{F}_p^{R,h}(J).$$

Using the definition of $\tilde{h}$ we have

$$\hat{h}(0) = h(0) - \tilde{h}(0) - \sigma^u v(0) = \beta^u \partial_\nu u_0 + \sigma^u u_0 - \beta^u \partial_\nu u_0 - \sigma^u u_0 = 0.$$

Thus, the data $(0, \hat{h}, 0)$ satisfies the compatibility condition $(C3)_R$. If $w(0) = 0$, the estimate

$$\begin{aligned}
\|R(w)\|_{{}_0\mathbb{F}_p^{R,h}(J)} &= \|\sigma^u w\|_{{}_0W_p^{1/2-1/2p}(J,L_p(\Gamma))\cap L_p(J,W_p^{1-1/p}(\Gamma))} \\
&\leqslant C\|w\|_{{}_0H_p^{1/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))} \\
&\leqslant C|J|^\tau \|w\|_{{}_0H_p^1(J,L_p(\Omega))\cap {}_0H_p^{1/2}(J,H_p^1(\Omega)} \\
&\leqslant C|J|^\tau \|w\|_{{}_0\mathbb{E}_p^z(J)}
\end{aligned}$$

holds true for some constants $C,\ \tau > 0$, which are independent of J. According to step 1 system (2.2.3) has the property of maximal regularity for every data $(f, h, u_0) \in \mathbb{F}_p^{P,R}(J)$. Thus, all assumptions of Lemma 2.5 are satisfied and we infer maximal regularity for system (2.2.4).

Combining step 1 and 2 we obtain maximal regularity for $(P|J)_R$ on the time interval $J = (0, T)$. □

2.3 Parabolic Problems: Neumann-Dirichlet Boundary Conditions

We are again interested in parabolic problems on cylindrical domains. More precisely, in this section we study systems of the form

$$\begin{aligned}
\partial_t u - \mu\Delta u &= f && \text{in } J \times \Omega, \\
u \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\
\delta \partial_\nu P_\Gamma u &= P_\Gamma h && \text{on } J \times \Gamma, \\
u(0) &= u_0 && \text{in } \Omega,
\end{aligned} \qquad (P|J)_{ND}$$

where $\Omega \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$ denotes a cylindrical domain and $J = (0, T)$, $T > 0$, a time interval. The coefficient $\delta > 0$ is assumed to be constant throughout this section, but we study $(P|J)_{ND}$ with both constant and variable coefficient μ. In addition, we assume the given data (f, h, u_0) to satisfy all necessary regularity and compatibility conditions for system $(P|J)_{ND}$. That is,

$$(f, h, u_0) \in \mathbb{F}_p^{P,ND}(J),$$

where the data space $\mathbb{F}_p^{P,ND}(J)$ is defined to consist of all

$$(f, h, u_0) \in \mathbb{F}_p^f(J) \times \mathbb{F}_p^h(J) \times \mathbb{F}_p^0$$

that satisfy the compatibility condition

$$\begin{aligned} u_0 \cdot \nu &= h(0) \cdot \nu, && \text{if } p > \tfrac{3}{2}, \\ \delta \partial_\nu P_\Gamma(u_0) &= P_\Gamma h(0), && \text{if } p > 3, \end{aligned} \tag{$(C3)_{ND}$}$$

as well as the compatibility condition

$$\begin{aligned} \delta[(\partial_{\nu_{\Gamma_{\text{top}}}} h^\Sigma)]_{\mathcal{R}^{\text{top}}} \cdot \nu_\Sigma &= [h^{\Gamma_{\text{top}}}]_{\mathcal{R}^{\text{top}}} \cdot \nu_\Sigma && \text{if } p > 2, \\ \delta[(\partial_{\nu_\Sigma} h^{\text{top}})]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Gamma_{\text{top}}} &= [h^\Sigma]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Gamma_{\text{top}}} && \text{if } p > 2, \end{aligned} \tag{$(C4)_{ND}$}$$

which arises from the boundary condition on the upper edge of Ω, and the condition

$$\begin{aligned} \delta[(\partial_{\nu_{\Gamma_{\text{bot}}}} h^\Sigma)]_{\mathcal{R}^{\text{bot}}} \cdot \nu_\Sigma &= [h^{\Gamma_{\text{bot}}}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_\Sigma && \text{if } p > 2, \\ \delta[(\partial_{\nu_\Sigma} h^{\text{bot}})]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Gamma_{\text{bot}}} &= [h^\Sigma]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Gamma_{\text{bot}}} && \text{if } p > 2, \end{aligned} \tag{$(C5)_{ND}$}$$

which arises from the boundary condition on the lower edge of Ω. In Section 1.4 the necessity of these regularity and compatibility conditions for $(P|J)_{ND}$ was shown.

We aim to find a unique solution

$$u = u(t, x) \in \mathbb{E}_p^u(J)$$

to system $(P|J)_{ND}$, since we use it in the following section to solve parabolic problems with perfect slip boundary conditions. This finally allows us to show in Chapter 3 that the Stokes equations on cylindrical domains have the property of maximal regularity.

Constant Coefficients

Within this paragraph we assume the coefficient

$$\mu > 0$$

of system $(P|J)_{ND}$ to be constant. To prove maximal regularity for the parabolic problem $(P|J)_{ND}$, we first prove the solvability of the analogous problem with homogeneous Neumann-Dirichlet boundary conditions. Using the retraction property of the trace operator with respect to the Neumann-Dirichlet boundary conditions, see Section 1.5, it is then possible to show the existence of a unique solution $u \in \mathbb{E}_p^u(J)$ to $(P|J)_{ND}$.

Lemma 2.7. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0, T)$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $1 < p < \infty$ and δ, $\mu > 0$ to be constant. Then the parabolic system*

$$\begin{aligned} \partial_t u - \mu \Delta u &= f && \text{in } J \times \Omega, \\ u \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ \delta \partial_\nu P_\Gamma u &= 0 && \text{on } J \times \Gamma, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{$(P|J)_{ND}^{h=0}$}$$

has a unique solution $u \in \mathbb{E}_p^u(J)$, for every data $(f, 0, u_0) \in \mathbb{F}_p^{P,ND}(J)$.

Proof. We split each of the functions u, f and u_0, into two components such that

$$u = (v, w) \text{ with } v\colon J \times \Omega \to \mathbb{R}^{n-1} \text{ and } w\colon J \times \Omega \to \mathbb{R},$$

$$f = (f_v, f_w) \in L_p(J \times \Omega)^{n-1} \times L_p(J \times \Omega)^1,$$

and

$$u_0 = (v_0, w_0) \in W_p^{2-2/p}(\Omega)^{n-1} \times W_p^{2-2/p}(\Omega)^1,$$

respectively. In addition, we consider the outer normal vector on the boundary of the cylindrical domain Ω, where we have

$$\nu = \pm \mathrm{e}_n \quad \text{on } A \times \{\pm a\},$$

on top and bottom of the boundary of Ω and

$$\nu \perp \mathrm{e}_n \quad \text{on } \partial A \times (-a,a) \text{ with } \nu = (\nu_A, 0),\ \nu_A \in \mathbb{R}^{n-1} \text{ the outer normal vector on } \partial A,$$

on the lateral boundary of Ω. The vector e_n defines the unit vector in the n-th direction. According to this decomposition, $(P|J)^{h=0}_{ND}$ decouples into the two systems

$$\begin{aligned} \partial_t v - \mu\Delta v &= f_v && \text{in } J \times \Omega, \\ \partial_\nu v &= 0 && \text{on } J \times A \times \{\pm a\}, \\ v \cdot \nu_A &= 0 && \text{on } J \times \partial A \times (-a,a), \\ \partial_{\nu_A} P_{\partial A} v &= 0 && \text{on } J \times \partial A \times (-a,a), \\ v(0) &= v_0 && \text{in } \Omega, \end{aligned} \tag{a}$$

and

$$\begin{aligned} \partial_t w - \mu\Delta w &= f_w && \text{in } J \times \Omega, \\ w &= 0 && \text{on } J \times A \times \{\pm a\}, \\ \partial_{\nu_A} w &= 0 && \text{on } J \times \partial A \times (-a,a), \\ w(0) &= w_0 && \text{in } \Omega. \end{aligned} \tag{b}$$

It is therefore sufficient to show maximal regularity for each of the systems (a) and (b) in order to prove maximal regularity for $(P|J)^{h=0}_{ND}$. We proceed in two steps.

Step 1. In order to prove maximal regularity for system (a) it is sufficient to prove maximal regularity for

$$\begin{aligned} \partial_t v + Tv &= f_v && \text{in } J \times \Omega, \\ v(0) &= v_0 && \text{in } \Omega, \end{aligned}$$

with operator $T\colon D(T) \subseteq L_p(\Omega)^{n-1} \to L_p(\Omega)^{n-1}$, which is defined through $T\,v := -\mu\Delta v$, $D(T) = \{v \in W_p^2(\Omega)^{n-1} : \partial_\nu v = 0 \text{ on } A \times \{\pm a\},\ v \cdot \nu_A = 0 \text{ and } \partial_{\nu_A} P_{\partial A} v = 0 \text{ on } \partial A \times (-a,a)\}$. Because of [31, Corollary 6.4, Theorem 6.5] it is sufficient to prove

$$\lambda + T \in \mathcal{RH}^\infty(L_p(\Omega)) \text{ with } \phi^\infty_{\lambda+T} < \tfrac{\pi}{2}, \text{ and for some } \lambda > 0.$$

In order to proceed, we split $T = T_1 + T_2$ into the two parts

$$\begin{aligned} &T_1\colon D(T_1) \subseteq L_p(A)^{n-1} \to L_p(A)^{n-1},\ T_1\,v := -\mu(\partial_1^2 + \partial_2^2 + \cdots + \partial_{n-1}^2)v \\ &\text{with } D(T_1) = \{v \in W_p^2(A)^{n-1} : v \cdot \nu_A = 0,\ \partial_{\nu_A} P_{\partial A} v = 0 \text{ on } \partial A\}, \end{aligned}$$

and

$$\begin{aligned} &T_2\colon D(T_2) \subseteq L_p(-a,a)^{n-1} \to L_p(-a,a)^{n-1},\ T_2\,v := -\mu\partial_n^2 v \\ &\text{with } D(T_2) = \{v \in W_p^2(-a,a)^{n-1} : \partial_\nu v = 0 \text{ on } \partial(-a,a) = \{\pm a\}\}. \end{aligned}$$

We have

$$\lambda_2 + T_2 \in \mathcal{H}^\infty \text{ for some } \lambda_2 > 0 \text{ and with } \phi^\infty_{\lambda_2+T_2} = 0,$$

see e. g. [15]. With a proof similar to that of [29, Theorem 6.1] we obtain

$$\lambda_1 + T_1 \in \mathcal{H}^\infty \text{ for some } \lambda_1 > 0 \text{ and with } \phi^\infty_{\lambda_1+T_1} = 0.$$

We can combine the result for $T = T_1 + T_2$ as follows

$$\lambda + T \in \mathcal{RH}^\infty \text{ with } \lambda = \lambda_1 + \lambda_2 > 0,\ \phi^\infty_{\lambda+T} = 0 \text{ and } D(T) = D(T_1) \cap D(T_2),$$

due to [43, Proposition 3.7]. This implies maximal regularity for system (a).

Step 2. To prove maximal regularity for system (b) it is sufficient to prove maximal regularity for

$$\begin{aligned} \partial_t w + Tw &= f_w && \text{in } J \times \Omega, \\ w(0) &= w_0 && \text{in } \Omega, \end{aligned} \tag{2.3.1}$$

with operator $T\colon D(T) \subseteq L_p(\Omega) \to L_p(\Omega)$, $T\,w := -\mu\Delta w$, $D(T) = \{w \in W^2_p(\Omega) : w = 0 \text{ on } A \times \{\pm a\},\ \partial_{\nu_A} w = 0 \text{ on } \partial A \times (-a,a)\}$. System (2.3.1) has the property of maximal regularity, due to [39, Theorem 8.10]. □

Now, using the retraction property with respect to the Neumann-Dirichlet boundary condition (Proposition 1.24), we obtain.

Proposition 2.8. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, δ, $\mu > 0$ to be constant, and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then the parabolic system $(P|J)_{ND}$ has a unique solution $u \in \mathbb{E}^u_p(J)$ for every data $(f, h, u_0) \in \mathbb{F}^{P,ND}_p(J)$.*

Proof. We choose $u_1 \in \mathbb{E}^u_p$, such that

$$\begin{aligned} u_1 \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\ \delta\partial_\nu P_\Gamma u_1 &= P_\Gamma h && \text{on } J \times \Gamma, \end{aligned}$$

which exists by Proposition 1.24. Setting $u_2 := u - u_1$, we obtain from $(P|J)_{ND}$ the following equations

$$\begin{aligned} \partial_t u_2 - \mu\Delta u_2 &= f - \partial_t u_1 + \mu\Delta u_1 && \text{in } J \times \Omega, \\ u_2 \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ \delta\partial_\nu P_\Gamma u_2 &= 0 && \text{on } J \times \Gamma, \\ u_2(0) &= u_0 - u_1(0) && \text{in } \Omega. \end{aligned} \tag{2.3.2}$$

Using Lemma 2.7, we obtain maximal regularity for (2.3.2) and thus maximal regularity for $(P|J)_{ND}$. □

Remark 2.9. In contrast to the parabolic problem with homogeneous Neumann-Dirichlet boundary conditions (Lemma 2.7) we cannot prove the existence of a unique solution $u \in \mathbb{E}^u_p(J)$ for all $1 < p < \infty$ for the parabolic problem with inhomogeneous Neumann-Dirichlet boundary conditions. The constraints $p \neq \frac{3}{2}$, $p \neq 2$ and $p \neq 3$ are due to the inhomogeneous boundary conditions. More precisely, we have to pay special attention to the compatibility conditions $(C3)_{ND}$, $(C4)_{ND}$ and $(C5)_{ND}$. Since the compatibility condition $(C3)_{ND}$ holds for $p > \frac{3}{2}$ and $p > 3$, respectively, and $(C4)_{ND}$ and $(C5)_{ND}$ hold for $p > 2$, we were not able to show the retraction property of the trace operator with respect to the Neumann-Dirichlet boundary conditions for the limiting cases $p = \frac{3}{2}$, $p = 2$ and $p = 3$ with our methods.

Variable Coefficients

In the following paragraph we assume

$$\mu \in \mathcal{BUC}(\Omega;\mathbb{R}) \text{ with } \inf_{\Omega} \mu > 0.$$

We extend the maximal regularity of parabolic problems with constant coefficients in a cylindrical domain to those with variable coefficients, applying a localisation argument utilized by Denk, Hieber and Prüss [15, Theorem 5.7]. Following their approach, which is used again in Proposition 3.8 and is treated there in detail, we have the following result.

Corollary 2.10. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $(f,h,u_0) \in \mathbb{F}_p^{P,ND}(J)$, $\mu \in \mathcal{BUC}(\Omega;\mathbb{R})$ with $\inf_\Omega \mu > 0$, and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then system $(P|J)_{ND}$ has a unique solution $u \in \mathbb{E}_p^u(J)$.*

2.4 Parabolic Problems: Perfect Slip Boundary Conditions

The parabolic problem on cylindrical domains with perfect slip boundary conditions we are interested in is given as

$$\begin{aligned} \partial_t u - \mu\Delta u &= f && \text{in } J \times \Omega, \\ u \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J \times \Gamma, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \qquad (P|J)_{S-}$$

We study this system with both constant and variable coefficients μ. For the constant coefficient case, we also show that the unique solution of $(P|J)_{S-}$ is a solenoidal function, if f is solenoidal. Considering parabolic problems with perfect slip boundary conditions is interesting, since we use them in the next section to solve parabolic problems with free slip boundary conditions. Moreover, in Chapter 3, using the Helmholtz projection and the unique solenoidal solution of $(P|J)_{S-}$, we can show that the Stokes equations on cylindrical domains with constant coefficients have the property of maximal regularity.

Let $J = (0,T)$, $T > 0$, be a time interval and $\Omega \subseteq \mathbb{R}^n$ a cylindrical domain. We are interested in proving the existence of a unique solution

$$u = u(t,x) \in \mathbb{E}_p^u(J)$$

to system $(P|J)_{S-}$ for every data

$$(f,h,u_0) \in \mathbb{F}_p^{P,S-}(J)$$

which fulfil the necessary regularity and compatibility conditions. The data space $\mathbb{F}_p^{P,S-}(J)$ is defined to consist of all

$$(f,h,u_0) \in \mathbb{F}_p^f(J) \times \mathbb{F}_p^h(J) \times \mathbb{F}_p^0$$

that satisfy the following compatibility conditions. For system $(P|J)_{S-}$ there are compatibility conditions, which arise from the compatibility between data h and initial data u_0, which are

$$\begin{aligned} u_0 \cdot \nu &= h(0) \cdot \nu, && \text{if } p > \tfrac{3}{2}, \\ -P_\Gamma D_-(u_0)\nu &= P_\Gamma h(0), && \text{in } p > 3, \end{aligned} \qquad (C3)_{S-}$$

compatibility conditions which arise from the boundary condition on the upper edge of Ω

$$\begin{aligned}
[h^{\Sigma}]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Gamma_{\text{top}}} &= -[h^{\text{top}}]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Sigma} && \text{if } p > 2, \\
\tfrac{1}{2}[\partial_{\nu_{\Gamma_{\text{top}}}} h^{\Sigma}]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Sigma} - \tfrac{1}{2}[\partial_{\nu_{\Sigma}} h^{\text{top}}]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Gamma_{\text{top}}} &= [h^{\text{top}}]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Sigma} && \text{if } p > 2,
\end{aligned} \qquad (C4)_{S-}$$

and conditions which arise from the boundary condition on the lower edge of Ω

$$\begin{aligned}
[h^{\Sigma}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Gamma_{\text{bot}}} &= -[h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Sigma} && \text{if } p > 2, \\
\tfrac{1}{2}[\partial_{\nu_{\Gamma_{\text{bot}}}} h^{\Sigma}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Sigma} - \tfrac{1}{2}[\partial_{\nu_{\Sigma}} h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Gamma_{\text{bot}}} &= [h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Sigma} && \text{if } p > 2.
\end{aligned} \qquad (C5)_{S-}$$

In Section 1.4 we have shown in detail that these are the necessary compatibility conditions for $(P|J)_{S-}$.

Constant Coefficients

In this paragraph we assume the coefficient

$$\mu > 0$$

of system $(P|J)_{S-}$ to be constant. To prove maximal regularity for the parabolic system $(P|J)_{S-}$, we decompose $(P|J)_{S-}$ into a system containing the inhomogeneous perfect slip boundary conditions and a parabolic system with homogeneous perfect slip boundary conditions. Using the retraction property of the trace operator with respect to the perfect slip boundary conditions, see Section 1.5, we are able to show that the system containing the inhomogeneous perfect slip boundary conditions has a unique solution. The remainder of the proof is then devoted to maximal regularity of the parabolic system with homogeneous perfect slip boundary conditions.

Proposition 2.11. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $\mu > 0$ to be a constant and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then the parabolic system $(P|J)_{S-}$, with data $(f,h,u_0) \in \mathbb{F}_p^{P,S-}(J)$ has a unique solution $u \in \mathbb{E}_p^u(J)$.*

Proof. Let us split the velocity into $u = \hat{u} + \tilde{u}$. According to this decomposition, $(P|J)_{S-}$ decouples into the two systems

$$\begin{aligned}
\hat{u} \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\
-P_{\Gamma} D_{-}(\hat{u})\nu &= P_{\Gamma} h && \text{on } J \times \Gamma,
\end{aligned} \qquad (2.4.1)$$

and

$$\begin{aligned}
\partial_t \tilde{u} - \mu \Delta \tilde{u} &= \tilde{f} && \text{in } J \times \Omega, \\
\tilde{u} \cdot \nu &= 0 && \text{on } J \times \Gamma, \\
-P_{\Gamma} D_{-}(\tilde{u})\nu &= 0 && \text{on } J \times \Gamma, \\
\tilde{u}(0) &= \tilde{u}_0 && \text{in } \Omega.
\end{aligned} \qquad (2.4.2)$$

The data $\tilde{f} := f - (\partial_t \hat{u} - \mu \Delta \hat{u})$ in $J \times \Omega$ and $\tilde{u}_0 := u_0 - \hat{u}(0)$ in Ω fulfil the necessary compatibility conditions and consequently, we have $(\tilde{f}, 0, \tilde{u}_0) \in \mathbb{F}_p^{P,S-}(J)$. From Proposition 1.21 we already know that the corresponding trace-operator is a retraction and thus system (2.4.1) has a solution $\hat{u} \in \mathbb{E}_p^u(J)$.

To prove maximal regularity for (2.4.2) we proceed in two steps.
Step 1. In this step we show that

$$\begin{aligned} \partial_t\tilde{u} - \mu\Delta\tilde{u} &= \tilde{f} && \text{in } J\times\Omega, \\ \tilde{u}\cdot\nu &= -\tfrac{1}{2}(\nabla_\Gamma\nu)\tilde{u}\cdot\nu = 0 && \text{on } J\times\Gamma, \\ \tfrac{1}{2}\partial_\nu P_\Gamma\tilde{u} &= -\tfrac{1}{2}(\nabla_\Gamma\nu)\tilde{u} && \text{on } J\times\Gamma, \\ \tilde{u}(0) &= \tilde{u}_0 && \text{in } \Omega, \end{aligned} \tag{2.4.3}$$

is equivalent to system (2.4.2) and well-posed. For that, consider the zero-th order operator

$$\mu \mapsto -\tfrac{1}{2}(\nabla_\Gamma\nu)\tilde{u}$$

and prove that $-\frac{1}{2}(\nabla_\Gamma\nu)\tilde{u}$ as well as $\partial_\nu P_\Gamma\tilde{u}$ are tangential on $J\times\Gamma$. The outer normal vector ν as well as the projection P_Γ are extended canonically to a tubular neighbourhood of Γ that means we extend ν constantly in the normal direction with $\partial_\nu\nu = 0$. For a comprehensive analysis of tubular neighbourhoods see [46, Section 2.3]. Therefore, $\nabla_\Gamma\nu = P_\Gamma\nabla\nu$ on Γ. From this, we see that

$$-\frac{1}{2}(\nabla_\Gamma\nu)\tilde{u} \tag{2.4.4}$$

is tangential. Next, by assuming $\tilde{u}\cdot\nu = 0$ on $J\times\Gamma$, we infer that

$$\begin{aligned} P_\Gamma\partial_\nu P_\Gamma\tilde{u} &= P_\Gamma\partial_\nu(\tilde{u} - (\nu\otimes\nu)\tilde{u}) = P_\Gamma\partial_\nu(\tilde{u} - (\tilde{u}\cdot\nu)\nu) \\ &= P_\Gamma(\nabla\tilde{u}^T\nu) - P_\Gamma\left(\nabla\left((\tilde{u}\cdot\nu)\,\nu\right)^T\nu\right) \\ &= P_\Gamma\partial_\nu\tilde{u} - [(\nabla_\Gamma(\tilde{u}\cdot\nu))\otimes\nu]^T\nu - (\tilde{u}\cdot\nu)(\nabla_\Gamma\nu^T)\nu. \end{aligned}$$

Note, that we use $\tilde{u}\cdot\nu = 0$ in the following equation, since we know it is valid on the boundary Γ and therefore on the tangential part. We do not know however how $\tilde{u}\cdot\nu$ behaves in the normal direction. For this reason, we could not use $\tilde{u}\cdot\nu = 0$ in the first line. Thus, we infer from the equations above

$$\begin{aligned} P_\Gamma\partial_\nu P_\Gamma\tilde{u} = P_\Gamma\partial_\nu\tilde{u} &= \partial_\nu\tilde{u} - (\nu\otimes\nu)(\partial_\nu\tilde{u}) = \partial_\nu\tilde{u} - \left((\nabla\tilde{u})^T\nu\cdot\nu\right)\nu \\ &= (\nabla\tilde{u})^T\nu - \left((\nabla\tilde{u})^T\nu\cdot\nu)\nu - ((\nabla\nu)\tilde{u}\cdot\nu\right)\nu + ((\nabla\nu)\tilde{u}\cdot\nu)\nu \\ &= \partial_\nu P_\Gamma\tilde{u} + ((\nabla\nu)\tilde{u}\cdot\nu)\nu = \partial_\nu P_\Gamma\tilde{u} + (\tilde{u}\cdot\partial_\nu\nu)\nu \\ &= \partial_\nu P_\Gamma\tilde{u}. \end{aligned}$$

This is why $-\frac{1}{2}(\nabla_\Gamma\nu)\tilde{u}$ and $\partial_\nu P_\Gamma\tilde{u}$ are tangential, if $\tilde{u}\cdot\nu = 0$ on $J\times\Gamma$. Moreover, we can conclude $\nabla_\Gamma\nu \in \mathcal{BUC}^1(\Sigma, \mathbb{R}^{n\times n})$ and $\nabla_\Gamma\nu_{|\Gamma_{\text{top}}} = \nabla_\Gamma\nu_{|\Gamma_{\text{bot}}} = 0$, since $A \subseteq \mathbb{R}^{n-1}$ is a bounded $\mathcal{C}^3$-domain. Together with $\tilde{u}\in\mathbb{E}_p^u(J)$ it follows that $-\frac{1}{2}(\nabla_\Gamma\nu)\tilde{u} \in W_p^{1/2-1/2p}(J, L_p(\Gamma))^n \cap L_p(J, W_p^{1-1/p}(\Gamma))^n$. Combined with $-\frac{1}{2}(\nabla\nu)\tilde{u}$ being tangential, we obtain $-\frac{1}{2}(\nabla\nu)\tilde{u} \in \mathbb{F}_p^h(J)$. Analogously we obtain

$$\begin{aligned}
&\tfrac{1}{2}\partial_\nu P_\Gamma \tilde{u} + \tfrac{1}{2}(\nabla_\Gamma \nu)\tilde{u} \\
&\quad = \tfrac{1}{2}\nabla(P_\Gamma\tilde{u})\nu + \tfrac{1}{2}P_\Gamma(\nabla\nu)\tilde{u} \\
&\quad = \tfrac{1}{2}\nabla(\tilde{u} - (\nu\otimes\nu)\tilde{u})^T\nu + \tfrac{1}{2}P_\Gamma^T(\nabla\nu)\tilde{u} \\
&\quad = \tfrac{1}{2}\nabla\tilde{u}^T\nu - \tfrac{1}{2}\nabla[(\tilde{u}\cdot\nu)\nu]^T\nu + \tfrac{1}{2}[\tilde{u}^T\nabla\nu^T P_\Gamma]^T \\
&\quad = \tfrac{1}{2}\nabla\tilde{u}^T\nu - \tfrac{1}{2}[\nabla(\tilde{u}\cdot\nu)\otimes\nu]^T\nu - \tfrac{1}{2}(\tilde{u}\cdot\nu)\nabla\nu^T\nu + \tfrac{1}{2}[\tilde{u}^T\nabla\nu^T P_\Gamma + \tilde{u}^T\partial_\nu\nu\nu^T]^T \\
&\quad = \tfrac{1}{2}\nabla\tilde{u}^T\nu - \tfrac{1}{2}[\nu\otimes\nabla(\tilde{u}\cdot\nu)]\nu + \tfrac{1}{2}[\tilde{u}^T\nabla\nu^T P_\Gamma + \tilde{u}^T\nabla\nu^T\nu\nu^T]^T \qquad (2.4.5)\\
&\quad = \tfrac{1}{2}\nabla\tilde{u}^T\nu - \tfrac{1}{2}[\nu\otimes((\nabla\tilde{u})\nu + (\nabla\nu)\tilde{u})] + \tfrac{1}{2}[\tilde{u}^T\nabla\nu^T P_\Gamma + \tilde{u}^T\nabla\nu^T(\nu\otimes\nu)]^T \\
&\quad = \tfrac{1}{2}\nabla\tilde{u}^T - \tfrac{1}{2}((\nabla\tilde{u})^T\nu\cdot\nu)\nu - \tfrac{1}{2}((\nabla\nu)\tilde{u}\cdot\nu)\nu + \tfrac{1}{2}[\tilde{u}^T(\nabla\nu)^T]^T \\
&\quad = \tfrac{1}{2}\nabla\tilde{u}^T\nu - \tfrac{1}{2}(\nu\otimes\nu)\nabla\tilde{u}^T\nu - \tfrac{1}{2}(\nu\otimes\nu)(\nabla\nu)\tilde{u} + \tfrac{1}{2}(\nabla\nu)\tilde{u} \\
&\quad = \tfrac{1}{2}P_\Gamma(\nabla\tilde{u}^T\nu) + \tfrac{1}{2}P_\Gamma((\nabla\nu)\tilde{u}) \\
&\quad = \tfrac{1}{2}P_\Gamma(\nabla\tilde{u}^T\nu + (\nabla\nu)\tilde{u} - \nabla(\tilde{u}\cdot\nu)) \\
&\quad = \tfrac{1}{2}P_\Gamma(\nabla\tilde{u}^T - \nabla\tilde{u})\nu \\
&\quad = -P_\Gamma D_-(\tilde{u})\nu
\end{aligned}$$

and see that system (2.4.2) is equivalent to (2.4.3).

Step 2. In this final step we show that (2.4.3) has the property of maximal regularity for the given time interval J. To this end, we establish the linear function $R\colon {}_0\mathbb{E}_p^u(J) \to {}_0\mathbb{F}_p^h(J)$ with $R(\tilde{u}) := -\frac{1}{2}(\nabla_\Gamma\nu)\tilde{u}$. It is $R(\tilde{u})(0) = 0$, if $u(0) = 0$. Now, we split $\tilde{u} := v + w$ and choose a $\tilde{h} \in \mathbb{F}_p^h(J)$ with

$$\begin{aligned}
\tilde{h}(0)\cdot\nu &= 0 && \text{on } \Gamma \text{ if } p > \tfrac{3}{2}, \\
P_\Gamma\tilde{h}(0) &= \tfrac{1}{2}\partial_\nu P_\Gamma\tilde{u}_0 && \text{on } \Gamma \text{ if } p > 3.
\end{aligned}$$

Thus, the system (2.4.3) can be rewritten as the following systems:

$$\begin{aligned}
\partial_t v - \mu\Delta v &= 0 && \text{in } J\times\Omega, \\
v\cdot\nu &= 0 && \text{on } J\times\Gamma, \\
\tfrac{1}{2}\partial_\nu P_\Gamma v &= P_\Gamma\tilde{h} && \text{on } J\times\Gamma, \\
v(0) &= \tilde{u}_0 && \text{in } \Omega,
\end{aligned} \qquad (2.4.6)$$

and

$$\begin{aligned}
\partial_t w - \mu\Delta w &= \tilde{f} && \text{in } J\times\Omega, \\
w\cdot\nu &= 0 && \text{on } J\times\Gamma, \\
\tfrac{1}{2}\partial_\nu P_\Gamma w &= \hat{h} - R(w) && \text{on } J\times\Gamma, \\
w(0) &= 0 && \text{in } \Omega.
\end{aligned} \qquad (2.4.7)$$

Where $\hat{h} := -\frac{1}{2}(\nabla_\Gamma\nu)v - P_\Gamma\tilde{h}$. Let $v \in \mathbb{E}_p^u(J)$ be the solution of system (2.4.6), which exists according to Proposition 2.8.

To prove maximal regularity for system (2.4.7), we want to use Lemma 2.5. Therefore, we have to check if all assumptions of Lemma 2.5 are satisfied. System $(P|J)_{ND}$ has the property of maximal regularity for every data $(f, h, u_0) \in \mathbb{F}_p^{P,ND}(J)$, due to Proposition 2.8. The data $(\tilde{f}, \hat{h}, 0)$ satisfies the compatibility conditions $(C3)_{ND}$, since we can conclude

$$\hat{h}(0) = -\tfrac{1}{2}(\nabla_\Gamma\nu)v(0) - P_\Gamma\tilde{h}(0) = -\tfrac{1}{2}(\nabla_\Gamma\nu)u_0 - \tfrac{1}{2}\partial_\nu P_\Gamma u_0 = P_\Gamma D_-(u_0)\nu = 0$$

from equation (2.4.5) and the definition of $\tilde{h}$. Note that $\nabla_\Gamma \nu \in \mathcal{BUC}^1(\Sigma, \mathbb{R}^{n\times n})$ and $\nabla_\Gamma \nu_{|\Gamma_{\mathrm{top}}} = \nabla_\Gamma \nu_{|\Gamma_{\mathrm{bot}}} = 0$. Therefore, the estimate

$$\begin{aligned}\|-R(w)\|_{{}_0\mathbb{F}_p^h(J)} &= \left\|\tfrac{1}{2}(\nabla_\Gamma \nu)w\right\|_{{}_0W_p^{1/2-1/2p}(J,L_p(\Gamma))\cap L_p(J,W_p^{1-1/p}(\Gamma))} \\ &\leqslant C\|w\|_{{}_0W_p^{1/2-1/2p}(J,L_p(\Gamma))\cap L_p(J,W_p^{1-1/p}(\Gamma))} \\ &\leqslant C|J|^\tau\|w\|_{{}_0W_p^{1-1/2p}(J,L_p(\Omega))\cap L_p(J,W_p^{2-1/p}(\Omega))} \\ &\leqslant C|J|^\tau\|w\|_{{}_0\mathbb{E}_p^u(J)}\end{aligned}$$

holds true, if $w(0) = 0$ in Ω. Here, $C, \tau > 0$ are constants, which are independent of J. Now, all assumptions of Lemma 2.5 are satisfied and thus system (2.4.7) has the property of maximal regularity. With this strategy we obtain a unique solution to (2.4.3) on the time interval J.

These two steps imply maximal regularity for (2.4.2) and thus maximal regularity for $(P|J)_{S-}$ and our initial assertion. □

Now, with the help of Proposition 2.11 and the Helmholtz decomposition, cf. Section 1.3, we can prove that system $(P|J)_{S-}$ with constant coefficients and homogeneous boundary conditions even has a unique solenoidal solution, if the given function f is solenoidal as well.

Corollary 2.12. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $\mu > 0$, $D_-(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then the parabolic system*

$$\begin{aligned}\partial_t u - \mu\Delta u &= f && \text{in } J\times\Omega,\\ u\cdot\nu &= 0 && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(u)\nu &= 0 && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } \Omega,\end{aligned} \tag{2.4.8}$$

has a unique solution $u \in \mathbb{E}_p^u(J) \cap L_p(J, L_{p,\sigma}(\Omega))$ for every set of data $(f, 0, 0, u_0) \in L_p(J, L_{p,\sigma}(\Omega)) \times \{0\} \times \{0\} \times \mathbb{F}_p^0 \cap L_{p,\sigma}(\Omega)$ fulfilling the compatibility conditions $(C3)_{S-}$.

Proof. Let $u \in \mathbb{E}_p^u(J)$ be the unique solution to system (2.4.8) for the given data $(f, 0, 0, u_0)$, which exists due to Proposition 2.11. We are thus left to show that $u \in L_p(J, L_{p,\sigma}(\Omega))$. There is an L_p-Helmholtz projection on finite cylinders, due to [41]. Thus, it is sufficient to prove $Hu = u$, where $H\colon L_p(\Omega) \to L_{p,\sigma}(\Omega)$ denotes the Helmholtz projection on Ω. Note that $-P_\Gamma D_-(u)\nu = -D_-(u)\nu$ on the boundary $J\times\Gamma$ and that $D_-(u)$ is a skew-symmetric matrix. Since $-P_\Gamma D_-(u)\nu = 0$, Lemma 1.11 yields $\operatorname{div}(D_-(u)) \in L_p(J, L_{p,\sigma}(\Omega))$. Also the equation

$$\begin{aligned}D_-(\nabla p)_{i,j} &= \tfrac{1}{2}(\partial_i(\nabla p)_j - \partial_j(\nabla p)_i)\\ &= \tfrac{1}{2}(\partial_i\partial_j p - \partial_j\partial_i p) = 0,\end{aligned} \tag{2.4.9}$$

holds true (in the sense of distributions) for all $p \in \dot{H}_p^1(\Omega, \mathbb{R}^n)$. Since $(1-H)u$ is a gradient, we obtain (in the distributional sense) by using (2.4.9) that $D_-(1-H)u) = 0$.

From this we conclude

$$\begin{aligned}\mu\Delta Hu &= 2\mu\mathrm{div}(D_-(Hu)) + \mu\nabla\mathrm{div}(Hu)\\ &= 2\mu\mathrm{div}(D_-(Hu))\\ &= 2\mu\mathrm{div}\left(D_-(Hu) + D_-((1-H)u)\right)\\ &= 2\mu\mathrm{div}(D_-(u))\\ &= 2\mu H\mathrm{div}(D_-(u))\\ &= H(\mathrm{div}(2\mu D_-(u)) + \mu\nabla\mathrm{div}(u))\\ &= H(\mu\Delta u).\end{aligned} \tag{2.4.10}$$

However, due to Proposition 1.10 we have $Hu \in \mathbb{E}_p^u(J) \cap L_p(J, L_{p,\sigma}(\Omega))$. In particular, $\mu\Delta Hu = H(\mu\Delta u)$ in $L_p(J \times \Omega)$.

In the left-hand side of (2.4.8), let us for the moment replace u with Hu. Using (2.4.9) and (2.4.10) we obtain

$$\begin{aligned}\partial_t Hu - \mu\Delta Hu &= H\partial_t u - H\mu\Delta u &&\\ &= H(\partial_t u - \mu\Delta u) &&\\ &= Hf = f && \text{in } J \times \Omega,\\ Hu \cdot \nu &= 0 && \text{on } J \times \Omega,\\ -P_\Gamma D_-(Hu)\nu &= -P_\Gamma D_-(Hu + (1-H)u)\nu &&\\ &= -P_\Gamma D_-(u)\nu = 0 && \text{on } J \times \Gamma,\\ Hu(0) &= Hu_0 = u_0 && \text{in } \Omega.\end{aligned}$$

Thus, in combination with Proposition 2.11 we conclude that Hu and u both are the unique solution to (2.4.8). Consequently, $Hu = u$. □

Remark 2.13. Corollary 2.12 generalises to the cases $p = \frac{3}{2}$, $p = 2$ and $p = 3$ by using interpolation results.

Remark 2.14. Our proof of Corollary 2.12 is only valid for constant coefficients as equation (2.4.10) does in general not hold for variable coefficients $\mu \in \mathcal{BUC}(\Omega; \mathbb{R})$.

Variable Coefficients

Denk, Hieber and Prüss proved in [15, Theorem 5.7] maximal regularity for elliptic operators in a Banach space of class $\mathcal{HT}$ with variable coefficients. For this purpose, they applied a localisation argument on elliptic operators and used then the maximal regularity of these elliptic operators with constant coefficients. Since we proved maximal regularity for $(P|J)_{ND}$ with constant coefficient $\mu > 0$ in Proposition 2.11, we can use the strategy of Denk, Hieber and Prüss to prove maximal regularity for $(P|J)_{ND}$ with variable coefficient

$$\mu \in \mathcal{BUC}(\Omega; \mathbb{R}) \text{ with } \inf_\Omega \mu > 0.$$

This strategy is used again to prove maximal regularity of the Stokes equations with perfect slip boundary conditions (Proposition 3.8) and is treated there in detail. Since the proof of maximal regularity for the Stokes equations is more difficult than for parabolic problem, we omit it for $(P|J)_{ND}$ with variable μ here.

Corollary 2.15. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0, T)$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $(f, h, u_0) \in \mathbb{F}_p^{P,S-}(J)$, $\mu \in \mathcal{BUC}(\Omega; \mathbb{R})$ with $\inf_\Omega \mu > 0$ and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then system $(P|J)_{S-}$ has a unique solution $u \in \mathbb{E}_p^u(J)$.*

2.5 Parabolic Problems: Free Slip Boundary Conditions

Using the maximal regularity of parabolic problems $(P|J)_{S-}$ with perfect slip boundary conditions from Section 2.4, we are able to prove maximal regularity for parabolic problems on cylindrical domains with free slip boundary conditions of the form

$$\begin{aligned} \partial_t u - \mu\Delta u &= f && \text{in } J\times\Omega, \\ u\cdot\nu &= h\cdot\nu && \text{on } J\times\Gamma, \\ P_\Gamma D_+(u)\nu &= P_\Gamma h && \text{on } J\times\Gamma, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \qquad (P|J)_{S+}$$

with $\Omega \subseteq \mathbb{R}^n$ a cylindrical domain and $J = (0, T)$, $T > 0$, a time interval. We study this system directly for variable coefficient μ, since we can prove the existence of a unique solution

$$u = u(t, x) \in \mathbb{E}_p^u(J)$$

to system $(P|J)_{S+}$ with variable μ straight without needing maximal regularity of system $(P|J)_{S+}$ with constant coefficient beforehand. The maximal regularity for $(P|J)_{S+}$ with constant coefficient then follows immediately from considering the system with variable coefficient. The study of parabolic problems with free slip boundary conditions is interesting, since we use them in Chapter 3 to show that the Stokes equations on cylindrical domains have the property of maximal regularity.

The data

$$(f, h, u_0) \in \mathbb{F}_p^{P,S+}(J)$$

have to satisfy the necessary regularity and compatibility conditions. The necessity of the regularity and compatibility conditions for $(P|J)_{S+}$ can be seen in Section 1.4. Then, the data space $\mathbb{F}_p^{P,S+}$, is defined to consist of all

$$(g, h, u_0) \in \mathbb{F}_p^f(J) \times \mathbb{F}_p^h(J) \times \mathbb{F}_p^0$$

that satisfy the compatibility condition

$$\begin{aligned} u_0\cdot\nu &= h(0)\cdot\nu, && \text{if } p > \tfrac{3}{2}, \\ P_\Gamma D_+(u_0)\nu &= P_\Gamma h(0), && \text{in } p > 3, \end{aligned} \qquad (C3)_{S+}$$

which arises from the compatibility between the data h and the initial data u_0, the compatibility condition

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\text{top}}}\cdot\nu_{\Gamma_{\text{top}}} &= [h^{\text{top}}]_{\mathcal{R}^{\text{top}}}\cdot\nu_\Sigma && \text{if } p > 2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\text{top}}}} h^\Sigma]_{\mathcal{R}^{\text{top}}}\cdot\nu_\Sigma + \tfrac{1}{2}[\partial_{\nu_\Sigma} h^{\text{top}}]_{\mathcal{R}^{\text{top}}}\cdot\nu_{\Gamma_{\text{top}}} &= [h^{\text{top}}]_{\mathcal{R}^{\text{top}}}\cdot\nu_\Sigma && \text{if } p > 2, \end{aligned} \qquad (C4)_{S+}$$

which arises from the boundary condition on the upper edge of Ω, and the condition

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\text{bot}}}\cdot\nu_{\Gamma_{\text{bot}}} &= [h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}}\cdot\nu_\Sigma && \text{if } p > 2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\text{bot}}}} h^\Sigma]_{\mathcal{R}^{\text{bot}}}\cdot\nu_\Sigma + \tfrac{1}{2}[\partial_{\nu_\Sigma} h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}}\cdot\nu_{\Gamma_{\text{bot}}} &= [h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}}\cdot\nu_\Sigma && \text{if } p > 2, \end{aligned} \qquad (C5)_{S+}$$

which arises from the boundary condition on the lower edge of Ω.

Variable Coefficients

The coefficient

$$\mu \in \mathcal{BUC}(\Omega;\mathbb{R}) \text{ with } \inf_{\Omega} \mu > 0.$$

of system $(P|J)_{S+}$ is assumed to be variable throughout this paragraph. To prove maximal regularity for the parabolic system $(P|J)_{S+}$, we use the same strategy as in the proof of Proposition 2.11. The difference is that we need the trace result with respect to free slip boundary conditions instead of the trace result with respect to perfect slip boundary conditions. Also we use the maximal regularity of parabolic problems with perfect slip boundary conditions instead of the maximal regularity of parabolic problems with Neumann-Dirichlet boundary conditions.

Theorem 2.16. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $\mu \in \mathcal{BUC}(\Omega;\mathbb{R})$ with $\inf_\Omega \mu > 0$ and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then a unique solution $u \in \mathbb{E}_p^u(J)$ of system $(P|J)_{S+}$ exists for every data $(f, h, u_0) \in \mathbb{F}_p^{P,S+}(J)$.*

Proof. As mentioned before, we proceed similarly to Proposition 2.11. That is, we split the velocity into $u = \hat{u} + \tilde{u}$, such that $(P|J)_{S+}$ decouples into two systems; one that contains the inhomogeneous free slip boundary conditions of $(P|J)_{S+}$ and one that is the parabolic system with homogeneous boundary condition:

$$\begin{aligned} \hat{u}\cdot\nu &= h\cdot\nu && \text{on } J\times\Gamma,\\ P_\Gamma D_+(\hat{u})\nu &= P_\Gamma h && \text{on } J\times\Gamma, \end{aligned} \tag{2.5.1}$$

and

$$\begin{aligned} \partial_t\tilde{u} - \mu\Delta\tilde{u} &= \tilde{f} && \text{in } J\times\Omega,\\ \tilde{u}\cdot\nu &= 0 && \text{on } J\times\Gamma,\\ P_\Gamma D_+(\tilde{u})\nu &= 0 && \text{on } J\times\Gamma,\\ \tilde{u}(0) &= \tilde{u}_0 && \text{in } \Omega. \end{aligned} \tag{2.5.2}$$

The data $\tilde{f} := f - (\partial_t\hat{u} - \mu\Delta\hat{u})$ in $J\times\Omega$ and $\tilde{u}_0 := u_0 - \hat{u}_0$ in Ω accomplish the necessary compatibility conditions $(C3)_{S+}$–$(C5)_{S+}$ and consequently we have $(\tilde{f}, 0, \tilde{u}_0) \in \mathbb{F}_p^{P,S+}(J)$. Set $\hat{u} \in \mathbb{E}_p^u(J)$ as the solution of (2.5.1), which exists according to Proposition 1.25.

It is left to prove maximal regularity for (2.5.2). To this end, we proceed in two steps. In the first step we show that (2.5.2) can be rewritten as a perturbed parabolic system with perfect slip boundary conditions and in the second step we prove maximal regularity for this perturbed system using Lemma 2.5.

Step 1. In this step we show that

$$\begin{aligned} \partial_t\tilde{u} - \mu\Delta\tilde{u} &= \tilde{f} && \text{in } J\times\Omega,\\ \tilde{u}\cdot\nu &= P_\Gamma(\nabla\tilde{u})\nu\cdot\nu = 0 && \text{on } J\times\Omega,\\ -P_\Gamma D_-(\tilde{u})\nu &= -P_\Gamma(\nabla\tilde{u})\nu && \text{on } J\times\Gamma,\\ \tilde{u}(0) &= \tilde{u}_0 && \text{in } \Omega, \end{aligned} \tag{2.5.3}$$

is equivalent to (2.5.2). Obviously $-P_\Gamma D_-(\tilde{u})\nu$ and $P_\Gamma(\nabla\tilde{u})\nu$ are tangential and therefore system (2.5.3) is well-posed. Considering

$$P_\Gamma D_+(\tilde{u})\nu = -P_\Gamma D_-(\tilde{u})\nu + P_\Gamma(\nabla\tilde{u})\nu, \tag{2.5.4}$$

we see that (2.5.3) is equivalent to (2.5.2).

Step 2. In this final step we show that (2.5.3) has the property of maximal regularity for a given time interval J. System (2.5.3) is a perturbed parabolic problem with perfect slip boundary conditions and a linear perturbation $R\colon {}_0\mathbb{E}_p^u(J) \longrightarrow {}_0\mathbb{F}_p^h(J)$ which is defined through $R(\tilde{u}) := P_\Gamma(\nabla\tilde{u})\nu$. Obviously it is $R(\tilde{u})(0) = 0$, if $\tilde{u}(0) = 0$ in Ω. To prove maximal regularity for (2.5.3) we want to use Lemma 2.5. But since the initial value $\tilde{u}_0$ of (2.5.3) is in general not zero, we have to rewrite (2.5.3) into two systems; one parabolic system without the perturbation R and with initial value $\tilde{u}_0$ and one parabolic system with perturbation R and initial value zero. In addition, the data of these two systems should satisfy the necessary compatibility conditions of a parabolic system with perfect slip boundary conditions, which is $(C3)_{S-}$–$(C5)_{S-}$. To this end, we split $\tilde{u} := v + w$ and choose a $\tilde{h} \in \mathbb{F}_p^h(J)$ with

$$\begin{aligned} \tilde{h}(0)\cdot\nu &= \tilde{u}_0\cdot\nu = 0 && \text{on } \Gamma \text{ if } p > \tfrac{3}{2}, \\ P_\Gamma\tilde{h}(0) &= -P_\Gamma D_-(\tilde{u}_0)\nu && \text{on } \Gamma \text{ if } p > 3. \end{aligned}$$

The equation $\tilde{u}_0\cdot\nu = 0$ is valid, since we have $(\tilde{f}, 0, \tilde{u}_0) \in \mathbb{F}_p^{P,S+}(J)$. Thus, we obtain for (2.5.3) the according systems

$$\begin{aligned} \partial_t v - \mu\Delta v &= 0 && \text{in } J\times\Omega, \\ v\cdot\nu &= 0 && \text{on } J\times\Gamma, \\ -P_\Gamma D_-(v)\nu &= P_\Gamma\tilde{h} && \text{on } J\times\Gamma, \\ v(0) &= \tilde{u}_0 && \text{in } \Omega, \end{aligned} \tag{2.5.5}$$

and

$$\begin{aligned} \partial_t w - \mu\Delta w &= \tilde{f} && \text{in } J\times\Omega, \\ w\cdot\nu &= (\hat{h} - R(w))\cdot\nu = 0 && \text{on } J\times\Gamma, \\ -P_\Gamma D_-(w)\nu &= \hat{h} - R(w) && \text{on } J\times\Gamma, \\ w(0) &= 0 && \text{in } \Omega, \end{aligned} \tag{2.5.6}$$

with $\hat{h} := -P_\Gamma(\nabla v)\nu - P_\Gamma\tilde{h}$. Obviously $\hat{h} - R(w) = -P_\Gamma\big((\nabla v)\nu + \tilde{h} + (\nabla w)\nu\big)$ is tangential, such that $(\hat{h} + R(w))\cdot\nu = 0$ in the normal direction. Due to the construction of $\tilde{h}$, the data $(0, \tilde{h}, \tilde{u}_0)$ fulfils the necessary compatibility conditions $(C3)_{S-}$–$(C5)_{S-}$ of system (2.5.5) and we have $(0, \tilde{h}, \tilde{u}_0) \in \mathbb{F}_p^{P,S-}(J)$. According to Corollary 2.15, it exists a unique solution $v \in \mathbb{E}_p^u(J)$ to system (2.5.5).

Now, we have to check all necessary assumptions of Lemma 2.5 to prove maximal regularity for (2.5.6). Considering the construction of $\tilde{h}$, the equation (2.5.4), the compatibility of the data $(\tilde{f}, 0, 0, \tilde{u}_0)$ with respect to system (2.5.2) and the compatibility of the data $(0, \tilde{h}, \tilde{u}_0)$ with respect to system (2.5.5), we have

$$\hat{h}(0) = -P_\Gamma(\nabla v(0))\nu - P_\Gamma\tilde{h}(0) = -P_\Gamma(\nabla\tilde{u}_0)\nu + P_\Gamma D_-(\tilde{u}_0)\nu = -P_\Gamma D_+(\tilde{u}_0)\nu = 0.$$

Thus, the data $(\tilde{f}, \hat{h}, 0)$ satisfies the compatibility conditions $(C3)_{S-}$–$(C5)_{S-}$ of system (2.5.6). Let $\gamma_1, \ldots, \gamma_{n-1}$ be a orthonormal basis of the tangent space $T_x\Gamma$ for $x \in \Gamma$, then for $w_0 = 0$ the inequality

$$\begin{aligned}
\|R(w)\|_{P_\Gamma 0\mathbb{F}_p^h(J)} &= \|P_\Gamma(\nabla w)\nu\|_{P_\Gamma 0\mathbb{F}_p^h(J)} \\
&= \left\| \sum_{i=1}^{n-1} \partial_{\gamma_i}(w\cdot\nu)\gamma_i - \sum_{i=1}^{n-1}(w\cdot\partial_{\gamma_i}\nu)\gamma_i \right\|_{P_\Gamma 0\mathbb{F}_p^h(J)} \\
&= \|\nabla_\Gamma(w\cdot\nu) - L_\Gamma(w)\|_{P_\Gamma 0\mathbb{F}_p^h(J)} \\
&= C\|L_\Gamma(w)\|_{P_\Gamma 0\mathbb{F}_p^h(J)}
\end{aligned}$$

holds, with $L_\Gamma(w) := \sum_i (w\cdot\partial_{\gamma_i}\nu)\gamma_i$. Since $A \subseteq \mathbb{R}^{n-1}$ is a bounded $\mathcal{C}^3$-domain, we have in addition

$$\begin{aligned}
\|L_\Gamma(w)\|_{P_\Gamma 0\mathbb{F}_p^h(J)} &\leqslant C\|L_\Gamma\|_{W^1_\infty(\Gamma,\mathcal{L}(\mathbb{R}^n,T\Gamma))}\|w\|_{P_\Gamma 0\mathbb{F}_p^h(J)} \\
&\leqslant C\|w\|_{{}_0H_p^{1/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))} \\
&\leqslant C|J|^\tau\|w\|_{{}_0H_p^1(J,L_p(\Omega))\cap {}_0H_p^{1/2}(J,H_p^1(\Omega))} \\
&\leqslant C|J|^\tau\|w\|_{{}_0\mathbb{E}_p^u(J)}
\end{aligned}$$

with constants $C, \tau > 0$ independent of T. Moreover, parabolic problems with perfect slip boundary conditions $(P|J)_{S-}$ have the property of maximal regularity for every data $(f, h, u_0) \in \mathbb{F}_p^{P,S-}(J)$ according to Corollary 2.15. Thus, all assumptions of Lemma 2.5 are satisfied and we obtain maximal regularity for (2.5.6). Maximal regularity of (2.5.5) and (2.5.6) implies maximal regularity for (2.5.3).

These steps prove maximal regularity for (2.5.2) and thus maximal regularity for $(P|J)_{S+}$ and our initial assertion.

□

3 Maximal L_p-Regularity of the Stokes Equations

The Stokes equations have been subject of much scientific research, e. g. [20, 21, 25, 52, 51, 48]. In this chapter we investigate the L_p-theory of the Stokes equations on cylinders in detail. More precisely, we prove maximal regularity of the Stokes equations with perfect slip and free slip boundary conditions. We are not only interested in the Stokes equations with constant coefficients ρ, μ and α, but also in the Stokes equations with variable coefficients. Considering the Stokes equations with free slip boundary conditions and variable coefficients prove to be valuable. We use these results in Chapter 4 to show the existence of a local-in-time strong solution to a model on the mechanisms of tropical storms in an L_p-setting, which comprises optimal restrictions on the integrability parameter p.

We aim to prove maximal regularity of the Stokes equations

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J\times\Omega,\\ \operatorname{div}(\rho u) &= g && \text{in } J\times\Omega,\\ \mathcal{B}^V(u,q) &= h && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \qquad (S|J)_V$$

For the boundary operators $\mathcal{B}^V$, $V\in\{\,S\pm\}$, we take $\mathcal{B}^V$ to be either the perfect slip or the free slip boundary operator, which are defined as on page 23 in Section 1.4. We denote by $\Omega := A\times(-a,a)\subseteq\mathbb{R}^n$ a cylindrical domain consisting of a bounded $\mathcal{C}^3$-domain A and an interval $(-a,a)$ with $a>0$. In addition, $J=(0,T)$, $T>0$, denotes a time interval. Note that the boundary conditions are imposed on the smooth part $\Gamma=\Gamma_{\text{top}}\,\dot\cup\,\Gamma_{\text{bot}}\,\dot\cup\,\Sigma$ of the boundary of Ω. The compatibility conditions on the edges are imposed below. Here, Γ_{top} and Γ_{bot} denote the boundary of top and bottom of Ω, respectively, and Σ denotes the lateral boundary. For a comprehensive study of cylindrical domains and their boundary we refer to Section 1.3.

For the Stokes equations we use the data spaces

$$\begin{aligned} \mathbb{F}_p^f(J) &= L_p(J\times\Omega)^n,\\ \mathbb{F}_p^g(J) &= H_p^{1/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega)),\\ \mathbb{F}_p^\Lambda(J) &= \{h\in W_p^{1/2-1/2p}(J,L_p(\Lambda))^n\cap L_p(J,W_p^{1-1/p}(\Lambda))^n :\\ &\qquad h\cdot\nu\in W_p^{1-1/2p}(J,L_p(\Lambda))\cap L_p(J,W_p^{2-1/p}(\Lambda))\},\ \Lambda\in\{\Gamma_{\text{top}},\Gamma_{\text{bot}},\Sigma\}\\ \mathbb{F}_p^h(J) &= \{h\colon\Gamma\longrightarrow\mathbb{R}^n : h_{|\Gamma_{\text{top}}} =: h^{\text{top}}\in\mathbb{F}_p^{\Gamma_{\text{top}}}(J),\\ &\qquad h_{|\Sigma} =: h^\Sigma\in\mathbb{F}_p^\Sigma(J),\ h_{|\Gamma_{\text{bot}}} =: h^{\text{bot}}\in\mathbb{F}_p^{\Gamma_{\text{bot}}}(J)\},\\ \mathbb{F}_p^0 &= W_p^{2-2/p}(\Omega)^n, \end{aligned}$$

and the solution space

$$\begin{aligned}\mathbb{E}_p(J) &:= \mathbb{E}_p^u(J) \times \mathbb{E}_p^q(J)\\ &= H_p^1(J, L_p(\Omega))^n \cap L_p(J, H_p^2(\Omega))^n \times \{q \in L_p(J, H_p^1(\Omega)) : (q)_\Omega = 0\},\end{aligned}$$

as defined on page 24 in Section 1.4. If the coefficients ρ, α and μ are constant, we assume the momentum equation of $(S|J)_V$ to take the form

$$\partial_t u - \mu\Delta u + \alpha\nabla q = f \quad \text{in } J \times \Omega,$$

and the divergence equation of $(S|J)_V$ as

$$\operatorname{div}(u) = g \quad \text{in } J \times \Omega,$$

cf. Remark 1.12

3.1 Stokes Equations: Perfect Slip Boundary Conditions

This section is devoted to the study of the Stokes equations on a cylindrical domain with perfect slip boundary conditions, i. e.

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J \times \Omega,\\ \operatorname{div}(\rho u) &= g && \text{in } J \times \Omega,\\ u \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J \times \Gamma,\\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{$(S|J)_{S-}$}$$

We study this system with both constant and variable coefficients ρ, α and μ. Considering the Stokes equations with perfect slip boundary conditions is interesting, since with their help and a perturbation argument we are able to show in Section 3.2 that the Stokes equations with free slip boundary conditions have the property of maximal regularity. This eventually allows us to prove the existence of a local-in-time strong solution to a model on the mechanisms of tropical storms in Chapter 4.

Again $\Omega \subseteq \mathbb{R}^n$ is a cylindrical domain and $J = (0, T)$, $T > 0$, a time interval. We aim to prove the existence of a unique solution

$$(u, q) = (u, q)(t, x)) \in \mathbb{E}_p(J)$$

to system $(S|J)_{S-}$ for every data

$$(f, g, h, u_0) \in \mathbb{F}_p^{S-}(J)$$

which meet the necessary regularity and compatibility conditions. On this account we introduce the data space $\mathbb{F}_p^{S-}(J)$, which is defined to consist of all

$$(f, g, h, u_0) \in \mathbb{F}_p^f(J) \times \mathbb{F}_p^g(J) \times \mathbb{F}_p^h(J) \times \mathbb{F}_p^0$$

that satisfy the following four compatibility conditions. The first of these conditions is given by

$$\operatorname{div}(u_0) = g(0) \quad \text{if } p \geqslant 2, \tag{$C1$}$$

which arises by touching time trace of the divergence equation. Next, the condition

$$\mathcal{F}(g, h\cdot\nu) \in H^1_p(J, {}_0\dot{H}^{-1}_p(\Omega)), \tag{$C2$}$$

compelled by the divergence condition and the normal boundary condition. The compatibility condition

$$\begin{aligned} u_0\cdot\nu &= h(0)\cdot\nu, &&\text{if } p>\tfrac{3}{2}, \\ -P_\Gamma D_-(u_0)\nu &= P_\Gamma h(0), &&\text{if } p>3, \end{aligned} \tag{$C3$}_{S-}$$

which arises from the boundary conditions of $(S|J)_{S-}$ on the smooth part of the boundary. The compatibility condition

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\mathrm{top}}}\cdot\nu_{\Gamma_{\mathrm{top}}} &= -[h^{\mathrm{top}}]_{\mathcal{R}^{\mathrm{top}}}\cdot\nu_\Sigma &&\text{if } p>2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\mathrm{top}}}}h^\Sigma]_{\mathcal{R}^{\mathrm{top}}}\cdot\nu_\Sigma - \tfrac{1}{2}[\partial_{\nu_\Sigma}h^{\mathrm{top}}]_{\mathcal{R}^{\mathrm{top}}}\cdot\nu_{\Gamma_{\mathrm{top}}} &= [h^{\mathrm{top}}]_{\mathcal{R}^{\mathrm{top}}}\cdot\nu_\Sigma &&\text{if } p>2, \end{aligned} \tag{$C4$}_{S-}$$

which arises from the boundary condition on the upper edge of Ω. The condition

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\mathrm{bot}}}\cdot\nu_{\Gamma_{\mathrm{bot}}} &= -[h^{\mathrm{bot}}]_{\mathcal{R}^{\mathrm{bot}}}\cdot\nu_\Sigma &&\text{if } p>2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\mathrm{bot}}}}h^\Sigma]_{\mathcal{R}^{\mathrm{bot}}}\cdot\nu_\Sigma - \tfrac{1}{2}[\partial_{\nu_\Sigma}h^{\mathrm{bot}}]_{\mathcal{R}^{\mathrm{bot}}}\cdot\nu_{\Gamma_{\mathrm{bot}}} &= [h^{\mathrm{bot}}]_{\mathcal{R}^{\mathrm{bot}}}\cdot\nu_\Sigma &&\text{if } p>2, \end{aligned} \tag{$C5$}_{S-}$$

which arises from the boundary condition on the lower edge of Ω. The necessity of these conditions was shown in Section 1.4.

3.1.1 Constant Coefficients

Throughout this subsection we assume the coefficients

$$\rho > 0,\ \alpha > 0,\ \mu > 0$$

of system $(S|J)_{S-}$ to be constant. As mentioned before, we can then consider system $(S|J)_{S-}$ as

$$\begin{aligned} \partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J\times\Omega, \\ \operatorname{div}(u) &= g && \text{in } J\times\Omega, \\ u\cdot\nu &= h\cdot\nu && \text{on } J\times\Gamma, \\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J\times\Gamma, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{$(S|J)^C_{S-}$}$$

according to Remark 1.12 (ii). The proof of maximal regularity of the Stokes equations $(S|J)^C_{S-}$ is proceeded in Proposition 3.2. To do so, we split $(S|J)^C_{S-}$ into three systems: the first is composed of the inhomogeneous perfect slip boundary conditions, the second is composed of the divergence condition and homogeneous perfect slip boundary conditions, and the third is composed of the Stokes equations containing a homogeneous divergence condition as well as a homogeneous perfect slip boundary condition. We prove the existence of a unique solution to the first system by using the retraction property of the trace operator with respect to the perfect slip boundary conditions, see Section 1.4. By using a result of Nau [39], we are able to prove the existence of a unique solution to the second system. Proving maximal regularity of the Stokes equations with homogeneous divergence condition as well as homogeneous perfect slip boundary conditions are more involved. We deal with this type of problem in more detail in the following corollary.

Corollary 3.1. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0, T)$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$, $(f, 0, 0, u_0) \in \mathbb{F}_p^{S-}(J)$, $D_-(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ and $\alpha,\ \mu > 0$. Then system*

$$\begin{aligned} \partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J \times \Omega, \\ \operatorname{div}(u) &= 0 && \text{in } J \times \Omega, \\ u \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u)\nu &= 0 && \text{on } J \times \Gamma, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{3.1.1}$$

has a unique solution $(u, q) \in \mathbb{E}_p(J)$.

Proof. It is known that the L_p-Helmholtz projection exists on finite cylinders, cf. [41]. To prove that (3.1.1) has the property of maximal regularity, we let $u \in \mathbb{E}_p^u(J) \cap L_p(J, L_{p,\sigma}(\Omega))$ be the unique solution of

$$\begin{aligned} \partial_t u - \mu\Delta u &= Hf && \text{in } J \times \Omega, \\ u \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ \mu P_\Gamma D_-(u)\nu &= 0 && \text{on } J \times \Gamma, \\ u(0) &= Hu_0 && \text{in } \Omega, \end{aligned} \tag{3.1.2}$$

which exists according to Corollary 2.12. Let us define $\nabla q := (1 - H)f$. Then, u and ∇q satisfy the equation

$$\partial_t u - \mu\Delta u + \nabla q = Hf + (1 - H)f = f,$$

in $J \times \Omega$. Since $u \in L_p(J, L_{p,\sigma}(\Omega))$, also the equation

$$\operatorname{div}(u) = 0 \quad \text{in } J \times \Omega,$$

holds. Furthermore, we have $\operatorname{div}(u) = 0$ in $J \times \Omega$ and $u_0 \cdot \nu = 0$ in $J \times \Gamma$, due to the compatibility conditions $(C1)$ and $(C3)_{S-}$. Therefore, it is $u_0 \in W_p^{2-2/p}(\Omega)^n \cap L_{p,\sigma}(\Omega)$ and we have $Hu_0 = u_0$. Hence, $(u, q) \in \mathbb{E}_p(J)$ is a solution to (3.1.1).

To prove that $(u, q) \in \mathbb{E}_p(J)$ is the unique solution to (3.1.1), let $(v, p) \in \mathbb{E}_p(J)$ be a solution to (3.1.1) with data $f = 0$ and $u_0 = 0$. Assume $(u, q) \in \mathbb{E}_p(J)$ to be the solution to (3.1.1) with data $f = 0$ and $u_0 = 0$ as constructed above. We have $v \in L_p(J, L_{p,\sigma}(\Omega))$, since $\operatorname{div}(v) = 0$ in $J \times \Omega$ and $v \cdot \nu = 0$ on $J \times \Gamma$. It follows that $Hv = v$. Using (2.4.10), we obtain

$$\partial_t v - \mu\Delta v = H(\partial_t v - \mu\Delta v) = H(\partial_t v - \mu\Delta v + \alpha\nabla p) = H(0) = 0$$

in $J \times \Omega$ and $v(0) = 0$ in Ω. Thus, v is a solution to (3.1.2) with data $f = 0$ and $u_0 = 0$. Due to Corollary 2.12 the system (3.1.2) has a unique solution. Therefore, it is $v = u$. The pressure $q \in \mathbb{E}_p^q(J)$ is also unique, since its divergence $\nabla q = (1 - H)f$ is defined through u and it is assumed to have mean value zero. Thus, (u, q) is the unique solution to (3.1.1). □

With the help of this proposition we are now able to prove maximal regularity of $(S|J)^C_{S-}$.

Proposition 3.2. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0, T)$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$, $\alpha,\ \mu > 0$ and $D_-(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$. Then system $(S|J)^C_{S-}$ has a unique solution $(u, q) \in \mathbb{E}_p(J)$ for every data $(f, g, h, u_0) \in \mathbb{F}_p^{S-}(J)$.*

Proof. We decompose the velocity $u = u_1 + u_2 + u_3$ according to the three systems

$$\begin{aligned} u_1 \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u_1)\nu &= P_\Gamma h && \text{on } J \times \Gamma, \end{aligned} \tag{3.1.3}$$

$$\begin{aligned} \operatorname{div}(u_2) &= g - \operatorname{div}(u_1) && \text{in } J \times \Omega, \\ u_2 \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u_2)\nu &= 0 && \text{on } J \times \Gamma, \end{aligned} \tag{3.1.4}$$

and

$$\begin{aligned} \partial_t u_3 - \mu\Delta u_3 + \alpha\nabla q &= f - (\partial_t - \mu\Delta)u_1 - (\partial_t - \mu\Delta)u_2 && \text{in } J \times \Omega, \\ \operatorname{div}(u_3) &= 0 && \text{in } J \times \Omega, \\ u_3 \cdot \nu &= 0 && \text{on } J \times \Omega, \\ -P_\Gamma D_-(u_3)\nu &= 0 && \text{on } J \times \Omega, \\ u_3(0) &= u_0 - u_1(0) - u_2(0) && \text{in } \Omega. \end{aligned} \tag{3.1.5}$$

Our strategy is to find a solution to (3.1.3) and (3.1.4). Then, we are left to deal with (3.1.5), but where the right-hand side depends on the solutions to (3.1.3) and (3.1.4) only. Using Corollary 3.1 then yields a unique solution (u_3, q) to system (3.1.5) and thus maximal regularity of $(S|J)^C_{S-}$.

Concerning the solvability of (3.1.3):
From Proposition 1.21 we know that the trace operator with respect to perfect slip boundary condition is a retraction and thus system (3.1.3) has a solution $u_1 \in \mathbb{E}^u_p(J)$.

Concerning the solvability (3.1.4):
To obtain a solution $u_2 \in \mathbb{E}^u_p(J)$, let us first consider the problem

$$\begin{aligned} \Delta p &= g - \operatorname{div}(u_1) && \text{in } J \times \Omega, \\ \partial_\nu p &= 0 && \text{on } J \times \Gamma. \end{aligned} \tag{3.1.6}$$

Making use of [39, Theorem 8.22] gives us a solution $\nabla p \in \mathbb{E}^u_p(J)$ to (3.1.6). By using $u_2 := \nabla p \in \mathbb{E}^u_p(J)$, we obtain a solution that solves the system

$$\begin{aligned} \operatorname{div}(u_2) &= \operatorname{div}(\nabla p) = \Delta p &&= g - \operatorname{div}(u_1) && \text{in } J \times \Omega, \\ u_2 \cdot \nu &= \nabla p \cdot \nu = \partial_\nu p &&= 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u_2)\nu &= -P_\Gamma D_-(\nabla p)\nu &&= 0 && \text{on } J \times \Gamma. \end{aligned}$$

This follows from equation (2.4.9) and the fact that the Theorem of Schwarz is valid. This implies that $u_2 \in \mathbb{E}^u_p(J)$ is a solution to (3.1.4).

Concerning the solvability of (3.1.5):
From Corollary 3.1 we obtain a unique solution $(u_3, q) \in \mathbb{E}_p(J)$ of system (3.1.5).

Combining the solutions of (3.1.3), (3.1.4) and (3.1.5), we obtain a unique solution $(u, q) = (u_1 + u_2 + u_3, q_3) \in \mathbb{E}_p(J)$ of $(S|J)^C_{S-}$. □

3.1.2 Variable Coefficients

Within this subsection we assume the coefficients of $(S|J)_{S-}$ to be variable, i. e.

$$\rho \in W^2_\infty(\Omega, (0,\infty)) \text{ with } \tfrac{1}{\rho} \in W^2_\infty(\Omega, (0,\infty))$$

and

$$\alpha \in \mathcal{BUC}^1(\Omega),\ \mu \in \mathcal{BUC}^1(\Omega) \text{ with } \inf_\Omega \alpha,\ \inf_\Omega \mu > 0.$$

We prove that the Stokes equations $(S|J)_{S-}$ with these coefficients have the property of maximal regularity. More precisely, the main result of this subsection is:

Theorem 3.3. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$ with $T > 0$. Assume $\Omega := A \times (-a,a)$ to be cylindrical domain, $\rho \in W^2_\infty(\Omega,(0,\infty))$ with $\frac{1}{\rho} \in W^2_\infty(\Omega,(0,\infty))$, $\alpha \in \mathcal{BUC}^1(\Omega)$, $\mu \in \mathcal{BUC}^1(\Omega)$ with $\inf_\Omega \alpha$, $\inf_\Omega \mu > 0$, and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then the Stokes equations $(S|J)_{S-}$ have a unique solution $(u,q) \in \mathbb{E}_p(J)$ for every data $(f,g,h,u_0) \in \mathbb{F}^{S-}_p(J)$.*

The proof of this theorem is provided at the end of this subsection. In the following paragraph, we state our strategy for proving this theorem.

Strategy

To prove Theorem 3.3, i. e. to show maximal regularity of $(S|J)_{S-}$, we progressively simplify system $(S|J)_{S-}$. That is, in the proof of Theorem 3.3, with the help of a substitution and a perturbation argument we show that it is sufficient to establish maximal regularity for the system

$$\begin{aligned} \partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J\times\Omega,\\ \operatorname{div}(u) &= g && \text{in } J\times\Omega,\\ u\cdot\nu &= h\cdot\nu && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{3.1.7}$$

in order to prove maximal regularity of $(S|J)_{S-}$. Then, in Proposition 3.9 we see that it is sufficient to prove maximal regularity for

$$\begin{aligned} \partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J\times\Omega,\\ \operatorname{div}(u) &= 0 && \text{in } J\times\Omega,\\ u\cdot\nu &= 0 && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J\times\Gamma,\\ u(0) &= 0 && \text{in } \Omega, \end{aligned} \tag{3.1.8}$$

in order to prove maximal regularity of (3.1.7). So we reduce system $(S|J)_{S-}$ to the much simpler problem (3.1.8). The actual proof, then, is to show maximal regularity of (3.1.8), which is done in Proposition 3.8. To this end, we use the already mentioned localization argument for variable coefficients, which is also used in Denk, Hieber and Prüss [15, Theorem 5.7]. However, for this we need additional time regularity property for the pressure q (Proposition 3.4) and maximal regularity of the Stokes equations with variable coefficients, which are assumed to be small with respect to the L_∞-norm (Proposition 3.7).

Regularity of the pressure

The following result shows that it is possible to obtain additional time regularity for the pressure, provided the data meets some additional assumptions.

Proposition 3.4. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$ with $T > 0$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $\alpha \in \mathcal{BUC}^1(\Omega)$, $\mu \in \mathcal{BUC}^1(\Omega)$ with $\inf_\Omega \alpha,\ \inf_\Omega \mu > 0$, and $(u,q) \in \mathbb{E}_p(J)$ to be a solution to the system*

$$\begin{aligned} \partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J\times\Omega,\\ \operatorname{div}(u) &= 0 && \text{in } J\times\Omega,\\ u\cdot\nu &= 0 && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J\times\Gamma,\\ u(0) &= 0 && \text{in } \Omega, \end{aligned} \tag{3.1.9}$$

where $(f,0,h,0) \in \mathbb{F}_p^{S-}(J)$ satisfy the additional regularity property

$$f \in {}_0H_p^{\vartheta}(J, L_p(\Omega)),$$

for some $\vartheta \in (0, \frac{1}{2} - \frac{1}{2p})$. Then

$$\alpha q \in {}_0H_p^{\vartheta}(J, L_p(\Omega))$$

and the estimate

$$\|\alpha q\|_{{}_0H_p^{\vartheta}((J,L_p(\Omega))} \leqslant C\left(\|u\|_{{}_0\mathbb{E}_p^u(J)} + \|f\|_{{}_0H_p^{\vartheta}(J,L_p(\Omega))}\right)$$

is valid with constant $C > 0$.

Proof. Given $\psi \in L_{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Elliptic problems with variable coefficients have maximal L_p-regularity due to Theorem 2.1. Thus, we take $\phi \in H_{p'}^2(\Omega)$ with $(\phi)_\Omega = 0$ to be a solution of

$$\begin{aligned} -\operatorname{div}(\alpha\nabla\phi) &= \alpha\psi_0 && \text{in } \Omega,\\ \partial_\nu\phi &= 0 && \text{on } \Gamma. \end{aligned} \tag{3.1.10}$$

Here, $\alpha\psi_0 := \alpha\psi - (\alpha\psi)_\Omega$, where $(\varphi)_\Omega := \frac{1}{|\Omega|}\int_\Omega \varphi \,\mathrm{d}x$ denotes the mean value of $\varphi \in L_p(\Omega)$. Due to the fact that the pressure q is assumed to have mean value zero, as well as (3.1.10), we can deduce by using partial integration

$$\begin{aligned} (\alpha q, \psi)_\Omega &= (q, \alpha\psi)_\Omega\\ &= (q, \alpha\psi_0)_\Omega + (q, (\alpha\psi)_\Omega)_\Omega\\ &= (q, \alpha\psi_0)_\Omega\\ &= -\left(q, \operatorname{div}(\alpha\nabla\phi)\right)_\Omega\\ &= -\int_{\partial\Omega} q\nu\cdot\alpha\nabla\phi \,\mathrm{d}\sigma + \int_\Omega \alpha\nabla q\cdot\nabla\phi \,\mathrm{d}x\\ &= (\alpha\nabla q, \nabla\phi)_\Omega. \end{aligned}$$

In view of the momentum equation, this leads to

$$(\alpha\nabla q, \nabla\phi)_\Omega = (f - \partial_t u + \mu\Delta u, \nabla\phi)_\Omega\,.$$

Partial integration furthermore implies

$$(\partial_t u, \nabla\phi)_\Omega = \partial_t (u, \nabla\phi)_\Omega = 0$$

and

$$\begin{aligned}(\mu\Delta u, \nabla\phi)_\Omega &= \int_{\partial\Omega} (\nabla u)^T \nu \cdot \mu\nabla\phi \, \mathrm{d}\sigma - \int_\Omega \nabla u : \nabla(\mu\nabla\phi) \, \mathrm{d}x \\ &= (\partial_\nu u, \mu\nabla\phi)_{\partial\Omega} - (\nabla u, \nabla(\mu\nabla\phi))_\Omega \, .\end{aligned}$$

In summary, we have obtained

$$(\alpha q, \psi)_\Omega = (\partial_\nu u, \mu\nabla\phi)_{\partial\Omega} - (\nabla u, \nabla\left(\mu\nabla\phi\right))_\Omega + (f, \nabla\phi)_\Omega \, .$$

Next, we observe that

$$\begin{aligned}\|\mu\nabla\phi\|_{L_{p'}(\Omega)} &\leqslant \|\mu\|_{L_\infty(\Omega)} \|\nabla\phi\|_{L_{p'}(\Omega)} \leqslant C\|\nabla\phi\|_{H^1_{p'}(\Omega)} \leqslant C\|\nabla^2\phi\|_{L_{p'}(\Omega)} \\ &\leqslant C\|\alpha\psi_0\|_{L_{p'}(\Omega)} \leqslant C\|\alpha\psi\|_{L_{p'}(\Omega)} \leqslant C\|\psi\|_{L_{p'}(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\|\nabla\left(\mu\nabla\phi\right)\|_{L_{p'}(\Omega)} &\leqslant \|\nabla\mu\nabla\phi\|_{L_{p'}(\Omega)} + \left\|\mu\nabla^2\phi\right\|_{L_{p'}(\Omega)} \\ &\leqslant \|\nabla\mu\|_{L_\infty(\Omega)} \|\nabla\phi\|_{L_{p'}(\Omega)} + \|\mu\|_{L_\infty(\Omega)} \left\|\nabla^2\phi\right\|_{L_{p'}(\Omega)} \\ &\leqslant C\,\|\psi\|_{L_{p'}(\Omega)} + C\,\|\alpha\psi_0\|_{L_{p'}(\Omega)} \\ &\leqslant C\,\|\psi\|_{L_{p'}(\Omega)} \, ,\end{aligned}$$

with constant $C > 0$, which only depends on Ω, α and μ. Now, the operator $\partial_t \colon D(\partial_t) \subseteq L_p(J) \to L_p(J)$ with $D(\partial_t) = {}_0H^1_p(J)$ has an $\mathcal{H}^\infty$-calculus with angle $\frac{\pi}{2}$ and the fractional power $\partial_t^\vartheta \in \mathcal{L}_{is}(D(\partial_t^\vartheta), L_p(J))$ with $D(\partial_t^\vartheta) = [L_p(J), D(\partial_t)]_\vartheta = H^\vartheta_p(J)$ for $0 < \vartheta < \frac{1}{2} - \frac{1}{2p}$, cf. [46]. Hence, we can estimate $\|\partial_t^\vartheta(\alpha q)\|_{L_p(\Omega)}$ as

$$\begin{aligned}&\|\partial_t^\vartheta(\alpha q)\|_{L_p(\Omega)} \\ &\quad= \sup_{\substack{\psi\in L_{p'}(\Omega)\\ \|\psi\|_{p'}=1}} (\partial_t^\vartheta(\alpha q), \psi)_\Omega \\ &\quad\leqslant \sup_{\substack{\psi\in L_{p'}(\Omega)\\ \|\psi\|_{p'}=1}} (\partial_t^\vartheta \partial_\nu u, \mu\nabla\phi)_\Gamma + \sup_{\substack{\psi\in L_{p'}(\Omega)\\ \|\psi\|_{p'}=1}} (\partial_t^\vartheta \nabla u, \nabla(\mu\nabla\phi))_\Omega + \sup_{\substack{\psi\in L_{p'}(\Omega)\\ \|\psi\|_{p'}=1}} (\partial_t^\vartheta f, \nabla\phi)_\Omega \\ &\quad\leqslant C\left(\|\partial_t^\vartheta \partial_\nu u\|_{L_p(\Gamma)} + \|\partial_t^\vartheta \nabla u\|_{L_p(\Omega)} + \|\partial_t^\vartheta \nabla u\|_{L_p(\Omega)}\right),\end{aligned}$$

for almost all $t \in J$ and $C > 0$. We can then infer that

$$\begin{aligned}&\left(\textstyle\int_J \|\partial_t^\vartheta(\alpha q)\|^p_{L_p(\Omega)} \, \mathrm{d}t\right)^{1/p} \\ &\leqslant C\left(\left(\textstyle\int_J \|\partial_t^\vartheta \partial_\nu u\|^p_{L_p(\Omega)} \, \mathrm{d}t\right)^{1/p} + \left(\textstyle\int_J \|\partial_t^\vartheta \nabla u\|^p_{L_p(\Omega)} \, \mathrm{d}t\right)^{1/p} + \left(\textstyle\int_J \|\partial_t^\vartheta f\|^p_{L_p(\Omega)} \, \mathrm{d}t\right)^{1/p}\right) \\ &\leqslant C\left(\|\partial_t^\vartheta \partial_\nu u\|_{L_p(J,L_p(\Gamma))} + \|\partial_t^\vartheta \nabla u\|_{L_p(J,L_p(\Omega))} + \|\partial_t^\vartheta f\|_{L_p(J,L_p(\Omega))}\right) \\ &\leqslant C\left(\|u\|_{{}_0\mathbb{E}^u_p(J)} + \|f\|_{{}_0H^\vartheta_p(J,L_p(\Omega))}\right),\end{aligned}$$

with constant $C > 0$, which only depends on Ω, α and μ. In conclusion, note that

$$\|\alpha q\|_{{}_0H_p^{\vartheta}(J,L_p(\Omega))} \leqslant C\|\partial_t^{\vartheta}(\alpha q)\|_{L_p(J,L_p(\Omega))} = C\left(\int_J \|\partial_t^{\vartheta}(\alpha q)\|_{L_p(\Omega)}^p \,\mathrm{dt}\right)^{1/p}.$$

Thus, we proved our initial assertion. □

Maximal Regularity of System (3.1.7)

We begin by proving a perturbation argument, which is used several times throughout this chapter.

Lemma 3.5. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $(f,g,h,u_0) \in \mathbb{F}_p^{S-}(J)$, $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Let also*

$$L\colon \mathbb{E}_p(J) \longrightarrow \mathbb{F}_p^{S-}(J)$$

be a linear operator, where we use the notation ${}_0L := L_{|{}_0\mathbb{E}_p(J)}\colon {}_0\mathbb{E}_p(J) \longrightarrow {}_0\mathbb{F}_p^{S-}(J)$. Let $(f,h) \in \mathbb{F}_p^f(J) \times \mathbb{F}_p^h(J)$ satisfy the compatibility conditions $(C4)_{S-}$–$(C5)_{S-}$ and $h(0) = 0$ in Ω. Let $R_1\colon \mathbb{E}_p(J) \longrightarrow \mathbb{F}_p^f(J)$, $R_2\colon {}_0\mathbb{E}_p(J) \longrightarrow {}_0\mathbb{F}_p^h(J)$ be linear functions with $R_2(u,q)(0) = 0$ for $u \in \mathbb{E}_p^u(J)$ with $u(0) = 0$ in Ω. If L is an isomorphism, and

$$\|R_1(u,q)\|_{\mathbb{F}_p^f(J)},\ \|R_2(u,q)\|_{{}_0\mathbb{F}_p^h(J)} \leqslant C|J|^{\tau}\|(u,q)\|_{{}_0\mathbb{E}_p(J)}, \tag{3.1.11}$$

or

$$\|R_1(u,q)\|_{\mathbb{F}_p^f(J)},\ \|R_2(u,q)\|_{{}_0\mathbb{F}_p^h(J)} < \frac{1}{\|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S-}(J)\longrightarrow {}_0\mathbb{E}_p(J)}}\|(u,q)\|_{{}_0\mathbb{E}_p(J)}, \tag{3.1.12}$$

for every $(u,q) \in \mathbb{E}_p(J)$ with $u(0) = 0$ in Ω and some constant $C, \tau > 0$, which are independent of J, then

$$L(u,q) = (f + R_1(u,q), 0, h + R_2(u,q), 0) \tag{3.1.13}$$

has a unique solution $(u,q) \in \mathbb{E}_p(J)$.

Proof. The functions $(f,h) \in \mathbb{F}_p^f(J) \times \mathbb{F}_p^h(J)$ satisfy the necessary compatibility conditions, and R_1, R_2 are linear with $R_2(u,q)(0) = 0$, if $u(0) = 0$ in Ω. Therefore, we can rewrite (3.1.13) as

$$L(u,q) = (f,0,h,0) + (R_1(u,q),0,R_2(u,q),0).$$

This equation is equivalent to

$$(u,q) = {}_0L^{-1}(f,0,h) + {}_0L^{-1}(R_1(u,q),0,R_2(u,q)),$$

since L is an isomorphism due to our assumptions. Subtraction yields

$$(\mathrm{Id} - {}_0L^{-1}(R_1,0,R_2))(u,q) = {}_0L^{-1}(f,0,h).$$

In the following step we prove that $(\mathrm{Id} - {}_0L^{-1}(R_1,0,R_2))$ is bijective, because then

$$(u,q) = (\mathrm{Id} - {}_0L^{-1}(R_1,0,R_2))^{-1}{}_0L^{-1}(f,0,h)$$

would be the unique solution to (3.1.13). In order to do so, we show that

$$\|{}_0L^{-1}(R_1, 0, R_2)\|_{{}_0\mathbb{E}_p(J) \longrightarrow {}_0\mathbb{E}_p(J)} < 1.$$

We have the bounds (3.1.11) or (3.1.12) for every $(u, q) \in \mathbb{E}_p(J)$ with $u(0) = 0$ in Ω and some constant $C, \tau > 0$, which are independent of J. In the case of (3.1.12) we have

$$\begin{aligned} \|{}_0L^{-1}(R_1, 0, R_2)\|_{{}_0\mathbb{E}_p(J)\to{}_0\mathbb{E}_p(J)} &\leqslant \|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S-}(J)\to{}_0\mathbb{E}_p(J)} \|(R_1, 0, R_2)\|_{{}_0\mathbb{E}_p(J)\to{}_0\mathbb{F}_p(J)} \\ &< \|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S-}(J)\to{}_0\mathbb{E}_p(J)} \frac{1}{\|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S-}(J)\to{}_0\mathbb{E}_p(J)}} \\ &= 1. \end{aligned}$$

In the case of (3.1.11), we can make the terms $\|R_1\|$ and $\|R_2\|$ so small that the estimate

$$\|{}_0L^{-1}(R_1, 0, R_2)\|_{{}_0\mathbb{E}_p(J)\to{}_0\mathbb{E}_p(J)} \leqslant \|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S-}(J)\to{}_0\mathbb{E}_p(J)} \|(R_1, 0, R_2)\|_{{}_0\mathbb{E}_p(J)\to{}_0\mathbb{F}_p^{S-}(J)} < 1,$$

holds. We do this by choosing a sufficiently small time interval J. By applying a Neumann series argument we obtain the existence of $(\mathrm{Id} - {}_0L^{-1}(R_1, 0, R_2))^{-1}$ and thus the unique solvability of (3.1.13) on a small time interval. We can also show the unique solvability of (3.1.13) for any given time interval J, since the admitted length of the time interval J does not depend on the data. This is done by successively solving the equation on small time intervals of fixed length, cf. Lemma 2.5, where a similar argument has been used. □

Remark 3.6. Let the same assumptions as in Lemma 3.5 apply, with $R_2 \equiv 0$. Then, we can conclude that

$$L(u, q) = (f + R_1(u, q), g, h, u_0)$$

has a unique solution $(u, q) \in \mathbb{E}_p(J)$ for every data $(f, g, h, u_0) \in \mathbb{F}_p^{S-}(J)$ by using the same arguments as in Lemma 3.5. Since no perturbations exists on the boundary due to $R_2 \equiv 0$, no problems can arise regarding the compatibilities between the boundary and the initial value.

As mentioned in *Strategy*, the most difficult part of proving solvability of (3.1.7) is to prove maximal regularity for system (3.1.8), which is done in Proposition 3.8. Before we address this issue, however, we first have to prove maximal regularity of the Stokes equations with variable coefficients which are assumed to be small with respect to the L_∞-norm, since we use it in the proof of Proposition 3.8.

Proposition 3.7. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0, T)$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $(f, g, h, u_0) \in \mathbb{F}_p^{S-}(J)$, $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$ and $\alpha_0,\ \mu_0 > 0$. Then there exists an $\varepsilon > 0$, such that for all $\alpha_1 \in \mathcal{BUC}^1(\Omega)$ and $\mu_1 \in \mathcal{BUC}(\Omega)$ with $\inf_\Omega \alpha,\ \inf_\Omega \mu > 0$ and $\|\mu_1\|_\infty, \|\alpha_1\|_\infty < \varepsilon$ the system*

$$\begin{aligned} \partial_t u - (\mu_0 + \mu_1)\,\Delta u + (\alpha_0 + \alpha_1)\,\nabla q &= f && \text{in } J \times \Omega, \\ \operatorname{div}(u) &= g && \text{in } J \times \Omega, \\ u \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J \times \Gamma, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{3.1.14}$$

has a unique solution $(u, q) \in \mathbb{E}_p(J)$.

Proof. System (3.1.14) is equivalent to

$$\begin{aligned} \partial_t u - \mu_0 \Delta u + \alpha_0 \nabla q &= f + \mu_1 \Delta u - \alpha_1 \nabla q && \text{in } J \times \Omega, \\ \operatorname{div}(u) &= g && \text{in } J \times \Omega, \\ u \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J \times \Gamma, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{3.1.15}$$

Let us use $L_c \colon \mathbb{E}_p(J) \to \mathbb{F}_p^{S-}(J)$ to denote the operator defined by the left-hand side of the above system. Making use of Proposition 3.2 with $q = \alpha q$, the Stokes equations with constant coefficients have maximal regularity. Therefore, the operator L_c is an isomorphism. Let $R \colon {}_0\mathbb{E}_p(J) \longrightarrow \mathbb{F}_p^f(J)$ with $R(u,q) := \mu_1 \Delta u - \alpha_1 \nabla q$ define the linear perturbation in the momentum equation of (3.1.15). Obviously it is $R(u,q)(0) = 0$, if $u(0) = 0$ in Ω. Then, we have the following bound on R:

$$\begin{aligned} \|R(u,q)\|_{\mathbb{F}_p^f(J)} &= \|\mu_1 \Delta u - \alpha_1 \nabla q\|_{L_p(J\times\Omega)} \\ &\leqslant \|\mu_1\|_\infty \|\Delta u\|_{L_p(J\times\Omega)} + \|\alpha_1\|_\infty \|\nabla q\|_{L_p(J\times\Omega)} \\ &\leqslant \|\mu_1\|_\infty \|u\|_{L_p(J,H_p^2(\Omega))} + \|\alpha_1\|_\infty \|q\|_{L_p(J,H_p^1(\Omega))} \\ &\leqslant (\|\mu_1\|_\infty + \|\alpha_1\|_\infty) \|(u,q)\|_{{}_0\mathbb{E}_p(J)} \\ &< 2\varepsilon \|(u,q)\|_{{}_0\mathbb{E}_p(J)} . \end{aligned}$$

By choosing ε, such that $\varepsilon < \frac{1}{2\|{}_0L_c^{-1}\|}$, we obtain

$$\|R(u,q)\|_{\mathbb{F}_p^f(J)} < \frac{1}{\|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S-}(J)\longrightarrow {}_0\mathbb{E}_p(J)}} \|(u,q)\|_{{}_0\mathbb{E}_p(J)}.$$

Thus, all assumptions of Remark 3.6 are satisfied and we obtain a unique solution $(u,q) \in \mathbb{E}_p(J)$ to

$$L(u,q) = (f + R(u,q), g, h, u_0), \tag{3.1.16}$$

by applying Remark 3.6. Since (3.1.15) can be rewritten as (3.1.16), $(u,q) \in \mathbb{E}_p(J)$ is also a unique solution to (3.1.15). □

Using maximal regularity of the Stokes equations with variable coefficients, that are assumed to be small with respect to the L_∞-norm, and some additional time regularity property for the pressure q, we are now able to prove maximal regularity of system (3.1.8).

Proposition 3.8. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $\alpha \in \mathcal{BUC}^1(\Omega)$, $\mu \in \mathcal{BUC}^1(\Omega)$, $\inf_\Omega \alpha$, $\inf_\Omega \mu > 0$, and $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then system*

$$\begin{aligned} \partial_t u - \mu \Delta u + \alpha \nabla q &= f && \textit{in } J \times \Omega, \\ \operatorname{div}(u) &= 0 && \textit{in } J \times \Omega, \\ u \cdot \nu &= 0 && \textit{on } J \times \Gamma, \\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \textit{on } J \times \Gamma, \\ u(0) &= 0 && \textit{in } \Omega, \end{aligned} \tag{3.1.8}$$

has a unique solution $(u,q) \in \mathbb{E}_p(J)$ for every data $f \in {}_0W_p^{1/2}(J, L_p(\Omega))$ and $h \in \mathbb{F}_p^h(J)$ that fulfils the compatibility conditions $(C1)$–$(C8)_+$.

Proof. This proof is based on the proof of [15, Theorem 5.7], where maximal regularity of elliptic operators in a Banach space of class $\mathcal{HT}$ with variable coefficients was demonstrated.

We begin by constructing constants that satisfy the assumptions of Proposition 3.7 for variable coefficients, so that we can use the former result to prove that system (3.1.8) has the property of maximal regularity. The given coefficients μ and α are uniformly bounded and continuous. Since $\bar{\Omega}$ is compact, we can cover $\bar{\Omega}$ by a finite number $N \in \mathbb{N}$ of balls $U_j = B_{r_j}(x_j)$, such that

$$\|\mu(x) - \mu(x_j)\|_\infty \leqslant \varepsilon \qquad \text{and} \qquad \|\alpha(x) - \alpha(x_j)\|_\infty \leqslant \varepsilon, \tag{3.1.17}$$

for all $|x - x_j| \leqslant r_j$ and $j \in \{1, \dots, N\}$. Now, we can define the desired coefficients by reflection, i. e.

$$\mu_j(x) := \begin{cases} \mu(x), & x \in \bar{U}_j, \\ \mu\left(x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}\right), & x \notin \bar{U}_j, \end{cases}$$

and

$$\alpha_j(x) := \begin{cases} \alpha(x), & x \in \bar{U}_j, \\ \alpha\left(x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}\right), & x \notin \bar{U}_j, \end{cases}$$

for $j \in \{1, \dots, N\}$. Hence for every $x \notin \bar{U}_j$ we have that $r_j < |x - x_j|$. From this we obtain

$$\left|x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2} - x_j\right| = \left|r_j^2 \frac{x - x_j}{|x - x_j|^2}\right| < \left|r_j |x - x_j| \frac{x - x_j}{|x - x_j|^2}\right| = r_j.$$

Using (3.1.17), it follows that

$$\left\|\mu\left(x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}\right) - \mu(x_j)\right\|_\infty \leqslant \varepsilon \quad \text{and} \quad \left\|\alpha\left(x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}\right) - \alpha(x_j)\right\|_\infty \leqslant \varepsilon.$$

On the other hand, for every $x \in \bar{U}_j$ we have $|x - x_j| < r_j$. Therefore, it follows again from (3.1.17) that

$$\|\mu(x) - \mu(x_j)\|_\infty \leqslant \varepsilon \quad \text{and} \quad \|\alpha(x) - \alpha(x_j)\|_\infty \leqslant \varepsilon$$

for every $x \in \bar{U}_j$. By definition of α_j and μ_j we can then conclude that $\|\mu_j(x) - \mu(x_j)\|_\infty \leqslant \varepsilon$ and $\|\alpha_j(x) - \alpha(x_j)\| \leqslant \varepsilon$ for all $x \in \bar{\Omega}$ and $j \in \{1, \dots, N\}$. Due to Proposition 3.7, we now obtain the property of maximal regularity for systems of the form

$$\begin{aligned} \partial_t u - \mu_j(x)\Delta u + \alpha_j(x)\nabla q &= f && \text{in } J \times \Omega, \\ \operatorname{div}(u) &= 0 && \text{in } J \times \Omega, \\ u \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J \times \Gamma, \\ u(0) &= 0 && \text{in } \Omega, \end{aligned}$$

for $j \in \{1, \dots, N\}$. However, maximal regularity of the systems above is not immediately applicable to our system (3.1.8), since $\mu_j(x) = \mu(x)$ and $\alpha_j(x) = \alpha(x)$ only holds for $x \in \bar{U}_j$, $j \in \{1, \dots, N\}$. To be able to utilise maximal regularity of the above systems for (3.1.8) we choose a partition of unity $\varphi_j \in \mathcal{C}^\infty(\mathbb{R}^n)$, such that $0 \leqslant \varphi_j(x) \leqslant 1$ and $\operatorname{supp}(\varphi_j) \subset U_j$ for $j \in \{1, \ldots, N\}$. Then, we have $\operatorname{supp}(\varphi_j u) \subset U_j$ and $\operatorname{supp}(\varphi_j q) \subset U_j$. Therefore, the equalities $\mu(\varphi_j u) = \mu_j(\varphi_j u)$ and $\alpha(\varphi_j q) = \alpha_j(\varphi_j q)$ apply here for $j \in \{1, \ldots, N\}$. By multiplying system (3.1.8) with φ_j we then arrive at

$$\begin{aligned} \partial_t(\varphi_j u) - \mu_j \Delta(\varphi_j u) + \alpha_j \nabla(\varphi_j q) &= \varphi_j f - \mu \Delta\varphi_j \cdot u - 2\mu \nabla\varphi_j \nabla u + \alpha \nabla\varphi_j \cdot q, \\ \operatorname{div}(\varphi_j u) &= \nabla\varphi_j \cdot u, \\ \varphi_j u \cdot \nu &= 0, \\ -P_\Gamma D_-(\varphi_j u)\nu &= P_\Gamma(\varphi_j h) - \tfrac{1}{2} P_\Gamma(\nabla\varphi_j \otimes u - u \otimes \nabla\varphi_j)\nu, \\ \varphi_j u(0) &= 0. \end{aligned} \tag{3.1.18}$$

In a next step we would like to prove maximal regularity for the systems (3.1.18), since this directly implies maximal regularity of (3.1.8). Because of the way μ_j and α_j were constructed, the systems (3.1.18) are perturbed version of the systems discussed in Proposition 3.7. To prove maximal regularity of (3.1.18), we split it into three systems. Let us set $\varphi_j u = u_j + \bar{u}_j + \nabla\eta_j$ and $\varphi_j q = q_j + \bar{q}_j - \partial_t \eta_j + \mu_j \Delta\eta_j$, where

$$\eta_j \in {}_0H^1_p(J, H^2_p(\Omega)) \cap {}_0H^{1/2}_p(J, H^3_p(\Omega))$$

are solutions to elliptic problems of the form

$$\begin{aligned} -\Delta\eta_j &= -\nabla\varphi_j \cdot u && \text{in } J \times \Omega, \\ \partial_\nu \eta_j &= 0 && \text{on } J \times \Gamma, \end{aligned}$$

by using the Lemmas 2.1 and 2.3 with $X = {}_0H^1_p(J)$ and $X = {}_0H^{1/2}_p(J)$, respectively. Then, we obtain the two systems

$$\begin{aligned} \partial_t \bar{u}_j - \mu_j \Delta\bar{u}_j + \alpha_j \nabla\bar{q}_j &= \varphi_j f && \text{in } J \times \Omega, \\ \operatorname{div}(\bar{u}_j) &= 0 && \text{in } J \times \Omega, \\ \bar{u}_j \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(\bar{u}_j)\nu &= P_\Gamma(\varphi_j h) && \text{on } J \times \Gamma, \\ \bar{u}_j(0) &= 0 && \text{in } \Omega, \end{aligned} \tag{3.1.19}$$

and

$$\begin{aligned} \partial_t u_j - \mu_j \Delta u_j + \alpha_j \nabla q_j &= -\mu \Delta\varphi_j \cdot u - 2\mu \nabla\varphi_j \nabla u + \alpha(x) \nabla\varphi_j \cdot q && \text{in } J \times \Omega, \\ \operatorname{div}(u_j) &= 0 && \text{in } J \times \Omega, \\ u_j \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(u_j)\nu &= -\tfrac{1}{2} P_\Gamma(\nabla\varphi_j \otimes u - u \otimes \nabla\varphi_j)\nu + P_\Gamma D_-(\nabla\eta_j)\nu && \text{on } J \times \Gamma, \\ u_j(0) &= 0 && \text{in } \Omega. \end{aligned} \tag{3.1.20}$$

Thus, we have separated system (3.1.18) for every $j \in \{1, \ldots, N\}$ into the Stokes equations dealing with the given data (f, h) and the Stokes equations dealing with the perturbations in the momentum equation and on the boundary resulting from the multiplication of (3.1.8) with φ_j. Both systems have homogeneous divergence equations according to the existence of η_j defined as above. This splitting allows us to apply Proposition 3.7 to the systems (3.1.19) and (3.1.20). Now, we proceed in three steps. The first two steps establish maximal regularity for the systems (3.1.19) and (3.1.20), whereas in the third step we estimate η_j appropriately and infer maximal regularity for (3.1.18) and thus, for our initial system (3.1.8).

Step 1. According to Proposition 3.7, we get unique solutions $(\bar{u}_j, \bar{q}_j) \in {}_0\mathbb{E}_p(J)$ to the systems (3.1.19) which satisfy

$$\|(\bar{u}_j, \bar{q}_j)\|_{{}_0\mathbb{E}_p(J)} \leqslant C' \|(f,h)\|_{\mathbb{F}_p^f(J)\times\mathbb{F}_p^h(J)}. \tag{3.1.21}$$

This implies

$$\|\bar{q}_j\|_{{}_0H_p^\vartheta(J,L_p(\Omega))} \leqslant C'(\|\bar{u}_j\|_{{}_0\mathbb{E}_p^u(J)} + \|f\|_{{}_0H_p^\vartheta(J,L_p(\Omega))}) \leqslant C'\|(f,h)\|_{\mathbb{F}_p^f(J)\times\mathbb{F}_p^h(J)}$$

for $j \in \{1,\dots,N\}$ and $\vartheta \in (0, \frac{1}{2} - \frac{1}{2p})$ due to Proposition 3.4.

Step 2. Let us call $L_j \colon {}_*\mathbb{E}_p(J) \to \mathbb{F}_p^f(J) \times P_\Gamma {}_0\mathbb{F}_p^h(J)$ the operator defined by the left-hand side of the first and fourth equation of (3.1.20) with ${}_*\mathbb{E}_p(J) := \{(v,p) \in {}_0\mathbb{E}_p(J) : \mathrm{div}(v) = 0,\ v\cdot\nu = 0 \text{ on } \Gamma\}$. Then we can rewrite (3.1.20) as

$$\begin{aligned} &L_j(u_j, q_j) \\ &\quad = (-\mu\Delta\varphi_j\cdot u - 2\mu\nabla\varphi_j\nabla u + \alpha\nabla\varphi_j, -\tfrac{1}{2}P_\Gamma(\nabla\varphi_j\otimes u - u\otimes\nabla\varphi_j)\nu + P_\Gamma D_-(\nabla\eta_j)\nu). \end{aligned}$$

Due to Proposition 3.7 the operator L_j has the property of maximal regularity for every data $(f,h) \in \mathbb{F}_p^f(J) \times P_\Gamma {}_0\mathbb{F}_p^h(J)$ and $j \in \{1,\dots,N\}$. Therefore, L_j^{-1} exists and multiplication by L_j^{-1} leads to

$$\begin{aligned} (u_j, q_j) = {}& L_j^{-1}(0, P_\Gamma D_-(\nabla\eta_j)\nu) \\ & - L_j^{-1}\left(\mu\Delta\varphi_j\cdot u + 2\mu\nabla\varphi_j\nabla u - \alpha\nabla\varphi_j\cdot q, \tfrac{1}{2}P_\Gamma(\nabla\varphi_j\otimes u - u\otimes\nabla\varphi_j)\nu\right). \end{aligned}$$

We want all terms of the equation to exist locally on U_j. Therefore, for $j \in \{1,\dots,N\}$ we fix some $\psi_j \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\psi_j \equiv 1$ on $\mathrm{supp}(\varphi_j)$ and $\mathrm{supp}(\psi_j) \subset U_j$. Then, we multiply the equation above with ψ_j to obtain

$$(u_j, q_j) = \psi_j L_j^{-1}(0, P_\Gamma D_-(\nabla\eta_j)\nu) - \psi_j L_j^{-1} C_j(u_j, q_j),$$

because $\mathrm{supp}(u_j),\ \mathrm{supp}(q_j) \subset \mathrm{supp}(\varphi_j)$, since $\varphi_j u = u_j + \bar{u}_j + \nabla\eta_j$ and $\varphi_j q = q_j + \bar{q}_j - \partial_t\eta_j + \mu_j\Delta\eta_j$. Let

$$\begin{aligned} C_j(u_j,q_j) := \Big(&\mu\Delta\varphi_j\cdot u(u_j) + 2\mu\nabla\varphi_j\nabla u(u_j) - \alpha\nabla\varphi_j\cdot q(q_j), \\ &\tfrac{1}{2}P_\Gamma(\nabla\varphi_j\otimes u(u_j) - u(u_j)\otimes\nabla\varphi_j)\nu\Big) \end{aligned}$$

be the differential operators for $j \in \{1,\dots,N\}$, which depend only on (u_j, q_j). This is so, because $(u,q) = \sum_j \psi_j(u_j + \bar{u}_j + \nabla\eta_j)$, where $\bar{u}_j$, η_j are as defined above. By rearranging the terms we get

$$\left(\mathrm{Id} + \psi_j L_j^{-1} C_j\right)(u_j, q_j) = \psi_j L_j^{-1}(0, P_\Gamma D_-(\nabla\eta_j)\nu).$$

In case the inverse $(\mathrm{Id} + \psi_j L_j^{-1} C_j)^{-1}$ exists, we would obtain

$$(u_j, q_j) = (\mathrm{Id} + \psi_j L_j^{-1} C_j)^{-1} \psi_j L_j^{-1}(0, P_\Gamma D_-(\nabla\eta_j)\nu)$$

and therefore the property of maximal regularity for (3.1.20). In order to prove existence of the inverse $(\mathrm{Id} + \psi_j L_j^{-1} C_j)^{-1}$, we show that $\|\psi_j L_j^{-1} C_j\| < 1$ and apply a Neumann series argument. Regarding the maximal regularity of the operators L_j for $j \in \{1, \dots, N\}$ we obtain that all L_j are bounded, since the operators are defined for $(v, p) \in {}_*\mathbb{E}_p(J)$ which by definition fulfil $v(0) = 0$ in Ω. Thus, we infer

$$\begin{aligned}\|\psi_j L_j^{-1} C_j\|_{{}_*\mathbb{E}_p(J) \to {}_*\mathbb{E}_p(J)} &\leqslant \|\psi_j\|_\infty \|L_j^{-1}\|_{\mathbb{F}_p^f(J) \times P_\Gamma {}_0\mathbb{F}_p^h(J) \to {}_*\mathbb{E}_p(J)} \|C_j\|_{{}_*\mathbb{E}_p(J) \to \mathbb{F}_p^f(J) \times P_\Gamma {}_0\mathbb{F}_p^h(J)} \\ &\leqslant 1 \cdot C \|C_j\|_{{}_*\mathbb{E}_p(J) \to \mathbb{F}_p^f(J) \times P_\Gamma {}_0\mathbb{F}_p^h(J)}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\|C_j(u_j, q_j)\|_{\mathbb{F}_p^f(J) \times P_\Gamma {}_0\mathbb{F}_p^h(J)} \\ &\quad \leqslant \|\mu \Delta\varphi_j \cdot u + 2\mu \nabla\varphi_j \nabla u - \alpha \nabla\varphi_j q\|_{\mathbb{F}_p^f(J)} + \|\tfrac{1}{2} P_\Gamma (\nabla\varphi_j \otimes u - u \otimes \nabla\varphi_j)\nu\|_{P_\Gamma {}_0\mathbb{F}_p^h(J)}.\end{aligned}$$

By using the regularity of the pressure q from Proposition 3.4 for the first term of the sum, we obtain

$$\begin{aligned}&\|\mu \Delta\varphi_j \cdot u + 2\mu \nabla\varphi_j \nabla u - \alpha \nabla\varphi_j q\|_{\mathbb{F}_p^f(J)} \\ &\qquad = \|\mu \Delta\varphi_j \cdot u + 2\mu \nabla\varphi_j \nabla u - \alpha \nabla\varphi_j q\|_{L_p(J \times \Omega)} \\ &\qquad \leqslant C|J|^\tau \|\mu \Delta\varphi_j \cdot u + 2\mu \nabla\varphi_j \nabla u - \alpha \nabla\varphi_j q\|_{{}_0H_p^\vartheta(J, L_p(\Omega))} \\ &\qquad \leqslant C|J|^\tau \big(\|u\|_{{}_0H_p^\vartheta(J, L_p(\Omega))} + \|\nabla u\|_{{}_0H_p^\vartheta(J, L_p(\Omega))} + \|\alpha q\|_{{}_0H_p^\vartheta(J, L_p(\Omega))}\big) \\ &\qquad \leqslant C|J|^\tau \big(\|u\|_{{}_0H_p^\vartheta(J, H_p^1(\Omega))} + \|u\|_{{}_0H_p^\vartheta(J, H_p^1(\Omega))} + \|\alpha q\|_{{}_0H_p^\vartheta(J, L_p(\Omega))}\big) \\ &\qquad \leqslant C|J|^\tau \Big(\|u\|_{{}_0E_p^u} + \|f\|_{{}_0H_p^\vartheta(J, L_p(\Omega))}\Big).\end{aligned}$$

Here, $j \in \{1, \dots, N\}$ and $\vartheta \in (0, \frac{1}{2} - \frac{1}{2p})$. The second term can be estimated as follows

$$\begin{aligned}\|\tfrac{1}{2} P_\Gamma (\nabla\varphi_j \otimes u - u \otimes \nabla\varphi_j)\nu\|_{P_\Gamma {}_0\mathbb{F}_p^h(J)} &\leqslant C \|P_\Gamma (\nabla\varphi_j \otimes u)\|_{P_\Gamma {}_0\mathbb{F}_p^h(J)} \\ &\leqslant C \|\nabla\varphi_j \otimes u\|_{{}_0H_p^{1/2}(J, L_p(\Omega)) \cap L_p(J, H_p^1(\Omega))} \\ &\leqslant C \|\nabla\varphi_j\|_{W_\infty^1(J, \Omega)} \|u\|_{{}_0H_p^{1/2}(J, L_p(\Omega)) \cap L_p(J, H_p^1(\Omega))} \\ &\leqslant C \|u\|_{{}_0H^{1/2}(J, L_p(\Omega)) \cap L_p(J, H_p^1(\Omega))} \\ &\leqslant C|J|^\tau \|u\|_{H_p^1(J, L_p(\Omega)) \cap {}_0H_p^{1/2}(J, H_p^1(\Omega))} \\ &\leqslant C|J|^\tau \|u\|_{{}_0\mathbb{E}_p^u(J)}.\end{aligned}$$

Here, C, $\tau > 0$ are constants which are independent of J. Then, we can estimate $\|C_j\| \leqslant \varepsilon$ for every fixed $\varepsilon > 0$ provided that J is sufficiently small. Since the admitted length of the time interval J does not depend on the data, we can show $\|C_j\| \leqslant \varepsilon$ for every fixed $\varepsilon > 0$ for any given time interval. This is done by successively validating this on small time intervals of fixed length, cf. Lemma 2.5 where a similar argument has been used. Now we can guarantee

$$\|L_j^{-1} C_j\| < 1$$

and obtain unique solutions $(u_j, q_j) \in \mathbb{E}_p(J)$ of the systems (3.1.20) for $j \in \{1, \dots, N\}$. Moreover, they satisfy

$$\begin{aligned} \|(u_j, q_j)\|_{{}_0\mathbb{E}_p(J)} &\leqslant \|P_\Gamma D_-(\nabla\eta_j)\nu\|_{P_\Gamma {}_0\mathbb{F}_p^h(J)} \\ &\leqslant C\|\nabla^2\eta_j\|_{{}_0W_p^{1/2-1/p}(J,L_p(\Gamma))\cap L_p(J,W_p^{1-1/p}(\Gamma))} \\ &\leqslant C\|\nabla\varphi_j \cdot u\|_{{}_0W_p^{1/2-1/p}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))} \\ &\leqslant C|J|^\tau\|u\|_{{}_0H_p^1(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))} \\ &\leqslant C|J|^\tau\|u\|_{{}_0\mathbb{E}_p^u(J)}. \end{aligned} \tag{3.1.22}$$

For the pressure term Proposition 3.4 furthermore implies

$$\begin{aligned} \|q_j\|_{{}_0H_p^\vartheta(J,L_p(\Omega))} &\leqslant C\Big(\|u_j\|_{{}_0\mathbb{E}_p^u(J)} + \|-\mu\Delta\varphi_j\cdot u - 2\mu\nabla\varphi_j\nabla u + \alpha\nabla\varphi_j q\|_{{}_0H_p^\vartheta(J,L_p(\Omega))}\Big) \\ &\leqslant C\left(\|u\|_{{}_0\mathbb{E}_p^u(J)} + \|f\|_{\mathbb{F}_p^f(J)}\right), \end{aligned}$$

for $\vartheta \in \left(0, \frac{1}{2} - \frac{1}{2p}\right)$.

Step 3. Using Proposition 3.4 once again, we can estimate η_j as follows

$$\begin{aligned} \|\mu_j\Delta\eta_j - \partial_t\eta_j\|_{{}_0H_p^\vartheta(J,L_p(\Omega))} &= \|\varphi_j q - q_j - \bar{q}_j\|_{{}_0H_p^\vartheta(J,L_p(\Omega))} \\ &\leqslant C\|u\|_{{}_0\mathbb{E}_p^u(J)} + C'\|(f,h)\|_{\mathbb{F}_p^f(J)\times\mathbb{F}_p^h(J)}. \end{aligned}$$

Additionally using

$$\|\Delta\eta_j\|_{{}_0H_p^1(J,L_p(\Omega))} \leqslant C\|u\|_{{}_0\mathbb{E}_p^u(J)},$$

which holds by construction, we conclude

$$\|\partial_t\eta_j\|_{{}_0H_p^\vartheta(J,L_p(\Omega))},\ \|\Delta\eta_j\|_{{}_0H_p^\vartheta(J,L_p(\Omega))} \leqslant C\|u\|_{{}_0\mathbb{E}_p^u(J)} + C'\|(f,h)\|_{\mathbb{F}_p^f(J)\times\mathbb{F}_p^h(J)}.$$

In the above equations j is always in $\{1, \dots, N\}$. Using

$$\|\partial_t\eta_j\|_{L_p(J,H_p^2(\Omega))},\ \|\Delta\eta_j\|_{{}_0H_p^1(J,L_p(\Omega))\cap L_p(J,H_p^2(\Omega))} \leqslant C\|u\|_{{}_0\mathbb{E}_p^u(J)},$$

we also infer

$$\begin{aligned} \|\partial_t\nabla\eta_j\|_{L_p(J\times\Omega)} &= \|\partial_t\nabla\eta_j\|_{L_p(J,L_p(\Omega))} \\ &\leqslant C\|\partial_t\eta_j\|_{{}_0H_p^{\vartheta/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))} \\ &\leqslant C|J|^\tau\|\partial_t\eta_j\|_{{}_0H_p^\vartheta(J,L_p(\Omega))\cap L_p(J,H_p^2(\Omega))} \\ &\leqslant |J|^\tau\big(C\|u\|_{{}_0\mathbb{E}_p^u(J)} + C'\|(f,h)\|_{\mathbb{F}_p(J)^f\times\mathbb{F}_p^h(J)}\big) \end{aligned}$$

and

$$\begin{aligned} \|\Delta\nabla\eta_j\|_{L_p(J\times\Omega)} &= \|\Delta\nabla\eta_j\|_{L_p(J,L_p(\Omega))} \\ &\leqslant C\|\Delta\eta_j\|_{{}_0H_p^{1/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))} \\ &\leqslant C|J|^\tau\|\Delta\eta_j\|_{{}_0H_p^1(J,L_p(\Omega))\cap L_p(J,H_p^2(\Omega))} \\ &\leqslant C|J|^\tau\|u\|_{{}_0\mathbb{E}_p^u(J)}, \end{aligned}$$

for $j \in \{1, \dots, N\}$. Hence, we have

$$\|(\nabla\eta_j, \mu_j\Delta\eta_j - \rho\partial_t\eta_j)\|_{{}_0\mathbb{E}_p(J)} \leqslant C|J|^\tau\|u\|_{{}_0\mathbb{E}_p^u(J)} + C'\|(f,h)\|_{\mathbb{F}_p^f(J)\times\mathbb{F}_p^h(J)}. \tag{3.1.23}$$

Finally, (3.1.21), (3.1.22) and (3.1.23) imply that

$$\begin{aligned}(u,q) &= \textstyle\sum_j(\varphi_j u, \varphi_j q) = \sum_j(u_j + \bar{u}_j + \nabla\eta_j, q_j + \bar{q}_j - \partial_t\eta_j + \mu_j\Delta\eta_j) \\ &= S(u,q) + T(f,h) \in {}_0\mathbb{E}_p(J).\end{aligned} \tag{3.1.24}$$

Here, S and T are two linear operators that satisfy

$$\|S(u,q)\|_{{}_0\mathbb{E}_p(J)} \leqslant C|J|^\tau\|u\|_{{}_0\mathbb{E}_p^u(J)} \quad \text{and} \quad \|T(f,h)\|_{{}_0\mathbb{E}_p(J)} \leqslant C'\|(f,h)\|_{\mathbb{F}_p^f(J)\times\mathbb{F}_p^h(J)}.$$

With the help of a Neumann series argument we can then show that $(\mathrm{Id} - S)$ is invertible and thus that

$$(u,q) = (\mathrm{Id} - S)^{-1}T(f,h)$$

is the unique solution to system (3.1.8). □

Now, using a perturbation argument and Proposition 3.8 we can prove maximal regularity of (3.1.7).

Proposition 3.9. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0,T)$. Assume $\Omega := A \times (-a,a)$ to be a cylindrical domain, $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$ and $\alpha \in \mathcal{BUC}^1(\Omega)$, $\mu \in \mathcal{BUC}^1(\Omega)$ with $\inf_\Omega \alpha$, $\inf_\Omega \mu > 0$. Then system*

$$\begin{aligned} \partial_t u - \mu\Delta u + \alpha\nabla q &= f && \text{in } J\times\Omega,\\ \mathrm{div}(u) &= g && \text{in } J\times\Omega,\\ u\cdot\nu &= h\cdot\nu && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(u)\nu &= P_\Gamma h && \text{on } J\times\Gamma,\\ u(0) &= u_0 && \text{in } \Omega,\end{aligned} \tag{3.1.7}$$

has a unique solution $(u,q) \in \mathbb{E}_p(J)$, for every data $(f,g,h,u_0) \in \mathbb{F}_p^{S-}(J)$.

Proof. To prove maximal regularity of (3.1.7) our strategy is to define appropriate (u_1,q_1), (u_2,q_2), $(u_3,q_3) \in \mathbb{E}_p(J)$, such that their sums $u := u_1 + u_2 + u_3$ and $q := q_1 + q_2 + q_3$ are unique solutions to (3.1.7).

Concerning the solvability of (u_1,q_1):
Let $H\colon L_p(\Omega)^n \longrightarrow L_p(\Omega)^n$ denote the Helmholtz projection as introduced in Section 1.3. We define $\nabla q_1 := (1-H)(1/\alpha\cdot f)$ with $(q_1)_\Omega = 0$. Then, let $u_1 \in \mathbb{E}_p^u(J)$ be the unique solution to

$$\begin{aligned} \partial_t u_1 - \mu\Delta u_1 &= f - \alpha\nabla q_1 && \text{in } J\times\Omega,\\ u_1\cdot\nu &= h\cdot\nu && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(u_1)\nu &= P_\Gamma h && \text{on } J\times\Gamma,\\ u_1(0) &= u_0 && \text{in } \Omega,\end{aligned} \tag{3.1.25}$$

which exists according to Corollary 2.15.

Concerning the solvability of (u_2, q_2)*:*
For every $t \in J$ we choose $\varphi(t,\cdot) \in H^2_p(\Omega)$ with $(\varphi)_\Omega = 0$ to be a solution of the elliptic problem

$$\begin{aligned} -\mathrm{div}(\alpha\nabla\varphi(t,\cdot)) &= g(t,\cdot) - \mathrm{div}(u_1)(t,\cdot) && \text{in } \Omega, \\ \partial_\nu\varphi &= 0 && \text{in } \Omega, \end{aligned} \tag{3.1.26}$$

which exists due to Theorem 2.1. Next, let us define

$$u_2 := -\alpha\nabla\varphi \qquad \text{and} \qquad q_2 := \partial_t\varphi - \mu\Delta\varphi.$$

Since $g \in \mathbb{F}^g_p(J)$ and $\mathrm{div}(u_1) \in {}_0W^{1/2}_p(J, L_p(\Omega))$, we obtain $\nabla^2\varphi \in {}_0W^{1/2}_p(J, L_p(\Omega))$ and $(u_2, q_2) \in \mathbb{E}_p(J)$. Now, we define data, such that (u_2, q_2) is the unique solution to the Stokes equations with respect to these data. For that purpose, we define

$$\begin{aligned} f_3 &:= -\partial_t u_2 + \mu\Delta u_2 - \alpha\nabla q_2 \\ &= -\partial_t(-\alpha\nabla\varphi) + \mu\Delta(-\alpha\nabla\varphi) - \alpha\nabla q_2 \\ &= \alpha\nabla\partial_t\varphi - \alpha\mu\nabla\Delta\varphi - 2\mu(\nabla^2\varphi)^T\nabla\alpha - \mu(\Delta\alpha)\nabla\varphi - \alpha\nabla q_2 \\ &= \alpha(\nabla\mu)\Delta\varphi - 2\mu(\nabla^2\varphi)^T\nabla\alpha - \mu(\Delta\alpha)\nabla\varphi, \end{aligned}$$

and $h_3 \in \mathbb{F}^h_p(J)$ with

$$\begin{aligned} h_3 \cdot \nu &:= -u_2\cdot\nu = -\alpha\nabla\varphi\cdot\nu = -\alpha\partial_\nu = 0 && \text{on } J\times\Gamma, \\ P_\Gamma(h_3) &:= P_\Gamma D_-(u_2)\nu && \text{on } J\times\Gamma. \end{aligned}$$

We then obtain $f_3 \in {}_0W^{1/2}_p(J, L_p(\Omega))$, since $\nabla^2\varphi \in {}_0W^{1/2}_p(J, L_p(\Omega))$, and $h_3 \in \mathbb{F}^h_p(J)$, because of Proposition 1.21. Since the given data g and u_0 satisfy the compatibility condition $(C1)$, we have

$$(g - \mathrm{div}(u_1))(0) = g(0) - \mathrm{div}(u_0) = 0.$$

Thus, we obtain $\varphi(0) = 0$ and therefore $f_3(0) = 0$ and $u_2(0) = 0$. Consequently, we get that $(u_2, q_2) \in \mathbb{E}_p(J)$ is the solution to

$$\begin{aligned} \partial_t u_2 - \mu\Delta u_2 + \alpha\nabla q_2 &= -f_3 && \text{in } J\times\Omega, \\ \mathrm{div}(u_2) &= g - \mathrm{div}(u_1) && \text{in } J\times\Omega, \\ u_2\cdot\nu &= -h_3\cdot\nu = 0 && \text{on } J\times\Gamma, \\ -P_\Gamma D_-(u_2)\nu &= P_\Gamma(-h_3) && \text{on } J\times\Gamma, \\ u_2(0) &= 0 && \text{in } \Omega. \end{aligned}$$

The data $(-f_3, g - \mathrm{div}(u_1), -h_3, 0)$ satisfies the necessary compatibility conditions by construction.

Concerning the solvability of (u_3, q_3)*:*
Finally, let $(u_3, q_3) \in \mathbb{E}_p(J)$ be the unique solution to

$$\begin{aligned} \partial_t u_3 - \mu\Delta u_3 + \alpha\nabla q_3 &= f_3 && \text{in } J\times\Omega, \\ \mathrm{div}(u_3) &= 0 && \text{in } J\times\Omega, \\ u_3\cdot\nu &= 0 && \text{on } J\times\Gamma, \\ P_\Gamma D_+(u_3)\nu &= P_\Gamma h_3 && \text{on } J\times\Omega, \\ u_3(0) &= 0 && \text{in } \Omega, \end{aligned}$$

with data $(f_3, h_3) \in {}_0W_p^{1/2}(J, L_p(\Omega)) \times \mathbb{F}_p^h(J)$ that satisfy the necessary compatibility conditions $(C1)$–$(C5)_{S-}$. Such a solution exists because of Proposition 3.8.

Combining $u := u_1 + u_2 + u_3 \in \mathbb{E}_p^u$ and $q := q_1 + q_2 + q_3 \in \mathbb{E}_p^q$, we find $(u, q) \in \mathbb{E}_p$ to be the unique solution to (3.1.7). □

Proof of Theorem 3.3

To prove maximal regularity of the Stokes equations $(S|J)_{S-}$, we split them into two systems, the first system is composed of the inhomogeneous perfect slip boundary conditions, and the second system is composed of the Stokes equations with homogeneous perfect slip boundary conditions. Using the retraction property of the trace operator with respect to the perfect slip boundary conditions, cf. Section 1.5, we show the existence of a solution to the latter system. The remainder of the proof is then devoted to maximal regularity of the Stokes equations with homogeneous perfect slip boundary conditions. To this end, we use a substitution to reduce the Stokes equations with homogeneous boundary conditions to a perturbed version of system (3.1.7). Using Proposition 3.9 and Lemma 3.5, we then infer maximal regularity of $(S|J)_{S-}$.

As a first step, we split u into two parts $u = \hat{u} + \tilde{u}$. Using this decomposition, $(S|J)_{S-}$ decouples into the two subsystems

$$\begin{aligned} \hat{u} \cdot \nu &= h \cdot \nu && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(\hat{u})\nu &= P_\Gamma h && \text{on } J \times \Gamma, \end{aligned} \tag{3.1.27}$$

and

$$\begin{aligned} \rho\partial_t \tilde{u} - \mu\Delta\tilde{u} + \alpha\nabla q &= \tilde{f} && \text{in } J \times \Omega, \\ \operatorname{div}(\rho\tilde{u}) &= \tilde{g} && \text{in } J \times \Omega, \\ \tilde{u} \cdot \nu &= 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(\tilde{u})\nu &= 0 && \text{on } J \times \Gamma, \\ \tilde{u}(0) &= \tilde{u}_0 && \text{in } \Omega. \end{aligned} \tag{3.1.28}$$

The data $\tilde{f} := f - (\rho\partial_t\hat{u} - \mu\Delta\hat{u})$ in $J \times \Omega$, $\tilde{g} := g - \operatorname{div}(\rho\hat{u})$ in $J \times \Omega$ and $\tilde{u}_0 := u_0 - \hat{u}_0$ in Ω satisfy the necessary compatibility conditions $(C1)$–$(C5)_{S-}$ by construction, because these conditions are linear. Consequently we have $(\tilde{f}, \tilde{g}, 0, \tilde{u}_0) \in \mathbb{F}_p^{S-}(J)$. Due to Proposition 1.21 it is known already that the trace operator with respect to perfect slip boundary conditions is a retraction. Thus, system (3.1.27) has a solution $\hat{u} \in \mathbb{E}_p^u(J)$.

To prove maximal regularity of (3.1.28) we proceed in two steps.
Step 1. In this step we show that

$$\begin{aligned} \partial_t v - \eta\Delta v + \alpha\nabla q &= \tilde{f} + 2\rho\eta\nabla v \cdot \nabla\tfrac{1}{\rho} + \rho\eta\left(\nabla\tfrac{1}{\rho}\right)v && \text{in } J \times \Omega, \\ \operatorname{div}(v) &= \tilde{g} && \text{in } J \times \Omega, \\ v \cdot \nu &= P_\Gamma\left(\tfrac{\rho}{2}\left(\nabla\tfrac{1}{\rho} \otimes v - v \otimes \nabla\tfrac{1}{\rho}\right)\right)\nu \cdot \nu = 0 && \text{on } J \times \Gamma, \\ -P_\Gamma D_-(v)\nu &= P_\Gamma\left(\tfrac{\rho}{2}\left(\nabla\tfrac{1}{\rho} \otimes v - v \otimes \nabla\tfrac{1}{\rho}\right)\right)\nu && \text{on } J \times \Gamma, \\ v(0) &= v_0 && \text{in } \Omega, \end{aligned} \tag{3.1.29}$$

with $\eta := \frac{1}{\rho}\mu \in \mathcal{BUC}^1(\Omega)$ and $\inf_\Omega \eta > 0$ is equivalent to (3.1.28). Obviously $-P_\Gamma D_-(v)\nu$ and $P_\Gamma(\frac{\rho}{2}(\nabla\frac{1}{\rho} \otimes v - v \otimes \nabla\frac{1}{\rho}))\nu$ are tangential. Therefore, (3.1.29) is well-posed. Now, let us consider the equations

$$\begin{aligned}\partial_t v - \eta\Delta v + \alpha\nabla q - 2\rho\eta\nabla v \cdot \nabla\tfrac{1}{\rho} - \rho\eta\left(\nabla\tfrac{1}{\rho}\right)v \\ &= \partial_t v - \mu\left(\tfrac{1}{\rho}\Delta v - \nabla v \cdot \nabla\tfrac{1}{\rho} - \left(\nabla\tfrac{1}{\rho}\right)v\right) + \alpha\nabla q \\ &= \rho\partial_t\left(\tfrac{1}{\rho}v\right) - \mu\Delta\left(\tfrac{1}{\rho}v\right) + \alpha\nabla q,\end{aligned}$$

and

$$\begin{aligned}-P_\Gamma D_-(v)\nu - P_\Gamma\left(\tfrac{\rho}{2}\left(\nabla\tfrac{1}{\rho}\otimes v - v\otimes\nabla\tfrac{1}{\rho}\right)\right)\nu \\ &= -P_\Gamma\tfrac{1}{2}\left(\nabla v - (\nabla v)^T\right)\nu - P_\Gamma\left(\tfrac{\rho}{2}\left(\nabla\tfrac{1}{\rho}\otimes v - v\otimes\nabla\tfrac{1}{\rho}\right)\right)\nu \\ &= -P_\Gamma\tfrac{\rho}{2}\left(\tfrac{1}{\rho}\nabla v + \nabla\tfrac{1}{\rho}\otimes v - \tfrac{1}{\rho}(\nabla v)^T - v\otimes\nabla\tfrac{1}{\rho}\right) \\ &= -\rho P_\Gamma\tfrac{1}{2}\left(\nabla\left(\tfrac{1}{\rho}v\right) - \left(\nabla\tfrac{1}{\rho}v\right)^T\right)\nu \\ &= -\rho P_\Gamma D_-\left(\tfrac{1}{\rho}v\right)\nu.\end{aligned} \tag{3.1.30}$$

Then, system (3.1.29) is equivalent to

$$\begin{aligned}\rho\partial_t\left(\tfrac{1}{\rho}v\right) - \mu\Delta\left(\tfrac{1}{\rho}v\right) + \alpha\nabla q &= f && \text{in } J\times\Omega,\\ \operatorname{div}\left(\rho\cdot\tfrac{1}{\rho}v\right) &= g && \text{in } J\times\Omega,\\ \tfrac{1}{\rho}v\cdot\nu &= 0 && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(\tfrac{1}{\rho}v)\nu &= 0 && \text{on } J\times\Gamma,\\ \rho\tfrac{1}{\rho}v(0) &= v_0 && \text{in } \Omega.\end{aligned}$$

By substituting $\tilde{u} = \frac{1}{\rho}v$ and $\tilde{u}_0 = \frac{1}{\rho}v_0$ we finally have established equivalence between (3.1.28) and (3.1.29).

Step 2. In this step we show that (3.1.29) has the property of maximal regularity for any given time interval J. Let us define the functions $R_1\colon {}_0\mathbb{E}_p(J) \longrightarrow \mathbb{F}_p^f(J)$ and $R_2\colon {}_0\mathbb{E}_p(J) \longrightarrow {}_0\mathbb{F}_p^h(J)$ with $R_1(v,q) := 2\rho\eta\nabla v\cdot\nabla\frac{1}{\rho} + \rho\eta(\nabla\frac{1}{\rho})v$ and $R_2(v,q) := P_\Gamma\left(\frac{\rho}{2}\left(\nabla\frac{1}{\rho}\otimes v - v\otimes\nabla\frac{1}{\rho}\right)\right)\nu$. Obviously it is, $R_2(v,q)(0) = 0$ for all $(v,q)\in\mathbb{E}_p(J)$ with $v(0) = 0$. Now, we split $v := v_1 + v_2$ and $q := q_1 + q_2$. Furthermore, we take a $\tilde{h}\in\mathbb{F}_p^h(J)$ such that

$$\begin{aligned}\tilde{h}(0)\cdot\nu &= v_0\cdot\nu = 0 && \text{on } \Gamma \text{ if } p > \tfrac{3}{2},\\ P_\Gamma\tilde{h}(0) &= -P_\Gamma D_-(v_0)\nu && \text{on } \Gamma \text{ if } p > 3.\end{aligned}$$

The equation $v_0\cdot\nu = 0$ holds true, since we have $(\tilde{f},\tilde{g},0,v_0)\in\mathbb{F}_p^{S-}(J)$ and therefore, compatibility condition $(C3)_{S-}$ is satisfied. Thus, for (3.1.29) we obtain the systems

$$\begin{aligned}\partial_t v_1 - \eta\Delta v_1 + \alpha q_1 &= 0 && \text{in } J\times\Omega,\\ \operatorname{div}(v_1) &= \tilde{g} && \text{in } J\times\Omega,\\ v_1\cdot\nu &= 0 && \text{on } J\times\Gamma,\\ -P_\Gamma D_-(v_1)\nu &= P_\Gamma\tilde{h} && \text{on } J\times\Gamma,\\ v_1(0) &= v_0 && \text{in } \Omega,\end{aligned} \tag{3.1.31}$$

and

$$\begin{aligned}
\partial_t v_2 - \eta\Delta v_2 + \alpha q_2 &= \hat{f} + R_1(v_2, q_2) && \text{in } J \times \Omega,\\
\operatorname{div}(v_2) &= 0 && \text{in } J \times \Omega,\\
v_2 \cdot \nu &= (\hat{h} + R_2(v_2, q_2)) \cdot \nu = 0 && \text{on } J \times \Gamma,\\
-P_\Gamma D_-(v_2)\nu &= \hat{h} + R_2(v_2, q_2) && \text{on } J \times \Gamma,\\
v_2(0) &= 0 && \text{in } \Omega.
\end{aligned} \tag{3.1.32}$$

Here, $\hat{f} := \tilde{f} + 2\rho\eta\nabla v_1 \cdot \nabla\frac{1}{\rho} + \rho\eta(\nabla\frac{1}{\rho})$ and $\hat{h} := -P_\Gamma\tilde{h} + P_\Gamma\left(\frac{\rho}{2}\left(\nabla\frac{1}{\rho} \otimes v_1 - v_1 \otimes \nabla\frac{1}{\rho}\right)\right)\nu$. Obviously $\hat{h}+R_2(v_2, q_2)$ is tangential to the boundary, such that $(\hat{h}+R(v_2, q_2))\cdot\nu = 0$. By construction of $\tilde{h}$, the data $(0, \tilde{g}, \tilde{h}, v_0)$ fulfils the necessary compatibility conditions $(C1)$–$(C5)_{S-}$ of system (3.1.31) and we have $(0, \tilde{g}, \tilde{h}, v_0) \in \mathbb{F}_p^{S-}(J)$. Thanks to Proposition 3.9, there exists a unique solution $(v_1, q_1) \in \mathbb{E}_p(J)$ to system (3.1.31).

Now, it is left to prove maximal regularity for (3.1.32). In order to do so, we want to apply Lemma 3.5. Therefore, we have to check all necessary assumptions of this lemma. Considering the construction of $\tilde{h}$, the equation (3.1.30), the substitution $v = \rho\tilde{u}$, the compatibility of the data $(\tilde{f}, \tilde{g}, 0, 0, \tilde{u}_0)$ and the compatibility of the data $(0, \tilde{g}, \tilde{h}, v_0)$, we have

$$\begin{aligned}
\hat{h}(0) &= -P_\Gamma\tilde{h}(0) + P_\Gamma\left(\frac{\rho}{2}\left(\nabla\frac{1}{\rho} \otimes v_1(0) - v_1(0) \otimes \nabla\frac{1}{\rho}\right)\right)\nu\\
&= P_\Gamma D_-(v_0)\nu + P_\Gamma\left(\frac{\rho}{2}\left(\nabla\frac{1}{\rho} \otimes v_0 - v_0 \otimes \nabla\frac{1}{\rho}\right)\right)\nu\\
&= \rho P_\Gamma D_-\left(\frac{1}{\rho}v_0\right)\nu\\
&= \rho P_\Gamma D_-(\tilde{u}_0)\nu\\
&= 0.
\end{aligned}$$

Then, the data $(\hat{f}, 0, \hat{h}, 0)$ satisfies the compatibility conditions $(C1)$–$(C5)_{S-}$ of system (3.1.32). Let $L\colon \mathbb{E}_p(J) \longrightarrow \mathbb{F}_p^{S-}(J)$ be the operator defined by the left-hand side of (3.1.32). From Proposition 3.9 it follows that the operator L is an isomorphism. If $v_2(0) = 0$ in Ω, the estimates

$$\begin{aligned}
\|R_1(v_2, q_2)\|_{\mathbb{F}_p^f} &= \left\|2\rho\eta\nabla v_2 \cdot \nabla\frac{1}{\rho} + \rho\eta(\Delta\frac{1}{\rho})v_2\right\|_{L_p(J\times\Omega)}\\
&\leqslant \left\|2\rho\eta\nabla v_2 \cdot \nabla\frac{1}{\rho}\right\|_{L_p(J\times\Omega)} + \left\|\rho\eta(\Delta\frac{1}{\rho})v_2\right\|_{L_p(J\times\Omega)}\\
&\leqslant 2\|\rho\|_\infty \|\eta\|_\infty \|\nabla v_2\|_{L_p(J\times\Omega)} \|\nabla\tfrac{1}{\rho}\|_\infty + \|\rho\|_\infty \|\eta\|_\infty \|\Delta\tfrac{1}{\rho}\|_\infty \|v_2\|_{L_p(J\times\Omega)}\\
&\leqslant C\|v_2\|_{L_p(J,H_p^1(\Omega))} + C'\|v_2\|_{L_p}\\
&\leqslant C\|v_2\|_{L_p(J,H_p^1(\Omega))} + C'\|v_2\|_{L_p(J,H_p^1(\Omega))}\\
&\leqslant C|J|^\tau \|v_2\|_{{}_0H_p^{1/2}(J,H_p^1(\Omega))}\\
&\leqslant C|J|^\tau \|(v_2, q_2)\|_{{}_0\mathbb{E}_p(J)},
\end{aligned}$$

and

$$\begin{aligned}\|R_2(v_2,q_2)\|_{P_\Gamma {}_0\mathbb{F}_p^h} &= \left\| P_\Gamma\left(\tfrac{\rho}{2}\left(\nabla\tfrac{1}{\rho}\otimes v_2 + v_2\otimes\nabla\tfrac{1}{\rho}\right)\right)\nu\right\|_{{}_0W_p^{1/2-1/2p}(J,L_p(\Gamma))\cap L_p(J,W_p^{1-1/p}(\Gamma))}\\
&\leqslant C\|P_\Gamma(\nabla\tfrac{1}{\rho}\otimes v_2)\|_{P_\Gamma {}_0\mathbb{F}_p^h}\\
&\leqslant C\|\nabla\tfrac{1}{\rho}\otimes v_2\|_{{}_0H_p^{1/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))}\\
&\leqslant C\|\nabla\tfrac{1}{\rho}\|_{W_\infty^1(J,\Omega)}\|v_2\|_{{}_0H_p^{1/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))}\\
&\leqslant C\|v_2\|_{{}_0H_p^{1/2}(J,L_p(\Omega))\cap L_p(J,H_p^1(\Omega))}\\
&\leqslant C|J|^\tau\|v_2\|_{{}_0H_p^1(J,L_p(\Omega))\cap H_p^{1/2}(J,H_p^1(\Omega))}\\
&\leqslant C|J|^\tau\|(v_2,q_2)\|_{{}_0\mathbb{E}_p(J)}\end{aligned}$$

hold true with constants $C,\tau>0$ that are independent of the time interval J. Therefore, all assumptions of Lemma 3.5 are satisfied and we obtain a unique solution $(v_2,q_2)\in\mathbb{E}_p(J)$ to (3.1.32) by applying this lemma. Thus, (3.1.29) has the property of maximal regularity.

Combining the above two steps implies maximal regularity of (3.1.28) and thus maximal regularity of $(S|J)_{S-}$, which was our initial assertion. □

3.2 Stokes Equations: Free Slip Boundary Conditions

In this section we study the Stokes equations on cylindrical domains with free slip boundary conditions and variable coefficients ρ, α and μ, i.e. we study equations of the form

$$\begin{aligned}\rho\partial_t u-\mu\Delta u+\alpha\nabla q &= f &&\text{in } J\times\Omega,\\
\operatorname{div}(\rho u) &= g &&\text{in } J\times\Omega,\\
u\cdot\nu &= h\cdot\nu &&\text{on } J\times\Gamma,\\
P_\Gamma D_+(u)\nu &= P_\Gamma h &&\text{on } J\times\Gamma,\\
u(0) &= u_0 &&\text{in } \Omega.\end{aligned}\qquad (S|J)_{S+}$$

Using maximal regularity of the Stokes equations with perfect slip boundary conditions and variable coefficients, see Theorem 3.3, and a perturbation argument (Lemma 3.5) we are able to deduce maximal regularity of $(S|J)_{S+}$. Considering the Stokes equations with free slip boundary conditions is rewarding, since with their help we are able to prove the existence of a local-in-time strong solution to a model on the mechanisms of tropical storms in Chapter 4.

As in the sections before, $\Omega\subseteq\mathbb{R}^n$ denotes a cylindrical domain and $J=(0,T)$, $T>0$, a time interval. The corresponding data space $\mathbb{F}_p^{S+}(J)$ to $(S|J)_{S+}$ is defined to consist of all

$$(f,g,h,u_0)\in\mathbb{F}_p^f(J)\times\mathbb{F}_p^g(J)\times\mathbb{F}_p^h(J)\times\mathbb{F}_p^0$$

that satisfy the necessary compatibility conditions. These are according to Section 1.4 the conditions

$$\operatorname{div}(u_0)=g(0)\quad\text{if } p\geqslant 2, \qquad (C1)$$

$$\mathcal{F}(g,h\cdot\nu)\in H_p^1(J,{}_0\dot{H}_p^{-1}(\Omega)), \qquad (C2)$$

$$\begin{aligned} u_0 \cdot \nu &= h(0) \cdot \nu, && \text{if } p > \tfrac{3}{2}, \\ P_\Gamma D_+(u_0)\nu &= P_\Gamma h(0), && \text{if } p > 3, \end{aligned} \qquad (C3)_{S+}$$

and the condition

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Gamma_{\text{top}}} &= [h^{\text{top}}]_{\mathcal{R}^{\text{top}}} \cdot \nu_\Sigma && \text{if } p > 2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\text{top}}}} h^\Sigma]_{\mathcal{R}^{\text{top}}} \cdot \nu_\Sigma + \tfrac{1}{2}[\partial_{\nu_\Sigma} h^{\text{top}}]_{\mathcal{R}^{\text{top}}} \cdot \nu_{\Gamma_{\text{top}}} &= [h^{\text{top}}]_{\mathcal{R}^{\text{top}}} \cdot \nu_\Sigma && \text{if } p > 2, \end{aligned} \qquad (C4)_{S+}$$

which arises from the boundary condition on the upper edge of Ω, as well as the condition

$$\begin{aligned} [h^\Sigma]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Gamma_{\text{bot}}} &= [h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_\Sigma && \text{if } p > 2, \\ \tfrac{1}{2}[\partial_{\nu_{\Gamma_{\text{bot}}}} h^\Sigma]_{\mathcal{R}^{\text{bot}}} \cdot \nu_\Sigma + \tfrac{1}{2}[\partial_{\nu_\Sigma} h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_{\Gamma_{\text{bot}}} &= [h^{\text{bot}}]_{\mathcal{R}^{\text{bot}}} \cdot \nu_\Sigma && \text{if } p > 2, \end{aligned} \qquad (C5)_{S+}$$

which arises from the boundary condition on the lower edge of Ω. The aim of this section is to find a unique solution

$$(u, q) = (u(t, x), q(t, x)) \in \mathbb{E}_p(J)$$

to system $(S|J)_{S+}$ for every data

$$(f, g, h, u_0) \in \mathbb{F}_p^{S+}(J).$$

Variable Coefficients

Within this paragraph we assume the coefficients of $(S|J)_{S+}$ to be variable, i. e.

$$\rho \in W^2_\infty(\Omega, (0, \infty)) \text{ with } \tfrac{1}{\rho} \in W^2_\infty(\Omega, (0, \infty))$$

and

$$\alpha \in \mathcal{BUC}^1(\Omega), \; \mu \in \mathcal{BUC}^1(\Omega) \text{ with } \inf_\Omega \alpha, \; \inf_\Omega \mu > 0.$$

The proof of maximal regularity of the Stokes equations $(S|J)_{S+}$ with free slip boundary conditions can be obtained in an analogous manner as the proof of maximal regularity for parabolic problems with free slip boundary conditions (Theorem 2.16). But instead of parabolic problems we have to consider Stokes equations. Moreover, we apply Theorem 3.3 instead of Corollary 2.15 and Lemma 3.5 instead of Lemma 2.5 in order to prove the maximal regularity of the perturbed system.

Theorem 3.10. *Let $A \subseteq \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ and $J = (0, T)$. Assume $\Omega := A \times (-a, a)$ to be a cylindrical domain, $1 < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$, $\rho \in W^2_\infty(\Omega, (0, \infty))$ with $\frac{1}{\rho} \in W^2_\infty(\Omega, (0, \infty))$, and $\alpha \in \mathcal{BUC}^1(\Omega)$, $\mu \in \mathcal{BUC}^1(\Omega)$ with $\inf_\Omega \alpha$, $\inf_\Omega \mu > 0$. Then system $(S|J)_{S+}$ has a unique solution $(u, q) \in \mathbb{E}_p(J)$ for every data $(f, g, h, u_0) \in \mathbb{F}_p^{S+}(J)$.*

A Model on the Mechanisms of Tropical Storms

4 Well-Posedness of a Model on the Mechanisms of Tropical Storms

Tornadoes, hurricanes and other tropical storms are among the most fascinating natural phenomena, especially in terms of their power and unpredictability. In order to understand the behaviour of such phenomena, e. g. where and with what intensity a hurricane hits the coast of a country, one studies velocity, pressure, temperature and moisture of tropical storms.

In order to do this, we analyse a mathematical model, which is explained in detail in the following section. This model consists of a basic tropical storm model describing the dynamics of tropical storms and was introduced by Nolan and Montgomery in [44]. However, it mainly models velocity, temperature and pressure. The work of Hill and Lackmann [26], as well as the work of Wu, Su, Fovell, Dunkerton, Wang and Kahn [56] shows in contrast that moisture has an enormous influence on tropical storms, e. g. regarding the size of a tropical storm. In [27] Hittmeir, Klein, Li and Titi show how moisture dynamics with phase changes can be coupled to an already existing model, the Primitive Equations. In addition, Hittmeir, Klein, Li and Titi introduce their model of moisture dynamics with phase changes in [28] and proved its well-posedness. This model includes not only moisture originated from rain water, but also moisture originated from water vapour and moisture, which was previously bounded in clouds. Therefore, we decide to couple the nonlinear moisture dynamics from [27] and the basic tropical storm model from [44] in the same way as in [28]. In addition, we slightly adapt the coefficients of the basic tropical storm model from [44] to the setting considered by Novotný, Růžička and Thäter [45]. With this adjustment, the model satisfies the 2nd Law of Thermodynamics. This makes the model physically more meaningful.

In conclusion, the general model considered in this thesis which is based on the basic topical storm model [44] is capable of including moisture dynamics and is thermodynamically consistent. Our aim is to prove the existence of a unique solution to this model.

In Section 4.1 we first present the model in detail. In Section 4.2 we lay out our strategy to prove existence and uniqueness of a local-in-time, strong solution to this model. Finally, we show the existence and uniqueness of a solution to this model in Section 4.3.

4.1 The Model

The basic tropical storm model we consider in this thesis is given by

$$\begin{aligned}
\rho\partial_t u + \rho(u\cdot\nabla)u - \mu\Delta u - \lambda\nabla\mathrm{div}u + \alpha\nabla q &= \rho\tfrac{\bar\theta-\theta}{\theta}\nabla F - \omega \mathrm{e}_3\times\rho u && \text{in } J\times\Omega,\\
\mathrm{div}(\rho u) &= 0 && \text{in } J\times\Omega,\\
\rho\partial_t\theta + \rho(u\cdot\nabla)\theta - p_0\,\mathrm{div}(\rho u F) - \kappa\Delta\theta &= \rho(u\cdot\nabla)\bar\theta - \kappa\Delta\bar\theta && \text{in } J\times\Omega,\\
u\cdot\nu = h_u\cdot\nu,\quad \beta^u P_\Gamma D_+(u)\nu &= P_\Gamma h_u && \text{on } J\times\Gamma,\\
\beta^\theta\partial_\nu\theta + \sigma^\theta\theta &= h_\theta && \text{on } J\times\Gamma,\\
u(0) &= u_0 && \text{in } \Omega,\\
\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned} \tag{$TS.1|J$}$$

This system has been introduced by Nolan and Montgomery [44] in order to describe the dynamics of tropical storms such as tornadoes or hurricanes. For a first rigorous analytical research see [49]. In comparison to [44], model ($TS.1|J$) is adapted slightly to the setting of Novotný, Růžička and Thäter [45]. This is done in order to make the model thermodynamically consistent, see Remark 4.1. To obtain a more precise representation of the actual conditions inside a tornado or hurricane, the model ($TS.1|J$) is coupled to nonlinear moisture dynamics with phase changes, given by

$$\begin{aligned}
\partial_t m_v + (u\cdot\nabla)m_v - \eta_v\Delta m_v - S_{ev} + S_{cd} &= 0 && \text{in } J\times\Omega,\\
\partial_t m_c + (u\cdot\nabla)m_c - \eta_c\Delta m_c - S_{cd} + S_{ac} + S_{cr} &= 0 && \text{in } J\times\Omega,\\
\partial_t m_r + (u\cdot\nabla)m_r - \eta_r\Delta m_r - S_{ac} - S_{cr} + S_{ev} &= \tfrac{V}{g\rho_m}\mathrm{e}_3\cdot\nabla(\rho_m m_r) && \text{in } J\times\Omega,\\
\beta^{m_v}\partial_\nu m_v + \sigma^{m_v}m_v &= h_v && \text{on } J\times\Gamma,\\
\beta^{m_c}\partial_\nu m_c + \sigma^{m_c}m_c &= h_c && \text{on } J\times\Gamma,\\
\beta^{m_r}\partial_\nu m_r + \sigma^{m_r}m_r &= h_r && \text{on } J\times\Gamma,\\
m_v(0) = m_{v,0},\quad m_c(0) = m_{c,0},\quad m_r(0) &= m_{r,0} && \text{in } \Omega.
\end{aligned} \tag{$TS.2|J$}$$

This system has been introduced by Hittmeir, Klein, Li and Titi in [27]. The combination of both systems ($TS|J$) = ($TS.1|J$) + ($TS.2|J$), i. e. the basic tropical storm model coupled to nonlinear moisture dynamics, represents the model we study in this chapter.

Since the model ($TS|J$) is supposed to describe the dynamics of a tropical storm, we assume all above equations to depend both on location and time. In the following, we assume time to be a positive 1-dimensional variable, and the location to be described by a 3-dimensional variable, since we are interested in modelling the behaviour on earth. Hence, by the positive time interval $J = (0,T) \subseteq \mathbb{R}$ with $T > 0$ we denote the time domain, and by $\Omega \subseteq \mathbb{R}^3$ the spatial domain. Since the shape of, for example, hurricanes resembles a cylinder, we would like to study the dynamics of tropical storms in cylinders. With equal right, one could also study tropical storms on an upper half-space, which could serve as a simplified model for the surface of earth, or on a sphere, representing an approximation of earth as a whole, but these two domains go beyond the scope of the present thesis. That is, we assume Ω to be a cylindrical domain. Then, model ($TS|J$) is considered on the cartesian product $J\times\Omega$. For a comprehensive introduction to cylindrical domains, their boundary, the deformation tensor and the projection P_Γ, we refer back to Section 1.3.

The first equation of $(TS.1|J)$ represents the anelastic equation of momentum, where u denotes the velocity and q its corresponding pressure. The atmospheric density is denoted by ρ and is assumed to be a given, time-independent, positive function with positive inverse $1/\rho$. The symbol ω stands for twice the angular velocity of earth's rotation, where we assume rotation to be performed around $\mathrm{e}_3 = (0,0,1)^{\mathrm{T}}$. Therefore, the term $\omega \mathrm{e}_3 \times u$ represents the Coriolis force. The term $\frac{\bar{\theta}-\theta}{\bar{\theta}} \nabla F$ on the other hand represents the buoyancy, with $F \in L_p(J, H_p^1(\Omega))$ being the potential of the external forces, for example gravity $-gx_3$, where g stands for the earth's constant gravitational acceleration. The temperature θ is assumed to be varying around a given mean value $\bar{\theta} = \bar{\theta}(x)$. The second equation of $(TS.1|J)$ is the anelastic incompressibility condition, which arises from the law of conservation of mass. The third equation of $(TS.1|J)$ arises from the law of the conservation of energy, and the first three equations of $(TS.2|J)$ represent moisture balances. They can be modelled by the method of Hittmeir, Klein, Li and Titi [27]. The water vapour mixing ratio m_v, the cloud water mixing ratio m_c, and the rain water mixing ratio m_r are considered in order to include moisture dynamics for warm clouds where also phase changes are modelled. Here, moisture is represented as vapour, cloud water and rain water. Furthermore, the terminal velocity of falling rain V is assumed to be constant, and ρ_m denotes the positive density of rain water. By the term S_{ev} the rate of evaporation of rain water is denoted. Whereas S_{cr} stands for the rate of auto-conversion of cloud water into rainwater by accumulation of microscopic droplets. The rate of the collection of cloud water by falling rain is denoted by S_{ac}. The rate of the condensation of water vapour to cloud water and the inverse evaporation process is denoted by S_{cd}. They are represented as

$$\begin{aligned}
S_{ev} &= C_{ev}\theta(m_r^+)^{\xi}(m_{vs}(\theta) - m_v)^+, \quad \xi \in (0,1], \\
S_{cr} &= C_{cr} m_c m_r, \\
S_{ac} &= C_{ac}(m_c - m_{ac}^*)^+, \\
S_{cd} &= C_{cd}(m_v - m_{vs}(\theta)) m_c + C_{cn}(m_v - m_{vs}(\theta))^+,
\end{aligned}$$

where C_{ev}, C_{cr}, C_{ac}, C_{cd}, C_{cn} are constant rates. Furthermore, the threshold value for the cloud water mixing ratio, beyond which auto-conversion of cloud water into precipitation becomes active,is denoted by the constant m_{ac}^*. The quantity m_{vs} represents the saturation mixing ratio, which satisfies

$$m_{vs}(\theta) = \frac{E e_s(\theta)}{R\rho\theta - e_s(\theta)},$$

where $E = R/R_v$ is the ratio of the individual gas constants of dry air and water vapour and c_s the saturation vapour pressure given by the Clausius-Clapeyron equation

$$c_s(\theta) = c_s(\theta_{\text{Start}}) \exp\left(\frac{L}{R_v}\left(\frac{1}{\theta_{\text{start}}} - \frac{1}{\theta}\right)\right).$$

Here, L stands for the latent heat per unit mass of water vapour, which we assume to be constant. Typically, the reference temperature $\theta_{\text{start}} = 273.15\text{K}$ is used. Moreover m_{vs} is positive, and bounded by zero and a positive constant m_{vs}^*

$$0 \leqslant m_{vs}(q,\theta) \leqslant m_{vs}^*.$$

The variable coefficients in system $(TS.1|J)$ are considered in two different settings, one with an anelastic limit [45] and one as given in [44, 49]. Therefore, for the coefficients in the anelastic limit, we have

$$\mu = \bar\mu, \quad \lambda = \bar\lambda + \tfrac{1}{3}\bar\mu, \quad \alpha = \rho, \quad p_0 = \frac{\bar p_0}{\bar\theta}, \quad \kappa = \bar\kappa, \tag{4.1.2}$$

with positive constants $\bar\mu$, $\bar\lambda$, and $\bar\kappa$. These are the shear viscosity, the bulk viscosity and the heat conductivity at temperature $\bar\theta$, respectively. The term $\frac{\bar p_0}{\bar\theta}\operatorname{div}(\rho u F)$ represents the heat production due to volume work.
The coefficients in the setting of [44, 49] are given by

$$\mu = \epsilon\rho, \quad \lambda = \epsilon\rho, \quad \alpha = 1, \quad p_0 = 0, \quad \kappa = \epsilon\rho \tag{4.1.3}$$

where the positive constant ϵ denotes the eddy viscosity. By combining these two settings we can obtain any possible combination of constant or variable coefficients μ, λ, α, p_0 and κ. The variable coefficients of the moisture balances are valid for both settings and given by

$$\eta_j = \bar\eta_j \rho_m, \quad j \in \{v, c, r\},$$

where the positive constants $\bar\eta_j$ are the viscosity of water vapour, cloud water and rain water, respectively, measured at temperature $\bar\theta$.

Remark 4.1. We note that in the coefficient setting of (4.1.2) every sufficiently smooth solution to $(TS.1|J)$ satisfies the energy balance

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u\|^2_{L_2(\Omega)} + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{\bar p_0}\|\theta-\bar\theta\|^2_{L_2(\Omega)}\\
&\quad + \bar\mu\|\nabla u\|^2_{L_2(\Omega)} + (\bar\lambda + \tfrac{1}{3}\bar\mu)\|\operatorname{div} u\|^2_{L_2(\Omega)} + \tfrac{\bar\kappa}{\bar p_0}\|\nabla(\theta-\bar\theta)\|^2_{L_2(\Omega)}\\
&\qquad = \int_\Gamma (u\cdot\nu)\,(\tfrac{1}{2}|\sqrt{\rho}u|^2 + \tfrac{1}{2}\tfrac{\rho}{\bar p_0}|(\theta-\bar\theta)|^2 + (\bar\lambda+\tfrac{1}{3}\bar\mu)\operatorname{div} u - \rho q)\,\mathrm{d}\sigma\\
&\qquad\quad + \int_\Gamma \bar\mu u\cdot\partial_\nu u\,\mathrm{d}\sigma + \int_\Gamma \tfrac{\bar\kappa}{\bar p_0}(\theta-\bar\theta)\partial_\nu(\theta-\bar\theta)\,\mathrm{d}\sigma.
\end{aligned}$$

This follows by multiplying the momentum balance with u, the heat balance with $\theta - \bar\theta$ and integrating over Ω.

The boundary conditions are represented by the equations 4–5 of $(TS.1|J)$ and by the equations 4–6 of $(TS.2|J)$. The boundary conditions for the velocity u are chosen in such a way that each component of the boundary – top, bottom, lateral boundary – is impermeable in case of $h_u \cdot \nu = 0$, and that there is no friction on the boundary in case of $P_\Gamma h_u = 0$. For $h_u \neq 0$, these boundary conditions can be used to introduce a flux through the boundary, and a friction on the boundary, respectively. Moreover, the boundary condition for temperature and moisture are Robin boundary conditions. We assume the boundary coefficients to be variable, more precisely

$$\beta^k, \beta^{m_j} \in \mathcal{BC}^1(J\times\Gamma, (0,\infty)) \text{ with } \inf_\Gamma \beta^k, \inf_\Gamma \beta^{m_j} > 0$$

and

$$\sigma^\theta, \sigma^{m_j} \in \mathcal{BC}^2(J\times\Gamma, [0,\infty)),$$

where $k \in \{u, \theta\}$ and $j \in \{v, c, r\}$.
The last two equations of $(TS.1|J)$ and the last equation of $(TS.2|J)$ represent the initial conditions with initial data $u_0, \theta_0, m_{j,0}$, $j \in \{v, c, r\}$.

4.2 Strategy

The main result of this thesis is the proof of existence and uniqueness of a local-in-time, strong solution to $(TS|J)$, i. e. Theorem 4.2, which we discuss shortly. The model $(TS|J)$ is given by system $(TS.1|J)$ coupled to system $(TS.2|J)$ as already explained in Section 4.1. To prove existence and uniqueness of a local-in-time, strong solution to $(TS|J)$, we first prove the same for $(TS.1|J)$ alone, after which we show existence and uniqueness of a local-in-time strong solution to $(TS.2|J)$ in a second step. System $(TS.1|J)$ is completely independent of the unknown water vapour mixing ratio m_v, the unknown cloud water mixing ratio m_c and the unknown rain water mixing ratio m_r of $(TS.2|J)$. Thus, there is no obstacle in decoupling $(TS.1|J)$ and $(TS.2|J)$ and proving existence and uniqueness of a local-in-time strong solution to $(TS.1|J)$ independent of $(TS.2|J)$. However, system $(TS.2|J)$ depends on the unknown velocity u and the temperature θ of system $(TS.1|J)$. But we can in fact prove existence and uniqueness of a local-in-time, strong solution to $(TS.2|J)$ independent of $(TS.1|J)$ by assuming the velocity and the temperature to be given functions. Then, we can obtain a local-in-time, strong solution to the combined system $(TS|J)$ by solving $(TS.1|J)$ first, and then solving $(TS.2|J)$, using the solutions for the velocity u and the temperature θ obtained from system $(TS.1|J)$.
In order to investigate system $(TS.1|J)$ first, we denote by

$$
\begin{aligned}
f_u &= -(\rho u \cdot \nabla)u + \lambda \nabla \mathrm{div} u + \rho \tfrac{\bar{\theta}-\theta}{\bar{\theta}} \nabla F - \omega \mathrm{e}_3 \times \rho u, \\
f_\theta &= -(\rho u \cdot \nabla)\theta - p_0 \, \mathrm{div}(\rho u F) + (\rho u \cdot \nabla)\bar{\theta} - \kappa \Delta \bar{\theta},
\end{aligned}
$$

all nonlinear terms and linear terms of lower order of system $(TS.1|J)$, and can thus rewrite $(TS.1|J)$ as

$$
\begin{aligned}
\rho \partial_t u - \mu \Delta u + \alpha \nabla q &= f_u && \text{in } J \times \Omega, \\
\mathrm{div}(\rho u) &= 0 && \text{in } J \times \Omega, \\
u \cdot \nu &= h_u \cdot \nu && \text{on } J \times \Gamma, \qquad (S|J)_{S+} \\
\beta^u P_\Gamma D_+(u)\nu &= P_\Gamma h_u && \text{on } J \times \Gamma, \\
u(0) &= u_0 && \text{in } \Omega,
\end{aligned}
$$

$$
\begin{aligned}
\rho \partial_t \theta - \kappa \Delta \theta &= f_\theta && \text{in } J \times \Omega, \\
\beta^\theta \partial_\nu \theta + \sigma^\theta \theta &= h_\theta && \text{on } J \times \Gamma, \qquad (H|J) \\
\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned}
$$

In this form, $(S|J)_{S+}$ resembles the Stokes equations with free slip boundary conditions, which we already studied with variable coefficients on cylindrical domains in Section 3.2. Furthermore, system $(H|J)$ resembles the heat equation. This is a parabolic problem with Robin boundary conditions, which we studied with variable coefficients on cylindrical domains in Section 2.2. We split $(TS.1|J)$ into a linear operator L and a nonlinear operator N. We define L to consist of all linear terms of highest order and N to consist of all nonlinear terms and linear terms of lower order. We do not repeat the definition of data and solution spaces here, in order to not interrupt the flow of reading. Their definitions may be found in the Sections 2.2 and 3.2, respectively. By

$$
L \colon \mathbb{E}_p(J) \times \mathbb{E}_p^z(J) \longrightarrow \mathbb{F}_p^{S+}(J) \times \mathbb{F}_p^{P,R}(J)
$$

we denote the linear operator which is defined by the left-hand side of systems $(S|J)_{S+}$ and $(H|J)$, and by N the nonlinear operator which is given as

$$N(v,p,\vartheta) := \begin{pmatrix} (-\rho(v\cdot\nabla)v + \lambda\nabla\mathrm{div}v - \rho\frac{\vartheta}{\bar\theta}\nabla F - \omega \mathrm{e}_3\times\rho v \\ 0 \\ 0 \\ -\rho(v\cdot\nabla)\vartheta + p_0\mathrm{div}(\rho v F) + \rho(v\cdot\nabla)\bar\theta \\ 0 \end{pmatrix}^T,$$

for $(v,p,\vartheta)\in\mathbb{E}_p(J)\times\mathbb{E}_p^z(J)$. It is convenient to consider the systems $(S|J)_{S+}$ and $(H|J)$ and thus $(TS.1|J)$ in the form

$$\begin{aligned} L(u,q,\theta) &= N(u,0,\theta) + (\rho\nabla F, 0, h_u, -\kappa\Delta\bar\theta, h_\theta), \\ (u,\theta)(0) &= (u_0,\theta_0). \end{aligned} \tag{4.2.1}$$

According to Theorem 2.6 and Theorem 3.10 there exists a bounded inverse of the linear operator L, which is the solution operator

$$L^{-1}\colon \mathbb{F}_p^{S,+}(J)\times\mathbb{F}_p^{P,R}(J)\to\mathbb{E}_p(J)\times\mathbb{E}_p^z(J)$$

of the Stokes equations with free slip boundary conditions and of a parabolic problem with Robin boundary conditions. So let (u^*,q^*,θ^*) be the solution of

$$\begin{aligned} L(u^*,q^*,\theta^*) &= (\rho\nabla F, 0, h_u, -\kappa\Delta\bar\theta, h_\theta), \\ (u^*,\theta^*)(0) &= (u_0,\theta_0). \end{aligned}$$

Thus, $(TS.1|J)$ is equivalent to

$$\begin{aligned} (u,q,\theta) &= (\tilde u,\tilde q,\tilde\theta) + (u^*,q^*,\theta^*), \\ (\tilde u,\tilde q,\tilde\theta) &= {}_0L^{-1}N(\tilde u + u^*, 0, \tilde\theta+\theta^*) \\ &=: K(\tilde u,\tilde q,\tilde\theta). \end{aligned} \tag{4.2.2}$$

By using the Contraction Principle we then obtain a unique local-in-time strong solution to $(TS.1|J)$.

To prove existence and uniqueness of a local-in-time, strong solution to system $(TS.2|J)$ with given u, $\theta\in L_\infty(J\times\Omega)$ we proceed analogously. By

$$\begin{aligned} f_v &= -(u\cdot\nabla)m_v + S_{ev} - S_{cd}, \\ f_c &= -(u\cdot\nabla)m_c + S_{cd} - S_{ac} - S_{cr}, \\ f_r &= -(u\cdot\nabla)m_r + S_{ac} + S_{cr} - S_{ev} + \frac{V}{g\rho_m}\mathrm{e}_3\cdot\nabla(\rho_m m_r), \end{aligned}$$

we denote all nonlinear terms and linear terms of lower order of system $(TS.2|J)$, and can thus rewrite $(TS.2|J)$ as

$$\begin{aligned} \partial_t m_j - \eta_j\Delta m_j &= f_j && \text{in } J\times\Omega, \\ \beta^{m_j}\partial_\nu m_j + \sigma^{m_j} m_j &= h_j && \text{on } J\times\Gamma, \\ m_j(0) &= m_{j,0} && \text{in } \Omega, \end{aligned} \tag{$(M|J)_j$}$$

with $j\in\{v,c,r\}$. Note that $(M|J)_j$ represents three systems of equations, one for each $j\in\{v,c,r\}$. In this form $(M|J)_j$ resembles parabolic systems with Robin boundary conditions, which we already studied in Section 2.2. We separate $(TS.2|J)$ into a linear operator L_m and a nonlinear operator N_m and define L_m to consist of all linear terms of highest order and N_m to consist of all nonlinear terms and linear terms of lower order. By

$$L_m\colon \mathbb{E}_p^z(J)\times\mathbb{E}_p^z(J)\times\mathbb{E}_p^z(J)\longrightarrow \mathbb{F}_p^{P,R}(J)\times\mathbb{F}_p^{P,R}(J)\times\mathbb{F}_p^{P,R}(J)$$

we denote the linear operator, which is defined by the left-hand side of systems $(M|J)_j$ for $j\in\{v,c,r\}$ and by N_m the nonlinear operator, which is given by

$$N_m(n_v,n_c,n_r):=$$

$$\begin{pmatrix} -(u\cdot\nabla)n_c+S_{ev}(n_v,n_r)-S_{cd}(n_v,n_c)\\ 0\\ 0\\ -(u\cdot\nabla)n_c+S_{cd}(n_v,n_c)-S_{ac}(n_c)-S_{cr}(n_c,n_r)\\ 0\\ 0\\ -(u\cdot\nabla)n_r+S_{ac}(n_c)+S_{cr}(n_c,n_r)-S_{ev}(n_v,n_r)+\frac{V}{g\rho_m}\mathrm{e}_3\cdot\nabla(\rho_m n_r)\\ 0\\ 0 \end{pmatrix}^T,$$

for $(n_v,n_c,n_r)\in\mathbb{E}_p^z(J)\times\mathbb{E}_p^z(J)\times\mathbb{E}_p^z(J)$. It is convenient to consider the systems $(M|J)_j$ for $j\in\{v,c,r\}$ and thus $(TS.2|J)$ in the form

$$\begin{aligned} L_m(m_v,m_c,m_r) &= N_m(m_v,m_c,m_r)+(0,h_v,0,h_c,0,h_r),\\ (m_v,m_c,m_r)(0) &= (m_{v,0},m_{c,0},m_{r,0}). \end{aligned} \tag{4.2.3}$$

According to Theorem 2.6 there exists a bounded inverse of the linear operator L_m, which is the solution operator

$$L_m^{-1}\colon \mathbb{F}_p^{P,R}(J)\times\mathbb{F}_p^{P,R}(J)\times\mathbb{F}_p^{P,R}(J)\to\mathbb{E}_p^z(J)\times\mathbb{E}_p^z(J)\times\mathbb{E}_p^z(J)$$

for the three parabolic systems with Robin boundary conditions and variable coefficients. Let (m_v^*,m_c^*,m_r^*) be the solution of

$$\begin{aligned} L_m(m_v^*,m_c^*,m_r^*) &= (0,h_v,0,h_c,0,h_r),\\ (m_v^*,m_c^*,m_r^*)(0) &= (m_{v,0},m_{c,0},m_{r,0}). \end{aligned}$$

Then, $(TS.2|J)$ is equivalent to

$$\begin{aligned} (m_v,m_c,m_r) &= (\tilde m_v,\tilde m_c,\tilde m_r)+(m_v^*,m_c^*,m_r^*),\\ (\tilde m_v,\tilde m_c,\tilde m_r) &= {}_0L_m^{-1}N_m(\tilde m_v+m_v^*,\tilde m_c+m_c^*,\tilde m_r+m_r^*)\\ &=: K_m(\tilde m_v,\tilde m_c,\tilde m_r). \end{aligned} \tag{4.2.4}$$

Using the Contraction Principle we then obtain a unique local-in-time, strong solution to $(TS.2|J)$.

Using this strategy, we are able to show existence and uniqueness of a local-in-time, strong solution to $(TS|J)$ and thus the main result of this chapter.

Theorem 4.2. *Let $A\subset\mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a>0$, $J=(0,T)$ a time interval with $T>0$, and $\Omega:=A\times(-a,a)$ a cylindrical domain. Assume α, $\mu\in\mathcal{BUC}^1(\Omega)$, κ, $\eta_j\in\mathcal{BUC}(\Omega)$, $\inf_\Omega\alpha$, $\inf_\Omega\mu$, $\inf_\Omega\kappa$, $\inf_\Omega\eta_j>0$, $j\in\{v,c,r\}$, β^u, β^θ, $\beta^j\in\mathcal{BC}^1(J\times\Gamma)$, σ^θ,*

$\sigma^j \in \mathcal{BC}^2(J \times \Gamma)$, $\inf_\Gamma \beta^u$, $\inf_\Gamma \beta^\theta$, $\inf_\Gamma \beta^j > 0$, ρ, $\rho_m \in W^2_\infty(\Omega)$ with $\inf_\Omega \rho$, $\inf_\Omega \rho_m > 0$ and $\frac{n+2}{2} < p < \infty$ with $p \neq \frac{3}{2}$, $p \neq 2$, $p \neq 3$. Then for every data

$$(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar{\theta}, h_\theta, \theta_0, 0, h_v, m_{v,0}, 0, h_c, m_{c,0}, 0, h_r, m_{r,0}) \\ \in \mathbb{F}^{S+}_p(J) \times \mathbb{F}^{P,R}_p(J) \times \mathbb{F}^{P,R}_p(J) \times \mathbb{F}^{P,R}_p(J) \times \mathbb{F}^{P,R}_p(J)$$

with $\inf_{x\in\Omega} m_{r,0} > 0$ there is a unique local-in-time, strong solution $(u, q, \theta, m_v, m_c, m_r)$ to $(TS|J)$ on a maximal time interval $(0, T^*)$ with

$$T^* = T^*(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar{\theta}, h_\theta, \theta_0, 0, h_v, m_{v,0}, 0, h_c, m_{c,0}, 0, h_r, m_{r,0}) \in J.$$

The solution satisfies

$$(u, q, \theta, m_v, m_c, m_r) \in \mathbb{E}_p(\bar{J}) \times \mathbb{E}^z_p(\bar{J}) \times \mathbb{E}^z_p(\bar{J}) \times \mathbb{E}^z_p(\bar{J}) \times \mathbb{E}^z_p(\bar{J})$$

for all $\bar{J} = (0, \bar{T})$ with $\bar{T} \in (0, T^*)$. Furthermore, the solution depends continuously on the data.

4.3 Well-Posedness of $(TS|J)$

The proof of the existence and uniqueness of a local-in-time, strong solution to the model $(TS|J)$ is provided in the end of this section. In order to show this, we begin by proving existence and uniqueness of a local-in-time, strong solution to $(TS.1|J)$, followed by proving the same for $(TS.2|J)$. Then, we are able to prove solvability of the entire model $(TS|J)$.

4.3.1 Well-Posedness of $(TS.1|J)$

In this subsection we study the basic tropical storm model $(TS.1|J)$ without nonlinear moisture dynamics:

$$\begin{aligned}
\rho\partial_t u + \rho(u\cdot\nabla)u - \mu\Delta u - \lambda\nabla\mathrm{div}u + \alpha\nabla q &= \rho\tfrac{\bar{\theta}-\theta}{\bar{\theta}}\nabla F - \omega \mathrm{e}_3 \times \rho u && \text{in } J\times\Omega,\\
\mathrm{div}(\rho u) &= 0 && \text{in } J\times\Omega,\\
u\cdot\nu &= h_u\cdot\nu && \text{on } J\times\Gamma,\\
\beta^u P_\Gamma D_+(u)\nu &= P_\Gamma h_u && \text{on } J\times\Gamma,\\
u(0) &= u_0 && \text{in } \Omega,\\
\rho\partial_t\theta + \rho(u\cdot\nabla)\theta - p_0\,\mathrm{div}(\rho u F) - \kappa\Delta\theta &= \rho(u\cdot\nabla)\bar{\theta} - \kappa\Delta\bar{\theta} && \text{in } J\times\Omega,\\
\beta^\theta\partial_\nu\theta + \sigma^\theta\theta &= h_\theta && \text{on } J\times\Gamma,\\
\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned} \qquad (TS.1|J)$$

We aim to prove existence and uniqueness of a local-in-time, strong solution of system $(TS.1|J)$. This is shown for two different situations: first for arbitrary data and small time intervals, and second for arbitrary time intervals and small data. The following two propositions comprise our results.

Proposition 4.3. *Let $A \subset \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$ be a constant, $J = (0, T)$ a time interval with $T > 0$ and $\Omega := A\times(-a, a)$ a cylindrical domain. Let α, $\mu \in \mathcal{BUC}^1(\Omega)$, $\kappa \in \mathcal{BUC}(\Omega)$, $\inf_\Omega\alpha$, $\inf_\Omega\mu$, $\inf_\Omega\kappa > 0$, β^u, $\beta^\theta \in \mathcal{BC}^1(J\times\Gamma)$, $\sigma^\theta \in \mathcal{BC}^2(J\times\Gamma)$,*

$\inf_\Gamma \beta^u$, $\inf_\Gamma \beta^\theta > 0$, $\frac{n+2}{3} < p < \infty$ *with* $p \neq \frac{3}{2}$, $p \neq 3$ *and* $\rho \in W^2_\infty(\Omega)$ *with* $\inf_\Omega \rho > 0$. *Then for*

$$(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar{\theta}, h_\theta, \theta_0) \in \mathbb{F}_p^{S+}(J) \times \mathbb{F}_p^{P,R}(J)$$

there exists a unique local-in-time strong solution (u, q, θ) *to system (TS.1|J) on a maximal time interval* $(0, T^*)$ *with*

$$T^* = T^*(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar{\theta}, h_\theta, \theta_0) \in J.$$

The solution satisfies

$$(u, q, \theta) \in \mathbb{E}_p(\bar{J}) \times \mathbb{E}_p^z(\bar{J})$$

for all $\bar{J} = (0, \bar{T})$ *with* $\bar{T} \in (0, T^*)$. *Furthermore, the solution depends continuously on the data.*

Proposition 4.4. *Let* $A \subset \mathbb{R}^{n-1}$ *be a bounded* $\mathcal{C}^3$*-domain,* $a > 0$ *a constant,* $J = (0, T)$ *a time interval with* $T > 0$ *and* $\Omega := A \times (-a, a)$ *a cylindrical domain. Let* α, $\mu \in \mathcal{BUC}^1(\Omega)$, $\kappa \in \mathcal{BUC}(\Omega)$, $\inf_\Omega \alpha$, $\inf_\Omega \mu$, $\inf_\Omega \kappa > 0$, β^u, $\beta^\theta \in \mathcal{BC}^1(J \times \Gamma)$, $\sigma^\theta \in \mathcal{BC}^2(J \times \Gamma)$, $\inf_\Gamma \beta^u$, $\inf_\Gamma \beta^\theta > 0$, $\frac{n+2}{3} < p < \infty$ *with* $p \neq \frac{3}{2}$, $p \neq 3$ *and* $\rho \in W^2_\infty(\Omega)$ *with* $\inf_\Omega \rho > 0$. *Then there is an* $\varepsilon = \varepsilon(J) > 0$, *such that system (TS.1|J) admits a unique solution*

$$(u, q, \theta) \in \mathbb{E}_p(J) \times \mathbb{E}_p^z(J)$$

for every data $(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar{\theta}, h_\theta, \theta_0) \in \mathbb{F}_p^{S+}(J) \times \mathbb{F}_p^{P,R}(J)$ *that satisfies the condition*

$$\|(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar{\theta}, h_\theta, \theta_0)\|_{\mathbb{F}_p^{S+}(J)\times\mathbb{F}_p^{P,R}(J)} \leqslant \varepsilon.$$

Furthermore, the solution depends continuously on the data.

In the following, we prove Propositions 4.3 and 4.4 simultaneously.

Proof. We start by fixing $J = (0, T)$ with $T > 0$, $\bar{J} = (0, \bar{T})$ with $\bar{T} \in (0, T]$ and

$$(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar{\theta}, h_\theta, \theta_0) \in \mathbb{F}_p^{S+}(J) \times \mathbb{F}_p^{P,R}(J).$$

The Stokes equations with variable coefficients and given data $(\rho\nabla F, 0, h_u, u_0)$, and the heat equations with variable coefficients and given data $(-\kappa\Delta\bar{\theta}, h_\theta, \theta_0)$ have the property of maximal regularity due to Theorems 3.10 and 2.6. We proceed in two steps.

Step 1. In the first step we show, that the perturbed system

$$\begin{aligned}
\rho\partial_t u - \mu\Delta u + \alpha\nabla q + \tfrac{\theta}{\bar{\theta}}\nabla F + \omega e_3 \times \rho u - \lambda\nabla \mathrm{div} u &= \rho\nabla F && \text{in } J \times \Omega,\\
\mathrm{div}(\rho u) &= 0 && \text{in } J \times \Omega,\\
u \cdot \nu &= h_u \cdot \nu && \text{on } J \times \Gamma,\\
\beta^u P_\Gamma D_+(u)\nu &= P_\Gamma h_u && \text{on } J \times \Gamma,\\
u(0) &= u_0 && \text{in } \Omega,\\
\rho\partial_t\theta - \kappa\Delta\theta - \rho(u \cdot \nabla)\bar{\theta} - p_0\,\mathrm{div}(\rho u F) &= -\kappa\Delta\bar{\theta} && \text{in } J \times \Omega,\\
\beta^\theta\partial_\nu\theta + \sigma^\theta\theta &= h_\theta && \text{on } J \times \Gamma,\\
\theta(0) &= \theta_0 && \text{in } \Omega,
\end{aligned} \tag{4.3.1}$$

has the property of maximal regularity by using maximal regularity of the Stokes equations $(S|J)_{S+}$ and maximal regularity of the heat equation. The heat equation is a parabolic problem with Robin boundary conditions $(P|J)_R$. We establish

$$L\colon \mathbb{E}_p(J) \times \mathbb{E}_p^z(J) \to \mathbb{F}_p^{S+}(J) \times \mathbb{F}_p^{P,R}(J)$$

as the operator defined by the left-hand side of (4.3.1) and the functions

$$R_1\colon {}_0\mathbb{E}_p(J) \times {}_0\mathbb{E}_p^z(J) \to \mathbb{F}_p^f(J) \ \text{ with } \ R_1(u,q,\theta) := -\rho\frac{\theta}{\bar\theta}\nabla F - \omega e_3 \times \rho u + \lambda\nabla \mathrm{div} u$$

and

$$R_2\colon {}_0\mathbb{E}_p(J) \times {}_0\mathbb{E}_p^z(J) \to \mathbb{F}_p^f(J) \ \text{ with } \ R_2(u,q,\theta) := \rho(u\cdot\nabla)\bar\theta + p_0\mathrm{div}(\rho u F).$$

Therefore, and because of linearity, we can write (4.3.1) as

$$\begin{aligned} L(u,q,\theta) &= (\rho\nabla F + R_1(u,q,\theta), 0, h_u, u_0, -\kappa\Delta\bar\theta + R_2(u,q,\theta), h_\theta, \theta_0) \\ &= (\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar\theta, h_\theta, \theta_0) + (R_1(u,q,\theta), 0,0,0, R_2(u,q,\theta), 0, 0). \end{aligned}$$

which is equivalent to

$$(u,q,\theta) = L^{-1}(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar\theta, h_\theta, \theta_0) + {}_0L^{-1}(R_1(u,q,\theta), 0,0,0, R_2(u,q,\theta), 0, 0),$$

since $(S|J)_{S+}$ and $(P|J)_R$ have the property of maximal regularity. Subtraction of ${}_0L^{-1}(R_1, 0,0,0,R_2,0,0)(u,q,\theta)$ leads to

$$\left(\mathrm{Id} - {}_0L^{-1}(R_1,0,0,0,R_2,0,0)\right)(u,q,\theta) = L^{-1}(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar\theta, h_\theta, \theta_0).$$

There is only left to show that $\left(\mathrm{Id} - {}_0L^{-1}(R_1,0,0,0,R_2,0,0)\right)^{-1}$ exists, because then

$$(u,q,\theta) = \left(\mathrm{Id} - {}_0L^{-1}(R_1,0,0,0,R_2,0,0)\right)^{-1} L^{-1}(\rho\nabla F, 0, h_u, u_0, -\kappa\Delta\bar\theta, h_\theta, \theta_0)$$

would be the unique solution to (4.3.1). If $\|{}_0L^{-1}(R_1,0,0,0,R_2,0,0)\| < 1$, the Neumann series argument provides us the existence of $\left(\mathrm{Id} - {}_0L^{-1}(R_1,0,0,0,R_2,0,0)\right)^{-1}$. We know already that

$$\begin{aligned} &\|{}_0L^{-1}(R_1,0,0,0,R_2,0,0)\|_{{}_0\mathbb{E}_p(J)\times{}_0\mathbb{E}_p^z(J)\to{}_0\mathbb{E}_p(J){}_0\mathbb{E}_p^z(J)} \\ &\qquad\leqslant \|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S+}(J)\times{}_0\mathbb{F}_p^{P,R}(J)\to{}_0\mathbb{E}_p(J)\times{}_0\mathbb{E}_p^z(J)} \\ &\qquad\qquad\cdot \|(R_1,0,0,0,R_2,0,0)\|_{{}_0\mathbb{E}_p(J)\times{}_0\mathbb{E}_p^z(J)\to{}_0\mathbb{F}_p^{S+}(J)\times{}_0\mathbb{F}_p^{P,R}(J)}\,. \end{aligned}$$

Next, we want to show that $\|R_1(u,q,\theta)\|_{\mathbb{F}_p^f(J)}$ and $\|R_2(u,q,\theta)\|_{\mathbb{F}_p^f(J)}$ can be made arbitrarily small. Then $\|{}_0L^{-1}(R_1,0,0,0,R_2,0,0)\|_{{}_0\mathbb{E}_p(J)\times{}_0\mathbb{E}_p^z(J)\to{}_0\mathbb{E}_p(J)\times{}_0\mathbb{E}_p^z(J)}$ can be assumed to be smaller than one, since $\|{}_0L^{-1}\|_{{}_0\mathbb{F}_p^{S+}(J)\times{}_0\mathbb{F}_p^{P,R}(J)\to{}_0\mathbb{E}_p(J)\times{}_0\mathbb{E}_p^z(J)}$ is bounded. We can estimate

$$\begin{aligned} \|\omega e_3 \times \rho u\|_{L_p(J\times\Omega)^n} &\leqslant C\|\rho u\|_{L_p(J\times\Omega)^n} \\ &\leqslant C\|\rho\|_\infty \|u\|_{L_p(J,L_p(\Omega))^n\cap L_p(J,L_p(\Omega))^n} \\ &\leqslant C|J|^\tau \|u\|_{{}_0H_p^{1/2}(J,L_p(\Omega))^n\cap L_p(J,H_p^1(\Omega))^n} \\ &\leqslant C|J|^\tau \|u\|_{{}_0\mathbb{E}_p^u(J)}, \end{aligned}$$

$$\begin{aligned}
\|\lambda\nabla \mathrm{div} u\|_{L_p(J\times\Omega)^n} &= \left\|\lambda\nabla \mathrm{div}\left(\tfrac{1}{\rho}\rho u\right)\right\|_{L_p(J\times\Omega)^n}\\
&= \left\|\lambda\nabla\left(\nabla(\tfrac{1}{\rho})\cdot\rho u+\tfrac{1}{\rho}\mathrm{div}(\rho u)\right)\right\|_{L_p(J\times\Omega)^n}\\
&= \left\|\lambda\left(\nabla\rho^{-1}\cdot\rho u\right)\right\|_{L_p(J\times\Omega)^n}\\
&\leqslant \|\lambda\|_\infty\left\|\nabla\rho^{-1}\right\|_\infty\ \|\rho\|_\infty\|u\|_{L_p(J\times\Omega)^n}\\
&\leqslant C\|u\|_{L_p(J,L_p(\Omega))^n\cap L_p(J,L_p(\Omega))^n}\\
&\leqslant C|J|^\tau\|u\|_{{}_0\mathbb{E}_p^u(J)},
\end{aligned}$$

$$\begin{aligned}
\|\rho\tfrac{\tilde\vartheta}{\bar\theta}\nabla F\|_{L_p(J\times\Omega)} &\leqslant \|\rho\|_\infty\|\bar\theta^{-1}\|_\infty\|\tilde\vartheta\|_{L_p(J\times\Omega)}\|\nabla F\|_\infty\\
&\leqslant C\|\tilde\vartheta\|_{L_p(J\times\Omega)}\\
&\leqslant C|J|^\tau\|\tilde\vartheta\|_{{}_0\mathbb{E}_p^z(J)},
\end{aligned}$$

$$\begin{aligned}
\|(\rho u\cdot\nabla)\bar\theta\|_{L_p(J\times\Omega)} &\leqslant \|\rho\|_\infty\|u\|_{L_p(J\times\Omega)^n}\|\nabla\bar\theta\|_\infty\\
&\leqslant C|J|^\tau\|u\|_{{}_0\mathbb{E}_p^u(J)},
\end{aligned}$$

and

$$\begin{aligned}
\|p_0\mathrm{div}(\rho uF)\|_{L_p(J\times\Omega)^n} &\leqslant \|p_0\|_\infty\|\mathrm{div}(\rho uF)\|_{L_p(J\times\Omega)^n}\\
&\leqslant C\|\rho uF\|_{{}_0H_p^{1/2}(J,L_p(\Omega))^n\cap L_p(J,H_p^1(\Omega))^n}\\
&\leqslant C\|\rho\|_\infty\|F\|_\infty\|u\|_{{}_0H_p^{1/2}(J,L_p(\Omega))^n\cap L_p(J,H_p^1(\Omega))^n}\\
&\leqslant C|J|^\tau\|u\|_{{}_0\mathbb{E}_p^u(J)},
\end{aligned}$$

with constants $C,\ \tau>0$, where C is independent of J. Also we have $\bar\theta\in W^2_\infty(\Omega)$, $\frac{1}{\bar\theta}\in L_\infty(\Omega)$, $F\in W^1_\infty(\Omega)$ and $\rho\in W^2_\infty(\Omega)$. Therefore we can make the two terms $\|-\rho\frac{\theta}{\bar\theta}-\omega e_3\times\rho u+\lambda\nabla\mathrm{div}u\|_{\mathbb{F}_p^f(J)}$ and $\|\rho(u\cdot\nabla)\bar\theta+p_0\mathrm{div}(\rho uF)\|_{\mathbb{F}_p^f(J)}$ small by choosing J sufficiently small. Since the admitted length of the time interval J does not depend on the data, we can show maximal regularity of (4.3.1) for any given time interval by successively solving them on sufficiently small time intervals of fixed length, cf. Lemma 2.5 where a similar argument has been used. Therefore, we can established maximal regularity of (4.3.1) for any time interval $J=(0,T)$ with $T>0$.

Step 2. By (u^*,q^*,θ^*) we denote the unique, maximal regular solution of the perturbed system (4.3.1), whose components satisfy the inequalities

$$\|(u^*,q^*)\|_{\mathbb{E}_p(J)}\leqslant M_1\left\|(\rho\nabla F,0,h_u,u_0,-\kappa\Delta\bar\theta,h_\theta,\theta_0)\right\|_{\mathbb{F}_p^{S+}(J)\times\mathbb{F}_p^{P,R}(J)},\tag{4.3.2}$$

and

$$\|\theta^*\|_{\mathbb{E}_p^z(J)}\leqslant M_2\left\|(\rho\nabla F,0,h_u,u_0,-\kappa\Delta\bar\theta,h_\theta,\theta_0)\right\|_{\mathbb{F}_p^{S+}(J)\times\mathbb{F}_p^{P,R}(J)},\tag{4.3.3}$$

with constants $M_1,\ M_2>0$, due to step 1. To deal with the remaining nonlinear terms and terms of lower order of system $(TS.1|J)$, it is convenient to rewrite them into the operator equation

$$\begin{aligned}
L_P(u,q,\theta) &= N(u,0,\theta)+(\rho\nabla F,0,h_u,-\kappa\Delta\bar\theta,h_\theta)\\
(u,\theta)(0) &= (u_0,\theta_0),
\end{aligned}\tag{4.3.4}$$

where $L_P\colon\mathbb{E}_p(J)\times\mathbb{E}_p^z(J)\to\mathbb{F}_p^{S+}(J)\times\mathbb{F}_p^{P,R}(J)$ denotes the linear operator defined by the left-hand side of (4.3.1). According to step 1, L_P is an isomorphism. The nonlinear operator N is given by

$$N(v,p,\vartheta):=\big(-\rho(v\cdot\nabla)v,0,0,-\rho(v\cdot\nabla)\vartheta,0\big),$$

for $(v,p,\vartheta)\in \mathbb{E}_p(J)\times\mathbb{E}_p^z(J)$. Therefore, system $(TS.1|J)$ is equivalent to

$$\begin{aligned}(u,q,\theta) &= (\tilde u,\tilde q,\tilde\theta)+(u^*,q^*,\theta^*)(w),\\ (\tilde u,\tilde q,\tilde\theta) &= {}_0L_P^{-1}N(\tilde u+u^*(w),0,\tilde\theta+\theta^*(w))\\ &=: K(\tilde u,\tilde q,\tilde\theta),\end{aligned}\tag{4.3.5}$$

where

$${}_0L_P^{-1}\colon {}_0\mathbb{F}_p^{S+}(J)\times {}_0\mathbb{F}_p^{P,R}(J)\to {}_0\mathbb{E}_p(J)\times {}_0\mathbb{E}_p^z(J)$$

denotes the bounded linear inverse of ${}_0L_P$. Now, we prove existence and uniqueness of a solution to

$$(\tilde u,\tilde q,\tilde\theta)=K(\tilde u,\tilde q,\tilde\theta),\quad (\tilde u,\tilde q,\tilde\theta)\in {}_0\mathbb{E}_p(\bar J)\times {}_0\mathbb{E}_p^z(\bar J),$$

where $\bar J=(0,\bar T)$ with $\bar T\in(0,T]$. Note, that

$$N(\cdot+u^*,0,\cdot+\theta^*)\colon {}_0\mathbb{E}_p^u(\bar J)\times {}_0\mathbb{E}_p^z(\bar J)\to \mathbb{F}_p^{S,+}(\bar J)\times\mathbb{F}_p^{\theta}(\bar J),$$

is Fréchet differentiable with

$$DN(\tilde u+u^*,0,\tilde\theta+\theta^*)(\tilde v,0,\tilde\vartheta)=\begin{pmatrix}-\rho((\tilde u+u^*)\cdot\nabla)\tilde v-\rho(\tilde v\cdot\nabla)(\tilde u+u^*)\\ 0\\ 0\\ -\rho((\tilde u+u^*)\cdot\nabla)\tilde\vartheta-\rho(\tilde v\cdot\nabla)(\tilde\theta+\theta^*)\\ 0\end{pmatrix}^T,$$

where $\tilde u,\ \tilde v\in {}_0\mathbb{E}_p^u(\bar J)$ and $\tilde\theta,\ \tilde\vartheta\in {}_0\mathbb{E}_p^z(\bar J)$. For $p>\frac{n+2}{3}$ set $\epsilon:=p-\frac{n+2}{3}>0$. Then we have an embedding

$$H_p^{1-\epsilon/3p,(2,1)}(\bar J)\cdot H_p^{2-\epsilon/3p,(2,1)}(\bar J)\hookrightarrow H_p^{0,(2,1)}(\bar J)$$

due to [34, Remark 1.8] with anistropic function spaces

$$\begin{aligned}H_p^{0,(2,1)}(\bar J) &= L_p(\bar J\times\Omega),\\ H_p^{2,(2,1)}(\bar J) &= H_p^1(\bar J,L_p(\Omega))\cap L_p(\bar J,H_p^2(\Omega)),\\ H_p^{1-\epsilon/3p,(2,1)}(\bar J) &= H_p^{1/2-\epsilon/6p}(\bar J,L_p(\Omega))\cap L_p(\bar J,H_p^{1-\epsilon/3p}(\Omega)),\\ H_p^{2-\epsilon/3p,(2,1)}(\bar J) &= H_p^{1-\epsilon/6p}(\bar J,L_p(\Omega))\cap L_p(\bar J,H_p^{2-\epsilon/3p}(\Omega)).\end{aligned}$$

For the definition of anisotropic function spaces, we refer the reader to Section 1.1. We thus have

$$\begin{aligned}&\|\rho((\tilde u+u^*)\cdot\nabla)\tilde v+\rho(\tilde v\cdot\nabla)(\tilde u+u^*)\|_{L_p(\bar J\times\Omega,\mathbb{R}^n)}\\ \leqslant\ &\|\rho\|_\infty\big(\|\nabla\tilde v\|_{{}_0H_p^{1-\epsilon/3p,(2,1)}(\bar J)}\|\tilde u+u^*\|_{H_p^{2-\epsilon/3p,(2,1)}(\bar J)}\\ &\qquad+\|\nabla(\tilde u+u^*)\|_{H_p^{1-\epsilon/3p,(2,1)}(\bar J)}\|\tilde v\|_{{}_0H_p^{2-\epsilon/3p,(2,1)}(\bar J)}\big)\\ \leqslant\ &C\|\tilde v\|_{{}_0H_p^{2-\epsilon/3p,(2,1)}(\bar J)}\|\tilde u+u^*\|_{H_p^{2,(2,1)}(\bar J)}+C\|\tilde u+u^*\|_{H_p^{2-\epsilon/3p,(2,1)}(\bar J)}\|\tilde v\|_{{}_0H_p^{2-\epsilon/3p,(2,1)}(\bar J)}\\ \leqslant\ &C|\bar J|^\tau\|\tilde v\|_{{}_0H_p^{2,(2,1)}(\bar J)}\|\tilde u+u^*\|_{H_p^{2,(2,1)}(\bar J)}+C\|\tilde u+u^*\|_{H_p^{2,(2,1)}(\bar J)}\|\tilde v\|_{{}_0H_p^{2-\epsilon/3p,(2,1)}(\bar J)}\\ \leqslant\ &C|\bar J|^\tau\|\tilde v\|_{{}_0\mathbb{E}_p^u(\bar J)}\|\tilde u+u^*\|_{\mathbb{E}_p^u(\bar J)}\end{aligned}$$

for all $p > \frac{n+2}{3}$, where the constant $C > 0$ is independent of $\bar{J}$ thanks to the homogeneous initial conditions. We obtain in the same way

$$\begin{aligned}\|\rho((\tilde{u}+u^*)\cdot\nabla)\tilde{\vartheta}+&\rho(\tilde{v}\cdot\nabla)(\tilde{\theta}+\theta^*)\|_{L_p(\bar{J}\times\Omega,\mathbb{R}^n)}\\ &\leqslant\ C|\bar{J}|^\tau\|\tilde{u}+u^*\|_{\mathbb{E}_p^u(\bar{J})}\|\tilde{\vartheta}\|_{{}_0\mathbb{E}_p^z(\bar{J})}+C|\bar{J}|^\tau\|\tilde{v}\|_{{}_0\mathbb{E}_p^u(\bar{J})}\|\tilde{\theta}+\theta^*\|_{\mathbb{E}_p^z(\bar{J})}.\end{aligned}$$

We infer for all $\tilde{u}\in {}_0\mathbb{E}_p^u(\bar{J})$, $\tilde{\theta}\in {}_0\mathbb{E}_p^z(\bar{J})$ the estimate

$$\begin{aligned}\|DN(\tilde{u}+u^*,0,\tilde{\theta}+&\theta^*)\|_{\mathcal{L}(\mathbb{E}_p^u(\bar{J})\times\mathbb{E}_p^z(\bar{J}),\mathbb{F}_p^{S,+}(\bar{J})\times\mathbb{F}_p^\theta(\bar{J}))}\\ &\leqslant C|\bar{J}|^\tau\Big(\|\tilde{u}+u^*\|_{\mathbb{E}_p^u(\bar{J})}+\|\tilde{\theta}+\theta^*\|_{\mathbb{E}_p^z(\bar{J})}\Big),\end{aligned}$$

where the constant $C > 0$ is independent of $\bar{J}$. Due to (4.3.2) and (4.3.3) we have

$$\|u^*\|_{\mathbb{E}_p^u(\bar{J})}\leqslant\|(u^*,q^*)\|_{\mathbb{E}_p(J)}\leqslant M_1\|(\rho\nabla F,0,h_u,u_0,-\kappa\Delta\bar{\theta},h_\theta,\theta_0)\|_{\mathbb{F}_p^{S+}(J)\times\mathbb{F}_p^{P,R}(J)}$$

and

$$\|\theta^*\|_{\mathbb{E}_p^z(\bar{J})}\leqslant M_2\|(\rho\nabla F,0,h_u,u_0,-\kappa\Delta\bar{\theta},h_\theta,\theta_0)\|_{\mathbb{F}_p^{S+}(J)\times\mathbb{F}_p^{P,R}(J)}.$$

Note, that $M_1,\ M_2 > 0$ are independent of $\bar{J}$, since (u^*,q^*,θ^*) are defined on the entire interval J. We define $\varepsilon > 0$ such that it fulfils

$$\|(\rho\nabla F,0,h_u,u_0,-\kappa\Delta\bar{\theta},h_\theta,\theta_0)\|_{\mathbb{F}_p^{S+}(J)\times\mathbb{F}_p^{P,R}(J)}\leqslant\varepsilon,$$

and obtain

$$\begin{aligned}\|DN(\tilde{u}+u^*,0,\tilde{\theta}+&\theta^*)\|_{\mathcal{B}({}_0\mathbb{E}_p^u(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J}),{}_0\mathbb{F}_p^{S,+}(\bar{J})\times{}_0\mathbb{F}_p^{P,R}(\bar{J}))}\\ &\leqslant C|\bar{J}|^\tau(\|\tilde{u}\|_{{}_0\mathbb{E}_p^u(\bar{J})}+\|\tilde{\theta}\|_{{}_0\mathbb{E}_p^z(\bar{J})}+\varepsilon M_1+\varepsilon M_2)\end{aligned}$$

for $\tilde{u}\in {}_0\mathbb{E}_p^u(\bar{J})$ and $\tilde{\theta}\in {}_0\mathbb{E}_p^z(\bar{J})$. We then infer

$$\begin{aligned}\|K(\tilde{u},\tilde{q},\tilde{\theta})-&K(\tilde{v},\tilde{p},\tilde{\vartheta})\|_{{}_0\mathbb{E}_p(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})}\\ &\leqslant C|\bar{J}|^\tau\bar{C}(\delta+\varepsilon M_1+\varepsilon M_2)\|(\tilde{u}-\tilde{v},\tilde{\theta}-\tilde{\vartheta})\|_{{}_0\mathbb{E}_p^u(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})}\end{aligned}\tag{4.3.6}$$

for $(\tilde{u},\tilde{q},\tilde{\theta})$, $(\tilde{v},\tilde{p},\tilde{\vartheta})\in {}_0\mathbb{E}_p(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})$, which satisfy $\|\tilde{u}\|_{{}_0\mathbb{E}_p^u(\bar{J})}+\|\tilde{\theta}\|_{{}_0\mathbb{E}_p^z(\bar{J})}$, $\|\tilde{v}\|_{{}_0\mathbb{E}_p^u(\bar{J})}+\|\tilde{\vartheta}\|_{{}_0\mathbb{E}_p^z(\bar{J})}\leqslant\delta$. We define

$$\bar{C}:=\sup\{\|{}_0L^{-1}\|_{\mathcal{B}({}_0\mathbb{F}_p^{S,+}(\bar{J})\times{}_0\mathbb{F}_p^\theta(\bar{J}),{}_0\mathbb{E}_p(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J}))}:\bar{J}\subset J\}.$$

Finally,

$$\begin{aligned}\|K(0,0,0)\|_{{}_0\mathbb{E}_p(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})}\leqslant&\bar{C}\|N(u^*,0,\theta^*)\|_{{}_0\mathbb{F}_p^{S,+}(\bar{J})\times{}_0\mathbb{F}_p^{P,R}(\bar{J})}\\ \leqslant&\varepsilon^2\bar{C}(M_1+M_2)^2\end{aligned}$$

implies

$$\|K(\tilde{u},\tilde{q},\tilde{\theta})\|_{{}_0\mathbb{E}_p(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})}\leqslant C|\bar{J}|^\tau\delta\bar{C}(\delta+\varepsilon M_1+\varepsilon M_2)+\varepsilon^2\bar{C}(M_1+M_2)^2\tag{4.3.7}$$

for $(\tilde{u},\tilde{q},\tilde{\theta})\in {}_0\mathbb{E}_p(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})$ with $\|\tilde{u}\|_{{}_0\mathbb{E}_p^u(\bar{J})}+\|\tilde{\theta}\|_{{}_0\mathbb{E}_p^z(\bar{J})}\leqslant\delta$.

For *Proposition 4.3*, $\varepsilon > 0$ is constituted by the data. Set $\frac{\delta}{2} := \varepsilon^2 \bar{C}(M_1 + M_2)^2$ and choose a $\bar{J} \subset J$ such that

$$C|\bar{J}|^\tau \bar{C}(\delta + \varepsilon M_1 + \varepsilon M_2) \leqslant \tfrac{1}{2}.$$

Therefore, we obtain

$$\|K(\tilde{u}, \tilde{q}, \tilde{\theta})\|_{{}_0\mathbb{E}_p(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})} \leqslant \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta$$

by using (4.3.7). Combining this with (4.3.6), we get that K is a contraction on a closed ball with radius δ. Now, using the Contraction Mapping Principle, Proposition 4.3 is proven.

For *Proposition 4.4*, we set $\bar{J} = J$ and choose $\delta > 0$, such that

$$\tfrac{3}{2} C|J|^\tau \bar{C}\delta \leqslant \tfrac{1}{2}$$

and furthermore $\varepsilon > 0$, such that

$$\varepsilon M_1 + \varepsilon M_2 \leqslant \tfrac{\delta}{2}, \; \varepsilon^2 \bar{C}(M_1 + M_2)^2 \leqslant \tfrac{\delta}{2}.$$

By applying ε and δ to (4.3.6) and (4.3.7), we get that K is a contraction on a closed ball with radius δ. Now, using the Contraction Mapping Principle, Proposition 4.4 is proven. □

4.3.2 Well-Posedness of $(TS.2|J)$

In this subsection we study the nonlinear moisture dynamics $(TS.2|J)$:

$$\begin{aligned}
\partial_t m_v + (u \cdot \nabla) m_v - \eta_v \Delta m_v - S_{ev} + S_{cd} &= 0 && \text{in } J \times \Omega,\\
\beta^{m_v} \partial_\nu m_v + \sigma^{m_v} m_v &= h_v, && \text{on } J \times \Gamma,\\
m_v(0) &= m_{v,0} && \text{in } \Omega,\\
\partial_t m_c + (u \cdot \nabla) m_c - \eta_c \Delta m_c - S_{cd} + S_{ac} + S_{cr} &= 0 && \text{in } J \times \Omega,\\
\beta^{m_c} \partial_\nu m_c + \sigma^{m_c} m_c &= h_c, && \text{on } J \times \Gamma,\\
m_c(0) &= m_{c,0} && \text{in } \Omega,\\
\partial_t m_r + (u \cdot \nabla) m_r - \eta_r \Delta m_r - S_{ac} - S_{cr} + S_{ev} &= \tfrac{V}{g\rho_m} \mathrm{e}_3 \cdot \nabla(\rho_m m_r) && \text{in } J \times \Omega,\\
\beta^{m_r} \partial_\nu m_r + \sigma^{m_r} m_r &= h_r, && \text{on } J \times \Gamma,\\
m_r(0) &= m_{r,0} && \text{in } \Omega,
\end{aligned} \qquad (TS.2|J)$$

for given $u,\ \theta \in L_\infty(\Omega)$. By

$$L_m \colon \mathbb{E}_p^z(J) \times \mathbb{E}_p^z(J) \times \mathbb{E}_p^z(J) \longrightarrow \mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J)$$

we denote the linear operator, which is defined by the left-hand side of the system $(TS.2|J)$ and by N_m the nonlinear operator, which is given by

$$\begin{aligned}
N_m(n_v, n_c, n_r) := \Big(&- (u \cdot \nabla) n_c + S_{ev}(n_v, n_r) - S_{cd}(n_v, n_c), 0, 0,\\
&- (u \cdot \nabla) n_c + S_{cd}(n_v, n_c) - S_{ac}(n_c) - S_{cr}(n_c, n_r), 0, 0\\
&- (u \cdot \nabla) n_r + S_{ac}(n_c) + S_{cr}(n_c, n_r) - S_{ev}(n_v, n_r) + \frac{V}{g\rho_m} \mathrm{e}_3 \cdot \nabla(\rho_m n_r), 0, 0 \Big),
\end{aligned}$$

for $(n_v, n_c, n_r) \in \mathbb{E}_p^z(J) \times \mathbb{E}_p^z(J) \times \mathbb{E}_p^z(J)$. Using the strategy explained in Section 4.2 leads to the following result:

Proposition 4.5. *Let $A \subset \mathbb{R}^{n-1}$ be a bounded $\mathcal{C}^3$-domain, $a > 0$, $J = (0,T)$ a time interval with $T > 0$ and $\Omega := A \times (-a, a)$ a cylindrical domain. Assume u, $\theta \in L_\infty(J \times \Omega)$, $\eta_j \in \mathcal{BUC}(\Omega)$, $\inf_\Omega \eta_j > 0$, $j \in \{v, c, r\}$, $\beta^j \in \mathcal{BC}^1(J \times \Gamma)$, $\sigma^j \in \mathcal{BC}^2(J \times \Gamma)$, $\inf_\Gamma \beta^j > 0$, $\frac{n+2}{2} < p < \infty$ with $p \neq 3$ and $\rho_m \in W_\infty^2(\Omega)$ with $\inf_\Omega \rho_m > 0$. Then for every data*

$$(0, h_v, m_{v,0}, 0, h_c, m_{c,0}, 0, h_r, m_{r,0}) \in \mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J)$$

with $\inf_{x \in \Omega} m_{r,0} > 0$ there is a unique local-in-time strong solution (m_v, m_c, m_r) to the nonlinear moisture dynamics (TS.2|J) on a maximal time interval $(0, T^*)$ with

$$T^* = T^*(0, h_v, m_{v,0}, 0, h_c, m_{c,0}, 0, h_r, m_{r,0}) \in J.$$

The solution satisfies

$$(m_v, m_c, m_r) \in \mathbb{E}_p^z(\bar{J}) \times \mathbb{E}_p^z(\bar{J}) \times \mathbb{E}_p^z(\bar{J})$$

for all $\bar{J} = (0, \bar{T})$ with $\bar{T} \in (0, T^*)$. Furthermore, the solution depends continuously on the data.

Proof. We begin by fixing the data $(0, h_v, m_{m,0,}, 0, h_c, m_{c,0}, 0, h_r, m_{r,0}) \in \mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J)$. Note, that the embedding

$$m_{r,0} \in W_p^{2-2/p}(\Omega) \hookrightarrow \mathcal{BUC}(\Omega) \hookrightarrow \mathcal{C}(\bar{\Omega}),$$

is valid due to Sobolev's Embedding Theorem for $p > \frac{n+2}{2}$. We require that

$$\zeta := \inf_{x \in \Omega} m_{r,0}(x) > 0.$$

Let $m_r^* \in \mathbb{E}_p^z(J)$ be the unique solution to the parabolic system

$$\begin{aligned} \partial_t m_r^* - \eta_r \Delta m_r^* &= 0 && \text{in } J \times \Omega, \\ \beta^{m_r} \partial_\nu m_r^* + \sigma^{m_r} m_r^* &= h_r, && \text{on } J \times \Gamma, \\ m_r^*(0) &= m_{r,0} && \text{in } \Omega, \end{aligned}$$

which exists due to Theorem 2.6. Now,

$$m_r^* \in \mathcal{C}([0,T] \times \bar{\Omega}),$$

since

$$H_p^1(J, L_p(\Omega)) \cap L_p(J, H_p^2(\Omega)) \hookrightarrow \mathcal{BUC}(J \times \Omega) \hookrightarrow \mathcal{C}([0,T] \times \bar{\Omega})$$

for $p > \frac{n+2}{2}$, cf. [8, Theorem 3.9.1]. The fact that m_r^* is continuous on $[0,T] \times \Omega$ implies that there exists a $T_1 \in (0, T]$, such that

$$\inf_{(t,x) \,\in\, (0,T_1) \times \Omega} m_r^*(t,x) \geqslant \frac{2\zeta}{3}.$$

We denote for $J_1 = (0, T_1)$ by $m_j^* \in H_p^1(J_1, L_p(\Omega)) \cap L_p(J_1, H_p^2(\Omega))$ the unique solution of the parabolic system

$$\begin{aligned} \partial_t m_j^* - \eta_j \Delta m_j^* &= 0 && \text{in } J_1 \times \Omega, \\ \beta^{m_j} \partial_\nu m_j^* + \sigma^{m_j} m_j^* &= h_j, && \text{on } J_1 \times \Gamma, \\ m_j^*(0) &= m_{j,0} && \text{in } \Omega, \end{aligned}$$

for $j \in \{v, c, r\}$ with $\inf_{J_1 \times \Omega} m_r^* \geqslant \frac{2\zeta}{3}$. Such solutions exists according to Theorem 2.6 and they satisfy the estimates

$$\|m_j^*\|_{\mathbb{E}_p^z(J)} \leqslant M_j \|(0, h_v, m_{m,0,}, 0, h_c, m_{c,0}, 0, h_r, m_{r,0})\|_{\mathbb{F}_p^{P,R}(J_1) \times \mathbb{F}_p^{P,R}(J_1) \times \mathbb{F}_p^{P,R}(J_1)} \tag{4.3.8}$$

with constants $M_j > 0$. We set

$$\|(0, h_v, m_{v,0}, 0, h_c, m_{c,0}, 0, h_r, m_{r,0})\|_{\mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J) \times \mathbb{F}_p^{P,R}(J)} =: \mathfrak{w}.$$

Let $\bar{J} = (0, \bar{T}) \subset J_1$ with $\bar{T} \in (0, T_1]$. Due to (4.2.4), it is sufficient to prove existence and uniqueness of a solution to

$$(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r) = K_m(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r), \quad (\tilde{m}_v, \tilde{m}_c, \tilde{m}_r) \in {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}).$$

For that purpose we show that K_m is a contraction on a closed ball of radius δ. For such a ball we have the embeddings

$$\begin{aligned} \bar{B}_\delta(0) \subset {}_0H_p^1(\bar{J}, L_p(\Omega)) \cap L_p(\bar{J}, H_p^2(\Omega)) &\hookrightarrow {}_0\mathcal{BUC}(\bar{J} \times \Omega) \\ &\hookrightarrow {}_0\mathcal{C}([0, \bar{T}] \times \bar{\Omega}). \end{aligned}$$

Therefore, we infer

$$\|\tilde{m}_j\|_{L_\infty(\bar{J} \times \Omega)} \leqslant C \|\tilde{m}_j\|_{{}_0\mathbb{E}_p^z(\bar{J})} \leqslant C^* \|(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r)\|_{{}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})} \leqslant C^* \delta \tag{4.3.9}$$

for $(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r) \in {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})$, $\|(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r)\|_{{}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})} \leqslant \delta$, and $j \in \{v, c, r\}$. The constants C, $C^* > 0$ are independent of $\bar{J} \subset J_1$ thanks to the homogeneous initial conditions.
Choose $\delta < \frac{\zeta}{3C^*}$. Thus we obtain

$$\inf_{(t,x) \in \bar{J} \times \Omega} m_r^* + \tilde{m}_r \geqslant \frac{\zeta}{3} > 0, \tag{4.3.10}$$

for all $\tilde{m}_r \in \bar{B}_\delta(0)$. Now, we have

$$\begin{aligned} &\|S_{ev}(m_v^* + \tilde{m}_v, m_r^* + \tilde{m}_r) - S_{ev}(m_v^* + \tilde{n}_v, m_r^* + \tilde{n}_r)\|_{L_p(\bar{J} \times \Omega)} \\ &\quad \leqslant C_{ev} \|\theta\|_\infty \|(m_r^* + \tilde{m}_r)^\xi (m_{vs} - m_v^* - \tilde{m}_v)^+ - (m_r^* + \tilde{n}_r)^\xi (m_{vs} - m_v^* - \tilde{n}_v)^+\|_p \\ &\quad \leqslant C_{ev} \|\theta\|_\infty \|((m_r^* + \tilde{m}_r)^\xi - (m_r^* + \tilde{n}_r)^\xi)(m_{vs} - m_v^* - \tilde{m}_v)^+\|_p \\ &\qquad + C_{ev} \|\theta\|_\infty \|(m_r^* + \tilde{n}_r)^\xi ((m_{vs} - m_v^* - \tilde{m}_v)^+ - (m_{vs} - m_v^* - \tilde{n}_v)^+)\|_p \\ &\quad \leqslant C_{ev} \|\theta\|_\infty (C' \|\tilde{m}_r - \tilde{n}_r\|_p \|m_{vs} - m_v^* - \tilde{m}_v\|_\infty + \|m_r^* + \tilde{n}_r\|_\infty^\xi \|\tilde{m}_v - \tilde{n}_v\|_p) \\ &\quad \leqslant C_{ev} \|\theta\|_\infty (C' \tilde{C} \|\tilde{m}_v - \tilde{n}_v\|_p + C''' \|\tilde{m}_r - \tilde{n}_r\|_p) \\ &\quad \leqslant C |\bar{J}|^\tau \|(\tilde{m}_v - \tilde{n}_v, \tilde{m}_c - \tilde{n}_c, \tilde{m}_r - \tilde{n}_r)\|_{\mathbb{E}_p^z(\bar{J}) \times \mathbb{E}_p^z(\bar{J}) \times \mathbb{E}_p^z(\bar{J})}, \end{aligned}$$

where we used for the third estimate

$$|a^+ - b^+| \leqslant |a-b|, \text{ for } a,b \in \mathbb{R}. \tag{4.3.11}$$

This estimate is obvious if $a,\ b \geqslant 0$ or $a,\ b < 0$. For $a \geqslant 0$ and $b < 0$ we have

$$|a^+ - b^+| = |a| = a = a - b + b \leqslant a - b \leqslant |a-b|,$$

which is analogous for $a < 0$ and $b \geqslant 0$. We also used

$$|s^\xi - \tilde{s}^\xi| \leqslant \sup_{\bar{s} \geqslant \frac{\zeta}{3}} \xi \bar{s}^{\xi-1}|s-\tilde{s}| \leqslant \xi(\tfrac{\zeta}{3})^{\xi-1}|s-\tilde{s}| =: C'|s-\tilde{s}|$$

for $s,\ \tilde{s} \geqslant \frac{\zeta}{3}$ and $\xi \in (0,1]$. This holds true, since $(m_r^* + \tilde{m}_r)$, $(m_r^* + \tilde{n}_r) \geqslant \frac{\zeta}{3}$, due to (4.3.10). For the fourth inequality we used

$$\|m_j^* + \tilde{m}_j\|_\infty \leqslant \|m_j^*\|_\infty + \tfrac{\zeta}{3} \leqslant \|m_j^*\|_{\mathbb{E}_p^z(T_1)} + \tfrac{\zeta}{3} \leqslant M_j\mathfrak{w} + \tfrac{\zeta}{3} =: C'' > 0,\ j \in \{v,c,r\}, \tag{4.3.12}$$

which holds due to (4.3.9) and our choice of δ. We can then conclude

$$\|m_r^* + \tilde{m}_r\|_\infty^\xi \leqslant (C'')^\xi =: C'''$$

and

$$\|m_v^* + \tilde{m}_v - m_{vs}\|_\infty \leqslant \|m_v^* + \tilde{m}_v\|_\infty + \|m_{vs}\|_\infty \leqslant C'' + \|m_{vs}\|_\infty =: \tilde{C} \tag{4.3.13}$$

for $m_v^* \in \mathbb{E}_p^z(\bar{J})$, $\tilde{m}_v \in {}_0\mathbb{E}_p^z(\bar{J})$, m_{vs} constant. Note that $\tilde{C}$, C', C'' , $C''' > 0$ are also independent of $\bar{J}$. With (4.3.11), (4.3.13) and (4.3.12) we obtain as well for $j \in \{v,r,c\}$ and $(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r) \in {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})$ with $\|\tilde{m}_v\|_{{}_0\mathbb{E}_p^z(\bar{J})} + \|\tilde{m}_c\|_{{}_0\mathbb{E}_p^z(\bar{J})} + \|\tilde{m}_r\|_{{}_0\mathbb{E}_p^z(\bar{J})} \leqslant \delta$ that

$$\begin{aligned}
&\|S_{cd}(m_v^* + \tilde{m}_v, m_c^* + \tilde{m}_c) - S_{cd}(m_v^* + \tilde{n}_v, m_c^* + \tilde{n}_c)\|_{L_p(\bar{J}\times\Omega)},\\
&\left\|(u\cdot\nabla)(m_j^* + \tilde{m}_j) - (u\cdot\nabla)(m_j^* + \tilde{m}_j)\right\|_{L_p(\bar{J}\times\Omega)},\\
&\left\|\tfrac{V}{g\rho_m}\cdot\nabla(\rho_m(m_r^* + \tilde{m}_r)) - \tfrac{V}{g\rho_m}(\rho_m(m_r^* + \tilde{n}_r))\right\|_{L_p(\bar{J}\times\Omega)}\\
&\qquad\qquad \leqslant\ C|\bar{T}|^\tau\,\|(\tilde{m}_v - \tilde{n}_v, \tilde{m}_c - \tilde{n}_c, \tilde{m}_r - \tilde{n}_r)\|_{{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})}.
\end{aligned}$$

The constant $C > 0$ is independent of $\bar{J} \subset J_1$ again. This implies

$$\begin{aligned}
&\|K_m(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r) - K_m(\tilde{n}_v, \tilde{n}_c, \tilde{n}_r)\|_{{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})}\\
&\qquad \leqslant C|\bar{J}|^\tau \bar{C}\|(\tilde{m}_v - \tilde{n}_v, \tilde{m}_c - \tilde{n}_c, \tilde{m}_r - \tilde{n}_r)\|_{{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})},
\end{aligned} \tag{4.3.14}$$

for $(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r)$, $(\tilde{n}_v, \tilde{n}_c, \tilde{n}_r) \in {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})$ that satisfy $\|\tilde{m}_v\|_{{}_0\mathbb{E}_p^z(\bar{J})} + \|\tilde{m}_c\|_{{}_0\mathbb{E}_p^z(\bar{J})} + \|\tilde{m}_r\|_{{}_0\mathbb{E}_p^z(\bar{J})} \leqslant \delta$ and $\|\tilde{n}_v\|_{{}_0\mathbb{E}_p^z(\bar{J})} + \|\tilde{n}_c\|_{{}_0\mathbb{E}_p^z(\bar{J})} + \|\tilde{n}_r\|_{{}_0\mathbb{E}_p^z(\bar{J})} \leqslant \delta$. Here,

$$\bar{C} := \sup\{\|{}_0L_m^{-1}\|_{\mathcal{L}({}_0\mathbb{F}_p^{P,R}(\bar{J})\times{}_0\mathbb{F}_p^{P,R}(\bar{J})\times{}_0\mathbb{F}_p^{P,R}(\bar{J})\,,\,{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J}))} : \bar{J} \subset J_1\}.$$

Finally, we have

$$\begin{aligned}
\|K_m(0,0,0)\|_{{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})\times{}_0\mathbb{E}_p^z(\bar{J})} &\leqslant \bar{C}\|N(m_v^*, m_c^*, m_r^*)\|_{{}_0\mathbb{F}_p^{P,R}(\bar{J})\times{}_0\mathbb{F}_p^{P,R}(\bar{J})\times{}_0\mathbb{F}_p^{P,R}(\bar{J})}\\
&\leqslant \bar{C}\|w\|_{L_p(\bar{J}\times\Omega)^3} \longrightarrow_{\bar{T}\to 0} 0
\end{aligned}$$

where

$$w := \begin{pmatrix} -(u \cdot \nabla)m_c^* + S_{ev}(m_v^*, m_r^*) - S_{cd}(m_v^*, m_c^*) \\ -(u \cdot \nabla)m_c^* + S_{cd}(m_v^*, m_c^*) - S_{ac}(m_c^*) - S_{cr}(m_c^*, m_r^*) \\ -(u \cdot \nabla)m_r^* + S_{ac}(m_c^*) + S_{cr}(m_c^*, m_r^*) - S_{ev}(m_v^*, m_r^*) + \frac{v}{\rho_m} \mathrm{e}_3 \cdot \nabla(\rho_m m_r^*) \end{pmatrix}^T .$$

In summary, this implies

$$\|K_m(\tilde{m}_v, \tilde{m}_c, \tilde{m}_c)\|_{{}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})} \leqslant C|T|^\tau \bar{C}\delta + \bar{C}\|w\|_{L_p(\bar{J}\times\Omega)^3}, \tag{4.3.15}$$

for $(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r) \in {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})$, $\|(\tilde{m}_v, \tilde{m}_c, \tilde{m}_r)\|_{{}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})} \leqslant \delta$. The value $\mathfrak{w} > 0$ is constituted by the data and we already set $\delta < \frac{\zeta}{3C^*}$. Now choose a $\bar{J} \subset J_1$ with $\bar{T} \in (0, T_1]$, such that

$$C|\bar{J}|^\tau \bar{C} < \tfrac{1}{2}$$

and

$$\bar{C}\|w\|_{L_p(\bar{J}\times\Omega)^3} < \tfrac{\delta}{2}.$$

Therefore, we obtain

$$\|K_m(\tilde{m}_v, \tilde{m}_c, \tilde{m}_c)\|_{{}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J}) \times {}_0\mathbb{E}_p^z(\bar{J})} \leqslant \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta$$

by using (4.3.15). Combining this with (4.3.14) we get that K_m is a contraction on a closed ball of radius δ. Now, using the Contraction Mapping Principle, Proposition 4.5 is proven. □

Remark 4.6. For the nonlinear moisture dynamics $(TS.2|J)$ we could prove the existence and uniqueness of a solution for arbitrary data, but small time intervals only. This is due to the nonlinear term

$$S_{ev} = C_{ev}\theta(m_r^+)^\xi(m_{vs}(\theta - m_v))^+, \quad \xi \in (0, 1],$$

which occurs in model $(TS.2|J)$. This term is not Fréchet differentiable, since $(m_r^+)^\xi$ is not differentiable, for instance, at zero for $\xi = \frac{1}{2}$. Therefore, we had to fix the solution to the linearisation of $(TS.2|J)$ on small time intervals and were thus only able to prove solvability of $(TS.2|J)$ for small time intervals.

4.3.3 Proof of Theorem 4.2

Finally we consider the entire system $(TS|J)$. By using Propositions 4.3 we obtain the existence and uniqueness of a local-in-time, strong solution (u, q, θ) to $(TS.1|J)$. Then, we obtain a solution (m_v, m_c, m_r) to $(TS.2|J)$ by using Proposition 4.5 and the solutions (u, θ) obtained from $(TS.1|J)$. Thus, $(u, q, \theta, m_v, m_c, m_r)$ is the unique local-in-time, strong solution to the entire model $(TS|J)$.

Conclusions

In this thesis we extend the model on the dynamics of tropical storms of Nolan and Montgomery [44] to a physically more satisfactory description with the goal of proving the existence and uniqueness of a solution in a general L_p-setting. To make the model thermodynamically consistent, we adapt the coefficients of the model by Nolan and Montgomery to the setting considered by Novotný, Růžička and Thäter [45] and by coupling this system to nonlinear moisture dynamics as introduced in Hittmeir, Klein, Li and Titi [27], we are also able to take the humidity into account. The improved model is of the form

$$\begin{aligned}
\rho\partial_t u + \rho(u\cdot\nabla)u - \mu\Delta u - \lambda\nabla\mathrm{div}u + \alpha\nabla q &= \rho\tfrac{\bar\theta-\theta}{\bar\theta}\nabla F - \omega \mathrm{e}_3\times\rho u && \text{in } J\times\Omega,\\
\mathrm{div}(\rho u) &= 0 && \text{in } J\times\Omega,\\
\rho\partial_t\theta + \rho(u\cdot\nabla)\theta - p_0\,\mathrm{div}(\rho u F) - \kappa\Delta\theta &= \rho(u\cdot\nabla)\bar\theta - \kappa\Delta\bar\theta && \text{in } J\times\Omega,\\
\partial_t m_v + (u\cdot\nabla)m_v - \eta_v\Delta m_v - S_{ev} + S_{cd} &= 0 && \text{in } J\times\Omega,\\
\partial_t m_c + (u\cdot\nabla)m_c - \eta_c\Delta m_c - S_{cd} + S_{ac} + S_{cr} &= 0 && \text{in } J\times\Omega,\\
\partial_t m_r + (u\cdot\nabla)m_r - \eta_r\Delta m_r - S_{ac} - S_{cr} + S_{ev} &= \tfrac{V}{g\rho_m}\mathrm{e}_3\cdot\nabla(\rho_m m_r) && \text{in } J\times\Omega,\\
u\cdot\nu = h_u\cdot\nu,\quad \beta^u P_\Gamma D_+(u)\nu &= P_\Gamma h_u && \text{on } J\times\Gamma,\\
\beta^\theta\partial_\nu\theta + \sigma^\theta\theta &= h_\theta && \text{on } J\times\Gamma,\\
\beta^{m_v}\partial_\nu m_v + \sigma^{m_v}m_v &= h_v && \text{on } J\times\Gamma,\\
\beta^{m_c}\partial_\nu m_c + \sigma^{m_c}m_c &= h_c && \text{on } J\times\Gamma,\\
\beta^{m_r}\partial_\nu m_r + \sigma^{m_r}m_r &= h_r && \text{on } J\times\Gamma,\\
u(0) = u_0,\quad \theta(0) &= \theta_0 && \text{in } \Omega\\
m_v(0) = m_{v,0},\quad m_c(0) = m_{c,0},\quad m_r(0) &= m_{r,0} && \text{in } \Omega.
\end{aligned} \tag{$TS|J$}$$

Here, $\Omega\subseteq\mathbb{R}^n$ denotes a cylindrical domain and $J=(0,T)$ some time interval. The vector fields u, θ, m_v, m_r, m_c and the gradient ∇q are unknown quantities. An interesting aspect of our model is the fact that all coefficients are assumed to be variable, in order to fit both the setting of Nolan and Montgomery and the setting of Novotný, Růžička and Thäter. The main focus of this thesis is the proof of solvability of $(TS|J)$. On that account, we use a linearisation argument to obtain five uncoupled systems: the Stokes equations with free slip boundary conditions and four parabolic systems with Robin boundary conditions. All these systems are defined on cylindrical domains and have variable coefficients.

In Chapter 2 we prove maximal L_p-regularity for parabolic problems with Robin boundary conditions and variable coefficients on cylindrical domains for the cases $1<p<\infty$ and $p\neq 3$. In the same chapter, we prove maximal L_p-regularity for parabolic problems with Neumann-Dirichlet boundary conditions, perfect slip and free slip boundary conditions for $1<p<\infty$ and $p\notin\left\{\frac{3}{2},2,3\right\}$. We have to exclude the cases of $p=\frac{3}{2}$, $p=2$ and $p=3$, because we allow for inhomogeneous boundary conditions. To prove maximal regularity for parabolic problems with inhomogeneous boundary conditions we use the retraction property of trace operators with respect to Robin boundary conditions, Neumann-Dirichlet boundary conditions, perfect slip and free slip boundary conditions, which was shown in Section 1.5. Our method does not provide the cases $p = \frac{3}{2}$, $p = 2$ and $p = 3$, which are the critical values for the trace spaces, the way it provides the cases $1 < p < \infty$ with $p \notin \left\{\frac{3}{2}, 2, 3\right\}$.

In Chapter 3 we are able to prove maximal L_p-regularity of the Stokes equations with free slip boundary conditions and variable coefficients on cylindrical domains for $1 < p < \infty$ and $p \notin \left\{\frac{3}{2}, 2, 3\right\}$. For that purpose, we use the maximal L_p-regularity of the Stokes equations with perfect slip boundary conditions, which is shown earlier in that chapter. By applying a localisation argument similar to the one used in Denk, Hieber and Prüss, and using maximal L_p-regularity of parabolic problems with perfect slip boundary conditions we show maximal L_p-regularity of the Stokes equations with perfect slip boundary conditions for $1 < p < \infty$ and $p \notin \{\frac{3}{2}, 2, 3\}$. For the same reason as mentioned above, we had to exclude the cases $p \in \left\{\frac{3}{2}, 2, 3\right\}$.

Using the results of Chapters 2 and 3, we are able to show maximal L_p-regularity for the linearisation of the model $(TS|J)$ for $1 < p < \infty$ and $p \notin \{\frac{3}{2}, 2, 3\}$ in Chapter 4. In order to prove the existence and uniqueness of a solution to the entire model $(TS|J)$, we split it into a system $(TS.1|J)$ containing u, q and θ, and a system $(TS.2|J)$ containing m_r, m_v and m_c. The existence and uniqueness of a solution to $(TS.1|J)$ are shown for the limiting cases of arbitrary data and small time intervals, as well as for arbitrary time intervals and small data. For the nonlinear moisture dynamics $(TS.2|J)$ we prove the existence and uniqueness of a solution for arbitrary data, but small time intervals only. By proving existence and uniqueness of a solution to $(TS.2|J)$ we also have to set $p > \frac{n+2}{2}$ for the integrability parameter. This is because we use the embedding

$$W_p^{2-2/p}(\Omega) \hookrightarrow \mathcal{BUC}(\Omega) \hookrightarrow \mathcal{C}(\bar{\Omega}),$$

which is only valid for $p > \frac{n+2}{2}$, because of Sobolev's embedding theorem, where $n \in \mathbb{N}$ denotes the dimension in the spatial direction. We need the embedding into $\mathcal{C}(\bar{\Omega})$ and thus, the restriction $p > \frac{n+2}{2}$, because the initial data of the rain water mixing ratio has to be strictly positive. Using the solvability of $(TS.1|J)$ and $(TS.2|J)$ we then infer existence and uniqueness of a local-in-time, strong solution to $(TS|J)$ on cylindrical domains for $\frac{n+2}{2} < p < \infty$ with $p \notin \left\{\frac{3}{2}, 2, 3\right\}$ and small time intervals.

Investigations of the behaviour of $(TS|J)$ on different domains and its solvability for arbitrary time intervals and small data are left for future research. Moreover, the stability of the model is an interesting aspect for further analyses, as it plays a major role for numerical considerations.

Contributions

The content of this thesis is based on a joint work with Jürgen Saal and Matthias Köhne. The essential parts will be published in [33]. All authors contributed equally to [33].

I proved that parabolic problems with constant coefficients in cylindrical domains (Chapter 2), as well as that Stokes equations with variable coefficients in cylindrical domains (Chapter 3) have the property of maximal regularity.

The results concerning the maximal regularity of elliptic operators in cylindrical domains (Chapter 2) and the additional regularity of the Helmholtz-projection (Chapter 1) were developed by Matthias Köhne and me. The proof of the retraction property for a trace operator with respect to perfect slip boundary conditions in Chapter 1 goes back to lively discussions amongst Jürgen Saal, Matthias Köhne and me. Also the development of a model describing the dynamics of tropical storms (Chapter 4), which is based on already existing models, goes back to a number of working session of these three. I implemented the proof of solvability to this model and the proof concerning the retraction property of the aforementioned trace operator.

I created the figures on page 19 and on page 38 of this thesis using Ti*k*Z. The PNGs of the tornadoes next to each page number were created from the GIF [13].

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Eidesstattliche Versicherung

Ich versichere an Eides Statt, dass die Dissertation von mir selbstständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist.

Düsseldorf,