Numerically Trivial Dualizing Sheaves

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André Schell
aus Duisburg

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Referent: Prof. Dr. Stefan Schröer

Korreferent: Juniorprof. Dr. Marcus Zibrowius

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Summary

Motivation for this work was the question to what extent the Beauville–Bogomolov decomposition, or parts and variants of it, holds true in positive characteristic. The statement itself applies to compact Kähler manifolds and its proof uses differential geometry. It can be transferred to smooth projective schemes X over the complex numbers by the Riemann existence theorem. Under the assumption that the dualizing sheaf ω_X is numerically trivial, it states that X admits a finite étale covering $X' \to X$ such that the total space X' satisfies $\omega_{X'} \simeq \mathcal{O}_{X'}$ and decomposes in a special way as a product, uniquely up to permutation.

During the first, preparatory part of this thesis, the notion of numerical triviality is treated and characterized. The Albanese morphism serves as a central technical tool in what follows, so its core properties are developed in a self-contained way, also verifying its existence under more general assumptions than what seems to be recorded in literature.

The second, main part begins to answer the question whether $\omega_X \in \mathrm{Pic}(X)$ has finite order as a consequence of its numerical triviality. More generally, this question can be asked if $\omega_X \simeq \mathcal{O}_X(K_X)$ is not necessarily invertible, as long as the canonical divisor K_X is a Weil divisor. The main result is an affirmative answer for $\mathbb{Q}$-Gorenstein surfaces with the property that the pullback of K_X along a resolution of singularities continues to be a Weil divisor. The proof adapts ideas used by Sakai to extend the Enriques classification of smooth surfaces to normal Gorenstein surfaces over the complex numbers. Without the $\mathbb{Q}$-Gorenstein assumption, the conclusion fails to hold true. Counterexamples of surfaces are given such that K_X is numerically trivial of infinite order.

Once it is known that $\omega_X \in \mathrm{Pic}(X)$ has finite order d, there always exists a finite flat covering $X' \to X$ such that $\omega_{X'} \simeq \mathcal{O}_{X'}$. It is étale if the characteristic p of the ground field is not dividing d. Here the main result is that for normal X, such an étale covering conversely cannot exist if the latter condition is violated. If the value of d is known, this especially gives a criterion for excluding the existence of a decomposition on an étale covering of X. For smooth surfaces, the situation is analyzed in detail afterwards. A softened variant of the decomposition, holding on a finite flat covering, is suggested. Such a decomposition is in turn proved to be unique, using a result by Fujita, and it now exists for all smooth surfaces in arbitrary characteristic.

Zusammenfassung

Motivation für diese Arbeit war die Frage, inwiefern die Beauville-Bogomolov-Zerlegung, oder Teile und Varianten von ihr, weiterhin in positiver Charakteristik gültig ist. Die Aussage selbst gilt für kompakte Kähler-Mannigfaltigkeiten und ihr Beweis benutzt Differentialgeometrie. Sie kann auf glatte projektive Schemata X über den komplexen Zahlen mithilfe des Riemannschen Existenzsatzes übertragen werden. Unter der Annahme, dass die dualisierende Garbe ω_X numerisch trivial ist, sagt sie die Existenz einer endlichen étalen Überlagerung $X' \to X$ mit $\omega_{X'} \simeq \mathcal{O}_{X'}$ aus, sodass der Totalraum X' eine spezielle Produktzerlegung erlaubt, welche eindeutig bis auf Permutation ist.

Im ersten, vorbereitenden Teil dieser Dissertation wird der Begriff der numerischen Trivialität behandelt und charakterisiert. Der Albanese-Morphismus dient im Weiteren als zentrales technisches Werkzeug, sodass seine Kerneigenschaften in eigenständiger Weise entwickelt werden. Seine Existenz kann dabei auch unter allgemeineren Voraussetzungen, als es anscheinend in der Literatur festgehalten ist, nachgewiesen werden.

Der Hauptteil dieser Arbeit beginnt die Frage zu beantworten, ob $\omega_X \in \operatorname{Pic}(X)$ stets endliche Ordnung als Konsequenz ihrer numerischen Trivialität besitzt. Allgemeiner kann diese Frage gestellt werden, falls $\omega_X \simeq \mathcal{O}_X(K_X)$ nicht notwendigerweise invertierbar ist, solange der kanonische Divisor K_X ein Weil-Divisor ist. Das Hauptresultat ist eine positive Antwort für $\mathbb{Q}$-Gorenstein-Flächen mit der Eigenschaft, dass der Pullback von K_X entlang einer Auflösung der Singularitäten ein Weil-Divisor bleibt. Der Beweis verwendet angepasste Ideen von Sakai, welche dieser zur Erweiterung der Enriques-Klassifikation auf normale Gorenstein-Flächen über den komplexen Zahlen verwendet hat. Ohne die $\mathbb{Q}$-Gorenstein-Eigenschaft bleibt die Schlussfolgerung nicht gültig. Es werden Gegenbeispiele von Flächen mit numerisch trivialem K_X von unendlicher Ordnung gegeben.

Sobald bekannt ist, dass $\omega_X \in \operatorname{Pic}(X)$ endliche Ordnung d besitzt, existiert eine endliche flache Überlagerung $X' \to X$ mit der Eigenschaft $\omega_{X'} \simeq \mathcal{O}_{X'}$. Diese ist étale, wenn die Charakteristik p des Grundkörpers kein Teiler von d ist. Hier ist das Hauptresultat, dass für normale X eine solche étale Überlagerung umgekehrt nicht existieren kann, falls letztere Bedingung verletzt ist. Ist der Wert d bekannt, bildet dies ein Ausschlusskriterium für die Existenz der Zerlegung nach einer étalen Überlagerung von X. Für glatte Flächen wird die Situation anschließend detailliert analysiert. Es wird eine mildere Variante der Zerlegung vorgeschlagen, welche nach einer endlichen flachen Überlagerung existiert. Eine derartige Zerlegung wird wiederum als eindeutig bewiesen, unter Verwendung eines Resultates von Fujita, und sie existiert nun für alle glatten Flächen in beliebiger Charakteristik.

Contents

Introduction

Algebraic geometry originated from the challenge to study the space of common zeros $X = V(f_1, \ldots, f_r)$ of polynomials $f_1, \ldots, f_r$ in several indeterminates with coefficients in a field k. Methods from commutative algebra and geometry are applied in synergy to make progress, and also these single branches profit by this interplay. In the 1960s, Grothendieck introduced the notion of a scheme X, which drastically enlarged the framework. The additional datum of a structure sheaf $\mathcal{O}_X$ on X associates in a compatible way to each open subset $U \subset X$ a ring $H^0(U, \mathcal{O}_X)$, reflecting this connection. One central object on proper X is the dualizing sheaf ω_X, which contains both algebraic and geometric information: Assuming that X is sufficiently regular, then on the algebraic side, it yields a duality $H^i(X, \mathcal{E}^\vee \otimes \omega_X) = H^{n-i}(X, \mathcal{E})^\vee$ on the cohomology groups of locally free sheaves $\mathcal{E}$ on X. Cohomology groups are substantial invariants of a scheme, for example their disparity on two schemes already implicates that both schemes themselves must be different. Moreover, $\omega_X = \det(\Omega^1_X)$ is the determinant of the cotangent sheaf Ω^1_X, where the latter includes essential geometric data.

The coincidence $\omega_X \simeq \mathcal{O}_X$ of the dualizing sheaf and the structure sheaf is a special situation with multilayer consequences for the scheme X. For instance, ω_X can be omitted from the duality above. In the case that the dualizing sheaf is different from the structure sheaf, it is desirable to measure the discrepancy and also to slightly modify X in order to move both sheaves closer together. A weaker variant is to ask if some power $\omega_X^{\otimes d} \simeq \mathcal{O}_X$ of the dualizing sheaf is trivial. Taking another step back, ω_X can be numerically trivial, meaning that its intersection number with every curve $C \subset X$ is zero.

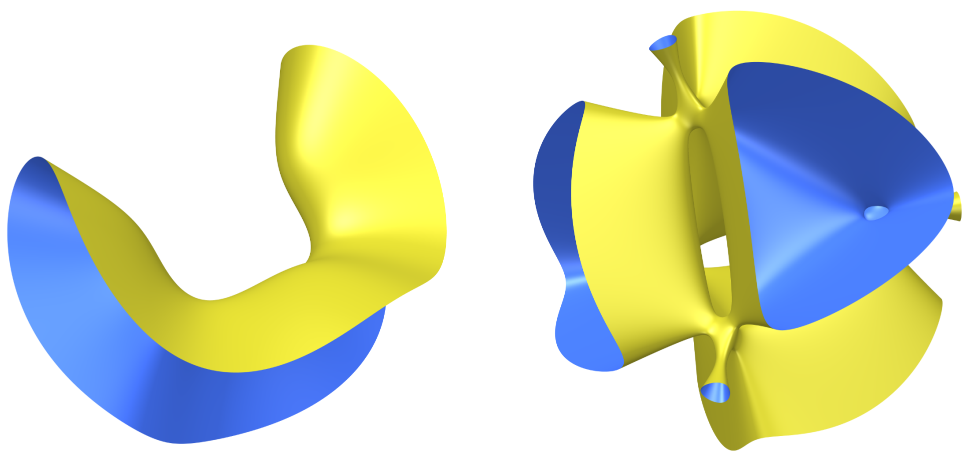

Figure 1: Real points around the origin of two smooth surfaces X with $\omega_X \simeq \mathcal{O}_X$.

Over the field $k = \mathbb{C}$ of complex numbers, the situation was clarified using methods from differential geometry. Although the techniques available in this different branch of mathematics cannot be directly applied in algebraic geometry, it is possible to transfer the conclusion. First, consider the result itself.

The *Beauville–Bogomolov decomposition* states that every connected, compact Kähler manifold Y with $c_1^{\mathrm{an}}(\omega_Y) = 0$ in $H^2(Y;\mathbb{Q})$ admits a finite étale covering $Y' \to Y$ such that the total space $Y' = \prod_{i=1}^r Y_i'$ decomposes as a product, where each factor is a complex torus, a Calabi–Yau manifold or a Hyperkähler manifold. The last two kinds of factors are unique up to permutation. It was established by Beauville [11], [12] who extends prior results due to Bogomolov [16] and Calabi [23].

A complex manifold Y is *Kähler* if it admits an Hermitian metric whose associated differential 2-form is closed. The quotient $\mathbb{C}^g/\Lambda$ by a lattice Λ of rank $2g$ is a *complex torus*. The notion *Calabi–Yau manifold* means a connected, compact Kähler manifold C of dimension at least 3 which is simply connected, satisfies $\omega_C \simeq \mathcal{O}_C$ and has Hodge numbers $h^{r,0}(C) = 0$ for $0 < r < \dim(C)$. A *Hyperkähler manifold* is a connected, compact Kähler manifold H which is simply connected with $H^0(H, \Omega_H^2) = \mathbb{C}\,\sigma$ for a symplectic form σ. The latter means σ induces a non-degenerate alternating pairing on the tangent spaces of all points of H. Consequences are that $\omega_H \simeq \mathcal{O}_H$ is trivial and $\dim(H)$ is even.

The *Riemann existence theorem* states that for a scheme X locally of finite type over $\mathbb{C}$, the functor which associates to every finite étale covering $X' \to X$ its analytification $X'^{\mathrm{an}} \to X^{\mathrm{an}}$ is an equivalence between the categories of finite étale coverings of X and X^{an}, respectively.

Given a smooth, connected, proper scheme X over $\mathbb{C}$, its analytification X^{an} is a connected, compact complex manifold. Moreover, if X is projective over $\mathbb{C}$, then X^{an} is Kähler. Thus if $c_1^{\mathrm{an}}(\omega_{X^{\mathrm{an}}}) = 0$ holds, the Beauville–Bogomolov decomposition exists for the total space of an étale covering of X^{an}, and in turn transfers to a corresponding decomposition of the total space of an étale covering of X.

A compact complex manifold Y is called *algebraic* if $Y \simeq X^{\mathrm{an}}$ for a smooth, proper scheme X. It seems reasonable to transfer the definition of the three possible factors, occurring in the decomposition, to the category of schemes over a field of arbitrary characteristic. Algebraic complex tori are exactly the abelian varieties. The term Calabi–Yau scheme means a smooth, connected, projective scheme C of dimension at least 3 which admits no non-trivial finite étale coverings, satisfies $\omega_C \simeq \mathcal{O}_C$ and has Hodge numbers $h^{r,0}(C) = 0$ for $0 < r < \dim(C)$. To the author's knowledge, there is no common agreement on a definition for a scheme to be Hyperkähler, see [40], Sections 1.2 and 3.1. As Hyperkähler surfaces over $\mathbb{C}$ are exactly the K3-surfaces, this at least suggests a natural answer in dimension two.

The motivation for this work is to investigate the realm of the Beauville–Bogomolov decomposition from an algebro-geometric point of view. Particular focus will be directed towards phenomena in positive characteristic. Moreover, the ground field is assumed to be arbitrary whenever it is achievable. The approach can be subdivided into several steps. To do so, note that both Calabi–Yau manifolds and Hyperkähler manifolds are simply connected. Therefore they satisfy $H^1(Y;\mathbb{C}) = 0$ and so especially $H^1(Y, \mathcal{O}_Y) = 0$ by Hodge theory. This yields a weak form of the decomposition into complex tori and manifolds with trivial canonical sheaf and vanishing first sheaf cohomology of the structure sheaf. Let X be an integral, proper scheme over a field k of characteristic $p \geq 0$ with ω_X invertible and numerically trivial. The following questions reflect stages on the path to an analogue of the Beauville–Bogomolov decomposition.

(Q1) Does the dualizing sheaf ω_X have finite order in $\mathrm{Pic}(X)$?

(Q2) Does there exist a finite étale covering $X' \to X$ such that $\omega_{X'} \simeq \mathcal{O}_{X'}$?

(Q3) Does there exist a finite étale covering $X' \to X$ such that $X' \simeq A \times B$, where A is an abelian variety and B is integral with $H^1(B, \mathcal{O}_B) = 0$ and $\omega_B \simeq \mathcal{O}_B$?

Each question answered in the affirmative implies the previous ones to be, too. The fact that question (Q3) can be answered in the negative in positive characteristic is known, for instance classical and supersingular Enriques surfaces are counterexamples. Hence the following softened variant is suggested.

(Q4) Does there exist a finite flat covering $X' \to X$ such that $X' \simeq A \times B$, where A is an abelian variety and B is integral of Albanese dimension zero with $\omega_B \simeq \mathcal{O}_B$?

These four questions will be researched consecutively throughout this thesis. The guideline is to cover the case of surfaces, both smooth and singular, as completely as possible. Whenever the methods used allow it, results are of course extended to higher dimensions.

There are generalizations of the Beauville–Bogomolov decomposition over $\mathbb{C}$ into different directions. On the one hand, for singular schemes, Höring and Peternell [66] completed the decomposition for normal, irreducible, projective X with at most klt singularities. They admit a quasi-étale covering such that the total space decomposes as a product of an abelian variety, singular Calabi–Yau schemes and singular Hyperkähler schemes.

On the other hand, it is possible to lessen the assumption on ω_Y. Cao and Höring [24] proved that the universal cover of a projective, connected, compact Kähler manifold Y with $\omega_Y^\vee$ nef decomposes as a product of $\mathbb{C}^n$, Calabi–Yau manifolds, Hyperkähler manifolds and a rationally connected manifold.

Question (Q1) is a special case of the *abundance conjecture*, which is an important conjecture for the minimal model program. In a simple form, it states that on a klt, projective scheme X, the canonical divisor K_X is nef, if it is semi-ample.

In positive characteristic, Das and Waldron [32], Theorem 3.3, answered question (Q1) in the affirmative for suitable threefolds in characteristic $p > 5$, using the minimal model program. More precisely, they proved that on a non-uniruled, klt, projective threefold X over an algebraically closed field of characteristic $p > 5$, if the $\mathbb{Q}$-divisor K_X is numerically trivial, then K_X is semi-ample. For terminal threefolds, the assumption that X is non-uniruled can be omitted by [133], Theorem 1.1.

The $C_{n,m}$*-conjecture* proposes that a fibration $f\colon X \to Y$ between smooth, integral, projective schemes of dimensions $n = \dim(X)$ and $m = \dim(Y)$ over an algebraically closed field, such that the geometric generic fiber $X_{\overline{\eta}}$ is smooth and integral, satisfies the inequality $\mathrm{kod}(X) \geq \mathrm{kod}(X_{\overline{\eta}}) + \mathrm{kod}(Y)$ of Kodaira dimensions. Proposed by Iitaka over $\mathbb{C}$, the conjecture has been verified in several cases, initiated by results due to Viehweg and Kawamata over $\mathbb{C}$. See [26] for an overview and a proof of $C_{2,m}$ as well as [38] for a proof of $C_{3,m}$ in the case that $p > 5$. In characteristic zero, the assumption on the generic fiber is automatically fulfilled. As a corollary to his proof of a variant of the $C_{n,m}$-conjecture, Kawamata used the Albanese morphism to answer question (Q1) over $\mathbb{C}$, in [73], Theorem 8.2, also allowing X to have canonical singularities. Indeed, if ω_X is numerically trivial, then so is $\omega_{X_{\overline{\eta}}}$, and induction on $\dim(X)$ can be applied. This approach prompted the author to employ the Albanese morphism, which will play an important role in different parts of this work.

The structure of this thesis is arranged as follows: It is subdivided into two parts, each consisting of three chapters, and an appendix. Part I provides the background and technical tools to treat questions (Q1) to (Q4) afterwards in Part II.

In Chapter 1, the condition $c_1^{\mathrm{an}}(\mathscr{L}^{\mathrm{an}}) = 0$ in $H^2(X^{\mathrm{an}}; \mathbb{Q})$ for an invertible sheaf $\mathscr{L}$ on a scheme X over $\mathbb{C}$ is translated via ℓ-adic cohomology to $c_1(\mathscr{L}) = 0$ in $H^2(X_{\text{ét}}, \mathbb{Q}_\ell)$, which can now be expressed in arbitrary characteristic. Another equivalent description is that $\mathscr{L}$ is *numerically trivial*, which means that its intersection number with every closed curve on X is zero. Intersection numbers are introduced using the Grothendieck group $C(X)$ of coherent sheaves on X, or alternatively via the intersection product of the Chow ring $\mathrm{CH}(X)$. Along the way, fundamentals for subsequent chapters are established.

The *dualizing sheaf* ω_X is discussed in Chapter 2, first in the absolute case of Serre duality for a scheme over a field and afterwards in the relative setting of Grothendieck duality for a morphism of schemes.

Chapter 3 is dedicated to give a self-contained development of the *Albanese morphism*, which is an essential tool in various parts of this work. An Albanese morphism means a universal morphism into the category of abelian varieties if a k-rational point is fixed, and otherwise into the category of principal homogeneous spaces under abelian varieties. Its existence is known, there are several papers dealing with it along the way, but there seems to be no thorough treatment over arbitrary fields. The core result in this context is the following, which holds true for suitable families $X \to S$. It also generalizes the existence of the Albanese morphism beyond what seems to be recorded in literature.

Theorem (Theorem 3.35)**.** *Let X be a proper scheme over an arbitrary field k with $h^0(\mathcal{O}_X) = 1$. Then the Albanese morphism* $\mathrm{alb}\colon X \to \mathrm{Alb}^1_{X/k}$ *exists and its formation commutes with noetherian base change.*

Chapter 4 addresses question (Q1). The first main result in this chapter is a direct proof that for a smooth, proper surface over an arbitrary field, the property of the dualizing sheaf to be numerically trivial implies that it is of finite order. The Enriques classification of surfaces, which is one of the fundamental achievements in algebraic geometry, implies this statement, but a direct proof might be of interest by itself. This is the proof of Theorem 4.17.

The conclusion that the invertible sheaf ω_X has finite order can be extended to suitable normal surfaces. In this situation, the dualizing sheaf is not necessarily invertible, but it is the reflexive sheaf $\omega_X \simeq \mathcal{O}_X(K_X)$ of rank 1 associated to the canonical divisor K_X that is a Weil divisor. So the natural transfer of (Q1) is to ask if K_X has finite order as a Weil divisor. As the second main achievement in Chapter 4, the following theorem could successfully be deduced.

Theorem (Theorem 4.22)**.** *Let X be a $\mathbb{Q}$-Gorenstein, geometrically normal, proper surface over an arbitrary field k with numerically trivial canonical divisor K_X. In the case that X is not Gorenstein, assume the existence of a perfect extension field $k \subset L$ and a resolution of singularities $r\colon \widetilde{X_L} \to X_L$ such that the $\mathbb{Q}$-divisor $r^*(K_{X_L})$ has integral coefficients. Then K_X has finite order.*

The proof adapts arguments used by Sakai [106], [107] in his extension of the Enriques classification to normal Gorenstein surfaces over $\mathbb{C}$. Moreover, it provides upper bounds for the order of K_X, depending on the type of singularities on X. The chapter ends with examples of non-smooth, normal, integral, projective surfaces which have numerically trivial canonical divisor. Depending on a certain parameter, these surfaces either satisfy the assumptions imposed in the preceding theorem, or they turn out to be non-$\mathbb{Q}$-Gorenstein and their canonical divisor has infinite order. Thus (Q1) cannot be answered in the affirmative for arbitrary normal surfaces.

In Chapter 5, question (Q2) is covered, under the necessary assumption that ω_X has finite order. This can be done for arbitrary invertible sheaves, asking whether to an invertible sheaf $\mathscr{L}$ of finite order on X, there exists a finite étale covering $X' \to X$ such that the pullback of $\mathscr{L}$ to X' is trivial. This in fact only depends on the divisibility of $\operatorname{ord}(\mathscr{L})$ by the characteristic p of the ground field k.

Theorem (Theorem 5.6)**.** *Let X be an integral, proper scheme over an algebraically closed field k and let $\mathscr{L} \in \operatorname{Pic}(X)$ be of finite order d. A finite étale covering $g\colon X' \to X$ with $g^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$ exists if and only if $p \nmid d$.*

Specializing to $\mathscr{L} = \omega_X$ and using $g^*(\omega_X) = \omega_{X'}$, this completely answers question (Q2), once the order of ω_X is known. Moreover if X is only assumed to be $\mathbb{Q}$-Gorenstein, this characterizes when there exists an étale covering $X' \to X$ such that the index of $K_{X'}$ equals its order.

At first in Chapter 6, the affirmative answer to the uniqueness of a decomposition as in questions (Q3) and (Q4) can be presented.

Theorem (Theorem 6.5)**.** *Let A, A', B, B' be schemes over an arbitrary field k such that*

(i) A and A' become abelian varieties after base change to $\overline{k}$,

(ii) B and B' are of Albanese dimension zero, geometrically integral and proper over k.

Suppose that $A \times B \simeq A' \times B'$. Then $A \simeq A'$ as well as $B_{\overline{k}} \simeq B'_{\overline{k}}$.

An essential ingredient to the proof is the work of Fujita [41], besides the properties developed previously for the Albanese morphism. Afterwards, a detailed analysis of smooth surfaces with dualizing sheaf of finite order follows, with respect to questions (Q2) to (Q4) from above. Whereas (Q3) must in general be answered in the negative in positive characteristic, the weaker variant (Q4) now always has a positive answer.

The appendix covers various topics and technical tools, which were outsourced to improve readability. Basic group schemes are introduced in Section A.1 and principal homogeneous spaces in Section A.2. An overview of the analytification of schemes locally of finite type over $\mathbb{C}$ and GAGA is provided in Section A.3. Cohomology and base change is the content of Section A.4 and properties of fibrations are contained in Section A.5. The subsequent Section A.6 treats the connection between curves of fiber type on a surface and fibrations from the surface onto a curve. The Picard scheme and a collection of its core properties is covered in Section A.7. Finally, a possible approach using models to deal with question (Q1) is included in Section A.8, being still at an early stage. Except for Section A.8, the Appendix is self-contained and does logically not depend on other chapters.

Conventions. The content of this thesis is addressed to be comprehensible for a reader familiar with the material presented during a standard graduate course in algebraic geometry, following for instance Hartshorne [64], Liu [88] or Görtz and Wedhorn [43]. For each topic beyond, the author always tried to include a short outline, covering its key features for its subsequent application, in order to be broadly accessible.

Let S be a scheme. For an S-scheme X and a morphism of schemes $b\colon S' \to S$, denote the base change of X along b by $X_{S'} = X \times_S S'$. In the case that $S' = \operatorname{Spec}(A)$ is affine, the notation $X_{S'} = X_A$ will be used interchangeably. For objects defined relatively to a morphism $f\colon X \to S$, like the sheaf of Kähler differentials, also use the notation $\Omega^1_{X/S}$, which suppresses the concrete underlying morphism. If the base scheme S is fixed, all products without index are defined over S, and similarly $\Omega^1_X = \Omega^1_{X/S}$ is abbreviated. The function field of an integral scheme X with generic point $\eta \in X$ is written as $K(X) = \mathcal{O}_{X,\eta}$.

If k is a ground field, its characteristic is always denoted by $p = \operatorname{char}(k)$. A *curve* or *surface* over k means an equidimensional k-scheme of dimension 1 or 2, respectively. Deviate from this, a curve $C \subset X$ on a normal, proper surface X over k means a closed subscheme which is a curve in the above sense, but without embedded points. Those correspond to effective Weil divisors on X by [58], Proposition 21.7.2.

A quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is also simply called a *quasicoherent sheaf* on X. Every *locally free sheaf* $\mathcal{E}$ on X is assumed to have finite rank. Especially, an *invertible sheaf* $\mathscr{L}$ on X is a locally free sheaf of rank 1. The Picard group $\text{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X, with group law defined by the tensor product. The subgroup $\text{Pic}^\tau(X) \subset \text{Pic}(X)$ is generated by isomorphism classes of invertible sheaves $\mathscr{L}$ which are τ-equivalent to $\mathcal{O}_X$. For X proper over a field k, the latter exactly means that $\mathscr{L}$ is numerically trivial. Moreover, $\text{Pic}^0(X) \subset \text{Pic}^\tau(X)$ denotes the subgroup generated by isomorphism classes of invertible sheaves $\mathscr{L}$ which are algebraically equivalent to $\mathcal{O}_X$. Details are given in Section 1.7 and Section A.7. Also, if there is no ambiguity and it simplifies the presentation, sometimes invertible sheaves $\mathscr{L}$ on X are notationally identified with its isomorphism class, so $\mathscr{L} \in \text{Pic}(X)$ will be used instead of $[\mathscr{L}] \in \text{Pic}(X)$. For any $\mathcal{O}_X$-module $\mathcal{F}$, set $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $\mathcal{F}$ is a coherent sheaf on a proper k-scheme X, abbreviate $h^i(\mathcal{F}) = \dim_k H^i(X, \mathcal{F})$, which is a natural number by the finiteness theorem. The *Euler characteristic* of $\mathcal{F}$ is the integer $\chi(\mathcal{F}) = \sum_{i\geq 0}(-1)^i h^i(\mathcal{F})$.

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Part I

Chapter 1

Numerically Trivial Sheaves

In order to approach questions related to the Beauville–Bogomolov decomposition for schemes directly, especially over a field of characteristic $p > 0$, the vanishing condition $c_1^{\mathrm{an}}(\omega_Y) = 0$ of the first Chern class has to be transferred to an algebraic analog. On a compact complex manifold Y, the first Chern class $c_1^{\mathrm{an}}(\omega_Y)$ can either be considered as an element of the abelian sheaf cohomology group $H^2(Y, \mathbb{Q}_Y)$ with coefficients in the constant sheaf $\mathbb{Q}_Y$ or of the singular cohomology group $H^2(Y; \mathbb{Q})$. Since Y is locally contractible, both groups can be identified.

Recall that the *singular cohomology* of a topological space X with coefficients in an abelian group G is the cohomology $H^i(X; G) = \ker(\delta^i) / \operatorname{Im}(\delta^{i-1})$ of the singular cochain complex $(C_i^\vee, \delta^i)$, where C_i is the free abelian group generated by the singular i-simplices, $C_i^\vee = \operatorname{Hom}_{\mathbb{Z}}(C_i, G)$ and $\delta^i \colon C_i^\vee \to C_{i+1}^\vee,\ \varphi \mapsto (-1)^{i+1} \varphi \circ \partial_{i+1}$ for the boundary map $\partial_i \colon C_i \to C_{i-1}$, defined by $\sigma \mapsto \sum (-1)^j \sigma(t_1, \ldots, \widehat{t_j}, \ldots, t_i)$.

If X is the underlying topological space of an irreducible scheme, then its higher singular cohomology groups are all zero. Indeed, let X be any topological space with a generic point, that is $\overline{\{\eta\}} = X$ for some $\eta \in X$. Then X is contractible, a strong deformation retraction $X \times [0,1] \to X$ of X onto $\{\eta\}$ is given by $(x, 0) \mapsto x$ and $(x, t) \mapsto \eta$ for $0 < t \leq 1$, since there are no proper closed subsets of X containing η. Singular cohomology is homotopy invariant, so $H^i(X; G) = 0$ for all $i > 0$. The vanishing also holds for the abelian sheaf cohomology groups $H^i(X, G_X)$, as every constant sheaf on X is flabby.

A replacement for the singular cohomology of a scheme X is the *étale cohomology* and the induced *ℓ-adic cohomology*. They comprise more information and the latter can be identified with the abelian sheaf cohomology of X^{an} in the case that X is of finite type over $\mathbb{C}$.

This chapter begins with a review of different regularity conditions in Section 1.1, in order to fix the notions, the connections between them and their basic properties, which will be used frequently throughout the following. The next sections introduce étale cohomology in Section 1.2 and ℓ-adic cohomology in Section 1.3. Over the complex numbers, the comparison between ℓ-adic, singular and abelian sheaf cohomology is discussed in Section 1.4. Then in Section 1.5, the first ℓ-adic Chern class $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Q}_\ell(1))$ is defined and its vanishing is characterized. One equivalent description is that $\mathscr{L}$ is numerically trivial. Therefore intersection numbers are treated in Section 1.6 using the Grothendieck group of coherent sheaves, and afterwards numerical triviality is characterized in Section 1.7.

Intersection numbers can also be defined via the Chow ring that is briefly introduced in Section 1.8. The fact that both variants coincide is the content of Section 1.9, which moreover discusses higher Chern classes with values in the Chow ring. This section also includes some corollaries of the Grothendieck–Riemann–Roch theorem that will be needed in the subsequent chapters. Finally in Section 1.10, higher ℓ-adic Chern classes and Poincaré duality are covered.

1.1 Regularity Conditions

Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Recapitulate the following definitions from commutative algebra, see Bourbaki [20], [21] for details.

- $\operatorname{depth}(R) = \max\{\, l \geq 0 \mid \exists\, R\text{-regular sequence } a_1, \dots, a_l \in \mathfrak{m} \,\}$ is the *depth* of R.
- $\dim(R) = \max\{\, l \geq 0 \mid \exists\,\text{prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_l \text{ in } R \,\}$ is the *dimension* of R.
- $\operatorname{edim}(R) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ is the *embedding dimension* of R.

The inequalities $\operatorname{depth}(R) \leq \dim(R) \leq \operatorname{edim}(R)$ always hold.

- R is *regular* if $\dim(R) = \operatorname{edim}(R)$.
- R is *Cohen–Macaulay* if $\operatorname{depth}(R) = \dim(R)$.
- R is *Gorenstein* if $\operatorname{Ext}^i_R(\kappa, R) = 0$ for $i < \dim(R)$ and $\operatorname{Ext}^{\dim(R)}_R(\kappa, R) \simeq \kappa$.
- R is *normal* if R is an integrally closed domain.
- R is *reduced* if R contains no non-zero nilpotent elements.

If R is regular, then R is Gorenstein. Note for the appearance of being Gorenstein in Grothendieck duality that R is Gorenstein if and only if R has finite injective dimension, which is the minimal length of an injective resolution. This means that R is a dualizing complex for itself, see [63], Chapter V, Theorem 9.1. An equivalent description of depth is given by $\operatorname{depth}(R) = \min\{\, l \geq 0 \mid \operatorname{Ext}^l_R(\kappa, R) \neq 0 \,\}$. Hence if R is Gorenstein, then R is Cohen–Macaulay. If R is regular, then R is normal. It is immediate that if R is normal, then R is reduced.

A locally noetherian scheme X is *regular*, *Cohen–Macaulay*, *Gorenstein*, *normal* or *reduced* if all its local rings $\mathcal{O}_{X,x}$ for $x \in X$ have the corresponding property. For a more detailed distinction, introduce for $i \geq 0$ Serre's conditions:

(S_i) $\operatorname{depth}(\mathcal{O}_{X,x}) \geq \min\{i, \dim(\mathcal{O}_{X,x})\}$ for all $x \in X$.

(R_i) $\mathcal{O}_{X,x}$ is regular for all $x \in X$ with $\dim(\mathcal{O}_{X,x}) \leq i$.

(G_i) $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$ with $\dim(\mathcal{O}_{X,x}) \leq i$.

Condition (S_1) means that X has no embedded points. The scheme X is reduced if and only if it satisfies (S_1) and (R_0), and X is normal if and only if it satisfies (S_2) and (R_1) by [56], Proposition 5.8.5 and Théorème 5.8.6. Let S be locally noetherian and let $f\colon X \to S$ be a morphism locally of finite type.

- f is *of relative dimension* r if X is non-empty and $\dim(X_s) = r$ for all $s \in S$ such that X_s is non-empty.
- f is *of relative equidimension* r if f is of relative dimension r and all non-empty fibers X_s are equidimensional, that is, each irreducible component of X_s has dimension r.
- f is *flat* if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,f(x)}$-module for all $x \in X$.
- f is *faithfully flat* if f is flat and surjective.
- f is *regular*, *Cohen–Macaulay*, *Gorenstein*, *normal*, *geometrically normal*, *reduced* or *geometrically reduced* if f is flat and all fibers X_s for $s \in S$ have the corresponding property.
- f is *smooth* if f is flat and all fibers X_s for $s \in S$ are geometrically regular.

Here, a certain property $\mathcal{P}$ for a k-scheme T holds *geometrically* if for all field extensions $k \subset E$, the E-scheme X_E has property $\mathcal{P}$. Note that this definition differs from [56], Définition 6.8.1, where a regular, normal or reduced morphism is defined to have geometrically regular, geometrically normal or geometrically reduced fibers, respectively. If in contrast the fibers are Cohen–Macaulay or Gorenstein, then they are already geometrically Cohen–Macaulay or geometrically Gorenstein, see [56], Corollaire 6.7.8 and [9], Tag 0C03. Thus a Cohen–Macaulay or Gorenstein morphism is stable under base change, whereas regular, normal or reduced morphisms are in general not.

Every flat morphism $f\colon X \to S$ locally of finite presentation is universally open due to [56], Théorème 2.4.6. Particularly, if X is non-empty, S is connected and f is additionally closed—for instance proper—then f is faithfully flat.

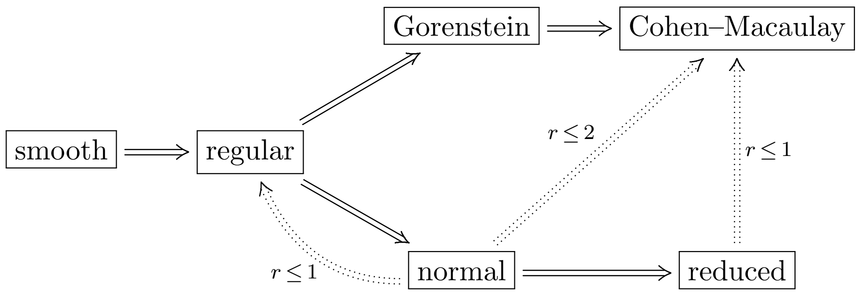
Proposition 1.1. *Let S be locally noetherian and let $f\colon X \to S$ be a faithfully flat morphism locally of finite type.*

(i) X is Cohen–Macaulay or Gorenstein if and only if both f and S have the corresponding property.

(ii) If X is regular, normal or reduced, then S has the corresponding property. Moreover, if S and f are regular, normal or reduced, then X has the corresponding property.

References for the preceding proposition are [63], Chapter V, Proposition 9.6 and [9], Tag 0C12, [56], Corollaire 6.3.5 and [43], Proposition 14.57.

The following diagram sums up the relations between the properties of f covered above. If all fibers of f are of dimension at most r, then the dotted implications are valid.



For Cohen–Macaulay and normal schemes, the following proposition presents additional properties. They refer to [43], Proposition 14.124 and Remark 6.37.

Proposition 1.2. *Let X be a locally noetherian scheme.*

(i) If X is Cohen–Macaulay, connected and locally of finite type over an arbitrary field k, then X is equidimensional.

(ii) If X is normal and connected, then X is irreducible and thus integral.

Let S be locally noetherian and let $f\colon X \to S$ be a morphism locally of finite type.

- f is *unramified* if $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a finite separable field extension of $\kappa(s)$ for all $x \in X$, where $s = f(x)$.
- f is *étale* if f is flat and unramified.
- f is *purely inseparable* if f is injective and for every $x \in X$, the field extension $\kappa(f(x)) \subset \kappa(x)$ is purely inseparable.

A morphism f is unramified if and only if for every $s \in S$, the fiber $X_s \simeq \coprod_{i \in I} \operatorname{Spec}(E_i)$ is a disjoint union for finite separable field extensions $\kappa(s) \subset E_i$. The reference for this is [58], Corollaire 17.4.2. Especially, every unramified morphism f is locally quasi-finite. Thus if f is additionally proper, then f is finite by [57], Théorème 8.11.1.

A morphism f is étale if and only if f is smooth of relative dimension 0, see [58], Théorème 17.6.1. In the case that f is a morphism of schemes over a separably closed field k, then f is étale if and only if the induced homomorphisms $\widehat{\mathcal{O}}_{S,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on formal completions of the local rings are bijective for all $x \in X$ due to [58], Proposition 17.6.3.

An étale morphism $f\colon X \to S$ satisfies $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{S,f(x)})$ for all $x \in X$, see [58], Proposition 17.6.4. Particularly if $f\colon X \to S$ is an étale morphism of irreducible schemes of finite type over an arbitrary field k, then f is dominant and $\dim(X) = \dim(S)$. If f is additionally proper, then f is finite and surjective.

For f to be purely inseparable, it is equivalent to demand that the base change $X_{S'} \to S'$ is injective for every morphism $S' \to S$ by [50], Proposition 3.5.8 and Remarque 3.5.11. So a purely inseparable morphism is also called a *universally injective* morphism.

1.2 Étale Cohomology

To define étale cohomology, it is necessary to extend the notion of a topology from a topology on a space to a *Grothendieck topology* on a category. Note that a section of a sheaf corresponds by definition to its compatible restrictions to an open covering. Thus to define a sheaf, it is only necessary to know what open coverings are, instead of open subsets themselves. This notion can be generalized in a meaningful way. For instance, open subschemes $U, V \subset X$ can be regarded as open embeddings $U \hookrightarrow X$ and $V \hookrightarrow X$. Then their intersection is the fiber product $U \cap V = U \times_X V$. Hence if instead of open embeddings, a wider class of morphisms should be considered, the fiber product is a natural candidate to replace the intersection.

The subsequent lines form a brief survey of the topic. For an elaboration, see Milne [90], Chapter II, based on the fundamentals provided in SGA 4 [7]. Let X be a locally noetherian scheme. A *site* $(\mathcal{C}/X)$ over X consists of a full subcategory $(\mathcal{C}/X)$ of (Sch/X) and a *Grothendieck topology* $\mathcal{T}$ on $(\mathcal{C}/X)$. The latter is defined in [7] to be a collection of *sieves* for objects of $(\mathcal{C}/X)$. A tangible way to obtain a topology is the use of a *pretopology*, which induces a Grothendieck topology. This is the notion used in the following. A pretopology is a collection of families $(g_i \colon U_i \to Y)_{i\in I}$ of morphisms in $(\mathcal{C}/X)$ for objects Y in $(\mathcal{C}/X)$ such that $Y = \bigcup g_i(U_i)$, called *coverings*, which satisfies suitable axioms recreating the behavior of a usual topology.

To obtain the *Zariski site* (Zar/X), set $(\mathcal{C}/X)$ as the category of locally noetherian X-schemes as well as the pretopology to be the collection of all open embeddings. In the same manner, define the *étale site* $(\mathrm{\acute{E}t}/X)$ and the *flat site* (Fppf/X), where étale morphisms and flat morphisms locally of finite type, respectively, are used instead of open embeddings.

Remark 1.3. Those sites are called the *big* sites, whereas for the *small* sites, the same condition imposed on the morphisms forming $\mathcal{T}$ is also used to define $(\mathcal{C}/X)$. The underlying category (zar/X) of the small Zariski site is the category of X-schemes whose structure morphism is an open embedding. In the case of the small étale site, the underlying category $(\mathrm{\acute{e}t}/X)$ is the category of X-schemes whose structure morphism is étale and of finite type—now including quasicompactness—and the pretopology is also given by étale morphisms of finite type.

The small and big sites defined above will both yield the same cohomology groups according to [90], Chapter III, Proposition 3.1 and Proposition 3.3. Hence often the small sites are chosen to work with, being less complicated. This usually happens if one other reason is satisfied, namely if the morphisms between objects in $(\mathcal{C}/X)$ inherit the condition imposed on their structure morphism to X. In the example of the small étale site, a morphism between X-schemes with étale structure morphism of finite type is automatically étale by [58], Proposition 17.3.4. The same holds true for open embeddings, but in general not for flat morphisms. Nevertheless, to be consistent with the treatment of the different Picard functors in Section A.7, the standard sites in the following are the big sites. Then the Picard functors can be defined on the same category, and they can therefore be compared directly.

Remark 1.4. The term "fppf" abbreviates *fidèlement plat de présentation finie*. Since all X-schemes considered above are locally noetherian, every morphism to one of them which is locally of finite type is automatically locally of finite presentation. Thus a faithfully flat morphism locally of finite presentation to a locally noetherian scheme will also be called an *fppf morphism*. Every covering $(U_i \to Y)_{i \in I}$ for the flat site yields actually an fppf morphism $\coprod U_i \to Y$.

A *sheaf* $\mathcal{F}$ on a site $(\mathcal{C}/X)$ is a presheaf, that is, a contravariant functor $\mathcal{F}\colon (\mathcal{C}/X) \to \mathcal{A}$ to a category $\mathcal{A}$, which additionally satisfies the sheaf axiom for all coverings in $\mathcal{T}$. Usually and if not mentioned otherwise, the category $\mathcal{A}$ will be the category (Ab) of abelian groups. Most constructions and properties of sheaves in the usual sense can be transferred analogously to sheaves on sites.

Example 1.5. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module in the usual sense. Then assign to any $g\colon U \to X$ in $(\mathcal{C}/X)$ the group $\Gamma(U, g^*(\mathcal{F}))$. Together with the natural restrictions, this defines sheaves $\mathcal{F}_{\text{ét}}$ on the site $(\text{Ét}/X)$ and $\mathcal{F}_{\text{fppf}}$ on the site (Fppf/X) by [90], Chapter II, Corollary 1.6. Those sheaves are again $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ also denotes the sheaf of rings given by the definition in the case $\mathcal{F} = \mathcal{O}_X$ above.

Example 1.6. Let G be a commutative group scheme over X, which can be expressed in saying that its functor of points $h_G\colon (\text{Sch}/X) \to (\text{Set})$ factorizes through (Ab). Thus to any $g\colon U \to X$ in $(\mathcal{C}/X)$, assign the group $h_G(U) = \text{Hom}_X(U, G) = G(U)$. This naturally yields an abelian sheaf G on $(\text{Ét}/X)$ and (Fppf/X) due to [90], Chapter II, Corollary 1.7.

Remark 1.7. Following Grothendieck's philosophy, [7], Exposé IV, the central object to study is a *topos*, which is a category equivalent to the category of sheaves on a site. The notion of topos can be seen as the suitable enlargement of the notion of topological space.

As for usual sheaves, write $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ for the sections of $\mathcal{F}$ over $U \to X$. The category $\text{S}(\mathcal{C}/X)$ of sheaves on a site $(\mathcal{C}/X)$ contains enough injectives by [90], Chapter III, Proposition 1.1, and the functor $\text{S}(\mathcal{C}/X) \to (\text{Ab})$, $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ is left-exact, see [90], Chapter II, Theorem 2.15. Its right derived functors $\text{S}(\mathcal{C}/X) \to (\text{Ab})$, $\mathcal{F} \mapsto H^i(X, \mathcal{F})$ yield the *i-th cohomology group* $H^i(X, \mathcal{F})$ of $(\mathcal{C}/X)$ with values in $\mathcal{F}$ for $i \geq 0$. In the case of the étale or fppf site, use the notation $H^i(X_{\text{ét}}, \mathcal{F})$ and $H^i(X_{\text{fppf}}, \mathcal{F})$ for the *i-th étale cohomology group* and *i-th fppf cohomology group* of X with values in $\mathcal{F}$, respectively.

1.3 ℓ-adic Cohomology

Let X be a locally noetherian scheme over an arbitrary field k of characteristic $p \geq 0$ and ℓ a prime number different from p. For $n \geq 1$, let $\mathcal{F}_n$ be the associated sheaf on the étale site of X to the constant group scheme $(\mathbb{Z}/\ell^n\mathbb{Z})_X$ or to the group scheme

$\mu_{\ell^n,X}$. The cohomology groups $H^i(X_{\text{ét}}, \mathcal{F}_n)$, computed for the category of sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, are again $\mathbb{Z}/\ell^n\mathbb{Z}$-modules and they coincide with the cohomology groups computed for the category of sheaves of abelian groups. Indeed, a Godement resolution by flabby sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules can be used to derive cohomology and the notion of being flabby is independent of the module structure, so it also yields the cohomology groups as abelian groups. For $m \geq n$, the projections $(\mathbb{Z}/\ell^m\mathbb{Z})_X \to (\mathbb{Z}/\ell^n\mathbb{Z})_X$ and multiplications $\mu_{\ell^m,X} \to \mu_{\ell^n,X}$ by ℓ^{m-n} produce an inverse system $(\mathcal{F}_n)_{n\geq 1}$. The induced homomorphisms $H^i(X_{\text{ét}}, \mathcal{F}_m) \to H^i(X_{\text{ét}}, \mathcal{F}_n)$ mount up to an inverse system and its limit $\varprojlim H^i(X_{\text{ét}}, \mathcal{F}_n)$ becomes a $\mathbb{Z}_\ell$-module. Here $\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$ denotes the ℓ-adic integers and $\mathbb{Q}_\ell = \operatorname{Frac}(\mathbb{Z}_\ell)$ the ℓ-adic numbers.

Definition 1.8. The $\mathbb{Z}_\ell$-module $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1)) := \varprojlim H^i(X_{\text{ét}}, \mu_{\ell^n,X})$ is defined to be the *i-th ℓ-adic cohomology group* of X *with integral coefficients*. Similarly, the $\mathbb{Q}_\ell$-vector space $H^i(X_{\text{ét}}, \mathbb{Q}_\ell(1)) = H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is the *i-th ℓ-adic cohomology group* of X *with rational coefficients*. The *i-th Betti number of* X is $b_i(X) = \operatorname{rank}_{\mathbb{Z}_\ell} H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1))$. Also abbreviate $b_i = b_i(X)$ if the dependence is obvious.

Furthermore, define the $\mathbb{Z}_\ell$-module $H^i(X_{\text{ét}}, \mathbb{Z}_\ell) = \varprojlim H^i(X_{\text{ét}}, (\mathbb{Z}/\ell^n\mathbb{Z})_X)$ and the $\mathbb{Q}_\ell$-vector space $H^i(X_{\text{ét}}, \mathbb{Q}_\ell) = H^i(X_{\text{ét}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Remark 1.9. If k is separably closed, then the choice of a primitive ℓ^n-th root of unity yields a non-canonical isomorphism $(\mathbb{Z}/\ell^n\mathbb{Z})_X \simeq \mu_{\ell^n,X}$. Hence there are induced bijections $H^i(X_{\text{ét}}, \mu_{\ell^n,X}) \simeq H^i(X_{\text{ét}}, (\mathbb{Z}/\ell^n\mathbb{Z})_X)$. Inductively, primitive roots of unity can be chosen such that

$$\begin{array}{ccc} \mu_{\ell^{n+1},X} & \xrightarrow{\simeq} & (\mathbb{Z}/\ell^{n+1}\mathbb{Z})_X \\ \ell \downarrow & & \downarrow \text{pr} \\ \mu_{\ell^n,X} & \xrightarrow[\simeq]{} & (\mathbb{Z}/\ell^n\mathbb{Z})_X \end{array}$$

commutes for all $n \geq 1$. Then $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1)) \simeq H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$ non-canonically. One reason to set the former to be the ℓ-adic cohomology groups of X is that Chern classes can naturally be defined to take values therein. This will be explicated in Section 1.5 and also Section 1.10 below.

Remark 1.10. The notion of a *sheaf of $\mathbb{Z}_\ell$-modules* $\mathcal{F} = (\mathcal{F}_n)_{n\geq 1}$ is a formalization of the two cases above. See [90], page 163f. for details.

Recall that $\mathbb{Z}_\ell$ is a principal ideal domain and its non-zero ideals are all generated by some ℓ^n with quotient $\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell = \mathbb{Z}/\ell^n\mathbb{Z}$. Thus if $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ is a finitely generated $\mathbb{Z}_\ell$-module, the classification of finitely generated modules over a principal ideal domain yields $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1)) \simeq \mathbb{Z}_\ell^{\oplus b_i} \oplus T$ for some finite module T. Hence the Betti numbers can alternatively be computed as $b_i = \dim_{\mathbb{Q}_\ell} H^i(X_{\text{ét}}, \mathbb{Q}_\ell(1))$.

For all $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ to be finitely generated, it is sufficient that all $H^i(X_{\text{ét}}, \mu_{\ell^n,X})$ are finite groups due to [90], Chapter V, Lemma 1.11. This holds for instance if X is of finite type over a separably closed field by [33], [Th. finitude], Corollaire 1.10, page 236. The following finiteness theorem is [90], Chapter VI, Theorem 1.1.

Proposition 1.11. *Let X be of finite type over a separably closed field k and $n = \dim(X)$. Then $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1)) = 0$ for $i > 2n$.*

Under the assumptions above, the *ℓ-adic Euler characteristic* of X is defined to be the integer $e(X) = \sum_{i=0}^{2n}(-1)^i b_i(X)$. Reinterpretations of this term will be given in Proposition 1.46 and Corollary 1.47 below.

Example 1.12. Observe that the ℓ-adic cohomology groups in general differ from the cohomology groups with coefficients in the constant sheaves $(\mathbb{Z}_\ell)_X$ or $(\mathbb{Q}_\ell)_X$. For instance, if X is normal, integral, noetherian and G_X is the constant sheaf defined by a torsion-free abelian group G, then $H^1(X_{\text{ét}}, G_X) = 0$. This can be seen as follows:

Let $\eta = \text{Spec}(K(X))$ and set $i\colon \eta \to X$ for the canonical morphism. On the small étale site $(\text{ét}/X)$, there is a natural identification $G_X \xrightarrow{\simeq} i_*(G_\eta)$. Indeed, let $U \to X$ be étale of finite type. The quasicompactness ensures that U is noetherian, so it has finitely many connected components. Hence to show the identification of sheaves above, reduce to U connected. Since U is normal, it is even integral. Then the generic fiber $U' = U_\eta$ is again integral, and the map on sections over U is the natural map $H^0(U_{\text{ét}}, G_U) \to H^0(U'_{\text{ét}}, G_{U'})$, which is the identity $G \to G$.

Moreover, $R^1 i_*(G_\eta)$ is zero: This sheaf is the sheafification of $U \mapsto H^1(U'_{\text{ét}}, G_{U'})$. Here $U' = U_\eta$ is a finite disjoint union of $\text{Spec}(E)$, where $K(X) \subset E$ is a finite separable field extension. Thus again, to show that $H^1(U'_{\text{ét}}, G_{U'}) = 0$, reduce to $U' = \text{Spec}(E)$. Then, as $G_{U'}$ is constant, $H^1(U'_{\text{ét}}, G_{U'})$ is isomorphic to $\text{Hom}_{\text{cont}}(H, G)$, where H is the absolute Galois group of E. But this homomorphism group is trivial, since H is profinite and G is discrete and torsion-free. So $R^1 i_*(G_\eta)$ is the sheafification of the trivial presheaf, hence trivial itself.

The Leray spectral sequence $E_2^{a,b} = H^a(X_{\text{ét}}, R^b i_*(G_\eta)) \Rightarrow H^{a+b}(\eta_{\text{ét}}, G_\eta)$, see [90], Chapter III, Theorem 1.18, eventually yields $H^1(X_{\text{ét}}, G_X) = H^1(\eta_{\text{ét}}, G_\eta)$. The right-hand side is zero, using the same argument as before. This proves the claim.

The preceding reasoning also indicates that for G finite, the group $H^1(X_{\text{ét}}, G_X)$ will comprise essentially more information. This motivates the use of ℓ-adic cohomology and the next section will exhibit that it is an expedient replacement for singular cohomology.

1.4 Comparison to Other Cohomology Theories

Let X be a scheme of finite type over $\mathbb{C}$. Its closed points $X(\mathbb{C})$ can naturally be equipped with the classical topology and a sheaf of rings. This yields a complex analytic space X^{an}, the *analytification* of X. See Section A.3 for an overview of this technique.

This section reviews that the ℓ-adic cohomology of X coincides with the ℓ-adic abelian sheaf cohomology of X^{an}, and under suitable assumptions also with the singular cohomology of X^{an} with coefficients in $\mathbb{Z}_\ell$. There is a natural choice $\zeta_{\ell^n} = \exp(\frac{2\pi \text{i}}{\ell^n})$ of a primitive complex ℓ^n-th root of unity, so identify $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(1)) = H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$ and consider the latter groups in the following, as they can be defined in the same manner on X^{an}.

Notationally, given a topological space Y and an abelian group G, its abelian sheaf cohomology groups with coefficients in the constant sheaf G_Y are $H^i(Y, G_Y)$. As in the definition of $H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$, set $H^i(Y, \mathbb{Z}_\ell) = \varprojlim H^i(Y, (\mathbb{Z}/\ell^n\mathbb{Z})_Y)$. The singular cohomology groups with coefficients in G are denoted by $H^i(Y; G)$. The notation $H^i(Y; \mathbb{Z}_\ell)$ will not be used, but the next proposition shows that $\varprojlim H^i(Y; \mathbb{Z}/\ell^n\mathbb{Z}) = H^i(Y; \varprojlim \mathbb{Z}/\ell^n\mathbb{Z})$ is true quite generally.

Proposition 1.13. *Let ℓ be a prime number and $i \geq 0$.*

(i) Let X be a scheme of finite type over $\mathbb{C}$. There is a natural identification

$$H^i(X_{\text{ét}}, \mathbb{Z}_\ell) = H^i(X^{\text{an}}, \mathbb{Z}_\ell).$$

If X^{an} is moreover locally contractible, then $H^i(X^{\text{an}}, \mathbb{Z}_\ell) = H^i(X^{\text{an}}, (\mathbb{Z}_\ell)_{X^{\text{an}}})$.

(ii) Let Y be a locally contractible topological space and G an arbitrary abelian group. Then

$$H^i(Y, G_Y) = H^i(Y; G).$$

(iii) Let Y be a topological space such that all singular homology groups $H_*(Y; \mathbb{Z})$ are finitely generated. Then

$$\varprojlim H^i(Y; \mathbb{Z}/\ell^n\mathbb{Z}) = H^i(Y; \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}) = H^i(Y; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

(iv) All assumptions above hold for a smooth, proper scheme X over $\mathbb{C}$ and $Y = X^{\text{an}}$.

The proof is conducted throughout the remainder of this section. Its crucial part is clearly the first isomorphism: For a finite abelian group G, the étale cohomology groups $H^i(X_{\text{ét}}, G_X)$ and abelian sheaf cohomology groups $H^i(X^{\text{an}}, G_{X^{\text{an}}})$ can naturally be identified by [8], Exposé XVI, Théorème 4.1. For $G = \mathbb{Z}/\ell^n\mathbb{Z}$, the functoriality of this identification guarantees that the conclusion holds true in the limit. This shows the first identification in (i).

On a locally contractible topological space Y, abelian sheaf cohomology and singular cohomology coincide, see [115]. Thus (ii) is valid. To complete the proof of (i), consider the case where $X^{\text{an}} = Y$ is locally contractible. The assertion is an application of [53], Chapitre 0, Proposition 13.3.1 and Remarques 13.3.2. Verify the assumptions imposed there: First, note that on any contractible open subset $U \subset Y$, the identification $H^i(U, (\mathbb{Z}/\ell^n\mathbb{Z})_U) = H^i(U; \mathbb{Z}/\ell^n\mathbb{Z})$ holds. For $i > 0$, this group is zero. Second, as stated in Section 1.3, the groups $H^i(X^{\text{an}}, (\mathbb{Z}/\ell^n\mathbb{Z})_{X^{\text{an}}}) = H^i(X_{\text{ét}}, (\mathbb{Z}/\ell^n\mathbb{Z})_X)$ are all finite, and so the inverse system $(H^i(X^{\text{an}}, (\mathbb{Z}/\ell^n\mathbb{Z})_{X^{\text{an}}})_{n\geq 1}$ satisfies the Mittag–Leffler condition. Eventually, the proposition can be applied, which yields the identification $H^i(X^{\text{an}}, \mathbb{Z}_\ell) = H^i(X^{\text{an}}, (\mathbb{Z}_\ell)_{X^{\text{an}}})$.

Prior to the proof of (iii), observe that every compact topological manifold has the homotopy type of a finite CW complex according to [75]. Hence its singular homology groups are finitely generated. Thus if X is a smooth, proper scheme over $\mathbb{C}$, then X^{an} is a compact complex manifold, so X^{an} is locally contractible with finitely generated singular homology groups. Consequently, (iv) holds once (iii) does.

Finally to verify (iii), assume that all singular homology groups $H_*(Y;\mathbb{Z})$ are finitely generated. This also implies that the singular cohomology groups are finitely generated and the universal coefficient theorem, [128], Proposition 11.9.6, holds: For any abelian group G, the sequence

$$0 \longrightarrow H^i(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} G \longrightarrow H^i(Y;G) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H^{i+1}(Y;\mathbb{Z}), G) \longrightarrow 0 \tag{1.1}$$

is exact, splits, and is natural in Y and G.

Thus $H^i(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^i(Y;\mathbb{Z}_\ell)$ is bijective if $\mathrm{Tor}_1^{\mathbb{Z}}(H^{i+1}(Y;\mathbb{Z}),\mathbb{Z}_\ell) = 0$. This is true: $H^{i+1}(Y;\mathbb{Z})$ is a finitely generated abelian group and Tor commutes with direct sums, so the assertion reduces to $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_\ell) = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\,\mathbb{Z},\mathbb{Z}_\ell) = \mathbb{Z}_\ell[m] = 0$ for all $m \geq 1$. Here $G[m] = \{\, g \in G \mid mg = 0 \,\}$ is the m-torsion subgroup of an additively denoted group G.

It remains to verify that $H^i(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\simeq} \varprojlim H^i(Y;\mathbb{Z}/\ell^n\,\mathbb{Z})$ naturally. The cases $G = \mathbb{Z}/\ell^n\,\mathbb{Z}$ for $n \geq 1$ in the universal coefficient theorem yield an exact sequence of inverse systems. The transition maps in $(H^i(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n\,\mathbb{Z})_{n\geq 1}$ are surjective, so the system satisfies the Mittag–Leffler condition. Hence the inverse limits form again an exact sequence. To deduce the claimed bijection, two arguments are required. First,

$$\varprojlim \mathrm{Tor}_1^{\mathbb{Z}}(H^{i+1}(Y;\mathbb{Z}),\mathbb{Z}/\ell^n\,\mathbb{Z}) = \varprojlim H^{i+1}(Y;\mathbb{Z})[\ell^n]$$

has to be zero. The transition maps on the right-hand side are multiplication by ℓ. In fact, compute the Tor groups by tensoring the rows in

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ell^{n+1}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^{n+1}\,\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{pr}} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\ell^{n}]{} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^{n}\,\mathbb{Z} & \longrightarrow & 0
\end{array}$$

with $H^{i+1}(Y;\mathbb{Z})$. The subsequent lemma now finishes this first argument.

Lemma 1.14. *Let G be a finite group, denoted additively, and ℓ a prime number. Then $\varprojlim G[\ell^n] = 0$, where the transition maps are multiplication by ℓ.*

Proof. Assume by contradiction that $(g_n) \in \varprojlim G[\ell^n]$ is non-zero. Then fix an $s \geq 1$ such that $g_s \neq 0$. For all $r > s$ also $g_r \neq 0$ holds, and the relations $\ell^r g_r = 0$ as well as $\ell^{r-s} g_r = g_s \neq 0$ are valid. So $\mathrm{ord}(g_r)$ divides ℓ^r, but does not divide ℓ^{r-s}. Hence $\mathrm{ord}(g_r) = \ell^m$ for some $m > r - s$. This is not possible for all $r > s$, since G is finite. □

Second, the identification $H^i(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \varprojlim (H^i(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n\,\mathbb{Z})$ has to be verified. This is the following lemma, which thereby concludes the proof of Proposition 1.13.

Lemma 1.15. *Let R be a ring, M a finitely presented R-module and (N_i) an inverse system of R-modules, which satisfies the Mittag–Leffler condition. Then the canonical map*

$$M \otimes_R \big(\varprojlim N_i\big) \longrightarrow \varprojlim(M \otimes_R N_i), \quad m \otimes (n_i) \longmapsto (m \otimes n_i)$$

is bijective.

Proof. Choose a finite presentation $R^{\oplus r} \to R^{\oplus s} \to M \to 0$. Tensoring with $\varprojlim N_i$ yields an exact sequence

$$(\varprojlim N_i)^{\oplus r} \longrightarrow (\varprojlim N_i)^{\oplus s} \longrightarrow M \otimes_R (\varprojlim N_i) \longrightarrow 0. \tag{1.2}$$

On the other hand, the tensor product with each N_i gives the following exact sequence:

$$N_i^{\oplus r} \longrightarrow N_i^{\oplus s} \longrightarrow M \otimes_R N_i \longrightarrow 0.$$

They form an exact sequence of inverse systems with the naturally induced transition maps. The inverse system $(N_i^{\oplus r})$ and in turn also its image in $(N_i^{\oplus s})$ continues to satisfy the Mittag–Leffler condition. Hence the inverse limit is the exact sequence

$$(\varprojlim N_i)^{\oplus r} \longrightarrow (\varprojlim N_i)^{\oplus s} \longrightarrow \varprojlim(M \otimes_R N_i) \longrightarrow 0. \tag{1.3}$$

The five lemma applied to the natural commutative diagram formed by (1.2) and (1.3) eventually yields the assertion. □

1.5 First Chern Classes

Let X be a locally noetherian scheme over an arbitrary field k of characteristic $p \geq 0$ and ℓ a prime number different from p. By [8], Exposé IX, Théorème 3.3, there are canonical identifications

$$\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^\times) = H^1(X_{\text{ét}}, \mathbb{G}_{m,X}).$$

For every integer $n \geq 1$, the Kummer sequence

$$1 \longrightarrow \mu_{\ell^n,X} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{\ell^n} \mathbb{G}_{m,X} \longrightarrow 1 \tag{1.4}$$

is exact on the étale site of X. Indeed, exactness can be verified on stalks and for a strictly henselian local ring R with residue field of characteristic p, multiplication $R^\times \to R^\times$ by ℓ^n is surjective due to Hensel's lemma. The following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_{\ell^{n+1},X} & \longrightarrow & \mathbb{G}_{m,X} & \xrightarrow{\ell^{n+1}} & \mathbb{G}_{m,X} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \mu_{\ell^n,X} & \longrightarrow & \mathbb{G}_{m,X} & \xrightarrow[\ell^n]{} & \mathbb{G}_{m,X} & \longrightarrow & 1
\end{array}$$

shows that the boundary maps $H^1(X_{\text{ét}}, \mathbb{G}_{m,X}) \xrightarrow{\delta} H^2(X_{\text{ét}}, \mu_{\ell^n,X})$ are compatible with the induced morphisms by $\mu_{\ell^{n+1},X} \to \mu_{\ell^n,X}$ on the second cohomology. This yields a map

$$c_1\colon \operatorname{Pic}(X) \longrightarrow H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1)).$$

Definition 1.16. The *first Chern class* of an invertible sheaf $\mathscr{L}$ on X is defined as $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$. The same name and notation is used for its canonical image $c_1(\mathscr{L}) \in H^i(X_{\text{ét}}, \mathbb{Q}_\ell(1))$.

A characterization of what it means for the first Chern class of an invertible sheaf to be torsion or zero will be given below. Before, in the case that X is of finite type over $k = \mathbb{C}$, note that the target $H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1)) = H^2(X_{\text{ét}}, \mathbb{Z}_\ell)$ can be identified with $H^2(X^{\text{an}}, \mathbb{Z}_\ell)$ due to Proposition 1.13. Here, c_1 can be reinterpreted as follows: The long exact sequence of cohomology groups to the exponential sequence

$$0 \longrightarrow \mathbb{Z}_{X^{\text{an}}} \xrightarrow{t \mapsto 2\pi i t} \mathcal{O}_{X^{\text{an}}} \xrightarrow{\exp} \mathcal{O}^\times_{X^{\text{an}}} \longrightarrow 1 \tag{1.5}$$

induces a map $c_1^{\text{an}}\colon H^1(X^{\text{an}}, \mathcal{O}^\times_{X^{\text{an}}}) \to H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}})$, which associates to every invertible sheaf a variant of the first Chern class in the analytic category. The canonical maps $H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}}) \to H^2(X^{\text{an}}, (\mathbb{Z}/\ell^n\mathbb{Z})_{X^{\text{an}}})$ define $\operatorname{pr}\colon H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}}) \to H^2(X^{\text{an}}, \mathbb{Z}_\ell)$. The data above is connected through the commutative diagram

$$\begin{array}{ccccccc} & & H^1(X_{\text{ét}}, \mathbb{G}_{m,X}) & & \xrightarrow{\quad c_1 \quad} & & H^2(X_{\text{ét}}, \mathbb{Z}_\ell) \\ & & \| & & & & \| \\ \operatorname{Pic}(X) & = & H^1(X^{\text{an}}, \mathcal{O}^\times_{X^{\text{an}}}) & \xrightarrow[c_1^{\text{an}}]{} & H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}}) & \xrightarrow[\operatorname{pr}]{} & H^2(X^{\text{an}}, \mathbb{Z}_\ell) \end{array} \tag{1.6}$$

due to [60], Exposé VII, Partie 3.8. Call an invertible sheaf $\mathscr{L}$ on a scheme *ℓ-divisible* if its class $[\mathscr{L}] \in \operatorname{Pic}(X)$ is ℓ-divisible, that is, for every $n \geq 1$, there exists an invertible sheaf $\mathscr{M}$ such that $\mathscr{M}^{\otimes \ell^n} \simeq \mathscr{L}$.

Proposition 1.17. *Let X be a scheme of finite type over an arbitrary field k of characteristic $p \geq 0$. Let $\ell \neq p$ be a prime number and $\mathscr{L}$ an invertible sheaf on X. Consider the following statement:*

(i) $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ is torsion.

Each one of the subsequent assertions is equivalent to (i) under the additional assumption imposed in the respective parenthesis.

(ii) There exists an $m \geq 1$ such that $\mathscr{L}^{\otimes m}$ is ℓ-divisible.

(iii) $\mathscr{L} \in \operatorname{Pic}^\tau(X)$. *($X$ is proper with $h^0(\mathcal{O}_X) = 1$ over k algebraically closed).*

(iv) $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Q}_\ell(1))$ is zero. *(k is separably closed).*

(v) $c_1^{\text{an}}(\mathscr{L}) \in H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}})$ is torsion. *($k = \mathbb{C}$ and $H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}})$ is fin. gen.).*

Proof. The equivalence of (i) and (ii) is a consequence of the stronger statement $c_1(\mathscr{L}) = 0$ if and only if $\mathscr{L}$ is ℓ-divisible. This can be verified as follows: First, note that the choice of an injective resolution of $\mathbb{G}_{m,X}$ shows that the map induced by $\ell^n \colon \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$ on $H^1(X_{\text{ét}}, \mathbb{G}_{m,X}) = \operatorname{Pic}(X)$ is again the multiplication by ℓ^n. The kernel of c_1 is the intersection over all kernels of the boundary homomorphisms δ, defined by the exact sequences

$$\operatorname{Pic}(X) \xrightarrow{\ell^n} \operatorname{Pic}(X) \xrightarrow{\delta} H^2(X_{\text{ét}}, \mu_{\ell^n,X}).$$

Hence $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Z}_\ell)$ is zero if and only if $\mathscr{L}$ is contained in the images of $\ell^n \colon \operatorname{Pic}(X) \to \operatorname{Pic}(X)$ for all $n \geq 1$, which means $\mathscr{L}$ is ℓ-divisible.

Next, show—more precisely than stated—that (ii) implies (iii) if X is proper over k, and that the converse holds if additionally $h^0(\mathcal{O}_X) = 1$ and k is algebraically closed. The properness of X ensures that its Néron–Severi group $\operatorname{NS}(X) = \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ is finitely generated, and thus it decomposes as $\operatorname{NS}(X) \simeq \mathbb{Z}^{\oplus \rho} \oplus T$ for the finite group $T = \operatorname{Pic}^\tau(X)/\operatorname{Pic}^0(X)$, see Section A.7. Assume (ii), so some multiple $\mathscr{M} = \mathscr{L}^{\otimes m}$ is ℓ-divisible. This divisibility remains true for the image of $\mathscr{M}$ in $\operatorname{NS}(X)$, so it has to be contained in the torsion part T. Thus a multiple of $\mathscr{M}$, and thereby of multiple of $\mathscr{L}$, is contained in the kernel $\operatorname{Pic}^0(X)$ of the projection to $\operatorname{NS}(X)$. This means $\mathscr{L} \in \operatorname{Pic}^\tau(X)$. Conversely, assume $\mathscr{L} \in \operatorname{Pic}^\tau(X)$. Then replace $\mathscr{L}$ by some multiple to reduce to the case that $\mathscr{L} \in \operatorname{Pic}^0(X)$. By Theorem A.30, the identity component $G = \operatorname{Pic}^0_X$ of the Picard scheme of X is of finite type, hence an algebraic group, and $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0_X(k)$ is bijective. The structure theory of algebraic groups, [101], Lemma 6.1, yields the existence of subgroups $0 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that each G_i/G_{i-1} for $1 \leq i \leq n$ is the multiplicative group, the additive group, an abelian variety or a finite group. Since k is algebraically closed and $\ell \neq p$, the k-valued points of both $\mathbb{G}_m$ and $\mathbb{G}_a$ are ℓ-divisible. The same holds for abelian varieties by [95], Section 6, Application 2. Now let m be the least common multiple of all orders of finite groups appearing as subquotients. Then $\mathscr{L}^{\otimes m^n}$ is ℓ-divisible. In fact by induction, for every $x \in G_i(k)$, its multiple $m^i x$ is ℓ-divisible. Hence (ii) holds.

Assertion (i) always implies (iv). For the converse, assume that k is separably closed. As explained in Section 1.3, the group $H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ is a finitely generated $\mathbb{Z}_\ell$-module, so $H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1)) \simeq \mathbb{Z}_\ell^{\oplus b_2} \oplus T$ decomposes for some finite group T. Since $\mathbb{Z}_\ell \to \mathbb{Q}_\ell$ is injective and $H^2(X_{\text{ét}}, \mathbb{Q}_\ell(1)) \simeq \mathbb{Q}_\ell^{\oplus b_2}$, also (iv) implies (i).

Finally, prove that (i) and (v) are equivalent if $k = \mathbb{C}$ and $H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}}) \simeq \mathbb{Z}^{\oplus r} \oplus S$ for a finite abelian group S. Assume (v). Then (1.6) yields that some $\mathscr{L}^{\otimes m}$ is contained in the kernel of c_1, and hence (i) follows. Conversely, assume that $\mathscr{L}^{\otimes m}$ is ℓ-divisible. Since no non-zero element of $\mathbb{Z}^{\oplus r}$ is ℓ-divisible, necessarily $m \cdot c_1^{\text{an}}(\mathscr{L}) \in S$ must hold. So $c_1^{\text{an}}(\mathscr{L})$ is torsion. □

Remark 1.18. The proof above has in fact shown that (i) implies (iii) whenever X is proper over k. Moreover, the equivalence stated in subsequent proposition was already proven.

Proposition 1.19. *In the situation of Proposition 1.17, the following two assertions are equivalent:*

(i) $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ *is zero.*

(ii) $\mathscr{L}$ *is* ℓ*-divisible.*

If X *is normal, proper with* $h^0(\mathcal{O}_X) = 1$ *over* k *algebraically closed, then* $\mathscr{L} \in \operatorname{Pic}^0(X)$ *implies (i) and (ii).*

Proof. Only the last statement remains to be verified. Under the additional assumptions, let $\mathscr{L} \in \operatorname{Pic}^0(X)$. Since X is normal, Theorem A.30 ensures that Pic^0_X is proper and $A = (\operatorname{Pic}^0_X)_{\text{red}}$ is an abelian variety. The group $A(k)$ is a divisible, as k is algebraically closed. Because $\operatorname{Pic}^0(X) = \operatorname{Pic}^0_X(k) = A(k)$, the invertible sheaf $\mathscr{L}$ is ℓ-divisible. Thus (ii) holds. □

Example 1.20. The contrary conclusion that an invertible sheaf $\mathscr{L}$ with $c_1(\mathscr{L}) = 0$ in $H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ always satisfies $\mathscr{L} \in \operatorname{Pic}^0(X)$ is not true in general. For instance, let k be algebraically closed and X a classical Enriques surface. Then $\operatorname{Pic}^\tau_X = \mathbb{Z}/2\mathbb{Z}$ and so $\operatorname{Pic}^0_X = \operatorname{Spec}(k)$ is trivial. The class of the dualizing sheaf ω_X is the non-trivial element of $\operatorname{Pic}^\tau(X)$, that means $\omega_X \notin \operatorname{Pic}^0(X)$. But for every odd prime $\ell \neq p$, clearly $\omega_X^{\otimes \ell} \simeq \omega_X$. Hence ω_X is ℓ-divisible and therefore $c_1(\omega_X) = 0$.

Remark 1.21. In contrast, the condition $c_1^{\text{an}}(\mathscr{L}) = 0$ for the analytic first Chern class is equivalent to $\mathscr{L} \in \operatorname{Pic}^0(X)$. Indeed, let $k = \mathbb{C}$ and X a smooth, integral, projective scheme over $\mathbb{C}$. Note that by Proposition 1.13, singular and abelian sheaf cohomology coincide on X^{an}. There exists a natural identification $\operatorname{Pic}^0(X) = \ker(c_1^{\text{an}})$ as follows: Set $Y = X^{\text{an}}$. The long exact sequence induced by the exponential sequence (1.5) yields

$$0 \longrightarrow H^1(Y, \mathcal{O}_Y)/H^1(Y, \mathbb{Z}_Y) \longrightarrow H^1(Y, \mathcal{O}_Y^\times) \xrightarrow{c_1^{\text{an}}} H^2(Y, \mathbb{Z}_Y).$$

There exists a natural identification $\operatorname{Pic}^0(X) = H^1(Y, \mathcal{O}_Y)/H^1(Y, \mathbb{Z}_Y)$ according to [14], Proposition 11.11.3. Thus $\operatorname{Pic}^0(X) = \ker(c_1^{\text{an}})$.

At this point, summarize what has been deduced during this section so far. If X is a smooth, proper scheme over $\mathbb{C}$, then c_1^{an} maps to $H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}})$. This group is finitely generated and can be identified with the singular cohomology group $H^2(X^{\text{an}}; \mathbb{Z})$ due to Proposition 1.13. For any field F, the universal coefficient theorem (1.1) implies that the kernel of the natural map $H^2(X^{\text{an}}; \mathbb{Z}) \to H^2(X^{\text{an}}; F)$ is the torsion part of $H^2(X^{\text{an}}; \mathbb{Z})$. So the condition $c_1^{\text{an}}(\mathscr{L}) = 0$ in $H^2(X^{\text{an}}; F)$, which is imposed for the Beauville–Bogomolov decomposition on $\mathscr{L} = \omega_X$, means that $c_1^{\text{an}}(\mathscr{L}) \in H^2(X^{\text{an}}, \mathbb{Z}_{X^{\text{an}}})$ is torsion.

Equivalent descriptions were given Proposition 1.17, which can be demanded from an invertible sheaf $\mathscr{L}$ on a proper scheme X over an arbitrary field k. A consequence is always that $\mathscr{L} \in \operatorname{Pic}^\tau(X)$, which exactly means that $\mathscr{L}$ is numerically trivial. This will be discussed in Section 1.7 below, after some basic intersection theory has been introduced in Section 1.6. The numerical triviality of ω_X will be the condition imposed on X to study questions related to the Beauville–Bogomolov decomposition over an arbitrary field k. It is the most tangible notion for practical applications.

This section is concluded with two well-known results about the Betti numbers $b_1(X)$ and $b_2(X)$, which can be deduced from the methods used above.

Proposition 1.22. *Let X be a normal, integral, proper scheme over an algebraically closed field k and $\ell \neq p$. Set $g = \dim(\operatorname{Pic}^0_X)$. Then*

$$b_1(X) = 2g \qquad \textit{and} \qquad b_2(X) \geq \rho(X).$$

Proof. Since X is proper, the Néron–Severi group $\operatorname{NS}(X)$ is finitely generated, and thus $T = \operatorname{Pic}^\tau(X)/\operatorname{Pic}^0(X)$ is a finite group. Consider the abelian variety $A = (\operatorname{Pic}^0_X)_{\mathrm{red}}$. The abelian group $\operatorname{Pic}^0(X) = A(k)$ is a divisible, as k is algebraically closed. Divisible abelian groups are exactly the injective $\mathbb{Z}$-modules, thus $\operatorname{Pic}^\tau(X) \simeq \operatorname{Pic}^0(X) \oplus T$. The Kummer sequence (1.4) shows that $H^1(X_{\text{ét}}, \mu_{\ell^n,X}) = \operatorname{Pic}(X)[\ell^n] = \operatorname{Pic}^\tau(X)[\ell^n]$, as k is separably closed. Lemma 1.14 implies that $\varprojlim T[\ell^n] = 0$. Moreover $A(k)[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{\oplus 2g}$ holds by for instance [95], Section 6, Proposition on page 64. Consequently, $H^1(X_{\text{ét}}, \mathbb{Z}_\ell(1)) \simeq \mathbb{Z}_\ell^{\oplus 2g}$ and especially $b_1(X) = 2g$.

For the inequality, consider now the part $\operatorname{Pic}(X) \xrightarrow{\ell^n} \operatorname{Pic}(X) \xrightarrow{\delta} H^2(X_{\text{ét}}, \mu_{\ell^n,X})$ induced by the Kummer sequence. The target of δ is a $\mathbb{Z}/\ell^n\mathbb{Z}$-module, so the divisibility of $\operatorname{Pic}^0(X)$ implies $\operatorname{Pic}^0(X) \subset \ker(\delta)$. Thus δ induces an injection from $\operatorname{NS}(X)/\ell^n \operatorname{NS}(X)$ into $H^2(X_{\text{ét}}, \mu_{\ell^n,X})$, which yields a morphism of inverse systems indexed over $n \geq 1$. Since $\operatorname{NS}(X)$ is finitely generated, $\operatorname{NS}(X)/\ell^n \operatorname{NS}(X) = \operatorname{NS}(X) \otimes \mathbb{Z}/\ell^n\mathbb{Z}$. Now Lemma 1.15 shows that the induced map on the limit is $\operatorname{NS}(X) \otimes \mathbb{Z}_\ell \hookrightarrow H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$. Eventually, forming ranks yields $\rho(X) \leq b_2(X)$. □

1.6 Intersection Numbers

Fix an arbitrary ground field k and let X be a proper k-scheme. The purpose of this section is to define and summarize basic properties of the intersection number $(\mathscr{L}_1 \cdots \mathscr{L}_r \mid \mathcal{F}) \in \mathbb{Z}$ of invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_r$ and a coherent sheaf $\mathcal{F}$ on X. The subsequent outline follows Kleiman [78], Appendix B, based upon work by Snapper. Proofs of the statements in this section can be found in this source.

Let $\operatorname{Coh}(X)$ be the abelian category of coherent sheaves on X. Denote by $\operatorname{Coh}_r(X)$ its full subcategory, consisting of those coherent sheaves $\mathcal{F}$ with $\dim(\operatorname{Supp}(\mathcal{F})) \leq r$. The *Grothendieck group* $C(X)$ of $\operatorname{Coh}(X)$ is the free abelian group on all $\mathcal{F} \in \operatorname{Coh}(X)$, modulo relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ for every short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. Its subgroup $C_r(X)$ is generated by $\operatorname{Coh}_r(X)$. The properness of X ensures that the Euler characteristic $\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i h^i(\mathcal{F})$ is a finite sum by [53], Théorème 3.2.1. As it is compatible with the relations in $C(X)$, there is an induced homomorphism $\chi\colon C(X) \to \mathbb{Z}$. If $Y \subset X$ is a closed subscheme and $\mathcal{F} = \mathcal{O}_Y$, also write $[Y] = [\mathcal{O}_Y]$ in $C(X)$.

For every invertible sheaf $\mathscr{L}$ on X, the assignment $[\mathcal{F}] \mapsto [\mathcal{F}] - [\mathscr{L}^\vee \otimes \mathcal{F}]$ yields an endomorphism of $C(X)$ denoted by $c_1(\mathscr{L})$. To shed some light on this definition, consider the case where $\mathcal{F} = \mathcal{O}_Y$ for a closed subscheme $Y \subset X$, for instance $Y = X$. If $D \subset Y$ is an effective Cartier divisor such that $\mathcal{O}_Y(D) \simeq \mathscr{L}|_Y$, then $c_1(\mathscr{L})[Y] = [D]$. This follows immediately from the defining sequence $0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$ noting that $\mathcal{O}_Y(-D) \simeq \mathscr{L}^\vee|_Y$. The connection to Chern classes as elements of the Chow ring will explain the notation c_1 and is illustrated in Section 1.9. The fact that the image of $C_r(X)$ under $c_1(\mathscr{L})$ is contained in $C_{r-1}(X)$ is an immediate consequence of the following:

Lemma 1.23. *Let $Y_1, \ldots, Y_s$ be the r-dimensional integral components of* $\mathrm{Supp}(\mathcal{F})$ *and denote the multiplicity of $\mathcal{F}$ at the generic point $\eta_j \in Y_j$ by $m_j = \mathrm{length}_{\mathcal{O}_{X,\eta_j}} \mathcal{F}_{\eta_j}$. Then $[\mathcal{F}] \equiv \sum_{j=1}^s m_j [Y_j] \mod C_{r-1}(X)$.*

Definition 1.24. The *intersection number* of invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_r$ on X with $\Sigma \in C_r(X)$ is the integer

$$(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \Sigma) = \chi(c_1(\mathscr{L}_1) \cdots c_1(\mathscr{L}_r)(\Sigma)).$$

For a coherent sheaf $\mathcal{F} \in \mathrm{Coh}_r(X)$, the intersection number with $[\mathcal{F}] \in C_r(X)$ is denoted by $(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F})$. In the case that $\mathcal{F} = \mathcal{O}_X$, abbreviate $(\mathscr{L}_1 \cdots \mathscr{L}_r) = (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{O}_X)$ and if $\mathscr{L}_i = \mathcal{O}_X(D_i)$ for a Cartier divisor D_i, also write $(D_1 \cdots D_r \,|\, \mathcal{F}) = (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F})$. Furthermore, if $\mathscr{L} := \mathscr{L}_1 = \cdots = \mathscr{L}_r$, use the notation $(\mathscr{L}^r \,|\, \mathcal{F}) = (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F})$. If $k \subset E$ is a field extension, denote by $(\mathscr{L}'_1 \cdots \mathscr{L}'_r \,|\, \Sigma')_E$ the intersection number of invertible sheaves $\mathscr{L}'_1, \ldots, \mathscr{L}'_r$ on the E-scheme X_E with $\Sigma' \in C_r(X_E)$.

Example 1.25. Let $C \subset X$ be a closed curve, for instance if $C = X$ is a curve itself. Then for an invertible sheaf $\mathscr{L}$ on X, the intersection number

$$(\mathscr{L} \,|\, \mathcal{O}_C) = \chi(\mathcal{O}_C) - \chi(\mathscr{L}^\vee|_C) = -\deg(\mathscr{L}^\vee|_C) = \deg(\mathscr{L}|_C)$$

is the degree of $\mathscr{L}$ restricted to C.

Example 1.26. Let $S \subset X$ be a closed surface. Again the important case is when $S = X$ is a surface. Then the definition yields for invertible sheaves $\mathscr{L}, \mathscr{N}$ on X the formula

$$(\mathscr{L} \cdot \mathscr{N} \,|\, \mathcal{O}_S) = \chi(\mathcal{O}_S) - \chi(\mathscr{L}^\vee|_S) - \chi(\mathscr{N}^\vee|_S) + \chi(\mathscr{L}^\vee|_S \otimes \mathscr{N}^\vee|_S),$$

which is the common direct definition of the intersection number on surfaces.

Proposition 1.27. *The intersection number satisfies the following properties:*

(i) $(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F})$ *is symmetric and multilinear in* $\mathscr{L}_1, \ldots, \mathscr{L}_r$.

(ii) $(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F}) = 0$ *if* $\mathcal{F} \in \mathrm{Coh}_{r-1}(X)$.

(iii) Let $D_1, \ldots, D_r$ be effective Cartier divisors on X and denote their schematic intersection by $Z = D_1 \cap \cdots \cap D_r$. If $Z \cap \mathrm{Supp}(\mathcal{F})$ is finite and $\mathcal{F}_z$ is a Cohen–Macaulay module over $\mathcal{O}_{X,z}$ for every $z \in Z$, then $(D_r \cdots D_r \,|\, \mathcal{F}) = h^0(\mathcal{F}|_Z)$.

(iv) *Let D be an effective Cartier divisor on X and $C \subset X$ an integral, closed curve such that $C \not\subset D$. Then $(D \,|\, \mathcal{O}_C) = h^0(\mathcal{O}_{C \cap D})$.* *(see Figure 2)*

(v) *Let $Y_1, \ldots, Y_s$ be the r-dimensional integral components of $\operatorname{Supp}(\mathcal{F})$ and denote the multiplicity of $\mathcal{F}$ at the generic point $\eta_j \in Y_j$ by $m_j = \operatorname{length}_{\mathcal{O}_{X,\eta_j}} \mathcal{F}_{\eta_j}$. Then $(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F}) = \sum_{j=1}^{s} m_j (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{O}_{Y_j})$.*

(vi) *Let $Y \subset X$ be a closed subscheme and $D \subset Y$ an effective Cartier divisor with associated sheaf $\mathcal{O}_Y(D) \simeq \mathscr{L}_r|_Y$. Then $(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{O}_Y) = (\mathscr{L}_1 \cdots \mathscr{L}_{r-1} \,|\, \mathcal{O}_D)$.*

(vii) *Let $k \subset E$ be a field extension. Then $(\mathscr{L}_{1,E} \cdots \mathscr{L}_{r,E} \,|\, \mathcal{F}_E)_E = (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F})$.*

(viii) *Let $g \colon X' \to X$ be a morphism of proper k-schemes and let $\mathcal{F}' \in \operatorname{Coh}_r(X')$. Then the equality $(g^*(\mathscr{L}_1) \cdots g^*(\mathscr{L}_r) \,|\, \mathcal{F}') = (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, g_*(\mathcal{F}'))$ holds.*

(ix) *Let $g \colon X' \to X$ be a morphism of proper k-schemes and $Y' \subset X'$ an integral, closed subscheme of dimension $\leq r$. Let $Y = g(Y')$ be its schematic image. Then the equality $(g^*(\mathscr{L}_1) \cdots g^*(\mathscr{L}_r) \,|\, \mathcal{O}_{Y'}) = \deg(g|_{Y'}) \cdot (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{O}_Y)$ holds, where $\deg(g|_{Y'})$ equals $[K(Y') : K(Y)]$ if this value is finite and 0 otherwise.*

(x) *Let $k \subset E$ be a finite field extension and $Y' \subset X_E$ an integral, closed subscheme of dimension $\leq r$. Let $Y \subset X$ be the schematic image of Y' under the natural projection $X_E \to X$. Then $(\mathscr{L}_{1,E} \cdots \mathscr{L}_{r,E} \,|\, \mathcal{O}_{Y'})_E = \frac{[K(Y'):K(Y)]}{[E:k]} \cdot (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{O}_Y)$.*

Proof. Only assertions (iv) and (x) are not stated directly in the source [78], Appendix B. The former is a special case of (iii). For (x), note that $\chi(\mathcal{F}') = [E : k] \cdot \chi_E(\mathcal{F}')$ for every coherent sheaf $\mathcal{F}'$ on X_E. Thus $(\mathscr{L}_{1,E} \cdots \mathscr{L}_{r,E} \,|\, \mathcal{O}_{Y'}) = [E : k] \cdot (\mathscr{L}_{1,E} \cdots \mathscr{L}_{r,E} \,|\, \mathcal{O}_{Y'})_E$ and (ix) yields the claim. □

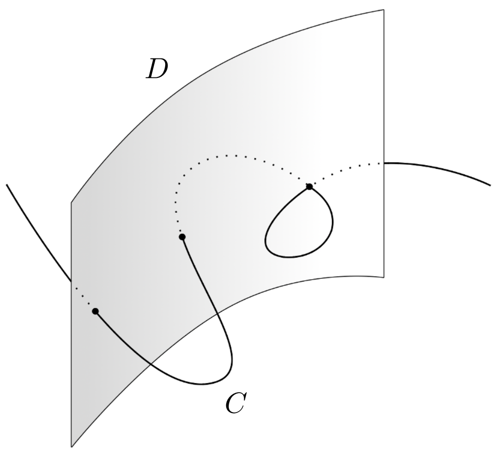


Figure 2: Intersection of an effective Cartier divisor D with a curve C.

Remark 1.28. The intersection number is uniquely determined by the properties above according to [76], Chapter I, Section 2, Proposition 7. Furthermore, $(\mathscr{L}_1 \cdots \mathscr{L}_r \mid \mathcal{F})$ is the coefficient in the numerical polynomial $\chi(\mathscr{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathscr{L}_r^{\otimes n_r} \otimes \mathcal{F}) \in \mathbb{Q}[n_1, \ldots, n_r]$ of the monomial $n_1 \cdots n_r$.

Proposition 1.29 (Riemann–Roch for Surfaces)**.** *Let X be a Cohen–Macaulay, reduced, proper surface over k and $\mathcal{K} = [\omega_X] - [\mathcal{O}_X]$. Then $\mathcal{K} \in C_1(X)$ and for every invertible sheaf $\mathscr{L}$ on X, the formula*

$$\chi(\mathscr{L}) = \frac{(\mathscr{L}^2) - (\mathscr{L} \mid \mathcal{K})}{2} + \chi(\mathcal{O}_X)$$

holds. Moreover, if X is additionally Gorenstein, then

$$\chi(\mathscr{L}) = \frac{(\mathscr{L}^2) - (\mathscr{L} \cdot \omega_X)}{2} + \chi(\mathcal{O}_X).$$

Observe that the proof of this statement given in [78], Proposition B.26, works under the lessened assumptions imposed above. Indeed, the dualizing sheaf exists for proper schemes and Proposition 2.21 below will show that ω_X is generically trivial, so Lemma 1.23 yields that $\mathcal{K} \in C_1(X)$.

Definition 1.30. Let $\mathscr{L}$ and $\mathscr{N}$ be invertible sheaves on X. They are *numerically equivalent* if $(\mathscr{L} \mid \mathcal{O}_C) = (\mathscr{N} \mid \mathcal{O}_C)$ for all integral, closed curves $C \subset X$. As seen in Example 1.25, this equality means $\deg(\mathscr{L}|_C) = \deg(\mathscr{N}|_C)$. The invertible sheaf $\mathscr{L}$ is *numerically trivial* if $\mathscr{L}$ is numerically equivalent to $\mathcal{O}_X$, that is, $\deg(\mathscr{L}|_C) = 0$ for all integral, closed curves $C \subset X$.

Numerical equivalence actually defines an equivalence relation $\mathscr{L} \equiv \mathscr{N}$ on the set of invertible sheaves on X and on $\mathrm{Pic}(X)$. The linearity of the intersection number yields that the equivalence relation is compatible with tensor products in the sense that $\mathscr{L} \equiv \mathscr{N}$ and $\mathscr{L}' \equiv \mathscr{N}'$ imply $\mathscr{L} \otimes \mathscr{L}' \equiv \mathscr{N} \otimes \mathscr{N}'$. Moreover, an invertible sheaf $\mathscr{L}$ is numerically trivial if and only if a tensor power $\mathscr{L}^{\otimes d}$ is numerically trivial for some non-zero integer d, which then in turn also holds true for all non-zero integers d.

Although numerical equivalence is only defined using intersection numbers with curves, a consequence is that numerically equivalent invertible sheaves always yield the same intersection numbers in the following sense:

Proposition 1.31. *Let $\mathscr{L}_1, \ldots, \mathscr{L}_r, \mathscr{N}_1, \ldots, \mathscr{N}_r$ be invertible sheaves on X and let $\Sigma \in C_r(X)$. If $\mathscr{L}_i$ and $\mathscr{N}_i$ are numerically equivalent for each $1 \leq i \leq r$, then also $(\mathscr{L}_1 \cdots \mathscr{L}_r \mid \Sigma) = (\mathscr{N}_1 \cdots \mathscr{N}_r \mid \Sigma)$.*

Numerical equivalence and especially numerical triviality behave well with regard to morphisms. This essential fact is the content of the next proposition, and it will be used frequently in what follows.

Proposition 1.32. *Let $g\colon X \to Y$ be a morphism of proper k-schemes and let $\mathscr{L}, \mathscr{N}$ be invertible sheaves on Y. The following statements hold:*

(i) If $\mathscr{L}$ and $\mathscr{N}$ are numerically equivalent, then $g^*(\mathscr{L})$ and $g^*(\mathscr{N})$ are numerically equivalent.

(ii) If $g^*(\mathscr{L})$ and $g^*(\mathscr{N})$ are numerically equivalent and g is surjective, then $\mathscr{L}$ and $\mathscr{N}$ are numerically equivalent.

In particular, if $\mathscr{L}$ is numerically trivial, then $g^*(\mathscr{L})$ is also numerically trivial. The converse is true if g is a surjection.

The Nakai–Moishezon criterion expresses how the ampleness of an invertible sheaf is determined by its numerical behavior:

Proposition 1.33 (Nakai–Moishezon Criterion)**.** *Let X be a proper scheme over an arbitrary field k and $\mathscr{L}$ an invertible sheaf on X. The following two conditions are equivalent:*

(i) $\mathscr{L}$ is ample.

(ii) $(\mathscr{L}^{\dim(Y)} \,|\, \mathcal{O}_Y) > 0$ for every integral, closed subscheme of $Y \subset X$.

Proof. Kleiman [76], Chapter III, Section 1, Theorem 1, proves the statement for integral X over an algebraically closed field k. Denote $F = \overline{k}$ and reduce to this case in several steps. First of all, recall that $\mathscr{L}$ is ample if and only if its restrictions to all integral components $X_i \subset X$ are ample. Moreover, $\mathscr{L}$ on X is ample if and only if its base change $\mathscr{L}_F$ on X_F is ample. Also, every integral, closed subscheme $Y \subset X$ of dimension d is contained in some X_i and the equality $(\mathscr{L}^d \,|\, \mathcal{O}_Y) = (\mathscr{L}|_{X_i}^d \,|\, \mathcal{O}_Y)$ holds due to Proposition 1.27 (ix). Hence assume without loss of generality that X is integral.

Assume (i) and let $Y \subset X$ be an integral, closed subscheme of dimension d and $Y_{F,1}, \ldots, Y_{F,r}$ the integral components of $Y_F \subset X_F$ with multiplicities $m_1, \ldots, m_r$. Then Proposition 1.27 yields $(\mathscr{L}^d \,|\, \mathcal{O}_Y) = (\mathscr{L}_F^d \,|\, \mathcal{O}_{Y_F})_F = \sum_{i=1}^r m_i \cdot (\mathscr{L}_F^d \,|\, \mathcal{O}_{Y_{F,i}})_F > 0$. Hence (ii) holds.

Conversely, now assume (ii) and show that $\mathscr{L}_F$ is ample. Let $Z \subset X_F$ be an integral, closed subscheme of dimension d. Its schematic image $Y \subset X$ has also dimension d. By [56], Corollaire 4.5.11, the relative separable closure of k in $K(Y)$ is a finite field extension $k \subset L$ and all irreducible components of Y_L are geometrically irreducible. So one integral component $Y' \subset Y_L$ satisfies $(Y'_F)_{\mathrm{red}} = Z$. Let $m \geq 1$ be the multiplicity of $\mathcal{O}_{Y'_F}$ at the generic point. Then Proposition 1.27 yields

$$(\mathscr{L}_F^d \,|\, \mathcal{O}_Z)_F = \frac{1}{m} \cdot (\mathscr{L}_F^d \,|\, \mathcal{O}_{Y'_F})_F = \frac{1}{m} \cdot (\mathscr{L}_L^d \,|\, \mathcal{O}_{Y'})_L = \frac{[K(Y') : K(Y)]}{m \cdot [L : k]} \cdot (\mathscr{L}^d \,|\, \mathcal{O}_Y) > 0.$$

Therefore $\mathscr{L}_F$ is ample, and eventually also $\mathscr{L}$. □

1.7 Characterization of Numerical Triviality

Let X be a proper scheme over an arbitrary field k and $\mathscr{L}$ an invertible sheaf on X. If $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ is torsion, then always $\mathscr{L} \in \operatorname{Pic}^\tau(X)$ holds, see Remark 1.18. The latter means that $\mathscr{L}$ is numerically trivial:

Proposition 1.34. *Let X be a proper scheme over an arbitrary field k and $\mathscr{L}$ an invertible sheaf on X. Fix a field extension $k \subset E$. Then the following statements are equivalent:*

(i) $\mathscr{L}_E \in \operatorname{Pic}^\tau(X_E)$.

(ii) $\mathscr{L}_E$ *is numerically trivial.*

(iii) $\chi_E(\mathcal{F} \otimes \mathscr{L}_E) = \chi_E(\mathcal{F})$ *for every coherent sheaf* $\mathcal{F}$ *on* X_E.

Moreover, if the statements above hold for the fixed field extension $k \subset E$, then they hold true for every other field extension $k \subset F$.

Proof. The equivalence over an algebraically closed field is proven in [78], Theorem 6.3, which holds for proper, not necessarily projective X according to op. cit., Remark 6.14. Reduce to this case, using essentially the same reasoning as in the proof of Proposition 1.33.

First, show that if (i) holds for $k \subset E$, then it holds true for every field extension $k \subset F$. Let $\lambda \in \operatorname{Pic}_{X/k}$ be the rational point coming from $\mathscr{L}$. By Theorem A.30, forming Pic^τ commutes with extending k, so $\lambda_E \in \operatorname{Pic}^\tau_{X_E/E}$ if and only if $\lambda \in \operatorname{Pic}^\tau_{X/k}$ if and only if $\lambda_F \in \operatorname{Pic}^\tau_{X_F/F}$.

Second, verify similarly if (ii) holds, then it holds true for every $k \subset F$. So assume that $\mathscr{L}_E$ is numerically trivial. Let $Y \subset X$ be an integral curve and $Y_{E,1}, \ldots, Y_{E,r}$ the integral components of $Y_E \subset X_E$ with multiplicities $m_1, \ldots, m_r$. Then

$$(\mathscr{L} \,|\, \mathcal{O}_Y) = \sum m_i (\mathscr{L}_E \,|\, \mathcal{O}_{Y_{E,i}})_E = 0 \tag{1.7}$$

shows that $\mathscr{L}$ is numerically trivial. To deduce that $\mathscr{L}_F$ is numerically trivial, let $Z \subset X_F$ be an integral, closed curve and $Y \subset X$ its schematic image. The relative separable closure of k in $K(Y)$ is a finite field extension $k \subset L$ and all irreducible components of Y_L are geometrically irreducible. So one integral component $Y' \subset Y_L$ satisfies $(Y'_F)_{\text{red}} = Z$. Let $m \geq 1$ be the multiplicity of $\mathcal{O}_{Y'_F}$ at the generic point, then as in the previous proof

$$(\mathscr{L}_F \,|\, \mathcal{O}_Z)_F = \frac{[K(Y') : K(Y)]}{m \cdot [L : k]} \cdot (\mathscr{L} \,|\, \mathcal{O}_Y) = 0.$$

Thus $\mathscr{L}_F$ is numerically trivial.

Combined, the equivalence of (i) and (ii) follows, as it holds in the case $F = \overline{k}$.

Assertion (iii) implies (ii), as it is the special case $\mathcal{F} = \mathcal{O}_{Y'}$ for an integral curve $Y' \subset X_E$. Hence to complete the proof, it is sufficient to verify that if (ii) is valid, then $\chi_F(\mathcal{F} \otimes \mathscr{L}_F) = \chi_F(\mathcal{F})$ follows for any field extension $k \subset F$ and coherent sheaf $\mathcal{F}$ on X_F. So assume (ii). Then, as seen above, also $\mathscr{L}_{\overline{F}}$ is numerically trivial. Thus over the algebraically closed field $\overline{F}$, the equality $\chi_{\overline{F}}(\mathcal{F}_{\overline{F}} \otimes \mathscr{L}_{\overline{F}}) = \chi_{\overline{F}}(\mathcal{F}_{\overline{F}})$ holds, which eventually implicates that also $\chi_F(\mathcal{F} \otimes \mathscr{L}_F) = \chi_F(\mathcal{F})$ is true. □

Any invertible sheaf $\mathscr{L}$ on X with $\mathscr{L}^{\otimes d} \simeq \mathcal{O}_X$ is numerically trivial by linearity of the intersection number. The next lemma shows that conversely, if $\mathscr{L}$ is numerically trivial and one of its powers $\mathscr{L}^{\otimes d}$ has a non-zero global section, then this power is also trivial. This is an important criterion to deduce that a numerically trivial invertible sheaf has finite order in the Picard group.

Lemma 1.35. *Let X be an integral scheme such that $H^0(X, \mathcal{O}_X)$ is a field and let $\mathscr{L}$ be an invertible sheaf on X.*

(i) If $H^0(X, \mathscr{L}) \neq 0$ and $H^0(X, \mathscr{L}^\vee) \neq 0$, then $\mathscr{L} \simeq \mathcal{O}_X$.

(ii) If $H^0(X, \mathscr{L}) \neq 0$ and $\mathscr{L}^{\otimes d} \simeq \mathcal{O}_X$ for some $d \geq 1$, then $\mathscr{L} \simeq \mathcal{O}_X$.

(iii) If X is proper over an arbitrary field k and $\mathscr{L}$ is numerically trivial with $h^0(\mathscr{L}) \neq 0$, then $\mathscr{L} \simeq \mathcal{O}_X$.

Note that for every reduced, connected, proper scheme over an arbitrary field k, the global sections $H^0(X, \mathcal{O}_X)$ are a finite field extension of k.

Proof. A non-zero global section $s \in H^0(X, \mathscr{L})$ defines an injection $\varphi\colon \mathcal{O}_X \to \mathscr{L}$. Indeed, it is injective at the generic point of X, so its kernel is a torsion sheaf, but $\mathcal{O}_X$ is torsion-free on the integral scheme X. Under assumption (i), there also exists an injective map $\mathscr{L} \to \mathcal{O}_X$. Their composition $\mathcal{O}_X \xrightarrow{\varphi} \mathscr{L} \to \mathcal{O}_X$ then is multiplication by a unit. The induced maps on all stalks then show that φ is also surjective, thus an isomorphism.

For assertion (ii), tensoring $\mathcal{O}_X \to \mathscr{L}$ with powers of $\mathscr{L}$ yields for every $i \geq 1$ injections $\mathscr{L}^{\otimes i-1} \to \mathscr{L}^{\otimes i}$. Altogether, there is an injective map from $\mathcal{O}_X$ to $\mathscr{L}^{\otimes d-1} \simeq \mathscr{L}^\vee$, that means, there exists a non-zero global section of $\mathscr{L}^\vee$. Hence the first part implies the claim.

To prove (iii), assume by contradiction that $\mathscr{L} \not\simeq \mathcal{O}_X$. For a non-zero $s \in H^0(X, \mathscr{L})$, denote the corresponding effective Cartier divisor by D. There exists an integral curve $C \subset X$ which is not contained in D with $C \cap D \neq \varnothing$, as explained in the following paragraph. Then the intersection $C \cap D$ has to be zero-dimensional, so $h^0(\mathcal{O}_{C \cap D}) \geq 1$. Now the contradiction $0 = (\mathscr{L} \mid \mathcal{O}_C) = h^0(\mathcal{O}_{C\cap D}) \geq 1$ follows.

In order to verify the existence of C, prove the following stronger claim: For every closed subset $D \subsetneq X$ and closed point $x \in D$, there exists an integral curve $C \subset X$ such that $C \not\subset D$ and $x \in C$. First, observe that the question is purely topological. Furthermore, replace X by an affine open neighborhood U of x, which is possible because the closure of a curve in U is again a curve in X. Write $X = \operatorname{Spec}(R)$ and prove the claim by induction on $n = \dim(X)$. The case $n = 0$ is the empty statement, and in the case $n = 1$ take $C = X$. Now suppose $n \geq 2$ and that the claim holds for $n - 1$. Let $\mathfrak{m} \subset R$ be the maximal ideal corresponding to $x \in X$. Denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_l \subset R$ for $l \geq 0$ the prime ideals which correspond to the generic points of the $(n-1)$-dimensional irreducible components

$D_1, \ldots, D_l$ of D containing x. So then $\bigcup \mathfrak{p}_i \subsetneq \mathfrak{m}$ holds, and the inclusion is in fact proper by prime avoidance, since $\mathfrak{m} = \mathfrak{p}_j$ for some j cannot hold as $n \geq 2$. Hence there exists some non-zero $r \in \mathfrak{m}$ such that $r \notin \mathfrak{p}_i$ for all $1 \leq i \leq l$. This means that the vanishing set $V(r)$ contains x but none of the D_i. Choose an irreducible component $Z \subset V(r)$ containing x. Then Z is $(n-1)$-dimensional by Krull's principal ideal theorem. Moreover, $Z \not\subset D$ by choice, so $Z \cap D \subsetneq Z$. Now the induction hypothesis applied to $D' = Z \cap D$ and $X' = Z$ gives the existence of an integral curve $C \subset Z$ such that $C \not\subset Z \cap D$ and $x \in C$. So $C \not\subset D$ must hold. This proves the claim, which in turn completes the entire proof. □

Remark 1.36. The lemma above shows that a numerically trivial invertible sheaf $\mathscr{L}$ on an integral, proper scheme over a field has infinite order if and only if $h^0(\mathscr{L}^{\otimes t}) = 0$ for all non-zero integers t. Those sheaves exist abundantly. For instance, let E be an elliptic curve over $\mathbb{C}$. Then $\mathscr{L}$ on E is numerically trivial if and only if $\deg(\mathscr{L}) = 0$. But these sheaves correspond to the closed points of E. Hence each point of E of infinite order corresponds to a numerically trivial invertible sheaf of infinite order. Those sheaves also exist in higher dimensions on schemes which are not necessarily abelian varieties: If X is a normal, proper $\mathbb{C}$-scheme with $h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = g$ non-zero, then $\operatorname{Pic}^0_{X/\mathbb{C}}$ is a g-dimensional abelian variety by Theorem A.30. Each element of infinite order in $\operatorname{Pic}^0_{X/\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}^g \,/\, \mathbb{Z}^{2g}$ yields a numerically trivial invertible sheaf on X of infinite order.

Over an arbitrary algebraically closed field k of characteristic $p \geq 0$, it may happen in the case $\dim(X) \geq 2$ and $p > 0$ that $\operatorname{Pic}^0_{X/k}$ is non-reduced, so that $\dim(\operatorname{Pic}^0_{X/k}) = 0$, although $g \geq 1$. Examples are ordinary or supersingular Enriques surfaces, which will be discussed in Section 4.4. But under the direct assumption that $\dim(\operatorname{Pic}^0_{X/k}) \geq 1$, the reduction $A = (\operatorname{Pic}^0_{X/k})_{\mathrm{red}}$ is again a non-trivial abelian variety. Then, there exist k-valued points of infinite order on A if and only if k is not the algebraic closure of a finite field. In fact, in Proposition 4.8 below, it will be deduced that if $k = \overline{\mathbb{F}}_p$, then $A(k)$ is a torsion group. For all other fields, it was shown in [39], Theorem 10.1, that the rank of $A(k)$ equals the cardinality of k, so there exist plenty of non-torsion points.

1.8 Chow Ring

Intersection numbers of invertible sheaves can also be defined using the Chow ring and products of Chern classes inside of it. Let X be a smooth, integral scheme of finite type over an arbitrary field k. In this section, the Chow ring $\mathrm{CH}(X)$ is introduced. This is an extensive topic, the range of its subsequent treatment is geared towards its application in the upcoming sections. References are Fulton [42], Eisenbud and Harris [37] as well as Hartshorne [64], Appendix A.

Let $Z^i(X)$ be the free abelian group generated by the integral, closed subschemes of codimension i in X. An element of $Z^i(X)$ is called a *cycle of codimension* i on X. The graded group of cycles on X is $Z(X) = \bigoplus_{i=0}^n Z^i(X)$, where $n = \dim(X)$. Given a closed subscheme $Y \subset X$ with integral components $Y_1, \ldots, Y_s$, denote by $\eta_j \in Y_j$ the generic point, and the multiplicity of Y_j in Y by $m_j = \text{length}_{\mathcal{O}_{Y,\eta_j}} \mathcal{O}_{Y,\eta_j}$. Then set the class $[Y] = \sum_{j=1}^{s} m_j[Y_j]$ in $Z(X)$. The subgroup $\text{Rat}(X) \subset Z(X)$ is generated by cycles $[Y_0] - [Y_\infty]$. Here, $Y \subset X \times \mathbb{P}^1$ is an integral, closed subscheme which projects dominantly to $\mathbb{P}^1$ and Y_z denotes the fiber of $Y \to \mathbb{P}^1$ over $z \in \mathbb{P}^1$. Two cycles $\alpha, \beta \in Z(X)$ are *rationally equivalent* if $\alpha - \beta \in \text{Rat}(X)$.

The *Chow group* of X is defined to be $\text{CH}(X) = Z(X)/\text{Rat}(X)$. It inherits a grading via the projection $Z(X) \to \text{CH}(X)$. As X is integral, the class $[X]$ generates $\text{CH}^0(X) = \mathbb{Z}$. If the scheme X is proper, the degree homomorphism $\deg_n\colon \text{CH}^n(X) \to \mathbb{Z}$ given by $\sum n_x[x] \mapsto \sum n_x[\kappa(x)\colon k]$ is well-defined, see [42], Theorem 1.4. Denote its extension by zero by $\deg\colon \text{CH}(X) \to \mathbb{Z}$. Since X is integral, $\text{Pic}(X)$ can be identified with the group of Cartier divisors on X modulo linear equivalence. Moreover, as X is locally factorial, that is, all local rings $\mathcal{O}_{X,x}$ are factorial, the group of Cartier divisors corresponds to the group $Z^1(X)$ of Weil divisors on X. The induced natural group homomorphism $c_1\colon \text{Pic}(X) \to \text{CH}^1(X)$, $\mathcal{O}_X(D) \mapsto [D]$ is indeed bijective. This will be readopted and generalized from invertible to locally free sheaves in the subsequent section on higher Chern classes.

An intersection product $\text{CH}^i(X) \times \text{CH}^j(X) \to \text{CH}^{i+j}(X)$ induces a ring structure on $\text{CH}(X)$. In the case that $V, W \subset X$ are smooth, integral, closed subschemes meeting generically transversely, then $[V] \cdot [W] = [V \cap W]$, see [42], Example 8.1.11. Generic transversality means that for all generic points $\eta \in V \cap W$, the tangent spaces $T_\eta V$ and $T_\eta W$ inside $T_\eta X$ satisfy $\text{codim}(T_\eta V \cap T_\eta W) = \text{codim}(T_\eta V) + \text{codim}(T_\eta W)$. Equivalently, the equality $T_\eta X = T_\eta V + T_\eta W$ holds. The geometric idea behind the intersection product for general cycles is to change their classes' representatives to cycles intersecting generically transversely, compare to the moving lemma [37], Theorem 1.6. Here the smoothness of X is crucial.

The *Chow ring* $\text{CH}(X)$ is commutative and defines a contravariant functor from the category of smooth, integral schemes of finite type over k to the category of rings, see [42], Proposition 8.3 (a). For proper morphisms $\varphi\colon X \to Y$, there is a pushforward homomorphism $\varphi_*\colon \text{CH}(X) \to \text{CH}(Y)$, which satisfies the projection formula with respect to the pullback homomorphism $\varphi^*\colon \text{CH}(Y) \to \text{CH}(X)$ given by functoriality due to [42], Theorem 1.4 and Proposition 8.3 (c). For an extended list of properties which in turn determine the intersection product, see [64], Appendix A, Theorem 1.1.

1.9 Higher Chern Classes

Let X be a smooth, integral scheme of finite type over an arbitrary field k. The definition of intersection numbers by means of the Chow ring uses Chern classes, and the latter will be covered during the first part of this section. They are also involved in the Grothendieck–Riemann–Roch theorem which is discussed afterwards, as some of its corollaries are important for the subsequent chapters. At the end of this section, it will be shown that both definitions of intersection numbers coincide.

The following survey introduces Chern classes, based on Grothendieck [47], Section 3, and follows Eisenbud and Harris [37], Chapters 5, 9, 14. A more general and in-depth treatment is again Fulton [42], Chapter 3, and for a concise overview, see Hartshorne [64], Appendix A.3.

Let $\mathcal{E}$ be a locally free sheaf of rank r on X. Denote $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}\mathcal{E})$ following Grothendieck's convention. Consider the associated projective bundle $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ and the class $\xi = [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)]$ in $\operatorname{CH}^1(\mathbb{P}(\mathcal{E}))$. The homomorphism $\pi^*\colon \operatorname{CH}(X) \to \operatorname{CH}(\mathbb{P}(\mathcal{E}))$ is injective and $\operatorname{CH}(\mathbb{P}(\mathcal{E}))$ becomes a free $\operatorname{CH}(X)$-module generated by $1, \xi, \xi^2, \ldots, \xi^{r-1}$ according to [42], Example 8.3.4. The *i-th Chern class* $c_i(\mathcal{E}) \in \operatorname{CH}^i(X)$ of $\mathcal{E}$ is defined by $c_0(\mathcal{E}) = 1$, $c_i(\mathcal{E}) = 0$ for $i > r$, and the relation

$$\sum_{i=0}^{r} (-1)^i \pi^*(c_i(\mathcal{E}))\xi^{r-i} = 0.$$

The equation has a unique solution, and hence it defines the higher Chern classes $c_i(\mathcal{E})$ for $1 \le i \le r$ all at once. Denote by $c(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E}) \in \operatorname{CH}(X)^\times$ the *total Chern class* of $\mathcal{E}$.

Proposition 1.37. *Chern classes satisfy the following properties:*

(i) Let $\mathscr{L} = \mathcal{O}_X(D)$ be an invertible sheaf. Then $c(\mathscr{L}) = 1 + [D]$.

(ii) Let $\varphi\colon X' \to X$ be a morphism of smooth, integral schemes of finite type over k and let $\mathcal{E}$ be a locally free sheaf on X. Then $c(\varphi^*(\mathcal{E})) = \varphi^*(c(\mathcal{E}))$.

(iii) (Whitney Sum Formula). Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of locally free sheaves. Then $c(\mathcal{E}) = c(\mathcal{E}') \cdot c(\mathcal{E}'')$.

(iv) Let $\mathcal{E}$ be a locally free sheaf of rank r on X. Then $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$ for $0 \le i \le r$. Furthermore, the invertible sheaf $\det(\mathcal{E}) = \wedge^r(\mathcal{E})$ satisfies $c_1(\mathcal{E}) = c_1(\det(\mathcal{E}))$.

(v) Let $\sigma_0, \ldots, \sigma_{r-i} \in H^0(X, \mathcal{E})$ for some $1 \le i \le r$ such that $V := V(\sigma_0 \wedge \cdots \wedge \sigma_{r-i})$ has codimension $\dim(X) - \dim(V) = i$. Then $c_i(\mathcal{E}) = [V]$ in $\operatorname{CH}^i(X)$.

In (v), the scheme structure on V is given by dualizing $\sigma_0 \wedge \cdots \wedge \sigma_{r-i}\colon \mathcal{O}_X \to \wedge^{r-i+1}(\mathcal{E})$. Properties (i) to (iii) uniquely determine Chern classes by [47], Théorème 1. For properties (iv) and (v), see [37], Section 5, with a proof of (v) given in [42], Example 14.4.2. The *i-th Chern class* of the scheme X is defined to be $c_i(X) = c_i(\Theta_X) = (-1)^i c_i(\Omega^1_X)$. Abbreviate $c_i = c_i(X)$ if the dependence is obvious.

To define certain expressions in Chern classes, the splitting construction, see [42], Section 3.2, page 51ff., is helpful: For every locally free sheaf $\mathcal{E}$ of rank r on X, there exists a smooth, integral k-scheme X' of finite type and a smooth morphism $\varphi\colon X' \to X$ such that the map $\varphi^*\colon \operatorname{CH}(X) \to \operatorname{CH}(X')$ is injective and the pullback $\varphi^*(\mathcal{E})$ admits a filtration $0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_r = \varphi^*(\mathcal{E})$ such that the subquotients $\mathscr{L}_i = \mathcal{E}_i / \mathcal{E}_{i-1}$ are all invertible. The morphism φ arises as a sequence of projective bundles, and it also exists collectively for any finite set of locally free sheaves on X. The properties given in Proposition 1.37 yield $c(\mathscr{L}_i) = 1 + \alpha_i$ for some $\alpha_i \in \mathrm{CH}^1(X')$ and result in the equality

$$\sum_{m=0}^{r} \varphi^*(c_m(\mathcal{E})) = \varphi^*(c(\mathcal{E})) = \prod_{i=1}^{r}(1+\alpha_i) = \sum_{m=0}^{r} e_m(\alpha_1, \ldots, \alpha_r),$$

where $e_m(T_1, \ldots, T_r)$ is the m-th elementary symmetric polynomial in r variables. Thus to define an expression which is a symmetric polynomial in the Chern classes of $\mathcal{E}$, it is sufficient and often convenient to define the term in the *Chern roots* $\alpha_1, \ldots, \alpha_r$ of $\mathcal{E}$. The latter are regarded as formal symbols in the above sense.

Denote $\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The *Chern character* of $\mathcal{E}$ is $\mathrm{ch}(\mathcal{E}) = \sum_{i=1}^{r} \exp(\alpha_i)$ in $\mathrm{CH}(X)_{\mathbb{Q}}$ for the exponential $\exp(\alpha_i) = \sum_{m \geq 0} \frac{\alpha_i{}^m}{m!}$. Note that the power series aborts at $\dim(X')$ for every splitting morphism $\varphi\colon X' \to X$, and moreover, only the terms up to degree r are relevant for the definition of $\mathrm{ch}(\mathcal{E})$. The *Todd class* of $\mathcal{E}$ is defined by $\mathrm{td}(\mathcal{E}) = \prod_{i=1}^{r} Q(\alpha_i)$ in $\mathrm{CH}(X)_{\mathbb{Q}}$, where $Q(T) = \sum_{m \geq 0} (-1)^m \frac{B_m^-}{m!} T^m$ is the Taylor series expansion of $T \mapsto \frac{T}{1-\exp(-T)}$ in 0 with Bernoulli numbers $B_0^- = 1$, $B_1^- = -\frac{1}{2}$ and so forth. For explicit expressions in low degrees, see [42], Examples 3.2.3 and 3.2.4. With the notation as in Proposition 1.37 (iii), the splitting construction yields relations

$$\begin{aligned} \mathrm{ch}(\mathcal{E}) &= \mathrm{ch}(\mathcal{E}') + \mathrm{ch}(\mathcal{E}''), \\ \mathrm{ch}(\mathcal{E} \otimes \widetilde{\mathcal{E}}) &= \mathrm{ch}(\mathcal{E}) \cdot \mathrm{ch}(\widetilde{\mathcal{E}}), \\ \mathrm{td}(\mathcal{E}) &= \mathrm{td}(\mathcal{E}') \cdot \mathrm{td}(\mathcal{E}''). \end{aligned}$$

Note that the Chern roots of $\mathcal{E} \otimes \widetilde{\mathcal{E}}$ are given by $\alpha_i + \widetilde{\alpha}_j$ for $1 \leq i \leq r$ and $1 \leq j \leq \widetilde{r}$ with the obvious notation.

Every coherent sheaf $\mathcal{F}$ on the smooth scheme X admits a finite resolution

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \tag{1.8}$$

by locally free sheaves $\mathcal{E}_i$, as conducted in [42], B.8.3. Chern classes, the Chern character and the Todd class can, thereby, also be defined for $\mathcal{F}$ via

$$c(\mathcal{F}) = \prod_{i=0}^{n} c(\mathcal{E}_i)^{(-1)^i} \qquad \text{and} \qquad \mathrm{ch}(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \, \mathrm{ch}(\mathcal{E}_i), \tag{1.9}$$

and analogously for the Todd class. This is the only possible way to achieve that the respective Whitney sum formulas hold true for coherent sheaves.

Remark 1.38. The well-definedness of (1.9) is a consequence of the fact that the definitions of Chern classes, the Chern character and the Todd class naturally extend to the Grothendieck group $L(X)$ of the abelian category of locally free sheaves on X. The assignments $c(\sum a_i[\mathcal{E}_i]) = \prod c(\mathcal{E}_i)^{a_i}$ and $\mathrm{ch}(\sum a_i[\mathcal{E}_i]) = \sum a_i \, \mathrm{ch}(\mathcal{E}_i)$ respect the relations of $L(X)$, and similarly for the Todd class. Recall that the Grothendieck group of coherent sheaves on X was denoted by $C(X)$. The natural homomorphism $L(X) \to C(X)$ defined on generators by $[\mathcal{E}] \mapsto [\mathcal{E}]$ is bijective. Indeed, since X is smooth, the homomorphism

$C(X) \to L(X)$ given by $[\mathcal{F}] \mapsto \sum_{i=0}^{n}(-1)^i[\mathcal{E}_i]$ for any finite locally free resolution of $\mathcal{F}$ as in (1.8) is well-defined and it is the inverse according to [42], B.8.3. Consequently, the assignments (1.9) do not depend on the chosen resolution.

The connection between the Chow ring $\mathrm{CH}(X)$ and the Grothendieck groups $C(X)$ and $L(X)$ will be covered in more detail at the end of this section. Prior to that, consider the following fundamental result of Grothendieck, [42], Theorem 15.2, and some of its corollaries.

Theorem 1.39 (Grothendieck–Riemann–Roch)**.** *Let $\varphi\colon X \to Y$ be a proper morphism of smooth, integral schemes of finite type over an arbitrary field k. Then in $\mathrm{CH}(Y)_{\mathbb{Q}}$, for all coherent sheaves $\mathcal{F}$ on X, the subsequent equality holds:*

$$\left(\sum_{i\geq 0}(-1)^i \operatorname{ch}(R^i\varphi_*\mathcal{F})\right)\cdot \operatorname{td}(\Theta_Y) = \varphi_*(\operatorname{ch}(\mathcal{F})\cdot \operatorname{td}(\Theta_X)).$$

The theorem exactly shows to what extent the Chern character $\operatorname{ch}\colon C(X) \to \mathrm{CH}(X)_{\mathbb{Q}}$ does not commute with the pushforward maps for proper $\varphi\colon X \to Y$. The pushforward $\varphi_*\colon C(X) \to C(Y)$ is defined by $\varphi_*[\mathcal{F}] = \sum(-1)^i[R^i\varphi_*\mathcal{F}]$ and the formula above then states that $\operatorname{ch}(\varphi_*\mathcal{F})\cdot\operatorname{td}(\Theta_Y) = \varphi_*(\operatorname{ch}(\mathcal{F})\cdot\operatorname{td}(\Theta_X))$. In other words, the twisted Chern character map $C(X) \to \mathrm{CH}(X)_{\mathbb{Q}}$ defined by $[\mathcal{F}] \mapsto \operatorname{ch}(\mathcal{F})\cdot\operatorname{td}(\Theta_X)$ commutes with pushforwards.

In the case that $\varphi\colon X \to \operatorname{Spec}(k)$ is the structure morphism, Grothendieck–Riemann–Roch specializes to Hirzebruch–Riemann–Roch:

Corollary 1.40 (Hirzebruch–Riemann–Roch)**.** *Let X be a smooth, integral, proper scheme over an arbitrary field k and $\mathcal{F}$ a coherent sheaf on X. Then*

$$\chi(\mathcal{F}) = \deg(\operatorname{ch}(\mathcal{F})\cdot\operatorname{td}(\Theta_X)).$$

The usual Riemann–Roch theorem for smooth surfaces, Proposition 1.29, is a consequence of the above, see [42], Example 15.2.2. Furthermore, Hirzebruch–Riemann–Roch additionally provides the following formula. Note that it is common to identify n-cycles with their degree, but this convention will not be used in this work.

Corollary 1.41 (Noether's Formula)**.** *Let X be a smooth, integral, proper surface over an arbitrary field k. Then*

$$\chi(\mathcal{O}_X) = \frac{\deg(c_1^2 + c_2)}{12}.$$

The Hirzebruch–Riemann–Roch formula yields the following additional corollary, see [42], Example 18.3.9.

Corollary 1.42. *Let $\varphi\colon X' \to X$ be an étale morphism between smooth, integral, proper schemes over an arbitrary field k. Then*

$$\chi(\mathcal{O}_{X'}) = \deg(\varphi)\cdot\chi(\mathcal{O}_X).$$

Proof. By Hirzebruch–Riemann–Roch, the equality $\deg(\mathrm{td}(\Theta_{X'})) = \deg(\varphi) \cdot \deg(\mathrm{td}(\Theta_X))$ has to be shown. In fact, the relation $\mathrm{td}(\Theta_{X'}) = \varphi^*(\mathrm{td}(\Theta_X))$ holds and consequently $\varphi_*(\mathrm{td}(\Theta_{X'})) = \deg(\varphi) \cdot \mathrm{td}(\Theta_X)$. Now compute the degree on both sides by pushing forward to $\mathrm{CH}(\mathrm{Spec}(k))_{\mathbb{Q}} = \mathbb{Q}$. □

Return to Remark 1.38 and the connection between the Chow ring $\mathrm{CH}(X)$ and the Grothendieck groups $C(X)$ and $L(X)$ for a smooth, integral, proper k-scheme X. Since every short exact sequence of locally free sheaves $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ splits, the relation imposed on $L(X)$ can be reformulated as $[\mathcal{E}' \oplus \mathcal{E}''] = [\mathcal{E}'] + [\mathcal{E}'']$. Because locally free sheaves are flat, the group $L(X)$ becomes a ring with multiplication given by the tensor product $[\mathcal{E}] \cdot [\widetilde{\mathcal{E}}] = [\mathcal{E} \otimes \widetilde{\mathcal{E}}]$. The Chern character defines a ring homomorphism $\mathrm{ch}\colon L(X) \to \mathrm{CH}(X)_{\mathbb{Q}}$ resting upon the relations $\mathrm{ch}(\mathcal{E}' \oplus \mathcal{E}'') = \mathrm{ch}(\mathcal{E}') + \mathrm{ch}(\mathcal{E}'')$ and $\mathrm{ch}(\mathcal{E} \otimes \widetilde{\mathcal{E}}) = \mathrm{ch}(\mathcal{E}) \cdot \mathrm{ch}(\widetilde{\mathcal{E}})$. The induced map

$$\mathrm{ch}\colon L(X)_{\mathbb{Q}} \xrightarrow{\simeq} \mathrm{CH}(X)_{\mathbb{Q}}$$

is an isomorphism of $\mathbb{Q}$-algebras by [42], Example 15.2.16 (b). Eventually, the identification $C(X) = L(X)$ induces an isomorphism $\mathrm{ch}\colon C(X)_{\mathbb{Q}} \xrightarrow{\simeq} \mathrm{CH}(X)_{\mathbb{Q}}$.

In Section 1.6, on a proper k-scheme X, the intersection number of invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_r$ with an integral, closed subscheme $Y \subset X$ of dimension r was defined using $C(X)$. In the case that X is additionally smooth and integral, an intersection number can also be defined using $\mathrm{CH}(X)$. To achieve this, consider Y as a cycle of codimension $n - r$ on X. Then $c_1(\mathscr{L}_1) \cdot \ldots \cdot c_1(\mathscr{L}_r)[Y]$ is contained in $\mathrm{CH}^n(X)$, so compute its degree. It can directly be seen how both definitions coincide, using the Chern character $\mathrm{ch}\colon C(X) \to \mathrm{CH}(X)_{\mathbb{Q}}$. This relies on the subsequent lemma.

Lemma 1.43. *Let X be a smooth, integral, proper scheme over an arbitrary field k. Let $\mathscr{L}$ be an invertible sheaf on X, and $\mathcal{F}$ a coherent sheaf with r-dimensional support $Y = \mathrm{Supp}(\mathcal{F})$. Denote by $Y_1, \ldots, Y_s$ the integral components of Y with multiplicities $m_1, \ldots, m_s$. Then the images of $[\mathcal{F}] \in C_r(X)$ under both compositions in the diagram*

$$\begin{array}{ccc} C(X) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}(X)_{\mathbb{Q}} \\ {\scriptstyle c_1(\mathscr{L})}\downarrow & & \downarrow{\scriptstyle c_1(\mathscr{L})} \\ C(X) & \xrightarrow[\mathrm{ch}]{} & \mathrm{CH}(X)_{\mathbb{Q}} \end{array}$$

are contained in $\mathrm{CH}^{\geq n-r+1}(X)$. Their components in degree $n - r + 1$ coincide, and they are equal to $\sum_{j=r}^{s} m_j c_1(\mathscr{L})[Y_j]$.

Proof. First, consider the special case $\mathcal{F} = \mathcal{O}_Y$ for an integral, closed subscheme $Y \subset X$. Note that $\mathrm{ch}([\mathcal{O}_Y]) = [Y] + \alpha$ is contained in $\mathrm{CH}^{\geq n-r}(X)$ for some $\alpha \in \mathrm{CH}^{\geq n-r+1}(X)$ by [42], Example 15.2.16 (a). Hence on the one hand,

$$c_1(\mathscr{L})\,\mathrm{ch}([\mathcal{O}_Y]) = c_1(\mathscr{L})[Y] + c_1(\mathscr{L})\alpha, \tag{1.10}$$

where the first summand is an element of $\mathrm{CH}^{n-r+1}(X)$ and the second one is contained in $\mathrm{CH}^{\geq n-r+2}(X)$. On the other hand, the element $c_1(\mathscr{L})[\mathcal{O}_Y] = [\mathcal{O}_Y] - [\mathscr{L}^\vee \otimes \mathcal{O}_Y]$ of $C(X)$ gets mapped under the Chern character to

$$\mathrm{ch}([\mathcal{O}_Y]) - \mathrm{ch}([\mathscr{L}^\vee \otimes \mathcal{O}_Y]) = \mathrm{ch}([\mathcal{O}_Y]) \cdot (1 - \mathrm{ch}([\mathscr{L}^\vee])).$$

Now $\mathrm{ch}(\mathscr{L}^\vee) = \exp(c_1(\mathscr{L}^\vee)) = 1 - c_1(\mathscr{L}) + \beta$ for some $\beta \in \mathrm{CH}^{\geq 2}(X)$. As a consequence,

$$\mathrm{ch}(c_1(\mathscr{L})[\mathcal{O}_Y]) = ([Y] + \alpha) \cdot (c_1(\mathscr{L}) - \beta) = c_1(\mathscr{L})[Y] + c_1(\mathscr{L})\alpha + \Delta \tag{1.11}$$

for $\Delta = -([Y] + \alpha)\beta$ contained in $\mathrm{CH}^{\geq n-r+2}(X)$. Comparison of (1.10) and (1.11) proves the assertion in the special case.

In the general case, Lemma 1.23 yields that $[\mathcal{F}] = \sum_{j=1}^s m_j[\mathcal{O}_{Y_j}] + R$ holds for some $R \in C_{r-1}(X)$. Inductively, $R = \sum n_i[\mathcal{O}_{Z_i}]$ for integral, closed subschemes Z_i of dimension at most $r-1$. The special case applied to every summand finally completes the proof. □

Proposition 1.44. *Let X be a smooth, integral, proper scheme over an arbitrary field k and $\mathscr{L}_1, \dots, \mathscr{L}_r$ invertible sheaves on X. Then the diagram*

$$\begin{array}{ccc} C_r(X) & \xrightarrow{\ \mathrm{ch}\ } & \mathrm{CH}(X)_{\mathbb{Q}} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ C_0(X) & \xrightarrow{\ \mathrm{ch}\ } & \mathrm{CH}(X)_{\mathbb{Q}} \\ & {\scriptstyle\chi}\searrow \quad \swarrow{\scriptstyle\deg} & \\ & \mathbb{Q} & \end{array}$$

commutes, where $\mu = c_1(\mathscr{L}_1) \cdots c_1(\mathscr{L}_r)$. Especially, let $[\mathcal{F}] \in C_r(X)$ be the class of a coherent sheaf $\mathcal{F}$, and denote by $Y_1, \dots, Y_s$ the integral components of $\mathrm{Supp}(\mathcal{F})$ *with multiplicities $m_1, \dots, m_s$. Then the following equalities hold:*

$$\begin{array}{ccc} (\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F}) & = & \deg(c_1(\mathscr{L}_r) \cdots c_1(\mathscr{L}_r)\,\mathrm{ch}(\mathcal{F})) \\ \| & & \| \\ \sum_{j=1}^s m_j(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{O}_{Y_j}) & & \sum_{j=1}^s m_j \deg(c_1(\mathscr{L}_r) \cdots c_1(\mathscr{L}_r)[Y_j]). \end{array}$$

Proof. Inductively apply the lemma to show that the upper square commutes. For the commutativity of the lower triangle, let $[\mathcal{G}] \in C_0(X)$ and denote $\mathrm{Supp}(\mathcal{G}) = \{z_1, \dots, z_t\}$ with multiplicities $n_1, \dots, n_t$. Thus $[\mathcal{G}] = \sum_{l=1}^t n_l\{z_l\}$. Then $\chi([\mathcal{G}]) = \sum_{l=1}^t n_l[\kappa(z_l) \colon k]$. On the other hand, the degree of $\mathrm{ch}([\mathcal{G}]) = \sum_{l=1}^t n_l[z_l]$ is by definition the same integer. So the diagram commutes. The commutativity of the outer trapezoid exactly means that $(\mathscr{L}_1 \cdots \mathscr{L}_r \,|\, \mathcal{F}) = \deg(c_1(\mathscr{L}_r) \cdots c_1(\mathscr{L}_r)\,\mathrm{ch}(\mathcal{F}))$ holds, as claimed. The additional assertions are immediate consequences of Lemma 1.23 and Lemma 1.43. □

1.10 Cycle Map

The aim of this section is to embed the direct definition $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ of the first Chern class for an invertible sheaf $\mathscr{L}$ into the larger picture of the Chow ring illustrated in the previous sections. Consequently, higher Chern classes $c_i(\mathcal{E}) \in H^{2i}(X_{\text{ét}}, \mathbb{Z}_\ell(i))$ for locally free sheaves $\mathcal{E}$ can also be defined as elements of the ℓ-adic cohomology groups, and its formation is compatible with the Chern classes defined as elements in the Chow ring. References are Jouanolou [60], Exposé VI, Partie 3 and Grothendieck [33], [Cycle], page 129ff. as well as Milne [90], Chapter VI, Section 9.

Let X be a locally noetherian scheme over an arbitrary field k of characteristic $p \geq 0$ and ℓ a prime number different from p. To start with, recall from Section 1.3 that $\mu_{\ell^n,X}$ is a sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on the étale site, and the inverse system $\mathbb{Z}_\ell(1) = (\mu_{\ell^n,X})_{n\geq 1}$ is a sheaf of $\mathbb{Z}_\ell$-modules. The same holds true for the tensor powers $\mathbb{Z}_\ell(j) := (\mu_{\ell^n,X}^{\otimes j})_{n\geq 1}$ for $j \geq 1$. Their cohomology groups are defined as $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(j)) = \varprojlim H^i(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes j})$. In the case that X is a scheme over a separably closed field k, then as before, choices of ℓ^n-th roots of unity yield bijections $H^i(X_{\text{ét}}, \mathbb{Z}_\ell(j)) \simeq H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$.

Let X be a smooth, integral, d-dimensional scheme of finite type over an algebraically closed field k. For every cycle $Y \in Z^i(X)$ and $n \geq 1$, there exists an associated class $\mathrm{cl}^n(Y) \in H^{2i}(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes i})$. In the case that Y is a smooth, integral subscheme of codimension i, the class can be defined as the image of the *fundamental class* $s_{Y/X}$ under $H^{2i}_Y(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes i}) \to H^{2i}(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes i})$. Here the source is the corresponding *cohomology group with support on* Y, defined as the right derived functor of $\mathrm{S}(\text{Ét}/X) \to (\mathrm{Ab})$, $\mathcal{F} \mapsto \ker(\Gamma(X, \mathcal{F}) \to \Gamma(X \smallsetminus Y, \mathcal{F}))$. See [90], Chapter VI, Section 6, for a survey of the fundamental class. Specializing further to codimension $i = 1$, identify $H^1_Y(X_{\text{ét}}, \mathbb{G}_{m,X}) \simeq \mathbb{Z}$ so that $H^1_Y(X_{\text{ét}}, \mathbb{G}_{m,X}) \to H^1(X_{\text{ét}}, \mathbb{G}_{m,X})$ maps 1 to $\mathcal{O}_X(Y)$. Then $s_{Y/X} \in H^2_Y(X_{\text{ét}}, \mu_{\ell^n,X})$ arises from the Kummer sequence as the image of $1 \in H^1_Y(X_{\text{ét}}, \mathbb{G}_{m,X})$ under the boundary map $H^1_Y(X_{\text{ét}}, \mathbb{G}_{m,X}) \to H^2_Y(X_{\text{ét}}, \mu_{\ell^n,X})$, similar to the definition of $c_1(\mathscr{L})$ given in Section 1.5.

The graded group $H(X_{\text{ét}}, \ell^n) := \bigoplus_{i=0}^d H^{2i}(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes i})$ becomes a ring using the cup product

$$\cup\colon H^{2i}(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes i}) \times H^{2j}(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes j}) \to H^{2(i+j)}(X_{\text{ét}}, \mu_{\ell^n,X}^{\otimes i+j}),$$

by [90], Chapter V, Proposition 1.16. The multiplication is commutative, since only cohomology groups of even degree appear. The definition of the associated cycle extends linearly to a map $Z(X) \to H(X_{\text{ét}}, \ell^n)$. It is constant on rational equivalence classes and induces the cycle map $\mathrm{cl}^n\colon \mathrm{CH}(X) \to H(X_{\text{ét}}, \ell^n)$. This cycle map is a homomorphism of graded rings for the intersection product on $\mathrm{CH}(X)$ and the cup product on $H(X_{\text{ét}}, \ell^n)$. The inverse limit process over $n \geq 1$ is compatible with the constructions above. Denote $H(X_{\text{ét}}) = \bigoplus_{i=0}^d H^{2i}(X_{\text{ét}}, \mathbb{Z}_\ell(i))$ and for the induced *cycle map*:

$$\mathrm{cl}\colon \mathrm{CH}(X) \to H(X_{\text{ét}}).$$

Completely analogous to Chern classes $c(\mathcal{E}) \in \mathrm{CH}(X)$ of locally free sheaves $\mathcal{E}$ on X, the theory yields Chern classes $c^\ell(\mathcal{E}) \in H(X_{\text{ét}})$ satisfying the properties of Proposition 1.37, see [60], Exposé VII, Partie 3, for details. Due to the uniqueness of properties (i) to (iii), the ℓ-adic Chern classes factorize through $\mathrm{CH}(X)$, that is, the relation $c^\ell(\mathcal{E}) = \mathrm{cl}(c(\mathcal{E}))$ holds for every locally free sheaf $\mathcal{E}$ on X. In particular, the elementary definition of the first Chern class $c_1(\mathscr{L}) \in H^2(X_{\text{ét}}, \mathbb{Z}_\ell(1))$ given in Section 1.5 is regained. Chern classes can also be defined for coherent sheaves using a locally free resolution as described in Section 1.9. Furthermore, the tensor product with $\mathbb{Q}_\ell$ yields Chern classes with values in $\bigoplus_{i=0}^d H^{2i}(X_{\text{ét}}, \mathbb{Q}_\ell(i))$.

To conclude this chapter and prelude the subsequent one, the ℓ-adic cohomology satisfies *Poincaré duality*, which is the following statement:

Theorem 1.45 (Poincaré Duality)**.** *Let X be a smooth, proper d-dimensional scheme over a separably closed field k. There is an isomorphism* $\mathrm{tr}\colon H^{2d}(X_{\text{ét}}, \mathbb{Z}_\ell(d)) \xrightarrow{\simeq} \mathbb{Z}_\ell$ *such that the pairing*

$$H^i(X_{\text{ét}}, \mathbb{Z}_\ell(j)) \times H^{2d-i}(X_{\text{ét}}, \mathbb{Z}_\ell(d-j)) \xrightarrow{\cup} H^{2d}(X_{\text{ét}}, \mathbb{Z}_\ell(d)) \xrightarrow{\mathrm{tr}} \mathbb{Z}_\ell$$

is non-degenerate for all $0 \le i \le 2d$ and $0 \le j \le d$.

A proof for $\mathbb{Z}/\ell^n\mathbb{Z}$ in place of $\mathbb{Z}_\ell$ can be found in [90], Chapter VI, Theorem 11.1f. and passing to the limit shows the proposition above. Especially, the Betti numbers fulfill the symmetry $b_i = b_{2n-i}$ for all $0 \le i \le 2n$.

Proposition 1.46. *Let X be a smooth, connected, proper d-dimensional scheme over a separably closed field k. Then $\deg(c_d) = e(X)$ is the ℓ-adic Euler characteristic.*

This proposition is [60], Exposé VII, Corollaire 4.9. In the case that $k = \mathbb{C}$, Proposition 1.13 in turn shows that the topological Euler characteristic equals $\deg(c_n)$. Poincaré duality in combination with Noether's formula leads to the following relation on surfaces, which involves several numerical invariants. It appears in the course of the Enriques classification of surfaces, and it will also be used in Section 4.5.

Corollary 1.47. *Let X be a smooth, integral, proper surface over a separably closed field k. Then*

$$b_2 - 2b_1 + \deg(c_1^2) = 10 - 12h^1(\mathcal{O}_X) + 12h^2(\mathcal{O}_X).$$

By Proposition 1.22, the estimate $b_2 \ge \rho(X)$ and $b_1 = 2\dim(\mathrm{Pic}^0_{X/k})$ hold. The latter equality and this corollary show that the ℓ-adic Betti numbers are independent of the chosen prime $\ell \ne p$ for surfaces satisfying the assumptions above. This independence in general is discussed in [69], Section 1.4. It is true for smooth, proper schemes of arbitrary dimension as a consequence of the *Weil conjectures*, and it is an open question under lessened assumptions.

Furthermore, the value $\deg(c_1^2) = (K_X^2)$ is the self-intersection number of the dualizing sheaf $\omega_X \simeq \mathcal{O}_X(K_X)$, which is the central object in the next chapter.

Chapter 2

Dualizing Sheaves

The purpose of this chapter is to introduce the dualizing sheaf, first, on a proper scheme over a field, and afterwards for suitable proper morphisms of schemes. In the former case, the associated duality is known as *Serre duality*, which dates back to the work of Serre [116], [117]. Its generalization to the relative setting of morphisms, called *Grothendieck duality*, has its origin in Grothendieck's papers [48], [49].

2.1 Serre Duality

Fix an n-dimensional, proper scheme X over an arbitrary field k. A *dualizing sheaf* on X is a coherent sheaf ω_X together with a k-linear trace map $\mathrm{tr}\colon H^n(X, \omega_X) \to k$ such that the pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{\mathrm{tr}} k \tag{2.1}$$

given by functoriality of cohomology and the trace is non-degenerate for all $\mathcal{F} \in \mathrm{Coh}(X)$. This means that the k-linear maps

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \longrightarrow H^n(X, \mathcal{F})^\vee \tag{2.2}$$

and $H^n(X, \mathcal{F}) \to \mathrm{Hom}(\mathcal{F}, \omega_X)^\vee$ given by the pairing are injective for all coherent sheaves $\mathcal{F}$ on X. The finiteness theorem shows that the dimensions $h^i(\mathcal{F})$ are finite for all $i \geq 0$, and therefore both maps are bijective. In this case, the functor

$$\mathrm{Coh}(X) \longrightarrow (k\text{-Vect}), \quad \mathcal{F} \longmapsto H^n(X, \mathcal{F})^\vee \tag{2.3}$$

is represented by ω_X. Yoneda's lemma now gives the uniqueness of (ω_X, tr) up to unique isomorphism.

Proposition 2.1. *There exists a dualizing sheaf ω_X on X.*

The existence of ω_X is stated in [64], Chapter III, Section 7, page 241f. with a proof given for projective X in loc. cit., Proposition 7.5. For proper, not necessarily projective schemes, the existence will result from the theory of Grothendieck duality, see Proposition 2.20 below.

Morphisms or isomorphisms as in (2.2) can be obtained, under suitable assumptions, for the derived functors of the appearing ones:

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \longrightarrow H^{n-i}(X, \mathcal{F})^\vee. \tag{2.4}$$

Here, $\mathcal{F} \mapsto \mathrm{Ext}^i(\mathcal{F}, \omega_X)$ and $\mathcal{F} \mapsto H^{n-i}(X, \mathcal{F})^\vee$ form contravariant δ-functors. The first one is universal, so the maps in (2.2) can be extended to maps in (2.4), see [64], Chapter III, Theorem 7.6 (a). This general version of duality holds in the following situation according to [28], Chapter 5.1 and particularly (5.1.11).

Proposition 2.2. *If X is Cohen–Macaulay and equidimensional, then there are natural identifications*

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) = H^{n-i}(X, \mathcal{F})^\vee$$

for every coherent sheaf $\mathcal{F}$ on X and every $0 \leq i \leq n$.

In the case of projective schemes over an algebraically closed field k, this generalized version of duality holds if and only if X is Cohen–Macaulay and equidimensional, see [64], Chapter III, Theorem 7.6. Observe that every connected, Cohen–Macaulay scheme locally of finite type over a field is equidimensional by Proposition 1.2.

If $\mathcal{F} = \mathcal{E}$ is locally free, the theory of universal δ-functors shows that the left-hand side in (2.4) can be expressed more tangibly due to [64], Chapter III, Propositions 6.3 and 6.7.

Proposition 2.3. *Let $\mathcal{E}$ be locally free sheaf and $\mathcal{G}$ a coherent sheaf on X. Then for $i \geq 0$:*

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{G}) = H^i(X, \mathcal{E}^\vee \otimes \mathcal{G}).$$

Thus the general version of Serre duality for locally free sheaves $\mathcal{E}$ yields the equalities

$$h^i(\mathcal{E}^\vee \otimes \omega_X) = h^{n-i}(\mathcal{E}).$$

Remark 2.4. Suppose that X has the *resolution property*, that is, every coherent sheaf $\mathcal{F}$ on X admits a surjection $\mathcal{E} \to \mathcal{F}$ from some locally free sheaf $\mathcal{E}$. Then it is sufficient to demand that the maps in (2.2) or (2.4) are bijective for all locally free sheaves $\mathcal{E}$. Indeed, for $\mathcal{F}$ coherent, choose a partial resolution $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ of locally free sheaves, apply both functors involved and use the five lemma. For instance, the resolution property is known to hold if X is projective by [13], Exposé II, Proposition 2.2.3, or if X is a surface according to [45], Theorem 5.2. For more general results, see [129].

Example 2.5. Let X be zero-dimensional, so $X = \mathrm{Spec}(A)$ for a finitely generated k-algebra A. Then $\omega_X = \widetilde{D}$ for the A-module $D = \mathrm{Hom}_k(A, k)$ and the trace is the evaluation at $1 \in A$. To see from the definition that

$$\mathrm{Hom}_A(M, D) \times M \longrightarrow D \stackrel{\mathrm{tr}}{\longrightarrow} k, \quad (\varphi, m) \longmapsto \varphi_m \longmapsto \varphi_m(1)$$

is non-degenerate for all finitely generated A-modules M, first let $\varphi \neq 0$. Choose $x \in M$ and $a \in A$ such that $\varphi_x(a) \neq 0$. Then $m := ax$ satisfies $\varphi_m(1) = a\varphi_x(1) = \varphi_x(a) \neq 0$. Now let M be a free A-module of finite rank, and $m \in M$ non-zero. Choose projections $p\colon M \to A$ and $q\colon A \to k$ such that $q(p(m)) \neq 0$. Define $\varphi\colon M \to D$, $x \mapsto \varphi_x$ by $\varphi_x(a) = q(a \cdot p(x))$. Then $\varphi_m(1) \neq 0$. By Remark 2.4, this suffices to verify that $\omega_X = \widetilde{D}$.

Example 2.6. On $X = \mathbb{P}^n$, explicitly computing the cohomology groups of the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$ using Čech cohomology for the open cover $\mathbb{P}^n = \bigcup D_+(T_i)$ shows that $H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) = k\, T_0^{-1} \cdots T_n^{-1}$ is the only one-dimensional top cohomology group. In fact, $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$ with $\operatorname{tr}\colon H^n(\mathbb{P}^n, \omega_{\mathbb{P}^n}) \to k$, $T_0^{-1} \cdots T_n^{-1} \mapsto 1$ forms the dualizing sheaf on $\mathbb{P}^n$, see [64], Chapter III, Theorem 7.1.

This concrete example can be used to show the existence of a dualizing sheaf for projective schemes. More generally, let P be a Cohen–Macaulay, equidimensional, proper k-scheme such that $X \subset P$ is closed of codimension $d = \dim(P) - \dim(X)$. Then

$$\omega_X = \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P) \tag{2.5}$$

is a dualizing sheaf on X. The proof is essentially [64], Chapter III, Proposition 7.5, where instead of loc. cit., Lemma 7.3, the more general version [89], Theorem 17.1, is used if there exists no ample sheaf on P.

Proposition 2.7 (Adjunction Formula). *Let X be a Cohen–Macaulay, equidimensional, proper scheme over an arbitrary field k. Let $D \subset X$ be an effective Cartier divisor. Then:*

(i) The scheme D is Cohen–Macaulay and equidimensional.

(ii) The adjunction formula $\omega_D = (\mathcal{O}_X(D) \otimes \omega_X)|_D$ holds.

Proof. The closed subscheme $D \subset X$ is regularly immersed of codimension 1, so the first assertion follows from [21], AC X.27, Proposition 4. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

An application of $\mathcal{H}om(\,\cdot\,, \omega_X)$ as well as the tensor product with $\mathcal{O}_X(D) \otimes \omega_X$ yield the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{H}om(\mathcal{O}_X, \omega_X) & \longrightarrow & \mathcal{H}om(\mathcal{O}_X(-D), \omega_X) & \longrightarrow & \mathcal{E}xt^1(\mathcal{O}_D, \omega_X) & \longrightarrow & 0 \\
\| & & \| & & \downarrow \simeq & & \\
\omega_X & \longrightarrow & \mathcal{O}_X(D) \otimes \omega_X & \longrightarrow & (\mathcal{O}_X(D) \otimes \omega_X)|_D & \longrightarrow & 0
\end{array}$$

using $\mathcal{E}xt^1(\mathcal{O}_X, \omega_X) = 0$. With $\omega_D = \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X)$, the five lemma guarantees that the induced arrow $\omega_D \to (\mathcal{O}_X(D) \otimes \omega_X)|_D$ is an isomorphism. □

If X is additionally smooth over k and n-equidimensional, then the dualizing sheaf ω_X coincides with the canonical sheaf $\Omega^n_X = \det(\Omega^1_X)$, which is the determinant of the sheaf of Kähler differentials due to Proposition 2.13 below. The next section briefly overviews this notion.

2.2 Kähler Differentials

Let $f\colon X \to S$ be a morphism of schemes. The diagonal morphism $\Delta\colon X \to X \times_S X$ is an immersion by [55], Errata 10. Thus $\Delta(X)$ is a closed subscheme of an open $U \subset X \times_S X$. Denote by $\mathcal{I}$ the corresponding ideal sheaf. The $\mathcal{O}_X$-module $\Omega^1_{X/S} = \Delta^*(\mathcal{I}\,/\,\mathcal{I}^2)$ is the sheaf of *Kähler differentials* to f.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *derivation* of $\mathcal{O}_X$ to $\mathcal{F}$ relative to f is a morphism of sheaves of groups $d\colon\ \mathcal{O}_X \to \mathcal{F}$ which satisfies the following two properties:

(i) For all sections $\sigma_1, \sigma_2 \in H^0(U, \mathcal{O}_U)$ over an open subset $U \subset X$, the Leibniz rule $d(\sigma_1\sigma_2) = \sigma_1 d(\sigma_2) + d(\sigma_1)\sigma_2$ holds.

(ii) For all sections $\tau \in H^0(V, \mathcal{O}_V)$ and $\sigma \in H^0(U, \mathcal{O}_U)$ over open subsets $V \subset S$ and $U \subset f^{-1}(V)$, the equality $d(\tau|_U \cdot \sigma) = \tau|_U \cdot d(\sigma)$ holds.

Denote by $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$ the $H^0(X, \mathcal{O}_X)$-module of derivations of $\mathcal{O}_X$ to $\mathcal{F}$ relative to f. The assignment $U \mapsto \mathrm{Der}_S(\mathcal{O}_U, \mathcal{F}|_U)$ yields an $\mathcal{O}_X$-module $\mathcal{D}er_S(\mathcal{O}_X, \mathcal{F})$. The sheaf of Kähler differentials $\Omega^1_{X/S}$ is equipped with a universal derivation $d\colon\ \mathcal{O}_X \to \Omega^1_{X/S}$, which means that

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{F}) \longrightarrow \mathcal{D}er_S(\mathcal{O}_X, \mathcal{F}), \quad h \longmapsto h \circ d$$

is an isomorphism of $\mathcal{O}_X$-modules, see [58], Corollaire 16.5.5. In the special case $\mathcal{F} = \mathcal{O}_X$, the identification shows that the dual to $\Omega^1_{X/S}$ is the *tangent sheaf*

$$\Theta_{X/S} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) = \mathcal{D}er_S(\mathcal{O}_X, \mathcal{O}_X)$$

to f. Hence the sheaf of Kähler differentials is also called the *cotangent sheaf* to f. If $\Omega^1_{X/S}$ is quasicoherent and of finite type, $x \in X$ and $s = f(x)$ such that $\kappa(s) \to \kappa(x)$ is bijective, then $\Theta_{X/S}(x) = \mathrm{Hom}_{\kappa(s)}(\mathfrak{m}'_x/\mathfrak{m}'^2_x, \kappa(s))$, where $\mathfrak{m}'_x$ is the maximal ideal of $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ by [58], (16.5.13). In the case that $S = \mathrm{Spec}(k)$ and $x \in X(k)$, this yields the usual definition $\Theta_X(x) = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$ of the Zariski tangent space.

Proposition 2.8. *The sheaf of Kähler differentials satisfies the following properties:*

(i) If $S' \to S$ is a morphism of schemes and $X' = X \times_S S'$ the base change, then there is a natural identification $(\Omega^1_{X/S})_{X'} = \Omega^1_{X'/S'}$.

(ii) If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are morphisms of schemes, then there is a natural exact sequence $\varphi^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$. If φ is additionally smooth, then $0 \to \varphi^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$ is exact.

(iii) If $g\colon Y \to S$ is another morphism of schemes, then there is a natural identification $\Omega^1_{(X\times Y)/S} = \mathrm{pr}_X^*(\Omega^1_{X/S}) \oplus \mathrm{pr}_Y^*(\Omega^1_{Y/S})$.

(iv) *If f is locally of finite presentation, then $\Omega^1_{X/S}$ is quasicoherent of finite presentation.*

(v) *If f is locally of finite presentation and $x \in X$, then f is smooth at x if and only if f is flat at x and $\Omega^1_{X/S}$ is locally free of rank $n = \dim_x(X_{f(x)})$ in a neighborhood of x.*

(vi) *If f is locally of finite presentation, then $\Omega^1_{X/S} = 0$ if and only if f is unramified.*

The reference is [58] and specifically: (i) Proposition 16.4.5; (ii) Corollaire 16.4.19 and [59], Exposé II, Théorème 4.3; (iii) Proposition 16.4.23; (iv) Corollaire 16.4.22; (v) Proposition 17.15.15; (vi) Corollaire 17.4.2.

For $n \geq 0$ the $\mathcal{O}_X$-module of *n-differentials* to f is defined to be $\Omega^n_{X/S} = \wedge^n(\Omega^1_{X/S})$. Of particular importance is the case when f is a smooth morphism with equidimensional fibers of dimension n, since then $\Omega^n_{X/S} = \det(\Omega^1_{X/S})$ is invertible.

Definition 2.9. Let $f\colon X \to S$ be a smooth morphism with equidimensional fibers of dimension n. The invertible sheaf $\Omega^n_{X/S}$ is the *canonical sheaf* to f. In the case that the base $S = \operatorname{Spec}(k)$ is a fixed ground field, call Ω^n_X the *canonical sheaf* of X.

Example 2.10. Let G be a group scheme over an arbitrary field k with structure morphism $f\colon G \to \operatorname{Spec}(k)$ and neutral element $e\colon \operatorname{Spec}(k) \to G$. There is a natural identification $\Omega^1_G = f^*e^*(\Omega^1_G)$ by [9], Tag 047I, so Ω^1_G is free. In the special case that $A = G$ is a g-dimensional abelian variety, the sheaf $\Omega^1_A = \mathcal{O}_A^{\oplus g}$ has rank g and $\Omega^g_A = \mathcal{O}_A$ is trivial.

2.3 Grothendieck Duality

Following the mindset of Grothendieck, it is desirable to generalize the notion of Serre duality from the absolute setting of a scheme X over a ground field k to the relative setting of a morphism $f\colon X \to Y$ of schemes, known as *Grothendieck duality.* This aim lead onto a challenging journey and in this section, some important results are summarized. The starting point was laid by Grothendieck [48], [49] and elaborated by Hartshorne [63] and later Conrad [28]. Quite recently, a fresh approach to this topic by Neeman [98] appeared, and also Lipman [87] published new material.

In the following outline, only noetherian schemes are considered for the sake of presentation. A lot of results hold for locally noetherian or arbitrary schemes, and especially the references given to [28] will reveal where assumptions can be lessened.

This section is based on Hartshorne [63] and Conrad [28], where the latter elaborates several details in the former source, particularly concerning entire compatibility of the dualizing sheaf and its trace with base change. The procedure is top-down in the sense that Grothendieck duality is first established on the level of derived categories, and afterwards results for quasicoherent sheaves are deduced. For Cohen–Macaulay, proper morphisms of fixed relative equidimension, duality can then be expressed in terms of a dualizing sheaf, rather than by an abstract functor.

Note also that there is a much more direct approach by Kleiman [77], which avoids derived categories. Here, the approach is bottom-up and resembles more the absolute setting of Serre duality. First, duality in degree zero is treated, with the existence of a dualizing sheaf for locally projective morphisms. Afterwards, full duality for Cohen–Macaulay, locally projective morphisms of fixed relative equidimension is inferred.

Let X be a noetherian scheme and denote by $D(X)$ the derived category of $\mathcal{O}_X$-modules. The full subcategories of complexes $\mathcal{F}^\bullet \in D(X)$ which are bounded below, that is, $\mathcal{F}^n = 0$ for $n \ll 0$, is denoted by $D^+(X)$. Similarly, define $D^-(X)$, $D^b(X)$ to be the full subcategories of complexes $\mathcal{F}^\bullet \in D(X)$ which are bounded above, and bounded below and above, respectively. For the full subcategories of complexes $\mathcal{F}^\bullet \in D(X)$ whose cohomologies are quasicoherent or coherent, write $D_{\mathrm{qc}}(X)$ and $D_{\mathrm{c}}(X)$, respectively. For details on derived categories, see for instance [63], Chapter I.

Let $f\colon X \to Y$ be a morphism of finite type between noetherian schemes. Under suitable assumptions, the statement of *Grothendieck duality* is the existence of an isomorphism

$$\begin{array}{ccc} Rf_* R\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, f^! \mathcal{G}^\bullet) & \xrightarrow{\simeq} & R\mathcal{H}om^\bullet_{\mathcal{O}_Y}(Rf_* \mathcal{F}^\bullet, \mathcal{G}^\bullet) \\ \downarrow & \nearrow_{\mathrm{tr}_f} & \\ R\mathcal{H}om^\bullet_{\mathcal{O}_Y}(Rf_* \mathcal{F}^\bullet, Rf_* f^! \mathcal{G}^\bullet) & & \end{array} \tag{2.6}$$

for all $\mathcal{F}^\bullet \in D^-_{\mathrm{qc}}(X)$ and $\mathcal{G}^\bullet \in D^+_{\mathrm{c}}(Y)$. Here, $f^!\colon D^+_{\mathrm{c}}(Y) \to D^+_{\mathrm{c}}(X)$ is a right adjoint functor to Rf_* and the *trace* $\mathrm{tr}_f\colon Rf_* f^! \to \mathrm{id}_{D^+_{\mathrm{c}}(Y)}$ is a morphism of δ-functors, see [28], Sections 3.3 and 3.4, based on [63], Chapter VII, Corollary 3.4, for details and their properties.

An essential ingredient is the existence of a *dualizing complex* $\mathcal{Q}^\bullet \in D^b_{\mathrm{c}}(Y)$. By definition, this is a complex $\mathcal{Q}^\bullet$ which has finite injective dimension, that means, it is isomorphic in $D(Y)$ to a bounded complex of injective sheaves, and the natural homomorphism

$$\mathcal{F}^\bullet \longrightarrow R\mathcal{H}om^\bullet_{\mathcal{O}_Y}(R\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{Q}^\bullet), \mathcal{Q}^\bullet)$$

defined in [63], Chapter V, Lemma 1.2, is an isomorphism for all $\mathcal{F}^\bullet \in D_{\mathrm{c}}(Y)$. It is sufficient to check that it is an isomorphism for $\mathcal{F}^\bullet = \mathcal{O}_X[0]$ by [63], Chapter V, Proposition 2.1. The existence of a dualizing complex is ensured in the following cases, see [63], Chapter V, Section 10.

(i) If Y is Gorenstein, then $\mathcal{O}_Y[0]$ is a dualizing complex.

(ii) If $f\colon X \to Y$ is of finite type and Y has a dualizing complex $\mathcal{Q}^\bullet$, then $f^!(\mathcal{Q}^\bullet)$ is a dualizing complex for X.

Thus especially every scheme of finite type over a field has a dualizing complex. The existence of a dualizing complex in turn yields a *residual complex*, see [28], Lemma 3.1.4 and the subsequent discussion based on [63], Chapter VI, Proposition 1.1. A residual complex is used to define the functor $f^!$ and, if f is additionally proper, also the trace tr_f. Now the central result is the following, the reference is [63], Chapter VII, Theorem 3.3 and [28], Theorem 3.4.4.

Theorem 2.11. *Let $f\colon X \to Y$ be a proper morphism between noetherian schemes, where Y admits a dualizing complex. Then Grothendieck duality holds.*

The next step is to translate Grothendieck duality into a more tangible version similar to Serre duality in the absolute case. The general approach is to simplify $f^!$ and tr_f by using a relative dualizing sheaf $\omega_{X/Y}$ and a morphism $R^r f_*(\omega_{X/Y}) \to \mathcal{O}_Y$ of sheaves.

Let $f\colon X \to Y$ be a morphism of finite type between noetherian schemes where Y admits a dualizing complex. The functor $f^!$ can be defined differently in two cases, namely when f is smooth and separated, or finite. In the situation that f is finite, denote by $\overline{f}\colon (X, \mathcal{O}_X) \to (Y, f_* \mathcal{O}_X)$ the induced morphism of ringed spaces. Then naturally $f^! = f^\flat$, where $f^\flat\colon D^+(Y) \to D^+(Y)$ is defined as $f^\flat \mathcal{G}^\bullet = \overline{f}^* R\,\mathcal{H}om_{\mathcal{O}_Y}(f_*(\mathcal{O}_X), \mathcal{G}^\bullet)$ according to [28], (2.2.8) and (3.3.19).

If f is smooth and separated, then naturally $f^! = f^\sharp$, where $f^\sharp\colon D(Y) \to D(X)$ is defined by $f^\sharp \mathcal{G}^\bullet = \Omega^r_{X/Y}[r] \otimes^{\mathrm{L}} f^* \mathcal{G}^\bullet$ due to [28], (2.2.7) and (3.3.21). Here r denotes the possibly varying value of the locally constant function $X \to \mathbb{N}$, $x \mapsto \dim_x(X_{f(x)})$. So the functor $f^! = f^\sharp$ relies essentially only on the canonical sheaf $\Omega^r_{X/Y}$. If r is constant, that is, f is of relative equidimension r, and f is additionally proper, then it is feasible to define the duality isomorphism (2.6) only in terms of $\Omega^r_{X/Y}$ and a trace $R^r f_*(\Omega^r_{X/Y}) \to \mathcal{O}_Y$, see [28], Section 3.4, pages 150 to 152. In what follows, this will be generalized to the situation where a relative dualizing sheaf $\omega_{X/Y}$ exists.

Note that both $f^\sharp$ and $f^\flat$ are defined without the assumption that a dualizing complex exists. Now let $f\colon X \to Y$ be a morphism of noetherian schemes which admits a factorization

$$f = s \circ i \tag{2.7}$$

for a closed embedding i and a smooth, separated morphism s of finite type. It is then possible to define $f^!\colon D^+_{\mathrm{qc}}(Y) \to D^+_{\mathrm{qc}}(X)$ by $f^! = i^\flat s^\sharp$ without assuming the existence of a dualizing complex on Y, see [28], Section 2.8 and particularly Theorem 2.8.1. In the case that a dualizing complex on Y exists, this definition can naturally be identified with the previous one by [28], Section 3.5, page 153f.

If now f itself is a smooth and separated morphism of relative equidimension r, then $f^! \mathcal{O}_Y = f^\sharp \mathcal{O}_Y = \Omega^r_{X/Y}[r]$ is simply the canonical sheaf shifted down to degree $-r$. Keeping this in mind, consider the following statement, which is [28], Theorem 3.5.1.

Proposition 2.12. *Let $f\colon X \to Y$ be a flat morphism of noetherian schemes such that $f = s \circ i$ for a closed embedding i and a smooth, separated morphism s of finite type. Then the following conditions are equivalent:*

(i) f is Cohen–Macaulay of relative equidimension r.

(ii) $H^j(f^! \mathcal{O}_Y) = 0$ for $j \neq -r$ and $H^{-r}(f^! \mathcal{O}_Y)$ is flat over Y.

When these conditions hold, then f is Gorenstein if and only if $H^{-r}(f^! \mathcal{O}_Y)$ is invertible.

The theorem suggests that f being Cohen–Macaulay of relative equidimension r is the suitable framework to define a relative dualizing sheaf and obtain a theory similar to the smooth case.

Any morphism $f\colon X \to Y$ of finite type between noetherian schemes can locally be factorized as a closed embedding followed by a smooth, separated morphism of finite type: Given $x \in X$, choose an affine open neighborhood $U = \operatorname{Spec}(A)$ such that $f(U) \subset V$ for an affine open subset $V = \operatorname{Spec}(R)$ of Y. Since the R-algebra A is finitely generated, there exists a factorization $A \twoheadleftarrow R[T_1, \dots, T_N] \hookleftarrow R$, which induces $U \hookrightarrow \mathbb{A}^N_V \twoheadrightarrow V$.

Let $f\colon X \to Y$ be a Cohen–Macaulay morphism of relative equidimension r between noetherian schemes. The *relative dualizing sheaf* $\omega_{X/Y}$ of f is the $\mathcal{O}_X$-module defined by gluing the $\mathcal{O}_U$-modules $H^{-r}(f|^!_U \mathcal{O}_Y)$ for a cover $X = \bigcup U$ where U are chosen as above, see [28], Section 3.5, page 157. Summarize what has been explicated up to this point:

Proposition 2.13. *Let $f\colon X \to Y$ be a Cohen–Macaulay morphism of relative equidimension r between noetherian schemes. Then the following holds:*

(i) $\omega_{X/Y}$ is coherent and flat over Y.

(ii) $\omega_{X/Y}$ is invertible if and only if f is Gorenstein.

(iii) $\omega_{X/Y} = \Omega^r_{X/Y}$ if f is smooth.

(iv) $\omega_{X/Y} = \mathcal{O}_X$ if f is étale.

(v) $\omega_{X/Y}|_U = \mathcal{E}xt^{N-r}_{\mathcal{O}_P}(\mathcal{O}_U, \omega_{P/Y})$ whenever $f|_U = s \circ i$ as in (2.7) with $s\colon P \to Y$ of relative equidimension N.

Observe that (v) generalizes (2.5) and refers to [28], (3.5.3). By [28], Theorem 3.6.1, the relative dualizing sheaf is compatible with base change in Y:

Proposition 2.14. *Let $f\colon X \to Y$ be a Cohen–Macaulay morphism of relative equidimension r between noetherian schemes. Let $Y' \to Y$ be a morphism of noetherian schemes. Consider the cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{b'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[b]{} & Y. \end{array}$$

Then naturally $b'^*(\omega_{X/Y}) = \omega_{X'/Y'}$ and this identification is compatible with the natural identifications stated in Proposition 2.13 (iii) to (v).

Add the assumption that f is proper. In this case, recreating the situation where f is smooth, a trace morphism $\gamma_f\colon R^r f_*(\omega_{X/Y}) \to \mathcal{O}_Y$ is definable as explained in [28], Corollary 3.6.6, which is compatible with base change. As a consequence, the functor $f^\sharp\colon D^+_{\mathrm{qc}}(Y) \to D^+_{\mathrm{qc}}(X)$ given by $f^\sharp \mathcal{G}^\bullet = \omega_{X/Y}[r] \otimes^{\mathrm{L}} f^* \mathcal{G}^\bullet$ can be used to define a trace

$\mathrm{tr}_f\colon Rf_*f^\sharp \to \mathrm{id}_{D^b_{\mathrm{qc}}(Y)}$ according to [28], (4.3.1) and (4.3.4). This leads to a restatement of the duality morphism

$$Rf_*R\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, f^\sharp\mathcal{G}^\bullet) \longrightarrow R\mathcal{H}om^\bullet_{\mathcal{O}_Y}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

for all $\mathcal{F}^\bullet \in D^b_{\mathrm{c}}(X)$ and $\mathcal{G}^\bullet \in D^b_{\mathrm{qc}}(Y)$. It is an isomorphism by [28], Theorem 4.3.1. The substantial difference compared to its former definition is that the role of a dualizing complex and its existence have dissolved into the sheaf $\omega_{X/Y}$. The trace γ_f is always surjective and it is an isomorphism if f has geometrically reduced and geometrically connected fibers according to [28], Corollary 4.4.5. By [28], Theorem 4.3.3, the formation of $\omega_{X/Y}$ is compatible with compositions:

Proposition 2.15. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be Cohen–Macaulay morphisms of relative equidimension r and r', respectively, between noetherian schemes. Then there is a natural identification*

$$\omega_{X/Z} = \omega_{X/Y} \otimes f^*(\omega_{Y/Z}).$$

The identification above itself is compatible with base change in Z due to [28], Theorem 4.4.4. A consequence of Proposition 2.14 and Proposition 2.15 is the following:

Corollary 2.16. *Let $f\colon X \to S$ and $g\colon Y \to S$ be Cohen–Macaulay morphisms of relative equidimension r and r', respectively, between noetherian schemes. Then*

$$\omega_{X\times Y/S} = \mathrm{pr}_X^*(\omega_{X/S}) \otimes \mathrm{pr}_Y^*(\omega_{Y/S}).$$

For a discussion to what extent this identification satisfies several compatibilities, see [28], Section 4.4, page 215f. The next theorem generalizes Proposition 2.2 to the relative situation, and can be found in [28], Theorem 5.1.2. Note that the formulation of Theorem 5.1.2 contains a typing error, which has been corrected in [29].

Theorem 2.17. *Let $f\colon X \to Y$ be a Cohen–Macaulay, proper morphism of relative equidimension r between noetherian schemes. Let $\mathcal{E}$ be a locally free sheaf on X and $m \geq 0$ such that $R^jf_*(\mathcal{E})$ is locally free for all $j > m$. Then for every quasicoherent sheaf $\mathcal{G}$ on Y, there is a natural isomorphism*

$$R^{r-j}f_*(\mathcal{E}^\vee \otimes \omega_{X/Y} \otimes f^*\mathcal{G}) \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_Y}(R^jf_*(\mathcal{E}), \mathcal{G}) \tag{2.8}$$

for every $j \geq m$. For $\mathcal{G} = \mathcal{O}_Y$, this isomorphism becomes

$$R^{r-j}f_*(\mathcal{E}^\vee \otimes \omega_{X/Y}) \xrightarrow{\simeq} R^jf_*(\mathcal{E})^\vee. \tag{2.9}$$

If the locally free sheaf $\mathcal{E}$ is replaced by a coherent sheaf $\mathcal{F}$ which is flat over Y, then the theorem continues to be valid in a more abstract version, see [28], (5.1.11).

Remark 2.18. Rewrite the isomorphism (2.8) from $\mathcal{A} \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{B}, \mathcal{C})$ into $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ and identify $\mathcal{E}^\vee \otimes \omega_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \omega_{X/Y})$ to obtain

$$R^{r-j} f_*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \omega_{X/Y}) \otimes f^* \mathcal{G}) \otimes R^j f_*(\mathcal{E}) \longrightarrow \mathcal{G} .$$

This morphism is induced by the trace γ_f, see [28], (5.1.7). In the special case that $Y = \mathrm{Spec}(k)$ and $\mathcal{G} = k$ as well as $j = r$, this recovers the original definition (2.1) of the dualizing sheaf, as it holds for all quasicoherent $\mathcal{F}$ in place of $\mathcal{E}$ by the preceding remark.

Remark 2.19. In the case that $j = r$, the identification (2.8) can also be rewritten to

$$f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_{X/Y} \otimes f^* \mathcal{G}) \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_Y}(R^r f_*(\mathcal{F}), \mathcal{G}),$$

which holds in addition for all quasicoherent $\mathcal{F}$ in place of $\mathcal{E}$ by [28], Corollary 5.1.3. Now insert $\mathcal{G} = \mathcal{O}_Y$ and take global sections to deduce a bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_{X/Y}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{O}_Y}(R^r f_*(\mathcal{F}), \mathcal{O}_Y)$$

that is natural in all quasicoherent $\mathcal{F}$ and maps $\mathrm{id}_{\omega_{X/Y}}$ to the trace γ_f. Hence $\omega_{X/Y}$ represents the functor

$$\mathrm{QCoh}(X) \longrightarrow (\Gamma(Y, \mathcal{O}_Y)\text{-Mod}), \quad \mathcal{F} \longmapsto \mathrm{Hom}_{\mathcal{O}_Y}(R^r f_*(\mathcal{F}), \mathcal{O}_Y),$$

which in turn generalizes (2.3).

As an application of the abstract theory of Grothendieck duality, the existence of the dualizing sheaf on a proper k-scheme is conducted in the following:

Proposition 2.20. *Let X be an n-dimensional, proper scheme over an arbitrary field k. Denote the structure morphism by $f\colon X \to \mathrm{Spec}(k)$. Then $\omega_X = H^{-n}(f^! k[0])$ is the dualizing sheaf.*

Proof. The proof follows [9], Tag 0AWP. Since $\mathrm{Spec}(k)$ is Gorenstein, a dualizing complex for $\mathrm{Spec}(k)$ is given by k, and hence $f^!(k[0])$ is a dualizing complex for X. Let $\mathcal{F}$ be a coherent sheaf on X and deduce an identification $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = \mathrm{Hom}_k(H^n(X, \mathcal{F}), k)$ which is functorial in $\mathcal{F}$. Write $\mathcal{W}^\bullet = f^! k[0]$, so

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = \mathrm{Hom}_{\mathcal{O}_X}(H^0(\mathcal{F}[0]), H^{-n}(\mathcal{W}^\bullet))$$

by definition. Note that $H^i(W^\bullet) = 0$ for $i < -n$ due to [9], Tag 0AWN. Hence there is an identification

$$\mathrm{Hom}_{\mathcal{O}_X}(H^0(\mathcal{F}[0]), H^{-n}(\mathcal{W}^\bullet)) = \mathrm{Ext}^{-n}_{\mathcal{O}_X}(\mathcal{F}[0], \mathcal{W}^\bullet)$$

by [9], Tag 06XS. The latter is defined as

$$\mathrm{Ext}^{-n}_{\mathcal{O}_X}(\mathcal{F}[0], \mathcal{W}^\bullet) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}[n], \mathcal{W}^\bullet).$$

Now $f^!$ is a right adjoint functor to Rf_*, and thereby

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}[n],\mathcal{W}^\bullet) &= \mathrm{Hom}_k(Rf_*(\mathcal{F}[n]),k[0])\\ &= \mathrm{Hom}_k(Rf_*(\mathcal{F})[n],k[0])\\ &= \mathrm{Hom}_k(H^0(Rf_*(\mathcal{F})[n]),k),\end{aligned}$$

where the last identification uses once again [9], Tag 06XS. Finally,

$$\begin{aligned}\mathrm{Hom}_k(H^0(Rf_*(\mathcal{F})[n]),k) &= \mathrm{Hom}_k(H^n(Rf_*(\mathcal{F})),k)\\ &= \mathrm{Hom}_k(R^nf_*(\mathcal{F}),k)\\ &= \mathrm{Hom}_k(H^n(X,\mathcal{F}),k)\end{aligned}$$

completes the identification $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_X) = \mathrm{Hom}_k(H^n(X,\mathcal{F}),k)$. Define the trace tr as the image of $\mathrm{id}_{\omega_X} \in \mathrm{Hom}_{\mathcal{O}_X}(\omega_X,\omega_X)$ in $H^n(X,\omega_X)^\vee$. Then the functoriality of the identification yields the non-degenerate pairing (2.1). □

For the next proposition, note that an $\mathcal{O}_X$-module $\mathcal{F}$ is called *reflexive* if the natural morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism. For more information about this notion, see [65].

Proposition 2.21. *Let X be a reduced, proper scheme over an arbitrary field k. There exists a dense open subset $U \subset X$ such that $\omega_X|_U$ is invertible. If X is additionally normal, then ω_X is reflexive.*

Proof. The regular locus of X is open according to [56], Corollaire 6.12.5. As X is reduced, its regular locus contains all generic points, so it is dense. Since every regular local noetherian ring is Gorenstein, the first claim follows. Moreover, if X is normal, then ω_X is reflexive due to [80], Lemma 3.7.5. □

Let X be an n-dimensional, normal, proper scheme over an arbitrary field k. There exists a correspondence between Weil divisors and reflexive sheaves which have rank 1 at every generic point of X by [65], Propositions 2.7 and 2.8. Thus there exists a Weil divisor K_X on X with associated sheaf $\mathcal{O}_X(K_X) \simeq \omega_X$. Such a Weil divisor K_X is called a *canonical divisor*, as ω_X is uniquely determined by its restriction $\omega_X|_U = \Omega^n_U$ to the regular locus $U \subset X$, where it coincides with the canonical sheaf. This fact is a consequence of [65], Theorem 1.12, which also states that for the inclusion $i\colon U \hookrightarrow X$, the dualizing sheaf is obtained as $i_*(\Omega^n_U) = \omega_X$.

Chapter 3

Albanese Morphisms

The main aim of this chapter is to give a self-contained treatment of the Albanese scheme and Albanese torsor. This will lead to existence results under quite general assumptions for schemes X over an arbitrary ground field, and also for suitable X over a general base scheme S. If X admits a section, then the Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^0_{X/S}$ is a pointed morphism to a projective abelian scheme $\mathrm{Alb}^0_{X/S}$, called the *Albanese scheme*, such that each pointed morphism $X \to A$ to some projective abelian scheme A has a unique factorization through alb. In the case that no section exists, principal homogeneous spaces under projective abelian schemes are considered instead, and the Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^1_{X/S}$ maps to the *Albanese torsor* $\mathrm{Alb}^1_{X/S}$.

The construction of the Albanese morphism is based on the Picard scheme and a Poincaré sheaf. The latter is introduced in Section 3.1. This approach yields good properties, for instance compatibility with base change and products. Afterwards in Section 3.2, projective abelian schemes and their duality is outlined. Then the Albanese scheme is the topic of Section 3.3, and the Albanese torsor is covered in Section 3.4.

The Albanese morphism provides a central technical tool for the subsequent chapters. For instance, it occurs as the projection to the abelian factor in the Beauville–Bogomolov decomposition. Under suitable assumptions, the pullback $\mathrm{Pic}^0(\mathrm{Alb}^1_{X/k}) \to \mathrm{Pic}^0(X)$ is bijective. Thus a tensor power of every numerically trivial invertible sheaf on X arises as a pullback from $\mathrm{Alb}^1_{X/k}$. Moreover, as the dimension of $\mathrm{Alb}^1_{X/k}$ is less or equal to $h^1(\mathscr{O}_X)$, its size is connected to the cohomology of ω_X by Serre duality.

One primary literature reference for this topic is [54], Théorème 2.1, Corollaire 3.2, Théorème 3.3 (iii), where Grothendieck establishes the existence of the Albanese torsor for geometrically normal, proper schemes with $h^0(\mathscr{O}_X) = 1$ over an arbitrary field k, and also for families under suitable assumptions. The proof is rather short.

In Appendix A of [131] and the references given there, Wittenberg adapts the work of Serre [120], [121], to extend the existence of the Albanese torsor to geometrically integral schemes over an arbitrary ground field. Moreover, Brion [22], Chapter 4, shows the existence of the Albanese scheme for smooth connected algebraic groups over an arbitrary ground field k.

The discussion [100] provided crucial insights to refine the treatment below. The idea to use the maximal abelian subscheme of the Picard scheme is based on this source, and the proofs of Proposition 3.9 and Proposition 3.10 are adopted from there.

3.1 Poincaré Sheaves

Let $f\colon X \to S$ be a separated morphism of finite type to a locally noetherian scheme S. Fix the base scheme S and suppress the dependence on S in most notations in the course of this chapter, following the conventions determined in the introduction, to significantly improve readability. In particular, all products without index are defined over S. Assume that the Picard scheme $\mathrm{Pic}_X = \mathrm{Pic}_{X/S}$ exists; see Section A.7 for basic properties of Picard schemes. The relative Picard functor $\mathrm{Pic}_{(X)} = \mathrm{Pic}_{(X/S)}$ is defined by $\mathrm{Pic}_{(X)}(T) = \mathrm{Pic}(X \times T)/\,\mathrm{Pic}(T)$ for locally noetherian S-schemes T. If Pic_X represents $\mathrm{Pic}_{(X)}$, then the class $[\mathcal{P}]$ of a Poincaré sheaf $\mathcal{P} \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ is the universal object for this functor. This notion will help to maintain an overview in explicit calculations in the subsequent sections.

Definition 3.1. A *Poincaré sheaf* is an invertible sheaf $\mathcal{P} \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ with the following universal property: For any S-scheme T and $\mathscr{L} \in \mathrm{Pic}(X \times T)$, there exists a unique morphism $h\colon T \to \mathrm{Pic}_X$ such that

$$\mathscr{L} \equiv (\mathrm{id}_X \times h)^*(\mathcal{P}) \mod \mathrm{Pic}(T).$$

Proposition 3.2. *A Poincaré sheaf* $\mathcal{P} \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ *exists if and only if* Pic_X *represents* $\mathrm{Pic}_{(X)}$. *Moreover, the base change of a Poincaré sheaf along an arbitrary morphism* $S' \to S$ *of locally noetherian schemes yields a Poincaré sheaf for* $X_{S'}$.

Proof. The subsequent lines are based on [78], Exercise 4.3. First, assume that a Poincaré sheaf $\mathcal{P} \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ exists. Then the natural transformation $\mathrm{Pic}_X \to \mathrm{Pic}_{(X)}$ given by

$$\mathrm{Pic}_X(T) \longrightarrow \mathrm{Pic}_{(X)}(T), \quad h \longmapsto [(\mathrm{id}_X \times h)^*(\mathcal{P})]$$

is a natural isomorphism by the universal property of $\mathcal{P}$. The converse is true by Yoneda's lemma: If the Picard scheme Pic_X represents the functor $\mathrm{Pic}_{(X)}$, choose a natural isomorphism $\Phi\colon \mathrm{Pic}_X \to \mathrm{Pic}_{(X)}$. Then any representative $\mathcal{P} \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ of the equivalence class of the universal element $\Phi_{\mathrm{Pic}_X}(\mathrm{id}_{\mathrm{Pic}_X}) = [\mathcal{P}]$ of $\mathrm{Pic}_{(X)}(\mathrm{Pic}_X)$ is a Poincaré sheaf. In fact, given an invertible sheaf $\mathscr{L} \in \mathrm{Pic}(X \times T)$, set $h = \Phi_T^{-1}([\mathscr{L}])$. Then the morphism $h\colon T \to \mathrm{Pic}_X$ yields the commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}_X(\mathrm{Pic}_X) & \xrightarrow{\ \psi \mapsto \psi \circ h\ } & \mathrm{Pic}_X(T) \\
{\scriptstyle \Phi_{\mathrm{Pic}_X}}\downarrow \simeq & & \simeq \downarrow {\scriptstyle \Phi_T} \\
\mathrm{Pic}_{(X)}(\mathrm{Pic}_X) & \xrightarrow[\ (\mathrm{id}_X \times h)^*\ ]{} & \mathrm{Pic}_{(X)}(T)
\end{array}$$

mapping $\mathrm{id}_{\mathrm{Pic}_X}$ as follows:

$$\begin{array}{ccc}
\mathrm{id}_{\mathrm{Pic}_X} & \longmapsto & h \\
\downarrow & & \downarrow \\
[\mathcal{P}] & \longmapsto & [\mathscr{L}].
\end{array}$$

This shows that $\mathcal{P}$ satisfies the universal property. Instead of $\mathrm{id}_{\mathrm{Pic}_X}$, for any automorphism φ of Pic_X, it is also possible to choose a sheaf $\mathcal{P}'$ such that $\Phi_{\mathrm{Pic}_X}(\varphi) = [\mathcal{P}']$. Then set $h' = \varphi^{-1} \circ \Phi_T^{-1}([\mathscr{L}])$ in place of h to deduce that $\mathcal{P}'$ is a Poincaré sheaf.

Conversely, let $\mathcal{P}'$ be any Poincaré sheaf whose class $[\mathcal{P}'] \in \mathrm{Pic}_{(X)}(\mathrm{Pic}_X)$ corresponds to some $\varphi \in \mathrm{Pic}_X(\mathrm{Pic}_X)$. Then choose $\mathscr{L} = \mathcal{P}'$ in the universal property of $\mathcal{P}$ to obtain $h = \varphi$. Now reverse the roles of $\mathcal{P}$ and $\mathcal{P}'$ to get an endomorphism g. By the uniqueness applied to both compositions, φ is an automorphism with $\varphi^{-1} = g$. So Poincaré sheaves are representatives of classes corresponding to automorphisms φ of Pic_X.

This point of view shows the asserted compatibility with base change along : For every morphism $S' \to S$, the pullback of a Poincaré sheaf $\mathcal{P} \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ to $X_{S'} \times_{S'} \mathrm{Pic}_{X_{S'}/S'}$ yields a Poincaré sheaf for $X_{S'}$, as the base change of an automorphism φ of Pic_X is an automorphism $\varphi_{S'}$ of $\mathrm{Pic}_{X_{S'}/S'}$. □

3.2 Abelian Schemes

Let k be an arbitrary field. An *abelian variety* A over k is a geometrically reduced, connected and proper k-group scheme. Then A is smooth and geometrically integral by [78], Lemma 5.1. Furthermore, A is projective, see [95], Section 6, page 62.

This notion can be generalized to families. Now let S be a noetherian base scheme. An *abelian S-scheme* is an S-group scheme $f \colon A \to S$, where f is flat and proper with geometrically reduced, connected fibers of dimension g for some $g \geq 0$. So the fibers of f are abelian varieties and in turn smooth, geometrically integral and projective. Hence f itself is smooth.

Some further properties of abelian S-schemes A are the following: It is indeed valid that A is a commutative group scheme, see [96], Corollary 6.5. In op.cit., Corollary 6.4, it is shown that every S-morphism $A \to G$ to an S-group scheme G which maps the identity to the identity is a homomorphism of S-group schemes. In turn, any morphism $A \to G$ can be decomposed into a homomorphism followed by a translation. If $S' \to S$ is any morphism of noetherian schemes, then the base change $f_{S'} \colon A_{S'} \to S'$ is an abelian S'-scheme.

The topic of the remainder of this section is the *dual abelian S-scheme* $A^\vee = \mathrm{Pic}^0_A$. This procedure of dualizing, combined with Poincaré sheaves, constitute the base for the construction of the Albanese morphism in the subsequent sections. The fibers A_s of $f \colon A \to S$ are abelian varieties, so the dual $A_s^\vee$ always exists and it is again an abelian variety according to [95], Section 13. It satisfies $A_s = A_s^{\vee\vee}$ naturally. Prior to the general case, first clarify the following notion: Let G be a group scheme locally of finite type over S. Its *connected component of the identity* G^0 is defined as a subfunctor of G by

$$G^0(T) = \{\, t \in G(T) \mid t_s \colon T_s \to G_s \text{ factors through } (G_s)^0 \text{ for all } s \in S \,\}$$

on S-schemes T. See also Section A.1 for some more details. If A is an abelian S-scheme, then $A^0 = A$ is immediate, since A has connected fibers. In the case that $f\colon A \to S$ is additionally projective, the existence of the Picard scheme Pic_A is ensured by Theorem A.26, and furthermore Pic^0_A is representable by a subscheme.

Remark 3.3. Observe that the projectivity is not automatically fulfilled, although all fibers A_s are abelian varieties and consequently projective. A counterexample can be found in [104], Chapitre XII. If S is normal, then it is always true that f is projective by op. cit., Corollaire XIII 2.7.

Now assume that A is projective over S so that Pic^0_A exists. Write $A^\vee = \operatorname{Pic}^0_A$. The dual $A^\vee$ is a projective S-scheme due to Theorem A.26, which is smooth by [96], Proposition 6.7. Thus $A^\vee$ is in fact a projective abelian S-scheme. In what follows, it will be reviewed that the natural duality $A = A^{\vee\vee}$ continues to be valid over the general noetherian base S. In doing so, some explicit computations are conducted, which will be useful later on.

First, note that according to Lemma A.11, the structure morphism $f\colon A \to S$ of a projective abelian S-scheme is a fibration and cohomologically flat in degree 0. Also, f has the identity section, so the Picard scheme Pic_A represents the relative Picard functor $\operatorname{Pic}_{(A)}$. As seen in the previous section, this means that a Poincaré sheaf for A exists.

Let $\varphi\colon A' \to A$ be a morphism of projective abelian S-schemes. There is an induced one $\varphi^\vee\colon A^\vee \to A'^\vee$, which is the restriction of $\varphi^\vee\colon \operatorname{Pic}_A \to \operatorname{Pic}_{A'}$ given by pullback of invertible sheaves. Namely, for any S-scheme T, the map $\operatorname{Pic}_A(T) \to \operatorname{Pic}_{A'}(T)$ corresponds to

$$\operatorname{Pic}(A \times T)/\operatorname{Pic}(T) \longrightarrow \operatorname{Pic}(A' \times T)/\operatorname{Pic}(T), \quad [\mathscr{L}] \longmapsto [(\varphi \times \mathrm{id}_T)^*(\mathscr{L})] \tag{3.1}$$

as both Pic_A and $\operatorname{Pic}_{A'}$ represent the relative Picard functor. In order to define this morphism in terms of Poincaré sheaves $\mathcal{P}_A \in \operatorname{Pic}(A \times \operatorname{Pic}_A)$ and $\mathcal{P}_{A'} \in \operatorname{Pic}(A' \times \operatorname{Pic}_{A'})$, consider the class of $(\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{P}_A) \in \operatorname{Pic}(A' \times \operatorname{Pic}_A)$ modulo $\operatorname{Pic}(\operatorname{Pic}_A)$. Denote the morphism corresponding to this class by $\varphi^\vee\colon \operatorname{Pic}_A \to \operatorname{Pic}_{A'}$ and verify that it actually coincides with the previous definition: By the universal property of $\mathcal{P}_{A'}$, it satisfies

$$(\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{P}_A) \quad\equiv\quad (\mathrm{id}_{A'} \times \varphi^\vee)^*(\mathcal{P}_{A'}) \mod \operatorname{Pic}(\operatorname{Pic}_A). \tag{3.2}$$

Now let T be an S-scheme and $h \in \operatorname{Pic}_A(T)$ correspond to the class of the invertible sheaf $\mathscr{L} = (\mathrm{id}_A \times h)^*(\mathcal{P}_A) \in \operatorname{Pic}(A \times T)$. Its image $\varphi^\vee \circ h$ in $\operatorname{Pic}_{A'}(T)$ is given by the class of

$$\begin{aligned}(\mathrm{id}_{A'} \times (\varphi^\vee \circ h))^*(\mathcal{P}_{A'}) &\equiv (\mathrm{id}_{A'} \times h)^*(\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{P}_A) \quad \text{[by (3.2)]}\\ &\equiv (\varphi \times \mathrm{id}_T)^*(\mathrm{id}_A \times h)^*(\mathcal{P}_A)\\ &\equiv (\varphi \times \mathrm{id}_T)^*(\mathscr{L})\end{aligned}$$

modulo $\operatorname{Pic}(T)$. So indeed, this coincides with the first definition (3.1) of $\varphi^\vee$.

In fact, $\varphi^\vee\colon \operatorname{Pic}_A \to \operatorname{Pic}_{A'}$ is a homomorphism, as $\varphi^*(\mathcal{O}_A) = \mathcal{O}_{A'}$. Thus it can be restricted to $\varphi^\vee\colon A^\vee \to A'^\vee$. Because the definition (3.1) does not involve Poincaré sheaves, the relation (3.2) holds for all choices of $\mathcal{P}_A$ and $\mathcal{P}_{A'}$.

Introduce some notation. Let R and T be S-schemes whose structure morphisms $f_R\colon R \to S$ and $f_T\colon T \to S$ have sections $e_R\colon S \to R$ and $e_T\colon S \to T$. Then define the two inclusions $\iota_R = \mathrm{id}_R \times (e_T \circ f_R) \colon R \to R \times T$ and $\iota_T = (e_R \circ f_T) \times \mathrm{id}_T \colon T \to R \times T$. They are used to *normalize* invertible sheaves in this sense: For $\mathscr{L} \in \mathrm{Pic}(R \times T)$, define

$$\mathscr{L}_{(R)} = \mathscr{L} \otimes \mathrm{pr}_R^* \iota_R^*(\mathscr{L}^\vee) \otimes \mathrm{pr}_R^* \iota_R^* \mathrm{pr}_T^* \iota_T^*(\mathscr{L}). \tag{3.3}$$

Observe that $(\iota_T \circ \mathrm{pr}_T) \circ (\iota_R \circ \mathrm{pr}_R) = (\iota_R \circ \mathrm{pr}_R) \circ (\iota_T \circ \mathrm{pr}_T) = e_{R \times T} \circ f_{R \times T}$ holds for the induced structure morphism $f_{R\times T}$ and section $e_{R\times T}$ of $R \times T$. The invertible sheaf $\mathscr{L}_{(R)}$ satisfies $[\mathscr{L}_{(R)}] = [\mathscr{L}]$ in $\mathrm{Pic}(R \times T)/\mathrm{Pic}(R)$ and $[\iota_R^*(\mathscr{L}_{(R)})] = [\mathcal{O}_R]$ in $\mathrm{Pic}(R)/\mathrm{Pic}(S)$. The last tensor factor in (3.3) ensures that $\iota_T^*(\mathscr{L}_{(R)}) = \iota_T^*(\mathscr{L})$, so the pullback to T remains unchanged. In the special case that already $\iota_T^*(\mathscr{L}) \simeq \mathcal{O}_T$ is trivial, then also the pullback $\iota_R^*(\mathscr{L}_{(R)}) \simeq \mathcal{O}_R$ to R is trivial.

Now let R' and T' be two further S-schemes, whose structure morphisms $f_{R'} \colon R' \to S$ and $f_{T'} \colon T' \to S$ have sections $e_{R'} \colon S \to R'$ and $e_{T'} \colon S \to T'$. Let $\varphi \colon R' \to R$ and $\psi \colon T' \to T$ be *pointed morphisms* of S-schemes, which means that they are compatible with the sections, so precisely $\varphi \circ e_{R'} = e_R$ and $\psi \circ e_{T'} = e_T$. In this situation, the equality $(\varphi \times \psi)^*(\mathscr{L}_{(R)}) = (\varphi \times \psi)^*(\mathscr{L})_{(R')}$ holds. This can be directly deduced from the equalities $\iota_R \circ \mathrm{pr}_R \circ (\varphi \times \psi) = (\varphi \times \psi) \circ \iota_{R'} \circ \mathrm{pr}_{R'}$ and $\iota_T \circ \mathrm{pr}_T \circ (\varphi \times \psi) = (\varphi \times \psi) \circ \iota_{T'} \circ \mathrm{pr}_{T'}$.

By means of symmetry, $\mathscr{L}_{(T)}$ is defined similarly and has the corresponding properties.

Proposition 3.4. *Let A be a projective abelian S-scheme. Then there is an isomorphism $\alpha \colon A \xrightarrow{\simeq} A^{\vee\vee}$ which is compatible with homomorphisms $\varphi \colon A' \to A$ of projective abelian S-schemes.*

Proof. The normalized Poincaré sheaf $\mathcal{Q} := \mathcal{P}_{(\mathrm{Pic}_A)}$ satisfies by its definition the identity $[\iota_{\mathrm{Pic}_A}^*(\mathcal{Q})] = [\mathcal{O}_{\mathrm{Pic}_A}]$ in $\mathrm{Pic}(\mathrm{Pic}_A)/\mathrm{Pic}(S)$. The class $[\mathcal{Q}|_{A \times A^\vee}] \in \mathrm{Pic}(A \times A^\vee)/\mathrm{Pic}(A)$ yields a morphism $A \to \mathrm{Pic}_{A^\vee}$. Since $\mathcal{Q}$ is normalized with respect to Pic_A, the composition with $e_A \colon S \to A$ is the identity section of $\mathrm{Pic}_{A^\vee}$, so a homomorphism has been defined. For every $s \in S$, there is an induced homomorphism $A_s \to \mathrm{Pic}_{A_s^\vee/\kappa(s)}$. Moreover, since the pullback of $\mathcal{P} \in \mathrm{Pic}(A \times \mathrm{Pic}_A)$ to $\mathrm{Pic}(A_s \times_{\kappa(s)} \mathrm{Pic}_{A_s/\kappa(s)})$ yields a Poincaré sheaf for A_s, this morphism is given in the same manner by a normalized Poincaré sheaf $\mathcal{Q}_s$ for A_s. Since A_s is connected, there is a factorization through $A_s^{\vee\vee}$.

As a consequence, also $A \to \mathrm{Pic}_{A^\vee}$ factorizes through $A^{\vee\vee} = \mathrm{Pic}^0_{A^\vee}$. So the restriction to its image yields a homomorphism $\alpha \colon A \to A^{\vee\vee}$. Its base change to geometric fibers $\alpha_{\bar{s}} \colon A_{\bar{s}} \to A_{\bar{s}}^{\vee\vee}$ is the biduality morphism for abelian varieties over an algebraically closed field, which is an isomorphism by [95], Section 13, Corollary on page 132. Faithfully flat descent implies that $\alpha_s \colon A_s \to A_s^{\vee\vee}$ is an isomorphism, see [56], Proposition 2.7.1. Then [53], Proposition 4.6.7 (ii) eventually shows that α itself is an isomorphism.

The fact that α is compatible with homomorphisms $\varphi \colon A' \to A$ of projective abelian S-schemes will be shown in the next lemma. □

Moreover, the subsequent lemma implies that α is independent of the chosen Poincaré sheaf $\mathcal{P}$ by using $\varphi = \mathrm{id}_A$ and two different Poincaré sheaves to define two morphisms α.

Lemma 3.5. *Let* $\varphi\colon A' \to A$ *be a homomorphism of projective abelian* S*-schemes. Let* $\mathcal{P}_A \in \operatorname{Pic}(A \times \operatorname{Pic}_A)$ *and* $\mathcal{P}_{A'} \in \operatorname{Pic}(A' \times \operatorname{Pic}_{A'})$ *be Poincaré sheaves,* $\alpha_A\colon A \xrightarrow{\simeq} A^{\vee\vee}$ *and* $\alpha_{A'}\colon A' \xrightarrow{\simeq} A'^{\vee\vee}$ *the associated isomorphisms defined as in the preceding proof. Then the diagram*

$$\begin{array}{ccc} A' & \xrightarrow{\varphi} & A \\ {\scriptstyle\alpha_{A'}}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\alpha_A} \\ A'^{\vee\vee} & \xrightarrow[\varphi^{\vee\vee}]{} & A^{\vee\vee} \end{array}$$

is commutative.

Proof. The commutativity of the diagram is the statement that in $\operatorname{Pic}(A' \times A^\vee)$, the equivalence $(\varphi \times \mathrm{id}_{A^\vee})^*(\mathcal{Q}_A|_{A\times A^\vee}) \equiv (\mathrm{id}_{A'} \times \varphi^\vee)^*(\mathcal{Q}_{A'}|_{A'\times A'^\vee})$ holds modulo $\operatorname{Pic}(A')$. This is a consequence of the equivalence in $\operatorname{Pic}(A' \times \operatorname{Pic}_A)$ of

$$(\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{Q}_A) \equiv (\mathrm{id}_{A'} \times \varphi^\vee)^*(\mathcal{Q}_{A'}) \tag{3.4}$$

modulo $\operatorname{Pic}(A')$. In fact, (3.4) is even an equality in $\operatorname{Pic}(A' \times \operatorname{Pic}_A)/\operatorname{Pic}(S)$. To see this, look back at (3.2) and choose an invertible sheaf $\mathscr{N}$ on Pic_A such that

$$(\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{P}_A) \otimes \mathrm{pr}^*_{\operatorname{Pic}_A}(\mathscr{N}) = (\mathrm{id}_{A'} \times \varphi^\vee)^*(\mathcal{P}_{A'})$$

in $\operatorname{Pic}(A' \times \operatorname{Pic}_A)$. Now finally compute

$$\begin{aligned} & (\mathrm{id}_{A'} \times \varphi^\vee)^*(\mathcal{Q}_{A'}) \\ = \; & (\mathrm{id}_{A'} \times \varphi^\vee)^*(\mathcal{P}_{A'(\operatorname{Pic}_{A'})}) \\ = \; & (\mathrm{id}_{A'} \times \varphi^\vee)^*(\mathcal{P}_{A'})_{(\operatorname{Pic}_A)} \\ = \; & (\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{P}_A)_{(\operatorname{Pic}_A)} \otimes \mathrm{pr}^*_{\operatorname{Pic}_A}(\mathscr{N})_{(\operatorname{Pic}_A)} \\ = \; & (\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{P}_{A(\operatorname{Pic}_A)}) \otimes f^*_{A'\times\operatorname{Pic}_A} e^*_{A'\times\operatorname{Pic}_A} \mathrm{pr}^*_{\operatorname{Pic}_A}(\mathscr{N}) \\ = \; & (\varphi \times \mathrm{id}_{\operatorname{Pic}_A})^*(\mathcal{Q}_A) \otimes f^*_{A'\times\operatorname{Pic}_A} e^*_{\operatorname{Pic}_A}(\mathscr{N}) \end{aligned}$$

to see that (3.4) is valid, as claimed. □

3.3 Albanese Schemes

Let S be a noetherian ground scheme. Given an S-scheme X, the construction of the Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^0_{X/S}$ relies on the existence of the Picard scheme $\operatorname{Pic}_{X/S}$ and its maximal abelian subscheme. The latter and its compatibility with base change is the content of the first part in this section. Afterwards, the Albanese morphism can be constructed. During the remainder of the section, properties of the Albanese will be inferred.

Definition 3.6. Let G be a commutative group scheme locally of finite type over S. A closed subgroup scheme $M \subset G$ is called *maximal abelian subscheme* if M is an abelian S-scheme and every homomorphism $A \to G$ from an abelian S-scheme A has a factorization $A \to M \hookrightarrow G$.

Example 3.7. If G^0 is representable and moreover an abelian S-scheme, then $M = G^0$ is the maximal abelian subscheme of G. It is beyond that compatible with arbitrary noetherian base change $S' \to S$.

In fact, let $A \to G$ be a homomorphism from an abelian S-scheme A. For every point $s \in S$, the induced map $A_s \to G_s$ factorizes through $(G_s)^0$, because A_s is connected. So by definition of G^0, there exists a factorization $A \to G^0 \to G$. The base change compatibility of M holds, as it is valid for G^0 and an abelian S-scheme becomes an abelian S'-scheme.

If a maximal abelian subscheme exists, then it is unique by its defining property. In particular, when the base $S = \mathrm{Spec}(k)$ is the spectrum of a field, its existence and base change compatibility is valid in general. Prior to the verification of this technical statement, consider the following preparatory lemma.

Lemma 3.8. *Assume that G^0 is quasi-projective over S, and that $G^0_{\mathrm{red}} \subset G^0$ is a subgroup scheme, which is in addition an abelian S-scheme. Then $M = G^0_{\mathrm{red}}$ is the maximal abelian subscheme, which is compatible with arbitrary noetherian base change in S.*

Proof. Let S' be noetherian, $S' \to S$ a morphism, A' an abelian S'-scheme and $A' \to G_{S'}$ a homomorphism. The latter has a factorization $A' \to G^0_{S'} \hookrightarrow G_{S'}$, as A' has connected fibers. The fppf quotient G^0/G^0_{red} is representable by a scheme, and its formation is compatible with base change. This follows from [35], Exposé V, Théorème 7.1, and also [103], Partie 5, Théorème 1. Here, the quasi-projectivity of G^0 and the properness of G^0_{red} are used. Write $Q = G^0/G^0_{\mathrm{red}}$ and consider the composition

$$q\colon A' \longrightarrow G^0_{S'} \longrightarrow Q_{S'}.$$

Assume for the moment that q is the zero homomorphism, which will be verified in the subsequent paragraph. Then, since $0 \to (G^0_{\mathrm{red}})_{S'} \to G^0_{S'} \to Q_{S'} \to 0$ is an exact sequence of commutative group schemes, the morphism $A' \to G^0_{S'}$ has a factorization through $(G^0_{\mathrm{red}})_{S'}$, as sought.

To deduce that q is the zero homomorphism, fix a point $x \in S'$ and base change along $\mathrm{Spec}(\kappa(x)) \to S'$. This yields

$$q_x\colon A'_x \longrightarrow G^0_x \longrightarrow Q_x.$$

The fiber A'_x is an abelian variety and so it is reduced. Hence $A'_x \to G^0_x$ has a factorization through $(G^0_x)_{\mathrm{red}}$. Since $(G^0_{\mathrm{red}})_x \subset G^0_x$ is a closed subscheme, the reduction $(G^0_x)_{\mathrm{red}}$ is a closed subscheme of $(G^0_{\mathrm{red}})_x$. So $A'_x \to G^0_x$ also factorizes through $(G^0_{\mathrm{red}})_x$, which shows that q_x is the zero homomorphism. Now assume without loss of generality that S' is connected, and apply [96], Corollary 6.2, to the two morphisms q and $r := e_{Q_{S'}} \circ f_{A'}$ from A' to $Q_{S'}$. Since $q_x = r_x$, the conclusion of loc. cit. yields the existence of a section $s\colon S' \to Q_{S'}$ such that $q = (s \circ f_{A'}) \cdot r = s \circ f_{A'}$. This means that q is constant, and thus necessarily $q = r$ is the zero homomorphism. $\square$

Proposition 3.9. *Let $S = \mathrm{Spec}(k)$. Then the maximal abelian subscheme $M \subset G$ exists.*

Proof. Replace G by G^0 to assume without loss of generality that G is connected. According to [78], Lemma 5.1, the connected group scheme G locally of finite type over the field k is geometrically irreducible separated and of finite type. The schematic image $H \subset G$ of a homomorphism $A \to G$ from an abelian variety A is again an abelian variety:

Since $S = \mathrm{Spec}(k)$, the image H is a subgroup scheme of G by [35], Exposé $\mathrm{VI_B}$, Partie 1, page 319. As A is proper over k, the map $A \to H$ is a proper surjection. This implicates that H inherits irreducibility and universally closedness from A. Hence H is proper over k. Because A is geometrically reduced, the schematic image H commutes with field extensions. Thus H is also geometrically reduced, and thus an abelian variety.

In conclusion, it suffices to verify the existence of an abelian subvariety $M \subset G$ which contains all other abelian subvarieties of G. Choose an abelian subvariety $A \subset G$ of maximal dimension. To show that $M = A$ is the maximal abelian subscheme, assume by contradiction that there exists another abelian variety $A' \subset G$ such that $A' \not\subset A$.

As the base is a field, the fppf quotient G/A is representable by a scheme according to [35], Exposé $\mathrm{VI_A}$, Théorème 3.2. Furthermore, as A is smooth and proper, the morphism $G \to G/A$ is smooth and proper. This is a consequence of the two facts that $G \to G/A$ is faithfully flat and that the base change along itself is $G \times A$ due to loc. cit.

Consider the homomorphism $\varphi\colon A' \hookrightarrow G \to G/A$. The quotient $Q := A'/\ker(\varphi)$ is again an abelian variety: Since $A' \to Q$ is faithfully flat and proper, and Q is of finite type over k, the reasoning used at the beginning of this proof can be applied again.

The induced homomorphism $Q \to G/A$ is a monomorphism, and as G/A is separated, it is proper. Thus it has to be a closed embedding according to [58], Corollaire 18.12.6. So $Q \subset G/A$ is a closed subgroup scheme. As before, the quotient $(G/A)/Q$ is representable by a scheme and $G/A \to (G/A)/Q$ is smooth and proper.

Consider the composition $G \to G/A \to (G/A)/Q$ and denote its kernel by C, so $C \subset G$ is a closed subgroup scheme. It is smooth and proper, since it is the fiber of morphism satisfying the two properties. By construction, both A and A' are contained in C, thus also in C^0. Altogether, C^0 is smooth, proper and geometrically irreducible. So C^0 is an abelian variety that contains both A and A', a contradiction to the maximality of A. □

Proposition 3.10. *Let $S = \mathrm{Spec}(k)$. The maximal abelian subscheme $M \subset G$ is compatible with arbitrary noetherian base change in S.*

Proof. Without loss of generality let G be connected. Let $S' \to S$ be a morphism from a noetherian scheme S', let A' be an abelian S'-scheme and $A' \to G_{S'}$ a homomorphism. The fppf quotient G/M is representable by a scheme and compatible with base change, so $(G/M)_{S'} = G_{S'}/M_{S'}$. If the composition

$$A' \longrightarrow G_{S'} \longrightarrow (G/M)_{S'}$$

is zero, then $A' \to G_{S'}$ factors through $M_{S'}$. This is the case if the maximal abelian subscheme of $(G/M)_{S'}$ is trivial. Thus it is sufficient to prove the following: If $M = 0$ and $M' \subset G_{S'}$ is the maximal abelian subscheme, then $M' = 0$.

The first step is to show this statement in the special case that k is separably closed and $S' = \mathrm{Spec}(E)$ for an arbitrary field extension $k \subset E$. Let $m \geq 1$ such that $p \nmid m$. Consider the m-torsion subgroup $G[m] \subset G$, which is the kernel of the multiplication by m, denoted by $[m] \colon G \to G$. By definition, $G[m]$ is compatible with base change, and hence $G[m]_E = G_E[m]$. The scheme $G[m]$ is finite and étale over $\mathrm{Spec}(k)$.

To see that it is finite, consider the Lie algebra $\mathrm{Lie}(G) = \ker(G(k[\varepsilon]) \to G(k))$ of G. The induced map $\mathrm{Lie}([m]) \colon \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is the multiplication by m due to [35], Exposé II, Partie 4, page 60, and its kernel is $\mathrm{Lie}(G[m])$. Since $p \nmid m$, the multiplication by m is bijective, thus $\mathrm{Lie}(G[m]) = 0$. As a vector space, $\mathrm{Lie}(G[m]) = (\mathfrak{m}_e/\mathfrak{m}_e^2)^\vee$ is the tangent space at the neutral element. Hence $\mathfrak{m}_e = 0$ by Nakayama's lemma. So $\dim(G[m]) = 0$ and $G[m]$ is finite over $\mathrm{Spec}(k)$.

Write $H = G[m]$ and verify that H is étale over $\mathrm{Spec}(k)$. The cokernel of $H^0 \to H$ is étale and $\mathrm{ord}(H^0) = c^r$ is power of the characteristic exponent c of k by [125], Section 3.7, (I) and (II). Hence on T-valued points, the elements of $H^0(T)$ have order dividing c^r, whereas the elements of $H(T)$ have order dividing m. As a consequence, $H^0 = 0$ and H is étale.

Now since k is separably closed, the étale group scheme $H = G[m]$ is constant. Thus also $G[m]_E = G_E[m]$ and $M'[m] \subset G_E[m]$ have to be constant. Therefore the subgroup scheme $A_m \subset G[m]$ corresponding to the same abstract group as $M'[m]$ satisfies $(A_m)_E = M'[m]$. Consider the reduced subscheme $A := \overline{\bigcup A_m}$ of G, where the union is taken over all $m > 1$ not divisible by p. It satisfies $M' = A_E$, which can be seen as follows: First of all, verify in the subsequent paragraph that $M' = \overline{\bigcup M'[m]}$ holds.

It is sufficient to show that the equality is already valid for the union over all $m = \ell^s$ for a prime $\ell \neq p$ and $s \geq 1$. Set $U = \bigcup M'[\ell^s]$. During this paragraph, assume without loss of generality that k is algebraically closed, because $\overline{U} \subsetneq M'$ implies $\overline{U_{\overline{k}}} \subsetneq M'_{\overline{k}}$. Note that $U \subset M'$ is a subgroup scheme, since the product of closed points in U lies again in U. As $U^0 \subset M'$ is an abelian subvariety, now show that their dimensions coincide. By definition of U, the order of $U[\ell^s](k)$ is $\ell^{2s\dim(M')}$. On the other hand, if U consists of d connected components, then this order is at most $d\ell^{2s\dim(U)}$. The limit for s towards infinity eventually yields $\dim(U) = \dim(M')$, so indeed $\overline{U} = M'$ holds, as claimed.

Now write $N = \bigcup A_m$, so the original claim $M' = A_E$ means that $\overline{N_E} = \overline{N}_E$. As $N_E \subset \overline{N}_E$, the inclusion $\overline{N_E} \subset \overline{N}_E$ follows immediately. For the other inclusion, observe that the schematic image of $\overline{N_E} \hookrightarrow \overline{N}_E \to \overline{N}$ contains N, and hence it has to equal $\overline{N}$. Since $\overline{N_E} = M'$ is irreducible, also $\overline{N}$ has to be irreducible. As k is separably closed, $\overline{N}$ is even geometrically irreducible, and thus $\overline{N}_E$ remains to be irreducible. The equality $\dim(\overline{N_E}) = \dim(\overline{N}) = \dim(\overline{N}_E)$ now shows that the inclusion $\overline{N_E} \subset \overline{N}_E$ of integral schemes has to be an equality, that is, $M' = A_E$.

Moreover, the closed subscheme $A \subset G$ is a subgroup scheme. Taking this for granted for the moment, then as A is irreducible, geometrically reduced and proper by faithfully flat descent, it is an abelian variety. To obtain the group structure on A, consider the restriction $A \times A \hookrightarrow G \times G \to G$ of the multiplication on G to the subscheme A. This morphism is proper, since G is separated. Consequently, as $A \times A$ is reduced, its schematic image $I \subset G$ is its set-theoretic image with the reduced subscheme structure. Base changing to E shows that $I_E = M' = A_E$, so already $I = A$ and the multiplication $G \times G \to G$ restricts to $A \times A \to A$. The same arguments show that the inverse $G \to G$ restricts to $A \to A$, and by definition of A, the identity section of G factors through A. Hence $A \subset G$ is actually a subgroup scheme, so it is an abelian variety. But the maximal abelian subscheme $M \subset G$ is trivial, so consequently $A = 0$. This implies $M' = A_E = 0$, which proves the special case $S' = \operatorname{Spec}(E)$ and k separably closed.

In the second step, let k be an arbitrary field and $S' = \operatorname{Spec}(E)$ for an arbitrary field extension $k \subset E$. Clearly, the field E can be replaced by its separable closure, so assume $k^{\mathrm{sep}} \subset E$. If it was known that the maximal abelian subscheme of $G_{k^{\mathrm{sep}}}$ is trivial, then the first step would imply $M' = 0$. Thus it is sufficient to prove the case that $E = k^{\mathrm{sep}}$. Consider $k^{\mathrm{sep}} = \bigcup L$ as the union of all finite separable field extensions $k \subset L$. There exists some L such that $M' \subset G_{k^{\mathrm{sep}}}$ descends to a subscheme $A \subset G_L$ by [57], Théorème 8.8.2. As before, the scheme A is an abelian subvariety. Consequently, the situation can be further reduced to the case that $k \subset E$ is a finite separable extension, and then by possibly enlarging E, to the case that $k \subset E$ is finite Galois. Denote $G = \operatorname{Gal}(E/k)$. For every $\sigma \in G$, the induced automorphism $\Phi_\sigma \colon G_E \to G_E$ maps $M' \subset G_E$ to an abelian variety $\Phi_\sigma(M') \subset G_E$ of the same dimension as M'. Hence by construction of the maximal abelian subscheme, necessarily $\Phi_\sigma(M') = M'$. This means that the induced action of G on M' is compatible with the action of G on $\operatorname{Spec}(E)$. Since M' is projective, Galois descent is effective. So there exists an abelian variety $A \subset G$ such that $A_E = M'$. But $A = 0$ by assumption, and hence $M' = 0$. In conclusion, the maximal abelian subscheme $M \subset G$ is compatible with arbitrary field extensions.

In the third and final step, it remains to show that M is compatible with arbitrary noetherian base change $S' \to \operatorname{Spec}(k)$. Write $Q = G/M$ for the fppf quotient and consider the composition

$$q \colon M' \longrightarrow G_{S'} \longrightarrow Q_{S'}.$$

As inferred in the proof of Lemma 3.8, it is sufficient to show that this is the zero homomorphism. Fix a point $x \in S'$ and base change along $\operatorname{Spec}(\kappa(x)) \to S'$. This yields

$$q_x \colon M'_x \longrightarrow G_x \longrightarrow Q_x.$$

The fiber M'_x is an abelian variety. Thus $M'_x \to G_x$ has a factorization through M_x according to the result of the second step: M is compatible with base change along arbitrary field extensions. Since $M = 0$ by assumption, this shows that q_x is the zero homomorphism. Now again [96], Corollary 6.2, implies that q is the zero homomorphism. Eventually, it follows that $M' \subset M_{S'} = 0$ is trivial, which completes the entire proof. □

Let $f\colon X \to S$ be a separated morphism of finite type between noetherian schemes. For the following reasoning, it has to be ensured that a certain set of assumptions is fulfilled. Start by defining the following properties:

(A1) f has a section $e\colon S \to X$.

(A2) f has connected fibers.

(A2*) f has geometrically connected fibers.

(A2**) f has geometrically integral fibers.

(A3) $\mathrm{Pic}_{X/S}$ exists representing the relative Picard functor $\mathrm{Pic}_{(X/S)}$.

(A4) There exists a projective maximal abelian subscheme $M \subset \mathrm{Pic}_{X/S}$.

(A4*) There exists a projective maximal abelian subscheme $M \subset \mathrm{Pic}_{X/S}$, which is compatible with arbitrary noetherian base change in S.

The morphism f is defined to satisfy property (A) if properties (A1), (A2), (A3), (A4) are true. Similarly, f is said to satisfy property (A*) if additionally (A2*) and (A4*) are valid, and f satisfies (A**) if (A*) and (A2**) hold.

Property (A) ensures the existence of the Albanese morphism, whereas (A*) will be necessary to guarantee its compatibility with base change. If f is proper and fulfills (A**), then the Albanese morphism is compatible with products. Apart from those rather technical properties, define another pair of more restrictive—but also more tangible—ones:

(AA) $S = \mathrm{Spec}(k)$ and X is proper with $h^0(\mathcal{O}_X) = 1$ and $X(k) \neq \varnothing$.

(AR) f is smooth and projective with geometrically integral fibers, f has a section, all $\mathrm{Pic}^0_{X_s/\kappa(s)}$ for $s \in S$ are smooth and proper of the same dimension.

Here (AA) abbreviates "Albanese Absolute" and (AR) stands for "Albanese Relative". The following implications hold: In the relative setting, (AR) implies (A**). In the absolute one, (AA) yields (A*).

To see this, use Lemma A.11 to ensure that all necessary assumptions in Theorem A.26 are satisfied to show that (AR) implies (A3) and that $\mathrm{Pic}^0_{X/S}$ is a projective abelian S-scheme. Hence also (A4*) holds as seen in Example 3.7. In the absolute situation (AA), a direct consequence is (A2*), and it results from Theorem A.30 and Proposition 3.10 that (A3), (A4*) are also valid.

For the next definition, suppose that f satisfies (A1) and consider X as a pointed S-scheme via the section e.

Definition 3.11. A pointed morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^0_{X/S}$ to a projective abelian S-scheme $\mathrm{Alb}^0_{X/S}$ is an *Albanese morphism* of X if it is universal among all pointed S-morphisms $X \to A$ to a projective abelian S-scheme A. This means that each such $X \to A$ has a factorization through alb and a unique homomorphism of abelian S-schemes $\mathrm{Alb}^0_{X/S} \to A$.

Here, abelian S-schemes are considered as pointed via their identity section. The universal property defining an Albanese morphism immediately implies that it is unique up to a unique isomorphism. Thus, once its existence is proven, the expression "the" Albanese morphism is used. The projective abelian S-scheme $\mathrm{Alb}^0_{X/S}$ is called the *Albanese scheme* of X. In the case that $S = \mathrm{Spec}(k)$, also the term *Albanese variety* is common. If the base S is fixed, abbreviate $\mathrm{Alb}^0_X = \mathrm{Alb}^0_{X/S}$. Prior to the central existence result of the Albanese scheme in this section, consider the following example:

Example 3.12 (Jacobian of a Smooth Curve)**.** Let C be a smooth, proper curve over an arbitrary ground field k with $h^0(\mathcal{O}_C) = 1$, and let $x \in C(k)$. Define $\Delta \subset C \times C$ to be the diagonal and $\{x\} \times C \subset C \times C$ to be the fiber of $x \in C$ under the first projection. Consider the invertible sheaf $\mathscr{L} = \mathcal{O}_{C\times C}(\Delta - \{x\} \times C)$ on $C \times C$. By Theorem A.30, the Picard scheme Pic_C is smooth and represents the relative Picard functor. The sheaf $\mathscr{L}$ defines a morphism $C \to \mathrm{Pic}_C$ with the property that a closed point $y \in C$ maps to the point of Pic_C corresponding to the invertible sheaf $(\mathrm{id}_C \times y)^*(\mathscr{L}) = \mathcal{O}_C(y-x)$. Since C is connected and the image of $x \in C$ is the neutral element of Pic_C, there is a factorization $j\colon C \to \mathrm{Pic}^0_C$. Note that $\mathrm{Pic}^0_C = \mathrm{Pic}^\tau_C$ by Example A.31, so the abelian variety $J = \mathrm{Pic}^0_C$ is actually the fine moduli space of invertible sheaves of degree zero on C. Hence $j\colon C \to J$ is the classical morphism to the *Jacobian variety* J of C. This is the Albanese morphism of C, which is a closed embedding, see [91], Propositions 6.1 and 2.3. Observe that here, the maximal abelian subscheme of $M \subset \mathrm{Pic}_C$ is $M = J$. In general, the Albanese scheme defined in the following will be its dual $\mathrm{Alb}^0_C = M^\vee$. Thus in this case, the well-known result $J^\vee \xrightarrow{\simeq} J$ follows. The isomorphism can be taken to be $j^{\vee\vee}$ by Proposition 3.19 below. In general, $M^\vee$ and M can be non-isomorphic, see for instance [67], Theorem 1.1.

As in the previous sections, fix the base scheme S from now on and suppress the dependence on S notationally to improve readability. In particular, all products without index are given over S and all Picard schemes are defined relative to S unless stated otherwise.

Theorem 3.13. *Assume $f\colon X \to S$ satisfies (A). Then there exists an Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^0_X$, where $\mathrm{Alb}^0_X = M^\vee$ and $M \subset \mathrm{Pic}_X$ is the maximal abelian subscheme.*

Proof. The following arguments are based on [78], Remark 5.25. Let A be a projective abelian S-scheme. Denote its dual by $B = A^\vee$. There is a natural correspondence between homomorphisms $B \to M$ of abelian S-schemes and pointed morphisms $X \to A$ as follows:

At first, let $B \to M$ be a homomorphism of abelian S-schemes. Its composition with the inclusion $B \to M \hookrightarrow \mathrm{Pic}_X$ corresponds to a class $[\mathscr{L}] \in \mathrm{Pic}(X \times B)/\mathrm{Pic}(B)$ for some $\mathscr{L} \in \mathrm{Pic}(X \times B)$ by assumption (A3). The fact that the morphism is a homomorphism translates to $[\iota_X^*(\mathscr{L})] = [\mathcal{O}_X]$ in $\mathrm{Pic}(X)/\mathrm{Pic}(S)$. Consider $[\mathscr{L}] \in \mathrm{Pic}(X \times B)/\mathrm{Pic}(X)$, this time modulo $\mathrm{Pic}(X)$ instead. There is a corresponding morphism $X \to \mathrm{Pic}_B$. This would be a pointed morphism if $[\iota_B^*(\mathscr{L})] = [\mathcal{O}_B]$ in $\mathrm{Pic}(B)/\mathrm{Pic}(S)$. In order to achieve this, choose the normalized representative $\mathscr{L}_{(B)}$ instead of $\mathscr{L}$ at the beginning. Now the obtained pointed morphism $X \to \mathrm{Pic}_B$ factors through $X \to \mathrm{Pic}^0_B = A$ by (A2). Here the identification $\mathrm{Pic}^0_B = B^\vee = A^{\vee\vee} = A$ is the inverse to the natural biduality isomorphism.

On the other hand, let $X \to A$ be a pointed morphism. Then the construction above can be reversed analogously: Consider $X \to A = A^{\vee\vee} \hookrightarrow \mathrm{Pic}_B$ which corresponds to a class $[\mathscr{N}] \in \mathrm{Pic}(X \times B)/\mathrm{Pic}(X)$ and satisfies $[\iota_B^*(\mathscr{N})] = [\mathcal{O}_B]$ in $\mathrm{Pic}(B)/\mathrm{Pic}(S)$. If necessary, choose $\mathscr{N}_{(X)}$ instead of $\mathscr{N}$ to achieve that $[\iota_X^*(\mathscr{N})] = [\mathcal{O}_X]$ holds in $\mathrm{Pic}(X)/\mathrm{Pic}(S)$. As a consequence, the class $[\mathscr{N}] \in \mathrm{Pic}(X \times B)/\mathrm{Pic}(B)$ corresponds to a homomorphism $B \to \mathrm{Pic}_X$. By definition of the maximal abelian subscheme $M \subset \mathrm{Pic}_X$, which exists by (A4), it factors through a homomorphism $B \to M$.

The two constructions are inverse to each other, because if both constructions are executed consecutively, it is possible to choose $\mathscr{L} = \mathscr{N}$.

Now let A' be another projective abelian S-scheme and set $B' = A'^{\vee}$ to be its dual. Then the constructions above are compatible with homomorphisms of abelian S-schemes in the following sense: If $B \to M$ has a factorization $B \xrightarrow{h} B' \to M$ for some homomorphism h, then the corresponding morphism $X \to A$ has a factorization $X \to A' \xrightarrow{h^\vee} A$, and vice versa. Here $h^\vee$ is the natural morphism $B'^{\vee} \to B^{\vee}$ induced by pullback on all T-valued points, using the natural identifications of A' and A with their bidual.

Indeed, let $B \to M$ have a factorization $B \xrightarrow{h} B' \to M \hookrightarrow \mathrm{Pic}_X$. Then $B' \to \mathrm{Pic}_X$ corresponds to some $[\mathscr{L}'] \in \mathrm{Pic}(X \times B')/\mathrm{Pic}(B')$ and thus $B \to \mathrm{Pic}_X$ corresponds to $[(\mathrm{id}_X \times h)^*(\mathscr{L}')] \in \mathrm{Pic}(X \times B)/\mathrm{Pic}(B)$. Normalize $\mathscr{L}'$ if necessary and consider its class $[\mathscr{L}'] \in \mathrm{Pic}(X \times B')/\mathrm{Pic}(X)$ to obtain a morphism $X \to \mathrm{Pic}_{B'}$ as explained above. Its composition with $h^\vee \colon \mathrm{Pic}_{B'} \to \mathrm{Pic}_B$ corresponds by definition of the dual morphism to the class $[(\mathrm{id}_X \times h)^*(\mathscr{L}')] \in \mathrm{Pic}(X \times B)/\mathrm{Pic}(X)$. Now the commutative diagram

$$
\begin{array}{ccccc}
 & & \mathrm{Pic}_{B'} & \xrightarrow{h^\vee} & \mathrm{Pic}_B \\
 & & \uparrow & & \uparrow \\
 & & A'^{\vee\vee} & & A^{\vee\vee} \\
 & & = \uparrow & & \uparrow = \\
X & \longrightarrow & A' & \xrightarrow[h^\vee]{} & A
\end{array}
$$

shows that the restriction of this morphism $X \to \mathrm{Pic}_B$ to $X \to A$ has a factorization through $h^\vee$, as claimed. The other implication, starting with $X \to A' \xrightarrow{j} A$ to obtain $B \xrightarrow{j^\vee} B' \to M$ works analogously.

Once this is established, define $\mathrm{Alb}^0_X = M^\vee$. First, consider the special case $B = M^{\vee\vee}$ and the natural identification $B \to M$. The construction yields a pointed morphism $\mathrm{alb} \colon X \to \mathrm{Alb}^0_X$. Return to a general A, then for any pointed morphism $X \to A$, the corresponding homomorphism $B \to M$ can be extended artificially to

$$
B \longrightarrow M \xrightarrow{=} M^{\vee\vee} \xrightarrow{=} M. \qquad (h \colon B \to M^{\vee\vee})
$$

This yields the factorization $X \xrightarrow{\mathrm{alb}} \mathrm{Alb}^0_X \xrightarrow{h^\vee} A$. Its uniqueness is seen by applying the construction once again, since the Albanese morphism corresponds to an isomorphism. □

Remark 3.14. The extension $X \xrightarrow{\text{alb}} \text{Alb}^0_X \hookrightarrow \text{Pic}_M$ of the Albanese morphism is given by the class $[\mathcal{Q}|_{X\times M}] \in \text{Pic}(X \times M)/\,\text{Pic}(X)$ for a normalized Poincaré sheaf $\mathcal{Q} = \mathcal{P}_{(\text{Pic}_X)}$, where $\mathcal{P} \in \text{Pic}(X \times \text{Pic}_X)$ is a Poincaré sheaf corresponding to the identity $\text{Pic}_X \to \text{Pic}_X$.

Proposition 3.15. *Assume that f satisfies (A). Let $S' \to S$ be a morphism of noetherian schemes such that the base change $f_{S'}\colon X_{S'} \to S'$ continues to satisfy (A). Then the Albanese morphism* alb: $X \to \text{Alb}^0_X$ *is compatible with base change along $S' \to S$. Hence if f satisfies (A*), then the Albanese morphism is compatible with arbitrary noetherian base change in S.*

Proof. It has to be verified that the normalized Poincaré sheaf $\mathcal{P}_{(\text{Pic}_X)} \in \text{Pic}(X \times \text{Pic}_X)$ is compatible with base change by Remark 3.14. This follows from Proposition 3.2. Furthermore, property (A*) is preserved under base change. □

Remark 3.16. In [22], Example 4.2.7, it is shown that the Albanese morphism of a smooth, connected, commutative group scheme G of finite type over an imperfect field can be incompatible with base change. Thus assumption (A3), the existence of the Picard scheme representing the relative Picard functor, is necessary to ensure compatibility with base change. Briefly outline the construction:

Start with an imperfect field k and a finite, purely inseparable field extension $k \subset k'$ of degree $d \geq 2$. Let A' be an abelian variety over k' of dimension $g \geq 1$. Its *Weil restriction* along $k \subset k'$ is the k-scheme G defined by the universal property $G(T) = A'(T_{k'})$ for all k-schemes T. The scheme G in fact exists, and it is a smooth, connected, commutative group scheme of finite type over k. Furthermore, G is not proper over k and of dimension $d{\cdot}g$. The identity $G \to G$ corresponds to a morphism $G_{k'} \to A'$, which is the Albanese morphism of $G_{k'}$ and has non-trivial kernel $H' \subset G_{k'}$. Also, the Albanese morphism $\text{alb}_G\colon G \to \text{Alb}^0_G$ exists, but it is not compatible with base change:

Assume by contradiction that this is true. Then its kernel $H \subset G$ has to be non-trivial, too. But the inclusion $H \subset G$ corresponds to a some $H' \to A'$. By functoriality in T of the universal property defining G, the latter factorizes as $H' \to G_{k'} \to A'$, and hence it is constant by definition of H'. It follows that $H = 0$, but then G is proper, a contradiction.

Proposition 3.17. *Assume that f satisfies (A). Let $\varphi\colon X \to A$ be a morphism of S-schemes, not necessarily pointed, to a projective abelian S-scheme A. Then φ has a unique factorization through the Albanese morphism* alb: $X \to \text{Alb}^0_X$*. It is given by a homomorphism h:* $\text{Alb}^0_X \to A$ *of abelian S-schemes and a translation $\tau\colon A \to A$. Specifically, τ is the translation by $\varphi \circ e_X$.*

Proof. Define $t_A = \varphi \circ e_X$ and $\tau\colon A \to A$ to be the translation by t_A, so $\tau^{-1}\colon A \to A$ is the translation by the inverse of t_A. Then $\tau^{-1} \circ \varphi\colon X \to A$ is a pointed morphism, and thus it has a unique factorization

$$\begin{CD} X @>{\mathrm{alb}}>> \mathrm{Alb}^0_X \\ @V{\varphi}VV @VV{h}V \\ A @. \underset{\tau}{\overset{\tau^{-1}}{\rightleftarrows}} @. A \end{CD}$$

through the Albanese morphism. This yields the factorization $\varphi = \tau \circ h \circ \mathrm{alb}$. For its uniqueness, suppose that there exists some $g\colon \mathrm{Alb}^0_X \to A$ such that $\varphi = g \circ \mathrm{alb}$. Then the equality $\tau^{-1} \circ \varphi = \tau^{-1} \circ g \circ \mathrm{alb}$ of pointed morphisms follows. This results in $\tau^{-1} \circ g = h$, which means that in fact $g = \tau \circ h$ is unique. □

Proposition 3.18. *Assume that f satisfies (A). Let $s_X\colon S \to X$ be a further section of X. Considered as a pointed scheme via s_X, the Albanese morphism of X is given by* $\sigma^{-1} \circ \mathrm{alb}$, *where $\sigma\colon$* $\mathrm{Alb}^0_X \to \mathrm{Alb}^0_X$ *is the translation by* $\mathrm{alb} \circ s_X$.

Proof. Let $\varphi\colon X \to A$ be a morphism of S-schemes to a projective abelian S-scheme, which maps s_X to the identity e_A. Consider the diagram

$$\begin{CD} X @>{\mathrm{alb}}>> \mathrm{Alb}^0_X @>{\sigma^{-1}}>> \mathrm{Alb}^0_X \\ @V{\varphi}VV @VV{h}V @VV{h}V \\ A @>>{\tau^{-1}}> A @>>{\tau}> A, \end{CD}$$

where the square on the left-hand side is the same as in the preceding proof, hence commutative. Since both $h \circ \sigma^{-1}$ and $\tau \circ h$ map the point $\mathrm{alb} \circ s_X \in \mathrm{Alb}^0_X(S)$ to e_A, also the square on the right-hand side must be commutative. In conclusion, there is a factorization $\varphi = h \circ \sigma^{-1} \circ \mathrm{alb}$, which shows that φ factorizes through $\sigma^{-1} \circ \mathrm{alb}$. As $h \circ \sigma^{-1} \circ \mathrm{alb} = \tau \circ h \circ \mathrm{alb}$ holds, its uniqueness is a consequence of Proposition 3.17. □

Let $\varphi\colon X \to Y$ be a morphism between S-schemes satisfying (A3). Consider the induced homomorphism $\varphi^\vee\colon \mathrm{Pic}_Y \to \mathrm{Pic}_X$ given by pullback of invertible sheaves, which restricts to $\varphi^\vee\colon \mathrm{Pic}^0_Y \to \mathrm{Pic}^0_X$. If X and Y additionally satisfy (A4), then it restricts further to $\varphi^\vee\colon M_Y \to M_X$. It fulfills the relation (3.2) stated in Section 3.2 for the dual of a morphism between projective abelian S-schemes by exactly the same computation. Similarly, Lemma 3.5 can be transferred to this more general situation as follows:

Proposition 3.19. *Let $\varphi\colon X \to Y$ be a pointed morphism between two S-schemes satisfying (A). Then the diagram*

$$\begin{CD} X @>{\varphi}>> Y \\ @V{\mathrm{alb}_X}VV @VV{\mathrm{alb}_Y}V \\ \mathrm{Alb}^0_X @>>{\varphi^{\vee\vee}}> \mathrm{Alb}^0_Y \end{CD}$$

is commutative. Furthermore, $\varphi^{\vee\vee}$ is the only morphism $\mathrm{Alb}^0_X \to \mathrm{Alb}^0_Y$ rendering the diagram commutative.

Proof. By Remark 3.14, the extension $X \xrightarrow{\mathrm{alb}_X} \mathrm{Alb}^0_X \hookrightarrow \mathrm{Pic}_{M_X}$ is given by the class $[\mathcal{Q}_X|_{X\times M_X}] \in \mathrm{Pic}(X \times M_X)/\mathrm{Pic}(X)$ for $\mathcal{Q}_X = \mathcal{P}_{X(\mathrm{Pic}_X)}$, where $\mathcal{P}_X \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ is a Poincaré sheaf corresponding to the identity $\mathrm{Pic}_X \to \mathrm{Pic}_X$. Similarly for Y instead of X. Thus the proof of Lemma 3.5 also works here. The uniqueness in the statement is a direct consequence of the universal property for alb_X applied to $\mathrm{alb}_Y \circ \varphi$. □

Remark 3.20. Consider the special case $\varphi = \mathrm{alb}_X$. The conclusion yields $\mathrm{alb}_X^{\vee\vee} = \mathrm{id}_{\mathrm{Alb}^0_X}$. So already $\mathrm{alb}_X^\vee \colon M_{\mathrm{Alb}^0_X} \to M_X$ is the identity. Specialize further to $S = \mathrm{Spec}(k)$ for an arbitrary field k and X geometrically normal and proper with $h^0(\mathcal{O}_X) = 1$. Then $M_X = (\mathrm{Pic}^0_X)_{\mathrm{red}}$. Thus the pullback $\mathrm{alb}_X^\vee(k) \colon \mathrm{Pic}^0(\mathrm{Alb}^0_X) \to \mathrm{Pic}^0(X)$ is bijective.

Proposition 3.21. *Let $\varphi \colon X \to Y$ be a morphism, not necessarily pointed, between two S-schemes satisfying (A). Then the diagram*

$$\begin{array}{ccccc} X & & \xrightarrow{\quad\varphi\quad} & & Y \\ \downarrow{\scriptstyle \mathrm{alb}_X} & & & & \downarrow{\scriptstyle \mathrm{alb}_Y} \\ \mathrm{Alb}^0_X & \xrightarrow[\varphi^{\vee\vee}]{} & \mathrm{Alb}^0_Y & \xrightarrow[\sigma]{\simeq} & \mathrm{Alb}^0_Y \end{array}$$

is commutative, where σ is the translation by $\mathrm{alb}_Y \circ \varphi \circ e_X$. Furthermore, $\sigma \circ \varphi^{\vee\vee}$ is the unique morphism $\mathrm{Alb}^0_X \to \mathrm{Alb}^0_Y$ which can be inserted into the lower row of the diagram to render it commutative.

Proof. Denote $s_Y = \varphi \circ e_X$. Then the morphism $\sigma^{-1} \circ \mathrm{alb}_Y$ is an Albanese morphism of Y considered as a pointed scheme via s_Y due to Proposition 3.18. Therefore Proposition 3.19 can be applied and it gives the commutativity of the diagram. The uniqueness in the statement is a direct consequence of Proposition 3.17 applied to $\mathrm{alb}_Y \circ \varphi$. □

Proposition 3.22. *Let $f \colon X \to S$ and $g \colon Y \to S$ as well as $f \times g \colon X \times Y \to S$ be proper morphisms satisfying (A**). Then a natural isomorphism $M_{X\times Y} \xrightarrow{\simeq} M_X \times M_Y$ exists.*

Proof. Define $\Phi \colon M_{X\times Y} \to M_X \times M_Y$ first: To the inclusion $\iota_X \colon X \to X \times Y$, consider $\iota_X^\vee \colon \mathrm{Pic}_{X\times Y} \to \mathrm{Pic}_X$ and its restriction $\iota_X^\vee \colon M_{X\times Y} \to M_X$. Similarly, obtain $\iota_Y^\vee \colon M_{X\times Y} \to M_Y$. Denote the morphism into the product by $\Phi \colon M_{X\times Y} \to M_X \times M_Y$.

Now define $\Psi \colon M_X \times M_Y \to M_{X\times Y}$ so that it is the inverse. For any S-scheme T, the group homomorphism $(\mathrm{Pic}_X \times \mathrm{Pic}_Y)(T) \to \mathrm{Pic}_{X\times Y}(T)$ determined by

$$\begin{array}{ccc} \mathrm{Pic}(X_T)/\mathrm{Pic}(T) \times \mathrm{Pic}(Y_T)/\mathrm{Pic}(T) & \longrightarrow & \mathrm{Pic}(X_T \times Y_T)/\mathrm{Pic}(T), \\ ([\mathscr{L}], [\mathscr{N}]) & \longmapsto & [\mathrm{pr}^*_{X_T}(\mathscr{L}) \otimes \mathrm{pr}^*_{Y_T}(\mathscr{N})] \end{array}$$

gives rise to a homomorphism $\mathrm{Pic}_X \times \mathrm{Pic}_Y \to \mathrm{Pic}_{X\times Y}$. It can also be obtained by multiplication of $\mathrm{pr}_X^\vee \colon \mathrm{Pic}_X \to \mathrm{Pic}_{X\times Y}$ and $\mathrm{pr}_Y^\vee \colon \mathrm{Pic}_Y \to \mathrm{Pic}_{X\times Y}$, where the latter are induced by $\mathrm{pr}_X\colon X \times Y \to X$ and $\mathrm{pr}_Y\colon X \times Y \to Y$. Then set $\Psi\colon M_X \times M_Y \to M_{X\times Y}$ as its restriction.

By definition, the composition $\Phi\circ\Psi$ is the identity on $M_X\times M_Y$. To show that $\Psi\circ\Phi$ is the identity on $M_{X\times Y}$, note that the definitions of Φ and Ψ are compatible with base change in S by (A4*). Use this to reduce to the case that $S = \mathrm{Spec}(k)$ for an algebraically closed field k. Actually, once this is established, faithfully flat descent, [56], Proposition 2.7.1, yields the statement for arbitrary fields k. Then [53], Proposition 4.6.7 (ii), shows that $\Psi \circ \Phi$ is an isomorphism for general S. So let $S = \mathrm{Spec}(k)$ for an algebraically closed field k. Then

$$M_{X\times Y} \xrightarrow{\Psi\circ\Phi} M_{X\times Y} \hookrightarrow \mathrm{Pic}_{X\times Y} \tag{3.5}$$

is given by some $[\mathscr{L}] \in \mathrm{Pic}(X \times Y \times M_{X\times Y})/\,\mathrm{Pic}(M_{X\times Y})$. To show that the composition $\Psi \circ \Phi$ is the identity, it is sufficient to verify that $[\mathscr{L}] = [\mathcal{P}|_{X\times Y\times M_{X\times Y}}]$ for the Poincaré sheaf $\mathcal{P} \in \mathrm{Pic}(X \times Y \times \mathrm{Pic}_{X\times Y})$. To do so, use the theorem of the cube, [95], Section 10, Theorem on page 91, and observe that at this point the additional assumption (A2**) enters. Consequently, it is sufficient to ensure that both sheaves coincide after pullback to the three closed subschemes $\{x\}\times Y\times M_{X\times Y}$ and $X\times\{y\}\times M_{X\times Y}$ as well as $X\times Y\times\{z\}$. Here $\{x\} = \mathrm{Spec}(k)$ is the image of $e_X\colon \mathrm{Spec}(k) \to X$ and similarly for $\{y\}$ and $\{z\}$. The pullback of $\mathscr{L}$ to $\{x\} \times Y \times M_{X\times Y}$ yields by (3.5) the morphism

$$M_{X\times Y} \xrightarrow{\Psi\circ\Phi} M_{X\times Y} \xrightarrow{\iota_Y^\vee} M_Y \hookrightarrow \mathrm{Pic}_Y\,.$$

Since the equality $\iota_Y^\vee \circ \Psi \circ \Phi = \mathrm{pr}_{M_Y} \circ \Phi = \iota_Y^\vee = \iota_Y^\vee \circ \mathrm{id}_{M_{X\times Y}}$ holds, the pullbacks of $\mathscr{L}$ and $\mathcal{P}|_{X\times Y\times M_{X\times Y}}$ to $\{x\} \times Y \times M_{X\times Y}$ coincide. By means of symmetry, the same conclusion is true for both pullbacks to $X \times \{y\} \times M_{X\times Y}$. Finally, the pullback of $\mathscr{L}$ to $X \times Y \times \{z\}$ yields

$$\mathrm{Spec}(k) \xrightarrow{e} M_{X\times Y} \xrightarrow{\Psi\circ\Phi} M_{X\times Y} \hookrightarrow \mathrm{Pic}_{X\times Y}\,.$$

Being derived from a homomorphism of group scheme, this is the identity section of $\mathrm{Pic}_{X\times Y}$. Clearly, the same holds for $\mathrm{id}_{M_{X\times Y}}$ instead of $\Psi \circ \Phi$, so eventually, the pullbacks of $\mathscr{L}$ and $\mathcal{P}|_{X\times Y\times M_{X\times Y}}$ to $X \times Y \times \{z\}$ also coincide. This completes the proof. □

Remark 3.23. If instead of (A4), the assumption is imposed that Pic^0_X is a flat and proper S-scheme, then there similarly is an isomorphism $\mathrm{Pic}^0_{X\times Y} \xrightarrow{\simeq} \mathrm{Pic}^0_X \times \mathrm{Pic}^0_Y$. Here, the assumption on Pic^0_X is necessary to apply [53], Proposition 4.6.7 (ii).

Remark 3.24. The analogous statement for $\mathrm{Pic}_{X\times Y}$ instead of $M_{X\times Y}$ or $\mathrm{Pic}^0_{X\times Y}$ is not valid in general. Suppose for instance that $\mathrm{Pic}^0_X(k) = \mathrm{Pic}^0(X)$ is non-zero, and consider $Y = \mathrm{Pic}^0_X$. Then $\mathcal{P}_X|_{X\times \mathrm{Pic}^0_X} \in \mathrm{Pic}(X \times \mathrm{Pic}^0_X)$ is not in the image of $\Psi(k)$, which can be seen as follows: Every invertible sheaf in the image is of the form $\mathrm{pr}_X^*(\mathscr{N}_1)\otimes \mathrm{pr}^*_{\mathrm{Pic}^0_X}(\mathscr{N}_2)$, and thereby its pullback along the morphism $(\mathrm{id}_X \times h)$ for all $h\colon \mathrm{Spec}(k) \to \mathrm{Pic}^0_X$ is isomorphic to $\mathscr{N}_1$. But $\mathcal{P}_X|_{X\times \mathrm{Pic}^0_X} \in \mathrm{Pic}(X \times \mathrm{Pic}_X)$ has by definition of a Poincaré sheaf the property that for every invertible sheaf $\mathscr{L} \in \mathrm{Pic}^0(X)$, there exists some k-valued point $h\colon \mathrm{Spec}(k) \to \mathrm{Pic}^0_X$ such that $\mathscr{L} \simeq (\mathrm{id}_X \times h)^*(\mathcal{P}_X|_{X\times \mathrm{Pic}^0_X})$.

The cokernel of $\Psi(k)$ was computed by Ischebeck [70], Satz 1.7. If $S = \mathrm{Spec}(k)$ for an algebraically closed field k and the k-schemes X, Y are normal and connected, then there is a natural exact sequence $0 \to \mathrm{Pic}(X) \times \mathrm{Pic}(Y) \to \mathrm{Pic}(X \times Y) \to \mathrm{Pic}(K(X) \otimes K(Y))$. Note that if X or Y is rational, then $\mathrm{Pic}(K(X)\otimes K(Y))$ is trivial as mentioned in op. cit., Bemerkungen 1.8.

Proposition 3.25. *Let $f\colon X \to S$ and $g\colon Y \to S$ as well as $f \times g\colon X \times Y \to S$ be proper morphisms satisfying property (A**). Then $\mathrm{alb}_X \times \mathrm{alb}_Y \colon X \times Y \to \mathrm{Alb}^0_X \times \mathrm{Alb}^0_Y$ is the Albanese morphism of $X \times Y$.*

Proof. By Proposition 3.22, there are identifications $\Psi_{X,Y}\colon M_X \times M_Y \xrightarrow{\simeq} M_{X\times Y}$ and $\Phi_{M_X,M_Y}\colon (M_X\times M_Y)^\vee \xrightarrow{\simeq} {M_X}^\vee\times {M_Y}^\vee = \mathrm{Alb}^0_X \times \mathrm{Alb}^0_Y$. Combining the two of them yields the natural isomorphism $\Phi_{M_X,M_Y} \circ \Psi^\vee_{X,Y}\colon\ \mathrm{Alb}^0_{X\times Y} \xrightarrow{\simeq} \mathrm{Alb}^0_X \times \mathrm{Alb}^0_Y$. Its composition with $\mathrm{alb}_{X\times Y}\colon X\times Y \to \mathrm{Alb}^0_{X\times Y}$ is consequently an Albanese morphism. By means of symmetry in X and Y, it is now enough to show that the diagram

$$\begin{array}{ccccccc}
X \times Y & \xrightarrow{\mathrm{alb}_{X\times Y}} & \mathrm{Alb}^0_{X\times Y} & \xrightarrow{\Psi^\vee_{X,Y}} & (M_X \times M_Y)^\vee & \xrightarrow{\Phi_{M_X,M_Y}} & \mathrm{Alb}^0_X \times \mathrm{Alb}^0_Y \\
\downarrow{\scriptstyle \mathrm{pr}_X} & & & & & & \downarrow{\scriptstyle \mathrm{pr}_{\mathrm{Alb}^0_X}} \\
X & & & \xrightarrow{\mathrm{alb}_X} & & & \mathrm{Alb}^0_X
\end{array}$$

is commutative. Observe that $\mathrm{pr}_{\mathrm{Alb}^0_X} \circ \Phi_{M_X,M_Y} = \iota^\vee_{M_X}$ is valid by definition of Φ_{M_X,M_Y}, which simplifies the diagram to

$$\begin{array}{ccccc}
X \times Y & \xrightarrow{\mathrm{alb}_{X\times Y}} & \mathrm{Alb}^0_{X\times Y} & \xrightarrow{\Psi^\vee_{X,Y}} & (M_X \times M_Y)^\vee \\
\downarrow{\scriptstyle \mathrm{pr}_X} & & & & \searrow{\scriptstyle \iota^\vee_{M_X}} \\
X & & \xrightarrow{\mathrm{alb}_X} & & \mathrm{Alb}^0_X\,.
\end{array}$$

Next, the composition $\iota^\vee_{M_X} \circ \Psi^\vee_{X,Y}$ is the dual to $\Psi_{X,Y} \circ \iota_{M_X} = \mathrm{pr}^\vee_X\colon M_X \to M_{X\times Y}$, and thus the diagram reduces further to

$$\begin{array}{ccc}
X \times Y & \xrightarrow{\mathrm{alb}_{X\times Y}} & \mathrm{Alb}^0_{X\times Y} \\
\downarrow{\scriptstyle \mathrm{pr}_X} & & \searrow{\scriptstyle \mathrm{pr}^{\vee\vee}_X} \\
X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb}^0_X\,.
\end{array}$$

The commutativity now is a consequence of Proposition 3.19. $\square$

3.4 Albanese Torsors

Let S be a noetherian scheme. One of the assumptions imposed on $f\colon X \to S$ to admit an Albanese morphism was (A1), the existence of a section. After omitting this requirement, the Albanese morphism continues to exist if "projective abelian S-schemes" are replaced by "principal homogeneous spaces under projective abelian S-schemes". At least if the base S is the spectrum of a field, this holds without restrictions. The first short part of this section is a comment on morphisms between such principal homogeneous spaces. Next, descent theory is covered, which will be utilized subsequently to derive the existence of the Albanese torsor. Applications of the theory form the conclusion of this chapter, which are oriented towards the pullback of numerically trivial invertible sheaves along the Albanese morphism.

Let J be a principal homogeneous space under a projective abelian S-scheme A. A brief introduction to principal homogeneous spaces is given in Section A.2. There is a canonical isomorphism $A^\vee \xrightarrow{\simeq} \operatorname{Pic}^0_J$ by [104], Proposition XIII 1.1 (ii). The following observation is a consequence of this fact:

Let A and B be projective abelian S-schemes, let J be a principal homogeneous space under A and let L be a principal homogeneous space under B. A morphism of S-schemes $\varphi\colon J \to L$ is *equivariant with respect to a homomorphism* $h\colon A \to B$ if the diagram

$$\begin{array}{ccc} J \times A & \xrightarrow{\varphi \times h} & L \times B \\ \downarrow & & \downarrow \\ J & \xrightarrow[\varphi]{} & L \end{array} \tag{3.6}$$

commutes, where the vertical arrows are the group actions.

Lemma 3.26. *Given a morphism of S-schemes φ as above, there exists a unique homomorphism $h\colon A \to B$ such that φ is equivariant with respect to h. Specifically, $h = \varphi^{\vee\vee}$.*

Proof. First of all, show that h is unique. By faithfully flat descent, assume without loss of generality that J has a section e. Let T be an S-scheme and $a \in A(T)$. Consider the T-valued point (e, a) of $J \times A$. The commutativity of diagram (3.6) implies that the equality $\varphi(e) + h(a) = \varphi(e + a)$ holds in $L(T)$. Hence h is uniquely determined by φ.

As stated above, the morphism $\varphi^\vee$ can canonically be identified with $B^\vee \to A^\vee$. Dualizing once again yields a homomorphism $h\colon A \to B$. Since Pic^0 commutes with base change, it is sufficient to show that (3.6) commutes after a faithfully flat base change. Thus assume that $J \simeq A$ and $L \simeq B$ are both trivial as homogeneous spaces. Then, after identifying both abelian schemes with their double dual, Proposition 3.21 shows that $\varphi = \tau \circ h$ for a translation τ. Now it is immediate that the diagram commutes. □

Define the *category of principal homogeneous spaces under projective abelian S-schemes*. An object in this category is a principal homogeneous space J under a projective abelian S-scheme A. A morphism $\varphi\colon J \to L$ in this category is simply a morphism of S-schemes.

By the preceding lemma, there exists a unique homomorphism $h\colon A \to B$ of projective abelian S-schemes such that φ is equivariant with respect to h, where J is a principal homogeneous space under A and L is a principal homogeneous space under B.

Descent Theory. Let $b\colon S' \to S$ be a morphism of schemes and let X' be a S'-scheme. Define $S'' = S' \times S'$ and $S''' = S' \times S' \times S'$. Denote by $\pi_i\colon S'' \to S'$ and $\mathrm{pr}_i\colon S''' \to S'$ the projection to the i-th factor and by $\mathrm{pr}_{ij}\colon S''' \to S''$ the projection to factors i and j for $i < j$. For a morphism $t\colon T' \to S'$ and a morphism $\varphi\colon X' \to Y'$ of S'-schemes, write $t^*(X') = X' \times_{S',t} T'$ for the base change of X' along t as well as $t^*(\varphi)\colon t^*(X') \to t^*(Y')$ for morphism induced by φ. There are canonical identifications

$$\begin{array}{ccccc} \mathrm{pr}_{12}^*\,\pi_1^*(X') & = & \mathrm{pr}_1^*(X') & = & \mathrm{pr}_{13}^*\,\pi_1^*(X'), \\ \mathrm{pr}_{12}^*\,\pi_2^*(X') & = & \mathrm{pr}_2^*(X') & = & \mathrm{pr}_{23}^*\,\pi_1^*(X'), \\ \mathrm{pr}_{13}^*\,\pi_2^*(X') & = & \mathrm{pr}_3^*(X') & = & \mathrm{pr}_{23}^*\,\pi_2^*(X'). \end{array}$$

A *descent datum* on X' for b is an isomorphism of S''-schemes

$$\delta\colon \pi_1^*(X') \longrightarrow \pi_2^*(X')$$

such that the *cocycle condition* $\mathrm{pr}_{23}^*(\delta) \circ \mathrm{pr}_{12}^*(\delta) = \mathrm{pr}_{13}^*(\delta)$ is satisfied. This means that

$$\begin{array}{ccccc} \mathrm{pr}_1^*(X') & \xrightarrow{\mathrm{pr}_{12}^*(\delta)} & \mathrm{pr}_2^*(X') & \xrightarrow{\mathrm{pr}_{23}^*(\delta)} & \mathrm{pr}_3^*(X') \\ & \searrow & \underset{\mathrm{pr}_{13}^*(\delta)}{\longrightarrow} & \nearrow & \end{array}$$

is commutative, using the identifications above. The pair (X', δ) is called a *scheme with descent datum* for b. An open subscheme $U' \subset X'$ is *stable under* δ if δ restricts to an isomorphism $\pi_1^*(U') \to \pi_2^*(U')$, in turn yielding a descent datum for b on U'.

Let (Y', γ) be another scheme with descent datum for b. A *morphism* $(X', \delta) \to (Y', \gamma)$ *of schemes with descent data* for b is a morphism $\varphi\colon X' \to Y'$ of S'-schemes such that

(3.7)
$$\begin{array}{ccc} \pi_1^*(X') & \xrightarrow{\pi_1^*(\varphi)} & \pi_1^*(Y') \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \pi_2^*(X') & \xrightarrow[\pi_2^*(\varphi)]{} & \pi_2^*(Y') \end{array}$$

commutes. This yields the category $(\mathrm{Sch}/S', b)$ of S'-schemes with descent data for b.

In the special case that $X' = b^*(X)$ is the base change of an S-scheme X, the canonical identification $\pi_1^* b^*(X) = \pi_2^* b^*(X)$ yields the *canonical descent datum* δ_{can} on X' for b. Concretely, $\delta_{\mathrm{can}} = (\mathrm{pr}_X, \pi_2 \circ \mathrm{pr}_{S''}, \mathrm{pr}_{S''})$. The canonical descent datum is compatible with morphisms $X \to Y$ of S-schemes and induces a functor $b^*\colon (\mathrm{Sch}/S) \to (\mathrm{Sch}/S', b)$ which maps X to $(b^*(X), \delta_{\mathrm{can}})$. A scheme with descent datum for b is *effective* if it is isomorphic to a scheme with canonical descent datum for b. The central result in descent theory is the following, referring to [59], Exposé VIII, Théorème 5.2, Proposition 7.2 and Corollaire 7.9.

Proposition 3.27. *Let $b\colon S' \to S$ be a faithfully flat, quasicompact morphism of schemes. Then the functor b^* is fully faithful. A scheme with descent datum (X', δ) for b is effective if and only if X' can be covered by quasi-affine open subschemes U' which are stable under δ.*

Consider the case of canonical descent data: Let $X' = b^*(X)$ be the pullback of an S-scheme X. Then for $S_i \subset S$ affine open, $U_i \subset X$ and $V_i \subset S'$ both affine open mapping to S_i, the affine open subset $U_i \times V_i \subset X \times S' = X'$ is stable under δ_{can} by its definition.

Remark 3.28 (Base Change of Descent Data)**.** Let (X', δ) be a descent datum for b and let $z\colon Z \to S$ be a morphism. Then there is an induced descent datum $(X'_{Z'}, \delta_z)$ for the base change b_z of b along z, where $Z' = Z \times_S S'$ and $X'_{Z'} = X' \times_{S'} Z'$. This defines a functor $z^*\colon (\mathrm{Sch}/S', b) \to (\mathrm{Sch}/Z', b_z)$. To obtain the induced descent datum, define the base changes Z' and Z'' of S' and S'' by

(3.8)
$$\begin{array}{ccc} Z'' & \longrightarrow & S'' \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ Z' & \longrightarrow & S' \\ {\scriptstyle b_z}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \underset{z}{\longrightarrow} & S \end{array}$$

for $1 \le i \le 2$. Here in fact $Z'' = Z' \times_Z Z'$ holds, and π_i is the projection to the i-th factor. Now base change δ along $Z'' \to S''$. Different paths in (3.8) yield the natural identification

$$\begin{aligned} \pi_i^*(X') \times_{S''} Z'' &= (X' \times_{S',\pi_i} S'') \times_{S''} Z'' \\ &= X' \times_{S'} Z' \times_{Z',\pi_i} Z' \quad = \quad \pi_i^*(X'_{Z'}), \end{aligned}$$

and $\delta_z\colon \pi_1^*(X'_{Z'}) \to \pi_2^*(X'_{Z'})$ corresponds to $\delta \times \mathrm{id}_{Z''}$ under these identifications. The same reasoning shows that the cocycle condition is satisfied. Furthermore, the functor z^* is compatible with compositions in the sense that $(z_2 \circ z_1)^* = z_1^* \circ z_2^*$.

Remark 3.29 (Products of Descent Data)**.** Let (X', δ) and (Y', γ) be descent data for b. In virtue of the natural identification $\pi_i^*(X') \times_{S''} \pi_i^*(Y') = \pi_i^*(X' \times_{S'} Y')$, the isomorphism $\delta \times \gamma$ is a descent datum on $X' \times_{S'} Y'$ for b. In the case that both descent data are canonical, then $\delta_{\mathrm{can}} \times \gamma_{\mathrm{can}}$ is the canonical descent datum on $b^*(X) \times_{S'} b^*(Y') = b^*(X \times Y)$.

Example 3.30 (Gluing Data)**.** Let S be an arbitrary scheme and $S = \bigcup_{i=1}^n U_i$ an open cover. Define the S-scheme $S' = \coprod_{i=1}^n U_i$ with structure morphism $b\colon S' \to S$ induced by the inclusions. In this case, Proposition 3.27 recovers gluing of morphisms and schemes: A morphism $f\colon X \to Y$ of S-schemes is uniquely determined by a collection of morphisms $f_{U_i}\colon X_{U_i} \to Y_{U_i}$ of U_i-schemes restricting to the same morphisms over the intersections $U_i \cap U_j$. An S-scheme X can be defined by affine U_i-schemes X_i being isomorphic over the intersections $U_i \cap U_j$ such that the isomorphisms $X_i \times_{U_i} U_i \cap U_j \xrightarrow{\simeq} X_j \times_{U_j} U_i \cap U_j$ satisfy the cocycle condition.

Example 3.31 (Galois Descent)**.** For a finite Galois extension $k \subset E$, set $S = \operatorname{Spec}(k)$ and $S' = \operatorname{Spec}(E)$. Then the induced morphism $b\colon S' \to S$ is a principal homogeneous space under $\operatorname{Gal}(E/k)$, see Example A.1. This can be extended as follows: Let G be a finite constant S-group scheme and let S' be a principal homogeneous space under G. Then $b\colon S' \to S$ is called a *Galois morphism*. Let X' be an S'-scheme and $\delta\colon \pi_1^*(X') \to \pi_2^*(X')$ a descent datum for b. This can be reinterpreted as follows:

The isomorphism $S' \times G \xrightarrow{\simeq} S''$, $(t, \sigma) \mapsto (t, t\sigma)$ induces $X' \times G \xrightarrow{\simeq} \pi_1^*(X')$. Composition with $\operatorname{pr}_{X'}$ gives a morphism $\Phi\colon X' \times G \to X'$, which defines a group action of G on X'. Denote by $\Phi_\sigma\colon X' \to X'$ the automorphism induced by $\sigma \in G$. Similarly, write $\xi\colon S' \times G \to S'$ for the G-action and $\xi_\sigma\colon S' \to S'$ for the induced automorphism. Then the action of G on X' is *compatible with the action of* G *on* S', meaning that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Phi_\sigma} & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow[\xi_\sigma]{} & S' \end{array}$$

is commutative. For detailed computations to the above, see [19], Section 6.2, Example B, where it is also shown that conversely, a compatible G-action gives rise to a descent datum.

A morphism $(X', \delta) \to (Y', \gamma)$ of S'-schemes with descent data for b is a G-equivariant morphism $\varphi\colon X' \to Y'$. Indeed, first denote by $\Psi\colon Y' \times G \to Y'$ the group action of G on Y', obtained from the descent datum γ. Now Diagram (3.7) translates—using the identifications $X' \times G \xrightarrow{\simeq} \pi_i^*(X')$ given at the beginning of this example—to

$$\begin{array}{ccc} X' \times G & \xrightarrow{\varphi \times \mathrm{id}_G} & Y' \times G \\ {\scriptstyle (\Phi, \mathrm{id}_G)}\downarrow & & \downarrow{\scriptstyle (\Psi, \mathrm{id}_G)} \\ X' \times G & \xrightarrow[\varphi \times \mathrm{id}_G]{} & Y' \times G. \end{array}$$

The commutativity of this diagram exactly means that φ is G-equivariant.

Example 3.32. In the previous example, in the case that X' is quasi-projective over S', the descent datum δ is always effective by [59], Exposé VIII, Corollaire 7.6. According to this source, the effectiveness of (X', δ) for quasi-projective X' holds more generally along every surjection $b\colon S' \to S$ which is finite locally free. For example, a finite field extension $k \subset E$ induces a finite locally free surjection $\operatorname{Spec}(E) \to \operatorname{Spec}(k)$.

Example 3.33. Now let $b\colon S' \to S$ be a purely inseparable, faithfully flat, quasicompact morphism. For instance, $b\colon \operatorname{Spec}(E) \to \operatorname{Spec}(k)$ is induced by an arbitrary purely inseparable field extension $k \subset E$. According to [59], Exposé VIII, Corollaire 7.5, every scheme with descent datum (X', δ) is effective in this case, too.

Let $f\colon X \to S$ be a separated morphism of finite type between noetherian schemes. Recall that an Albanese morphism exists for f if properties (A1) to (A4) are fulfilled.

Without assumption (A1), the existence of a section of f, a naturally altered definition of the Albanese morphism is the following:

Definition 3.34. A morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^1_{X/S}$ to a principal homogeneous space $\mathrm{Alb}^1_{X/S}$ under a projective abelian S-scheme $\mathrm{Alb}^0_{X/S}$ is an *Albanese morphism* of X if it is universal among all S-morphisms $X \to J$ to a principal homogeneous space J under a projective abelian S-scheme A. This means that each such $X \to J$ has a factorization through alb and a unique S-morphism $\mathrm{Alb}^1_{X/S} \to J$.

As for the usual Albanese morphism, the universal property implies its uniqueness up to a unique isomorphism, and the expression "the" Albanese morphism will be used here, too. The principal homogeneous space $\mathrm{Alb}^1_{X/S}$ is the *Albanese torsor* of the S-scheme X. See Remark A.3 for a comment on the notions "principal homogeneous space" and "torsor", where the latter is used here for entirely practical reasons. If the base S is fixed, abbreviate $\mathrm{Alb}^1_X = \mathrm{Alb}^1_{X/S}$. The subsequent existence theorem is the central result in this chapter.

Theorem 3.35. *Let $f\colon X \to S$ be a separated morphism of finite type between noetherian schemes. Suppose that one of the following two assumptions is satisfied:*

(i) X is proper over $S = \mathrm{Spec}(k)$ for an arbitrary field k, and $h^0(\mathcal{O}_X) = 1$.

(ii) f is smooth and projective with geometrically integral fibers, all $\mathrm{Pic}^0_{X_s/\kappa(s)}$ for $s \in S$ are smooth, proper and of the same dimension. There exists a morphism $S' \to S$ which decomposes as a finite locally free surjection followed by a purely inseparable, faithfully flat morphism of finite type, such that the S'-scheme $X_{S'}$ admits a section.

Then there exists an Albanese morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^1_X$*, where* Alb^1_X *is a principal homogeneous space under* $\mathrm{Alb}^0_X = M^\vee$ *and* $M \subset \mathrm{Pic}_X$ *is the maximal abelian subscheme. The formation of* Alb^1_X *commutes with arbitrary noetherian base change in* S*.*

Remark 3.36. If there exists a section $e\colon S \to X$, the principal homogeneous space Alb^1_X can be trivialized along the section $\mathrm{alb} \circ e$. Then $\mathrm{alb}\colon X \to \mathrm{Alb}^1_X$ is a pointed morphism and satisfies the universal property of the Albanese morphism in the original sense. In particular, there is an isomorphism $\mathrm{Alb}^0_X \simeq \mathrm{Alb}^1_X$ depending on the choice of e. Moreover, note that conditions (i) and (ii) recreate (AA) and (AR).

Proof. Let $X \to J$ be a morphism of S-schemes to a principal homogeneous space J under a projective abelian S-scheme A. Depending on the additional assumption (i) or (ii), choose a suitable faithfully flat morphism $b\colon S' \to S$ of finite type such that the base change $X' := b^*(X) = X_{S'}$ along b has a section $e_{X'} = (\iota, \mathrm{id}_{S'})$. In situation (i), choose a closed point $x \in X$, set $S' = \mathrm{Spec}(\kappa(x))$ and let $b\colon S' \to S$ be the induced finite locally free morphism. In situation (ii), let $b\colon S' \to S$ be the one given by assumption.

Now both situations (i) and (ii) can be unified. The respective assumptions imply the subsequent technical conditions, which follow from Lemma A.11, Theorem A.26, Theorem A.30, Example 3.7 and Proposition 3.10.

- Pic_X exists.
- (A2*): f has geometrically connected fibers.
- (A4*): There exists a projective maximal abelian subscheme $M \subset \mathrm{Pic}_X$, which is compatible with arbitrary noetherian base change in S.
- X' satisfies (A3): $\mathrm{Pic}_{X'/S'}$ represents the relative Picard functor $\mathrm{Pic}_{(X'/S')}$.

Consequently, the S'-scheme X' satisfies property (A*). The section $e_{X'}$ yields a section of $J' := b^*(J)$, and thus J' is the trivial principal homogeneous space, that is, a projective abelian S'-scheme. Therefore the base change $X' \to J'$ has a unique factorization through the Albanese morphism $\mathrm{alb}' \colon X' \to \mathrm{Alb}^0_{X'/S'}$. As stated in Proposition 3.15, the Albanese morphism of X' commutes with noetherian base change.

Step 1: Define a descent datum on $\mathrm{Alb}^0_{X'/S'}$. Denote the canonical descent datum on X' by $\delta \colon \pi_1^*(X') \xrightarrow{\simeq} \pi_2^*(X')$. Observe that the morphism δ is not necessarily pointed, as it maps $e_{\pi_1^*(X')} = (\iota \circ \pi_1, \pi_1, \mathrm{id}_{S''})$ to $(\iota \circ \pi_1, \pi_2, \mathrm{id}_{S''})$, which does in general not coincide with $e_{\pi_2^*(X')} = (\iota \circ \pi_2, \pi_2, \mathrm{id}_{S''})$. Consider the diagram

$$\begin{array}{ccc} \pi_1^*(X') & \xrightarrow{\pi_1^*(\mathrm{alb}')} & \pi_1^*(\mathrm{Alb}^0_{X'/S'}) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \pi_2^*(X') & \xrightarrow[\pi_2^*(\mathrm{alb}')]{} & \pi_2^*(\mathrm{Alb}^0_{X'/S'}), \end{array} \tag{3.9}$$

where α exists by Proposition 3.21. Switch the rows in the diagram and consider δ^{-1} instead of δ to obtain a morphism $\alpha^{-1} \colon \pi_2^*(\mathrm{Alb}^0_{X'/S'}) \to \pi_1^*(\mathrm{Alb}^0_{X'/S'})$. This is in fact the inverse to α, as seen by combining both diagrams, because of the uniqueness in the universal property of the Albanese morphisms $\pi_1^*(\mathrm{alb}')$ and $\pi_2^*(\mathrm{alb}')$, respectively.

A similar approach guarantees that α fulfills the cocycle condition. Indeed, in the following diagram

$$\begin{array}{ccccccc} & \mathrm{pr}_1^*(X') & \longrightarrow & \mathrm{pr}_1^*(\mathrm{Alb}^0_{X'/S'}) & \\ & \downarrow{\scriptstyle \mathrm{pr}_{12}^*(\delta)} & & {\scriptstyle \mathrm{pr}_{12}^*(\alpha)}\downarrow & \\ \mathrm{pr}_{13}^*(\delta) & \mathrm{pr}_2^*(X') & \longrightarrow & \mathrm{pr}_2^*(\mathrm{Alb}^0_{X'/S'}) & \mathrm{pr}_{13}^*(\alpha) \\ & \downarrow{\scriptstyle \mathrm{pr}_{23}^*(\delta)} & & {\scriptstyle \mathrm{pr}_{23}^*(\alpha)}\downarrow & \\ & \mathrm{pr}_3^*(X') & \longrightarrow & \mathrm{pr}_3^*(\mathrm{Alb}^0_{X'/S'}) & , \end{array}$$

the two rectangles in the center and the outer circle commute by definition of α. The cocycle condition for δ is the commutativity of the triangle on the left-hand side. Now the uniqueness in Proposition 3.21 applied to both $\mathrm{pr}_{23}^*(\delta) \circ \mathrm{pr}_{12}^*(\delta) = \mathrm{pr}_{13}^*(\delta)$ implies that the triangle on the right-hand side is also commutative. This is the cocycle condition for α. In conclusion, $(\mathrm{Alb}^0_{X'/S'}, \alpha)$ is a scheme with descent datum for b.

Step 2: The descent datum is effective. Distinguish cases (i) and (ii). In situation (i), descent for quasi-projective schemes with descent data is already effective along the finite locally free surjection $b\colon S' \to S$ due to Example 3.32.

In situation (ii), there exists a decomposition $S' \to \widetilde{S} \to S$ where $S' \to \widetilde{S}$ is a finite locally free surjection, and $\widetilde{S} \to S$ is a purely inseparable, faithfully flat morphism of finite type. The descent will take two stages. First, descent is effective along $S' \to \widetilde{S}$, since $\mathrm{Alb}^0_{X'/S'}$ is projective. The subsequent Step 3 and Step 4 will then yield the existence of the Albanese morphism $\mathrm{alb}_{\widetilde{S}}\colon \widetilde{X} \to \mathrm{Alb}^1_{\widetilde{X}/\widetilde{S}}$ of $\widetilde{X} := X_{\widetilde{S}}$. It also commutes with base change by Step 5.

Independent of the fact that $\widetilde{X}$ may admit no section, $\mathrm{alb}_{\widetilde{S}}$ satisfies its universal property. So with its base change compatibility, it is possible to proceed as in Step 1 and define a descent datum for $\mathrm{Alb}^1_{\widetilde{X}/\widetilde{S}}$ along $\widetilde{S} \to S$. Then in the second stage, use that every scheme with descent datum is effective along a purely inseparable, faithfully flat morphism of finite type by Example 3.33.

Step 3: The Albanese torsor exists. The fact that the descent datum $(\mathrm{Alb}^0_{X'/S'}, \alpha)$ is effective means that there exists and S-scheme Alb^1_X and an isomorphism of descent data $(\mathrm{Alb}^0_{X'/S'}, \alpha) \simeq (\mathrm{Alb}^1_X \times S', \alpha_{\mathrm{can}})$ for the canonical descent datum α_{can} on $\mathrm{Alb}^1_X \times S'$. Hence the morphism of canonical descent data

$$(X', \delta) \longrightarrow (\mathrm{Alb}^0_{X'/S'}, \alpha) \xrightarrow{\simeq} (\mathrm{Alb}^1_X \times S', \alpha_{\mathrm{can}})$$

originates from a morphism $\mathrm{alb}\colon X \to \mathrm{Alb}^1_X$ by Proposition 3.27. To see that Alb^1_X is a principal homogeneous space under $M^\vee$, first note that there is a natural identification

$$\mathrm{Alb}^0_{X'/S'} = M'^\vee = M^\vee \times S', \tag{3.10}$$

using the identification $M' = M \times S'$ and compatibility of Pic^0 with base change. Second, due to Proposition 3.21, the descent datum α on $\mathrm{Alb}^0_{X'/S'}$ can be decomposed as

$$\pi_1^*(\mathrm{Alb}^0_{X'/S'}) \xrightarrow{\delta^{\vee\vee}} \pi_2^*(\mathrm{Alb}^0_{X'/S'}) \xrightarrow{\sigma} \pi_2^*(\mathrm{Alb}^0_{X'/S'}),$$

where σ is the translation by $\pi_2^*(\mathrm{alb}') \circ \delta \circ e_{\pi_1^*(X')}$, see (3.9). The morphism $\delta^{\vee\vee}$ is the canonical descent datum on $\mathrm{Alb}^0_{X'/S'} = M^\vee \times S'$. In fact, as δ is the canonical descent datum coming from the identification

$$X \times S' \times_{S',\pi_1} S'' = X \times S' \times_{S',\pi_2} S'',$$

the application of the Picard functor over S'' is compatible with the base change $S'' \to S$, and thus it yields

$$M \times S' \times_{S',\pi_1} S'' = M \times S' \times_{S',\pi_2} S'',$$

which is the canonical descent datum on $M \times S'$. The same procedure applied once again now shows the claim about $\delta^{\vee\vee}$.

To define an action of $M^\vee$ on Alb^1_X, keep the decomposition $\alpha = \sigma \circ \delta^{\vee\vee}$ in mind and consider

$$\begin{array}{ccc} \pi_1^*(\mathrm{Alb}^0_{X'/S'}) \times_{S''} \pi_1^*(\mathrm{Alb}^0_{X'/S'}) & \xrightarrow{\pi_1^*(m')} & \pi_1^*(\mathrm{Alb}^0_{X'/S'}) \\ \downarrow{\scriptstyle \alpha\times\delta^{\vee\vee}} & & \downarrow{\scriptstyle \alpha} \\ \pi_2^*(\mathrm{Alb}^0_{X'/S'}) \times_{S''} \pi_2^*(\mathrm{Alb}^0_{X'/S'}) & \xrightarrow[\pi_2^*(m')]{} & \pi_2^*(\mathrm{Alb}^0_{X'/S'}), \end{array}$$

where m' is the group law on $\mathrm{Alb}^0_{X'/S'} = M^\vee \times S'$ derived via base change from the group law on $M^\vee$. The diagram is commutative, which can be directly verified on T''-valued points: As the S'-scheme $\mathrm{Alb}^0_{X'/S'}$ is obtained via base change from an S-scheme, a T''-valued point of $\pi_i^*(\mathrm{Alb}^0_{X'/S'})$ corresponds to an element of $M^\vee(T'')$. Now the commutativity is simply the statement that the group law on $M^\vee(T'')$ is associative.

Hence $m'\colon (\mathrm{Alb}^0_{X'/S'} \times_{S'} \mathrm{Alb}^0_{X'/S'}, \alpha \times \delta^{\vee\vee}) \to (\mathrm{Alb}^0_{X'/S'}, \alpha)$ is a morphism of schemes with descent data. It descends to some $\mu\colon \mathrm{Alb}^1_X \times M^\vee \to \mathrm{Alb}^1_X$. This defines a group action on Alb^1_X. In fact, unitality and associativity are expressed in the commutativity of two diagrams of S-schemes. As $\mu_{S'} = m'$, they become commutative after base change to S', and thus by descent, they had to be commutative before. The same argument ensures that Alb^1_X is a principal homogeneous space under $M^\vee$, since

$$\mathrm{Alb}^1_X \times M^\vee \longrightarrow \mathrm{Alb}^1_X \times \mathrm{Alb}^1_X, \quad (j, g) \longmapsto (j, jg)$$

becomes an isomorphism after base change to S'.

Step 4: Verify the factorization through the Albanese torsor. The factorization of the given morphism $X' \to J'$ as $X' \xrightarrow{\mathrm{alb}'} \mathrm{Alb}^0_{X'/S'} \xrightarrow{h'} J'$ now yields an extension of (3.9) to

$$\begin{array}{ccccc} \pi_1^*(X') & \xrightarrow{\pi_1^*(\mathrm{alb}')} & \pi_1^*(\mathrm{Alb}^0_{X'/S'}) & \xrightarrow{\pi_1^*(h')} & \pi_1^*(J') \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ \pi_2^*(X') & \xrightarrow[\pi_2^*(\mathrm{alb}')]{} & \pi_2^*(\mathrm{Alb}^0_{X'/S'}) & \xrightarrow[\pi_2^*(h')]{} & \pi_2^*(J'), \end{array}$$

where γ denotes the canonical descent datum on J'. Here the left square is (3.9) and thus commutative. The outer square is commutative, as $X' \to J'$ is obtained by base change from $X \to J$. An application of Proposition 3.17 to the Albanese morphism $\pi_1^*(\mathrm{alb}')$ guarantees that the right square is commutative, too.

Thus the canonical morphism of descent data $(X', \delta) \to (J', \gamma)$ has a factorization $(X', \delta) \to (\mathrm{Alb}^0_{X'/S'}, \alpha) \to (J', \gamma)$. The identification $(\mathrm{Alb}^0_{X'/S'}, \alpha) \simeq (\mathrm{Alb}^1_X \times S', \alpha_{\mathrm{can}})$ yields the factorization

$$(X', \delta) \longrightarrow (\mathrm{Alb}^1_X \times S', \alpha_{\mathrm{can}}) \longrightarrow (J', \gamma), \tag{3.11}$$

where now all three descent data are canonical. Thus there is a morphism $h\colon \mathrm{Alb}^1_X \to J$ such that its composition with $\mathrm{alb}\colon X \to \mathrm{Alb}^1_X$ is the given morphism $X \to J$.

The factorization is indeed unique, since two different morphisms $h_1, h_2\colon \mathrm{Alb}^1_X \to J$ would yield two different factorizations as in (3.11). But this cannot happen by the universal property of the Albanese morphism $X' \to \mathrm{Alb}^1_X \times S'$.

Step 5: Compatibility with base change holds. Let $z\colon Z \to S$ be a morphism and define Z' by the cartesian square

$$\begin{array}{ccc} Z' & \longrightarrow & S' \\ {\scriptstyle b_z}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ Z & \underset{z}{\longrightarrow} & S. \end{array}$$

Denote the base changes of X along these morphisms by

$$\begin{array}{ccc} X'_{Z'} & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow \\ X_Z & \longrightarrow & X. \end{array} \tag{3.12}$$

Here $X'(S') \neq \varnothing$ and thus $X'_{Z'}(Z') \neq \varnothing$. The Albanese morphism of $X'_{Z'}$ is the base change of that of X'. An application of base change for descent data, see Remark 3.28, to (3.9) gives an induced descent datum on $\mathrm{Alb}^0_{X'_{Z'}/Z'}$ for b_z, along with a morphism $(X'_{Z'}, \delta_z) \to (\mathrm{Alb}^0_{X'_{Z'}/Z'}, \alpha_z)$. Then descent yields $\mathrm{alb}_{X_Z}\colon X_Z \to \mathrm{Alb}^1_{X_Z/Z}$. But in Diagram (3.12), another path to obtain $X'_{Z'} \to \mathrm{Alb}^0_{X'_{Z'}/Z'}$ is to initially base change $X \to \mathrm{Alb}^1_X$ to $z^*(\mathrm{alb})\colon X_Z \to \mathrm{Alb}^1_X \times Z$, then proceeding along $Z' \to Z$. Hence $\mathrm{alb}_Z = z^*(\mathrm{alb})$ follows from the faithfulness of the latter. □

Example 3.37. Let X be as in Theorem 3.35 (i) and assume further that $X_{\overline{k}}$ is an abelian variety. Then by faithfully flat descent, already $\mathrm{alb}\colon X \to \mathrm{Alb}^1_X$ is an isomorphism and X is a principal homogeneous space under $\mathrm{Alb}^0_X = (\mathrm{Pic}^0_X)^\vee$.

As a concrete example, consider the *Selmer curve* $C = V(3u^3 + 4v^3 + 5w^3) \subset \mathbb{P}^2_{\mathbb{Q}}$. It is an irreducible curve, which is smooth by the Jacobi criterion. The genus formula yields $h^1(\mathcal{O}_C) = \frac{(3-1)(3-2)}{2} = 1$. According to [25], Chapter 18, Theorem 1, there exists no $\mathbb{Q}$-rational point on C, so the abelian variety Alb^0_C cannot be isomorphic to $\mathrm{Alb}^1_C = C$. As mentioned in [27], Section 8, an explicit description as a vanishing set is given by $\mathrm{Alb}^0_C = V(u^3 + v^3 + 60w^3) \subset \mathbb{P}^2_{\mathbb{Q}}$. This curve has only the single rational point $(1 : -1 : 0)$ due to [25], Chapter 18, Lemma 2.

Proposition 3.38. *Let $f\colon X \to S$ and $g\colon Y \to S$ as well as $f \times g\colon X \times Y \to S$ be proper morphisms satisfying the assumptions made in Theorem 3.35 and additionally (A2**). Then $\mathrm{alb}_X \times \mathrm{alb}_Y\colon X \times Y \to \mathrm{Alb}^1_X \times \mathrm{Alb}^1_Y$ is the Albanese morphism of $X \times Y$.*

Proof. Set $S' = X \times Y$ to achieve that $X_{S'}$ and $Y_{S'}$ each have a canonical section. Combined, both yield the canonical section of $(X \times Y)_{S'} = X_{S'} \times_{S'} Y_{S'}$. Then

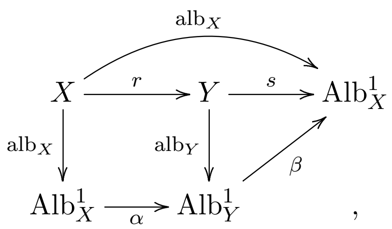
$$(\mathrm{alb}_X \times \mathrm{alb}_Y)_{S'}\colon X_{S'} \times_{S'} Y_{S'} \longrightarrow \mathrm{Alb}^1_{X_{S'}} \times_{S'} \mathrm{Alb}^1_{Y_{S'}} = \mathrm{Alb}^0_{X_{S'}} \times_{S'} \mathrm{Alb}^0_{Y_{S'}}$$

is the product $\mathrm{alb}_{X_{S'}} \times \mathrm{alb}_{Y_{S'}}$ of the usual Albanese morphisms. As a consequence of Proposition 3.25, it coincides with $\mathrm{alb}_{X_{S'} \times_{S'} Y_{S'}}$. But this morphism is also obtained by base changing $\mathrm{alb}_{X \times Y}$ to S'. Hence by descent, already $\mathrm{alb}_X \times \mathrm{alb}_Y = \mathrm{alb}_{X \times Y}$ is true. □

Proposition 3.39. *Let X be a proper scheme over an arbitrary field k with $h^0(\mathcal{O}_X) = 1$. Denote by $X \to Y \to \mathrm{Alb}^1_X$ the Stein factorization of the Albanese morphism. Then the induced arrow $\mathrm{Alb}^1_X \to \mathrm{Alb}^1_Y$ is an isomorphism and pullback of invertible sheaves gives further isomorphisms $M_{\mathrm{Alb}^1_X} \to M_Y$ and $M_Y \to M_X$.*

If X is additionally geometrically normal and $k \subset E$ is a field extension such that $X(E)$ is non-empty, then $\mathrm{Pic}^0(\mathrm{Alb}^1_{X_E}) \to \mathrm{Pic}^0(Y_E)$ and $\mathrm{Pic}^0(Y_E) \to \mathrm{Pic}^0(X_E)$ are both bijective.

Proof. First, observe that Y is proper with $h^0(\mathcal{O}_Y) = 1$. This shows that Y satisfies the assumptions of Theorem 3.35, and thus the Albanese morphism $\mathrm{alb}_Y \colon Y \to \mathrm{Alb}^1_Y$ exists. Consider the following commutative diagram



,

where α and β are obtained by the universal properties of alb_X and alb_Y, respectively. The uniqueness in the universal property of alb_X applied to $\mathrm{alb}_X = \beta \circ \alpha \circ \mathrm{alb}_X$ yields $\beta \circ \alpha = \mathrm{id}$. On the other hand, the diagram shows

$$\alpha \circ \beta \circ \mathrm{alb}_Y \circ r = \alpha \circ \mathrm{alb}_X = \mathrm{alb}_Y \circ r.$$

Since r is an epimorphism by [59], Exposé VIII, Proposition 5.1, the equation above now gives $\alpha \circ \beta \circ \mathrm{alb}_Y = \mathrm{alb}_Y$. As before, $\alpha \circ \beta = \mathrm{id}$ follows, so α and β are in fact inverse to each other.

Consider $s^\vee \colon M_{\mathrm{Alb}^1_X} \to M_Y$ and $r^\vee \colon M_Y \to M_X$. To show that they are isomorphisms, the diagram implicates that it is sufficient to verify that $\mathrm{alb}_X^\vee$ and $\mathrm{alb}_Y^\vee$ have inverses. The reasoning is the same for each of them, so consider the former. By faithfully flat descent, the bijectivity of $\mathrm{alb}_X^\vee$ can be verified after base change to $\overline{k}$. After fixing a $\overline{k}$-rational point of $X_{\overline{k}}$, this means to show that the dual $\mathrm{alb}_{X_{\overline{k}}}^\vee$ of the usual Albanese morphism $\mathrm{alb}_{X_{\overline{k}}} \colon X_{\overline{k}} \to \mathrm{Alb}^0_{X_{\overline{k}}}$ is bijective. But this map is the identity according to Remark 3.20. Thus $s^\vee$ and $r^\vee$ are indeed isomorphisms.

Now assume that X is geometrically normal. Then also Y is geometrically normal by Proposition A.12. This shows that $M_X = (\mathrm{Pic}^0_X)_{\mathrm{red}}$ and $M_Y = (\mathrm{Pic}^0_Y)_{\mathrm{red}}$ according to Theorem A.30 and Lemma 3.8. For the principal homogeneous space Alb^1_X under $\mathrm{Alb}^0_X = M_X^\vee$, there is a natural identification $\mathrm{Pic}^0_{\mathrm{Alb}^1_X} = M_X^{\vee\vee} = M_X$, as reviewed prior to Lemma 3.26. This is an abelian variety, so $M_{\mathrm{Alb}^1_X} = \mathrm{Pic}^0_{\mathrm{Alb}^1_X}$. Hence the identifications from above become $s^\vee \colon \mathrm{Pic}^0_{\mathrm{Alb}^1_X} \to (\mathrm{Pic}^0_Y)_{\mathrm{red}}$ and $r^\vee \colon (\mathrm{Pic}^0_Y)_{\mathrm{red}} \to (\mathrm{Pic}^0_X)_{\mathrm{red}}$.

Consider a field extension $k \subset E$ such that X has an E-rational point. Its images yield E-rational points of Y and Alb^1_X. So the Picard schemes of X_E, Y_E, $\mathrm{Alb}^1_{X_E}$ represent their respective relative Picard functors. Thus the maps above, on E-valued points, give bijections $\mathrm{Pic}^0(\mathrm{Alb}^1_{X_E}) \to \mathrm{Pic}^0(Y_E)$ and $\mathrm{Pic}^0(Y_E) \to \mathrm{Pic}^0(X_E)$. □

Note that on a proper k-scheme X, a sufficient condition for $h^0(\mathcal{O}_X) = 1$ to hold is that X is geometrically reduced and geometrically connected by Lemma A.10.

Definition 3.40. Let X be a proper scheme over an arbitrary field k with $h^0(\mathcal{O}_X) = 1$. The *Albanese dimension* of X is the dimension m of the image of the Albanese morphism $X \to \mathrm{Alb}^1_X$. The scheme X is of *maximal Albanese dimension* if $m = \dim(X)$.

Example 3.41. Let A be an abelian variety of dimension g over k. Then clearly A is of maximal Albanese dimension $m = g$. The next proposition shows that the projective space $\mathbb{P}^n$ for $n \geq 0$ has Albanese dimension $m = 0$.

Proposition 3.42. *Let X be a proper scheme over an arbitrary field k with $h^0(\mathcal{O}_X) = 1$, which is of Albanese dimension $m \geq 0$. Then $m \leq h^1(\mathcal{O}_X)$ holds. Also, the following two conditions are equivalent:*

(i) $m = 0$.

(ii) $\mathrm{Alb}^1_X = \mathrm{Alb}^0_X = \mathrm{Spec}(k)$.

Furthermore, $h^1(\mathcal{O}_X) = 0$ implies the equivalent conditions above. The converse holds for geometrically normal X under the additional assumption that $h^2(\mathcal{O}_X) = 0$ or $p = 0$.

Proof. By definition, $m \leq \dim(\mathrm{Alb}^1_X)$. The latter equals $\dim(\mathrm{Alb}^0_X)$, which in turn is the dimension of the maximal abelian subscheme $M_X \subset \mathrm{Pic}_X$. As $\dim(\mathrm{Pic}_X) \leq h^1(\mathcal{O}_X)$, this proves the first claim $m \leq h^1(\mathcal{O}_X)$. In particular if $h^1(\mathcal{O}_X) = 0$, then (i) follows.

To deduce the equivalence of (i) and (ii), notice that (ii) clearly implies (i). Now assume (i). The schematic image $\mathrm{Im}(\mathrm{alb}) \subset \mathrm{Alb}^1_X$ is connected, zero-dimensional and proper over k. Hence, $\mathrm{Im}(\mathrm{alb}) = \mathrm{Spec}(R)$ for an artinian local k-algebra R. Since $h^0(\mathcal{O}_X) = 1$, the morphism $X \to \mathrm{Spec}(R)$ has a factorization through $\mathrm{Spec}(k)$. But then $\mathrm{Im}(\mathrm{alb}) = \mathrm{Spec}(k)$. This is an abelian variety and $X \to \mathrm{Spec}(k)$ consequently satisfies the universal property of the Albanese morphism. Therefore already $\mathrm{Alb}^1_X = \mathrm{Spec}(k)$. Eventually, (ii) holds.

Finally, let X be geometrically normal. In this case, $\mathrm{Alb}^0_X = (\mathrm{Pic}^0_X)^\vee_{\mathrm{red}}$ holds. So (ii) implies $\dim(\mathrm{Pic}_X) = 0$. The additional assumption ensures that Pic_X is smooth, and thereby $h^1(\mathcal{O}_X) = \dim(\mathrm{Pic}_X) = 0$. □

Both the geometric normality and the additional assumption imposed in the preceding proposition are necessary for the converse of the statement to hold:

Example 3.43. A supersingular Enriques surface X in characteristic $p = 2$ has $\mathrm{Pic}^0_X = \alpha_2$. Hence the reduction is $\mathrm{Spec}(k)$ and (ii) holds, but $h^1(\mathcal{O}_X) = 1$.

Example 3.44. If X is not geometrically normal, then X can be of Albanese dimension $m = 0$ with $h^2(\mathcal{O}_X) = 0$, but $h^1(\mathcal{O}_X) \geq 1$. A concrete example is the rational curve with one cusp $X = V_+(v^2w - u^3) \subset \mathbb{P}^2$. The adjunction formula shows that $\omega_X \simeq \mathcal{O}_X$, so $h^1(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1$. Its normalization is $\mathbb{P}^1 \to X$, which is a universal homeomorphism by [43], Example 12.47. Since $\mathbb{P}^1$ has $h^1(\mathcal{O}_{\mathbb{P}^1}) = 0$, its Albanese dimension is zero. The surjectivity of $\mathbb{P}^1 \to X$ directly shows that the Albanese dimension of X has to be zero, too. As $h^2(\mathcal{O}_X) = 0$, the identity component Pic^0_X of the Picard scheme is smooth of dimension $h^1(\mathcal{O}_X) = 1$. In fact if k is perfect, then this is a unipotent algebraic group according to [19], Section 9.2, Proposition 9, so $\mathrm{Pic}^0_X = \mathbb{G}_a$.

To conclude this chapter, consider the following proposition, which states that a k-scheme naturally admits a—in general non-trivial—fibration to a projective scheme, once the existence of the Albanese morphism is known.

Proposition 3.45. *Let X be a proper scheme over an arbitrary field k with $h^0(\mathcal{O}_X) = 1$, which is of Albanese dimension $m \geq 0$. Then there exists a fibration $X \to Y$ to a projective k-scheme Y of dimension* $\dim(Y) = m$.

Proof. Consider the Stein factorization $X \xrightarrow{r} Y \xrightarrow{s} \mathrm{Alb}^1_X$ of the Albanese morphism. Being an abelian variety, Alb^0_X is projective. Then also Alb^1_X is projective, because projectivity descends along field extensions, see [43], Proposition 14.55. Since s is finite, Y is projective, too. Eventually, $\dim(Y) = m$ is a consequence of the fact that s is finite. □

Part II

Chapter 4

Dualizing Sheaves of Finite Order

The aim of this chapter is to address question (Q1) posed in the introduction: If ω_X is invertible and numerically trivial, does $\omega_X \in \mathrm{Pic}(X)$ have finite order? The aim is to cover the case of surfaces X as completely as possible. Some introductory remarks start Section 4.1, followed by several examples to initiate the topic. In essentially two special situations, the question can be answered in the affirmative by a systematic approach. Namely, Section 4.2 examines cases in which every numerically trivial invertible sheaf has finite order, including the situation where X has Albanese dimension $m = 0$. Then Section 4.3 treats the other extreme when the Albanese dimension $m = \dim(X)$ is maximal. The cases in between are in general the more intricate ones.

The affirmative answer to question (Q1) for smooth surfaces is known as a consequence of the Enriques classification of surfaces. An overview of this classification, which is a fundamental result achieved by Mumford and Bombieri [94], [18], [17] in arbitrary characteristic, is the content of Section 4.4. This also lays the foundation for the later Section 6.2, where all smooth surfaces with numerically trivial dualizing sheaves are studied with regard to questions (Q2) to (Q4). Then the first main result in Section 4.5 is a direct proof that on smooth surfaces X, the numerical triviality of ω_X implies its finite order. The motivation for this is to gain a better understanding of what specifically forces the dualizing sheaf to be torsion. It turns out that in the intermediate case of Albanese dimension $m = 1$, the existence of a certain curve is crucial. This leads to a criterion which might be of independent interest, as it holds true in higher dimensions.

Afterwards in Section 4.6, singular surfaces are considered. If X is normal but not necessarily Gorenstein, then ω_X is at least a reflexive sheaf of rank 1, associated to the Weil divisor K_X. So the natural variant of question (Q1) is to ask if K_X has finite order as a Weil divisor. Here the arguments used by Sakai [106], [107] in his studies of Gorenstein surfaces over $\mathbb{C}$ can be adapted to conclude that a suitable class of $\mathbb{Q}$-Gorenstein surfaces X in arbitrary characteristic has the property that K_X is of finite order if it is numerically trivial. This is the second main result in this chapter. The final Section 4.7 consists of examples of $\mathbb{Q}$-Gorenstein surfaces with K_X of different finite orders. Also, the examples can be altered slightly to produce non-$\mathbb{Q}$-Gorenstein surfaces on which K_X is numerically trivial, but of infinite order.

4.1 Introductory Examples

To begin with, the subsequent lemma ensures that in answering question (Q1), it is possible to reduce the situation to the case of an algebraically closed ground field.

Lemma 4.1. *Let X be a Gorenstein, proper scheme over an arbitrary field k. For every algebraic field extension $k \subset E$ and every integer d, the following assertions hold:*

(i) $(\omega_X)_E = \omega_{X_E}$.

(ii) ω_X *is numerically trivial if and only if* ω_{X_E} *is numerically trivial.*

(iii) $\omega_X^{\otimes d} \simeq \mathcal{O}_X$ *if and only if* $\omega_{X_E}^{\otimes d} \simeq \mathcal{O}_{X_E}$.

Proof. The first assertion is a consequence of the compatibility of ω_X with base change, Proposition 2.14. Numerical triviality can be verified after base change due to Proposition 1.34, so the second part follows. Finally, the pullback induced on Picard groups by an algebraic field extension is injective by Proposition A.25, showing the third statement. □

Let X be a Gorenstein, integral, proper scheme of dimension n over an arbitrary field k. Suppose that ω_X is numerically trivial. In the course of the remainder of this page, collect some immediate consequences of this assumption. Proposition 1.34 implies that $\chi(\mathcal{F}) = \chi(\mathcal{F} \otimes \omega_X)$ for all coherent sheaves $\mathcal{F}$ on X. For $t \in \mathbb{Z}$ and $\mathcal{F} = \omega_X^{\otimes t}$, this yields

$$\chi(\omega_X^{\otimes t}) = \chi(\mathcal{O}_X). \tag{4.1}$$

Given a locally free sheaf $\mathcal{E}$, Serre duality implicates $\chi(\mathcal{E} \otimes \omega_X) = (-1)^n \chi(\mathcal{E}^\vee)$. Consequently, the relation $\chi(\mathcal{E}) = (-1)^n \chi(\mathcal{E}^\vee)$ follows. In the case that the dimension n of X is odd, this especially shows that $\chi(\mathcal{O}_X) = 0$.

Every other numerically trivial invertible sheaf $\mathscr{N} \not\simeq \omega_X$ satisfies $h^n(\mathscr{N}) = 0$. In fact, otherwise $h^0(\mathscr{N}^\vee \otimes \omega_X) \geq 1$ and Lemma 1.35 would imply $\mathscr{N} \simeq \omega_X$. Now let $\mathscr{M}$ be any invertible sheaf on X with $h^0(\mathscr{M}) \geq 1$. In the case that

$$h^n(\mathscr{M} \otimes \omega_X^{\otimes m}) = h^0(\mathscr{M}^\vee \otimes \omega_X^{\otimes 1-m}) \geq 1 \tag{4.2}$$

for some integer $m \neq 1$, there is an injection $\mathscr{M} \to \omega_X^{\otimes 1-m}$ showing that $h^0(\omega_X^{\otimes 1-m}) \geq 1$, so ω_X has finite order.

Before giving some examples, recall the divergence between properness and projectivity of k-schemes with increasing dimension. Any proper curve over a field k is projective by for instance [9], Tag 09NZ. For surfaces, it is still true that regular, proper surfaces are projective due to [76], Chapter IV, Section 2, Corollary 4, which traces back to Zariski–Goodman. But there exist normal, non-projective, proper surfaces according to [111]. Smooth, proper threefolds can be non-projective as Hironaka's example shows, see the survey [127].

Example 4.2 (Points)**.** Let X be a Gorenstein, proper scheme over an arbitrary field k of dimension $n = 0$. Then every connected component of X is $\mathrm{Spec}(A)$ for an artinian local k-algebra A, and hence simply a point. Thereby $\mathrm{Pic}(X)$ is trivial, and in particular $\omega_X \simeq \mathcal{O}_X$. It is immediate that if X is additionally integral, then all questions (Q1) to (Q4) from the introduction can be answered in the affirmative.

Example 4.3 (Curves)**.** Let X be a Gorenstein, integral, proper curve over an arbitrary field k. For instance, the adjunction formula shows that every integral curve on a regular, integral, proper surface is Gorenstein. Assume that ω_X is numerically trivial. Since the dimension of X is odd, the equation $0 = \chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X)$ holds. Now $h^0(\mathcal{O}_X) \geq 1$, so also $h^0(\omega_X) = h^1(\mathcal{O}_X) \geq 1$. Therefore $\omega_X \simeq \mathcal{O}_X$.

In the case that $h^0(\mathcal{O}_X) = 1$, the curve X has genus $h^1(\mathcal{O}_X) = 1$. In fact, it is conversely valid that every integral, proper curve X with $h^0(\mathcal{O}_X) = h^1(\mathcal{O}_X) = 1$ satisfies $\omega_X \simeq \mathcal{O}_X$. Indeed, a non-zero global section of ω_X gives an injection $\mathcal{O}_X \to \omega_X$, since X is integral. Its cokernel has to be trivial, as it is a torsion sheaf and $\chi(\mathcal{O}_X) = \chi(\omega_X)$.

If X is additionally smooth, its base change $X_{\overline{k}}$ is an elliptic curve after the choice of a rational point, thus an abelian variety. So for $X_{\overline{k}}$, all questions (Q1) to (Q4) can be answered in the affirmative. The same conclusion holds true for X if questions (Q3) and (Q4) are altered to allow principal homogeneous spaces under abelian varieties as factors in the decomposition of the total space, which appears to be reasonable if the ground field is not algebraically closed. It is in effect true that X is a principal homogeneous space under an abelian variety by Example 3.37.

Example 4.4 (Abelian Varieties)**.** The sheaf of Kähler differentials on an abelian variety A of dimension g is $\Omega^1_A = \mathcal{O}_A^{\oplus g}$ by Example 2.10 and as a consequence, its determinant $\omega_A = \mathcal{O}_A$ is trivial.

Example 4.5 (Projective Spaces)**.** The dualizing sheaf $\omega_{\mathbb{P}^n}$ is not numerically trivial for every $n \geq 1$. This follows immediately from the fact that $\omega^\vee_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(n+1)$ is ample, so its restriction to an integral curve stays ample, and consequently it has positive degree by the Nakai–Moishezon criterion. Also, $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ contains no non-trivial divisible elements at all.

Example 4.6 (Hypersurfaces $X \subset \mathbb{P}^n$ of Degree $n+1$)**.** Since $\mathcal{O}_{\mathbb{P}^n}(X) = \mathcal{O}_{\mathbb{P}^n}(n+1)$, the adjunction formula implies that $\omega_X = \mathcal{O}_{\mathbb{P}^n}(n+1) \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1)|_X = \mathcal{O}_X$ is trivial. The long exact sequence in cohomology to $0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$ yields that $h^i(\mathcal{O}_X) = 0$ for $1 \leq i \leq n-2$ and $h^{n-1}(\mathcal{O}_X) = 1$ provided that $n \geq 2$. In the case $n = 3$, this gives a Gorenstein surface $X \subset \mathbb{P}^3$. If k is algebraically closed and X is smooth, it turns out to be a K3-surface by the Enriques classification of surfaces described below in Section 4.4.

A first concrete example is the Fermat surface $V_+(T_0^4 + T_1^4 + T_2^4 + T_3^4) \subset \mathbb{P}^3$, which is smooth by the Jacobi criterion if $p \neq 2$. In the case that $p \notin \{2, 3\}$, another example is the surface $X = V_+(T_0^4 + T_0T_1^3 + T_2^3T_3 + T_3^4) \subset \mathbb{P}^3$. Figure 1 on the first page of the introduction shows on its left-hand side the real image within a ball around the origin of the affine chart $X \cap D_+(T_3) = V(f)$, where $f = x^4 + xy^3 + z^3 + 1$ and $x = \frac{T_0}{T_3}, y = \frac{T_1}{T_3}, z = \frac{T_2}{T_3}$. Now suppose $\sqrt{2} \in k$ and $p = 0$ for simplicity. For parameters $b, c \in k$, consider the polynomials

$$g = (1 - z + \sqrt{2}x)(1 - z - \sqrt{2}x)(1 + z + \sqrt{2}x)(1 + z - \sqrt{2}x),$$

$$f = (x^2 + y^2 + z^2 - c)^2 - \frac{3c - 1}{3 - c} g - b.$$

The homogenization $F \in k[T_0, T_1, T_2, T_3]$ of f defines a surface $X = V_+(F) \subset \mathbb{P}^3$. For suitable choices of b and c, it is a Kummer surface, either singular or smoothed. Choosing $b = \frac{1}{2}$ and $c = 4$ gives the surface $V(f)$ on the right-hand side in Figure 1, in the introduction. For $b = \frac{1}{12}$ and $c = \frac{4}{3}$, the smooth surface in the subsequent Figure 3 on the left-hand side is obtained. If $b = 0$ and $c = \frac{4}{3}$, then the singular surface on the right-hand side arises, which has 16 singular points.

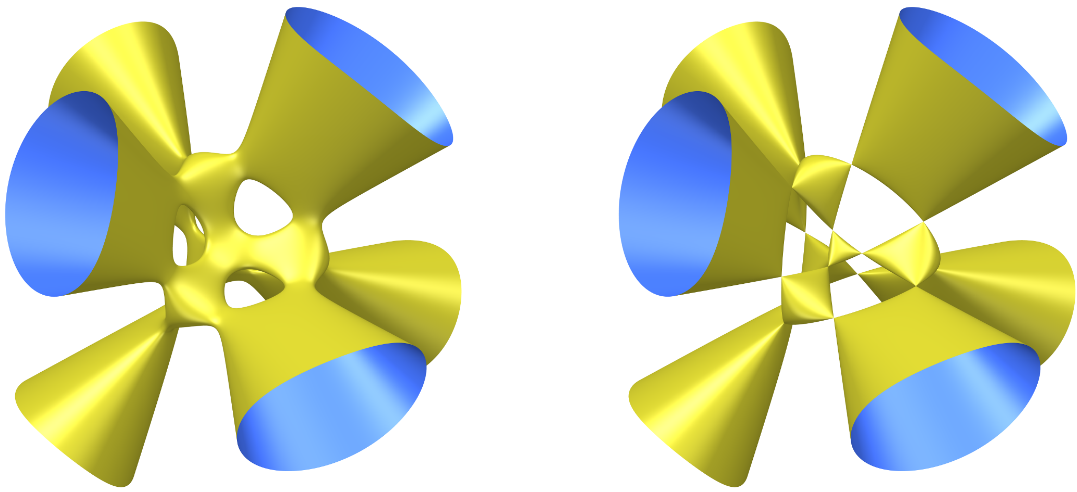


Figure 3: Real points around the origin of a smooth and a singular surface.

Example 4.7 (Complete Intersection of Three Quadrics in $\mathbb{P}^5$)**.** Suppose $p \neq 2$ and that k is algebraically closed for simplicity. In order to give an example of a scheme with numerically trivial but non-trivial dualizing sheaf, construct a K3-surface with an involution first. Its quotient will then have the queried property. Define the three quadrics in $\mathbb{P}^5$ by

$$\begin{aligned} Q_1 &= V_+(T_0^2 + T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2), \\ Q_2 &= V_+(T_0^2 - T_1^2 \qquad\qquad\qquad + T_4^2 - T_5^2), \\ Q_3 &= V_+(T_0^2 - T_1^2 + T_2^2 - T_3^2 \qquad\qquad\qquad), \end{aligned}$$

and set $X = Q_1 \cap Q_2 \cap Q_3$. To show that X is smooth, it is sufficient to verify smoothness at all closed points, which will be checked via the projective Jacobi criterion. The formal Jacobi matrix to the three equations defining X is

$$J = \begin{pmatrix} 2T_0 & 2T_1 & 2T_2 & 2T_3 & 2T_4 & 2T_5 \\ 2T_0 & -2T_1 & 0 & 0 & 2T_4 & -2T_5 \\ 2T_0 & -2T_1 & 2T_2 & -2T_3 & 0 & 0 \end{pmatrix}.$$

Denote by $d(j_1, j_2, j_3)$ the determinant of the 3×3 minor which is obtained by selecting columns $0 \leq j_1 < j_2 < j_3 \leq 5$. Then $d(j_1, j_2, j_3) = \pm u \cdot T_{j_1} T_{j_2} T_{j_3}$ where u is a power of 2, and thereby $u \in k^\times$. Now if $x = (\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : \lambda_5)$ is a closed point of X, then by its definition, it has to have at least three non-zero entries. The corresponding determinant thus satisfies $d(j_1, j_2, j_3) \neq 0$, which shows that the Jacobi matrix $J(x)$ at x has rank $3 = 5 - 2$. Thus X is smooth of dimension 2 over k, which especially ensures that X is a complete intersection.

As for the hypersurface in the previous example, consecutively using the defining exact sequences for $\mathcal{O}_{Q_1}, \mathcal{O}_{Q_1 \cap Q_2}, \mathcal{O}_X$ shows that the complete intersection X has $h^1(\mathcal{O}_X) = 0$. The adjunction formula applied three times yields that $\omega_X = \mathcal{O}_{\mathbb{P}^5}(3 \cdot 2 - 6)|_X = \mathcal{O}_X$. In conclusion, it turns out that X is a K3-surface by the Enriques classification of surfaces.

Define an involution $\iota\colon \mathbb{P}^5 \to \mathbb{P}^5$ by $T_i \mapsto (-1)^i T_i$. Solving the equation

$$(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : \lambda_5) \quad = \quad (\lambda_0 : -\lambda_1 : \lambda_2 : -\lambda_3 : \lambda_4 : -\lambda_5)$$

shows that the closed fixed points are exactly of the form $(0 : \lambda_1 : 0 : \lambda_3 : 0 : \lambda_5)$ or $(\lambda_0 : 0 : \lambda_2 : 0 : \lambda_4 : 0)$. Their closure is $Z = Z_1 \cup Z_2$, where $Z_1 = V_+(T_0, T_2, T_4)$ and $Z_2 = V_+(T_1, T_3, T_5)$.

The action of the constant group scheme $G = \mathbb{Z}/2\,\mathbb{Z}$ via ι on $\mathbb{P}^5$ leaves all the quadrics invariant, which in turn induces an action on their intersection X. The next step is to show that $X \cap Z = \varnothing$, which then implies that G acts freely on X. The intersection $X \cap Z_1$ is the vanishing set of the polynomials

$$T_0, \quad T_2, \quad T_4, \quad T_1^2 + T_3^2 + T_5^2, \quad T_1^2 + T_5^2, \quad T_1^2 + T_3^2,$$

which is empty. The same conclusion holds for the intersection $X \cap Z_2$, which is the vanishing set of

$$T_1, \quad T_3, \quad T_5, \quad T_0^2 + T_2^2 + T_4^2, \quad T_0^2 + T_4^2, \quad T_0^2 + T_2^2.$$

So G acts freely on X. Hence the quotient $Y = X/G$ by the constant group scheme G exists due to [95], Section 7, Theorem on page 66 and the subsequent remark. The canonical morphism $q\colon X \to Y$ is finite and étale, since G acts freely on all closed points of X. Since $\chi(\mathcal{O}_X) = \deg(q) \cdot \chi(\mathcal{O}_Y)$ holds by Corollary 1.42, inserting $\deg(q) = 2$ and $\chi(\mathcal{O}_X) = 2$ yields $\chi(\mathcal{O}_Y) = 1$. Therefore the Enriques classification implies that Y is actually an Enriques surface, so ω_Y is numerically trivial with $\omega_Y \not\simeq \mathcal{O}_Y$ and $\omega_Y^{\otimes 2} \simeq \mathcal{O}_Y$.

4.2 $\mathrm{Pic}^\tau(X)$ is a Torsion Group

Proposition 4.8. *Let X be a proper scheme over an arbitrary field k. In each of the following cases, the group $\mathrm{Pic}^\tau(X)$ is torsion:*

(i) $h^1(\mathcal{O}_X) = 0$.

(ii) X is geometrically normal of Albanese dimension $m = 0$ with $h^0(\mathcal{O}_X) = 1$.

(iii) X is geometrically normal and k is an algebraic extension of $\mathbb{F}_p$.

Proof. Assume (i). By Theorem A.30, the scheme Pic^τ_X is an open and closed subgroup scheme of Pic_X which is of finite type over k. Furthermore, $\dim(\mathrm{Pic}_X) \leq h^1(\mathcal{O}_X) = 0$, so Pic^τ_X is finite and has the discrete topology. The injection $\mathrm{Pic}^\tau(X) \hookrightarrow \mathrm{Pic}^\tau_X(k)$ yields that $\mathrm{Pic}^\tau(X)$ is finite and particularly a torsion group.

Now assume (ii). Proposition 3.42 then shows that $(\mathrm{Pic}^0_X)_{\mathrm{red}} = \mathrm{Spec}(k)$. As above, the injection $\mathrm{Pic}^0(X) \hookrightarrow \mathrm{Pic}^0_X(k)$ shows that $\mathrm{Pic}^0(X)$ is trivial. Consequently, also $\mathrm{Pic}^\tau(X)$ is a torsion group.

Finally suppose (iii). Assume without loss of generality that k is algebraically closed, since $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{k}})$ is injective and preserves numerical triviality. Suppose for the moment that the k-valued points of every abelian variety A over k form a torsion group, which will be verified in the next paragraph. Then as the reduction $A = (\mathrm{Pic}^0_X)_{\mathrm{red}}$ is an abelian variety and $A(k) = \mathrm{Pic}^0_X(k)$, the latter is a torsion group. Now as before, the homomorphism $\mathrm{Pic}^0(X) \hookrightarrow \mathrm{Pic}^0_X(k)$ is injective, and so $\mathrm{Pic}^\tau(X)$ is torsion.

To see that $A(k)$ is torsion, choose some $a \in A(k)$ and consider $k = \bigcup \mathbb{F}_{p^n}$ as the union of its finite subfields. According to [57], Théorème 8.8.2, the scheme A and the morphism $a\colon \mathrm{Spec}(k) \to A$ descend to some finite field. Furthermore, the data of the group scheme A consists of morphisms satisfying certain compatibility conditions. Hence it also descends, where faithfully flat descent, [56], Proposition 2.7.1, guarantees that the compatibility conditions are already fulfilled over the smaller field. Thus there exists a finite field $k' = \mathbb{F}_{p^n}$, an abelian variety A' over k' and a k'-valued point $a'\colon \mathrm{Spec}(k') \to A'$ such that after base change to k, this point becomes the given point a. So it is sufficient to show that a' has finite order. To achieve this, choose a closed embedding $A' \to \mathbb{P}^n_{k'}$ into some projective space. Then $\mathbb{P}^n_{k'}(k')$ is a finite set, which contains a' as well as all its multiples. So a' indeed has to have finite order. □

4.3 Maximal Albanese Dimension

In the preceding section, it was deduced that on a sufficient regular scheme X of minimal Albanese dimension $m = 0$, every numerically trivial invertible sheaf has finite order. In higher Albanese dimensions, this clearly fails to be valid for every invertible sheaf, as already discussed in Remark 1.36. But in the other extreme case that X has maximal Albanese dimension $m = \dim(X)$, the result still holds true for the dualizing sheaf ω_X.

This situation has been studied intensively in recent years with remarkable results, and an overview of what is known will be given in the following. For smooth surfaces, the result will be deduced directly with the techniques introduced previously.

Lemma 4.9. *Let $f\colon X \to Z$ be a morphism of integral, proper schemes over an arbitrary field k, where X is geometrically normal and Z is smooth. Decompose f as $X \xrightarrow{g} Y \hookrightarrow Z$ such that $Y \subset Z$ is the schematic image of f. Denote by $U \subset X$ the smooth locus of X and $n = \dim(X)$, $m = \dim(Y)$. Assume that $K(Y) \subset K(X)$ is separable. Then:*

(i) $h^0(\omega_X) = h^0(\Omega^n_U)$.

(ii) If $f^*(\Omega^1_Z)$ is free, then the estimates $h^0(\Omega^1_U) \geq m$ and $h^0(\Omega^m_U) \geq 1$ hold.

(iii) If f is surjective, then the natural morphism $f^*(\Omega^1_Z)|_U \to \Omega^1_U$ is injective, and especially $H^0(U, f^*(\Omega^1_Z)) \to H^0(U, \Omega^1_U)$ is an injection.

Remark 4.10. The separability assumption on $K(Y) \subset K(X)$ means that g is generically smooth on the source in the sense that there exists a non-empty open subset $U \subset X$ such that the restriction of g to U is smooth. Indeed, as it will also be used in the proof, then $\Omega^1_{X/Y}$ is generically free of rank $n - m$. Thus Proposition 2.8 can be applied to show generic smoothness on the source, since generic flatness holds in any case by [56], Théorème 6.9.1. Another equivalent description is that the generic fiber X_η is geometrically reduced, see [56], Proposition 4.6.1. This separability assumption is automatically fulfilled in characteristic $p = 0$, or in the case that k is perfect and g is a fibration to a curve, see Remark A.13.

Proof. Since U and Z are smooth, their sheaves of Kähler differentials are locally free of rank $n = \dim(X)$ and $d = \dim(Z)$, respectively. By assumption, X is geometrically normal, so the complement of the smooth locus of $X_{\overline{k}}$ has codimension at least 2. Consequently, the same conclusion holds for the complement of $U \subset X$. The dualizing sheaf ω_X is reflexive by Proposition 2.21, so according to [65], Proposition 1.11, the restriction map $H^0(X, \omega_X) \to H^0(U, \omega_X)$ is bijective. As $\omega_X|_U = \Omega^n_U$ holds, assertion (i) follows.

Consider the natural exact sequence $f^*(\Omega^1_Z) \to \Omega^1_X \to \Omega^1_{X/Z} \to 0$. Let $\xi \in X$ and $\eta \in Y$ be the generic points and write $R = \mathcal{O}_{Z,\eta}$ for the local ring with residue field $K(Y)$. Then $\Omega^1_{X/Z,\xi} = \Omega^1_{K(X)/R}$ holds. The maps $R \to K(Y) \to K(X)$ induce the exact sequence $\Omega^1_{K(Y)/R} \otimes_{K(Y)} K(X) \to \Omega^1_{K(X)/R} \to \Omega^1_{K(X)/K(Y)} \to 0$. Since $R \to K(Y)$ is surjective, every $y \in K(Y)$ has a preimage $r \in R$, and thereby $d(y) = d(r) = rd(1) = 0$ in $\Omega^1_{K(Y)/R}$. Hence $\Omega^1_{K(Y)/R} = 0$, and eventually $\Omega^1_{K(X)/R} = \Omega^1_{K(X)/K(Y)}$. The separable field extension $K(Y) \subset K(X)$ decomposes as a transcendental extension of transcendence degree $\dim(X_\eta) = n - m$, followed by a finite separable extension. Thus [88], Chapter 6, Proposition 1.15 yields $\dim_{K(X)}(\Omega^1_{X/Z,\xi}) = n - m$. Consider the commutative diagram

$$\begin{array}{ccccccc} H^0(U, f^*(\Omega^1_Z)) & \xrightarrow{\varphi} & H^0(U, \Omega^1_U) & \xrightarrow{\psi} & H^0(U, \Omega^1_{X/Z}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K(X)^{\oplus d} & \xrightarrow[\varphi_\xi]{} & K(X)^{\oplus n} & \xrightarrow[\psi_\xi]{} & K(X)^{\oplus n-m} & \longrightarrow & 0, \end{array} \tag{4.3}$$

where the vertical arrows are the maps to the stalk at $\xi \in X$.

For assertion (ii), assume that $f^*(\Omega^1_Z)$ is free. Then the vertical arrow on the left-hand side maps bases to bases. Choose a basis of $H^0(U, f^*(\Omega^1_Z))$ and denote their images under φ by $s_1, \ldots, s_g \in H^0(U, \Omega^1_U)$. Because $\mathrm{Im}(\varphi_\xi)$ has dimension m, assume without loss of generality that $s_{1,\xi}, \ldots, s_{m,\xi}$ form a basis of $\mathrm{Im}(\varphi_\xi)$. Observe that since X is integral, the map $H^0(U, \mathcal{E}) \to \mathcal{E}_\xi \simeq K(X)^{\oplus \operatorname{rank}(\mathcal{E})}$ is injective for every locally free sheaf $\mathcal{E}$ on X. Therefore the sections $s_1, \ldots, s_m \in H^0(U, \Omega^1_U)$ are linearly independent.

Now consider the section $s_1 \wedge \ldots \wedge s_m \in H^0(U, \Omega^m_U)$, which is defined locally by the non-zero compatible elements $s_1|_V \wedge \ldots \wedge s_m|_V \in H^0(V, \Omega^m_V)$ for $V \subset U$ affine open such that $\Omega^1_U|_V \simeq \mathcal{O}_V^{\oplus n}$. Then $s_1 \wedge \ldots \wedge s_m \neq 0$, since locally $s_1|_V \wedge \ldots \wedge s_m|_V \neq 0$. This completes the proof of (ii).

Finally, consider assertion (iii), in which case $Y = Z$ and so $d = m$. Thus φ_ξ has to be an injection. Hence the morphism $f^*(\Omega^1_Z)|_U \to \Omega^1_U$ of locally free sheaves on the integral scheme is injective, because it is generically injective. □

Now let k be an arbitrary field of characteristic $p = 0$, and X a Gorenstein, geometrically normal, proper k-scheme of maximal Albanese dimension $m = \dim(X)$ with $h^0(\mathcal{O}_X) = 1$. The lemma can be applied to the Albanese morphism $\mathrm{alb}\colon X \to A$ of X, since Ω^1_A is free. Parts (i) and (ii) yield $h^0(\omega_X) \geq 1$. So if ω_X is numerically trivial, it is trivial. This yields the following:

Proposition 4.11. *Let k be a field of characteristic $p = 0$. Let X be a Gorenstein, geometrically normal, proper k-scheme with $h^0(\mathcal{O}_X) = 1$. If ω_X is numerically trivial and X is of maximal Albanese dimension, then $\omega_X \simeq \mathcal{O}_X$.*

Lemma 4.9 fails in general in characteristic $p > 0$. Take for example the relative Frobenius $F\colon E \to E^{(p)}$ of an elliptic curve E, given by an affine Weierstraß equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Then $E^{(p)}$ is given by the induced equation, where all coefficients are raised to the power of p. On this affine chart $V \subset E^{(p)}$, the invariant differential $\omega = \frac{dx}{2y + a_1^p x + a_3^p} \in H^0(V, \Omega^1_{E^{(p)}})$ is a generator. For more details to this, see for instance Section 6.2 below, where the invariant differential will be used to determine the order of the dualizing sheaf on bielliptic surfaces. Now $F^*(\omega) = \frac{pdx^{p-1}}{2y^p + a_1^p x^p + a_3^p} = 0$, so the induced map $\Omega^1_{E^{(p)}} \to F_*(\Omega^1_E)$ restricted to V is zero. Thus it has to be zero everywhere, since $\Omega^1_{E^{(p)}}$ is locally free. Hence the corresponding morphism $F^*(\Omega^1_{E^{(p)}}) \to \Omega^1_E$ by adjointness is also zero, and in particular zero on global sections.

But in the case of the Albanese morphism $\mathrm{alb}\colon X \to A$, it is nevertheless still true that $H^0(X, \mathrm{alb}^*(\Omega^1_A)) \to H^0(X, \Omega^1_X)$ is injective. This was proved by Igusa [68] and also Serre [119], Section 6. As a consequence, Proposition 4.11 also holds in characteristic

$p > 0$ for smooth X. In this situation, denote $g = \dim(A)$. Then the injection above gives moreover the estimate $h^0(\Omega^1_X) \geq g$. It implies that if ω_X is numerically trivial, then $n \geq g$. Otherwise, $h^0(\Omega^1_X) \geq n+1$ would yield $h^0(\Omega^n_X) \geq \binom{n+1}{n} = n+1$, but this is impossible when $\Omega^n_X = \omega_X$ is numerically trivial and $n \geq 1$. So in the case that X is additionally of maximal Albanese dimension, the Albanese morphism has to be surjective.

Despite the conclusion in the preceding paragraph, even if $X \to A$ is surjective, it is for general X not true that $K(A) \subset K(X)$ is separable. Counterexamples are certain surfaces of general type, see [84], Theorem 8.7. But if ω_X is numerically trivial and X is of maximal Albanese dimension, then this is again the case by [62], Proposition 1.4. Moreover, op. cit., Theorem 3.2, shows that under the additional assumption that X is smooth, the Albanese morphism is even birational. Note that the last two results continue to hold under lessened assumptions imposed on X, and statements in the non-smooth case are deduced in [61] and also [32], Appendix A.

Proposition 4.12. *Let X be a smooth, proper surface over a separably closed field k with $h^0(\mathcal{O}_X) = 1$. Assume that $(\omega_X^2) = 0$ and $h^0(\omega_X) = 0$. Then the Albanese dimension m of X satisfies $m = h^1(\mathcal{O}_X) \leq 1$.*

Proof. Corollary 1.47 yields the equality $b_2 - 2b_1 + \deg(c_1^2) = 10 - 12h^1(\mathcal{O}_X) + 12h^2(\mathcal{O}_X)$. First, Serre duality gives $h^2(\mathcal{O}_X) = h^0(\omega_X) = 0$. This also ensures that Pic^0_X is smooth and of dimension $g = h^1(\mathcal{O}_X)$. Second, $\deg(c_1^2) = (\omega_X^2) = 0$ by assumption. Third, the identification $b_1 = 2g$ holds due to Proposition 1.22. Altogether, the original equality above simplifies to $b_2 = 10 - 8g$. This value is non-negative, and hence only the cases $g = 0$ and $g = 1$ are possible. The case $g = 0$ is equivalent to $m = 0$ by Proposition 3.42. Otherwise if $g = 1$, then $\dim(\mathrm{Alb}^1_X) = 1$ holds, so $m \leq 1$. But the equivalence in the case $g = 0$ shows that then in fact $m = 1$ must be true. □

Corollary 4.13. *Let X be a smooth, proper surface over a separably closed field k with $h^0(\mathcal{O}_X) = 1$. If X is of maximal Albanese dimension and ω_X is numerically trivial, then $\omega_X \simeq \mathcal{O}_X$.*

4.4 Kodaira Dimension and the Enriques Classification

Let X be an integral, proper scheme of dimension n over an arbitrary field k. Every invertible sheaf $\mathscr{L}$ on X yields a number $\mathrm{kod}(\mathscr{L}) \in \{-\infty, 0, 1, \ldots, n\}$, called *Kodaira dimension* of $\mathscr{L}$. Synonyms are *Kodaira-Iitaka dimension*, *ℓ-dimension* or *D-dimension*. It can be defined in various equivalent ways, which will briefly be described in the following, based on Cutkosky [31].

Let $\mathrm{Bs}(\mathscr{L}^{\otimes t}) = \{\, x \in X \mid s(x) = 0 \text{ for all } s \in H^0(X, \mathscr{L}^{\otimes t}) \,\}$ be the *base locus* of $\mathscr{L}^{\otimes t}$, and $\Phi_t \colon X \smallsetminus \mathrm{Bs}(\mathscr{L}^{\otimes t}) \to \mathbb{P}^{n_t}$ the induced natural morphism, where $n_t = h^0(\mathscr{L}^{\otimes t}) - 1$. Denote by Z_t the closure of the image of Φ_t. Then define

$$\mathrm{kod}(\mathscr{L}) = \sup_{t \geq 1} \{\dim(Z_t)\}.$$

The base locus is invariant under base change of X along field extensions $k \subset E$. As a consequence, also $\operatorname{kod}(\mathscr{L}) = \operatorname{kod}(\mathscr{L}_E)$ holds.

Lemma 4.14. *In the situation above, assume that X is geometrically integral. Then:*

(i) $\operatorname{kod}(\mathscr{L}) = -\infty \iff h^0(\mathscr{L}^{\otimes t}) = 0$ *for all* $t \geq 1$.

(ii) $\operatorname{kod}(\mathscr{L}) = 0 \iff h^0(\mathscr{L}^{\otimes t}) \leq 1$ *for all* $t \geq 1$ *and* $h^0(\mathscr{L}^{\otimes t_0}) = 1$ *for some* $t_0 \geq 1$.

(iii) $\operatorname{kod}(\mathscr{L}) \geq 1 \iff h^0(\mathscr{L}^{\otimes t_0}) \geq 2$ *for some* $t_0 \geq 1$.

Proof. Clearly, (i) is immediate by definition. Then (iii) follows once (ii) is also proven. Since both $\operatorname{kod}(\mathscr{L})$ and $h^0(\mathscr{L}^{\otimes t})$ are invariant under field extensions, assume without loss of generality that k is algebraically closed. To prove (ii), first assume that $h^0(\mathscr{L}^{\otimes t}) \leq 1$ for all $t \geq 1$ and $h^0(\mathscr{L}^{\otimes t_0}) = 1$ for some $t_0 \geq 1$. Whenever $h^0(\mathscr{L}^{\otimes t}) = 1$, then $n_t = 0$ holds. This directly yields $\dim(Z_t) = 0$. Hence $\operatorname{kod}(\mathscr{L}) = 0$ follows. Now for the other implication, suppose $\operatorname{kod}(\mathscr{L}) = 0$ and assume by contradiction that $h^0(\mathscr{L}^{\otimes t}) \geq 2$ for some $t \geq 1$. Choose $\sigma_0, \sigma_1 \in H^0(X, \mathscr{L}^{\otimes t})$ linearly independent over k, extend them to a basis and denote by $\Psi_t \colon X \smallsetminus \operatorname{Bs}(\mathscr{L}^{\otimes t}) \to Z_t$ the restriction of the induced morphism Φ_t. As the zero-dimensional scheme Z_t is integral and k is algebraically closed, this necessarily yields $Z_t = \operatorname{Spec}(k)$. Hence $\operatorname{Pic}(Z_t) = 0$ and the sections $T_0, T_1 \in H^0(\mathbb{P}^{n_t}, \mathcal{O}_{\mathbb{P}^{n_t}}(1))$ given by the first two indeterminates must become linearly dependent after restriction to Z_t. Thus $\sigma_0 = \Psi_t^*(T_0|_{Z_t})$ and $\sigma_1 = \Psi_t^*(T_1|_{Z_t})$ are also linearly dependent, a contradiction. □

Consider the graded ring $R(\mathscr{L}) = \bigoplus_{t \geq 0} H^0(X, \mathscr{L}^{\otimes t})$ and define the *D-model* of $\mathscr{L}$ to be $P(\mathscr{L}) = \operatorname{Proj}(R(\mathscr{L}))$. Following [52], Partie 3.7, the open subset

$$U_{\mathscr{L}} = \{ x \in X \mid s(x) \neq 0 \text{ for some } s \in H^0(X, \mathscr{L}^{\otimes t}) \text{ and } t \geq 1 \} \subset X$$

is the source of a canonical morphism $\varphi_{\mathscr{L}} \colon U_{\mathscr{L}} \to P(\mathscr{L})$. It satisfies $\varphi_*(\mathcal{O}_{U_{\mathscr{L}}}) = \mathcal{O}_{P(\mathscr{L})}$. Indeed, this is a local problem, solved by [53], Proposition 1.4.5. Another interpretation of the Kodaira dimension is

$$\operatorname{kod}(\mathscr{L}) = \dim(P(\mathscr{L}))$$

according to [31], Theorem 7.2. This also yields that $\operatorname{kod}(\mathscr{L}) = \operatorname{kod}(\mathscr{L}^{\otimes t})$ for all $t \geq 1$, since $\dim(P(\mathscr{L})) = \dim(P(\mathscr{L}^{\otimes t}))$ holds by [52], Proposition 3.1.8. Finally, the value $\operatorname{kod}(\mathscr{L})$ equals the natural number κ such that the function $t \mapsto h^0(\mathscr{L}^{\otimes t})$, if it is not zero, is bounded by two polynomials of degree κ for t sufficiently divisible, see [31], Lemma 6.1 and Theorem 9.2. In fact, moreover the limit

$$\lim_{t \to \infty} \frac{h^0(\mathscr{L}^{\otimes t_0 t})}{t^\kappa}$$

exists and is non-zero for some $t_0 \geq 1$, again due to [31], Theorem 7.2. Thereby $\operatorname{kod}(\mathscr{L})$ is equal to the number κ such that the limit above exists and is non-zero for some $t_0 \geq 1$.

In the special case that X is regular over k and $\mathscr{L} = \omega_X$, define the *Kodaira dimension* of X as $\operatorname{kod}(X) = \operatorname{kod}(\omega_X)$. Note that it is common in the singular case to define the Kodaira dimension of X as $\operatorname{kod}(\widetilde{X})$ for a resolution of singularities $\widetilde{X} \to X$, its existence presumed. This value may differ from $\operatorname{kod}(\omega_X)$ if X is Gorenstein, see [83], Example 2.1.6, as well as Section 4.6 and Section 4.7 below. The Kodaira dimension is a birational invariant of smooth, integral, proper k-schemes by the proof of [64], Chapter II, Theorem 8.19.

The value $\operatorname{kod}(X) \in \{-\infty, 0, 1, \dots, n\}$ thus naturally sorts X in one of $n+2$ classes. If $\operatorname{kod}(X) = n$ is maximal, then X is called *of general type*, otherwise *of special type*. In the case that ω_X is numerically trivial, then either $\operatorname{kod}(X) = -\infty$ or $\operatorname{kod}(X) = 0$ holds by Lemma 1.35, and the problem to solve in this chapter is to exclude the former case.

Classification of Curves. Before turning towards surfaces, briefly survey curves. So let X be a regular, integral, proper curve over an algebraically closed field k. Then:

$$\begin{aligned} \operatorname{kod}(X) = -\infty \quad &\iff \quad h^1(\mathcal{O}_X) = 0. \\ \operatorname{kod}(X) = 0 \quad &\iff \quad h^1(\mathcal{O}_X) = 1. \\ \operatorname{kod}(X) = 1 \quad &\iff \quad h^1(\mathcal{O}_X) \geq 2. \end{aligned}$$

This follows directly from the following facts, see for instance [88], Section 7.4, as a reference. If $h^1(\mathcal{O}_X) = 0$, then $X \simeq \mathbb{P}^1$ and so $\operatorname{kod}(X) = -\infty$. In the case that $h^1(\mathcal{O}_X) = 1$, the choice of a point $x \in X(k)$ turns X into an elliptic curve, so $\operatorname{kod}(X) = 0$. Finally, the assumption $h^1(\mathcal{O}_X) \geq 2$ results in the estimate $\deg(\omega_X) = 2 \cdot h^1(\mathcal{O}_X) - 2 > 0$, and thus ω_X is ample, showing $\operatorname{kod}(X) = 1$. The isomorphism classes of elliptic curves are parameterized by elements of $k = \mathbb{A}^1(k)$ via the j-invariant, and their coarse moduli scheme is $\mathbb{A}^1$. In the case of curves of general type, the classification problem becomes more difficult, which can be summarized in saying that their "moduli space has dimension $3g - 3$", see [96], Chapter 5. An overview of coarse moduli schemes of curves is given in Abramovich and Oort [2], Part II, whereas a treatment in the larger category of algebraic stacks is [9], Tag 0DZY, which traces back to the fundamental work of Deligne and Mumford [34].

Classification of Surfaces. Now let X be a regular, integral, proper surface over a fixed algebraically closed ground field k. Then X is Gorenstein and $\omega_X \simeq \mathcal{O}_X(K_X)$ is the invertible sheaf associated to a canonical divisor K_X, which is a Cartier divisor. The classification of surfaces can be divided into two steps. An integral curve C on X is called a (-1)-*curve*, if $C \simeq \mathbb{P}^1$ and $(C^2) = -1$. Since k is algebraically closed, this is equivalent to $(C^2) < 0$ and $(K_X \cdot C) < 0$ by the adjunction formula. Similarly, a (-2)-*curve* is a curve C such that $C \simeq \mathbb{P}^1$ and $(C^2) = -2$. This means that $(C^2) < 0$ and $(K_X \cdot C) = 0$. The surface X is called *minimal* if there are no (-1)-curves on X. The first step is to pass from X to a minimal surface and the second step is the classification of minimal surfaces. A reference with a concise overview of the subject is Liedtke [84].

This paragraph outlining the first step refers to Hartshorne [64], Chapter V, Section 5. The exceptional divisor of the blowup at a closed point on a regular, integral, proper surface is a (-1)-curve, and it is conversely valid that every (-1)-curve C on X is contractible to a regular point. This means that there exists a surface X_1 and a morphism $X \to X_1$

which is the blowup at a closed point $x_1 \in X_1$ such that its exceptional divisor is C. Successively contracting (-1)-curves on X will terminate after a finite number of steps, since the Picard numbers fulfill $\rho(X_1) = \rho(X) - 1$. This leads to a birational morphism $X \to X_n$ onto a minimal surface $X_{\min} = X_n$. It is actually also true that every birational morphism between regular, integral, proper surfaces is a finite sequence of contractions of (-1)-curves. The minimal surface $X_{\min}$ is unique if $\mathrm{kod}(X) \geq 0$ and in general not unique if $\mathrm{kod}(X) = -\infty$. For instance, $\mathbb{P}^2$ and the Hirzebruch surfaces $F_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ for $e \geq 0$ and $e \neq 1$ are minimal and birational to $\mathbb{P}^2$, but non-isomorphic. Observe that the surface F_1 is not minimal, since the section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}) \subset F_1$ is a (-1)-curve. Also, those surfaces listed above cover all minimal models of $\mathbb{P}^2$, see [10], Theorem 12.8.

The second step is the *Enriques classification* of minimal surfaces, which was accomplished by Mumford and Bombieri [94], [18], [17] in arbitrary characteristic $p \geq 0$. From now on, suppose additionally that X is minimal. The following list characterizes X with regard to $\mathrm{kod}(X)$, and equivalent descriptions of each of the four possible cases are given:

(a1) $\mathrm{kod}(X) = -\infty$.

(a2) There exists an integral curve C on X such that $(K_X \cdot C) < 0$.

(a3) X is either $\mathbb{P}^2$ or a ruled surface.

(a4) $h^0(\omega_X^{\otimes 12}) = 0$.

(b1) $\mathrm{kod}(X) = 0$.

(b2) ω_X is numerically trivial.

(b3) X is either a K3-surface, Enriques surface, abelian surface or bielliptic surface.

(b4) $\omega_X^{\otimes 4} \simeq \mathcal{O}_X$ or $\omega_X^{\otimes 6} \simeq \mathcal{O}_X$.

(c1) $\mathrm{kod}(X) = 1$.

(c2) K_X is nef with $(K_X^2) = 0$ and $(K_X \cdot H) > 0$ for all ample H.

(c3) For some $n \geq 1$, the sheaf $\omega_X^{\otimes n}$ yields a genus-one fibration.

(c4) $(K_X^2) = 0$ and $4K_X$ or $6K_X$ is linear equivalent to a curve.

(d1) $\mathrm{kod}(X) = 2$.

(d2) K_X is nef with $(K_X^2) > 0$ and $(K_X \cdot H) > 0$ for all ample H.

(d3) ω_X is semi-ample and the canonical morphism $\varphi_{\omega_X}\colon X \to P(\omega_X)$ is a birational morphism onto a normal surface, which is the contraction of all (-2)-curves on X.

For (d3), see [84], Theorem 8.1. A *ruled surface* is isomorphic to $\mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ of $\operatorname{rank}(\mathcal{E}) = 2$ on a regular, integral, proper curve C. A *genus-one fibration* is a fibration $X \to Y$ onto a curve Y such that the generic fiber satisfies $h^1(\mathcal{O}_{X_\eta}) = 1$. Note that X_η is regular and geometrically integral by Remark A.13. So the generic fiber is either smooth and as a consequence $X_{\overline{\eta}} := (X_\eta)_{\overline{K(Y)}}$ becomes an elliptic curve after the choice of a rational point, or $X_{\overline{\eta}}$ is a rational curve with one cusp. The latter means it is isomorphic to $V_+(v^2w - u^3) \subset \mathbb{P}^2$, see for instance [10], Theorem 7.18. This can only happen if $p \in \{2, 3\}$. In fact, since $h^1(\mathcal{O}_{X_{\overline{\eta}}}) = 1$, there is only one singular point of multiplicity 1 by [88], Chapter 7, Proposition 5.4. For the normalization $X'_{\overline{\eta}}$ of $X_{\overline{\eta}}$, there is the condition that $1 = h^1(\mathcal{O}_{X_{\overline{\eta}}}) - h^1(\mathcal{O}_{X'_{\overline{\eta}}})$ is an integral multiple of $(p-1)/2$, yielding only the possibilities $p \in \{2, 3\}$. This divisibility relation dates back to a result of Tate [124] in terms of function fields, whereas a proof in the language of schemes was given by Schröer [113].

If the generic fiber X_η is smooth, the fibration is called *elliptic*, otherwise *quasielliptic*. Given an elliptic fibration, the smoothness of the generic fiber passes on to almost all closed fibers X_y by Proposition A.12. Similarly, almost all fibers of a quasielliptic fibration are rational with one cusp, again due to [10], Theorem 7.18.

Surfaces in class (b) with numerically trivial dualizing sheaf are divided into four subclasses, according to their second Betti number, as displayed in the table below:

	b_2	b_1	$\deg(c_2)$	h^1	h^2	χ
K3	22	0	24	0	1	2
Enriques	10	0	12	0	0	1
				1	1	1
abelian	6	4	0	2	1	0
bielliptic	2	2	0	1	0	0
				2	1	0

Table 1
Invariants of the four classes of surfaces in $\operatorname{kod}(X) = 0$.

The order of ω_X is always dividing 4 or 6, and all possibilities actually occur. Details will be given in Section 6.2. The four classes are related to each other by essentially two constructions, which will be outlined in the following, in the course of presenting some of their basic properties. Consider the chart outlining the connections (I), (II), (III), which will be described afterwards:

$$\begin{array}{ccc} \text{abelian} & \overset{\text{(II)}}{\cdots\cdots} & \text{K3} \\ \text{(III)}\ \vdots & & \vdots\ \text{(I)} \\ \text{bielliptic} & & \text{Enriques.} \end{array}$$

Beforehand, note that every finite commutative group scheme G over k has a *Cartier dual* G^D, defined by the group valued point functor $G^D(T) = \mathrm{Hom}_{(\text{Grp-Sch})}(G_T, \mathbb{G}_{m,T})$. Details are given in Section A.1. There is a natural isomorphism

$$H^1(X_{\mathrm{fppf}}, G_X) \xrightarrow{\simeq} \mathrm{Hom}_{(\text{Grp-Sch})}(G^D, \mathrm{Pic}_X) \tag{4.4}$$

according to Raynaud [105], Proposition 6.2.1. The left-hand side classifies isomorphism classes of principal homogeneous spaces under G_X, as reviewed in Remark A.3. If G_X is smooth, every principal homogeneous space already becomes trivial on an étale covering of X. An explicit construction in the case of $G_X = \mu_{m,X}$ will be studied later in Section 5.1.

(I) Let X be an Enriques surface. The following explanations are based on [17] as well as Cossec and Dolgachev [30], Chapters 0 and I. The group scheme Pic^τ_X is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, μ_2 or α_2. In characteristic $p \neq 2$ only $\mathrm{Pic}^\tau_X = \mathbb{Z}/2\mathbb{Z}$ is possible, in which case there is an isomorphism $\mathbb{Z}/2\mathbb{Z} \simeq \mu_2$. For $G = \mathbb{Z}/2\mathbb{Z}$, μ_2, α_2 its dual is $G^D = \mu_2$, $\mathbb{Z}/2\mathbb{Z}$, α_2 and the corresponding principal homogeneous space $\widetilde{X} \to X$ to the inclusion $\mathrm{Pic}^\tau_X \subset \mathrm{Pic}_X$ is called the *K3-cover* of X. The subsequent table contains basic properties in the three different cases. Here $F\colon H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is the k-linear map induced by the absolute Frobenius.

	Pic^τ_X	$h^1(\mathcal{O}_X)$	$h^2(\mathcal{O}_X)$	F	p	$\widetilde{X} \to X$
classical	$\mathbb{Z}/2\mathbb{Z}$	0	0		$\neq 2$ 2	étale inseparable
ordinary	μ_2	1	1	bijective	2	étale
supersingular	α_2	1	1	zero	2	inseparable

Table 2
Invariants of Enriques surfaces.

The integral surface $\widetilde{X}$ is always Gorenstein with $\omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}$ and satisfies $h^1(\mathcal{O}_{\widetilde{X}}) = 0$ as well as $h^2(\mathcal{O}_{\widetilde{X}}) = 1$ like a K3-surface. But $\widetilde{X}$ is only a K3-surface in the case that $\mathrm{Pic}^\tau_X \simeq \mu_2$, that is, if X is classical and $p \neq 2$ or if X is ordinary and $p = 2$. In those two cases, the covering $\widetilde{X} \to X$ is étale since $G = \mathbb{Z}/2\mathbb{Z}$ is smooth over k. In general, $\widetilde{X}$ is only birational to a K3-surface or a rational surface, and can be non-normal.

(II) Abelian surfaces are exactly the two-dimensional abelian varieties. The Kummer construction is a method to pass from an abelian surface to a K3-surface. Take the minimal resolution of singularities of the quotient of an abelian surface by the sign involution to obtain a K3-surface, see [10], Theorem 10.6. In characteristic $p = 2$, the method yields a K3-surface if and only if the abelian surface is not supersingular, see [71] for a detailed treatment. An abelian variety is called *supersingular* if it is isogenous to a product of elliptic curves, where each of them does not have a point of order $p > 0$. For supersingular abelian surfaces, the Kummer construction yields a rational surface.

(III) Bielliptic surfaces always possess two genus-one fibrations. One is the Albanese morphism onto the Albanese scheme, which is an elliptic curve. The other one maps onto $\mathbb{P}^1$ and is transversal to the first one. Bielliptic surfaces are also known as *hyperelliptic* or *quasi-hyperelliptic* surfaces, depending on the shape of the Albanese fibration. Bielliptic surfaces are always quotients of a product of two curves, precisely: Every bielliptic surface X is isomorphic to $(E \times C)/G$, where E is an elliptic curve, the curve C is either elliptic or rational with one cusp, and $G \subset E$ is a finite subgroup scheme. The group scheme G acts faithfully on both factors, whereat it acts by translations on E. The action on $E \times C$ is the diagonal action, which is free as already the action of G on the first factor E is free. Furthermore, the two genus-one fibrations are induced by the projections on the two factors:

$$\begin{array}{ccc} (E \times C)/G \simeq X & \longrightarrow & C/G \simeq \mathbb{P}^1 \\ \downarrow \scriptstyle{\text{alb}} & & \\ E/G \simeq \mathrm{Alb}^1_X & & . \end{array}$$

The detailed list of all possible cases was established in [18], [17]. The canonical morphism $A = E \times C \to (E \times C)/G = X$ is finite and flat, see [95], Section 12, Theorem 1 on page 111. In the case that G is a constant group scheme, it is even étale by op. cit., Section 7, Theorem on page 66. Similar to K3-covers of Enriques surfaces, this yields a connection between abelian surfaces and bielliptic surfaces if $C = E'$ is elliptic. It may happen that the order of ω_X equals the degree of $A \to X$. Then, if this value is not divisible by p, the subsequent Corollary 5.8 will show that this is the étale covering canonically attached to ω_X. But this is not always the case. Especially in the quasielliptic case, it is even possible that $\omega_X \simeq \mathcal{O}_X$. This will be investigated further in Section 6.2.

4.5 Smooth Surfaces with Numerically Trivial Dualizing Sheaf

Let X be a regular, integral, proper surface over an algebraically closed field k with ω_X numerically trivial. So X contains no (-1)-curves, that is, it is a minimal surface. The Enriques classification then shows that $\omega_X \in \mathrm{Pic}(X)$ must have finite order: The numerical triviality of ω_X is equivalent to $\mathrm{kod}(X) = 0$, which exactly means that some power $\omega_X^{\otimes d}$ admits a non-zero global section. So this power is trivial due to Lemma 1.35.

Independent of this conclusion using the classification, a direct proof of this fact will be given below in order to get a more accurate understanding of what specifically causes the dualizing sheaf to have finite order. This is one of the two main results in this chapter. One central task in the proof will be to show that the following criterion is applicable. It can be formulated in arbitrary dimensions, and might be of independent interest.

Lemma 4.15. *Let X be a Gorenstein, integral, proper scheme over an arbitrary field k such that ω_X is numerically trivial. Let $D \subset X$ be an integral effective Cartier divisor. Suppose that there exists a proper surjection $f\colon X \to Y$ of k-schemes and natural numbers $m_1, m_2, m_3 \geq 1$ satisfying the following conditions:*

(i) $\omega_D^{\otimes m_1} \simeq \mathcal{O}_D$ *and* $h^0(\mathcal{O}_D(m_2D)) \geq 1$.

(ii) $\omega_X^{\otimes m_3} \simeq f^*(\mathscr{N})$ *for some* $\mathscr{N} \in \mathrm{Pic}(Y)$.

(iii) If $D \xrightarrow{\alpha} Z \xrightarrow{\beta} Y$ *denotes the Stein factorization of* $f|_D$*, then* β *is flat.*

Then $\omega_X \in \mathrm{Pic}(X)$ *has finite order, which divides* $\mathrm{rank}(\beta_*(\mathcal{O}_Z)) \cdot \mathrm{lcm}(m_1, m_2, m_2)$.

Remark 4.16. (1) Observe that in (i), a sufficient condition for $h^0(\mathcal{O}_D(m_2D)) \geq 1$ to hold is that $\mathcal{O}_X(D)$ is semi-ample.

(2) In the case that f is the fibration obtained from the Albanese morphism, and additionally k is algebraically closed and X is normal, then condition (ii) is automatically satisfied according to Proposition 3.39.

(3) Condition (iii) forces the restriction $f|_D$ to be surjective. Indeed, the finite and flat part β has to be finite locally free, which means that $\beta_*(\mathcal{O}_Z)$ is a locally free sheaf by the structure theorem of finitely generated projective modules over a ring.

If on the other hand $f|_D$ is surjective and Y is a normal curve, then condition (iii) is automatically fulfilled: Here β is a surjection from an integral scheme to a Dedekind scheme, and thus flat, see for instance [43], Proposition 14.14.

Proof. Set $m = \mathrm{lcm}(m_1, m_2, m_2)$ so that (i) and (ii) are valid for m in place of m_1, m_2, m_3, respectively. The adjunction formula and (i) yield $\mathcal{O}_D \simeq \omega_X^{\otimes m} \otimes \mathcal{O}_X(mD)|_D$. Then it follows from (ii) that $f^*(\mathscr{N}^\vee)|_D \simeq \mathcal{O}_D(mD)$. Therefore $h^0(f^*(\mathscr{N}^\vee)|_D) \geq 1$ is true by assumption (i). The projection formula applied to the fibration α yields

$$h^0(\beta^*(\mathscr{N}^\vee)) = h^0(\alpha^*\beta^*(\mathscr{N}^\vee)) = h^0(f^*(\mathscr{N}^\vee)|_D) \geq 1.$$

Because ω_X is numerically trivial and f is surjective, also $\mathscr{N}$ must be numerically trivial. Thus this conclusion holds true for $\beta^*(\mathscr{N}^\vee)$. Now D is integral by assumption, so also Z is integral. Consequently, it follows that $\beta^*(\mathscr{N}^\vee) \simeq \mathcal{O}_Z$ is trivial, as a non-zero global section exists.

Now the projection formula yields $\beta_*(\mathcal{O}_Z) \simeq \beta_*\beta^*(\mathscr{N}^\vee) = \beta_*(\mathcal{O}_Z) \otimes \mathscr{N}^\vee$. Denote the rank of the locally free sheaf $\beta_*(\mathcal{O}_Z)$ by r, and set $\mathscr{L} = \det(\beta_*(\mathcal{O}_Z))$. Taking determinants gives $\mathscr{L} \simeq \mathscr{L} \otimes \mathscr{N}^{\otimes -r}$. Hence $\mathscr{N}^{\otimes r} \simeq \mathcal{O}_Y$ follows, so $\omega_X^{\otimes mr} \simeq \mathcal{O}_X$ is trivial. □

In the proof below, this criterion will be applied to the Albanese morphism of X. The proof is structured as follows: Except for the case of Albanese dimension 1, the other two cases can directly be completed by previous results in this chapter. In the former case, the aforementioned criterion is applied, and the challenge is to verify that its assumptions are fulfilled. Having the Enriques classification in mind, this case will in the course of the proof turn out to mean that the surface has the cohomological invariants of a bielliptic surface. Here the proof uses arguments from the analysis of bielliptic surfaces, [18], Theorem 3, and in doing so, some details will be added to a certain argument which was only briefly addressed in the source.

Theorem 4.17. *Let X be a smooth, proper surface over an arbitrary field k with numerically trivial dualizing sheaf ω_X. Then $\omega_X \in \mathrm{Pic}(X)$ has finite order.*

Proof. First of all, by Lemma 4.1, it is sufficient to consider the case of an algebraically closed ground field k. Furthermore, reduce to a connected component and assume without loss of generality that X is connected. Then X is integral as a consequence of its normality. The Albanese dimension of X has the three possibilities $m = 0, 1, 2$. The case $m = 0$ was treated in Proposition 4.8: Then $\mathrm{Pic}^{\tau}(X)$ is a torsion group, so in particular ω_X has finite order. If X is of maximal Albanese dimension $m = 2$, then Corollary 4.13 implies that $\omega_X \simeq \mathscr{O}_X$ is trivial. Thereby it remains to consider the case that X is of Albanese dimension $m = 1$.

If $h^0(\omega_X) = 1$, then $\omega_X \simeq \mathscr{O}_X$. Thus suppose $h^0(\omega_X) = 0$ from now on. In this situation, Proposition 4.12 yields that $h^1(\mathscr{O}_X) = m = 1$. Denote the Stein factorization of the Albanese morphism by $X \xrightarrow{f} Y \to \mathrm{Alb}^1_X$. Since f is a fibration, the properties of fibrations collected in Proposition A.12 show that the curve Y is normal, integral, and hence regular. As already mentioned in Remark 4.16 above, the morphism f to the Dedekind scheme Y is flat. Thus all its fibers are equidimensional of dimension 1. Because X is normal, Proposition 3.39 yields that $\dim(\mathrm{Pic}_X) = \dim(\mathrm{Pic}_Y)$ and $\omega_X^{\otimes l} \simeq f^*(\mathscr{N})$ for some $\mathscr{N} \in \mathrm{Pic}^0(Y)$ and $l \geq 1$. Actually, $l = 1$ is achievable:

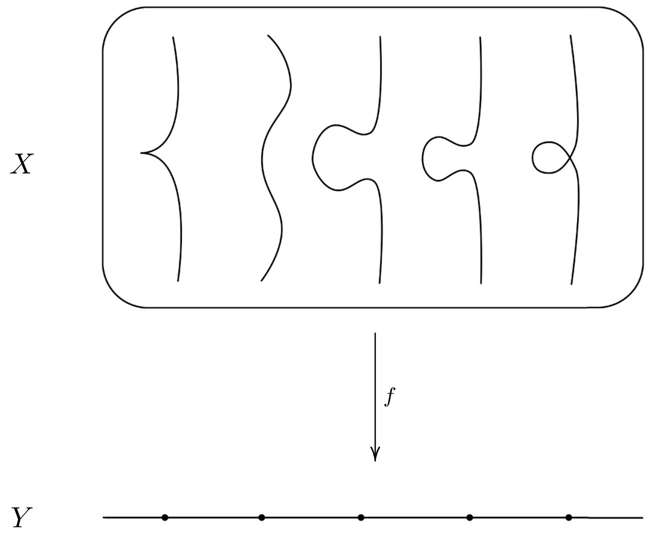


Figure 4: A visualization of the fibration f.

Step 1: The dualizing sheaf $\omega_X = f^*(\mathscr{N})$ is the pullback of an invertible sheaf. The vanishing $h^0(\omega_X) = 0$ means $h^2(\mathscr{O}_X) = 0$ by Serre duality. Hence both Pic_X and Pic_Y are smooth due to Theorem A.30, and of the same dimension $h^1(\mathscr{O}_Y) = h^1(\mathscr{O}_X) = 1$. This implicates that Y becomes an elliptic curve after the choice of a k-rational point. Particularly, it follows that $\omega_Y = \mathscr{O}_Y$ and already $Y = \mathrm{Alb}^1_X$.

Because f is a fibration and all sheaf cohomology groups on the curve Y are zero in degrees at least 2, the Leray spectral sequence $E_2^{a,b} = H^a(Y, R^b f_*(\mathcal{O}_X)) \Rightarrow H^{a+b}(X, \mathcal{O}_X)$, see [46], Théorème 3.7.3, yields the exact sequence in low degrees

$$0 \longrightarrow H^1(Y, \mathcal{O}_Y) \xrightarrow{\simeq} H^1(X, \mathcal{O}_X) \longrightarrow H^0(Y, R^1 f_*(\mathcal{O}_X)) \longrightarrow 0. \tag{4.5}$$

Consequently, the coherent sheaf $\mathcal{F} = R^1 f_*(\mathcal{O}_X)$ satisfies $h^0(\mathcal{F}) = 0$. Consider the torsion subsheaf $\mathcal{T} \subset \mathcal{F}$, defined by the sections whose image in the stalk at the generic point $\eta \in Y$ is zero. Thus also $h^0(\mathcal{T}) = 0$ holds. As $\mathcal{T}_\eta = 0$ and Y is a curve, the support of the coherent sheaf $\mathcal{T}$ is a closed subset of Y and at most zero-dimensional. It thus consists of finitely many closed points and therefore $\mathcal{T} = \bigoplus \mathcal{T}_x$ is the sum of its non-zero stalks. But $h^0(\mathcal{T}) = 0$, so $\mathcal{T} = 0$. Hence $\mathcal{F}$ is torsion-free. As every torsion-free finitely generated module over a Dedekind ring is free, $\mathcal{F}$ is locally free.

Now $X_\eta \to \operatorname{Spec}(K(Y))$ is a flat base change of f, and hence $h^0(\mathcal{O}_{X_\eta}) = 1$. The dualizing sheaves satisfy the relation $\omega_X = \omega_{X/Y} \otimes f^*(\omega_Y) = \omega_{X/Y}$ and the compatibility of $\omega_{X/Y}$ with base change implies that $\omega_X|_{X_\eta} = \omega_{X_\eta}$ holds. As a consequence, the dualizing sheaf ω_{X_η} of the integral curve X_η is numerically trivial. By Example 4.3, then $\omega_{X_\eta} \simeq \mathcal{O}_{X_\eta}$. Thereby $h^1(\mathcal{O}_{X_\eta}) = 1$, which shows that f is a genus-one fibration. Since the fibers of f are one-dimensional, cohomology and base change ensures that $\mathcal{O}_X$ is cohomologically flat in degree 1. Now $\mathcal{F} = R^1 f_*(\mathcal{O}_X)$ is locally free, so its rank can be computed at the generic point. This shows that $\operatorname{rank}(\mathcal{F}) = h^1(\mathcal{O}_{X_\eta}) = 1$, and hence $\mathcal{F}$ is invertible.

The latter means that f has no wild fibers. Using this fact together with $h^1(\mathcal{O}_Y) = 1$ and $\chi(\mathcal{O}_X) = 0$, the canonical bundle formula, [18], Theorem 2, implies that $\deg(\mathcal{F}^\vee) = 0$, that is, $\mathcal{F}^\vee$ is numerically trivial. Furthermore, it gives that $\omega_X = f^*(\mathcal{F}^\vee) \otimes \mathcal{O}_X(D)$, where either $D = 0$ and f has no multiple fibers, or D is a curve supported on all multiple fibers of f. Recall that a fiber F is multiple if $F = rF'$ for an indecomposable curve of canonical type F' and a natural number $r \geq 2$. For the terminology and basic properties of curves of fiber type and of canonical type, which will be used throughout the proof, see Section A.6. As both ω_X and $f^*(\mathcal{F}^\vee)$ are numerically trivial, also D is numerically trivial. So $D = 0$ must hold. Eventually, set $\mathcal{N} = \mathcal{F}^\vee$ so that $\omega_X = f^*(\mathcal{N})$, as claimed.

Step 2: All fibers of f are integral. As seen in Step 1, the fibration f has no multiple fibers. Thus to verify that all fibers of f are integral, it suffices to deduce that all fibers are irreducible. Note that the regularity of X implies that X is locally factorial, meaning that all local rings $\mathcal{O}_{X,x}$ are factorial. This implies that Cartier divisors and Weil divisors on X coincide, so every curve on X is Cartier. Let $C \subset X$ be an arbitrary integral curve. The adjunction formula and the numerical triviality of ω_X yield

$$\left(C^2\right) = \deg(\mathcal{O}_C(C)) = \deg(\omega_C) = 2 \cdot h^1(\mathcal{O}_C) - 2. \tag{4.6}$$

This value is an even integer ≥ -2. If $(C^2) = -2$, then $C \simeq \mathbb{P}^1$ and so C is a (-2)-curve. Actually, there exist no (-2)-curves on X. The subsequent reasoning to show this fact follows the proof of [10], Lemma 8.7. The estimate

$$\rho(X) \leq b_2 = 4\dim(\mathrm{Pic}^0_X) - (K_X^2) + 10 - 12h^1(\mathcal{O}_X) + 12h^2(\mathcal{O}_X) = 2$$

is obtained from Proposition 1.22 and Corollary 1.47. Here the smoothness of X enters, and for all remaining arguments in this proof, it is sufficient for X to be locally factorial. To see that the estimate is an equality, consider the classes $H, F \in \mathrm{Num}(X)$ of an ample invertible sheaf and a fiber of f. Their classes in $\mathrm{Num}(X) \otimes \mathbb{Q}$ are non-zero, and they are linearly independent since $(H^2) > 0$ and $(F^2) = 0$. So necessarily $\rho(X) = 2$.

Suppose there exists a (-2)-curve C on X. If C is contained in a fiber F of f, decompose $F = \sum_{i=1}^r m_i C_i$ into its integral components C_i for some $r \geq 2$ such that $C_1 = C$. Then F and C_1 are also linearly independent in $\mathrm{Num}(X) \otimes \mathbb{Q}$. Since $\rho(X) = 2$, they thus have to form a basis, which gives a linear combination $H = \alpha F + \beta C_1$. But then $(H \cdot F) = 0$ follows, since F is a curve of fiber type, contradicting the Nakai–Moishezon criterion. So C is not contained in a fiber of f, and hence $f(C) = Y$. This means that the Albanese dimension of C is one. But $C \simeq \mathbb{P}^1$, so this is a contradiction. Consequently, there are no (-2)-curves on X.

The implies that every curve of fiber type on X has to be irreducible. Indeed, each integral component of a reducible curve of fiber type would have negative self-intersection by Proposition A.21. In conclusion, all fibers of f are integral curves.

Step 3: There exists an integral curve $C \subset X$ of canonical type surjecting onto Y. In order to apply Lemma 4.15 to the fibration f, the following properties will show that its assumptions are verified: There exists an integral curve C on X with $\omega_C \simeq \mathcal{O}_C$ such that $f(C) = Y$ and $\mathcal{O}_X(C)$ is semi-ample. Except for the semi-ampleness of $\mathcal{O}_X(C)$, the existence will be shown in this step. Afterwards, the curve C can be altered slightly to see that $\mathcal{O}_X(C)$ is actually semi-ample. As there are no integral curves with negative self-intersection on X, every connected curve with $(C^2) = 0$ has to be irreducible. Then C is of fiber type, and as ω_X is numerically trivial, this means that C is of canonical type. Hence $\omega_C \simeq \mathcal{O}_C$ and $h^1(\mathcal{O}_C) = 1$ hold by Proposition A.24. Thus the task in this step is to find an integral curve C on X with $(C^2) = 0$ such that $f(C) = Y$.

The argumentation is following the proof of [18], Theorem 3. Also, details will be added to a certain argument which was only briefly addressed in the source. Let H be an ample curve on X. Denote by $X_y = f^{-1}(y)$ the fiber over a closed point $y \in Y$. Observe that all X_y are numerically equivalent, since all closed points $y \in Y$ have degree one, so they are numerically equivalent. Also, fix a point $e \in Y$. Consider the divisor $D = 2(H \cdot X_e)H - (H^2)X_e$. By construction, $(D^2) = 0$.

The intersection number $(D \cdot X_y) = 2(H \cdot X_y)^2 > 0$ is positive. Define $D_y = D + X_y - X_e$ for $y \in Y$ closed. If for some y the global sections $H^0(X, \mathcal{O}_X(D_y))$ are non-zero, there then exists a curve $C \sim D_y$, and thus $C \equiv D_y \equiv D$ satisfies $(C^2) = 0$, as desired. Assume by contradiction that $h^0(\mathcal{O}_X(D_y)) = 0$ for all $y \in Y$ closed. Note that $h^2(\mathcal{O}_X(D_y)) = 0$ for all $y \in Y$ closed, since otherwise there would be an effective divisor $D' \sim -D_y + K_X$, which in turn implies the contradiction $(D' \cdot X_y) = -(D \cdot X_y) < 0$. Riemann–Roch consequently yields $h^1(\mathcal{O}_X(D_y)) = 0$, because $\chi(\mathcal{O}_X) = 0$ and $(D_y^2) = 0$. Using that $\mathcal{O}_{X_e}(X_y) = \mathcal{O}_{X_e}$ is trivial, the exact sequence $0 \to \mathcal{O}_X(-X_e) \to \mathcal{O}_X \to \mathcal{O}_{X_e} \to 0$ induces the exact sequence $0 \to \mathcal{O}_X(D_y) \to \mathcal{O}_X(D + X_y) \to \mathcal{O}_{X_e}(D) \to 0$. Now as $h^0(\mathcal{O}_X(D_y)) = 0 = h^1(\mathcal{O}_X(D_y))$, there is an induced bijection

$$r_y \colon H^0(X, \mathcal{O}_X(D + X_y)) \xrightarrow{\simeq} H^0(X_e, \mathcal{O}_{X_e}(D)). \tag{4.7}$$

The target is non-zero, as $\chi(\mathcal{O}_{X_e}(D)) = (D \cdot X_e) > 0$ holds on the curve X_e. Choose a non-zero section $\sigma \in H^0(X_e, \mathcal{O}_{X_e}(D))$ to obtain $s_y = r_y^{-1}(\sigma)$ as well as curves $C_y = V(s_y)$ on X. Consider $Y \xrightarrow{j} \mathrm{Pic}^0_Y \xrightarrow{f^*} \mathrm{Pic}^0_X$, $y \mapsto \mathcal{O}_X(X_y - X_e)$ and compose it with the translation by $\mathcal{O}_X(D + X_e)$ to obtain

$$Y \longrightarrow \mathrm{Pic}_X, \quad y \longmapsto \mathcal{O}_X(D + X_y).$$

Since Pic_X represents the relative Picard functor, this morphism corresponds to the class of some $\mathscr{L} \in \mathrm{Pic}(X \times Y)$ modulo $\mathrm{Pic}(Y)$. By construction, $\mathscr{L}|_{X \times \{y\}} = \mathcal{O}_X(D + X_y)$ holds for $y \in Y$ closed. The restriction of $\mathscr{L}|_{X_e \times Y}$ induces $Y \to \mathrm{Pic}_{X_e}$ which maps every closed point $y \in Y$ to $\mathscr{L}|_{X_e \times \{y\}} = \mathcal{O}_{X_e}(D + X_y) = \mathcal{O}_{X_e}(D)$, so it is constant. Hence this morphism is also given by $\mathrm{pr}_{X_e}^*(\mathcal{O}_{X_e}(D))$, where $\mathrm{pr}_{X_e} \colon X_e \times Y \to X_e$. This means that $\mathscr{L}|_{X_e \times Y} = \mathrm{pr}_{X_e}^*(\mathcal{O}_{X_e}(D)) \otimes \mathrm{pr}_Y^*(\mathscr{M})$ for some invertible sheaf $\mathscr{M}$ on Y. Replace $\mathscr{L}$ by $\mathscr{L} \otimes \mathrm{pr}_Y^*(\mathscr{M})$ to achieve that $\mathscr{L}|_{X_e \times Y} = \mathrm{pr}_{X_e}^*(\mathcal{O}_{X_e}(D))$. The fact that pr_{X_e} is a fibration, see Example A.17, gives a bijection

$$\mathrm{pr}_{X_e}^* \colon H^0(X_e, \mathcal{O}_{X_e}(D)) \xrightarrow{\simeq} H^0(X_e \times Y, \mathscr{L}|_{X_e \times Y}).$$

Set $\tau = \mathrm{pr}_{X_e}^*(\sigma)$. The pullback of τ to any closed fiber $X_e \times \{y\}$ of the projection to Y is again σ.

Now the next aim is to extend $\tau \in H^0(X_e \times Y, \mathscr{L}|_{X_e \times Y})$ to some $t \in H^0(X \times Y, \mathscr{L})$, similar to (4.7). This will be shown by proving that $(\mathrm{pr}_Y)_*(\mathscr{L}) \to (\mathrm{pr}_Y)_*(\mathscr{L}|_{X_e \times Y})$ is an isomorphism. To do so, first observe that the restriction

$$H^0(X \times \{y\}, \mathscr{L}|_{X \times \{y\}}) \longrightarrow H^0(X_e \times \{y\}, \mathscr{L}|_{X_e \times \{y\}}) \tag{4.8}$$

regains the map r_y from (4.7) for all $y \in Y$ closed. Thereby it is bijective and both sides' dimensions are constant in $y \in Y$ closed. This value then also has to apply to $y = \eta$ as cohomological dimensions are upper-semicontinuous for flat sheaves. Now cohomology and base change gives the extension of (4.8) to the diagram

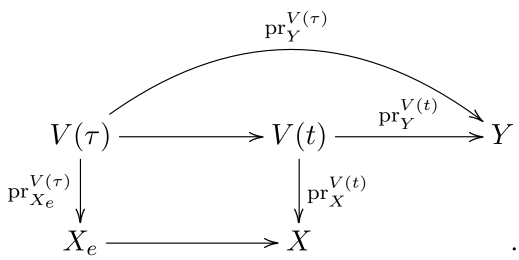
$$\begin{array}{ccc} (\mathrm{pr}_Y)_*(\mathscr{L}) \otimes \kappa(y) & \longrightarrow & (\mathrm{pr}_Y)_*(\mathscr{L}|_{X_e \times Y}) \otimes \kappa(y) \\ \simeq \downarrow & & \downarrow \simeq \\ H^0(X \times \{y\}, \mathscr{L}|_{X \times \{y\}}) & \xrightarrow[\simeq]{} & H^0(X_e \times \{y\}, \mathscr{L}|_{X_e \times \{y\}}) \end{array}$$

for all $y \in Y$, and ensures that both $(\mathrm{pr}_Y)_*(\mathscr{L})$ and $(\mathrm{pr}_Y)_*(\mathscr{L}|_{X_e \times Y})$ are locally free. Therefore the top arrow in the diagram is bijective for all closed points $y \in Y$. Nakayama's lemma implies that then $(\mathrm{pr}_Y)_*(\mathscr{L})_y \to (\mathrm{pr}_Y)_*(\mathscr{L}|_{X_e \times Y})_y$ is still surjective. As both sheaves are free of the same rank, it has to be a bijection. But the points $y \in Y$ with this property are constructible by [57], Proposition 9.4.4, so also $\eta \in Y$ is one of them. Hence $(\mathrm{pr}_Y)_*(\mathscr{L}) \to (\mathrm{pr}_Y)_*(\mathscr{L}|_{X_e \times Y})$ is an isomorphism, as claimed.

So there exists $t \in H^0(X \times Y, \mathscr{L})$ which restricts to $s_y \in H^0(X \times \{y\}, \mathcal{O}_X(D + X_y))$ as well as to $\tau \in H^0(X_e \times Y, \mathscr{L}|_{X_e \times Y})$ and further to $\sigma \in H^0(X_e \times \{y\}, \mathcal{O}_{X_e}(D))$. Notice that t is in fact independent of $y \in Y$, because the assignments $t \mapsto s_y \mapsto \sigma$ are induced by bijections and σ does not depend on y. This argument also yields that all those sections above are non-zero. Geometrically speaking, the vanishing set $V(t) \subset X \times Y$ corresponds to a morphism

$$Y \longrightarrow \mathrm{Hilb}_{X/k},$$

where a closed point $y \in Y$ maps to the subset $C_y = V(s_y)$, regarded as a point in the Hilbert scheme. Furthermore, the induced morphism $Y \to \mathrm{Hilb}_{X_e/k}$ is given by the vanishing set $V(\tau) \subset X_e \times Y$, and it maps all closed points $y \in Y$ to $V(\sigma)$. Now consider the commutative diagram



.

Since $(\mathrm{pr}_Y^{V(t)})^{-1}(y) = C_y \times \{y\}$ for all closed points $y \in Y$, the image of $\mathrm{pr}_X^{V(t)}$ contains the union $Z = \bigcup C_y$ over all closed points $y \in Y$. The projection to X is proper, so also $\overline{Z}$ has to be a subset of its image. Now show that $\overline{Z} = X$. Assume by contradiction that this is not the case. Then $\dim(\overline{Z}) = 1$, and thus only finitely many integral curves can appear in all C_y. Recall that $C_y \sim D + X_y$ and it can be assumed that there is no fiber $X_{y'}$ among the integral components of any C_y. Indeed, otherwise $C := C_y - X_{y'}$ is a curve numerically equivalent to D, which completes the task in this step. Hence $C_y \cap X_e = V(\sigma)$ shows that the multiplicity of C_y at each of its integral components is bounded by the degree of $V(\sigma)$. Thus there can only be finitely many different curves C_y. But then $C_y = C_{y'}$ has to hold for two points $y \neq y'$. Consequently, $X_y \sim X_{y'}$ follows, which cannot be true because $y \not\sim y'$ on the genus-one curve Y and f^* is injective.

Thus $\mathrm{pr}_X^{V(t)}$ is surjective, and thus its base change $\mathrm{pr}_{X_e}^{V(\tau)}$ has to be, too. But on the other hand, $V(\tau)$ is a curve on $X_e \times Y$, and each of its closed points has to be contained in some $(\mathrm{pr}_Y^{V(\tau)})^{-1}(y) = V(\sigma) \times \{y\}$. So the image of the set of all closed points maps via $\mathrm{pr}_{X_e}^{V(\tau)}$ to the finite set $V(\sigma) \subset X_e$. Therefore $\mathrm{pr}_{X_e}^{V(\tau)}$ cannot be surjective. This finally contradicts the assumption that $h^0(\mathcal{O}_X(D_y)) = 0$.

Choose a curve C linear equivalent to $D_y = D + X_y - X_e$. Then C is numerically equivalent to D, so $(C^2) = 0$ and $(C \cdot X_y) > 0$ for all closed fibers X_y. As seen in Step 2, there are no integral curves with negative self-intersection number on X, so every connected component of C has to be irreducible. Replace C by one of its integral components which surjects onto Y. Then still $(C^2) = 0$ holds, as required.

Step 4: The sheaf $\mathscr{O}_X(C)$ is semi-ample. This step's arguments are based on [94], page 333ff. Use that $\mathscr{O}_C(2K_X + 2C) = \omega_C^{\otimes 2}$ is trivial and consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X(2K_X + C) \longrightarrow \mathscr{O}_X(2K_X + 2C) \longrightarrow \mathscr{O}_C \longrightarrow 0.$$

Recall from (4.2) at the beginning of this section for the following reasoning that every curve M on X satisfies $h^2(\mathscr{O}_X(2K_X + M)) = 0$, as a consequence of $h^0(\mathscr{O}_X(-K_X)) = 0$. So especially $h^2(\mathscr{O}_X(2K_X + C)) = 0$. The long exact cohomology sequence derived from the short exact sequence above, and $h^1(\mathscr{O}_C) = 1$, show that $h^1(\mathscr{O}_X(2K_X + 2C)) \geq 1$. Since the self-intersection number of $2K_X + 2C$ is zero and $\chi(\mathscr{O}_X) = 0$, Riemann–Roch then gives that $h^0(\mathscr{O}_X(2K_X + 2C)) \geq 1$.

Choose a curve $A \sim 2K_X + 2C$. This curve is of canonical type: As $(A \cdot C) = 0$ and C is integral, decompose $A = \alpha C + \sum \beta_i B_i$ for integral curves B_i disjoint to C. Thus also $(A \cdot B_i) = 2(K_X \cdot B_i) + 2(C \cdot B_i) = 0$. Some coefficient β_i has to be non-zero, since otherwise $2K_X \sim \gamma C$ for some $\gamma \in \mathbb{Z}$, but then $(C \cdot X_y) > 0$ could not be true. In conclusion, there exists an indecomposable curve $C' = \frac{1}{\gcd(\beta_j)} \sum \beta_j B_j$ of canonical type which is disjoint to C, where $\sum \beta_j B_j$ denotes some connected component of $\sum \beta_i B_i$.

The subsequent arguments will be similar to the previous ones in this step, now for $C + C'$ instead of C. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_X(2K_X + C + C') \longrightarrow \mathscr{O}_X(2K_X + 2C + 2C') \longrightarrow \mathscr{O}_C \oplus \mathscr{O}_{C'} \longrightarrow 0.$$

This time, $h^1(\mathscr{O}_C \oplus \mathscr{O}_{C'}) = 2$ implies $h^1(\mathscr{O}_X(2K_X + 2C + 2C')) \geq 2$ and Riemann–Roch yields $h^0(\mathscr{O}_X(2K_X + 2C + 2C')) \geq 2$. Choose a curve $D' \sim 2K_X + 2C + 2C'$. Then by Proposition A.23, there is a fibration $g\colon X \to Y'$ onto a curve Y' such that each connected component of D' is a rational multiple of a fiber of g. Since $(C \cdot D') = 0$, the curve C has to be contained in a fiber of g. So Proposition A.21 shows that a multiple of C is a multiple of fiber. In conclusion, $\mathscr{O}_X(C)$ is semi-ample. Therefore, all assumptions for the application of Lemma 4.15 were verified to be fulfilled. Eventually, the criterion yields that $\omega_X \in \mathrm{Pic}(X)$ has finite order. □

The first step in the preceding proof can always be bypassed in characteristic $p = 0$ by the following lemma, which also holds true for higher-dimensional X. Let X be a Gorenstein, normal, integral, proper scheme over an algebraically closed field k of characteristic $p \geq 0$. The normality of X implies according to Proposition 3.39 that $\mathrm{Pic}^0(\mathrm{Alb}_X^1) \to \mathrm{Pic}^0(X)$ is bijective. Suppose that ω_X is numerically trivial. Then $\omega_X^{\otimes l} \simeq \mathrm{alb}_X^*(\mathscr{M})$ is valid for an invertible sheaf $\mathscr{M}$ on Alb_X^1 and some $l \geq 1$.

Lemma 4.18. *In the situation above, assume that $p \nmid l$. Then there exists an integral scheme X', a finite étale morphism $g\colon X' \to X$ and an invertible sheaf $\mathscr{M}'$ on $\mathrm{Alb}^1_{X'}$ such that $\omega_{X'} \simeq \mathrm{alb}^*_{X'}(\mathscr{M}')$. Furthermore, $\omega_{X'} \in \mathrm{Pic}(X')$ is numerically trivial and has the same order as $\omega_X \in \mathrm{Pic}(X)$.*

Proof. The group $\mathrm{Pic}^0(\mathrm{Alb}^1_X) = \mathrm{Pic}^0_X(k)$ is divisible, since $(\mathrm{Pic}^0_X)_{\mathrm{red}}$ is an abelian variety and k is algebraically closed. Choose some $\mathscr{N}$ satisfying $\mathscr{N}^{\otimes l} \simeq \mathscr{M}$ and consider the invertible sheaf $\mathscr{L} = \omega_X \otimes \mathrm{alb}^*_X(\mathscr{N})^\vee$ of order l.

There exists a canonical covering $g\colon X' \to X$ associated to $\mathscr{L}$ with the property that $g^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$. Those coverings will be studied in detail in the next chapter, see specifically Proposition 5.3. Here X' is integral and g is finite étale, since $p \nmid l$ by assumption. Consider the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\quad g \quad} & X \\ {\scriptstyle \mathrm{alb}_{X'}}\downarrow & & \downarrow{\scriptstyle \mathrm{alb}_X} \\ \mathrm{Alb}^1_{X'} & \xrightarrow[\quad h \quad]{} & \mathrm{Alb}^1_X\,. \end{array}$$

The étaleness of g implies that $\omega_{X'} = g^*(\omega_X)$ holds, so the invertible sheaf $\mathscr{M}' = h^*(\mathscr{N})$ now satisfies $\omega_{X'} = \mathrm{alb}^*_{X'}(\mathscr{M}')$, as claimed. Because g is finite locally free, its pullback induced on Picard groups is injective. This proves the assertion. □

4.6 $\mathbb{Q}$-Gorenstein Surfaces with Numerically Trivial Canonical Divisor

The aim of this section is to generalize Theorem 4.17 to singular surfaces. This will be accomplished in Theorem 4.22 below, which is the second main result in this chapter. Its proof relies on arguments used by Sakai [106], [107] in his extension of the Enriques classification to normal Gorenstein surfaces over $\mathbb{C}$. They can be adapted to the situation of suitable $\mathbb{Q}$-Gorenstein surfaces over a ground field of arbitrary characteristic, showing that K_X has finite order if it is numerically trivial. Specifically, the ideas of [106], Section 2, and [107], Section 1, are utilized.

Fix an arbitrary ground field k. Let X be a normal, proper k-scheme. Every $x \in X$ of codimension $\dim(\mathcal{O}_{X,x}) = 1$ is a discrete valuation ring. Hence the group homomorphism $\mathrm{Div}(X) \to Z^1(X),\ D \mapsto \sum \mathrm{val}_x(D)\overline{\{x\}}$ from the group of Cartier divisors to the group of Weil divisors on X is injective. It induces an injection $\mathrm{DivCl}(X) \to \mathrm{Cl}(X)$ between both groups modulo linear equivalence. Furthermore, X is locally factorial, that means all $\mathcal{O}_{X,x}$ are factorial rings, if and only if $\mathrm{Div}(X) \to Z^1(X)$ is bijective. The bijectivity then holds true for $\mathrm{DivCl}(X) \to \mathrm{Cl}(X)$. Since X is reduced and noetherian, the natural group homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X),\ D \mapsto \mathcal{O}_X(D)$ is surjective and induces a bijection $\mathrm{DivCl}(X) \to \mathrm{Pic}(X)$. For details to the above, see [58], Théorème 21.6.9 and Proposition 21.3.4.

On the other hand, the group $\mathrm{Cl}(X)$ of Weil divisors modulo linear equivalence is isomorphic to the group of isomorphism classes of reflexive sheaves which have rank 1 at every generic point of X, according to [65], Section 2. Denote by $\mathcal{O}_X(D)$ the reflexive sheaf associated to a Weil divisor D. This gives an extension of the map $D \mapsto \mathcal{O}_X(D)$ from above to Weil divisors. Two Weil divisors D_1 and D_2 satisfy

$$\mathcal{O}_X(D_1 + D_2) = (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee},$$

and the right-hand side defines the composition law of two reflexive sheaves $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$ of rank 1. Linear equivalence of Weil divisors $D_1 \sim D_2$ exactly means that $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$. Moreover, $\mathcal{O}_X(D)$ is uniquely determined by its restriction $\mathcal{O}_X(D)|_V$ to the regular locus $V \subset X$ due to [65], Theorem 1.12.

A $\mathbb{Q}$-*divisor* is an element of the group $Z^1(X)_\mathbb{Q} := Z^1(X) \otimes_\mathbb{Z} \mathbb{Q}$. Thus there are inclusions $\mathrm{Div}(X) \subset Z^1(X) \subset Z^1(X)_\mathbb{Q}$ and Weil divisors are exactly the $\mathbb{Q}$-divisors with integral coefficients. A $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-*Cartier* if mD is a Cartier divisor for some natural number $m \geq 1$. The smallest $m \geq 1$ with this property is the *index* of D. Furthermore, $D = \sum q_i D_i$ is *effective* if $q_i \geq 0$ holds for all coefficients. The integral curves D_i form the *integral components* of D. The *support* of D is $\mathrm{Supp}(D) = \bigcup \mathrm{Supp}(D_i)$, where $\mathrm{Supp}(D_i)$ is the underlying topological space of D_i.

The definition of integer-valued intersection numbers $(D_1 \cdot \ldots \cdot D_n)$ for Cartier divisors can be extended to $\mathbb{Q}$-Cartier divisors, with values in the rational numbers: Choose m_i such that $m_i D_i$ is Cartier and set $(D_1 \cdot \ldots \cdot D_n) = \frac{1}{m_1 \cdot \ldots \cdot m_n} \cdot (m_1 D_1 \cdot \ldots \cdot m_n D_n)$. As discussed at the end of Section 2.3, the dualizing sheaf ω_X is reflexive of rank 1. A Weil divisor K_X with $\omega_X \simeq \mathcal{O}_X(K_X)$ is a *canonical divisor* on X. Keeping in mind that such a divisor is only unique up to linear equivalence, also call K_X “the” canonical divisor. A proper k-scheme is $\mathbb{Q}$-*Gorenstein* if it is normal and its canonical divisor is $\mathbb{Q}$-Cartier.

Now suppose that X is a normal, proper surface over k. Its singular locus is closed in X of codimension at least 2, so there are only finitely many singular points, each one of them closed in X. Thus X has only isolated singularities. A *resolution of singularities* is a proper morphism

$$r\colon \widetilde{X} \to X$$

where $\widetilde{X}$ is a regular surface and r is an isomorphism over the regular locus of X. By Zariski's main theorem, r is a fibration. The existence of resolutions for surfaces was proved by Lipman [85], [86] for excellent, reduced, noetherian surfaces, generalizing previous results due to Zariski [132], Abhyankar [1] and others. A resolution can be obtained by iterating blow-ups and normalizations. Its fibers are at most one-dimensional and the exceptional locus $E \subset \widetilde{X}$, that is, the preimage of the singular locus, is a curve. Denote the integral components of E by E_i. Then the intersection matrix $((E_i \cdot E_j))_{i,j}$ is negative definite. There exists a unique minimal resolution, meaning that all the others factor through it. Its exceptional locus contains no (-1)-curves. Over an arbitrary ground field, a (-1)-*curve* is a curve $C \simeq \mathbb{P}^1_L$ for a finite field extension $k \subset L$ such that $(C^2) = -[L : k]$.

A singular point $x \in X$ is called *rational* if $R^1 r_*(\mathcal{O}_{\widetilde{X}})_x = 0$ for a resolution of singularities $r\colon \widetilde{X} \to X$. This condition is independent of the choice of r, which relies on the fact that for a blow-up at a regular point, the higher direct images of the structure sheaf are zero, and a birational morphism of regular, proper surfaces is composed of those.

On the normal surface X, intersection numbers with rational coefficients can actually be defined for all Weil divisors, as introduced in [93], Section II (b). For a generalization to higher dimensions, see [114]. Let $r\colon \widetilde{X} \to X$ be a resolution of singularities. Given an integral curve $C \subset X$ with generic point $\xi \in C$, its *strict transform* under r is the integral curve $C' = \overline{r^{-1}(\xi)}$. Extend this linearly to define the strict transform D' of an arbitrary Weil divisor D on X. The negative definite intersection matrix $((E_i \cdot E_j))_{i,j}$ is invertible over $\mathbb{Q}$. So there exist unique numbers $q_i \in \mathbb{Q}$ such that $r^*(D) = D' + \sum q_i E_i$ satisfies $(r^*(D) \cdot E_j) = 0$ for all E_j. This defines a pullback map

$$r^*\colon Z^1(X) \to Z^1(\widetilde{X})_{\mathbb{Q}},$$

which respects the group structures as well as linear equivalence. In the case that D is Cartier, the definition of $r^*(D)$ coincides with the usual one for Cartier divisors. The intersection number of two Weil divisors on X is then set to be $(D_1 \cdot D_2) = (r^*(D_1) \cdot r^*(D_2))$. Since the morphism r is of degree 1, this definition is also in accordance with the given one for Cartier divisors. Thereby it moreover extends the definition of intersection numbers for $\mathbb{Q}$-Cartier Weil divisors, discussed above. The intersection number $(D_1 \cdot D_2)$ of two Weil divisors on X is bilinear and independent of the chosen resolution r. If D_1 and D_2 are effective without common integral components, then $(D_1 \cdot D_2) = 0$ holds if and only if D_1 and D_2 are disjoint.

A Weil divisor D on X is *numerically trivial* if $(D \cdot C) = 0$ for all integral curves C on X. The demand that D has finite order means that $dD \sim 0$ as Weil divisors for some $d \geq 1$, or equivalently, that the d-fold composition $\mathcal{O}_X(dD)$ of the reflexive sheaf $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$. If this holds for $D = K_X$, notice that then X necessarily has to be $\mathbb{Q}$-Gorenstein. The following lemma is the version of Lemma 4.1 for $\mathbb{Q}$-Gorenstein X about the compatibility of the relevant terms with algebraic field extensions.

Lemma 4.19. *Let X be a $\mathbb{Q}$-Gorenstein, geometrically normal, proper k-scheme. For every algebraic field extension $k \subset L$ and every integer d, the following assertions hold:*

(i) The natural map $\pi\colon X_L \to X$ satisfies $\pi^*(\mathcal{O}_X(dK_X)) = \mathcal{O}_{X_L}(dK_{X_L})$.

(ii) K_X is numerically trivial if and only if K_{X_L} is numerically trivial.

(iii) $dK_X \sim 0$ if and only if $dK_{X_L} \sim 0$.

Proof. Since π is flat, every $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation satisfies $\pi^*(\mathcal{F}^\vee) = \pi^*(\mathcal{F})^\vee$. Thus π^* is compatible with the composition law of rank 1 reflexive sheaves, as π^* always commutes with tensor products. Now $\pi^*(\omega_X) = \omega_{X_L}$ holds, so this results in $\pi^*(\mathcal{O}_X(dK_X)) = \mathcal{O}_{X_L}(dK_{X_L})$.

Let $m \geq 1$ be an integer such that mK_X is Cartier. By definition, K_X is numerically trivial if and only if mK_X has this property. Hence by Proposition 1.34 and (i), the invertible sheaf $\mathcal{O}_X(mK_X)$ is numerically trivial if and only if its pullback $\mathcal{O}_{X_L}(mK_{X_L})$ has this property. This verifies the second assertion.

For the third claim, first suppose that $dK_X \sim 0$. This means that $\mathcal{O}_X(dK_X) \simeq \mathcal{O}_X$, and then applying π^* yields $\mathcal{O}_{X_L}(dK_{X_L}) \simeq \mathcal{O}_{X_L}$. Thus also $dK_{X_L} \sim 0$. On the other hand, if the latter is assumed, then $\pi^*(\mathcal{O}_X(dK_X)) \simeq \mathcal{O}_{X_L}$ follows. The faithful flatness of π implies that already $\mathcal{O}_X(dK_X)$ has to be invertible. Because the pullback $\mathrm{Pic}(X) \to \mathrm{Pic}(X_L)$ is injective due to Proposition A.25, this gives that $\mathcal{O}_X(dK_X) \simeq \mathcal{O}_X$, thus $dK_X \sim 0$. □

Lemma 4.20. *Let X be a $\mathbb{Q}$-Gorenstein, proper surface over an algebraically closed field k, $r\colon \widetilde{X} \to X$ its minimal resolution of singularities. Then $r^*(K_X) = K_{\widetilde{X}} + C$ for an effective $\mathbb{Q}$-divisor C. Each closed point $x \in X$ satisfies one of the following two properties:*

(i) $\mathrm{Supp}(C) \cap \mathrm{Supp}(r^{-1}(x)) = \varnothing$.

(ii) $\mathrm{Supp}(C) \cap \mathrm{Supp}(r^{-1}(x)) = \mathrm{Supp}(r^{-1}(x))$.

Moreover, (i) holds if and only if $x \in X$ is regular or a rational Gorenstein singularity.

Proof. Denote by E the exceptional locus of r. As the restriction of r to $\widetilde{X} \smallsetminus E$ is an isomorphism onto the regular locus of X, there exists some $\mathbb{Q}$-divisor C supported on E such that $r^*(K_X) = K_{\widetilde{X}} + C$. Decompose $C = C' - C''$ into effective $\mathbb{Q}$-divisors C' and C'' without common integral components.

Assume by contradiction that C'' is non-zero and let D be an integral component of C''. Then D sits in the exceptional locus and so $(D^2) < 0$. Thus $(K_{\widetilde{X}} \cdot D) \geq 0$, since there are no (-1)-curves in E. Let $m \geq 1$ be the index of the $\mathbb{Q}$-Cartier divisor K_X. Thus $\mathcal{O}_X(mK_X)$ is invertible. Every singular point $x \in X$ has an open neighborhood W such that $\mathcal{O}_X(mK_X)|_W \simeq \mathcal{O}_W$. Consider the open neighborhood $U = r^{-1}(W)$ of the fiber $r^{-1}(x)$. Then $r^*(\mathcal{O}_X(mK_X))|_U \simeq \mathcal{O}_U$ is trivial, and varying the singular point $x \in X$ implicates $r^*(\mathcal{O}_X(mK_X))|_E \simeq \mathcal{O}_E$. Hence especially $(r^*(mK_X) \cdot D) = 0$ holds. Use this to compute $(mC'' \cdot D) = (mK_{\widetilde{X}} \cdot D) + (mC' \cdot D) \geq 0$, as C' and D have no common component. So $(mC'' \cdot D) \geq 0$ for all integral components D of mC'', and thus $(mC'')^2 \geq 0$. This is a contradiction, as the curve mC'' is supported on E. Eventually, this shows that $C'' = 0$ and, in turn, the effectiveness of C follows.

Next, let $x \in X$ be a closed point. Suppose that $\mathrm{Supp}(C) \cap \mathrm{Supp}(r^{-1}(x))$ is not empty and verify that then (ii) has to be valid. If the latter would be false, then due to the connectedness of $r^{-1}(x)$, this curve must contain an integral component D such that $(C \cdot D) > 0$. Now $(r^*(K_X) \cdot D) = 0$ results in $(K_{\widetilde{X}} \cdot D) < 0$, which in turn implicates that D is a (-1)-curve, a contradiction. Hence (ii) is true.

For the final part of the assertion, let again $x \in X$ be a closed point. First, assume that (i) holds. Without loss of generality, let $X \smallsetminus \{x\}$ be regular: The claim is a local property and X has only isolated singularities. So factorize $r = r_1 \circ r_2$ where r_1 is the minimal resolution of the singular points in $X \smallsetminus \{x\}$ and r_2 is the minimal resolution of the point $r_1^{-1}(x)$. Now replace r by r_2. If $x \in X$ is not regular, then $r^{-1}(x)$ is a curve. Let D be one of its integral components. Then assumption (i) yields $(C \cdot D) = 0$. As the intersection number of $K_{\widetilde{X}} + C = r^*(K_X)$ with D is trivial, this implies $(K_{\widetilde{X}} \cdot D) = 0$. So each integral component D of $r^{-1}(x)$ is a (-2)-curve. Now [4], Theorem 2.7, shows that $x \in X$ is Gorenstein and $\chi(\mathcal{O}_{\widetilde{X}}) = \chi(\mathcal{O}_X)$. Because $H^0(X, R^2 r_*(\mathcal{O}_{\widetilde{X}})) = 0$, the Leray spectral sequence $E_2^{a,b} = H^a(X, R^b r_*(\mathcal{O}_{\widetilde{X}})) \Rightarrow H^{a+b}(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ gives the exactness of $0 \to H^1(X, \mathcal{O}_X) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to H^0(X, R^1 r_*(\mathcal{O}_{\widetilde{X}})) \to H^2(X, \mathcal{O}_X) \to H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to 0$. Hence $h^0(R^1 r_*(\mathcal{O}_{\widetilde{X}})) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_{\widetilde{X}}) = 0$ holds. As $R^1 r_*(\mathcal{O}_{\widetilde{X}})$ is a skyscraper sheaf, it in turn has to be zero. So $x \in X$ is a rational Gorenstein singularity.

Conversely, let $x \in X$ be regular or a rational Gorenstein singularity. In both cases, the local ring $A = R^1 r_*(\mathcal{O}_{\widetilde{X}})_x$ is zero, so especially $\widehat{A} = 0$. The theorem on formal functions yields $\widehat{A} = \varprojlim H^1(E, \mathcal{O}_{nE})$. All curves Z supported on E index another inverse system, and the multiples nE of E are cofinal therein. Consequently, also $\widehat{A} = \varprojlim H^1(E, \mathcal{O}_Z)$ is true. All transition maps in this system are surjective: If $Z_1 \leq Z_2$, then the surjection $\mathcal{O}_{Z_2} \to \mathcal{O}_{Z_1}$ shows that the induced map $H^1(E, \mathcal{O}_{Z_2}) \to H^1(E, \mathcal{O}_{Z_1})$ is onto. Thus $\widehat{A} = 0$ implies that all $H^1(E, \mathcal{O}_Z)$ are zero.

Now assume by contradiction that (ii) holds and decompose $C = Z + Z'$ such that Z is supported on $r^{-1}(x)$ and Z' is disjoint to $r^{-1}(x)$. Let W be an open neighborhood of x containing no singular point of $X \smallsetminus \{x\}$ such that $\mathcal{O}_X(K_X)|_W \simeq \mathcal{O}_W$. Then on the open neighborhood $U := r^{-1}(W)$ of $r^{-1}(x)$, also the sheaf $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + Z)|_U = r^*(\mathcal{O}_X(K_X))|_U$ is trivial. In particular, Z is Cartier and the adjunction formula implies $\omega_Z \simeq \mathcal{O}_Z$. So $h^1(\mathcal{O}_Z) = h^0(\mathcal{O}_Z) \geq 1$ contradicts $\widehat{A} = 0$. Eventually, (i) follows, as claimed. □

Remark 4.21. Let X be a normal, proper surface over an algebraically closed field k. Rational Gorenstein singularities on X are also known as *rational double points* or *Du Val singularities* and they were studied by Artin [4], [5], [6] in arbitrary characteristic $p \geq 0$. The reduction E_{red} of the exceptional divisor E on the minimal resolution of a rational double point is an *ADE-curve*, which is a reduced curve with only (-2)-curves as integral components $E_i \subset E$ and negative definite intersection matrix $((E_i \cdot E_j))_{i,j}$. Their dual graphs are exactly the simply laced Dynkin diagrams A_n, D_n, E_6, E_7, E_8 appearing in the ADE classification. See [10], Theorem 3.32, for an explicit computation. Conversely, every ADE-curve on a regular, proper surface over k is contractible to a rational Gorenstein singularity, which was already used in the preceding proof.

Let $g \colon \widetilde{X} \to Y$ be a birational morphism between smooth, integral, proper surfaces over an algebraically closed field k. So g is a sequence of contractions of (-1)-curves. Consider a $\mathbb{Q}$-divisor $D = \sum \delta_i D_i$ on $\widetilde{X}$ with distinct integral components D_i. Since g is closed, the schematic image $g(D_i)$ is the set-theoretic image with its reduced subscheme structure. Hence $g(D_i)$ is either an integral curve or a closed point. Define the *image of* D under g to be the $\mathbb{Q}$-divisor $g(D) = \sum \delta_i g(D_i)$ on Y, where the sum is taken only over all curves $g(D_i)$.

Now the core achievement of this section follows, which is one of the central results in this thesis. An essential ingredient to the proof is that every non-rational minimal surface in Kodaira dimension $-\infty$ is a ruled surface. Moreover, the Enriques classification can be applied to obtain upper bounds for the order of K_X in the cases in which X has only rational Gorenstein singularities. For this purpose, note that the classification yields that the order of ω_X on a smooth, integral, proper surface X is always dividing 4 or 6. All different values occurring here will be pictured in detail later in the course of Section 6.2. For the sake of completeness, the smooth case is included in the table below.

Theorem 4.22. *Let X be a $\mathbb{Q}$-Gorenstein, geometrically normal, proper surface over an arbitrary field k with numerically trivial canonical divisor K_X. In the case that X is not Gorenstein, assume the existence of a perfect extension field $k \subset L$ and a resolution of singularities $r\colon \widetilde{X_L} \to X_L$ such that the $\mathbb{Q}$-divisor $r^*(K_{X_L})$ has integral coefficients. Then K_X has finite order. Furthermore, suppose that $h^0(\mathcal{O}_X) = 1$ and let $m \geq 1$ be the index of K_X. Then $dK_X \sim 0$ holds for the values of d displayed in the subsequent table.*

X	d
smooth	4 or 6
not smooth, but Gorenstein	2
not Gorenstein	$m(m-1)$

Table 3
Upper bounds for the order of K_X.

See Remark 4.27 below for a comment on when the additional assumption in the non-Gorenstein case over $k = \mathbb{C}$ is fulfilled, and when it can be violated.

Remark 4.23. Moreover, the proof will show the following: In the case that the ground field k is algebraically closed, X satisfies $h^0(\mathcal{O}_X) = 1$ and is not Gorenstein, and r is the minimal resolution of singularities, then $r^*(K_X) = K_{\widetilde{X}} + C$ for a curve C by Lemma 4.20. If C is integral, then $dK_X \sim 0$ holds for $d = \operatorname{lcm}(2, m)$. This gives a sharper bound than $m(m-1) = \operatorname{lcm}(m-1, m)$.

Proof. Conduct several steps of reduction to begin with. First, reduce by Lemma 4.19 to the case that $k = L$ is perfect. Second, it is moreover possible to assume that k is algebraically closed. For this purpose, show that $r_{\overline{k}}^*(K_{X_{\overline{k}}})$ continues to have integral coefficients: As k is perfect, the regular scheme $\widetilde{X}$ is smooth, so the base change $r_{\overline{k}}$ is a resolution of singularities for $X_{\overline{k}}$, which sits in the following natural commutative diagram:

$$\begin{array}{ccc} \widetilde{X}_{\overline{k}} & \xrightarrow{r_{\overline{k}}} & X_{\overline{k}} \\ \downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow[r]{} & X. \end{array}$$

Let $m \geq 1$ be the index of K_X. The Cartier divisor mK_X pulls back to $mK_{X_{\overline{k}}}$, since their associated invertible sheaves do likewise. By assumption, $D := r^*(K_X)$ is Cartier. Then $r^*(mK_X) = mD$ is the usual pullback of Cartier divisors. The diagram's commutativity yields that on $\widetilde{X}_{\overline{k}}$, the equality $r^*_{\overline{k}}(mK_{X_{\overline{k}}}) = mD_{\overline{k}}$ of Cartier divisors holds. Considered as an equality in $Z^1(\widetilde{X}_{\overline{k}})_{\mathbb{Q}}$, use $m \cdot r^*_{\overline{k}}(K_{X_{\overline{k}}}) = r^*_{\overline{k}}(mK_{X_{\overline{k}}})$ to deduce that $r^*_{\overline{k}}(K_{X_{\overline{k}}}) = D_{\overline{k}}$ has integral coefficients. Hence assume without loss of generality that k is algebraically closed.

Next, if $r^*(K_X)$ has integral coefficients for a given resolution of singularities, then this conclusion holds true for any other resolution, in particular for the minimal resolution $s\colon M \to X$. To verify this, it is sufficient to show that $r^*(K_X)$ has integral coefficients if and only if $s^*(K_X)$ does. Consider the factorizations

$$\widetilde{X} \xrightarrow{b} M \xrightarrow{s} X \quad (r = s \circ b) \qquad \text{and} \qquad Z^1(\widetilde{X})_{\mathbb{Q}} \xleftarrow{b^*} Z^1(M)_{\mathbb{Q}} \xleftarrow{s^*} Z^1(X). \quad (r^* = b^* \circ s^*)$$

Here b^* is induced by the usual pullback $Z^1(M) \to Z^1(\widetilde{X})$ of divisors between the locally factorial schemes $\widetilde{X}$ and M. Decompose the $\mathbb{Q}$-divisor $s^*(K_X) = \sum q_i C_i$ into its integral components. Apply b^* to obtain $r^*(K_X) = \sum q_i C_i' + F$, where C_i' denotes the strict transform of C_i and F is a $\mathbb{Q}$-divisor supported on the exceptional locus of b. Both summands $\sum q_i C_i'$ and F have no common integral components. If $r^*(K_X)$ has only integral coefficients, then all q_i are integers, which means that $s^*(K_X)$ has only integral coefficients. Conversely, if the latter is assumed, then also $r^*(K_X) = b^* s^*(K_X)$ has integral coefficients. So assume from now on that r is the minimal resolution of singularities.

Moreover, to verify that K_X has finite order, it is sufficient to consider one connected component of X. Note that the assumption $h^0(\mathcal{O}_X) = 1$, imposed for the additional part of the assertion, guarantees from the outset that X is connected. As X is normal, each connected component is irreducible, so assume further that X is integral. According to Lemma 4.20, there exists an effective $\mathbb{Q}$-divisor C such that $r^*(K_X) = K_{\widetilde{X}} + C$. By assumption, $r^*(K_X)$ is numerically trivial and Cartier. As $K_{\widetilde{X}}$ is Cartier in any case, also C is Cartier. To deduce that $dK_X \sim 0$ for some multiple $d = nm \geq 1$ of the index m, the task is to find a non-zero global section of the invertible sheaf

$$\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC) = r^*(\mathcal{O}_X(mK_X)^{\otimes n})$$

or of its dual. Then, since r is a fibration by Zariski's main theorem, the projection formula yields that also $\mathcal{O}_X(mK_X)^{\otimes n}$ or its dual has a non-zero global section, so this numerically trivial invertible sheaf must be trivial.

Distinguish the following three cases: $C = 0$ or C is a curve with either $m = 1$ or $m \geq 2$. The first case $C = 0$ means that X has only rational Gorenstein singularities. In the second case, in which $m = 1$ and C is a curve, now X is Gorenstein but at least one non-rational singularity exists. The third case, where C is a curve and $m \geq 2$, is the situation in which X is not Gorenstein.

In the first case, the divisor $K_{\widetilde{X}} = r^*(K_X)$ is numerically trivial. Due to Theorem 4.17, there exists some $d \geq 1$ such that $h^0(\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}})) = 1$. If X is not smooth, then $d = 2$ can be chosen: In fact, the Enriques classification yields that the order of $K_{\widetilde{X}}$ can only be higher in the case that $\widetilde{X}$ is bielliptic. But the arguments of Step 2 in the proof of Theorem 4.17 have shown that then, if $K_{\widetilde{X}}$ is not trivial, $\widetilde{X}$ does not contain any (-2)-curve. This is not possible, since then $\widetilde{X} = X$ would be smooth, a contradiction. So here if $\widetilde{X}$ is bielliptic, then its canonical divisor is necessarily trivial.

To treat the second and third case in which C is a curve, consider the following common preparatory arguments: Every singular point $x \in X$ has an open neighborhood W such that $\mathcal{O}_X(mK_X)|_W \simeq \mathcal{O}_W$. So on the open neighborhood $U = r^{-1}(W)$ of the fiber $r^{-1}(x)$, the invertible sheaf $r^*(\mathcal{O}_X(mK_X))|_U$ is trivial. Vary the singular point $x \in X$ to see that $\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + mC)|_Z = r^*(\mathcal{O}_X(mK_X))|_Z$ is trivial for all curves Z supported on the exceptional locus.

In order to deduce that $h^0(\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC)) \geq 1$ for some multiple $d = nm \geq 1$ of m, consider a curve C' supported on C. This curve will be chosen adequately in the different cases later on, so that C' satisfies properties (I) and (II) discussed below, which will thereby prove the assertion. Since $\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC)|_{C'} = \mathcal{O}_{C'}$ is trivial, the exact sequence $0 \to \mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC - C') \to \mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC) \to \mathcal{O}_{C'} \to 0$ follows. Suppose that the following vanishing holds:

(I) $h^2(\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC - C')) = h^0(\mathcal{O}_{\widetilde{X}}((1-d)K_{\widetilde{X}} - dC + C')) = 0.$

Then the long exact sequence of cohomology groups yields $h^1(\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC)) \geq h^1(\mathcal{O}_{C'})$ and moreover $h^2(\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC)) = 0$. The numerical triviality of $\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC)$ gives the equality $\chi(\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC)) = \chi(\mathcal{O}_{\widetilde{X}})$. Inserting the information obtained before yields the estimate $h^0(\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC)) \geq \chi(\mathcal{O}_{\widetilde{X}}) + h^1(\mathcal{O}_{C'})$. Therefore it is sufficient to show that $\chi(\mathcal{O}_{\widetilde{X}}) + h^1(\mathcal{O}_{C'}) \geq 1$.

Observe that $h^2(\mathcal{O}_{\widetilde{X}}) = h^0(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})) = 0$ holds, and furthermore $h^0(\mathcal{O}_{\widetilde{X}}(rK_{\widetilde{X}})) = 0$ for all $r \geq 1$. Otherwise $h^0(\mathcal{O}_{\widetilde{X}}(rK_{\widetilde{X}} + rC)) \geq 1$ is valid, so $rK_{\widetilde{X}} + rC \sim 0$ follows. But then $rK_{\widetilde{X}} \sim -rC$ contradicts $h^0(\mathcal{O}_{\widetilde{X}}(-rC)) = 0$, which holds for the curve C. Especially, this shows that $\operatorname{kod}(\widetilde{X}) = -\infty$.

So it suffices to verify that $h^1(\mathcal{O}_{C'}) \geq h^1(\mathcal{O}_{\widetilde{X}})$. If the right-hand side is zero, the conclusion is trivial, so assume that $h^1(\mathcal{O}_{\widetilde{X}}) \geq 1$. Let $g \colon \widetilde{X} \to Y$ be a sequence of contractions of (-1)-curves, until Y is minimal. Each contraction can be seen as the blow-up at a regular point, so $h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_{\widetilde{X}})$ is non-zero. Thus Y is not isomorphic to $\mathbb{P}^2$. Hence Y must be a ruled surface, equipped with a fibration $f \colon Y \to B$ to a regular curve B such that all fibers of f are isomorphic to $\mathbb{P}^1$. Now it remains to show:

(II) There exists an integral component $C'_i \subset C'$ such that $C'_i \to B$ is finite.

Indeed, then the subsequent Lemma 4.24 implies $h^1(\mathcal{O}_{C'}) \geq h^1(\mathcal{O}_B) = h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_{\widetilde{X}})$, as sought. To summarize, the task is to find for some $d = nm$ a suitable curve C' which satisfies (I) and (II).

Now consider the second case, that is, $m = 1$. Here choose $d = 2$ and $C' = C$. Then condition (I) means that $h^0(\mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}} - C)) = 0$. If this was not the case, then already $K_X \sim 0$ follows, so assume that (I) holds. To verify (II), it is by induction sufficient to ensure that the image $D_1 = g_1(C)$ of C under one contraction $g_1 : \widetilde{X} \to Y_1$ of a (-1)-curve is again a curve with $\mathcal{O}_{Y_1}(K_{Y_1} + D_1)$ numerically trivial. This will be shown in Lemma 4.25 below. Taking this for granted for the moment, consider the image $D = g(C)$ under the sequence $g : \widetilde{X} \to Y$ of contractions onto the minimal surface Y. No connected component $D' \subset D$ can purely be supported on a single fiber of the ruled fibration $f : Y \to B$. To see this, let F be such a fiber. Then, since $F \simeq \mathbb{P}^1$ and $(F^2) = 0$, necessarily $(K_Y \cdot F) = -2$ follows. But the numerical triviality of $\mathcal{O}_Y(K_Y + D)$ implies $(D'^2) = (D \cdot D') = -(K_Y \cdot D')$, so D' cannot be a multiple of a fiber. So at least one integral component of D must map surjectively onto B, which then in turn also holds for C. This completes the second case, modulo both lemmas used above.

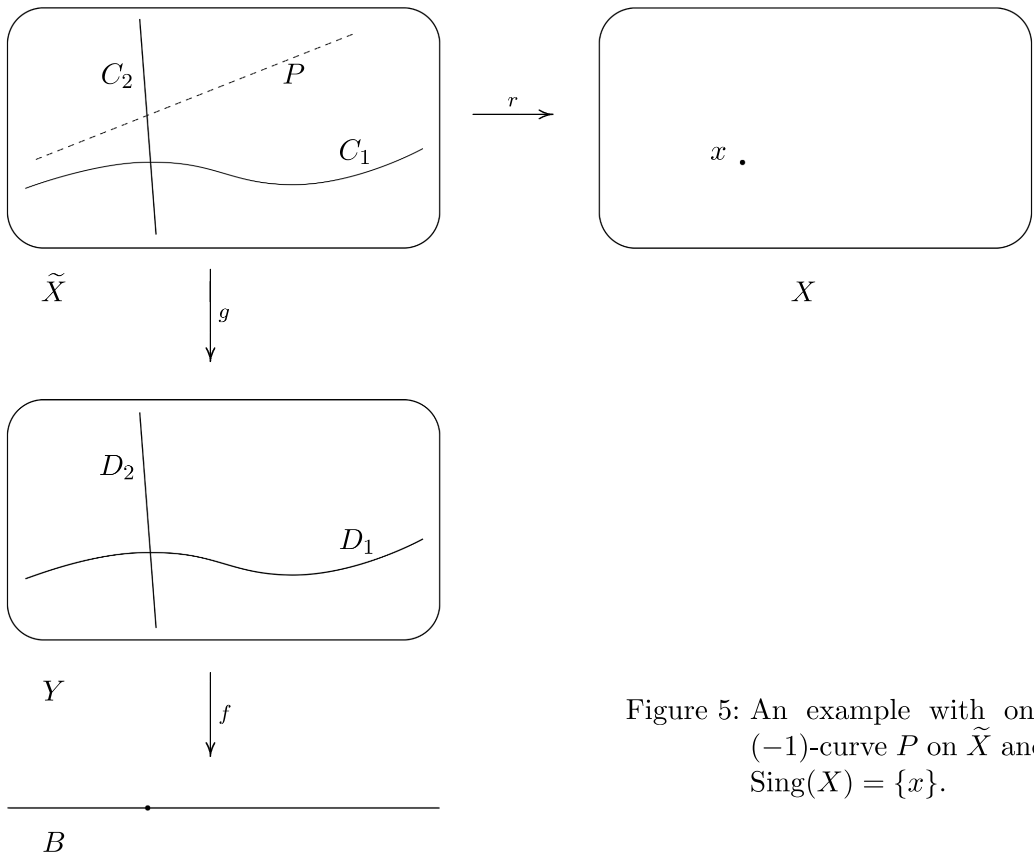


Figure 5: An example with one (-1)-curve P on $\widetilde{X}$ and $\mathrm{Sing}(X) = \{x\}$.

Notice that these arguments also apply if $C' = C$ is integral and $m \geq 2$. In fact, then $\omega_C^{\otimes m} = \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + mC)|_C = \mathcal{O}_C$ is trivial, which particularly implies that ω_C is numerically trivial. Thus already $\omega_C \simeq \mathcal{O}_C$. So $d = 2$ can be chosen, although d is not necessarily a multiple of m. Then the preceding reasoning yields $h^0(\mathcal{O}_{\widetilde{X}}(2K_{\widetilde{X}} + 2C)) = 1$. For $d = \mathrm{lcm}(2, m)$, consequently $\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dC) = r^*(\mathcal{O}_X(mK_X)^{\otimes \frac{d}{m}})$ has a non-zero global section, which gives $dK_X \sim 0$.

In the remaining third case, C is a curve and $m \geq 2$. Choose $C' = C$ again, but now $d = m$. If (I) was not satisfied, then $h^0(\mathcal{O}_{\widetilde{X}}((1-m)K_{\widetilde{X}} + (1-m)C)) \geq 1$, which means that $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + C)^{\otimes 1-m}$ has a non-zero global section. This conclusion in turn holds true for $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + C)^{\otimes m(1-m)} = r^*(\mathcal{O}_X(mK_X))^{\otimes 1-m}$. Thereby $m(m-1)K_X \sim 0$ follows. So assume that (I) is valid. To verify (II), the arguments in the second step can be adopted directly. Eventually, $mK_X \sim 0$ is true. □

Lemma 4.24. *Let $C \to B$ be a morphism of connected, proper curves over an algebraically closed field k. Suppose that B is regular and that there exists an integral component $C_i \subset C$ such that the induced morphism $C_i \to B$ is finite. Then $h^1(\mathcal{O}_C) \geq h^1(\mathcal{O}_{C_i}) \geq h^1(\mathcal{O}_B)$.*

Proof. The proof consists of three steps. To see the first estimate, note that the closed embedding $C_i \hookrightarrow C$ gives the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_C \to \mathcal{O}_{C_i} \to 0$. Since $h^2(\mathcal{I}) = 0$ on the curve C, the assertion $h^1(\mathcal{O}_C) \geq h^1(\mathcal{O}_{C_i})$ follows. Hence, replace C by C_i and assume without loss of generality that C is integral.

Second, consider the normalization $\varphi\colon \widetilde{C} \to C$, which is a finite surjection. The induced sequence $0 \to \mathcal{O}_C \to \varphi_*(\mathcal{O}_{\widetilde{C}}) \to \mathcal{S} \to 0$ is exact and $\mathcal{S}$ is supported on the at most zero-dimensional singular locus of C. Thereby $h^1(\mathcal{O}_C) \geq h^1(\varphi_*(\mathcal{O}_{\widetilde{C}}))$, and since φ is affine, the right-hand side equals $h^1(\mathcal{O}_{\widetilde{C}})$. So moreover replace C by $\widetilde{C}$ and assume without loss of generality that C is regular.

Third, decompose the finite field extension $K(B) \subset K(C)$ into a separable $K(B) \subset L$ and a purely inseparable $L \subset K(C)$ subextension. The category of finitely generated field extensions $k \subset F$ of transcendence degree 1 is equivalent to the category of regular, integral, proper curves over k with non-constant morphisms. Hence there exists such a curve D with function field $K(D) = L$ and finite morphisms $C \to D$ and $D \to B$. Then by [64], Chapter IV, Example 2.5.4, the claim $h^1(\mathcal{O}_C) \geq h^1(\mathcal{O}_B)$ follows. In fact, as $D \simeq C^{(p^r)}$ is a Frobenius twist in the case $p > 0$, the equality $h^1(\mathcal{O}_C) = h^1(\mathcal{O}_D)$ holds. The Riemann–Hurwitz formula implies $h^1(\mathcal{O}_D) \geq h^1(\mathcal{O}_B)$. This proves the assertion. □

Lemma 4.25. *Let X be a smooth, integral, proper surface over an algebraically closed field k. Suppose there exists a (-1)-curve $E \subset X$ and let $g\colon X \to Y$ be its contraction. Let $C \subset X$ be a curve such that $\mathcal{O}_X(rK_X + C)$ is numerically trivial for some $r \geq 1$. Then the image $D = g(C)$ is a curve and $\mathcal{O}_Y(rK_Y + D)$ is again numerically trivial.*

Proof. To see that D is a curve, note that each connected component $C' \subset C$ satisfies $(C'^2) = (C \cdot C') = -r(K_X \cdot C')$, and therefore C' cannot be a multiple of a (-1)-curve. Hence no connected component of C gets contracted by g to a point.

Observe that E must intersect C, since $(C \cdot E) = -r(K_X \cdot E) = r$. For some integer n, the equality $g^*(D) = C + nE$ is true, because g restricts to an isomorphism on $X \smallsetminus E$. Compute $0 = (g^*(D) \cdot E) = (C \cdot E) - n$ to deduce that $n = r$. This means that $g^*(D) = C + rE$. As $g^*(K_Y) = K_X - E$ holds, this results in $g^*(rK_Y + D) = rK_X + C$. So the pullback $g^*(\mathcal{O}_Y(rK_Y + D)) = \mathcal{O}_X(rK_X + C)$ is numerically trivial. The surjectivity of g guarantees that also $\mathcal{O}_Y(rK_Y + D)$ is numerically trivial, which completes the proof. □

Remark 4.26. Without the assumption that $r^*(K_X)$ or rather C has integral coefficients, the method used in the proof breaks down at the following point: Suppose for instance that $C' = rC$ is a curve for some $r \geq 2$ and d is a multiple of m. The vanishing in (I) then says that $h^0(\mathcal{O}_{\widetilde{X}}((1-d)K_{\widetilde{X}}+(r-d)C)) = 0$. Denote $\mathscr{L} = \mathcal{O}_{\widetilde{X}}((r-d)K_{\widetilde{X}}+(r-d)C)$. Then the vanishing means that $\omega_{\widetilde{X}}^{\otimes 1-r} \otimes \mathscr{L}$ has only trivial global sections. But there seems to be no a priori reason for this to happen, as $\mathscr{L}$ is numerically trivial and $(1-r)K_{\widetilde{X}} \equiv (r-1)C$ tends to behave numerically like an effective Cartier divisor. Also, it may be possible to proceed by choosing C' as a suitable integral curve supported on C.

Remark 4.27. Over the complex numbers, Sakai [110], Theorem 4.1, proved the conclusion that K_X has finite order on a $\mathbb{Q}$-Gorenstein surface X without the additional assumption on $r^*(K_X)$. It seems to be an interesting question for further investigation to what extent this holds true over arbitrary fields. The approach roughly goes as follows, using the notation from the preceding proof. First, Sakai studies and classifies all ruled surfaces Y according to their anti-Kodaira dimension $\mathrm{kod}(\omega_Y^\vee)$ in [108]. On this basis, he classifies all possible pairs (Y, D), where $D = g(C)$ is the image of $C = r^*(K_X) - K_{\widetilde{X}}$ under the sequence g of contractions of (-1)-curves, in [109], Theorem 5.3. Then a refined lemma in the spirit of Lemma 4.25 shows that in all cases where K_X would have infinite order, a contradiction to the classification of those pairs would follow.

This classification also highlights that over $k = \mathbb{C}$, the assumption in Theorem 4.22 that $r^*(K_X)$ has integral coefficients is always fulfilled if $\mathrm{kod}(\omega_Y^\vee) \in \{-\infty, 0, 1\}$ and can only fail to be valid in the case that $\mathrm{kod}(\omega_Y^\vee) = 2$.

4.7 Examples and Counterexamples of Singular Surfaces

Let k be an algebraically closed field of arbitrary characteristic $p \geq 0$. Consider a normal, integral, proper surface over k with numerically trivial canonical divisor K_X. In the case that X is smooth, the order of K_X divides 4 or 6 by the Enriques classification. The subsequent Section 6.2 will provide more details in this situation. Here in this section, consider non-smooth X.

In the situation of Theorem 4.22, the canonical divisor K_X has order dividing 2 if X is not smooth, but Gorenstein. Both cases occur by Example 4.28 below. If X is not Gorenstein and K_X has index m, then its order divides $m(m-1)$. The subsequent Example 4.29 shows that for every $m \geq 2$, there exists such X with $\mathrm{ord}(K_X) = m$. Moreover, these examples can be modified slightly to yield non-$\mathbb{Q}$-Gorenstein X with $\mathrm{ord}(K_X) = \infty$.

Example 4.28. Let $\widetilde{X}$ be a smooth, integral, proper surface over k with numerically trivial $\omega_{\widetilde{X}}$ such that there exists an ADE-curve C on X. Consider the contraction $r\colon \widetilde{X} \to X$ of C. Then X is a non-smooth Gorenstein surface with only rational double points as singularities. Thus $r^*(K_X) = K_{\widetilde{X}}$ and the orders of K_X and $K_{\widetilde{X}}$ coincide. By Theorem 4.22, the order can be either 1 or 2. There are examples for both cases:

Note that the surface $\widetilde{X}$ cannot be abelian, since $\mathbb{P}^1$ has Albanese dimension $m = 0$ and so there especially cannot exist (-2)-curves on abelian surfaces. Also, observe that each curve on $\widetilde{X}$ which is isomorphic to $\mathbb{P}^1$ is automatically a (-2)-curve, as the dualizing sheaf is numerically trivial. The arguments of Step 2 in the proof of Theorem 4.17 have shown that bielliptic surfaces also do not contain any (-2)-curve. There exist Enriques surfaces $\widetilde{X}$ with (-2)-curves on them in every characteristic $p \geq 0$, although this is not the generic case according to [36], Section 5, and [72], Table 1. If then $\widetilde{X}$ is classical, this gives an example with K_X of order 2. The Kummer construction mentioned in Section 4.4 yields a K3-surface $\widetilde{X}$ containing (-2)-curves. Here K_X has order 1.

Now construct further non-smooth, normal, integral, projective surfaces X over k with K_X numerically trivial. The construction depends on a chosen invertible sheaf $\mathcal{A}$. If $\mathcal{A}$ has finite order $m \geq 1$, then X will turn out to be $\mathbb{Q}$-Gorenstein of index m and K_X has order m. On the other hand, if $\mathcal{A}$ is numerically trivial of infinite order, then also K_X has infinite order. In this case, X is not $\mathbb{Q}$-Gorenstein.

Example 4.29. To construct X, start with a smooth, integral, proper curve B of genus $g \geq 2$ over k. Suppose that there exists a numerically trivial invertible sheaf $\mathcal{A}$ on B. The existence of such an $\mathcal{A}$ of infinite order is equivalent to the assumption that k is not the algebraic closure of a finite field, as discussed in Remark 1.36. For each $m \geq 1$ not divisible by p, the abelian variety $J = \operatorname{Pic}^0_B$ has m-torsion $J[m](k) \simeq (\mathbb{Z}/m\mathbb{Z})^{\oplus 2g}$, and each point of order m yields an invertible sheaf $\mathcal{A}$ of order m on B. If p divides m, then there may be no k-rational points of order m. Nevertheless, over every algebraically closed field k, there exist for instance hyperelliptic curves B with $J[p](k) \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus i}$ for each $1 \leq i \leq g$ according to [3]. Then also $J[p^r](k) \simeq (\mathbb{Z}/p^r\mathbb{Z})^{\oplus i}$ is non-zero for all $r \geq 1$ by [95], Section 6, Proposition on page 64. So for every natural number $m \geq 1$, there exists at least one curve B as above which admits an invertible sheaf $\mathcal{A}$ of order m.

Let $\mathscr{L} = \mathcal{O}_B$ and $\mathscr{N} = \omega_B \otimes \mathcal{A}^\vee$. Define $\mathcal{E} = \mathscr{N} \oplus \mathscr{L}$ and consider the resulting split short exact sequence

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathcal{E} \longrightarrow \mathscr{L} \longrightarrow 0.$$

Define the ruled surface $\widetilde{X} = \mathbb{P}(\mathcal{E})$ with structure morphism $f\colon \widetilde{X} \to B$. For general properties of ruled surfaces see [64], Chapter V, Section 2. The dualizing sheaf on $\widetilde{X}$ is given by

$$\omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-2) \otimes f^*(\omega_B \otimes \det(\mathcal{E})) = \mathcal{O}_{\widetilde{X}}(-2) \otimes f^*(\omega_B^{\otimes 2} \otimes \mathcal{A}^\vee).$$

The homomorphism

$$\operatorname{Pic}(B) \oplus \mathbb{Z} \longrightarrow \operatorname{Pic}(\widetilde{X}), \quad (\mathscr{M}, n) \longmapsto f^*(\mathscr{M}) \otimes \mathcal{O}_{\widetilde{X}}(n)$$

is bijective and $\operatorname{Num}(\widetilde{X}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ is generated by the classes of a closed fiber F and the section $E = \mathbb{P}(\mathscr{L}) \subset \mathbb{P}(\mathcal{E}) = \widetilde{X}$. To keep the notation simple, identify an invertible sheaf $\mathscr{M}$ on B with its pullback $f^*(\mathscr{M})|_E$ to $E \simeq B$. Consider $\mathcal{B} = \omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(2E)$ and compute

$$(\mathcal{B} \cdot F) = (\mathcal{O}_{\widetilde{X}}(-2) \cdot F) + 2(E \cdot F) = -2 + 2 = 0.$$

Since $\mathcal{O}_{\widetilde{X}}(E) = \mathcal{O}_{\widetilde{X}}(1) \otimes f^*(\mathscr{N}^\vee)$ and $\mathcal{O}_{\widetilde{X}}(1)|_E = \mathscr{L} = \mathcal{O}_E$, the adjunction formula yields

$$\mathcal{B}|_E = \omega_E \otimes \mathcal{O}_{\widetilde{X}}(E)|_E = \mathcal{A}\,.$$

These two computations show that $\mathcal{B} = f^*(\mathcal{A})$, and hence $\omega_{\widetilde{X}} = f^*(\mathcal{A}) \otimes \mathcal{O}_{\widetilde{X}}(-2E)$. Also, $\mathcal{B}$ inherits the property of being numerically trivial from $\mathcal{A}$, so $K_{\widetilde{X}} \equiv -2E$. The splitting of $\mathcal{E} = \mathscr{N} \oplus \mathscr{L}$ gives a second section $E' = \mathbb{P}(\mathscr{N})$, which is disjoint to E. The self-intersection number

$$(E^2) = \deg(\mathscr{L}) - \deg(\mathscr{N}) = -\deg(\omega_B) = -2g + 2$$

is negative and in turn $(E'^2) = \deg(\mathscr{N}) - \deg(\mathscr{L}) = -(E^2)$ is positive. In this situation, $\mathcal{O}_{\widetilde{X}}(E')$ is semi-ample and a globally generated power defines a contraction of the curve $E \subset \widetilde{X}$, see [112], Section 3. Let $r\colon \widetilde{X} \to X$ be the contraction to a normal, projective surface X and $x = r(E)$ the singular point. Denote by $U = \widetilde{X} \smallsetminus E$ the complement of the exceptional locus and by $V = X \smallsetminus \{x\}$ the regular locus. Then $r|_U\colon U \to V$ is an isomorphism.

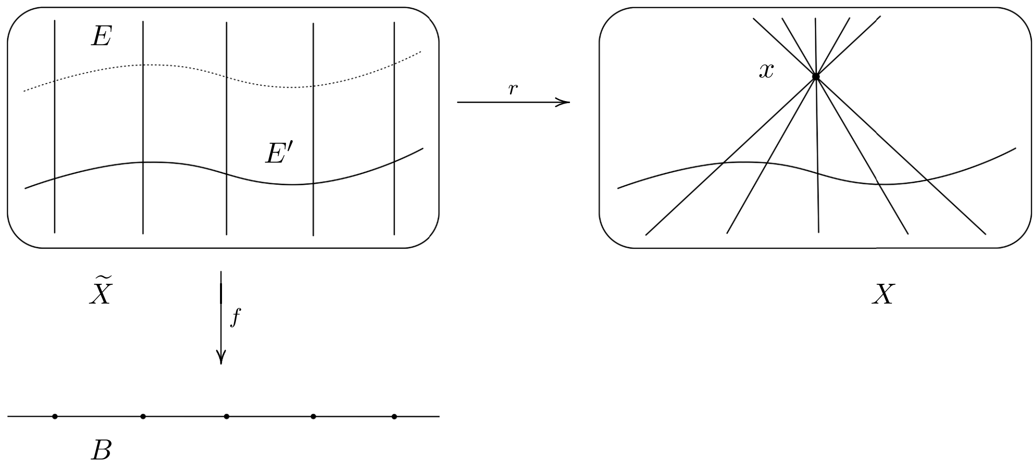


Figure 6: Contraction of the section $E \subset \widetilde{X}$.

As explained in the preceding section, the dualizing sheaf $\omega_X \simeq \mathcal{O}_X(K_X)$ is reflexive of rank 1 and uniquely determined by its restriction $\omega_X|_V = \omega_{\widetilde{X}}|_U$. The previous computation $\omega_{\widetilde{X}} = f^*(\mathcal{A}) \otimes \mathcal{O}_{\widetilde{X}}(-2E)$ shows that $K_{\widetilde{X}} = f^*(A) - 2E$ for a divisor A on B such that $\mathcal{O}_B(A) \simeq \mathcal{A}$. Thus $K_V = K_U = f^*(A)|_U$. Since E is not contained in any fiber of f, both $f^*(A)$ and its restriction $f^*(A)|_U$ have the same integral components and multiplicities. This implies that the strict transform of K_X under r is the divisor $D := f^*(A)$. As D is numerically trivial, consequently $r^*(K_X) = D$ holds by definition of the pullback. In turn, this also yields that K_X is numerically trivial.

Now show that the order of $\mathcal{A} \in \mathrm{Pic}(B)$ coincides with the order of K_X. Consider the case in which $\mathcal{A}$ has finite order $m \geq 1$ first. This means that $mA \sim 0$ and so also $r^*(mK_X) = mD = f^*(mA)$ is linearly equivalent to zero. So $r^*(\mathcal{O}_X(mK_X)) \simeq \mathcal{O}_{\widetilde{X}}$ is trivial, and then its restriction to U shows that $mK_V \sim 0$ must hold, and thus also $mK_X \sim 0$. To determine that K_X is in fact of order m, assume that $dK_X \sim 0$ for some $d \geq 1$. Hence dK_X is in particular Cartier. Then $r^*(dK_X) = dD$ is the usual pullback of Cartier divisors, so $f^*(\mathcal{A}^{\otimes d}) \simeq \mathcal{O}_{\widetilde{X}}(dD)$ is trivial. Now the injectivity of $f^*\colon \mathrm{Pic}(B) \to \mathrm{Pic}(\widetilde{X})$ implies that d is a multiple of m. Therefore indeed $\mathrm{ord}(K_X) = m$. The same reasoning shows in the case that $\mathcal{A}$ has infinite order, then also K_X must have infinite order.

This argument can be extended slightly to deduce in the case that $\mathcal{A}$ has finite order $m \geq 1$, then m is actually the index of K_X, and that X is not $\mathbb{Q}$-Gorenstein if $\mathcal{A}$ has infinite order. Indeed, suppose that dK_X is Cartier for some $d \geq 1$. Then $r^*(dK_X) = dD$ and $K_{\widetilde{X}} = D - 2E$ show $r^*(dK_X) = dK_{\widetilde{X}} + 2dE$. The restriction $r^*(\mathcal{O}_X(dK_X))|_E = \mathcal{O}_E$ is trivial, since $E = r^{-1}(x)$ is a fiber. This means that $\mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}})|_E = \mathcal{O}_{\widetilde{X}}(-2dE)|_E$ holds, and the adjunction formula yields

$$\omega_E^{\otimes d} = \mathcal{O}_{\widetilde{X}}(dK_{\widetilde{X}} + dE)|_E = \mathcal{O}_{\widetilde{X}}(-dE)|_E = \mathscr{N}^{\otimes d} = \omega_E^{\otimes d} \otimes \mathcal{A}^{\otimes -d}\,.$$

So $\mathcal{A}^{\otimes d} \simeq \mathcal{O}_E$ follows. As a result, d is a multiple of the order m of $\mathcal{A}$. Because the latter equals the order of K_X, also m must be the index of K_X. If $\mathcal{A}$ has infinite order, this is a contradiction.

Chapter 5

Invertible Sheaves and Coverings

Once it is known that the dualizing sheaf ω_X has finite order, the next problem to address is (Q2), the question concerning the existence of a finite étale covering $f\colon X' \to X$ with $\omega_{X'} \simeq \mathcal{O}_{X'}$. The answer to this question forms the content of this chapter.

The assumption that the dualizing sheaf ω_X has finite order is a necessary condition for f to exist by Corollary 5.2 below. Note that an étale covering f always satisfies $\omega_{X'} = f^*(\omega_X)$, so it is equivalent to demand whether $\omega_{X'} \simeq \mathcal{O}_{X'}$ or $f^*(\omega_X) \simeq \mathcal{O}_{X'}$. The former condition is the one which appears for the total space of the Beauville–Bogomolov decomposition, but the latter can be asked for arbitrary invertible sheaves $\mathscr{L}$ in place of ω_X. Given an invertible sheaf $\mathscr{L}$ of order d, there is a well-known natural way to obtain a finite flat covering $g\colon \widetilde{X} \to X$ with $g^*(\mathscr{L}) = \mathcal{O}_{\widetilde{X}}$. Its construction and the calculation of some relevant properties is executed in Section 5.1. The covering g is étale if and only if the characteristic p of the ground field is not dividing d.

If this divisiblity is valid, there might in the first instance still exist some finite étale covering f with $\omega_{X'} = f^*(\omega_X)$. For normal X, this possibility is excluded in Section 5.2, constituting the main result in this chapter. At the end, the results obtained before are applied specifically to $\mathscr{L} = \omega_X$, and also to the situation in which X is only $\mathbb{Q}$-Gorenstein.

5.1 Canonical Coverings Associated to Invertible Sheaves

Let X be a connected, locally noetherian scheme. A finite flat morphism $f\colon X' \to X$ where X' is connected is called a *finite flat covering* of X. The *total space* of f is X'. If f is moreover étale, then it is called a *finite étale covering* of X. The structure theorem of finitely generated projective modules over a ring implies that a finite flat covering f is finite locally free, which means that f is affine and $f_*(\mathcal{O}_{X'})$ is a locally free sheaf. The *degree* of f is $\deg(f) = \operatorname{rank}(f_*(\mathcal{O}_{X'}))$.

Notice that f has to be surjective, because otherwise, as f is proper, the complement of its image is open. So it contains an affine open subset on which the locally free sheaf $f_*(\mathcal{O}_{X'})$ has trivial local sections, which is a contradiction.

Proposition 5.1. *Let $f\colon X' \to X$ be a finite flat covering of connected, locally noetherian schemes and $\mathscr{L}$ an invertible sheaf on X such that $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$. Then $\mathscr{L} \in \operatorname{Pic}(X)$ has finite order, which is dividing the degree of f.*

Proof. By assumption, $\mathcal{E} = f_*(\mathcal{O}_{X'})$ is a locally free sheaf on X. On the one hand, the fact that $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$ implies $f_*f^*(\mathscr{L}) \simeq \mathcal{E}$. On the other hand, the projection formula gives the equality $\mathcal{E} \otimes \mathscr{L} = f_*f^*(\mathscr{L})$. Combined, both show that $\mathcal{E} \simeq \mathcal{E} \otimes \mathscr{L}$. The determinant then yields $\mathscr{N} \simeq \mathscr{N} \otimes \mathscr{L}^{\otimes r}$, where $\mathscr{N} = \det(\mathcal{E})$ and $r = \operatorname{rank}(\mathcal{E})$. Hence $\mathscr{L}^{\otimes r} \simeq \mathcal{O}_X$ is true. □

In particular, the following corollary can be drawn:

Corollary 5.2. *Let $f\colon X' \to X$ be a finite étale covering between Gorenstein, connected, proper schemes over an arbitrary field k such that $\omega_{X'} \simeq \mathcal{O}_{X'}$. Then ω_X has finite order, which is dividing the degree of f.*

Back to the case of an arbitrary invertible sheaf $\mathscr{L}$. Proposition 5.1 shows that if $\mathscr{L}$ has order d in $\operatorname{Pic}(X)$, the degree of each finite étale covering $f\colon X' \to X$ with $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$ has to be a multiple of d. Associated to $\mathscr{L}$, there is a natural way to obtain a finite flat covering of degree d, which will be discussed in the following. It is étale if and only if the characteristic p of X is not dividing d.

Let X be a scheme. Given an invertible sheaf $\mathscr{L}$ on X and for some $d \geq 1$ a global section $s \in H^0(X, \mathscr{L}^{\otimes d})$, there exists an associated *branched covering* $g\colon \widetilde{X} \to X$, constructed as follows: Define the locally free $\mathcal{O}_X$-algebra

$$\mathcal{A} = \mathcal{O}_X \oplus \mathscr{L}^{\otimes -1} \oplus \mathscr{L}^{\otimes -2} \oplus \cdots \oplus \mathscr{L}^{\otimes -d+1},$$

with multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ induced by

$$\begin{array}{lll} \mathscr{L}^{\otimes -i} \otimes \mathscr{L}^{\otimes -j} \xrightarrow{\text{can}} \mathscr{L}^{\otimes -(i+j)} & \text{if} & i+j \leq d-1, \\ \mathscr{L}^{\otimes -i} \otimes \mathscr{L}^{\otimes -j} \xrightarrow{\text{can}} \mathscr{L}^{\otimes -(i+j)} \xrightarrow{s} \mathscr{L}^{\otimes d-(i+j)} & \text{if} & i+j \geq d, \end{array}$$

for $0 \leq i, j \leq d-1$. There is an equivalence between the category of quasicoherent $\mathcal{O}_X$-algebras and the category of affine X-schemes, given by $\mathcal{B} \mapsto (\operatorname{Spec}(\mathcal{B}) \to X)$ with inverse $(g\colon \widetilde{X} \to X) \mapsto g_*(\mathcal{O}_{\widetilde{X}})$, see [52], Proposition 1.3.1. Denote $\widetilde{X} = \operatorname{Spec}(\mathcal{A})$ and define the branched covering to be the canonical morphism $g\colon \widetilde{X} \to X$.

Proposition 5.3. *Let X be an integral, noetherian scheme such that $H^0(X, \mathcal{O}_X)$ is a field of characteristic $p \geq 0$ and characteristic exponent $e \geq 1$. Let $\mathscr{L} \in \operatorname{Pic}(X)$ be of finite order d, and $s \in H^0(X, \mathscr{L}^{\otimes d})$ a global section defining an isomorphism $\mathcal{O}_X \to \mathscr{L}^{\otimes d}$. The following statements hold for $g\colon \widetilde{X} \to X$ constructed as above:*

(i) g is a finite flat covering of degree d, and $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^0(X, \mathcal{O}_X)$.

(ii) If every element of $H^0(X, \mathcal{O}_X^\times)$ has a d-th root, then the X-scheme $\widetilde{X}$ depends on s only up to isomorphism.

(iii) $g^*(\mathscr{L}^{\otimes d-1}) = \omega_{\widetilde{X}/X}$ and $g^*(\mathscr{L}) = \mathcal{O}_{\widetilde{X}}$, so combined also $\omega_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}$.

(iv) g is étale if and only if $p \nmid d$.

(v) g is purely inseparable if and only if d is an e-power.

(vi) Let X' be a scheme such that every element of $H^0(X', \mathcal{O}_{X'}^\times)$ has a d-th root. Then every morphism $f\colon X' \to X$ satisfying $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$ factorizes through g.

(vii) The X-scheme $\widetilde{X}$ is a principal homogeneous space under $\mu_{d,X}$.

Proof. By definition, g is affine. Since also $g_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{A}$ is locally free, the structure theorem of finitely generated projective modules over a ring implies that g is finite and flat. The ring $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^0(X, \mathcal{A})$ is canonically isomorphic to the field $H^0(X, \mathcal{O}_X)$, since $H^0(X, \mathscr{L}^{\otimes -i}) = 0$ for $1 \leq i \leq d-1$ by Lemma 1.35. Hence it contains only the trivial idempotents, which means that X' is connected. This shows (i).

Now verify (ii). Each other isomorphism $\mathcal{O}_X \to \mathscr{L}^{\otimes d}$ is given by $vs \in H^0(X, \mathscr{L}^{\otimes d})$ for a unit $v \in H^0(X, \mathcal{O}_X^\times)$. Choose a d-th root w of v. Let $\mathcal{A}'$ be $\mathcal{A}$ as an $\mathcal{O}_X$-module, but with multiplication induced by vs instead of s. The isomorphism of $\mathcal{O}_X$-modules $\mathcal{A}' \to \mathcal{A}$, defined by $\mathscr{L}^{\otimes -i} \xrightarrow{w^i} \mathscr{L}^{\otimes -i}$ for $0 \leq i \leq d-1$, is actually a morphism of $\mathcal{O}_X$-algebras. Hence $\mathrm{Spec}(\mathcal{A})$ and $\mathrm{Spec}(\mathcal{A}')$ are isomorphic X-schemes.

For assertion (iii), note that the finite morphism g induces an equivalence between the category of locally free sheaves $\mathcal{E}$ on $\widetilde{X}$ and the category of locally free $\mathcal{A}$-modules, given by $\mathcal{E} \mapsto g_*(\mathcal{E})$ and preserving ranks. To show that $g^*(\mathscr{L}^{\otimes d-1}) = \omega_{\widetilde{X}/X}$, it is therefore sufficient to verify that their pushforwards under g are isomorphic $\mathcal{A}$-modules. Now Theorem 2.17 implicates that $g_*(\omega_{\widetilde{X}/X}) = \mathcal{A}^\vee = \mathcal{O}_X \oplus \mathscr{L} \oplus \mathscr{L}^{\otimes 2} \oplus \cdots \oplus \mathscr{L}^{\otimes d-1}$. The $\mathcal{O}_X$-submodule $\mathscr{L}^{\otimes d-1}$ generates this $\mathcal{A}$-module, because $\mathscr{L}^{\otimes -i} \otimes \mathscr{L}^{\otimes d-1} \xrightarrow{\simeq} \mathscr{L}^{\otimes d-i-1}$ for $0 \leq i \leq d-1$. On the other hand, $g_*g^*(\mathscr{L}^{\otimes d-1}) = \mathcal{A} \otimes \mathscr{L}^{\otimes d-1} = \mathscr{L}^{\otimes d-1} \oplus \mathscr{L}^{\otimes d-2} \oplus \cdots \oplus \mathcal{O}_X$ follows from the projection formula, and this $\mathcal{A}$-module is canonically isomorphic to the former. Similarly, $g_*g^*(\mathscr{L}^{\otimes -1}) = \mathscr{L}^{\otimes -1} \oplus \mathscr{L}^{\otimes -2} \oplus \cdots \oplus \mathscr{L}^{\otimes -d}$. At this point, the isomorphism $\mathcal{O}_X \to \mathscr{L}^{\otimes d}$ enters, which yields $\mathscr{L}^{\otimes -d} \xrightarrow{\simeq} \mathcal{O}_X$. The global section $s' \in H^0(X, \mathscr{L}^{\otimes -d})$ corresponding to $1 \in H^0(X, \mathcal{O}_X)$ defines an identification $\mathcal{A} \to g_*g^*(\mathscr{L}^{\otimes -1})$, $1 \mapsto s'$. So also $\mathcal{O}_{\widetilde{X}} = g^*(\mathscr{L}^{\otimes -1})$ holds naturally, and thereby as well $g^*(\mathscr{L}) = \mathcal{O}_{\widetilde{X}}$.

To show the equivalence in (iv), note that the formation of the relative spectrum $\mathrm{Spec}(\mathcal{A}) \to X$ is compatible with base change in X. So the fiber $\widetilde{X}_x$ over a point $x \in X$ with residue field $E = \kappa(x)$ is the spectrum of $E[S]/(S^d - u) \xrightarrow{\simeq} \mathcal{A}(x)$. The bijection is defined by mapping $[S]$ to a generator $l \in \mathscr{L}^{\otimes -1}(x)$ and $u = s(x) \cdot l^d$. The polynomial $S^d - u$ is separable—which means that $\mathcal{A}(x)$ is an étale E-algebra—if and only if $p \nmid d$.

Similarly reason for (v). First, assume that $d = e^r$ is an e-power. Then $S^d - u$ is purely inseparable and thus it is a power of an irreducible purely inseparable polynomial $S^{e^{r'}} - u'$. So $(\widetilde{X}_x)_{\mathrm{red}} = \mathrm{Spec}(E[S]/(S^{e^{r'}} - u'))$ is the spectrum of a purely inseparable field extension of E. As g is consequently especially injective, it is a purely inseparable morphism. On the other hand, now assume that $d = e^r m$ for $m \geq 2$ not divisible by p. Then over the algebraic closure of E, there is a factorization $S^d - u = (S^{e^r} - u_1) \cdot \ldots \cdot (S^{e^r} - u_m)$ for distinct m-th roots $u_1, \ldots, u_m \in \overline{E}$ of u. The Chinese remainder theorem implies that the geometric fiber $(\widetilde{X}_x)_{\overline{E}}$ consists of m points, and thus g is not universally injective.

Next, treat assertion (vi). Let $f\colon X' \to X$ be a morphism satisfying $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$. Consider the fiber product

$$\begin{array}{ccccccc}
\operatorname{Spec}(f^*\mathcal{A}) & = & X' \times_X \widetilde{X} & \xrightarrow{\ \mathrm{pr}_2\ } & \widetilde{X} & = & \operatorname{Spec}(\mathcal{A}) \\
 & & h \uparrow \downarrow \mathrm{pr}_1 & & \downarrow g & & \\
 & & X' & \xrightarrow[\ f\ ]{} & X & &
\end{array}$$

and observe that factorizations of f through g correspond to sections h of pr_1. Those are given by the set

$$\operatorname{Hom}_{X'}(X', \operatorname{Spec}(f^*\mathcal{A})) = \operatorname{Hom}_{(\mathcal{O}_{X'}\text{-}\mathrm{Alg})}(f^*\mathcal{A}, \mathcal{O}_{X'}),$$

as $\operatorname{Spec}(f^*\mathcal{A})$ represents the contravariant functor $(\mathrm{Sch}/X') \to (\mathrm{Set})$ given on objects by $(\varphi\colon T' \to X') \mapsto \operatorname{Hom}_{(\mathcal{O}_{X'}\text{-}\mathrm{Alg})}(f^*\mathcal{A}, \varphi_*(\mathcal{O}_{T'}))$. Let the isomorphism $\mathcal{O}_{X'} \to f^*(\mathscr{L})^{\otimes -1}$ be given by $l \in H^0(X', f^*(\mathscr{L})^{\otimes -1})$. Then there is an induced isomorphism of $\mathcal{O}_{X'}$-algebras $\mathcal{O}_{X'}[S]/(S^d - u) \xrightarrow{\simeq} f^*(\mathcal{A})$, $[S] \mapsto l$, where $u = f^*(s) \cdot l^d \in H^0(X', \mathcal{O}_{X'})$. The choice of a d-th root w of u finally yields a homomorphism $\mathcal{O}_{X'}[S]/(S^d - u) \to \mathcal{O}_{X'}$, $[S] \mapsto w$ of $\mathcal{O}_{X'}$-algebras, and thus a section of pr_1 as claimed.

Finally, cover the last statement (vii). Fix X as a base scheme and abbreviate $\mu_d = \mu_{d,X}$. To define the action $\widetilde{X} \times \mu_d \to \widetilde{X}$ of μ_d on $\widetilde{X}$, note that it is sufficient to define compatible actions on an open cover of $\widetilde{X}$. Let $U \subset X$ be open such that there exists a trivialization $\mathcal{O}_U \xrightarrow{\simeq} \mathscr{L}^{\otimes -1}|_U$ given by some section $l \in H^0(U, \mathscr{L}^{\otimes -1})$. It induces an identification $\mathcal{O}_U[S]/(S^d - u) \xrightarrow{\simeq} \mathcal{A}|_U$, $[S] \mapsto l$, where $u = s_U \cdot l^d \in H^0(U, \mathcal{O}_U)$, as seen before. To define the action on $\widetilde{X}_U$, use the description $\widetilde{X}_U(T) = \operatorname{Hom}_{(\mathcal{O}_U\text{-}\mathrm{Alg})}(\mathcal{A}|_U, \varphi_*(\mathcal{O}_T))$ for every U-scheme T with structure morphism $\varphi\colon T \to U$. Combined, there is an identification

$$\widetilde{X}_U(T) \simeq \operatorname{Hom}_{(\mathcal{O}_U\text{-}\mathrm{Alg})}(\mathcal{O}_U[S]/(S^d - u), \varphi_*(\mathcal{O}_T)) = \{\, x \in H^0(T, \mathcal{O}_T) \mid x^d = u \,\},$$

where $u \in H^0(T, \mathcal{O}_T)$ also denotes its image induced by $\mathcal{O}_U \to \varphi_*(\mathcal{O}_T)$ on global sections. On this set, the group $\mu_d(T) = \mu_d(H^0(T, \mathcal{O}_T))$ acts by multiplication. This action is independent of the chosen trivialization $\mathcal{O}_U \xrightarrow{\simeq} \mathscr{L}^{\otimes -1}|_U$. To see this, let there be another one given by $a \cdot l \in H^0(U, \mathscr{L}^{\otimes -1})$ for a unit $a \in H^0(U, \mathcal{O}_U^\times)$. Then $\mathcal{O}_U[S]/(S^d - v) \xrightarrow{\simeq} \mathcal{A}|_U$, where now $v = s_U \cdot a^d \cdot l^d = a^d \cdot u$, and $\widetilde{X}_U(T) \simeq \{\, y \in H^0(T, \mathcal{O}_T) \mid y^d = a^d \cdot u \,\}$. So $x \mapsto a \cdot x = y$ is a natural bijection between the two descriptions of $\widetilde{X}_U(T)$, compatible with the action of $\mu_d(T)$. Eventually, the action of μ_d on $\widetilde{X}_U$ is compatible with restriction, as claimed. It is evident that the action on T-valued points is simply transitive, and thus $\widetilde{X}$ is in fact a principal homogeneous space. □

Let X be an integral, proper scheme over an algebraically closed field k. The assumption in statement (ii) above is satisfied because $H^0(X, \mathcal{O}_X) = k$. So the X-scheme $\widetilde{X}$ does up to isomorphism not depend on the chosen trivialization $\mathscr{L}^{\otimes d} \simeq \mathcal{O}_X$. Consequently, call $g\colon \widetilde{X} \to X$ the *canonical covering* associated to the invertible sheaf $\mathscr{L}$ of finite order.

Remark 5.4. Let X be as in Proposition 5.3 and additionally proper over an arbitrary field k. Then $\widetilde{X}$ is also proper, so the dualizing sheaves on both schemes exist. By statement (iii), the covering g satisfies $\omega_{\widetilde{X}} = g^*(\omega_X)$, even in the case that g is not étale. If X is also Gorenstein and $\mathscr{L} = \omega_X$ has finite order in $\mathrm{Pic}(X)$, then $\widetilde{X}$ is Gorenstein with $g^*(\omega_X) = \mathcal{O}_{\widetilde{X}}$ and in turn also $\omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}$.

Back to the general situation of Proposition 5.3. Consider a factorization $d = qm$. Then $\mathscr{L}^{\otimes q}$ has order m in $\mathrm{Pic}(X)$. Let $g_1 \colon \widetilde{X}_1 \to X$ be the finite flat covering of degree m, given by the locally free $\mathcal{O}_X$-algebra $\mathcal{B} = \mathcal{O}_X \oplus \mathscr{L}^{\otimes -q} \oplus \mathscr{L}^{\otimes -2q} \oplus \cdots \oplus \mathscr{L}^{\otimes -(m-1)q}$ associated to $\mathscr{L}^{\otimes q}$ and the same isomorphism $\mathcal{O}_X \to \mathscr{L}^{\otimes mq}$ as for $\mathcal{A}$. There exists a natural factorization $g = g_1 \circ g_2$. Indeed, each X-morphism $g_2 \colon \mathrm{Spec}(\mathcal{A}) \to \mathrm{Spec}(\mathcal{B})$ corresponds to a homomorphism $\mathcal{B} \to \mathcal{A}$ of $\mathcal{O}_X$-algebras, and there is a natural inclusion $\mathcal{B} \subset \mathcal{A}$. More concretely, consider the $\mathcal{O}_{\widetilde{X}_1}$-algebra $\widetilde{\mathcal{C}} = \mathcal{O}_{\widetilde{X}_1} \oplus g_1^*(\mathscr{L})^{\otimes -1} \oplus g_1^*(\mathscr{L})^{\otimes -2} \oplus \cdots \oplus g_1^*(\mathscr{L})^{\otimes -q+1}$, defined by the invertible sheaf $g_1^*(\mathscr{L})$ and the natural identification $g_1^*(\mathscr{L}^{\otimes q}) = \mathcal{O}_{\widetilde{X}_1}$. The associated finite flat covering can naturally be identified with $g_2 \colon \widetilde{X} \to \widetilde{X}_1$ defined above. To see this, recall that $(g_1)_*(\mathcal{O}_{\widetilde{X}_1}) = \mathcal{B}$. Now use the equivalence between the category of locally free sheaves $\mathcal{E}$ on $\widetilde{X}_1$ and the category of locally free $\mathcal{B}$-modules, which is given by $\mathcal{E} \mapsto (g_1)_*(\mathcal{E})$. Thus to deduce at first that $\widetilde{\mathcal{C}}$ and $\mathcal{A}$ are isomorphic as $\mathcal{O}_{X_1}$-modules, it has to be verified that the finite locally free $\mathcal{B}$-algebras $(g_1)_*(\widetilde{\mathcal{C}})$ and $\mathcal{A}$ coincide. Now the projection formula yields

$$(g_1)_*(\widetilde{\mathcal{C}}) = \mathcal{B} \oplus (\mathscr{L}^{\otimes -1} \otimes \mathcal{B}) \oplus (\mathscr{L}^{\otimes -2} \otimes \mathcal{B}) \oplus \cdots \oplus (\mathscr{L}^{\otimes -q+1} \otimes \mathcal{B}),$$

and this $\mathcal{B}$-algebra canonically identifies with $\mathcal{A}$. The same reasoning shows that the $\mathcal{O}_{X_1}$-algebra structures of $\widetilde{\mathcal{C}}$ and $\mathcal{A}$ are the same. In particular, this shows the following:

Proposition 5.5. *In the situation of Proposition 5.3, let $d = e^r m$ for $m \geq 1$ not divisible by p. Then $g \colon \widetilde{X} \to X$ naturally decomposes into a finite étale covering $g_1 \colon \widetilde{X}_1 \to X$ of degree m associated to $\mathscr{L}^{\otimes e^r}$ and a purely inseparable finite flat covering $g_2 \colon \widetilde{X} \to \widetilde{X}_1$ of degree e^r associated to $g_1^*(\mathscr{L})$.*

Consider the case where $g \colon \widetilde{X} \to X$ is étale. Let X' be a scheme such that every element of $H^0(X', \mathcal{O}_{X'}^\times)$ has a d-th root. Then every finite flat covering $f \colon X' \to X$ with the property $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$ satisfies the property that $\deg(g)$ divides $\deg(f)$. Moreover, if the degrees coincide, then $\widetilde{X}$ and X' are isomorphic as X-schemes. So g is in this sense the minimal finite flat covering which trivializes $\mathscr{L}$.

In fact, consider a factorization $f = g \circ g'$. As g is unramified and separated, the morphism g' is also a finite flat covering. If f is étale, then g' is also étale. In any case, $\deg(f) = \deg(g) \cdot \deg(g')$ holds. The equality $\deg(f) = \deg(g)$ means $\deg(g') = 1$, in which case g' is an isomorphism: Any locally free $\mathcal{O}_{\widetilde{X}}$-algebra $\widetilde{\mathcal{A}}$ of rank 1 is isomorphic to $\mathcal{O}_{\widetilde{X}}$, since the canonical morphism $\mathcal{O}_{\widetilde{X}} \to \widetilde{\mathcal{A}}$ maps 1 to 1, and therefore it is locally an isomorphism.

5.2 Non-Existence of Étale Coverings

In contrast to the situation described at the end of the previous section, in the case that the order of $\mathscr{L}$ is divisible by p, then a finite étale covering $f\colon X' \to X$ with $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$ cannot exist at all if X is additionally normal. This is the following theorem, which constitutes this chapter's main result.

Theorem 5.6. *Let X and $\mathscr{L}$ be as in Proposition 5.3. Suppose additionally that X is normal and assume that $p \mid d$. Let X' be a scheme such that every element of $H^0(X', \mathcal{O}_{X'}^\times)$ has a p-th root. Then there does not exist a finite étale covering $f\colon X' \to X$ satisfying $f^*(\mathscr{L}) \simeq \mathcal{O}_{X'}$.*

Proof. Assume by contradiction that such an f exists. Observe that the assumption that p divides $d \geq 1$ necessarily implicates $p > 0$. Furthermore, X is normal and integral, so the étaleness of f yields that the same holds for X'. Write $d = qm$ for $q = p^r$ and $r \geq 1$ such that $p \nmid m$. The sheaf $\mathscr{L}^{\otimes m}$ has order q and also satisfies $f^*(\mathscr{L}^{\otimes m}) \simeq \mathcal{O}_{X'}$. Replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes m}$ makes it possible to assume that $d = q$ is a p-power.

Let $g\colon \widetilde{X} \to X$ be a purely inseparable finite flat covering associated to $\mathscr{L}$, as constructed in Section 5.1. By Proposition 5.3, there exists a factorization of f through g. Extend it as

$$X' \longrightarrow Y \longrightarrow \widetilde{X} \longrightarrow X,$$

where Y is the schematic image of X' in $\widetilde{X}$. Let $U = \operatorname{Spec}(R)$ be an affine open subset of X over which $\mathscr{L}$ is trivial. Over $U \subset X$, the morphisms above are given by ring homomorphisms

$$A' \longleftarrow A \longleftarrow R[S]/(S^q - u) \longleftarrow R$$

for some $u \in R^\times$, where $R \to R[S]/(S^q - u),\ a \mapsto [a]$. Write $E = K(X)$, $L = K(Y)$ and $L' = K(X')$. The sequence above induces a commutative diagram

$$\begin{array}{ccccccc} A' & \longleftarrow & A & \longleftarrow & R[S]/(S^q - u) & \longleftarrow & R \\ \downarrow & & \downarrow & & & & \downarrow \\ L' & \longleftarrow & L & \longleftarrow & \longleftarrow & \longleftarrow & E, \end{array}$$

where the vertical maps are the inclusions into the corresponding fields of fractions. The field extension $E \subset L'$ is finite and separable, since f is étale. Hence $E \subset L$ is also finite and separable.

Now let x be the class of S in $R[S]/(S^q - u)$. Then x^q is a unit, so x is a unit. Write x and u also for their images in L, as well as in E in the case of u. The separable minimal polynomial $\mu(T)$ of $x \in L$ over E divides $T^q - u \in E[T]$. Over $\overline{E}$, there is the decomposition $T^q - u = (T - w)^q$ for the q-th root $w \in \overline{E}$ of u. As $\mu(T)$ is a separable polynomial, it has to equal $T - w$ over $\overline{E}$, so also $\mu(T) = T - w$ in $E[T]$. Therefore $x = w$ is contained in E. Consider $w \in E$ as a zero of $T^q - u \in R[T]$. Since X is normal and integral, the ring R is an integrally closed domain, thus already $w \in R$.

Consequently, $R[S]/(S^q - u) = R[S]/((S-w)^q)$. Since $(S-w)/((S-w)^q)$ is contained in the nilradical of $R[S]/((S-w)^q)$ and the quotient $R[S]/(S-w)$ is isomorphic to the reduced ring R, it has to be equal to the nilradical. So the reduction of $R[S]/(S^q - u)$ is isomorphic to R. Varying $U \subset X$ in an open cover of X in turn shows that $\widetilde{X}_{\text{red}} \hookrightarrow \widetilde{X} \xrightarrow{g} X$ is an isomorphism. Thus the pullback of $\mathscr{L}$ to $\widetilde{X}_{\text{red}}$ is not trivial, contradicting the fact that already $g^*(\mathscr{L})$ is trivial. $\square$

Example 5.7. In general, it is possible that an étale morphism $f\colon X' \to X$ has a factorization through a purely inseparable finite flat covering $g\colon \widetilde{X} \to X$ of degree $d \geq 2$. For instance, let X be a reduced scheme and consider the $\mathcal{O}_X$-algebra $\mathcal{A} = \mathcal{O}_X[T]/(T^n)$. Denote $\widetilde{X} = \operatorname{Spec}(\mathcal{A})$ and write $g\colon \widetilde{X} \to X$ for the natural morphism. Then every fiber $\widetilde{X}_x = \operatorname{Spec}(\kappa(x)[T]/(T^n))$ is a singleton. So g is purely inseparable, since also $\kappa(g(\widetilde{x})) = \kappa(\widetilde{x})$ holds for every $\widetilde{x} \in \widetilde{X}$. Because $\widetilde{X}_{\text{red}} = X$, there exists a canonical section of g, and thus every morphism $f\colon X' \to X$ factorizes through g.

In the special case $\mathscr{L} = \omega_X$ on a Gorenstein, proper k-scheme X, the subsequent corollary summarizes the results obtained in the course of this chapter.

Corollary 5.8. *Let X be a Gorenstein, integral, proper scheme over an algebraically closed field k of characteristic $p \geq 0$ such that ω_X has finite order d in* $\operatorname{Pic}(X)$.

(i) The canonical covering $g\colon \widetilde{X} \to X$ is a finite flat covering of degree d, whose total space $\widetilde{X}$ is Gorenstein with $h^0(\mathcal{O}_{\widetilde{X}}) = 1$. The equalities $g^*(\omega_X) = \omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}$ hold. Every finite flat covering $f\colon X' \to X$ satisfying $f^*(\omega_X) \simeq \mathcal{O}_{X'}$ and $h^0(\mathcal{O}_{X'}) = 1$ factorizes through g.

(ii) If $p \nmid d$, then g is étale. Among all finite étale coverings $f\colon X' \to X$ with $\omega_{X'} \simeq \mathcal{O}_{X'}$, g is up to isomorphism the unique one of minimal degree d.

(iii) If $p \mid d$, then g is not étale, and decomposes as a purely inseparable finite flat covering followed by a finite étale covering. If X is additionally normal, then there does not exist a finite étale covering $f\colon X' \to X$ with $\omega_{X'} \simeq \mathcal{O}_{X'}$.

Example 5.9. Let k be of characteristic $p = 2$ and X a classical Enriques surface. Since $\operatorname{Pic}^\tau_X = \mathbb{Z}/2\mathbb{Z}$ and $h^0(\omega_X) = h^2(\mathcal{O}_X) = 0$, the dualizing sheaf ω_X has order $d = 2$ in $\operatorname{Pic}(X)$. By the result above, since $p = d$, there does not exist a finite étale covering trivializing the dualizing sheaf.

Remark 5.10. Now let X be a $\mathbb{Q}$-Gorenstein, integral, proper scheme over an algebraically closed field k of characteristic $p \geq 0$ such that K_X has index $m \geq 2$ and order d. Then it is impossible that there exists a finite flat covering $f\colon X' \to X$ with $K_{X'} \sim 0$. Otherwise X' is Gorenstein and the faithful flatness of f implies that X has to be Gorenstein, too.

Decompose $d = lm$. Nevertheless, it is possible to remove the factor l after a finite flat covering. To achieve this, consider the invertible sheaf $\mathscr{L} = \mathcal{O}_X(mK_X)$ of order l and the associated canonical covering $g\colon \widetilde{X} \to X$. Note that the flatness of g implies that every

$\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation satisfies $g^*(\mathcal{F}^\vee) = g^*(\mathcal{F})^\vee$. So g^* is compatible with the composition law $\mathcal{O}_X(D_1 + D_2) = (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee}$ of rank 1 reflexive sheaves. This results in $g^*(\mathcal{O}_X(tK_X)) = \mathcal{O}_{\widetilde{X}}(tK_{\widetilde{X}})$ for all $t \geq 1$, since $\omega_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}$ is trivial and thereby $g^*(\mathcal{O}_X(K_X)) = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ holds. For $t = m$, this gives $\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}})$ and so $mK_{\widetilde{X}} \sim 0$. Especially, the total space $\widetilde{X}$ is $\mathbb{Q}$-Gorenstein of index at most m. In the case that $p \nmid l$, the covering g is étale. On the other hand, if $p \mid l$, there does not exist a finite étale covering $f\colon X' \to X$ with $\mathcal{O}_{X'} \simeq f^*(\mathcal{O}_X(mK_X)) = \mathcal{O}_{X'}(mK_{X'})$. Hence $mK_{X'} \not\sim 0$ must hold on the total space X' of every finite étale covering $X' \to X$.

One particular consequence of Corollary 5.8 is that the Beauville–Bogomolov decomposition happens to fail in positive characteristic. In Section 6.2, the order of the dualizing sheaf for all classes of smooth, integral, proper surfaces with ω_X of finite order will be specified. Therefore question (Q2) will be answered in detail. This is postponed to the upcoming chapter, so that in the course of working through the classification of those surfaces, their behavior with regard to questions (Q3) and (Q4) can be investigated likewise.

Chapter 6

Total Spaces of Coverings

The content of this chapter is twofold: During the first Section 6.1, the uniqueness of a decomposition, as stated in questions (Q3) and (Q4) for the total space of a covering, is proved. Its verification comprises two essential ingredients, namely the results of Fujita [41] and compatibility properties of the Albanese morphism.

In Section 6.2, a detailed analysis of smooth surfaces with dualizing sheaf of finite order follows. Here the focus lies on the interplay between the fact whether ω_X is already trivial, question (Q2), which can now be answered by the result of the previous chapter, and question (Q3). For instance, in characteristic $p = 0$, a smooth surface with $\omega_X \simeq \mathcal{O}_X$ is already in Beauville–Bogomolov decomposition, but this does not hold true in positive characteristic. Question (Q4) in return is always answered in the affirmative. Although the Enriques classification has already clarified the situation to a large extent, there remain some details which appear to be worth investigating.

6.1 Uniqueness of a Decomposition

The starting point of this section is to prove over an algebraically closed field that if $X = A \times B$ is the product of an abelian variety A and an integral, proper scheme B with $h^1(\mathcal{O}_B) = 0$, then the decomposition as a product of this kind is unique up to isomorphism of the individual factors. The result below will actually hold under more general assumptions. Roughly speaking, the point is that B has to be in some sense "far away" from being an abelian variety. The crucial tool for its proof is the subsequent result, which refers to [41], Theorem 6.

Proposition 6.1. *Let X, Y, Z be integral, proper schemes over an algebraically closed field k, where Z is projective. Assume that $X \times Z \simeq Y \times Z$. If X and Z are Picard independent, then $X \simeq Y$.*

The notion of *Picard independency* can be defined in the following, more general situation. Let $f_X \colon X \to S$ and $f_Y \colon Y \to S$ be separated morphisms of finite type to a locally noetherian scheme S such that $\mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{Y/S}$ exist and represent the relative Picard functor. Then it is true that every S-morphism $Y \to \mathrm{Pic}_{X/S}$ has a factorization through S if and only if every invertible sheaf $\mathscr{L}$ on $X \times_S Y$ is of the form $\mathscr{L} = \mathrm{pr}_X^*(\mathcal{A}) \otimes \mathrm{pr}_Y^*(\mathcal{B})$ for invertible sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y. This can be deduced as follows:

For the first implication, let $\mathscr{L}$ be an invertible sheaf on $X \times_S Y$, and assume that the induced morphism $Y \to \mathrm{Pic}_{X/S}$ admits a factorization $Y \xrightarrow{f_Y} S \xrightarrow{g} \mathrm{Pic}_{X/S}$. This means that

$$\mathscr{L} \equiv (\mathrm{id}_X \times f_Y)^*(\mathcal{A}) = \mathrm{pr}_X^*(\mathcal{A})$$

holds modulo $\mathrm{Pic}(Y)$ for an invertible sheaf $\mathcal{A}$ on $X = X \times_S S$, whose class modulo $\mathrm{Pic}(S)$ corresponds to g. Hence $\mathscr{L} = \mathrm{pr}_X^*(\mathcal{A}) \otimes \mathrm{pr}_Y^*(\mathcal{B})$ for an invertible sheaf $\mathcal{B}$ on Y.

Conversely, let $Y \to \mathrm{Pic}_{X/S}$ be an S-morphism corresponding to the class modulo $\mathrm{Pic}(Y)$ of an invertible sheaf $\mathscr{L} = \mathrm{pr}_X^*(\mathcal{A}) \otimes \mathrm{pr}_Y^*(\mathcal{B})$ on $X \times_S Y$. But another representative of this class is simply $\mathrm{pr}_X^*(\mathcal{A})$, which corresponds to the morphism $Y \xrightarrow{f_Y} S \xrightarrow{g} \mathrm{Pic}_{X/S}$, where g is the morphism obtained by $\mathcal{A}$. Thereby the factorization through S exists.

The second statement in the equivalence above is symmetric in X and Y, so the first one has to be, too. This leads to the next definition.

Definition 6.2. Two S-schemes X and Y as above are called *Picard independent* if the following equivalent conditions are satisfied:

(i) Every S-morphism $Y \to \mathrm{Pic}_{X/S}$ has a factorization through S.

(ii) Every S-morphism $X \to \mathrm{Pic}_{Y/S}$ has a factorization through S.

(iii) Every invertible sheaf $\mathscr{L}$ on $X \times_S Y$ is of the form $\mathscr{L} = \mathrm{pr}_X^*(\mathcal{A}) \otimes \mathrm{pr}_Y^*(\mathcal{B})$ for invertible sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y.

In order to apply the above to the decomposition occurring both in (Q3) and (Q4), fix $S = \mathrm{Spec}(k)$ for an arbitrary field k and consider the subsequent two lemmata.

Lemma 6.3. *Let J be a k-scheme such that $J_{\overline{k}}$ is an abelian variety and B a proper k-scheme of Albanese dimension $m = 0$ with $h^0(\mathcal{O}_B) = 1$. Then every morphism $B \to \mathrm{Pic}_J$ has a factorization through* $\mathrm{Spec}(k)$.

Proof. Let $B \to \mathrm{Pic}_J$ be a morphism of k-schemes. To show that it factorizes through $\mathrm{Spec}(k)$, it is sufficient to verify that it is constant: If the latter was the case, then its schematic image in Pic_J equals $\mathrm{Spec}(R)$ for an artinian local k-algebra R. Since $h^0(\mathcal{O}_B) = 1$, the morphism $X \to \mathrm{Spec}(R)$ has a factorization through $\mathrm{Spec}(k)$. But then necessarily $R = k$.

Now to prove that $B \to \mathrm{Pic}_J$ is constant, assume without loss of generality that k is algebraically closed. Choose a k-rational point of B and compose with a translation $\mathrm{Pic}_J \to \mathrm{Pic}_J$ to achieve that the neutral element of Pic_J is contained in the image of $B \to \mathrm{Pic}_J$. As B is connected, this morphism admits a factorization through the abelian variety Pic_J^0. But B has Albanese dimension $m = 0$, and hence the image has to be a point. □

Lemma 6.4. *Let J be a k-scheme such that $J_{\overline{k}}$ is an abelian variety, and B a geometrically integral, proper k-scheme of Albanese dimension $m = 0$. Then $\mathrm{pr}_J \colon J \times B \to J$ is the Albanese morphism of $J \times B$.*

Proof. The Albanese morphism exists for J, B and also for $J \times B$ by assumption. Moreover, Proposition 3.38 shows that $\mathrm{alb}_J \times \mathrm{alb}_B$ is the Albanese morphism of $J \times B$. As explained in Example 3.37, J is a principal homogeneous space under an abelian variety and $\mathrm{alb}_J = \mathrm{id}_J$. Proposition 3.42 yields that alb_B is the structure morphism of B. □

Now the main result of this section can be derived directly. Beforehand, recall that a proper k-scheme X with $h^0(\mathcal{O}_X) = 1$ is of Albanese dimension $m = 0$ if $h^1(\mathcal{O}_X) = 0$, according to Proposition 3.42. Hence the conclusion indeed applies to both (Q3) and (Q4).

Theorem 6.5. *Let J, J', B, B' be schemes over an arbitrary field k such that*

(i) J and J' become abelian varieties after base change to $\overline{k}$,

(ii) B and B' are of Albanese dimension $m = 0$, geometrically integral and proper over k.

Suppose that $J \times B \simeq J' \times B'$. Then $J \simeq J'$ as well as $B_{\overline{k}} \simeq B'_{\overline{k}}$.

Proof. First, Lemma 6.4 shows that $J = \mathrm{Alb}^1_{J\times B}$ and $J' = \mathrm{Alb}^1_{J'\times B'}$ are isomorphic. So $J_{\overline{k}} \times B_{\overline{k}} \simeq J_{\overline{k}} \times B'_{\overline{k}}$ holds. Over $\overline{k}$, the Picard schemes of $J_{\overline{k}}$ and $B_{\overline{k}}$ represent the relative Picard functor. Then $J_{\overline{k}}$ and $B_{\overline{k}}$ are Picard independent by Lemma 6.3, so eventually Proposition 6.1 yields that $B_{\overline{k}} \simeq B'_{\overline{k}}$. □

Remark 6.6. In general, it is not valid that the existence of an isomorphism $X \times Z \simeq Y \times Z$ implies that $X \simeq Y$. Counterexamples, where all occurring schemes are elliptic curves, were given by Shioda [122].

6.2 Surfaces and the Beauville–Bogomolov Decomposition

There are four classes of smooth surfaces in Kodaira dimension zero: K3-surfaces, Enriques surfaces, abelian surfaces and bielliptic surfaces. As seen for classical Enriques surfaces in Example 5.9, it actually happens in characteristic $p > 0$ that ω_X has finite order, but there does not exist a finite étale covering $X' \to X$ such that $\omega_{X'}$ is trivial. Moreover, there are Enriques and bielliptic surfaces with $\omega_X \simeq \mathcal{O}_X$ such that X does not decompose into a product as in the Beauville–Bogomolov decomposition.

Let X be a smooth, integral, proper surface over a fixed algebraically closed field k of characteristic $p \geq 0$ with ω_X numerically trivial. To examine the differences in characteristic $p > 0$ to the situation in characteristic $p = 0$, define the following four properties:

(P1) The dualizing sheaf $\omega_X \simeq \mathcal{O}_X$ is trivial.

(P2) There exists a finite étale covering $X' \to X$ such that $\omega_{X'} \simeq \mathcal{O}_{X'}$.

(P3) There exists a finite étale covering $X' \to X$ such that $X' \simeq A \times B$, where A is an abelian variety and B is integral with $h^1(\mathcal{O}_B) = 0$ and $\omega_B \simeq \mathcal{O}_B$.

(P4) There exists a finite flat covering $X' \to X$ such that $X' \simeq A \times B$, where A is an abelian variety and B is integral of Albanese dimension $m = 0$ with $\omega_B \simeq \mathcal{O}_B$.

Here properties (P2) to (P4) directly echo questions (Q2) to (Q4) from the introduction, now formulated as statements. As question (Q1) is always answered in the affirmative for smooth surfaces, it is replaced by the stronger property (P1).

Clearly, the implications (P1) $\Rightarrow$ (P2) and (P3) $\Rightarrow$ (P2) as well as (P3) $\Rightarrow$ (P4) hold. Property (P4) is always fulfilled, which will be discussed during the course of this section. Taking this for granted for the time being, there remain five possible combinations: Let "y" abbreviate "yes" and "n" abbreviate "no". The first two patterns are (n, y, y, y) and (y, y, y, y), which are the only ones possible when $p = 0$. The three additional possibilities are (y, y, n, y) and (n, y, n, y) as well as (n, n, n, y). It turns out that all combinations occur in positive characteristic.

In characteristic $p = 0$, recall that if B is normal, it has Albanese dimension $m = 0$ if and only if $h^1(\mathcal{O}_B) = 0$ by Proposition 3.42. So another variant for (P3) in positive characteristic is to only demand that the Albanese dimension of B is zero, as imposed in (P4). But the effect of this change is minor and it is discussed below, too.

Now suppose that $X' \to X$ is finite étale and X' decomposes as in (P3). Then it is not possible to have a decomposition $X' \simeq A \times B$ as a product of two curves, where A is an abelian variety and B satisfies $h^1(\mathcal{O}_B) = 0$. Indeed as X' is smooth, this would imply that A is an elliptic curve and $B \simeq \mathbb{P}^1$. But then $X' = \mathbb{P}(\mathcal{O}_A \oplus \mathcal{O}_A)$ is a ruled surface, and thereby $\operatorname{kod}(X') = -\infty$. Observe that as B is a smooth curve, the condition $h^1(\mathcal{O}_B) = 0$ is again equivalent to the demand that B is of Albanese dimension $m = 0$.

So whenever a surface $X' \simeq A \times B$ appears in (P3), it is either abelian or it satisfies $h^1(\mathcal{O}_{X'}) = 0$ and $\omega_{X'} \simeq \mathcal{O}_{X'}$, in which case it is a K3-surface. In this sense, the term "decomposition" appears to be improper for surfaces, but no different notion is introduced as it is in alignment with the general situation. The two classes of abelian surfaces and K3-surfaces have pattern (y, y, y, y). If $p \notin \{2, 3\}$, then the converse is also true, but in the case $p \in \{2, 3\}$, there also exist Enriques and bielliptic surfaces with this pattern.

Note in this context that the total space of a finite étale covering of an abelian variety is an abelian variety itself due to Serre–Lang, [95], Section 18, Theorem on page 167. All K3-surfaces are simply connected, that is, they admit no non-trivial finite étale coverings, which follows from the Enriques classification and Corollary 1.42.

Enriques Surfaces. Let $p = 2$, since this is the only case where new phenomena in comparison to the classical situation appear for Enriques surfaces. First, let X be an ordinary or supersingular Enriques surface, that means $\operatorname{Pic}^\tau_X = \mu_2$ or $\operatorname{Pic}^\tau_X = \alpha_2$. Both finite group schemes consist of a single point and are non-reduced, which yields $\omega_X \simeq \mathcal{O}_X$. This also follows from the fact that $h^0(\omega_X) = h^2(\mathcal{O}_X) = 1$ and ω_X is numerically trivial. Here $X = X'$ as in (P3) in not valid, as $h^1(\mathcal{O}_X) = 1$ is positive.

If X is an ordinary Enriques surface, the K3-cover $\widetilde{X} \to X$ is étale and $\widetilde{X}$ is a K3-surface. In the case that X is classical or supersingular, there exists no non-trivial étale covering of X. In fact, by Corollary 1.42, a non-trivial étale morphism $f\colon X' \to X$ would satisfy the inequality

$$\chi(\mathcal{O}_{X'}) = \deg(f) \cdot \chi(\mathcal{O}_X) = \deg(f) \geq 2. \tag{6.1}$$

The Enriques classification shows that only $\chi(\mathcal{O}_{X'}) = 2 = \deg(f)$ is possible. This would mean that X' has to be a K3-surface. But a non-trivial étale covering of degree 2 corresponds to a principal homogeneous space under $(\mathbb{Z}/2\mathbb{Z})_X$, see [30], page 20. Then Raynaud's correspondence (4.4) implies that there exists an associated non-trivial homomorphism $\mu_2 \to \mathrm{Pic}^\tau_X$. But for $\mathrm{Pic}^\tau_X = \mathbb{Z}/2\mathbb{Z}$, it has to be constant as already the underlying map of topological spaces has to be constant. For $\mathrm{Pic}^\tau_X = \alpha_2$, Cartier duality yields $\alpha_2 \to \mathbb{Z}/2\mathbb{Z}$ and the same reasoning works. So if X is classical or supersingular, then X is actually simply connected.

As mentioned in Section 4.4, the K3-cover $X' \to X$ of an Enriques surface is always an integral surface X' with $\omega_{X'} \simeq \mathcal{O}_{X'}$ and $h^1(\mathcal{O}_{X'}) = 0$. The latter condition implies that X' is of Albanese dimension $m = 0$, and hence (P4) holds. All possible property patterns for Enriques surfaces in arbitrary characteristic $p \geq 0$ are displayed below.

	p	(P1)	(P2)	(P3)	(P4)
classical	$\neq 2$	n	y	y	y
classical	2	n	n	n	y
ordinary	2	y	y	y	y
supersingular	2	y	y	n	y

Table 4
Property patterns of Enriques surfaces.

To conclude the discussion of Enriques surfaces, consider the variant of (P3) where B is only demanded to have Albanese dimension $m = 0$, instead of $h^1(\mathcal{O}_B) = 0$. The effect of this change is the following: The factor B is additionally allowed to be an ordinary or supersingular Enriques surface, apart from being a K3-surface. Hence it improves the situation for supersingular Enriques surfaces, which now satisfy (P3). But it does not affect classical Enriques surfaces in characteristic $p = 2$. So the impact of this variant is rather minor and it does not solve the problem that property (P3) may fail to be fulfilled. This variant does not at all affect the situation for bielliptic surfaces, which is discussed now.

Bielliptic Surfaces. The identification of bielliptic surfaces $X \simeq (E \times C)/G$ as quotients described in Section 4.4 makes it possible to derive the order of $\omega_X \in \mathrm{Pic}(X)$ from the concrete shape of the G-action.

To do so, start by outlining the general approach as it was executed in [18], [17] in the elliptic and quasielliptic case, respectively. Denote $A = E \times C$ and $q\colon A \to X$ for the induced finite flat covering. Note that C is always Gorenstein with $\omega_C \simeq \mathcal{O}_C$ by Example 4.4 and Example 4.6. Hence also A is Gorenstein with $\omega_A \simeq \mathcal{O}_A$, and q is Gorenstein.

Proposition 6.7. *The relative dualizing sheaf $\omega_{A/X} \simeq \mathcal{O}_A$ is trivial, so $q^*(\omega_X) \simeq \mathcal{O}_A$. Moreover, $q\colon A \to X$ is étale if and only if G is a constant group scheme.*

Proof. Write $P = G \times A$ and $\rho\colon P \to A$ for the group action. The image of the closed embedding $(\rho, \mathrm{pr}_A)\colon P \to A \times A$ is $A \times_{A/G} A$ by [95], Section 12, Theorem 1 on page 111. Let $\iota\colon A \to P$ be the inclusion at the neutral element. Consider the commutative diagram

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_A}{\frown\!\!\!\longrightarrow} & & \\ A & \xrightarrow{\iota} & P & \xrightarrow{\rho} & A \\ & & {\scriptstyle \mathrm{pr}_A}\downarrow & & \downarrow{\scriptstyle q} \\ & & A & \xrightarrow[q]{} & X. \end{array}$$

Proposition 1.1 shows that every scheme and every morphism, forming the cartesian square in the diagram above, is Gorenstein. This in turn yields that G is Gorenstein. Because G is zero-dimensional, its dualizing sheaf has to be trivial. Hence also $\omega_{\mathrm{pr}_A} \simeq \mathcal{O}_P$. Since $\rho^*(\omega_{A/X}) = \omega_{\mathrm{pr}_A}$ and $\rho \circ \iota = \mathrm{id}_A$, eventually $\omega_{A/X}$ is trivial. Thus the natural identification $\omega_A = q^*(\omega_X) \otimes \omega_{A/X}$ becomes $\omega_A \simeq q^*(\omega_X)$. As $\omega_A \simeq \mathcal{O}_A$, the first statement is proved.

For the second assertion, observe that G is constant if and only if $G \to \mathrm{Spec}(k)$ is étale, as k is separably closed. If this is the case, then [95], Section 7, Theorem on page 66, yields that q is étale. Now assume that q is étale. Then pr_A is étale by base change. Faithfully flat descent [58], Corollaire 17.7.3, finally shows that $G \to \mathrm{Spec}(k)$ is étale. $\square$

For all $i \geq 1$ there is a natural isomorphism $\omega_X^{\otimes i} \simeq q_*(q^*(\omega_X^{\otimes i}))^G \simeq q_*(\omega_A^{\otimes i})^G$ by [95], Section 12, page 114. Here $\mathcal{F}^G$ denotes the sheaf of G-invariant sections of a G-linearized sheaf $\mathcal{F}$ on a scheme X with G-action. See [15], Chapter 3, for a detailed elaboration of this notion. The G-invariant sections are defined as follows: The *G-linearization* on $\mathcal{F}$ is a collection of isomorphisms $\gamma_g^*(\mathcal{F}_T) \xrightarrow{\simeq} \mathcal{F}_T$ indexed by $g \in G(T)$, where $\gamma_g\colon X_T \to X_T$ is the automorphism corresponding to g. Those isomorphisms satisfy certain natural compatibility conditions, see [15], Proposition 3.29. A global section $\sigma \in H^0(X, \mathcal{F})$ is called *G-invariant* if for all $g \in G(T)$, the pullback $\sigma_T \in H^0(X_T, \mathcal{F}_T)$ gets mapped to itself under the bijection $H^0(X_T, \mathcal{F}_T) \to H^0(X_T, \gamma_g^*(\mathcal{F}_T)) \to H^0(X_T, \mathcal{F}_T)$. In the special situation of the induced G-linearization on $\mathcal{F} = q_*(\omega_A^{\otimes i})$, this map is simply

$$\nu_g^*\colon H^0(A_T, \omega_{A_T}^{\otimes i}) \longrightarrow H^0(A_T, \omega_{A_T}^{\otimes i}), \tag{6.2}$$

where $\nu_g\colon A_T \to A_T$ is the automorphism corresponding to g. The reason is that the G-linearization on ω_A is the natural identification given by uniqueness of the dualizing sheaf, because of its universal property. When G is a constant group scheme, it is sufficient to consider only $T = \mathrm{Spec}(k)$.

As ω_X is numerically trivial, one of its powers is isomorphic to $\mathcal{O}_X$ if and only if this power has a non-zero global section. Thus

$$\operatorname{ord}(\omega_X) = \min\{\, i \geq 1 \mid \omega_A^{\otimes i} \text{ has a } G\text{-invariant non-zero global section}\,\}.$$

Since $\omega_A \simeq \mathcal{O}_A$ and $h^0(\omega_A) = 1$, it is by k-linearity sufficient to choose a non-zero global section $\sigma \in H^0(A, \omega_A)$ and determine the minimal $i \geq 1$ such that $\sigma^{\otimes i}$ is G-invariant.

To do the computation, first only look at the elliptic curve E. Let it be given in variables u, v, w by the Weierstraß equation

$$v^2w + a_1uvw + a_3vw^2 = u^3 + a_2u^2w + a_4uw^2 + a_6w^3$$

with neutral element $(0 : 1 : 0)$. On the affine open subset $W = E \cap D_+(w)$ of E, the section $\omega = \frac{dx}{2y+a_1x+a_3} \in H^0(W, \omega_E)$ is a generator, and lifts to a non-zero global section $\omega \in H^0(E, \omega_E)$, called the *invariant differential*, as computed in [88], Chapter 6, Proposition 1.26. Here the elements $x = \frac{u}{w}$ and $y = \frac{v}{w}$ of $H^0(W, \mathcal{O}_W) \subset K(E)$ are the variables on W where E is given by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

It is indeed true that ω is invariant under all translations, see [123], Chapter III, Proposition 5.1. Every group automorphism of E is given by

$$x \longmapsto a^2x + r, \quad y \longmapsto a^3y + a^2sx + t$$

for some $a \in k^\times$ and $r, s, t \in k$, see [81], Appendix 1, (1.13). A direct computation reveals that $\omega \mapsto a^{-1}\omega$ and thus $\omega^{\otimes i} \mapsto a^{-i}\omega^{\otimes i}$ for the section $\omega^{\otimes i} \in H^0(E, \omega_E^{\otimes i})$. In conclusion, a group automorphism leaves $\omega^{\otimes i}$ invariant as soon as $a^i = 1$.

Back to bielliptic surfaces $X \simeq A/G$ as quotients of surfaces $A = E \times C$. The dualizing sheaf of A is $\omega_A = \mathrm{pr}_E^*(\omega_E) \otimes \mathrm{pr}_C^*(\omega_C)$. An application of the Künneth formula, [74], Theorem 14, yields $H^0(A, \omega_A^{\otimes i}) = H^0(E, \omega_E^{\otimes i}) \otimes_k H^0(C, \omega_C^{\otimes i})$ for all $i \geq 0$. Now the group scheme G acts via the diagonal action on A for distinct actions on E an C. Note that the automorphisms of an abelian variety consist of group automorphisms and translations. The action on E is always given by translations, which fix the invariant differential on E. In turn, the order of ω_X is only determined by the action on the second factor C, so

$$\operatorname{ord}(\omega_X) = \min\{\, i \geq 1 \mid \omega_C^{\otimes i} \text{ has a } G\text{-invariant non-zero global section}\,\}.$$

In the elliptic case, the curve $C = E'$ is an elliptic curve, too. All possible group schemes G and their actions on $A = E \times E'$ such that $X = (E \times E')/G$ is a bielliptic surface are given in [18], page 37. The procedure to compute the order of ω_X in each case is the following: Identify by which group automorphisms G acts on E' and match them with the classification in [81], Appendix 1, Section 2. Then $\operatorname{ord}(\omega_X)$ is the least common multiple of all orders of $a \in k^\times$ appearing in those automorphisms. This was already stated in [18] for their different classes (a), (b), (c), (d) of G-actions, and leads to the following table:

	ord(ω_X)			
p	(a)	(b)	(c)	(d)
$\neq 2, 3$	2	3	4	6
3	2	1	4	2
2	1	3	1	3

Table 5
Values of ord(ω_X) for bielliptic surfaces, elliptic cases.

In the quasielliptic case, the curve C is rational with one cusp. Here the additional possibility that $p \mid \operatorname{ord}(\omega_X)$ appears. The approach to determine $\operatorname{ord}(\omega_X)$ in [17] is similar: A computation of the automorphism scheme of C leads to a list of all possible group schemes G acting on C and on $A = E \times C$ such that $X = A/G$ is quasielliptic. Then the order of ω_X is derived by looking at the action of G on some non-zero global section of $H^0(C, \omega_C^{\otimes i})$, as before. This was done in [17], Proposition 8, albeit the explicit values of $\operatorname{ord}(\omega_X)$ were not written down. Some more details on how to treat this case, in which G is never constant, will be given in the following.

Write $C = V_+(v^2w - u^3) \subset \mathbb{P}^2$ in variables u, v, w. The cusp is the point $c = (0 : 0 : 1)$. Set $V = C \cap D_+(v)$ and $W = C \cap D_+(w)$, so that $C = V \cup W$ and $C \smallsetminus V = \{c\}$. The affine open set V is the spectrum of $k[s,t]/(s - t^3) = k[\,t\,]$, where $s = \frac{w}{v}$ and $t = \frac{u}{v}$. As V is smooth, and thereby $\omega_C|_V = \Omega^1_V$ is the canonical sheaf, it is generated by $\omega = dt$ in $H^0(V, \omega_C|_V)$. Now ω lifts to a generator $\omega \in H^0(C, \omega_C)$ of the invertible sheaf $\omega_C \simeq \mathcal{O}_C$, because every non-zero global section has to map to some generator of $H^0(V, \omega_C|_V)$ as a $k[\,t\,]$-module, and hence to some $a \cdot dt$ for $a \in k^\times$. The above is compatible with base change along $T \to \operatorname{Spec}(k)$ for $T = \operatorname{Spec}(R)$ affine, and thus also for general T. So the induced global section $\omega \in H^0(C_T, \omega_{C_T})$ continues to be determined by its restriction $dt \in H^0(V_T, \omega_{C_T})$.

Consider the automorphism scheme $H = \operatorname{Aut}_{C/k}$ of C. Every $\varphi \in H(T)$ leaves $V_T \subset C_T$ invariant. This holds for $T = \operatorname{Spec}(k)$, since $V \subset C$ is the smooth locus, and therefore it is true over all closed points of T, so that V_T and its image have the same closed points. Nevertheless, it is possible that φ is not contained in the stabilizer subgroup scheme $H_c \subset H$ of the singular point c. Actually, those automorphisms have to exist in $G \subset H$ to render it possible that $X = (E \times C)/G$ becomes smooth, see [17], Proposition 7. The restriction of an automorphism $\nu_g \colon C_T \to C_T$ for $g \in G(T)$, thereby, induces an automorphism of $H^0(V_T, \omega_{C_T}^{\otimes i})$ for $i \geq 1$ as in (6.2). Consequently, it is equivalent to determine whether $\omega^{\otimes i} \in H^0(C_T, \omega_{C_T}^{\otimes i})$ is invariant under ν_g or $dt^{\otimes i} \in H^0(V_T, \omega_{C_T}^{\otimes i})$ is invariant under $\nu_g|_{V_T}$.

The list of possible group schemes G acting on C such that X becomes bielliptic was computed in [17], preceding to Proposition 8. The group action on C is determined by its effect on $t \in H^0(V_T, \mathcal{O}_{V_T})$, so the effect on $dt^{\otimes i} \in H^0(V_T, \omega_{C_T}^{\otimes i})$ can be inferred as a result. Using the notation of [17], this yields the subsequent tables:

(a)	(b)	(c)	(d)	(e)
3	6	3	1	2

Table 6
Values of $\mathrm{ord}(\omega_X)$ for bielliptic surfaces, quasielliptic cases, in characteristic $p = 3$.

The values above in characteristic $p = 3$ were also derived using a different approach by Lang [82], Section 3.B, where he points out that case (f) in [17] cannot exist.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
2	6	2	4	4	1	3	1

Table 7
Values of $\mathrm{ord}(\omega_X)$ for bielliptic surfaces, quasielliptic cases, in characteristic $p = 2$.

Once the order d of ω_X is known, property (P2) can be determined by Corollary 5.8: If $d \geq 2$, there is a finite étale covering $\widetilde{X} \to X$ such that $\omega_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}$ if and only if $p \nmid d$. In the cases $p \notin \{2, 3\}$, the curve $C = E'$ is always elliptic and $d \geq 2$ is never divisible by p. So there always exists such a finite étale covering $g\colon \widetilde{X} \to X$, where $\widetilde{X}$ has to be an abelian surface. In fact, as $p \notin \{2, 3\}$, only abelian surfaces satisfy $\omega_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}$ and $\chi(\mathcal{O}_{\widetilde{X}}) = \deg(g) \cdot \chi(\mathcal{O}_X) = 0$.

Staying in the elliptic case, the situation becomes more versatile if $p \in \{2, 3\}$. It remains to be true that d is never divisible by p. But it is additionally possible that $d = 1$. Nevertheless, the natural morphism $A = E \times E' \to (E \times E')/G = X$ is almost always étale, with an exceptional case in characteristic $p = 2$, when the group scheme G is not constant. But also in this case, treated in the subsequent paragraph, there still exists a finite étale covering $\widetilde{X} \to X$ such that $\omega_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}$, where $\widetilde{X}$ is an abelian surface.

The exceptional case is (a3) in [18], page 37, where $G = \mathbb{Z}/2\mathbb{Z} \times \mu_2$ is non-reduced in characteristic $p = 2$. Here $A \to A/G$ is not étale, since $G \to \mathrm{Spec}(k)$ is not constant by Proposition 6.7. However, the subgroup scheme $H = \mu_2$ of G acts by translation on A, so it can be identified with a subgroup $H \subset A$. Thus A/H is an abelian variety, as explained in the proof of Proposition 3.9. There is a factorization $A \to A/H \to A/G = X$ where $A/H \to X$ is étale as a quotient by the constant group scheme $G/H = \mathbb{Z}/2\mathbb{Z}$. In summary, all bielliptic surfaces in the elliptic case satisfy properties (P2) and (P3). Property (P1) is never fulfilled if $p \notin \{2, 3\}$, but can hold otherwise, see the table above.

In the quasielliptic case, the curve C is rational with one cusp and necessarily $p \in \{2, 3\}$. Here there never exists an étale morphism $\varphi\colon A \to X$ from an abelian surface A. Assume by contradiction that this was the case. Let $X \to B$ be the Albanese morphism of X. Choose a closed point $b \in B$ such that the fiber X_b is isomorphic to C. Then choose a second closed point $x \in X_b$ and a third one $e \in A$ mapping to x. Now regard A and B as abelian varieties with neutral element $e \in A$ and $b \in B$. Consider the diagram

$$\begin{array}{ccc} D & \longrightarrow & A \\ {\scriptstyle \text{étale}}\downarrow & & \downarrow{\scriptstyle \text{étale}} \\ C & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\kappa(b)) & \longrightarrow & B, \end{array}$$

where $D := A_b$ is the fiber over $b \in B$. The composition $A \to B$ is by choice a homomorphism. Thus the curve $D = \ker(A \to B)$ is of canonical type and a subgroup scheme of A. If some connected component of D was reducible, then each of its integral components would be a (-2)-curve and in turn isomorphic to $\mathbb{P}^1$. But this is impossible, as $\operatorname{Alb}^1_{\mathbb{P}^1} = \operatorname{Spec}(k)$. Hence the identity component D^0 is irreducible. As both C and $D \to C$ are reduced, also D^0 is reduced. So D^0 is an integral, proper group scheme over the algebraically closed field k, and thus an elliptic curve. But as $D^0 \to C$ is faithfully flat, also C is regular, a contradiction. This has shown that no étale morphism $A \to X$ from an abelian surface A can exist if X is quasielliptic.

On the other hand, there always exists the finite flat covering $E \times C \to X$ to the quotient $X = (E \times C)/G$. Here E is an elliptic curve and C is integral of Albanese dimension $m = 0$ with $\omega_C \simeq \mathcal{O}_C$ due to Example 3.44. Hence property (P4) is fulfilled.

The next table summarizes all possible patterns, where $d = \operatorname{ord}(\omega_X)$. Here (n, y, n, y), which cannot occur for an Enriques surface, appears. If $p \notin \{2, 3\}$, the pattern is always (n, y, y, y) and for each $p \in \{2, 3\}$, every possibility listed in the table actually occurs.

		(P1)	(P2)	(P3)	(P4)
elliptic	$d \geq 2$	n	y	y	y
elliptic	$d = 1$	y	y	y	y
quasielliptic	$d \geq 2$ and $p \mid d$	n	n	n	y
quasielliptic	$d \geq 2$ and $p \nmid d$	n	y	n	y
quasielliptic	$d = 1$	y	y	n	y

Table 8
Property patterns of bielliptic surfaces.

Remark 6.8. For a bielliptic surface X, the dualizing sheaf $\omega_X \in \operatorname{Pic}^\tau(X)$ already has the property $\omega_X \in \operatorname{Pic}^0(X)$. Indeed, the first step in the proof of Theorem 4.17 can be applied just as well here and shows that $\omega_X \simeq f^*(\mathscr{N})$ for some invertible sheaf $\mathscr{N}$ on Y, where $f\colon X \to Y$ is the Albanese morphism of X. Now $\mathscr{N} \in \operatorname{Pic}^\tau(Y)$ is numerically trivial and the equality $\operatorname{Pic}^0(Y) = \operatorname{Pic}^\tau(Y)$ holds according to Example A.31. Hence Proposition 3.39 implies $f^*(\mathscr{N}) \in \operatorname{Pic}^0(X)$, as claimed.

It appears to be an interesting question to what extent property (P4) holds true for singular surfaces or in higher dimensions.

Appendix A

A.1 Basic Group Schemes

Settle the notation: Let $G_{\mathbb{Z}}$ be a group scheme over $\mathbb{Z}$. Denote its base change to a scheme S by G_S. In the case that the ground scheme S is fixed, also abbreviate $G = G_S$.

(i) Let Γ be an abstract group and $\Gamma_{\mathbb{Z}} = \coprod_{\gamma\in\Gamma} \operatorname{Spec}(\mathbb{Z})$. Given a connected scheme T, a morphism $T \to \Gamma_{\mathbb{Z}}$ corresponds to the choice of a connected component of $\Gamma_{\mathbb{Z}}$, and thus to an element $\gamma \in \Gamma$. Hence Γ induces a group structure on $\Gamma_{\mathbb{Z}}(T)$. It extends to T-valued points for not necessarily connected schemes T by decomposing T into its connected components. This defines the *constant group scheme* $\Gamma_{\mathbb{Z}}$.

(ii) Set $\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[x, x^{-1}])$. The identification $\mathbb{G}_{m,\mathbb{Z}}(T) = H^0(T, \mathcal{O}_T^\times)$ for every scheme T induces a group structure on $\mathbb{G}_{m,\mathbb{Z}}(T)$ by multiplication. This yields the *multiplicative group scheme* $\mathbb{G}_{m,\mathbb{Z}}$.

(iii) Define $\mathbb{G}_{a,\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[x])$. For any scheme T, a group structure on $\mathbb{G}_{a,\mathbb{Z}}(T)$ is given by the identification $\mathbb{G}_{a,\mathbb{Z}}(T) = H^0(T, \mathcal{O}_T)$ and addition. This yields the *additive group scheme* $\mathbb{G}_{a,\mathbb{Z}}$.

(iv) For $n \geq 1$, set $\mu_{n,\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[x]/(x^n - 1))$. Given an arbitrary scheme T, the identification $\mu_{n,\mathbb{Z}}(T) = \{\, r \in H^0(T, \mathcal{O}_T) \mid r^n = 1 \,\}$ induces a group structure on $\mu_{n,\mathbb{Z}}(T)$ by multiplication. This yields the group scheme $\mu_{n,\mathbb{Z}}$ of *n-th roots of unity.* It can also be seen as the kernel of the multiplication $[n]\colon \mathbb{G}_m \to \mathbb{G}_m$.

(v) Let S be a scheme of characteristic $p > 0$, that is, $H^0(S, \mathcal{O}_S)$ is a ring of characteristic p. Define the group scheme $\alpha_{p,S}$ as the kernel of the homomorphism $\mathbb{G}_{a,S} \to \mathbb{G}_{a,S}$ induced by the Frobenius. Thus $\alpha_{p,S}(T) = \{\, r \in H^0(T, \mathcal{O}_T) \mid r^p = 0 \,\}$ for all S-schemes T. If $S = \operatorname{Spec}(R)$ is affine, then $\alpha_{p,S} = \operatorname{Spec}(R[x]/(x^p))$.

Let G be a group scheme over a base scheme S. If $G \to S$ is finite locally free of constant degree d, then the *order* of G is defined to be $\operatorname{ord}(G) = d$. If G is commutative, then multiplication by d is constant according to [126], Theorem on page 4.

Group Schemes of Prime Order. Now let k be an algebraically closed ground field. Each finite group scheme G over k of order $\ell = \dim_k H^0(G, \mathcal{O}_G)$ for a prime number ℓ is isomorphic to the constant group scheme $\mathbb{Z}/\ell\mathbb{Z}$, to μ_ℓ or to α_ℓ by [126], Lemma 1. Here α_ℓ only exists in the case that ℓ equals the characteristic p of k.

Cartier Duality. Every finite commutative group scheme G over a field k of characteristic p has a *Cartier dual* G^D, see [125], Section 3.8, for details. It is defined by the group valued point functor $G^D(T) = \mathrm{Hom}_{(\text{Grp-Sch})}(G_T, \mathbb{G}_{m,T})$, which is actually representable by a scheme. Evaluation induces a natural isomorphism $G \xrightarrow{\simeq} G^{DD}$ and the assignment $G \mapsto G^D$ defines a contravariant auto-equivalence of the category of finite commutative group schemes over k. For any prime ℓ, duality yields $(\mathbb{Z}/\ell\mathbb{Z})^D \simeq \mu_\ell$ and $\alpha_p^D \simeq \alpha_p$.

The first isomorphism is a direct consequence of the definition. For the self-duality of α_p, observe that for a given k-algebra R, a morphism of R-schemes $\alpha_{p,R} \to \mathbb{G}_{m,R}$ corresponds to a homomorphism of R-algebras $R[x, x^{-1}] \to R[x]/(x^p)$, and hence to a unit $\varphi(x)$ in the latter ring. If $\alpha_{p,R} \to \mathbb{G}_{m,R}$ is additionally a homomorphism of group schemes, then moreover $\varphi(x_1 + x_2) = \varphi(x_1)\varphi(x_2)$ has to hold in $R[x_1, x_2]/(x_1^p, x_2^p)$. Write $\varphi(x) = \sum_{i=0}^{p-1} \lambda_i x^i$ and equate coefficients at 1, $x_1^0 x_2$, $x_1^1 x_2$, $x_1^2 x_2$, $\ldots$, $x_1^{p-1} x_2$. It follows that $\varphi(x) = \exp(\lambda_1 x)$, where $\lambda_1^p = 0$ and $\exp(\lambda_1 x) = \sum_{i=0}^{p-1} \frac{(\lambda_1 x)^i}{i!}$. This defines an isomorphism $\alpha_p \xrightarrow{\simeq} \alpha_p^D$, which is given on R-valued points by $\lambda \mapsto \exp(\lambda x)$. Due to the functional equation of the exponential function, this isomorphism is in fact compatible with the group scheme structure.

If ℓ differs from the characteristic p of k, a choice of a primitive root of unity in k yields a non-canonical isomorphism $\mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\simeq} \mu_\ell$. Although $\mu_p \simeq \alpha_p$ as k-schemes, given by $x \mapsto x - 1$, they are not isomorphic as group schemes over k, as their duals already have different underlying topological spaces.

Connected Component G^0 and Torsion Component G^τ of the Identity. Let G be a commutative group scheme locally of finite type over a scheme S. If S is the spectrum of a field, then G^0 denotes the *connected component of the identity*. It is an open and closed subgroup scheme of G according to [78], Lemma 5.1, which is geometrically irreducible, of finite type over k, and its formation commutes with field extensions $k \subset E$. Also, define the *torsion component of the identity* $G^\tau \subset G$ to be the union of the preimages of G^0 under multiplication by all $n \geq 1$. Due to op. cit., Lemma 6.9, this set is open in G, so it inherits a scheme structure, and it is moreover a subgroup scheme of G. The formation of G^τ also commutes with field extensions $k \subset E$.

In the general situation of a noetherian ground scheme S, the definitions of G^0 and G^τ can be deviated as follows, based on [35], Exposé $\mathrm{VI_B}$, Partie 3, where details are provided. Consider the subfunctor G^0 of G defined on T-valued points for S-schemes T by

$$G^0(T) = \{\, t \in G(T) \mid t_s \colon T_s \to G_s \text{ factors through } (G_s)^0 \text{ for all } s \in S \,\}.$$

Without further assumptions, there is no reason that this functor is representable by a subgroup scheme of G. At least in the case that the set-theoretic union $\bigcup (G_s)^0$ is open in G, then the union is naturally a subscheme of G and, indeed, represents the functor G^0. In the same manner, define the subfunctor G^τ of G. Again, if the set-theoretic union $\bigcup (G_s)^\tau \subset G$ is open, then this is the subscheme of G representing the functor G^τ. Both G^0 and G^τ are compatible with base change in S.

A.2 Principal Homogeneous Spaces

Let S be a locally noetherian ground scheme, and G an S-group scheme whose structure morphism is fppf. A *principal homogeneous space* under G is an fppf S-scheme J with a right G-action $\rho\colon J \times G \to J$ such that $\Phi = (\mathrm{pr}_J, \rho)$ is an isomorphism. This means that

$$\Phi\colon J \times G \longrightarrow J \times J, \quad (j, g) \longmapsto (j, jg),$$

described by its induced maps on T-valued points for all S-schemes T, is a collection of bijections. So the action on T-valued points is simply transitive.

In the case that $J(S) \neq \varnothing$, every section $e\colon S \to J$ induces an isomorphism $\psi\colon G \xrightarrow{\simeq} J$ of G-schemes, given on T-valued points by $g \mapsto eg$. Here e also denotes the composition of the structure morphism $T \to S$ and $e\colon S \to J$. The simple transitivity of the action ρ implicates that ψ is an isomorphism.

Example A.1. Let $k \subset E$ be a finite Galois extension of degree d. Write $S = \mathrm{Spec}(k)$, $S' = \mathrm{Spec}(E)$ and $b\colon S' \to S$ for the natural morphism. Denote by G the constant S-group scheme $\mathrm{Gal}(E/k) = \{\sigma_1, \ldots, \sigma_d\}$, which acts by $S'(T) \times G(T) \to S'(T)$, $(t, \sigma) \mapsto \mathrm{Spec}(\sigma) \circ t$ on S' for connected S-schemes T. Then S' is a principal homogeneous space under G, because the map $S'(T) \times G(T) \to S'(T) \times S'(T)$, $(t, \sigma) \mapsto (t, t\sigma)$ corresponds to the homomorphism $\varphi\colon E \otimes_k E \to \prod_{i=1}^d E$, $x \otimes y \mapsto (x \cdot \sigma_i(y))$, which is bijective:

Choose a basis $y_1, \ldots, y_d \in E$ as a vector space over k. Denote by $V = \prod_{i=1}^d E$ the target of φ, considered as a vector space over E. The family $(\sigma_i(y_j))_i \in V$ of the images of $1 \otimes y_j$ under φ, indexed by $1 \leq j \leq d$, is linearly independent over E. Indeed, otherwise the resulting quadratic matrix $(\sigma_i(y_j))_{i,j}$ had linearly dependent columns, and therefore also linearly dependent rows. The latter yields non-trivial linear combinations $\sum_{i=1}^d \lambda_i \sigma_i(y_j) = 0$ for all $1 \leq j \leq d$. But then $\sum_{i=1}^d \lambda_i \sigma_i = 0$ follows, which contradicts the linear independence of characters.

Remark A.2. A principal homogeneous space J under G becomes isomorphic to G after a faithfully flat base change, by definition. Therefore J inherits every property of G for which faithfully flat descent is valid, see [56], Proposition 2.7.1, and also [58], Proposition 17.7.4. If J is quasicompact over S, this includes being proper, finite, flat, smooth or étale.

Remark A.3. The set of isomorphism classes of principal homogeneous spaces under a fixed S-group scheme G can be embedded into the first cohomology group $H^1(S_{\mathrm{fppf}}, G)$ if G is commutative. In the case that G is not commutative, the non-abelian Čech cohomology group $\check{H}^1(S_{\mathrm{fppf}}, G)$ can be used instead of $H^1(S_{\mathrm{fppf}}, G)$. The in general non-trivial cokernel of the embedding arises as there are potentially *sheaf principal homogeneous spaces*, also called *torsors*, which are not representable by a scheme. There are several cases in which every torsor is representable by a scheme, for instance if G is affine or a regular abelian S-scheme. For details, see [90], Chapter III, Section 4.

A.3 Analytification and GAGA

There exists a functor $X \mapsto X^{\mathrm{an}}$ from the category of schemes locally of finite type over $\mathbb{C}$ to the category of analytic spaces over $\mathbb{C}$, the *analytification*. Given X or $f\colon X \to Y$, also X^{an} and $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ are called the analytification of X and f, respectively. An *analytic space* is a locally ringed space which is locally isomorphic to the vanishing set of finitely many analytic functions, that is, holomorphic functions on an open subset in $\mathbb{C}^r$. For a treatment of complex analytic spaces, see Grauert and Remmert [44].

This section presents some essential properties of the analytification. The foundation was laid by Serre in his article "Géométrie algébrique et géométrie analytique" [118], abbreviated as GAGA. This summary mainly follows Grothendieck and Raynaud [59], Exposé XII. A down-to-earth approach to this topic is Neeman [97], and for an overview, see also Hartshorne [64], Appendix B.

Let X be a scheme locally of finite type over $\mathbb{C}$. It is covered by affine open subsets U where each of them is the spectrum of a $\mathbb{C}$-algebra $\mathbb{C}[T_1, \ldots, T_r]/(f_1, \ldots, f_s)$. The polynomials $f_1, \ldots, f_s$ define analytic functions on $D = \mathbb{C}^r$, equipped with the classical topology as a metric space, and their vanishing set is the closed subset $U^{\mathrm{an}} = V(f_1, \ldots, f_s)$. Its structure sheaf is the quotient $\mathcal{O}_{U^{\mathrm{an}}} = \mathcal{O}_D/(f_1, \ldots, f_s)$ of the sheaf $\mathcal{O}_D$ of analytic functions on D, analogous to $\mathcal{O}_U$. For an affine open cover $(U_i)_{i\in I}$ of X, the enclosed gluing datum provides a gluing datum for the family $(U_i^{\mathrm{an}})_{i\in I}$, which in turn yields X^{an}. In the same manner, a morphism $f\colon X \to Y$ is locally given by polynomials, which define the analytification $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$, and this construction is functorial.

For every scheme X locally of finite type over $\mathbb{C}$, the natural morphism $\alpha\colon X^{\mathrm{an}} \to X$ of locally ringed spaces over $\mathbb{C}$ is flat. The underlying continuous map is the inclusion of the closed points into X, that means $X^{\mathrm{an}} = X(\mathbb{C})$. It is compatible with fiber products and for every $x \in X^{\mathrm{an}}$, the induced map on completions $\widehat{\alpha}_x\colon \widehat{\mathcal{O}}_{X,\alpha(x)} \to \widehat{\mathcal{O}}_{X^{\mathrm{an}},x}$ is bijective.

Various topological and regularity properties hold for X if and only if they hold for X^{an}. For instance, this is true for Serre's conditions (S_i) and (R_i), connectedness, irreducibility, and $\dim(X) = \dim(X^{\mathrm{an}})$. Similarly, a morphism f of finite type is an open or closed embedding, an isomorphism, separated, proper, finite, flat, smooth or étale if and only if its analytification f^{an} has the corresponding property.

Remark A.4. The underlying topological space of the fiber product $X^{\mathrm{an}} \times X^{\mathrm{an}}$ in the category of analytic spaces is the usual product of sets. As X^{an} is separated over $\mathbb{C}$ if the diagonal $\Delta \subset X^{\mathrm{an}} \times X^{\mathrm{an}}$ is closed, this is equivalent to X^{an} being Hausdorff. Furthermore, X^{an} is proper over $\mathbb{C}$ if and only if X^{an} is compact, and X^{an} is smooth over $\mathbb{C}$ if and only if X^{an} is a complex manifold.

Étaleness of a morphism between complex analytic spaces means that the induced homomorphisms on the formal completions of all local rings are bijective. Recall that this characterizes étaleness for a morphism of schemes over a separably closed field. In the category of analytic spaces, this property exactly means that it is a local isomorphism due to [51], Proposition 1.9. For schemes, because of the coarser topology, étale morphisms are in general not local isomorphisms, since a local isomorphism is often a global isomorphism. Indeed, a separated morphism $f\colon X \to Y$ of irreducible schemes is a local isomorphism if and only if it is an open embedding, by [50], Proposition 8.2.8. Thus if f is additionally closed, it is an isomorphism.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}^{\mathrm{an}} := \alpha^*(\mathcal{F})$ is an $\mathcal{O}_{X^{\mathrm{an}}}$-module, and the induced functor between the categories of modules under the structure sheaf is exact and faithful. If f is proper, then there exist natural morphisms $(R^i f_* \mathcal{F})^{\mathrm{an}} \to R^i f_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}})$ for all $i \geq 0$, which are bijective if $\mathcal{F}$ is coherent. Applied to the structure morphism, this yields the first part of the following central theorem:

Theorem A.5 (Serre)**.** *Let $\mathcal{F}$ be a coherent sheaf on a proper $\mathbb{C}$-scheme X. Then naturally $H^i(X, \mathcal{F}) \xrightarrow{\simeq} H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ for all $i \geq 0$. Moreover, the functor induced by $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ between the categories of coherent sheaves on X and X^{an} is an equivalence of categories.*

Furthermore, $\mathcal{F}$ is locally free of rank r if and only if $\mathcal{F}^{\mathrm{an}}$ is locally free of rank r, by [97], Lemma A1.5.1. Hence especially $\mathrm{Pic}(X) = \mathrm{Pic}(X^{\mathrm{an}})$. A consequence of Serre's theorem is that $X \mapsto X^{\mathrm{an}}$ defines a faithfully flat functor from the category of proper $\mathbb{C}$-schemes to the category of compact complex analytic spaces.

Analytification behaves well with respect to coverings. If X is a proper $\mathbb{C}$-scheme, then the category of finite X-schemes is equivalent to the category of analytic spaces finite over X^{an}, and analogously for finite étale morphisms. In the latter case, the *Riemann existence theorem* states that the same conclusion holds under more general assumptions. Concerning the terminology, a *finite covering* of analytic spaces is a finite morphism such that every irreducible component of the total space maps onto an irreducible component of the base.

Theorem A.6 (Riemann Existence Theorem)**.** *Let X be a scheme locally of finite type over $\mathbb{C}$. Analytification induces an equivalence between the category of finite étale morphisms $X' \to X$ and the category of finite étale coverings ${X'}^{\mathrm{an}} \to X^{\mathrm{an}}$.*

A corollary of this theorem is that for a connected scheme X locally of finite type over $\mathbb{C}$, the profinite completion of its topological fundamental group $\widehat{\pi}_1^{\mathrm{top}}(X^{\mathrm{an}}, x)$ at a point $x \in X^{\mathrm{an}}$ can be canonically identified with the étale fundamental group $\pi_1(X, \alpha(x))$. The latter, introduced by Grothendieck [59], is the automorphism group $\mathrm{Aut}(F)$ of the fiber functor $F\colon (\mathrm{Fin\,\acute{E}t}/X) \to (\mathrm{Fin\,Set})$, $U \mapsto U(\Omega)$ from the Galois category of finite étale X-schemes to the category of finite sets, where $\mathrm{Spec}(\Omega) \to X$ is a geometric point.

This section is concluded with some brief remarks on the essential image of the analytification, based on the survey [64], Appendix B, Section 3f. and the references given there. A compact complex analytic space Y is called *algebraic* if $Y \simeq X^{\mathrm{an}}$ for a proper $\mathbb{C}$-scheme X. If Y is a manifold of dimension 1, that is, a Riemann surface, then Y is algebraic. In higher dimensions, there always exist compact manifolds which are not algebraic. Closed subspaces of an algebraic Y are again algebraic. This relies on *Oka's coherence theorem*, [44], Chapter 2, Section 5, which ensures that $\mathcal{O}_Y$ is coherent. A closed subspace is consequently given by coherent ideal sheaf, so Serre's theorem can be applied. In particular, projective complex analytic spaces are algebraic. Moreover, a compact manifold Y is projective if and only if it is *Kähler* and *Moishezon*. The latter means that the field of meromorphic functions $M(Y)$ has transcendence degree over $\mathbb{C}$ equal to $n = \dim(Y)$. This value is always at most n. The Moishezon condition is necessary for Y to be algebraic, since $K(X) = M(X^{\mathrm{an}})$ has transcendence degree $\dim(X) = \dim(X^{\mathrm{an}})$.

A.4 Cohomology and Base Change

Let $f\colon X \to S$ be a proper morphism of locally noetherian schemes, $\mathcal{F}$ a quasicoherent sheaf on X and $d \geq 0$. Given a morphism $b\colon S' \to S$, consider the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\;c\;} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\;b\;]{} & S. \end{array}$$

There is a natural morphism $\varphi_b^d\colon b^* R^d f_*(\mathcal{F}) \to R^d g_*(c^*\mathcal{F})$ due to [53], (1.4.15.4). For flat b, it is always true that φ_b^d is an isomorphism by [53], Proposition 1.4.15. If this holds for all b, then the sheaf $\mathcal{F}$ is called *cohomologically flat in degree d over S*. Define f to be *cohomologically flat in degree d* if $\mathcal{O}_X$ is cohomologically flat in degree d over S.

Criteria for cohomological flatness were elaborated in vast generality by Grothendieck in [55], specifically Section 7.7f. Mumford made simplifications in [95], Section 5, which were included by Hartshorne in [64], Chapter III, Section 12. Moreover, Osserman [102] assembled an overview with extended results, which served as the basis for what follows.

Given a point $s \in S$, denote by $\iota_s\colon \operatorname{Spec}(\kappa(s)) \to S$ the natural morphism and by $\mathcal{F}_s$ the pullback of $\mathcal{F}$ to the fiber X_s. For convenience, define every $\mathcal{F}$ to be cohomologically flat in degree -1 over S.

Theorem A.7 (Cohomology and Base Change)**.** *Let $f\colon X \to S$ be a proper morphism of locally noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf on X which is flat over S. The subsequent two statements are equivalent for each fixed $d \geq 0$.*

- *The sheaf $\mathcal{F}$ is cohomologically flat in degree d over S.*
- *For every point $s \in S$, the map $\varphi_{\iota_s}^d\colon R^d f_*(\mathcal{F})(s) \to H^d(X_s, \mathcal{F}_s)$ is surjective.*

Furthermore, consider the following statements for a fixed $d \geq 0$.

(i) The sheaf $\mathcal{F}$ is cohomologically flat in degrees $d-1$ and d over S.

(ii) The map $S \to \mathbb{N}$, $s \mapsto h^d(\mathcal{F}_s)$ is constant of value v.

(iii) The sheaf $R^d f_*(\mathcal{F})$ is locally free of rank r.

Then (i) implies both (ii) and (iii) as well as $v = r$. If S is additionally reduced, then conversely (ii) implies (i), and in turn also (iii) with $v = r$.

Observe that if (iii) holds and additionally $\varphi^d_{\iota_s}$ is surjective for all $s \in S$, then it is an isomorphism by the first equivalence, and thus (ii) holds.

In the situation of the theorem, the Euler characteristic $S \to \mathbb{Z},\ s \mapsto \chi(\mathcal{F}_s)$ is locally constant, whereas the map $S \to \mathbb{N},\ s \mapsto h^d(\mathcal{F}_s)$ is always upper semi-continuous. The latter means that for every $s_0 \in S$, there exists an open neighborhood U such that $h^d(\mathcal{F}_{s_0}) \geq h^d(\mathcal{F}_s)$ for all $s \in U$. It is possible to set limits to what happens if this is not an equality:

Proposition A.8. *Let S be an irreducible, noetherian scheme and η its generic point. Let $f\colon X \to S$ be a proper morphism and $\mathcal{F}$ a coherent sheaf on X which is flat over S. Suppose $s_0 \in S$ and $d \geq 0$ such that $h^d(\mathcal{F}_{s_0}) > h^d(\mathcal{F}_\eta)$. Then also $h^{d-1}(\mathcal{F}_{s_0}) > h^{d-1}(\mathcal{F}_\eta)$ or $h^{d+1}(\mathcal{F}_{s_0}) > h^{d+1}(\mathcal{F}_\eta)$.*

If S is additionally regular and $\dim(S) = 1$, then $R^{d+1}f_*(\mathcal{F})$ has torsion if and only if there exists an $s_0 \in S$ such that $h^d(\mathcal{F}_{s_0}) > h^d(\mathcal{F}_\eta)$ and $h^{d+1}(\mathcal{F}_{s_0}) > h^{d+1}(\mathcal{F}_\eta)$.

A.5 Fibrations and Zariski's Main Theorem

Definition A.9. A morphism $f\colon X \to Y$ of schemes is called a *fibration* if the associated homomorphism $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is an isomorphism. If X and Y are schemes over some base scheme S, also demand f to be a morphism of S-schemes.

The additional request that a fibration f is compatible with base change in S, that is, for every morphism $Y' \to Y$, the base change $f_{Y'}\colon X_{Y'} \to Y'$ continues to be a fibration, exactly means that f is additionally cohomologically flat in degree 0 over S. Notice in this context that every proper morphism $f\colon X \to \operatorname{Spec}(k)$ is cohomologically flat in all degrees $d \geq 0$.

Lemma A.10. *Let X be a proper scheme over an arbitrary field k, which is geometrically reduced and geometrically connected. Then the structure morphism $f\colon X \to \operatorname{Spec}(k)$ is a fibration, that is, $h^0(\mathcal{O}_X) = 1$.*

Proof. As X is proper over k, the ring $A = H^0(X, \mathcal{O}_X)$ is a finite-dimensional k-algebra. Hence A is a product of artinian local k-algebras, where each factor corresponds to a connected component of X. By assumption, there is only one factor. Furthermore, as X is geometrically reduced, the ring A is a finite separable field extension of k. Thus $A \otimes_k \overline{k} \simeq \overline{k}[T]/(q)$ for a separable polynomial q of degree $[A : k]$. Since X is geometrically connected, the Chinese remainder theorem implies $[A : k] = 1$. So $h^0(\mathcal{O}_X) = 1$ holds. □

The following lemma shows that the property of being a fibration can often be verified fiberwise, which is valid according to [55], Proposition 7.8.6 and Corollaire 7.8.8.

Lemma A.11. *Let $f\colon X \to S$ be a flat, proper morphism of locally noetherian schemes such that all fibers $f_s\colon X_s \to \operatorname{Spec}(\kappa(s))$ for $s \in S$ are fibrations. Then f is a fibration and cohomologically flat in degree 0.*

Given any proper morphism $h\colon X \to Z$, its *Stein factorization*, [53], Théorème 4.3.1, naturally yields an associated fibration: There exists a decomposition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of h into a fibration f and a finite morphism g, where $Y = \mathrm{Spec}(h_*(\mathcal{O}_X))$. The next proposition summarizes some fundamental properties of fibrations.

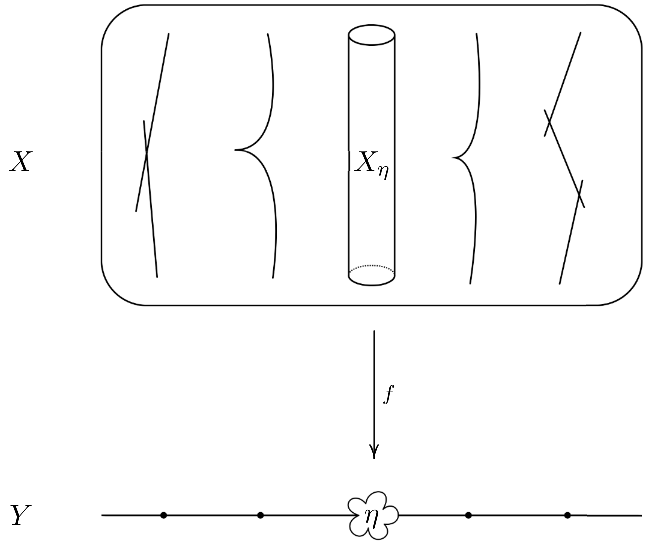


Figure 7: A fibration f from a surface X to a curve Y.

Proposition A.12. *Let $f\colon X \to Y$ be a fibration between proper schemes over an arbitrary field k. The following statements hold:*

(i) f is surjective and for all $y \in Y$, the fiber X_y is connected.

(ii) If X is integral, then Y is integral and f is generically flat.

(iii) If X is normal, geometrically normal or geometrically integral, then Y has the same property.

(iv) The map $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$, $\mathscr{N} \mapsto f^*(\mathscr{N})$ is injective.

(v) There exists a natural injection $H^1(Y, \mathcal{O}_Y) \hookrightarrow H^1(X, \mathcal{O}_X)$.

(vi) Assume that X is irreducible. Then Y is irreducible, and the generic fiber X_η is irreducible, geometrically connected and of dimension $\dim(X) - \dim(Y)$. If additionally X is reduced, normal, Cohen–Macaulay, Gorenstein or regular, then X_η has the same property.

(vii) Assume that Y is irreducible. If X_η is geometrically irreducible, geometrically integral, geometrically reduced, geometrically normal or smooth, then there exists a non-empty open subset $V \subset Y$ such that X_y has the same property for all $y \in V$.

Proof. To see that f is surjective, note that $\mathrm{Im}(f) \subset Y$ is closed as a consequence of the properness of f. Consider $U \subset Y$ open and observe that $H^0(U, \mathcal{O}_Y) = H^0(f^{-1}(U), \mathcal{O}_X)$. Now suppose that $V = Y \smallsetminus \mathrm{Im}(f)$ is non-empty. Then choose $U \subset V$ affine, open and non-empty. Hence $H^0(U, \mathcal{O}_Y) = H^0(\varnothing, \mathcal{O}_X) = 0$, which implies $U = \varnothing$, a contradiction. Thus f is surjective. Its fibers are connected according to [53], Corollaire 4.3.2.

To deduce (ii), let X be integral. Then $H^0(U, \mathcal{O}_Y) = H^0(f^{-1}(U), \mathcal{O}_X)$ is an integral domain for every non-empty open $U \subset Y$, so Y is integral. Eventually, [56], Théorème 6.9.1 shows that f is generically flat.

For assertion (iii), note that normality passes on from X to Y by the same reasoning as for the integrality. The base change $f_{\overline{k}}\colon X_{\overline{k}} \to Y_{\overline{k}}$ continues to be a fibration, because $Y_{\overline{k}} \to Y$ is flat. So this shows that $Y_{\overline{k}}$ is integral or normal, whenever $X_{\overline{k}}$ has this property. Therefore in such cases, Y is geometrically integral or geometrically normal, respectively.

As $\mathcal{O}_Y = f_*(\mathcal{O}_X)$, the projection formula $f_* f^*(\mathscr{N}) = f_*(\mathcal{O}_X) \otimes \mathscr{N}$ yields assertion (iv).

In statement (v), the injection is given by the exact sequence of low-degree terms in the Leray spectral sequence $E_2^{a,b} = H^a(Y, R^b f_*(\mathcal{O}_X)) \Rightarrow H^{a+b}(X, \mathcal{O}_X)$ using $\mathcal{O}_Y = f_*(\mathcal{O}_X)$.

Next, cover (vi). The surjectivity of f implies that also Y is irreducible. The generic fiber of f is irreducible by [50], Chapitre 0, (2.1.8). The assertion about its dimension is a consequence of [56], Corollaire 5.6.6. To see its geometric connectedness, consider the flat base change $f_{\overline{\eta}}\colon X_{\overline{\eta}} \to \mathrm{Spec}(\overline{K(Y)})$ of f, which is again a fibration, so $h^0(\mathcal{O}_{X_{\overline{\eta}}}) = 1$. As a result, X_η is geometrically connected.

The property of being reduced, normal, Cohen–Macaulay, Gorenstein or regular is stable under localization, and consequently transfers to X_η. Indeed, the generic fiber can be covered by $\mathrm{Spec}(A \otimes_R \mathrm{Frac}(R)) = \mathrm{Spec}(T^{-1}A)$ for affine open subsets $\mathrm{Spec}(A) \subset X$ mapping to $\mathrm{Spec}(R) \subset Y$, where T is the image of the multiplicative system $S = R \smallsetminus \{0\}$ under $R \to A$. Finally, assertion (vii) follows from [57], Théorème 9.7.7 and Théorème 12.2.4. □

Remark A.13. In the case that k is a perfect field, X is integral and $\dim(Y) = 1$, it is always true that the generic fiber X_η is geometrically integral according to [56], Corollaire 4.6.3, [10], Lemma 7.2, and the proof of Proposition A.16 (i) below. If the base Y is higher-dimensional, this is no longer ensured. For instance, *wild hypersurface bundles of degree* p are counterexamples over a field of characteristic $p > 0$. Those are morphisms $f\colon X \to Y$ of smooth, integral, proper k-schemes with an embedding of X into some projective bundle $\mathbb{P}(\mathcal{E}) \to Y$ such that each geometric fiber $X_{\overline{y}} = X \times_Y \mathrm{Spec}(\overline{\kappa(y)})$ is defined by x^p for some non-zero $x \in \mathcal{E}_y$. For a treatment of *wild conic bundles* on Fano threefolds X, namely wild hypersurface bundles $f\colon X \to Y$ of degree $p = 2$ and relative dimension 1, see [92].

The next statement, one variant of *Zariski's main theorem*, follows from the proof of [64], Chapter III, Corollary 11.4, together with [53], Corollaire 4.4.9.

Proposition A.14 (Zariski's Main Theorem)**.** *Let $f\colon X \to Y$ be a birational, proper morphism of integral, noetherian schemes, where Y is normal. Then f is a fibration. If additionally f is quasi-finite, then f is an isomorphism.*

Before deducing a partial converse of this theorem and a stronger criterion for f to be a fibration if f is a morphism of k-schemes, first consider the following remark.

Remark A.15 (Resolution of Indeterminacy)**.** Let $f\colon X \dashrightarrow Y$ be a rational map between integral, proper schemes over an arbitrary field k. The *graph* Γ_f of f is defined as follows: For any representative $f_U\colon U \to Y$ of f, let $\Gamma_{f_U} \subset U \times Y$ be the usual graph of the morphism f_U, that is, the image of the closed embedding $(\mathrm{id}_U, f_U)\colon U \to U \times Y$. Define Γ_f to be the reduced closure of Γ_{f_U} inside $X \times Y$. Since X is integral, this is a well-defined, integral, proper k-scheme. The projection $\mathrm{pr}_X\colon \Gamma_f \to X$ restricted to $U = \Gamma_{f_U}$ yields the inclusion $U \hookrightarrow X$, so pr_X is birational, and the diagram

$$\begin{array}{ccc} & \Gamma_f & \\ {}^{\mathrm{pr}_X}\swarrow & & \searrow^{\mathrm{pr}_Y} \\ X & \overset{}{\underset{f}{\dashrightarrow}} & Y \end{array}$$

is commutative. This means that restricting to Γ_{f_U} yields the same morphism to Y on both paths. Also if X is normal, then Zariski's main theorem shows that pr_X is a fibration. In conclusion, after passing from X to Γ_f via the birational morphism pr_X, now the map to Y is defined everywhere.

Proposition A.16. *Let $f\colon X \to Y$ be a surjection between integral, proper schemes over an arbitrary field k, where Y is normal. Then the following statements hold:*

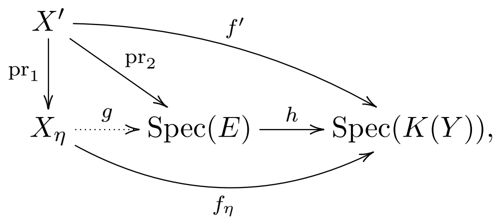
(i) f is a fibration if and only if $K(Y)$ is algebraically closed in $K(X)$.

(ii) Suppose $\dim(X) = \dim(Y)$*. Then f is a fibration if and only if f is birational.*

Note that the equality $K(X_\eta) = K(X)$ and (i) show that f is a fibration if and only if its flat base change f_η is a fibration. The latter holds for instance if X_η is geometrically reduced and geometrically connected according to Lemma A.10.

Proof. Let $X \xrightarrow{\varphi} Y' \xrightarrow{\psi} Y$ be the Stein factorization of f. Consider assertion (i). First, assume that $K(Y)$ is algebraically closed in $K(X)$. The finite morphism ψ yields a finite field extension $K(Y) \subset K(Y')$ inside $K(X)$. Since every element of $K(Y')$ is algebraic over $K(Y)$, the assumption implies that $K(Y) = K(Y')$. Hence ψ is birational and Zariski's main theorem shows that $\psi_*(\mathcal{O}_{Y'}) = \mathcal{O}_Y$, and therefore $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

Conversely, assume that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and by contradiction that $K(Y)$ is not algebraically closed in $K(X)$. Then there exists a finite field extension $K(Y) \subset E$ of degree ≥ 2 inside $K(X)$. The generic fiber $f_\eta\colon X_\eta \to \mathrm{Spec}(K(Y))$ is again a fibration as a flat base change of f, that means $H^0(X_\eta, \mathcal{O}_{X_\eta}) = K(Y)$. The inclusion of fields $E \subset K(X) = K(X_\eta)$ gives rise to a dominant rational map $g\colon X_\eta \dashrightarrow \mathrm{Spec}(E)$. Resolution of indeterminacy yields the commutative diagram



where $X' = \Gamma_g$ with $K(X') = K(X_\eta)$ and $(\mathrm{pr}_1)_*(\mathcal{O}_{X'}) = \mathcal{O}_{X_\eta}$. By the commutativity of the diagram, also f' is a fibration, so $K(Y) \to H^0(X', \mathcal{O}_{X'})$ is bijective. But on the other hand, this map has a factorization through the field extension $K(Y) \subset E$ of degree ≥ 2, which is a contradiction. Hence $K(Y)$ is algebraically closed in $K(X)$. This completes the proof of (i).

To show assertion (ii), assume that $\dim(X) = \dim(Y)$. If f is birational, that means $K(Y) = K(X)$, then f is a fibration by (i). Conversely, suppose that f is a fibration. As the value of $\mathrm{trdeg}(K(X)/k) = \dim(X)$ equals $\mathrm{trdeg}(K(Y)/k) = \dim(Y)$, the additivity of transcendence degrees implies that the field extension $K(Y) \subset K(X)$ is finite, in particular algebraic. Now since $K(Y)$ is algebraically closed in $K(X)$, this gives $K(X) = K(Y)$, so f is birational. □

Example A.17. Let X and X' be proper schemes over an arbitrary field k. If $h^0(\mathcal{O}_X) = 1$, then $\mathrm{pr}_{X'} \colon X \times X' \to X'$ is a fibration: Every fiber of this projection is a base change of X to an extension field of k, so it remains to be a fibration by flat base change. Thus Lemma A.11 can be applied.

Remark A.18. Let X be a proper scheme over an arbitrary field k. Suppose that X is reduced and connected. In general for $h^0(\mathcal{O}_X) = 1$ to hold, it is with regard to Lemma A.10 necessary that X is both geometrically reduced and geometrically connected.

First, the purely topological connectedness of X is not sufficient: Consider $X = \mathrm{Spec}(\mathbb{C})$ in the natural way as a scheme over $\mathbb{R}$. Then $h^0(\mathcal{O}_X) = 2$, although X is geometrically reduced and connected. Here, as computed in Example A.1, the base change to the algebraic closure $X_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}) \amalg \mathrm{Spec}(\mathbb{C})$ is not connected, but reduced.

Similarly, let k now be an imperfect, separably closed field of characteristic $p > 0$, for instance the separable closure of $\mathbb{F}_p(T)$. Choose $\alpha \in k$ without p-th root, and consider the purely inseparable field extension $E = k[T]/(T^p - \alpha)$. Then the k-scheme $X = \mathrm{Spec}(E)$ satisfies $h^0(\mathcal{O}_X) = p$, although X is reduced and geometrically connected. In this case, X_E is the spectrum of $E \otimes_k E$, which is non-reduced. Indeed, let $\beta \in E$ be a p-th root of α, so that $1, \beta, ..., \beta^{p-1}$ form a k-basis of E. Then $1 \otimes \beta - \beta \otimes 1 \in E \otimes_k E$ is non-zero, since it is a non-trivial linear combination of two basis vectors. But its p-th power is zero, so X_E is non-reduced. But as k is separably closed, the connected k-scheme X is geometrically connected.

A.6 Curves of Fiber Type

Fix a normal, integral, proper surface X over an arbitrary field k. In this section, curves of fiber type on X are introduced, as a tool to obtain a fibration from X to a curve. Recall that a *curve* on X means an effective Weil divisor, that is, a closed subscheme without embedded points which is equidimensional of dimension one.

Definition A.19. A curve $C = \sum_{i=1}^d m_i C_i$ on X with integral components C_i and multiplicities $m_i \geq 1$ as well as $d \geq 1$ is called *of fiber type* if $(C \cdot C_i) = 0$ for every $1 \leq i \leq d$. It is called *of canonical type* if additionally $(K_X \cdot C_i) = 0$ for every $1 \leq i \leq d$. The curve C is *indecomposable* if C is connected and $\gcd(m_i) = 1$.

Clearly, a curve is of fiber type if and only if each of its connected components is of fiber type. If K_X is numerically trivial, then all curves of fiber type are automatically of canonical type.

Example A.20. Let $f\colon X \to Y$ be a surjection onto a regular curve Y, for instance a fibration. Then for every closed points $y \in Y$, the fiber $X_y \subset X$ is a curve of fiber type: In fact, $\deg(\mathcal{O}_Y(y)) > 0$ and the Nakai–Moishezon criterion show that the invertible sheaf $\mathcal{O}_Y(y)$ is ample. Thereby $\mathcal{O}_X(X_y) = f^*(\mathcal{O}_Y(y))$ is still semi-ample and especially nef. Furthermore,

$$(X_y^2) = \deg(f^*(\mathcal{O}_Y(y))|_{X_y}) = \deg(\mathcal{O}_{X_y}) = 0.$$

So X_y is nef and has trivial self-intersection number, which exactly means that X_y is of fiber type.

The intersection matrix $((C_i \cdot C_j))_{1 \leq i,j \leq d}$ of a curve of fiber type is always negative semi-definite. This is the core of the subsequent statement, which refers to [18], Lemma on page 28.

Proposition A.21. *Let $C = \sum_{i=1}^d m_i C_i$ be a connected curve of fiber type on X. Consider a Weil divisor $D = \sum_{i=1}^d n_i C_i$ for $n_i \in \mathbb{Z}$. Then $(D^2) \leq 0$ holds, with equality if and only if D is a rational multiple of C.*

Let $C = \sum_{i=1}^d m_i C_i$ be a curve on X. Decompose each curve $C' = \sum_{i=1}^d m_i' C_i + C''$ which is linear equivalent to C such that no C_i appears as an integral component of C''. Set $n_i = \min\{m_i'\}$, where the minimum is taken over all such C'. Define the *fixed part* of C to be $F = \sum_{i=1}^d n_i C_i$. Its *movable part* is $M = C - F$. The subsequent lemma follows directly from this definition.

Lemma A.22. *The canonical inclusion $H^0(X, \mathcal{O}_X(M)) \hookrightarrow H^0(X, \mathcal{O}_X(C))$ is bijective and either $M = 0$ or M is a nef curve without fixed part.*

Now the following proposition characterizes how curves of fiber type on X are connected with fibrations from X to a curve. Set $k' = H^0(X, \mathcal{O}_X)$ for the field of global sections.

Proposition A.23. *Let $C \subset X$ be a curve of fiber type. The following are equivalent:*

(i) There exists some $n \geq 1$ such that $\dim_{k'} H^0(X, \mathcal{O}_X(nC)) \geq 2$.

(ii) There exists a fibration $f: X \to Y$ onto a curve Y such that each connected component of C is a rational multiple of a closed fiber of f.

Proof. The preceding example shows that (ii) implies (i). Now for the other implication, assume that $\dim_{k'} H^0(X, \mathcal{O}_X(nC)) \geq 2$ holds for some $n \geq 1$. Let $M = nC - F$ be the movable part of nC. Apply Proposition A.21 to the nef curve M to see that M inherits to be a curve of fiber type. As a consequence, the at most zero-dimensional base locus of M must be empty, so M is actually base-point-free. Since the points where M is not Cartier are contained in the base locus, M is Cartier. So $\mathcal{O}_X(M)$ is a globally generated invertible sheaf.

Let $f: X \to Y$ be the fibration obtained from the Stein factorization of the induced morphism to some $\mathbb{P}^n$. Then $\mathcal{O}_X(M) = f^*(\mathscr{M})$ holds for an ample invertible sheaf $\mathscr{M}$ on Y. Because M is a curve, necessarily $\dim(Y) \geq 1$ is valid. In order to exclude the possibility $\dim(Y) = 2$, compute

$$0 = (M^2) = \deg(f) \cdot (\mathscr{M}^2).$$

This shows that $\deg(f) = 0$. So f is a fibration to a regular, integral curve Y. Now M must be a rational multiple of the pullback of some effective divisor on Y. Hence each connected component of M has to be a rational multiple of a closed fiber. This property then also has to hold true for the original curve C. □

The dualizing sheaf and cohomology groups of an indecomposable curve of canonical type are described in the next proposition, see [94], page 332f., for a proof.

Proposition A.24. *Assume that k is algebraically closed and X is additionally Gorenstein. Let C be an indecomposable curve of canonical type on X. Then $\omega_C \simeq \mathcal{O}_C$ and $h^0(\mathcal{O}_C) = h^1(\mathcal{O}_C) = 1$.*

Observe that indecomposable curves of canonical type C on regular surfaces over an algebraically closed field k are classified, see for instance [30], Proposition 3.1.1. If C is integral, then C is either an elliptic curve, a rational curve with one cusp or a rational curve with one node. In the case that $C = \sum_{j=1}^{r} m_j C_j$ is reducible, each integral component $C_i \subset C$ is a (-2)-curve. To compute all possible dual graphs of C, it is useful to note that by Proposition A.21 for all $1 \leq i \leq d$, the reduced curve $\sum_{j \neq i} C_j$ is an ADE-curve. Those are exactly the possible exceptional divisors of the minimal resolution of a rational double point, see Remark 4.21. The complete classification of indecomposable curves of canonical type dates back to Kodaira [79] and Néron [99].

A.7 Picard Schemes

Proposition A.25. *Let X be a proper scheme over an arbitrary field k and $k \subset E$ an algebraic field extension. Then the natural map* $\operatorname{Pic}(X) \to \operatorname{Pic}(X_E)$ *is injective.*

Proof. The concerning map can be extended further to $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{\overline{k}})$, which is injective by [130], Proposition 2.2, whose proof works for proper schemes. □

The subsequent treatment of the Picard scheme refers to Kleiman [78]. Let $f \colon X \to S$ be a separated morphism of finite type between locally noetherian schemes. Denote by $(\mathcal{C}/S)$ the category of locally noetherian S-schemes. The relative *Picard functor* is the contravariant functor defined by $\operatorname{Pic}_{(X/S)} \colon (\mathcal{C}/S) \to (\mathrm{Ab})$, $T \mapsto \operatorname{Pic}(X_T)/\operatorname{Pic}(T)$ with the natural extension of this assignment, given by pullback, to morphisms. It is a presheaf on (Zar/S), $(\mathrm{\acute{E}t}/S)$ and (Fppf/S). Denote its sheafifications on the respective sites by $\operatorname{Pic}_{(X/S)(\mathrm{zar})}$, $\operatorname{Pic}_{(X/S)(\mathrm{\acute{e}t})}$ and $\operatorname{Pic}_{(X/S)(\mathrm{fppf})}$. If any of the four functors

$$\operatorname{Pic}_{(X/S)} \longrightarrow \operatorname{Pic}_{(X/S)(\mathrm{zar})} \longrightarrow \operatorname{Pic}_{(X/S)(\mathrm{\acute{e}t})} \longrightarrow \operatorname{Pic}_{(X/S)(\mathrm{fppf})}$$

is representable by a scheme $\operatorname{Pic}_{X/S}$, then call it the *Picard scheme* of X over S. As discussed in [78], Section 4, there is at most one Picard scheme: If any of the four functors is representable by $\operatorname{Pic}_{X/S}$, then it also represents the other functors on its right-hand side in the chain above. Observe that in [78], the different notation $\operatorname{Pic}_{X/S}$ for the relative Picard functor and $\mathbf{Pic}_{X/S}$ for the Picard scheme is used.

In what follows, two theorems summarize core properties of the Picard scheme $\operatorname{Pic}_{X/S}$. As defined in Section A.1, if the group scheme $\operatorname{Pic}_{X/S}$ exists and it is locally of finite type over S, then there are natural ways to define two subfunctors $\operatorname{Pic}^0_{X/S}$ and $\operatorname{Pic}^\tau_{X/S}$. Their representability as a subgroup scheme is a priori unclear, and the two theorems below give affirmative answers under suitable assumptions. The first one treats the general situation over a noetherian base scheme S, and the second one covers the special case $S = \operatorname{Spec}(k)$ for a field k, where sharper statements can be derived.

Theorem A.26. *Let $f \colon X \to S$ be a separated morphism of finite type between noetherian schemes.*

(i) If S is integral and f is proper, then there exists a non-empty open subscheme $V \subset S$ such that $\operatorname{Pic}_{X_V/V}$ *exists. It is a disjoint union of open subschemes which are quasi-projective over V.*

(ii) If f is projective, flat and has geometrically integral fibers, then $\operatorname{Pic}_{X/S}$ *exists representing* $\operatorname{Pic}_{(X/S)(\mathrm{\acute{e}t})}$. *It is separated and locally of finite type over S.*

(iii) If f is a fibration and cohomologically flat in degree 0*, then the four Picard functors are related as follows:* $\operatorname{Pic}_{(X/S)} \hookrightarrow \operatorname{Pic}_{(X/S)(\mathrm{zar})} \hookrightarrow \operatorname{Pic}_{(X/S)(\mathrm{\acute{e}t})} \xrightarrow{\simeq} \operatorname{Pic}_{(X/S)(\mathrm{fppf})}$.

(iv) If f is a fibration, cohomologically flat in degree 0 *and admits a section, then furthermore the first three coincide:* $\operatorname{Pic}_{(X/S)} \xrightarrow{\simeq} \operatorname{Pic}_{(X/S)(\mathrm{zar})} \xrightarrow{\simeq} \operatorname{Pic}_{(X/S)(\mathrm{\acute{e}t})}$.

(v) *Assume that* $\mathrm{Pic}_{X/S}$ *exists and that* $\mathrm{Pic}^0_{X_s/\kappa(s)}$ *is smooth of the same dimension for all* $s \in S$*. Then* $\mathrm{Pic}^0_{X/S} \subset \mathrm{Pic}_{X/S}$ *is an open subgroup scheme of finite type over* S*. If in addition all* $\mathrm{Pic}^0_{X_s/\kappa(s)}$ *are proper and* $\mathrm{Pic}_{X/S}$ *is separated, then* $\mathrm{Pic}^0_{X/S} \subset \mathrm{Pic}_{X/S}$ *is closed and proper over* S*. Moreover, if* f *is smooth and projective with geometrically irreducible fibers, then* $\mathrm{Pic}^0_{X/S}$ *is projective over* S*.*

(vi) *Assume that* f *is proper and that* $\mathrm{Pic}_{X/S}$ *exists. Then* $\mathrm{Pic}^\tau_{X/S} \subset \mathrm{Pic}_{X/S}$ *is an open subgroup scheme of finite type over* S*. If* f *is projective and flat with geometrically integral fibers, then* $\mathrm{Pic}^\tau_{X/S} \subset \mathrm{Pic}_{X/S}$ *is also closed and quasi-projective over* S*. If in this case* f *is additionally smooth, then* $\mathrm{Pic}^\tau_{X/S}$ *is projective over* S*.*

(vii) *Forming* $\mathrm{Pic}_{X/S}$*,* $\mathrm{Pic}^\tau_{X/S}$ *and* $\mathrm{Pic}^0_{X/S}$ *commutes with base change in* S*.*

(viii) *If* $\mathrm{Pic}_{X/S}$ *exists representing* $\mathrm{Pic}_{(X/S)(\text{ét})}$ *and* $h^2(\mathcal{O}_{X_s}) = 0$ *for some* $s \in S$*, then* $\mathrm{Pic}_{X/S}$ *is smooth over an open neighborhood of* s*.*

All statements refer to [78] and specifically: (i) Theorem 4.18.2; (ii) Theorem 4.8; (iii) Theorem 2.5 and the bottom of page 21; (iv) Theorem 2.5; (v) Proposition 5.20 and Answer 5.7; (vi) Remark 6.19, Theorem 6.16 and Answer 5.7; (vii) Answer 4.4 and the proof of Theorem 6.16; (viii) Proposition 5.19.

Introduce some notation for the next definition. Let $S = \mathrm{Spec}(k)$ and $k \subset E$ be a field extension, let $s\colon \mathrm{Spec}(E) \to T$ be a morphism to some k-scheme T and $\mathscr{M}$ an invertible sheaf on X_T. Denote by $\mathscr{M}_s$ the pullback of $\mathscr{M}$ along $(\mathrm{id}_X \times s)\colon X_E \to X_T$.

Definition A.27. Two invertible sheaves $\mathscr{L}$ and $\mathscr{N}$ on X are *algebraically equivalent* if there is an $n \geq 1$ and for every $1 \leq i \leq n$ there exist

(i) a field extension $k \subset E_i$,

(ii) a connected k-scheme T_i,

(iii) morphisms $s_i, t_i\colon \mathrm{Spec}(E_i) \to T_i$,

(iv) an invertible sheaf $\mathscr{M}_i$ on X_{T_i},

such that $\mathscr{L}_{E_1} \simeq \mathscr{M}_{1,s_1}$ and $\mathscr{M}_{i,t_i} \simeq \mathscr{M}_{i+1,s_{i+1}}$ for $1 \leq i \leq n-1$, as well as $\mathscr{M}_{n,t_n} \simeq \mathscr{N}_{E_n}$. Two invertible sheaves $\mathscr{L}$ and $\mathscr{N}$ on X are τ*-equivalent* if there exists some $m \geq 1$ such that $\mathscr{L}^{\otimes m}$ and $\mathscr{N}^{\otimes m}$ are algebraically equivalent.

It follows directly from the definition that algebraic equivalence actually defines an equivalence relation $\mathscr{L} \sim_{\mathrm{alg}} \mathscr{N}$ on the set of invertible sheaves and on $\mathrm{Pic}(X)$. Furthermore, it is compatible with tensor products in the sense that $\mathscr{L} \sim_{\mathrm{alg}} \mathscr{N}$ and $\mathscr{L}' \sim_{\mathrm{alg}} \mathscr{N}'$ imply $\mathscr{L} \otimes \mathscr{L}' \sim_{\mathrm{alg}} \mathscr{N} \otimes \mathscr{N}'$. To see the latter, note that replacing all $\mathscr{M}_i$ by $\mathscr{M}_i \otimes \mathscr{L}'_{T_i}$ in the definition above shows that $\mathscr{L} \otimes \mathscr{L}' \sim_{\mathrm{alg}} \mathscr{N} \otimes \mathscr{L}'$, so by the same reasoning also $\mathscr{N} \otimes \mathscr{L}' \sim_{\mathrm{alg}} \mathscr{N} \otimes \mathscr{N}'$. Consequently, also τ-equivalence is an equivalence relation and compatible with tensor products.

Definition A.28. Let $\mathrm{Pic}^\tau(X) \subset \mathrm{Pic}(X)$ be the subgroup of isomorphism classes of invertible sheaves which are τ-equivalent to $\mathcal{O}_X$. Define $\mathrm{Pic}^0(X) \subset \mathrm{Pic}^\tau(X)$ to be the subgroup of classes of invertible sheaves which are algebraically equivalent to $\mathcal{O}_X$.

Remark A.29. Note that according to Proposition 1.34, an invertible sheaf $\mathscr{L}$ on a proper k-scheme X is τ-equivalent to $\mathcal{O}_X$ if and only if it is numerically trivial, which means $\deg(\mathscr{L}|_C) = 0$ for every integral curve $C \subset X$.

The following theorem covers the special situation of Theorem A.26 in which the base scheme S is the spectrum of a field. In this case, more results can be obtained.

Theorem A.30. *Let X be a separated scheme of finite type over an arbitrary field k.*

(i) If X is proper, then $\mathrm{Pic}_{X/k}$ exists, and is separated and locally of finite type over k. It is a disjoint union of open subschemes which are quasi-projective over k.

(ii) If $h^0(\mathcal{O}_X) = 1$, then $\mathrm{Pic}_{(X/k)} \hookrightarrow \mathrm{Pic}_{(X/k)(\mathrm{zar})} \hookrightarrow \mathrm{Pic}_{(X/k)(\text{ét})} \xrightarrow{\simeq} \mathrm{Pic}_{(X/k)(\mathrm{fppf})}$.

(iii) If $h^0(\mathcal{O}_X) = 1$ and $X(k) \neq \varnothing$, then also $\mathrm{Pic}_{(X/k)} \xrightarrow{\simeq} \mathrm{Pic}_{(X/k)(\mathrm{zar})} \xrightarrow{\simeq} \mathrm{Pic}_{(X/k)(\text{ét})}$. In particular if $\mathrm{Pic}_{X/k}$ additionally exists, then $\mathrm{Pic}(X) = \mathrm{Pic}_{X/k}(k)$.

(iv) Assume that $\mathrm{Pic}_{X/k}$ exists. Then $\mathrm{Pic}^0_{X/k} \subset \mathrm{Pic}_{X/k}$ is an open and closed subgroup scheme of finite type over k, which is geometrically irreducible.

(v) Assume that $\mathrm{Pic}_{X/k}$ exists. Then $\mathrm{Pic}^\tau_{X/k} \subset \mathrm{Pic}_{X/k}$ is an open subgroup scheme. If X is proper, then it is also closed and of finite type over k.

(vi) Forming $\mathrm{Pic}_{X/k}$, $\mathrm{Pic}^\tau_{X/k}$ and $\mathrm{Pic}^0_{X/k}$ commutes with extending k.

(vii) If X is proper with $h^0(\mathcal{O}_X) = 1$, then $\dim(\mathrm{Pic}_{X/k}) \leq h^1(\mathcal{O}_X)$. Equality holds if and only if $\mathrm{Pic}_{X/k}$ is smooth at the neutral element. In this case, $\mathrm{Pic}_{X/k}$ is smooth everywhere.

(viii) Let X be proper with $h^0(\mathcal{O}_X) = 1$. If $h^2(\mathcal{O}_X) = 0$ or $p = 0$, then $\mathrm{Pic}_{X/k}$ is smooth.

(ix) Assume that $\mathrm{Pic}_{X/k}$ exists. Let $\lambda \in \mathrm{Pic}_{X/k}$ be a rational point coming from an invertible sheaf $\mathscr{L}$ on X. Then $\lambda \in \mathrm{Pic}^0_{X/k}$ if and only if $\mathscr{L}$ is algebraically equivalent to $\mathcal{O}_X$.

(x) Assume that $\mathrm{Pic}_{X/k}$ exists. Let $\lambda \in \mathrm{Pic}_{X/k}$ be a rational point coming from an invertible sheaf $\mathscr{L}$ on X. Then $\lambda \in \mathrm{Pic}^\tau_{X/k}$ if and only if $\mathscr{L}$ is τ-equivalent to $\mathcal{O}_X$.

(xi) If X is geometrically normal and proper, then $\mathrm{Pic}^0_{X/k}$ is proper and $(\mathrm{Pic}^0_{X/k})_{\mathrm{red}}$ is an abelian variety.

Unless mentioned otherwise, the statements all refer to [78]. Specifically: (i) Corollary 4.18.3 and Proposition 4.17; (ii) Theorem 2.5 and the bottom of page 21; (iii) Theorem 2.5; (iv) Proposition 5.3; (v) Proposition 6.12 and Remark 6.14; (vi) Answer 4.4, Proposition 6.12 and Proposition 5.3; (vii) Corollary 5.13 and (ii); (viii) Corollary 5.14, Proposition 5.19 and (ii); (ix) Proposition 5.10; (x) Answer 6.11. Assertion (xi) refers to [54], Théorème 2.1 and Corollaire 3.2.

Let X be a proper scheme over an arbitrary field k. Then $\mathrm{Pic}(X) = \mathrm{Pic}_{(X/k)}(k)$. Consider the group homomorphism $\iota\colon \mathrm{Pic}(X) \to \mathrm{Pic}_{X/k}(k)$. By Theorem A.30, the map ι is injective if $h^0(\mathcal{O}_X) = 1$, and an isomorphism if additionally $X(k) \neq \varnothing$. Furthermore, $\mathrm{Pic}^0(X) = \iota^{-1}(\mathrm{Pic}^0_{X/k}(k))$ and $\mathrm{Pic}^\tau(X) = \iota^{-1}(\mathrm{Pic}^\tau_{X/k}(k))$.

The *Néron–Severi group* $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ of X is a finitely generated abelian group. If k is algebraically closed, this holds due to [13], Exposé XIII, Théorème 5.1. The case of an arbitrary field follows, since $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{k}})$ is injective and the preimage of $\mathrm{Pic}^0(X_{\overline{k}})$ is $\mathrm{Pic}^0(X)$. The *Picard number* of X is the rank $\rho(X) = \mathrm{rank}(\mathrm{NS}(X))$ of its Néron–Severi group. Hence, $\mathrm{NS}(X) \simeq \mathbb{Z}^{\oplus\rho(X)} \oplus T$ holds for the finite torsion group $T = \mathrm{Pic}^\tau(X)/\mathrm{Pic}^0(X)$. Finally, $\mathrm{Num}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$ is the group of invertible sheaves modulo numerical equivalence, so $\mathrm{Num}(X) = \mathrm{NS}(X)/T \simeq \mathbb{Z}^{\oplus\rho(X)}$.

Example A.31. Let C be a smooth, proper curve over an arbitrary field k, which satisfies $h^0(\mathcal{O}_C) = 1$. Then $\mathrm{Pic}^0_C = \mathrm{Pic}^\tau_C$ and hence $\mathrm{NS}(C)$ is torsion-free.

To verify that $\mathrm{Pic}^0_C \subset \mathrm{Pic}^\tau_C$ is an equality, assume without loss of generality that k is algebraically closed, by faithfully flat descent. Also, it is sufficient to deduce that the inclusion is bijective on closed points, since then Pic^τ_C has to be connected. Hence let $\mathscr{L}$ be a numerically trivial invertible sheaf on C, and show that $\mathscr{L}$ is algebraically equivalent to $\mathcal{O}_C$. As C is locally factorial, the class of $\mathscr{L}$ in $\mathrm{Pic}(C)$ corresponds to a Weil divisor $D = \sum n_i x_i$ with $n_i \in \mathbb{Z}$, up to linear equivalence. So $\mathscr{L} \simeq \bigotimes \mathcal{O}_C(x_i)^{\otimes n_i}$, where $0 = \sum n_i$. This shows that it is sufficient to deduce that $\mathcal{O}_C(x)$ and $\mathcal{O}_C(y)$ are algebraically equivalent for every two closed points $x, y \in C$.

Consider the diagonal $\Delta \subset C \times C$ and the associated invertible sheaf $\mathscr{M} = \mathcal{O}_{C\times C}(\Delta)$. Let $x \in C$ be a closed point. The induced morphism $x\colon \mathrm{Spec}(k) \to C$ defines another copy $C' \subset C \times C$ of the curve C as the image of $(\mathrm{id}_C \times x)\colon C \to C \times C$. The intersection $\Delta \cap C' = \Delta \times_{C\times C} C'$ is the point $x\colon \mathrm{Spec}(k) \to C'$. The associated invertible sheaf to the divisor x on $C' = C$ is therefore $\mathcal{O}_C(x) = \mathscr{M}_x$. Indeed, this equality means by definition that $\mathcal{O}_{C'}(\Delta \cap C') = \mathcal{O}_{C\times C}(\Delta)|_{C'}$, which is a consequence of the fact that C' is integral and not contained in Δ. Hence $\mathcal{O}_C(x) = \mathscr{M}_x$ and $\mathscr{M}_y = \mathcal{O}_C(y)$ for all closed points $x, y \in C$, so $\mathcal{O}_C(x)$ and $\mathcal{O}_C(y)$ are in fact algebraically equivalent.

More generally, given any proper curve C over an algebraically closed field k, its Néron–Severi group $\mathrm{NS}(C) \simeq \mathbb{Z}^{\oplus r}$ is torsion-free of rank equal to the number r of irreducible components of C according to [19], Section 9.2, Corollary 14.

For abelian varieties, the following proposition yields the same result. The reference is [95], Section 19, Corollary 2 on page 178.

Proposition A.32. *Let k be an arbitrary field and let A be an abelian variety over k. Then* $\mathrm{Pic}^0_A = \mathrm{Pic}^\tau_A$.

A.8 Approach Using Models

In this section, the aim is to show how a model of a scheme can be used to approach question (Q1) from a different perspective. To do so, fix a natural number $n \geq 0$ and a prime number $p > 0$. Consider the following statement:

(†) If X is a Gorenstein, geometrically normal, proper n-equidimensional scheme over a field k of characteristic p such that ω_X is numerically trivial, then $\omega_X \in \mathrm{Pic}(X)$ has finite order.

Alternatively, X can additionally assumed to be smooth or projective. The statement is true for all finite fields, since in those cases, every numerically trivial invertible sheaf has finite order due to Proposition 4.8. The idea is to use this special case to progress in proving (†) for all fields. Roughly speaking, the approach is the following: Choose a suitable *model* of X, that is, a morphism $X_0 \to S_0$ to an irreducible scheme S_0 with generic fiber X. The subsequent diagram illustrates the situation.

(A.1)
$$\begin{array}{ccccc} X_s & \longrightarrow & X_0 & \longleftarrow & X \\ \downarrow{\scriptstyle \pi_s} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_\eta} \\ \mathrm{Spec}(\kappa(s)) & \longrightarrow & S_0 & \longleftarrow & \mathrm{Spec}(k) \\ & & \downarrow & & \\ & & \mathrm{Spec}(L) & & . \end{array}$$

If (†) holds for all fibers X_s for $s \in S_0$ closed, then the idea is that this implies (†) for the generic fiber $X = X_{0,\eta}$. In this setting, *cohomology and base change* and *Grothendieck duality* are new available tools which might help to achieve this goal.

Start with a series of reduction steps and deduce the existence of a model X_0. In the first step, reduce to the case that X is additionally geometrically integral and k is finitely generated over its prime field $F \subset k$. Note that it is possible to enlarge k to $\overline{k}$ by Lemma 4.1. Then decompose X into its connected components. As X is normal, each component is irreducible and hence integral. Thus is it sufficient to prove (†) for an algebraically closed field k, where X is additionally integral. Now consider:

Proposition A.33. *Let X be a scheme of finite type over an arbitrary field k and let F be the prime field of k. Then there exists an intermediate field $F \subset L \subset k$ which is finitely generated over F and a scheme Y of finite type over L such that $X \simeq Y_k$ as k-schemes.*

Proof. Consider k as the union of its subfields which are finitely generated over F. Then apply [57], Théorème 8.8.2 (ii). □

If X is geometrically normal, geometrically integral, Gorenstein and proper over k, then the same holds for Y over L by faithfully flat descent. Furthermore, if $\omega_X = (\omega_Y)_k$ is numerically trivial, then this is also true for ω_Y and both sheaves have the same order in their respective Picard groups. Hence to prove (†), it is possible to replace X by Y, so reduce to:

- The field k is finitely generated over its prime field F.
- The scheme X is geometrically integral.

As a second step of reduction, use induction on $d = \operatorname{trdeg}(k/F)$. If $d = 0$, then $F \subset k$ is an algebraic extension and Proposition 4.8 shows the validity of (†). So the induction step remains to be shown. Thus the following additional assumption can be imposed.

- For all fields $F \subset E \subset k$ with $\operatorname{trdeg}(E/F) < \operatorname{trdeg}(k/F)$, statement (†) is true.

Choose some intermediate field $F \subset L \subset k$ of $\operatorname{trdeg}(k/L) = 1$. Then $\operatorname{trdeg}(L/F) = d - 1$ holds. Hence every algebraic extension $L \subset E$ has transcendence degree $d - 1$ over F, so (†) is valid for schemes over E. Write $k = L(t_1, \ldots, t_m)$ and consider the L-algebra $A = L[t_1, \ldots, t_m]$. Let $S_0 \subset \operatorname{Spec}(A)$ be the regular locus, which contains the generic point and is open by [56], Proposition 7.8.6 (iii). Thus S_0 is a regular, integral, one-dimensional scheme of finite type over L such that $K(S_0) = k$.

There exists a non-empty affine open subset $S_\lambda \subset S_0$ and a scheme X_λ of finite type over S_λ such that X is the generic fiber of $X_\lambda \to S_\lambda$ by [57], Théorème 8.8.2 (ii). Replace S_0 by S_λ and X_0 by X_λ to keep the notation, and consider Diagram (A.1) again. For all closed points $s \in S_0$, the field extension $L \subset \kappa(s)$ is finite. Hence by the reduction in the previous paragraph, (†) is true for all closed fibers X_s which satisfy the assumptions imposed in (†).

To achieve that this is the case for all fibers, conduct a series of steps, each shrinking S_0 to some non-empty affine open subset $S_\lambda \subset S_0$, and replacing X_0 by its base change to S_λ. In doing so, it is possible to achieve that $\pi\colon X_0 \to S_0$ satisfies the following properties:

- π is proper, of relative dimension n ([57], Théorème 8.10.5 and Corollaire 13.1.5).
- π is flat ([56], Théorème 6.9.1).
- π is geom. integral, geom. normal, Cohen–Macaulay ([57], Théorème 12.2.4).
- π is a fibration and cohomologically flat in degree 0 (Lemma A.11).
- X_0 is integral.

In fact, X_0 is integral, since π is flat and its generic fiber X as well as S_0 are integral. To see that π can also assumed to be Gorenstein, note that the dualizing sheaf $\omega_{X_\lambda/S_\lambda}$ exists for every open subscheme $S_\lambda \subset S_0$. It is coherent and for every open $S_\mu \subset S_\lambda$, the pullback of $\omega_{X_\lambda/S_\lambda}$ to X_μ is ω_{X_μ/S_μ}. The pullback of $\omega_{X_\lambda/S_\lambda}$ to the generic fiber X is the invertible sheaf ω_X. Thus by [57], Proposition 8.5.5, there exists some non-empty affine open subset S_λ such that $\omega_{X_\lambda/S_\lambda}$ is invertible, which means that $X_\lambda \to S_\lambda$ is Gorenstein. So assume further without loss of generality:

- π is Gorenstein.

After possibly shrinking S_0 once again, the Picard scheme Pic_{X_0/S_0} exists and represents the functor $\mathrm{Pic}_{(X/S)(\mathrm{fppf})}$ by Theorem A.26. Furthermore, $\mathrm{Pic}^\tau_{X_0/S_0}$ is an open subgroup scheme, whose formation commutes with base change in S_0. The next step is to deduce:

- $[\omega_{X_0/S_0}] \in \mathrm{Pic}^\tau_{X_0/S_0}(S_0)$ and ω_{X_s} is numerically trivial for all $s \in S_0$.

Clearly, the former part of the statement implies the latter, as the formation of Pic^τ commutes with base change. Consider $\alpha_0 \colon S_0 \to \mathrm{Pic}_{X_0/S_0}$ coming from ω_{X_0/S_0}. The assumption $\omega_X \in \mathrm{Pic}^\tau(X)$ means that the base change $\alpha \colon \mathrm{Spec}(k) \to \mathrm{Pic}_{X/k}$ factorizes through $\mathrm{Pic}^\tau_{X/k}$. An application of [57], Théorème 8.8.2 (i), shows that there exists a dense open subscheme $S_\lambda \subset S_0$ such that the base change $\alpha_\lambda \colon S_\lambda \to \mathrm{Pic}_{X_\lambda/S_\lambda}$ factorizes through $\mathrm{Pic}^\tau_{X_\lambda/S_\lambda}$, that is, $[\omega_{X_\lambda/S_\lambda}] \in \mathrm{Pic}^\tau_{X_\lambda/S_\lambda}(S_\lambda)$. So replace S_0 by S_λ one final time.

This completes the setup of the model $\pi \colon X_0 \to S_0$. Recall from Section A.4 that for all integers t and $i \geq 0$, the function $S_0 \to \mathbb{N}$, $s \mapsto h^i(\omega_{X_s}^{\otimes t})$ is upper semi-continuous, and $S_0 \to \mathbb{Z}$, $s \mapsto \chi(\omega_{X_s}^{\otimes t})$ is constant. Now $\omega_X^{\otimes t} \simeq \mathcal{O}_X$ means that $h^0(\omega_X^{\otimes t}) = 1$, which is in turn equivalent to $h^0(\omega_{X_s}^{\otimes t}) = 1$ for all closed points s on the curve S_0. So in order to show (†), it has to be verified that the natural numbers $\mathrm{ord}(\omega_{X_s})$ for closed $s \in S_0$ are bounded by some common value. For instance, it would be sufficient to prove that if X is as in (†) with ω_X of finite order d_X, then $d_X \leq c_n$ for a constant c_n only depending on the dimension n.

Clearly, up to this point, the crucial idea to progress is still missing, since the setup works similarly for any numerically trivial invertible sheaf $\mathscr{L}$ in place of ω_X, and there exist such $\mathscr{L}$ of infinite order. In the case of the dualizing sheaf, there is a possible interplay between the use of cohomology and base change, and Grothendieck duality, Theorem 2.17. For example, since $s \mapsto h^n(\omega_{X_s})$ is constant of value 1, the sheaf $R^n\pi_*(\omega_{X_0/S_0})$ is invertible. Then Grothendieck duality implies that $R^1\pi_*(\mathcal{O}_{X_0}) = R^{n-1}\pi_*(\omega_{X_0/S_0})^\vee$ is torsion-free, and hence locally free.

As mentioned above, the assertion that ω_X has finite order means that $s \mapsto h^0(\omega_{X_s}^{\otimes t})$ is constant of value 1 for some non-zero integer t. This is by Proposition A.8 equivalent to $R^1\pi_*(\omega_{X_0/S_0}^{\otimes t})$ or $R^n\pi_*(\omega_{X_0/S_0}^{\otimes -t+1})$ being torsion-free. If this could be done for an integer t which is a multiple of some $\mathrm{ord}(\omega_{X_s})$, then (†) would follow. Nevertheless, there still needs to be a source of additional information to proceed in this direction.

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`https://imaginary.org/program/surfer`

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Ich versichere an Eides statt, dass die Dissertation von mir selbstständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist.

André Schell, Düsseldorf, Dezember 2019