The Stokes and Navier-Stokes Equations in Triebel-Lizorkin-Lorentz Spaces and on Uniform $C^{2,1}$-Domains

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Introduction

In mathematical fluid dynamics the Navier-Stokes equations

$$\begin{cases} \rho\partial_t u - \mu\Delta u + \nabla p + \rho(u\cdot\nabla)u &= \rho f \quad \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T)\times\Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega, \end{cases} \tag{0.1}$$

play a central role. They describe the behavior of a moving incompressible Newtonian fluid with velocity field u and pressure p inside a domain $\Omega \subset \mathbb{R}^n$. The vector field f is the external body force affecting the fluid and u_0 is the observed velocity field at a starting time $t = 0$, while the interval $(0,T)$ is the remaining (finite or infinite) timeframe under consideration. The appearing constants are the density ρ and the viscosity μ.

From a physical point of view, the space dimensions $n = 2$ and $n = 3$ are of greatest interest. In this case, the domain Ω may be seen as filled with a moving fluid, described by the mentioned quantities. Depending on the choice of the domain Ω, it is further possible to use the Navier-Stokes equations to describe the flow around a fixed body. This is the case if Ω is an exterior domain, i.e., the complement of Ω is a compact nonempty set (see [52]).

From a mathematical point of view it is convenient to reformulate (0.1) such that one may concentrate on the nonconstant quantities. Dividing the first line of (0.1) by $\rho > 0$ and considering $\nabla(p/\rho)$ as the unknown gradient field instead of ∇p, yields that the quotient μ/ρ is the only remaining quantity which is constant in the model. Now for the mathematical theory the actual size of the quantity μ/ρ is not essential. Hence, for simplicity in the notation, it is common to set $\mu/\rho = 1$ and we receive

$$\begin{cases} \partial_t u - \Delta u + \nabla p + (u\cdot\nabla)u &= f \quad \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T)\times\Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{cases} \tag{0.2}$$

Still, note that μ/ρ is physically not a dimensionless quantity so that, strictly spoken, the physical unit is skipped by setting $\mu/\rho = 1$. Nevertheless, for the mathematical theory the assumptions are made without loss of generality, the problem (0.2) is meaningful and any obtained mathematical result can be reformulated as a statement with precise physical meaning. Moreover, as this is common as well, we will call the quantity p in (0.2), which now stands for the pressure divided by the (constant) density, the pressure again. Since p in (0.2) is proportional to the physical pressure, it has the same behavior in all aspects that we are going to consider in this thesis.

Note that uniqueness of solutions to the Stokes and Navier-Stokes equations can only be achieved for a couple $(u, \nabla p)$ of the velocity and the gradient of the pressure but not for the pressure p itself, since for any solution (u, p) and an arbitrary constant C also $(u, p + C)$ is a solution.

We will further use $1 < p < \infty$ as a parameter, appearing in some function spaces, but keep denoting the pressure in (0.2) by p as well, since it will appear as a gradient ∇p only, so there will not be any notational confusion.

The linearized version of the Navier-Stokes equations,

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \quad \text{in } (0,T) \times \Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T) \times \Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega, \end{cases} \tag{0.3}$$

called the Stokes equations, is on the one hand of great interest itself in fluid dynamics. On the other hand, a treatment of (0.3) is usually the starting point in order to obtain properties of the Navier-Stokes equations, subject to certain boundary conditions, like well-posedness, regularity or stability.

In case $\partial\Omega \neq \varnothing$ one has to add conditions on the boundary to the equations. Interpreting the domain Ω as filled with a fluid gives that $\partial\Omega$ may be seen as a rigid wall and the boundary conditions state the expected behavior of the fluid at this wall. One reasonable condition is that the fluid may not penetrate the wall, i.e.,

$$\nu \cdot u = 0$$

shall hold at the boundary, where ν is the outward unit normal vector at $\partial\Omega$. In this thesis, we will consider the Stokes and Navier-Stokes equations subject to *partial slip type boundary conditions* of the form

$$\begin{cases} \Pi_\tau(\alpha u + (\nabla u^T \pm \nabla u)\nu) &= 0 \quad \text{on } (0,T) \times \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{0.4}$$

where $\alpha \in \mathbb{R}$ and we write $(0.4)_+$ or $(0.4)_-$, depending on whether $\nabla u^T + \nabla u$ or $\nabla u^T - \nabla u$ is considered. Here Π_τ is the projection onto the tangent space at $\partial\Omega$.

The case $(0.4)_+$ for $\alpha > 0$ is called *Navier condition*, describing the situation that the fluid slips along the wall and is stressed in tangential direction, where $\alpha > 0$ is the related friction parameter. If we (formally) let $\alpha \to \infty$ then we end up with the *no slip boundary condition* (or *Dirichlet boundary condition*), i.e.,

$$u = 0 \quad \text{on } (0,T) \times \partial\Omega,$$

meaning that the fluid does not slip along the wall. The case $(0.4)_-$ for $\alpha = 0$ is called *perfect slip boundary condition* (cf. [48]). In the physically interesting case $n = 3$ this equals the *vorticity condition*

$$\begin{cases} \nu \times \operatorname{curl} u &= 0 \quad \text{on } (0,T) \times \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } (0,T) \times \partial\Omega. \end{cases}$$

In investigations of the Stokes equations the considered class of domains $\Omega \subset \mathbb{R}^n$ mostly consists of Helmholtz domains, i.e., for the Lebesgue space L_q, the classical Helmholtz decomposition

$$L_q(\Omega)^n = L_{q,\sigma}(\Omega) \oplus G_q(\Omega) \tag{0.5}$$

holds for all $1 < q < \infty$. Here $L_{q,\sigma}(\Omega) \subset L_q(\Omega)^n$ is the closure of the space of smooth functions with compact support and vanishing divergence in Ω and $G_q(\Omega)$ is the space of gradient fields $\nabla p \in L_q(\Omega)^n$, where $p \in L_{q,\mathrm{loc}}(\Omega)$.

For the no slip boundary condition on domains with compact boundary but also on bent and perturbed half spaces it has been proved by Farwig and Sohr (see [25]) that the boundary regularity $C^{1,1}$ is sufficient for well-posedness in L_q (for $1 < q < \infty$) of the Stokes equations, where analyticity of the Stokes semigroup has been established. Geissert, Heck, Hieber and Sawada (see [30]) have proved that, on domains with uniform C^3-boundary, validity of the Helmholtz decomposition (0.5) is sufficient for the Stokes equations to be well-posed, where maximal L_q-regularity of the Stokes operator has been proved. Nevertheless, Bolkart, Giga, Miura, Suzuki and Tsutsui have shown that validity of the Helmholtz decomposition is not a necessary condition for well-posedness of the Stokes equations (see [11]).

First mathematical approaches on the Stokes equations with first order boundary conditions are due to Miyakawa and Giga (see [51], [32]; cf. [60]). Investigations concerning Robin boundary conditions are due to Saal, Shibata and Shimada (see [58], [57], [59], [61]). For further investigations on that topic we refer to [12], [19] and [31] and for a general overview of the state of research for the Stokes equations in the L_p-setting see [37].

One aim in this thesis is to obtain results concerning the Stokes equations and the Stokes resolvent problem subject to partial slip type boundary conditions on a large class of domains including particularly

- domains with noncompact boundary and
- non-Helmholtz domains.

The localization technique that we are going to apply has already been utilized in a similar way by Kunstmann for second order elliptic operators subject to no slip boundary conditions (see [46]). It turned out that uniform $C^{2,1}$-regularity of the boundary $\partial\Omega$ is suitable for our purposes and our methods, due to the structure of the boundary conditions (0.4).

Using the special structure of perfect slip boundary conditions, we first establish existence and $L_{q,\sigma}$-invariance of the Laplace resolvent (Theorem 6.5 and Proposition 8.1). This feature was utilized in [50] already (cf. [48], [45]) to study the Stokes operator subject to Neumann type boundary conditions on domains with Lipschitz boundary. We further make use of a suitable generalization of the Helmholtz decomposition, given by

$$L_q(\Omega)^n = L_{q,\sigma}(\Omega) \oplus \mathcal{G}_q(\Omega) \tag{0.6}$$

(Lemma 4.1 and (4.1)), where $\mathcal{G}_q(\Omega)$ is a proper subspace of $G_q(\Omega)$ in case the decomposition (0.5) does not hold. Note that in case the intersection $L_{q,\sigma}(\Omega) \cap G_q(\Omega)$ has finite dimension the concept of generalized Helmholtz decompositions has already been established by Farwig, Simader, Sohr and Varnhorn (see [24]). We apply the results for the Laplace resolvent to obtain a unique solution to the Stokes resolvent problem

$$\begin{cases} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \end{cases}$$

subject to perfect slip boundary conditions as well as the corresponding resolvent estimate, where ∇p is contained in the space $\mathcal{G}_q(\Omega)$ (Theorem 9.1 and Theorem 9.2). In case the gradient field ∇p is assumed to be contained in the space $G_q(\Omega)$, we prove that solutions of the Stokes resolvent problem are no longer unique if $\mathcal{G}_q(\Omega)$ is a proper subspace of $G_q(\Omega)$. In spite of this fact, note that this can not occur in the ground space

$$\widetilde{L}_q(\Omega) = \begin{cases} L_q(\Omega) + L_2(\Omega), & 1 < q < 2 \\ L_q(\Omega) \cap L_2(\Omega), & 2 \leqslant q < \infty, \end{cases}$$

which has been utilized by Farwig, Kozono and Sohr and later also by Rosteck (see [22], [23], [55]). A main difference to the L_q-approach is that for $\widetilde{L}_q(\Omega)$ the related Helmholtz decomposition holds on arbitrary domains with uniform C^2-boundary.

We generalize the results about perfect slip boundary conditions to the boundary conditions (0.4) by utilizing a perturbation argument (Theorem 10.2) and deduce that the related Stokes operator is the generator of a strongly continuous analytic semigroup (Theorem 11.3). Our results about the Stokes equations on uniform $C^{2,1}$-domains are finally applied to the Navier-Stokes equations to obtain corresponding local mild solutions (Theorem 12.1).

Since there seem to be no results in the literature about the Stokes equations in L_q $(1 < q < \infty)$ on a general class of non-Helmholtz domains (in the sense of (0.5)), the main results on that topic, stated in this thesis, are new.

A second aim in this thesis is to develop the theory of maximal regularity for the Stokes equations and to apply this to the Navier-Stokes equations in the whole space for the scale of Triebel-Lizorkin-Lorentz spaces (TLL spaces) $F^{s,r}_{p,q}$.

TLL spaces $F^{s,r}_{p,q}$ may be seen as a unification of Triebel-Lizorkin spaces $F^{s}_{p,q}$ and Lorentz spaces $L_{p,r}$ and were introduced by Cheng, Peng and Yang (see [14]) in 2005. The admissible parameters are $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leqslant r \leqslant \infty$. Implicitly the spaces $F^{s,r}_{p,q}$ appear in the pertinent monograph of Triebel (see [66], Sec. 2.4.2) already. In 2011, Xiang and Yan (see [67]) already considered TLL spaces in the context of partial differential equations and established local well-posedness of a quasi-geostrophic equation.

Depending on the choice of the parameters s, r, p, q, the scale $F^{s,r}_{p,q}$ contains many important function spaces, e.g.,

- Bessel-potential spaces H^s_p,
- Sobolev-Slobodeckiĭ spaces W^s_p,
- Lorentz spaces $L_{p,r}$ and particularly
- Lebesgue spaces L_p.

Local well-posedness in L_p in the whole space under suitable conditions for the parameters is due to Kato (see [44]). For mild solutions in $L_{p,r}$ in the whole space see [68]. As initiators for investigations in the subject, concerning classical function spaces, we should mention Leray, Hopf, Fujita, Kato, Solonnikov and Giga. We refrain from trying to give a complete list. Instead, we refer to the monographs [29] and [64] and the references therein.

The main result (Theorem 14.7) gives existence and uniqueness of local strong solutions on a maximal time interval in $\Omega = \mathbb{R}^n$ for (0.2) in TLL spaces. Since the result is valid for general TLL spaces, this finally yields corresponding outcomes simultaneously in all the function spaces listed above.

We will make use both of the analytic semigroup theory and the theory of sectorial operators in this thesis. Considering the Stokes and Navier-Stokes equations in TLL spaces, we will focus on sectoriality and maximal regularity while in classical Lebesgue spaces on domains we will focus on analytic semigroups. Of course, for an operator $A : \mathscr{D}(A) \subset X \to X$ it is well-known that generating a bounded analytic strongly continuous semigroup is equivalent to $-A$ being pseudo-sectorial, if the spectral angle is smaller than $\frac{\pi}{2}$. Still, the approaches and notation are different: An operator A, generating an analytic semigroup, usually relates to the Cauchy problem

$$\begin{cases} u'(t) - Au(t) &= f(t), \quad t \in (0, T) \\ u(0) &= 0 \end{cases} \tag{0.7}$$

and the focus is on the corresponding resolvent set, lying in a sector with angle θ. Here the related result becomes stronger the bigger the angle θ can be chosen. A sectorial operator A usually relates to

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in (0,T) \\ u(0) &= 0, \end{cases} \tag{0.8}$$

the focus is on the corresponding spectrum and the related spectral angle φ_A shall be preferably small. In the Chapters II and III, where the focus is on analytic semigroup theory, we define the Laplace operator, including the related boundary conditions, via the mapping $u \mapsto \Delta u$. In Chapter IV, where the focus is on sectoriality, we define the Laplace operator in TLL spaces via $u \mapsto -\Delta u$, for the sake of convenient notation in that subject.

This thesis is organized as follows. In Chapter I we begin by introducing general notation and basic function spaces. We proceed with some preliminary results about domains with noncompact boundary, including related trace operators and the versions of Gauß's theorem and Green's formula that we plan to make use of. In the common literature, results of that kind for noncompact boundaries are hard to find. Therefore, some of the proofs are adapted versions from results that one can find in the literature in a more restrictive setting. We further state and discuss our main assumptions for the results concerning Stokes and Navier-Stokes equations on uniform $C^{2,1}$-domains. Eventually, we present the main tools and notation in the context of maximal regularity and bounded H^∞-calculus that we are going to make use of for the main results concerning Navier-Stokes equations in TLL spaces.

In Chapter II we apply a localization method for domains with noncompact boundary to the Laplace resolvent problem subject to perfect slip boundary conditions. We further prove $L_{q,\sigma}$-invariance for the related resolvent, which serves as a main tool in our considerations about the Stokes equations subject to partial slip type boundary conditions.

We state and prove our main results concerning the Stokes and Navier-Stokes equations on uniform $C^{2,1}$-domains in Chapter III. For this purpose, we start with the Stokes resolvent problem, where we make use of the results in Chapter II in order to treat perfect slip boundary conditions. A perturbation argument is used subsequently to transfer the obtained result to general partial slip type boundary conditions. Afterwards, we prove existence and suitable L_p-L_q-estimates for the Stokes semigroup in order to obtain existence and uniqueness of local mild solutions for the Navier-Stokes equations on uniform $C^{2,1}$-domains.

In Chapter IV we begin by defining and investigating TLL spaces. We establish fundamental properties of TLL spaces, such as property (α) and their affiliation to the class $\mathcal{HT}$ and we further prove a multiplier result of Mikhlin type. We prove that the Laplace and the Stokes operator in TLL spaces admit a bounded H^∞-calculus. Finally, this is applied to obtain unique maximal strong solutions of the Navier-Stokes equations in TLL spaces. We further prove that for the obtained maximal solution we either have a blow-up at finite time or the solution exists globally in time.

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I Preliminaries

1 Essentials

1.1 General Notation

For $x \in \mathbb{R}^n$ we denote the components by x_j, $j = 1, \dots, n$ and we write x' for the vector of the first $n-1$ components. We denote the components of a vector field u in $\mathbb{R}^n$ by u^j, so $u = (u^1, \dots, u^n)^T$. The identity matrix is $I := (\delta_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$. We also denote the identity map between normed vector spaces by I if no confusion seems likely. The transposed of some vector or matrix v is v^T. We denote $x \cdot y := \sum_{i=1}^n x_i y_i$ for two vectors and $A \cdot B := \sum_{i,j=1}^n A_{ij} B_{ij}$ for matrices, respectively.

For a linear continuous operator $T : X \to Y$ and two normed spaces X, Y we write $\mathscr{R}(T)$ for its range and $\mathscr{N}(T)$ for its kernel as well as $\|T\|_{X\to Y}$ for the operator norm in $\mathscr{L}(X,Y)$, the space of continuous linear operators from X to Y. We further denote $\mathscr{L}(X) := \mathscr{L}(X,X)$. For any normed space X the related dual space is denoted by X' and the duality pairing is $\langle \cdot, \cdot \rangle_{X,X'}$. We denote $\|\cdot\| \sim \|\cdot\|'$ for two equivalent norms $\|\cdot\|$ and $\|\cdot\|'$, as well as $\|\cdot\| \lesssim \|\cdot\|'$ in case there is a constant $C > 0$ such that $\|\cdot\| \leqslant C\|\cdot\|'$.

The outward unit normal vector at the boundary of some sufficiently regular domain $\Omega \subset \mathbb{R}^n$ is $\nu : \partial\Omega \to \mathbb{R}^n$. As usual, λ_n denotes the Lebesgue measure on the Lebesgue σ-algebra (i.e., the completion of the Borel σ-algebra) of $\mathbb{R}^n$ and σ denotes the related surface measure.

By the gradient of a function u we mean the column vector $\nabla u = (\partial_1 u, \dots, \partial_n u)^T$ and by the gradient ∇u of a vector field with m components we mean the matrix $\nabla u = (\nabla u^1, \dots, \nabla u^m)$, i.e., ∇u^T is the Jacobian matrix of u. We further denote $\mathrm{D}_\pm(u) := \nabla u^T \pm \nabla u$ in case $m = n$. The vector containing all partial derivatives of order $k \geqslant 2$ of a function u is $\nabla^k u$ (with n^k entries) and similarly we define $\nabla^k u$ (with mn^k entries) if u is a vector field with m components.

Divergence and Laplace operator are denoted by div and Δ, respectively. For a matrix-valued function v with components $v_{i,j}$ $(i, j = 1, \dots, n)$ we denote by $\operatorname{div} v$ the column vector with entries $\operatorname{div}(v_{i,1}, \dots, v_{i,n})$ for $i = 1, \dots, n$.

The half space is $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ and the bent half space is $H_\omega := \{x = (x', x_n)^T \in \mathbb{R}^n : x_n > \omega(x')\}$ for $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$. For a function ω on $\mathbb{R}^{n-1}$ its gradient with respect to the $n-1$ components is $\nabla'\omega$ and its matrix of second derivatives is $\nabla'^2\omega$. Similarly we use the notation $\nabla'^k\omega$ for higher derivatives $k \in \mathbb{N}$ and we write Δ' for the Laplace operator with respect to the first $n-1$ components.

Given any parameters $a, b, c, \dots$ we write $C = C(a, b, c, \dots)$ to express that C is a constant depending on (and only on) these parameters. We further make use of the index notation $C_{a,b,\dots}$ to emphasize dependencies of the constant on certain parameters. In general, $C, C', C'', \dots$ are positive constants that may change from line to line. We primarily denote constants by C and make use of $C', C'', \dots$ where it is relevant to emphasize that the constant is now a different one.

The natural numbers $\mathbb{N}$ do not contain zero and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote the Euclidean norm on $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ by $|\cdot|$. The ball with respect to the Euclidean norm with radius $r > 0$ and center a is always denoted by $B_r(a)$. The sector in the complex plane with angle $0 < \theta < \pi$ is $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \theta\}$.

1.2 The Functional Analytic Setting

As usual, $C^k(\Omega)$ is the space of k-times continuously differentiable functions on some open subset $\Omega \subset \mathbb{R}^n$ for $k \in \mathbb{N}_0$ and $C^{k,1}(\Omega)$ is the subspace of functions with a Lipschitz continuous k-th derivative.

For $1 \leqslant q \leqslant \infty$, any open subset $\Omega \subset \mathbb{R}^n$ and a Banach space X, the usual Lebesgue space is denoted by $L_q(\Omega, X)$, i.e., the space of measurable (i.e., also separable-valued) functions $f : \Omega \to X$ satisfying

$$\int_\Omega \|f\|_X^q \, d\lambda_n < \infty$$

if $q < \infty$ and

$$\operatorname*{ess-sup}_{y \in \Omega} \|f(y)\|_X < \infty \tag{1.1}$$

if $q = \infty$, where two functions are considered equal if they coincide except on a null set. We further use the notation $L_q(\Omega, X)$ in case Ω is any measure space with some measure μ and a related σ-algebra. We refer to [7] for the definition of X-valued measurable functions and the Bochner-Lebesgue integral.

The Sobolev space $W_q^k(\Omega, X)$ on a domain $\Omega \subset \mathbb{R}^n$ for some $k \in \mathbb{N}_0$ consists of those functions $f \in L_q(\Omega, X)$ satisfying $\partial_\alpha u \in L_q(\Omega, X)$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant k$. In case the underlying domain Ω is clear from the context, we often write $\|\cdot\|_q$ for the Lebesgue norm and $\|\cdot\|_{k,q}$ for the Sobolev norm. Otherwise, we write $\|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{k,q,\Omega}$, respectively. The space of q-summable sequences for $1 \leqslant q < \infty$ in a Banach space X is $l_q(X)$, i.e., the space of $(x_k)_{k\in\mathbb{N}_0} \subset X$ such that

$$\|(x_k)_{k\in\mathbb{N}_0}\|_{l_q(X)} = \Big(\sum_{k\in\mathbb{N}_0} \|x_k\|_X^q\Big)^{\frac{1}{q}}$$

is finite and in case each element of the sequence shall be allowed to be contained in a different Banach space X_i, where $i \in I$ comes from a countable index set I, we write $l_q(\bigoplus_{i\in I} X_i)$. Furthermore, in case X_i is a function space $F(\Omega_i)$ or $F(\partial\Omega_i)$ of functions on some domain Ω_i or on its boundary $\partial\Omega_i$ (e.g., $F = W_q^k$ for $k \in \mathbb{N}_0$ and $1 \leqslant q \leqslant \infty$), we make use of the short form $\|\cdot\|_{l_q(F)}$ for the norm in $l_q(\bigoplus_{i\in I} X_i)$. In addition, for some Banach space X and $s \in \mathbb{R}$ let $l_q^s(X) := l_q\big(\bigoplus_{k\in\mathbb{N}_0} X_k\big)$, where $X_k := (X, \|2^{ks}\cdot\|_X)$. In case $X = \mathbb{R}$ or $X = \mathbb{C}$ we denote $l_q^s = l_q^s(X)$.

The Sobolev-Slobodeckiĭ space $W_q^s(\Omega)$ for $s = k + \lambda$, $k \in \mathbb{N}_0$, $0 < \lambda < 1$ can be defined as the space of functions $u \in W_q^k(\Omega)$ such that

$$\|u\|_{W_q^s(\Omega)} := \|u\|_{W_q^k(\Omega)} + \sum_{|\alpha|=k} \left(\int_\Omega \int_\Omega \frac{|\partial_\alpha u(y) - \partial_\alpha u(x)|^q}{|y-x|^{n+\lambda q}} \, dy \, dx\right)^{\frac{1}{q}}$$

is finite (cf. [49]). We will further need Sobolev-Slobodeckiĭ spaces on the boundary $W_q^s(\partial\Omega)$ for $s = 1 - \frac{1}{q}$, constituted by the image of the trace operator

$$\operatorname{tr} : W_q^1(\Omega) \to L_q(\partial\Omega), \quad \operatorname{tr} u = u|_{\partial\Omega} \ \forall u \in C_c^\infty(\overline{\Omega}).$$

For a treatment of the trace operator and a concrete definition of Sobolev-Slobodeckiĭ spaces on the boundary we refer to [49].[a] See also [66], Thm. 4.7.1 for the special case of bounded smooth domains.

[a] Note that the Besov scale $B_q^s(\partial\Omega)$ from [49] coincides with the Sobolev-Slobodeckiĭ scale, since in our considerations s never is an integer, except $s = 0$.

On the whole space, via real and complex interpolation, we receive Sobolev-Slobodeckiĭ spaces
$$W_q^s(\mathbb{R}^n, X) = \big(L_q(\mathbb{R}^n, X), W_q^k(\mathbb{R}^n, X)\big)_{\frac{s}{k},q}$$
(in case $s \notin \mathbb{N}$) and Bessel-potential spaces
$$H_q^s(\mathbb{R}^n, X) = \big[L_q(\mathbb{R}^n, X), W_q^k(\mathbb{R}^n, X)\big]_{\frac{s}{k}},$$
respectively, where $k \in \mathbb{N}$, $1 < q < \infty$, $0 < s < k$ and X is a complex Banach space. Moreover, $W_q^s(\mathbb{R}^n, X) = H_q^s(\mathbb{R}^n, X)$ if $s \in \mathbb{N}$. For an introduction of the real interpolation functor $(\cdot,\cdot)_{\theta,q}$ and the complex interpolation functor $[\cdot,\cdot]_\theta$ we refer to [9] and [66].

We set $C_c^\infty(\Omega) := \{u \in C^\infty(\Omega) : \operatorname{spt}(u) \subset \Omega \text{ compact}\}$ and $C_{c,\sigma}^\infty(\Omega) := \{u \in C_c^\infty(\Omega)^n : \operatorname{div} u = 0\}$, where $\operatorname{spt}(u)$ is the support of some function u. The space of solenoidal functions is $L_{q,\sigma}(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}$, where the completion is taken in $L_q(\Omega)^n$. The homogeneous Sobolev space $\widehat{W}_q^1(\Omega) := \{p \in L_{q,\mathrm{loc}}(\Omega) : \nabla p \in L_q(\Omega)^n\}$ is endowed with the seminorm $|p|_{\widehat{W}_q^1(\Omega)} = \|\nabla p\|_q$. We further define the space of gradient fields $G_q(\Omega) := \{\nabla p : p \in \widehat{W}_q^1(\Omega)\}$, endowed with the L_q-norm (subspace topology of $L_q(\Omega)^n$). The dual exponent of $1 \leqslant q \leqslant \infty$ is q', i.e., $\frac{1}{q} + \frac{1}{q'} = 1$. As usual, for some domain Ω and some $1 < q < \infty$, we say that the *Helmholtz decomposition* holds if the direct decomposition
$$L_q(\Omega)^n = L_{q,\sigma}(\Omega) \oplus G_q(\Omega)$$
is valid.

We denote $\langle f, g\rangle_{q,q'} := \int_\Omega fg \, d\lambda_n$ for $f \in L_q(\Omega)$, $g \in L_{q'}(\Omega)$ and $\langle f, g\rangle_{q,q'} := \int_\Omega f \cdot g \, d\lambda_n$ for $f \in L_q(\Omega)^n$, $g \in L_{q'}(\Omega)^n$. Now $\langle f, \varphi\rangle$ is the application of a distribution $f \in \mathscr{D}'(\Omega)$ to a test function $\varphi \in C_c^\infty(\Omega)$, in particular $\langle f, \varphi\rangle = \int_\Omega f\varphi \, d\lambda_n$ in case $f \in L^1_{\mathrm{loc}}(\Omega)$ (similar for $f \in \mathscr{D}'(\Omega)^n$ and $\varphi \in C_c^\infty(\Omega)^n$).

The Laplace operator subject to partial slip type boundary conditions in $L_q(\Omega)^n$ (for a sufficiently regular boundary $\partial\Omega$) is
$$\Delta^\pm_{\alpha,q} : \mathscr{D}(\Delta^\pm_{\alpha,q}) \subset L_q(\Omega)^n \to L_q(\Omega)^n, \quad u \mapsto \Delta u$$
on $\mathscr{D}(\Delta^\pm_{\alpha,q}) := \{u \in W_q^2(\Omega)^n : \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) = 0 \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\}$, where $1 < q < \infty$, $\alpha \in \mathbb{R}$ and Π_τ is the projection onto the tangent space at some point on $\partial\Omega$. The Laplace operator subject to perfect slip boundary conditions is
$$\Delta_{\mathrm{PS}} = \Delta_{\mathrm{PS},q} : \mathscr{D}(\Delta_{\mathrm{PS},q}) \subset L_q(\Omega)^n \to L_q(\Omega)^n, \quad u \mapsto \Delta u \tag{1.2}$$
on $\mathscr{D}(\Delta_{\mathrm{PS},q}) := \{u \in W_q^2(\Omega)^n : \mathrm{D}_-(u)\nu = 0 \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\}$.

The space of Schwartz functions is denoted by $\mathscr{S}(\mathbb{R}^n)$ and thus $\mathscr{S}'(\mathbb{R}^n)$ is the space of tempered distributions. The corresponding space of X-valued Schwartz functions (where X is a Banach space) is $\mathscr{S}(\mathbb{R}^n, X)$ and we set $\mathscr{S}'(\mathbb{R}^n, X) = \mathscr{L}(\mathscr{S}(\mathbb{R}^n), X)$, i.e., the space of continuous linear operators $T : \mathscr{S}(\mathbb{R}^n) \to X$.

For a Banach space X and a measure space $(\Omega, \mathcal{A}, \mu)$ let $\mathcal{M}(\Omega, X)$ be the space of measurable (i.e., also separable-valued) functions $f : \Omega \to X$. The Lorentz space $L_{q,r}(X) = L_{q,r}(\Omega, X) \subset \mathcal{M}(\Omega, X)$ with parameters $1 \leqslant q, r \leqslant \infty$ consists of those functions whose Lorentz quasinorm
$$|||f|||_{L_{q,r}(\Omega,X)} := \begin{cases} \left(\displaystyle\int_0^\infty \left[t^{\frac{1}{q}} f^*(t)\right]^r \frac{dt}{t}\right)^{\frac{1}{r}}, & r < \infty \\ \displaystyle\sup_{t>0} t^{\frac{1}{q}} f^*(t), & r = \infty \end{cases}$$

is finite, where

$$f^*(t) = \inf\{\alpha \geqslant 0 : d_f(\alpha) \leqslant t\}, \quad t \geqslant 0$$

is the decreasing rearrangement and

$$d_f(\alpha) = \mu(\{z \in \Omega : \|f(z)\| > \alpha\}), \quad \alpha \geqslant 0$$

is the distribution function of $f \in \mathcal{M}(\Omega, X)$. Two functions in $L_{q,r}(\Omega, X)$ are considered equal if they are equal except on a null set (with respect to μ).

For $1 < q_0, q_1, q < \infty$, $q_0 \neq q_1$, $1 \leqslant r_0, r_1, r \leqslant \infty$ and $0 < \theta < 1$ such that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ we have

$$\big(L_{q_0,r_0}(X), L_{q_1,r_1}(X)\big)_{\theta,r} = L_{q,r}(X)$$

(see [66], Rem. 1.18.6/4). In view of the identity $L_{q,q}(X) = L_q(X)$ the Lorentz spaces are identified as real interpolation spaces of the Lebesgue spaces $L_q(X)$. Note that $L_{q,r}(\Omega, X)$ is hence normable in case $q > 1$. We denote the corresponding norm by $\|\cdot\|_{L_{q,r}(X)}$ for X-valued functions and we denote $\|\cdot\|_{L_{q,r}}$ for the Lorentz norm of scalar functions.

In the following we assume that X is of class $\mathcal{HT}$ (we give one possible definition for spaces of class $\mathcal{HT}$ in Section 5). In case $s = m \in \mathbb{N}_0$ the Bessel-potential spaces are given by the Sobolev spaces, so $H^m_q(\mathbb{R}^n, X) = W^m_q(\mathbb{R}^n, X)$. For $s \in \mathbb{R}$ and $1 < q < \infty$ we will also use the representation

$$H^s_q(\mathbb{R}^n, X) = \big\{u \in \mathscr{S}'(\mathbb{R}^n, X) : \mathscr{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}}\mathscr{F}u \in L_q(\mathbb{R}^n, X)\big\},$$

where $\|\mathscr{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}}\mathscr{F}\cdot\|_{L_q(\mathbb{R}^n,X)}$ is an equivalent norm in $H^s_q(\mathbb{R}^n, X)$ and $\mathscr{F}u = \widehat{u}$ denotes the Fourier transform of some function u. We refer to [3] and [8] for the Fourier transform of vector-valued functions. For the case of scalar functions see also [34]. Moreover, the continuous embeddings

$$H^s_q(\mathbb{R}^n, X) \subset W^{s-\epsilon}_q(\mathbb{R}^n, X) \subset H^{s-2\epsilon}_q(\mathbb{R}^n, X)$$

hold for any $\epsilon > 0$. We refer to [40] and [4] for a detailed treatise of Bessel-potential and Sobolev-Slobodeckiĭ spaces.

In general, if $F(\mathbb{R}, X)$ is some normed function space (e.g., $F = H^s_p$ or $F = W^s_p$) and $U \subset \mathbb{R}$ open, then we denote by $F(U, X)$ the space of restrictions of functions $u \in F(\mathbb{R}, X)$ to U, equipped with the norm $\|u\|_{F(U,X)} = \inf\{\|v\|_{F(\mathbb{R},X)} : v \in F(\mathbb{R}, X), v|_U = u\}$.

2 Uniform $C^{2,1}$-Boundaries

2.1 The Definition

Let $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain with $C^{2,1}$-boundary, so we can cover $\overline{\Omega}$ with open balls B_l, $l \in \Gamma$ and a countable index set Γ such that, writing $\Gamma_0 := \{l \in \Gamma : B_l \subset \Omega\}$ and $\Gamma_1 := \{l \in \Gamma : B_l \cap \partial\Omega \neq \varnothing\}$, for each $l \in \Gamma_1$ we can find a compactly supported function $\omega_l \in C^{2,1}(\mathbb{R}^{n-1})$ which describes the boundary locally in B_l after rotating and shifting the coordinates. The latter precisely means that for $l \in \Gamma_1$ we can find a rotation matrix $Q_l \in \mathbb{R}^{n\times n}$ and a translation vector $\tau_l \in \mathbb{R}^n$ so that

$$\Omega \cap B_l = H_l \cap B_l \quad \text{and} \quad \partial\Omega \cap B_l = \partial H_l \cap B_l,$$

where $H_l := Q_l^T H_{\omega_l} + \tau_l$ is the rotation and translation of the bent half space H_{ω_l}.

We say that Ω has a *uniform $C^{2,1}$-boundary* (or Ω is a *uniform $C^{2,1}$-domain*) if we can choose the cover B_l, $l \in \Gamma$ in such a way that the radii are all bigger or equal to some fixed $\rho > 0$ and if there is a constant $M \geqslant 1$ such that

$$\|\nabla'\omega_l\|_\infty, \|\nabla'^2\omega_l\|_\infty, \|\nabla'^3\omega_l\|_\infty \leqslant M \tag{2.1}$$

for all $l \in \Gamma_1$ (note that $\omega_l \in W^3_\infty(\Omega)$).

Now, without loss of generality, we can assume that all of the balls B_l, $l \in \Gamma$ have the same radius $\rho > 0$ and that there is $\bar{N} \in \mathbb{N}$ so that at most $\bar{N}$ of the balls B_l have nonempty intersection. Moreover, for arbitrary $\kappa > 0$ we can assume that

$$\|\nabla'\omega_l\|_\infty \leqslant \kappa \tag{2.2}$$

holds for all $l \in \Gamma_1$. This can be achieved by choosing the radius ρ small enough and the rotations Q_l in such a way that the plane $\{x_n = 0\}$ is rotated into the tangent plane of some point on $\partial\Omega \cap B_l$.

For two indices $l, m \in \Gamma$ we write $m \sim l$ if $B_m \cap B_l \neq \varnothing$ and we write $m \approx l$ if $m \sim l$ and $l, m \in \Gamma_1$. Note that for any $l \in \Gamma$ we have $\#\{m \sim l\} \leqslant \bar{N}$.

In order to handle uniform $C^{2,1}$-domains on a local level, we introduce the following partition of unity. Let $(\varphi_l)_{l\in\Gamma} \subset C^\infty(\mathbb{R}^n)$ so that $0 \leqslant \varphi_l \leqslant 1$, $\operatorname{spt}(\varphi_l) \subset B_l$ and

$$\sum_{l\in\Gamma} \varphi_l^2 = 1. \tag{2.3}$$

Since the B_l have a fixed radius ρ, we can choose $(\varphi_l)_{l\in\Gamma}$ in such a way that

$$\sup_{l\in\Gamma} \|\nabla\varphi_l\|_\infty < \infty \quad \text{and} \quad \sup_{l\in\Gamma} \|\nabla^2\varphi_l\|_\infty < \infty. \tag{2.4}$$

2.2 Parametrization of the Boundary

Fix some $l \in \Gamma_1$. A $C^{2,1}$-diffeomorphism between H_{ω_l} and $\mathbb{R}^n_+$ is given by

$$\Phi_l : H_{\omega_l} \xrightarrow{\cong} \mathbb{R}^n_+, \quad x \mapsto \begin{pmatrix} x' \\ x_n - \omega_l(x') \end{pmatrix}$$

with the inverse

$$\Phi_l^{-1} : \mathbb{R}^n_+ \xrightarrow{\cong} H_{\omega_l}, \quad x \mapsto \begin{pmatrix} x' \\ x_n + \omega_l(x') \end{pmatrix}.$$

We obtain

$$\nabla\Phi_l{}^T = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ -\partial_1\omega_l & \dots & -\partial_{n-1}\omega_l & 1 \end{pmatrix}, \quad (\nabla\Phi_l{}^T)^{-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ \partial_1\omega_l & \dots & \partial_{n-1}\omega_l & 1 \end{pmatrix}.$$

Now $\Psi_l(x) := \Phi_l(Q_l(x - \tau_l))$ defines a $C^{2,1}$-diffeomorphism $\Psi_l : H_l \xrightarrow{\cong} \mathbb{R}^n_+$. Using the canonical extension of Φ_l to $\mathbb{R}^n$ and therefore of Ψ_l as well, we receive functions $\Phi_l : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ and $\Psi_l : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$, respectively. Restriction to B_l gives

$$\Psi_l : B_l \xrightarrow{\cong} V_l, \quad x \mapsto \Phi_l(Q_l(x - \tau_l))$$

onto some open subset $V_l \subset \mathbb{R}^n$ and its inverse

$$\Psi_l^{-1} : V_l \xrightarrow{\cong} B_l, \quad x \mapsto Q_l^T \Phi_l^{-1}(x) + \tau_l.$$

The set of diffeomorphisms Ψ_l, $l \in \Gamma_1$ characterizes the $C^{2,1}$-manifold $\partial\Omega$. The related parametrization is given by $\phi_l(\xi) := \Psi_l^{-1}\binom{\xi}{0}$, i.e.,

$$\phi_l : U_l \longrightarrow \partial\Omega \cap B_l, \quad \xi \mapsto Q_l^T \Psi_l^{-1}\begin{pmatrix}\xi\\0\end{pmatrix} + \tau_l = Q_l^T \begin{pmatrix}\xi\\ \omega_l(\xi)\end{pmatrix} + \tau_l, \tag{2.5}$$

where $U_l := \{\xi \in \mathbb{R}^{n-1} : \binom{\xi}{0} \in V_l\}$ (see [27]). We have

$$\nabla\phi_l{}^T = Q_l^T \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \partial_1\omega_l & \dots & \partial_{n-1}\omega_l \end{pmatrix} \tag{2.6}$$

and therefore, since $Q_l Q_l^T = I$,

$$(\nabla\phi_l)\nabla\phi_l{}^T = \begin{pmatrix} 1 & & & \partial_1\omega_l \\ & \ddots & & \vdots \\ & & 1 & \partial_{n-1}\omega_l \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \partial_1\omega_l & \dots & \partial_{n-1}\omega_l \end{pmatrix}.$$

Using the theorem of Binet-Cauchy, we obtain $\det\big((\nabla\phi_l)\nabla\phi_l{}^T\big) = 1 + |\nabla'\omega_l|^2$, in particular

$$\|\det\big((\nabla\phi_l)\nabla\phi_l{}^T\big)\|_\infty \geqslant 1. \tag{2.7}$$

Equation (2.6) further yields

$$\|\nabla\phi_l\|_\infty \leqslant C \quad \forall l \in \Gamma_1 \tag{2.8}$$

with a constant $C = C(n, M) > 0$ and $M > 0$ from (2.1). Using Cramer's rule, we obtain

$$\begin{aligned} ((\nabla\phi_l)\nabla\phi_l{}^T)^{-1} &= \frac{1}{\det\big((\nabla\phi_l)\nabla\phi_l{}^T\big)} \big((-1)^{i+j}\det[(\nabla\phi_l)\nabla\phi_l{}^T]_{i,j}\big)^T_{i,j=1,\dots,n-1} \\ &= \frac{1}{1+|\nabla'\omega_l|^2}\big((-1)^{i+j}\det[(\nabla\phi_l)\nabla\phi_l{}^T]_{i,j}\big)^T_{i,j=1,\dots,n-1}, \end{aligned} \tag{2.9}$$

where $[A]_{i,j}$ means cancellation of the i-th row and j-th column of some matrix A. Together with (2.7) and (2.6) this yields

$$\|\big((\nabla\phi_l)\nabla\phi_l{}^T\big)^{-1}\|_\infty \leqslant C \quad \forall l \in \Gamma_1$$

with a constant $C = C(n, M) > 0$. Since we can estimate

$$\|\nabla' \frac{1}{1+|\nabla'\omega_l|^2}\|_\infty \leqslant C\|\nabla'\omega_l\|_\infty \|\nabla'^2\omega_l\|_\infty$$

with $C = C(n) > 0$, (2.9) together with (2.8) also yields

$$\|\big((\nabla\phi_l)\nabla\phi_l{}^T\big)^{-1}\|_{1,\infty} \leqslant C \quad \forall l \in \Gamma_1 \tag{2.10}$$

where $C = C(n, M) > 0$.

2.3 The Outward Unit Normal Vector

Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain and $n \geqslant 2$. The outward unit normal vector at $\partial\Omega$ is denoted by $\nu : \partial\Omega \to \mathbb{R}^n$. Let $\widehat{\nu}_l : \partial H_{\omega_l} \to \mathbb{R}^n$ be the outward unit normal vector at ∂H_{ω_l} for $l \in \Gamma_1$, which is given by

$$\widehat{\nu}_l = \frac{1}{\sqrt{|\nabla'\omega_l|^2 + 1}}(\partial_1\omega_l, \ldots, \partial_{n-1}\omega_l, -1)^T, \tag{2.11}$$

and let $\nu_l : \partial H_l \to \mathbb{R}^n$ be the outward unit normal vector at ∂H_l, i.e., ν_l arises from rotating and translating $\widehat{\nu}_l$. Then we have $\nu = \nu_l$ on $\partial\Omega \cap B_l = \partial H_l \cap B_l$. The representation (2.11) gives that we can extend $\widehat{\nu}_l$ constantly to a function in $W^2_\infty(H_{\omega_l})^n$ and therefore we can also extend ν_l to a function $\bar{\nu}_l \in W^2_\infty(H_l)^n$. This trivial extension yields a constant $C = C(n, M) > 0$ so that

$$\|\bar{\nu}_l\|_{2,\infty,H_l} \leqslant C \tag{2.12}$$

for all $l \in \Gamma_1$, where M is the constant from (2.1). Now

$$\bar{\nu} := \sum_{l\in\Gamma_1} \varphi_l^2 \bar{\nu}_l \in W^2_\infty(\Omega)^n \tag{2.13}$$

is an extension of ν, since we have

$$\|\bar{\nu}\|_\infty = \sup_{m\in\Gamma_1} \|\mathcal{X}_{B_m} \sum_{l\approx m} \varphi_l^2 \bar{\nu}_l\|_\infty \leqslant \sup_{m\in\Gamma_1} \sum_{l\approx m} \|\bar{\nu}_l\|_\infty \leqslant \bar{N}C$$

and the analogous estimates for $\|\nabla\bar{\nu}\|_\infty$ and $\|\nabla^2\bar{\nu}\|_\infty$. In total we receive

$$\|\bar{\nu}\|_{2,\infty,\Omega} \leqslant C \tag{2.14}$$

for $C = C(n, M) > 0$.

2.4 Boundary Operators

For $n \geqslant 2$ let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain and $\nu : \partial\Omega \to \mathbb{R}^n$ its outward unit normal vector. For a vector field u on Ω with n components the normal and tangential projections of u on $\partial\Omega$ are given by

- $\Pi_\nu u := (\nu\nu^T)u$ and
- $\Pi_\tau u := (I - \nu\nu^T)u$,

respectively. We have $\Pi_\nu u = (\nu \cdot u)\nu$, in particular

$$\Pi_\nu u = 0 \text{ on } \partial\Omega \quad \Leftrightarrow \quad \nu \cdot u = 0 \text{ on } \partial\Omega$$

and in dimension $n = 3$ we also have $\Pi_\tau u = -\nu \times (\nu \times u)$ as well as

$$\Pi_\tau u = 0 \text{ on } \partial\Omega \quad \Leftrightarrow \quad \nu \times u = 0 \text{ on } \partial\Omega.$$

Consider the two boundary operators $\mathrm{D}_\pm(u) = \nabla u^T \pm \nabla u = (\partial_j u^i \pm \partial_i u^j)_{i,j=1,\ldots,n}$. In dimension $n = 3$ we have $\mathrm{D}_-(u)\nu = -\nu \times \operatorname{curl} u$ on $\partial\Omega$. Also note that $\Pi_\nu \mathrm{D}_-(u)\nu = \nu\nu^T(\nabla u^T)\nu - \nu\nu^T(\nabla u)\nu = \nu\nu^T(\nabla u)\nu - \nu\nu^T(\nabla u)\nu = 0$ and therefore

$$\Pi_\tau \mathrm{D}_-(u)\nu = \mathrm{D}_-(u)\nu \quad \text{on } \partial\Omega. \tag{2.15}$$

Lemma 2.1. *Consider a scalar function φ on Ω and some vector fields u, v, w on Ω with n components, respectively. We then have the following calculation rules (in case the product rule for derivatives is applicable).*

(i) $\operatorname{div}(\mathrm{D}_-(u)v) = (\nabla \operatorname{div} u - \Delta u) \cdot v + (\nabla u^T - \nabla u) \cdot \nabla v.$

(ii) $\operatorname{div}(\mathrm{D}_-(u)\nabla\varphi) = (\nabla \operatorname{div} u - \Delta u) \cdot \nabla\varphi.$

(iii) $v \cdot \mathrm{D}_-(u)w = -w \cdot \mathrm{D}_-(u)v.$

Proof. Simple computations yield (i) and (iii) while (ii) follows from (i), since $v := \nabla\varphi$ implies $(\nabla u^T - \nabla u) \cdot \nabla v = \sum_{i,j=1}^n (\partial_j u^i - \partial_i u^j)\partial_i\partial_j\varphi = 0.$ □

3 Traces and Gauß's Theorem

Definition 3.1. Let $1 < q < \infty$, let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain and $n \geqslant 2$. We define
$$E_q(\Omega) := \{f \in L_q(\Omega)^n : \operatorname{div} f \in L_q(\Omega)\}$$
with norm $\|f\|_{E_q(\Omega)} := \|f\|_q + \|\operatorname{div} f\|_q$ and
$$W_q^{-\frac{1}{q}}(\partial\Omega) := [W_{q'}^{1-\frac{1}{q'}}(\partial\Omega)]'.$$

Lemma 3.2. *Let $1 < q < \infty$, $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the segment property (cf. [2]). Then $C_c^\infty(\overline{\Omega})^n \subset E_q(\Omega)$ is dense.*

Proof. Step 1. Let $J_\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the mollifier from [2], Sec. 2.17, i.e., $J_\epsilon(x) := \frac{1}{\epsilon^n} J(\frac{x}{\epsilon})$ for $\epsilon > 0$ and a function $J \in C_c^\infty(\mathbb{R}^n)$ satisfying $J(x) \geqslant 0$ for all $x \in \mathbb{R}^n$, $J(x) = 0$ for $|x| \geqslant 1$ and $\int_{\mathbb{R}^n} J(x)dx = 1$. Following the arguments in the proof of [2], Lem. 3.15, we aim to show that for $u \in E_q(\Omega)$ and any subdomain $\Omega' \subset\subset \Omega$ (i.e., $\overline{\Omega'}$ is compact and $\overline{\Omega'} \subset \Omega$)
$$J_\epsilon * u \xrightarrow{\epsilon \searrow 0} u \quad \text{in } E_q(\Omega') \tag{3.1}$$
holds, where $J_\epsilon * u$ means convolution of J_ϵ with the trivial extension of u to $\mathbb{R}^n$. Due to [2], Lem. 2.18 (c) we have
$$\|J_\epsilon * v\|_q \leqslant \|v\|_q \quad \text{and} \quad J_\epsilon * v \xrightarrow{\epsilon \searrow 0} v \text{ in } L_q(\Omega) \quad \forall v \in L_q(\Omega). \tag{3.2}$$
Now let $\Omega' \subset\subset \Omega$, $u \in E_q(\Omega)$ and $0 < \epsilon < \operatorname{dist}(\Omega', \partial\Omega)$. Writing $\tilde{u}$ for the trivial extension of u to $\mathbb{R}^n$, we have for any $\phi \in C_c^\infty(\Omega')$
$$\begin{aligned}
\int_{\Omega'} (J_\epsilon * u) \cdot \nabla\phi \, d\lambda_n &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{u}(x-y)J_\epsilon(y) \cdot \nabla\phi(x) \, dx \, dy \\
&= \sum_{j=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{u}^j(x-y)J_\epsilon(y)\partial_j\phi(x) \, dx \, dy \\
&= -\sum_{j=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_j\tilde{u}^j(x-y)J_\epsilon(y)\phi(x) \, dx \, dy \\
&= -\sum_{j=1}^n \int_{\Omega'} (J_\epsilon * \partial_j u^j)\phi \, d\lambda_n \\
&= -\int_{\Omega'} (J_\epsilon * \operatorname{div} u)\phi \, d\lambda_n,
\end{aligned}$$

so $\operatorname{div}(J_\epsilon * u) = J_\epsilon * \operatorname{div} u$ holds in the sense of distributions in Ω'. Therefore, (3.2) gives

$$\|\operatorname{div}(J_\epsilon * u) - \operatorname{div} u\|_{L_q(\Omega')} = \|J_\epsilon * \operatorname{div} u - \operatorname{div} u\|_{L_q(\Omega')} \xrightarrow{\epsilon \searrow 0} 0$$

and (3.1) is proved.

Step 2. Following the arguments in the proof of [2], Thm. 3.16, we establish density and continuity of the embedding

$$W_q^1(\Omega)^n \cap C^\infty(\Omega)^n \subset E_q(\Omega) \tag{3.3}$$

by using (3.1). Continuity of (3.3) is obvious, since we can estimate the E_q-norm directly by the W_q^1-norm. Now let $u \in E_q(\Omega)$ and $\delta > 0$. Set $\Omega_k := \{x \in \Omega : |x| < k,\ \operatorname{dist}(x, \partial\Omega) > \frac{1}{k}\}$ for $k \in \mathbb{N}$ as well as $\Omega_0 = \Omega_{-1} = \varnothing$ and $U_k := \Omega_{k+1} \cap (\overline{\Omega}_{k-1})^c$. Then the U_k, $k \in \mathbb{N}$ form an open cover of Ω. Let $(\psi_k)_{k\in\mathbb{N}}$ be a related partition of unity, i.e., $\psi_k \in C_c^\infty(U_k)$, $0 \leqslant \psi_k \leqslant 1$ and $\sum_{k=1}^\infty \psi_k = 1$ on Ω. For $0 < \epsilon < \frac{1}{(k+1)(k+2)}$ we have

$$\operatorname{spt}(J_\epsilon * (\psi_k u)) \subset \Omega_{k+2} \cap (\Omega_{k-2})^c =: V_k \subset\subset \Omega.$$

We apply (3.1) to $\Omega' = V_k$ now: Starting with some $k \in \mathbb{N}$, let $0 < \epsilon_k < \frac{1}{(k+1)(k+2)}$ such that

$$\|J_{\epsilon_k} * (\psi_k u) - \psi_k u\|_{E_q(\Omega)} = \|J_{\epsilon_k} * (\psi_k u) - \psi_k u\|_{E_q(V_k)} < \frac{\delta}{2^k}.$$

Set $\Phi := \sum_{k=1}^\infty J_{\epsilon_k} * (\psi_k u)$ and note that on any $\Omega' \subset\subset \Omega$ there is only a finite number of nonzero summands. For $x \in \Omega_k$ we have

$$u(x) = \sum_{j=1}^{k+2} \psi_j(x) u(x) \quad \text{and} \quad \Phi(x) = \sum_{j=1}^{k+2} J_{\epsilon_j} * (\psi_j u)(x).$$

Hence $\Phi \in C^\infty(\Omega)$ and

$$\|u - \Phi\|_{E_q(\Omega_k)} \leqslant \sum_{j=1}^{k+2} \|J_{\epsilon_j} * (\psi_j u) - \psi_j u\|_{E_q(\Omega)} \leqslant \delta.$$

By use of the monotone convergence theorem we conclude

$$\|u - \Phi\|_{E_q(\Omega)} = \lim_{k\to\infty} \|u - \Phi\|_{E_q(\Omega_k)} \leqslant \delta,$$

so embedding (3.3) is dense.

Step 3. The embedding $C_c^\infty(\overline{\Omega}) \subset W_q^1(\Omega) \cap C^\infty(\Omega)$ is dense, due to [2], Thm. 3.18. So, using the density and continuity of (3.3), we obtain the statement. □

Lemma 3.3 (Trace). *Let $1 \leqslant q < \infty$ and let Ω be a uniform $C^{2,1}$-domain. Then the trace*

$$\operatorname{tr} = \operatorname{tr}_{\partial\Omega} : W_q^1(\Omega) \to W_q^{1-\frac{1}{q}}(\partial\Omega), \quad \operatorname{tr} u = u|_{\partial\Omega}\ \forall u \in C_c^\infty(\overline{\Omega})$$

is continuous. For $q > 1$ it is surjective with a continuous linear right inverse

$$\mathrm{R}_{\partial\Omega} : W_q^{1-1/q}(\partial\Omega) \to W_q^1(\Omega).$$

Proof. In case $1 < q < \infty$ we refer to [49], Thm. 2, where the assumption is only a uniform Lipschitz domain and the additional existence of a continuous linear right inverse $\mathrm{R}_{\partial\Omega} : W_q^{1-1/q}(\partial\Omega) \to W_q^1(\Omega)$ is proved. In case $q = 1$ we make use of the trace for bounded C^1-domains, constructed in [20], Thm. 5.5/1. Choosing for all parts of the boundary $\partial\Omega \cap B_l$, $l \in \Gamma_1$ a bounded C^1-domain U_l such that $\partial U_l \cap B_l = \partial\Omega \cap B_l$ and denoting by tr_l the trace operator for U_l, we can define the trace of $u \in W_q^1(\Omega)$ as

$$\mathrm{tr}\, u := \sum_{l\in\Gamma_1} \mathrm{tr}_l(\varphi_l^2 u).^{\mathrm{b}}$$

Looking at the construction of tr_l in the proof of [20], Thm. 5.5/1, we observe that the uniformity of the boundary $\partial\Omega$ yields that the continuity of tr_l is uniform in $l \in \Gamma_1$. Therefore, we obtain a uniform estimate of the operators tr_l in their operator norm and hence

$$\begin{aligned}
\|\mathrm{tr}\, u\|_{L_1(\partial\Omega)} &= \int_{\partial\Omega} \Big| \sum_{l\in\Gamma_1} \mathrm{tr}_l(\varphi_l^2 u) \Big| \, d\sigma \\
&\leqslant \sum_{l\in\Gamma_1} \int_{\partial\Omega\cap B_l} |\mathrm{tr}_l(\varphi_l^2 u)| \, d\sigma \\
&= \sum_{l\in\Gamma_1} \|\mathrm{tr}_l(\varphi_l^2 u)\|_{L_1(\partial\Omega\cap B_l)} \\
&\leqslant C \sum_{l\in\Gamma_1} \|\varphi_l^2 u\|_{W_1^1(\Omega\cap B_l)} \\
&\leqslant C' \|u\|_{W_1^1(\Omega)},
\end{aligned}$$

with constants $C = C(n,\Omega) > 0$ and $C' = C'(n,\Omega) > 0$, where in the last estimate we made use of (2.1) and of the condition that $\bar{N}$ of the balls B_l have nonempty intersection at most. □

We will write $u|_{\partial\Omega} = \mathrm{tr}\, u$ also for $u \in W_q^1(\Omega)$. Furthermore, for the surface integral we will write $\int_{\partial\Omega} u \, d\sigma = \int_{\partial\Omega} u|_{\partial\Omega} \, d\sigma$ for $u \in W_1^1(\Omega)$ if no confusion seems likely.

Lemma 3.4 (Gauß's theorem in W_1^1)**.** *Let Ω be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $u \in W_1^1(\Omega)^n$. Then we have*

$$\int_\Omega \mathrm{div}\, u \, d\lambda_n = \int_{\partial\Omega} \nu \cdot u \, d\sigma. \tag{3.4}$$

Proof. In case $u \in C_c^\infty(\overline{\Omega})^n$ see, e.g., [7], Thm. XII.3.15. Since $C_c^\infty(\overline{\Omega}) \subset W_1^1(\Omega)$ is dense (see [2], Thm. 3.18), starting with some $u \in W_1^1(\Omega)^n$, we can find a sequence $(u_k)_{k\in\mathbb{N}} \subset C_c^\infty(\overline{\Omega})^n$ converging to u in $W_1^1(\Omega)^n$. Now, replacing u in (3.4) by u_k, we see that the left-hand side converges to $\int_\Omega \mathrm{div}\, u \, d\lambda_n$ and, thanks to Lemma 3.3, the right-hand side converges to $\int_{\partial\Omega} \nu \cdot u \, d\sigma$. □

Lemma 3.5 (Green's formula in W_q^1)**.** *Let $\Omega \subset \mathbb{R}^n$ be uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$. Then*

$$\int_\Omega u(\mathrm{div}\, v) \, d\lambda_n = \int_{\partial\Omega} u(\nu \cdot v) \, d\sigma - \int_\Omega \nabla u \cdot v \, d\lambda_n$$

holds for all $u \in W_q^1(\Omega)$ and $v \in W_{q'}^1(\Omega)^n$.

[b] The trace still does not depend on the specific choice of the partition of unity $(\varphi_l)_{l\in\Gamma}$. This is obvious for $u \in C_c^\infty(\overline{\Omega})$ and the continuity of the trace yields the same for $u \in W_q^1(\Omega)$.

Proof. Lemma 3.4 yields $\int_\Omega \operatorname{div}(uv)\, d\lambda_n = \int_{\partial\Omega} \nu \cdot (uv)\, d\sigma$. Using the representation $\operatorname{div}(uv) = \nabla u \cdot v + u(\operatorname{div} v)$, we obtain the statement. □

Lemma 3.6 (Trace of the normal component)**.** *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$. Then there exists a bounded linear operator*

$$\operatorname{tr}_\nu : E_{q'}(\Omega) \longrightarrow W_{q'}^{-\frac{1}{q'}}(\partial\Omega)$$

such that for any $v \in W^1_{q'}(\Omega)^n$ we have $\operatorname{tr}_\nu v = \nu \cdot v|_{\partial\Omega}$ in $W_{q'}^{-1/q'}(\partial\Omega)$, i.e.,

$$\operatorname{tr}_\nu v = \Big[W_q^{1-\frac{1}{q}}(\partial\Omega) \ni g \mapsto \int_{\partial\Omega} g(\nu \cdot v)\, d\sigma\Big].$$

For $v \in E_{q'}(\Omega)$, we denote by $\langle u, \nu \cdot v\rangle_{\partial\Omega} := \langle \operatorname{tr} u, \operatorname{tr}_\nu v\rangle_{\partial\Omega}$ the application of $\operatorname{tr}_\nu v$ to some $g = \operatorname{tr} u \in W_q^{1-1/q}(\partial\Omega)$, $u \in W^1_q(\Omega)$.

Proof. We follow the arguments in [64], Sec. II.1.2 to construct the trace of the normal component on uniform $C^{2,1}$-domains. Let $g \in W_q^{1-1/q}(\partial\Omega)$ and $v \in W^1_{q'}(\Omega)^n$. Then we have $\mathrm{R}_{\partial\Omega}\, g \in W^1_q(\Omega)$, so, using Lemma 3.5, we obtain

$$\langle \mathrm{R}_{\partial\Omega}\, g, \operatorname{div} v\rangle_{q,q'} = \langle g, \nu \cdot v\rangle_{\partial\Omega} - \langle \nabla \mathrm{R}_{\partial\Omega}\, g, v\rangle_{q,q'}.$$

Therefore, we can estimate

$$\begin{aligned} |\langle g, \nu \cdot v\rangle_{\partial\Omega}| &\leqslant |\langle \nabla \mathrm{R}_{\partial\Omega}\, g, v\rangle_{q,q'}| + |\langle \mathrm{R}_{\partial\Omega}\, g, \operatorname{div} v\rangle_{q,q'}| \\ &\leqslant \|\nabla \mathrm{R}_{\partial\Omega}\, g\|_q \|v\|_{q'} + \|\mathrm{R}_{\partial\Omega}\, g\|_q \|\operatorname{div} v\|_{q'} \\ &\leqslant \|\mathrm{R}_{\partial\Omega}\, g\|_{W^1_q(\Omega)} \|v\|_{E_{q'}(\Omega)} \\ &\leqslant C\|g\|_{W_q^{1-1/q}(\partial\Omega)} \|v\|_{E_{q'}(\Omega)}, \end{aligned}$$

where $C = C(n, q, \Omega) > 0$. We obtain

$$\operatorname{tr}_\nu v := \Big[W_q^{1-\frac{1}{q}}(\partial\Omega) \ni g \mapsto \langle g, \nu \cdot v\rangle_{\partial\Omega}\Big] \in W_{q'}^{-\frac{1}{q'}}(\partial\Omega)$$

with $\|\operatorname{tr}_\nu v\|_{W_{q'}^{-1/q'}(\partial\Omega)} \leqslant C\|v\|_{E_{q'}(\Omega)}$. Consequently,

$$\operatorname{tr}_\nu : \big(W^1_{q'}(\Omega)^n, \|\cdot\|_{E_{q'}(\Omega)}\big) \longrightarrow W_{q'}^{-\frac{1}{q'}}(\partial\Omega), \quad \operatorname{tr}_\nu v = \Big[g \mapsto \int_{\partial\Omega} g(\nu \cdot v)\, d\sigma\Big]$$

is continuous. Since $W^1_{q'}(\Omega)^n \subset E_{q'}(\Omega)$ is dense (see Lemma 3.2), there exists a unique continuous extension

$$\operatorname{tr}_\nu : E_{q'}(\Omega) \longrightarrow W_{q'}^{-\frac{1}{q'}}(\partial\Omega).$$ □

Lemma 3.7 (Green's formula in E_q)**.** *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$. Then we have for $u \in W^1_q(\Omega)$ and $v \in E_{q'}(\Omega)$*

$$\int_\Omega u(\operatorname{div} v)\, d\lambda_n = \langle u, \nu \cdot v\rangle_{\partial\Omega} - \int_\Omega \nabla u \cdot v\, d\lambda_n. \tag{3.5}$$

Proof. Due to Lemma 3.2 we can choose a sequence $(v_k)_{k\in\mathbb{N}} \subset W^1_{q'}(\Omega)^n$ converging to v in $E_{q'}(\Omega)$. Now Lemma 3.5 gives that (3.5) is true for v_k instead of v. It is not hard to see that, for $k \to \infty$, the two terms $\int_\Omega u(\operatorname{div} v_k)\, d\lambda_n$ and $\int_\Omega \nabla u \cdot v_k \, d\lambda_n$ converge to $\int_\Omega u(\operatorname{div} v)\, d\lambda_n$ and $\int_\Omega \nabla u \cdot v \, d\lambda_n$, respectively. Using the continuity of $\operatorname{tr} : W^1_q(\Omega) \to W^{1-1/q}_q(\partial\Omega)$ and $\operatorname{tr}_\nu : E_{q'}(\Omega) \to W^{-1/q'}_{q'}(\partial\Omega)$, we obtain the third term $\langle u, \nu \cdot v_k \rangle_{\partial\Omega}$ converging to $\langle u, \nu \cdot v \rangle_{\partial\Omega}$ as well, for $k \to \infty$. □

Lemma 3.8 (Extended Gauß theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$. Then for $u \in W^1_q(\Omega)$ and $v \in E_{q'}(\Omega)$ we have*

$$\int_\Omega \operatorname{div}(uv)\, d\lambda_n = \langle u, \nu \cdot v \rangle_{\partial\Omega}.$$

Proof. The conditions for u and v give that $\operatorname{div}(uv) = \nabla u \cdot v + u(\operatorname{div} v)$ is a function in $L_1(\Omega)$, so the left-hand side of the formula is well-defined. Lemma 3.7 yields the statement. □

4 The Spaces $L_{q,\sigma}(\Omega)$ and $G_q(\Omega)$

4.1 Main Assumptions

The following abstract statement is the starting point for the methods that we are going to apply in Chapter III.

Lemma 4.1. *Let E be a normed vector space and let $E_1, E_2 \subset E$ be subspaces with $E = E_1 + E_2$ such that $U := E_1 \cap E_2$ is a complemented subspace of E, i.e., there is a continuous linear projection $Q : E \to E$ with $\mathscr{R}(Q) = U$. Let $\widetilde{E}_1 := (I - Q)E_1$ and $\widetilde{E}_2 := (I - Q)E_2$. Then we have the following.*

(i) *For $j \in \{1, 2\}$, $\widetilde{E}_j$ is a closed subspace of E in case $E_j \subset E$ is closed.*

(ii) *$E_1 = \widetilde{E}_1 \oplus U$ and $E_2 = \widetilde{E}_2 \oplus U$.*

(iii) *$E = \widetilde{E}_1 \oplus E_2 = E_1 \oplus \widetilde{E}_2 = \widetilde{E}_1 \oplus \widetilde{E}_2 \oplus U$.*

Note that the algebraic decompositions in (ii) *and* (iii) *are topological ones in case the appearing subspaces of E are closed.*

Proof. First, note that $\widetilde{E}_1 \subset E_1$ and $\widetilde{E}_2 \subset E_2$. Since $\widetilde{E}_j$ is a closed subspace of E_j for $j = 0, 1$, we obtain (i).

In order to prove (ii), note that the definition of Q yields $U = QE_1 = QE_2$, since any $x \in U$ fulfills $x = Qx \in QE_j$ for $j = 1, 2$. Therefore, $E_j = QE_j \oplus (I - Q)E_j = U \oplus \widetilde{E}_j$ holds for $j = 1, 2$.

It remains to verify (iii). We have $\widetilde{E}_1 \cap \widetilde{E}_2 \subset E_1 \cap E_2 = \mathscr{R}(Q)$ and on the other hand $\widetilde{E}_1 \cap \widetilde{E}_2 \subset \widetilde{E}_1 = (I - Q)E_1 \subset (I - Q)E = \mathscr{N}(Q)$. Thus $\widetilde{E}_1 \cap \widetilde{E}_2 = \{0\}$. Now let $x \in E$. Writing $x = x_1 + x_2$ with $x_1 \in E_1$ and $x_2 \in E_2$ (using the assumption that $E = E_1 + E_2$), we obtain $x = (I - Q)x_1 + (I - Q)x_2 + Q(x_1 + x_2) \in \widetilde{E}_1 + \widetilde{E}_2 + U$. Thus, $E = \widetilde{E}_1 \oplus \widetilde{E}_2 \oplus U$ holds. The remaining equalities in (iii) are consequences of (ii). □

For any domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ and $1 < q < \infty$ consider the following assumptions about the spaces $L_{q,\sigma}(\Omega)$ and $G_q(\Omega)$. These assumptions will be in the focus of our main results in Chapter III.

Assumption 4.2.

(i) $U_q(\Omega) := L_{q,\sigma}(\Omega) \cap G_q(\Omega)$ is a complemented subspace of $L_q(\Omega)^n$.

(ii) $L_{q,\sigma}(\Omega) + G_q(\Omega)$ is a closed subspace of $L_q(\Omega)^n$.

Assumption 4.3. $L_q(\Omega)^n = L_{q,\sigma}(\Omega) + G_q(\Omega)$.

Assumption 4.4. $C_c^\infty(\overline{\Omega}) \subset \widehat{W}^1_{q'}(\Omega)$ is dense.

In case Assumption 4.2(i) is valid, we denote the continuous linear projection onto $U_q(\Omega)$ by $\mathbb{Q}_q : L_q(\Omega)^n \to L_q(\Omega)^n$ and we can define a closed subspace of $G_q(\Omega)$ by setting $\mathcal{G}_q(\Omega) := (I - \mathbb{Q}_q)G_q(\Omega)$. If Assumptions 4.2(i) and 4.3 are both valid, we have the decomposition

$$L_q(\Omega)^n = L_{q,\sigma}(\Omega) \oplus \mathcal{G}_q(\Omega) \tag{4.1}$$

(see Lemma 4.1) and we denote the related continuous linear projection onto $L_{q,\sigma}(\Omega)$ by $\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_q$. If only Assumption 4.2(i) is valid, we still have

$$L_{q,\sigma}(\Omega) + G_q(\Omega) = L_{q,\sigma}(\Omega) \oplus \mathcal{G}_q(\Omega),^{\text{c}} \tag{4.2}$$

but note that the direct decomposition (4.2) may not be a topological one. This can only be guaranteed if additionally Assumption 4.2(ii) holds.

Decomposition (4.1) may be seen as a generalized Helmholtz decomposition. In case $U_q(\Omega)$ is finite-dimensional we refer to [24] for an abstract setting of generalized Helmholtz decompositions.

4.2 The Space $L_{q,\sigma}(\Omega)$

Lemma 4.5. *For an arbitrary domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ and $1 < q < \infty$ we have*

$$L_{q,\sigma}(\Omega) = \{f \in L_q(\Omega)^n : \langle f, \nabla\varphi\rangle_{q,q'} = 0 \ \forall \varphi \in \widehat{W}^1_{q'}(\Omega)\}. \tag{4.3}$$

Proof. See [29], Lem. III.1.1. □

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$. Then we have*

$$L_{q,\sigma}(\Omega) \subset \{f \in L_q(\Omega)^n : \operatorname{div} f = 0,\ \nu \cdot f|_{\partial\Omega} = 0\}, \tag{4.4}$$

where $\nu \cdot f|_{\partial\Omega} = \operatorname{tr}_\nu f \in W_q^{-1/q}(\partial\Omega)$ is the trace of the normal component (see Lemma 3.6). If additionally Assumption 4.4 is valid, then we have

$$L_{q,\sigma}(\Omega) = \{f \in L_q(\Omega)^n : \operatorname{div} f = 0,\ \nu \cdot f|_{\partial\Omega} = 0\}. \tag{4.5}$$

Proof. Let $f \in L_{q,\sigma}(\Omega)$. For any $\varphi \in C_c^\infty(\Omega)$ we have $\langle \operatorname{div} f, \varphi\rangle = -\langle f, \nabla\varphi\rangle = 0$, due to (4.3), and therefore $\operatorname{div} f = 0$ in the sense of distributions. We now aim to show that $\langle g, \operatorname{tr}_\nu f\rangle_{\partial\Omega} = 0$ holds for $g \in W_{q'}^{1-1/q'}(\partial\Omega)$. We can write $g = \operatorname{tr} u$ with some $u \in W^1_{q'}(\Omega)$, since the trace is surjective from $W^1_{q'}(\Omega)$ to $W_{q'}^{1-1/q'}(\partial\Omega)$. We use Lemma 3.8 (note that $f \in E_q(\Omega)$) and (4.3) to obtain

$$\langle g, \operatorname{tr}_\nu f\rangle_{\partial\Omega} = \langle u, \nu \cdot f\rangle_{\partial\Omega} = \int_\Omega \operatorname{div}(uf)\, d\lambda_n = \int_\Omega \nabla u \cdot f\, d\lambda_n = 0.$$

[c] The inclusion "$\supset$" as well as directness of the sum are obvious but also any function $f = f_0 + \nabla\pi \in L_{q,\sigma}(\Omega) + G_q(\Omega)$ can be written as $f = f_0 + \mathbb{Q}_q\nabla\pi + (I - \mathbb{Q}_q)\nabla\pi$ so that $(I - \mathbb{Q}_q)\nabla\pi \in \mathcal{G}_q(\Omega)$ and $f_0 + \mathbb{Q}_q\nabla\pi \in L_{q,\sigma}(\Omega)$.

Let now conversely $f \in L_q(\Omega)^n$ with $\operatorname{div} f = 0$ and $\nu \cdot f|_{\partial\Omega} = 0$ and additionally assume that Assumption 4.4 is valid. For $\varphi \in C_c^\infty(\overline{\Omega})$ we have, using Lemma 3.8,

$$\langle f, \nabla\varphi\rangle_{q,q'} = \int_\Omega \operatorname{div}(\varphi f)\, d\lambda_n = \langle \varphi, \nu \cdot f\rangle_{\partial\Omega} = 0.$$

Since $C_c^\infty(\overline{\Omega}) \subset \widehat{W}^1_{q'}(\Omega)$ is dense, this holds for $\varphi \in \widehat{W}^1_{q'}(\Omega)$ as well. Hence, (4.3) gives that $f \in L_{q,\sigma}(\Omega)$. □

Remark 4.7. Note that without Assumption 4.4 the right-hand side of (4.4) can in fact be larger than $L_{q,\sigma}(\Omega)$. An aperture domain as considered in [26] and [21] is an example of a Helmholtz domain with uniform $C^{2,1}$-boundary for which (4.5) does not hold if $q > \frac{n}{n-1}$. Here we have

$$L_{q,\sigma}(\Omega) = \{f \in L_q(\Omega)^n : \operatorname{div} f = 0,\ \nu \cdot f|_{\partial\Omega} = 0,\ \Phi(f) = 0\},$$

where $\Phi(f) = \int_M \nu \cdot f\, d\sigma$ denotes the flux of a function f through the aperture of the domain and M is an $(n-1)$-dimensional manifold shutting the aperture.

4.3 Discussion of the Main Assumptions

Since the Assumptions 4.2, 4.3 and 4.4 will be essential for the main results in Chapter III, we first show that there is in fact a large class of (Helmholtz and non-Helmholtz) domains, satisfying these assumptions.

Definition 4.8. For $n \geqslant 2$ we call a domain $\Omega \subset \mathbb{R}^n$, satisfying the segment property (cf. [2]), a *perturbed cone* if there exists a (convex or concave) cone $\Omega_C \subset \mathbb{R}^n$ (where we assume the apex to be at the origin, w.l.o.g.) and $R > 0$ so that $\Omega \backslash B_R(0) = \Omega_C \backslash B_R(0)$, where the maximal cone $\Omega_C = \mathbb{R}^n$ and the minimal cone $\Omega_C = \varnothing$ are admitted.

We now prove that domains in the class of perturbed cones, which contains also non-Helmholtz domains as we will discuss in Remark 4.10, satisfy Asspumption 4.4.

Lemma 4.9. *Let $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a perturbed cone. Then $C_c^\infty(\overline{\Omega}) \subset \widehat{W}^1_q(\Omega)$ is dense for all $1 \leqslant q < \infty$. Hence, Assumption 4.4 is valid for Ω and for all $1 < q < \infty$.*

Proof. We first convince ourselves that it is sufficient to prove that $\widehat{W}^1_{c,q}(\Omega)$, consisting of those functions in $\widehat{W}^1_q(\Omega)$ having compact support in $\overline{\Omega}$, is a dense subspace of $\widehat{W}^1_q(\Omega)$. In fact, the (algebraic) inclusion $\widehat{W}^1_{c,q}(\Omega) \subset W^1_q(\Omega)$ and the density of $C_c^\infty(\overline{\Omega}) \subset W^1_q(\Omega)$ (see [2], Thm. 3.18; Ω is assumed to have the segment property) yield that $C_c^\infty(\overline{\Omega}) \subset \widehat{W}^1_{c,q}(\Omega)$ is dense. Hence, for some given function $p \in \widehat{W}^1_q(\Omega)$ it remains to find a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\widehat{W}^1_{c,q}(\Omega)$ such that $\|\nabla\psi_k - \nabla p\|_q \xrightarrow{k\to\infty} 0$.

Let $\mathcal{X} \in C^\infty(\mathbb{R}^n)$ so that $\mathcal{X} = 1$ in $\bar{B}_{1/2}(0)$, $\mathcal{X} = 0$ in $\mathbb{R}^n\backslash B_1(0)$ and $0 \leqslant \mathcal{X} \leqslant 1$. Let $\mathcal{X}_k(x) := \mathcal{X}(\frac{x}{k})$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Then we have $\mathcal{X}_k = 1$ in $\bar{B}_{k/2}(0)$, $\mathcal{X} = 0$ in $\mathbb{R}^n\backslash B_k(0)$ and $0 \leqslant \mathcal{X} \leqslant 1$. Setting $M := \|\nabla\mathcal{X}\|_\infty$, we further have

$$\|\nabla\mathcal{X}_k\|_\infty \leqslant \frac{M}{k}. \tag{4.6}$$

Let $R_k := B_k(0)\backslash\bar{B}_{k/2}(0)$ be the k-th annulus. Due to the assumption on Ω there exists $N \in \mathbb{N}$ so that for the scaling $\phi_k : \Omega \cap R_N \to \Omega \cap R_{kN}$, $x \mapsto kx$ we have

$$\phi_k(\Omega \cap R_N) = \Omega \cap R_{kN} \tag{4.7}$$

for all $k \in \mathbb{N}$.

Now for $p \in \widehat{W}^1_q(\Omega)$ we define $\psi_k := \mathcal{X}_{kN}\big(p - \frac{1}{\lambda_n(\Omega\cap R_{kN})}\int_{\Omega\cap R_{kN}} p\,d\lambda_n\big)$. Then ψ_k is a function in $\widehat{W}^1_{c,q}(\Omega)$ for all $k \in \mathbb{N}$ and we have

$$\begin{aligned}\|\nabla\psi_k - \nabla p\|_q \leqslant{}& \|\nabla\mathcal{X}_{kN}\|_\infty \Big\|p - \frac{1}{\lambda_n(\Omega\cap R_{kN})}\int_{\Omega\cap R_{kN}} p\,d\lambda_n\Big\|_{q,\Omega\cap R_{kN}} \\ &+ \|1-\mathcal{X}_{kN}\|_\infty \|\nabla p\|_{q,\Omega\backslash B_{kN/2}(0)}.\end{aligned} \tag{4.8}$$

Now, using (4.6), we can estimate $\|1-\mathcal{X}_{kN}\|_\infty \leqslant 1$ and $\|\nabla\mathcal{X}_{kN}\|_\infty \leqslant \frac{M}{kN}$ as well as

$$\begin{aligned}&\Big\|p - \frac{1}{\lambda_n(\Omega\cap R_{kN})}\int_{\Omega\cap R_{kN}} p\,d\lambda_n\Big\|^q_{q,\Omega\cap R_{kN}} \\ &= k^n \int_{\Omega\cap R_N}\Big|p\circ\phi_k - \frac{k^n}{\lambda_n(\Omega\cap R_{kN})}\int_{\Omega\cap R_N} p\circ\phi_k\,d\lambda_n\Big|^q d\lambda_n \\ &= k^n \Big\|p\circ\phi_k - \frac{k^n}{\lambda_n(\Omega\cap R_{kN})}\int_{\Omega\cap R_N} p\circ\phi_k\,d\lambda_n\Big\|^q_{q,\Omega\cap R_N} \\ &\leqslant k^n C^q \|\nabla(p\circ\phi_k)\|^q_{q,\Omega\cap R_N} \\ &= k^n C^q \int_{\Omega\cap R_N} |k(\nabla p\circ\phi_k)|^q\,d\lambda_n \\ &= k^n k^q \frac{1}{k^n} C^q \int_{\Omega\cap R_{kN}} |\nabla p|^q\,d\lambda_n \\ &= k^q C^q \|\nabla p\|^q_{q,\Omega\cap R_{kN}},\end{aligned}$$

using (4.7), were $C = C(n,q,\Omega\cap R_N) > 0$ is the constant from the Poincaré inequality (see [29], Thm. II.5.4). In total we have

$$\|\nabla\psi_k - \nabla p\|_q \leqslant \frac{MC}{N}\|\nabla p\|_{q,\Omega\cap R_{kN}} + \|\nabla p\|_{q,\Omega\backslash B_{kN/2}} \xrightarrow{k\to\infty} 0,$$

since $\nabla p \in L_q(\Omega)^n$. □

We gather some remarks about our main assumptions.

Remark 4.10.

(a) Obviously any Helmholtz domain in the classical sense fulfills the Assumptions 4.2 and 4.3 with $U_q(\Omega) = \{0\}$ and $L_{q,\sigma}(\Omega) + G_q(\Omega) = L_q(\Omega)^n$.

(b) For domains $\Omega \subset \mathbb{R}^n$ with uniform $C^{2,1}$-boundary, Assumption 4.4 is known to be valid for

- $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ and perturbed half spaces, i.e., there exists some $R > 0$ such that $\Omega\backslash B_R(0) = \mathbb{R}^n_+\backslash B_R(0)$ (Lemma 4.9; cf. [29], Thm. II.7.8 for the half space),
- bent half spaces $\Omega = H_\omega$ (see [25], Lem. 5.1; alternatively one could check that bent half spaces are (ϵ,∞)-domains, see the definition in Lemma 4.11),
- bounded domains (Lemma 4.9: choose $R > 0$ such that $\Omega \subset B_R(0)$; cf. [29], Thm. II.7.2, Def. II.1.1),
- exterior domains, i.e., Ω is the complement of some compact set in $\mathbb{R}^n$ (Lemma 4.9: choose $R > 0$ such that $\Omega\backslash B_R(0) = \mathbb{R}^n\backslash B_R(0)$; cf. [25], Lem. 5.1 and [29], Thm. II.7.2, Def. II.1.1),

- asymptotically flat domains, i.e., Ω is a layer-like domain $\Omega = \{x \in \mathbb{R}^n : \gamma_-(x') < x_n < \gamma_+(x')\}$ which is delimited by two functions $\gamma_+, \gamma_- \in C^{2,1}(\mathbb{R}^{n-1})$ with the asymptotic behavior $\lim_{|x'|\to\infty} \gamma_\pm(x') = c_\pm$, where $c_- < c_+$ and $\lim_{|x'|\to\infty} \nabla\gamma_\pm(x') = 0$ (see [1], Lem. 2.6 and Cor. 6.4), and
- (ϵ, ∞)-domains, as treated in [15] and [42] (see Lemma 4.11 below).

With the exception of general (ϵ, ∞)-domains, all of the mentioned domains are Helmholtz domains in the classical sense and therefore satisfy the Assumptions 4.2 and 4.3 as well. We refer to [28] and [52] for perturbed half spaces, bounded domains and exterior domains and we refer to [1] for asymptotically flat domains. See also [62] for bent half spaces and [29] for the whole space and the half space.

(c) Assumption 4.4 will be an important tool for all our results regarding the Stokes equations. Even for the key statement concerning our main results, Proposition 8.1 in Chapter II, finding a proof without this condition seems hopeless. Note that, e.g., the natural identity

$$L_{q,\sigma}(\Omega) = \{f \in L_q(\Omega)^n : \operatorname{div} f = 0,\ \nu \cdot f = 0 \text{ on } \partial\Omega\} \tag{4.9}$$

is a consequence of Assumption 4.4 (see Lemma 4.6). An aperture domain, as treated in [26] and [21], is an example of a domain for which Assumption 4.4 does not hold for all $1 < q < \infty$. The identity (4.9) is not satisfied in this case as well (as mentioned in Remark 4.7). An approach to circumvent this problem and to include also domains not satisfying Assumption 4.4 might be to define the space $G_q(\Omega)$ as the closure of $C_c^\infty(\bar{\Omega})^n$ in $\widehat{W}^1_{q'}(\Omega)$ instead. Then Proposition 8.1 had to be proved for a larger space $\bar{L}_{q,\sigma}(\Omega)$ such that $\bar{L}_{q,\sigma}(\Omega) \oplus G_q(\Omega) = L_q(\Omega)^n$ holds but there seems to be no reason for Proposition 8.1 to hold in general.

(d) In case $1 < q \leqslant 2$ we have $U_q(\Omega) = L_{q,\sigma}(\Omega) \cap G_q(\Omega) = \{0\}$ for any uniform $C^{2,1}$-domain Ω so Assumption 4.2(i) holds and we have $\mathcal{G}_q(\Omega) = G_q(\Omega)$. This is due to [23], Thm. 1.2 (see also [22], Thm. 2.1 for the 3-dimensional case), from which we receive the direct decomposition

$$L_q(\Omega)^n + L_2(\Omega)^n = [L_{q,\sigma}(\Omega) + L_{2,\sigma}(\Omega)] \oplus [G_q(\Omega) + G_2(\Omega)]$$

and therefore $L_{q,\sigma}(\Omega) \cap G_q(\Omega) \subset [L_{q,\sigma}(\Omega) + L_{2,\sigma}(\Omega)] \cap [G_q(\Omega) + G_2(\Omega)] = \{0\}$.

(e) Obviously, in case $U_q(\Omega)$ has finite dimension, Assumption 4.2(i) is valid and, in case $L_{q,\sigma}(\Omega) + G_q(\Omega)$ has finite codimension, Assumption 4.2(ii) is valid. We refer to [24] for an elaboration of generalized Helmholtz decompositions in this situation.

(f) A sector-like domain with opening angle $\beta > \pi$ and a smoothed vertex, as considered by Bogovskiĭ and Maslennikova (see [10]), is an example of a non-Helmholtz domain (for q either small or large enough) to which our main theorems in Chapter III apply: Lemma 4.9 gives that Assumption 4.4 is valid for sector-like domains. For these domains Assumptions 4.2 and 4.3 are valid if $q > \frac{2}{1-\pi/\beta}$. We have $\dim U_q(\Omega) = 1$ in this case. If $\frac{2}{1+\pi/\beta} < q < \frac{2}{1-\pi/\beta}$, Assumptions 4.2 and 4.3 hold and we have $\mathcal{G}_q(\Omega) = G_q(\Omega)$. If $q < \frac{2}{1+\pi/\beta}$, Assumption 4.2 holds, but 4.3 does not. We have $\operatorname{codim}(L_{q,\sigma}(\Omega) + G_q(\Omega)) = 1$ in this case. In the special cases $q = \frac{2}{1\pm\pi/\beta}$, Assumption 4.2(i) is still valid, but 4.2(ii) is not. Hence, Theorem 9.1 is applicable to these domains for any $q \in (1, \infty)\backslash\{\frac{2}{1\pm\pi/\beta}\}$ (merely the assertion (iv) is not known to hold for the special cases $q = \frac{2}{1\pm\pi/\beta}$). Theorems 9.2 and 10.2 are applicable for $q \in (\frac{2}{1+\pi/\beta}, \infty)\backslash\{\frac{2}{1-\pi/\beta}\}$ and Theorem 12.1 is applicable for $q \in (\frac{4}{1+\pi/\beta}, \infty)\backslash\{\frac{2}{1-\pi/\beta}, \frac{4}{1-\pi/\beta}\}$.

Lemma 4.11. *Let $n \geqslant 2$, $1 \leqslant q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an (ϵ, ∞)-domain for some $\epsilon > 0$, i.e., for all $x, y \in \Omega$ there exists a rectifiable curve γ in Ω with length $l(\gamma)$, connecting x and y, such that*

$$l(\gamma) < \frac{|x-y|}{\epsilon} \tag{4.10}$$

and

$$\operatorname{dist}(z, \partial\Omega) > \frac{\epsilon|x-z||y-z|}{|x-y|} \quad \forall z \in \gamma. \tag{4.11}$$

Condition (4.11) *says that there is a tube around γ, lying in Ω, such that in some point $z \in \gamma$ the tube's width is of the order of* $\min\{|x-z|, |y-z|\}$ *(cf.* [15] *and* [42]*). Then $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_q^1(\Omega)$ is dense.*

Proof. Due to [15], Thm. 1.2, the conditions on Ω yield a continuous extension operator $\Lambda : \widehat{W}_q^1(\Omega) \longrightarrow \widehat{W}_q^1(\mathbb{R}^n)$, where we choose the weight $w = 1$. Now, using the density of $C_c^\infty(\mathbb{R}^n) \subset \widehat{W}_q^1(\mathbb{R}^n)$, we obtain the statement. □

5 $\mathcal{R}$-boundedness, Maximal Regularity and H^∞-Calculus

In order to deal with operator-valued Fourier multipliers we employ the following concept. Let X, Y be complex Banach spaces. Let $\mathcal{E}_P$ denote the set of families of random variables $(\epsilon_i)_{i\in I}$ on a probability space $P = (\Omega, \mathcal{A}, \mu)$ (i.e., μ is a probability measure, defined on the σ-algebra $\mathcal{A}$ of all possible events, and Ω is the underlying sample space) with values in $\{\pm 1\}$, which are independent and symmetrically distributed. We say that a family of continuous linear operators $\mathcal{T} \subset \mathscr{L}(X, Y)$ is $\mathcal{R}$-*bounded* if there is a probability space $P = (\Omega, \mathcal{A}, \mu)$ with $\mathcal{E}_P \neq \varnothing$, $p \in [1, \infty)$ and a constant $C > 0$ such that for all $N \in \mathbb{N}$, $(\epsilon_1, \ldots, \epsilon_N) \in \mathcal{E}_P$, $T_i \in \mathcal{T}$ and $x_i \in X$ (for $1 \leqslant i \leqslant N$)

$$\left\| \sum_{i=1}^N \epsilon_i T_i x_i \right\|_{L_p(\Omega, Y)} \leqslant C \left\| \sum_{i=1}^N \epsilon_i x_i \right\|_{L_p(\Omega, X)}. \tag{5.1}$$

In this case we call $\mathcal{R}_p(\mathcal{T}) := \inf\{C > 0 : (5.1) \text{ holds}\}$ the $\mathcal{R}$-*bound* or the $\mathcal{R}_p$-*bound*. Note that $\mathcal{R}$-boundedness implies boundedness of $\mathcal{T} \subset \mathscr{L}(X, Y)$. If a family $\mathcal{T} \subset \mathscr{L}(X, Y)$ is $\mathcal{R}_p$-bounded for some $p \in [1, \infty)$ then it is also $\mathcal{R}_q$-bounded for any $q \in (1, \infty)$ (see, e.g., [47]). In this case (5.1) also holds for a (possibly different) constant $C > 0$ if we replace P by an arbitrary probability space Q with $\mathcal{E}_Q \neq \varnothing$. Also note that, in view of Lebesgue's dominated convergence theorem, it is sufficient to claim (5.1) for x_i in a dense subspace of X. The following result is useful to extend boundedness to $\mathcal{R}$-boundedness in some concrete cases (see [17], Lem. 3.5).

Theorem 5.1 (Kahane's contraction principle). *Let X be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $P = (\Omega, \mathcal{A}, \mu)$ a probability space and $1 \leqslant p < \infty$. Let $N \in \mathbb{N}$ and $a_j, b_j \in \mathbb{F}$ with $|a_j| \leqslant |b_j|$ for $j = 1, \ldots, N$. Then we have for all $x_1, \ldots, x_N \in X$ and $\epsilon_1, \ldots, \epsilon_N \in \mathcal{E}_P$*

$$\Big\| \sum_{i=1}^N a_i \epsilon_i x_i \Big\|_{L_p(\Omega, X)} \leqslant C_{\mathbb{F}} \Big\| \sum_{i=1}^N b_i \epsilon_i x_i \Big\|_{L_p(\Omega, X)},$$

where $C_{\mathbb{R}} = 1$ and $C_{\mathbb{C}} = 2$.

We call a linear and densely defined operator $A : \mathscr{D}(A) \subset X \to X$ *pseudo-sectorial* if its spectrum $\sigma(A)$ is contained in a closed sector $\overline{\Sigma}_\varphi$ with angle $\varphi \in (0,\pi)$ and the family $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\varphi}\} \subset \mathscr{L}(X)$ is bounded. If $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\varphi}\} \subset \mathscr{L}(X)$ is even $\mathcal{R}$-bounded, A is called *pseudo-$\mathcal{R}$-sectorial.* We omit the prefix "pseudo-" if the range $\mathscr{R}(A) \subset X$ is dense and so we receive a *sectorial* or *$\mathcal{R}$-sectorial* operator, respectively. We denote the infimum over all $\varphi \in (0,\pi)$ such that $\sigma(A) \subset \overline{\Sigma}_\varphi$ and such that the family $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\varphi}\} \subset \mathscr{L}(X)$ is bounded, by φ_A (*spectral angle*) if A is a (pseudo-)sectorial operator and likewise $\varphi_A^{\mathcal{R}}$ is the infimum over all $\varphi \in (0,\pi)$ such that this family is $\mathcal{R}$-bounded if A is a (pseudo-)$\mathcal{R}$-sectorial operator.

For a pseudo-sectorial operator A and a fixed angle $\varphi < \varphi_A$ we will make use of the Dunford calculus

$$f \longmapsto f(A),$$

which maps a function $f \in \mathscr{H}_0(\Sigma_\varphi) = \bigcup_{\alpha,\beta<0} \mathscr{H}_{\alpha,\beta}(\Sigma_\varphi)$ to a bounded operator on X, as well as of its extension to $\mathscr{H}_p(\Sigma_\varphi) = \bigcup_{\alpha\in\mathbb{R}} \mathscr{H}_{\alpha,\alpha}(\Sigma_\varphi)$ if A is sectorial. Here $\mathscr{H}_{\alpha,\beta}(\Sigma_\varphi)$ is the space of holomorphic functions $f : \Sigma_\varphi \to \mathbb{C}$ such that

$$\|f\|_{\varphi,\alpha,\beta} = \sup_{|z|\leqslant 1} |z^\alpha f(z)| + \sup_{|z|>1} |z^{-\beta} f(z)|$$

is finite. We refer to [17] and [36] for a precise definition and treatise of this functional calculus. Note that for a function $f \in \mathscr{H}_{\alpha,\alpha}(\Sigma_\varphi)$ we receive a bounded operator $f(A)$ in case $\alpha < 0$ but in general only a closed operator on the domain

$$\mathscr{D}(f(A)) = \{x \in X : (g^k f)(A)x \in \mathscr{D}(A^k) \cap \mathscr{R}(A^k)\}. \tag{5.2}$$

Here $k > \alpha$ is a nonnegative integer and $g \in \mathscr{H}_0(\Sigma_\varphi)$ is the function $g(z) := \frac{z}{(1+z)^2}$, which leads to a bijective mapping $g(A)^{-k} : \mathscr{D}(A^k) \cap \mathscr{R}(A^k) \longrightarrow X$. A slight modification of this functional calculus leads to the following well-known characterization of sectorial operators (see [36], Prop. 3.4.4).

Proposition 5.2. *An operator $A : \mathscr{D}(A) \subset X \to X$ is pseudo-sectorial with angle $\varphi_A < \frac{\pi}{2}$ if and only if $-A$ is the generator of a bounded analytic strongly continuous semigroup.*

The functional calculus is also used to describe the following important property of an operator A. Let $A : \mathscr{D}(A) \subset X \to X$ be a sectorial operator. Then A has a *bounded H^∞-calculus* in X if for some $\varphi \in (\varphi_A, \pi)$ there is a constant $C_\varphi > 0$ such that for any $f \in \mathscr{H}_0(\Sigma_\varphi)$ we have

$$\|f(A)\|_{X\to X} \leqslant C_\varphi \|f\|_{\infty,\Sigma_\varphi}. \tag{5.3}$$

In this case (5.3) also holds for all bounded holomorphic functions f on Σ_φ. The infimum over all angles $\varphi \in (\varphi_A, \pi)$, such that (5.3) holds with a constant $C_\varphi > 0$, is called *H^∞-angle* and is denoted by φ_A^∞. Likewise we say that A has an *$\mathcal{R}$-bounded H^∞-calculus* in X if the set

$$\{f(A) : f \in \mathscr{H}_0(\Sigma_\varphi), \|f\|_{\infty,\Sigma_\varphi} \leqslant 1\} \subset \mathscr{L}(X)$$

is $\mathcal{R}$-bounded and the related $\mathcal{R}H^\infty$-angle is denoted by $\varphi_A^{\mathcal{R},\infty}$. If A has a bounded H^∞-calculus, then

$$\mathscr{D}(A^\alpha) = [X, \mathscr{D}(A)]_\alpha \tag{5.4}$$

holds for all $0 < \alpha < 1$ (see [17]), where the fractional power $A^\alpha : \mathscr{D}(A^\alpha) \subset X \longrightarrow X$ is defined via the functional calculus above with the function $z \mapsto z^\alpha$.

We recall the following two assertions that frequently occur in the context of the H^∞-calculus. A proof can be found in [18] (see the proofs of [18], Prop. 2.9 and [18], Prop. 2.7, respectively).

Lemma 5.3. (i) *Let $0 < \varphi < \pi$. Then for all $\alpha \in \mathbb{N}_0^n$ there is a constant $C_{\alpha,\varphi} > 0$ so that for every holomorphic and bounded function $h : \Sigma_\varphi \to \mathbb{C}$ we have*

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{|\alpha|} |\partial_\alpha h(|\xi|^2)| \leqslant C_{\alpha,\varphi} \|h\|_{\infty,\Sigma_\varphi}.$$

(ii) *Let $\frac{\pi}{2} < \varphi < \pi$. Then for $k = 0, 1$ there is a constant $C_\varphi > 0$ so that for every $h \in \mathscr{H}_0(\Sigma_\varphi)$ we have*

$$\sup_{\xi \in \mathbb{R} \setminus \{0\}} |\xi|^k |\partial_k h(i\xi)| \leqslant C_\varphi \|h\|_{\infty,\Sigma_\varphi}.$$

Next, we give the definition of maximal regularity (due to [47]) for an operator $A : \mathscr{D}(A) \subset X \to X$, which is the generator of a bounded analytic strongly continuous semigroup $(e^{tA})_{t \geqslant 0}$ on a complex Banach space X. Therefore, we fix $1 < p < \infty$ and $0 < T \leqslant \infty$. The operator A has *maximal L_p-regularity* on $(0,T)$ if for all $f \in L_p((0,T), X)$ the solution

$$u(t) = \int_0^t e^{(t-s)A} f(s) ds$$

of the Cauchy problem

$$\begin{cases} u'(t) - Au(t) &= f(t), \quad t \in (0,T) \\ u(0) &= 0 \end{cases}$$

is Fréchet differentiable a.e., takes its values in $\mathscr{D}(A)$ a.e., and we have

$$u', Au \in L_p((0,T), X).$$

In this case we receive

$$\|u'\|_{L_p((0,T),X)} + \|Au\|_{L_p((0,T),X)} \leqslant C \|f\|_{L_p((0,T),X)} \tag{5.5}$$

by an application of the closed graph theorem. We write $A \in \mathrm{MR}(X, C)$ if A has maximal L_p-regularity for one (or equivalently for all; see [63]) $1 < p < \infty$ on some $(0,T)$ so that (5.5) holds with a constant $C = C(T) > 0$. If $A \in \mathrm{MR}(X, C)$ and C does not depend on T (i.e., (5.5) holds for $T = \infty$), we write $A \in \mathrm{MR}(X)$.

Now we take a look at the advantages of maximal regularity. Again for a complex Banach space X, let $A : \mathscr{D}(A) \subset X \to X$ be the generator of a bounded analytic strongly continuous semigroup. For $1 < p < \infty$ and $T \in (0, \infty]$ we set

$$\mathbb{E}_T := H_p^1((0,T), X) \cap L_p((0,T), \mathscr{D}(A))$$

(the solution space of the related Cauchy problem) and

$$\mathbb{F}_T \times \mathbb{I} := L_p((0,T), X) \times \big\{ x = u(0) : u \in \mathbb{E}_T \big\} \tag{5.6}$$

(the data space). Note that $\mathbb{I}$ is a Banach space, where $\|x\|_{\mathbb{I}} = \inf_{u(0)=x} \|u\|_{\mathbb{E}_T}$ is the related norm, $\mathbb{I}$ is independent of T and we have

$$\mathbb{I} = \big(X, \mathscr{D}(A)\big)_{1-\frac{1}{p},p} \tag{5.7}$$

(see [54], Prop. 3.4.4). If A has maximal L_p-regularity on a finite interval $(0,T)$, then the solution operator L^{-1}, where

$$L : \mathbb{E}_T \longrightarrow \mathbb{F}_T \times \mathbb{I}, \quad u \longmapsto \Big(\Big(\frac{d}{dt} - A \Big) u, u(0) \Big), \tag{5.8}$$

exists and is an isomorphism. This leads to the estimate

$$\|u\|_{H_p^1((0,T),X)\cap L_p((0,T),\mathscr{D}(A))} \leqslant C(T)\big(\|f\|_{L_p((0,T),X)} + \|x\|_{\mathbb{I}}\big),$$

when $u := L^{-1}(f,x)$ is the solution to $(f,x) \in \mathbb{F}_T \times \mathbb{I}$.

Lemma 5.4. *Let X be a Banach space, $1 < p < \infty$, $0 < T_0 < \infty$ and let $A \in \mathrm{MR}(X, C(T_0))$. Then there exists a constant $C' = C'(T_0) > 0$ such that*

$$\|L^{-1}(f,0)\|_{H_p^1((0,T),X)\cap L_p((0,T),\mathscr{D}(A))} \leqslant C'(T_0)\|f\|_{L_p((0,T),X)}$$

holds for all $T \in (0,T_0]$ and for all $f \in L_p((0,T),X)$, where L^{-1} is the solution operator, i.e., the inverse of L given in (5.8).

Proof. By the trivial extension of $f \in L_p((0,T),X)$ to $(0,T_0)$ we obtain the estimate (5.5) with a constant independent of $T \in (0,T_0]$. Now the assertion follows from the fact that the Poincaré inequality $\|u\|_{L_p((0,T),X)} \leqslant K\|u'\|_{L_p((0,T),X)}$ holds with a constant $K > 0$, which is independent of $T \in (0,T_0]$ as well. □

Lemma 5.5. *Let $1 < p < \infty$ and $T \in (0,\infty]$. Then, with the notation above, we have the continuous embedding*

$$\mathbb{E}_T \subset BUC([0,T),\mathbb{I})$$

(where BUC, as usual, means bounded and uniformly continuous). Here the operator $A : \mathscr{D}(A) \subset X \to X$ only needs to be closed and densely defined in a Banach space X.

Proof. The case $T = \infty$ follows essentially from the strong continuity of the translation semigroup. Then, by a standard extension and retraction argument we obtain the case $T < \infty$ as a consequence. See [3], Prop. 1.4.2 for details. □

In the theory of partial differential equations, the notions of class $\mathcal{HT}$ and property (α) for Banach spaces turned out to be significant. A Banach space X is *of class $\mathcal{HT}$* if the *Hilbert transform*

$$H : \mathscr{S}(\mathbb{R},X) \longrightarrow \mathcal{M}(\mathbb{R},X), \quad Hf(t) = \lim_{\epsilon\searrow 0}\int_{|s|>\epsilon}\frac{f(t-s)}{s}ds$$

has an extension $H \in \mathscr{L}(L_p(\mathbb{R},X))$ for one (or equivalently for all; see [3]) $1 < p < \infty$. A complex Banach space X has *property* (α) if there exist $1 \leqslant p < \infty$, two probability spaces $P = (\Omega,\mathcal{A},\mu)$, $P' = (\Omega',\mathcal{A}',\mu')$ with $\mathcal{E}_P, \mathcal{E}_{P'} \neq \varnothing$ and a constant $\alpha > 0$ such that for all $N \in \mathbb{N}$, $x_{ij} \in X$, $a_{ij} \in \mathbb{C}$, $|a_{ij}| \leqslant 1$ $(i,j = 1,\dots,N)$ and for all $(\epsilon_1,\dots,\epsilon_N) \in \mathcal{E}_P$, $(\epsilon_1',\dots,\epsilon_N') \in \mathcal{E}_{P'}$ we have

$$\Big\|\sum_{i,j=1}^N \epsilon_i\epsilon_j' a_{ij}x_{ij}\Big\|_{L_p(\Omega\times\Omega',X)} \leqslant \alpha\Big\|\sum_{i,j=1}^N \epsilon_i\epsilon_j' x_{ij}\Big\|_{L_p(\Omega\times\Omega',X)}.$$

A useful application of property (α) is the following one, which is a direct consequence of the Kalton-Weis theorem (see [54], Thm. 4.5.6).

Theorem 5.6. *Let X be a Banach space with property (α) and let $A : D(A) \subset X \to X$ be an operator with a bounded H^∞-calculus. Then A has an $\mathcal{R}$-bounded H^∞-calculus with $\varphi_A^\infty = \varphi_A^{\mathcal{R},\infty}$.*

The following operator-valued version of Mikhlin's theorem is important for our purposes as well as the subsequent characterization of maximal L_p-regularity. The results are due to Girardi and Weis (see [33] or [54], Thm. 4.3.9, Thm. 4.4.4).

Theorem 5.7. *Let X and Y be complex Banach spaces of class $\mathcal{HT}$ having property (α) and let $1 < p < \infty$. For $m_\lambda \in C^n(\mathbb{R}^n \backslash \{0\}, \mathscr{L}(X,Y))$, $\lambda \in \Lambda$, assume that $\kappa_\alpha := \mathcal{R}_p\{\xi^\alpha \partial_\alpha m_\lambda(\xi) : \xi \in \mathbb{R}^n\backslash\{0\}, \lambda \in \Lambda\} < \infty$ for each $\alpha \in \{0,1\}^n$, where Λ is some index set. Then the operator*

$$\mathscr{F}^{-1} m_\lambda \mathscr{F} : \mathscr{S}(\mathbb{R}^n, X) \longrightarrow \mathscr{S}'(\mathbb{R}^n, Y)$$

has a unique extension $T_\lambda \in \mathscr{L}(L_p(\mathbb{R}^n, X), L_p(\mathbb{R}^n, Y))$ for every $\lambda \in \Lambda$ and we have

$$\mathcal{R}_p\{T_\lambda : \lambda \in \Lambda\} \leqslant C_{p,n} \sum_{\alpha \in \{0,1\}^n} \kappa_\alpha =: C.$$

In particular, we have $\|\mathscr{F}^{-1} m_\lambda \mathscr{F} f\|_{L_p(Y)} \leqslant C \|f\|_{L_p(X)}$ for $f \in \mathscr{S}(\mathbb{R}^n, X)$, $\lambda \in \Lambda$.

Theorem 5.8. *Let X be a Banach space of class $\mathcal{HT}$, $1 < p < \infty$ and let $A : \mathscr{D}(A) \subset X \to X$ be the generator of a bounded analytic strongly continuous semigroup. Then the following conditions are equivalent.*

(i) *A has maximal L_p-regularity on $(0, \infty)$.*

(ii) *$-A$ is pseudo-$\mathcal{R}$-sectorial with $\varphi_A^{\mathcal{R}} < \frac{\pi}{2}$.*

II The Laplace Resolvent on Uniform $C^{2,1}$-Domains

We aim to obtain unique solvability of the resolvent problem

$$\begin{cases} \lambda u - \Delta u &= f \quad \text{in } \Omega \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial\Omega \\ \Pi_\nu u &= \Pi_\nu h \quad \text{on } \partial\Omega \end{cases}$$

as well as certain properties of the resolvent that will allow us to carry over the result to the Stokes resolvent problem. A starting point for this investigation is a localization technique with an infinite number of local neighborhoods under consideration.

The basic idea is in principle from [46], where a localization method for domains with noncompact boundary is applied. For the (countably many) parameters $l \in \Gamma$ we multiply the resolvent problem by the smooth cut-off functions φ_l in order to receive a system of local equations (one equation for each of the $l \in \Gamma$) where now a sequence $(u_l)_{l\in\Gamma}$ of the form $u_l = \varphi_l u$ is the potential solution. Introducing a suitable Banach space X for the sequence $(u_l)_{l\in\Gamma}$ as well as a Banach space Y containing the right-hand side functions of the local equations, the purpose is to obtain unique solvability on a local level in space and finally to carry over this result to the original problem. In comparison to [46], where Dirichlet boundary conditions have been investigated, the localization of the boundary conditions here is somewhat more intricate.

6 Perfect Slip Boundary Conditions for the Laplace Resolvent

We begin by treating the half space $\mathbb{R}^n_+$ and, via perturbation arguments, bent rotated and shifted versions of the half space. These special domains in turn, serve as auxiliary domains that occur when the general domain Ω is considered on a local level.

6.1 The Half Space

Proposition 6.1. *Let $n \geqslant 2$, $1 < q < \infty$ and $0 < \theta < \pi$. Then for $f \in L_q(\mathbb{R}^n_+)^n$, $g \in W^1_q(\mathbb{R}^n_+)^n$, $h \in W^2_q(\mathbb{R}^n_+)^n$ and any $\lambda \in \Sigma_\theta$ there exists a unique solution $u \in W^2_q(\mathbb{R}^n_+)^n$ of*

$$\begin{cases} \lambda u - \Delta u &= f \quad \text{in } \mathbb{R}^n_+ \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial\mathbb{R}^n_+ \\ \Pi_\nu u &= \Pi_\nu h \quad \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{6.1}$$

and this solution fulfills the resolvent estimate

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_q \leqslant C\|(f, \sqrt{\lambda}g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_q \tag{6.2}$$

where $C = C(n, q, \theta) > 0$.

Proof. In the half space the outward unit normal vector is $\nu = (0, \dots, 0, -1)^T$, and the tangential and normal projections are given by $\Pi_\tau g = (g^1, \dots, g^{n-1}, 0)^T$ and $\Pi_\nu h = (0, \dots, 0, h^n)^T$, respectively. Then (6.1) reads

$$\begin{cases} \lambda u - \Delta u &= f & \text{in } \mathbb{R}^n_+ \\ \partial_1 u^n - \partial_n u^1 &= g^1 & \text{on } \partial\mathbb{R}^n_+ \\ \partial_2 u^n - \partial_n u^2 &= g^2 & \text{on } \partial\mathbb{R}^n_+ \\ & \vdots & \\ \partial_{n-1} u^n - \partial_n u^{n-1} &= g^{n-1} & \text{on } \partial\mathbb{R}^n_+ \\ u^n &= h^n & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Hence we can solve the inhomogeneous Dirichlet boundary problem

$$\begin{cases} \lambda u^n - \Delta u^n &= f^n & \text{in } \mathbb{R}^n_+ \\ u^n &= h^n & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

first and then, after inserting the solution $u^n \in W^2_q(\mathbb{R}^n_+)$, solve the decoupled Neumann boundary problems

$$\begin{cases} \lambda u^j - \Delta u^j &= f^j & \text{in } \mathbb{R}^n_+ \\ -\partial_n u^j &= g^j - \partial_j u^n & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

for $j = 1, \dots, n-1$. See [47], Thm. 7.7 and Sec. 7.18 for a detailed treatment of the problems with Dirichlet and Neumann boundary conditions. Thus, we obtain unique solvability of (6.1) as well as estimate (6.2). $\square$

6.2 The Bent Half Space

Theorem 6.2. *Let $\omega \in W^3_\infty(\mathbb{R}^{n-1})$, $n \geqslant 2$, $1 < q < \infty$ and $0 < \theta < \pi$. Choose some $M \geqslant 1$ such that*

$$\|\nabla'\omega\|_\infty, \|\nabla'^2\omega\|_\infty, \|\nabla'^3\omega\|_\infty \leqslant M \tag{6.3}$$

holds. Then there exist $\kappa = \kappa(n, q, \theta) > 0$ and $\lambda_0 = \lambda_0(n, q, \kappa, M) > 0$ such that in case $\|\nabla'\omega\|_\infty \leqslant \kappa$, $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ for $f \in L_q(H_\omega)^n$, $g \in W^1_q(H_\omega)^n$ and $h \in W^2_q(H_\omega)^n$ there exists a unique solution $u \in W^2_q(H_\omega)^n$ of

$$\begin{cases} \lambda u - \Delta u &= f & \text{in } H_\omega \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g & \text{on } \partial H_\omega \\ \Pi_\nu u &= \Pi_\nu h & \text{on } \partial H_\omega \end{cases} \tag{6.4}$$

and this solution fulfills the resolvent estimate

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_q \leqslant C\|(f, \sqrt{\lambda} g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_q \tag{6.5}$$

where $C = C(n, q, \theta, M) > 0$.

In order to prove Theorem 6.2 we use the change of coordinates from $x \in H_\omega$ to $\tilde{x} \in \mathbb{R}^n_+$, given by

$$\Phi : H_\omega \xrightarrow{\cong} \mathbb{R}^n_+, \quad x \mapsto \begin{pmatrix} x' \\ x_n - \omega(x') \end{pmatrix} =: \tilde{x},$$

and we write $u \circ \Phi^{-1} =: J_\omega^{-1} u =: \tilde{u}$ for a function u on H_ω. For the partial derivatives we have the following behavior under the change of coordinates, from which we obtain particularly that

$$J_\omega : W^k_q(\mathbb{R}^n_+) \xrightarrow{\cong} W^k_q(H_\omega), \quad \tilde{u} \mapsto u \tag{6.6}$$

is an isomorphism for $k = 0, 1, 2$ such that the continuity constants of J_ω and J_ω^{-1} only depend on M from (2.1) and on n.

- $\widetilde{\partial_i u} = \partial_i \widetilde{u} - (\partial_i \omega)\partial_n \widetilde{u}$ for $i = 1, \ldots, n-1$.
- $\widetilde{\partial_n u} = \partial_n \widetilde{u}$.
- $\widetilde{\partial_j \partial_i u} = \partial_j \partial_i \widetilde{u} - (\partial_j \omega)\partial_i \partial_n \widetilde{u} - (\partial_j \partial_i \omega)\partial_n \widetilde{u} - (\partial_i \omega)\partial_j \partial_n \widetilde{u} + (\partial_i \omega)(\partial_j \omega)\partial_n^2 \widetilde{u}$ for $i, j = 1, \ldots, n-1$.
- $\widetilde{\partial_n \partial_i u} = \partial_i \partial_n \widetilde{u} + (\partial_i \omega)\partial_n^2 \widetilde{u}$ for $i = 1, \ldots, n-1$.
- $\widetilde{\partial_n^2 u} = \partial_n^2 \widetilde{u}$.
- $\widetilde{\Delta u} = \Delta \widetilde{u} - 2(\nabla' \omega^T, 0) \cdot \nabla \partial_n \widetilde{u} - (\Delta' \omega)\partial_n \widetilde{u} + |\nabla' \omega|^2 \partial_n^2 \widetilde{u}$ when u is a scalar function.
- $\widetilde{\nabla u^T} = \nabla \widetilde{u}^T - (\nabla' \omega^T, 0)\partial_n \widetilde{u}$ when u is a scalar function.
- $\widetilde{\nabla u^T} = \nabla \widetilde{u}^T - \begin{pmatrix} (\partial_1 \omega)\partial_n \widetilde{u}^1 & \ldots & (\partial_{n-1}\omega)\partial_n \widetilde{u}^1 & 0 \\ \vdots & & \vdots & \vdots \\ (\partial_1 \omega)\partial_n \widetilde{u}^n & \ldots & (\partial_{n-1}\omega)\partial_n \widetilde{u}^n & 0 \end{pmatrix} =: \nabla \widetilde{u}^T - E(\widetilde{u})$ when u is a vector field.
  Setting $\partial_n \omega := 0$, we can write $E(\widetilde{u}) = \big((\partial_j \omega)\partial_n \widetilde{u}^i\big)_{i,j=1,\ldots,n}$.

Hence, we can write

$$(\widetilde{\lambda - \Delta})u = (\lambda - \Delta)\widetilde{u} + B\widetilde{u}, \tag{6.7}$$

where

$$B\widetilde{u} := 2(\nabla' \omega^T, 0) \cdot \nabla \partial_n \widetilde{u} + (\Delta' \omega)\partial_n \widetilde{u} - |\nabla' \omega|^2 \partial_n^2 \widetilde{u} \tag{6.8}$$

for a scalar function $\widetilde{u}$ and we define $B\widetilde{u}$ componentwise if $\widetilde{u}$ is a vector field. For the boundary condition operator we further have

$$\widetilde{\mathrm{D}_-(u)}\nu = \mathrm{D}_-(\widetilde{u})\widetilde{\nu} + (E(\widetilde{u})^T - E(\widetilde{u}))\widetilde{\nu}. \tag{6.9}$$

Now (6.9) gives that (2.15) holds for $\widetilde{\mathrm{D}_-(u)}\nu$ instead of $\mathrm{D}_-(u)\nu$ as well, i.e., $(I - \widetilde{\nu}\widetilde{\nu}^T)\widetilde{\mathrm{D}_-(u)}\nu = \widetilde{\mathrm{D}_-(u)}\nu$. Also note that transporting the normal vector $\nu : \partial H_\omega \to \mathbb{R}^n$ of the bent half space via a change of coordinates to $\widetilde{\nu} : \partial \mathbb{R}^n_+ \to \mathbb{R}^n$ does not yield the normal vector of the half space. In fact, since $\nu(x)$ does not depend on the last component x_n, which can be seen in the concrete representation

$$\widetilde{\nu} = \frac{1}{\sqrt{|\nabla' \omega|^2 + 1}}(\partial_1 \omega, \ldots, \partial_{n-1}\omega, -1)^T, \tag{6.10}$$

we can identify $\nu = \widetilde{\nu}$ and even consider it as a function on the whole space, i.e.,

$$\nu = \widetilde{\nu} : \mathbb{R}^n \to \mathbb{R}^n.$$

In this case (6.10) gives that

$$\|\nu\|_{2,\infty} \leqslant C_n \|(\nabla' \omega, \nabla'^2 \omega, \nabla'^3 \omega)\|_\infty \tag{6.11}$$

holds with a constant $C_n > 0$ depending only on the space dimension n. We denote the outward unit normal vector of the half space by $\nu_+ := (0, \ldots, 0, -1)^T$.

The boundary condition

$$\begin{cases} \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial H_\omega \\ \Pi_\nu u &= \Pi_\nu h \quad \text{on } \partial H_\omega \end{cases} \tag{6.12}$$

can be written equivalently as

$$\mathrm{D}_-(u)\nu + \Pi_\nu u = \Pi_\tau g + \Pi_\nu h \quad \text{on } \partial H_\omega$$

due to separation of the tangential and the normal part in (6.12) and utilizing (2.15). Now (6.7) and (6.9) give that a change of coordinates in (6.4) yields the equivalent[a] problem

$$\begin{cases} \lambda\widetilde{u} - \Delta\widetilde{u} + B\widetilde{u} &= \widetilde{f} \quad \text{in } \mathbb{R}^n_+ \\ \mathrm{D}_-(\widetilde{u})\widetilde{\nu} + (E(\widetilde{u})^T - E(\widetilde{u}))\widetilde{\nu} + \Pi_\nu\widetilde{u} &= \Pi_\tau\widetilde{g} + \Pi_\nu\widetilde{h} \quad \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{6.13}$$

with $\widetilde{f} \in L_q(\mathbb{R}^n_+)^n$, $\widetilde{g} \in W^1_q(\mathbb{R}^n_+)^n$ and $\widetilde{h} \in W^2_q(\mathbb{R}^n_+)^n$.

We apply the matrix

$$\nabla\Phi^T = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -\partial_1\omega & -\partial_2\omega & \dots & -\partial_{n-1}\omega & 1 \end{pmatrix}$$

to the boundary condition of (6.13). The matrix $\nabla\Phi^T$ satisfies $\det\nabla\Phi^T = 1$, $(\nabla\Phi^T)^{-1} = 2I - \nabla\Phi^T$ and it maps the tangent space at any point $x \in \partial H_\omega$ into the tangent space at $\partial\mathbb{R}^n_+$.[b] Therefore we have:

- $(\nabla\Phi^T)\mathrm{D}_-(\widetilde{u})\nu = (I - \nu_+\nu_+^T)(\nabla\Phi^T)\mathrm{D}_-(\widetilde{u})\nu$
 $= (I-\nu_+\nu_+^T)\mathrm{D}_-(\widetilde{u})\nu_+ + (I-\nu_+\nu_+^T)(\nabla\Phi^T - I)\mathrm{D}_-(\widetilde{u})\nu_+ + (I-\nu_+\nu_+^T)(\nabla\Phi^T)\mathrm{D}_-(\widetilde{u})(\nu - \nu_+)$
 $= \mathrm{D}_-(\widetilde{u})\nu_+ + (I - \nu_+\nu_+^T)(\nabla\Phi^T - I)\mathrm{D}_-(\widetilde{u})\nu_+ + (I - \nu_+\nu_+^T)(\nabla\Phi^T)\mathrm{D}_-(\widetilde{u})(\nu - \nu_+)$.
- $(\nabla\Phi^T)\Pi_\nu\widetilde{u} = \nu_+\nu_+^T\widetilde{u} + (I-\nu_+\nu_+^T)((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u} + \nu_+\nu_+^T((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u}$.
- $(\nabla\Phi^T)(E(\widetilde{u})^T - E(\widetilde{u}))\nu = (I - \nu_+\nu_+^T)(\nabla\Phi^T)(E(\widetilde{u})^T - E(\widetilde{u}))\nu$.[c]
- $(\nabla\Phi^T)\Pi_\tau\widetilde{g} = (I - \nu_+\nu_+^T)(\nabla\Phi^T)(I - \nu\nu^T)\widetilde{g}$.
- $(\nabla\Phi^T)\Pi_\nu\widetilde{h} = \nu_+\nu_+^T(\nabla\Phi^T)\nu\nu^T\widetilde{h} + (I - \nu_+\nu_+^T)(\nabla\Phi^T)\nu\nu^T\widetilde{h}$.

Hence, (6.13) becomes

$$\begin{cases} \lambda\widetilde{u} - \Delta\widetilde{u} + B\widetilde{u} &= \widetilde{f} \quad \text{in } \mathbb{R}^n_+ \\ \mathrm{D}_-(\widetilde{u})\nu_+ + \nu_+\nu_+^T\widetilde{u} + B'\widetilde{u} &= (I - \nu_+\nu_+^T)\widetilde{G} + \nu_+\nu_+^T\widetilde{H} \quad \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{6.14}$$

where

$$\begin{aligned} B'\widetilde{u} :=& (I - \nu_+\nu_+^T)[(\nabla\Phi^T - I)\mathrm{D}_-(\widetilde{u})\nu_+ + (\nabla\Phi^T)\mathrm{D}_-(\widetilde{u})(\nu - \nu_+) \\ &+ ((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u} + (\nabla\Phi^T)(E(\widetilde{u})^T - E(\widetilde{u}))\nu] \\ &+ \nu_+\nu_+^T((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u} \end{aligned} \tag{6.15}$$

[a] Since the change of coordinates $u \mapsto \widetilde{u}$ is an isomorphism $W^k_q(H_\omega) \to W^k_q(\mathbb{R}^n_+)$ for $k = 0, 1, 2$ (see (6.6)), this problem is in fact an equivalent one.

[b] The normal and tangential projections in the half space are given by $\nu_+\nu_+^T$ and $I - \nu_+\nu_+^T$, respectively. A vector in the tangent space at some point $x \in \partial H_\omega$ can be written as $(I - \nu\nu^T)u$. Now we have $\nu_+\nu_+^T\nabla\Phi^T = \sqrt{|\nabla'\omega|^2 + 1}\nu_+\nu^T$ and therefore $\nu_+\nu_+^T(\nabla\Phi^T)(I - \nu\nu^T)u = 0$, since $\nu^T\nu = 1$.

[c] $(E(\widetilde{u})^T - E(\widetilde{u}))\nu$ is contained in the tangent space at ∂H_ω.

and

$$\begin{aligned} \widetilde{G} &:= (\nabla\Phi^T)(I-\nu\nu^T)\widetilde{g} + (\nabla\Phi^T)\nu\nu^T\widetilde{h} \in W_q^1(\mathbb{R}^n_+)^n, \\ \widetilde{H} &:= (\nabla\Phi^T)\nu\nu^T\widetilde{h} \in W_q^2(\mathbb{R}^n_+)^n. \end{aligned} \tag{6.16}$$

Now $\widetilde{G}$ and $\widetilde{H}$ are the new right-hand side functions in the boundary condition. Regarding (6.16), we observe that the invertible matrix $\nabla\Phi^T$ is exactly the right matrix to receive the intended regularity for $\widetilde{G}$ and $\widetilde{H}$. Consequently, the proof of Theorem 6.2 is reduced to the following perturbed version of Proposition 6.1.

Proposition 6.3. *Let $n \geqslant 2$, $1 < q < \infty$, $0 < \theta < \pi$, $\omega \in W^3_\infty(\mathbb{R}^{n-1})$ and let $M > 0$ such that (6.3) holds. Then there exist $\kappa = \kappa(n,q,\theta) > 0$ and $\lambda_0 = \lambda_0(n,q,\kappa,M) > 0$ such that in case $\|\nabla'\omega\|_\infty \leqslant \kappa$, $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ for $\widetilde{f} \in L_q(\mathbb{R}^n_+)^n$, $\widetilde{G} \in W^1_q(\mathbb{R}^n_+)^n$ and $\widetilde{H} \in W^2_q(\mathbb{R}^n_+)^n$ there exists a unique solution $\widetilde{u} \in W^2_q(\mathbb{R}^n_+)^n$ of (6.14) and this solution fulfills the resolvent estimate*

$$\|(\lambda\widetilde{u}, \sqrt{\lambda}\nabla\widetilde{u}, \nabla^2\widetilde{u})\|_q \leqslant C\|(\widetilde{f}, \sqrt{\lambda}\widetilde{G}, \nabla\widetilde{G}, \lambda\widetilde{H}, \sqrt{\lambda}\nabla\widetilde{H}, \nabla^2\widetilde{H})\|_q \tag{6.17}$$

where $C = C(n,q,\theta,M) > 0$.

Proof. We prove the statement using a perturbation argument via the Neumann series, where the version we make use of is [47], Lem. 7.10. Therefore, we define the spaces

$$\begin{aligned} X &:= W^2_q(\mathbb{R}^n_+)^n, \\ Y &:= L_q(\mathbb{R}^n_+)^n \times \big\{(I-\nu_+\nu_+^T)\widetilde{G} + \nu_+\nu_+^T\widetilde{H} : \widetilde{G} \in W^1_q(\mathbb{R}^n_+)^n, \widetilde{H} \in W^2_q(\mathbb{R}^n_+)^n\big\} \\ &= L_q(\mathbb{R}^n_+)^n \times \big\{(\widetilde{G}^1, \dots, \widetilde{G}^{n-1}, \widetilde{H}^n)^T : \widetilde{G} \in W^1_q(\mathbb{R}^n_+)^n, \widetilde{H} \in W^2_q(\mathbb{R}^n_+)^n\big\}, \\ Z &:= L_q(\mathbb{R}^n_+)^n \times L_q(\partial\mathbb{R}^n_+)^n \end{aligned}$$

with norms (depending on $\lambda \in \Sigma_\theta$)

$$\begin{aligned} \|\widetilde{u}\|_X &:= \|(\lambda\widetilde{u}, \sqrt{\lambda}\nabla\widetilde{u}, \nabla^2\widetilde{u})\|_q, \\ \|(\widetilde{f}, (I-\nu_+\nu_+^T)\widetilde{G} + \nu_+\nu_+^T\widetilde{H})\|_Y &:= \|(\widetilde{f}, \sqrt{\lambda}\widetilde{G}^1, \dots, \sqrt{\lambda}\widetilde{G}^{n-1}, \\ &\qquad \nabla\widetilde{G}^1, \dots, \nabla\widetilde{G}^{n-1}, \lambda\widetilde{H}^n, \sqrt{\lambda}\nabla\widetilde{H}^n, \nabla^2\widetilde{H}^n)\|_q, \\ \|\cdot\|_Z &:= \|\cdot\|_{L_q(\mathbb{R}^n_+)^n \times L_q(\partial\mathbb{R}^n_+)^n} \end{aligned}$$

as well as the continuous linear operators

$$\begin{aligned} S &: X \longrightarrow Y, \quad \widetilde{u} \mapsto ((\lambda-\Delta)\widetilde{u}, \mathrm{D}_-(\widetilde{u})\nu_+ + \nu_+\nu_+^T\widetilde{u}), \\ P &: X \longrightarrow Y, \quad \widetilde{u} \mapsto (B\widetilde{u}, B'\widetilde{u}), \\ Q &: Y \longrightarrow Z, \quad (\widetilde{f}, \widetilde{k}) \mapsto (\widetilde{f}, \mathrm{tr}_{\partial\mathbb{R}^n_+}\widetilde{k}) \end{aligned}$$

(where B and B' are the operators from (6.8) and (6.15)). By standard arguments we obtain that the space Y is complete so X, Y and Z are Banach spaces. Due to Proposition 6.1, for any $(\widetilde{f}, \widetilde{k}) \in Y$ there exists a unique $\widetilde{u} \in X$ satisfying

$$QS\widetilde{u} = Q(\widetilde{f}, \widetilde{k})$$

and there exists $C = C(n,q,\theta) > 0$ such that

$$\|\widetilde{u}\|_X \leqslant C\|(\widetilde{f}, \widetilde{k})\|_Y.$$

We now aim to show that we can choose $\lambda_0 = \lambda_0(n, q, \kappa, M) > 0$ and a constant $C' = C'(n, M) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ and $\|\nabla'\omega\|_\infty \leqslant \kappa < 1$ we have

$$\|P\widetilde{u}\|_Y \leqslant C'\kappa\|\widetilde{u}\|_X \tag{6.18}$$

for all $\widetilde{u} \in X$. Then, prescribing $\kappa < \frac{1}{2CC'}$, we deduce

$$\|P\|_{X\to Y} \leqslant \frac{1}{2C}$$

and as a consequence (see [47], Lem. 7.10) we receive: For any $(\widetilde{f}, \widetilde{k}) \in Y$ there exists a unique $\widetilde{u} \in X$ satisfying

$$Q(S + P)\widetilde{u} = Q(\widetilde{f}, \widetilde{k})$$

and we have

$$\|\widetilde{u}\|_X \leqslant 2C\|(\widetilde{f}, \widetilde{k})\|_Y.$$

This is exactly the claim of the proposition.

It remains to prove (6.18). For this purpose, we assume $M \geqslant 1$, $\kappa < 1$ and $\lambda_0 \geqslant \frac{M^2}{\kappa^2}$. Let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$, $\|\nabla'\omega\|_\infty \leqslant \kappa$ and $\widetilde{u} \in X$. Then, for the operator B, we have

$$\begin{aligned}\|B\widetilde{u}\|_q &= \Big\| \big(2(\nabla'\omega^T, 0)\cdot\nabla\partial_n\widetilde{u}^j + (\Delta'\omega)\partial_n\widetilde{u}^j - |\nabla'\omega|^2\partial_n^2\widetilde{u}^j\big)_{j=1,\dots,n}\Big\|_q \\ &\leqslant C\Big(\kappa\|\nabla^2\widetilde{u}\|_q + \frac{M}{\sqrt{|\lambda|}}\|\sqrt{\lambda}\nabla\widetilde{u}\|_q + \kappa^2\|\nabla^2\widetilde{u}\|_q\Big) \\ &\leqslant C'\kappa\|\widetilde{u}\|_X\end{aligned}$$

with some constants $C = C(n) > 0$ and $C' = C'(n, M) > 0$. For the operator B' we have (denoting $Y = Y_1 \times Y_2$)

$$\begin{aligned}\|B'\widetilde{u}\|_{Y_2} \leqslant &\Big\|\sqrt{\lambda}\big[(\nabla\Phi^T - I)\mathrm{D}_-(\widetilde{u})\nu_+ + (\nabla\Phi^T)\mathrm{D}_-(\widetilde{u})(\nu - \nu_+) \\ &\quad + ((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u} + (\nabla\Phi^T)(E(\widetilde{u})^T - E(\widetilde{u}))\nu\big]\Big\|_q \\ &+ \Big\|\nabla\big[(\nabla\Phi^T - I)\mathrm{D}_-(\widetilde{u})\nu_+ + (\nabla\Phi^T)\mathrm{D}_-(\widetilde{u})(\nu - \nu_+) \\ &\quad + ((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u} + (\nabla\Phi^T)(E(\widetilde{u})^T - E(\widetilde{u}))\nu\big]\Big\|_q \\ &+ \Big\|\lambda\big[((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u}\big]\Big\|_q \\ &+ \Big\|\sqrt{\lambda}\nabla\big[((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u}\big]\Big\|_q \\ &+ \Big\|\nabla^2\big[((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u}\big]\Big\|_q.\end{aligned} \tag{6.19}$$

Via the triangle inequality we receive eleven different summands in (6.19). Now each of the summands can be estimated by $C'\kappa\|\widetilde{u}\|_X$ with a constant $C' = C'(n, M) > 0$, where all of the estimates can be done in a similar way. One essentially needs that $\|\nu - \nu_+\|_\infty$ and $\|\nabla\Phi^T - I\|_\infty$ may be estimated by κ up to a constant depending only on n, as well as (6.3) and the condition $\lambda_0 \geqslant \frac{M^2}{\kappa^2}$. Then (6.18) is verified. We exemplarily treat three of the terms in (6.19):

$$\begin{aligned}\Big\|\sqrt{\lambda}((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u}\Big\|_q &\leqslant \frac{1}{\sqrt{|\lambda|}}(\|\nabla\Phi\|_\infty\|\nu\|_\infty^2 + 1)\|\lambda\widetilde{u}\|_q \\ &\leqslant C'(n, M)\kappa\|\lambda\widetilde{u}\|_q \\ &\leqslant C'(n, M)\kappa\|\widetilde{u}\|_X,\end{aligned}$$

$$
\begin{aligned}
\Big\|\nabla\big[(\nabla\Phi^T)(E(\widetilde{u})^T - E(\widetilde{u}))\nu\big]\Big\|_q &\leqslant C'(n)\Big(\|(\nabla^2\Phi)(E(\widetilde{u})^T - E(\widetilde{u}))\nu\|_q \\
&\qquad + \sum_{k=1}^n \|(\nabla\Phi^T)[\partial_k(E(\widetilde{u})^T - E(\widetilde{u}))]\nu\|_q \\
&\qquad + \|(\nabla\Phi^T)(E(\widetilde{u})^T - E(\widetilde{u}))\nabla\nu\|_q\Big) \\
&\leqslant C'(n,M)\Big(\kappa\|\nabla\widetilde{u}\|_q + \kappa\|\nabla^2\widetilde{u}\|_q + \|\nabla\widetilde{u}\|_q + \kappa\|\nabla\widetilde{u}\|_q\Big) \\
&\leqslant C'(n,M)\Big(\frac{1}{\sqrt{|\lambda|}}\|\sqrt{\lambda}\nabla\widetilde{u}\|_q + \kappa\|\nabla^2\widetilde{u}\|_q\Big) \\
&\leqslant C'(n,M)\kappa\Big(\|\sqrt{\lambda}\nabla\widetilde{u}\|_q + \|\nabla^2\widetilde{u}\|_q\Big) \\
&\leqslant C'(n,M)\kappa\|\widetilde{u}\|_X, \\
\Big\|\nabla^2\big[((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\widetilde{u}\big]\Big\|_q &\leqslant C'(n)\Big(\frac{1}{|\lambda|}\|\nabla^2((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\|_\infty\|\lambda\widetilde{u}\|_q \\
&\qquad + \frac{1}{\sqrt{|\lambda|}}\|\nabla((\nabla\Phi^T)\nu\nu^T - \nu_+\nu_+^T)\|_\infty\|\sqrt{\lambda}\nabla\widetilde{u}\|_q \\
&\qquad + \|(\nabla\Phi^T)(\nu\nu^T - \nu_+\nu_+^T)\|_\infty\|\nabla^2\widetilde{u}\|_q \\
&\qquad + \|(\nabla\Phi^T - I)\nu_+\nu_+^T\|_\infty\|\nabla^2\widetilde{u}\|_q\Big) \\
&\leqslant C'(n,M)\kappa\|\widetilde{u}\|_X. \qquad \square
\end{aligned}
$$

Proof of Theorem 6.2. For $f \in L_q(H_\omega)^n$, $g \in W_q^1(H_\omega)^n$ and $h \in W_q^2(H_\omega)^n$ we have $\widetilde{f} \in L_q(\mathbb{R}_+^n)^n$, $\widetilde{g} \in W_q^1(\mathbb{R}_+^n)^n$ and $\widetilde{h} \in W_q^2(\mathbb{R}_+^n)^n$ and we define $\widetilde{G} \in W_q^1(\mathbb{R}_+^n)^n$ and $\widetilde{H} \in W_q^2(\mathbb{R}_+^n)^n$ as in (6.16). Choose κ and λ_0 as in Proposition 6.3. Then for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ and $\|\nabla'\omega\|_\infty \leqslant \kappa$ there exists a unique solution $\widetilde{u} \in W_q^2(\mathbb{R}_+^n)^n$ of (6.14), satisfying (6.17). The calculations above give that $u = J_\omega\widetilde{u}$ is the unique solution of (6.4).

Now, assuming $|\lambda| \geqslant 1$, the isomorphism (6.6) gives that $u = J_\omega\widetilde{u}$ fulfills

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_{q,H_\omega} \leqslant C\|(\lambda\widetilde{u}, \sqrt{\lambda}\nabla\widetilde{u}, \nabla^2\widetilde{u})\|_{q,\mathbb{R}_+^n}, \tag{6.20}$$

where $C = C(n,M) > 0$ and on the other hand

$$\|(\widetilde{f}, \sqrt{\lambda}\widetilde{g}, \nabla\widetilde{g}, \lambda\widetilde{h}, \sqrt{\lambda}\nabla\widetilde{h}, \nabla^2\widetilde{h})\|_{q,\mathbb{R}_+^n} \leqslant C\|(f, \sqrt{\lambda}g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_{q,H_\omega}. \tag{6.21}$$

Using (6.11) and (6.6) and assuming $|\lambda| \geqslant 1$ again, we further obtain

$$\|(\widetilde{f}, \sqrt{\lambda}\widetilde{G}, \nabla\widetilde{G}, \lambda\widetilde{H}, \sqrt{\lambda}\nabla\widetilde{H}, \nabla^2\widetilde{H})\|_{q,\mathbb{R}_+^n} \leqslant C\|(\widetilde{f}, \sqrt{\lambda}\widetilde{g}, \nabla\widetilde{g}, \lambda\widetilde{h}, \sqrt{\lambda}\nabla\widetilde{h}, \nabla^2\widetilde{h})\|_{q,\mathbb{R}_+^n}, \tag{6.22}$$

where $C = C(n,M) > 0$. Now (6.17), (6.20), (6.21) and (6.22) yield (6.5). $\square$

6.3 The Bent, Rotated and Shifted Half Space

Theorem 6.4. *Let $Q^T H_\omega + \tau$ be a bent, rotated and shifted half space, i.e., $Q \in \mathbb{R}^{n\times n}$ is a rotation matrix ($Q^T Q = 1$ and $\det Q = 1$) and $\tau \in \mathbb{R}^n$ is some shifting vector. Let $\omega \in W_\infty^3(\mathbb{R}^{n-1})$, $n \geqslant 2$, $1 < q < \infty$ and $0 < \theta < \pi$. Fix $M \geqslant 1$ such that*

$$\|\nabla'\omega\|_\infty, \|\nabla'^2\omega\|_\infty, \|\nabla'^3\omega\|_\infty \leqslant M \tag{6.23}$$

holds. Then there exist $\kappa = \kappa(n,q,\theta) > 0$ and $\lambda_0 = \lambda_0(n,q,\kappa,M) > 0$ such that in case $\|\nabla'\omega\|_\infty \leqslant \kappa$, $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ for $f \in L_q(Q^T H_\omega + \tau)^n$, $g \in W_q^1(Q^T H_\omega + \tau)^n$ and

$h \in W_q^2(Q^T H_\omega + \tau)^n$ there exists a unique solution $u \in W_q^2(Q^T H_\omega + \tau)^n$ of

$$\begin{cases} \lambda u - \Delta u &= f \quad \text{in } Q^T H_\omega + \tau \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial(Q^T H_\omega + \tau) \\ \Pi_\nu u &= \Pi_\nu h \quad \text{on } \partial(Q^T H_\omega + \tau) \end{cases} \tag{6.24}$$

and this solution fulfills the resolvent estimate

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_q \leqslant C\|(f, \sqrt{\lambda} g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_q \tag{6.25}$$

where $C = C(n, q, \theta, M) > 0$.

Proof. We begin by observing that shifting the problem in direction of some vector $\tau \in \mathbb{R}^n$ does not cause any extra difficulties: Using the coordinate shifting $x^\tau := x - \tau$ for $x \in Q^T H_\omega + \tau$ and $u^\tau(x^\tau) := u(x^\tau + \tau)$ for functions u on $Q^T H_\omega + \tau$ we obtain that $x \mapsto x^\tau$ commutes with arbitrary derivative operators and that

$$\|u\|_{k,q,Q^T H_\omega + \tau} = \|u^\tau\|_{k,q,Q^T H_\omega}$$

holds for $k \in \mathbb{N}_0$. In addition, ν^τ is the outward unit normal vector at the boundary of $Q^T H_\omega$ when ν is the outward unit normal vector at the boundary of $Q^T H_\omega + \tau$. Hence, applying $u \mapsto u^\tau$ to (6.24), the resulting problem is an equivalent one. We therefore may assume $\tau = 0$ in (6.24) without loss of generality.

It remains to treat

$$\begin{cases} \lambda u - \Delta u &= f \quad \text{in } Q^T H_\omega \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial(Q^T H_\omega) \\ \Pi_\nu u &= \Pi_\nu h \quad \text{on } \partial(Q^T H_\omega). \end{cases} \tag{6.26}$$

For a vector field u on $Q^T H_\omega$ and $x \in H_\omega$ the chain rule gives

$$\frac{\partial}{\partial x_j} u^k(Q^T x) = \nabla u^k(Q^T x) \cdot q_j$$

for $j, k = 1, \dots, n$, where q_j is the j-th column vector of $Q^T = (q_1, \dots, q_n)$. Consequently

$$\frac{\partial}{\partial x_j} u(Q^T x) = \nabla u(Q^T x)^T q_j$$

for $j = 1, \dots, n$ and therefore, writing $u^Q(x) := u(Q^T x)$, we have

$$\nabla u^Q(x)^T = (\nabla u(Q^T x)^T q_1, \dots, \nabla u(Q^T x)^T q_n) = \nabla u(Q^T x)^T Q^T,$$

i.e., $\left(\nabla u^Q\right)^T = \left([\nabla u]^Q\right)^T Q^T$ and hence

$$\nabla u^Q = Q[\nabla u]^Q. \tag{6.27}$$

Besides, we have

$$\nabla(Q u^Q)^T = Q\left(\nabla u^Q\right)^T. \tag{6.28}$$

Moreover, when ν is the outward unit normal vector at the boundary of $Q^T H_\omega$, then $Q\nu^Q$ is the outward unit normal vector at the boundary of H_ω. Thus, the normal and tangential projections of a vector field u^Q on ∂H_ω are given by

- $\Pi_\nu^Q u^Q := Q(\nu^Q)(\nu^Q)^T Q^T u^Q$ and

- $\Pi_\tau^Q u^Q := Q\big(I - (\nu^Q)(\nu^Q)^T\big)Q^T u^Q$,

respectively. Applying $u \mapsto u^Q$ to the boundary term

$$\mathrm{D}_-(u)\nu = \Pi_\tau \mathrm{D}_-(u)\nu = (I - \nu\nu^T)(\nabla u^T - \nabla u)\nu$$

in (6.26), we obtain, using (6.27),

$$\begin{aligned}
[\mathrm{D}_-(u)\nu]^Q &= [(I - \nu\nu^T)(\nabla u^T - \nabla u)\nu]^Q \\
&= Q^T \Pi_\tau^Q Q\big(\big([\nabla u]^Q\big)^T - [\nabla u]^Q\big)\nu^Q \\
&= Q^T \Pi_\tau^Q Q\big(\big(\nabla u^Q\big)^T Q - \big(\big(\nabla u^Q\big)^T Q\big)^T\big)Q^T Q \nu^Q.
\end{aligned}$$

Now (6.28) gives that we can write

$$\big(\nabla u^Q\big)^T = Q^T \nabla \big(Q u^Q\big)^T,$$

so we receive

$$[\mathrm{D}_-(u)\nu]^Q = Q^T \Pi_\tau^Q \big(\nabla \big(Q u^Q\big)^T - \nabla(Q u^Q)\big)Q\nu^Q.$$

Applying $u \mapsto u^Q$ to the boundary term $\Pi_\nu u$ in (6.26) gives

$$[\Pi_\nu u]^Q = [\nu\nu^T u]^Q = Q^T \Pi_\nu^Q Q u^Q.$$

Furthermore, the orthogonality of Q yields

$$[\Delta u]^Q = \Delta u^Q.$$

In total, application of $u \mapsto u^Q$ and Q to (6.26) yields the equivalent problem

$$\left\{\begin{aligned}
\lambda(Qu^Q) - \Delta(Qu^Q) &= Qf^Q && \text{in } H_\omega \\
\Pi_\tau^Q \big(\nabla \big(Qu^Q\big)^T - \nabla(Qu^Q)\big)Q\nu^Q &= \Pi_\tau^Q Q g^Q && \text{on } \partial H_\omega \\
\Pi_\nu^Q Q u^Q &= \Pi_\nu^Q Q h^Q && \text{on } \partial H_\omega .
\end{aligned}\right. \tag{6.29}$$

Theorem 6.2 yields some $\kappa = \kappa(n, q, \theta) > 0$ and $\lambda_0 = \lambda_0(n, q, \kappa, M) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ and $\|\nabla'\omega\|_\infty \leqslant \kappa$, problem (6.29) has a unique solution $Qu^Q \in W_q^2(H_\omega)^n$ satisfying the related resolvent estimate. Now the transformation $u \mapsto Qu^Q$ is an isomorphism $W_q^k(Q^T H_\omega)^n \xrightarrow{\cong} W_q^k(H_\omega)^n$ for $k = 0, 1, 2$, where the related continuity constants only depend on n, since we need an upper bound for powers of $\|Q\|_\infty$ only. Consequently, $u \in W_q^2(Q^T H_\omega)^n$ is the unique solution of (6.26) and (6.25) holds. □

6.4 The General Case

Theorem 6.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain whose boundary is uniformly $C^{2,1}$, $n \geqslant 2$, $1 < q < \infty$ and $0 < \theta < \pi$. Then there exist $\lambda_0 = \lambda_0(n, q, \theta, \Omega) > 0$ and $C = C(n, q, \theta, \Omega) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ the problem*

$$\left\{\begin{aligned}
\lambda u - \Delta u &= f && \text{in } \Omega \\
\mathrm{D}_-(u)\nu &= \Pi_\tau g && \text{on } \partial\Omega \\
\Pi_\nu u &= \Pi_\nu h && \text{on } \partial\Omega
\end{aligned}\right. \tag{6.30}$$

has a unique solution $u \in W_q^2(\Omega)^n$ for any $f \in L_q(\Omega)^n$, $g \in W_q^1(\Omega)^n$ and $h \in W_q^2(\Omega)^n$ and this solution fulfills the resolvent estimate

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_q \leqslant C\|(f, \sqrt{\lambda} g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_q. \tag{6.31}$$

In particular, the operator Δ_{PS}, defined in (1.2)*, is the generator of a strongly continuous analytic semigroup on $L_q(\Omega)^n$.*

Proof. Due to separation of tangential and normal part on the boundary, we can rewrite (6.30), using (2.15), as

$$\begin{cases} \lambda u - \Delta u &= f \qquad\qquad \text{in } \Omega \\ \mathrm{D}_-(u)\nu + \Pi_\nu u &= \Pi_\tau g + \Pi_\nu h \quad \text{on } \partial\Omega. \end{cases} \tag{6.32}$$

Moreover, we introduce the Banach space for the boundary functions in (6.32),

$$\begin{aligned} \mathrm{BF}_q(\partial\Omega) &= \mathrm{BF}_{q,\lambda}(\partial\Omega) \\ &:= \big\{ a \in L_q(\partial\Omega)^n : a = \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h,\ g \in W_q^1(\Omega)^n,\ h \in W_q^2(\Omega)^n \big\}, \end{aligned}$$

with norm

$$\|a\|_{\mathrm{BF}_{q,\lambda}(\partial\Omega)} := \inf_{g,h} \|(\sqrt{\lambda} g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_q,$$

where the infimum runs over all $g \in W_q^1(\Omega)^n$, $h \in W_q^2(\Omega)^n$ such that $a = \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h$. For $\lambda = 1$ the space $\mathrm{BF}_q(\partial\Omega)$ is therefore equipped with the natural norm for the range of the continuous linear operator $T : W_q^1(\Omega)^n \times W_q^2(\Omega)^n \to L_q(\partial\Omega)^n$, $(g,h) \mapsto \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h$. We allow arbitrary $\lambda \in \Sigma_\theta$ in the definition of $\|\cdot\|_{\mathrm{BF}_{q,\lambda}(\partial\Omega)}$, since we will need this for a perturbation argument later on.

Step 1: Local coordinates. For the sake of consistent notation we denote

$$\Omega_l := \begin{cases} H_l, & l \in \Gamma_1 \\ \mathbb{R}^n, & l \in \Gamma_0 \end{cases}$$

and hence by the space $\mathrm{BF}_q(\partial\Omega_l)$ we mean $\mathrm{BF}_q(\partial\Omega_l) = \mathrm{BF}_q(\partial H_l)$ for $l \in \Gamma_1$ and $\mathrm{BF}_q(\partial\Omega_l) := \{0\}$ for $l \in \Gamma_0$. We introduce the Banach spaces

$$\begin{aligned} X &:= l_q\Big(\bigoplus_{l\in\Gamma} W_q^2(\Omega_l)^n\Big), \\ Y &:= l_q\Big(\bigoplus_{l\in\Gamma} L_q(\Omega_l)^n\Big) \times l_q\Big(\bigoplus_{l\in\Gamma} \mathrm{BF}_q(\partial\Omega_l)\Big) \end{aligned}$$

with norms (depending on $\lambda \in \Sigma_\theta$)

$$\begin{aligned} \|(u_l)_{l\in\Gamma}\|_X &:= \|(\lambda u_l, \sqrt{\lambda}\nabla u_l, \nabla^2 u_l)_{l\in\Gamma}\|_{l_q(L_q)}, \\ \|(f_l, a_l)_{l\in\Gamma}\|_Y &:= \|(f_l)_{l\in\Gamma}\|_{l_q(L_q)} + \|(a_l)_{l\in\Gamma}\|_{l_q(\mathrm{BF}_{q,\lambda})} \end{aligned}$$

as well as the linear and continuous operator

$$S : X \longmapsto Y, \quad (u_l)_{l\in\Gamma} \longmapsto \big((\lambda - \Delta)u_l, \operatorname{tr}_{\partial\Omega_l} \mathrm{D}_-(u_l)\nu_l + \nu_l\nu_l^T \operatorname{tr}_{\partial\Omega_l} u_l\big)_{l\in\Gamma},$$

where we set $\operatorname{tr}_{\partial\Omega_l} \mathrm{D}_-(u_l)\nu_l + \nu_l\nu_l^T \operatorname{tr}_{\partial\Omega_l} u_l := 0$ in case $l \in \Gamma_0$.

For the bent, rotated and shifted half space $H_l = Q_l^T H_{\omega_l} + \tau_l$, $l \in \Gamma_1$ and the related constant $M \geqslant 1$ from (2.1), let initially $\kappa = \kappa(n,q,\theta) > 0$ and $\lambda_0 = \lambda_0(n,q,\kappa,M) > 0$ such that the conditions of Theorem 6.4 are satisfied. We further assume $\kappa < 1$ and $\lambda_0 \geqslant \frac{M^2}{\kappa^2}$. Let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ and note that (2.2) gives $\|\nabla'\omega_l\|_\infty \leqslant \kappa$ for all $l \in \Gamma_1$.

Now we can deduce from Theorem 6.4 that we have an isomorphism

$$S : X \xrightarrow{\cong} Y \tag{6.33}$$

and the continuity constants of S and S^{-1} depend on q, n, θ and M only. For this purpose, fix some $(f_l, a_l)_{l\in\Gamma} \in Y$. Then, for all $l \in \Gamma_1$, Theorem 6.4 yields a unique $u_l \in W_q^2(H_l)^n$, such that $(\lambda-\Delta)u_l = f_l$ and $\operatorname{tr}_{\partial H_l} \mathrm{D}_-(u_l)\nu_l + \nu_l\nu_l^T \operatorname{tr}_{\partial H_l} u_l = a_l$. For $l \in \Gamma_0$, existence and uniqueness of the solution $u_l \in W_q^2(\mathbb{R}^n)^n$ to $(\lambda - \Delta)u_l = f_l$ is due to the heat equation admitting a strongly continuous bounded analytic semigroup in the whole space (see, e.g., [47]). In addition, we have a constant $C = C(n, q, \theta, M) > 0$ such that

$$\|(\lambda u_l, \sqrt{\lambda}\nabla u_l, \nabla^2 u_l)\|_{q,H_l} \leqslant C\|(f_l, \sqrt{\lambda} g_l, \nabla g_l, \lambda h_l, \sqrt{\lambda}\nabla h_l, \nabla^2 h_l)\|_{q,H_l} \tag{6.34}$$

holds for all $l \in \Gamma_1$ and arbitrary $g_l \in W_q^1(H_l)^n$, $h_l \in W_q^2(H_l)^n$ such that $a_l = \Pi_\tau \operatorname{tr}_{\partial H_l} g_l + \Pi_\nu \operatorname{tr}_{\partial H_l} h_l$ as well as for all $l \in \Gamma_0$ by putting $g_l = h_l = 0$. Consequently, for $l \in \Gamma$, we have

$$\|(\lambda u_l, \sqrt{\lambda}\nabla u_l, \nabla^2 u_l)\|_{q,\Omega_l} \leqslant C\big(\|f_l\|_{q,\Omega_l} + \|a_l\|_{\mathrm{BF}_{q,\lambda}(\partial\Omega_l)}\big). \tag{6.35}$$

Thus

$$\begin{aligned}\|(u_l)_{l\in\Gamma}\|_X^q &= \sum_{l\in\Gamma} \|(\lambda u_l, \sqrt{\lambda}\nabla u_l, \nabla^2 u_l)\|_{q,\Omega_l}^q \\ &\leqslant C^q \sum_{l\in\Gamma} \big(\|f_l\|_{q,\Omega_l} + \|a_l\|_{\mathrm{BF}_{q,\lambda}(\partial\Omega_l)}\big)^q \\ &\leqslant C_S^q \|(f_l, a_l)_{l\in\Gamma}\|_Y^q,\end{aligned} \tag{6.36}$$

where $C_S = C_S(n, q, \theta, M) > 0$. On the other hand, it is not hard to see that

$$\|S(u_l)_{l\in\Gamma}\|_Y \leqslant C'\|(u_l)_{l\in\Gamma}\|_X$$

holds for arbitrary $(u_l)_{l\in\Gamma} \in X$, where $C' = C'(n, q) > 0$. Hence, (6.33) is verified.

Step 2: Localizing (6.30). We now multiply (6.32) by the functions φ_l, $l \in \Gamma$ in order to receive corresponding local equations. If φ is a scalar function and u is a vector field, then the product rule yields the matrix identity[d]

$$\nabla(\varphi u)^T = u\nabla\varphi^T + \varphi\nabla u^T \tag{6.37}$$

and the vector identity

$$\Delta(\varphi u) = (\Delta\varphi)u + 2(\nabla u^T)\nabla\varphi + \varphi\Delta u. \tag{6.38}$$

Thus, writing $u_m = \varphi_m u$ and using (2.3), we have

$$\begin{aligned}&\varphi_l(\lambda - \Delta)u \\ &= (\lambda - \Delta)(\varphi_l u) + 2(\nabla u^T)\nabla\varphi_l + (\Delta\varphi_l)u \\ &= (\lambda - \Delta)(\varphi_l u) + 2\Big(\nabla \sum_{m\in\Gamma} \varphi_m^2 u\Big)^T \nabla\varphi_l + (\Delta\varphi_l) \sum_{m\in\Gamma} \varphi_m^2 u \\ &= (\lambda - \Delta)u_l + \sum_{m\sim l} \big[2u_m(\nabla\varphi_m{}^T)\nabla\varphi_l + 2\varphi_m(\nabla u_m{}^T)\nabla\varphi_l + (\Delta\varphi_l)\varphi_m u_m\big].\end{aligned}$$

For the tangential boundary condition in (6.32) we have (note that $\nu = \nu_l$ on $\operatorname{spt}(\varphi_l)$ for $l \in \Gamma_1$), using (2.3), (6.37) and writing $u_m = \varphi_m u$ again,

$$\begin{aligned}\varphi_l \mathrm{D}_-(u)\nu &= (\varphi_l\nabla u^T - \varphi_l\nabla u)\nu_l \\ &= (\nabla u_l{}^T - \nabla u_l)\nu_l - u(\nabla\varphi_l{}^T)\nu_l + (\nabla\varphi_l)u^T\nu_l \\ &= \mathrm{D}_-(u_l)\nu_l - \sum_{m\in\Gamma} \varphi_m^2 u(\nabla\varphi_l{}^T)\nu_l + \sum_{m\in\Gamma} \varphi_m^2(\nabla\varphi_l)u^T\nu_l \\ &= \mathrm{D}_-(u_l)\nu_l - \sum_{m\sim l} \big[\varphi_m u_m(\nabla\varphi_l{}^T)\nu_l - \varphi_m(\nabla\varphi_l)u_m^T\nu_l\big] \\ &= \mathrm{D}_-(u_l)\nu_l - \sum_{m\sim l} \varphi_m\big[u_m\nabla\varphi_l{}^T - (u_m\nabla\varphi_l{}^T)^T\big]\nu_l\end{aligned}$$

[d] $\nabla(\varphi u)^T = \big(\partial_j(\varphi u^i)\big)_{i,j=1,\dots,n} = \big(u^i\partial_j\varphi + \varphi\partial_j u^i\big)_{i,j=1,\dots,n} = u\nabla\varphi^T + \varphi\nabla u^T$

and for the normal boundary condition we have

$$\varphi_l \Pi_\nu u = \varphi_l \nu_l \nu_l^T u = \nu_l \nu_l^T u_l$$

for $l \in \Gamma_1$. In total, multiplying (6.32) by φ_l for $l \in \Gamma$ and writing $u_m = \varphi_m u$ yields the local equations

$$\begin{cases} \lambda u_l - \Delta u_l + \sum_{m\sim l} \big[2u_m (\nabla \varphi_m{}^T) \nabla \varphi_l + 2\varphi_m (\nabla u_m{}^T) \nabla \varphi_l + (\Delta \varphi_l) \varphi_m u_m \big] \\ = f_l \quad \text{in } \Omega_l \text{ for all } l \in \Gamma, \\ \mathrm{D}_-(u_l)\nu_l + \nu_l \nu_l^T u_l - \sum_{m \approx l} \varphi_m \big[u_m \nabla \varphi_l{}^T - (u_m \nabla \varphi_l{}^T)^T \big] \nu_l \\ = (I - \nu_l \nu_l^T) g_l + \nu_l \nu_l^T h_l \quad \text{on } \partial\Omega_l \text{ for all } l \in \Gamma_1. \end{cases} \tag{6.39}$$

Therefore, we define the perturbation operator $P : X \longrightarrow Y$ by

$$\begin{aligned} (u_l)_{l\in\Gamma} \longmapsto \Big(\sum_{m\sim l} & \big[2u_m (\nabla \varphi_m{}^T) \nabla \varphi_l + 2\varphi_m (\nabla u_m{}^T) \nabla \varphi_l + (\Delta \varphi_l) \varphi_m u_m \big], \\ & - \operatorname{tr}_{\partial\Omega_l} \sum_{m\approx l} \varphi_m \big[u_m \nabla \varphi_l{}^T - (u_m \nabla \varphi_l{}^T)^T \big] \nu_l \Big)_{l\in\Gamma}, \end{aligned}$$

where in case $l \in \Gamma_0$ we set $\operatorname{tr}_{\partial\Omega_l} \sum_{m\approx l} \varphi_m \big[u_m \nabla \varphi_l{}^T - (u_m \nabla \varphi_l{}^T)^T \big] \nu_l := 0$ again.

Step 3: Well-posedness of local equations. We now aim to verify that there exists $C_P = C_P(n, q, \Omega) > 0$ such that

$$\| P(u_l)_{l\in\Gamma} \|_Y \leqslant \frac{C_P}{\sqrt{|\lambda|}} \| (u_l)_{l\in\Gamma} \|_X \tag{6.40}$$

holds for all $(u_l)_{l\in\Gamma} \in X$ and for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$. For this purpose, let $(u_l)_{l\in\Gamma} \in X$. Then for all $l \in \Gamma$ we have, using (2.4),

$$\begin{aligned} \Big\| \sum_{m\sim l} 2u_m (\nabla \varphi_m{}^T) \nabla \varphi_l \Big\|_{q,\Omega_l}^q &= \int_{\Omega_l} \Big| \sum_{m\sim l} 2u_m (\nabla \varphi_m{}^T) \nabla \varphi_l \Big|^q \, d\lambda_n \\ &\leqslant C \sum_{m\sim l} \int_{\Omega_l \cap B_l \cap B_m} \Big| u_m (\nabla \varphi_m{}^T) \nabla \varphi_l \Big|^q \, d\lambda_n \\ &= C \sum_{m\sim l} \int_{\Omega_m \cap B_l \cap B_m} \Big| u_m (\nabla \varphi_m{}^T) \nabla \varphi_l \Big|^q \, d\lambda_n \\ &\leqslant C' \sum_{m\sim l} \int_{\Omega_m \cap B_m} |u_m|^q \, d\lambda_n \\ &= C' \sum_{m\sim l} \| u_m \|_{q,\Omega_m \cap B_m}^q \end{aligned} \tag{6.41}$$

with constants $C = C(n,q) > 0$ and $C' = C'(n,q,\Omega) > 0$, where we also used (2.4) and that the support of the function $u_m (\nabla \varphi_m{}^T) \nabla \varphi_l$ is contained in $B_m \cap B_l$. Since at most $\bar{N}$ of the balls B_l have nonempty intersection, we deduce

$$\begin{aligned} \Big\| \Big(\sum_{m\sim l} 2u_m (\nabla \varphi_m{}^T) \nabla \varphi_l \Big)_{l\in\Gamma} \Big\|_{l_q(L_q)}^q &\leqslant C' \sum_{l\in\Gamma} \sum_{m\sim l} \| u_m \|_{q,\Omega_m \cap B_m}^q \\ &\leqslant C'' \sum_{l\in\Gamma} \| u_l \|_{q,\Omega_l \cap B_l}^q \\ &\leqslant C'' \sum_{l\in\Gamma} \| u_l \|_{q,\Omega_l}^q, \end{aligned}$$

where $C'' = C''(n,q,\Omega) > 0$. In the same way we obtain

$$\Big\|\Big(\sum_{m\sim l}(\Delta\varphi_l)\varphi_m u_m\Big)_{l\in\Gamma}\Big\|_{l_q(L_q)}^q \leqslant C''\sum_{l\in\Gamma}\|u_l\|_{q,\Omega_l}^q$$

and

$$\Big\|\Big(\sum_{m\sim l}2\varphi_m(\nabla {u_m}^T)\nabla\varphi_l\Big)_{l\in\Gamma}\Big\|_{l_q(L_q)}^q \leqslant C''\sum_{l\in\Gamma}\|\nabla u_l\|_{q,\Omega_l}^q.$$

In total, by the definition of the norm in X, we have a constant $C_P = C_P(n,q,\Omega) > 0$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{m\sim l}\big[2u_m(\nabla{\varphi_m}^T)\nabla\varphi_l + 2\varphi_m(\nabla{u_m}^T)\nabla\varphi_l + (\Delta\varphi_l)\varphi_m u_m\big]\Big)_{l\in\Gamma}\Big\|_{l_q(L_q)}\\
&\leqslant C_P\|(u_l,\nabla u_l)_{l\in\Gamma}\|_{l_q(L_q)}\\
&\leqslant C_P\|(\sqrt{\lambda}u_l,\nabla u_l)_{l\in\Gamma}\|_{l_q(L_q)}\\
&\leqslant \frac{C_P}{\sqrt{|\lambda|}}\|(u_l)_{l\in\Gamma}\|_X
\end{aligned}$$

(note that the condition $\lambda_0 \geqslant \frac{M^2}{\kappa^2}$ yields $|\lambda| \geqslant 1$). In order to treat the boundary term of P, we make use of the extension $\bar\nu_l \in W^2_\infty(H_l)^n$ of the outward unit normal vector ν_l for H_l, which satisfies (2.12): For $l \in \Gamma_1$, a function $g_l \in W^1_q(H_l)^n$ satisfying $\operatorname{tr}_{\partial H_l} g_l = \operatorname{tr}_{\partial H_l}\sum_{m\approx l}\varphi_m\big[u_m\nabla{\varphi_l}^T - (u_m\nabla{\varphi_l}^T)^T\big]\nu_l$ is given by $g_l := \sum_{m\approx l}\varphi_m\big[u_m\nabla{\varphi_l}^T - (u_m\nabla{\varphi_l}^T)^T\big]\bar\nu_l$. Note that $\operatorname{tr}_{\partial H_l} g_l$ is contained in the tangent space at ∂H_l, since

$$\nu_l\nu_l^T\operatorname{tr}_{\partial H_l} g_l = 0.$$

We further obtain, similar to (6.41) but additionally using (2.12),

$$\|(\sqrt{\lambda}g_l,\nabla g_l)\|_{q,H_l}^q \leqslant C\sum_{m\approx l}\int_{H_m\cap B_m}(|\sqrt{\lambda}u_m|^q + |\nabla u_m|^q)\,d\lambda_n,$$

where again $C = C(n,q,\Omega) > 0$. Consequently, we receive

$$\begin{aligned}
&\Big\|\Big(\operatorname{tr}_{\partial\Omega_l}\sum_{m\approx l}\varphi_m\big[u_m\nabla{\varphi_l}^T - (u_m\nabla{\varphi_l}^T)^T\big]\nu_l\Big)_{l\in\Gamma}\Big\|_{l_q(BF_{q,\lambda})}^q\\
&= \sum_{l\in\Gamma_1}\Big\|\operatorname{tr}_{\partial H_l}\sum_{m\approx l}\varphi_m\big[u_m\nabla{\varphi_l}^T - (u_m\nabla{\varphi_l}^T)^T\big]\nu_l\Big\|_{BF_{q,\lambda}(\partial H_l)}^q\\
&\leqslant \sum_{l\in\Gamma_1}\|(\sqrt{\lambda}g_l,\nabla g_l)\|_{q,H_l}^q\\
&\leqslant C\sum_{l\in\Gamma_1}\sum_{m\approx l}\int_{H_m\cap B_m}(|\sqrt{\lambda}u_m|^q + |\nabla u_m|^q)\,d\lambda_n\\
&\leqslant C'\sum_{l\in\Gamma_1}\int_{H_l\cap B_l}(|\sqrt{\lambda}u_l|^q + |\nabla u_l|^q)\,d\lambda_n
\end{aligned}\tag{6.42}$$

with some constant $C' = C'(n,q,\Omega) > 0$ and therefore

$$\Big\|\Big(\operatorname{tr}_{\partial\Omega_l}\sum_{m\approx l}\varphi_m\big[u_m\nabla{\varphi_l}^T - (u_m\nabla{\varphi_l}^T)^T\big]\nu_l\Big)_{l\in\Gamma}\Big\|_{l_q(BF_{q,\lambda})} \leqslant \frac{C_P}{\sqrt{|\lambda|}}\|(u_l)_{l\in\Gamma}\|_X,$$

where $C_P = C_P(n,q,\Omega) > 0$. Hence (6.40) is proved.

We now increase $\lambda_0 = \lambda_0(n, q, \theta, \Omega)$[e] such that $\lambda_0 \geqslant (2C_S C_P)^2$, where C_P is the constant from (6.40) and C_S is the constant from (6.36). Then we have

$$\|P\|_{X\to Y} \leqslant \frac{1}{2C_S} \tag{6.43}$$

so the Neumann series gives that we have an isomorphism

$$S + P : X \xrightarrow{\cong} Y \tag{6.44}$$

so that

$$\|(S+P)^{-1}\|_{Y\to X} \leqslant C_S \frac{1}{1 - C_S\|P\|_{X\to Y}} \leqslant 2C_S. \tag{6.45}$$

Now (6.44) gives that (6.39) is uniquely solvable for any right-hand side functions $f_l \in L_q(\Omega_l)^n$, $g_l \in W_q^1(\Omega)^n$ and $h_l \in W_q^2(\Omega)^n$ such that $(f_l)_{l\in\Gamma} \in l_q(\bigoplus_{l\in\Gamma} L_q(\Omega_l)^n)$ and $(a_l)_{l\in\Gamma} \in l_q(\bigoplus_{l\in\Gamma} \mathrm{BF}_q(\partial\Omega_l))$ for $a_l := (I - \nu_l\nu_l^T)\operatorname{tr}_{\partial H_l} g_l + \nu_l\nu_l^T \operatorname{tr}_{\partial H_l} h_l$ $(l \in \Gamma_1)$ and $a_l := 0$ $(l \in \Gamma_0)$, respectively. Moreover, (6.45) yields the related resolvent estimate.

Step 4: Uniqueness and resolvent estimate. We briefly convince ourselves that we have proved uniqueness for (6.30) as well as the related resolvent estimate (6.31). For any solution $u \in W_q^2(\Omega)$ of (6.30) we have seen that $(u_l)_{l\in\Gamma} := (\varphi_l u)_{l\in\Gamma}$ solves the local equations (6.39) with right-hand side functions $(f_l)_{l\in\Gamma} := (\varphi_l f)_{l\in\Gamma}$, $(g_l)_{l\in\Gamma_1} := (\varphi_l g)_{l\in\Gamma_1}$ and $(h_l)_{l\in\Gamma_1} := (\varphi_l h)_{l\in\Gamma_1}$. Now (6.39) is uniquely solvable, so we have uniqueness for (6.30). For $a_l := (I - \nu_l\nu_l^T)\operatorname{tr}_{\partial H_l} g_l + \nu_l\nu_l^T \operatorname{tr}_{\partial H_l} h_l$ if $l \in \Gamma_1$ and $a_l := 0$ if $l \in \Gamma_0$, we have $(S+P)(u_l)_{l\in\Gamma} = (f_l, a_l)_{l\in\Gamma}$. Estimate (6.45) therefore implies

$$\|(u_l)_{l\in\Gamma}\|_X \leqslant 2C_S\|(f_l, a_l)_{l\in\Gamma}\|_Y \leqslant 2C_S\|(f_l, \sqrt{\lambda}g_l, \nabla g_l, \lambda h_l, \sqrt{\lambda}\nabla h_l, \nabla^2 h_l)_{l\in\Gamma}\|_{l_q(L_q)}. \tag{6.46}$$

It remains to prove existence of some constant $C = C(n, q, \Omega) > 0$ so that

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_{q,\Omega} \leqslant C\|(u_l)_{l\in\Gamma}\|_X \tag{6.47}$$

and

$$\|(f_l, \sqrt{\lambda}g_l, \nabla g_l, \lambda h_l, \sqrt{\lambda}\nabla h_l, \nabla^2 h_l)_{l\in\Gamma}\|_{l_q(L_q)} \leqslant C\|(f, \sqrt{\lambda}g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_{q,\Omega}. \tag{6.48}$$

For $u \in W_q^2(\Omega)^n$ and $u_m := \varphi_m u$ we have

$$\begin{aligned}
\|\lambda u\|_{q,\Omega}^q &= |\lambda|^q \int_\Omega \Big|\sum_{m\in\Gamma} \varphi_m u_m\Big|^q \, d\lambda_n \\
&= |\lambda|^q \int_\Omega \sum_{l\in\Gamma} \varphi_l^2 \Big|\sum_{m\sim l} \varphi_m u_m\Big|^q \, d\lambda_n \\
&\leqslant C|\lambda|^q \int_\Omega \sum_{l\in\Gamma}\sum_{m\sim l} |\varphi_m u_m|^q \, d\lambda_n \\
&\leqslant C'|\lambda|^q \int_\Omega \sum_{l\in\Gamma} |\varphi_l u_l|^q \, d\lambda_n \\
&= C'|\lambda|^q \sum_{l\in\Gamma} \int_{\Omega_l \cap B_l} |\varphi_l u_l|^q \, d\lambda_n \\
&\leqslant C'|\lambda|^q \sum_{l\in\Gamma} \int_{\Omega_l} |u_l|^q \, d\lambda_n \\
&= C'\|(\lambda u_l)_{l\in\Gamma}\|_{l_q(L_q)}^q,
\end{aligned} \tag{6.49}$$

[e] The constants M and κ only depend on the domain Ω so we do not need to specify the dependence of λ_0 on M and κ.

where $C = C(n, q, \Omega) > 0$ and $C' = C'(n, q, \Omega) > 0$. Similarly, using (2.4), we obtain

$$\|\sqrt{\lambda}\nabla u\|_{q,\Omega}^q \leqslant C\|(\sqrt{\lambda}u_l, \sqrt{\lambda}\nabla u_l)_{l\in\Gamma}\|_{l_q(L_q)}^q$$

and

$$\|\nabla^2 u\|_{q,\Omega}^q \leqslant C\|(u_l, \nabla u_l, \nabla^2 u_l)_{l\in\Gamma}\|_{l_q(L_q)}^q$$

with some constant $C = C(n, q, \Omega) > 0$. In total we obtain (6.47), since $|\lambda| \geqslant 1$.

For $f \in L_q(\Omega)^n$ and $f_l := \varphi_l f$ we have

$$\begin{aligned}\|(f_l)_{l\in\Gamma}\|_{l_q(L_q)}^q &= \sum_{l\in\Gamma}\int_{\Omega_l} |\varphi_l f|^q \, d\lambda_n \\ &\leqslant \sum_{l\in\Gamma}\int_{\Omega_l\cap B_l} |f|^q \, d\lambda_n \\ &= \sum_{l\in\Gamma}\int_{\Omega\cap B_l} |f|^q \, d\lambda_n \\ &\leqslant C\|f\|_{q,\Omega}^q,\end{aligned}$$

where $C = C(n, q, \Omega) > 0$. Using $|\lambda| \geqslant 1$ and (2.4) again, we obtain similarly

$$\|(\sqrt{\lambda}g_l, \nabla g_l, \lambda h_l, \sqrt{\lambda}\nabla h_l, \nabla^2 h_l)_{l\in\Gamma}\|_{l_q(L_q)}^q \leqslant C\|(\sqrt{\lambda}g, \nabla g, \lambda h, \sqrt{\lambda}\nabla h, \nabla^2 h)\|_{q,\Omega}^q$$

with some constant $C = C(n, q, \Omega) > 0$. Hence (6.48) is proved. In total, (6.46), (6.47) and (6.48) imply (6.31).

Step 5: Existence. As a last step we need to prove existence of a solution to (6.30). For this purpose we introduce the notation $\bar{D}v := (\varphi_l v)_{l\in\Gamma}$ for functions v on Ω and $\bar{C}(v_l)_{l\in\Gamma} := \sum_{l\in\Gamma}\varphi_l v_l$ for sequences $(v_l)_{l\in\Gamma}$ of functions v_l on Ω_l. If v is a function on $\partial\Omega$, then we still write $\varphi_l v$ for the restriction $(\varphi_l|_{\partial\Omega})v$ so that $\bar{D}v$ is a sequence of functions on $\partial\Omega$ and similarly, if v_l, $l \in \Gamma$ are functions on $\partial\Omega_l$ (in particular $v_l = 0$ for $l \in \Gamma_0$), then $\bar{C}(v_l)_{l\in\Gamma}$ is a function on $\partial\Omega$. We further denote $R_\Omega u := \operatorname{tr}_{\partial\Omega}\mathrm{D}_-(u)\nu + \Pi_\nu \operatorname{tr}_{\partial\Omega} u$.

Note that there is no way to deduce existence of a solution to (6.30) from the verified unique solvability of the local equations (6.39) by abstract means. In fact we have proved that for any solution $u \in W_q^2(\Omega)^n$ of (6.30) the representation

$$(f, \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h) = \bar{C}(S + P)\bar{D}u \tag{6.50}$$

holds and $S + P$ is invertible. Still, at this point, the lack of invertibility for $\bar{C}$ makes it untransparent to decide whether it is possible to invert $\bar{C}(S+P)\bar{D}$. The identity $\bar{C}\bar{D}v = v$ holds for arbitrary functions v on Ω but, conversely, the identity $\bar{D}\bar{C}(v_l)_{l\in\Gamma} = (v_l)_{l\in\Gamma}$ can only be guaranteed if the sequence $(v_l)_{l\in\Gamma}$ is of the form $v_l = \varphi_l v$ for some function v on Ω. Unfortunately, for $u \in W_q^2(\Omega)$, we do not know the sequence $(f_l, a_l)_{l\in\Gamma} := (S + P)\bar{D}u$ to be of the form $(f_l, a_l) = (\varphi_l f, \varphi_l a)$. Hence, the ansatz $\bar{C}(S + P)^{-1}\bar{D}(f, \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h)$ might not lead to a solution. Instead, we have to find a substitute P' for P such that the converse identity for (6.50), i.e.,

$$u = \bar{C}(S + P')^{-1}\bar{D}(f, \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h) \tag{6.51}$$

holds. In order to find such a perturbation $P' : X \to Y$, we assume for a moment that the (unknown) operator $S + P' : X \to Y$ is an isomorphism. Then our purpose (6.51) gives

$$\begin{aligned}&(\lambda - \Delta, R_\Omega)\bar{C}(S + P')^{-1}\bar{D}(f, \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h) \\ &= (f, \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h) \\ &= \bar{C}(S + P')(S + P')^{-1}\bar{D}(f, \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h),\end{aligned}$$

so that consequently

$$(\lambda - \Delta, R_\Omega)\bar{C} = \bar{C}(S + P') \tag{6.52}$$

must be satisfied. We therefore compute $\bar{C}S - ((\lambda - \Delta), R_\Omega)\bar{C}$ now: For $(u_l)_{l\in\Gamma} \in X$ we have, using (6.38),

$$\begin{aligned}
\sum_{l\in\Gamma} \varphi_l(\lambda - \Delta)u_l - (\lambda - \Delta)\sum_{l\in\Gamma} \varphi_l u_l &= \sum_{l\in\Gamma} \big[\varphi_l(\lambda - \Delta)u_l - (\lambda - \Delta)(\varphi_l u_l)\big] \\
&= \sum_{l\in\Gamma} \big[(\Delta\varphi_l)u_l + 2(\nabla u_l)\nabla\varphi_l^T\big] \\
&= \sum_{m\in\Gamma} \varphi_m^2 \sum_{l\sim m} \big[(\Delta\varphi_l)u_l + 2(\nabla u_l{}^T)\nabla\varphi_l\big] \\
&= \bar{C}\Big(\varphi_l \sum_{m\sim l} \big[(\Delta\varphi_m)u_m + 2(\nabla u_m{}^T)\nabla\varphi_m\big]\Big)_{l\in\Gamma}
\end{aligned}$$

and the identity $\nu = \nu_l$ on $\partial\Omega \cap B_l$ as well as (6.37) yield

$$\begin{aligned}
&\sum_{l\in\Gamma} \big[\operatorname{tr}_{\partial\Omega_l} \varphi_l \mathrm{D}_-(u_l)\nu_l + \nu_l\nu_l^T \operatorname{tr}_{\partial\Omega_l} \varphi_l u_l\big] - \big[\operatorname{tr}_{\partial\Omega} \mathrm{D}_-\Big(\sum_{l\in\Gamma}\varphi_l u_l\Big)\nu + \nu\nu^T \operatorname{tr}_{\partial\Omega} \sum_{l\in\Gamma}\varphi_l u_l\big] \\
&= \sum_{l\in\Gamma} \operatorname{tr}_{\partial\Omega_l} \varphi_l \mathrm{D}_-(u_l)\nu_l - \operatorname{tr}_{\partial\Omega} \mathrm{D}_-\Big(\sum_{l\in\Gamma}\varphi_l u_l\Big)\nu \\
&= \sum_{l\in\Gamma_1} \big[\operatorname{tr}_{\partial\Omega_l} \varphi_l(\nabla u_l{}^T - \nabla u_l)\nu_l - \operatorname{tr}_{\partial\Omega}(\nabla(\varphi_l u_l)^T - \nabla(\varphi_l u_l))\nu\big] \\
&= -\sum_{l\in\Gamma_1} \operatorname{tr}_{\partial\Omega_l}(u_l\nabla\varphi_l{}^T - (u_l\nabla\varphi_l{}^T)^T)\nu_l \\
&= -\sum_{m\in\Gamma}\sum_{l\approx m} \operatorname{tr}_{\partial\Omega_l} \varphi_m^2(u_l\nabla\varphi_l{}^T - (u_l\nabla\varphi_l{}^T)^T)\nu_l \\
&= \bar{C}\Big(-\sum_{m\approx l} \operatorname{tr}_{\partial\Omega_m} \varphi_l(u_m\nabla\varphi_m{}^T - (u_m\nabla\varphi_m{}^T)^T)\nu_m\Big)_{l\in\Gamma} \\
&= \bar{C}\Big(-\operatorname{tr}_{\partial\Omega_l} \sum_{m\approx l} \varphi_l(u_m\nabla\varphi_m{}^T - (u_m\nabla\varphi_m{}^T)^T)\nu_m\Big)_{l\in\Gamma}.
\end{aligned}$$

Therefore, we define $P' : X \longrightarrow Y$ by

$$\begin{aligned}
(u_l)_{l\in\Gamma} \longmapsto \Big(-\varphi_l \sum_{m\sim l} &\big[(\Delta\varphi_m)u_m + 2(\nabla u_m{}^T)\nabla\varphi_m\big], \\
&\operatorname{tr}_{\partial\Omega_l} \sum_{m\approx l} \varphi_l(u_m\nabla\varphi_m{}^T - (u_m\nabla\varphi_m{}^T)^T)\nu_m\Big)_{l\in\Gamma}.
\end{aligned}$$

Then (6.52) is true and consequently (6.51) must be the solution of (6.30) in case $(S + P')^{-1}$ exists. Therefore, it remains to verify that P' is a perturbation of S so that $S + P' : X \longrightarrow Y$ is an isomorphism.

In the same way as in (6.40) we obtain existence of some $C_{P'} = C_{P'}(n, q, \Omega) > 0$ such that

$$\|P'(u_l)_{l\in\Gamma}\|_Y \leqslant \frac{C_{P'}}{\sqrt{|\lambda|}}\|(u_l)_{l\in\Gamma}\|_X \tag{6.53}$$

holds for all $(u_l)_{l\in\Gamma} \in X$ and for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ (Note that $\operatorname{tr}_{\partial H_l} \sum_{m\approx l} \varphi_l(u_m\nabla\varphi_m{}^T - (u_m\nabla\varphi_m{}^T)^T)\nu_m$ is contained in the tangent space at ∂H_l for every $l \in \Gamma_1$ again). Thus, by increasing $\lambda_0 = \lambda_0(n, q, \theta, \Omega)$ such that $\lambda_0 \geqslant (2C_S C_{P'})^2$, where C_S and $C_{P'}$ are the constants from (6.36) and (6.53), we achieve as in (6.43) that

$$\|P'\|_{X \to Y} \leqslant \frac{1}{2C_S}$$

holds, so again the Neumann series yields that we have an isomorphism

$$S + P' : X \xrightarrow{\cong} Y. \qquad \square$$

7 Neumann Boundary Conditions for the Laplace Resolvent

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{1,1}$-domain, $n \geqslant 2$, $1 < q < \infty$ and $0 < \theta < \pi$. Then there exist $\lambda_0 = \lambda_0(n, q, \theta, \Omega) > 0$ and $C = C(n, q, \theta, \Omega) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ the problem*

$$\begin{cases} \lambda u - \Delta u &= f \quad \text{in } \Omega \\ \qquad \partial_\nu u &= g \quad \text{on } \partial\Omega \end{cases} \tag{7.1}$$

for all $f \in L_q(\Omega)$ and $g \in W_q^1(\Omega)$ has a unique solution $u \in W_q^2(\Omega)$ and this solution fulfills the resolvent estimate

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_q \leqslant C\|(f, \sqrt{\lambda} g, \nabla g)\|_q. \tag{7.2}$$

Proof. The proof is very similar to the proof of Theorem 6.5 but a little simpler, since the Neumann boundary condition is a condition only for scalar functions instead of vector fields. One could also take vector fields with componentwise Neumann boundary conditions into account but this would not cause any extra difficulties. In particular, there is no distinction between boundary conditions in tangential and normal direction. Consequently there is no necessity to include a strategy for the components of u like an additional application of the matrix $\nabla\Phi^T$ in the proof of Theorem 6.2 to the boundary terms to be able to apply a perturbation argument. Therefore, we only sketch the main differences from the proof of Theorem 6.5.

First, we obtain that Theorem 7.1 is true for the half space $\Omega = \mathbb{R}^n_+$ (with $\lambda_0 = 0$). This is due to [47], Thm. 7.7, since (7.1) satisfies the Lopatinski-Shapiro condition.

Next, let $\Omega = H_\omega$ be a bent half space, where $\omega \in W^2_\infty(\mathbb{R}^{n-1})$ such that

$$\|\nabla'\omega\|_\infty, \|\nabla'^2\omega\|_\infty \leqslant M \tag{7.3}$$

holds for some $M > 0$. Application of the change of coordinates $x \mapsto \tilde{x}$ (we use the same notation as in the proof of Theorem 6.2) to (7.1) yields the equivalent problem

$$\begin{cases} \lambda\tilde{u} - \Delta\tilde{u} + B\tilde{u} &= \tilde{f} \quad \text{in } \mathbb{R}^n_+ \\ \nu_+ \cdot \nabla\tilde{u} + B'\tilde{u} &= \tilde{g} \quad \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{7.4}$$

where B is the same operator as in the proof of Theorem 6.2 and

$$B'\tilde{u} := (\nu - \nu_+) \cdot \nabla\tilde{u} - \nu \cdot (\nabla'\omega^T, 0)\partial_n\tilde{u}.$$

Again, we can apply [47], Lem. 7.10, defining the Banach spaces

$$\begin{aligned} X &:= W_q^2(\mathbb{R}^n_+), \\ Y &:= L_q(\mathbb{R}^n_+) \times W_q^1(\mathbb{R}^n_+), \\ Z &:= L_q(\mathbb{R}^n_+) \times L_q(\partial\mathbb{R}^n_+) \end{aligned}$$

with norms

$$\begin{aligned}
\|\widetilde{u}\|_X &:= \|(\lambda\widetilde{u}, \sqrt{\lambda}\nabla\widetilde{u}, \nabla^2\widetilde{u})\|_q,\\
\|(\widetilde{f},\widetilde{g})\|_Y &:= \|(\widetilde{f}, \sqrt{\lambda}\widetilde{g}, \nabla\widetilde{g})\|_q,\\
\|\cdot\|_Z &:= \|\cdot\|_{L_q(\mathbb{R}^n_+)\times L_q(\partial\mathbb{R}^n_+)}
\end{aligned}$$

and the operators

$$\begin{aligned}
S : X &\longrightarrow Y, \quad \widetilde{u} \mapsto ((\lambda-\Delta)\widetilde{u}, \nu_+\cdot\nabla\widetilde{u}),\\
P : X &\longrightarrow Y, \quad \widetilde{u} \mapsto (B\widetilde{u}, B'\widetilde{u}),\\
Q : Y &\longrightarrow Z, \quad (\widetilde{f},\widetilde{g}) \mapsto (\widetilde{f}, \operatorname{tr}_{\partial\mathbb{R}^n_+}\widetilde{g})
\end{aligned}$$

this time. Comparing the definition of the space Y to the related definition in the proof of Theorem 6.2, we observe that only zero and first order derivatives are needed for the Neumann boundary conditions. Therefore, uniform $C^{1,1}$-regularity of the boundary is sufficient in order to receive the same result that we have proved for perfect slip boundary conditions. We obtain that for sufficiently small $\kappa = \kappa(n,q,\theta) > 0$ and for sufficiently large $\lambda_0 = \lambda_0(n,q,\kappa,M) > 0$, the problem (7.1) for $\Omega = H_\omega$ is uniquely solvable if $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ and $\|\nabla'\omega\|_\infty \leqslant \kappa$ and also (7.2) holds with some $C = C(n,q,\theta,M) > 0$. Hence, Theorem 7.1 holds for bent half spaces.

We easily transfer the latter result to the bent, rotated and shifted half space by using similar arguments as in the proof of Theorem 6.4.

As a last step we apply the localization procedure to (7.1) for arbitrary $C^{2,1}$-domains Ω, where we use the same notation as in the proof of Theorem 6.5. Multiplication of (7.1) by φ_l for $l \in \Gamma_1$ yields the local equations

$$\begin{cases}
\lambda u_l - \Delta u_l + \sum_{m\sim l}\left[2u_m(\nabla\varphi_m{}^T)\nabla\varphi_l + 2\varphi_m(\nabla u_m{}^T)\nabla\varphi_l + (\Delta\varphi_l)\varphi_m u_m\right]\\
= f_l \quad \text{in } \Omega_l \text{ for all } l \in \Gamma,\\
\nu_l\cdot\nabla u_l - \nu_l\cdot\sum_{m\approx l}(\nabla\varphi_l)\varphi_m u_m\\
= g_l \quad \text{on } \partial\Omega_l \text{ for all } l \in \Gamma_1
\end{cases} \tag{7.5}$$

for $u_m = \varphi_m u$. Similar to the proof of Theorem 6.5 we obtain that (7.5) is uniquely solvable, where we apply the same perturbation argument, defining $\mathrm{BF}_q(\partial\Omega) = \mathrm{BF}_{q,\lambda}(\partial\Omega)$ as the trace of $W^1_q(\Omega)$ (i.e., as $W_q^{1-1/q}(\partial\Omega)$) with norm

$$\|a\|_{\mathrm{BF}_{q,\lambda}(\partial\Omega)} = \inf_{g\in W^1_q(\Omega),\ \operatorname{tr} g = a} \|(\sqrt{\lambda}g, \nabla g)\|_q$$

and defining the Banach spaces

$$\begin{aligned}
X &:= l_q(\bigoplus_{l\in\Gamma} W^2_q(\Omega_l)),\\
Y &:= l_q(\bigoplus_{l\in\Gamma} L_q(\Omega_l)) \times l_q(\bigoplus_{l\in\Gamma} \mathrm{BF}_q(\partial\Omega_l))
\end{aligned}$$

with norms

$$\begin{aligned}
\|(u_l)_{l\in\Gamma}\|_X &:= \|(\lambda u_l, \sqrt{\lambda}\nabla u_l, \nabla^2 u_l)_{l\in\Gamma}\|_{l_q(L_q)},\\
\|(f_l, a_l)_{l\in\Gamma}\|_Y &:= \|(f_l)_{l\in\Gamma}\|_{l_q(L_q)} + \|(a_l)_{l\in\Gamma}\|_{l_q(\mathrm{BF}_{q,\lambda})}
\end{aligned}$$

and the operators

$$\begin{aligned} S : X \longrightarrow Y, &\quad (u_l)_{l\in\Gamma} \longmapsto \big((\lambda-\Delta)u_l, \nu_l \cdot \operatorname{tr}_{\partial\Omega_l} \nabla u_l\big)_{l\in\Gamma}, \\ P : X \longrightarrow Y, &\quad (u_l)_{l\in\Gamma} \longmapsto \Big(\sum_{m\sim l} \big[2u_m(\nabla\varphi_m{}^T)\nabla\varphi_l + 2\varphi_m(\nabla u_m{}^T)\nabla\varphi_l + (\Delta\varphi_l)\varphi_m u_m\big], \\ &\qquad\qquad - \nu_l \cdot \operatorname{tr}_{\partial\Omega_l} \sum_{m\approx l} (\nabla\varphi_l)\varphi_m u_m \Big)_{l\in\Gamma} \end{aligned}$$

this time. This gives uniqueness for solutions of (7.1) and the resolvent estimate (7.2). In order to obtain existence of a solution to (7.1), we use a representation

$$u = \bar{C}(S+P')^{-1}\bar{D}(f, \operatorname{tr} g)$$

as in the proof of Theorem 6.5, where in this case we have to consider

$$\begin{aligned} P' : X \longrightarrow Y, \quad (u_l)_{l\in\Gamma} \longmapsto \Big(&-\varphi_l \sum_{m\sim l} \big[(\Delta\varphi_m)u_m + 2(\nabla u_m{}^T)\nabla\varphi_m\big], \\ &\nu_l \cdot \operatorname{tr}_{\partial\Omega_l} \sum_{m\approx l} \varphi_l(\nabla\varphi_m)u_m \Big)_{l\in\Gamma}. \end{aligned}$$

□

Proposition 7.2. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{1,1}$-domain, $n \geqslant 2$, $1 < q < \infty$ and $0 < \theta < \pi$. Let $\lambda_0 > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ the conditions of Theorem* 7.1 *are satisfied for q and q'. Let $w \in W_q^2(\Omega)^n$ such that*

$$\begin{cases} (\lambda-\Delta)\operatorname{div} w &= 0 \quad \text{in } \Omega \\ \partial_\nu \operatorname{div} w &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7.6}$$

Then $\operatorname{div} w = 0$.[f]

Proof. Let $\Delta_{\mathrm{N},q} : \mathscr{D}(\Delta_{\mathrm{N},q}) \subset L_q(\Omega) \to L_q(\Omega),\ u \mapsto \Delta u$ be the Neumann-Laplace operator, i.e., $\mathscr{D}(\Delta_{\mathrm{N},q}) = \{u \in W_q^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$, and let $\Delta_{\mathrm{N},q}^* : L_{q'}(\Omega) \to \mathscr{D}(\Delta_{\mathrm{N},q})'$ be the continuous dual operator (endowing $\mathscr{D}(\Delta_{\mathrm{N},q})$ with the graph norm). Note that we can regard $L_{q'}(\Omega)$ as a subspace of $\mathscr{D}(\Delta_{\mathrm{N},q})'$, since $\mathscr{D}(\Delta_{\mathrm{N},q}) \subset L_q(\Omega)$ is dense.

We aim to prove that $(\lambda - \Delta_{\mathrm{N},q'}^*)\operatorname{div} w = 0$. For this purpose, fix some $\varphi \in \mathscr{D}(\Delta_{\mathrm{N},q'})$. Then the Neumann boundary conditions $\nu \cdot \nabla\varphi = 0$ and $\nu \cdot \nabla \operatorname{div} w = 0$ on $\partial\Omega$ yield

$$\begin{aligned} &\langle(\lambda - \Delta_{\mathrm{N},q'}^*)\operatorname{div} w, \varphi\rangle_{\mathscr{D}(\Delta_{\mathrm{N},q'})',\mathscr{D}(\Delta_{\mathrm{N},q'})} \\ &= \langle \operatorname{div} w, (\lambda-\Delta)\varphi\rangle_{q,q'} \\ &= \langle \operatorname{div} w, \lambda\varphi\rangle_{q,q'} - \langle \operatorname{div} w, \operatorname{div}\nabla\varphi\rangle_{q,q'} \\ &= \langle \operatorname{div} w, \lambda\varphi\rangle_{q,q'} + \int_\Omega \nabla \operatorname{div} w \cdot \nabla\varphi \, d\lambda_n - \langle \operatorname{div} w, \nu\cdot\nabla\varphi\rangle_{\partial\Omega} \\ &= \langle \operatorname{div} w, \lambda\varphi\rangle_{q,q'} - \int_\Omega (\Delta \operatorname{div} w)\varphi \, d\lambda_n + \langle \varphi, \nu\cdot\nabla \operatorname{div} w\rangle_{\partial\Omega} \\ &= \langle(\lambda-\Delta)\operatorname{div} w, \varphi\rangle_{q,q'} \\ &= 0, \end{aligned}$$

where we made use of Lemma 3.7, one time with $\nabla\varphi \in E_{q'}(\Omega)$ and $\operatorname{div} w \in W_q^1(\Omega)$ as well as a second time with $\nabla \operatorname{div} w \in E_q(\Omega)$ and $\varphi \in W_{q'}^1(\Omega)$. Therefore $(\lambda - \Delta_{\mathrm{N},q'}^*)\operatorname{div} w = 0$. Now $\lambda - \Delta_{\mathrm{N},q'} : \mathscr{D}(\Delta_{\mathrm{N},q'}) \xrightarrow{\cong} L_{q'}(\Omega)$ is an isomorphism, due to Theorem 7.1, so its continuous dual operator, $\lambda - \Delta_{\mathrm{N},q'}^*$, is injective. Hence $\operatorname{div} w = 0$. □

[f] By $\operatorname{div} w$ solving (7.6) we mean $(\lambda-\Delta)\operatorname{div} w = 0$ in the sense of distributions and $\operatorname{tr}_\nu \nabla \operatorname{div} w = 0$ in $W_q^{-1/q}(\partial\Omega)$. Note that $(\lambda-\Delta)\operatorname{div} w = 0$ implies $\operatorname{div}\nabla\operatorname{div} w = \Delta \operatorname{div} w = \lambda \operatorname{div} w \in L_q(\Omega)$ and therefore $\nabla \operatorname{div} w \in E_q(\Omega)$.

8 $L_{q,\sigma}$-Invariance of the Laplace Resolvent

An important property of the Laplace resolvent for our purposes is the following, which is a special feature of the perfect slip boundary conditions.

Proposition 8.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$ such that Assumption 4.4 is valid. Let $0 < \theta < \pi$, choose $\lambda_0 = \lambda_0(n, q, \theta, \Omega) > 0$ so that the conditions of Theorem 6.5 and Proposition 7.2 are satisfied and let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$. Then the following implications hold.*

(i) $u \in \mathscr{D}(\Delta_{\mathrm{PS}}) \cap L_{q,\sigma}(\Omega) \;\Rightarrow\; \Delta u \in L_{q,\sigma}(\Omega)$.

(ii) $f \in L_{q,\sigma}(\Omega) \;\Rightarrow\; (\lambda - \Delta_{\mathrm{PS},q})^{-1} f \in L_{q,\sigma}(\Omega)$.

Proof. We will make use of both the $L_{q,\sigma}(\Omega)$-representations in Lemmas 4.5 and 4.6. Let $u \in \mathscr{D}(\Delta_{\mathrm{PS}}) \cap L_{q,\sigma}(\Omega)$ and $\varphi \in C_c^\infty(\overline{\Omega})$. Then we have

$$\begin{aligned}
\langle \Delta u, \nabla\varphi \rangle_{q,q'} &= -\int_\Omega (\nabla \operatorname{div} u - \Delta u) \cdot \nabla\varphi \, d\lambda_n \\
&= -\int_\Omega \operatorname{div}(\mathrm{D}_-(u)\nabla\varphi) \, d\lambda_n \\
&= -\int_{\partial\Omega} \nu \cdot \mathrm{D}_-(u)\nabla\varphi \, d\sigma \\
&= \int_{\partial\Omega} \nabla\varphi \cdot \mathrm{D}_-(u)\nu \, d\sigma \\
&= 0,
\end{aligned}$$

where we made use of Lemma 3.4 and of Lemma 2.1(ii) and (iii). Now this holds for $\varphi \in \widehat{W}_{q'}^1(\Omega)$ as well, since $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_{q'}^1(\Omega)$ is dense. Hence (i) is true.

In order to see (ii), let $f \in L_{q,\sigma}(\Omega)$. The function $u := (\lambda - \Delta_{\mathrm{PS},q})^{-1} f \in W_q^2(\Omega)^n$ is the solution of

$$\begin{cases}
\lambda u - \Delta u &= f \quad \text{in } \Omega \\
\mathrm{D}_-(u)\nu &= 0 \quad \text{on } \partial\Omega \\
\nu \cdot u &= 0 \quad \text{on } \partial\Omega,
\end{cases} \tag{8.1}$$

so, applying tr_ν to the first line of (8.1), we receive

$$\operatorname{tr}_\nu \Delta u = 0. \tag{8.2}$$

Furthermore, we obtain $(\lambda - \Delta) \operatorname{div} u = 0$ in the sense of distributions, applying div to the first line of (8.1). Now we aim to show that the boundary condition $\partial_\nu \operatorname{div} u = 0$ on $\partial\Omega$ holds: Let $k \in W_{q'}^{1-1/q'}(\partial\Omega)$ and choose $w \in W_{q'}^1(\Omega)$ so that $\operatorname{tr} w = k$. First note that $\nabla \operatorname{div} u \in E_q(\Omega)$, so $\operatorname{tr}_\nu \nabla \operatorname{div} u$ is well defined. We have

$$\begin{aligned}
\big\langle k, \operatorname{tr}_\nu \nabla \operatorname{div} u \big\rangle_{\partial\Omega} &= \big\langle w, \nu \cdot (\nabla \operatorname{div} u - \Delta u) \big\rangle_{\partial\Omega} \\
&= \int_\Omega \operatorname{div}(w(\nabla \operatorname{div} u - \Delta u)) \, d\lambda_n \\
&= \int_\Omega \nabla w \cdot (\nabla \operatorname{div} u - \Delta u) \, d\lambda_n,
\end{aligned} \tag{8.3}$$

using (8.2), Lemma 3.8 and $\operatorname{div}(\nabla \operatorname{div} u - \Delta u) = 0$. In case $w \in C_c^\infty(\overline{\Omega})$, we obtain for the last term of (8.3) that

$$\begin{aligned}
\int_\Omega \nabla w \cdot (\nabla \operatorname{div} u - \Delta u)\, d\lambda_n &= \int_\Omega \operatorname{div}(\mathrm{D}_-(u)\nabla w)\, d\lambda_n \\
&= \int_{\partial\Omega} \nu \cdot \mathrm{D}_-(u)\nabla w\, d\sigma \\
&= -\int_{\partial\Omega} \nabla w \cdot \mathrm{D}_-(u)\nu\, d\sigma \\
&= 0,
\end{aligned} \tag{8.4}$$

using Lemma 2.1(ii), (iii) and Lemma 3.4. The density of $C_c^\infty(\overline{\Omega}) \subset W_{q'}^1(\Omega)$ gives that (8.4) holds for $w \in W_{q'}^1(\Omega)$ as well. Therefore, (8.3) and (8.4) yield $\partial_\nu \operatorname{div} u = 0$ on $\partial\Omega$. We have in total

$$\begin{cases} (\lambda - \Delta)\operatorname{div} u &= 0 \quad \text{in } \Omega \\ \partial_\nu \operatorname{div} u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Consequently $\operatorname{div} u = 0$, due to Proposition 7.2. Lemma 4.6 yields $u \in L_{q,\sigma}(\Omega)$. $\square$

III Stokes and Navier-Stokes Equations on Uniform $C^{2,1}$-Domains

The main results concerning Stokes and Navier-Stokes equations on general domains are presented and proved in this chapter. For this purpose, we will take the Assumptions 4.2, 4.3 and 4.4 into consideration. A starting point will be the perfect slip boundary conditions, for which we have the useful properties of the Laplace resolvent from Chapter II available.

9 The Stokes Resolvent Problem: Perfect Slip Boundary Conditions

9.1 Homogeneous Boundary Conditions

Theorem 9.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$ and $1 < q < \infty$ such that Assumption 4.4 holds. Let $0 < \theta < \pi$ and denote $U_q(\Omega) = L_{q,\sigma}(\Omega) \cap G_q(\Omega)$ again. Then there exist $\lambda_0 = \lambda_0(n, q, \theta, \Omega) > 0$ and $C = C(n, q, \theta, \Omega) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ we have the following, concerning*

$$\begin{cases} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ \mathrm{D}_-(u)\nu &= 0 \quad \text{on } \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{9.1}$$

(i) *Provided that $f \in L_q(\Omega)^n$, problem (9.1) has a solution*

$$(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$$

if and only if $f \in L_{q,\sigma}(\Omega) + G_q(\Omega)$. In particular, there exists a solution of (9.1) for any $f \in L_q(\Omega)^n$ in case Assumption 4.3 is valid.

(ii) *The solution space $S_{\text{hom}} \subset [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$ of the homogeneous problem (9.1) (i.e., $f = 0$) is*

$$S_{\text{hom}} = \Big\{ \big((\lambda - \Delta_{\mathrm{PS},q})^{-1} \nabla\pi, -\nabla\pi \big) : \nabla\pi \in U_q(\Omega) \Big\}.$$

In particular, we obtain $\dim S_{\text{hom}} = \dim U_q(\Omega)$.

(iii) *In case Assumption 4.2(i) is valid, we obtain: For $f \in L_q(\Omega)^n$ there exists a unique solution*

$$(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$$

of (9.1) if and only if $f \in L_{q,\sigma}(\Omega) + G_q(\Omega)$. In particular, in case Assumption 4.3 is valid as well, there exists a unique solution of (9.1) in $[W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ for any $f \in L_q(\Omega)^n$.

(iv) *In case Assumption* 4.2 *(i.e.,* 4.2(i) *and* 4.2(ii)*) is valid, the solution in* (iii) *satisfies the resolvent estimate*

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u, \nabla p)\|_q \leqslant C\|f\|_q \tag{9.2}$$

for any $f \in L_{q,\sigma}(\Omega) + G_q(\Omega)$.

Proof. Choose $\lambda_0 = \lambda_0(n, q, \theta, \Omega)$ and $C = C(n, q, \theta, \Omega)$ such that the conditions of Theorem 6.5 and Proposition 7.2 are satisfied and let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$.

In order to prove (i), we decompose a given function $f \in L_{q,\sigma}(\Omega) + G_q(\Omega)$ into $f_0 \in L_{q,\sigma}(\Omega)$ and $\nabla\pi \in G_q(\Omega)$. Setting

$$(u, \nabla p) := ((\lambda - \Delta_{\mathrm{PS},q})^{-1} f_0, \nabla\pi),$$

we obtain a solution of (9.1), due to Proposition 8.1(ii). Conversely, if there exists a solution $(u, \nabla p)$ of (9.1) with right-hand side $f \in L_q(\Omega)^n$, then Proposition 8.1(i) gives that $f \in L_{q,\sigma}(\Omega) + G_q(\Omega)$.

A solution of the homogeneous problem (9.1) is given by $\big((\lambda - \Delta_{\mathrm{PS},q})^{-1}\nabla\pi, -\nabla\pi\big)$ with some $\nabla\pi \in U_q(\Omega)$, due to Proposition 8.1(ii). If, conversely, $(u, \nabla p) \in [W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$ solves (9.1) with $f = 0$ then we have $(\lambda - \Delta)u = -\nabla p \in G_q(\Omega)$. On the other hand Proposition 8.1(i) yields $(\lambda - \Delta)u \in L_{q,\sigma}(\Omega)$. Therefore, $\nabla p = -(\lambda - \Delta)u \in U_q(\Omega)$, so we have in total

$$(u, \nabla p) = ((\lambda - \Delta_{\mathrm{PS},q})^{-1}\nabla\pi, -\nabla\pi)$$

with $\nabla\pi := -\nabla p \in U_q(\Omega)$. This proves (ii).

Now let Assumption 4.2(i) be valid and $f \in L_{q,\sigma}(\Omega) + G_q(\Omega)$. Using the direct decomposition (4.2), we can decompose $f = f_0 + \nabla p$ into $f_0 \in L_{q,\sigma}(\Omega)$ and $\nabla p \in \mathcal{G}_q(\Omega)$. The solution

$$(u, \nabla p) := \big((\lambda - \Delta_{\mathrm{PS},q})^{-1} f_0, \nabla p\big)$$

of (9.1) is contained in $[W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$, thanks to Proposition 8.1(ii), so we only have to prove that there is at most one solution in this space to obtain uniqueness. Therefore, let $(v, \nabla\pi) \in [W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ be a solution of the homogeneous problem (9.1). Proposition 8.1(i) then yields $(\lambda - \Delta)v \in L_{q,\sigma}(\Omega)$, but on the other hand we have

$$(\lambda - \Delta)v = -\nabla\pi \in \mathcal{G}_q(\Omega).$$

Since $\mathcal{G}_q(\Omega) \cap L_{q,\sigma}(\Omega) = \{0\}$, we deduce $\nabla\pi = 0$ and $v = -(\lambda - \Delta_{\mathrm{PS},q})^{-1}\nabla\pi = 0$. Hence, solutions of (9.1) in $[W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ are unique. Therefore, the sufficiency in (iii) is proved. Conversely, for any right-hand side function $f \in L_q(\Omega)^n$ the condition $f \in L_{q,\sigma}(\Omega) + G_q(\Omega)$ is also necessary to obtain existence of the solution in (iii), since $[W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ is a subspace of $[W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$ and for the latter space we have seen necessity in (i) already. Hence, (iii) is proved.

Now let Assumption 4.2 be valid. Then the right-hand side of (4.2) is a Banach space, yielding a constant $C' = C'(n, q, \Omega) > 0$ so that for the decomposition $f = f_0 + \nabla p$ we have

$$\|(f_0, \nabla p)\|_q \leqslant C'\|f\|_q. \tag{9.3}$$

Hence, the functions $u = (\lambda - \Delta_{\mathrm{PS},q})^{-1} f_0$ and ∇p fulfill the resolvent estimate (9.2), due to (9.3) and Theorem 6.5. □

9.2 Inhomogeneous Boundary Conditions

In case of inhomogeneous boundary conditions the problem is a little more intricate. Nevertheless, in order to be able to make use of perturbation theory, an inhomogeneous version of Theorem 9.1 is needed.

Theorem 9.2. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$ such that Assumption 4.4 is valid and let $0 < \theta < \pi$. Then there exist $\lambda_0 = \lambda_0(n, q, \theta, \Omega) > 0$ and $C = C(n, q, \theta, \Omega) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ we have the following, concerning*

$$\begin{cases} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{9.4}$$

(i) *If Assumption 4.3 is valid, then for all $f \in L_q(\Omega)^n$ and $g \in W_q^1(\Omega)^n$ there exists a solution*
$$(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$$
of (9.4) (which may not be unique; see Theorem 9.1(ii)).

(ii) *If Assumptions 4.2 and 4.3 are valid, then for all $f \in L_q(\Omega)^n$ and $g \in W_q^1(\Omega)^n$ there exists a unique solution*
$$(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$$
of (9.4) and the resolvent estimate
$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u, \nabla p)\|_q \leqslant C\|(f, \sqrt{\lambda} g, \nabla g)\|_q \tag{9.5}$$
holds.

Proof. Fix $\lambda_0 = \lambda_0(n, q, \theta, \Omega)$ such that the conditions of Theorem 6.5 and Proposition 7.2 are satisfied and let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$.

Let $g \in W_q^1(\Omega)^n$ and assume initially $f \in L_{q,\sigma}(\Omega)$. Denote by $\tilde{u} \in W_q^2(\Omega)^n$ the unique solution of

$$\begin{cases} \lambda \tilde{u} - \Delta \tilde{u} &= 0 \quad \text{in } \Omega \\ \mathrm{D}_-(\tilde{u})\nu &= \Pi_\tau g \quad \text{on } \partial\Omega \\ \nu \cdot \tilde{u} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

(see Theorem 6.5). Let $\nabla p \in G_q(\Omega)$ be a solution of

$$\langle -\nabla p, \nabla\varphi \rangle_{q,q'} = \langle \nabla \operatorname{div} \tilde{u} - \Delta\tilde{u}, \nabla\varphi \rangle_{q,q'} \quad \forall \varphi \in \widehat{W}_{q'}^1(\Omega) \tag{9.6}$$

(Assumption 4.3 yields existence of such a solution). Using Theorem 6.5 again, we define $u \in W_q^2(\Omega)^n$ as the unique solution of

$$\begin{cases} \lambda u - \Delta u &= f - \nabla p \quad \text{in } \Omega \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{9.7}$$

We now aim to prove that $u \in L_{q,\sigma}(\Omega)$, where we use the representation of $L_{q,\sigma}(\Omega)$ from Lemma 4.6. Applying tr_ν to the first line of (9.7) gives

$$\mathrm{tr}_\nu \Delta u = \mathrm{tr}_\nu \nabla p, \tag{9.8}$$

where $\operatorname{tr}_\nu \nabla p$ is well-defined, since $\operatorname{div}\nabla p = -\operatorname{div}(\nabla \operatorname{div}\tilde{u} - \Delta\tilde{u}) = 0$ in the sense of distributions, due to (9.6). Also $\operatorname{tr}_\nu \nabla \operatorname{div} u$ is well-defined, since (9.7) yields $\operatorname{div}\nabla\operatorname{div} u = \operatorname{div}\Delta u = \lambda \operatorname{div} u \in L_q(\Omega)^n$. In order to obtain that $\operatorname{tr}_\nu \nabla \operatorname{div} u = 0$, let $k \in W^{1-1/q'}_{q'}(\partial\Omega)$ and fix any $w \in W^1_{q'}(\Omega)$ so that $\operatorname{tr} w = k$. Then we have, using (9.8), Lemma 3.8 and $\operatorname{div}\nabla p = 0$,

$$\begin{aligned} \langle k, \operatorname{tr}_\nu \nabla \operatorname{div} u\rangle_{\partial\Omega} &= \langle w, \nu\cdot(\nabla p + \nabla \operatorname{div} u - \Delta u)\rangle_{\partial\Omega} \\ &= \int_\Omega \operatorname{div}(w(\nabla p + \nabla\operatorname{div} u - \Delta u))\, d\lambda_n \\ &= \int_\Omega \nabla w \cdot (\nabla p + \nabla \operatorname{div} u - \Delta u)\, d\lambda_n \end{aligned} \tag{9.9}$$

as in (8.3). Now, in the last term of (9.9), we can replace $\nabla\operatorname{div} u - \Delta u$ by $\nabla\operatorname{div}\tilde{u} - \Delta\tilde{u}$. In fact, using Lemma 2.1(ii) and (iii) and Lemma 3.4, we obtain for $w \in C^\infty_c(\overline{\Omega})$

$$\begin{aligned} \int_\Omega \nabla w\cdot(\nabla\operatorname{div} u - \Delta u)\, d\lambda_n &= \int_\Omega \operatorname{div}(\mathrm{D}_-(u)\nabla w)\, d\lambda_n \\ &= \int_{\partial\Omega} \nu\cdot(\mathrm{D}_-(u)\nabla w)\, d\sigma \\ &= -\int_{\partial\Omega} \nabla w\cdot(\mathrm{D}_-(u)\nu)\, d\sigma \\ &= -\int_{\partial\Omega} \nabla w\cdot(\Pi_\tau g)\, d\sigma \end{aligned}$$

and the same for $\tilde{u}$ instead of u, so we have

$$\int_\Omega \nabla w\cdot(\nabla\operatorname{div} u - \Delta u)\, d\lambda_n = \int_\Omega \nabla w\cdot(\nabla\operatorname{div}\tilde{u} - \Delta\tilde{u})\, d\lambda_n \tag{9.10}$$

for $w \in C^\infty_c(\overline{\Omega})$. The density of $C^\infty_c(\overline{\Omega}) \subset W^1_{q'}(\Omega)$ yields that (9.10) holds for $w \in W^1_{q'}(\Omega)$ as well and therefore (9.6) gives that the right-hand side of (9.9) vanishes. Consequently we have $\operatorname{tr}_\nu \nabla\operatorname{div} u = 0$ in total. Moreover, applying div to the first line of (9.7), we obtain $(\lambda - \Delta)\operatorname{div} u = 0$. Hence

$$\begin{cases} (\lambda-\Delta)\operatorname{div} u &= 0 \quad \text{in } \Omega \\ \partial_\nu \operatorname{div} u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

holds. Proposition 7.2 then yields $\operatorname{div} u = 0$, so we receive $u \in L_{q,\sigma}(\Omega)$ and $(u, \nabla p) \in [W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$ is a solution of (9.4).

Now, in the general case $f \in L_q(\Omega)^n$, we can decompose $f = f_0 + \nabla\pi$, where $f_0 \in L_{q,\sigma}(\Omega)$ and $\nabla\pi \in G_q(\Omega)$ (Assumption 4.3). We have shown that there exists a solution $(u, \nabla p) \in [W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$ of (9.4) with right-hand side function f_0, so $(u, \nabla p + \nabla\pi)$ solves (9.4) with right-hand side function f. Thus, (i) is proved.

Let now Assumptions 4.2 and 4.3 be valid. Again, let initially $f \in L_{q,\sigma}(\Omega)$. As in the proof of (i) let $\tilde{u} \in W^2_q(\Omega)^n$ be the unique solution of

$$\begin{cases} \lambda\tilde{u} - \Delta\tilde{u} &= 0 \quad \text{in } \Omega \\ \mathrm{D}_-(\tilde{u})\nu &= \Pi_\tau g \quad \text{on } \partial\Omega \\ \nu\cdot\tilde{u} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Theorem 6.5 then yields

$$\|(\lambda\tilde{u}, \sqrt{\lambda}\nabla\tilde{u}, \nabla^2\tilde{u})\|_q \leqslant C\|(\sqrt{\lambda}g, \nabla g)\|_q \tag{9.11}$$

with a constant $C = C(n, q, \theta, \Omega) > 0$. The direct decomposition (4.1) gives that

$$\langle -\nabla p, \nabla \varphi \rangle_{q,q'} = \langle \nabla \operatorname{div} \tilde{u} - \Delta \tilde{u}, \nabla \varphi \rangle_{q,q'} \quad \forall \varphi \in \widehat{W}^1_{q'}(\Omega)$$

has a unique solution $\nabla p \in \mathcal{G}_q(\Omega)$, which relates to the decomposition of the function $\nabla \operatorname{div} \tilde{u} - \Delta \tilde{u} = v_0 - \nabla p$ into $v_0 \in L_{q,\sigma}(\Omega)$ and $-\nabla p \in \mathcal{G}_q(\Omega)$. Furthermore, (4.1) yields a constant $C' = C'(n, q, \Omega) > 0$ so that

$$\|\nabla p\|_q \leqslant C' \|\nabla \operatorname{div} \tilde{u} - \Delta \tilde{u}\|_q. \tag{9.12}$$

Again, we define $u \in W^2_q(\Omega)^n$ as the unique solution of

$$\begin{cases} \lambda u - \Delta u &= f - \nabla p \quad \text{in } \Omega \\ \mathrm{D}_-(u)\nu &= \Pi_\tau g \quad \text{on } \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

and obtain $u \in L_{q,\sigma}(\Omega)$ in the same way as in the proof of (i). Hence, $(u, \nabla p) \in [W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ is the unique solution of (9.4). Moreover, Theorem 6.5 yields

$$\|(\lambda u, \sqrt{\lambda} \nabla u, \nabla^2 u)\|_q \leqslant C \|(f - \nabla p, \sqrt{\lambda} g, \nabla g)\|_q \tag{9.13}$$

with a constant $C = C(n, q, \theta, \Omega) > 0$. The estimates (9.11), (9.12) and (9.13) imply (9.5).

Now, let $f \in L_q(\Omega)^n$. Decomposition (4.1) gives $f = f_0 + \nabla \pi$ with two unique functions $f_0 \in L_{q,\sigma}(\Omega)$ and $\nabla \pi \in \mathcal{G}_q(\Omega)$ as well as a constant $C' = C'(n, q, \Omega) > 0$ so that

$$\|(f_0, \nabla \pi)\|_q \leqslant C' \|f\|_q. \tag{9.14}$$

We have proved that (9.4) with right-hand side function f_0 admits a unique solution $(u, \nabla p) \in [W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ satisfying (9.5) with f_0 instead of f. Thus, $(u, \nabla p + \nabla \pi) \in [W^2_q(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ is the unique solution of (9.4) with right-hand side function f and (9.14) yields the related resolvent estimate (9.5) with $\nabla p + \nabla \pi$ instead of ∇p. Hence (ii) is proved. □

10 The Stokes Resolvent Problem: Partial Slip Type Boundary Conditions

In order to receive results similar to Theorem 9.1 and Theorem 9.2 subject to a more general class of boundary conditions, we first obtain that it is possible to perturb the perfect slip boundary conditions in such a way that we receive partial slip type boundary conditions.

Lemma 10.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$. Then there exists a matrix $A \in W^1_\infty(\Omega)^{n \times n}$ such that for all $u \in W^2_q(\Omega)^n$ with $\nu \cdot u = 0$ on $\partial\Omega$ we have*

$$\Pi_\tau \mathrm{D}_+(u)\nu = \mathrm{D}_-(u)\nu + \Pi_\tau A u \quad \text{on } \partial\Omega.$$

Proof. Let $T_x\partial\Omega \subset \mathbb{R}^{n-1}$ be the tangent space at some fixed point $x \in \partial\Omega$. Let $\tau_1, \ldots, \tau_{n-1}$ be a basis of $T_x\partial\Omega$. Then, with the outer unit normal $\tau_n := \nu = \nu(x)$, let $\tau^1, \ldots, \tau^n$ be the dual basis of $\tau_1, \ldots, \tau_n$ in $\mathbb{R}^n$ (i.e., $\tau_i \cdot \tau^j = \delta_{ij}$ for $i, j = 1, \ldots, n$). Then we have $\tau^n = \nu$, since $\tau^n \cdot \tau_j = 0$ for $j = 1, \ldots, n-1$ implies $\tau^n = \beta\nu$ for some $\beta \in \mathbb{R}$ but then $1 = \tau^n \cdot \nu = \beta \nu \cdot \nu = \beta$.

We first observe for the tangential projection $\Pi_\tau u = (I - \nu\nu^T)u$, the change of basis matrix $S := (\tau_1, \ldots, \tau_{n-1}, \nu)^T$ and the vector $[u]_{1,\ldots,n}$ of covariant components $[u]_i := u \cdot \tau_i = (Su)_i$ that

(a) $\Pi_\tau u = \sum_{k=1}^{n-1}(u\cdot\tau_k)\tau^k$,

(b) $S^{-1} = (\tau^1,\dots,\tau^{n-1},\nu)$ and

(c) $\Pi_\tau S^{-1}[u]_{1,\dots,n} = \Pi_\tau S^{-1}([u]_1,\dots,[u]_{n-1},0)^T$.

It is obvious that $(\tau^1,\dots,\tau^{n-1},\nu)$ is a right inverse of S but S also has full rank, so $(\tau^1,\dots,\tau^{n-1},\nu)$ must be the left inverse as well. Thus (b) is true. We receive from (b), using the representation $u = S^{-1}[u]_{1,\dots,n}$, that

$$\Pi_\tau u = (I-\nu\nu^T)(\tau^1,\dots,\tau^{n-1},\nu)[u]_{1,\dots,n} = (\tau^1,\dots,\tau^{n-1},0)[u]_{1,\dots,n} = \sum_{k=1}^{n-1}(u\cdot\tau_k)\tau^k,$$

so (a) is true. Now, using (a) and (b), we obtain (c), since

$$\begin{aligned}\Pi_\tau S^{-1}([u]_1,\dots,[u]_{n-1},0)^T &= (\tau^1,\dots,\tau^{n-1},0)([u]_1,\dots,[u]_{n-1},0)^T\\ &= \sum_{k=1}^{n-1}(u\cdot\tau_k)\tau^k\\ &= \Pi_\tau u\\ &= \Pi_\tau S^{-1}[u]_{1,\dots,n}.\end{aligned}$$

We now choose a concrete basis of $T_x\partial\Omega$ in an arbitrary point $x\in\partial\Omega$. For this purpose, let ϕ_l, $l\in\Gamma_1$ be the parametrization of the boundary $\partial\Omega$ chosen in (2.5). If, for some $l\in\Gamma_1$, the point $x\in\partial\Omega$ is contained in the part $\partial\Omega\cap B_l$ of the boundary, the functions $\partial_i\phi_l$, $i=1,\dots,n-1$ form a basis of $T_x\partial\Omega$. More precisely, we can define $\tau_i = \tau_i(x) := \partial_i\phi_l(\phi_l^{-1}(x))$ for $i=1,\dots,n-1$. Let $l\in\Gamma_1$ be fixed now. For a function v on $\partial\Omega\cap B_l$ and $i=1,\dots,n-1$ we define the i-th tangential derivative as

$$\partial_{\tau_i}v := \partial_i(v\circ\phi_l)\circ\phi_l^{-1}$$

and if v is a vector field, then $\partial_{\tau_i}v$ is defined componentwise. If $v\in W^1_q(\Omega\cap B_l)$, the chain rule gives $\partial_{\tau_i}v = \nabla v\cdot(\partial_i\phi_l\circ\phi_l^{-1}) = \nabla v\cdot\tau_i$, so the tangential derivative is exactly the directional derivative in direction of the tangential vector. In case $v\in W^1_q(\Omega\cap B_l)^n$, we have $\partial_{\tau_i}v = (\nabla v^T)\tau_i$. Therefore, for $u\in W^2_q(\Omega)^n$ we have

$$[(\nabla u)\nu]_i = \tau_i\cdot(\nabla u)\nu = \nu\cdot(\nabla u^T)\tau_i = \nu\cdot\partial_{\tau_i}u \quad \text{on } \partial\Omega\cap B_l, \tag{10.1}$$

where $i=1,\dots,n-1$. For $u\in W^2_q(\Omega)^n$ satisfying $\nu\cdot u = 0$ on $\partial\Omega$ we obtain

$$0 = \partial_{\tau_i}(\nu\cdot u) = u\cdot\partial_{\tau_i}\nu + \nu\cdot\partial_{\tau_i}u \quad \text{on } \partial\Omega\cap B_l. \tag{10.2}$$

Utilizing (10.1) and (10.2) and writing $(\nabla u^T)\nu = \partial_\nu u$, we receive

$$[\mathrm{D}_\pm(u)\nu]_i = [\partial_\nu u]_i \mp (\partial_{\tau_i}\nu)\cdot u \quad \text{on } \partial\Omega\cap B_l \text{ for } i=1,\dots,n-1. \tag{10.3}$$

Now (c) and (10.3) yield

$$\begin{aligned}\Pi_\tau\mathrm{D}_\pm(u)\nu &= \Pi_\tau S^{-1}[\mathrm{D}_\pm(u)\nu]_{1,\dots,n}\\ &= \Pi_\tau S^{-1}\big([\mathrm{D}_\pm(u)\nu]_1,\dots,[\mathrm{D}_\pm(u)\nu]_{n-1},0\big)^T\\ &= \Pi_\tau S^{-1}\Big(([\partial_\nu u]_1,\dots,[\partial_\nu u]_{n-1},0)^T \mp \big((\partial_{\tau_1}\nu)\cdot u,\dots,(\partial_{\tau_{n-1}}\nu)\cdot u,0\big)^T\Big)\\ &= \Pi_\tau(\partial_\nu u \mp S^{-1}Ru) \quad \text{on } \partial\Omega\cap B_l,\end{aligned}$$

where $R := (\partial_{\tau_1}\nu, \ldots, \partial_{\tau_{n-1}}\nu, 0)$. Hence, for $u \in W_q^2(\Omega)^n$ satisfying $\nu \cdot u = 0$ on $\partial\Omega$, we obtain (using (2.15))

$$\Pi_\tau \mathrm{D}_+(u)\nu = \mathrm{D}_-(u)\nu - 2\Pi_\tau S^{-1}Ru \quad \text{on } \partial\Omega.$$

It remains to prove that there exists an extension $A \in W_\infty^1(\Omega)^{n\times n}$ of $-2S^{-1}R$. Therefore, we first consider the entries of S^{-1}. We have shown in (2.13) that there exists an extension $\bar{\nu} \in W_\infty^2(\Omega)^n$ of ν. In the same way we can establish an extension $\bar{\tau}_i \in W_\infty^2(\Omega)^n$ of τ_i for $i = 1, \ldots, n-1$ and the corresponding extension $\partial_{\bar{\tau}_i}$ of the tangential derivative operator ∂_{τ_i}.

A representation of τ^i is given by $\tau^i = \sum_{k=1}^{n-1} g^{ik}\tau_k$, where $(g^{jk})_{j,k=1,\ldots,n-1} := G^{-1}$ is the inverse of the Gram matrix $G := (\tau_j \cdot \tau_k)_{j,k=1,\ldots,n-1}$ (cf. [54]). In (2.10) we have established a uniform upper bound for $\|G^{-1}\|_{1,\infty}$, so we also have an extension $\bar{\tau}^i \in W_\infty^2(\Omega)^n$ of τ^i for $i = 1, \ldots, n-1$. Now, considering the entries of R, we obtain that $\partial_{\tau_i}\nu$ for $i = 1, \ldots, n-1$ can be written as the directional derivative of the extension $\bar{\nu}$ in direction of τ_i. Since $\bar{\nu} \in W_\infty^2(\Omega)^n$, we receive $\partial_{\bar{\tau}_i}\bar{\nu} \in W_\infty^1(\Omega)^n$. Summarizing, we have extensions of S^{-1} and R, hence also of $-2S^{-1}R$, in $W_\infty^1(\Omega)^{n\times n}$. $\square$

Now we are able to take general partial slip type boundary conditions into consideration. The result reads the following.

Theorem 10.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$, $1 < q < \infty$ such that Assumptions 4.3 and 4.4 are valid. Let $0 < \theta < \pi$ and $\alpha \in \mathbb{R}$. Then there exist $\lambda_0 = \lambda_0(n, q, \theta, \Omega, \alpha) > 0$ and $C = C(n, q, \theta, \Omega) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ we have the following with regard to*

$$\begin{cases} \lambda u - \Delta u + \nabla p &= f & \text{in } \Omega \\ \operatorname{div} u &= 0 & \text{in } \Omega \\ \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) &= \Pi_\tau g & \text{on } \partial\Omega \\ \nu \cdot u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{10.4}$$

where we denote $(10.4)_+$ and $(10.4)_-$ for the respective boundary terms $\mathrm{D}_\pm$ again.

(i) *There exists $\epsilon = \epsilon(n, q, \Omega, \lambda) > 0$ so that in case $|\alpha| < \epsilon$ for any $f \in L_q(\Omega)^n$ and $g \in W_q^1(\Omega)^n$ there exists a solution*

$$(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$$

of $(10.4)_-$.

(ii) *Additionally, let Assumption 4.2 be valid. For any $f \in L_q(\Omega)^n$ and $g \in W_q^1(\Omega)^n$ there exists a unique solution*

$$(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$$

of $(10.4)_+$ and of $(10.4)_-$, respectively, and the estimate

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u, \nabla p)\|_q \leqslant C\|(f, \sqrt{\lambda}g, \nabla g)\|_q \tag{10.5}$$

holds in each case.

Remark 10.3. Note that Theorem 10.2(ii) yields that solutions $(u, \nabla p)$ of (10.4) in the class $[W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$ are not unique in case $[W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ is a proper subspace. In fact, if $\nabla\pi \in U_q(\Omega) = L_{q,\sigma}(\Omega) \cap G_q(\Omega)$ is a nonzero function, then Theorem 10.2(ii) yields a solution $(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ of (10.4) with $f = \nabla\pi$ and $g = 0$, so $(u, \nabla p - \nabla\pi) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$ is a solution of the homogeneous problem (10.2). This solution is nonzero, since $\nabla p - \nabla\pi = 0$ would yield $\nabla\pi = 0$, due to the definition of $\mathcal{G}_q(\Omega)$.

Proof. We start with proving (ii). Let initially $f \in L_{q,\sigma}(\Omega)$, choose $\lambda_0 = \lambda_0(n, q, \theta, \Omega)$ and $C = C(n, q, \theta, \Omega)$ such that the conditions of Theorem 9.2(ii) are satisfied and let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$. Let $A \in W_\infty^1(\Omega)^{n\times n}$ be the matrix from Lemma 10.1.

We define the Banach spaces

$$\begin{aligned} X &:= \big\{(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega) : (\lambda - \Delta)u + \nabla p \in L_{q,\sigma}(\Omega)\big\}, \\ Y &:= L_{q,\sigma}(\Omega) \times \big\{\Pi_\tau \operatorname{tr} g : g \in W_q^1(\Omega)^n\big\} \end{aligned}$$

with norms (depending on λ)

$$\begin{aligned} \|(u, \nabla p)\|_X &:= \|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u, \nabla p)\|_q, \\ \|(f, a)\|_Y &:= \|f\|_q + \inf\{\|(\sqrt{\lambda} g, \nabla g)\|_q : g \in W_q^1(\Omega)^n, a = \Pi_\tau \operatorname{tr} g\}. \end{aligned}$$

We further define the operators

$$\begin{aligned} S : X &\longrightarrow Y, \quad (u, \nabla p) \longmapsto ((\lambda - \Delta)u + \nabla p, \operatorname{tr} \mathrm{D}_-(u)\nu), \\ P_- : X &\longrightarrow Y, \quad (u, \nabla p) \longmapsto (0, \Pi_\tau \operatorname{tr} \alpha u), \\ P_+ : X &\longrightarrow Y, \quad (u, \nabla p) \longmapsto (0, \Pi_\tau \operatorname{tr}(Au + \alpha u)). \end{aligned}$$

The statement for $f \in L_{q,\sigma}(\Omega)$, $g \in W_q^1(\Omega)^n$ now means that

$$S + P_\pm : X \longrightarrow Y \tag{10.6}$$

is bijective such that $(S + P_\pm)^{-1}$ is bounded, uniformly in λ. More precisely, the related continuity constant of (10.6) is only allowed to depend on n, q, θ, Ω and we prescribe $|\lambda| \geqslant \lambda_0$. Besides, in (10.6) the operator $S + P_-$ relates to $(10.4)_-$ while $S + P_+$ corresponds to $(10.4)_+$.

Theorem 9.2(ii) gives that S is bijective and for $(f, a) \in Y$, $(u, \nabla p) := S^{-1}(f, a)$ and any $g \in W_q^1(\Omega)^n$ satisfying $a = \Pi_\tau \operatorname{tr} g$ we have

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u, \nabla p)\|_q \leqslant C\|(f, \sqrt{\lambda} g, \nabla g)\|_q.$$

Consequently

$$\|(u, \nabla p)\|_X \leqslant C\|(f, a)\|_Y. \tag{10.7}$$

Next, we prove that the two operators $P_\pm$ are continuous with

$$\|P_\pm\|_{X\to Y} \leqslant \frac{C' + |\alpha|}{\sqrt{|\lambda|}}, \tag{10.8}$$

where $C' = C'(n, q, \Omega) > 0$. The definition of the norm in Y directly gives

$$\|P_-(u, \nabla p)\|_Y \leqslant |\alpha|\|(\sqrt{\lambda} u, \nabla u)\|_q, \tag{10.9}$$

for all $(u, \nabla p) \in X$. We obtain the same for the second part of P_+, i.e.,

$$\|(0, \Pi_\tau \operatorname{tr} \alpha u)\|_Y \leqslant |\alpha| \|(\sqrt{\lambda} u, \nabla u)\|_q.$$

Lemma 10.1 yields for the first part

$$\|(0, \Pi_\tau \operatorname{tr} Au)\|_Y \leqslant \|A\|_{1,\infty} \|(\sqrt{\lambda} u, \nabla u)\|_q$$

if $|\lambda| \geqslant 1$. In total we obtain (10.8).

We now increase the constant $\lambda_0 = \lambda_0(n, q, \theta, \Omega)$ to some $\lambda_0 = \lambda_0(n, q, \theta, \Omega, \alpha)$ so that $\lambda_0 \geqslant 1$ and $\lambda_0 \geqslant (2C)^2(C' + |\alpha|)^2$, where C and C' are the constants from (10.7) and (10.8). Then (10.8) yields for $|\lambda| \geqslant \lambda_0$

$$\|P_\pm\|_{X \to Y} \leqslant \frac{1}{2C}.$$

Consequently, by use of the Neumann series, we receive that $S + P_\pm$ is bijective and

$$\|(u, \nabla p)\|_X \leqslant C \frac{1}{1 - C\|P_\pm\|_{X \to Y}} \|(f, a)\|_Y \leqslant 2C \|(f, a)\|_Y \tag{10.10}$$

for all $(f, a) \in Y$ and $(u, \nabla p) = (S + P_\pm)^{-1}(f, a)$. For any $g \in W_q^1(\Omega)^n$ we have $(f, a) \in Y$, where $a := \Pi_\tau \operatorname{tr} g \in$, so (10.10) implies (10.5) in the special case $f \in L_{q,\sigma}(\Omega)$.

Now let $f \in L_q(\Omega)^n$. Using (4.1), we decompose $f = f_0 + \nabla \pi$, $f_0 \in L_{q,\sigma}(\Omega)$, $\nabla \pi \in \mathcal{G}_q(\Omega)$, where we additionally have a constant $C'' = C''(n, q, \Omega) > 0$ so that

$$\|(f_0, \nabla \pi)\|_q \leqslant C'' \|f\|_q. \tag{10.11}$$

Problem $(10.4)_\pm$ with right-hand side function f_0 admits a unique solution $(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ satisfying (10.5) with f_0 instead of f. Hence, $(u, \nabla p + \nabla \pi) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times \mathcal{G}_q(\Omega)$ is the unique solution of $(10.4)_\pm$ with right-hand side function f and (10.11) yields the resolvent estimate (10.5) with $\nabla p + \nabla \pi$ instead of ∇p.

In order to prove (i), similar to the proof of (ii), consider the Banach spaces

$$\begin{aligned} X' &:= [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega), \\ Y' &:= L_q(\Omega)^n \times \{\Pi_\tau \operatorname{tr} g : g \in W_q^1(\Omega)^n\} \end{aligned} \tag{10.12}$$

with the same norms as for X and Y and the operators $S : X' \to Y'$ and $P_- : X' \to Y'$ as defined above. Then estimate (10.9) is still valid for X', Y' instead of X, Y, so we have

$$\|P_-\|_{X' \to Y'} \leqslant |\alpha| \tag{10.13}$$

if $\lambda_0 \geqslant 1$. Theorem 9.2(i) yields that $S : X' \to Y'$ is surjective. Hence, there exists $\epsilon = \epsilon(n, q, \Omega, \lambda) > 0$ so that in case $\|P_-\|_{X' \to Y'} < \epsilon$ the operator $S + P_- : X' \to Y'$ is surjective as well. Now (10.13) yields the statement. □

Corollary 10.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$ and $1 < q < \infty$ such that the Helmholtz decomposition*

$$L_q(\Omega)^n = L_{q,\sigma}(\Omega) \oplus G_q(\Omega) \tag{10.14}$$

holds and such that Assumption 4.4 is valid. Let $0 < \theta < \pi$. Then there exist $\lambda_0 = \lambda_0(n, q, \theta, \Omega, \alpha) > 0$ and $C = C(n, q, \theta, \Omega) > 0$ such that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$, problem $(10.4)_-$ and $(10.4)_+$, respectively, has a unique solution

$$(u, \nabla p) \in [W_q^2(\Omega)^n \cap L_{q,\sigma}(\Omega)] \times G_q(\Omega)$$

for any $f \in L_q(\Omega)^n$, $g \in W_q^1(\Omega)^n$ and this solution fulfills the resolvent estimate

$$\|(\lambda u, \sqrt{\lambda} \nabla u, \nabla^2 u, \nabla p)\|_q \leqslant C \|(f, \sqrt{\lambda} g, \nabla g)\|_q.$$

Remark 10.5. Corollary 10.4 shows that we can recover the main result in [30] (concerning the Stokes resolvent problem) for partial slip type boundary conditions instead of no slip. Still, note that in [30] the Assumption 4.4 is not required.

Remark 10.6. As the proof of Theorem 10.2 shows, we could further add another zero order boundary term of the form Au with some matrix $A \in W^1_\infty(\Omega)^{n\times n}$ to the partial slip type boundary conditions, i.e.,

$$\begin{cases} \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu + Au) &= \Pi_\tau g \quad \text{on } \partial\Omega \\ \nu\cdot u &= 0 \qquad \text{on } \partial\Omega \end{cases}$$

and the assertion is still valid (where the quantities now may additionally depend on the matrix A, of course).

11 The Stokes Operator

We aim to define a suitable Stokes operator such that solving the related Cauchy problem leads to well-posedness of the Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \quad \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T)\times\Omega \\ \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ \nu\cdot u &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{cases} \tag{11.1}$$

We begin by stating some auxiliary results, showing that the definition given afterwards is meaningful.

11.1 Projected and Non-Projected Equations

Under Assumptions 4.2 and 4.3 we can use the decomposition (4.1) to reformulate $(11.1)_+$ and $(11.1)_-$, respectively, with $f : (0,T) \to L_q(\Omega)^n$ and $u_0 \in L_{q,\sigma}(\Omega)$ as the equivalent problems

$$\begin{cases} \partial_t u - \widetilde{\mathbb{P}}\Delta u &= f_0 \quad \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T)\times\Omega \\ \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ \nu\cdot u &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega \end{cases} \tag{11.2}$$

with $f_0 : (0,T) \to L_{q,\sigma}(\Omega)$ and $u_0 \in L_{q,\sigma}(\Omega)$, where we denote the continuous linear projection onto $L_{q,\sigma}(\Omega)$, related to decomposition (4.1), by $\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_q : L_q(\Omega)^n \to L_q(\Omega)^n$. We obtain this reformulation by the following equivalence of the corresponding resolvent problems which is a direct consequence of the continuity of the projection $\widetilde{\mathbb{P}}$.

Lemma 11.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$ and $1 < q < \infty$ such that Assumptions 4.2 and 4.3 are valid and let $0 < \theta < \pi$ and $\alpha \in \mathbb{R}$. Then for any $\lambda \in \Sigma_\theta$ and $f \in L_q(\Omega)^n$ a couple*

$$(u, \nabla p) \in [L_{q,\sigma}(\Omega) \cap \mathscr{D}(\Delta^\pm_{\alpha,q})] \times \mathcal{G}_q(\Omega)$$

solves

$$\begin{cases} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) &= 0 \quad \text{on } \partial\Omega \\ \nu\cdot u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{11.3}$$

if and only if $u \in L_{q,\sigma}(\Omega) \cap \mathscr{D}(\Delta^{\pm}_{\alpha,q})$ *solves*

$$\begin{cases} \lambda u - \widetilde{\mathbb{P}}\Delta u &= \widetilde{\mathbb{P}} f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) &= 0 \quad \text{on } \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{11.4}$$

and $\nabla p = (I - \widetilde{\mathbb{P}})(f - \lambda u + \Delta u)$. *In this case, for a fixed* $\lambda_0 > 0$, *the validity of*

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u, \nabla p)\|_q \leqslant C\|f\|_q \tag{11.5}$$

for all $|\lambda| > \lambda_0$ *with some* $C = C(n, q, \theta, \Omega) > 0$ *is equivalent to the validity of*

$$\|(\lambda u, \sqrt{\lambda}\nabla u, \nabla^2 u)\|_q \leqslant C'\|f\|_q \tag{11.6}$$

for all $|\lambda| > \lambda_0$ *with some* $C' = C'(n, q, \theta, \Omega) > 0$.

Definition 11.2 (Stokes operator)**.** Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$ and $1 < q < \infty$ such that Assumptions 4.2 and 4.3 are valid and let $\alpha \in \mathbb{R}$. We define the *Stokes operator* as

$$A^{\pm}_{\mathrm{S},\alpha} = A^{\pm}_{\mathrm{S},\alpha,q} : \mathscr{D}(A^{\pm}_{\mathrm{S},\alpha}) \subset L_{q,\sigma}(\Omega) \to L_{q,\sigma}(\Omega), \quad u \longmapsto \widetilde{\mathbb{P}}_q \Delta u$$

on $\mathscr{D}(A^{\pm}_{\mathrm{S},\alpha,q}) := \mathscr{D}(\Delta^{\pm}_{\alpha,q}) \cap L_{q,\sigma}(\Omega)$.

11.2 Stokes Semigroup

Proposition 11.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a uniform* $C^{2,1}$*-domain,* $n \geqslant 2$ *and* $1 < q < \infty$ *such that Assumptions* 4.2, 4.3 *and* 4.4 *are valid and let* $\alpha \in \mathbb{R}$. *Then the Stokes operator* $A^{\pm}_{\mathrm{S},\alpha}$ *is the generator of a strongly continuous analytic semigroup*

$$(e^{tA^{\pm}_{\mathrm{S},\alpha}})_{t \geqslant 0}$$

on $L_{q,\sigma}(\Omega)$. *For arbitrary* $\omega \in (0, \frac{\pi}{2})$ *we can find* $d \geqslant 0$ *such that the semigroup, generated by the shifted Stokes operator* $A^{\pm}_{\mathrm{S},\alpha} - d$, *is a bounded analytic strongly continuous semigroup with angle* ω.

Proof. For $\theta := \omega + \frac{\pi}{2}$, Theorem 10.2 and Lemma 11.1 yield some constants $\lambda_0 = \lambda_0(n, q, \theta, \Omega, \alpha)$ and $C = C(n, q, \theta, \Omega)$ so that for $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$ the resolvent $(\lambda - A^{\pm}_{\mathrm{S},\alpha})^{-1} : L_{q,\sigma}(\Omega) \to L_{q,\sigma}(\Omega)$ exists and fulfills the resolvent estimate

$$\|\lambda(\lambda - A^{\pm}_{\mathrm{S},\alpha})^{-1} f\|_q \leqslant C\|f\|_q.$$

Let $\delta = \delta(\theta) \in (0, 1)$ so that $|a + b| \geqslant \delta(|a| + b)$ for all $a \in \Sigma_\theta$ and $b > 0$. Existence of such a constant δ can be deduced from a compactness argument by employing $\theta < \pi$. Then for any $\lambda \in \Sigma_\theta$ we have $|\lambda + \frac{1}{\delta}\lambda_0| \geqslant \lambda_0$ and $\lambda + \frac{1}{\delta}\lambda_0 \in \Sigma_\theta$ and consequently for $f \in L_{q,\sigma}(\Omega)$

$$\begin{aligned} &\Big\|\lambda\Big(\lambda + \frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha}\Big)^{-1} f\Big\|_q \\ &\leqslant \Big\|\Big(\lambda + \frac{1}{\delta}\lambda_0\Big)\Big(\lambda + \frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha}\Big)^{-1} f\Big\|_q + \frac{\frac{1}{\delta}\lambda_0}{|\lambda + \frac{1}{\delta}\lambda_0|}\Big\|\Big(\lambda + \frac{1}{\delta}\lambda_0\Big)\Big(\lambda + \frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha}\Big)^{-1} f\Big\|_q \\ &= \Big(1 + \frac{\frac{1}{\delta}\lambda_0}{|\lambda + \frac{1}{\delta}\lambda_0|}\Big)\Big\|\Big(\lambda + \frac{1}{\delta}\lambda_0\Big)\Big(\lambda + \frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha}\Big)^{-1} f\Big\|_q \\ &\leqslant \Big(1 + \frac{1}{\delta}\Big)C\|f\|_q. \end{aligned}$$

Hence, $A^{\pm}_{\mathrm{S},\alpha} - \frac{1}{\delta}\lambda_0$ is the generator of a strongly continuous bounded analytic semigroup with angle ω. □

Lemma 11.4. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{0,1}$-domain (i.e., a uniform Lipschitz domain) and $n \geqslant 2$. Then there exists a linear operator E, mapping functions on Ω to functions on $\mathbb{R}^n$, such that $Ef|_\Omega = f$ holds for any function f on Ω (i.e., E is an extension operator) and such that*

$$E : W^k_q(\Omega) \longrightarrow W^k_q(\mathbb{R}^n) \tag{11.7}$$

is continuous for all $1 \leqslant q \leqslant \infty$ and all $k \in \mathbb{N}_0$.

Proof. See [65], Thm. VI.3.1/5. The condition for Ω to be a uniform $C^{0,1}$-domain is exactly the condition in [65] for $\partial\Omega$ to be minimally smooth. □

The following semigroup estimates are the essential tools for application of a fixed-point argument to the Navier-Stokes equations in order to receive local well-posedness.

Proposition 11.5. *Let $1 < p \leqslant q < \infty$, $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain such that Assumptions 4.2, 4.3 and 4.4 are valid for p and for q. Let $\alpha \in \mathbb{R}$ and $T > 0$. Then for the semigroup of the Stokes operator $A^{\pm}_{\mathrm{S},\alpha,p}$ there exists a constant $C = C(n, q, p, \Omega, \alpha, T) > 0$ such that for all $t \in (0, T)$ and any $f \in L_{p,\sigma}(\Omega)$ the following inequalities hold.*

(i) $\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_q \leqslant Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p$ *if* $\frac{1}{p} - \frac{1}{q} < \frac{2}{n}$.

(ii) $\|\nabla e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_q \leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p$ *if* $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$.

Proof. Let $\beta := n(\frac{1}{p} - \frac{1}{q}) \in [0, 2)$. Then for the Bessel-potential space

$$H^\beta_p(\mathbb{R}^n) = [L_p(\mathbb{R}^n), W^2_p(\mathbb{R}^n)]_{\frac{\beta}{2}} \tag{11.8}$$

we have the Sobolev embedding $H^\beta_p(\mathbb{R}^n) \subset L_q(\mathbb{R}^n)$ (see, e.g., [66]) with some embedding constant $C_e = C_e(n, q, p) > 0$ since the condition $p \leqslant q$, $\frac{n}{p} - \beta \leqslant \frac{n}{q}$ is satisfied. Let $t \in (0, T)$ and $f \in L_{p,\sigma}(\Omega)$ and denote by E the extension operator from Lemma 11.4 (where, in case of a vector field v, by Ev we mean componentwise application of the operator E). Then we conclude

$$\begin{aligned}
\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{L_q(\Omega)^n} &\leqslant \|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{L_q(\mathbb{R}^n)^n} \\
&\leqslant C_e\|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{H^\beta_p(\mathbb{R}^n)^n} \\
&\leqslant C_e\|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{1-\frac{\beta}{2}}_{L_p(\mathbb{R}^n)^n}\|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{\frac{\beta}{2}}_{H^2_p(\mathbb{R}^n)^n} \\
&\leqslant C_e\|E\|\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{1-\frac{\beta}{2}}_{L_p(\Omega)^n}\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{\frac{\beta}{2}}_{H^2_p(\Omega)^n}
\end{aligned} \tag{11.9}$$

where $\|E\|$ denotes the maximum of the operator norms of (11.7) for $k \in \{0, 2\}$. If $\overline{\omega} = \overline{\omega}(n, p, \Omega, \alpha) \in \mathbb{R}$ denotes the growth bound of $(e^{tA^{\pm}_{\mathrm{S},\alpha,p}})_{t\geqslant 0}$ then for $\omega := |\overline{\omega}| + 1$ and some $M = M(n, p, \Omega, \alpha) > 0$ we have

$$\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{L_p(\Omega)^n} \leqslant Me^{\omega t}\|f\|_{L_p(\Omega)^n} \leqslant Me^{\omega T}\|f\|_{L_p(\Omega)^n}. \tag{11.10}$$

Since $(e^{tA^{\pm}_{\mathrm{S},\alpha,p}})_{t\geqslant 0}$ is an analytic semigroup, we have $e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f \in \mathscr{D}(A^{\pm}_{\mathrm{S},\alpha,p})$. Fix any $0 < \theta < \pi$ and choose $\lambda_0 = \lambda_0(n, p, \theta, \Omega, \alpha) \geqslant 1$ and $C = C(n, p, \theta, \Omega) > 0$ such that the conditions of Theorem 10.2 are satisfied. In the proof of Proposition 11.3 we have seen that for some appropriate $\delta = \delta(\theta) \in (0,1)$ the strongly continuous semigroup $(e^{t(A^{\pm}_{\mathrm{S},\alpha,p} - \frac{1}{\delta}\lambda_0)})_{t\geqslant 0}$, generated by $A^{\pm}_{\mathrm{S},\alpha,p} - \frac{1}{\delta}\lambda_0$, is analytic and bounded. We receive from Theorem 10.2 and Lemma 11.1 that

$$\Big\|\Big(\frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha,p}\Big)^{-1}\Big\|_{L_p(\Omega)^n \to H^2_p(\Omega)^n} \leqslant C$$

(since $\frac{1}{\delta}\lambda_0 \geqslant \lambda_0 \geqslant 1$). Consequently,

$$\begin{aligned}
\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{H^2_p(\Omega)^n} &= \Big\|\Big(\frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha,p}\Big)^{-1}\Big(\frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha,p}\Big) e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\Big\|_{H^2_p(\Omega)^n} \\
&\leqslant C e^{\frac{1}{\delta}\lambda_0 T}\Big\|\Big(\frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha,p}\Big) e^{t(A^{\pm}_{\mathrm{S},\alpha,p} - \frac{1}{\delta}\lambda_0)} f\Big\|_{L_p(\Omega)^n} \\
&\leqslant CC' e^{\frac{1}{\delta}\lambda_0 T}\frac{1}{t}\|f\|_{L_p(\Omega)^n},
\end{aligned} \tag{11.11}$$

where

$$C' = C'(n,p,\theta,\Omega,\alpha) = \sup_{t>0}\Big\|t\Big(\frac{1}{\delta}\lambda_0 - A^{\pm}_{\mathrm{S},\alpha,p}\Big) e^{t(A^{\pm}_{\mathrm{S},\alpha,p} - \frac{1}{\delta}\lambda_0)}\Big\|_{L_{p,\sigma}(\Omega) \to L_{p,\sigma}(\Omega)}$$

is a finite constant, since $(e^{t(A^{\pm}_{\mathrm{S},\alpha,p} - \frac{1}{\delta}\lambda_0)})_{t\geqslant 0}$ is bounded and analytic. Now (11.9), (11.10) and (11.11) yield

$$\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{L_q(\Omega)^n} \leqslant C_e\|E\|(Me^{\omega T})^{1-\frac{\beta}{2}}(CC'e^{\frac{1}{\delta}\lambda_0 T})^{\frac{\beta}{2}} t^{-\frac{\beta}{2}}\|f\|_{L_p(\Omega)^n}.$$

Hence (i) is proved.

In order to prove (ii), let $t \in (0,T)$ and $f \in L_{p,\sigma}(\Omega)$ again, where we have $\beta = n(\frac{1}{p} - \frac{1}{q}) \in [0,1)$ this time. The condition $p \leqslant q$, $\frac{n}{p} - \beta \leqslant \frac{n}{q}$ for Sobolev's embedding is still satisfied, so we have $H^{\beta+1}_p(\mathbb{R}^n) \subset H^1_q(\mathbb{R}^n)$ with some embedding constant $C_e = C_e(n,q,p) > 0$ as above. Furthermore, the condition $\beta < 1$ gives that (11.8) holds with $\beta + 1$ instead of β. Therefore we have

$$\begin{aligned}
\|\nabla e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{L_q(\Omega)^{n^2}} &\leqslant \|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{H^1_q(\Omega)^n} \\
&\leqslant \|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{H^1_q(\mathbb{R}^n)^n} \\
&\leqslant C_e\|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{H^{\beta+1}_p(\mathbb{R}^n)^n} \\
&\leqslant C_e\|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{1-\frac{\beta+1}{2}}_{L_p(\mathbb{R}^n)^n}\|Ee^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{\frac{\beta+1}{2}}_{H^2_p(\mathbb{R}^n)^n} \\
&\leqslant C_e\|E\|\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{1-\frac{\beta+1}{2}}_{L_p(\Omega)^n}\|e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|^{\frac{\beta+1}{2}}_{H^2_p(\Omega)^n}.
\end{aligned}$$

Applying (11.10) and (11.11) again, we conclude

$$\begin{aligned}
\|\nabla e^{tA^{\pm}_{\mathrm{S},\alpha,p}} f\|_{L_q(\Omega)^{n^2}} &\leqslant C_e\|E\|(Me^{\omega T}\|f\|_{L_p(\Omega)^n})^{1-\frac{\beta+1}{2}}\Big(CC'e^{\frac{1}{\delta}\lambda_0 T}\frac{1}{t}\|f\|_{L_p(\Omega)^n}\Big)^{\frac{\beta+1}{2}} \\
&= C_e\|E\|(Me^{\omega T})^{1-\frac{\beta+1}{2}}(CC'e^{\frac{1}{\delta}\lambda_0 T})^{\frac{\beta+1}{2}} t^{-\frac{\beta+1}{2}}\|f\|_{L_p(\Omega)^n}. \qquad \square
\end{aligned}$$

12 The Navier-Stokes Equations

We consider the Navier-Stokes equations subject to partial slip type boundary conditions

$$\begin{cases} \partial_t u - \Delta u + \nabla p + (u\cdot\nabla)u &= 0 \quad \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T)\times\Omega \\ \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ \nu\cdot u &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega, \end{cases} \tag{12.1}$$

where $(u\cdot\nabla)u = \sum_{j=1}^n u^j\partial_j u$. We denote $(12.1)_+$ and $(12.1)_-$ for the Navier-Stokes equations subject to boundary conditions with the related boundary operator D_+ and D_-, respectively. With the assumption $\operatorname{div} u = 0$ we also have $(u\cdot\nabla)u = \sum_{j=1}^n \partial_j(u^j u)$.

Theorem 12.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$ and $n < q < \infty$ such that Assumptions 4.2, 4.3 and 4.4 are valid for q and also for $\frac{q}{2}$. Again we denote the projections related to decomposition (4.1) by $\widetilde{\mathbb{P}}_q$ and $\widetilde{\mathbb{P}}_{q/2}$, respectively. Let $0 < \theta < \pi$, $\alpha \in \mathbb{R}$ and $u_0 \in L_{q,\sigma}(\Omega)$. Then there exists $T > 0$ such that the Navier-Stokes equations $(12.1)_+$ and $(12.1)_-$, respectively, admit a unique mild solution depending continuously on u_0, i.e., the integral equation*

$$u(t) = e^{tA^\pm_{\mathrm{S},\alpha,q}}u_0 - \int_0^t e^{(t-s)A^\pm_{\mathrm{S},\alpha,q/2}}\widetilde{\mathbb{P}}_{q/2}\sum_{j=1}^n \partial_j(u^j(s)u(s))ds, \quad t\in[0,T], \tag{12.2}$$

related to the projected Navier-Stokes equations

$$\begin{cases} \partial_t u - \widetilde{\mathbb{P}}_q\Delta u + \widetilde{\mathbb{P}}_{q/2}(u\cdot\nabla)u &= 0 \quad \textit{in } (0,T)\times\Omega \\ \operatorname{div} u &= 0 \quad \textit{in } (0,T)\times\Omega \\ \Pi_\tau(\alpha u + \mathrm{D}_\pm(u)\nu) &= 0 \quad \textit{on } (0,T)\times\partial\Omega \\ \nu\cdot u &= 0 \quad \textit{on } (0,T)\times\partial\Omega \\ u|_{t=0} &= u_0 \quad \textit{in } \Omega \end{cases} \tag{12.3}$$

admits a unique solution

$$u \in \mathrm{BC}\big([0,T], L_{q,\sigma}(\Omega)\big) \quad \textit{with} \quad [t \mapsto \sqrt{t}\nabla u(t)] \in \mathrm{BC}\big([0,T], L_q(\Omega)^{n\times n}\big). \tag{12.4}$$

Proof. Let $u_0 \in L_{q,\sigma}(\Omega)$. For $M > 0$ and $T > 0$ we define $X_{M,T}$ as the space of functions u satisfying (12.4) and $\|u\|_T \leqslant M\|u_0\|_q$, where $\|u\|_T := \sup_{t\in[0,T]}\|u(t)\|_q + \sup_{t\in[0,T]}\sqrt{t}\|\nabla u(t)\|_q$ and we set

$$H(u)(t) := e^{tA^\pm_{\mathrm{S},\alpha,q}}u_0 - \int_0^t e^{(t-s)A^\pm_{\mathrm{S},\alpha,q/2}}\widetilde{\mathbb{P}}_{q/2}\sum_{j=1}^n \partial_j(u^j(s)u(s))\,ds$$

for $u \in X_{M,T}$ and $t \in [0,T]$. We aim to prove that H is a contraction. Assuming $T < 1$, we can apply Proposition 11.5(i) with $T = 1$ and $p = \frac{q}{2}$ (and with $p = q$ for the first term) to receive a constant $C = C(n,q,\Omega,\alpha) > 0$ so that

$$\|H(u)(t)\|_q \leqslant C\Big(\|u_0\|_q + \sum_{j=1}^n \int_0^t (t-s)^{-\frac{n}{2q}}\|\widetilde{\mathbb{P}}_{q/2}\partial_j(u^j(s)u(s))\|_{\frac{q}{2}}\,ds\Big). \tag{12.5}$$

The continuity of $\widetilde{\mathbb{P}}_{q/2}$ on $L_{q/2}(\Omega)^n$ and Hölder's estimate yield some $C' = C'(n,q,\Omega) > 0$ so that

$$\begin{aligned}
&\sum_{j=1}^n \int_0^t (t-s)^{-\frac{n}{2q}} \|\widetilde{\mathbb{P}}_{q/2}\partial_j(u^j(s)u(s))\|_{\frac{q}{2}}\, ds \\
&\leqslant C' \int_0^t (t-s)^{-\frac{n}{2q}} \|\nabla u(s)\|_q \|u(s)\|_q\, ds \\
&\leqslant C' \Big(\sup_{\tau\in[0,T]} \|u(\tau)\|_q \Big) \Big(\sup_{\tau\in[0,T]} \sqrt{\tau}\|\nabla u(\tau)\|_q \Big) \int_0^t \frac{(t-s)^{-\frac{n}{2q}}}{\sqrt{s}}\, ds \\
&\leqslant C' \Big(\sup_{\tau\in[0,T]} \|u(\tau)\|_q + \sup_{\tau\in[0,T]} \sqrt{\tau}\|\nabla u(\tau)\|_q \Big)^2 \int_0^t \frac{(t-s)^{-\frac{n}{2q}}}{\sqrt{s}}\, ds \qquad (12.6)\\
&= C'\|u\|_T^2 \int_0^t \frac{(t-s)^{-\frac{n}{2q}}}{\sqrt{s}}\, ds \\
&\leqslant C' C_{n,q} \|u\|_T^2 t^{\frac{1}{2}-\frac{n}{2q}} \\
&\leqslant C' C_{n,q} \|u\|_T^2 T^{\frac{1}{2}-\frac{n}{2q}} \\
&\leqslant C' C_{n,q} M^2 \|u_0\|_q^2 T^{\frac{1}{2}-\frac{n}{2q}},
\end{aligned}$$

since $\frac{n}{2q} < 1$ gives that

$$\begin{aligned}
\int_0^t \frac{(t-s)^{-\frac{n}{2q}}}{\sqrt{s}}\, ds &= \int_0^{t/2} \frac{1}{\sqrt{s}} \frac{1}{(t-s)^{\frac{n}{2q}}}\, ds + \int_{t/2}^t \frac{1}{\sqrt{s}} \frac{1}{(t-s)^{\frac{n}{2q}}}\, ds \\
&\leqslant \int_0^{t/2} \frac{1}{\sqrt{s}} \frac{1}{(t-\frac{t}{2})^{\frac{n}{2q}}}\, ds + \int_{t/2}^t \frac{1}{\sqrt{\frac{t}{2}}} \frac{1}{(t-s)^{\frac{n}{2q}}}\, ds \\
&= \Big(2^{\frac{n}{2q}+\frac{1}{2}} + \frac{1}{1-\frac{n}{2q}} 2^{\frac{n}{2q}-\frac{1}{2}} \Big) t^{\frac{1}{2}-\frac{n}{2q}} \\
&=: C_{n,q} t^{\frac{1}{2}-\frac{n}{2q}}.
\end{aligned}$$

Therefore

$$\|H(u)(t)\|_q \leqslant C\|u_0\|_q + CC'C_{n,q}M^2 T^{\frac{1}{2}-\frac{n}{2q}} \|u_0\|_q^2 \qquad (12.7)$$

for $M > 0$, $0 < T < 1$, $u \in X_{M,T}$ and $t \in [0,T]$.

Now we aim to receive a similar estimate for

$$\nabla H(u)(t) = \nabla e^{tA^{\pm}_{S,\alpha,q}} u_0 - \int_0^t \nabla e^{(t-s)A^{\pm}_{S,\alpha,q/2}} \widetilde{\mathbb{P}}_{q/2} \sum_{j=1}^n \partial_j(u^j(s)u(s))\, ds^{\mathrm{a}}$$

for $u \in X_{M,T}$ and $t \in [0,T]$. We assume $T < 1$ again and apply Proposition 11.5(ii) with $T = 1$ and $p = \frac{q}{2}$ (and with $p = q$ for the first term) to receive a constant $C =$

[a] We can interchange the integral $\int_0^t$ and the gradient: Pairing with some test function $\varphi \in C_c^\infty(\Omega)^n$ yields that the gradient of $\int_0^t e^{(t-s)A^{\pm}_{S,\alpha,q/2}} f_0\, ds$ (where $f_0 := \widetilde{\mathbb{P}} \sum_{j=1}^n \partial_j(u^j(s)u(s)) \in L_{q/2,\sigma}(\Omega)$) in the distributional sense is in fact given by $\int_0^t \nabla e^{(t-s)A^{\pm}_{S,\alpha,q/2}} f_0\, ds$.

$C(n,q,\Omega,\alpha)>0$ so that

$$\begin{aligned}\sqrt{t}\|\nabla H(u)(t)\|_q &\leqslant \sqrt{t}C\Big(t^{-\frac{1}{2}}\|u_0\|_q+\sum_{j=1}^n\int_0^t (t-s)^{-\frac{n}{2q}-\frac{1}{2}}\|\widetilde{\mathbb{P}}_{q/2}\partial_j(u^j(s)u(s))\|_{\frac{q}{2}}\,ds\Big)\\ &= C\Big(\|u_0\|_q+\sum_{j=1}^n\int_0^t \sqrt{t}(t-s)^{-\frac{n}{2q}-\frac{1}{2}}\|\widetilde{\mathbb{P}}_{q/2}\partial_j(u^j(s)u(s))\|_{\frac{q}{2}}\,ds\Big).\end{aligned}\tag{12.8}$$

Again, using the continuity of $\widetilde{\mathbb{P}}_{q/2}$ on $L_{q/2}(\Omega)^n$ and Hölder's estimate, we receive a constant $C''=C''(n,q,\Omega)>0$ so that

$$\begin{aligned}&\sum_{j=1}^n\int_0^t \sqrt{t}(t-s)^{-\frac{n}{2q}-\frac{1}{2}}\|\widetilde{\mathbb{P}}_{q/2}\partial_j(u^j(s)u(s))\|_{\frac{q}{2}}\,ds\\ &\quad\leqslant C''\int_0^t\sqrt{t}(t-s)^{-\frac{n}{2q}-\frac{1}{2}}\|\nabla u(s)\|_q\|u(s)\|_q\,ds\\ &\quad\leqslant C''\Big(\sup_{\tau\in[0,T]}\|u(\tau)\|_q\Big)\Big(\sup_{\tau\in[0,T]}\sqrt{\tau}\|\nabla u(\tau)\|_q\Big)\int_0^t\sqrt{t}\frac{(t-s)^{-\frac{n}{2q}-\frac{1}{2}}}{\sqrt{s}}\,ds\\ &\quad\leqslant C''\Big(\sup_{\tau\in[0,T]}\|u(\tau)\|_q+\sup_{\tau\in[0,T]}\sqrt{\tau}\|\nabla u(\tau)\|_q\Big)^2\int_0^t\sqrt{t}\frac{(t-s)^{-\frac{n}{2q}-\frac{1}{2}}}{\sqrt{s}}\,ds\\ &\quad\leqslant C''C'_{n,q}\|u\|_T^2t^{\frac{1}{2}-\frac{n}{2q}}\\ &\quad\leqslant C''C'_{n,q}\|u\|_T^2T^{\frac{1}{2}-\frac{n}{2q}}\\ &\quad\leqslant C''C'_{n,q}M^2\|u_0\|_q^2T^{\frac{1}{2}-\frac{n}{2q}},\end{aligned}$$

since $\frac{n}{2q}<\frac{1}{2}$ gives that

$$\begin{aligned}\int_0^t\sqrt{t}\frac{(t-s)^{-\frac{n}{2q}-\frac{1}{2}}}{\sqrt{s}}\,ds &= \int_0^{t/2}\sqrt{t}\frac{(t-s)^{-\frac{n}{2q}-\frac{1}{2}}}{\sqrt{s}}\,ds+\int_{t/2}^t\sqrt{t}\frac{(t-s)^{-\frac{n}{2q}-\frac{1}{2}}}{\sqrt{s}}\,ds\\ &\leqslant \int_0^{t/2}\sqrt{t}\frac{(t-\frac{t}{2})^{-\frac{n}{2q}-\frac{1}{2}}}{\sqrt{s}}\,ds+\int_{t/2}^t\sqrt{t}\frac{(t-s)^{-\frac{n}{2q}-\frac{1}{2}}}{\sqrt{\frac{t}{2}}}\,ds\\ &= \Big(2^{\frac{n}{2q}+1}+\frac{1}{1-\frac{n}{q}}2^{\frac{n}{2q}+1}\Big)t^{\frac{1}{2}-\frac{n}{2q}}\\ &=: C'_{n,q}t^{\frac{1}{2}-\frac{n}{2q}}.\end{aligned}$$

Therefore

$$\sqrt{t}\|\nabla H(u)(t)\|_q\leqslant C\|u_0\|_q+CC''C'_{n,q}M^2T^{\frac{1}{2}-\frac{n}{2q}}\|u_0\|_q^2\tag{12.9}$$

for $M>0$, $0<T<1$, $u\in X_{M,T}$ and $t\in[0,T]$.

We further receive for fixed $t_0\in[0,T]$ and $u\in X_{M,T}$ that $\|Hu(t_0)-Hu(t)\|_q\xrightarrow{t\to t_0}0$ and $\|\sqrt{t_0}\nabla Hu(t_0)-\sqrt{t}\nabla Hu(t)\|_q\xrightarrow{t\to t_0}0$ hold, by establishing analogous estimates. Hence, for arbitrary $u\in X_{M,T}$, the functions Hu and $t\mapsto\sqrt{t}\nabla Hu(t)$ are continuous.

Now let $0<T<T'$, where $T':=\min\Big\{1,\Big(\frac{1}{4C^2C'C_{n,q}\|u_0\|_q}\Big)^{\frac{2}{1-\frac{n}{q}}},\Big(\frac{1}{4C^2C''C'_{n,q}\|u_0\|_q}\Big)^{\frac{2}{1-\frac{n}{q}}}\Big\}$ and $M\geqslant 2C$. Then (12.7) and (12.9) yield

$$H:X_{M,T}\to X_{M,T},\tag{12.10}$$

i.e., H maps $X_{M,T}$ into itself.

We proceed to prove that $H : X_{M,T} \to X_{M,T}$ satisfies a contraction estimate for $M \geqslant 2C$ and $T > 0$ small enough. Let $u, v \in X_{M,T}$ and $t \in [0,T]$. Then we have

$$H(u)(t) - H(v)(t) = \int_0^t e^{(t-s)A^{\pm}_{S,\alpha,q/2}} \widetilde{\mathbb{P}}_{q/2} \sum_{j=1}^n \partial_j (u^j(s)u(s) - v^j(s)v(s))\, ds.$$

As in (12.5) we obtain

$$\|H(u)(t) - H(v)(t)\|_q \leqslant C \sum_{j=1}^n \int_0^t (t-s)^{-\frac{n}{2q}} \|\widetilde{\mathbb{P}}_{q/2} \partial_j (u^j(s)u(s) - v^j(s)v(s))\|_{\frac{q}{2}}\, ds$$

and as in (12.8) we receive

$$\|\sqrt{t}\nabla\big(H(u)(t) - H(v)(t)\big)\|_q \leqslant C \sum_{j=1}^n \int_0^t \sqrt{t}(t-s)^{-\frac{n}{2q}-\frac{1}{2}} \|\widetilde{\mathbb{P}}_{q/2} \partial_j (u^j(s)u(s) - v^j(s)v(s))\|_{\frac{q}{2}}\, ds.$$

Now for $0 \leqslant s \leqslant t$ we can estimate

$$\begin{aligned}
&\|\partial_j\big(u^j(s)u(s) - v^j(s)v(s)\big)\|_{\frac{q}{2}} \\
&= \|\partial_j\big(u^j(s)[u(s) - v(s)] + [u^j(s) - v^j(s)]v(s)\big)\|_{\frac{q}{2}} \\
&\leqslant (\|u(s)\|_q + \|v(s)\|_q)\|\nabla\big(u(s) - v(s)\big)\|_q + (\|\nabla u(s)\|_q + \|\nabla v(s)\|_q)\|u(s) - v(s)\|_q \\
&\leqslant (\|u\|_T + \|v\|_T)\frac{1}{\sqrt{s}}\|u - v\|_T + \frac{1}{\sqrt{s}}(\|u\|_T + \|v\|_T)\|u - v\|_T \\
&\leqslant \frac{4M}{\sqrt{s}}\|u_0\|_q\|u - v\|_T.
\end{aligned} \tag{12.11}$$

The continuity of $\widetilde{\mathbb{P}}_{q/2}$ on $L_{q/2}(\Omega)^n$ and (12.11) yield some $C''' = C'''(n,q,\Omega) > 0$ so that

$$\begin{aligned}
\|H(u)(t) - H(v)(t)\|_q &\leqslant CC'''M\|u_0\|_q\|u - v\|_T \int_0^t \frac{(t-s)^{-\frac{n}{2q}}}{\sqrt{s}}\, ds \\
&\leqslant CC'''C_{n,q}MT^{\frac{1}{2}-\frac{n}{2q}}\|u_0\|_q\|u - v\|_T
\end{aligned}$$

and

$$\|\sqrt{t}\nabla\big(H(u)(t) - H(v)(t)\big)\|_q \leqslant CC'''C'_{n,q}MT^{\frac{1}{2}-\frac{n}{2q}}\|u_0\|_q\|u - v\|_T.$$

Thus, we have

$$\|H(u) - H(v)\|_T \leqslant \frac{1}{2}\|u - v\|_T \tag{12.12}$$

if $0 < T < T''$, where $T'' := \min\left\{1, \left(\frac{1}{4C^2C'''C_{n,q}\|u_0\|_q}\right)^{\frac{2}{1-\frac{n}{q}}}, \left(\frac{1}{4C^2C'''C'_{n,q}\|u_0\|_q}\right)^{\frac{2}{1-\frac{n}{q}}}\right\}$ and $M \geqslant 2C$.

In total, for $M \geqslant 2C$ and $0 < T < T_0$, where $T_0 := \min\{T', T''\}$, (12.10) and (12.12) yield that $H : X_{M,T} \to X_{M,T}$ is a contraction and therefore has a unique fixed point. □

Remark 12.2. Rewriting (12.3) as the original Navier-Stokes equations (12.1) might be not possible in case the projections $\widetilde{\mathbb{P}}_q$ and $\widetilde{\mathbb{P}}_{q/2}$ fail to coincide on $L_q(\Omega) \cap L_{q/2}(\Omega)$. In fact, we do not know the projection $\widetilde{\mathbb{P}}_q$ to be consistent with respect to $1 < q < \infty$ in general. Nevertheless, we can not replace $\widetilde{\mathbb{P}}_{q/2}$ in (12.2) by $\widetilde{\mathbb{P}}_q$ and guarantee that

(12.2) is meaningful, since the nonlinear term is not contained in $L_q(\Omega)^n$ in general but in $L_{q/2}(\Omega)^n$. Still, when applying Theorem 12.1 to a common setting, e.g., bent half spaces or domains with a compact boundary, then the obtained mild solution of (12.3) always conincides with the usual meaning of a mild solution of (12.1) (cf. [30]). In this case, $\widetilde{\mathbb{P}}_q$ equals the standard Helmholtz projection.

IV Stokes and Navier-Stokes Equations in TLL Spaces

The subject of this chapter, on the one hand, is the scale of Triebel-Lizorkin-Lorentz spaces (TLL spaces) $F^{s,r}_{p,q}$ and their properties. On the other hand, we apply these properties to solve the Stokes and Navier-Stokes equations in this scale. The obtained results are consequently valid for all function spaces that are included in the scale of TLL spaces $F^{s,r}_{p,q}$: By setting $r = p$, we obtain the Bessel-potential spaces H^s_p for $q = 2$ as well as the Sobolev-Slobodeckiĭ spaces W^s_p for $q = p$ in case $s \notin \mathbb{Z}$ and $q = 2$ in case $s \in \mathbb{Z}$. In particular, we obtain the Lebesgue spaces L_p by setting $s = 0$ as well as the Lorentz spaces $L_{p,r} = F^{0,r}_{p,2}$.

Concerning the Stokes equations we aim to prove that the Stokes operator in TLL spaces admits a bounded H^∞-calculus. Finally, we apply this to prove existence of unique maximal strong solutions of the Navier-Stokes equations

$$\left\{ \begin{aligned} \partial_t u - \Delta u + \nabla p + (u \cdot \nabla) u &= f && \text{in } (0,T) \times \mathbb{R}^n \\ \operatorname{div} u &= 0 && \text{in } (0,T) \times \mathbb{R}^n \\ u|_{t=0} &= u_0 && \text{in } \mathbb{R}^n \end{aligned} \right.$$

in TLL ground spaces, i.e., the solution is supposed to exist on a time interval $(0, T^*)$ which is not possible to be increased. This can be either the whole time line $(0, T^*) = (0, \infty)$ or we have a blow-up of the solution at finite time $T^* < \infty$.

13 TLL Spaces

13.1 Definition and Properties

For parameters $s \in \mathbb{R}$, $1 < p, q < \infty$, $1 \leqslant r \leqslant \infty$ we call

$$F^{s,r}_{p,q} := \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \|u\|_{F^{s,r}_{p,q}} < \infty \right\}$$

Triebel-Lizorkin-Lorentz space (as defined in [14]). The norm is given by

$$\|u\|_{F^{s,r}_{p,q}} := \|(\varphi_k * u)_{k \in \mathbb{N}_0}\|_{L_{p,r}(l^s_q)} = \left\| \Big(\sum_{k \in \mathbb{N}_0} [2^{sk} |\varphi_k * u|]^q \Big)^{\frac{1}{q}} \right\|_{L_{p,r}}$$

where $(\widehat{\varphi}_k)_{k \in \mathbb{N}_0}$ is a dyadic decomposition defined as follows (cf. [66], Def. 2.3.1/2).

Definition 13.1. Let $\boldsymbol{\Phi_N}$ (for $N \in \mathbb{N}$) denote the set of systems of functions $(\varphi_k)_{k \in \mathbb{N}_0} \subset \mathscr{S}(\mathbb{R}^n)$ with the following properties.

- $\widehat{\varphi}_k \geqslant 0$ for all $k \in \mathbb{N}_0$.
- $\operatorname{spt}(\widehat{\varphi}_k) \subset \{2^{k-N} \leqslant |x| \leqslant 2^{k+N}\}$ for $k \in \mathbb{N}$ and $\operatorname{spt}(\widehat{\varphi}_0) \subset \{|x| \leqslant 2^N\}$.
- There exist $D_1, D_2 > 0$ such that for all $\xi \in \mathbb{R}^n$

$$D_1 \leqslant \sum_{k=0}^{\infty} \widehat{\varphi}_k(\xi) \leqslant D_2. \tag{13.1}$$

- For any $\alpha \in \mathbb{N}_0^n$, there is $C_\alpha > 0$ such that for all $k \in \mathbb{N}_0$ and $\xi \in \mathbb{R}^n$

$$|\xi|^{|\alpha|}|\partial_\alpha \widehat{\varphi}_k(\xi)| \leqslant C_\alpha. \tag{13.2}$$

Additionally, we set $\mathbf{\Phi} := \bigcup_{N\in\mathbb{N}} \mathbf{\Phi}_N$ and call each family $(\widehat{\varphi}_k)_{k\in\mathbb{N}_0}$ with $(\varphi_k)_{k\in\mathbb{N}_0} \in \mathbf{\Phi}$ a *dyadic decomposition*.

Note that the constant C_α in (13.2) does not depend on the index k but on the selected $N \in \mathbb{N}$, i.e., on the radius 2^N of the dyadic decomposition. Also note that the existence of D_2 in (13.1) can be deduced from (13.2) with $\alpha = 0$ and the properties of $\operatorname{spt}(\widehat{\varphi}_k)$. We will often use the following more specific dyadic decomposition.

Example 13.2. Let $\phi \in C^\infty(\mathbb{R}^n)$ be radially symmetric with $\operatorname{spt}(\phi) \subset \{|x| \leqslant 1\}$, $\phi = 1$ on $\{|x| \leqslant \frac{1}{2}\}$ and $0 \leqslant \phi \leqslant 1$. We set $\widehat{\psi}(\xi) := \phi(\frac{\xi}{2}) - \phi(\xi)$ and $\widehat{\psi}_k(\xi) := \widehat{\psi}(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Now we set $\varphi_k := \psi_k$ for $k \geqslant 1$ and define $\varphi_0 \in \mathscr{S}(\mathbb{R}^n)$ by

$$\widehat{\varphi}_0(\xi) = \begin{cases} \sum\limits_{j\leqslant 0} \widehat{\psi}_j(\xi), & \text{if } \xi \neq 0 \\ 1, & \text{if } \xi = 0. \end{cases}$$

This implies $(\varphi_k)_{k\in\mathbb{N}_0} \in \mathbf{\Phi}_1$ with $\sum_{k\in\mathbb{N}_0} \varphi_k(\xi) = 1$ for all $\xi \in \mathbb{R}^n$. In addition, we have the following (easy to verify) properties:

(a) $\|\varphi_k\|_1 = \|\psi\|_1$ for all $k \in \mathbb{N}$.

(b) $\sum_{k=1}^N \widehat{\varphi}_k \xrightarrow{N\to\infty} 1$ locally uniformly on $\mathbb{R}^n$.

(c) $\sum_{j=0}^N \varphi_j * f \xrightarrow{N\to\infty} f$ in $\mathscr{S}(\mathbb{R}^n)$ for all $f \in \mathscr{S}(\mathbb{R}^n)$.

(d) $\sum_{j=0}^N \varphi_j * u \xrightarrow{N\to\infty} u$ in $\mathscr{S}'(\mathbb{R}^n)$ for all $u \in \mathscr{S}'(\mathbb{R}^n)$.

If we replace the Lorentz-norm $\|\cdot\|_{L_{p,r}(l_q^s)}$ by $\|\cdot\|_{L_p(l_q^s)}$, then we receive the well-known Triebel-Lizorkin spaces $F_{p,q}^s$. More precisely we have $F_{p,q}^{s,p} = F_{p,q}^s$. One can find the following result as [66], Rem. 2.4.2/1.

Proposition 13.3. *The TLL spaces are independent of the choice of the dyadic decomposition.*

The following result is due to Yang, Cheng and Peng (see [14]), where their proof is based on wavelet theory. We notice that it is possible to derive this property by L_p-interpolation and retraction and coretraction techniques as developed in [66], as well.

Theorem 13.4. *For $s \in \mathbb{R}$, $1 < p_0, p_1, q < \infty$, $1 \leqslant r_0, r_1, r \leqslant \infty$, $p_0 \neq p_1$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have*

$$\left(F_{p_0,q}^{s,r_0}, F_{p_1,q}^{s,r_1}\right)_{\theta,r} = F_{p,q}^{s,r}.$$

In particular, if $r_0 = p_0$ and $r_1 = p_1$, we have $\left(F_{p_0,q}^s, F_{p_1,q}^s\right)_{\theta,r} = F_{p,q}^{s,r}$.

Lemma 13.5. *Let $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leqslant r \leqslant \infty$. Then we have the following continuous embeddings.*

(i) $\mathscr{S}(\mathbb{R}^n) \subset F_{p,q}^{s,r} \subset \mathscr{S}'(\mathbb{R}^n)$ *and the first embedding is dense in case $r < \infty$.*

(ii) $F^{s+\tau,r}_{p,q} \subset F^{s,r}_{p,q}$ *for* $\tau \geqslant 0$.

(iii) $F^{s,r}_{p,q} \subset L_{p,r}$ *if* $s > 0$.

Proof. Assertion (i) follows from the corresponding fact for Triebel-Lizorkin spaces, since

$$\mathscr{S}(\mathbb{R}^n) \subset F^s_{p_0,q} \cap F^s_{p_1,q} \subset F^{s,r}_{p,q} \subset F^s_{p_0,q} + F^s_{p_1,q} \subset \mathscr{S}'(\mathbb{R}^n)$$

and since the intersection of an interpolation couple of Banach spaces is dense in their real interpolation space. Assertion (ii) is a consequence of $l^{s+\tau}_q \subset l^s_q$.

In order to prove (iii), we use the dyadic decomposition $(\widehat{\varphi}_k)_{k\in\mathbb{N}_0}$ given in Example 13.2. First we consider the estimate

$$\Big\| \sum_{k=0}^{\infty} |\varphi_k * u| \Big\|_{L_{p,r}} \leqslant C \Big\| \Big(\frac{1}{2^{sk}}\Big)_{k\in\mathbb{N}_0} \Big\|_{l_{q'}} \|u\|_{F^{s,r}_{p,q}} \tag{13.3}$$

that we receive from Hölder's inequality with $\frac{1}{q'} + \frac{1}{q} = 1$ and some $C = C(p, r, n) > 0$. Since $s > 0$, the right-hand side is finite for $u \in F^{s,r}_{p,q}$. Applying Example 13.2(d) we have $u = \sum_{k=0}^{\infty} \varphi_k * u$ where the convergence is in $\mathscr{S}'(\mathbb{R}^n)$. Now (13.3) gives that the series even converges pointwise a.e. and thus u is a measurable function. On the other hand (13.3) gives $\|u\|_{L_{p,r}} \leqslant C'\|u\|_{F^{s,r}_{p,q}}$ with some $C' = C'(p, r, n) > 0$. □

Proposition 13.6. $F^{s,r}_{p,q}$ *is of class* $\mathcal{HT}$ *for* $s \in \mathbb{R}$ *and* $1 < p, q, r < \infty$.

Proof. We need to show that the Hilbert transform

$$H : \mathscr{S}(\mathbb{R}, F^{s,r}_{p,q}) \longrightarrow \mathcal{M}(\mathbb{R}, F^{s,r}_{p,q}), \quad Hf(t) = \lim_{\epsilon \searrow 0} \int_{|s|>\epsilon} \frac{f(t-s)}{s} ds$$

has an extension $H \in \mathscr{L}(L_p(\mathbb{R}, F^{s,r}_{p,q}))$. For any $s \in \mathbb{R}$ and $1 < q < \infty$, Tonelli's theorem implies that $L_q(\mathbb{R}^n, l^s_q)$ is a space of class $\mathcal{HT}$ and so is $L_p(\mathbb{R}^n, l^s_q)$ for arbitrary $1 < p < \infty$. Since the Triebel-Lizorkin space $F^s_{p,q}$ is a retract of $L_p(\mathbb{R}^n, l^s_q)$ we can transfer the $\mathcal{HT}$-property to $F^s_{p,q}$ for any $s \in \mathbb{R}$ and $1 < p, q < \infty$.

Now for fixed parameters s, p, q, r as in the assertion we can use Theorem 13.4 to complete the proof. As a direct consequence of the interpolation property

$$L_r(\mathbb{R}, (X_0, X_1)_{\theta,r}) = (L_r(\mathbb{R}, X_0), L_r(\mathbb{R}, X_1))_{\theta,r}$$

we obtain that for an interpolation couple X_0, X_1 of spaces of class $\mathcal{HT}$ the real interpolation space $(X_0, X_1)_{\theta,r}$ is also of class $\mathcal{HT}$. Thus, $F^{s,r}_{p,q}$ is of class $\mathcal{HT}$. □

Corollary 13.7. $F^{s,r}_{p,q}$ *is reflexive for* $s \in \mathbb{R}$ *and* $1 < p, q, r < \infty$ *(due to* [56]*).*

Corollary 13.7 could also be obtained in a direct way, regarding the following result which is a conclusion of the corresponding result for Triebel-Lizorkin spaces (see [66], Thm. 2.6.2) and Theorem 13.4.

Proposition 13.8. *The dual space to* $F^{s,r}_{p,q}$ *is given by* $F^{-s,r'}_{p',q'}$ *for* $s \in \mathbb{R}$ *and* $1 < p, q, r < \infty$*, where* $1 < p', q', r' < \infty$ *are given by* $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $\frac{1}{r} + \frac{1}{r'} = 1$.

Proposition 13.9. $F^{s,r}_{p,q}$ *has property* (α) *for* $s \in \mathbb{R}$ *and* $1 < p, q, r < \infty$.

Proof. The Triebel-Lizorkin spaces $F^s_{p,q}$ have property (α), since there exists an isomorphism to a closed subspace of $L_p(\mathbb{R}^n, l^s_q)$. This implies the assertion, since property (α) is preserved under real interpolation (see [43], Thm. 4.5). □

Theorem 13.10 (Multiplier theorem for TLL spaces)**.** *Let $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leqslant r < \infty$. Let $(m_\lambda)_{\lambda\in\Lambda} \subset C^n(\mathbb{R}^n\backslash\{0\})$ such that*

$$C_\alpha := \sup_{\xi\in\mathbb{R}^n\backslash\{0\},\lambda\in\Lambda} |\xi^\alpha \partial_\alpha m_\lambda(\xi)| < \infty$$

for all $\alpha \in \{0,1\}^n$. Then for every $\lambda \in \Lambda$

$$\mathscr{F}^{-1} m_\lambda \mathscr{F} : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$$

has a (unique) continuous extension $T_\lambda : F^{s,r}_{p,q} \longrightarrow F^{s,r}_{p,q}$ such that

$$\|T_\lambda\|_{F^{s,r}_{p,q}\to F^{s,r}_{p,q}} \leqslant C \max_{\alpha\in\{0,1\}^n} C_\alpha,$$

where $C = C(n,s,r,p,q) > 0$. Furthermore, $(T_\lambda)_{\lambda\in\Lambda} \subset \mathscr{L}(F^{s,r}_{p,q})$ is $\mathcal{R}$-bounded in case $1 < r < \infty$.

Proof. We define $M_\lambda \in L_\infty(\mathbb{R}^n, \mathscr{L}(l^s_q))$ by setting $M_\lambda(\xi)x := (m_\lambda(\xi)x_k)_{k\in\mathbb{N}_0}$ for $\xi \in \mathbb{R}^n\backslash\{0\}$, $x = (x_k)_{k\in\mathbb{N}_0} \in l^s_q$ and $\lambda \in \Lambda$. By Kahane's contraction principle (see Theorem 5.1) we see that the assumption $C_\alpha < \infty$ implies $\mathcal{R}$-boundedness of $\{\xi^\alpha \partial_\alpha M_\lambda(\xi) : \xi \in \mathbb{R}^n\backslash\{0\},\ \lambda \in \Lambda\} \subset \mathscr{L}(l^s_q)$ and the $\mathcal{R}_q$-bound does not exceed $2\max_{\alpha\in\{0,1\}^n} C_\alpha$. Since l^s_q is of class $\mathcal{HT}$ (note that $1 < q < \infty$) and has property (α), Theorem 5.7 gives that M_λ is a Fourier multiplier, i.e.,

$$\mathscr{F}^{-1} M_\lambda \mathscr{F} : \mathscr{S}(\mathbb{R}^n, l^s_q) \longrightarrow \mathscr{S}'(\mathbb{R}^n, l^s_q)$$

has a (unique) continuous extension $S_\lambda : L_p(l^s_q) \longrightarrow L_p(l^s_q)$ such that

$$\mathcal{R}_q(\{S_\lambda : \lambda \in \Lambda\}) \leqslant C \max_{\alpha\in\{0,1\}^n} C_\alpha =: K \tag{13.4}$$

for all $\lambda \in \Lambda$, where $C = C(n,s,r,p,q)$. From the identity

$$(\varphi_k * \mathscr{F}^{-1} m_\lambda \mathscr{F} f)_{k\in\mathbb{N}_0} = \mathscr{F}^{-1} M_\lambda \mathscr{F} (\varphi_k * f)_{k\in\mathbb{N}_0} \tag{13.5}$$

we receive $\|\mathscr{F}^{-1} m_\lambda \mathscr{F} f\|_{F^s_{p,q}} \leqslant K\|f\|_{F^s_{p,q}}$ for $f \in \mathscr{S}(\mathbb{R}^n)$ and consequently we have a continuous extension $T_\lambda : F^s_{p,q} \longrightarrow F^s_{p,q}$ of $\mathscr{F}^{-1} m_\lambda \mathscr{F} : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$. Now (13.4) and (13.5) imply $\mathcal{R}_q(\{T_\lambda : \lambda \in \Lambda\}) \leqslant K$. Hence, the assertion is proved in case $p = r$.

In order to generalize the result, we select $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and receive $F^{s,r}_{p,q} = (F^s_{p_0,q}, F^s_{p_1,q})_{\theta,r}$. Thus, for

$$T_\lambda : (F^s_{p_0,q}, F^s_{p_1,q})_{\theta,r} \longrightarrow (F^s_{p_0,q}, F^s_{p_1,q})_{\theta,r}$$

we obtain the estimate $\|T_\lambda\|_{F^{s,r}_{p,q}\to F^{s,r}_{p,q}} \leqslant C' \max_{\alpha\in\{0,1\}^n} C_\alpha$, since the real interpolation method is exact of type θ, where $C' = C'(n,s,r,p,q) > 0$.

Since $F^s_{p_j,q}$ ist of class $\mathcal{HT}$ for $j = 0,1$ we receive $\mathcal{R}$-boundedness of

$$T_\lambda : (F^s_{p_0,q}, F^s_{p_1,q})_{\theta,r} \longrightarrow (F^s_{p_0,q}, F^s_{p_1,q})_{\theta,r}$$

for $1 < r < \infty$ as a consequence of the case $p = r$ proved above (see [43], Thm. 3.19). $\square$

Proposition 13.11. *For $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leqslant r \leqslant \infty$ the following representations hold.*

(i) $F^{s+\sigma,r}_{p,q} = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathscr{F}^{-1}(1+|\xi|^2)^{\frac{\sigma}{2}} \mathscr{F} u \in F^{s,r}_{p,q}\}$ *for $\sigma \in \mathbb{R}$.*

(ii) $F^{s+k,r}_{p,q} = \{u \in \mathscr{S}'(\mathbb{R}^n) : \partial_\alpha u \in F^{s,r}_{p,q} \ \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leqslant k\}$ *for* $k \in \mathbb{N}_0$ *and* $r < \infty$.

(iii) $F^{s+2m,r}_{p,q} = \{u \in \mathscr{S}'(\mathbb{R}^n) : \Delta^j u \in F^{s,r}_{p,q} \ \forall j \in \mathbb{N}_0, j \leqslant m\}$ *for* $m \in \mathbb{N}_0$ *and* $r < \infty$.

The corresponding norms are eqivalent, where the norm of the space on the right-hand side is given by $\|\mathscr{F}^{-1}(1+|\xi|^2)^{\frac{\sigma}{2}}\mathscr{F}u\|_{F^{s,r}_{p,q}}$ *in* (i), *by* $\sum_{|\alpha|\leqslant k}\|\partial_\alpha u\|_{F^{s,r}_{p,q}}$ *in* (ii) *and by* $\sum_{0\leqslant j\leqslant m}\|\Delta^j u\|_{F^{s,r}_{p,q}}$ *in* (iii).

Proof. We consider the Bessel-potential operator $B^\sigma u := \mathscr{F}^{-1}(1+|\xi|^2)^{\frac{\sigma}{2}}\mathscr{F}u$ for $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\sigma \in \mathbb{R}$. If we fix $(\varphi_k)_{k\in\mathbb{N}_0} \in \mathbf{\Phi}$ and $\sigma \in \mathbb{R}$, then by setting $\widehat{\psi}_k(\xi) = \frac{2^{k\sigma}}{(1+|\xi|^2)^{\frac{\sigma}{2}}}\widehat{\varphi}_k(\xi)$ we obtain $(\psi_k)_{k\in\mathbb{N}_0} \in \mathbf{\Phi}$ (cf. proof of [66], Thm. 2.3.4). Hence,

$$\|u\|_{F^{s,r}_{p,q}} \sim \|B^\sigma u\|_{F^{s-\sigma,r}_{p,q}} \tag{13.6}$$

and (i) is proved.

Now the special case $\sigma = 2m$ in (13.6) leads to $F^{s+2m,r}_{p,q} = \{u \in \mathscr{S}'(\mathbb{R}^n) : (I-\Delta)^m u \in F^{s,r}_{p,q}\}$ together with the equivalence $\|u\|_{F^{s+2m,r}_{p,q}} \sim \|(I-\Delta)^m u\|_{F^{s,r}_{p,q}}$. Hence, for (iii) it remains to show $\sum_{0\leqslant j\leqslant m}\|\Delta^j u\|_{F^{s,r}_{p,q}} \leqslant C\|(I-\Delta)^m u\|_{F^{s,r}_{p,q}}$, since the converse estimate is obvious. For this purpose we write

$$(-\Delta)^j u = \mathscr{F}^{-1}\frac{|\xi|^{2j}}{(1+|\xi|^2)^k}\mathscr{F}(I-\Delta)^k u.$$

The associated symbol $\frac{|\xi|^{2j}}{(1+|\xi|^2)^k}$ fulfills the conditions of Theorem 13.10. Therefore, we receive (iii).

In order to verify (ii) we write

$$\partial_\alpha u = i^{|\alpha|}\mathscr{F}^{-1}\frac{\xi^\alpha}{(1+|\xi|)^{\frac{|\alpha|}{2}}}\mathscr{F}B^{|\alpha|}u \quad \text{for } |\alpha| \leqslant k \tag{13.7}$$

and

$$B^k u = \mathscr{F}^{-1}\sum_{|\alpha|\leqslant k}\frac{k!}{\alpha!(k-|\alpha|)!}\xi^\alpha\frac{\xi^\alpha}{(1+|\xi|^2)^{\frac{k}{2}}}\mathscr{F}u \tag{13.8}$$

(cf. [35], Sec. 1.3.1). Applying Theorem 13.10 and (13.6) again, we receive $\|u\|_{F^{s+k,r}_{p,q}} \sim \sum_{|\alpha|\leqslant k}\|\partial_\alpha u\|_{F^{s,r}_{p,q}}$, where (13.8) gives the estimate "$\leqslant$" and (13.7) gives "$\geqslant$". □

13.2 The Laplace Operator in TLL Spaces

The Laplace operator in $F^{s,r}_{p,q}$ for $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leqslant r \leqslant \infty$ is defined as

$$\mathcal{A}_L = \mathcal{A}^{s,r}_{L,p,q} : \mathscr{D}(\mathcal{A}_L) \subset F^{s,r}_{p,q} \longrightarrow F^{s,r}_{p,q}, \quad u \longmapsto -\Delta u,$$

where the domain is $\mathscr{D}(\mathcal{A}_L) = F^{s+2,r}_{p,q}$. We will use the same notation, i.e., $\mathcal{A}_L$, for the Laplace operator defined in $(F^{s,r}_{p,q})^n$, which is then understood to be applied component-wise.

Proposition 13.12. *The operator* $\mathcal{A}_L$ *is* $\mathcal{R}$*-sectorial with* $\varphi^{\mathcal{R}}_{\mathcal{A}_L} = 0$ *for* $s \in \mathbb{R}$ *and* $1 < p, q, r < \infty$ *and we have* $(\mathcal{A}^{s,r}_{L,p,q})' = \mathcal{A}^{-s,r'}_{L,p',q'}$. *In particular,* $\mathcal{A}_L$ *is injective.*

Proof. Lemma 13.5(i) implies that $\mathcal{A}_L$ is densely defined. For $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ we aim to show $\lambda \in \rho(-\mathcal{A}_L)$ with

$$(\lambda + \mathcal{A}_L)^{-1} = \mathscr{F}^{-1} \frac{1}{\lambda + |\xi|^2} \mathscr{F}.$$

Therefore, we consider the symbols $\frac{1}{\lambda+|\xi|^2}$ and $\frac{|\xi|^2}{\lambda+|\xi|^2}$, which are smooth and fulfill the conditions of Theorem 13.10. Considering the first symbol we obtain that $\mathscr{F}^{-1}\frac{1}{\lambda+|\xi|^2}\mathscr{F}$ defines a bounded operator on $F^{s,r}_{p,q}$. Considering the second symbol and using Proposition 13.11 we obtain that this operator in fact takes values in $F^{s+2,r}_{p,q}$ and hence must be the inverse operator of $\lambda + \mathcal{A}_L$.

In order to prove the claimed $\mathcal{R}$-boundedness of $\{\lambda(\lambda+\mathcal{A}_L)^{-1} : \lambda \in \Sigma_{\pi-\varphi}\} \subset \mathscr{L}(F^{s,r}_{p,q})$ we show the uniform estimate

$$\sup_{\xi\in\mathbb{R}^n,\, \lambda\in\Sigma_\varphi} |\xi^\alpha \partial_\alpha m_\lambda(\xi)| < \infty$$

for all $\alpha \in \mathbb{N}_0^n$ and $\varphi > 0$, where $m_\lambda(\xi) := \frac{\lambda}{\lambda+|\xi|^2}$. This is a consequence of Lemma 5.3(i), so we can apply Theorem 13.10. Summarizing, $\mathcal{A}_L$ is pseudo-$\mathcal{R}$-sectorial with $\varphi^{\mathcal{R}}_{\mathcal{A}_L} = 0$.

Let now initially $s > -2$. Then we obtain in an elementary way that $\mathcal{A}_L$ is injective: For $u \in \mathscr{N}(\mathcal{A}_L)$ we have $\operatorname{spt}(\widehat{u}) \subset \{0\}$ and thus u is a polynomial (see [34], Cor. 2.4.2). Lemma 13.5 gives that $F^{s+2,r}_{p,q} \subset L_{p,\infty}$ and it is not hard to show that $L_{p,\infty}$ does not contain any nontrivial polynomials. Hence $u = 0$. Now we consider the decomposition $F^{s,r}_{p,q} = \mathscr{N}(\mathcal{A}_L) \oplus \overline{\mathscr{R}(\mathcal{A}_L)}$, which is a consequence of the pseudo-$\mathcal{R}$-sectoriality proved above and of the reflexivity of $F^{s,r}_{p,q}$ obtained in Corollary 13.7 (see, e.g., [36], Prop. 2.1.1). The injectivity of $\mathcal{A}_L$ then gives that $\mathscr{R}(\mathcal{A}_L) \subset F^{s,r}_{p,q}$ is dense.

By integration by parts we easily obtain $\mathcal{A}^{-s,r'}_{L,p',q'} \subset (\mathcal{A}^{s,r}_{L,p,q})'$. The fact that $-1 \in \rho(\mathcal{A}^{s,r}_{L,p,q})$ for all $s \in \mathbb{R}$ and $1 < p, q, r < \infty$ then gives $\mathcal{A}^{-s,r'}_{L,p',q'} = (\mathcal{A}^{s,r}_{L,p,q})'$. Since $(F^{s,r}_{p,q})' = F^{-s,r'}_{p',q'}$, the $\mathcal{R}$-sectoriality with $\varphi^{\mathcal{R}}_{\mathcal{A}_L} = 0$ for $s \leqslant -2$ now follows by standard permanence properties of $\mathcal{R}$-sectorial operators. □

Remark 13.13. The proof of Proposition 13.12 shows that for $r = 1$ we still have that $\mathcal{A}_L$ is pseudo-sectorial with $\varphi_{\mathcal{A}_L} = 0$ and, in case $s > -2$, $\mathcal{A}_L$ is injective.

Proposition 13.14. *Let $s \in \mathbb{R}$ and $1 < p, q, r < \infty$. Then $\mathcal{A}_L$ has an $\mathcal{R}$-bounded H^∞-calculus with $\varphi^{\mathcal{R},\infty}_{\mathcal{A}_L} = 0$.*

Proof. Thanks to Proposition 13.9 and Theorem 5.6 it is sufficient to prove that $\mathcal{A}_L$ has a bounded H^∞-calculus with $\varphi^\infty_{\mathcal{A}_L} = 0$. Let $\varphi \in (0, \pi)$ and $f \in \mathscr{H}_0(\Sigma_\varphi)$. The operator $\mathcal{A}_L$ is sectorial due to Proposition 13.12. Using Cauchy's integral formula we receive $f(\mathcal{A}_L)u = \mathscr{F}^{-1}f(|\xi|^2)\mathscr{F}u$ for all $u \in \mathscr{S}(\mathbb{R}^n)$. Now the symbol $f(|\xi|^2)$ fulfills the condition of Theorem 13.10 (due to Lemma 5.3(i)) which yields

$$\|f(\mathcal{A}_L)\|_{F^{s,r}_{p,q}\to F^{s,r}_{p,q}} \leqslant C_\varphi \|f\|_{\infty,\Sigma_\varphi}$$

with some $C_\varphi > 0$ independent of f. □

Note that Proposition 13.14 implies Proposition 13.12 if we only knew the sectoriality of $\mathcal{A}_L$. But, as the proof of Proposition 13.12 shows, $\mathcal{R}$-sectoriality can be obtained in a direct way at essentially the same cost.

Now, we consider an alternative representation for TLL spaces. We prove that $F^{s+2\alpha,r}_{p,q}$ is the domain of $(I - \Delta)^\alpha$ in $F^{s,r}_{p,q}$, where $\alpha \in [0, 1]$.

Proposition 13.15. *Let $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leqslant r < \infty$. Then*

$$A : \mathscr{D}(A) = \mathscr{D}(\mathcal{A}_L) \subset F^{s,r}_{p,q} \longrightarrow F^{s,r}_{p,q}, \quad u \longmapsto (I - \Delta)u$$

is sectorial with angle $\varphi_A = 0$ and for $\alpha \in [0,1]$

$$\mathscr{D}(A^\alpha) = \left\{u \in F^{s,r}_{p,q} : \mathscr{F}^{-1}(1 + |\xi|^2)^\alpha \mathscr{F} u \in F^{s,r}_{p,q}\right\} = F^{s+2\alpha,r}_{p,q} \tag{13.9}$$

holds with equivalent norms, i.e., $\|u\|_{\mathscr{D}(A^\alpha)} \sim \|\mathscr{F}^{-1}(1+|\xi|^2)^\alpha \mathscr{F} u\|_{F^{s,r}_{p,q}}$ for all $u \in \mathscr{D}(A^\alpha)$. Moreover, we have

$$A^\alpha u = \mathscr{F}^{-1}(1 + |\xi|^2)^\alpha \mathscr{F} u \tag{13.10}$$

for all $u \in \mathscr{D}(A^\alpha)$.

Proof. The second equality in (13.9) is Proposition 13.11(i). The Laplace operator $\mathcal{A}_L$ is pseudo-sectorial with angle $\varphi_{\mathcal{A}_L} = 0$ and so is A. Now $-1 \in \rho(\mathcal{A}_L)$, so A is bijective and thus sectorial.

We now assume $\alpha \in (0,1)$, since the cases $\alpha = 0$ and $\alpha = 1$ are obvious (due to Proposition 13.11). We set $g(z) := \frac{z}{(1+z)^2}$ and $h_\alpha(z) := z^\alpha$. Using Cauchy's integral formula, we obtain

$$(gh_\alpha)(A)f = \mathscr{F}^{-1}\frac{(1 + |\xi|^2)^{\alpha+1}}{(2 + |\xi|^2)^2}\mathscr{F} f \tag{13.11}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. Theorem 13.10 gives that (13.11) even holds for all $f \in F^{s,r}_{p,q}$. Now $A : \mathscr{D}(A) \to F^{s,r}_{p,q}$ is bijective with $A^{-1}f = \mathscr{F}^{-1}\frac{1}{1+|\xi|^2}\mathscr{F} f$ for $f \in F^{s,r}_{p,q}$ and thus we obtain

$$g(A)^{-1}f = (1 + A)^2 A^{-1} f = \mathscr{F}^{-1}\frac{(2 + |\xi|^2)^2}{1 + |\xi|^2}\mathscr{F} f \tag{13.12}$$

for all $f \in \mathscr{D}(A)$. Relations (13.11) and (13.12) yield (13.10) by the fact that A^α is given by $g(A)^{-1}(gh_\alpha)(A)$.

Now we verify (13.9) together with equivalence of the norms. For this purpose, let first $u \in F^{s,r}_{p,q}$ so that $\mathscr{F}^{-1}(1 + |\xi|^2)^\alpha \mathscr{F} u \in F^{s,r}_{p,q}$. Then, using (13.11) and Theorem 13.10, we obtain $(2 - \Delta)(gh_\alpha)(A)u \in F^{s,r}_{p,q}$. Consequently, we have $(gh_\alpha)(A)u \in \mathscr{D}(A)$. Now we also know $(gh_\alpha)(A)u \in \mathscr{R}(A) = F^{s,r}_{p,q}$, so we obtain $u \in \mathscr{D}(A^\alpha)$. Hence, we have $\left\{u \in F^{s,r}_{p,q} : \mathscr{F}^{-1}(1+|\xi|^2)^\alpha \mathscr{F} u \in F^{s,r}_{p,q}\right\} \subset \mathscr{D}(A^\alpha)$, so we can restrict ourselves to $u \in \mathscr{D}(A^\alpha)$ to show the equivalence

$$\|u\|_{\mathscr{D}(A^\alpha)} = \|u\|_{F^{s,r}_{p,q}} + \|A^\alpha u\|_{F^{s,r}_{p,q}} \sim \|\mathscr{F}^{-1}(1 + |\xi|^2)^\alpha \mathscr{F} u\|_{F^{s,r}_{p,q}}. \tag{13.13}$$

For $u \in \mathscr{D}(A^\alpha)$ we can apply (13.10), so we directly receive "$\geqslant$" in (13.13). Applying Theorem 13.10 to the symbol $\frac{1}{(1+|\xi|^2)^\alpha}$ and using (13.10) again, we receive the estimate $\|u\|_{F^{s,r}_{p,q}} \leqslant C\|\mathscr{F}^{-1}(1 + |\xi|^2)^\alpha \mathscr{F} u\|_{F^{s,r}_{p,q}}$ and consequently the converse inequality in (13.13). Hence, we have proved the equivalence (13.13) and this also shows

$$\mathscr{D}(A^\alpha) \subset \left\{u \in F^{s,r}_{p,q} : \mathscr{F}^{-1}(1 + |\xi|^2)^\alpha \mathscr{F} u \in F^{s,r}_{p,q}\right\}. \qquad \square$$

As a consequence of Propositions 13.14 and 13.15 and of (5.4) we infer the following result on complex interpolation of TLL spaces.

Corollary 13.16. *Let $-\infty < s_0 \leqslant s_1 < \infty$ and $1 < p, q, r < \infty$. Then for $\eta \in (0,1)$ we have*

$$[F^{s_0,r}_{p,q}, F^{s_1,r}_{p,q}]_\eta = F^{(1-\eta)s_0+\eta s_1,r}_{p,q}.$$

Proof. For $s \in \mathbb{R}$ we receive

$$[F^{s,r}_{p,q}, F^{s+2k\theta,r}_{p,q}]_\eta = F^{s+2k\theta\eta,r}_{p,q} \tag{13.14}$$

in case $k = 1$, $\theta = 1$ from (5.4) and from Propositions 13.14 and 13.15. Since for any $\beta \geqslant 0$ we can write $A^\beta = A^m A^\alpha$ for some $m \in \mathbb{N}_0$ and $\alpha \in [0,1]$, (13.14) holds for all $k \in \mathbb{N}_0$ and $\theta = 1$. An application of the reiteration theorem now gives (13.14) for all $\theta \in [0,1]$ and $k \in \mathbb{N}_0$. This proves the claim. □

13.3 The Stokes Operator in TLL Spaces

We first introduce the Helmholtz projection $\mathbb{P}$ on $(F^{s,r}_{p,q})^n$. Again $n \in \mathbb{N}$ is the dimension and $s \in \mathbb{R}$, $1 < p, q < \infty$, $1 \leqslant r < \infty$. For $u \in \mathscr{S}(\mathbb{R}^n)^n$ we set

$$\mathbb{P}u := \mathscr{F}^{-1}\Big[I - \frac{\xi\xi^T}{|\xi|^2}\Big]\mathscr{F}u = u - \Big(\sum_{j=1}^n \mathscr{F}^{-1}\frac{\xi_i\xi_j}{|\xi|^2}\mathscr{F}u^j\Big)_{1\leqslant i\leqslant n}. \tag{13.15}$$

By virtue of Theorem 13.10 we obtain $\mathbb{P} \in \mathscr{L}((F^{s,r}_{p,q})^n)$. The space of solenoidal functions is

$$(F^{s,r}_{p,q})^n_\sigma := \big\{u \in (F^{s,r}_{p,q})^n : \operatorname{div} u = 0\big\}$$

and the space of gradient fields in $(F^{s,r}_{p,q})^n$ is

$$\mathscr{G} := \big\{\nabla p : p \in \mathscr{D}'(\mathbb{R}^n), \nabla p \in (F^{s,r}_{p,q})^n\big\}.$$

Proposition 13.17. *Similar to the definition of the space of gradient fields we set*

$$\mathscr{G}^* := \big\{\nabla p : p \in \mathscr{S}'(\mathbb{R}^n), \nabla p \in (F^{s,r}_{p,q})^n\big\}.$$

Let $1 < p, q, r < \infty$ and $n \geqslant 2$. If $s > -2$, we additionally admit $r = 1$. Then the range and the kernel of the Helmholtz projection are given by $\mathscr{R}(\mathbb{P}) = (F^{s,r}_{p,q})^n_\sigma$ and $\mathscr{N}(\mathbb{P}) = \mathscr{G} = \overline{\mathscr{G}^*}$. In particular, we have the Helmholtz decomposition

$$(F^{s,r}_{p,q})^n = (F^{s,r}_{p,q})^n_\sigma \oplus \mathscr{G}.$$

Proof. We prove the claim in three steps and start with some general observations that we will make use of. First we remark that one can obtain the inclusion $\mathscr{R}(\mathbb{P}) \subset (F^{s,r}_{p,q})^n_\sigma$ by direct computation (and approximation). Second the injectivity of the Laplace operator (see Proposition 13.12 and Remark 13.13) yields

$$(F^{s,r}_{p,q})^n_\sigma \cap \mathscr{G} = \{0\}. \tag{13.16}$$

Furthermore, de Rham's theorem (see [16]; cf. [29] and the references therein) gives that $\mathscr{G}$ is a closed subspace of $(F^{s,r}_{p,q})^n$.

Step 1. We show $\mathscr{N}(\mathbb{P}) \subset \mathscr{G}$ in the special case that $F^{s,r}_{p,q}$ is a Lebesgue space. So, we fix $1 < \eta < 2$ and set $\mathscr{G}^*_\eta := \big\{\nabla p : p \in \mathscr{S}'(\mathbb{R}^n), \nabla p \in L_\eta(\mathbb{R}^n)^n\big\}$. Furthermore, let $\mathbb{P}_\eta$ denote the Helmholtz projection on $L_\eta(\mathbb{R}^n)^n$. Then the Hausdorff-Young theorem gives that $\mathbb{P}_\eta u = \mathscr{F}^{-1}\big[I - \frac{\xi\xi^T}{|\xi|^2}\big]\mathscr{F}u$ (which is a priori valid for Schwartz functions) is meaningful for all $u \in L_\eta(\mathbb{R}^n)^n$. Thus, for $u \in \mathscr{N}(\mathbb{P}_\eta)$ we have $\widehat{u} = \xi\frac{\xi^T}{|\xi|^2}\widehat{u}$ with $\frac{\xi^T}{|\xi|^2}\widehat{u} \in \mathscr{S}'(\mathbb{R}^n)$ (since $n \geqslant 2$) and receive $\mathscr{N}(\mathbb{P}_\eta) \subset \mathscr{G}^*_\eta$.

Step 2. We use the first step to show the inclusions $\mathscr{N}(\mathbb{P}) \subset \overline{\mathscr{G}^*} \subset \mathscr{G} \subset \mathscr{N}(\mathbb{P})$ (in the stated order). For a fixed $1 < \eta < 2$ we receive $(I - \mathbb{P})((F^{s,r}_{p,q})^n \cap L_\eta(\mathbb{R}^n)^n) \subset \mathscr{G}^*$

from the first step. Since $(F^{s,r}_{p,q})^n \cap L_\eta(\mathbb{R}^n)^n$ is dense in $(F^{s,r}_{p,q})^n$ we obtain $\mathscr{N}(\mathbb{P}) = (I-\mathbb{P})((F^{s,r}_{p,q})^n) \subset \overline{\mathscr{G}^*}$ as a consequence. The second inclusion $\overline{\mathscr{G}^*} \subset \mathscr{G}$ is valid, since $\mathscr{G}$ is closed. For the third inclusion we fix $u \in \mathscr{G}$. Since we have already shown $\mathscr{N}(\mathbb{P}) \subset \overline{\mathscr{G}^*} \subset \mathscr{G}$, we obtain $\mathbb{P}u = u - (I-\mathbb{P})u \in \mathscr{G}$. On the other hand, we have $\mathbb{P}u \in \mathscr{R}(\mathbb{P}) \subset (F^{s,r}_{p,q})^n_\sigma$. Consequently, (13.16) implies $\mathbb{P}u = 0$.

Step 3. It remains to prove $\mathscr{R}(\mathbb{P}) = (F^{s,r}_{p,q})^n_\sigma$. In view of what we have already seen we obtain $(F^{s,r}_{p,q})^n = \mathscr{R}(\mathbb{P}) \oplus \mathscr{N}(\mathbb{P}) = \mathscr{R}(\mathbb{P}) \oplus \mathscr{G} \subset (F^{s,r}_{p,q})^n_\sigma + \mathscr{G}$. Now the last inclusion must be an equality, since the converse inclusion is obvious. Besides, (13.16) yields directness of the sum, so $\mathscr{R}(\mathbb{P}) \oplus \mathscr{G} = (F^{s,r}_{p,q})^n_\sigma \oplus \mathscr{G}$ together with $\mathscr{R}(\mathbb{P}) \subset (F^{s,r}_{p,q})^n_\sigma$ gives $\mathscr{R}(\mathbb{P}) = (F^{s,r}_{p,q})^n_\sigma$. □

Remark 13.18. The space $(F^{s,r}_{p,q})^n_\sigma$ is of class $\mathcal{HT}$ for $1 < p,q,r < \infty$ and $s \in \mathbb{R}$. This is a consequence of Proposition 13.6: $F^{s,r}_{p,q}$ is of class $\mathcal{HT}$, so is $(F^{s,r}_{p,q})^n$ and hence $(F^{s,r}_{p,q})^n_\sigma$ as a closed subspace.

Now we are able to define the *Stokes operator* as

$$\mathcal{A}_S = \mathcal{A}^{s,r}_{S,p,q} : \mathscr{D}(\mathcal{A}_S) \subset (F^{s,r}_{p,q})^n_\sigma \longrightarrow (F^{s,r}_{p,q})^n_\sigma, \quad u \longmapsto -\mathbb{P}\Delta u$$

on the domain $\mathscr{D}(\mathcal{A}_S) = (F^{s+2,r}_{p,q})^n_\sigma$.

Proposition 13.19. *For $s \in \mathbb{R}$ and $1 < p,q,r < \infty$ we have $\mathcal{A}_S = \mathcal{A}_L|_{\mathscr{D}(\mathcal{A}_S)}$. Besides, we have $\rho(\mathcal{A}_L) \subset \rho(\mathcal{A}_S)$ with $(\lambda - \mathcal{A}_S)^{-1} = (\lambda - \mathcal{A}_L)^{-1}|_{(F^{s,r}_{p,q})^n_\sigma}$ for all $\lambda \in \rho(\mathcal{A}_L)$.*

Proof. For $u \in \mathscr{D}(\mathcal{A}_S)$ we have $u \in \mathscr{N}(I-\mathbb{P})$, since $\mathbb{P}$ is a projection and thus $\mathbb{P}u = u$. By Proposition 13.11 and the continuity of $\Delta : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ we receive $\mathbb{P}\Delta = \Delta\mathbb{P}$ on $(F^{s+2,r}_{p,q})^n$. This gives $\mathcal{A}_S u = \mathcal{A}_L u$ for $u \in \mathscr{D}(\mathcal{A}_S)$.

Now let $\lambda \in \rho(\mathcal{A}_L)$ and set $T_\lambda v := (\lambda - \mathcal{A}_L)^{-1} v$ for $v \in (F^{s,r}_{p,q})^n_\sigma$. Again we can use $\mathbb{P}\Delta = \Delta\mathbb{P}$ and receive $\mathbb{P}T_\lambda = T_\lambda$ by injectivity of $\lambda - \mathcal{A}_L$, i.e T_λ maps into $(F^{s,r}_{p,q})^n_\sigma$. Consequently, $T_\lambda = (\lambda - \mathcal{A}_S)^{-1}$ on $(F^{s,r}_{p,q})^n_\sigma$. □

Proposition 13.20. *Let $s \in \mathbb{R}$ and $1 < p,q,r < \infty$. Then $\mathcal{A}_S$ is $\mathcal{R}$-sectorial with $\varphi_{\mathcal{A}_S} = 0$. Hence $-\mathcal{A}_S \in \mathrm{MR}((F^{s,r}_{p,q})^n_\sigma)$. Furthermore, we have $(\mathcal{A}^{s,r}_{S,p,q})' = \mathcal{A}^{-s,r'}_{S,p',q'}$.*

Proof. Let $0 < \varphi < \pi$. Then we receive $\mathcal{R}$-boundedness of $\{\lambda(\lambda + \mathcal{A}_S)^{-1} : \lambda \in \Sigma_\varphi\} \subset \mathscr{L}((F^{s,r}_{p,q})^n_\sigma)$ as a direct consequence of Propositions 13.12 and 13.19. The space $(F^{s,r}_{p,q})^n_\sigma$ is reflexive (see Corollary 13.7). Consequently, we obtain density of $\mathscr{D}(\mathcal{A}_S) \subset (F^{s,r}_{p,q})^n_\sigma$ (see, e.g., [36], Prop. 2.1.1). The Laplace operator $\mathcal{A}_L$ is injective and so is $\mathcal{A}_S$. The remaining proof is thus completely analogous to the proof of Proposition 13.12. □

14 The Navier-Stokes Equations in TLL Spaces

14.1 The Time Derivative Operator

We consider the operator

$$B : \mathscr{D}(B) = H^1_\eta(\mathbb{R}, X) \subset L_\eta(\mathbb{R}, X) \longrightarrow L_\eta(\mathbb{R}, X), \quad u \longmapsto \Big(1 + \frac{d}{dt}\Big)u.$$

The proofs of the following two assertions work very similar to the proofs of Propositions 13.15 and 13.14, respectively. In addition, one can find a similar result in [18]. Therefore, we only sketch the main steps.

Proposition 14.1. *Let $1 < \eta < \infty$ and let X be a Banach space of class $\mathcal{HT}$ with property (α). Then B is sectorial with angle $\varphi_B \leqslant \frac{\pi}{2}$ and we have*

$$\mathscr{D}(B^\alpha) = H^\alpha_\eta(\mathbb{R}, X)$$

for $\alpha \in [0,1]$. The related norms are equivalent. Furthermore, we have

$$B^\alpha u = \mathscr{F}^{-1}(1+i\xi)^\alpha \mathscr{F} u \quad \forall u \in \mathscr{D}(B^\alpha).$$

Proof. First note that the Bessel-potential spaces have the representations

$$\begin{aligned} H^\alpha_\eta(\mathbb{R}, X) &= \big\{u \in \mathscr{S}'(\mathbb{R}, X) : \mathscr{F}^{-1}(1+\xi^2)^{\frac{\alpha}{2}} \mathscr{F} u \in L_\eta(\mathbb{R}, X)\big\} \\ &= \big\{u \in \mathscr{S}'(\mathbb{R}, X) : \mathscr{F}^{-1}(1+i\xi)^{\alpha} \mathscr{F} u \in L_\eta(\mathbb{R}, X)\big\}. \end{aligned}$$

This is a consequence of Theorem 5.7, where the symbols $\frac{(1+|\xi|^2)^{\alpha/2}}{(1+i\xi)^\alpha}$ and $\frac{(1+i\xi)^\alpha}{(1+|\xi|^2)^{\alpha/2}}$ need to be considered.

Now, using Theorem 5.7 several times, it is straightforward to see that the inverse operator of $\lambda - B$ for $\mathrm{Re}(\lambda) < 0$ is given by $T_\lambda := \mathscr{F}^{-1}\frac{1}{\lambda-1-i\xi}\mathscr{F}$ and to obtain the estimate $\|\lambda(\lambda-B)^{-1}\|_{L_\eta(\mathbb{R},X)\to L_\eta(\mathbb{R},X)} \leqslant C_\varphi$ for all $-\lambda \in \Sigma_\varphi$ and $\varphi < \frac{\pi}{2}$. Here we also need Lemma 5.3(ii) together with Kahane's contraction principle. Thus B is pseudo-sectorial with $\varphi_B \leqslant \frac{\pi}{2}$. Since those computations also work in case $\lambda = 0$, B is bijective and in particular sectorial.

Cauchy's integral formula yields a formula for $(gh_\alpha)(B)$ just as in (13.11). Now likewise the rest of the proof works in the same way as in Proposition 13.15 by using Theorem 5.7 instead of Theorem 13.10. □

Proposition 14.2. *Let X be a Banach space of class $\mathcal{HT}$ with property (α) and $1 < \eta < \infty$. Then B has an $\mathcal{R}$-bounded H^∞-calculus in $L_\eta(\mathbb{R}, X)$ with $\varphi_B^{\mathcal{R},\infty} \leqslant \frac{\pi}{2}$.*

Proof. Let $\varphi \in (0, \frac{\pi}{2})$ and $f \in \mathscr{H}_0(\Sigma_\varphi)$. We already know from Proposition 14.1 that B is sectorial. Cauchy's integral formula gives that

$$f(B)u = \mathscr{F}^{-1} f(i\xi) \mathscr{F} u \tag{14.1}$$

holds for all $u \in \mathscr{S}(\mathbb{R}^n, X)$. From Lemma 5.3(ii) and Kahane's contraction principle we receive that $f(i\xi)$ fulfills the condition of Theorem 5.7, so (14.1) even holds for all $u \in L_\eta(\mathbb{R}, X)$ and we have

$$\|f(B)\|_{L_\eta(\mathbb{R},X)\to L_\eta(\mathbb{R},X)} \leqslant C\|f\|_{\infty,\Sigma_\varphi}.$$ □

14.2 Continuous Embeddings and Multiplication Results

Lemma 14.3. *Let $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leqslant r \leqslant \infty$ such that $p > \frac{n}{2}$. Let $\delta > 0$ such that $\frac{n}{2p} + \delta < 1$. Then there exists $\overline{\epsilon} > 0$ s.t. for all $0 \leqslant \epsilon < \overline{\epsilon}$ we have the continuous embedding*

$$F^{s+2-\delta,r}_{p,q} \subset F^{s+1,r}_{2p-\epsilon,q}.$$

Proof. We use an embedding theorem for Triebel-Lizorkin spaces and deduce the result via interpolation. Select $1 < p_0 < p < p_1 < \infty$ such that $\frac{n}{2p_j} + \delta < 1$ for $j = 0,1$ and let $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then, setting $\epsilon_j := \frac{(1-\theta)p_1 + \theta p_0}{p_{1-j}}\epsilon$ for $j = 0,1$, we have $\frac{1}{2p-\epsilon} = \frac{1-\theta}{2p_0-\epsilon_0} + \frac{\theta}{2p_1-\epsilon_1}$. Using [66], Thm. 2.8.1, we obtain for small $\epsilon \geqslant 0$

$$F^s_{p_j,q} \subset F^{s-\frac{n}{2p_j}}_{2p_j-\epsilon_j,q}$$

and hence, using Theorem 13.4,

$$F^{s+2-\delta,r}_{p,q} = \big(F^{s+2-\delta}_{p_0,q}, F^{s+2-\delta}_{p_1,q}\big)_{\theta,r} \subset \big(F^{s+1}_{2p_0-\epsilon_0,q}, F^{s+1}_{2p_1-\epsilon_1,q}\big)_{\theta,r} = F^{s+1,r}_{2p-\epsilon,q}. \qquad \square$$

Lemma 14.4. *Let $s > 0$, $1 < p, q < \infty$ and $1 \leqslant r \leqslant \infty$. Then there exists $\bar{\epsilon} > 0$ s.t. for all $0 < \epsilon < \bar{\epsilon}$ the product $\pi : F^{s,r}_{2p-\epsilon,q} \times F^{s,r}_{2p-\epsilon,q} \to F^{s,r}_{p,q}$, $(u, v) \mapsto \pi(u, v) = uv$, induced by pointwise multiplication, is continuous.*

Proof. Again we make use of a corresponding fact for Triebel-Lizorkin spaces (which is included in [41]) and extend this to TLL spaces via interpolation. We fix parameters $1 < p_0 < p < p_1 < \infty$ such that $\frac{1}{p} = \frac{1}{2p_0} + \frac{1}{2p_1}$. For some small $\epsilon > 0$ and $\epsilon_j := \epsilon \frac{p_0+p_1}{2p_{1-j}}$ for $j = 0, 1$ we then have $\frac{1}{2(2p_0-\epsilon_0)} + \frac{1}{2(2p_1-\epsilon_1)} = \frac{1}{2p-\epsilon}$. Choosing ϵ and $|p_0 - p_1|$ small enough we obtain

$$F^{s,r}_{2p-\epsilon,q} = \big(F^{s}_{2p_0-\epsilon_0}, F^{s}_{2p_1-\epsilon_1}\big)_{\frac{1}{2},r} \tag{14.2}$$

(due to Theorem 13.4) as well as $\frac{1}{2p_i-\epsilon_i} < \frac{1}{p_j} \leqslant \frac{1}{2p_j-\epsilon_j} + \frac{1}{2p_i-\epsilon_i}$ which is, due to [41], Thm. 6.1, a sufficient condition for the product

$$\pi : F^{s}_{2p_j-\epsilon_j,q} \times F^{s}_{2p_i-\epsilon_i,q} \to F^{s}_{p_j,q}$$

to be continuous for $i = 0, 1$ and $j = 0, 1$. Hence $\pi(\,\cdot\,, u) : F^{s}_{2p_j-\epsilon_j,q} \to F^{s}_{p_j,q}$ is continuous for each $u \in F^{s}_{2p_i-\epsilon_i,q}$ for $i = 0, 1$ and $j = 0, 1$ and so is $\pi(\,\cdot\,, u) : F^{s,r}_{2p-\epsilon,q} \to F^{s,r}_{p,q}$, using (14.2). Thus, the whole product

$$\pi : F^{s,r}_{2p-\epsilon,q} \times F^{s}_{2p_i-\epsilon_i,q} \to F^{s,r}_{p,q}$$

is continuous for $i = 0, 1$.

Now by repeating an analogue argument with $\pi(v, \,\cdot\,) : F^{s}_{2p_i-\epsilon_i,q} \to F^{s,r}_{p,q}$ for $i = 0, 1$ we obtain continuity of $\pi : F^{s,r}_{2p-\epsilon,q} \times F^{s,r}_{2p-\epsilon,q} \to F^{s,r}_{p,q}$, where we made use of (14.2) again and the (simpler) fact that we receive $F^{s,r}_{p,q}$ by real interpolation with itself. $\square$

Consider a function space $\mathcal{F}$ of time-dependent functions on some time interval $(0, T)$ (or in other words on $[0, T]$, since we usually identify two functions differing on a null set). If $\mathcal{F}$ contains the smooth functions with compact support on $(0, T]$, then we denote their closure in $\mathcal{F}$ by ${}_0\mathcal{F}$. Observe that for the function spaces of time-dependent functions that appear in the sequel, ${}_0\mathcal{F}$ consists of those functions $u \in \mathcal{F}$ with $u|_{t=0} = 0$ if the trace in time exists in terms of a standard trace operator on $\mathcal{F}$. Further, note that we usually have ${}_0\mathcal{F} = \mathcal{F}$ in case the trace in time does not exist in that sense (cf. [66], Thm. 4.3.2/1(a)).

Lemma 14.5. *Let $s \in \mathbb{R}$, $1 < p, q, r < \infty$, $1 < \eta < \infty$ and $\alpha \in [0, 1]$. Then for $T \in (0, \infty]$ we have the continuous embeddings*

$$H^1_\eta\big(\mathbb{R}, F^{s,r}_{p,q}\big) \cap L_\eta\big(\mathbb{R}, F^{s+2,r}_{p,q}\big) \subset H^\alpha_\eta(\mathbb{R}, F^{s+2(1-\alpha),r}_{p,q}) \tag{14.3}$$

and

$$H^1_\eta\big((0,T), F^{s,r}_{p,q}\big) \cap L_\eta\big((0,T), F^{s+2,r}_{p,q}\big) \subset H^\alpha_\eta((0,T), F^{s+2(1-\alpha),r}_{p,q}). \tag{14.4}$$

For $T \in (0, \infty)$ we also have the continuous embedding

$${}_0H^1_\eta\big((0,T), F^{s,r}_{p,q}\big) \cap L_\eta\big((0,T), F^{s+2,r}_{p,q}\big) \subset {}_0H^\alpha_\eta((0,T), F^{s+2(1-\alpha),r}_{p,q}) \tag{14.5}$$

locally uniformly in time, i.e., for every $T_0 > 0$ there exists an embedding constant $C > 0$ for (14.5), which is independent of $T \in (0, T_0]$.

Proof. Let $A = I - \Delta$ in $F^{s,r}_{p,q}$ be the operator from Proposition 13.15 and $B = 1 + \frac{d}{dt}$ in $L_\eta(\mathbb{R}, F^{s,r}_{p,q})$ the operator from Proposition 14.1. Propositions 14.2 and 13.14 give that A and B admit a bounded H^∞-calculus with $\varphi^\infty_A + \varphi^\infty_B < \pi$. Note that A can be interpreted as an operator in $L_\eta(\mathbb{R}, F^{s,r}_{p,q})$ instead of $F^{s,r}_{p,q}$ in a trivial way, where it still admits a bounded H^∞-calculus with the same angle $\varphi^\infty_A = 0$. Obviously A and B are resolvent commuting operators. So all conditions of the mixed derivative theorem (in the version of [18], Lem. 4.1) are fulfilled. This yields that

$$\|A^{1-\alpha}B^\alpha u\|_{L_\eta(\mathbb{R},F^{s,r}_{p,q})} \leqslant C\|Au + Bu\|_{L_\eta(\mathbb{R},F^{s,r}_{p,q})}$$

holds for all $u \in \mathscr{D}(A) \cap \mathscr{D}(B)$ and all $\alpha \in [0,1]$. Now we use Propositions 14.1, 13.15 and 13.11 and receive for all $u \in \mathscr{S}(\mathbb{R}, F^{s+2,r}_{p,q}) \subset H^1_\eta(\mathbb{R}, F^{s,r}_{p,q}) \cap L_\eta(\mathbb{R}, F^{s+2,r}_{p,q})$

$$\begin{aligned}
\|u\|_{H^\alpha_\eta(\mathbb{R},F^{s+2(1-\alpha),r}_{p,q})} &\sim \|B^\alpha u\|_{L_\eta(\mathbb{R},F^{s+2(1-\alpha),r}_{p,q})} \\
&\sim \|B^\alpha u\|_{L_\eta(\mathbb{R},\mathscr{D}(A^{1-\alpha}))} \\
&\sim \|A^{1-\alpha}B^\alpha u\|_{L_\eta(\mathbb{R},F^{s,r}_{p,q})} \\
&\lesssim \|Au + Bu\|_{L_\eta(\mathbb{R},F^{s,r}_{p,q})} \\
&\lesssim \|u\|_{H^1_\eta(\mathbb{R},F^{s,r}_{p,q})\cap L_\eta(\mathbb{R},F^{s+2,r}_{p,q})}.
\end{aligned}$$

This proves (14.3).

We obtain (14.4) as a consequence of (14.3) by suitable retraction and extension. More precisely, we make use of (A.1), which yields an extension operator simultaneously on $H^1_\eta((0,T), F^{s,r}_{p,q})$ and on $L_\eta((0,T), F^{s+2,r}_{p,q})$.

In order to prove (14.5), we make use of the extension operator (A.3) in case $\beta = 1$. For a fixed $T_0 > 0$ we receive

$$\begin{aligned}
\|u\|_{H^\alpha_\eta((0,T),F^{s+2(1-\alpha),r}_{p,q})} &\leqslant \|E_{\infty,1}E_T u\|_{H^\alpha_\eta(\mathbb{R},F^{s+2(1-\alpha),r}_{p,q})} \\
&\leqslant C\|E_{\infty,1}E_T u\|_{H^1_\eta(\mathbb{R},F^{s,r}_{p,q})\cap L_\eta(\mathbb{R},F^{s+2,r}_{p,q})} \\
&\leqslant C'\|u\|_{H^1_\eta((0,T),F^{s,r}_{p,q})\cap L_\eta((0,T),F^{s+2,r}_{p,q})}
\end{aligned}$$

for all $u \in {}_0H^1_\eta\big((0,T), F^{s,r}_{p,q}\big) \cap L_\eta\big((0,T), F^{s+2,r}_{p,q}\big)$ with a constant $C' > 0$, independent of $T \in (0, T_0]$. □

We will additionally need the following embeddings for Bessel-potential spaces on a time-interval.

Lemma 14.6. *Let $1 < \eta < \infty$ and let X be a Banach space of class $\mathcal{HT}$. Then for $s > \frac{1}{2\eta}$ and $T \in (0, \infty]$ we have the continuous embedding*

$$H^s_\eta((0,T), X) \subset L_{2\eta}((0,T), X). \tag{14.6}$$

For $\alpha > \frac{1}{2\eta}$ and $T_0 > 0$ the continuous embedding

$${}_0H^\alpha_\eta((0,T), X) \subset L_{2\eta}((0,T), X) \tag{14.7}$$

holds with an embedding constant $C > 0$, which is independent of $T \in (0, T_0]$.

Proof. For (14.7) let first $\alpha \in (\frac{1}{\eta}, 1]$. We select $\epsilon > 0$ such that $\alpha - 2\epsilon > \frac{1}{\eta}$. The embedding constant of $H^\alpha_\eta((0,T), X) \subset W^{\alpha-\epsilon}_\eta((0,T), X)$ does not depend on $T \in (0, \infty]$ and for the extension operator

$$E_{\infty,1}E_T : {}_0W^{\alpha-\epsilon}_\eta((0,T), X) \longrightarrow {}_0W^{\alpha-\epsilon}_\eta(\mathbb{R}, X)$$

from (A.3) there exists a continuity constant independent of $T \in (0, T_0]$. Consequently, for $u \in {}_0H^\alpha_\eta((0,T), X)$ we conclude

$$\begin{aligned}
\|u\|_{L_{2\eta}((0,T),X)} &\leqslant \|E_{\infty,1}E_T u\|_{L_{2\eta}(\mathbb{R},X)} \\
&\leqslant C\|E_{\infty,1}E_T u\|_{H^{\alpha-2\epsilon}_\eta(\mathbb{R},X)} \\
&\leqslant C'\|E_{\infty,1}E_T u\|_{W^{\alpha-\epsilon}_\eta(\mathbb{R},X)} \\
&\leqslant C''\|u\|_{W^{\alpha-\epsilon}_\eta((0,T),X)} \\
&\leqslant C'''\|u\|_{H^\alpha_\eta((0,T),X)},
\end{aligned}$$

where $C''' > 0$ is a constant independent of $T \in (0, T_0]$.

Now let $\alpha \in (\frac{1}{2\eta}, \frac{1}{\eta}]$. In this case we have ${}_0H^\alpha_\eta((0,T), X) = H^\alpha_\eta((0,T), X)$ (see [66], Thm. 4.3.2/1(a)), so we can make use of an extension argument as well, where we have the trivial extension available this time. The case $\alpha > 1$ is an obvious consequence.

Relation (14.6) is a well-known Sobolev embedding. It can be obtained by an analogous extension argument as above, where we make use of (A.2) instead of (A.3). For $\mathbb{R}$ instead of $(0,T)$ see, e.g., [5], Thm. 3.7.5. □

14.3 Maximal Strong Solutions

Our main result reads the following.

Theorem 14.7. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $s > -1$ and let $1 < p, q, r < \infty$ and $1 < \eta < \infty$ such that $\frac{n}{2p} + \frac{1}{\eta} < 1$. Then for every $f \in L_\eta((0,\infty), (F^{s,r}_{p,q})^n)$ and every initial value $u_0 \in \left(F^{s,r}_{p,q}, F^{s+2,r}_{p,q}\right)^n_{1-1/\eta}$ with $\operatorname{div} f = 0$ and $\operatorname{div} u_0 = 0$ there is a maximal time $T^* > 0$ such that the Navier-Stokes equations*

$$\left\{\begin{aligned}
\partial_t u - \Delta u + \nabla p + (u\cdot\nabla)u &= f && \text{in } (0,T)\times\mathbb{R}^n \\
\operatorname{div} u &= 0 && \text{in } (0,T)\times\mathbb{R}^n \\
u|_{t=0} &= u_0 && \text{in } \mathbb{R}^n
\end{aligned}\right. \tag{14.8}$$

have a unique maximal strong solution $(u, \nabla p)$ on $(0, T^*)$ satisfying

$$\begin{aligned}
&u \in H^1_\eta\big((0,T), (F^{s,r}_{p,q})^n\big) \cap L_\eta\big((0,T), (F^{s+2,r}_{p,q})^n\big), \\
&\nabla p \in L_\eta((0,T), (F^{s,r}_{p,q})^n)
\end{aligned}$$

for every $T \in (0, T^*)$. If additionally $\frac{n}{2p} + \frac{2}{\eta} < 1$, then u is either a global solution or we have

$$T^* < \infty \quad \text{and} \quad \limsup_{t\nearrow T^*} \|u(t)\|_{\left(F^{s,r}_{p,q}, F^{s+2,r}_{p,q}\right)^n_{1-1/\eta,\eta}} = \infty. \tag{14.9}$$

Remark 14.8. The constraints on the parameters p, η, especially the more restrictive one for the additional property that u is either a global solution or (14.9) holds, rely on the use of the multiplication result for $F^{s,r}_{p,q}$-spaces given in Lemma 14.4. They might be improved to the standard contraints in classical function spaces such as L_p. This, however, requires optimal results on multiplication for $F^{s,r}_{p,q}$-spaces which by now are not available and would go beyond the scope of this thesis.

In order to give a proof, we consider the usual operatorial formulation relying on the use of Helmholtz projection and Stokes operator. We fix $s \in \mathbb{R}$, $1 < p, q, r < \infty$, $1 < \eta < \infty$

and $X_\sigma := (F^{s,r}_{p,q})^n_\sigma$ with $n \geqslant 2$. As above, $\mathcal{A}_S$ is the Stokes operator in X_σ. The space of maximal regularity for the Stokes equations is

$$\mathbb{E}_T := H^1_\eta\big((0,T), X_\sigma\big) \cap L_\eta\big((0,T), \mathscr{D}(\mathcal{A}_S)\big),$$

where $T \in (0,\infty]$. Next, as in (5.6), we set

$$\mathbb{F}_T := L_\eta((0,T), X_\sigma) \quad \text{and} \quad \mathbb{I} := \{u_0 = u(0) : u \in \mathbb{E}_T\},$$

equipped with the norm $\|u_0\|_{\mathbb{I}} = \inf_{u(0)=u_0} \|u\|_{\mathbb{E}_T}$, so $\mathbb{F}_T \times \mathbb{I}$ is the data space with right-hand side functions $f \in \mathbb{F}_T$ and initial values $u_0 \in \mathbb{I}$. Note that by (5.7), Proposition 13.17, and [66], Thm. 1.9.3/1 we obtain

$$\mathbb{I} = (X_\sigma, \mathscr{D}(\mathcal{A}_S))_{1-\frac{1}{\eta},\eta} = \mathbb{P}\left(F^{s,r}_{p,q}, F^{s+2,r}_{p,q}\right)^n_{1-\frac{1}{\eta},\eta},$$

where $\mathbb{P} \in \mathscr{L}((F^{s,r}_{p,q})^n)$ denotes the Helmholtz projection introduced in (13.15).

The solution operator L^{-1} for the Stokes equation is an isomorphism if $T < \infty$, due to Proposition 13.20, where

$$L : \mathbb{E}_T \xrightarrow{\cong} \mathbb{F}_T \times \mathbb{I}, \quad u \longmapsto \Big(\Big(\frac{d}{dt} + \mathcal{A}_S\Big)u, u(0)\Big).$$

The nonlinear term is

$$G(u) := -\mathbb{P}(u \cdot \nabla)u = -\mathbb{P}\operatorname{div}(uu^T), \quad u \in (F^{s,r}_{p,q})^n_\sigma.$$

Now, our main theorem concerning the Navier-Stokes equations in TLL spaces can be formulated as the following.

Theorem 14.9. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $s > -1$ and let $1 < p, q, r < \infty$ and $1 < \eta < \infty$ such that $\frac{n}{2p} + \frac{1}{\eta} < 1$. Then for all $(f, u_0) \in \mathbb{F}_\infty \times \mathbb{I}$*

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}(u \cdot \nabla)u &= f \quad \text{in } (0,T) \times \mathbb{R}^n, \\ u(0) &= u_0 \quad \text{in } \mathbb{R}^n \end{cases} \tag{14.10}$$

has a unique maximal strong solution $u : [0, T^*) \longrightarrow \mathbb{I}$ with $T^* \in (0,\infty]$ and $u \in \mathbb{E}_T$ for all $T \in (0, T^*)$. If additionally $\frac{n}{2p} + \frac{2}{\eta} < 1$, then u is either a global solution or we have $T^* < \infty$ and $\limsup_{t \nearrow T^*} \|u(t)\|_{\mathbb{I}} = \infty$.

We first convince ourselves that the two systems (14.8) and (14.10) are equivalent. This particularly shows that Theorem 14.9 implies Theorem 14.7. Indeed, when u is the solution of (14.10) given by Theorem 14.9, we receive the solution $(u, \nabla p)$ of (14.8) as claimed in Theorem 14.7 by setting $\nabla p = -(I - \mathbb{P})(u \cdot \nabla)u$. On the other hand, if $(u, \nabla p)$ is a solution of (14.8), then u solves (14.10) and consequently $\nabla p = -(I - \mathbb{P})(u \cdot \nabla)u$.

The proof of the additional statement in Theorem 14.9 will essentially make use of the following embedding for the space of initial values.

Lemma 14.10. *Let $s \in \mathbb{R}$, $1 < p, q, r < \infty$ and $1 < \eta < \infty$ such that $\frac{n}{2p} + \frac{2}{\eta} < 1$. Then there exists $\overline{\epsilon} > 0$ so that for all $0 \leqslant \epsilon < \overline{\epsilon}$ we have the continuous embedding*

$$\mathbb{I} \subset (F^{s+1,r}_{2p-\epsilon,q})^n.$$

Proof. Select $0 < \epsilon' < \min\{\eta - 1, \frac{\eta}{2}[1 - (\frac{n}{2p} + \frac{2}{\eta})]\}$ and $T \in (0, \infty)$. Then we have the continuous embeddings

$$\mathbb{E}_T \subset H_\eta^{\frac{1+\epsilon'}{\eta}}\Big((0,T), \Big(F_{p,q}^{s+2(1-\frac{1+\epsilon'}{\eta}),r}\Big)^n\Big) \subset C\Big([0,T], \Big(F_{p,q}^{s+2(1-\frac{1+\epsilon'}{\eta}),r}\Big)^n\Big),$$

where the first embedding follows from Lemma 14.5 and the second one can be deduced from standard Sobolev embedding in the same way as in the proof of Lemma 14.6. Now, setting $\delta := \frac{2(1+\epsilon')}{\eta}$, we deduce $\frac{n}{2p} + \delta < 1$, so Lemma 14.3 yields the continuous embedding

$$\Big(F_{p,q}^{s+2(1-\frac{1+\epsilon'}{\eta}),r}\Big)^n \subset \big(F_{2p-\epsilon,q}^{s+1,r}\big)^n$$

for small $\epsilon \geqslant 0$. This leads to $\|u_0\|_{(F_{2p-\epsilon,q}^{s+1,r})^n} \leqslant C\|u\|_{\mathbb{E}_T}$ for $u_0 \in \mathbb{I}$ and any $u \in \mathbb{E}_T$ with $u(0) = u_0$ so the assertion is proved. □

Proof of Theorem 14.9. Let $(f, u_0) \in \mathbb{F}_\infty \times \mathbb{I}$. We start with local existence and uniqueness, so we need to show that there is a unique solution $u \in \mathbb{E}_T$ of

$$Lu = \big(f + G(u), u_0\big)$$

on some time interval. First of all we note that it is possible to restrict ourselves to those solutions with $u(0) = 0$. In fact, by setting $u^* := L^{-1}(f, u_0)$, we can always consider $\bar{u} = u - u^* \in {}_0\mathbb{E}_T$ for $u \in \mathbb{E}_T$, so for any $T \in (0, \infty)$ the following assertions are equivalent:

(a) $Lu = (f + G(u), u_0)$ has a unique solution $u \in \mathbb{E}_T$.

(b) $L\bar{u} = (G(\bar{u} + u^*), 0)$ has a unique solution $\bar{u} \in {}_0\mathbb{E}_T$.

Before we are able to verify (b), it is necessary to have the continuous embedding

$$G(\mathbb{E}_T) \subset \mathbb{F}_T \tag{14.11}$$

for $T \in (0, \infty)$. For $u \in \mathbb{E}_T$, using Proposition 13.11, we have

$$\begin{aligned}\|G(u)\|_{\mathbb{F}_T} &= \|\mathbb{P}(u \cdot \nabla)u\|_{L_\eta((0,T),(F_{p,q}^{s,r})^n)} \\ &\leqslant C\|\operatorname{div}(uu^T)\|_{L_\eta((0,T),(F_{p,q}^{s,r})^n)} \\ &\leqslant C'\|uu^T\|_{L_\eta((0,T),(F_{p,q}^{s+1,r})^{n\times n})} \\ &\leqslant C''\|u\|^2_{L_{2\eta}((0,T),(F_{2p-\epsilon,q}^{s+1,r})^n)},\end{aligned}$$

where we applied Hölder's inequality together with Lemma 14.4 (note that $s + 1 > 0$ is assumed) to obtain the last inequality. Here we choose $\epsilon > 0$ small enough such that Lemmas 14.3 and 14.4 can be applied. Now it remains to prove $\mathbb{E}_T \subset L_{2\eta}\big((0,T), (F_{2p-\epsilon,q}^{s+1,r})^n\big)$ to obtain (14.11). Due to the condition $\frac{n}{2p} + \frac{1}{\eta} < 1$, we can select $\delta > \frac{1}{\eta}$ such that $\frac{n}{2p} + \delta < 1$. Then we have $F_{p,q}^{s+2-\delta,r} \subset F_{2p-\epsilon,q}^{s+1,r}$ according to Lemma 14.3. By setting $\alpha := \frac{\delta}{2}$ we receive the continuous embeddings

$$\begin{aligned}\mathbb{E}_T &\subset H_\eta^\alpha\big((0,T), (F_{p,q}^{s+2(1-\alpha),r})^n\big) \\ &\subset L_{2\eta}\big((0,T), (F_{p,q}^{s+2(1-\alpha),r})^n\big) \\ &\subset L_{2\eta}\big((0,T), (F_{2p-\epsilon,q}^{s+1,r})^n\big),\end{aligned} \tag{14.12}$$

where we used Lemma 14.5 for the first embedding, Lemma 14.6 for the second embedding and Lemma 14.3 for the last embedding. This yields (14.11).

In order to obtain (b), we define

$$N : {}_0\mathbb{E}_T \longrightarrow \mathbb{F}_T \times \{0\}, \quad \bar{u} \longmapsto L\bar{u} - \big(G(\bar{u} + u^*), 0\big)$$

for $T \in (0, \infty)$. The embedding (14.11) gives that N is well-defined, i.e., we have indeed $N(\bar{u}) \in \mathbb{F}_T \times \{0\}$ for all $\bar{u} \in {}_0\mathbb{E}_T$. Furthermore, N is continuously Fréchet-differentiable, where

$$DN(0)v = Lv - \big(DG(u^*)v, 0\big) = Lv + \big(\mathbb{P}(u^* \cdot \nabla)v + \mathbb{P}(v \cdot \nabla)u^*, 0\big) \quad \forall v \in {}_0\mathbb{E}_T$$

is the derivative at the origin. Our aim is to verify that there exists a unique $\bar{u} \in {}_0\mathbb{E}_T$ such that $N(\bar{u}) = 0$ for small time intervals $(0, T)$.

As a first step to see this, we prove that $DN(0) : {}_0\mathbb{E}_T \to \mathbb{F}_T \times \{0\}$ is an isomorphism when $T > 0$ is small enough. Similar to the verification of (14.11) we obtain for $T > 0$ and $v \in {}_0\mathbb{E}_T$ that

$$\begin{aligned}
\big\|\big(DG(u^*)v, 0\big)\big\|_{\mathbb{F}_T \times \{0\}} &= \|\mathbb{P}\operatorname{div}(u^* v^T) + \mathbb{P}\operatorname{div}(v(u^*)^T)\|_{L_\eta((0,T),(F^{s,r}_{p,q})^n)} \\
&\leqslant C\|\operatorname{div}(u^* v^T) + \operatorname{div}(v(u^*)^T)\|_{L_\eta((0,T),(F^{s,r}_{p,q})^n)} \\
&\leqslant C'\|u^* v^T + v(u^*)^T\|_{L_\eta((0,T),(F^{s+1,r}_{p,q})^{n\times n})} \\
&\leqslant C''\|u^*\|_{L_{2\eta}((0,T),(F^{s+1,r}_{2p-\epsilon,q})^n)}\|v\|_{L_{2\eta}((0,T),(F^{s+1,r}_{2p-\epsilon,q})^n)},
\end{aligned} \tag{14.13}$$

in view of Proposition 13.11, Lemma 14.4 and Hölder's inequality, where the constant $C'' > 0$ is independent of $T \in (0, \infty)$ and $\epsilon > 0$ is small enough. Again let $\delta > \frac{1}{\eta}$ such that $\frac{n}{2p} + \delta < 1$ and set $\alpha := \frac{\delta}{2}$. Then we have $F^{s+2-\delta,r}_{p,q} \subset F^{s+1,r}_{2p-\epsilon,q}$ and, since $\alpha > \frac{1}{2\eta}$, we obtain for any fixed $T_0 > 0$

$$\begin{aligned}
{}_0\mathbb{E}_T &\subset {}_0H^\alpha_\eta\big((0,T), (F^{s+2(1-\alpha),r}_{p,q})^n\big) \\
&\subset L_{2\eta}\big((0,T), (F^{s+2(1-\alpha),r}_{p,q})^n\big) \\
&\subset L_{2\eta}\big((0,T), (F^{s+1,r}_{2p-\epsilon,q})^n\big),
\end{aligned} \tag{14.14}$$

where the embeddings are continuous with an embedding constant independent of $T \in (0, T_0]$, due to Lemmas 14.5 and 14.6. Hence, we have in total

$$\big\|\big(DG(u^*)v, 0\big)\big\|_{\mathbb{F}_T \times \{0\}} \leqslant C_1\|u^*\|_{L_{2\eta}((0,T),(F^{s+1,r}_{2p-\epsilon,q})^n)}\|v\|_{\mathbb{E}_T} \tag{14.15}$$

for all $v \in {}_0\mathbb{E}_T$ and for all $T \in (0, T_0]$ with some $C_1 > 0$. Thanks to Lemma 5.4 there is also a constant $C_2 > 0$ such that $\|L^{-1}\|_{\mathbb{F}_T \times \{0\} \to {}_0\mathbb{E}_T} \leqslant C_2$ for all $T \in (0, T_0]$.

The size of the finite time interval $(0, T_0)$ was arbitrary up to this point. Proceeding from any finite $T_0 > 0$, we will shrink the interval $(0, T_0)$ in the following to receive a unique local solution. The constants C_1 and C_2 found above can be assumed to be fixed, so they do not change by shrinking $(0, T_0)$. First, let $(0, T_0)$ be small enough, so that

$$\|u^*\|_{L_{2\eta}((0,T_0),(F^{s+1,r}_{2p-\epsilon,q})^n)} \leqslant \frac{1}{2C_1C_2} \tag{14.16}$$

holds. Then we obtain from (14.15) and (14.16)

$$\big\|\big(DG(u^*), 0\big)\big\|_{{}_0\mathbb{E}_T \to \mathbb{F}_T \times \{0\}} < \frac{1}{\|L^{-1}\|_{\mathbb{F}_T \times \{0\} \to {}_0\mathbb{E}_T}}$$

for all $T \in (0, T_0]$. Hence, by use of the Neumann series, we obtain that $DN(0) : {}_0\mathbb{E}_T \to \mathbb{F}_T \times \{0\}$ is an isomorphism for all $T \in (0, T_0]$.

We apply the inverse function theorem (see, e.g., [6], Thm. VII.7.3) and receive open neighborhoods $0 \in U_T \subset {}_0\mathbb{E}_T$ and $N(0) \in V_T \subset \mathbb{F}_T \times \{0\}$ such that $N : U_T \to V_T$ is bijective. Following the idea in [13], we fix $T \in (0, T_0]$ and define for $0 < T' < T$ a function $F_{T'} \in \mathbb{F}_T$ by

$$F_{T'}(t) := \begin{cases} 0, & \text{if } t \in (0, T') \\ G(u^*)(t), & \text{if } t \in [T', T). \end{cases}$$

Then we have

$$\begin{aligned} \big\|\big(F_{T'}, 0\big) - \big(G(u^*), 0\big)\big\|^\eta_{\mathbb{F}_T \times \{0\}} &= \int_0^T \|F_{T'}(t) - G(u^*)(t)\|^\eta_{X_\sigma} dt \\ &= \int_0^{T'} \|G(u^*)(t)\|^\eta_{X_\sigma} dt \xrightarrow{T' \searrow 0} 0 \end{aligned}$$

and thus $(F_{T'}, 0) \xrightarrow{T' \searrow 0} N(0)$ in $\mathbb{F}_T \times \{0\}$. Since V_T is a neighborhood of $N(0)$, this yields $(F_{T'}, 0) \in V_T$ if $T' \in (0, T)$ is small enough and consequently for $\bar{u} := N^{-1}(F_{T'}, 0) \in U_T$ we have $N(\bar{u}) = (F_{T'}, 0) = (0, 0)$ on $(0, T')$. Hence, by restriction of $\bar{u}$ to $(0, T')$, we obtain a solution $\bar{u} \in {}_0\mathbb{E}_{T'}$ of (b). Since $N : U_T \to V_T$ is bijective, this solution is unique.

Having established local existence and uniqueness of a solution for (14.10), we now extend the solution to a maximal time interval $[0, T^*)$. First we note that uniqueness holds on any time interval: Considering two solutions $u, v \in \mathbb{E}_T$ of (14.10) on $[0, T)$ for given data f, u_0 and some $T \in (0, \infty]$, we know from the established local uniqueness that $u = v$ holds on some $[0, T') \subset [0, T)$. We assume that u and v do not coincide on $[0, T)$. Then Lemma 5.5 allows us to apply a continuity argument, which provides some $0 < t_1 < t_2 < T$ so that $u(t) = v(t)$ for all $t \in [0, t_1]$ and $u(t) \neq v(t)$ for all $t \in (t_1, t_2)$. Now, setting $u_1 := u(t_1)$ and $f_1 := f(t_1 + \cdot)$, we can apply local uniqueness of the solution of (14.10) with data f_1, u_1 and receive $u(t_1 + \cdot) = v(t_1 + \cdot)$ on some $[0, T'')$, a contradiction to $u(t) \neq v(t)$ for all $t \in (t_1, t_2)$.

In order to obtain a maximal time interval $[0, T^*)$ for the solution of (14.10), we define for fixed data f, u_0

$$\begin{aligned} &M := \big\{(J_T, u_T) : T \in (0, \infty),\ \exists \text{ solution } u_T \in \mathbb{E}_T \text{ of (14.10) on } J_T = [0, T)\big\}, \\ &J^* := \bigcup \big\{J_T : (J_T, u_T) \in M\big\} =: [0, T^*) \end{aligned}$$

and $u : [0, T^*) \to \mathbb{I}$, $u(t) := u_T(t)$ for $t \in J_T$. Due to the uniqueness proved above, u is well-defined and consequently the desired maximal solution.

Now, let additionally $\frac{n}{2p} + \frac{2}{\eta} < 1$. We assume $T^* < \infty$ and

$$\operatorname{limsup}_{t \nearrow T^*} \|u(t)\|_{\mathbb{I}} < \infty$$

for the maximal solution u. Then we have $u \in BC([0, T^*), \mathbb{I})$ (i.e., bounded and continuous). For $T \in (0, T^*]$ and $v \in \mathbb{E}_T$ we define the linear operator

$$Bv := \big(\mathbb{P} \operatorname{div}(uv^T), 0\big).$$

Then we have $(L + B)u = (f, u_0)$. As in (14.13) we obtain

$$\|Bv\|_{\mathbb{F}_T \times \mathbb{I}} \leqslant C \|u\|_{L_{2\eta}((0,T),(F^{s+1,r}_{2p-\epsilon,q})^n)} \|v\|_{L_{2\eta}((0,T),(F^{s+1,r}_{2p-\epsilon,q})^n)} \quad \forall v \in \mathbb{E}_T \tag{14.17}$$

with a constant $C > 0$ independent of T and $\epsilon > 0$ small enough such that Lemmas 14.3, 14.4 and 14.10 can be applied. Concerning (14.14) and Lemma 14.10 we obtain

$$\|Bv\|_{\mathbb{F}_T\times\mathbb{I}} \leqslant C'\left(\int_0^T \|u(t)\|^{2\eta}_{(F^{s+1,r}_{2p-\epsilon,q})^n)}\right)^{\frac{1}{2\eta}} \|v\|_{\mathbb{E}_T} \leqslant C''T^{\frac{1}{2\eta}}\|u\|_{BC([0,T^*),\mathbb{I})}\|v\|_{\mathbb{E}_T}$$

for all $v \in {}_0\mathbb{E}_T$ with a constant $C'' > 0$ independent of $T \in (0, T^*]$. Due to (14.12) we can also deduce $B \in \mathscr{L}(\mathbb{E}_T, \mathbb{F}_T \times \mathbb{I})$ from (14.17). Furthermore, Lemma 5.4 yields a constant $K > 0$, such that $\|L^{-1}\|_{\mathbb{F}_T\times\{0\}\to{}_0\mathbb{E}_T} \leqslant K$ holds for all $T \in (0, T^*]$. Consequently, we obtain for sufficiently small $T \in (0, T^*]$ that

$$\|B\|_{{}_0\mathbb{E}_T\to\mathbb{F}_T\times\{0\}} < \frac{1}{\|L^{-1}\|_{\mathbb{F}_T\times\{0\}\to{}_0\mathbb{E}_T}},$$

which yields that $L + B : {}_0\mathbb{E}_T \to \mathbb{F}_T \times \{0\}$ is an isomorphism. More precisely, we need to choose

$$T \leqslant \frac{1}{(2C''K\|u\|_{BC([0,T^*),\mathbb{I})})^{2\eta}}. \tag{14.18}$$

Now, for T as in (14.18), we can select $T_1 \in (0, T^*)$ and repeat the argument on $(T_1, T + T_1)$ instead of $(0, T)$. This yields that $L + B : {}_0\mathbb{E}_{(T_1,T+T_1)} \to \mathbb{F}_{(T_1,T+T_1)} \times \{0\}$ is an isomorphism, where ${}_0\mathbb{E}_{(T_1,T+T_1)}$ (and $\mathbb{F}_{(T_1,T+T_1)}$, respectively) consists of the translations of functions in ${}_0\mathbb{E}_T$ (and $\mathbb{F}_T$, respectively) by T_1. We repeat this argument k times on the interval $(kT_1, T + kT_1) \cap (0, T^*)$ until we reach $T + kT_1 \geqslant T^*$. Finally we have that $L + B : {}_0\mathbb{E}_{T^*} \to \mathbb{F}_{T^*} \times \{0\}$ is an isomorphism. Now it is not hard to deduce that

$$L + B : \mathbb{E}_{T^*} \xrightarrow{\cong} \mathbb{F}_{T^*} \times \mathbb{I} \tag{14.19}$$

is an isomorphism: Continuity and injectivity are obvious while we receive the surjectivity by setting $v^* := L^{-1}(0, v_0)$ and $v := (L + B)^{-1}(g - \mathbb{P}\operatorname{div}(v^*u^T), 0) + v^* \in \mathbb{E}_{T^*}$ for $(g, v_0) \in \mathbb{F}_T \times \mathbb{I}$. As a consequence of (14.19) and Lemma 5.5 we can achieve

$$u = (L + B)^{-1}(f, u_0) \in \mathbb{E}_{T^*} \subset BUC([0, T^*), \mathbb{I})$$

and hence $u(T^*) = \lim_{t\nearrow T^*} u(t) \in \mathbb{I}$. Application of the local existence and uniqueness now gives a solution of (14.10) for data $f(\cdot + T^*), u(T^*)$ on some time interval $[0, T'')$, which yields an extension of u to a solution of (14.10) with data f, u_0 on $[0, T^* + T'')$, in contradiction to the maximality of u. □

A Extension Operators

Let $1 < \eta < \infty$. For fixed $m \in \mathbb{N}$ and $T \in (0, \infty]$ there exists a mapping

$$u \longmapsto E_{T,m}u$$

for functions u (defined on $(0,T)$ with values in any vector space) such that for all $k \in \{0, 1, \dots, m\}$ and any Banach space X we have an extension operator

$$E_{T,m} : W^k_\eta((0,T), X) \longrightarrow W^k_\eta(\mathbb{R}, X). \tag{A.1}$$

A precise proof can be found in [2], Thm. 4.26 for the case of scalar-valued functions, but the given proof can be directly transferred to the vector-valued case. The operator $E_{T,m}$ is the coretraction of

$$R : W^k_\eta(\mathbb{R}, X) \longrightarrow W^k_\eta((0,T), X), \quad u \longmapsto u|_{(0,T)},$$

so, by the interpolation $W^s_\eta(\mathbb{R}, X) = \big(L_\eta(\mathbb{R}, X), W^k_\eta(\mathbb{R}, X)\big)_{\frac{s}{k},\eta}$ for $0 < s < k$, $s \notin \mathbb{N}$, we obtain

$$W^s_\eta((0,T), X) = \big(L_\eta((0,T), X), W^k_\eta((0,T), X)\big)_{\frac{s}{k},\eta}$$

and the extension operator

$$E_{T,m} : W^s_\eta((0,T), X) \longrightarrow W^s_\eta(\mathbb{R}, X) \tag{A.2}$$

(see [66], Thm. 1.2.4).

Now let $T \in (0, \infty)$, $1 < \eta < \infty$, and let X be a Banach space. For a function u defined on $(0,T)$ with values in any vector space we set

$$E_T u(\tau) := \begin{cases} u(\tau), & \text{if } 0 < \tau < T \\ u(2T - \tau), & \text{if } T \leqslant \tau < 2T \\ 0, & \text{if } 2T \leqslant \tau \end{cases}$$

(cf. [53]). Then, due to [53], Prop. 6.1, this leads to an extension operator

$$E_T : {}_0W^\beta_\eta\big((0,T), X\big) \longrightarrow {}_0W^\beta_\eta\big((0,\infty), X\big)$$

for $\beta \in (\frac{1}{\eta}, 1]$ such that for any fixed $T_0 \in (0, \infty)$ there is a constant $C > 0$ with $\|E_T\| \leqslant C$ for all $T \in (0, T_0]$. Now we use (A.2) in case $T = \infty$ and $m = 1$ and receive the extension operator

$$E_{\infty,1}E_T : {}_0W^\beta_\eta\big((0,T), X\big) \longrightarrow {}_0W^\beta_\eta\big(\mathbb{R}, X\big) \tag{A.3}$$

(for $\beta \in (\frac{1}{\eta}, 1]$), whose operator norms $\|E_{\infty,1}E_T\|$, $T \in (0, T_0]$ are bounded above for a fixed $T_0 > 0$ as well. The structure of E_T also gives that

$$\|E_T u\|_{L_\eta((0,\infty),X)} \leqslant 2\|u\|_{L_\eta((0,T),X)}.$$

B An Alternative Proof of Proposition 8.1(ii)

If we knew the solution u in Theorem 6.5 to be consistent in $q \in (1, \infty)$ and if we assume that $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_s^1(\Omega)$ is dense for $s = q'$ (Assumption 4.4) but also for $s = 2$, then for Proposition 8.1(ii), which is the key to the proof of our main results regarding the Stokes equations on uniform $C^{2,1}$-domains, we can give an alternative proof. This proof needs some further properties about $\Delta_{\mathrm{PS},q}$ (which makes it a little longer), but these are also of interest themselves and then the proof gets along with abstract functional analytic duality arguments. Therefore, we consider the following additional statement to Theorem 6.5.

Lemma B.1 (Consistency). *Under the conditions of Theorem* 6.5*, if $f \in L_q(\Omega)^n \cap L_r(\Omega)^n$, $g \in W_q^1(\Omega)^n \cap W_r^1(\Omega)^n$ and $h \in W_q^2(\Omega)^n \cap W_r^2(\Omega)^n$ for some $1 < q, r < \infty$, then for $\overline{\lambda}_0 := \max\{\lambda_0(n, q, \theta, \Omega), \lambda_0(n, r, \theta, \Omega)\}$ and $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \overline{\lambda}_0$ the solution of* (6.30) *fulfills $u \in W_q^2(\Omega)^n \cap W_r^2(\Omega)^n$. In particular, Δ_{PS} is the generator of a strongly continuous analytic semigroup with consistent resolvent $(\lambda - \Delta_{\mathrm{PS},q})^{-1} = (\lambda - \Delta_{\mathrm{PS},r})^{-1}$ on $L_q(\Omega)^n \cap L_r(\Omega)^n$ for $1 < q, r < \infty$.*

Proof. Following the proof of Theorem 6.5, we similarly deduce the statement for general uniform $C^{2,1}$-domains from the statement for bent half spaces and this in turn from the half space $\mathbb{R}_+^n$. We only sketch the main steps. In order to distinguish between the exponents q and r, we denote $X_q := l_q(\bigoplus_{l \in \Gamma} W_q^2(\Omega_l)^n)$ instead of X and similarly denote Y_q, Z_q instead of Y, Z as well as S_q, P_q, P_q', $\bar{C}_q$, $\bar{D}_q$ for the operators S, P, P', $\bar{C}$, $\bar{D}$ defined in the proof of Theorem 6.5.

As we have seen, the solution $u \in W_q^2(\Omega)^n$ of (6.30) for $f \in L_q(\Omega)^n$, $g \in W_q^1(\Omega)^n$, $h \in W_q^2(\Omega)^n$ satisfies the representation

$$u = \bar{C}_q(S_q + P_q')^{-1}\bar{D}_q(f, a),$$

where $a = \Pi_\tau \operatorname{tr} g + \Pi_\nu \operatorname{tr} h$. For the operator $\bar{D}_q$ we directly observe that $\bar{D}_q = \bar{D}_r$ holds on $(L_q(\Omega)^n \times \mathrm{BF}_{q,\lambda}(\partial\Omega)) \cap (L_r(\Omega)^n \times \mathrm{BF}_{r,\lambda}(\partial\Omega))$ and the same is true for $\bar{C}_q$. Therefore, it remains to see $(S_q + P_q')^{-1} = (S_r + P_r')^{-1}$ on $Y_q \cap Y_r$.

Now $(S_q + P_q')^{-1}$ is defined via the Neumann series, i.e., writing $S_q + P_q' = S_q(I + S_q^{-1}P_q')$, we receive

$$(S_q + P_q')^{-1} = (I + S_q^{-1}P_q')^{-1}S_q^{-1} = \left(\sum_{k=0}^{\infty}(-1)^k(S_q^{-1}P_q')^k\right)S_q^{-1}.$$

The identity $P_q' = P_r'$ on $X_q \cap X_r$ is obvious. Hence, it remains to see that $S_q^{-1} = S_r^{-1}$ holds on $Y_q \cap Y_r$, since convergence of the Neumann series in the operator norm on $X_q \cap X_r$ is eventually a consequence.

For this purpose, fix some $(f_l, a_l)_{l \in \Gamma} \in Y_q \cap Y_r$. Setting $(u_l)_{l \in \Gamma} = S_q^{-1}(f_l, a_l)_{l \in \Gamma}$ we receive that for all $l \in \Gamma$ the function u_l is the unique solution of $(\lambda - \Delta)u_l = f_l$, $\Pi_\tau \mathrm{D}_-(u_l)\nu_l + \Pi_\nu u_l = a_l$ on $\partial\Omega_l$ in $W_q^2(\Omega_l)^n$. Therefore, it remains to obtain that

$u_l \in W_r^2(\Omega)^n$ holds for all $l \in \Gamma$. In fact, the unique solvability of $(\lambda - \Delta)u_l = f_l$, $\Pi_\tau \mathrm{D}_-(u_l)\nu_l + \Pi_\nu u_l = a_l$ on $\partial\Omega_l$ in $W_r^2(\Omega_l)^n$ then implies the resolvent estimate

$$\|(\lambda u_l, \sqrt{\lambda}\nabla u_l, \nabla^2 u_l)\|_{r,\Omega_l} \leqslant C(\|f_l\|_{r,\Omega_l} + \|a_l\|_{\mathrm{BF}_{r,\lambda}(\partial\Omega_l)})$$

for all $l \in \Gamma$ and consequently $(u_l)_{l\in\Gamma} \in X_r$, as in the proof of Theorem 6.5.

In total, it remains to prove consistency for domains of the type Ω_l, i.e., the whole space $\mathbb{R}^n$ and bent, rotated and shifted half spaces. For the whole space we have a representation of the solution via Fourier transformation (cf. Proposition 13.12) which directly yields consistency. For the bent, rotated and shifted half spaces we can repeat the argument from above: Since unique solvability of (6.24) can be reduced to the half space (see the proof of Theorem 6.4), mainly via the Neumann series, we obtain the statement for bent rotated and shifted half spaces, in case constistency holds for the half space.

In order to obtain the statement for the half space $\Omega = \mathbb{R}^n_+$, consider the proof of Proposition 6.1. We have seen that for the half space we can separate the boundary conditions such that we receive a remaining Dirichlet boundary problem and $n-1$ Neumann boundary problems. These problems in turn fulfill the consistency condition, since there is a representation of the respective solution via Fourier transformation (in the whole space) and a reflection principle, which does not depend on the parameter q. $\square$

Proposition 8.1(ii) under the mentioned additional assumption reads the following.

Proposition B.2. *Let $\Omega \subset \mathbb{R}^n$ be a uniform $C^{2,1}$-domain, $n \geqslant 2$, $1 < q < \infty$, $0 < \theta < \pi$ such that $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_s^1(\Omega)$ is dense for $s \in \{q', 2\}$. Choose $\lambda_0 = \lambda_0(n, q, \theta, \Omega) > 0$ so that the conditions of Theorem 6.5 and Proposition 7.2 are satisfied and let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \lambda_0$. Then we have the implication*

$$f \in L_{q,\sigma}(\Omega) \;\Rightarrow\; (\lambda - \Delta_{\mathrm{PS},q})^{-1} f \in L_{q,\sigma}(\Omega).$$

Lemma B.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$ and $1 < q < \infty$. The operator $\Delta_{\mathrm{PS},q}$ is symmetric, i.e., for the dual operator*

$$\Delta'_{\mathrm{PS},q} : \mathscr{D}(\Delta'_{\mathrm{PS},q}) \subset L_{q'}(\Omega)^n \to L_{q'}(\Omega)^n,$$

defined on

$$\begin{aligned}\mathscr{D}(\Delta'_{\mathrm{PS},q}) = \{&v \in L_{q'}(\Omega)^n : \\ &\exists\ \Delta'_{\mathrm{PS},q} v \in L_{q'}(\Omega)^n : \langle \Delta_{\mathrm{PS},q} u, v\rangle_{q,q'} = \langle u, \Delta'_{\mathrm{PS},q} v\rangle_{q,q'}\ \forall u \in \mathscr{D}(\Delta_{\mathrm{PS},q})\},\end{aligned}$$

we have $\Delta_{\mathrm{PS},q'} \subset \Delta'_{\mathrm{PS},q}$.

Proof. Let $u \in \mathscr{D}(\Delta_{\mathrm{PS},q})$ and $v \in \mathscr{D}(\Delta_{\mathrm{PS},q'})$. Then we have

$$\langle \Delta_{\mathrm{PS},q} u, v\rangle_{q,q'} = \int_\Omega (\nabla \operatorname{div} u)\cdot v \, d\lambda_n - \int_\Omega (\nabla \operatorname{div} u - \Delta u)\cdot v\, d\lambda_n. \tag{B.1}$$

Regarding the first term, we write $(\nabla \operatorname{div} u)\cdot v = \operatorname{div}(v \operatorname{div} u) - (\operatorname{div} v)(\operatorname{div} u)$ and obtain

$$\int_\Omega (\nabla \operatorname{div} u)\cdot v\, d\lambda_n = \int_{\partial\Omega} \nu\cdot v \operatorname{div} u\, d\sigma - \int_\Omega (\operatorname{div} v)(\operatorname{div} u)\, d\lambda_n = \int_\Omega (\operatorname{div} v)(\operatorname{div} u)\, d\lambda_n,$$

where we made use of Gauß's theorem (Lemma 3.4; note that $v \operatorname{div} u \in W_1^1(\Omega)^n$). By interchanging u and v we conclude

$$\int_\Omega (\nabla \operatorname{div} u)\cdot v\, d\lambda_n = \int_\Omega (\nabla \operatorname{div} v)\cdot u\, d\lambda_n.$$

Regarding the second term, we make use of Lemma 2.1(i) to write $(\nabla \operatorname{div} u - \Delta u) \cdot v = \operatorname{div}(\mathrm{D}_-(u)v) - (\nabla u^T - \nabla u) \cdot \nabla v$ and obtain

$$\int_\Omega (\nabla \operatorname{div} u - \Delta u) \cdot v \, d\lambda_n = \int_{\partial\Omega} \nu \cdot \mathrm{D}_-(u)v \, d\sigma - \int_\Omega (\nabla u^T - \nabla u) \cdot \nabla v \, d\lambda_n,$$

where we made use of Gauß's theorem again (Lemma 3.4; note that $\mathrm{D}_-(u)v \in W_1^1(\Omega)^n$). For the first integrand we have $\nu \cdot \mathrm{D}_-(u)v = 0$ on $\partial\Omega$ (Lemma 2.1(iii)) and the second integrand, $(\nabla u^T - \nabla u) \cdot \nabla v$, is symmetric in u and v. Hence, interchanging u and v gives

$$\int_\Omega (\nabla \operatorname{div} u - \Delta u) \cdot v \, d\lambda_n = \int_\Omega (\nabla \operatorname{div} v - \Delta v) \cdot u \, d\lambda_n.$$

Summarizing, we are able to continue (B.1) and receive $\langle \Delta_{\mathrm{PS},q} u, v\rangle_{q,q'} = \langle u, \Delta_{\mathrm{PS},q'} v\rangle_{q,q'}$. $\square$

Lemma B.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$ and $1 < q < \infty$ such that the Helmholtz projection $\mathbb{P}_q : L_q(\Omega)^n \to L_q(\Omega)^n$ with range $L_{q,\sigma}(\Omega)$ and kernel $G_q(\Omega)$ exists. Consequently the Helmholtz projection exists for q' as well and we have $\mathbb{P}_q' = \mathbb{P}_{q'}$. Moreover we require that $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_s^1(\Omega)$ is dense for $s \in \{q, q'\}$. Let $\Delta_{\mathrm{PS},q}'$ be the dual operator of $\Delta_{\mathrm{PS},q}$ as in Lemma* B.3 *and let*

$$\Delta_{\mathrm{PS},q}^* : L_{q'}(\Omega)^n \to \mathscr{D}(\Delta_{\mathrm{PS},q})'$$

be the continuous dual operator of $\Delta_{\mathrm{PS},q} \in \mathscr{L}(\mathscr{D}(\Delta_{\mathrm{PS},q}), L_q(\Omega)^n)$, where $\mathscr{D}(\Delta_{\mathrm{PS},q})$ is endowed with the graph norm. Then we have

$$\mathbb{P}_q \Delta_{\mathrm{PS},q'}^* u = \Delta_{\mathrm{PS},q'}^* \mathbb{P}_q u$$

for all $u \in \mathscr{D}(\Delta_{\mathrm{PS},q})$. In particular, $\Delta_{\mathrm{PS},q'}^*$ maps $L_{q,\sigma}(\Omega)$ into $L_q(\Omega)^n$.

Proof. We have $\Delta_{\mathrm{PS},q'}' \subset \Delta_{\mathrm{PS},q'}^*$ and due to Lemma B.3 we also have $\Delta_{\mathrm{PS},q} \subset \Delta_{\mathrm{PS},q'}'$. Therefore $\Delta_{\mathrm{PS},q} \subset \Delta_{\mathrm{PS},q'}^*$. In particular, the expression $\mathbb{P}_q \Delta_{\mathrm{PS},q'}^* u$ $(= \mathbb{P}_q \Delta u)$ is meaningful. Also note that the density and continuity of the embedding $\mathscr{D}(\Delta_{\mathrm{PS},q}) \subset L_q(\Omega)^n$ give that we can interpret $L_{q'}(\Omega)^n$ as a subspace of $\mathscr{D}(\Delta_{\mathrm{PS},q})'$. Now let $u \in \mathscr{D}(\Delta_{\mathrm{PS},q})$ and $v \in \mathscr{D}(\Delta_{\mathrm{PS},q'})$. Then we have

$$\begin{aligned} \langle \Delta_{\mathrm{PS},q'}^* \mathbb{P}_q u, v\rangle_{\mathscr{D}(\Delta_{\mathrm{PS},q'})',\mathscr{D}(\Delta_{\mathrm{PS},q'})} &= \langle \mathbb{P}_q u, \Delta_{\mathrm{PS},q'} v\rangle_{q,q'} \\ &= \langle u, \mathbb{P}_{q'} \Delta v\rangle_{q,q'} \\ &= -\langle u, \mathbb{P}_{q'}(\nabla \operatorname{div} v - \Delta v)\rangle_{q,q'}. \end{aligned} \tag{B.2}$$

Now we obtain

$$\nabla \operatorname{div} v - \Delta v \in L_{q',\sigma}(\Omega), \ \nabla \operatorname{div} u - \Delta u \in L_{q,\sigma}(\Omega). \tag{B.3}$$

Indeed, using Lemma 2.1(ii) and (iii), we see that

$$\begin{aligned} \int_\Omega (\nabla \operatorname{div} v - \Delta v) \cdot \nabla\varphi \, d\lambda_n &= \int_\Omega \operatorname{div}(\mathrm{D}_-(v)\nabla\varphi) \, d\lambda_n \\ &= \int_{\partial\Omega} \nu \cdot (\mathrm{D}_-(v)\nabla\varphi) \, d\sigma \\ &= -\int_{\partial\Omega} \nabla\varphi \cdot (\mathrm{D}_-(v)\nu) \, d\sigma \\ &= 0 \end{aligned}$$

holds for all $\varphi \in C_c^\infty(\overline{\Omega})$ (Gauß's theorem, i.e., Lemma 3.4, is applicable) and thus for all $\varphi \in \widehat{W}_q^1(\Omega)$ as well, due to the density of $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_q^1(\Omega)$. Since $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_{q'}^1(\Omega)$ is dense, we can do the same for $\nabla \operatorname{div} u - \Delta u$. Besides, we have

$$\int_\Omega u \cdot (\nabla \operatorname{div} v - \Delta v)\, d\lambda_n = \int_\Omega v \cdot (\nabla \operatorname{div} u - \Delta u)\, d\lambda_n, \tag{B.4}$$

since application of Lemma 3.4 and Lemma 2.1(i) and 2.1(iii) gives

$$\begin{aligned}\int_\Omega u \cdot (\nabla \operatorname{div} v - \Delta v)\, d\lambda_n &= \int_\Omega \operatorname{div}(\mathrm{D}_-(v)u)\, d\lambda_n - \int_\Omega (\nabla v^T - \nabla v) \cdot \nabla u\, d\lambda_n \\ &= -\int_{\partial\Omega} u \cdot (\mathrm{D}_-(v)\nu)\, d\sigma - \int_\Omega (\nabla v^T - \nabla v) \cdot \nabla u\, d\lambda_n \\ &= \int_\Omega (\nabla v - \nabla v^T) \cdot \nabla u\, d\lambda_n\end{aligned}$$

and the same for $\int_\Omega v \cdot (\nabla \operatorname{div} u - \Delta u)\, d\lambda_n$. Now, continuing (B.2), we obtain

$$\begin{aligned}-\langle u, \mathbb{P}_{q'}(\nabla \operatorname{div} v - \Delta v)\rangle_{q,q'} &= -\langle u, \nabla \operatorname{div} v - \Delta v\rangle_{q,q'} \\ &= -\langle \nabla \operatorname{div} u - \Delta u, v\rangle_{q,q'} \\ &= -\langle \mathbb{P}_q(\nabla \operatorname{div} u - \Delta u), v\rangle_{q,q'} \\ &= \langle \mathbb{P}_q \Delta u, v\rangle_{q,q'} \\ &= \langle \mathbb{P}_q \Delta^*_{\mathrm{PS},q'} u, v\rangle_{q,q'}\end{aligned}$$

by use of (B.3) and (B.4), so we have in total

$$\langle \Delta^*_{\mathrm{PS},q'} \mathbb{P}_q u, v\rangle_{\mathscr{D}(\Delta_{\mathrm{PS},q'})', \mathscr{D}(\Delta_{\mathrm{PS},q'})} = \langle \mathbb{P}_q \Delta^*_{\mathrm{PS},q'} u, v\rangle_{q,q'}. \qquad \square$$

Lemma B.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniform $C^{2,1}$-boundary, $n \geqslant 2$ and $1 < q < \infty$ such that the Helmholtz projection $\mathbb{P}_q : L_q(\Omega)^n \to L_q(\Omega)^n$ exists and we require that $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_s^1(\Omega)$ is dense for $s \in \{q, q'\}$. Let $0 < \theta < \pi$, $\overline{\lambda}_0 := \max\{\lambda_0(\Omega, q, \theta, n), \lambda_0(\Omega, q', \theta, n)\}$ with λ_0 from Theorem 6.5 and let $\lambda \in \Sigma_\theta$, $|\lambda| \geqslant \overline{\lambda}_0$. Then we have the implication*

$$f \in L_{q,\sigma}(\Omega) \quad \Longrightarrow \quad (\lambda - \Delta_{\mathrm{PS},q})^{-1} f \in L_{q,\sigma}(\Omega).$$

Proof. We have $\Delta_{\mathrm{PS},q} \subset \Delta^*_{\mathrm{PS},q'}$ again, since we know that $\Delta_{\mathrm{PS},q} \subset \Delta'_{\mathrm{PS},q'}$ (due to Lemma B.3) and $\Delta'_{\mathrm{PS},q'} \subset \Delta^*_{\mathrm{PS},q'}$ are valid. Hence, for $u := (\lambda - \Delta_{\mathrm{PS},q})^{-1} f$ we have

$$(\lambda - \Delta^*_{\mathrm{PS},q'})u = (\lambda - \Delta)u = \mathbb{P}_q(\lambda - \Delta)u = \mathbb{P}_q(\lambda - \Delta^*_{\mathrm{PS},q'})u.$$

Due to Lemma B.4, we conclude

$$(\lambda - \Delta^*_{\mathrm{PS},q'})u = (\lambda - \Delta^*_{\mathrm{PS},q'})\mathbb{P}_q u.$$

Now $\lambda - \Delta_{\mathrm{PS},q'} : \mathscr{D}(\Delta_{\mathrm{PS},q'}) \to L_{q'}(\Omega)^n$ is an isomorphism and therefore $\lambda - \Delta^*_{\mathrm{PS},q'} : L_q(\Omega)^n \to \mathscr{D}(\Delta_{\mathrm{PS},q'})'$ is injective. Hence $u = \mathbb{P}_q u$. $\square$

Proof of Proposition B.2. Let initially $f \in L_{q,\sigma}(\Omega) \cap L_{2,\sigma}(\Omega)$ and let $\mathbb{P}_2 : L_2(\Omega)^n \to L_2(\Omega)^n$ be the Helmholtz projection. Due to Theorem 6.5 and the additional statement that the resolvent is consistent, we have

$$u := (\lambda - \Delta_{\mathrm{PS},q})^{-1} f = (\lambda - \Delta_{\mathrm{PS},2})^{-1} f \in \mathscr{D}(\Delta_{\mathrm{PS},q}) \cap \mathscr{D}(\Delta_{\mathrm{PS},2}). \tag{B.5}$$

Application of Lemma B.5 with $q = 2$ gives $u \in L_{2,\sigma}(\Omega)$. Therefore, we have $\operatorname{div} u = 0$ and this yields for any $\varphi \in C_c^\infty(\overline{\Omega})$

$$\langle u, \nabla\varphi\rangle_{q,q'} = \int_\Omega \operatorname{div}(\varphi u)\, d\lambda_n = \int_{\partial\Omega} \nu \cdot (\varphi u)\, d\sigma = \int_{\partial\Omega} \varphi(\nu \cdot u)\, d\sigma = 0, \tag{B.6}$$

using Gauß's theorem (Lemma 3.4; note that $\varphi u \in W_1^1(\Omega)^n$). Since $C_c^\infty(\overline{\Omega}) \subset \widehat{W}_{q'}^1(\Omega)$ is dense, we obtain (B.6) for $\varphi \in \widehat{W}_{q'}^1(\Omega)$ as well. Hence $u \in L_{q,\sigma}(\Omega)$.

Let now $f \in L_{q,\sigma}(\Omega)$. Since $L_{q,\sigma}(\Omega) \cap L_{2,\sigma}(\Omega) \subset L_{q,\sigma}(\Omega)$ is dense (note that $C_{c,\sigma}^\infty(\Omega) \subset L_{q,\sigma}(\Omega)$ is dense), there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset L_{q,\sigma}(\Omega) \cap L_{2,\sigma}(\Omega)$ such that

$$f_k \xrightarrow{k\to\infty} f \quad \text{in } L_q(\Omega)^n.$$

Therefore, we obtain

$$(\lambda - \Delta_{\mathrm{PS},q})^{-1} f = \lim_{k\to\infty}^{L_q} (\lambda - \Delta_{\mathrm{PS},q})^{-1} f_k \in L_{q,\sigma}(\Omega),$$

since $(\lambda - \Delta_{\mathrm{PS},q})^{-1} f_k \in L_{q,\sigma}(\Omega)$ holds for all $k \in \mathbb{N}$ and $L_{q,\sigma}(\Omega)$ is complete. □

Remark B.6. Note that the additional statement about the resolvent $(\lambda - \Delta_{\mathrm{PS},q})^{-1}$ to be consistent (Lemma B.1) was needed in (B.5) only.

Summary

In this thesis we considered the Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla p + (u \cdot \nabla)u &= f \quad \text{in } (0,T) \times \Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T) \times \Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a domain and $(0,T)$ is some time interval, as well as the Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \quad \text{in } (0,T) \times \Omega \\ \operatorname{div} u &= 0 \quad \text{in } (0,T) \times \Omega \\ u|_{t=0} &= u_0 \quad \text{in } \Omega, \end{cases}$$

which is the related linearization. The vector field u and the gradient field ∇p are the unknown quantities while f is a given vector field and u_0 is a given initial value. The Stokes and Navier-Stokes equations were treated subject to partial slip type boundary conditions

$$\begin{cases} \Pi_\tau(\alpha u + (\nabla u^T \pm \nabla u)\nu) &= 0 \quad \text{on } (0,T) \times \partial\Omega \\ \nu \cdot u &= 0 \quad \text{on } (0,T) \times \partial\Omega, \end{cases}$$

where ν is the outward unit normal vector, Π_τ is the projection onto the tangent space at $\partial\Omega$ and α is a real number. The partial slip type boundary conditions include the well-known Navier boundary conditions and the perfect slip boundary condition, which equals the vorticity condition in space dimension $n = 3$.

In Lebesgue ground spaces $L_q(\Omega)$ we have proved well-posedness of the Stokes equations, utilizing analytic semigroup theory for a general class of uniform $C^{2,1}$-domains. We discussed that this class includes non-Helmholtz domains, e.g., sector-like domains with a smoothed vertex and an opening angle $\beta > \pi$ as considered by Bogovskiĭ and Maslennikova in [10]. In addition, we established further results on the Stokes resolvent problem as well as applications to the Navier-Stokes equations on uniform $C^{2,1}$-domains.

We proved existence and uniqueness of maximal strong solutions to the Navier-Stokes equations for the case $\Omega = \mathbb{R}^n$ in the scale of Triebel-Lizorkin-Lorentz spaces (see [14] and [66]). Many important function spaces such as Sobolev-Slobodeckiĭ spaces W_p^s, Bessel-potential spaces H_p^s, Lorentz spaces $L_{p,r}$ and, in particular, Lebesgue spaces L_p are included in this scale. In total, the obtained results concerning Triebel-Lizorkin-Lorentz spaces now yield corresponding results simultaneously for all these function spaces.

Contributions

The content of this thesis is based on a joint work with Jürgen Saal. The main subject of Chapters II and III has been established in [38], where both authors contributed equally to the content of [38]. The author of this thesis implemented the computations for the localization of the Laplace resolvent problem and established the results concerning the Stokes semigroup and applications to the Navier-Stokes equations. The strategies for proving the assertions concerning the Stokes resolvent problem on uniform $C^{2,1}$-domains have been established in several mutual deliberations of Jürgen Saal and the author of this thesis.

The content of Chapter IV has been published in [39]. Both authors contributed equally to [39]. The author of this thesis significantly established the basic properties of TLL spaces and he stated and proved the multiplier result of Mikhlin type for TLL spaces. The results concerning the H^∞-calculus for the Stokes operator and applications to the Navier-Stokes equations in TLL spaces have been developed in a number of working sessions of Jürgen Saal and the author of this thesis. The result concerning the Helmholtz projection in TLL spaces particularly includes several hints of the research group of the Applied Analysis chair.

The preliminaries for this thesis, given in Chapter I, are mainly contained in the introductory parts of [39] and [38] concerning the content and the formulation.

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Ich versichere an Eides Statt, dass die Dissertation von mir selbstständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist.

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