Multivariate generalized Ornstein-Uhlenbeck processes and connections to semiselfsimilarity properties

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Abstract

This thesis investigates the multivariate generalized Ornstein-Uhlenbeck (MGOU) process introduced by Behme and Lindner and connections to stochastic objects that can model semiselfsimilarity such as semiselfsimilar processes and semistable hemigroups. The first part focuses on stochastic exponentials and MGOU processes both of which occur in a left and right version due to the non-commutativity of matrix multiplication. Under numerous commutativity assumptions it is proven that the left and right stochastic exponential of a semimartingale are equal and a closed form expression for both is obtained that in the case of a continuous semimartingale simplifies to a formula proven by Yan. Extending the work of Behme and Lindner left MGOU processes driven by semi-Lévy processes are studied and right MGOU processes are defined and studied when driven by either Lévy processes or semi-Lévy processes. Conditions for periodic stationarity are given and MGOU processes with real time parameter are defined. The second part introduces the notions of random semiselfsimilar processes and random semistable hemigroups which are able to model random space-time scaling properties and by their usage of stochastic exponentials allow for connections to MGOU processes. A random Lamperti transform creates a one-to-one correspondence between random semiselfsimilar processes and periodically stationary processes and is used to construct MGOU processes out of random semiselfsimilar processes. A one-to-one correspondence between random semistable hemigroups and random semiselfsimilar processes with independent increments as well as a random integral representation of random semistable hemigroups are proven and used to construct MGOU processes out of random semistable hemigroups.

Zusammenfassung

Diese Doktorarbeit untersucht den von Behme und Lindner eingeführten multivariaten verallgemeinerten Ornstein-Uhlenbeck-Prozess (MGOU-Prozess) und Verbindungen zu stochastischen Objekten, die Semiselbstähnlichkeit modellieren können, wie beispielsweise semiselbstähnliche Prozesse und semistabile Hemigruppen. Der erste Teil konzentriert sich auf stochastische Exponentiale und MGOU-Prozesse, die beide aufgrund der Nicht-Kommutativität der Matrizenmultiplikation in einer linken und rechten Version auftreten. Unter zahlreichen Kommutativitätsvoraussetzungen wird bewiesen, dass das linke und rechte stochastische Exponential eines Semimartingals übereinstimmen, und für beide wird eine geschlossene Darstellung hergeleitet, die sich für stetige Semimartingale zu einer von Yan bewiesenen Formel vereinfacht. Als Erweiterung der Arbeit von Behme und Lindner werden linke MGOU-Prozesse untersucht, die von semi-Lévy-Prozessen angetrieben werden, und es werden rechte MGOU-Prozesse definiert und untersucht, die von Lévy- oder semi-Lévy-Prozessen angetrieben werden. Bedingungen für periodische Stationarität werden angegeben und MGOU-Prozesse mit reellem Zeitparameter werden definiert. Der zweite Teil führt zufällig semiselbstähnliche Prozesse und zufällig semistabile Hemigruppen ein, welche zufällige Raum-Zeit-Skalierungen modellieren können und durch die Verwendung von stochastischen Exponentialen Verbindungen zu MGOU-Prozessen ermöglichen. Eine zufällige Lamperti-Transformation stellt eine eins-zu-eins Beziehung zwischen zufällig semiselbstähnlichen Prozessen und periodisch stationären Prozessen her und wird dazu verwendet, MGOU-Prozesse aus zufällig semiselbstähnlichen Prozessen zu konstruieren. Eine eins-zu-eins Beziehung zwischen zufällig semistabilen Hemigruppen und zufällig semiselbstähnlichen Prozessen sowie eine Darstellung zufällig semistabiler Hemigruppen durch ein Zufallsintegral werden bewiesen und dazu verwendet, MGOU-Prozesse aus zufällig semistabilen Hemigruppen zu konstruieren.

1 Introduction

The Ornstein-Uhlenbeck process originated from the physical problem of modelling the velocity of a free particle that underlies brownian motion with friction, and was first studied in [30]. From a mathematical perspective various generalizations of this process have been developed and found to be connected to semiselfsimilar processes and semistable hemigroups. For the multivariate generalized Ornstein-Uhlenbeck process introduced in [4], however, connections of this kind have not been established yet which will be the main aim of this thesis. In order to get familiar with the set of problems and our notation we first illustrate the simple setting of the Ornstein-Uhlenbeck process.

The Ornstein-Uhlenbeck (OU) process $V = (V_t)_{t\geq 0}$ is the unique strong solution of the stochastic differential equation

$$\mathrm{d}V_t = -HV_t\,\mathrm{d}t + \mathrm{d}B_t\,, \tag{1.1}$$

where $B = (B_t)_{t\geq 0}$ is a one-dimensional brownian motion, the initial value V_0 is a random variable independent of B, and H is a positive real constant that is interpreted as the friction coefficient in the physical model of particle movement. In closed form the OU process is given by the stochastic integral

$$V_t = e^{-Ht}\left(V_0 + \int_0^t e^{Hu}\,\mathrm{d}B_u\right). \tag{1.2}$$

Because of $H > 0$ the limit

$$V_\infty := \int_0^\infty e^{-Hu}\,\mathrm{d}B_u = \int_{-\infty}^0 e^{Hu}\,\mathrm{d}B_{-u} \tag{1.3}$$

is almost surely a finite random variable and choosing the initial value V_0 so that it has the same distribution as V_∞ results in the OU process being stationary ([26]). By extending brownian motion to real time parameters (1.2) can then be written in the form

$$V_t = e^{-Ht}\int_{-\infty}^t e^{Hu}\,\mathrm{d}B_u\,. \tag{1.4}$$

The stationary OU process can also be constructed in a second way. In fact, the Lamperti transform $V = \mathsf{Lam}(B)$ of brownian motion given by

$$V_t = \mathsf{Lam}(B_t) = e^{-t/2}B_{e^t} \tag{1.5}$$

is stationary because by the $\frac{1}{2}$-selfsimilarity of brownian motion we have for all $c > 1$

$$V_{t+\log(c)} = c^{-1/2}e^{-t/2}B_{ce^t} \stackrel{\mathrm{D}}{=} e^{-t/2}B_{e^t} = V_t,$$

and V solves the stochastic differential equation (1.1) with $H = \frac{1}{2}$ because by the integration by parts formula

$$\begin{aligned} V_0 + B_t - \frac{1}{2}\int_0^t V_u\,\mathrm{d}u &= B_1 + B_t - \frac{1}{2}\int_0^t e^{-u/2}B_{e^u}\,\mathrm{d}u \\ &= B_1 + B_t + \int_0^t B_{e^u}\,\mathrm{d}(e^{-u/2}) \end{aligned}$$

$$= B_1 + B_t + e^{-t/2} B_{e^t} - B_1 - \int_0^t e^{-u/2} \, \mathrm{d}B_{e^u} - \left[e^{-t/2}, B_{e^t}\right]$$

$$= B_t + V_t - \int_1^{e^t} u^{-1/2} \, \mathrm{d}B_u = V_t.$$

Here the last step follows from an application of the Dambis-Dubins-Schwarz theorem (Theorem II.42 in [31]) with the continuous local martingale $M = (M_t)_{t \geq 0}$ defined by

$$M_t := \int_1^{e^t} u^{-1/2} \, \mathrm{d}B_u$$

which yields $M_t = B_{[M,M]_t}$ where

$$[M, M]_t = \left[\int_1^{e^t} u^{-1/2} \, \mathrm{d}B_u \, , \int_1^{e^t} u^{-1/2} \, \mathrm{d}B_u \right] = \int_1^{e^t} \left(u^{-1/2}\right)^2 \mathrm{d}[B, B]_u$$

$$= \int_1^{e^t} u^{-1} \, \mathrm{d}u = \Big[\log(|u|) \Big]_{u=1}^{e^t} = t.$$

On the other hand, if V is the stationary OU process in (1.4) then its inverse Lamperti transform $Z = \mathsf{Lam}^{-1}(V)$ given by

$$Z_t = \mathsf{Lam}^{-1}(V_t) = t^H V_{\log(t)} \tag{1.6}$$

is H-selfsimilar because by the stationary increments of B we have for all $c > 1$

$$Z_{ct} = (ct)^H V_{\log(ct)} = (ct)^H e^{-H \log(ct)} \int_{-\infty}^{\log(ct)} e^{Hu} \, \mathrm{d}B_u$$

$$= \int_{-\infty}^{\log(t)} e^{H(u + \log(c))} \, \mathrm{d}\big(B_{u+\log(c)} - B_{\log(c)}\big)$$

$$\overset{\mathrm{D}}{=} c^H \int_{-\infty}^{\log(t)} e^{Hu} \, \mathrm{d}B_u = c^H t^H V_{\log(t)} = c^H Z_t.$$

Using (1.4) the increments of $Z = \mathsf{Lam}^{-1}(V)$ can be written as the stochastic integral

$$Z_{s,t} := Z_t - Z_s = t^H V_{\log(t)} - s^H V_{\log(s)} = \int_{\log(s)}^{\log(t)} e^{Hu} \, \mathrm{d}B_u \, . \tag{1.7}$$

Because of the properties of Brownian motion and the stochastic integral the random variables $Z_{q,r}$ and $Z_{s,t}$ are independent whenever $q \leq r \leq s \leq t$, it holds $Z_{r,s} + Z_{s,t} = Z_{r,t}$ for all $r \leq s \leq t$, and the map $(s, t) \longmapsto Z_{s,t}$ is continuous with respect to convergence in distribution. The family $(Z_{s,t})_{0 \leq s \leq t}$ is then called hemigroup, a notion that was introduced in [16]. This hemigroup is H-stable which means that for all $c > 1$ it holds

$$Z_{cs,ct} = \int_{\log(cs)}^{\log(ct)} e^{Hu} \, \mathrm{d}B_u = \int_{\log(s)}^{\log(t)} e^{H(u+\log(c))} \, \mathrm{d}\big(B_{u+\log(c)} - B_{\log(c)}\big)$$

$$\stackrel{\mathrm{D}}{=} c^H \int_{\log(s)}^{\log(t)} e^{Hu}\,\mathrm{d}B_u = c^H Z_{s,t}. \tag{1.8}$$

On the other hand, if $(Z_{s,t})_{0\le s\le t}$ is an H-stable hemigroup then $Z = (Z_t)_{t\ge 0}$ defined by $Z_t := Z_{0,t}$ is H-selfsimilar because

$$Z_{ct} = Z_{0,ct} \stackrel{\mathrm{D}}{=} c^H Z_{0,t} = c^H Z_t.$$

If additionally each $Z_{s,t}$ has the integral representation (1.7) then $V = \mathsf{Lam}(Z)$ is the stationary OU process in (1.4) because

$$V_t = \mathsf{Lam}(Z_{0,t}) = e^{-Ht} Z_{0,e^t} = e^{-Ht} \int_{-\infty}^{t} e^{Hu}\,\mathrm{d}B_u\,.$$

Various generalizations of the OU process have been developed and many of the connections to selfsimilar processes and stable hemigroups that were exemplified above have been proven to hold true in a more general setting as well.

The first step to generalize the OU process is to replace the driving BROWNian motion B in (1.2) with a more general stochastic process Y. Various choices for Y have been discussed in the literature. The fractional ORNSTEIN-UHLENBECK (FOU) process replaces B with a fractional BROWNian motion ([9], [20]), the ORNSTEIN-UHLENBECK type process or LÉVY-ORNSTEIN-UHLENBECK (LOU) process replaces B with a LÉVY process ([33], [29], [18]), and the fractional OU type process or fractional LÉVY-ORNSTEIN-UHLENBECK (FLOU) process replaces B with a fractional LÉVY process ([14], [27], [28]).

The second step to generalize the OU process is to also replace the deterministic exponent H in (1.2) with a stochastic process X so that the OU process is driven by a bivariate background driving process (X, Y). If X is a LÉVY process and Y is fractional BROWNian motion then V is called generalized fractional ORNSTEIN-UHLENBECK (GFOU) process ([12]). If (X, Y) is a bivariate LÉVY process then V is called generalized ORNSTEIN-UHLENBECK (GOU) process. It is given by

$$V_t = \exp(X_t)^{-1}\left(V_0 + \int_0^t \exp(X_{u-})\,\mathrm{d}Y_u\right) \tag{1.9}$$

and solves the stochastic differential equation $\mathrm{d}V_t = V_{t-}\,\mathrm{d}U_t + \mathrm{d}L_t$ where (U, L) is a bivariate LÉVY process that can be calculated from the background driving process (X, Y) with the formulas

$$U_t = -X_t + [X, X]^c_t + \sum_{0<s\le t} \left((1+\Delta X_s)^{-1} - 1 + \Delta X_s\right), \tag{1.10a}$$

$$L_t = Y_t - [X, Y]^c_t + \sum_{0<s\le t} \left((1+\Delta X_s)^{-1} - 1\right)\Delta Y_s. \tag{1.10b}$$

Here $[X, X]^c$ is the continuous part of the quadratic variation of X and $\Delta X_t = X_t - X_{t-}$ is the jump of X at time t of which there are at most countably many in any given finite time interval. The GOU process emerged in [10] as the solution of the continuous-time stochastic difference equation $V_t = A_{s,t}V_s + B_{s,t}$ where $(A_{s,t}, B_{s,t})_{0\le s\le t}$ is a family of real-valued random variables that satisfy the equations

$$A_{r,t} = A_{s,t}A_{r,s} \quad \text{and} \quad B_{r,t} = A_{s,t}B_{r,s} + B_{s,t} \tag{1.11}$$

for $0 \leq r \leq s \leq t$ alongside other independence, stationarity, and continuity assumptions. $A_{s,t}$ and $B_{s,t}$ can be expressed in terms of the background driving process (X, Y) with the formulas

$$A_{s,t} = \exp(X_t)^{-1} \exp(X_s), \tag{1.12a}$$

$$B_{s,t} = \exp(X_t)^{-1} \int_s^t \exp(X_{u-}) \,\mathrm{d}Y_u \,. \tag{1.12b}$$

As outlined in [23], [11], and [5] the stationarity of the GOU process is equivalent to the existence of the limit

$$V_\infty = \int_0^\infty \exp(X_{u-})^{-1} \,\mathrm{d}L_u \,.$$

Necessary and sufficient conditions for the existence of this limit are discussed in [13] in the case of Lévy processes and in [6] in the case of Markov additive processes. GOU processes have found application in option pricing, insurance and perpetuities, and risk theory. We point to the survey [26] and the introduction of [23] for references regarding these topics.

The multivariate generalized Ornstein-Uhlenbeck (MGOU) process is a multivariate extension of the GOU process in which X and Y are both matrix-valued Lévy processes and the deterministic exponential function in (1.3) is replaced with its stochastic analog, the stochastic exponential. The stochastic exponential $\mathsf{Exp}(X)$ of a real-valued semimartingale X is defined as the unique strong solution of the stochastic differential equation

$$\mathrm{d}\mathsf{Exp}(X_t) = \mathsf{Exp}(X_{t-}) \,\mathrm{d}X_t \tag{1.13a}$$

with initial value $\mathsf{Exp}(X_0) = 1$. Due to the non-commutativity of matrix multiplication there is a left stochastic exponential $\overleftarrow{\mathsf{Exp}}(X)$ and a right stochastic exponential $\overrightarrow{\mathsf{Exp}}(X)$ in dimension $n \geq 2$ which correspond to the stochastic differential equations

$$\mathrm{d}\overleftarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(X_{t-}) \,\mathrm{d}X_t \quad \text{and} \quad \mathrm{d}\overrightarrow{\mathsf{Exp}}(X_t) = \mathrm{d}X_t \,\overrightarrow{\mathsf{Exp}}(X_{t-}) \tag{1.13b}$$

with initial value $\overleftarrow{\mathsf{Exp}}(X_0) = \overrightarrow{\mathsf{Exp}}(X_0) = I$. This allows for two different versions of the MGOU process. The left version is given by

$$V_t = \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \left(V_0 + \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u \right) \tag{1.14}$$

and solves the stochastic differential equation $\mathrm{d}V_t = V_{t-} \,\mathrm{d}U_t + \mathrm{d}L_t$ where (U, L) is a bivariate Lévy process that can be calculated from the background driving process (X, Y) with the formulas

$$U_t = -X_t + [X, X]^c_t + \sum_{0<s\leq t} ((I + \Delta X_s)^{-1} - I + \Delta X_s), \tag{1.15a}$$

$$L_t = Y_t - [X, Y]^c_t + \sum_{0<s\leq t} ((I + \Delta X_s)^{-1} - I)\Delta Y_s. \tag{1.15b}$$

The left MGOU process (1.14) was first considered in [2] and [4] as the solution of the continuous-time stochastic difference equation $V_t = A_{s,t}V_s + B_{s,t}$ where $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ is a family of matrix-valued random variables that satisfy (1.11) alongside other independence, stationarity, and continuity assumptions. $A_{s,t}$ and $B_{s,t}$ can be expressed in terms of the background driving process (X,Y) with the formulas

$$A_{s,t} = \overleftarrow{\mathsf{Exp}}(X_t)^{-1}\overleftarrow{\mathsf{Exp}}(X_s), \tag{1.16a}$$

$$B_{s,t} = \overleftarrow{\mathsf{Exp}}(X_t)^{-1}\int_s^t \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\,. \tag{1.16b}$$

Stationarity conditions for the left MGOU process are also given in [2] and [4] and formulas for the first and second moments of stationary left MGOU processes are derived in [3]. Left MGOU processes appear as the state vector process of COGARCH processes (Example 3.6 in [4]) and RC-CARMA processes ([8]). So far the right MGOU process

$$V_t = \left(V_0 + \int_0^t \mathrm{d}Y_u\,\overrightarrow{\mathsf{Exp}}(X_{u-})\right)\overrightarrow{\mathsf{Exp}}(X_t)^{-1} \tag{1.17}$$

has not been studied in the literature although its treatment is similar to the left counterpart.

The relationships between the stationary OU process V, the $\frac{1}{2}$-selfsimilar BROWNian motion B, and the stable hemigroup $(Z_{s,t})_{0\le s\le t}$ formed by the increments of $Z = \mathsf{Lam}^{-1}(V)$ that were exemplified at the beginning extend to a more general setting in the following way.

A matrix-valued process $Z = (Z_t)_{t\ge 0}$ is called selfsimilar with exponent $H \in \mathbb{R}^{n\times n}$ or simply H-selfsimilar if for all $c > 1$

$$(Z_{ct})_{t\ge 0} \overset{\mathrm{D}}{=} \left(c^H Z_t\right)_{t\ge 0} \tag{1.18}$$

with $c^H = \exp\left(H\log(c)\right)$. If (1.18) only holds for some $c > 1$ then Z is called H-semiselfsimilar. If $Z = (Z_t)_{t>0}$ is H-selfsimilar then its LAMPERTI transform $V = (V_t)_{t\in\mathbb{R}}$ given by

$$V_t = \mathsf{Lam}(Z_t) = e^{-tH} Z_{e^t} \tag{1.19a}$$

is stationary. Conversely, if $V = (V_t)_{t\in\mathbb{R}}$ is stationary then its inverse LAMPERTI transform $Z = (Z_t)_{t>0}$ given by

$$Z_t = \mathsf{Lam}^{-1}(V_t) = t^H V_{\log(t)} \tag{1.19b}$$

is H-selfsimilar. This one-to-one correspondence between selfsimilar and stationary processes was proven in [22] and it has been shown in [24] that it also holds true for semiselfsimilar and periodically stationary processes. If $Z = (Z_t)_{t>0}$ is an H-selfsimilar process then $Y = (Y_t)_{t\in\mathbb{R}}$ given by

$$Y_t = \begin{cases} \displaystyle\int_1^{e^t} u^{-H}\,\mathrm{d}Z_u & , \quad t \ge 0 \\ \displaystyle -\int_{e^t}^1 u^{-H}\,\mathrm{d}Z_u & , \quad t < 0 \end{cases} \tag{1.20}$$

is a LÉVY process and $\mathsf{Lam}(Z)$ is a stationary ORNSTEIN-UHLENBECK type process with background driving process Y ([17]). Analogously, if $Z = (Z_t)_{t\ge 0}$ is an H-semiselfsimilar process then $Y = (Y_t)_{t\in\mathbb{R}}$ given by (1.20) is a semi-LÉVY process, that is a LÉVY process except that the increments are only periodically stationary, and $\mathsf{Lam}(Z)$ is a periodically stationary ORNSTEIN-UHLENBECK type process with background driving process Y. This was proven in [1], where the stochastic integral in (1.20) is defined as a random integral in the sense of [19], and independently in [25], where the stochastic integral in (1.20) is defined using a semimartingale approach in the sense of [32].

A family $(Z_{s,t})_{0\le s\le t}$ of matrix-valued random variables is called stable hemigroup with exponent $H\in\mathbb{R}^{n\times n}$ or simply H-stable hemigroup if $Z_{q,r}$ and $Z_{s,t}$ are independent whenever $q\le r\le s\le t$, it holds $Z_{r,s}+Z_{s,t}=Z_{r,t}$ for all $0\le r\le s\le t$, the map $(s,t)\longmapsto Z_{s,t}$ is continuous with respect to convergence in distribution and for all $c>1$ and all $0\le s\le t$ it holds

$$Z_{cs,ct}\overset{\mathrm{D}}{=}c^H Z_{s,t}. \tag{1.21}$$

If (1.21) only holds for some $c>1$ then $(Z_{s,t})_{0\le s\le t}$ is called H-semistable hemigroup. It has been proven in [1] that the increments $Z_{s,t}=Z_t-Z_s$ of an H-semiselfsimilar process $Z=(Z_t)_{t\ge0}$ with independent increments form an H-semistable hemigroup from which the semiselfsimilar process can be reobtained by $Z_t=Z_{0,t}$. Also, given an H-semistable hemigroup $(Z_{s,t})_{0\le s\le t}$ such that $t^H\longrightarrow0$ for $t\downarrow0$, each $Z_{s,t}$ has the random integral representation

$$Z_{s,t}=\int_{\log(s)}^{\log(t)} e^{Hu}\,\mathrm{d}Y_u\,, \tag{1.22}$$

where Y is defined as in (1.20) with the H-semiselfsimilar process $Z=(Z_{0,t})_{t\ge0}$ as the integrator. Conversely, (1.22) defines an H-semistable hemigroup if Y is a semi-Lévy process that satisfies a logarithmic moment condition and $t^H\longrightarrow0$ for $t\downarrow0$.

So far connections of this kind have not been established in the context of GOU or MGOU processes in neither the stationary nor the periodically stationary case.

In this thesis we extend results about MGOU processes originally obtained in [4] to the case that the background driving process is a semi-Lévy instead of a Lévy process, including conditions for periodic stationarity, and build connections between MGOU processes, semiselfsimilar processes, and semistable hemigroups by introducing the concept of random semiselfsimilarity and random semistability. These generalizations of semiselfsimilarity and semistability are able to model random scaling rather than just deterministic scaling and work nicely in conjunction with MGOU processes.

The basic idea for incorporating random scaling is to replace the deterministic exponent H in (1.18) and (1.21) with a Lévy process X, as in the transition from OU to GOU processes, and then also replace deterministic exponentials with stochastic exponentials, as in the transition from GOU to MGOU processes. More precisely, for random left semiselfsimilarity we write (1.18) in the form

$$(Z_{ct})_{t\ge0}\overset{\mathrm{D}}{=}\left(c^H Z_t\right)_{t\ge0}\quad\Longleftrightarrow\quad\left(\left(e^{H\log(c)}\right)^{-1}Z_{ct}\right)_{t\ge0}\overset{\mathrm{D}}{=}(Z_t)_{t\ge0}, \tag{1.23a}$$

replace the deterministic term $e^{H\log(c)}$ with the stochastic term $\overleftarrow{\mathsf{Exp}}(X_{\log(c)})$ and obtain the random scaling property

$$\left(\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}Z_{ct}\right)_{t\ge0}\overset{\mathrm{D}}{=}(Z_t)_{t\ge0}. \tag{1.23b}$$

For random left semistability we write (1.21) in the form

$$Z_{cs,ct}\overset{\mathrm{D}}{=}c^H Z_{s,t}\quad\Longleftrightarrow\quad\left(e^{H\log(c)}\right)^{-1}Z_{cs,ct}\overset{\mathrm{D}}{=}Z_{s,t}, \tag{1.24a}$$

replace the deterministic term $e^{H\log(c)}$ with the stochastic term $\overleftarrow{\mathsf{Exp}}(X_{\log(c)})$ and obtain the random scaling property

$$\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}Z_{cs,ct}\overset{\mathrm{D}}{=}Z_{s,t}. \tag{1.24b}$$

Right versions of (1.23b) and (1.24b) are defined similarly by utilizing right stochastic exponentials and respecting the inverted order of multiplication. Since in general no independence between X and Z is given it turns out that the random semiselfsimilarity in (1.23b) and the random semistability in (1.24b) need to be simultaneous with the periodic stationarity of the increments of X. This then necessitates a further distinction of the time parameter because our definition of periodically stationary increments differentiates between a positive period for positive time parameter and a negative period for negative time parameter in order to be compatible with Lévy processes with real time parameter.

The structure of this thesis is as follows. In chapter 2 we provide the necessary background for Lévy processes, quadratic covariations of semimartingales, and stochastic integrals. In chapter 3 we discuss properties of the left and right stochastic exponential of a matrix-valued semimartingale and prove a closed form expression that extends a corresponding result in [34] but under even more commutativity conditions since we do not assume the semimartingale to be continuous. These two chapters are mostly of preparative nature. In chapter 4 we take on the results about left MGOU processes in [4], prove corresponding results in the case that the background driving process is a semi-Lévy instead of a Lévy process, and construct left MGOU processes with real time parameter. We also study right MGOU processes, which are not considered in [4], and transfer all results from left MGOU processes in both the stationary and periodically stationary case. In the last two chapters we connect MGOU processes to random semiselfsimilar processes and random semistable hemigroups. In chapter 5 we prove that a generalization of the Lamperti transform creates a one-to-one correspondence between random semiselfsimilar processes and periodically stationary processes and show that this random Lamperti transform allows the construction of an MGOU process out of a random semiselfsimilar process and vice versa. In chapter 6 we prove a one-to-one correspondence between random semistable hemigroups and random semiselfsimilar processes with independent increments, derive a random integral representation of a random semistable hemigroup and show that this integral representation allows the construction of a periodically stationary process and an MGOU process out of a random semistable hemigroup and vice versa.

2 Stochastic Calculus

A matrix-valued stochastic process either takes values in the group $\mathbb{R}^{n\times m}$ of $n \times m$-matrices, with the group action being addition, or in the group $\mathsf{GL}_n(\mathbb{R})$ of invertible $n\times n$-matrices, with the group action being multiplication. In the first case the process has additive increments and in the second case the process has multiplicative increments. In each of these two cases a stochastic process and its increments may have the following properties.

Definition 2.1. Let either $I = [0, \infty)$ or $I = \mathbb{R}$. Let $X = (X_t)_{t\in I}$ be a stochastic process where each random variable X_t takes values in the group $(\mathbb{R}^{n\times m}, +)$ for some $n, m \in \mathbb{N}$. For $s < t$ the **additive increment** of X on the time interval $(s, t]$ is the random variable $X_t - X_s$.

(a) X has **independent increments** if $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_k} - X_{t_{k-1}}$ are independent for all $t_0 < \dots < t_k$, $k \in \mathbb{N}$.

(b) X has **stationary increments** if $X_t - X_s \overset{\mathrm{D}}{=} X_{t-s} - X_0$ for all $s < t$.

(c) X has **periodically stationary increments** with **period** $p > 0$ or simply p**-stationary increments** if

$$(X_{t+p} - X_p)_{t\geq 0} \overset{\mathrm{D}}{=} (X_t - X_0)_{t\geq 0} \quad \text{and} \quad (X_{t-p} - X_{-p})_{t\leq 0} \overset{\mathrm{D}}{=} (X_t - X_0)_{t\leq 0}. \tag{2.1}$$

(d) X is **stationary** if $X_{t+h} \overset{\mathrm{D}}{=} X_t$ for all $t \in I$ and $h > 0$.

(e) X is **periodically stationary** with **period** $p > 0$ or simply p**-stationary** if

$$(X_{t+p})_{t\geq 0} \overset{\mathrm{D}}{=} (X_t)_{t\geq 0} \quad \text{and} \quad (X_{t-p})_{t\leq 0} \overset{\mathrm{D}}{=} (X_t)_{t\leq 0}. \tag{2.2}$$

(f) X is **continuous in probability** if $P\big(\|X_t - X_s\| \geq \varepsilon\big) \xrightarrow{s\to t} 0$ for all $\varepsilon > 0$.

(g) X is **càdlàg/càglàd** if its paths are almost surely right/left continuous with limits from the left/right.

(h) X is called **semi-Lévy process** with **period** $p > 0$ or simply p**-semi-Lévy process** if $X_0 = \mathbf{0}$ and X is càdlàg and continuous in probability and has independent and p-stationary increments.

(i) X is called Lévy **process** if X is a p-semi-Lévy process for all $p > 0$.

Definition 2.2. Let either $I = [0, \infty)$ or $I = \mathbb{R}$. Let $X = (X_t)_{t\in I}$ be a stochastic process where each random variable X_t takes values in the group $(\mathsf{GL}_n(\mathbb{R}), \cdot)$ for some $n \in \mathbb{N}$. For $s < t$ the **multiplicative left/right increment** of X on the time interval $(s, t]$ is the random variable $X_t X_s^{-1}$ respectively $X_s^{-1} X_t$.

(a) X has **independent left/right increments** if $X_{t_0}, X_{t_1} X_{t_0}^{-1}, \dots, X_{t_k} X_{t_{k-1}}^{-1}$ respectively $X_{t_0}, X_{t_0}^{-1} X_{t_1}, \dots, X_{t_{k-1}}^{-1} X_{t_k}$ are independent for all $t_0 < \dots < t_k$, $k \in \mathbb{N}$.

(b) X has **stationary left/right increments** if $X_t X_s^{-1} \overset{\mathrm{D}}{=} X_{t-s} X_0^{-1}$ for all $s < t$ respectively $X_s^{-1} X_t \overset{\mathrm{D}}{=} X_0^{-1} X_{t-s}$ for all $s < t$.

(c) X has **periodically stationary left/right increments** with **period** $p > 0$ or simply **p-stationary left/right increments** if

$$(X_{t+p}X_p^{-1})_{t\geq 0} \stackrel{\mathrm{D}}{=} (X_tX_0^{-1})_{t\geq 0} \quad \text{and} \quad (X_{t-p}X_{-p}^{-1})_{t\leq 0} \stackrel{\mathrm{D}}{=} (X_tX_0^{-1})_{t\leq 0} \tag{2.3a}$$

respectively

$$(X_p^{-1}X_{t+p})_{t\geq 0} \stackrel{\mathrm{D}}{=} (X_0^{-1}X_t)_{t\geq 0} \quad \text{and} \quad (X_{-p}^{-1}X_{t-p})_{t\leq 0} \stackrel{\mathrm{D}}{=} (X_0^{-1}X_t)_{t\leq 0}. \tag{2.3b}$$

(d) X is **stationary** if $X_{t+h} \stackrel{\mathrm{D}}{=} X_t$ for all $t \in I$ and $h > 0$.

(e) X is **periodically stationary** with **period** $p > 0$ or simply **p-stationary** if

$$(X_{t+p})_{t\geq 0} \stackrel{\mathrm{D}}{=} (X_t)_{t\geq 0} \quad \text{and} \quad (X_{t-p})_{t\leq 0} \stackrel{\mathrm{D}}{=} (X_t)_{t\leq 0}. \tag{2.4}$$

(f) X is **continuous in probability** if $P\big(\|X_tX_s^{-1}\| \geq \varepsilon\big) \xrightarrow{s\to t} 0$ for all $\varepsilon > 0$.

(g) X is **càdlàg/càglàd** if its paths are almost surely right/left continuous with limits from the left/right.

(h) X is called **left/right semi-Lévy process** with **period** $p > 0$ or simply **left/right p-semi-Lévy process** if $X_0 = I$ and X is càdlàg and continuous in probability and has independent and p-stationary right/left increments.

(i) X is called **left/right Lévy process** if X is a left/right p-semi-Lévy process for all $p > 0$.

To clarify the group action we could speak of "additive Lévy processes" and "multiplicative Lévy processes" as Lévy processes with additive respectively multiplicative increments but this would lead to confusion with the term "additive process" used by Sato for a process that is continuous in probability and càdlàg and has independent increments but does not specify the group action. Instead we use the term "semi-Lévy process", which is an additive process that has periodically stationary increments, and have the group action be clear from the context.

In the definition of periodically stationary increments in (2.1), (2.3a), (2.3b) and periodic stationarity in (2.2), (2.4) we need to differentiate between negative and positive time parameter in order to be able to construct a Lévy process respectively semi-Lévy process with time parameter $t \in \mathbb{R}$ from two independent copies of the same Lévy process respectively semi-Lévy process with time parameter $t \geq 0$. This construction is needed in chapter 4 for the definition of MGOU processes with real time parameter.

Theorem 2.3. *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a stochastic process in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ and let (X', Y') be an independent copy of (X, Y). Let $\widetilde{X} = (\widetilde{X}_t)_{t\in\mathbb{R}}$ and $\widetilde{Y} = (\widetilde{Y}_t)_{t\in\mathbb{R}}$ be defined by*

$$\widetilde{X}_t := \begin{cases} X_t & \text{for } t \geq 0 \\ -X'_{(-t)-} & \text{for } t < 0 \end{cases} \quad \text{and} \quad \widetilde{Y}_t := \begin{cases} Y_t & \text{for } t \geq 0 \\ -Y'_{(-t)-} & \text{for } t < 0 \end{cases}. \tag{2.5}$$

(a) *If (X, Y) is a Lévy process then $\widetilde{X}$, $\widetilde{Y}$, and $(\widetilde{X}, \widetilde{Y})$ are also Lévy processes.*

(b) *If (X, Y) is a p-semi-Lévy process for some $p > 0$ then $\widetilde{X}$, $\widetilde{Y}$, and $(\widetilde{X}, \widetilde{Y})$ are also p-semi-Lévy processes.*

Proof. We only prove that $\widetilde{X}$ is a Lévy process respectively p-semi-Lévy process because the argumentation is the same for $\widetilde{Y}$ and $(\widetilde{X}, \widetilde{Y})$. The càdlàg property of X and X' and the left limit $X'_{(-t)-}$ in (2.5) ensure that $\widetilde{X}$ is càdlàg as well. (The left limit only ensures the càdlàg property of $\widetilde{X}$ but is not needed for distributional properties since at fixed times Lévy processes almost surely do not have jumps.) $\widetilde{X}_0 = X_0 = \mathbf{0}$ holds by definition. $\widetilde{X}$ has independent increments because for $t_1 < t_2 < 0 \leq t_3 < t_4$ the random variables

$$(\widetilde{X}_{t_4} - \widetilde{X}_{t_3}, \widetilde{X}_{t_3} - \widetilde{X}_{t_2}, \widetilde{X}_{t_2} - \widetilde{X}_{t_1}) = (X_{t_4} - X_{t_3}, X_{t_3} - X_0 + X'_{-t_2} - X'_0, X'_{-t_1} - X'_{-t_2})$$

are independent. If X has stationary increments then $\widetilde{X}$ also has stationary increments because in the case $0 \leq s < t$ the stationary increments of X yield

$$\widetilde{X}_t - \widetilde{X}_s = X_t - X_s \overset{\mathrm{D}}{=} X_{t-s} = \widetilde{X}_{t-s}$$

and in the case $s < t < 0$ the stationary increments of X' yield

$$\widetilde{X}_t - \widetilde{X}_s = -(X'_{-t} - X'_{-s}) \overset{\mathrm{D}}{=} -X'_{-(t-s)} = \widetilde{X}_{t-s}.$$

In the mixed case $s < 0 \leq t$ it holds

$$\widetilde{X}_{t-s} - \widetilde{X}_t = X_{t-s} - X_t \overset{\mathrm{D}}{=} X_{-s} \overset{\mathrm{D}}{=} X'_{-s} = -\widetilde{X}_s$$

and because $\widetilde{X}_t$ is independent of both $\widetilde{X}_{t-s} - \widetilde{X}_t$ and $\widetilde{X}_s$ this yields $\widetilde{X}_{t-s} \overset{\mathrm{D}}{=} \widetilde{X}_t - \widetilde{X}_s$. If X has p-stationary increments then $\widetilde{X}$ also has p-stationary increments because for $t \geq 0$ the p-stationary increments of X yield

$$\widetilde{X}_{t+p} - \widetilde{X}_p = X_{t+p} - X_p \overset{\mathrm{D}}{=} X_t = \widetilde{X}_t$$

and for $t < 0$ the p-stationary increments of X' yield

$$\widetilde{X}_{t-p} - \widetilde{X}_{-p} = -(X'_{-t+p} - X'_p) \overset{\mathrm{D}}{=} -X'_{-t} = \widetilde{X}_t.$$

□

2.1 One-dimensional stochastic integration

Our construction of a stochastic integral for one-dimensional stochastic processes follows chapter II in the book [31] of Protter. Given a filtered and complete probability space $(\Omega, \mathcal{A}, \mathcal{F}, P)$ such that $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is right-continuous and $\mathcal{F}_0$ is complete, the first step is to define a stochastic integral for simple predictable processes.

Definition 2.4. A process $H = (H_t)_{t\geq 0}$ is called **simple predictable** if each H_t is of the form

$$H_t = K_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{n} K_i \mathbf{1}_{(T_i, T_{i+1}]}(t) \tag{2.6}$$

where $0 = T_1 \leq \ldots \leq T_{n+1}$ are a.s. finite stopping times and $K_0, \ldots, K_n$ are a.s. finite random variables such that $\sigma(K_i) \subseteq \mathcal{F}_{T_i}$ for all $i = 0, \ldots, n$.

Let $\mathbf{S}$ be the set of all simple predictable processes and $\mathbf{S}_u$ the space $\mathbf{S}$ endowed with the topology of uniform convergence. Let $\mathbf{L}$ be the set of bounded random variables and $\mathbf{L}^0$ the space $\mathbf{L}$ endowed with the topology of convergence in probability. For a given process $X = (X_t)_{t\geq 0}$ a linear mapping $I_X : \mathbf{S}_u \longrightarrow \mathbf{L}^0$ is defined by

$$I_X(H) := K_0 X_0 + \sum_{i=1}^{n} K_i (X_{T_{i+1}} - X_{T_i}) \tag{2.7}$$

for $H \in \mathbf{S}$ with representation (2.6). The definition of a stochastic integral for more general integrands H than simple predictable processes requires the integrator X to be a semimartingale.

Definition 2.5. Let $X = (X_t)_{t\geq 0}$ be a stochastic process.

(a) X is called **total semimartingale** if X is càdlàg, adapted, and the mapping I_X defined in (2.7) is continuous.

(b) X is called **semimartingale** if for each constant $T > 0$ the process $X^T = (X_{t\wedge T})_{t\geq 0}$ is a total semimartingale.

(c) X is called **classical semimartingale** if X is càdlàg, adapted, and there exist a local martingale N and a finite variation process B such that $N_0 = B_0 = 0$ and

$$X_t = X_0 + N_t + B_t.$$

Theorem 2.6 gives basic properties of semimartingales. Notably finite sums of semimartingales are also semimartingales.

Theorem 2.6. [31, Theorem II.1, III.30, III.47]

(a) *The set of all semimartingales is a vector space.*

(b) *An adapted càdlàg process X is a semimartingale if and only if X is a classical semimartingale.*

Now let $\mathbb{D}$ be the set of adapted processes with càdlàg paths and let $\mathbb{L}$ be the set of adapted processes with càglàd paths. $\mathbf{S}$, $\mathbb{D}$, and $\mathbb{L}$ are endowed with the topology induced by the following type of convergence.

Definition 2.7. A sequence $(H^n)_{n\in\mathbb{N}}$ of processes $H^n = (H^n_t)_{t\geq 0}$ in $\mathbf{S}$, $\mathbb{D}$, or $\mathbb{L}$ converges **uniformly on compacts in probability** to a process $H = (H_t)_{t\geq 0}$, written $H^n \xrightarrow{\text{ucp}} H$, if

$$\sup_{0\leq s\leq t} |H^n_s - H_s| \xrightarrow{\text{P}} 0 \quad \text{for all } t > 0. \tag{2.8}$$

The spaces $\mathbf{S}$, $\mathbb{D}$ and $\mathbb{L}$ endowed with this ucp-topology are denoted $\mathbf{S}_{ucp}$, $\mathbb{D}_{ucp}$ and $\mathbb{L}_{ucp}$ respectively. $\mathbf{S}_{ucp}$ is dense in $\mathbb{L}_{ucp}$ and $\mathbb{D}_{ucp}$ is a complete metric space with respect to the metric

$$d(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{E}\Big[\min\Big(1, \sup_{0\leq s\leq n} |X_s - Y_s|\Big)\Big]. \tag{2.9}$$

Definition 2.8. For a simple predictable process $H \in \mathbf{S}$ with representation (2.6) and an adapted càdlàg process $X \in \mathbb{D}$ the **stochastic integral** of H with respect to X is defined as

$$H \cdot X := \int H_s \,\mathrm{d}X_s := J_X(H) := K_0 X_0 + \sum_{i=1}^{n} K_i \Big(X^{T_{i+1}} - X^{T_i}\Big). \tag{2.10}$$

Evaluating (2.10) at time t gives

$$\begin{aligned} \int_0^t H_s \,\mathrm{d}X_s := J_X(H)_t &= K_0 X_0 + \sum_{i=1}^{n} K_i \Big(X_{t\wedge T_{i+1}} - X_{t\wedge T_i}\Big) \\ &= K_0 X_0 + \sum_{i=1}^{n} K_i \Big(X^t_{T_{i+1}} - X^t_{T_i}\Big) = I_{X^t}(H). \end{aligned}$$

(2.10) induces a linear mapping $J_X : \mathbf{S} \longrightarrow \mathbb{D}$. In the case that X is a semimartingale the mapping $J_X : \mathbf{S}_{ucp} \longrightarrow \mathbb{D}_{ucp}$ is continuous by Theorem II.11 in [31]. Since $\mathbf{S}_{ucp}$ is dense in $\mathbb{L}_{ucp}$ and $\mathbb{D}_{ucp}$ is a complete metric space, there exists a unique extension of J_X to $\mathbb{L}_{ucp}$.

Definition 2.9. For a semimartingale X the unique continuous and linear extension $J_X : \mathbb{L}_{ucp} \longrightarrow \mathbb{D}_{ucp}$ is called **stochastic integral** with respect to X.

The above steps in the construction of the stochastic integral show that it is a pathwise construction.

Another important object in stochastic calculus is the quadratic covariation of two semimartingales.

Definition 2.10. Let X, Y be semimartingales in $\mathbb{R}$.

(a) The **quadratic covariation** of X and Y is the process $[X,Y] = ([X,Y]_t)_{t\geq 0}$ defined by

$$[X,Y]_t := X_t Y_t - X_0 Y_0 - \int_0^t X_{u-} \,\mathrm{d}Y_u - \int_0^t Y_{u-} \,\mathrm{d}X_u \,. \tag{2.11a}$$

(b) The **path-by-path continuous part** or simply **continuous part** of the quadratic covariation $[X,Y]$ is the process $[X,Y]^c = ([X,Y]_t^c)_{t\geq 0}$ defined by

$$[X,Y]_t^c := [X,Y]_t - \sum_{0<s\leq t} \Delta X_s \Delta Y_s. \tag{2.11b}$$

Theorem 2.11 gives properties of the one-dimensional quadratic covariation and its continuous part.

Theorem 2.11. *Let X, Y, Z be semimartingales in $\mathbb{R}$.*

(a) *The quadratic covariation $[X,Y]$ is a semimartingale and has paths of finite variation on compacts. It satisfies $[X,Y]_0 = 0$ and $\Delta[X,Y] = \Delta X \Delta Y$.*

(b) *If X is adapted, càdlàg and has paths of finite variation on compacts, then $[X,Z]^c = 0$.*

(c) *X, Y, Z satisfy the identities*

$$\Big[[X,Y],Z\Big]^c = \Big[[X,Y]^c,Z\Big]^c = \Big[[X,Y]^c,Z\Big] = 0, \tag{2.12a}$$

$$\Big[[X,Y],Z\Big]_t = \sum_{0<s\leq t} \Delta X_s \Delta Y_s \Delta Z_s. \tag{2.12b}$$

Proof. (a) These statements are Corollary II.6.1 and Theorem II.23(i) in [31]. Note that in [31] the quadratic covariation is defined without the term $-X_0 Y_0$ and thus Theorem II.23(i) in [31] states that $[X,Y]_0 = X_0 Y_0$ instead of $[X,Y]_0 = 0$.

(b) By Theorem II.26 in [31] it holds $[X,X]^c = 0$ and then by Theorem II.28 in [31] it also holds $[X,Z]^c = 0$.

(c) By (a) both $[X,Y]$ and $[X,Y]^c$ satisfy the assumptions of (b) which implies that $\Big[[X,Y],Z\Big]^c = 0$ and $\Big[[X,Y]^c,Z\Big]^c = 0$. Furthermore we have

$$\Big[[X,Y]^c,Z\Big]_t = \Big[[X,Y]^c,Z\Big]_t^c + \sum_{0<s\leq t} \Delta[X,Y]_s^c \Delta Z_s = 0$$

and by using $\Delta[X,Y] = \Delta X \Delta Y$ we have

$$\Big[[X,Y],Z\Big]_t = \Big[[X,Y],Z\Big]_t^c + \sum_{0<s\leq t} \Delta[X,Y]_s \Delta Z_s = \sum_{0<s\leq t} \Delta X_s \Delta Y_s \Delta Z_s.$$

□

Theorem 2.12 gives calculation rules for one-dimensional stochastic integrals and quadratic covariations of integral processes.

Theorem 2.12. [31, Corollary II.6.2, Theorem II.19, II.29] *Let X, Y be semimartingales in $\mathbb{R}$ and let $G, H \in \mathbb{L}$.*

(a) *XY is a semimartingale and $XY = X_0Y_0 + X_- \cdot Y + Y_- \cdot X + [X, Y]$ which in integral notation becomes the* **integration by parts formula**

$$X_tY_t = X_0Y_0 + \int_0^t X_{u-}\,\mathrm{d}Y_u + \int_0^t Y_{u-}\,\mathrm{d}X_u + [X, Y]_t. \tag{2.13a}$$

(b) *$G \cdot X$ is a semimartingale and $H \cdot (G \cdot X) = (HG) \cdot X$ which in integral notation becomes*

$$\int_0^t H_u\,\mathrm{d}\Big(\int_0^u G_v\,\mathrm{d}X_v\Big) = \int_0^t H_uG_u\,\mathrm{d}X_u\,. \tag{2.13b}$$

(c) *$G \cdot X$ and $H \cdot Y$ are semimartingales and $[G \cdot X, H \cdot Y] = (GH) \cdot [X, Y]$ which in integral notation becomes*

$$\Big[\int_0^t G_u\,\mathrm{d}X_u\,, \int_0^t H_u\,\mathrm{d}Y_u\Big] = \int_0^t G_uH_u\,\mathrm{d}[X, Y]_u\,. \tag{2.13c}$$

Again note that the integration by parts formula in [31] slightly differs from our integration by parts formula (2.13a) because the quadratic covariation in [31] is defined without the term $-X_0Y_0$.

Before we move on to the matrix-valued case we prove that for $H, H' \in \mathbb{L}_{ucp}$ and two semimartingales X, X' such that the bivariate processes (H, X) and (H', X') are equal in distribution also the stochastic integrals $J_X(H)$ and $J_{X'}(H')$ are equal in distribution. This result is implicitly used many times in chapters 4, 5, and 6 when we need the equality in distribution of stochastic integrals and we know that the pairs of integrand and integrator are equal in distribution. For the proof we need the following technical lemma.

Lemma 2.13. *Let $(Y^n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes with càdlàg paths that converges in ucp to a stochastic process Y with càdlàg paths. Then $(Y^n)_{n\in\mathbb{N}}$ also converges in distribution to Y.*

Proof. We apply Theorem 13.1 in [7] with $P_n \coloneqq P_{Y^n}$ and prove that the family $(P_n)_{n\in\mathbb{N}}$ of probability measures is tight and that the vector $(Y^n_{t_1}, \dots, Y^n_{t_k})$ converges in distribution to $(Y_{t_1}, \dots, Y_{t_k})$ for all $t_1 < \dots < t_k$, $k \in \mathbb{N}$. For the tightness of $(P_n)_{n\in\mathbb{N}}$ we verify the two conditions (13.4) and (13.5) in Theorem 13.2 in [7]. First, by the triangle inequality we have for all $T > 0$

$$\begin{aligned}
&\lim_{a\to\infty} \limsup_{n\to\infty} P_n(\|x\|_\infty \geq a) = \lim_{a\to\infty} \limsup_{n\to\infty} P\Big(\sup_{0\leq t\leq T} |Y^n_t| \geq a\Big) \\
&\leq \lim_{a\to\infty} \limsup_{n\to\infty} P\Big(\sup_{0\leq t\leq T} |Y^n_t - Y_t| + \sup_{0\leq t\leq T} |Y_t| \geq a\Big) \\
&\leq \underbrace{\lim_{a\to\infty} \limsup_{n\to\infty} P\left(\sup_{0\leq t\leq T} |Y^n_t - Y_t| \geq \frac{a}{2}\right)}_{=0} + \underbrace{\lim_{a\to\infty} P\left(\sup_{0\leq t\leq T} |Y_t| \geq \frac{a}{2}\right)}_{=0} = 0,
\end{aligned}$$

because $Y^n \xrightarrow{\text{ucp}} Y$, which shows (13.4) in [7]. Second, using the notation in (12.2),(12.6) and the inequality (12.7) in [7] we have

$$\begin{aligned} w'_{Y^n}(\delta) &\leq w_{Y^n}(2\delta) \\ &= \sup_{0\leq t\leq T-2\delta} \sup_{r,s\in[t,t+2\delta]} |Y_r^n - Y_s^n| \\ &\leq \sup_{0\leq t\leq T-2\delta} \sup_{r,s\in[t,t+2\delta]} \Big(|Y_r^n - Y_r| + |Y_r - Y_t| + |Y_t - Y_s| + |Y_s - Y_s^n|\Big) \\ &\leq 2 \sup_{0\leq s\leq T} |Y_s^n - Y_s| + 2 \sup_{0\leq t\leq T-2\delta} \sup_{s\in[t,t+2\delta]} |Y_s - Y_t| \end{aligned}$$

and thus for all $\varepsilon > 0$

$$\begin{aligned} &\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P_n(w'_x(\delta) \geq \varepsilon) = \lim_{\delta\downarrow 0} \limsup_{n\to\infty} P(w'_{Y^n}(\delta) \geq \varepsilon) \\ &\leq \lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\Big(2 \sup_{0\leq s\leq T} |Y_s^n - Y_s| + 2 \sup_{0\leq t\leq T-2\delta} \sup_{s\in[t,t+2\delta]} |Y_s - Y_t| \geq \varepsilon\Big) \\ &\leq \underbrace{\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\left(\sup_{0\leq s\leq T} |Y_s^n - Y_s| \geq \frac{\varepsilon}{4}\right)}_{=0} + \underbrace{\lim_{\delta\downarrow 0} P\left(\sup_{0\leq t\leq T-2\delta} \sup_{s\in[t,t+2\delta]} |Y_s - Y_t| \geq \frac{\varepsilon}{4}\right)}_{=0} \end{aligned}$$

because $Y^n \xrightarrow{\text{ucp}} Y$ and Y has càdlàg paths, which shows (13.5) in [7]. Finally, for all $\varepsilon > 0$

$$\begin{aligned} P\Big(\|(Y_{t_1}^n, \dots, Y_{t_k}^n) - (Y_{t_1}, \dots, Y_{t_k})\|_\infty \geq \varepsilon\Big) &= P\Big(\|(Y_{t_1}^n - Y_{t_1}, \dots, Y_{t_k}^n - Y_{t_k})\|_\infty \geq \varepsilon\Big) \\ &= P\Big(\max_{i=1,\dots,k} |Y_{t_i}^n - Y_{t_i}| \geq \varepsilon\Big) \\ &\leq P\Big(\sup_{0\leq s\leq t_k} |Y_s^n - Y_s| \geq \varepsilon\Big) \xrightarrow{n\to\infty} 0, \end{aligned}$$

because $Y^n \xrightarrow{\text{ucp}} Y$. Thus $(Y_{t_1}^n, \dots, Y_{t_k}^n) \xrightarrow{\text{P}} (Y_{t_1}, \dots, Y_{t_k})$ which implies $(Y_{t_1}^n, \dots, Y_{t_k}^n) \xrightarrow{\text{D}} (Y_{t_1}, \dots, Y_{t_k})$. □

Theorem 2.14. *Let $H, H' \in \mathbb{L}_{ucp}$ and let X, X' be semimartingales such that $(H, X) \overset{\text{D}}{=} (H', X')$. Then $J_X(H) \overset{\text{D}}{=} J_{X'}(H')$.*

Proof. Since $\mathbf{S}_{ucp}$ is dense in $\mathbb{L}_{ucp}$ there exist sequences $(H^n)_{n\in\mathbb{N}}, (H'^n)_{n\in\mathbb{N}}$ of simple predictable processes with $H^n \xrightarrow{\text{ucp}} H$ and $H'^n \xrightarrow{\text{ucp}} H'$. From the proof of Theorem II.10 in [31] it can be seen that the H^n and H'^n can be constructed directly from H and H' respectively. Thus $(H^n, X) \overset{\text{D}}{=} (H'^n, X')$ for all n and by (2.10) also $J_X(H^n) \overset{\text{D}}{=} J_{X'}(H'^n)$ for all n. On the other hand, since $J_X, J_{X'} : \mathbb{L}_{ucp} \longrightarrow \mathbb{D}_{ucp}$ are continuous, we have

$$J_X(H^n) \xrightarrow{\text{ucp}} J_X(H) \quad \text{and} \quad J_{X'}(H'^n) \xrightarrow{\text{ucp}} J_{X'}(H').$$

$J_X(H^n)$, $J_X(H)$ and $J_{X'}(H'^n)$, $J_{X'}(H')$ have càdlàg paths which by Lemma 2.13 implies

$$J_X(H^n) \xrightarrow{\text{D}} J_X(H) \quad \text{and} \quad J_{X'}(H'^n) \xrightarrow{\text{D}} J_{X'}(H')$$

Since $J_X(H^n) \overset{\text{D}}{=} J_{X'}(H'^n)$ for all n it follows that $J_X(H) \overset{\text{D}}{=} J_{X'}(H')$. □

2.2 Multi-dimensional stochastic integration

Stochastic integrals for and quadratic covariations of matrix-valued processes are reduced to the one-dimensional case by means of the matrix product. As in the one-dimensional case we denote by $\mathbb{D}$ the set of adapted processes in $\mathbb{R}^{n\times n}$ with càdlàg paths and by $\mathbb{L}$ the set of adapted processes in $\mathbb{R}^{n\times n}$ with càglàd paths.

Definition 2.15. A process X in $\mathbb{R}^{n\times n}$ is called **semimartingale** if each component $X^{(i,j)}$ is a semimartingale in the sense of Definition 2.5(b).

If X and Y are semimartingales in $\mathbb{R}^{n\times n}$ then the product XY is also a semimartingale because each component $(XY)^{(i,j)}$ is by definition of the matrix product and Theorem 2.12(a) the sum of one-dimensional semimartingales and thus a semimartingale by Theorem 2.6(a).

Definition 2.16. Let X be a semimartingale in $\mathbb{R}^{n\times n}$ and let $G, H \in \mathbb{L}$.

(a) The **left stochastic integral** is the process $G \cdot X = \left(\int_0^t G_u \, \mathrm{d}X_u\right)_{t\geq 0}$ whose (i,j)-component is defined by

$$\left(\int_0^t G_u \, \mathrm{d}X_u\right)^{(i,j)} := \sum_{k=1}^n \int_0^t G_u^{(i,k)} \, \mathrm{d}X_u^{(k,j)} \,. \tag{2.14a}$$

(b) The **right stochastic integral** is the process $X : H = \left(\int_0^t \mathrm{d}X_u \, H_u\right)_{t\geq 0}$ whose (i,j)-component is defined by

$$\left(\int_0^t \mathrm{d}X_u \, H_u\right)^{(i,j)} := \sum_{l=1}^n \int_0^t H_u^{(l,j)} \, \mathrm{d}X_u^{(i,l)} \,. \tag{2.14b}$$

(c) The **two-sided stochastic integral** is the process $G \cdot X : H = \left(\int_0^t G_u \, \mathrm{d}X_u \, H_u\right)_{t\geq 0}$ whose (i,j)-component is defined by

$$\left(\int_0^t G_u \, \mathrm{d}X_u \, H_u\right)^{(i,j)} := \sum_{k,l=1}^n \int_0^t G_u^{(i,k)} H_u^{(l,j)} \, \mathrm{d}X_u^{(k,l)} \,. \tag{2.14c}$$

$G \cdot X$ and $X : H$ and $G \cdot X : H$ are semimartingales because each of their components is by definition and Theorem 2.12(b) the sum of one-dimensional semimartingales and thus a semimartingale by Theorem 2.6(a).

Definition 2.17. Let X, Y be semimartingales in $\mathbb{R}^{n\times n}$.

(a) The **quadratic covariation** of X and Y is the process $[X,Y] = ([X,Y]_t)_{t\geq 0}$ whose (i,j)-component is defined by

$$[X,Y]_t^{(i,j)} := \sum_{k=1}^n \left[X^{(i,k)}, Y^{(k,j)}\right]_t \,. \tag{2.15a}$$

(b) The **path-by-path continuous part** or simply **continuous part** of the quadratic covariation $[X, Y]$ is the process $[X, Y]^c = ([X, Y]_t^c)_{t \geq 0}$ whose (i, j)-component is defined by

$$[X, Y]_t^{c,(i,j)} := \sum_{k=1}^{n} \left[X^{(i,k)}, Y^{(k,j)}\right]_t^c. \tag{2.15b}$$

Theorem 2.18 and Theorem 2.19 show that the defining formulas (2.11a) and (2.11b) for the one-dimensional quadratic covariation and its continuous part as well as the properties in Theorem 2.11 also hold true in the multi-dimensional case.

Theorem 2.18. *Let X, Y be semimartingales in $\mathbb{R}^{n \times n}$.*

(a) *The quadratic covariation of X and Y is for all $t \geq 0$ given by*

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{u-} \, \mathrm{d}Y_u - \int_0^t \mathrm{d}X_u \, Y_{u-}. \tag{2.16a}$$

(b) *The continuous part of the quadratic covariation $[X, Y]$ is for all $t \geq 0$ given by*

$$[X, Y]_t^c = [X, Y]_t - \sum_{0<s\leq t} \Delta X_s \Delta Y_s. \tag{2.16b}$$

Proof. (a) By (2.15a), (2.11a), (2.14a) and (2.14b) we have for all $i, j = 1, \ldots, n$

$$\begin{aligned}
[X, Y]_t^{(i,j)} &= \sum_{k=1}^{n} \left[X^{(i,k)}, Y^{(k,j)}\right]_t \\
&= \sum_{k=1}^{n} \left(X_t^{(i,k)} Y_t^{(k,j)} - X_0^{(i,k)} Y_0^{(k,j)} - \int_0^t X_{u-}^{(i,k)} \, \mathrm{d}Y_u^{(k,j)} - \int_0^t Y_{u-}^{(k,j)} \, \mathrm{d}X_u^{(i,k)} \right) \\
&= (X_t Y_t)^{(i,j)} - (X_0 Y_0)^{(i,j)} - \left(\int_0^t X_{u-} \, \mathrm{d}Y_u \right)^{(i,j)} - \left(\int_0^t \mathrm{d}X_u \, Y_{u-} \right)^{(i,j)} \\
&= \left(X_t Y_t - X_0 Y_0 - \int_0^t X_{u-} \, \mathrm{d}Y_u - \int_0^t \mathrm{d}X_u \, Y_{u-} \right)^{(i,j)}.
\end{aligned}$$

(b) By (2.15b) and (2.11b) we have for all $i, j = 1, \ldots, n$

$$\begin{aligned}
[X, Y]_t^{c,(i,j)} &= \sum_{k=1}^{n} \left[X^{(i,k)}, Y^{(k,j)}\right]_t^c \\
&= \sum_{k=1}^{n} \left(\left[X^{(i,k)}, Y^{(k,j)}\right]_t - \sum_{0<s\leq t} \Delta X_s^{(i,k)} \Delta Y_s^{(k,j)} \right) \\
&= \sum_{k=1}^{n} \left[X^{(i,k)}, Y^{(k,j)}\right]_t - \sum_{0<s\leq t} \sum_{k=1}^{n} (\Delta X_s)^{(i,k)} (\Delta Y_s)^{(k,j)} \\
&= [X, Y]_t^{(i,j)} - \sum_{0<s\leq t} (\Delta X_s \Delta Y_s)^{(i,j)} \\
&= \left([X, Y]_t - \sum_{0<s\leq t} \Delta X_s \Delta Y_s \right)^{(i,j)}.
\end{aligned}$$

□

Theorem 2.19. *Let X, Y, Z be semimartingales in $\mathbb{R}^{n\times n}$.*

(a) *The quadratic covariation $[X,Y]$ is a semimartingale and has paths of finite variation on compacts. It satisfies $[X,Y]_0 = \mathbf{0}$ and $\Delta[X,Y] = \Delta X \Delta Y$.*

(b) *If X is adapted, càdlàg and has paths of finite variation on compacts, then $[X,Z]^c = \mathbf{0}$.*

(c) *X, Y, Z satisfy the identities*

$$\Big[[X,Y],Z\Big]^c = \Big[[X,Y]^c,Z\Big]^c = \Big[[X,Y]^c,Z\Big] = \mathbf{0}, \tag{2.17a}$$

$$\Big[[X,Y],Z\Big]_t = \sum_{0<s\leq t} \Delta X_s \Delta Y_s \Delta Z_s. \tag{2.17b}$$

Proof. (a) $[X,Y]$ is a semimartingale because each component $[X,Y]^{(i,j)}$ is by definition and Theorem 2.11(a) the sum of one-dimensional semimartingales and thus a semimartingale by Theorem 2.6(a). For $i,j = 1,\dots,n$ we clearly have $[X,Y]_0^{(i,j)} = 0$ and

$$\begin{aligned}(\Delta[X,Y])^{(i,j)} = \Delta[X,Y]^{(i,j)} = \Delta \sum_{k=1}^n \Big[X^{(i,k)}, Y^{(k,j)}\Big] = \sum_{k=1}^n \Delta \Big[X^{(i,k)}, Y^{(k,j)}\Big] \\ = \sum_{k=1}^n \Delta X^{(i,k)} \Delta Y^{(k,j)} = \sum_{k=1}^n (\Delta X)^{(i,k)} (\Delta Y)^{(k,j)} = (\Delta X \Delta Y)^{(i,j)}.\end{aligned}$$

(b) Each component $X^{(i,k)}$ of X is adapted, càdlàg and has paths of finite variation on compacts and each component $Z^{(k,j)}$ of Z is by definition a semimartingale. Then by Theorem 2.11(b) we have $\Big[X^{(i,k)}, Z^{(k,j)}\Big]^c = 0$ which implies that

$$[X,Z]^{c,(i,j)} = \sum_{k=1}^n \Big[X^{(i,k)}, Z^{(k,j)}\Big]^c = 0.$$

(c) By (a) both $[X,Y]$ and $[X,Y]^c$ satisfy the assumptions of (b) which implies that $\Big[[X,Y],Z\Big]^c = \mathbf{0}$ and $\Big[[X,Y]^c,Z\Big]^c = \mathbf{0}$. Furthermore we have

$$\Big[[X,Y]^c,Z\Big]_t = \Big[[X,Y]^c,Z\Big]_t^c + \sum_{0<s\leq t} \Delta[X,Y]_s^c \Delta Z_s = \mathbf{0}$$

and by using $\Delta[X,Y] = \Delta X \Delta Y$ we have

$$\Big[[X,Y],Z\Big]_t = \Big[[X,Y],Z\Big]_t^c + \sum_{0<s\leq t} \Delta[X,Y]_s \Delta Z_s = \sum_{0<s\leq t} \Delta X_s \Delta Y_s \Delta Z_s.$$

□

Theorem 2.20 shows that the calculation rules for one-dimensional stochastic integrals and quadratic covariations of integral processes in Theorem 2.12 also hold true in the multi-dimensional case but the distinction between left and right stochastic integrals becomes important.

Theorem 2.20. *Let X, Y be semimartingales in $\mathbb{R}^{n\times n}$ and let $G, H \in \mathbb{L}$.*

(a) *$XY = X_0Y_0 + X_- \cdot Y + X : Y_- + [X,Y]$ which in integral notation becomes the* **integration by parts formula**

$$X_tY_t = X_0Y_0 + \int_0^t X_{u-}\,\mathrm{d}Y_u + \int_0^t \mathrm{d}X_u\,Y_{u-} + [X,Y]_t. \tag{2.18a}$$

(b) $H \cdot (G \cdot X) = (HG) \cdot X$ *and* $(Y : H) : G = Y : (HG)$ *which in integral notation become*

$$\int_0^t H_u \,\mathrm{d}\Big(\int_0^u G_v \,\mathrm{d}X_v\Big) = \int_0^t H_u G_u \,\mathrm{d}X_u\,, \tag{2.18b}$$

$$\int_0^t \mathrm{d}\Big(\int_0^u \mathrm{d}Y_v\, H_v\Big)\, G_u = \int_0^t \mathrm{d}Y_u\, H_u G_u. \tag{2.18c}$$

(c) $[G \cdot X, Y : H] = G \cdot [X, Y] : H$ *which in integral notation becomes*

$$\Big[\int_0^t G_u \,\mathrm{d}X_u\,, \int_0^t \mathrm{d}Y_u\, H_u\Big] = \int_0^t G_u \,\mathrm{d}[X,Y]_u\, H_u. \tag{2.18d}$$

(d) $[X : G, H \cdot Y] = [X : (GH), Y] = [X, (GH) \cdot Y]$ *which in integral notation becomes*

$$\Big[\int_0^t \mathrm{d}X_u\, G_u, \int_0^t H_u \,\mathrm{d}Y_u\Big] = \Big[\int_0^t \mathrm{d}X_u\, G_u H_u, Y_t\Big] = \Big[X_t, \int_0^t G_u H_u \,\mathrm{d}Y_u\Big]. \tag{2.18e}$$

Proof. (a) (2.18a) is equivalent to (2.16a).

(b) By (2.14a) and (2.13b) we have for all $i, j = 1\ldots, n$

$$\begin{aligned}
\Big(\int_0^t G_u \,\mathrm{d}\Big(\int_0^u H_v \,\mathrm{d}X_v\Big)\Big)^{(i,j)} &= \sum_{k=1}^n \int_0^t G_u^{(i,k)} \,\mathrm{d}\Big(\int_0^u H_v \,\mathrm{d}X_v\Big)^{(k,j)} \\
&= \sum_{k=1}^n \sum_{l=1}^n \int_0^t G_u^{(i,k)} \,\mathrm{d}\Big(\int_0^u H_v^{(k,l)} \,\mathrm{d}X_v^{(l,j)}\Big) \\
&= \sum_{l=1}^n \sum_{k=1}^n \int_0^t G_u^{(i,k)} H_u^{(k,l)} \,\mathrm{d}X_u^{(l,j)} \\
&= \sum_{l=1}^n \int_0^t (G_u H_u)^{(i,l)} \,\mathrm{d}X_u^{(l,j)} \\
&= \Big(\int_0^t G_u H_u \,\mathrm{d}X_u\Big)^{(i,j)}
\end{aligned}$$

and by (2.14b) and (2.13b) we have for all $i, j = 1, \ldots, n$

$$\begin{aligned}
\Big(\int_0^t \mathrm{d}\Big(\int_0^u \mathrm{d}Y_v\, H_v\Big)\, G_u\Big)^{(i,j)} &= \sum_{k=1}^n \int_0^t G_u^{(k,j)} \,\mathrm{d}\Big(\int_0^u \mathrm{d}Y_v\, H_v\Big)^{(i,k)} \\
&= \sum_{k=1}^n \sum_{l=1}^n \int_0^t G_u^{(k,j)} \,\mathrm{d}\Big(\int_0^u H_v^{(l,k)} \,\mathrm{d}Y_v^{(i,l)}\Big) \\
&= \sum_{l=1}^n \sum_{k=1}^n \int_0^t H_u^{(l,k)} G_u^{(k,j)} \,\mathrm{d}Y_u^{(i,l)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{n} \int_0^t (H_u G_u)^{(l,j)} \, \mathrm{d}Y_u^{(i,l)} \\
&= \left(\int_0^t \mathrm{d}Y_u \, H_u G_u \right)^{(i,j)} .
\end{aligned}
$$

(c) By (2.15a), (2.14a), (2.14b), (2.13c), and (2.14c) we have for all $i, j = 1, \ldots, n$

$$
\begin{aligned}
\left[\int_0^t G_u \, \mathrm{d}X_u \, , \int_0^t \mathrm{d}Y_u \, H_u \right]^{(i,j)} &= \sum_{k=1}^{n} \left[\left(\int_0^t G_u \, \mathrm{d}X_u \right)^{(i,k)} , \left(\int_0^t \mathrm{d}Y_u \, H_u \right)^{(k,j)} \right] \\
&= \sum_{k=1}^{n} \sum_{l,m=1}^{n} \left[\int_0^t G_u^{(i,l)} \, \mathrm{d}X_u^{(l,k)} \, , \int_0^t H_u^{(m,j)} \, \mathrm{d}Y_u^{(k,m)} \right] \\
&= \sum_{l,m=1}^{n} \sum_{k=1}^{n} \int_0^t G_u^{(i,l)} H_u^{(m,j)} \, \mathrm{d}\Big[X^{(l,k)}, Y^{(k,m)} \Big]_u \\
&= \sum_{l,m=1}^{n} \int_0^t G_u^{(i,l)} H_u^{(m,j)} \, \mathrm{d}[X, Y]_u^{(l,m)} \\
&= \left(\int_0^t G_u \, \mathrm{d}[X, Y]_u \, H_u \right)^{(i,j)} .
\end{aligned}
$$

(d) By (2.15a), (2.14a), (2.14b), and (2.13c) we have for all $i, j = 1, \ldots, n$

$$
\begin{aligned}
\left[\int_0^t \mathrm{d}X_u \, G_u , \int_0^t H_u \, \mathrm{d}Y_u \right]^{(i,j)} &= \sum_{k=1}^{n} \left[\left(\int_0^t \mathrm{d}X_u \, G_u \right)^{(i,k)} , \left(\int_0^t H_u \, \mathrm{d}Y_u \right)^{(k,j)} \right] \\
&= \sum_{k=1}^{n} \sum_{l,m=1}^{n} \left[\int_0^t G_u^{(l,k)} \, \mathrm{d}X_u^{(i,l)} \, , \int_0^t H_u^{(k,m)} \, \mathrm{d}Y_u^{(m,j)} \right] \\
&= \sum_{l,m=1}^{n} \sum_{k=1}^{n} \int_0^t G_u^{(l,k)} H_u^{(k,m)} \, \mathrm{d}\Big[X^{(i,l)}, Y^{(m,j)} \Big]_u \\
&= \sum_{l,m=1}^{n} \int_0^t (G_u H_u)^{(l,m)} \, \mathrm{d}\Big[X^{(i,l)}, Y^{(m,j)} \Big]_u \\
&= \sum_{l,m=1}^{n} \left[\int_0^t (G_u H_u)^{(l,m)} \, \mathrm{d}X_u^{(i,l)} \, , \int_0^t \mathrm{d}Y_u^{(m,j)} \right] \\
&= \sum_{m=1}^{n} \left[\left(\int_0^t \mathrm{d}X_u \, G_u H_u \right)^{(i,m)} , Y_t^{(m,j)} \right] \\
&= \left[\int_0^t \mathrm{d}X_u \, G_u H_u , Y_t \right]^{(i,j)}
\end{aligned}
$$

and analogously

$$
\begin{aligned}
\left[\int_0^t \mathrm{d}X_u\, G_u, \int_0^t H_u\, \mathrm{d}Y_u\right]^{(i,j)} &= \sum_{k=1}^n \left[\left(\int_0^t \mathrm{d}X_u\, G_u\right)^{(i,k)}, \left(\int_0^t H_u\, \mathrm{d}Y_u\right)^{(k,j)}\right] \\
&= \sum_{k=1}^n \sum_{l,m=1}^n \left[\int_0^t G_u^{(l,k)}\, \mathrm{d}X_u^{(i,l)}, \int_0^t H_u^{(k,m)}\, \mathrm{d}Y_u^{(m,j)}\right] \\
&= \sum_{l,m=1}^n \sum_{k=1}^n \int_0^t G_u^{(l,k)} H_u^{(k,m)}\, \mathrm{d}\left[X^{(i,l)}, Y^{(m,j)}\right]_u \\
&= \sum_{l,m=1}^n \int_0^t (G_u H_u)^{(l,m)}\, \mathrm{d}\left[X^{(i,l)}, Y^{(m,j)}\right]_u \\
&= \sum_{l,m=1}^n \left[\int_0^t \mathrm{d}X_u^{(i,l)}, \int_0^t (G_u H_u)^{(l,m)}\, \mathrm{d}Y_u^{(m,j)}\right] \\
&= \sum_{l=1}^n \left[X_t^{(i,l)}, \left(\int_0^t G_u H_u\, \mathrm{d}Y_u\right)^{(l,j)}\right] \\
&= \left[X_t, \int_0^t G_u H_u\, \mathrm{d}Y_u\right]^{(i,j)}.
\end{aligned}
$$

□

3 Stochastic Exponential

Chapter 3 deals with the stochastic exponential of a semimartingale which will be utilized throughout the entire thesis. It is a building block of the integral representation of MGOU processes, it describes the scaling of random selfsimilar processes and random stable hemigroups, and it converts random selfsimilar processes into stationary processes and vice versa by means of the LAMPERTI transform. In chapter 3.1 we give the definition and basic properties of the stochastic exponential and in chapter 3.2 we prove a closed form expression for the stochastic exponential of a matrix-valued semimartingale under commutativity conditions.

3.1 Definition and general properties

The stochastic exponential of a real-valued semimartingale X is motivated by the fact that the deterministic exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ solves the deterministic differential equation $\mathrm{d}y(t) = y(t)\,\mathrm{d}t$ with initial condition $y(0) = 1$. Replacing the real-valued function $y(t)$ with a real-valued stochastic process Y and the deterministic differential $\mathrm{d}t$ with the stochastic differential $\mathrm{d}X_t$ leads to the stochastic differential equation $\mathrm{d}Y_t = Y_t\,\mathrm{d}X_t$ with initial condition $Y_0 = 1$. The unique strong solution is the stochastic exponential $\mathsf{Exp}(X)$.

Due to the non-commutativity of matrix multiplication there are two types of stochastic exponentials in dimension $n \geq 2$ as one can consider the two stochastic differential equations $\mathrm{d}Y_t = Y_t\,\mathrm{d}X_t$ and $\mathrm{d}Y_t = \mathrm{d}X_t\,Y_t$.

Definition 3.1. Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a filtered and complete probability space such that $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{A}$. Let $X = (X_t)_{t\geq 0}$ be an $\mathcal{F}$-adapted semimartingale in $\mathbb{R}^{n\times n}$.

(a) The **left stochastic exponential** $\overleftarrow{\mathsf{Exp}}(X)$ of X is the $\mathcal{F}$-adapted càdlàg process that solves the stochastic differential equation

$$\mathrm{d}\overleftarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(X_{t-})\,\mathrm{d}X_t\ ,\ \overleftarrow{\mathsf{Exp}}(X_0) = I \quad\Longleftrightarrow\quad \overleftarrow{\mathsf{Exp}}(X_t) = I + \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}X_u\,. \tag{3.1a}$$

(b) The **right stochastic exponential** $\overrightarrow{\mathsf{Exp}}(X)$ of X is the $\mathcal{F}$-adapted càdlàg process that solves the stochastic differential equation

$$\mathrm{d}\overrightarrow{\mathsf{Exp}}(X_t) = \mathrm{d}X_t\,\overrightarrow{\mathsf{Exp}}(X_{t-})\ ,\ \overrightarrow{\mathsf{Exp}}(X_0) = I \quad\Longleftrightarrow\quad \overrightarrow{\mathsf{Exp}}(X_t) = I + \int_0^t \mathrm{d}X_u\,\overrightarrow{\mathsf{Exp}}(X_{u-}). \tag{3.1b}$$

By Theorem V.7 in [31] the solutions of (3.1a) and (3.1b) are unique in the strong sense and are $\mathcal{F}$-semimartingales. Thus $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ are semimartingales whenever X is a semimartingale.

Similar to a stochastic exponential one can define a stochastic logarithm although stochastic logarithms will not be as prevalent in this thesis as stochastic exponentials. Because of the non-commutativity of matrix multiplication there are also two types of stochastic logarithms in dimension $n \geq 2$.

Definition 3.2. Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a filtered and complete probability space such that $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{A}$. Let $X = (X_t)_{t\geq 0}$ be an $\mathcal{F}$-adapted semimartingale in $\mathsf{GL}_n(\mathbb{R})$.

(a) The **left stochastic logarithm** $\overleftarrow{\mathsf{Log}}(X)$ of X is defined by

$$\overleftarrow{\mathsf{Log}}(X_t) \coloneqq \int_0^t X_{u-}^{-1}\,\mathrm{d}X_u \quad \Longleftrightarrow \quad \mathrm{d}\overleftarrow{\mathsf{Log}}(X_t) = X_{t-}^{-1}\,\mathrm{d}X_t\ ,\ \overleftarrow{\mathsf{Log}}(X_0) = \mathbf{0}. \tag{3.2a}$$

(b) The **right stochastic logarithm** $\overrightarrow{\mathsf{Log}}(X)$ of X is defined by

$$\overrightarrow{\mathsf{Log}}(X_t) \coloneqq \int_0^t \mathrm{d}X_u\,X_{u-}^{-1} \quad \Longleftrightarrow \quad \mathrm{d}\overrightarrow{\mathsf{Log}}(X_t) = \mathrm{d}X_t\,X_{t-}^{-1}\ ,\ \overrightarrow{\mathsf{Log}}(X_0) = \mathbf{0}. \tag{3.2b}$$

By Theorem II.19 in [31] the integral processes $X_-^{-1}\cdot X$ and $X : X_-^{-1}$ are $\mathcal{F}$-semimartingales. Thus $\overleftarrow{\mathsf{Log}}(X)$ and $\overrightarrow{\mathsf{Log}}(X)$ are semimartingales whenever X is a semimartingale.

Oftentimes a stochastic exponential needs to be inverted which means that $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ need to be processes in $\mathsf{GL}_n(\mathbb{R})$. By Theorem 1 in [21] this holds true if and only if

$$\det(I + \Delta X_t) \neq 0 \quad \text{for all } t. \tag{3.3}$$

In this case the processes $\overleftarrow{\mathsf{Exp}}(X)^{-1}$ and $\overrightarrow{\mathsf{Exp}}(X)^{-1}$ as well as $\overleftarrow{\mathsf{Log}}\big(\overleftarrow{\mathsf{Exp}}(X)\big)$ and $\overrightarrow{\mathsf{Log}}\big(\overrightarrow{\mathsf{Exp}}(X)\big)$ are well-defined and the stochastic exponential and stochastic logarithm show properties that are known from the deterministic exponential and logarithm functions.

Lemma 3.3. (a) *Let $X = (X_t)_{t\geq 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ with $X_0 = \mathbf{0}$ which satisfies (3.3). Then for all $0 \leq s \leq t$*

$$\overleftarrow{\mathsf{Exp}}(X_t - X_s) = \overleftarrow{\mathsf{Exp}}(X_s)^{-1}\overleftarrow{\mathsf{Exp}}(X_t), \tag{3.4a}$$
$$\overrightarrow{\mathsf{Exp}}(X_t - X_s) = \overrightarrow{\mathsf{Exp}}(X_t)\overrightarrow{\mathsf{Exp}}(X_s)^{-1}, \tag{3.4b}$$
$$\overleftarrow{\mathsf{Log}}\big(\overleftarrow{\mathsf{Exp}}(X_t)\big) = X_t, \tag{3.4c}$$
$$\overrightarrow{\mathsf{Log}}\big(\overrightarrow{\mathsf{Exp}}(X_t)\big) = X_t. \tag{3.4d}$$

(b) *Let $X = (X_t)_{t\geq 0}$ be a semimartingale in $\mathsf{GL}_n(\mathbb{R})$ with $X_0 = I$. Then for all $0 \leq s \leq t$*

$$\overleftarrow{\mathsf{Log}}(X_s^{-1}X_t) = \overleftarrow{\mathsf{Log}}(X_t) - \overleftarrow{\mathsf{Log}}(X_s), \tag{3.5a}$$
$$\overrightarrow{\mathsf{Log}}(X_tX_s^{-1}) = \overrightarrow{\mathsf{Log}}(X_t) - \overrightarrow{\mathsf{Log}}(X_s), \tag{3.5b}$$
$$\overleftarrow{\mathsf{Exp}}\big(\overleftarrow{\mathsf{Log}}(X_t)\big) = X_t, \tag{3.5c}$$
$$\overrightarrow{\mathsf{Exp}}\big(\overrightarrow{\mathsf{Log}}(X_t)\big) = X_t. \tag{3.5d}$$

Proof. (a) To prove (3.4a) we write $Z \coloneqq \overleftarrow{\mathsf{Exp}}(X)$, $Z_{s,t} \coloneqq Z_s^{-1}Z_t = \overleftarrow{\mathsf{Exp}}(X_s)^{-1}\overleftarrow{\mathsf{Exp}}(X_t)$ and $t = s + h$ with $h \geq 0$. Then

$$\begin{aligned} Z_{s,s+h} &= \left(I + \int_0^s Z_{u-}\,\mathrm{d}X_u\right)^{-1}\left(I + \int_0^s Z_{u-}\,\mathrm{d}X_u + \int_s^{s+h} Z_{u-}\,\mathrm{d}X_u\right) \\ &= I + Z_s^{-1}\int_s^{s+h} Z_{u-}\,\mathrm{d}X_u = I + \int_0^h Z_s^{-1}Z_{(s+u)-}\,\mathrm{d}X_{s+u} \end{aligned}$$

$$= I + \int_0^h Z_{s,(s+u)-}\, \mathrm{d}(X_{s+u} - X_s)\,.$$

Since $Z_{s,s+0} = I$ and the left stochastic exponential is the unique solution of (3.1a) it follows that

$$\overleftarrow{\mathsf{Exp}}(X_s)^{-1}\overleftarrow{\mathsf{Exp}}(X_t) = Z_{s,t} = Z_{s,s+h} = \overleftarrow{\mathsf{Exp}}(X_{s+h} - X_s) = \overleftarrow{\mathsf{Exp}}(X_t - X_s).$$

To prove (3.4b) we write $Z := \overrightarrow{\mathsf{Exp}}(X)$, $Z_{s,t} := Z_t Z_s^{-1} = \overrightarrow{\mathsf{Exp}}(X_t)\overrightarrow{\mathsf{Exp}}(X_s)^{-1}$ and $t = s + h$ with $h \geq 0$. Then

$$\begin{aligned}
Z_{s,s+h} &= \Bigg(I + \int_0^s \mathrm{d}X_u\, Z_{u-} + \int_s^{s+h} \mathrm{d}X_u\, Z_{u-}\Bigg)\Bigg(I + \int_0^s \mathrm{d}X_u\, Z_{u-}\Bigg)^{-1} \\
&= I + \int_s^{s+h} \mathrm{d}X_u\, Z_{u-} Z_s^{-1} = I + \int_0^h \mathrm{d}X_{s+u}\, Z_{(s+u)-} Z_s^{-1} \\
&= I + \int_0^h \mathrm{d}(X_{s+u} - X_s)\, Z_{s,(s+u)-}.
\end{aligned}$$

Since $Z_{s,s+0} = I$ and the right stochastic exponential is the unique solution of (3.1b) it follows that

$$\overrightarrow{\mathsf{Exp}}(X_t)\overrightarrow{\mathsf{Exp}}(X_s)^{-1} = Z_{s,t} = Z_{s,s+h} = \overrightarrow{\mathsf{Exp}}(X_{s+h} - X_s) = \overrightarrow{\mathsf{Exp}}(X_t - X_s).$$

For the proof of (3.4c) the integral form of (3.2a) and the differential form of (3.1a) together yield

$$\begin{aligned}
\overleftarrow{\mathsf{Log}}\Big(\overleftarrow{\mathsf{Exp}}(X_t)\Big) &= \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})^{-1}\, \mathrm{d}\overleftarrow{\mathsf{Exp}}(X_u) = \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})^{-1}\overleftarrow{\mathsf{Exp}}(X_{u-})\, \mathrm{d}X_u \\
&= \int_0^t \mathrm{d}X_u = X_t - X_0 = X_t.
\end{aligned}$$

For the proof of (3.4d) the integral form of (3.2b) and the differential form of (3.1b) together yield

$$\begin{aligned}
\overrightarrow{\mathsf{Log}}\Big(\overrightarrow{\mathsf{Exp}}(X_t)\Big) &= \int_0^t \mathrm{d}\overrightarrow{\mathsf{Exp}}(X_u)\, \overrightarrow{\mathsf{Exp}}(X_{u-})^{-1} = \int_0^t \mathrm{d}X_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_{u-})^{-1} \\
&= \int_0^t \mathrm{d}X_u = X_t - X_0 = X_t.
\end{aligned}$$

(b) For the proof of (3.5a) the integral from of (3.2a) yields

$$\overleftarrow{\mathsf{Log}}(X_t) - \overleftarrow{\mathsf{Log}}(X_s) = \int_s^t X_{u-}^{-1}\, \mathrm{d}X_u = \int_0^{t-s} X_{(s+u)-}^{-1}\, \mathrm{d}X_{s+u}$$

$$= \int_0^{t-s} \big(X_s^{-1} X_{(s+u)-}\big)^{-1} \, \mathrm{d}\big(X_s^{-1} X_{s+u}\big)$$
$$= \overleftarrow{\mathsf{Log}}\big(X_s^{-1} X_{s+t-s}\big) = \overleftarrow{\mathsf{Log}}\big(X_s^{-1} X_t\big).$$

For the proof of (3.5b) the integral form of (3.2b) yields

$$\overrightarrow{\mathsf{Log}}(X_t) - \overrightarrow{\mathsf{Log}}(X_s) = \int_s^t \mathrm{d}X_u \, X_{u-}^{-1} = \int_0^{t-s} \mathrm{d}X_{s+u} \, X_{(s+u)-}^{-1}$$
$$= \int_0^{t-s} \mathrm{d}\big(X_{s+u} X_s^{-1}\big) \, \big(X_{(s+u)-} X_s^{-1}\big)^{-1}$$
$$= \overrightarrow{\mathsf{Log}}\big(X_{s+t-s} X_s^{-1}\big) = \overrightarrow{\mathsf{Log}}\big(X_t X_s^{-1}\big).$$

For the proof of (3.5c) the differential form of (3.2a) and the integral form of (3.1a) respectively yield

$$X_t = I + X_t - X_0 = I + \int_0^t \mathrm{d}X_u = I + \int_0^t X_{u-} X_{u-}^{-1} \, \mathrm{d}X_u = I + \int_0^t X_{u-} \, \mathrm{d}\overleftarrow{\mathsf{Log}}(X_u)$$

and

$$\overleftarrow{\mathsf{Exp}}\Big(\overleftarrow{\mathsf{Log}}(X_t)\Big) = I + \int_0^t \overleftarrow{\mathsf{Exp}}\Big(\overleftarrow{\mathsf{Log}}(X_{u-})\Big) \, \mathrm{d}\overleftarrow{\mathsf{Log}}(X_u)\,.$$

Since $X_0 = I$ and $\overleftarrow{\mathsf{Exp}}\Big(\overleftarrow{\mathsf{Log}}(X_0)\Big) = I$ it follows from the uniqueness of the solution of (3.1a) that $X_t = \overleftarrow{\mathsf{Exp}}\Big(\overleftarrow{\mathsf{Log}}(X_t)\Big)$ for all $t \geq 0$.
For the proof of (3.5d) the differential form of (3.2b) and the integral form of (3.1b) respectively yield

$$X_t = I + X_t - X_0 = I + \int_0^t \mathrm{d}X_u = I + \int_0^t \mathrm{d}X_u \, X_{u-}^{-1} X_{u-} = I + \int_0^t \mathrm{d}\overrightarrow{\mathsf{Log}}(X_u) \, X_{u-}$$

and

$$\overrightarrow{\mathsf{Exp}}\Big(\overrightarrow{\mathsf{Log}}(X_t)\Big) = I + \int_0^t \mathrm{d}\overrightarrow{\mathsf{Log}}(X_u) \, \overrightarrow{\mathsf{Exp}}\Big(\overrightarrow{\mathsf{Log}}(X_{u-})\Big).$$

Since $X_0 = I$ and $\overrightarrow{\mathsf{Exp}}\Big(\overrightarrow{\mathsf{Log}}(X_0)\Big) = I$ it follows from the uniqueness of the solution of (3.1b) that $X_t = \overrightarrow{\mathsf{Exp}}\Big(\overrightarrow{\mathsf{Log}}(X_t)\Big)$ for all $t \geq 0$.

□

The inverse of the left respectively right stochastic exponential of a semimartingale X is the right respectively left stochastic exponential of a semimartingale U which can be directly calculated from X.

Lemma 3.4. *Let $X = (X_t)_{t \geq 0}$ be a semimartingale in $\mathbb{R}^{n \times n}$ which satisfies (3.3) and let $U = (U_t)_{t \geq 0}$ be defined by*

$$U_t = -X_t + [X, X]_t^c + \sum_{0 < s \leq t} \Big((I + \Delta X_s)^{-1} - I + \Delta X_s \Big). \tag{3.6}$$

Then for all $t \geq 0$

$$\overleftarrow{\mathsf{Exp}}(X_t)^{-1} = \overrightarrow{\mathsf{Exp}}(U_t), \tag{3.7a}$$

$$\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overleftarrow{\mathsf{Exp}}(U_t), \tag{3.7b}$$

$$\det(I + \Delta U_t) = \frac{1}{\det(I + \Delta X_t)}. \tag{3.7c}$$

Proof. (3.7a) has already been proven in Theorem 1 in [21]. (3.7b) can be proven similarly: By definition of U we first have $[U, X]_t^c = -[X, X]_t^c$ and

$$\Delta U_t = -\Delta X_t + (I + \Delta X_t)^{-1} - I + \Delta X_t = (I + \Delta X_t)^{-1} - I. \tag{3.8}$$

From this it follows

$$\begin{aligned} X_t + U_t + [U, X]_t &= X_t - X_t + [X, X]_t^c + \sum_{0<s\leq t} \Big((I + \Delta X_s)^{-1} - I + \Delta X_s \Big) \\ &\quad + [U, X]_t^c + \sum_{0<s\leq t} (\Delta U_s)(\Delta X_s) \\ &= X_t - X_t + [X, X]_t^c - [X, X]_t^c \\ &\quad + \sum_{0<s\leq t} \Big((I + \Delta X_s)^{-1} - I + \Delta X_s + (I + \Delta X_s)^{-1} \Delta X_s - \Delta X_s \Big) \\ &= \sum_{0<s\leq t} \Big((I + \Delta X_s)^{-1} (I + \Delta X_s) - I + \Delta X_s - \Delta X_s \Big) = \mathbf{0}. \end{aligned}$$

The integration by parts formula (2.18a) together with (3.1a), (3.1b), and (2.18d) now yield

$$\begin{aligned} \overleftarrow{\mathsf{Exp}}(U_t)\overrightarrow{\mathsf{Exp}}(X_t) &= \overleftarrow{\mathsf{Exp}}(U_0)\overrightarrow{\mathsf{Exp}}(X_0) + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}\overrightarrow{\mathsf{Exp}}(X_u) + \int_0^t \mathrm{d}\overleftarrow{\mathsf{Exp}}(U_u)\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\ &\quad + \Big[\overleftarrow{\mathsf{Exp}}(U), \overrightarrow{\mathsf{Exp}}(X) \Big]_t \\ &= I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}X_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}U_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\ &\quad + \left[I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}U_u\, , I + \int_0^t \mathrm{d}X_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \right] \\ &= I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}X_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}U_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\ &\quad + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}[U, X]_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\ &= I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d} \underbrace{(X_u + U_u + [U, X]_u)}_{=\mathbf{0}} \overrightarrow{\mathsf{Exp}}(X_{u-}) = I \end{aligned}$$

which shows that $\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overleftarrow{\mathsf{Exp}}(U_t)$. (3.7c) follows from (3.8) because

$$\det(I + \Delta U_t) = \det\Big((I + \Delta X_t)^{-1} \Big) = \frac{1}{\det(I + \Delta X_t)}.$$

□

Behme and Lindner have shown that the stochastic exponential and stochastic logarithm preserve the properties of Lévy processes.

Proposition 3.5. [4, Proposition 2.4]

(a) *Let* $X = (X_t)_{t\geq 0}$ *be a* Lévy *process in* $\mathbb{R}^{n\times n}$ *which satisfies (3.3). Then* $\overleftarrow{\mathsf{Exp}}(X)$ *and* $\overrightarrow{\mathsf{Exp}}(X)$ *are left respectively right* Lévy *processes in* $\mathsf{GL}_n(\mathbb{R})$.

(b) *Let* $X = (X_t)_{t\geq 0}$ *be a left or right* Lévy *process in* $\mathsf{GL}_n(\mathbb{R})$. *Then* $\overleftarrow{\mathsf{Log}}(X)$ *and* $\overrightarrow{\mathsf{Log}}(X)$ *are* Lévy *processes in* $\mathbb{R}^{n\times n}$ *which satisfy (3.3).*

The stochastic exponential and stochastic logarithm also preserve the properties of semi-Lévy processes in that stationary increments of X can be weakened to periodically stationary increments and $\overleftarrow{\mathsf{Exp}}(X)$, $\overrightarrow{\mathsf{Exp}}(X)$, $\overleftarrow{\mathsf{Log}}(X)$, $\overrightarrow{\mathsf{Log}}(X)$ still have periodically stationary increments. This enables the study of semiselfsimilarity and semistable hemigroups in conjunction with semi-Lévy processes in chapters 5 and 6.

Theorem 3.6. (a) *Let* $X = (X_t)_{t\geq 0}$ *be a* p*-semi-*Lévy *process in* $\mathbb{R}^{n\times n}$ *for some* $p > 0$ *which satisfies (3.3). Then* $\overleftarrow{\mathsf{Exp}}(X)$ *and* $\overrightarrow{\mathsf{Exp}}(X)$ *are right respectively left* p*-semi-*Lévy *processes in* $\mathsf{GL}_n(\mathbb{R})$.

(b) *Let* $X = (X_t)_{t\geq 0}$ *be a right or left* p*-semi-*Lévy *process in* $\mathsf{GL}_n(\mathbb{R})$ *for some* $p > 0$. *Then* $\overleftarrow{\mathsf{Log}}(X)$ *and* $\overrightarrow{\mathsf{Log}}(X)$ *are* p*-semi-*Lévy *processes in* $\mathbb{R}^{n\times n}$ *which satisfy (3.3).*

Proof. In the proof of Proposition 2.4 in [4], which goes back to the proof of Proposition 5.5 in [2], the stationarity of the increments of X is only needed in order to show that $\overleftarrow{\mathsf{Exp}}(X)$, $\overrightarrow{\mathsf{Exp}}(X)$, $\overleftarrow{\mathsf{Log}}(X)$, $\overrightarrow{\mathsf{Log}}(X)$ also have stationary increments, but not for the proof of the remaining properties of a Lévy process. Thus we only need to prove that p-stationary increments of X result in p-stationary increments of $\overleftarrow{\mathsf{Exp}}(X)$, $\overrightarrow{\mathsf{Exp}}(X)$, $\overleftarrow{\mathsf{Log}}(X)$, $\overrightarrow{\mathsf{Log}}(X)$ while the remaining properties of a p-semi-Lévy process can be shown similarly to Proposition 5.5 in [2].

(a) (3.4a) and (3.4b) yield for all $t \geq 0$

$$\overleftarrow{\mathsf{Exp}}(X_p)^{-1}\overleftarrow{\mathsf{Exp}}(X_{t+p}) = \overleftarrow{\mathsf{Exp}}(X_{t+p} - X_p) \overset{\mathrm{D}}{=} \overleftarrow{\mathsf{Exp}}(X_t - X_0) = \overleftarrow{\mathsf{Exp}}(X_t),$$
$$\overrightarrow{\mathsf{Exp}}(X_{t+p})\overrightarrow{\mathsf{Exp}}(X_p)^{-1} = \overrightarrow{\mathsf{Exp}}(X_{t+p} - X_p) \overset{\mathrm{D}}{=} \overrightarrow{\mathsf{Exp}}(X_t - X_0) = \overrightarrow{\mathsf{Exp}}(X_t).$$

(b) (3.5a) and (3.5b) yield for all $t \geq 0$

$$\overleftarrow{\mathsf{Log}}(X_{t+p}) - \overleftarrow{\mathsf{Log}}(X_p) = \overleftarrow{\mathsf{Log}}(X_p^{-1}X_{t+p}) \overset{\mathrm{D}}{=} \overleftarrow{\mathsf{Log}}(X_0^{-1}X_t) = \overleftarrow{\mathsf{Log}}(X_t),$$
$$\overrightarrow{\mathsf{Log}}(X_{t+p}) - \overrightarrow{\mathsf{Log}}(X_p) = \overrightarrow{\mathsf{Log}}(X_{t+p}X_p^{-1}) \overset{\mathrm{D}}{=} \overrightarrow{\mathsf{Log}}(X_tX_0^{-1}) = \overrightarrow{\mathsf{Log}}(X_t).$$

□

3.2 Closed form expression

In dimension $n = 1$ there is no need to specify the order of multiplication and the stochastic exponential $\mathsf{Exp}(X)$ of a semimartingale X can be written in the closed form

$$\mathsf{Exp}(X_t) = \exp\left(X_t - \frac{1}{2}[X,X]_t\right) \prod_{0<s\leq t} (1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right).$$

See Theorem II.37 in [31] for a proof of this formula. Behme and Lindner mention in their paper [4] that for dimension $n \geq 2$ no such closed form expression is known. Yan however proved in [34] that the stochastic exponential of a continuous semimartingale in dimension $n \geq 2$, under the assumption that various processes commute in the sense that their left and right stochastic integral with respect to each other are equal, can be written in the closed form

$$\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) = \exp\left(X_t - \frac{1}{2}[X,X]_t\right).$$

We now prove a closed form expression for the stochastic exponential of a semimartingale in dimension $n \geq 2$ which generalizes both of the above mentioned results. We follow the proof of Yan but allow the semimartingale to also have jump parts which requires additional commutativity assumptions involving the jump parts of X.

Definition 3.7. [34, Definition 1.1] Let $H = (H_t)_{t\geq 0}$ and $Z = (Z_t)_{t\geq 0}$ be semimartingales in $\mathbb{R}^{n\times n}$. Then the pair (H, Z) is called **commutative** if for all $t \geq 0$

$$\int_0^t H_{u-}\,\mathrm{d}Z_u = \int_0^t \mathrm{d}Z_u\,H_{u-} \tag{3.9}$$

which can also be written as $(H_- \cdot Z)_t = (Z : H_-)_t$ or as $H_{t-}\,\mathrm{d}Z_t = \mathrm{d}Z_t\,H_{t-}$.

Note that since in [34] only continuous semimartingales are considered we have to integrate H_{u-} rather than H_u to ensure that we integrate a process with càglàd paths.

Definition 3.8. [34, Definition 6.1] Let $r \in (0,\infty) \cup \{\infty\}$, let $f : B_r(z_0) \longrightarrow \mathbb{C}$ be analytic with Taylor series $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ in $B_r(z_0)$ and let $A \in \mathbb{R}^{n\times n}$ with $\|A - z_0 I\| < r$. Then the matrix

$$f(A) := \sum_{k=0}^{\infty} a_k(A - z_0 I)^k \tag{3.10}$$

is well-defined because the series converges absolutely.

The following proposition is a collection of the results in [34] that lead to the closed form expression for the stochastic exponential of a continuous semimartingale in dimension $n \geq 2$.

Proposition 3.9. [34, Theorem 4.1,4.2,6.1] *Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale in $\mathbb{R}^{n\times n}$ such that (X,X) and $(X,[X,X])$ are commutative.*

(a) *For all $k > 1$*

$$\mathrm{d}X_t^k = kX_t^{k-1}\,\mathrm{d}X_t + \frac{k(k-1)}{2}X_t^{k-2}\,\mathrm{d}[X,X]_t\,, \tag{3.11a}$$

$$\mathrm{d}X_t^k = k\,\mathrm{d}X_t\,X_t^{k-1} + \frac{k(k-1)}{2}\,\mathrm{d}[X,X]_t\,X_t^{k-2}. \tag{3.11b}$$

(b) *For any analytic function $f : B_r(z_0) \longrightarrow \mathbb{C}$*

$$\mathrm{d}f(X_t) = f'(X_t)\,\mathrm{d}X_t + \frac{1}{2}f''(X_t)\,\mathrm{d}[X,X]_t\,, \tag{3.12a}$$

$$\mathrm{d}f(X_t) = \mathrm{d}X_t\,f'(X_t) + \frac{1}{2}\,\mathrm{d}[X,X]_t\,f''(X_t). \tag{3.12b}$$

(c) *If additionally* $X_0 = \mathbf{0}$ *and* $V := X + \frac{1}{2}[X,X]$*, then* $Y = \exp(X)$ *is the unique solution to both the stochastic integral equations*

$$Y_t = I + \int_0^t Y_u \, \mathrm{d}V_u \quad \textit{and} \quad Y_t = I + \int_0^t \mathrm{d}V_u \, Y_u. \tag{3.13}$$

(d) *If additionally* $([X,X],X)$ *and* $([X,X],[X,X])$ *are commutative as well, then*

$$\overleftarrow{\mathsf{Exp}}(X) = \overrightarrow{\mathsf{Exp}}(X) = \exp\left(X - \frac{1}{2}[X,X]\right). \tag{3.14}$$

The next four theorems generalize Proposition 3.9(a)-(d) and lead to a closed form expression for the stochastic exponential of a semimartingale in dimension $n \geq 2$.

Theorem 3.10. *Let* $X = (X_t)_{t\geq 0}$ *be a semimartingale in* $\mathbb{R}^{n\times n}$ *such that* (X,X) *and* $(X,[X,X]^c)$ *are commutative,* $X_0 = \mathbf{0}$*, and* $X_t X_{t-} = X_{t-} X_t$ *for all* $t \geq 0$*. Then for all* $k > 1$

$$X_t^k = k \int_0^t X_{u-}^{k-1} \, \mathrm{d}X_u + \frac{k(k-1)}{2} \int_0^t X_{u-}^{k-2} \, \mathrm{d}[X,X]_u^c + \sum_{0<s\leq t} \left(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s\right), \tag{3.15a}$$

$$X_t^k = k \int_0^t \mathrm{d}X_u \, X_{u-}^{k-1} + \frac{k(k-1)}{2} \int_0^t \mathrm{d}[X,X]_u^c \, X_{u-}^{k-2} + \sum_{0<s\leq t} \left(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s\right). \tag{3.15b}$$

Proof. We first prove (3.15a) by induction over k. For $k = 2$, since (X_-, X) is commutative, the integration by parts formula yields

$$\begin{aligned} X_t^2 &= \int_0^t X_{u-} \, \mathrm{d}X_u + \int_0^t \mathrm{d}X_u \, X_{u-} + [X,X]_t \\ &= \int_0^t X_{u-} \, \mathrm{d}X_u + \int_0^t X_{u-} \, \mathrm{d}X_u + [X,X]_t^c + \sum_{0<s\leq t} (\Delta X_s)^2 \\ &= 2 \int_0^t X_{u-} \, \mathrm{d}X_u + \int_0^t \mathrm{d}[X,X]_u^c + \sum_{0<s\leq t} \left(\Delta X_s^2 - 2X_{s-}\Delta X_s\right). \end{aligned}$$

In the last step we used the additional assumption $X_s X_{s-} = X_{s-} X_s$ for all $s \geq 0$ to obtain

$$\begin{aligned} (\Delta X_s)^2 &= (X_s - X_{s-})^2 = X_s^2 - X_s X_{s-} - X_{s-} X_s + X_{s-}^2 \\ &= X_s^2 - X_{s-}^2 + 2X_{s-}^2 - 2X_s X_{s-} = \Delta X_s^2 - 2X_{s-}\Delta X_s. \end{aligned} \tag{3.16}$$

For the induction step we assume that (3.15a) holds for some $k > 1$. Then

$$\begin{aligned} [X^k, X]_t^c = &\left[k \int_0^t X_{u-}^{k-1} \, \mathrm{d}X_u + \frac{k(k-1)}{2} \int_0^t X_{u-}^{k-2} \, \mathrm{d}[X,X]_u^c \, , X_t\right]^c \\ &+ \underbrace{\left[\sum_{0<s\leq t} \left(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s\right), X_t\right]^c}_{=\mathbf{0}} \end{aligned}$$

$$= k\int_0^t X_{u-}^{k-1}\,\mathrm{d}[X,X]_u^c + \frac{k(k-1)}{2}\int_0^t X_{u-}^{k-2}\,\underbrace{\mathrm{d}\Big[[X,X]^c, X\Big]_u^c}_{=\mathbf{0}}$$

$$= k\int_0^t X_{u-}^{k-1}\,\mathrm{d}[X,X]_u^c$$

and the integration by parts formula yields

$$\begin{aligned}
X_t^{k+1} = X_t^k X_t &= \int_0^t X_{u-}^k\,\mathrm{d}X_u + \int_0^t \mathrm{d}X_u^k\,X_{u-} + [X^k,X]_t \\
&= \int_0^t X_{u-}^k\,\mathrm{d}X_u + \int_0^t \left(kX_{u-}^{k-1}\,\mathrm{d}X_u + \frac{k(k-1)}{2}X_{u-}^{k-2}\,\mathrm{d}[X,X]_u^c\right)X_{u-} \\
&\quad + \int_0^t \mathrm{d}\left(\sum_{0<s\le u}\left(\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s\right)\right)X_{u-} + [X^k,X]_t^c + \sum_{0<s\le t}(\Delta X_s^k)(\Delta X_s) \\
&= \int_0^t X_{u-}^k\,\mathrm{d}X_u + k\int_0^t X_{u-}^k\,\mathrm{d}X_u + \frac{k(k-1)}{2}\int_0^t X_{u-}^{k-1}\,\mathrm{d}[X,X]_u^c \\
&\quad + \sum_{0<s\le t}\left(\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s\right)X_{s-} + k\int_0^t X_{u-}^{k-1}\,\mathrm{d}[X,X]_u^c + \sum_{0<s\le t}(\Delta X_s^k)(\Delta X_s) \\
&= (k+1)\int_0^t X_{u-}^k\,\mathrm{d}X_u + \frac{k(k+1)}{2}\int_0^t X_{u-}^{k-1}\,\mathrm{d}[X,X]_u^c \\
&\quad + \sum_{0<s\le t}\left((\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s)X_{s-} + (\Delta X_s^k)(\Delta X_s)\right) \\
&= (k+1)\int_0^t X_{u-}^k\,\mathrm{d}X_u + \frac{k(k+1)}{2}\int_0^t X_{u-}^{k-1}\,\mathrm{d}[X,X]_u^c \\
&\quad + \sum_{0<s\le t}\left(\Delta X_s^{k+1} - (k+1)X_{s-}^k\Delta X_s\right).
\end{aligned}$$

Here the last step follows from

$$\begin{aligned}
&(\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s)X_{s-} + (\Delta X_s^k)(\Delta X_s) \\
&= \left(X_s^k - X_{s-}^k - kX_{s-}^{k-1}(X_s - X_{s-})\right)X_{s-} + (X_s^k - X_{s-}^k)(X_s - X_{s-}) \\
&= X_s^k X_{s-} - X_{s-}^{k+1} - kX_{s-}^{k-1}X_sX_{s-} + kX_{s-}^{k+1} + X_s^{k+1} - X_s^kX_{s-} - X_{s-}^kX_s + X_{s-}^{k+1} \\
&= -kX_{s-}^{k-1}X_sX_{s-} + kX_{s-}^{k+1} + X_s^{k+1} - X_{s-}^kX_s \\
&= X_s^{k+1} - X_{s-}^{k+1} + (k+1)X_{s-}^{k+1} - kX_{s-}^{k-1}X_sX_{s-} - X_{s-}^kX_s \\
&= X_s^{k+1} - X_{s-}^{k+1} + (k+1)X_{s-}^{k+1} - (k+1)X_{s-}^kX_s \\
&= X_s^{k+1} - X_{s-}^{k+1} - (k+1)X_{s-}^k(X_s - X_{s-}) \\
&= \Delta X_s^{k+1} - (k+1)X_{s-}^k\Delta X_s. \qquad (3.17)
\end{aligned}$$

We now prove (3.15b) in a similar way. For $k = 2$ the integration by parts formula together with (3.16) yields

$$\begin{aligned}
X_t^2 &= \int_0^t X_{u-}\,\mathrm{d}X_u + \int_0^t \mathrm{d}X_u\,X_{u-} + [X,X]_t \\
&= \int_0^t \mathrm{d}X_u\,X_{u-} + \int_0^t \mathrm{d}X_u\,X_{u-} + [X,X]_t^c + \sum_{0<s\le t}(\Delta X_s)^2 \\
&= 2\int_0^t \mathrm{d}X_u\,X_{u-} + \int_0^t \mathrm{d}[X,X]_u^c + \sum_{0<s\le t}\Big(\Delta X_s^2 - 2X_{s-}\Delta X_s\Big).
\end{aligned}$$

For the induction step we assume that (3.15b) holds for some $k>1$. Then

$$\begin{aligned}
[X^k,X]_t^c &= \left[k\int_0^t \mathrm{d}X_u\,X_{u-}^{k-1} + \frac{k(k-1)}{2}\int_0^t \mathrm{d}[X,X]_u^c\,X_{u-}^{k-2}, X_t\right]^c \\
&\quad + \underbrace{\left[\sum_{0<s\le t}\Big(\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s\Big), X_t\right]^c}_{=\mathbf{0}} \\
&= k\int_0^t \mathrm{d}[X,X]_u^c\,X_{u-}^{k-1} + \frac{k(k-1)}{2}\int_0^t \underbrace{\mathrm{d}\Big[[X,X]^c, X\Big]_u^c}_{=\mathbf{0}} X_{u-}^{k-2} \\
&= k\int_0^t \mathrm{d}[X,X]_u^c\,X_{u-}^{k-1}
\end{aligned}$$

and the integration by parts formula together with (3.17) yields

$$\begin{aligned}
X_t^{k+1} = X_t^k X_t &= \int_0^t X_{u-}^k\,\mathrm{d}X_u + \int_0^t \mathrm{d}X_u^k\,X_{u-} + [X^k,X]_t \\
&= \int_0^t \mathrm{d}X_u\,X_{u-}^k + \int_0^t \left(k\,\mathrm{d}X_u\,X_{u-}^{k-1} + \frac{k(k-1)}{2}\mathrm{d}[X,X]_u^c\,X_{u-}^{k-2}\right) X_{u-} \\
&\quad + \int_0^t \mathrm{d}\left(\sum_{0<s\le u}\Big(\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s\Big)\right) X_{u-} + [X^k,X]_t^c + \sum_{0<s\le t}(\Delta X_s^k)(\Delta X_s) \\
&= \int_0^t \mathrm{d}X_u\,X_{u-}^k + k\int_0^t \mathrm{d}X_u\,X_{u-}^k + \frac{k(k-1)}{2}\int_0^t \mathrm{d}[X,X]_u^c\,X_{u-}^{k-1} \\
&\quad + \sum_{0<s\le t}\Big(\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s\Big)X_{s-} + k\int_0^t \mathrm{d}[X,X]_u^c\,X_{u-}^{k-1} + \sum_{0<s\le t}(\Delta X_s^k)(\Delta X_s) \\
&= (k+1)\int_0^t \mathrm{d}X_u\,X_{u-}^k + \frac{k(k+1)}{2}\int_0^t \mathrm{d}[X,X]_u^c\,X_{u-}^{k-1} \\
&\quad + \sum_{0<s\le t}\Big((\Delta X_s^k - kX_{s-}^{k-1}\Delta X_s)X_{s-} + (\Delta X_s^k)(\Delta X_s)\Big)
\end{aligned}$$

$$= (k+1)\int_0^t \mathrm{d}X_u\, X_{u-}^k + \frac{k(k+1)}{2}\int_0^t \mathrm{d}[X,X]_u^c\, X_{u-}^{k-1}$$
$$+ \sum_{0<s\le t}\Big(\Delta X_s^{k+1} - (k+1)X_{s-}^k \Delta X_s\Big).$$

□

Theorem 3.11. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic and let $Z = (Z_t)_{t\ge 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ such that (Z, Z) and $(Z, [Z, Z]^c)$ are commutative, $Z_0 = z_0 I$, and $Z_t Z_{t-} = Z_{t-} Z_t$ for all $t \ge 0$. Then*

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_{u-})\,\mathrm{d}Z_u + \frac{1}{2}\int_0^t f''(Z_{u-})\,\mathrm{d}[Z,Z]_u^c + \sum_{0<s\le t}\Big(\Delta f(Z_s) - f'(Z_{s-})\Delta Z_s\Big), \tag{3.18a}$$

$$f(Z_t) = f(Z_0) + \int_0^t \mathrm{d}Z_u\, f'(Z_{u-}) + \frac{1}{2}\int_0^t \mathrm{d}[Z,Z]_u^c\, f''(Z_{u-}) + \sum_{0<s\le t}\Big(\Delta f(Z_s) - f'(Z_{s-})\Delta Z_s\Big). \tag{3.18b}$$

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ be the Taylor series of f and let $X_t := Z_t - z_0 I$. Then

$$\mathrm{d}X_t = \mathrm{d}Z_t \quad , \quad \mathrm{d}[X,X]_t^c = \mathrm{d}[Z,Z]_t^c \quad , \quad \Delta X_t = \Delta Z_t.$$

X satisfies the assumptions of Theorem 3.10 because

$$\begin{aligned}
X_0 &= Z_0 - z_0 I = z_0 I - z_0 I = \mathbf{0},\\
X_t X_{t-} &= Z_t Z_{t-} - z_0 Z_t - z_0 Z_{t-} + z_0^2 I\\
&= Z_{t-} Z_t - z_0 Z_{t-} - z_0 Z_t + z_0^2 I = X_{t-} X_t,\\
X_{t-}\,\mathrm{d}X_t &= Z_{t-}\,\mathrm{d}Z_t - z_0 I\,\mathrm{d}Z_t\\
&= \mathrm{d}Z_t\, Z_{t-} - \mathrm{d}Z_t\, z_0 I = \mathrm{d}X_t\, X_{t-},\\
X_{t-}\,\mathrm{d}[X,X]_t^c &= Z_{t-}\,\mathrm{d}[Z,Z]_t^c - z_0 I\,\mathrm{d}[Z,Z]_t^c\\
&= \mathrm{d}[Z,Z]_t^c\, Z_{t-} - \mathrm{d}[Z,Z]_t^c\, z_0 I = \mathrm{d}[X,X]_t^c\, X_{t-}.
\end{aligned}$$

We can therefore apply (3.15a), which obviously also holds true for $k \in \{0, 1\}$, to every power $(Z_t - z_0 I)^k = X_t^k$ in the Taylor series of $f(Z_t)$. This results in

$$f(Z_t) = \sum_{k=0}^{\infty} a_k (Z_t - z_0 I)^k = \sum_{k=0}^{\infty} a_k X_t^k$$
$$= \sum_{k=0}^{\infty} a_k \left(k\int_0^t X_{u-}^{k-1}\,\mathrm{d}X_u + \frac{k(k-1)}{2}\int_0^t X_{u-}^{k-2}\,\mathrm{d}[X,X]_u^c + \sum_{0<s\le t}\Big(\Delta X_s^k - k X_{s-}^{k-1}\Delta X_s\Big)\right)$$
$$= \int_0^t \left(\sum_{k=1}^{\infty} k a_k X_{u-}^{k-1}\right)\mathrm{d}X_u + \frac{1}{2}\int_0^t \left(\sum_{k=2}^{\infty} k(k-1) a_k X_{u-}^{k-2}\right)\mathrm{d}[X,X]_u^c$$
$$+ \sum_{0<s\le t}\left(\sum_{k=0}^{\infty} a_k X_s^k - \sum_{k=0}^{\infty} a_k X_{s-}^k - \left(\sum_{k=1}^{\infty} k a_k X_{s-}^{k-1}\right)\Delta X_s\right)$$

$$
\begin{aligned}
&= \int_0^t \left(\sum_{k=1}^{\infty} k a_k (Z_{u-} - z_0 I)^{k-1} \right) \mathrm{d}Z_u + \frac{1}{2} \int_0^t \left(\sum_{k=2}^{\infty} k(k-1) a_k (Z_{u-} - z_0 I)^{k-2} \right) \mathrm{d}[Z,Z]^c_u \\
&\quad + \sum_{0<s\le t} \left(\sum_{k=0}^{\infty} a_k (Z_s - z_0 I)^k - \sum_{k=0}^{\infty} a_k (Z_{s-} - z_0 I)^k - \left(\sum_{k=1}^{\infty} k a_k (Z_{s-} - z_0 I)^{k-1} \right) \Delta Z_s \right) \\
&= \int_0^t f'(Z_{u-}) \, \mathrm{d}Z_u + \frac{1}{2} \int_0^t f''(Z_{u-}) \, \mathrm{d}[Z,Z]^c_u + \sum_{0<s\le t} \Big(\Delta f(Z_s) - f'(Z_{s-}) \Delta Z_s \Big)
\end{aligned}
$$

which is (3.18a). Similarly we can instead apply (3.15b), which obviously holds for all $k \in \mathbb{N}_0$ as well, and obtain

$$
\begin{aligned}
& f(Z_t) = \sum_{k=0}^{\infty} a_k (Z_t - z_0 I)^k = \sum_{k=0}^{\infty} a_k X_t^k \\
&= \sum_{k=0}^{\infty} a_k \left(k \int_0^t \mathrm{d}X_u \, X_{u-}^{k-1} + \frac{k(k-1)}{2} \int_0^t \mathrm{d}[X,X]^c_u \, X_{u-}^{k-2} + \sum_{0<s\le t} \Big(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s \Big) \right) \\
&= \int_0^t \mathrm{d}X_u \left(\sum_{k=1}^{\infty} k a_k X_{u-}^{k-1} \right) + \frac{1}{2} \int_0^t \mathrm{d}[X,X]^c_u \left(\sum_{k=2}^{\infty} k(k-1) a_k X_{u-}^{k-2} \right) \\
&\quad + \sum_{0<s\le t} \left(\sum_{k=0}^{\infty} a_k X_s^k - \sum_{k=0}^{\infty} a_k X_{s-}^k - \left(\sum_{k=1}^{\infty} k a_k X_{s-}^{k-1} \right) \Delta X_s \right) \\
&= \int_0^t \mathrm{d}Z_u \left(\sum_{k=1}^{\infty} k a_k (Z_{u-} - z_0 I)^{k-1} \right) + \frac{1}{2} \int_0^t \mathrm{d}[Z,Z]^c_u \left(\sum_{k=2}^{\infty} k(k-1) a_k (Z_{u-} - z_0 I)^{k-2} \right) \\
&\quad + \sum_{0<s\le t} \left(\sum_{k=0}^{\infty} a_k (Z_s - z_0 I)^k - \sum_{k=0}^{\infty} a_k (Z_{s-} - z_0 I)^k - \left(\sum_{k=1}^{\infty} k a_k (Z_{s-} - z_0 I)^{k-1} \right) \Delta Z_s \right) \\
&= \int_0^t \mathrm{d}Z_u \, f'(Z_{u-}) + \frac{1}{2} \int_0^t \mathrm{d}[Z,Z]^c_u \, f''(Z_{u-}) + \sum_{0<s\le t} \Big(\Delta f(Z_s) - f'(Z_{s-}) \Delta Z_s \Big)
\end{aligned}
$$

which is (3.18b). □

Theorem 3.12. *Let $Z = (Z_t)_{t\ge 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ such that (Z,Z), $(Z,[Z,Z]^c)$ are commutative, $Z_0 = \mathbf{0}$, and $Z_t Z_{t-} = Z_{t-} Z_t$ for all $t \ge 0$. Let*

$$
V_t := Z_t + \frac{1}{2} [Z,Z]^c_t + \sum_{0<s\le t} \Big(\exp(\Delta Z_s) - (I + \Delta Z_s) \Big). \tag{3.19}
$$

Then $Y = \exp(Z)$ is the unique solution to both of the stochastic integral equations

$$
Y_t = I + \int_0^t Y_{u-} \, \mathrm{d}V_u \quad \text{and} \quad Y_t = I + \int_0^t \mathrm{d}V_u \, Y_{u-}. \tag{3.20}
$$

Proof. We use Theorem 3.11 with the analytic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) := \exp(z)$ to show that $Y = \exp(Z) = f(Z)$ is a solution to both equations in (3.20). First, an application of (3.18a) to $f(Z_t)$ yields

$$
\exp(Z_t) = f(Z_t)
$$

$$= f(Z_0) + \int_0^t f'(Z_{u-})\,\mathrm{d}Z_u + \frac{1}{2}\int_0^t f''(Z_{u-})\,\mathrm{d}[Z,Z]^c_u + \sum_{0<s\le t}\Big(\Delta f(Z_s) - f'(Z_{s-})\Delta Z_s\Big)$$
$$= I + \int_0^t \exp(Z_{u-})\,\mathrm{d}Z_u + \frac{1}{2}\int_0^t \exp(Z_{u-})\,\mathrm{d}[Z,Z]^c_u + \sum_{0<s\le t}\Big(\Delta \exp(Z_s) - \exp(Z_{s-})\Delta Z_s\Big)$$
$$= I + \int_0^t \exp(Z_{u-})\,\mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u\right) + \sum_{0<s\le t}\Big(\exp(Z_s) - \exp(Z_{s-})(I+\Delta Z_s)\Big).$$

We now rewrite the remaining sum as a right stochastic integral of $\exp(Z_{u-})$ with respect to

$$\sum_{0<s\le u} M_s \quad \text{where} \quad M_s \coloneqq \exp(\Delta Z_s) - (I + \Delta Z_s). \tag{3.21}$$

Note that $M_s = \mathbf{0}$ if $\Delta Z_s = \mathbf{0}$. Since $\Delta \sum_{0<s\le u} M_s = M_u$ we have

$$\sum_{0<s\le t}\Big(\exp(Z_s) - \exp(Z_{s-})(I+\Delta Z_s)\Big)$$
$$= \sum_{0<s\le t} \exp(Z_{s-})\exp(Z_{s-})^{-1}\Big(\exp(Z_s) - \exp(Z_{s-})(I+\Delta Z_s)\Big)$$
$$= \sum_{0<s\le t} \exp(Z_{s-})\Big(\exp(Z_{s-})^{-1}\exp(Z_s) - (I+\Delta Z_s)\Big)$$
$$= \sum_{0<s\le t} \exp(Z_{s-})\Big(\exp(\Delta Z_s) - (I+\Delta Z_s)\Big)$$
$$= \sum_{0<s\le t} \exp(Z_{s-})M_s$$
$$= \sum_{0<s\le t} \exp(Z_{s-})\Delta\left(\sum_{0<r\le s} M_r\right)$$
$$= \int_0^t \exp(Z_{u-})\,\mathrm{d}\left(\sum_{0<s\le u} M_s\right).$$

Combining both terms into one integral then results in

$$\exp(Z_t) = I + \int_0^t \exp(Z_{u-})\,\mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u\right) + \sum_{0<s\le t}\Big(\exp(Z_s) - \exp(Z_{s-})(I+\Delta Z_s)\Big)$$
$$= I + \int_0^t \exp(Z_{u-})\,\mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u\right) + \int_0^t \exp(Z_{u-})\,\mathrm{d}\left(\sum_{0<s\le u} M_s\right)$$
$$= I + \int_0^t \exp(Z_{u-})\,\mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u + \sum_{0<s\le u} M_s\right) = I + \int_0^t \exp(Z_{u-})\,\mathrm{d}V_u$$

which is the first equation in (3.20). Similarly, an application of (3.18b) to $f(Z_t)$ yields

$$\exp(Z_t) = f(Z_t)$$

$$= f(Z_0) + \int_0^t \mathrm{d}Z_u\, f'(Z_{u-}) + \frac{1}{2}\int_0^t \mathrm{d}[Z,Z]^c_u\, f''(Z_{u-}) + \sum_{0<s\le t}\Big(\Delta f(Z_s) - f'(Z_{s-})\Delta Z_s\Big)$$

$$= I + \int_0^t \mathrm{d}Z_u \exp(Z_{u-}) + \frac{1}{2}\int_0^t \mathrm{d}[Z,Z]^c_u \exp(Z_{u-}) + \sum_{0<s\le t}\Big(\Delta \exp(Z_s) - \exp(Z_{s-})\Delta Z_s\Big)$$

$$= I + \int_0^t \mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u\right)\exp(Z_{u-}) + \sum_{0<s\le t}\Big(\exp(Z_s) - (I+\Delta Z_s)\exp(Z_{s-})\Big).$$

Again we rewrite the remaining sum as a left stochastic integral of $\exp(Z_{u-})$ with respect to $\sum_{0<s\le u} M_s$ with M_s defined as in (3.21). We now have

$$\sum_{0<s\le t}\Big(\exp(Z_s) - (I+\Delta Z_s)\exp(Z_{s-})\Big)$$
$$= \sum_{0<s\le t}\Big(\exp(Z_s) - (I+\Delta Z_s)\exp(Z_{s-})\Big)\exp(Z_{s-})^{-1}\exp(Z_{s-})$$
$$= \sum_{0<s\le t}\Big(\exp(Z_s)\exp(Z_{s-})^{-1} - (I+\Delta Z_s)\Big)\exp(Z_{s-})$$
$$= \sum_{0<s\le t}\Big(\exp(\Delta Z_s) - (I+\Delta Z_s)\Big)\exp(Z_{s-})$$
$$= \sum_{0<s\le t} M_s \exp(Z_{s-})$$
$$= \sum_{0<s\le t}\Delta\left(\sum_{0<r\le s} M_r\right)\exp(Z_{s-})$$
$$= \int_0^t \mathrm{d}\left(\sum_{0<s\le u} M_s\right)\exp(Z_{u-})$$

and combining both terms into one integral with respect to V_s results in

$$\exp(Z_t) = I + \int_0^t \mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u\right)\exp(Z_{u-}) + \sum_{0<s\le t}\Big(\exp(Z_s) - (I+\Delta Z_s)\exp(Z_{s-})\Big)$$
$$= I + \int_0^t \mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u\right)\exp(Z_{u-}) + \int_0^t \mathrm{d}\left(\sum_{0<s\le u} M_s\right)\exp(Z_{u-})$$
$$= I + \int_0^t \mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u + \sum_{0<s\le u} M_s\right)\exp(Z_{u-}) = I + \int_0^t \mathrm{d}V_u \exp(Z_{u-})$$

which is the second equation in (3.20). □

Theorem 3.13. *Let $X = (X_t)_{t\ge 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ such that $X_0 = \mathbf{0}$ and for all $s,t \ge 0$*

$$\|\Delta X_t\| < 1\,,\ X_t X_{t-} = X_{t-} X_t\,,\ \Delta X_t \Delta X_s = \Delta X_s \Delta X_t\,,\ [X,X]^c_t \Delta X_t = \Delta X_t [X,X]^c_t.$$

Additionally assume that (X,X), $(X,[X,X]^c)$, $([X,X]^c,X)$, $([X,X]^c,[X,X]^c)$ as well as

$$\left(\sum_{0<s\le \cdot -}\Big(\log(I+\Delta X_s) - \Delta X_s\Big), X\right) \text{ and } \left(\sum_{0<s\le \cdot -}\Big(\log(I+\Delta X_s) - \Delta X_s\Big), [X,X]^c\right)$$

are commutative. Then the left and right stochastic exponential of X are equal and given by

$$\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) = \exp\left(X_t - \frac{1}{2}[X,X]^c_t + \sum_{0<s\leq t} \Big(\log(I + \Delta X_s) - \Delta X_s \Big) \right). \tag{3.22}$$

Proof. Since $\|\Delta X_t\| < 1$ for all $t \geq 0$ the matrix logarithm $\log(I + \Delta X_t)$ is defined for all $t \geq 0$ by Theorem 2.8 in [15] and we can define

$$Z_t \coloneqq X_t - \frac{1}{2}[X,X]^c_t + \sum_{0<s\leq t} \Big(\log(I + \Delta X_s) - \Delta X_s \Big) = Z^{(1)}_t - \frac{1}{2} Z^{(2)}_t + Z^{(3)}_t$$

where $Z^{(1)}_t = X_t$, $Z^{(2)}_t = [X,X]^c_t$ and

$$Z^{(3)}_t = \sum_{0<s\leq t} \Big(\log(I + \Delta X_s) - \Delta X_s \Big) = \sum_{0<s\leq t} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} (\Delta X_s)^k$$

with increments $\Delta Z^{(1)}_t = \Delta X_t$, $\Delta Z^{(2)}_t = \mathbf{0}$ and

$$\Delta Z^{(3)}_t = \log(I + \Delta X_t) - \Delta X_t = \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} (\Delta X_t)^k.$$

We verify that $Z = (Z_t)_{t\geq 0}$ fulfills the assumptions of Theorem 3.12. First, $Z_0 = \mathbf{0}$ is clear by definition. Second, the pair (Z, Z) is commutative because

$$\begin{aligned}
(Z^{(1)}_- \cdot Z^{(1)})_t &= \int_0^t X_{u-}\, \mathrm{d}X_u = \int_0^t \mathrm{d}X_u\, X_{u-} = (Z^{(1)} : Z^{(1)}_-)_t, \\
(Z^{(1)}_- \cdot Z^{(2)})_t &= \int_0^t X_{u-}\, \mathrm{d}[X,X]^c_u = \int_0^t \mathrm{d}[X,X]^c_u\, X_{u-} = (Z^{(2)} : Z^{(1)}_-)_t, \\
(Z^{(1)}_- \cdot Z^{(3)})_t &= \sum_{0<s\leq t} X_{s-} \Big(\log(I + \Delta X_s) - \Delta X_s \Big) \\
&= \sum_{0<s\leq t} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} X_{s-} (\Delta X_s)^k \\
&= \sum_{0<s\leq t} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} (\Delta X_s)^k X_{s-} \\
&= \sum_{0<s\leq t} \Big(\log(I + \Delta X_s) - \Delta X_s \Big) X_{s-} = (Z^{(3)} : Z^{(1)}_-)_t, \\
(Z^{(2)}_- \cdot Z^{(1)})_t &= \int_0^t [X,X]^c_u\, \mathrm{d}X_u = \int_0^t \mathrm{d}X_u\, [X,X]^c_u = (Z^{(1)} : Z^{(2)}_-)_t, \\
(Z^{(2)}_- \cdot Z^{(2)})_t &= \int_0^t [X,X]^c_u\, \mathrm{d}[X,X]^c_u = \int_0^t \mathrm{d}[X,X]^c_u\, [X,X]^c_u = (Z^{(2)} : Z^{(2)}_-)_t, \\
(Z^{(2)}_- \cdot Z^{(3)})_t &= \sum_{0<s\leq t} [X,X]^c_s \Big(\log(I + \Delta X_s) - \Delta X_s \Big) \\
&= \sum_{0<s\leq t} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} [X,X]^c_s (\Delta X_s)^k
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{0<s\le t} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} (\Delta X_s)^k [X,X]_s^c \\
&= \sum_{0<s\le t} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) [X,X]_s^c = \big(Z^{(3)} : Z_-^{(2)}\big)_t,
\end{aligned}
$$

$$
\begin{aligned}
\big(Z_-^{(3)} \cdot Z^{(1)}\big)_t &= \int_0^t \sum_{0<s\le u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \, \mathrm{d}X_u \\
&= \int_0^t \mathrm{d}X_u \sum_{0<s\le u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) = \big(Z^{(1)} : Z_-^{(3)}\big)_t,
\end{aligned}
$$

$$
\begin{aligned}
\big(Z_-^{(3)} \cdot Z^{(2)}\big)_t &= \int_0^t \sum_{0<s\le u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \, \mathrm{d}[X,X]_u^c \\
&= \int_0^t \mathrm{d}[X,X]_u^c \sum_{0<s\le u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) = \big(Z^{(2)} : Z_-^{(3)}\big)_t,
\end{aligned}
$$

$$
\begin{aligned}
\big(Z_-^{(3)} \cdot Z^{(3)}\big)_t &= \sum_{0<s\le t} \sum_{0<r\le s-} \Big(\log(I+\Delta X_r) - \Delta X_r \Big)\Big(\log(I+\Delta X_s) - \Delta X_s \Big) \\
&= \sum_{0<s\le t} \sum_{0<r\le s-} \sum_{k,l=2}^{\infty} \frac{(-1)^{k+l+2}}{kl} (\Delta X_r)^k (\Delta X_s)^l \\
&= \sum_{0<s\le t} \sum_{0<r\le s-} \sum_{l,k=2}^{\infty} \frac{(-1)^{l+k+2}}{lk} (\Delta X_s)^l (\Delta X_r)^k \\
&= \sum_{0<s\le t} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \sum_{0<r\le s-} \Big(\log(I+\Delta X_r) - \Delta X_r \Big) \\
&= \big(Z^{(3)} : Z_-^{(3)}\big)_t,
\end{aligned}
$$

and thus

$$
\begin{aligned}
(Z_- \cdot Z)_t &= \big(Z_-^{(1)} \cdot Z^{(1)}\big)_t + \frac{1}{4}\big(Z_-^{(2)} \cdot Z^{(2)}\big)_t + \big(Z_-^{(3)} \cdot Z^{(3)}\big)_t + \big(Z_-^{(1)} \cdot Z^{(3)}\big)_t + \big(Z_-^{(3)} \cdot Z^{(1)}\big)_t \\
&\quad - \frac{1}{2}\Big(\big(Z_-^{(1)} \cdot Z^{(2)}\big)_t + \big(Z_-^{(2)} \cdot Z^{(1)}\big)_t + \big(Z_-^{(2)} \cdot Z^{(3)}\big)_t + \big(Z_-^{(3)} \cdot Z^{(2)}\big)_t \Big) \\
&= \big(Z^{(1)} : Z_-^{(1)}\big)_t + \frac{1}{4}\big(Z^{(2)} : Z_-^{(2)}\big)_t + \big(Z^{(3)} : Z_-^{(3)}\big)_t + \big(Z^{(3)} : Z_-^{(1)}\big)_t + \big(Z^{(1)} : Z_-^{(3)}\big)_t \\
&\quad - \frac{1}{2}\Big(\big(Z^{(2)} : Z_-^{(1)}\big)_t + \big(Z^{(1)} : Z_-^{(2)}\big)_t + \big(Z^{(3)} : Z_-^{(2)}\big)_t + \big(Z^{(2)} : Z_-^{(3)}\big)_t \Big) \\
&= (Z : Z_-)_t.
\end{aligned}
$$

Third, the pair $(Z, [Z,Z]^c)$ is commutative because $[Z,Z]_t^c = [X,X]_t^c = Z_t^{(2)}$ for all $t \ge 0$ and thus

$$
\begin{aligned}
(Z_- \cdot [Z,Z]^c)_t &= \big(Z_-^{(1)} \cdot Z^{(2)}\big)_t - \frac{1}{2}\big(Z_-^{(2)} \cdot Z^{(2)}\big)_t + \big(Z_-^{(3)} \cdot Z^{(2)}\big)_t \\
&= \big(Z^{(2)} : Z_-^{(1)}\big)_t - \frac{1}{2}\big(Z^{(2)} : Z_-^{(2)}\big)_t + \big(Z^{(2)} : Z_-^{(3)}\big)_t = ([Z,Z]^c : Z_-)_t.
\end{aligned}
$$

Fourth, $Z_t Z_{t-} = Z_{t-} Z_t$ for all $t \ge 0$ because

$$
Z_t^{(1)} Z_{t-}^{(1)} = X_t X_{t-} = X_{t-} X_t = Z_{t-}^{(1)} Z_t^{(1)},
$$

$$\begin{aligned}
Z_t^{(2)} Z_{t-}^{(2)} &= [X,X]_t^c [X,X]_{t-}^c = [X,X]_t^c [X,X]_t^c = [X,X]_{t-}^c [X,X]_t^c = Z_{t-}^{(2)} Z_t^{(2)}, \\
Z_t^{(3)} Z_{t-}^{(3)} &= \sum_{0<s\le t} \sum_{0<r\le t-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big)\Big(\log(I+\Delta X_r) - \Delta X_r \Big) \\
&= \sum_{0<s\le t} \sum_{0<r\le t-} \sum_{k,l=2}^{\infty} \frac{(-1)^{k+l+2}}{kl} (\Delta X_s)^k (\Delta X_r)^l \\
&= \sum_{0<r\le t-} \sum_{0<s\le t} \sum_{l,k=2}^{\infty} \frac{(-1)^{l+k+2}}{lk} (\Delta X_r)^l (\Delta X_s)^k \\
&= \sum_{0<r\le t-} \sum_{0<s\le t} \Big(\log(I+\Delta X_r) - \Delta X_r \Big)\Big(\log(I+\Delta X_s) - \Delta X_s \Big) = Z_{t-}^{(3)} Z_t^{(3)},
\end{aligned}$$

and for the mixed terms we have

$$\begin{aligned}
& Z_t^{(1)} Z_{t-}^{(2)} + Z_t^{(2)} Z_{t-}^{(1)} \\
&= Z_{t-}^{(2)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(2)} + \Delta Z_t^{(1)} Z_{t-}^{(2)} - Z_{t-}^{(2)} \Delta Z_t^{(1)} + \Delta Z_t^{(2)} Z_{t-}^{(1)} - Z_{t-}^{(1)} \Delta Z_t^{(2)} \\
&= Z_{t-}^{(2)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(2)} + \underbrace{\Big(\Delta X_t [X,X]_t^c - [X,X]_t^c \Delta X_t \Big)}_{=\mathbf{0}} + \underbrace{\Delta [X,X]_t^c Z_{t-}^{(1)}}_{=\mathbf{0}} - \underbrace{Z_{t-}^{(1)} \Delta [X,X]_t^c}_{=\mathbf{0}} \\
&= Z_{t-}^{(2)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(2)}
\end{aligned}$$

and

$$\begin{aligned}
& Z_t^{(1)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(1)} \\
&= Z_{t-}^{(3)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(3)} + \Delta Z_t^{(1)} Z_{t-}^{(3)} - Z_{t-}^{(3)} \Delta Z_t^{(1)} + \Delta Z_t^{(3)} Z_{t-}^{(1)} - Z_{t-}^{(1)} \Delta Z_t^{(3)} \\
&= Z_{t-}^{(3)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(3)} + \sum_{0<s\le t-} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \underbrace{\Big((\Delta X_t)(\Delta X_s)^k - (\Delta X_s)^k (\Delta X_t) \Big)}_{=\mathbf{0}} \\
&\quad + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \underbrace{\Big((\Delta X_t)^k X_{t-} - X_{t-} (\Delta X_t)^k \Big)}_{=\mathbf{0}} \\
&= Z_{t-}^{(3)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(3)}
\end{aligned}$$

and

$$\begin{aligned}
& Z_t^{(2)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(2)} \\
&= Z_{t-}^{(3)} Z_t^{(2)} + Z_{t-}^{(2)} Z_t^{(3)} + \Delta Z_t^{(2)} Z_{t-}^{(3)} - Z_{t-}^{(3)} \Delta Z_t^{(2)} + \Delta Z_t^{(3)} Z_{t-}^{(2)} - Z_{t-}^{(2)} \Delta Z_t^{(3)} \\
&= Z_{t-}^{(3)} Z_t^{(2)} + Z_{t-}^{(2)} Z_t^{(3)} + \underbrace{\Delta [X,X]_t^c Z_{t-}^{(3)}}_{=\mathbf{0}} - \underbrace{Z_{t-}^{(3)} \Delta [X,X]_t^c}_{=\mathbf{0}} \\
&\quad + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \underbrace{\Big((\Delta X_t)^k [X,X]_t^c - [X,X]_t^c (\Delta X_t)^k \Big)}_{=\mathbf{0}} \\
&= Z_{t-}^{(3)} Z_t^{(2)} + Z_{t-}^{(2)} Z_t^{(3)}.
\end{aligned}$$

and thus

$$\begin{aligned}
Z_t Z_{t-} = Z_t^{(1)} Z_{t-}^{(1)} + \frac{1}{4} Z_t^{(2)} Z_{t-}^{(2)} + Z_t^{(3)} Z_{t-}^{(3)} + \Big(Z_t^{(1)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(1)} \Big) \\
- \frac{1}{2} \Big(Z_t^{(1)} Z_{t-}^{(2)} + Z_t^{(2)} Z_{t-}^{(1)} \Big) - \frac{1}{2} \Big(Z_t^{(2)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(2)} \Big)
\end{aligned}$$

$$= Z^{(1)}_{t-}Z^{(1)}_t + \frac{1}{4}Z^{(2)}_{t-}Z^{(2)}_t + Z^{(3)}_{t-}Z^{(3)}_t + \left(Z^{(3)}_{t-}Z^{(1)}_t + Z^{(1)}_{t-}Z^{(3)}_t\right)$$
$$- \frac{1}{2}\left(Z^{(2)}_{t-}Z^{(1)}_t + Z^{(1)}_{t-}Z^{(2)}_t\right) - \frac{1}{2}\left(Z^{(3)}_{t-}Z^{(2)}_t + Z^{(2)}_{t-}Z^{(3)}_t\right) = Z_{t-}Z_t.$$

Therefore Z fulfills all the assumptions of Theorem 3.12 and $Y = \exp(Z)$ is the unique solution to both of the stochastic integral equations in (3.20) with V given by (3.19). The equations $[Z, Z]^c_t = [X, X]^c_t$ and

$$\Delta Z_t = \Delta Z^{(1)}_t + \Delta Z^{(2)}_t + \Delta Z^{(3)}_t = \Delta X_t + \log(I + \Delta X_t) - \Delta X_t = \log(I + \Delta X_t)$$

show that

$$\begin{aligned} V_t &= Z_t + \frac{1}{2}[Z, Z]^c_t + \sum_{0<s\le t}\Big(\exp(\Delta Z_s) - (I + \Delta Z_s)\Big) \\ &= X_t - \frac{1}{2}[X, X]^c_t + \sum_{0<s\le t}\Big(\log(I + \Delta X_s) - \Delta X_s\Big) \\ &\quad + \frac{1}{2}[X, X]^c_t + \sum_{0<s\le t}\Big(\exp\left(\log(I + \Delta X_s)\right) - \left(I + \log(I + \Delta X_s)\right)\Big) \\ &= X_t + \sum_{0<s\le t}\Big(\log(I + \Delta X_s) - \Delta X_s + I + \Delta X_s - I - \log(I + \Delta X_s)\Big) \\ &= X_t \end{aligned}$$

and we conclude that $\overleftarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(V_t) = Y_t = \exp(Z_t)$ and $\overrightarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(V_t) = Y_t = \exp(Z_t)$. Plugging in the definition of Z yields (3.22). □

Remark 3.14. Since $[X, X]^c_t = [X, X]_t - \sum_{0<s\le t} (\Delta X_s)^2$ we may also write (3.22) as

$$\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) = \exp\left(X_t - \frac{1}{2}[X, X]_t + \sum_{0<s\le t}\left(\log(I + \Delta X_s) - \Delta X_s + \frac{1}{2}(\Delta X_s)^2\right)\right).$$

In the special case $n = 1$ this simplifies to

$$\begin{aligned} \overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) &= \exp\left(X_t - \frac{1}{2}[X, X]_t\right)\prod_{0<s\le t}\exp\left(\log(1 + \Delta X_s) - \Delta X_s + \frac{1}{2}(\Delta X_s)^2\right) \\ &= \exp\left(X_t - \frac{1}{2}[X, X]_t\right)\prod_{0<s\le t}(1 + \Delta X_s)\exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right) \end{aligned}$$

which is the well-known formula for the one-dimensional stochastic exponential as in Theorem II.37 in [31].

To conclude this chapter we give an example of a semimartingale X for which the assumptions of Theorem 3.13 are fulfilled and the stochastic exponential can be computed with (3.22).

Example 3.15. Let $B = (B_t)_{t\ge 0}$ be a one-dimensional BROWNian motion and $N = (N_t)_{t\ge 0}$ a one-dimensional POISSON process with almost surely increasing jump times $(T_k)_{k\in\mathbb{N}}$ and increments

$$c_t := \Delta N_t = \sum_{k=1}^{\infty} \mathbf{1}_{\{T_k = t\}}.$$

We verify that $X = (X_t)_{t\ge 0}$ defined by

$$X_t := \begin{pmatrix} \frac{1}{2}N_t & B_t \\ -B_t & \frac{1}{2}N_t \end{pmatrix}$$

fulfills the assumptions of Theorem 3.13 and compute $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ with (3.22). For this purpose we need the identities

$$\Delta X_t = \begin{pmatrix} \frac{1}{2}\Delta N_t & \Delta B_t \\ -\Delta B_t & \frac{1}{2}\Delta N_t \end{pmatrix} = \begin{pmatrix} \frac{c_t}{2} & 0 \\ 0 & \frac{c_t}{2} \end{pmatrix} = \frac{c_t}{2} I$$

and

$$\begin{aligned} [X,X]^c_t &= \left[\begin{pmatrix} \frac{1}{2}N & B \\ -B & \frac{1}{2}N \end{pmatrix}, \begin{pmatrix} \frac{1}{2}N & B \\ -B & \frac{1}{2}N \end{pmatrix} \right]^c_t \\ &= \begin{pmatrix} \frac{1}{4}[N,N]^c_t - [B,B]^c_t & \frac{1}{2}[N,B]^c_t + \frac{1}{2}[B,N]^c_t \\ -\frac{1}{2}[B,N]^c_t - \frac{1}{2}[N,B]^c_t & -[B,B]^c_t + \frac{1}{4}[N,N]^c_t \end{pmatrix} \\ &= \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} = -tI. \end{aligned}$$

and

$$\begin{aligned} \sum_{0<s\le t} \Big(\log(I + \Delta X_s) - \Delta X_s \Big) &= \sum_{0<s\le t} \left(\log\left(I + \frac{c_s}{2} I \right) - \frac{c_s}{2} I \right) \\ &= \sum_{k=1}^{N_t} \left(\log\left(\frac{3}{2}\right) - \frac{1}{2} \right) I \\ &= \left(\log\left(\frac{3}{2}\right) - \frac{1}{2} \right) N_t I. \end{aligned}$$

Then $X_0 = \mathbf{0}$ by definition, $\|\Delta X_t\| = \dfrac{|c_t|}{2} < 1$, and

$$\begin{aligned} X_t X_{t-} &= X_t (X_t - \Delta X_t) = X_t \left(X_t - \frac{c_t}{2} I \right) \\ &= \left(X_t - \frac{c_t}{2} I \right) X_t = (X_t - \Delta X_t) X_t = X_{t-} X_t, \\ \Delta X_t \Delta X_s &= \left(\frac{c_t}{2} I \right) \left(\frac{c_s}{2} I \right) = \left(\frac{c_s}{2} I \right) \left(\frac{c_t}{2} I \right) = \Delta X_s \Delta X_t, \\ [X,X]^c_t \Delta X_t &= (-tI) \left(\frac{c_t}{2} I \right) = \left(\frac{c_t}{2} I \right) (-tI) = \Delta X_t [X,X]^c_t. \end{aligned}$$

For the various commutativity conditions we have

$$\begin{aligned} X_- \cdot X &= \begin{pmatrix} \frac{1}{2}N_- & B \\ -B & \frac{1}{2}N_- \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}N & B \\ -B & \frac{1}{2}N \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}N_- \cdot N - B \cdot B & \frac{1}{2}N_- \cdot B + \frac{1}{2}B \cdot N \\ -\frac{1}{2}B \cdot N - \frac{1}{2}N_- \cdot B & -B \cdot B + \frac{1}{4}N_- \cdot N \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}N : N_- - B : B & \frac{1}{2}B : N_- + \frac{1}{2}N : B \\ -\frac{1}{2}N : B - \frac{1}{2}B : N_- & -B : B + \frac{1}{4}N : N_- \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}N & B \\ -B & \frac{1}{2}N \end{pmatrix} : \begin{pmatrix} \frac{1}{2}N_- & B \\ -B & \frac{1}{2}N_- \end{pmatrix} = X : X_-, \\ (X_- \cdot [X,X]^c)_t &= \int_0^t X_{u-} \, \mathrm{d}[X,X]^c_u = \int_0^t X_{u-} \, \mathrm{d}(-uI) = \int_0^t \mathrm{d}(-uI) \, X_{u-} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \mathrm{d}[X,X]^c_u \, X_{u-} = ([X,X]^c : X_-)_t, \\
([X,X]^c \cdot X)_t &= \int_0^t [X,X]^c_u \, \mathrm{d}X_u = \int_0^t (-uI) \, \mathrm{d}X_u = \int_0^t \mathrm{d}X_u \, (-uI) \\
&= \int_0^t \mathrm{d}X_u \, [X,X]^c_u = (X : [X,X]^c)_t, \\
([X,X]^c \cdot [X,X]^c)_t &= \int_0^t [X,X]^c_u \, \mathrm{d}[X,X]^c_u = \int_0^t u \, \mathrm{d}u \, I = \frac{t^2}{2} I = \int_0^t \mathrm{d}u \, uI \\
&= \int_0^t \mathrm{d}[X,X]^c_u \, [X,X]^c_u = ([X,X]^c : [X,X]^c)_t
\end{aligned}
$$

as well as

$$
\begin{aligned}
\left(\sum_{0<s\leq \cdot -} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \cdot X \right)_t &= \int_0^t \sum_{0<s\leq u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \, \mathrm{d}X_u \\
&= \int_0^t \left(\log\left(\frac{3}{2}\right) - \frac{1}{2} \right) N_{u-} I \, \mathrm{d}X_u \\
&= \int_0^t \mathrm{d}X_u \left(\log\left(\frac{3}{2}\right) - \frac{1}{2} \right) N_{u-} I \\
&= \int_0^t \mathrm{d}X_u \sum_{0<s\leq u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \\
&= \left(X : \sum_{0<s\leq \cdot -} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \right)_t, \\
\left(\sum_{0<s\leq \cdot -} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \cdot [X,X]^c \right)_t &= \int_0^t \sum_{0<s\leq u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \, \mathrm{d}[X,X]^c_u \\
&= \int_0^t \left(\log\left(\frac{3}{2}\right) - \frac{1}{2} \right) N_{u-} I \, \mathrm{d}(-uI) \\
&= \int_0^t \mathrm{d}(-uI) \left(\log\left(\frac{3}{2}\right) - \frac{1}{2} \right) N_{u-} I \\
&= \int_0^t \mathrm{d}[X,X]^c_u \sum_{0<s\leq u-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \\
&= \left([X,X]^c : \sum_{0<s\leq \cdot -} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \right)_t .
\end{aligned}
$$

Thus X fulfills the assumption of Theorem 3.13 and by (3.22)

$$
\begin{aligned}
\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) &= \exp\left(X_t - \frac{1}{2}[X,X]_t^c + \sum_{0<s\leq t}\Big(\log(I+\Delta X_s) - \Delta X_s\Big)\right) \\
&= \exp\left(X_t + \frac{t}{2}I + \left(\log\left(\frac{3}{2}\right) - \frac{1}{2}\right) N_t I\right) \\
&= e^{t/2}\left(\frac{3}{2}\right)^{N_t} \exp\begin{pmatrix} 0 & B_t \\ -B_t & 0 \end{pmatrix} \\
&= e^{t/2}\left(\frac{3}{2}\right)^{N_t} \begin{pmatrix} \cos(B_t) & \sin(B_t) \\ -\sin(B_t) & \cos(B_t) \end{pmatrix}.
\end{aligned}
$$

4 Multivariate generalized Ornstein-Uhlenbeck processes

Chapter 4 outlines the theory of MGOU processes based on the work of Behme and Lindner and extends some of their results. We recap the motivation behind and the definition of MGOU processes and quote the characterization of MGOU processes as solutions of a random recurrence equation and of a stochastic differential equation as well as conditions under which MGOU processes are stationary. We prove that these results hold true in the periodically stationary case and construct MGOU processes with real time parameter.

Due to the non-commutativity of matrix multiplication there are two types of MGOU processes in dimension $n \geq 2$. Behme and Lindner only discussed left MGOU processes, which are covered in chapter 4.1, but they did not consider right MGOU processes, which are the topic of chapter 4.2. The theories of left and right MGOU processes closely resemble each other but key distinctions are the different order of multiplication and the prevalence of either left or right stochastic exponentials.

4.1 Left MGOU processes

In their paper [4] Behme and Lindner motivate the definition of a multivariate generalized Ornstein-Uhlenbeck process $V = (V_t)_{t\geq 0}$ by a family of $\mathsf{GL}_n(\mathbb{R}) \times \mathbb{R}^{n\times n}$-valued random variables $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$, which they call random functional, such that V satisfies the random recurrence equation

$$V_t = A_{s,t}V_s + B_{s,t} \tag{4.1}$$

almost surely for all $0 \leq s \leq t$. They impose the following assumptions on $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$.

(L0) $A_{t,t} = I$ and $B_{t,t} = \mathbf{0}$ for all $t \geq 0$.

(L1) $A_{r,t} = A_{s,t}A_{r,s}$ and $B_{r,t} = A_{s,t}B_{r,s} + B_{s,t}$ almost surely for all $0 \leq r \leq s \leq t$.

(L2) $(A_{s,t}, B_{s,t})_{a\leq s\leq t\leq b}$ and $(A_{s,t}, B_{s,t})_{c\leq s\leq t\leq d}$ are independent for all $0 \leq a \leq b \leq c \leq d$.

(L3) $(A_{s,t}, B_{s,t})_{0\leq s\leq t} \overset{\mathrm{D}}{=} (A_{0,t-s}, B_{0,t-s})_{0\leq s\leq t}$ for all $t \geq 0$.

(L4) $A_{0,t} \xrightarrow{\mathrm{P}} I$ and $B_{0,t} \xrightarrow{\mathrm{P}} \mathbf{0}$ for $t \downarrow 0$.

We refer to (L0),(L1),(L2),(L3),(L4) as the **L-assumptions**. There is a one-to-one correspondence between a family $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ of $\mathsf{GL}_n(\mathbb{R}) \times \mathbb{R}^{n\times n}$-valued random variables which satisfies the L-assumptions and a bivariate Lévy process (X, Y) such that X satisfies (3.3).

Theorem 4.1. [4, Theorem 3.1]

(a) *Let $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ satisfy the L-assumptions and let $(A_t := A_{0,t})_{t\geq 0}$, $(B_t := B_{0,t})_{t\geq 0}$ be càdlàg. Then the process $(X, Y) = (X_t, Y_t)_{t\geq 0}$ defined by*

$$X_t := \int_0^t A_{u-}\, \mathrm{d}A_u^{-1} = \overleftarrow{\mathsf{Log}}(A_t^{-1}), \tag{4.2a}$$

$$Y_t := \int_0^t A_{u-}\, \mathrm{d}(A_u^{-1}B_u) \tag{4.2b}$$

is the unique Lévy process such that X satisfies (3.3) and

$$A_{s,t} = \overleftarrow{\mathsf{Exp}}(X_t)^{-1}\overleftarrow{\mathsf{Exp}}(X_s), \tag{4.3a}$$

$$B_{s,t} = \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \int_s^t \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u \tag{4.3b}$$

almost surely for all $0 \leq s \leq t$.

(b) *Let* $(X, Y) = (X_t, Y_t)_{t\geq 0}$ *be a* Lévy *process such that* X *satisfies (3.3). Then the family* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *defined as in (4.3) satisfies the L-assumptions.*

The definition of a multivariate generalized Ornstein-Uhlenbeck process is motivated by the formulas for $A_{s,t}$ and $B_{s,t}$ in (4.3) and the random recurrence equation (4.1) as they together yield

$$\begin{aligned} V_t = A_{0,t}V_0 + B_{0,t} &= \overleftarrow{\mathsf{Exp}}(X_t)^{-1}V_0 + \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u \\ &= \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \left(V_0 + \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u \right). \end{aligned}$$

Definition 4.2. [4, Definition 3.2] Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a Lévy process such that X satisfies (3.3) and let V_0 be an $\mathbb{R}^{n\times n}$-valued random variable. Then the process $V = (V_t)_{t\geq 0}$ defined by

$$V_t := \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \left(V_0 + \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u \right) \tag{4.4}$$

is called **left multivariate generalized** Ornstein-Uhlenbeck **process** or simply **left MGOU process**. The process (X, Y) is called **background driving process**.

Example 4.3. Let $B = (B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and $N = (N_t)_{t\geq 0}$ a one-dimensional Poisson process with almost surely increasing jump times $(T_k)_{k\in\mathbb{N}}$. We compute the left MGOU process $V = (V_t)_{t\geq 0}$ with $V_0 = \mathbf{0}$ and background driving process $(X, Y) = (X_t, Y_t)_{t\geq 0}$ defined by

$$X_t := \begin{pmatrix} \frac{1}{2}N_t & B_t \\ -B_t & \frac{1}{2}N_t \end{pmatrix} \quad \text{and} \quad Y_t := \begin{pmatrix} B_t & 0 \\ 0 & B_t \end{pmatrix}.$$

In Example 3.15 we already computed that

$$\overleftarrow{\mathsf{Exp}}(X_t) = e^{t/2} \left(\frac{3}{2}\right)^{N_t} \begin{pmatrix} \cos(B_t) & \sin(B_t) \\ -\sin(B_t) & \cos(B_t) \end{pmatrix}$$

and thus by (4.4)

$$\begin{aligned} V_t &= \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u \\ &= e^{-t/2} \left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix} \int_0^t e^{u/2} \left(\frac{3}{2}\right)^{N_{u-}} \begin{pmatrix} \cos(B_u)\,\mathrm{d}B_u & \sin(B_u)\,\mathrm{d}B_u \\ -\sin(B_u)\,\mathrm{d}B_u & \cos(B_u)\,\mathrm{d}B_u \end{pmatrix}. \end{aligned}$$

In order to compute the stochastic integrals

$$\int_0^t e^{u/2} \left(\frac{3}{2}\right)^{N_{u-}} \cos(B_u)\,\mathrm{d}B_u \quad \text{and} \quad \int_0^t e^{u/2} \left(\frac{3}{2}\right)^{N_{u-}} \sin(B_u)\,\mathrm{d}B_u$$

we apply the multi-dimensional Ito formula (see Theorem II.33 in [31]) with the functions $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$f(x,y,z) := e^{x/2}\left(\frac{3}{2}\right)^{y}\sin(z) \quad \text{and} \quad g(x,y,z) := e^{x/2}\left(\frac{3}{2}\right)^{y}\cos(z).$$

An application of the multi-dimensional Ito formula with f yields

$$\begin{aligned}
&e^{t/2}\left(\frac{3}{2}\right)^{N_t}\sin(B_t) \\
&= f(t,N_t,B_t) - f(0,N_0,B_0) \\
&= \frac{1}{2}\int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\sin(B_u)\,\mathrm{d}u + \log\left(\frac{3}{2}\right)\int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\sin(B_u)\,\mathrm{d}N_u \\
&\quad + \int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}B_u - \frac{1}{2}\int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\sin(B_u)\,\mathrm{d}[B,B]_u^c \\
&\quad + \sum_{0<s\le t}\Delta\left(e^{s/2}\left(\frac{3}{2}\right)^{N_s}\sin(B_s)\right) - \log\left(\frac{3}{2}\right)e^{s/2}\left(\frac{3}{2}\right)^{N_{s-}}\sin(B_s)\Delta N_s \\
&= \log\left(\frac{3}{2}\right)\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^{k-1}\sin(B_{T_k}) + \int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}B_u \\
&\quad + \sum_{k=1}^{N_t} e^{T_k/2}\left(\left(\frac{3}{2}\right)^{k} - \left(\frac{3}{2}\right)^{k-1}\right)\sin(B_{T_k}) - \log\left(\frac{3}{2}\right)e^{T_k/2}\left(\frac{3}{2}\right)^{k-1}\sin(B_{T_k}) \\
&= \log\left(\frac{3}{2}\right)\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^{k-1}\sin(B_{T_k}) + \int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}B_u \\
&\quad + \frac{1}{3}\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^{k}\sin(B_{T_k}) - \log\left(\frac{3}{2}\right)\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^{k-1}\sin(B_{T_k}) \\
&= \int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}B_u + \frac{1}{3}\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^{k}\sin(B_{T_k})
\end{aligned}$$

so that

$$\int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}B_u = e^{t/2}\left(\frac{3}{2}\right)^{N_t}\sin(B_t) - \frac{1}{3}\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^{k}\sin(B_{T_k}).$$

An application of the multi-dimensional Ito formula with g yields

$$\begin{aligned}
&e^{t/2}\left(\frac{3}{2}\right)^{N_t}\cos(B_t) - 1 \\
&= g(t,N_t,B_t) - g(0,N_0,B_0) \\
&= \frac{1}{2}\int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}u + \log\left(\frac{3}{2}\right)\int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}N_u \\
&\quad - \int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\sin(B_u)\,\mathrm{d}B_u - \frac{1}{2}\int_0^t e^{u/2}\left(\frac{3}{2}\right)^{N_{u-}}\cos(B_u)\,\mathrm{d}[B,B]_u^c
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0<s\le t} \Delta \left(e^{s/2} \left(\frac{3}{2}\right)^{N_s} \cos(B_s) \right) - \log\left(\frac{3}{2}\right) e^{s/2} \left(\frac{3}{2}\right)^{N_{s-}} \cos(B_s)\Delta N_s \\
&= \log\left(\frac{3}{2}\right) \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k-1} \cos(B_{T_k}) - \int_0^t e^{u/2} \left(\frac{3}{2}\right)^{N_{u-}} \sin(B_u)\,\mathrm{d}B_u \\
&+ \sum_{k=1}^{N_t} e^{T_k/2} \left(\left(\frac{3}{2}\right)^{k} - \left(\frac{3}{2}\right)^{k-1} \right) \cos(B_{T_k}) - \log\left(\frac{3}{2}\right) e^{T_k/2} \left(\frac{3}{2}\right)^{k-1} \cos(B_{T_k}) \\
&= \log\left(\frac{3}{2}\right) \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k-1} \cos(B_{T_k}) - \int_0^t e^{u/2} \left(\frac{3}{2}\right)^{N_{u-}} \sin(B_u)\,\mathrm{d}B_u \\
&+ \frac{1}{3} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k} \cos(B_{T_k}) - \log\left(\frac{3}{2}\right) \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k-1} \cos(B_{T_k}) \\
&= -\int_0^t e^{u/2} \left(\frac{3}{2}\right)^{N_{u-}} \sin(B_u)\,\mathrm{d}B_u + \frac{1}{3} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k} \cos(B_{T_k})
\end{aligned}
$$

so that

$$
\int_0^t e^{u/2} \left(\frac{3}{2}\right)^{N_{u-}} \sin(B_u)\,\mathrm{d}B_u = 1 - e^{t/2} \left(\frac{3}{2}\right)^{N_t} \cos(B_t) + \frac{1}{3} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k} \cos(B_{T_k}).
$$

Inserting both stochastic integrals yields

$$
\begin{aligned}
\int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + e^{t/2} \left(\frac{3}{2}\right)^{N_t} \begin{pmatrix} \sin(B_t) & -\cos(B_t) \\ \cos(B_t) & \sin(B_t) \end{pmatrix} \\
- \frac{1}{3} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k} \begin{pmatrix} \sin(B_{T_k}) & -\cos(B_{T_k}) \\ \cos(B_{T_k}) & \sin(B_{T_k}) \end{pmatrix}
\end{aligned}
$$

and thus the left MGOU process is given by

$$
\begin{aligned}
V_t &= \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u \\
&= e^{-t/2} \left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&+ e^{-t/2} \left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix} \cdot e^{t/2} \left(\frac{3}{2}\right)^{N_t} \begin{pmatrix} \sin(B_t) & -\cos(B_t) \\ \cos(B_t) & \sin(B_t) \end{pmatrix} \\
&- e^{-t/2} \left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix} \cdot \frac{1}{3} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k} \begin{pmatrix} \sin(B_{T_k}) & -\cos(B_{T_k}) \\ \cos(B_{T_k}) & \sin(B_{T_k}) \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + e^{-t/2} \left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \sin(B_t) & \cos(B_t) \\ -\cos(B_t) & \sin(B_t) \end{pmatrix} \\
&- \frac{1}{3} e^{-t/2} \left(\frac{2}{3}\right)^{N_t} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2}\right)^{k} \begin{pmatrix} \sin(B_{T_k} - B_t) & -\cos(B_{T_k} - B_t) \\ \cos(B_{T_k} - B_t) & \sin(B_{T_k} - B_t) \end{pmatrix}.
\end{aligned}
$$

The left MGOU process is described by a stochastic differential equation driven by a bivariate Lévy process (U, L) which is constructed from the background driving process (X, Y). The process U already appeared in (3.6) when computing the inverse of a stochastic exponential.

Theorem 4.4. [4, Theorem 3.4]

(a) *Let $(X,Y) = (X_t, Y_t)_{t\geq 0}$ be a* LÉVY *process such that X satisfies (3.3) and let the process $(U,L) = (U_t, L_t)_{t\geq 0}$ be defined by*

$$U_t = -X_t + [X,X]_t^c + \sum_{0<s\leq t} ((I+\Delta X_s)^{-1} - I + \Delta X_s), \tag{4.5a}$$

$$L_t = Y_t - [X,Y]_t^c + \sum_{0<s\leq t} ((I+\Delta X_s)^{-1} - I)\Delta Y_s. \tag{4.5b}$$

Then (U,L) is a LÉVY *process such that U satisfies (3.3) and the left MGOU process V defined as in (4.4) solves the stochastic differential equation*

$$\mathrm{d}V_t = \mathrm{d}U_t\, V_{t-} + \mathrm{d}L_t\,. \tag{4.6}$$

(b) *Let $(U,L) = (U_t, L_t)_{t\geq 0}$ be a* LÉVY *process such that U satisfies (3.3) and let V_0 be an $\mathbb{R}^{n\times n}$-valued random variable. Then the solution $V = (V_t)_{t\geq 0}$ of the stochastic differential equation (4.6) is a left MGOU process and its background driving process $(X,Y) = (X_t, Y_t)_{t\geq 0}$ is the* LÉVY *process given by*

$$X_t = \overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U_t)^{-1}\right), \tag{4.7a}$$

$$Y_t = L_t + \left[\overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U)^{-1}\right), L\right]_t. \tag{4.7b}$$

Furthermore X satisfies (3.3).

The following relations between (X,Y) and (U,L) will be used in future proofs.

Proposition 4.5. [4, Proposition 3.5] *Let $(X,Y) = (X_t, Y_t)_{t\geq 0}$ be a* LÉVY *process such that X satisfies (3.3) and let the process $(U,L) = (U_t, L_t)_{t\geq 0}$ be defined as in (4.5). Then for all $t \geq 0$*

$$L_t = Y_t + [U,Y]_t, \tag{4.8a}$$

$$Y_t = L_t + [X,L]_t. \tag{4.8b}$$

We now prove that Theorem 4.1 and Theorem 4.4 also hold true when the stationarity property (L3) is weakened to a property (L3*) requiring only periodic stationarity and the stationary increments of the background driving process of the left MGOU process are weakened to periodically stationary increments. We thus replace (L3) by

(L3*) $(A_{p,p+t}, B_{p,p+t})_{t\geq 0} \overset{\mathrm{D}}{=} (A_{0,t}, B_{0,t})_{t\geq 0}$ for some $p > 0$

and refer to (L0),(L1),(L2),(L3*),(L4) as the **L*-assumptions**.

Theorem 4.6 and Theorem 4.7 show that there still is a one-to-one-correspondence between $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ and (X,Y) but imposing the L*-assumptions on the random functional results in a semi-LÉVY process as the background driving process.

Theorem 4.6. *Let $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ satisfy the L*-assumptions for some $p > 0$ and let $(A_t := A_{0,t})_{t\geq 0}$, $(B_t := B_{0,t})_{t\geq 0}$ be càdlàg. Then the process $(X,Y) = (X_t, Y_t)_{t\geq 0}$ defined as in (4.2) is the unique p-semi-*LÉVY *process such that X satisfies (3.3) and (4.3) holds almost surely.*

Proof. The proof is similar to the proofs of Theorem 3.1(a) and Proposition 6.1 in [4]. (X,Y) is càdlàg since $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are càdlàg, and $(X_0,Y_0)=(0,0)$ holds by definition. By (L1) we have for all $0\leq s\leq t$

$$\begin{aligned}
X_t - X_s &= \int_s^t A_{u-}\,\mathrm{d}A_u^{-1} = \int_s^t A_{s,u-}A_s\,\mathrm{d}(A_{s,u}A_s)^{-1} \\
&= \int_s^t A_{s,u-}A_s\,\mathrm{d}(A_s^{-1}A_{s,u}^{-1}) = \int_s^t A_{s,u-}\,\mathrm{d}A_{s,u}^{-1} \\
Y_t - Y_s &= \int_s^t A_{u-}\,\mathrm{d}(A_u^{-1}B_u) \\
&= \int_s^t A_{s,u-}A_s\,\mathrm{d}\big((A_{s,u}A_s)^{-1}(A_{s,u}B_s + B_{s,u})\big) \\
&= \int_s^t A_{s,u-}A_s\,\mathrm{d}(A_s^{-1}A_{s,u}^{-1}A_{s,u}B_s + A_s^{-1}A_{s,u}^{-1}B_{s,u}) \\
&= \int_s^t A_{s,u-}\,\mathrm{d}(A_{s,u}^{-1}B_{s,u})
\end{aligned}$$

so that (X,Y) has independent increments because by (L2) the random variables

$$(X_{t_{i+1}} - X_{t_i}, Y_{t_{i+1}} - Y_{t_i}) = \left(\int_{t_i}^{t_{i+1}} A_{t_i,u-}\,\mathrm{d}A_{t_i,u}^{-1}\,,\ \int_{t_i}^{t_{i+1}} A_{t_i,u-}\,\mathrm{d}(A_{t_i,u}^{-1}B_{t_i,u})\right)$$

are independent for all $t_0<\ldots<t_k$, $k\in\mathbb{N}$. (X,Y) has p-stationary increments because by (L2),(L3*)

$$\begin{aligned}
(X_{t+p}-X_p, Y_{t+p}-Y_p)_{t\geq 0} &= \left(\int_p^{t+p} A_{p,u-}\,\mathrm{d}A_{p,u}^{-1}\,,\ \int_p^{t+p} A_{p,u-}\,\mathrm{d}(A_{p,u}^{-1}B_{p,u})\right)_{t\geq 0} \\
&= \left(\int_0^t A_{p,(p+u)-}\,\mathrm{d}A_{p,p+u}^{-1}\,,\ \int_0^t A_{p,(p+u)-}\,\mathrm{d}(A_{p,p+u}^{-1}B_{p,p+u})\right)_{t\geq 0} \\
&\overset{\mathrm{D}}{=} \left(\int_0^t A_{u-}\,\mathrm{d}A_u^{-1}\,,\ \int_0^t A_{u-}\,\mathrm{d}(A_u^{-1}B_u)\right)_{t\geq 0} = (X_t,Y_t)_{t\geq 0}.
\end{aligned}$$

Thus (X,Y) is a p-semi-Lévy process. $X=\overleftarrow{\mathsf{Log}}(A^{-1})$ satisfies (3.3) by Theorem 3.6(b) and (4.3) is derived in exactly the same way as in the proof of Theorem 3.1 in [4]. ☐

Theorem 4.7. *Let $(X,Y)=(X_t,Y_t)_{t\geq 0}$ be a p-semi-Lévy process for some $p>0$ such that X satisfies (3.3). Then the family $(A_{s,t},B_{s,t})_{0\leq s\leq t}$ defined as in (4.3) satisfies the L*-assumptions.*

Proof. (L0), (L1), and (L4) are obtained in the same way as in the proof of Theorem 3.1(b) in [4]. (L2) can also be shown in the same way as in [4] since by Theorem 3.6(a) $\overleftarrow{\mathsf{Exp}}(X)$ is a p-semi-LÉVY process and therefore still has independent increments. (L3*) follows from

$$\begin{aligned}
&(A_{p,p+t}, B_{p,p+t})_{t\geq 0} \\
&= \left(\overleftarrow{\mathsf{Exp}}(X_{t+p})^{-1}\overleftarrow{\mathsf{Exp}}(X_p), \overleftarrow{\mathsf{Exp}}(X_{t+p})^{-1}\overleftarrow{\mathsf{Exp}}(X_p) \int_p^{t+p} \overleftarrow{\mathsf{Exp}}(X_p)^{-1}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u \right)_{t\geq 0} \\
&= \left(\overleftarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1}, \overleftarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1} \int_p^{t+p} \overleftarrow{\mathsf{Exp}}(X_{u-}-X_p)\,\mathrm{d}Y_u \right)_{t\geq 0} \\
&= \left(\overleftarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1}, \overleftarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1} \int_0^{t} \overleftarrow{\mathsf{Exp}}(X_{(u+p)-}-X_p)\,\mathrm{d}(Y_{u+p}-Y_p) \right)_{t\geq 0} \\
&\overset{\mathrm{D}}{=} \left(\overleftarrow{\mathsf{Exp}}(X_t)^{-1}, \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \int_0^{t} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u \right)_{t\geq 0} = (A_{0,t}, B_{0,t})_{t\geq 0}.
\end{aligned}$$

□

Theorem 4.8 and Theorem 4.9 show that the left MGOU process driven by a semi-LÉVY process is still described by the stochastic differential equation (4.6) but the process (U, L) driving this equation now also is a semi-LÉVY process.

Theorem 4.8. *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a p-semi-LÉVY process for some $p > 0$ such that X satisfies (3.3) and let the process $(U, L) = (U_t, L_t)_{t\geq 0}$ be defined as in (4.5). Then (U, L) is a p-semi-LÉVY process such that U satisfies (3.3) and the left MGOU process V defined as in (4.4) solves the stochastic differential equation (4.6).*

Proof. (U, L) is càdlàg and has independent increments because (X, Y) is càdlàg and has independent increments. $(U_0, L_0) = (0, 0)$ holds by definition. Therefore we only need to prove that (U, L) has p-stationary increments. As both U and L consist of summands depending only on X or Y, ΔX or ΔY, respectively $[X, X]$ or $[X, Y]$, we first look at these three summands separately. Since (X, Y) has p-stationary increments we have

$$\begin{aligned}
\Delta X_{s+p} &= X_{s+p} - X_{(s+p)-} = (X_{s+p} - X_p) - (X_{(s+p)-} - X_p) \overset{\mathrm{D}}{=} X_s - X_{s-} = \Delta X_s, \\
\Delta Y_{s+p} &= Y_{s+p} - Y_{(s+p)-} = (Y_{s+p} - Y_p) - (Y_{(s+p)-} - Y_p) \overset{\mathrm{D}}{=} Y_s - Y_{s-} = \Delta Y_s.
\end{aligned}$$

For the second summand of L we have

$$\begin{aligned}
&[X, Y]_{t+p} - [X, Y]_p \\
&= X_{t+p}Y_{t+p} - \int_0^{t+p} X_{u-}\,\mathrm{d}Y_u - \int_0^{t+p} \mathrm{d}X_u\,Y_{u-} - X_pY_p + \int_0^{p} X_{u-}\,\mathrm{d}Y_u + \int_0^{p} \mathrm{d}X_u\,Y_{u-} \\
&= X_{t+p}Y_{t+p} - X_pY_p - \int_p^{t+p} X_{u-}\,\mathrm{d}Y_u - \int_p^{t+p} \mathrm{d}X_u\,Y_{u-} \\
&= X_{t+p}Y_{t+p} - X_pY_p - \int_0^{t} X_{(u+p)-}\,\mathrm{d}Y_{u+p} - \int_0^{t} \mathrm{d}X_{u+p}\,Y_{(u+p)-} \\
&= X_{t+p}Y_{t+p} - X_pY_p - \int_0^{t} (X_{(u+p)-} - X_p)\,\mathrm{d}Y_{u+p} - \int_0^{t} X_p\,\mathrm{d}Y_{u+p}
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t \mathrm{d}X_{u+p}\,(Y_{(u+p)-} - Y_p) - \int_0^t \mathrm{d}X_{u+p}\,Y_p \\
&= X_{t+p}Y_{t+p} - X_pY_p - \int_0^t (X_{(u+p)-} - X_p)\,\mathrm{d}Y_{u+p} - X_p(Y_{t+p} - Y_p) \\
&\quad - \int_0^t \mathrm{d}X_{u+p}\,(Y_{(u+p)-} - Y_p) - (X_{t+p} - X_p)Y_p \\
&= (X_{t+p} - X_p)(Y_{t+p} - Y_p) - \int_0^t (X_{(u+p)-} - X_p)\,\mathrm{d}(Y_{u+p} - Y_p) \\
&\quad - \int_0^t \mathrm{d}(X_{u+p} - X_p)\,(Y_{(u+p)-} - Y_p) \\
&\overset{\mathrm{D}}{=} X_tY_t - \int_0^t X_{u-}\,\mathrm{d}Y_u - \int_0^t \mathrm{d}X_u\,Y_{u-} = [X,Y]_t
\end{aligned}
$$

and therefore

$$
\begin{aligned}
[X,Y]^c_{t+p} - [X,Y]^c_p &= [X,Y]_{t+p} - \sum_{0<s\le t+p} \Delta X_s \Delta Y_s - [X,Y]_p + \sum_{0<s\le p} \Delta X_s \Delta Y_s \\
&= [X,Y]_{t+p} - [X,Y]_p - \sum_{p<s\le t+p} \Delta X_s \Delta Y_s \\
&= [X,Y]_{t+p} - [X,Y]_p - \sum_{0<s\le t} \Delta X_{s+p} \Delta Y_{s+p} \\
&\overset{\mathrm{D}}{=} [X,Y]_t - \sum_{0<s\le t} \Delta X_s \Delta Y_s = [X,Y]^c_t.
\end{aligned}
$$

In the special case $Y = X$ we also have $[X,X]^c_{t+p} - [X,X]^c_p \overset{\mathrm{D}}{=} [X,X]^c_t$. Together this yields

$$
\begin{aligned}
U_{t+p} - U_p &= -(X_{t+p} - X_p) + [X,X]^c_{t+p} - [X,X]^c_p + \sum_{p<s\le t+p} ((I + \Delta X_s)^{-1} - I + \Delta X_s) \\
&= -(X_{t+p} - X_p) + [X,X]^c_{t+p} - [X,X]^c_p + \sum_{0<s\le t} ((I + \Delta X_{s+p})^{-1} - I + \Delta X_{s+p}) \\
&\overset{\mathrm{D}}{=} -X_t + [X,X]^c_t + \sum_{0<s\le t} ((I + \Delta X_s)^{-1} - I + \Delta X_s) = U_t, \\
L_{t+p} - L_p &= Y_{t+p} - Y_p - ([X,Y]^c_{t+p} - [X,Y]^c_p) + \sum_{p<s\le t+p} ((I + \Delta X_s)^{-1} - I)\Delta Y_s \\
&= Y_{t+p} - Y_p - ([X,Y]^c_{t+p} - [X,Y]^c_p) + \sum_{0<s\le t} ((I + \Delta X_{s+p})^{-1} - I)\Delta Y_{s+p} \\
&\overset{\mathrm{D}}{=} Y_t - [X,Y]^c_t + \sum_{0<s\le t} ((I + \Delta X_s)^{-1} - I)\Delta Y_s = L_t.
\end{aligned}
$$

Combining this we get $(U_{t+p} - U_p, L_{t+p} - L_p)_{t\ge 0} \overset{\mathrm{D}}{=} (U_t, L_t)_{t\ge 0}$. Thus (U, L) is a p-semi-Lévy process. (3.3) is clear from (3.7c) and the stochastic differential equation (4.6) is derived in exactly the same way as in the proof of Theorem 3.4(a) in [4]. □

Theorem 4.9. *Let $(U, L) = (U_t, L_t)_{t\geq 0}$ be a p-semi-Lévy process for some $p > 0$ such that U satisfies (3.3) and let V_0 be an $\mathbb{R}^{n\times n}$-valued random variable. Then the solution $V = (V_t)_{t\geq 0}$ of the stochastic differential equation (4.6) is a left MGOU process and its background driving process $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is the p-semi-Lévy process given by (4.7). Furthermore X satisfies (3.3).*

Proof. (X, Y) is càdlàg and has independent increments because (U, L) is càdlàg and has independent increments. $(X_0, Y_0) = (0, 0)$ holds by definition. X has p-stationary increments because by (3.5a), (3.4b), and the p-stationary increments of U we have for all $t \geq 0$

$$\begin{aligned}(X_{t+p} - X_p)_{t\geq 0} &= \left(\overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U_{t+p})^{-1}\right) - \overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U_p)^{-1}\right)\right)_{t\geq 0} \\ &= \left(\overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U_p)\overrightarrow{\mathsf{Exp}}(U_{t+p})^{-1}\right)\right)_{t\geq 0} \\ &= \left(\overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U_{t+p} - U_p)^{-1}\right)\right)_{t\geq 0} \\ &\overset{\mathrm{D}}{=} \left(\overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U_t)^{-1}\right)\right)_{t\geq 0} = (X_t)_{t\geq 0}.\end{aligned}$$

Inserting (4.7a) in (4.7b) yields $Y_t = L_t + [X, L]_t$ and since X and L have p-stationary increments it can be shown as in the proof of Theorem 4.8 that $[X, L]$ has p-stationary increments as well. Therefore Y has p-stationary increments because

$$(Y_{t+p} - Y_p)_{t\geq 0} = (L_{t+p} - L_p + [X, L]_{t+p} - [X, L]_p)_{t\geq 0} \overset{\mathrm{D}}{=} (L_t - [X, L]_t)_{t\geq 0} = (Y_t)_{t\geq 0}.$$

The fact that the solution of (4.6) is a left MGOU process driven by (X, Y) is proven in exactly the same way as in the proof of Theorem 3.4(b) in [4]. Furthermore, by (3.7b) and (3.4c)

$$X_t = \overleftarrow{\mathsf{Log}}\left(\overrightarrow{\mathsf{Exp}}(U_t)^{-1}\right) = \overleftarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(\widetilde{U}_t)\right) = \widetilde{U}_t$$

where $\widetilde{U} = (\widetilde{U}_t)_{t\geq 0}$ is constructed from U by (3.6), and then by (3.7c) we have for all $t \geq 0$

$$\det(I + \Delta X_t) = \det\left(I + \Delta\widetilde{U}_t\right) = \frac{1}{\det(I + \Delta U_t)} \neq 0.$$

□

Behme and Lindner also give conditions under which the left MGOU process is stationary.

Theorem 4.10. [4, Theorem 5.2] *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a Lévy process such that X satisfies (3.3) and let $V = (V_t)_{t\geq 0}$ be the left MGOU process driven by (X, Y). Let $(U, L) = (U_t, L_t)_{t\geq 0}$ be defined as in (4.5).*

(a) *Let* $\mathrm{P}\text{-}\lim_{t\to\infty} \overleftarrow{\mathsf{Exp}}(U_t) = \mathbf{0}$. *Then a finite random variable V_0 can be chosen such that V is stationary if and only if the integral $\int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u$ converges in distribution to a finite random variable. In this case V_0 can be chosen independently of (X, Y) with*

$$V_0 \overset{\mathrm{D}}{=} \mathrm{D}\text{-}\lim_{t\to\infty} \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u\,. \tag{4.9a}$$

(b) *Let* $\mathrm{P}\text{-}\lim_{t\to\infty} \overleftarrow{\mathsf{Exp}}(X_t) = \mathbf{0}$. *Then a finite random variable* V_0 *can be chosen such that* V *is stationary if and only if the integral* $\int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u$ *converges in probability to a finite random variable. In this case* V_0 *can be chosen as*

$$V_0 = -\mathrm{P}\text{-}\lim_{t\to\infty} \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\,. \tag{4.9b}$$

For the proof they need the following distributional equality of left stochastic integrals of stochastic exponentials.

Theorem 4.11. [4, Proposition 8.3] *Let* $(X,Y) = (X_t, Y_t)_{t\geq 0}$ *be a* LÉVY *process such that* X *satisfies (3.3) and let* $(U,L) = (U_t, L_t)_{t\geq 0}$ *be defined as in (4.5). Then for all* $t > 0$

$$\overrightarrow{\mathsf{Exp}}(U_t) \int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u \overset{\mathrm{D}}{=} \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u\,. \tag{4.10}$$

In the situation of Theorem 4.11 the left side of (4.10) can also be written in the form

$$\begin{aligned}\overrightarrow{\mathsf{Exp}}(U_t) \int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u &= -\int_0^t \overrightarrow{\mathsf{Exp}}(U_t - U_{u-})\,\mathrm{d}(Y_t - Y_u) \overset{\mathrm{D}}{=} -\int_0^t \overrightarrow{\mathsf{Exp}}(U_{(t-u)-})\,\mathrm{d}Y_{t-u}\\ &= -\int_t^0 \overrightarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u = \int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u\end{aligned}$$

and (4.10) then becomes

$$\int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u \overset{\mathrm{D}}{=} \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u\,.$$

We now prove an analogous result in the case of periodically stationary increments and use it to derive conditions under which the left MGOU process is periodically stationary.

Theorem 4.12. *Let* $(X,Y) = (X_t, Y_t)_{t\geq 0}$ *be a* p*-semi-*LÉVY *process for some* $p > 0$ *such that* X *satisfies (3.3) and let* $(U,L) = (U_t, L_t)_{t\geq 0}$ *be defined as in (4.5). Then for all* $n \in \mathbb{N}$

$$\int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u \overset{\mathrm{D}}{=} \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u\,. \tag{4.11}$$

Proof. The proof is similar to the proof of Proposition 8.3 in [4]. Let (U', Y') be an independent copy of (U,Y). Then (U,L) and (U',L') are p-semi-LÉVY processes by Theorem 4.8. For fixed $n \in \mathbb{N}$ and $0 \leq s \leq np$ now let

$$\begin{aligned}\widehat{U}_s &:= U_{np} - U_{np-s},\\ \widehat{Y}_s &:= Y_{np} - Y_{np-s}\end{aligned}$$

and for $s \geq 0$ let

$$\widehat{U}'_s := U'_{(s+np)-} - U'_{np-},$$

$$\widehat{Y}'_s := Y'_{(s+np)-} - Y'_{np-}.$$

Then

$$\begin{aligned}
(\widehat{U}_s, \widehat{Y}_s)_{0\le s\le np} &= (U_{np} - U_{np-s}, Y_{np} - Y_{np-s})_{0\le s\le np} \\
&\overset{\mathrm{D}}{=} (U_s, Y_s)_{0\le s\le np} \\
&\overset{\mathrm{D}}{=} (U'_s, Y'_s)_{0\le s\le np} \\
&\overset{\mathrm{D}}{=} (U'_{(s+np)-} - U'_{np-}, Y'_{(s+np)-} - Y'_{np-})_{0\le s\le np} \\
&= (\widehat{U}'_s, \widehat{Y}'_s)_{0\le s\le np}.
\end{aligned}$$

For partitions $\sigma_m^- = \left(\dots, \frac{-2np}{m}, \frac{-np}{m}, 0\right)$ of the negative real line and $\sigma_m^+ = \left(0, \frac{np}{m}, \frac{2np}{m}, \dots\right)$ of the positive real line let

$$\begin{aligned}
A^{\sigma_m^-} &:= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}'_{np(i+1)/m}\right)\left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right), \\
B^{\sigma_m^+} &:= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m-}\right)\left(\widehat{Y}_{np(i+1)/m-} - \widehat{Y}_{npi/m-}\right).
\end{aligned}$$

Then as $|\sigma_m^-| \xrightarrow{m\to\infty} 0$ we have by Theorems II.21 and II.23 in [31]

$$\begin{aligned}
A^{\sigma_m^-} &= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}'_{np(i+1)/m}\right)\left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \\
&= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}'_{npi/m}\right)\left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \\
&\quad + \sum_{i=0}^{m-1} \left(\overleftarrow{\mathsf{Exp}}\left(\widehat{U}'_{np(i+1)/m}\right) - \overleftarrow{\mathsf{Exp}}\left(\widehat{U}'_{npi/m}\right)\right)\left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \\
&\xrightarrow{\mathrm{P}} \int_0^{np} \overleftarrow{\mathsf{Exp}}(\widehat{U}'_{u-})\,\mathrm{d}\widehat{Y}'_u + \left[\overleftarrow{\mathsf{Exp}}(\widehat{U}'), \widehat{Y}'\right]_{np} \\
&\overset{\mathrm{D}}{=} \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u + \left[\overleftarrow{\mathsf{Exp}}(U), Y\right]_{np}.
\end{aligned}$$

This equals the right side in (4.11) because from (4.8a), (2.18d), and (3.1a) we obtain

$$\begin{aligned}
\int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u &= \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}(Y_u + [U,Y]_u) \\
&= \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u + \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}[U,Y]_u \\
&= \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u + \left[\int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}U_u\,, \int_0^{np} \mathrm{d}Y_u\right] \\
&= \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u + \left[\overleftarrow{\mathsf{Exp}}(U), Y\right]_{np}.
\end{aligned}$$

As $|\sigma_m^+| \xrightarrow{m\to\infty} 0$ we have by Lemma 8.2 in [4] and Theorem II.21 in [31]

$$\begin{aligned}
B^{\sigma_m^+} &= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m-}\right)\left(\widehat{Y}_{np(i+1)/m-} - \widehat{Y}_{npi/m-}\right) \\
&= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(U_{np} - U_{np(m-i-1)/m}\right)\left(Y_{np(m-i)/m} - Y_{np(m-i-1)/m}\right) \\
&= \sum_{i=0}^{m-1} \overrightarrow{\mathsf{Exp}}(U_{np})\overrightarrow{\mathsf{Exp}}\left(U_{np(m-i-1)/m}\right)^{-1}\left(Y_{np(m-i)/m} - Y_{np(m-i-1)/m}\right) \\
&= \overrightarrow{\mathsf{Exp}}(U_{np}) \sum_{i=1}^{m} \overrightarrow{\mathsf{Exp}}\left(U_{np(i-1)/m}\right)^{-1}\left(Y_{npi/m} - Y_{np(i-1)/m}\right) \\
&\xrightarrow{\mathrm{P}} \overrightarrow{\mathsf{Exp}}(U_{np}) \int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u\,.
\end{aligned}$$

Using the p-stationary increments of (U,Y) we can also write this in the form

$$\begin{aligned}
\overrightarrow{\mathsf{Exp}}(U_{np}) \int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u &= \overrightarrow{\mathsf{Exp}}(U_{np}) \int_{-np}^{0} \overrightarrow{\mathsf{Exp}}(U_{(np+u)-})^{-1}\,\mathrm{d}Y_{np+u} \\
&= -\int_{-np}^{0} \overrightarrow{\mathsf{Exp}}(U_{np} - U_{(np+u)-})\,\mathrm{d}(Y_{np} - Y_{np+u}) \\
&\overset{\mathrm{D}}{=} -\int_{-np}^{0} \overrightarrow{\mathsf{Exp}}(U_{(-u)-})\,\mathrm{d}Y_{-u} = \int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Y_u
\end{aligned}$$

which is the left side in (4.11). The equality in distribution now follows from the fact that

$$\begin{aligned}
B^{\sigma_m^+} &= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m-}\right)\left(\widehat{Y}_{np(i+1)/m-} - \widehat{Y}_{npi/m-}\right) \\
&= \sum_{i=0}^{m-1} \left(\overleftarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m}\right) - \Delta\overleftarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m}\right)\right) \\
&\qquad \cdot \left(\left(\widehat{Y}_{np(i+1)/m} - \Delta\widehat{Y}_{np(i+1)/m}\right) - \left(\widehat{Y}_{npi/m} - \Delta\widehat{Y}_{npi/m}\right)\right) \\
&= \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m}\right)\left(\widehat{Y}_{np(i+1)/m} - \widehat{Y}_{npi/m}\right) \\
&\overset{\mathrm{D}}{=} \sum_{i=0}^{m-1} \overleftarrow{\mathsf{Exp}}\left(\widehat{U}'_{np(i+1)/m}\right)\left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) = A^{\sigma_m^-}
\end{aligned}$$

because at fixed times both $\overleftarrow{\mathsf{Exp}}(\widehat{U})$ and $\widehat{Y}$ almost surely do not have jumps. □

Theorem 4.13. *Let $(X,Y) = (X_t,Y_t)_{t\geq 0}$ be a p-semi-Lévy process for some $p > 0$ such that X satisfies (3.3) and let $V = (V_t)_{t\geq 0}$ be the left MGOU process driven by (X,Y). Let $(U,L) = (U_t,L_t)_{t\geq 0}$ be defined as in (4.5).*

(a) *Let* $\text{P-}\lim_{n\to\infty} \overrightarrow{\mathsf{Exp}}(U_{np}) = \mathbf{0}$. *Then a finite random variable* V_0 *can be chosen such that* V *is p-stationary if and only if the integral* $\int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u$ *converges in distribution to a finite random variable. In this case* V_0 *can be chosen independently of* (X,Y) *with*

$$V_0 \overset{\mathrm{D}}{=} \text{D-}\lim_{n\to\infty} \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u\,. \tag{4.12a}$$

(b) *Let* $\text{P-}\lim_{n\to\infty} \overleftarrow{\mathsf{Exp}}(X_{np}) = \mathbf{0}$. *Then a finite random variable* V_0 *can be chosen such that* V *is p-stationary if and only if the integral* $\int_0^{np} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u$ *converges in probability to a finite random variable. In this case* V_0 *can be chosen as*

$$V_0 = -\text{P-}\lim_{n\to\infty} \int_0^{np} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\,. \tag{4.12b}$$

Proof. The proof is similar to the proof of Theorem 5.2 in [4].

(a) Assume that V is p-stationary. Then $V_{np} \overset{\mathrm{D}}{=} V_0$ for all $n \in \mathbb{N}$ and thus

$$\begin{aligned}
V_0 &\overset{\mathrm{D}}{=} \text{D-}\lim_{n\to\infty} V_{np} \\
&= \text{D-}\lim_{n\to\infty} \overrightarrow{\mathsf{Exp}}(U_{np})\left(V_0 + \int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u\right) \\
&\overset{\mathrm{D}}{=} \text{D-}\lim_{n\to\infty} \overrightarrow{\mathsf{Exp}}(U_{np})V_0 + \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u \\
&= \text{D-}\lim_{n\to\infty} \int_0^{np} \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u\,.
\end{aligned}$$

Now assume that (4.12a) holds. Then for fixed $t \geq 0$ and all $n \in \mathbb{N}$

$$\begin{aligned}
V_{t+np} &= \overrightarrow{\mathsf{Exp}}(U_{t+np})\left(V_0 + \int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u + \int_{np}^{t+np} \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u\right) \\
&= \overrightarrow{\mathsf{Exp}}(U_{t+np})\overrightarrow{\mathsf{Exp}}(U_{np})^{-1}\Bigg(\overrightarrow{\mathsf{Exp}}(U_{np})V_0 + \overrightarrow{\mathsf{Exp}}(U_{np})\int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u \\
&\quad + \overrightarrow{\mathsf{Exp}}(U_{np})\int_0^{t} \overrightarrow{\mathsf{Exp}}(U_{(u+np)-})^{-1}\,\mathrm{d}Y_{u+np}\Bigg) \\
&= \overrightarrow{\mathsf{Exp}}(U_{t+np} - U_{np})\Bigg(\overrightarrow{\mathsf{Exp}}(U_{np})V_0 + \overrightarrow{\mathsf{Exp}}(U_{np})\int_0^{np} \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u
\end{aligned}$$

$$+\int_0^t \overrightarrow{\mathsf{Exp}}(U_{(u+np)-}-U_{np})^{-1}\,\mathrm{d}(Y_{u+np}-Y_{np})\Bigg)$$

$$\stackrel{\mathrm{D}}{=}\overrightarrow{\mathsf{Exp}}(U_t)\Bigg(\overrightarrow{\mathsf{Exp}}(U_{np})V_0+\int_0^{np}\overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}L_u+\int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u\Bigg)$$

$$\xrightarrow{\mathrm{D}}\overrightarrow{\mathsf{Exp}}(U_t)\Bigg(V_0+\int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})^{-1}\,\mathrm{d}Y_u\Bigg)=V_t$$

as $n\to\infty$, where we have used (4.11) in the second to last step. Thus

$$V_{t+p}\stackrel{\mathrm{D}}{=}\text{D-}\lim_{n\to\infty}V_{t+p+np}=\text{D-}\lim_{n\to\infty}V_{t+(n+1)p}\stackrel{\mathrm{D}}{=}V_t.$$

(b) Assume that V is p-stationary. Then $V_{np}\stackrel{\mathrm{D}}{=}V_0$ for all $n\in\mathbb{N}_0$ and thus

$$V_0+\int_0^{np}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u=\overleftarrow{\mathsf{Exp}}(X_{np})V_{np}\xrightarrow{\mathrm{P}}0\iff V_0=-\text{P-}\lim_{n\to\infty}\int_0^{np}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\,.$$

Now assume that (4.12b) holds. Then for all $n\in\mathbb{N}$

$$\begin{aligned}
V_{np}&=\overleftarrow{\mathsf{Exp}}(X_{np})^{-1}\Bigg(V_0+\int_0^{np}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\Bigg)\\
&=\overleftarrow{\mathsf{Exp}}(X_{np})^{-1}\Bigg(-\int_0^{\infty}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u+\int_0^{np}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\Bigg)\\
&=-\overleftarrow{\mathsf{Exp}}(X_{np})^{-1}\int_{np}^{\infty}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\\
&=-\overleftarrow{\mathsf{Exp}}(X_{np})^{-1}\int_0^{\infty}\overleftarrow{\mathsf{Exp}}(X_{(u+np)-})\,\mathrm{d}Y_{u+np}\\
&=-\int_0^{\infty}\overleftarrow{\mathsf{Exp}}(X_{(u+np)-}-X_{np})\,\mathrm{d}(Y_{u+np}-Y_{np})\\
&\stackrel{\mathrm{D}}{=}-\int_0^{\infty}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u=V_0
\end{aligned}$$

and thus for all $t\geq 0$ by the independence of V_p and $(A_{p,p+t},B_{p,p+t})$

$$V_{t+p}=A_{p,p+t}V_p+B_{p,p+t}\stackrel{\mathrm{D}}{=}A_{0,t}V_0+B_{0,t}=V_t.$$

□

When studying the connection between left MGOU processes, left semiselfsimilar processes, and left semistable hemigroups in chapters 5.1 and 6.1 we need the notion of a left MGOU process with time parameter $t\in\mathbb{R}$ rather than just $t\geq 0$. In order to define a left MGOU process with real time parameter we make use of the construction of a Lévy process with real time parameter in Theorem 2.3.

Theorem 4.14. *Let $(X,Y) = (X_t, Y_t)_{t\geq 0}$ be a semimartingale such that X satisfies (3.3) and let (X', Y') be an independent copy of (X,Y). Let $(\widetilde{X}, \widetilde{Y})$ be defined as in (2.5). Assume that*

$$V_0 := \int_{-\infty}^{0} \overleftarrow{\mathsf{Exp}}\left(-X'_{(-u)-}\right) \mathrm{d}(-Y'_{-u}) := \text{P-}\lim_{t\to\infty} \int_{-t}^{0} \overleftarrow{\mathsf{Exp}}\left(-X'_{(-u)-}\right) \mathrm{d}(-Y'_{-u}) \tag{4.13a}$$

exists and let the process $V = (V_t)_{t\in\mathbb{R}}$ be defined by

$$V_t := \overleftarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\, \mathrm{d}\widetilde{Y}_u\,. \tag{4.13b}$$

(a) *If (X,Y) is a* Lévy *process then V is stationary.*

(b) *If (X,Y) is a p-semi-*Lévy *process for some $p > 0$ then V is p-stationary.*

Proof. First assume that (X,Y) is a Lévy process. Then $(\widetilde{X}, \widetilde{Y})$ is a Lévy process by Theorem 2.3(a) and therefore has stationary increments. By (3.4a) we have for all $t \in \mathbb{R}$ and $h > 0$

$$\begin{aligned}
V_{t+h} &= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t+h})^{-1} \int_{-\infty}^{t+h} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\, \mathrm{d}\widetilde{Y}_u \\
&= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t+h} - \widetilde{X}_h)^{-1} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_h)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{(u+h)-})\, \mathrm{d}\widetilde{Y}_{u+h} \\
&= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t+h} - \widetilde{X}_h)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{(u+h)-} - \widetilde{X}_h)\, \mathrm{d}(\widetilde{Y}_{u+h} - \widetilde{Y}_h) \\
&\overset{\mathrm{D}}{=} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\, \mathrm{d}\widetilde{Y}_u = V_t.
\end{aligned}$$

Now assume that (X,Y) is a p-semi-Lévy process. Then $(\widetilde{X}, \widetilde{Y})$ is a p-semi-Lévy process by Theorem 2.3(b) and therefore has p-stationary increments. By (3.4a) we have for all $t \geq 0$

$$\begin{aligned}
V_{t+p} &= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t+p})^{-1} \int_{-\infty}^{t+p} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\, \mathrm{d}\widetilde{Y}_u \\
&= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t+p} - \widetilde{X}_p)^{-1} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_p)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{(u+p)-})\, \mathrm{d}\widetilde{Y}_{u+p} \\
&= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t+p} - \widetilde{X}_p)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{(u+p)-} - \widetilde{X}_p)\, \mathrm{d}(\widetilde{Y}_{u+p} - \widetilde{Y}_p) \\
&\overset{\mathrm{D}}{=} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\, \mathrm{d}\widetilde{Y}_u = V_t
\end{aligned}$$

and for $t < 0$

$$\begin{aligned}
V_{t-p} &= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t-p})^{-1} \int_{-\infty}^{t-p} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\,\mathrm{d}\widetilde{Y}_u \\
&= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t-p} - \widetilde{X}_{-p})^{-1} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{-p})^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{(u-p)-})\,\mathrm{d}\widetilde{Y}_{u-p} \\
&= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{t-p} - \widetilde{X}_{-p})^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{(u-p)-} - \widetilde{X}_{-p})\,\mathrm{d}(\widetilde{Y}_{u-p} - \widetilde{Y}_{-p}) \\
&\overset{\mathrm{D}}{=} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\,\mathrm{d}\widetilde{Y}_u = V_t.
\end{aligned}$$

□

For $t > 0$ the process in (4.13b) can be written as

$$\begin{aligned}
V_t &= \overleftarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} \int_{-\infty}^{t} \overleftarrow{\mathsf{Exp}}(\widetilde{X}_{u-})\,\mathrm{d}\widetilde{Y}_u \\
&= \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \left(\int_{-\infty}^{0} \overleftarrow{\mathsf{Exp}}\left(-X'_{(-u)-}\right) \mathrm{d}(-Y'_{-u}) + \int_{0}^{t} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u \right) \\
&= \overleftarrow{\mathsf{Exp}}(X_t)^{-1} \left(V_0 + \int_{0}^{t} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u \right)
\end{aligned}$$

which is the usual integral form of a left MGOU process as in (4.4) but by (4.13a) V_0 can be chosen independently of the background driving process (X, Y). This fact together with Theorem 4.14 now motivates the following definition.

Definition 4.15. Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a Lévy respectively semi-Lévy process such that X satisfies (3.3) and let (X', Y') be an independent copy of (X, Y). Let $(\widetilde{X}, \widetilde{Y})$ be defined as in (2.5) and assume that V_0 as in (4.13a) exists. Then $V = (V_t)_{t\in\mathbb{R}}$ defined as in (4.13b) is called **stationary left MGOU process** respectively **periodically stationary left MGOU process**. The process $(\widetilde{X}, \widetilde{Y})$ is called **background driving process**.

4.2 Right MGOU processes

Given a family of $\mathsf{GL}_n(\mathbb{R}) \times \mathbb{R}^{n\times n}$-valued random variables $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ we now study the random recurrence equation

$$V_t = V_s A_{s,t} + B_{s,t} \tag{4.14}$$

which differs from (4.1) in that V_s is now multiplicated by $A_{s,t}$ from the right instead of from the left. In order to obtain a similar characterization of the solutions $V = (V_t)_{t\geq 0}$ of (4.14) as in the left case we need to impose the following assumptions on $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$.

(R0) $A_{t,t} = I$ and $B_{t,t} = \mathbf{0}$ for all $t \geq 0$.

(R1) $A_{r,t} = A_{r,s}A_{s,t}$ and $B_{r,t} = B_{r,s}A_{s,t} + B_{s,t}$ almost surely for all $0 \leq r \leq s \leq t$.

(R2) $(A_{s,t}, B_{s,t})_{a\le s\le t\le b}$ and $(A_{s,t}, B_{s,t})_{c\le s\le t\le d}$ are independent for all $0 \le a \le b \le c \le d$.

(R3) $(A_{s,t}, B_{s,t})_{0\le s\le t} \overset{\mathrm{D}}{=} (A_{0,t-s}, B_{0,t-s})_{0\le s\le t}$ for all $t \ge 0$.

(R4) $A_{0,t} \xrightarrow{\mathrm{P}} I$ and $B_{0,t} \xrightarrow{\mathrm{P}} \mathbf{0}$ for $t \downarrow 0$.

We refer to (R0),(R1),(R2),(R3),(R4) as the **R-assumptions**. In fact, the only difference between the R- and L-assumptions is (R1) which reflects the change in the order of multiplication in (4.14).

In Theorem 4.16 and Theorem 4.17 we prove a one-to-one correspondence between a family $(A_{s,t}, B_{s,t})_{0\le s\le t}$ of $\mathsf{GL}_n(\mathbb{R})\times\mathbb{R}^{n\times n}$-valued random variables which satisfies the R-assumptions and a bivariate Lévy process (X, Y) such that X satisfies (3.3). This result is similar to the left case in Theorem 4.1 which was proven by Behme and Lindner.

Theorem 4.16. *Let $(A_{s,t}, B_{s,t})_{0\le s\le t}$ satisfy the R-assumptions and let $(A_t := A_{0,t})_{t\ge0}$, $(B_t := B_{0,t})_{t\ge0}$ be càdlàg. Then the process $(X,Y) = (X_t, Y_t)_{t\ge0}$ defined by*

$$X_t := \int_0^t \mathrm{d}A_u^{-1}\, A_{u-} = \overrightarrow{\mathsf{Log}}(A_t^{-1}), \tag{4.15a}$$

$$Y_t := \int_0^t \mathrm{d}(B_u A_u^{-1})\, A_{u-} \tag{4.15b}$$

is the unique Lévy process such that X satisfies (3.3) and

$$A_{s,t} = \overrightarrow{\mathsf{Exp}}(X_s)\overrightarrow{\mathsf{Exp}}(X_t)^{-1}, \tag{4.16a}$$

$$B_{s,t} = \int_s^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_t)^{-1} \tag{4.16b}$$

almost surely for all $0 \le s \le t$.

Proof. (X,Y) is càdlàg since $(A_t)_{t\ge0}$ and $(B_t)_{t\ge0}$ are càdlàg, and $(X_0, Y_0) = (0,0)$ holds by definition. By (R1) we have for all $0 \le s \le t$

$$\begin{aligned}
X_t - X_s &= \int_s^t \mathrm{d}A_u^{-1}\, A_{u-} = \int_s^t \mathrm{d}(A_s A_{s,u})^{-1}\, A_s A_{s,u-} \\
&= \int_s^t \mathrm{d}(A_{s,u}^{-1} A_s^{-1})\, A_s A_{s,u-} = \int_s^t \mathrm{d}A_{s,u}^{-1}\, A_{s,u-} \\
Y_t - Y_s &= \int_s^t \mathrm{d}(B_u A_u^{-1})\, A_{u-} \\
&= \int_s^t \mathrm{d}\big((A_s A_{s,u})^{-1}(B_s A_{s,u} + B_{s,u})\big)\, A_s A_{s,u-} \\
&= \int_s^t \mathrm{d}(B_s A_{s,u} A_{s,u}^{-1} A_s^{-1} + B_{s,u} A_{s,u}^{-1} A_s^{-1})\, A_s A_{s,u-}
\end{aligned}$$

$$= \int_s^t \mathrm{d}(B_{s,u} A_{s,u}^{-1})\, A_{s,u-}$$

so that (X, Y) has independent increments because by (R2) the random variables

$$(X_{t_{i+1}} - X_{t_i}, Y_{t_{i+1}} - Y_{t_i}) = \left(\int_{t_i}^{t_{i+1}} \mathrm{d}A_{t_i,u}^{-1}\, A_{t_i,u-}, \int_{t_i}^{t_{i+1}} \mathrm{d}(B_{t_i,u} A_{t_i,u}^{-1})\, A_{t_i,u-} \right)$$

are independent for all $t_0 < \ldots < t_k$, $k \in \mathbb{N}$. (X, Y) has stationary increments because by (R2),(R3)

$$\begin{aligned}
(X_t - X_s, Y_t - Y_s) &= \left(\int_s^t A_{s,u-}\, \mathrm{d}A_{s,u}^{-1}, \int_s^t A_{s,u-}\, \mathrm{d}(A_{s,u}^{-1} B_{s,u}) \right) \\
&= \left(\int_0^{t-s} A_{s,(s+u)-}\, \mathrm{d}A_{s,s+u}^{-1}, \int_0^{t-s} A_{s,(s+u)-}\, \mathrm{d}(A_{s,s+u}^{-1} B_{s,s+u}) \right) \\
&\overset{\mathrm{D}}{=} \left(\int_0^{t-s} A_{u-}\, \mathrm{d}A_u^{-1}, \int_0^{t-s} A_{u-}\, \mathrm{d}(A_u^{-1} B_u) \right) = (X_{t-s}, Y_{t-s}).
\end{aligned}$$

Thus (X, Y) is a Lévy process. $X = \overrightarrow{\mathsf{Log}}(A^{-1})$ satisfies (3.3) by Proposition 3.5(b) and the calculations to derive (4.16) are similar to the proof of Theorem 3.1(a) in [4]. In fact, from the definition of X_t we get

$$X_t = \overrightarrow{\mathsf{Log}}(A_t^{-1}) \quad \Longleftrightarrow \quad \overrightarrow{\mathsf{Exp}}(X_t) = A_t^{-1} \quad \Longleftrightarrow \quad A_t = \overrightarrow{\mathsf{Exp}}(X_t)^{-1}$$

and (R1) then yields

$$A_{s,t} = A_s^{-1} A_t = \left(\overrightarrow{\mathsf{Exp}}(X_s)^{-1} \right)^{-1} \overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overrightarrow{\mathsf{Exp}}(X_s) \overrightarrow{\mathsf{Exp}}(X_t)^{-1}$$

for $0 \le s \le t$. Analogously, from the definition of Y_t we get

$$Y_t = \int_0^t \mathrm{d}(B_u A_u^{-1})\, A_{u-} \quad \Longleftrightarrow \quad \mathrm{d}Y_t\, A_{t-}^{-1} = \mathrm{d}(B_t A_t^{-1}) \quad \Longleftrightarrow \quad B_t = \int_0^t \mathrm{d}Y_u\, A_{u-}^{-1} A_t$$

and again (R1) yields

$$\begin{aligned}
B_{s,t} = B_t - B_s A_{s,t} &= \int_0^t \mathrm{d}Y_u\, A_{u-}^{-1} A_t - \int_0^s \mathrm{d}Y_u\, A_{u-}^{-1} A_s A_s^{-1} A_t \\
&= \int_s^t \mathrm{d}Y_u\, A_{u-}^{-1} A_t = \int_s^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \overrightarrow{\mathsf{Exp}}(X_t)^{-1}
\end{aligned}$$

for $0 \le s \le t$. □

Theorem 4.17. *Let $(X, Y) = (X_t, Y_t)_{t \ge 0}$ be a Lévy process such that X satisfies (3.3). Then the family $(A_{s,t}, B_{s,t})_{0 \le s \le t}$ defined as in (4.16) satisfies the R-assumptions.*

Proof. The proof is similar to the proof of Theorem 3.1(b) in [4]. (R0) is clear from the definition of $A_{s,t}$ and $B_{s,t}$ in (4.16). (R1) is obtained by direct calculation since for all $0 \le r \le s \le t$

$$A_{r,s}A_{s,t} = \overrightarrow{\mathsf{Exp}}(X_r)\overrightarrow{\mathsf{Exp}}(X_s)^{-1}\overrightarrow{\mathsf{Exp}}(X_s)\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overrightarrow{\mathsf{Exp}}(X_r)\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = A_{r,t}$$

and

$$\begin{aligned}
&B_{r,s}A_{s,t} + B_{s,t} \\
&= \int_r^s \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_s)^{-1}\overrightarrow{\mathsf{Exp}}(X_s)\overrightarrow{\mathsf{Exp}}(X_t)^{-1} + \int_s^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_t)^{-1} \\
&= \int_r^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = B_{r,t}.
\end{aligned}$$

(R2) is a result of the independent increments of (X, Y) and the fact that $(A_{s,t}, B_{s,t})$ can be written as

$$\begin{aligned}
(A_{s,t}, B_{s,t}) &= \left(\overrightarrow{\mathsf{Exp}}(X_s)\overrightarrow{\mathsf{Exp}}(X_t)^{-1}, \int_s^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_s)^{-1}\overrightarrow{\mathsf{Exp}}(X_s)\overrightarrow{\mathsf{Exp}}(X_t)^{-1}\right) \\
&= \left(\overrightarrow{\mathsf{Exp}}(X_t - X_s)^{-1}, \int_s^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-} - X_s)\overrightarrow{\mathsf{Exp}}(X_t - X_s)^{-1}\right).
\end{aligned}$$

(R3) follows from the same calculation since for fixed $t \ge 0$ by the stationarity of the increments of (X, Y)

$$\begin{aligned}
&(A_{s,t}, B_{s,t})_{0 \le s \le t} \\
&= \left(\overrightarrow{\mathsf{Exp}}(X_t - X_s)^{-1}, \int_s^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-} - X_s)\overrightarrow{\mathsf{Exp}}(X_t - X_s)^{-1}\right)_{0 \le s \le t} \\
&= \left(\overrightarrow{\mathsf{Exp}}(X_t - X_s)^{-1}, \int_0^{t-s} \mathrm{d}(Y_{u+s} - Y_s)\, \overrightarrow{\mathsf{Exp}}(X_{(u+s)-} - X_s)\overrightarrow{\mathsf{Exp}}(X_t - X_s)^{-1}\right)_{0 \le s \le t} \\
&\overset{\mathrm{D}}{=} \left(\overrightarrow{\mathsf{Exp}}(X_{t-s})^{-1}, \int_0^{t-s} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_{t-s})^{-1}\right)_{0 \le s \le t} \\
&= (A_{0,t-s}, B_{0,t-s})_{0 \le s \le t}.
\end{aligned}$$

(R4) follows from the fact that by Proposition 3.5(a) $\overrightarrow{\mathsf{Exp}}(X)^{-1}$ is a Lévy process and therefore continuous in probability at 0 so that $A_{0,t} = \overrightarrow{\mathsf{Exp}}(X_t)^{-1} \xrightarrow{\mathrm{P}} \overrightarrow{\mathsf{Exp}}(X_0)^{-1} = I$ for $t \downarrow 0$, as well as from the continuity of the stochastic integral which gives

$$B_{0,t} = \int_0^t \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_t)^{-1} \xrightarrow{\mathrm{P}} \mathbf{0} \quad \text{for } t \downarrow 0.$$

□

Theorem 4.17 motivates the definition of a right MGOU process as a counterpart to the left MGOU process in Definition 4.2 as the formulas for $A_{s,t}$ and $B_{s,t}$ in (4.16) and the random recurrence equation (4.14) together yield

$$V_t = V_0 A_{0,t} + B_{0,t} = V_0 \overleftarrow{\mathsf{Exp}}(X_t)^{-1} + \int_0^t \mathrm{d}Y_u \, \overleftarrow{\mathsf{Exp}}(X_{u-}) \overleftarrow{\mathsf{Exp}}(X_t)^{-1}$$
$$= \left(V_0 + \int_0^t \mathrm{d}Y_u \, \overleftarrow{\mathsf{Exp}}(X_{u-}) \right) \overleftarrow{\mathsf{Exp}}(X_t)^{-1}.$$

Definition 4.18. Let $(X, Y) = (X_t, Y_t)_{t \geq 0}$ be a LÉVY process such that X satisfies (3.3) and let V_0 be an $\mathbb{R}^{n \times n}$-valued random variable. Then the process $V = (V_t)_{t \geq 0}$ defined by

$$V_t := \left(V_0 + \int_0^t \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) \right) \overrightarrow{\mathsf{Exp}}(X_t)^{-1} \tag{4.17}$$

is called **right multivariate generalized** ORNSTEIN-UHLENBECK **process** or simply **right MGOU process**. The process (X, Y) is called **background driving process**.

Example 4.19. Let $B = (B_t)_{t \geq 0}$ be a one-dimensional BROWNian motion and $N = (N_t)_{t \geq 0}$ a one-dimensional POISSON process with almost surely increasing jump times $(T_k)_{k \in \mathbb{N}}$. We compute the right MGOU process $V = (V_t)_{t \geq 0}$ with $V_0 = \mathbf{0}$ and background driving process $(X, Y) = (X_t, Y_t)_{t \geq 0}$ defined by

$$X_t := \begin{pmatrix} \frac{1}{2} N_t & B_t \\ -B_t & \frac{1}{2} N_t \end{pmatrix} \quad \text{and} \quad Y_t := \begin{pmatrix} B_t & 0 \\ 0 & B_t \end{pmatrix}.$$

In Example 3.15 we already computed that

$$\overrightarrow{\mathsf{Exp}}(X_t) = e^{t/2} \left(\frac{3}{2} \right)^{N_t} \begin{pmatrix} \cos(B_t) & \sin(B_t) \\ -\sin(B_t) & \cos(B_t) \end{pmatrix}$$

and thus by (4.17)

$$V_t = \int_0^t \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) \overrightarrow{\mathsf{Exp}}(X_t)^{-1}$$
$$= \int_0^t e^{u/2} \left(\frac{3}{2} \right)^{N_{u-}} \begin{pmatrix} \cos(B_u)\,\mathrm{d}B_u & \sin(B_u)\,\mathrm{d}B_u \\ -\sin(B_u)\,\mathrm{d}B_u & \cos(B_u)\,\mathrm{d}B_u \end{pmatrix} \cdot e^{-t/2} \left(\frac{2}{3} \right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix}.$$

In Example 4.3 we already computed that

$$\int_0^t e^{u/2} \left(\frac{3}{2} \right)^{N_{u-}} \cos(B_u)\,\mathrm{d}B_u = e^{t/2} \left(\frac{3}{2} \right)^{N_t} \sin(B_t) - \frac{1}{3} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2} \right)^k \sin(B_{T_k}),$$
$$\int_0^t e^{u/2} \left(\frac{3}{2} \right)^{N_{u-}} \sin(B_u)\,\mathrm{d}B_u = 1 - e^{t/2} \left(\frac{3}{2} \right)^{N_t} \cos(B_t) + \frac{1}{3} \sum_{k=1}^{N_t} e^{T_k/2} \left(\frac{3}{2} \right)^k \cos(B_{T_k}).$$

Inserting both stochastic integrals yields

$$\int_0^t \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + e^{t/2}\left(\frac{3}{2}\right)^{N_t} \begin{pmatrix} \sin(B_t) & -\cos(B_t) \\ \cos(B_t) & \sin(B_t) \end{pmatrix}$$
$$- \frac{1}{3}\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^k \begin{pmatrix} \sin(B_{T_k}) & -\cos(B_{T_k}) \\ \cos(B_{T_k}) & \sin(B_{T_k}) \end{pmatrix}$$

and thus the right MGOU process is given by

$$\begin{aligned} V_t &= \int_0^t \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_t)^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot e^{-t/2}\left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix} \\ &\quad + e^{t/2}\left(\frac{3}{2}\right)^{N_t} \begin{pmatrix} \sin(B_t) & -\cos(B_t) \\ \cos(B_t) & \sin(B_t) \end{pmatrix} \cdot e^{-t/2}\left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix} \\ &\quad - \frac{1}{3}\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^k \begin{pmatrix} \sin(B_{T_k}) & -\cos(B_{T_k}) \\ \cos(B_{T_k}) & \sin(B_{T_k}) \end{pmatrix} \cdot e^{-t/2}\left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + e^{-t/2}\left(\frac{2}{3}\right)^{N_t} \begin{pmatrix} \sin(B_t) & \cos(B_t) \\ -\cos(B_t) & \sin(B_t) \end{pmatrix} \\ &\quad - \frac{1}{3}e^{-t/2}\left(\frac{2}{3}\right)^{N_t}\sum_{k=1}^{N_t} e^{T_k/2}\left(\frac{3}{2}\right)^k \begin{pmatrix} \sin(B_{T_k}-B_t) & -\cos(B_{T_k}-B_t) \\ \cos(B_{T_k}-B_t) & \sin(B_{T_k}-B_t) \end{pmatrix}. \end{aligned}$$

This result coincides with the left MGOU process in Example 4.3 which comes to no surprise since $\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t)$ by Example 3.15 and all matrices appearing in the calculations are either diagonal or belong to the commutative matrix group $\mathsf{SO}(2)$.

In Theorem 4.20 and Theorem 4.21 we prove that the right MGOU process is described by a stochastic differential equation driven by a bivariate Lévy process (U, L) which is constructed from the background driving process (X, Y). This result is similar to the left case in Theorem 4.4 which was proven by Behme and Lindner.

Theorem 4.20. *Let $(X,Y) = (X_t, Y_t)_{t\geq 0}$ be a Lévy process such that X satisfies (3.3) and let the process $(U, L) = (U_t, L_t)_{t\geq 0}$ be defined by*

$$U_t = -X_t + [X,X]_t^c + \sum_{0<s\leq t}\Big((I+\Delta X_s)^{-1} - I + \Delta X_s\Big), \tag{4.18a}$$

$$L_t = Y_t - [Y,X]_t^c + \sum_{0<s\leq t}\Delta Y_s\Big((I+\Delta X_s)^{-1} - I\Big). \tag{4.18b}$$

Then (U, L) is a Lévy process such that U satisfies (3.3) and the right MGOU process V defined as in (4.17) solves the stochastic differential equation

$$\mathrm{d}V_t = V_{t-}\,\mathrm{d}U_t + \mathrm{d}L_t\,. \tag{4.19}$$

Proof. (U, L) is càdlàg and has independent increments because (X, Y) is càdlàg and has independent increments. $(U_0, L_0) = (0, 0)$ holds by definition. Therefore we only need to prove that (U, L) has stationary increments. As both U and L consist of summands depending only on X or Y, ΔX or ΔY, respectively $[X,X]$ or $[X,Y]$, we first look at these three summands separately. Since (X,Y) has stationary increments we have $X_t - X_s \stackrel{\mathrm{D}}{=} X_{t-s}$ and $Y_t - Y_s \stackrel{\mathrm{D}}{=} Y_{t-s}$ and therefore

$$\begin{aligned}
\Delta X_{r+s} &= X_{r+s} - X_{(r+s)-} = (X_{r+s} - X_s) - (X_{(r+s)-} - X_s) \stackrel{\mathrm{D}}{=} X_r - X_{r-} = \Delta X_r, \\
\Delta Y_{r+s} &= Y_{r+s} - Y_{(r+s)-} = (Y_{r+s} - Y_s) - (Y_{(r+s)-} - Y_s) \stackrel{\mathrm{D}}{=} Y_r - Y_{r-} = \Delta Y_r.
\end{aligned}$$

For the second summand of L we have

$$\begin{aligned}
[Y,X]_t - [Y,X]_s &= Y_t X_t - \int_0^t Y_{u-}\, \mathrm{d}X_u - \int_0^t \mathrm{d}Y_u\, X_{u-} - Y_s X_s + \int_0^s Y_{u-}\, \mathrm{d}X_u + \int_0^s \mathrm{d}Y_u\, X_{u-} \\
&= Y_t X_t - Y_s X_s - \int_s^t Y_{u-}\, \mathrm{d}X_u - \int_s^t \mathrm{d}Y_u\, X_{u-} \\
&= Y_t X_t - Y_s X_s - \int_0^{t-s} Y_{(u+s)-}\, \mathrm{d}X_{u+s} - \int_0^{t-s} \mathrm{d}Y_{u+s}\, X_{(u+s)-} \\
&= Y_t X_t - Y_s X_s - \int_0^{t-s} (Y_{(u+s)-} - Y_s)\, \mathrm{d}X_{u+s} - \int_0^{t-s} Y_s\, \mathrm{d}X_{u+s} \\
&\quad - \int_0^{t-s} \mathrm{d}Y_{u+s}\, (X_{(u+s)-} - X_s) - \int_0^{t-s} \mathrm{d}Y_{u+s}\, X_s \\
&= Y_t X_t - Y_s X_s - \int_0^{t-s} (Y_{(u+s)-} - Y_s)\, \mathrm{d}X_{u+s} - Y_s(X_t - X_s) \\
&\quad - \int_0^{t-s} \mathrm{d}Y_{u+s}\, (X_{(u+s)-} - X_s) - (Y_t - Y_s) X_s \\
&= (Y_t - Y_s)(X_t - X_s) - \int_0^{t-s} (Y_{(u+s)-} - Y_s)\, \mathrm{d}(X_{u+s} - X_s) \\
&\quad - \int_0^{t-s} \mathrm{d}(Y_{u+s} - Y_s)\, (X_{(u+s)-} - X_s) \\
&\stackrel{\mathrm{D}}{=} Y_{t-s} X_{t-s} - \int_0^{t-s} Y_{u-}\, \mathrm{d}X_u - \int_0^{t} \mathrm{d}Y_u\, X_{u-} = [Y,X]_{t-s}
\end{aligned}$$

and therefore

$$\begin{aligned}
[Y,X]^c_t - [Y,X]^c_s &= [Y,X]_t - \sum_{0<r\le t} \Delta Y_r \Delta X_r - [Y,X]_s + \sum_{0<r\le s} \Delta Y_r \Delta X_r \\
&= [Y,X]_t - [Y,X]_s - \sum_{s<r\le t} \Delta Y_r \Delta X_r \\
&= [Y,X]_t - [Y,X]_s - \sum_{0<r\le t-s} \Delta Y_{r+s} \Delta X_{r+s} \\
&\stackrel{\mathrm{D}}{=} [Y,X]_{t-s} - \sum_{0<r\le t-s} \Delta Y_r \Delta X_r = [Y,X]^c_{t-s}.
\end{aligned}$$

In the special case $Y = X$ we also have $[X,X]_t^c - [X,X]_s^c \overset{\mathrm{D}}{=} [X,X]_{t-s}^c$. Together this yields

$$\begin{aligned}
U_t - U_s &= -(X_t - X_s) + [X,X]_t^c - [X,X]_s^c + \sum_{s<r\le t} \left((I+\Delta X_r)^{-1} - I + \Delta X_r\right) \\
&= -(X_t - X_s) + [X,X]_t^c - [X,X]_s^c + \sum_{0<r\le t-s} \left((I+\Delta X_{r+s})^{-1} - I + \Delta X_{r+s}\right) \\
&\overset{\mathrm{D}}{=} -X_{t-s} + [X,X]_{t-s}^c + \sum_{0<r\le t-s} \left((I+\Delta X_r)^{-1} - I + \Delta X_r\right) = U_{t-s}, \\
L_t - L_s &= Y_t - Y_s - ([Y,X]_t^c - [Y,X]_s^c) + \sum_{s<r\le t} \Delta Y_r\left((I+\Delta X_r)^{-1} - I\right) \\
&= Y_t - Y_s - ([Y,X]_t^c - [Y,X]_s^c) + \sum_{0<r\le t-s} \Delta Y_{r+s}\left((I+\Delta X_{r+s})^{-1} - I\right) \\
&\overset{\mathrm{D}}{=} Y_{t-s} - [Y,X]_{t-s}^c + \sum_{0<r\le t-s} \Delta Y_r\left((I+\Delta X_r)^{-1} - I\right) = L_{t-s}.
\end{aligned}$$

Combining this we get $(U_t - U_s, L_t - L_s) \overset{\mathrm{D}}{=} (U_{t-s}, L_{t-s})$. Thus (U, L) is a Lévy process. (3.3) is clear from (3.7c). In order to derive the stochastic differential equation (4.19) let $A_t = A_{0,t}$ and $B_t = B_{0,t}$ be defined as in (4.16). By (3.7b) we have $A_t = \overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overleftarrow{\mathsf{Exp}}(U_t)$ and thus

$$\mathrm{d}A_t = \mathrm{d}\overleftarrow{\mathsf{Exp}}(U_t) = \overleftarrow{\mathsf{Exp}}(U_{t-})\,\mathrm{d}U_t = A_{t-}\,\mathrm{d}U_t \quad\Longleftrightarrow\quad A_{t-}^{-1}\,\mathrm{d}A_t = \mathrm{d}U_t\,.$$

Equation (4.14) with $s = 0$ becomes $V_t = V_0 A_t + B_t$ and taking differentials yields

$$\begin{aligned}
\mathrm{d}V_t &= V_0\,\mathrm{d}A_t + \mathrm{d}B_t = V_0 A_{t-}\,\mathrm{d}U_t + \mathrm{d}B_t = (V_0 A_{t-} + B_{t-})\,\mathrm{d}U_t + \mathrm{d}B_t - B_{t-}\,\mathrm{d}U_t \\
&= V_{t-}\,\mathrm{d}U_t + \mathrm{d}B_t - B_{t-}A_{t-}^{-1}\,\mathrm{d}A_t = V_{t-}\,\mathrm{d}U_t + \mathrm{d}L'_t
\end{aligned}$$

where $L'_t := B_t - \int_0^t B_{u-}A_{u-}^{-1}\,\mathrm{d}A_u$. In order to show that $L'_t = L_t$ we use the integration by parts formula (2.18a) and (2.18e) to obtain

$$\begin{aligned}
L'_t &= B_t - \int_0^t B_{u-}A_{u-}^{-1}\,\mathrm{d}A_u \\
&= B_t - \left(B_t A_t^{-1} A_t - B_0 A_0^{-1} A_0 - \int_0^t \mathrm{d}(B_u A_u^{-1})\,A_{u-} - [BA^{-1}, A]_t\right) \\
&= \int_0^t \mathrm{d}(B_u A_u^{-1})\,A_{u-} + [BA^{-1}, A]_t \\
&= \int_0^t \mathrm{d}Y_u\,A_{u-}^{-1}A_{u-} + \left[\int_0^t \mathrm{d}Y_u\,\overrightarrow{\mathsf{Exp}}(X_{u-}), \overrightarrow{\mathsf{Exp}}(X_t)^{-1}\right] \\
&= Y_t + \left[Y_t, \int_0^t \overrightarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}\overrightarrow{\mathsf{Exp}}(X_u)^{-1}\right] \\
&= Y_t + \left[Y_t, \int_0^t A_{u-}^{-1}\,\mathrm{d}A_u\right]
\end{aligned}$$

$$= Y_t + \left[Y_t, \int_0^t \mathrm{d}U_u \right]$$
$$= Y_t + [Y, U]_t.$$

Finally, using (2.17a) and (2.17b) the covariation of Y and U can be calculated as

$$[Y,U]_t = -[Y,X]_t + \left[Y,[X,X]^c\right]_t + \left[Y_t, \sum_{0<s\le t} \left((I+\Delta X_s)^{-1} - I + \Delta X_s \right) \right]$$
$$= -[Y,X]_t^c - \sum_{0<s\le t} \Delta Y_s \Delta X_s + \sum_{0<s\le t} \Delta Y_s \Delta \left(\sum_{0<r\le s} \left((I+\Delta X_r)^{-1} - I + \Delta X_r \right) \right)$$
$$= -[Y,X]_t^c - \sum_{0<s\le t} \Delta Y_s \Delta X_s + \sum_{0<s\le t} \Delta Y_s \left((I+\Delta X_s)^{-1} - I + \Delta X_s \right)$$
$$= -[Y,X]_t^c + \sum_{0<s\le t} \Delta Y_s \left((I+\Delta X_t)^{-1} - I \right) = L_t - Y_t$$

so that $L'_t = Y_t + [Y,U]_t = Y_t + L_t - Y_t = L_t$. □

Theorem 4.21. *Let $(U,L) = (U_t, L_t)_{t\ge 0}$ be a* Lévy *process such that U satisfies (3.3) and let V_0 be an $\mathbb{R}^{n\times n}$-valued random variable. Then the solution $V = (V_t)_{t\ge 0}$ of the stochastic differential equation (4.19) is a right MGOU process and its background driving process is the* Lévy *process $(X,Y) = (X_t, Y_t)_{t\ge 0}$ given by*

$$X_t = \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_t)^{-1} \right), \tag{4.20a}$$
$$Y_t = L_t + \left[L, \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U)^{-1} \right) \right]_t. \tag{4.20b}$$

Furthermore X satisfies (3.3).

Proof. (X,Y) is càdlàg and has independent increments because (U,L) is càdlàg and has independent increments. $(X_0, Y_0) = (0,0)$ holds by definition. X has stationary increments because by (3.5b), (3.4a), and the stationary increments of U we have for all $0 \le s \le t$

$$X_t - X_s = \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_t)^{-1} \right) - \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_s)^{-1} \right)$$
$$= \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_t)^{-1} \overleftarrow{\mathsf{Exp}}(U_s) \right)$$
$$= \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_t - U_s)^{-1} \right)$$
$$\overset{\mathrm{D}}{=} \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_{t-s})^{-1} \right)$$
$$= X_{t-s}.$$

Inserting (4.20a) in (4.20b) yields $Y_t = L_t + [L,X]_t$ and since X and L have stationary increments it can be shown as in the proof of Theorem 4.20 that $[L,X]$ has stationary increments as well. Therefore Y has stationary increments because

$$Y_t - Y_s = L_t - L_s + [L,X]_t - [L,X]_s \overset{\mathrm{D}}{=} L_{t-s} - [L,X]_{t-s} = Y_{t-s}.$$

The calculations to derive (4.15) are similar to the proof of Theorem 3.4(b) in [4]. In fact, (4.20a) is equivalent to $\overrightarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(U_t)^{-1}$ which is (3.7b). (4.20b) can be written as

$$Y_t = L_t + [L,X]_t$$

$$
\begin{aligned}
&= L_t + [L,X]^c_t + \sum_{0<s\le t} \Delta L_s \Delta X_s \\
&= L_t + [L,X]^c_t - \sum_{0<s\le t} \Delta L_s (I+\Delta X_s)\Big((I+\Delta X_s)^{-1} - I\Big)
\end{aligned}
$$

from which we deduce that $[Y,X]^c_t = [L,X]^c_t$ and

$$\Delta Y_t = \Delta L_t - \Delta L_t (I+\Delta X_t)\Big((I+\Delta X_t)^{-1} - I\Big) = \Delta L_t (I+\Delta X_t).$$

Inserting this yields

$$Y_t = L_t + [Y,X]^c_t - \sum_{0<s\le t} \Delta Y_s \Big((I+\Delta X_s)^{-1} - I\Big)$$

which is equivalent to (4.18b). By Theorem 4.20 the right MGOU process V driven by (X,Y) is the unique solution of the stochastic differential equation (4.19). Furthermore, by (3.7a) and (3.4d)

$$X_t = \overrightarrow{\mathsf{Log}}\Big(\overleftarrow{\mathsf{Exp}}(U_t)^{-1}\Big) = \overrightarrow{\mathsf{Log}}\Big(\overrightarrow{\mathsf{Exp}}(\widetilde{U}_t)\Big) = \widetilde{U}_t$$

where $\widetilde{U} = (\widetilde{U}_t)_{t\ge 0}$ is constructed from U by (3.6), and then by (3.7c) we have for all $t \ge 0$

$$\det(I+\Delta X_t) = \det\Big(I + \Delta \widetilde{U}_t\Big) = \frac{1}{\det(I+\Delta U_t)} \neq 0.$$

□

The following relations between (X,Y) and (U,L) will be used in future proofs.

Proposition 4.22. *Let $(X,Y) = (X_t,Y_t)_{t\ge 0}$ be a* Lévy*-process such that X satisfies (3.3) and let the process $(U,L) = (U_t,L_t)_{t\ge 0}$ be defined as in (4.18). Then for all $t \ge 0$*

$$L_t = Y_t + [Y,U]_t, \tag{4.21a}$$
$$Y_t = L_t + [L,X]_t. \tag{4.21b}$$

Proof. The proof is similar to the proof of Proposition 3.5 in [4]. In fact, (4.21a) has already been shown in the last part of the proof of Theorem 4.20. For (4.21b), the definition of L in (4.18b) yields $[L,X]^c_t = [Y,X]^c_t$ and

$$\Delta L_t = \Delta Y_t + \Delta Y_t \Big((I+\Delta X_t)^{-1} - I\Big) = \Delta Y_t (I+\Delta X_t)^{-1}$$

so that

$$
\begin{aligned}
[L,X]_t &= [L,X]^c_t + \sum_{0<s\le t} \Delta L_s \Delta X_s \\
&= [Y,X]^c_t + \sum_{0<s\le t} \Delta Y_s (I+\Delta X_s)^{-1} \Delta X_s \\
&= [Y,X]^c_t + \sum_{0<s\le t} \Delta Y_s (I+\Delta X_s)^{-1} (I+\Delta X_s - I) \\
&= [Y,X]^c_t - \sum_{0<s\le t} \Delta Y_s \Big((I+\Delta X_s)^{-1} - I\Big) \\
&= Y_t - L_t.
\end{aligned}
$$

□

We now procede similarly to the left case and prove that Theorem 4.16, Theorem 4.17, Theorem 4.20, and Theorem 4.21 also hold true when the stationarity property (R3) is weakened to a property (R3*) requiring only periodic stationarity and the stationary increments of the background driving process of the right MGOU process are weakened to periodically stationary increments. We thus replace (R3) by

(R3*) $(A_{p,p+t}, B_{p,p+t})_{t\geq 0} \overset{\mathrm{D}}{=} (A_{0,t}, B_{0,t})_{t\geq 0}$ for some $p>0$

and refer to (R0),(R1),(R2),(R3*),(R4) as the **R*-assumptions**. In fact, (R3*) is the same as (L3*).

Theorem 4.23 and Theorem 4.24 show that there still is a one-to-one-correspondence between $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ and (X,Y) but as in the left case imposing the R*-assumptions on the random functional results in a semi-LÉVY process as the background driving process.

Theorem 4.23. *Let $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ satisfy the R*-assumptions for some $p>0$ and let $(A_t := A_{0,t})_{t\geq 0}$, $(B_t := B_{0,t})_{t\geq 0}$ be càdlàg. Then the process $(X,Y) = (X_t, Y_t)_{t\geq 0}$ defined as in (4.15) is the unique p-semi-LÉVY process such that X satisfies (3.3) and (4.16) holds almost surely.*

Proof. The proof is similar to the proof of Theorem 4.16. (X,Y) is càdlàg since $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are càdlàg, and $(X_0, Y_0) = (0,0)$ holds by definition. (X,Y) has independent increments because as in the proof of Theorem 4.16 the random variables

$$(X_{t_{i+1}} - X_{t_i}, Y_{t_{i+1}} - Y_{t_i}) = \left(\int_{t_i}^{t_{i+1}} \mathrm{d}A_{t_i,u}^{-1}\, A_{t_i,u-}, \int_{t_i}^{t_{i+1}} \mathrm{d}\big(B_{t_i,u} A_{t_i,u}^{-1}\big)\, A_{t_i,u-} \right)$$

are independent for all $t_0 < \ldots < t_k$, $k\in\mathbb{N}$. (X,Y) has p-stationary increments because by (R2),(R3*)

$$\begin{aligned}(X_{t+p} - X_p, Y_{t+p} - Y_p)_{t\geq 0} &= \left(\int_p^{t+p} \mathrm{d}A_{p,u}^{-1}\, A_{p,u-}, \int_p^{t+p} \mathrm{d}\big(B_{p,u} A_{p,u}^{-1}\big)\, A_{p,u-} \right)_{t\geq 0} \\ &= \left(\int_0^{t} \mathrm{d}A_{p,p+u}^{-1}\, A_{p,(p+u)-}, \int_0^{t} \mathrm{d}\big(B_{p,p+u} A_{p,p+u}^{-1}\big)\, A_{p,(p+u)-} \right)_{t\geq 0} \\ &\overset{\mathrm{D}}{=} \left(\int_0^{t} \mathrm{d}A_{u}^{-1}\, A_{u-}, \int_0^{t} \mathrm{d}\big(B_{u} A_{u}^{-1}\big)\, A_{u-} \right)_{t\geq 0} = (X_t, Y_t)_{t\geq 0}.\end{aligned}$$

Thus (X,Y) is a p-semi-LÉVY process. $X = \overrightarrow{\mathsf{Log}}(A^{-1})$ satisfies (3.3) by Theorem 3.6(b) and (4.16) is derived in exactly the same way as in the proof of Theorem 4.16. ☐

Theorem 4.24. *Let $(X,Y) = (X_t, Y_t)_{t\geq 0}$ be a p-semi-LÉVY process for some $p>0$ such that X satisfies (3.3). Then the family $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ defined as in (4.16) satisfies the R*-assumptions.*

Proof. (R0), (R1), (R2), and (R4) are obtained in the same way as in the proof of Theorem 4.17. (R3*) follows from

$$(A_{p,p+t}, B_{p,p+t})_{t\geq 0}$$

$$= \left(\overrightarrow{\mathsf{Exp}}(X_p)\overrightarrow{\mathsf{Exp}}(X_{t+p})^{-1}, \int_p^{t+p} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_p)^{-1}\overrightarrow{\mathsf{Exp}}(X_p)\overrightarrow{\mathsf{Exp}}(X_{t+p})^{-1}\right)_{t\geq 0}$$

$$= \left(\overrightarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1}, \int_p^{t+p} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}-X_p)\overrightarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1}\right)_{t\geq 0}$$

$$= \left(\overrightarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1}, \int_0^{t} \mathrm{d}(Y_{u+p}-Y_p)\, \overrightarrow{\mathsf{Exp}}(X_{(u+p)-}-X_p)\overrightarrow{\mathsf{Exp}}(X_{t+p}-X_p)^{-1}\right)_{t\geq 0}$$

$$\overset{\mathrm{D}}{=} \left(\overrightarrow{\mathsf{Exp}}(X_t)^{-1}, \int_0^{t} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_t)^{-1}\right)_{t\geq 0} = (A_{0,t}, B_{0,t})_{t\geq 0}.$$

□

Theorem 4.25 and Theorem 4.26 show that the right MGOU process driven by a semi-LÉVY process is still described by the stochastic differential equation (4.19) but as in the left case the process (U, L) driving this equation now also is a semi-LÉVY process.

Theorem 4.25. *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a p-semi-LÉVY process such that X satisfies (3.3) and let the process $(U, L) = (U_t, L_t)_{t\geq 0}$ be defined as in (4.18). Then (U, L) is a p-semi-LÉVY process such that U satisfies (3.3) and the right MGOU process V defined as in (4.17) solves the stochastic differential equation (4.19).*

Proof. Since (4.18a) is the same as (4.5a) and (4.18b) is similar to (4.5b), the proof of Theorem 4.8 carries over in almost the exact same way. Notably we again only need to prove that (U, L) has p-stationary increments. First, the p-stationary increments of (X, Y) yield $\Delta X_{s+p} \overset{\mathrm{D}}{=} \Delta X_s$ and $\Delta Y_{s+p} \overset{\mathrm{D}}{=} \Delta Y_s$. We then have

$$[Y, X]_{t+p} - [Y, X]_p \overset{\mathrm{D}}{=} [Y, X]_t \quad , \quad [Y, X]^c_{t+p} - [Y, X]^c_p \overset{\mathrm{D}}{=} [Y, X]^c_t$$

with the special case $[X, X]^c_{t+p} - [X, X]^c_p \overset{\mathrm{D}}{=} [X, X]^c_t$. This yields $U_{t+p} - U_p \overset{\mathrm{D}}{=} U_t$ and

$$\begin{aligned} L_{t+p} - L_p &= Y_{t+p} - Y_p - ([Y, X]^c_{t+p} - [Y, X]^c_p) + \sum_{p<s\leq t+p} \Delta Y_s((I + \Delta X_s)^{-1} - I) \\ &= Y_{t+p} - Y_p - ([Y, X]^c_{t+p} - [Y, X]^c_p) + \sum_{0<s\leq t} \Delta Y_{s+p}((I + \Delta X_{s+p})^{-1} - I) \\ &\overset{\mathrm{D}}{=} Y_t - [Y, X]^c_t + \sum_{0<s\leq t} \Delta Y_s((I + \Delta X_s)^{-1} - I) = L_t. \end{aligned}$$

Combining this we get $(U_{t+p} - U_p, L_{t+p} - L_p)_{t\geq 0} \overset{\mathrm{D}}{=} (U_t, L_t)_{t\geq 0}$. Thus (U, L) is a p-semi-LÉVY process. (3.3) is clear from (3.7c) and the stochastic differential equation (4.19) is derived in exactly the same way as in the proof of Theorem 4.20. □

Theorem 4.26. *Let $(U, L) = (U_t, L_t)_{t\geq 0}$ be a p-semi-LÉVY process for some $p > 0$ such that U satisfies (3.3) and let V_0 be an $\mathbb{R}^{n\times n}$-valued random variable. Then the solution $V = (V_t)_{t\geq 0}$ of the stochastic differential equation (4.19) is a right MGOU process and its background driving process $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is the p-semi-LÉVY process given by (4.15). Furthermore X satisfies (3.3).*

Proof. (X, Y) is càdlàg and has independent increments because (U, L) is càdlàg and has independent increments. $(X_0, Y_0) = (0, 0)$ holds by definition. X has p-stationary increments because by (3.5b), (3.4a), and the p-stationary increments of U we have for all $t \geq 0$

$$\begin{aligned}
(X_{t+p} - X_p)_{t\geq 0} &= \left(\overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_{t+p})^{-1}\right) - \overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_p)^{-1}\right)\right)_{t\geq 0} \\
&= \left(\overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_{t+p})^{-1}\overleftarrow{\mathsf{Exp}}(U_p)\right)\right)_{t\geq 0} \\
&= \left(\overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_{t+p} - U_p)^{-1}\right)\right)_{t\geq 0} \\
&\overset{\mathrm{D}}{=} \left(\overrightarrow{\mathsf{Log}}\left(\overleftarrow{\mathsf{Exp}}(U_t)^{-1}\right)\right)_{t\geq 0} = (X_t)_{t\geq 0}.
\end{aligned}$$

Inserting (4.20a) in (4.20b) yields $Y_t = L_t + [L, X]_t$ and since X and L have p-stationary increments it can be shown as in the proof of Theorem 4.25 that $[L, X]$ has p-stationary increments as well. Therefore Y has p-stationary increments because

$$(Y_{t+p} - Y_p)_{t\geq 0} = (L_{t+p} - L_p + [L, X]_{t+p} - [L, X]_p)_{t\geq 0} \overset{\mathrm{D}}{=} (L_t - [L, X]_t)_{t\geq 0} = (Y_t)_{t\geq 0}.$$

The facts that the solution of (4.19) is a right MGOU process driven by (X, Y) and that X satisfies (3.3) is proven in exactly the same way as in the proof of Theorem 4.21. □

We now give conditions under which the right MGOU process is stationary and state a distributional equality of right stochastic integrals of stochastic exponentials that is needed for the proof. These results are similar to the left case in Theorem 4.10 and Theorem 4.11 which were proven by Behme and Lindner. In fact, the proofs are analogous to the proofs of Theorem 5.2 and Proposition 8.3 in [4] and we just state the corresponding results.

Theorem 4.27. *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a Lévy process such that X satisfies (3.3) and let $V = (V_t)_{t\geq 0}$ be the right MGOU process driven by (X, Y). Let $(U, L) = (U_t, L_t)_{t\geq 0}$ be defined as in (4.18).*

(a) *Let $\mathrm{P}\text{-}\lim_{t\to\infty} \overrightarrow{\mathsf{Exp}}(U_t) = \mathbf{0}$. Then a finite random variable V_0 can be chosen such that V is stationary if and only if the integral $\int_0^t \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-})$ converges in distribution to a finite random variable. In this case V_0 can be chosen independently of (X, Y) with*

$$V_0 \overset{\mathrm{D}}{=} \mathrm{D}\text{-}\lim_{t\to\infty} \int_0^t \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}). \tag{4.22a}$$

(b) *Let $\mathrm{P}\text{-}\lim_{t\to\infty} \overrightarrow{\mathsf{Exp}}(X_t) = \mathbf{0}$. Then a finite random variable V_0 can be chosen such that V is stationary if and only if the integral $\int_0^t \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-})$ converges in probability to a finite random variable. In this case V_0 can be chosen as*

$$V_0 = -\mathrm{P}\text{-}\lim_{t\to\infty} \int_0^t \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}). \tag{4.22b}$$

Theorem 4.28. *Let* $(X,Y) = (X_t, Y_t)_{t\geq 0}$ *be a* LÉVY *process such that* X *satisfies (3.3) and let* $(U,L) = (U_t, L_t)_{t\geq 0}$ *be defined as in (4.18). Then for all* $t > 0$

$$\int_0^t \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1}\overleftarrow{\mathsf{Exp}}(U_t) \overset{\mathrm{D}}{=} \int_0^t \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}). \tag{4.23}$$

In the situation of Theorem 4.28 the left side of (4.23) can also be written in the form

$$\begin{aligned}\int_0^t \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1}\overleftarrow{\mathsf{Exp}}(U_t) &= -\int_0^t \mathrm{d}(Y_t - Y_u) \overleftarrow{\mathsf{Exp}}(U_t - U_{u-}) \overset{\mathrm{D}}{=} -\int_0^t \mathrm{d}Y_{t-u} \overleftarrow{\mathsf{Exp}}(U_{(t-u)-}) \\ &= -\int_t^0 \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-}) = \int_0^t \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})\end{aligned}$$

and (4.23) then becomes

$$\int_0^t \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-}) \overset{\mathrm{D}}{=} \int_0^t \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}).$$

We now prove an analogous result in the case of periodically stationary increments and use it to derive conditions under which the right MGOU process is periodically stationary.

Theorem 4.29. *Let* $(X,Y) = (X_t, Y_t)_{t\geq 0}$ *be a* p*-semi-*LÉVY *process for some* $p > 0$ *such that* X *satisfies (3.3) and let* $(U,L) = (U_t, L_t)_{t\geq 0}$ *be defined as in (4.18). Then for all* $n \in \mathbb{N}$

$$\int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-}) \overset{\mathrm{D}}{=} \int_0^{np} \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}). \tag{4.24}$$

Proof. The proof is similar to the proof of Theorem 4.12. Let (U', Y') be an independent copy of (U, Y). Then (U, L) and (U', L') are p-semi-LÉVY processes by Theorem 4.25. For fixed $n \in \mathbb{N}$ and $0 \leq s \leq np$ now let

$$\begin{aligned}\widehat{U}_s &:= U_{np} - U_{np-s}, \\ \widehat{Y}_s &:= Y_{np} - Y_{np-s}\end{aligned}$$

and for $s \geq 0$ let

$$\begin{aligned}\widehat{U}'_s &:= U'_{(s+np)-} - U'_{np-}, \\ \widehat{Y}'_s &:= Y'_{(s+np)-} - Y'_{np-}.\end{aligned}$$

Then

$$\begin{aligned}(\widehat{U}_s, \widehat{Y}_s)_{0\leq s\leq np} &= (U_{np} - U_{np-s}, Y_{np} - Y_{np-s})_{0\leq s\leq np} \\ &\overset{\mathrm{D}}{=} (U_s, Y_s)_{0\leq s\leq np} \\ &\overset{\mathrm{D}}{=} (U'_s, Y'_s)_{0\leq s\leq np} \\ &\overset{\mathrm{D}}{=} (U'_{(s+np)-} - U'_{np-}, Y'_{(s+np)-} - Y'_{np-})_{0\leq s\leq np} \\ &= (\widehat{U}'_s, \widehat{Y}'_s)_{0\leq s\leq np}.\end{aligned}$$

For partitions $\sigma_m^- = \left(\dots, \frac{-2np}{m}, \frac{-np}{m}, 0\right)$ of the negative real line and $\sigma_m^+ = \left(0, \frac{np}{m}, \frac{2np}{m}, \dots\right)$ of the positive real line let

$$A^{\sigma_m^-} := \sum_{i=0}^{m-1} \left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \overrightarrow{\mathsf{Exp}} \left(\widehat{U}'_{np(i+1)/m}\right),$$

$$B^{\sigma_m^+} := \sum_{i=0}^{m-1} \left(\widehat{Y}_{np(i+1)/m-} - \widehat{Y}_{npi/m-}\right) \overrightarrow{\mathsf{Exp}} \left(\widehat{U}_{np(i+1)/m-}\right).$$

Then as $|\sigma_m^-| \xrightarrow{m\to\infty} 0$ we have by Theorems II.21 and II.23 in [31]

$$\begin{aligned}
A^{\sigma_m^-} &= \sum_{i=0}^{m-1} \left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \overrightarrow{\mathsf{Exp}} \left(\widehat{U}'_{np(i+1)/m}\right) \\
&= \sum_{i=0}^{m-1} \left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \overrightarrow{\mathsf{Exp}} \left(\widehat{U}'_{npi/m}\right) \\
&\quad + \sum_{i=0}^{m-1} \left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \left(\overrightarrow{\mathsf{Exp}} \left(\widehat{U}'_{np(i+1)/m}\right) - \overrightarrow{\mathsf{Exp}} \left(\widehat{U}'_{npi/m}\right)\right) \\
&\xrightarrow{\mathrm{P}} \int_0^{np} \mathrm{d}\widehat{Y}'_u \, \overrightarrow{\mathsf{Exp}}(\widehat{U}'_{u-}) + \left[\widehat{Y}', \overrightarrow{\mathsf{Exp}}(\widehat{U}')\right]_{np} \\
&\overset{\mathrm{D}}{=} \int_0^{np} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(U_{u-}) + \left[Y, \overleftarrow{\mathsf{Exp}}(U)\right]_{np}.
\end{aligned}$$

This equals the right side in (4.24) because from (4.21a), (2.18d), and (3.1b) we obtain

$$\begin{aligned}
\int_0^{np} \mathrm{d}L_u \, \overrightarrow{\mathsf{Exp}}(U_{u-}) &= \int_0^{np} \mathrm{d}(Y_u + [Y,U]_u) \, \overrightarrow{\mathsf{Exp}}(U_{u-}) \\
&= \int_0^{np} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(U_{u-}) + \int_0^{np} \mathrm{d}[Y,U]_u \, \overrightarrow{\mathsf{Exp}}(U_{u-}) \\
&= \int_0^{np} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(U_{u-}) + \left[\int_0^{np} \mathrm{d}Y_u \, , \int_0^{np} \mathrm{d}U_u \, \overrightarrow{\mathsf{Exp}}(U_{u-})\right] \\
&= \int_0^{np} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(U_{u-}) + \left[Y, \overrightarrow{\mathsf{Exp}}(U)\right]_{np}.
\end{aligned}$$

As $|\sigma_m^+| \xrightarrow{m\to\infty} 0$ we have by Lemma 8.2 in [4] and Theorem II.21 in [31]

$$\begin{aligned}
B^{\sigma_m^+} &= \sum_{i=0}^{m-1} \left(\widehat{Y}_{np(i+1)/m-} - \widehat{Y}_{npi/m-}\right) \overrightarrow{\mathsf{Exp}} \left(\widehat{U}_{np(i+1)/m-}\right) \\
&= \sum_{i=0}^{m-1} \left(Y_{np(m-i)/m} - Y_{np(m-i-1)/m}\right) \overrightarrow{\mathsf{Exp}} \left(U_{np} - U_{np(m-i-1)/m}\right) \\
&= \sum_{i=0}^{m-1} \left(Y_{np(m-i)/m} - Y_{np(m-i-1)/m}\right) \overleftarrow{\mathsf{Exp}} \left(U_{np(m-i-1)/m}\right)^{-1} \overleftarrow{\mathsf{Exp}}(U_{np}) \\
&= \sum_{i=1}^{m} \left(Y_{npi/m} - Y_{np(i-1)/m}\right) \overleftarrow{\mathsf{Exp}} \left(U_{np(i-1)/m}\right)^{-1} \overleftarrow{\mathsf{Exp}}(U_{np})
\end{aligned}$$

$$\xrightarrow{\mathrm{P}} \int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1}\overleftarrow{\mathsf{Exp}}(U_{np}).$$

Using the p-stationary increments of (U,Y) we can also write this in the form

$$\begin{aligned}
\int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1}\overleftarrow{\mathsf{Exp}}(U_{np}) &= \int_{-np}^{0} \mathrm{d}Y_{np+u} \overleftarrow{\mathsf{Exp}}(U_{(np+u)-})^{-1}\overleftarrow{\mathsf{Exp}}(U_{np}) \\
&= -\int_{-np}^{0} \mathrm{d}(Y_{np} - Y_{np+u}) \overleftarrow{\mathsf{Exp}}(U_{np} - U_{(np+u)-}) \\
&\overset{\mathrm{D}}{=} -\int_{-np}^{0} \mathrm{d}Y_{-u} \overleftarrow{\mathsf{Exp}}(U_{(-u)-}) = \int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})
\end{aligned}$$

which is the left side in (4.24). The equality in distribution now follows from the fact that

$$\begin{aligned}
B^{\sigma_m^+} &= \sum_{i=0}^{m-1} \left(\widehat{Y}_{np(i+1)/m-} - \widehat{Y}_{npi/m-}\right) \overrightarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m-}\right) \\
&= \sum_{i=0}^{m-1} \left(\left(\widehat{Y}_{np(i+1)/m} - \Delta\widehat{Y}_{np(i+1)/m}\right) - \left(\widehat{Y}_{npi/m} - \Delta\widehat{Y}_{npi/m}\right)\right) \\
&\qquad \cdot \left(\overrightarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m}\right) - \Delta\overrightarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m}\right)\right) \\
&= \sum_{i=0}^{m-1} \left(\widehat{Y}_{np(i+1)/m} - \widehat{Y}_{npi/m}\right) \overrightarrow{\mathsf{Exp}}\left(\widehat{U}_{np(i+1)/m}\right) \\
&\overset{\mathrm{D}}{=} \sum_{i=0}^{m-1} \left(\widehat{Y}'_{np(i+1)/m} - \widehat{Y}'_{npi/m}\right) \overrightarrow{\mathsf{Exp}}\left(\widehat{U}'_{np(i+1)/m}\right) = A^{\sigma_m^-}
\end{aligned}$$

because at fixed times both $\overrightarrow{\mathsf{Exp}}(\widehat{U})$ and $\widehat{Y}$ almost surely do not have jumps. $\square$

Theorem 4.30. *Let $(X,Y) = (X_t,Y_t)_{t\geq 0}$ be a p-semi-Lévy process for some $p > 0$ such that X satisfies (3.3) and let $V = (V_t)_{t\geq 0}$ be the right MGOU process driven by (X,Y). Let $(U,L) = (U_t,L_t)_{t\geq 0}$ be defined as in (4.18).*

(a) *Let $\mathrm{P}\text{-}\lim_{n\to\infty} \overleftarrow{\mathsf{Exp}}(U_{np}) = \mathbf{0}$. Then a finite random variable V_0 can be chosen such that V is p-stationary if and only if the integral $\int_0^{np} \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-})$ converges in distribution to a finite random variable. In this case V_0 can be chosen independently of (X,Y) with*

$$V_0 \overset{\mathrm{D}}{=} \mathrm{D}\text{-}\lim_{n\to\infty} \int_0^{np} \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}). \tag{4.25a}$$

(b) *Let $\mathrm{P}\text{-}\lim_{n\to\infty} \overrightarrow{\mathsf{Exp}}(X_{np}) = \mathbf{0}$. Then a finite random variable V_0 can be chosen such that V is p-stationary if and only if the integral $\int_0^{np} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-})$ converges in probability* *to a finite random variable. In this case V_0 can be chosen as*

$$V_0 = -\text{P-}\lim_{n\to\infty} \int_0^{np} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}). \tag{4.25b}$$

Proof. The proof is similar to the proof of Theorem 4.13.

(a) Assume that V is p-stationary. Then $V_{np} \overset{\mathrm{D}}{=} V_0$ for all $n \in \mathbb{N}$ and thus

$$\begin{aligned} V_0 &\overset{\mathrm{D}}{=} \text{D-}\lim_{n\to\infty} V_{np} \\ &= \text{D-}\lim_{n\to\infty} \left(V_0 + \int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1} \right) \overleftarrow{\mathsf{Exp}}(U_{np}) \\ &\overset{\mathrm{D}}{=} \text{D-}\lim_{n\to\infty} V_0 \overleftarrow{\mathsf{Exp}}(U_{np}) + \int_0^{np} \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}) \\ &= \text{D-}\lim_{n\to\infty} \int_0^{np} \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}). \end{aligned}$$

Now assume that (4.25a) holds. Then for fixed $t \geq 0$ and all $n \in \mathbb{N}$

$$\begin{aligned} V_{t+np} &= \left(V_0 + \int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1} + \int_{np}^{t+np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1} \right) \overleftarrow{\mathsf{Exp}}(U_{t+np}) \\ &= \Bigg(V_0 \overleftarrow{\mathsf{Exp}}(U_{np}) + \int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1} \overleftarrow{\mathsf{Exp}}(U_{np}) \\ &\quad + \int_0^t \mathrm{d}Y_{np+u} \overleftarrow{\mathsf{Exp}}(U_{np+u-})^{-1} \overleftarrow{\mathsf{Exp}}(U_{np}) \Bigg) \overleftarrow{\mathsf{Exp}}(U_{np})^{-1} \overleftarrow{\mathsf{Exp}}(U_{t+np}) \\ &= \Bigg(V_0 \overleftarrow{\mathsf{Exp}}(U_{np}) + \int_0^{np} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1} \overleftarrow{\mathsf{Exp}}(U_{np}) \\ &\quad + \int_0^t \mathrm{d}(Y_{np+u} - Y_{np}) \overleftarrow{\mathsf{Exp}}(U_{np+u-} - U_{np})^{-1} \Bigg) \overleftarrow{\mathsf{Exp}}(U_{t+np} - U_{np}) \\ &\overset{\mathrm{D}}{=} \left(V_0 \overleftarrow{\mathsf{Exp}}(U_{np}) + \int_0^{np} \mathrm{d}L_u \overrightarrow{\mathsf{Exp}}(U_{u-}) + \int_0^t \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1} \right) \overleftarrow{\mathsf{Exp}}(U_t) \\ &\overset{\mathrm{D}}{\longrightarrow} \left(V_0 + \int_0^t \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(U_{u-})^{-1} \right) \overleftarrow{\mathsf{Exp}}(U_t) = V_t \end{aligned}$$

as $n \to \infty$, where we have used (4.24) in the second to last step. Thus

$$V_{t+p} \overset{\mathrm{D}}{=} \text{D-}\lim_{n\to\infty} V_{t+p+np} = \text{D-}\lim_{n\to\infty} V_{t+(n+1)p} \overset{\mathrm{D}}{=} V_t.$$

(b) Assume that V is p-stationary. Then $V_{np} \overset{\mathrm{D}}{=} V_0$ for all $n \in \mathbb{N}_0$ and thus

$$V_0 + \int_0^{np} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) = V_{np}\overrightarrow{\mathsf{Exp}}(X_{np}) \xrightarrow{\mathrm{P}} 0 \iff V_0 = -\mathrm{P}\text{-}\lim_{n\to\infty} \int_0^{np} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}).$$

Now assume that (4.25b) holds. Then for all $n \in \mathbb{N}$

$$\begin{aligned} V_{np} &= \left(V_0 + \int_0^{np} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) \right) \overrightarrow{\mathsf{Exp}}(X_{np})^{-1} \\ &= \left(-\int_0^{\infty} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) + \int_0^{np} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) \right) \overrightarrow{\mathsf{Exp}}(X_{np})^{-1} \\ &= -\int_{np}^{\infty} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) \overrightarrow{\mathsf{Exp}}(X_{np})^{-1} \\ &= -\int_0^{\infty} \mathrm{d}Y_{np+u} \overrightarrow{\mathsf{Exp}}(X_{np+u-}) \overrightarrow{\mathsf{Exp}}(X_{np})^{-1} \\ &= -\int_0^{\infty} \mathrm{d}(Y_{np+u} - Y_{np}) \overrightarrow{\mathsf{Exp}}(X_{np+u-} - X_{np}) \\ &\overset{\mathrm{D}}{=} -\int_0^{\infty} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) = V_0 \end{aligned}$$

and thus for all $t \geq 0$ by the independence of V_p and $(A_{p,p+t}, B_{p,p+t})$

$$V_{t+p} = V_p A_{p,p+t} + B_{p,p+t} \overset{\mathrm{D}}{=} V_0 A_{0,t} + B_{0,t} = V_t.$$

□

When studying the connection between right MGOU processes, right semiselfsimilar processes, and right semistable hemigroups in chapters 5.2 and 6.2 we need the notion of a right MGOU process with time parameter $t \in \mathbb{R}$ rather than just $t \geq 0$. In order to define a right MGOU process with real time parameter we make use of the construction of a Lévy process with real time parameter in Theorem 2.3.

Theorem 4.31. *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a semimartingale such that X satisfies (3.3) and let (X', Y') be an independent copy of (X, Y). Let $(\widetilde{X}, \widetilde{Y})$ be defined as in (2.5). Assume that*

$$V_0 := \int_{-\infty}^{0} \mathrm{d}(-Y'_{-u}) \overrightarrow{\mathsf{Exp}}\left(-X'_{(-u)-}\right) := \mathrm{P}\text{-}\lim_{t\to\infty} \int_{-t}^{0} \mathrm{d}(-Y'_{-u}) \overrightarrow{\mathsf{Exp}}\left(-X'_{(-u)-}\right) \tag{4.26a}$$

exists and let the process $V = (V_t)_{t\in\mathbb{R}}$ be defined by

$$V_t := \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_u \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1}. \tag{4.26b}$$

(a) *If (X, Y) is a* Lévy *process then V is stationary.*

(b) *If (X, Y) is a p-semi-*Lévy *process for some $p > 0$ then V is p-stationary.*

Proof. First assume that (X, Y) is a Lévy process. Then $(\widetilde{X}, \widetilde{Y})$ is a Lévy process by Theorem 2.3(a) and therefore has stationary increments. By (3.4b) we have for all $t \in \mathbb{R}$ and $h > 0$

$$\begin{aligned} V_{t+h} &= \int_{-\infty}^{t+h} \mathrm{d}\widetilde{Y}_u \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t+h})^{-1} \\ &= \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_{u+h} \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{(u+h)-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_h)^{-1} \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t+h} - \widetilde{X}_h)^{-1} \\ &= \int_{-\infty}^{t} \mathrm{d}(\widetilde{Y}_{u+h} - \widetilde{Y}_h) \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{(u+h)-} - \widetilde{X}_h) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t+h} - \widetilde{X}_h)^{-1} \\ &\overset{\mathrm{D}}{=} \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_u \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} = V_t. \end{aligned}$$

Now assume that (X, Y) is a p-semi-Lévy process. Then $(\widetilde{X}, \widetilde{Y})$ is a p-semi-Lévy process by Theorem 2.3(b) and therefore has p-stationary increments. By (3.4b) we have for all $t \geq 0$

$$\begin{aligned} V_{t+p} &= \int_{-\infty}^{t+p} \mathrm{d}\widetilde{Y}_u \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t+p})^{-1} \\ &= \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_{u+p} \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{(u+p)-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_p)^{-1} \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t+p} - \widetilde{X}_p)^{-1} \\ &= \int_{-\infty}^{t} \mathrm{d}(\widetilde{Y}_{u+p} - \widetilde{Y}_p) \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{(u+p)-} - \widetilde{X}_p) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t+p} - \widetilde{X}_p)^{-1} \\ &\overset{\mathrm{D}}{=} \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_u \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} = V_t \end{aligned}$$

and for $t < 0$

$$\begin{aligned} V_{t-p} &= \int_{-\infty}^{t-p} \mathrm{d}\widetilde{Y}_u \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t-p})^{-1} \\ &= \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_{u-p} \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{(u-p)-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{-p})^{-1} \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t-p} - \widetilde{X}_{-p})^{-1} \\ &= \int_{-\infty}^{t} \mathrm{d}(\widetilde{Y}_{u-p} - \widetilde{Y}_{-p}) \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{(u-p)-} - \widetilde{X}_{-p}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{t-p} - \widetilde{X}_{-p})^{-1} \\ &\overset{\mathrm{D}}{=} \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_u \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} = V_t. \end{aligned}$$

□

For $t > 0$ the process in (4.26b) can be written as

$$\begin{aligned} V_t &= \int_{-\infty}^{t} \mathrm{d}\widetilde{Y}_u \, \overrightarrow{\mathsf{Exp}}(\widetilde{X}_{u-}) \overrightarrow{\mathsf{Exp}}(\widetilde{X}_t)^{-1} \\ &= \left(\int_{-\infty}^{0} \mathrm{d}(-Y'_{-u}) \, \overrightarrow{\mathsf{Exp}} \left(-X'_{(-u)-} \right) + \int_{0}^{t} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) \right) \overrightarrow{\mathsf{Exp}}(X_t)^{-1} \\ &= \left(V_0 + \int_{0}^{t} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) \right) \overrightarrow{\mathsf{Exp}}(X_t)^{-1} \end{aligned}$$

which is the usual integral form of a right MGOU process as in (4.17) but by (4.26a) V_0 can be chosen independently of the background driving process (X, Y). This fact together with Theorem 4.31 now motivates the following definition.

Definition 4.32. Let $(X, Y) = (X_t, Y_t)_{t \geq 0}$ be a Lévy respectively semi-Lévy process such that X satisfies (3.3) and let (X', Y') be an independent copy of (X, Y). Let $(\widetilde{X}, \widetilde{Y})$ be defined as in (2.5) and assume that V_0 as in (4.26a) exists. Then $V = (V_t)_{t \in \mathbb{R}}$ defined as in (4.26b) is called **stationary right MGOU process** respectively **periodically stationary right MGOU process**. The process $(\widetilde{X}, \widetilde{Y})$ is called **background driving process**.

5 Connection between MGOU and semiselfsimilar processes

Chapter 5 introduces random semiselfsimilarity as a generalization of semiselfsimilarity that incorporates random scaling, and the random LAMPERTI transform as a generalization of the LAMPERTI transform that is compatible with random semiselfsimilarity and MGOU processes. We show how random semiselfsimilar processes are connected to periodically stationary processes and MGOU processes by means of the random LAMPERTI transform. These connections extend to the random selfsimilar and stationary case.

Due to the non-commutativity of matrix multiplication there are two types of random LAMPERTI transforms and two types of random semiselfsimilarity in dimension $n \geq 2$. Chapter 5.1 discusses the left random LAMPERTI transform, random left semiselfsimilarity, and the connection to left MGOU processes while chapter 5.2 discusses the right random LAMPERTI transform, random right semiselfsimilarity, and the connection to right MGOU processes.

5.1 Left MGOU and left semiselfsimilar processes

In the left case the generalization of the LAMPERTI transform and its inverse transform is defined as follows.

Definition 5.1. Let $X = (X_t)_{t\in\mathbb{R}}$ be a semimartingale which satisfies (3.3).

(a) The **left X-random LAMPERTI transform** of a process $Z = (Z_t)_{t>0}$ is the process $\overleftarrow{\mathsf{Lam}}(Z) = V = (V_t)_{t\in\mathbb{R}}$ defined by

$$\overleftarrow{\mathsf{Lam}}(Z_t) = V_t := \overleftarrow{\mathsf{Exp}}(X_t)^{-1} Z_{e^t}. \tag{5.1a}$$

(b) The **left X-random inverse LAMPERTI transform** of a process $V = (V_t)_{t\in\mathbb{R}}$ is the process $\overleftarrow{\mathsf{Lam}}^{-1}(V) = Z = (Z_t)_{t>0}$ defined by

$$\overleftarrow{\mathsf{Lam}}^{-1}(V_t) = Z_t := \overleftarrow{\mathsf{Exp}}(X_{\log(t)}) V_{\log(t)}. \tag{5.1b}$$

In the left case the generalization of selfsimilarity and semiselfsimilarity is defined as follows.

Definition 5.2. Let $X = (X_t)_{t\in\mathbb{R}}$ be a semimartingale with $X_0 = \mathbf{0}$ which satisfies (3.3).

(a) A process $Z = (Z_t)_{t>0}$ is called **random left selfsimilar** with **exponent** X or simply **left X-selfsimilar** if for all $c > 1$

$$\left(X_{\log(t)+\log(c)} - X_{\log(c)}, \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{ct}\right)_{t\geq 1} \overset{\mathrm{D}}{=} (X_{\log(t)}, Z_t)_{t\geq 1}, \tag{5.2a}$$

$$\left(X_{\log(t)-\log(c)} - X_{-\log(c)}, \overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}t}\right)_{0<t\leq 1} \overset{\mathrm{D}}{=} (X_{\log(t)}, Z_t)_{0<t\leq 1}. \tag{5.2b}$$

(b) A process $Z = (Z_t)_{t>0}$ is called **random left semiselfsimilar** with **exponent** $X_{\log(c)}$ or simply **left $X_{\log(c)}$-semiselfsimilar** if (5.2a) and (5.2b) hold for some $c > 1$.

In the definition of left semiselfsimilarity we can without loss of generality assume $c > 1$ because for $c = 1$ (5.2a) and (5.2b) are automatically fulfilled and for $0 < c < 1$ swapping (5.2a) and (5.2b) and rescaling the time parameter of Z leads to left semiselfsimilarity with $c^{-1} > 1$. In particular, left $X_{\log(c)}$-semiselfsimilarity for all $c > 1$ implies left X-selfsimilarity.

If the exponent X is a semi-LÉVY process the left random LAMPERTI transform and its inverse transform establish a one-to-one correspondence between left semiselfsimilar processes and periodically stationary processes. If the exponent X is a LÉVY process the one-to-one correspondence extends to left selfsimilar processes and stationary processes.

Theorem 5.3. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* $\log(c)$*-semi-*LÉVY *process for some $c > 1$ which satisfies (3.3).*

(a) *If $Z = (Z_t)_{t>0}$ is left $X_{\log(c)}$-semiselfsimilar, then its left X-random* LAMPERTI *transform $V = (V_t)_{t\in\mathbb{R}}$ is* $\log(c)$*-stationary.*

(b) *If $V = (V_t)_{t\in\mathbb{R}}$ is* $\log(c)$*-stationary and independent of X, then its left X-random inverse* LAMPERTI *transform $Z = (Z_t)_{t>0}$ is left $X_{\log(c)}$-semiselfsimilar.*

Proof. (a) Because Z is left $X_{\log(c)}$-semiselfsimilar and X has $\log(c)$-stationary increments, we have

$$\begin{aligned}
(V_{t+\log(c)})_{t\geq 0} &= \left(\overleftarrow{\mathsf{Exp}}(X_{t+\log(c)})^{-1} Z_{e^{t+\log(c)}}\right)_{t\geq 0} \\
&= \left(\overleftarrow{\mathsf{Exp}}(X_{\log(e^t)+\log(c)})^{-1}\overleftarrow{\mathsf{Exp}}(X_{\log(c)})\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{ce^t}\right)_{t\geq 0} \\
&= \left(\overleftarrow{\mathsf{Exp}}(X_{\log(e^t)+\log(c)} - X_{\log(c)})^{-1}\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{ce^t}\right)_{t\geq 0} \\
&\overset{\mathrm{D}}{=} \left(\overleftarrow{\mathsf{Exp}}(X_t)^{-1} Z_{e^t}\right)_{t\geq 0} = (V_t)_{t\geq 0}
\end{aligned}$$

and

$$\begin{aligned}
(V_{t-\log(c)})_{t\leq 0} &= \left(\overleftarrow{\mathsf{Exp}}(X_{t-\log(c)})^{-1} Z_{e^{t-\log(c)}}\right)_{t\leq 0} \\
&= \left(\overleftarrow{\mathsf{Exp}}(X_{\log(e^t)-\log(c)})^{-1}\overleftarrow{\mathsf{Exp}}(X_{-\log(c)})\overleftarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} Z_{c^{-1}e^t}\right)_{t\leq 0} \\
&= \left(\overleftarrow{\mathsf{Exp}}(X_{\log(e^t)-\log(c)} - X_{-\log(c)})^{-1}\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}e^t}\right)_{t\leq 0} \\
&\overset{\mathrm{D}}{=} \left(\overleftarrow{\mathsf{Exp}}(X_t)^{-1} Z_{e^t}\right)_{t\leq 0} = (V_t)_{t\leq 0}.
\end{aligned}$$

Thus V is $\log(c)$-stationary.

(b) Because V is $\log(c)$-stationary and X has $\log(c)$-stationary increments, we have for $t \geq 1$

$$\begin{aligned}
\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{ct} &= \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}\overleftarrow{\mathsf{Exp}}(X_{\log(ct)}) V_{\log(ct)} \\
&= \overleftarrow{\mathsf{Exp}}(X_{\log(t)+\log(c)} - X_{\log(c)}) V_{\log(t)+\log(c)} \\
&\overset{\mathrm{D}}{=} \overleftarrow{\mathsf{Exp}}(X_{\log(t)}) V_{\log(t)} = Z_t
\end{aligned}$$

and similarly for $0 < t \leq 1$

$$\begin{aligned}
\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}t} &= \overleftarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1}\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1}t)}) V_{\log(c^{-1}t)} \\
&= \overleftarrow{\mathsf{Exp}}(X_{\log(t)-\log(c)} - X_{-\log(c)}) V_{\log(t)-\log(c)} \\
&\overset{\mathrm{D}}{=} \overleftarrow{\mathsf{Exp}}(X_{\log(t)}) V_{\log(t)} = Z_t.
\end{aligned}$$

Together with the $\log(c)$-stationary increments of X this yields (5.2a) and (5.2b) and thus Z is left $X_{\log(c)}$-semiselfsimilar.

□

Corollary 5.4. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* LÉVY *process which satisfies (3.3).*

(a) *If $Z = (Z_t)_{t>0}$ is left X-selfsimilar, then its left X-random* LAMPERTI *transform $V = (V_t)_{t\in\mathbb{R}}$ is stationary.*

(b) *If $V = (V_t)_{t\in\mathbb{R}}$ is stationary and independent of X, then its left X-random inverse* Lamperti *transform $Z = (Z_t)_{t>0}$ is left X-selfsimilar.*

Proof. X is a $\log(c)$-semi-Lévy process for all $c > 1$ and we can apply Theorem 5.3.

(a) Since Z is left $X_{\log(c)}$-semiselfsimilar for all $c > 1$, V is $\log(c)$-stationary for all $c > 1$ by Theorem 5.3(a) and thus stationary.

(b) Since V is $\log(c)$-stationary for all $c > 1$, Z is left $X_{\log(c)}$-semiselfsimilar for all $c > 1$ by Theorem 5.3(b) and thus left X-selfsimilar.

□

Given a semi-Lévy process X and a left semiselfsimilar process Z we now construct a new semi-Lévy process Y by integrating the left stochastic exponential of X with respect to Z. The bivariate process (X, Y) will be the background driving process of a left MGOU process that can be constructed from Z. The subsequent corollary shows that Y is a Lévy process if X is a Lévy process and Z is left selfsimilar.

Lemma 5.5. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-*Lévy *process for some $c > 1$ which satisfies (3.3) and let $Z = (Z_t)_{t>0}$ be a left $X_{\log(c)}$-semiselfsimilar process with independent increments. Then the stochastic process $Y = (Y_t)_{t\in\mathbb{R}}$ defined by*

$$Y_t := \begin{cases} \displaystyle\int_1^{e^t} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u & , \quad t \geq 0 \\ \displaystyle -\int_{e^t}^{1} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u & , \quad t < 0 \end{cases} \tag{5.3}$$

*is a $\log(c)$-semi-*Lévy *process.*

Proof. By definition of the stochastic integral Y is continuous in probability, càdlàg, and satisfies $Y_0 = \mathbf{0}$. Because X and Z have independent increments, Y also has independent increments. Because Z is left semiselfsimilar and X has $\log(c)$-stationary increments, we have for $t \geq 0$

$$\begin{aligned} Y_{t+\log(c)} - Y_{\log(c)} &= \int_1^{ce^t} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u - \int_1^{c} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u \\ &= \int_c^{ce^t} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u \\ &= \int_1^{e^t} \overleftarrow{\mathsf{Exp}}(X_{\log(cu)-})^{-1}\,\mathrm{d}Z_{cu} \\ &= \int_1^{e^t} \overleftarrow{\mathsf{Exp}}(X_{(\log(u)+\log(c))-})^{-1}\overleftarrow{\mathsf{Exp}}(X_{\log(c)})\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}\,\mathrm{d}Z_{cu} \\ &= \int_1^{e^t} \overleftarrow{\mathsf{Exp}}(X_{(\log(u)+\log(c))-} - X_{\log(c)})^{-1}\,\mathrm{d}\Big(\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{cu}\Big) \end{aligned}$$

$$\stackrel{\mathrm{D}}{=} \int_{1}^{e^t} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u = Y_t$$

and for $t \leq 0$

$$\begin{aligned}
Y_{t-\log(c)} - Y_{-\log(c)} &= - \int_{c^{-1}e^t}^{1} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u + \int_{c^{-1}}^{1} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u \\
&= - \int_{c^{-1}e^t}^{c^{-1}} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u \\
&= - \int_{e^t}^{1} \overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1}u)-})^{-1}\,\mathrm{d}Z_{c^{-1}u} \\
&= - \int_{e^t}^{1} \overleftarrow{\mathsf{Exp}}(X_{(\log(u)-\log(c))-})^{-1}\overleftarrow{\mathsf{Exp}}(X_{-\log(c)})\overleftarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1}\,\mathrm{d}Z_{c^{-1}u} \\
&= - \int_{e^t}^{1} \overleftarrow{\mathsf{Exp}}(X_{(\log(u)-\log(c))-} - X_{-\log(c)})^{-1}\,\mathrm{d}\Big(\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}u}\Big) \\
&\stackrel{\mathrm{D}}{=} - \int_{e^t}^{1} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u = Y_t.
\end{aligned}$$

Together with the independent increments Y therefore has $\log(c)$-stationary increments. Thus Y is a $\log(c)$-semi-Lévy process. □

Corollary 5.6. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a Lévy process which satisfies (3.3) and let $Z = (Z_t)_{t>0}$ be a left X-selfsimilar process with independent increments. Then the stochastic process $Y = (Y_t)_{t\in\mathbb{R}}$ defined as in (5.3) is a Lévy process.*

Proof. X is a $\log(c)$-semi-Lévy process for all $c > 1$ and Z is left $X_{\log(c)}$-semiselfsimilar for all $c > 1$. Then Y is a $\log(c)$-semi-Lévy process for all $c > 1$ by Lemma 5.5 and thus a Lévy process. □

Theorem 5.7 and its subsequent corollary show how a left MGOU process can be constructed out of a left semiselfsimilar respectively left selfsimilar process by means of the left random Lamperti transform. Conversely, Theorem 5.9 and its subsequent corollary show how a left semiselfsimilar respectively left selfsimilar process can be constructed out of a left MGOU process by means of the left random inverse Lamperti transform.

Theorem 5.7. *Let $X = (X_t)_{t\geq 0}$ be a $\log(c)$-semi-Lévy process for some $c > 1$ which satisfies (3.3) and let $Z = (Z_t)_{t\geq 1}$ be left $X_{\log(c)}$-semiselfsimilar. By Lemma 5.5 $Y = (Y_t)_{t\geq 0}$ defined as in (5.3) is a $\log(c)$-semi-Lévy process. Then the left X-random Lamperti transform $\overleftarrow{\mathsf{Lam}}(Z) = V = (V_t)_{t\geq 0}$ is a $\log(c)$-stationary left MGOU process driven by (X, Y).*

Proof. By Theorem 5.3(a) $\overleftarrow{\mathsf{Lam}}(Z)$ is $\log(c)$-stationary. We prove that $\overleftarrow{\mathsf{Lam}}(Z)$ solves the stochastic integral equation

$$V_t = V_0 + L_t + \int_0^t \mathrm{d}U_u\, V_{u-},$$

with $(U,L) = (U_t, L_t)_{t\geq 0}$ defined as in (4.5), which is equivalent to the stochastic differential equation (4.6). By (3.7a) $\overleftarrow{\mathsf{Exp}}(X_t)^{-1} = \overrightarrow{\mathsf{Exp}}(U_t)$ for all $t \geq 0$ and thus the left X-random LAMPERTI transform of Z can be written as $V_t = \overleftarrow{\mathsf{Lam}}(Z_t) = \overleftarrow{\mathsf{Exp}}(X_t)^{-1} Z_{e^t} = \overrightarrow{\mathsf{Exp}}(U_t) Z_{e^t}$. By (3.1b) and the integration by parts formula we have for all $t \geq 0$

$$\begin{aligned}
V_0 + L_t + \int_0^t \mathrm{d}U_u\, V_{u-} &= Z_1 + L_t + \int_0^t \mathrm{d}U_u\, \overrightarrow{\mathsf{Exp}}(U_{u-}) Z_{e^{u-}} \\
&= Z_1 + L_t + \int_0^t \mathrm{d}\overrightarrow{\mathsf{Exp}}(U_u)\, Z_{e^{u-}} \\
&= Z_1 + L_t + \overrightarrow{\mathsf{Exp}}(U_t) Z_{e^t} - Z_1 - \int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Z_{e^u} - \left[\overrightarrow{\mathsf{Exp}}(U_t), Z_{e^t}\right] \\
&= L_t + \overleftarrow{\mathsf{Exp}}(X_t)^{-1} Z_{e^t} - \int_1^{e^t} \overleftarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1}\,\mathrm{d}Z_u - \left[\overrightarrow{\mathsf{Exp}}(U_t), Z_{e^t}\right] \\
&= L_t + V_t - Y_t - \left[\overrightarrow{\mathsf{Exp}}(U_t), Z_{e^t}\right] \\
&= V_t + [U_t, Y_t] - \left[\overrightarrow{\mathsf{Exp}}(U_t), Z_{e^t}\right] = V_t
\end{aligned}$$

since $L_t = Y_t + [U_t, Y_t]$ by (4.8a), and (2.18e) together with (3.1b) yields

$$\begin{aligned}
[U_t, Y_t] &= \left[U_t, \int_1^{e^t} \overrightarrow{\mathsf{Exp}}(U_{\log(u)-})\,\mathrm{d}Z_u\right] = \left[U_t, \int_0^t \overrightarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}Z_{e^u}\right] \\
&= \left[\int_0^t \mathrm{d}U_u\, \overrightarrow{\mathsf{Exp}}(U_{u-}), Z_{e^t}\right] = \left[\int_0^t \mathrm{d}\overrightarrow{\mathsf{Exp}}(U_u)\,, Z_{e^t}\right] = \left[\overrightarrow{\mathsf{Exp}}(U_t), Z_{e^t}\right].
\end{aligned}$$

By Theorem 4.9 $\overleftarrow{\mathsf{Lam}}(Z)$ is a left MGOU process driven by (X,Y). □

Corollary 5.8. *Let $X = (X_t)_{t\geq 0}$ be a LÉVY process which satisfies (3.3) and let $Z = (Z_t)_{t\geq 1}$ be left X-selfsimilar. By Corollary 5.6 $Y = (Y_t)_{t\geq 0}$ defined as in (5.3) is a LÉVY process. Then the left X-random LAMPERTI transform $\overleftarrow{\mathsf{Lam}}(Z) = V = (V_t)_{t\geq 0}$ is a stationary left MGOU process driven by (X,Y).*

Proof. X is a $\log(c)$-semi-LÉVY process for all $c > 1$ and Z is left $X_{\log(c)}$-semiselfsimilar for all $c > 1$. By Theorem 5.7 $\overleftarrow{\mathsf{Lam}}(Z)$ is a $\log(c)$-stationary left MGOU process driven by (X,Y) for all $c > 1$ and thus a stationary left MGOU process. □

Theorem 5.9. *Let $(X,Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-LÉVY process for some $c > 1$ such that X satisfies (3.3) and let $V = (V_t)_{t\in\mathbb{R}}$ be the $\log(c)$-stationary left MGOU process driven by (X,Y) as in (4.13b). Then the left X-random inverse LAMPERTI transform $\overleftarrow{\mathsf{Lam}}^{-1}(V)$ is left $X_{\log(c)}$-semiselfsimilar.*

Proof. Because (X,Y) has $\log(c)$-stationary increments we have for $t \geq 1$

$$\begin{aligned}
\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \overleftarrow{\mathsf{Lam}}^{-1}(V_{ct}) &= \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \overleftarrow{\mathsf{Exp}}(X_{\log(ct)}) V_{\log(ct)} \\
&= \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \int_{-\infty}^{\log(ct)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u
\end{aligned}$$

$$= \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \int_{-\infty}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{(u+\log(c))-}) \,\mathrm{d}Y_{u+\log(c)}$$
$$= \int_{-\infty}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{(u+\log(c))-} - X_{\log(c)}) \,\mathrm{d}(Y_{u+\log(c)} - Y_{\log(c)})$$
$$\overset{\mathrm{D}}{=} \int_{-\infty}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u = \overleftarrow{\mathsf{Exp}}(X_{\log(t)}) V_{\log(t)} = \overleftarrow{\mathsf{Lam}}^{-1}(V_t)$$

and for $0 < t \leq 1$

$$\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} \overleftarrow{\mathsf{Lam}}^{-1}(V_{c^{-1}t}) = \overleftarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1}t)}) V_{\log(c^{-1}t)}$$
$$= \overleftarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \int_{-\infty}^{\log(c^{-1}t)} \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u$$
$$= \overleftarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \int_{-\infty}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{(u-\log(c))-}) \,\mathrm{d}Y_{u-\log(c)}$$
$$= \int_{-\infty}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{(u-\log(c))-} - X_{-\log(c)}) \,\mathrm{d}(Y_{u-\log(c)} - Y_{-\log(c)})$$
$$\overset{\mathrm{D}}{=} \int_{-\infty}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-}) \,\mathrm{d}Y_u = \overleftarrow{\mathsf{Exp}}(X_{\log(t)}) V_{\log(t)} = \overleftarrow{\mathsf{Lam}}^{-1}(V_t).$$

Together with the $\log(c)$-stationary increments of X this yields (5.2a) and (5.2b) and thus $\overleftarrow{\mathsf{Lam}}^{-1}(V)$ is left $X_{\log(c)}$-semiselfsimilar. □

Corollary 5.10. *Let $(X, Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ be a* Lévy *process such that X satisfies (3.3) and let $V = (V_t)_{t\in\mathbb{R}}$ be the stationary left MGOU process driven by (X, Y) as in (4.13b). Then the left X-random inverse* Lamperti *transform $\overleftarrow{\mathsf{Lam}}^{-1}(V)$ is left X-selfsimilar.*

Proof. (X, Y) is a $\log(c)$-semi-Lévy process for all $c > 1$. Then $\overleftarrow{\mathsf{Lam}}^{-1}(V)$ is left $X_{\log(c)}$-semiselfsimilar for all $c > 1$ by Theorem 5.9 and thus left X-selfsimilar. □

5.2 Right MGOU and right semiselfsimilar processes

In the right case the generalization of the Lamperti transform and its inverse transform is defined as follows.

Definition 5.11. Let $X = (X_t)_{t\in\mathbb{R}}$ be a semimartingale with $X_0 = \mathbf{0}$ which satisfies (3.3).

(a) The **right X-random** Lamperti **transform** of a process $Z = (Z_t)_{t>0}$ is the process $\overrightarrow{\mathsf{Lam}}(Z) = V = (V_t)_{t\in\mathbb{R}}$ defined by

$$\overrightarrow{\mathsf{Lam}}(Z_t) = V_t := Z_{e^t} \overrightarrow{\mathsf{Exp}}(X_t)^{-1}. \tag{5.4a}$$

(b) The **right X-random inverse** Lamperti **transform** of a process $V = (V_t)_{t\in\mathbb{R}}$ is the process $\overrightarrow{\mathsf{Lam}}^{-1}(V) = Z = (Z_t)_{t>0}$ defined by

$$\overrightarrow{\mathsf{Lam}}^{-1}(V_t) = Z_t := V_{\log(t)} \overrightarrow{\mathsf{Exp}}(X_{\log(t)}). \tag{5.4b}$$

In the right case the generalization of selfsimilarity and semiselfsimilarity is defined as follows.

Definition 5.12. Let $X = (X_t)_{t\in\mathbb{R}}$ be a Lévy process which satisfies (3.3).

(a) A process $Z = (Z_t)_{t>0}$ is called **random right selfsimilar** with **exponent** X or simply **right** X**-selfsimilar** if for all $c > 1$

$$\left(X_{\log(t)+\log(c)} - X_{\log(c)}, Z_{ct}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}\right)_{t\geq 1} \stackrel{\mathrm{D}}{=} (X_{\log(t)}, Z_t)_{t\geq 1}, \tag{5.5a}$$

$$\left(X_{\log(t)-\log(c)} - X_{-\log(c)}, Z_{c^{-1}t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1}\right)_{0<t\leq 1} \stackrel{\mathrm{D}}{=} (X_{\log(t)}, Z_t)_{0<t\leq 1}. \tag{5.5b}$$

(b) A process $Z = (Z_t)_{t>0}$ is called **random right semiselfsimilar** with **exponent** $X_{\log(c)}$ or simply **right** $X_{\log(c)}$**-semiselfsimilar** if (5.5a) and (5.5b) hold for some $c > 1$.

In the definition of right semiselfsimilarity we can without loss of generality assume $c > 1$ because for $c = 1$ (5.5a) and (5.5b) are automatically fulfilled and for $0 < c < 1$ swapping (5.5a) and (5.5b) and rescaling the time parameter of Z leads to right semiselfsimilarity with $c^{-1} > 1$. In particular, right $X_{\log(c)}$-semiselfsimilarity for all $c > 1$ implies right X-selfsimilarity.

If the exponent X is a semi-Lévy process the right random Lamperti transform and its inverse transform establish a one-to-one correspondence between right semiselfsimilar processes and periodically stationary processes. If the exponent X is a Lévy process the one-to-one correspondence extends to right selfsimilar processes and stationary processes.

Theorem 5.13. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-Lévy process for some $c > 1$ which satisfies (3.3).*

(a) *If $Z = (Z_t)_{t>0}$ is right $X_{\log(c)}$-semiselfsimilar, then its right X-random Lamperti transform $V = (V_t)_{t\in\mathbb{R}}$ is $\log(c)$-stationary.*

(b) *If $V = (V_t)_{t\in\mathbb{R}}$ is $\log(c)$-stationary and independent of X, then its right X-random inverse Lamperti transform $Z = (Z_t)_{t>0}$ is right $X_{\log(c)}$-semiselfsimilar.*

Proof. (a) Because Z is right $X_{\log(c)}$-semiselfsimilar and X has $\log(c)$-stationary increments, we have

$$\begin{aligned}(V_{t+\log(c)})_{t\geq 0} &= \left(Z_{e^{t+\log(c)}}\overrightarrow{\mathsf{Exp}}(X_{t+\log(c)})^{-1}\right)_{t\geq 0}\\ &= \left(Z_{ce^t}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})\overrightarrow{\mathsf{Exp}}(X_{\log(e^t)+\log(c)})^{-1}\right)_{t\geq 0}\\ &= \left(Z_{ce^t}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}\overrightarrow{\mathsf{Exp}}(X_{\log(e^t)+\log(c)} - X_{\log(c)})^{-1}\right)_{t\geq 0}\\ &\stackrel{\mathrm{D}}{=} \left(Z_{e^t}\overrightarrow{\mathsf{Exp}}(X_t)^{-1}\right)_{t\geq 0} = (V_t)_{t\geq 0}\end{aligned}$$

and

$$\begin{aligned}(V_{t-\log(c)})_{t\leq 0} &= \left(Z_{e^{t-\log(c)}}\overrightarrow{\mathsf{Exp}}(X_{t-\log(c)})^{-1}\right)_{t\leq 0}\\ &= \left(Z_{c^{-1}e^t}\overrightarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1}\overrightarrow{\mathsf{Exp}}(X_{-\log(c)})\overrightarrow{\mathsf{Exp}}(X_{\log(e^t)-\log(c)})^{-1}\right)_{t\leq 0}\\ &= \left(Z_{c^{-1}e^t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1}\overrightarrow{\mathsf{Exp}}(X_{\log(e^t)-\log(c)} - X_{-\log(c)})^{-1}\right)_{t\leq 0}\\ &\stackrel{\mathrm{D}}{=} \left(Z_{e^t}\overrightarrow{\mathsf{Exp}}(X_t)^{-1}\right)_{t\leq 0} = (V_t)_{t\leq 0}.\end{aligned}$$

Thus V is $\log(c)$-stationary.

(b) Because V is $\log(c)$-stationary and X has $\log(c)$-stationary increments, we have for $t \geq 1$

$$\begin{aligned} Z_{ct}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} &= V_{\log(ct)}\overrightarrow{\mathsf{Exp}}(X_{\log(ct)})\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \\ &= V_{\log(t)+\log(c)}\overrightarrow{\mathsf{Exp}}(X_{\log(t)+\log(c)} - X_{\log(c)}) \\ &\overset{\mathrm{D}}{=} V_{\log(t)}\overrightarrow{\mathsf{Exp}}(X_{\log(t)}) = Z_t \end{aligned}$$

and similarly for $0 < t \leq 1$

$$\begin{aligned} Z_{c^{-1}t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} &= V_{\log(c^{-1}t)}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1}t)})\overrightarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \\ &= V_{\log(t)-\log(c)}\overrightarrow{\mathsf{Exp}}(X_{\log(t)-\log(c)} - X_{-\log(c)}) \\ &\overset{\mathrm{D}}{=} V_{\log(t)}\overrightarrow{\mathsf{Exp}}(X_{\log(t)}) = Z_t. \end{aligned}$$

Together with the $\log(c)$-stationary increments of X this yields (5.5a) and (5.5b) and thus Z is right $X_{\log(c)}$-semiselfsimilar.

□

Corollary 5.14. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* Lévy *process which satisfies (3.3).*

(a) *If $Z = (Z_t)_{t>0}$ is right X-selfsimilar, then its right X-random* Lamperti *transform $V = (V_t)_{t\in\mathbb{R}}$ is stationary.*

(b) *If $V = (V_t)_{t\in\mathbb{R}}$ is stationary and independent of X, then its right X-random inverse* Lamperti *transform $Z = (Z_t)_{t>0}$ is right X-selfsimilar.*

Proof. X is a $\log(c)$-semi-Lévy process for all $c > 1$ and we can apply Theorem 5.13.

(a) Since Z is right $X_{\log(c)}$-semiselfsimilar for all $c > 1$, V is $\log(c)$-stationary for all $c > 1$ by Theorem 5.13(a) and thus stationary.

(b) Since V is $\log(c)$-stationary for all $c > 1$, Z is right $X_{\log(c)}$-semiselfsimilar for all $c > 1$ by Theorem 5.13(b) and thus right X-selfsimilar.

□

Given a semi-Lévy process X and a right semiselfsimilar process Z we now construct a new semi-Lévy process Y by integrating the right stochastic exponential of X with respect to Z. The bivariate process (X, Y) will be the background driving process of a right MGOU process that can be constructed from Z. The subsequent corollary shows that Y is a Lévy process if X is a Lévy process and Z is right selfsimilar.

Lemma 5.15. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-*Lévy *process for some $c > 1$ which satisfies (3.3) and let $Z = (Z_t)_{t>0}$ be a right $X_{\log(c)}$-semiselfsimilar process with independent increments. Then the stochastic process $Y = (Y_t)_{t\in\mathbb{R}}$ defined by*

$$Y_t := \begin{cases} \displaystyle\int_1^{e^t} \mathrm{d}Z_u\,\overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} &, \quad t \geq 0 \\ \displaystyle -\int_{e^t}^1 \mathrm{d}Z_u\,\overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} &, \quad t < 0 \end{cases} \tag{5.6}$$

*is a $\log(c)$-semi-*Lévy *process.*

Proof. By definition of the stochastic integral Y is continuous in probability, càdlàg, and satisfies $Y_0 = \mathbf{0}$. Because X and Z have independent increments, Y also has independent increments. Because Z is right semiselfsimilar and X has $\log(c)$-stationary increments, we have for $t \geq 0$

$$\begin{aligned}
Y_{t+\log(c)} - Y_{\log(c)} &= \int_1^{ce^t} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} - \int_1^{c} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} \\
&= \int_c^{ce^t} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} \\
&= \int_1^{e^t} \mathrm{d}Z_{cu} \, \overrightarrow{\mathsf{Exp}}(X_{\log(cu)-})^{-1} \\
&= \int_1^{e^t} \mathrm{d}Z_{cu} \, \overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \overrightarrow{\mathsf{Exp}}(X_{\log(c)}) \overrightarrow{\mathsf{Exp}}(X_{(\log(u)+\log(c))-})^{-1} \\
&= \int_1^{e^t} \mathrm{d}\Big(Z_{cu} \overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}\Big) \, \overrightarrow{\mathsf{Exp}}(X_{(\log(u)+\log(c))-} - X_{\log(c)})^{-1} \\
&\overset{\mathrm{D}}{=} \int_1^{e^t} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} = Y_t
\end{aligned}$$

and for $t \leq 0$

$$\begin{aligned}
Y_{t-\log(c)} - Y_{-\log(c)} &= - \int_{c^{-1}e^t}^{1} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} + \int_{c^{-1}}^{1} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} \\
&= - \int_{c^{-1}e^t}^{c^{-1}} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} \\
&= - \int_{e^t}^{1} \mathrm{d}Z_{c^{-1}u} \, \overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1}u)-})^{-1} \\
&= - \int_{e^t}^{1} \mathrm{d}Z_{c^{-1}u} \, \overrightarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \overrightarrow{\mathsf{Exp}}(X_{-\log(c)}) \overrightarrow{\mathsf{Exp}}(X_{(\log(u)-\log(c))-})^{-1} \\
&= - \int_{e^t}^{1} \mathrm{d}\Big(Z_{c^{-1}u} \overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1}\Big) \, \overrightarrow{\mathsf{Exp}}(X_{(\log(u)-\log(c))-} - X_{-\log(c)})^{-1} \\
&\overset{\mathrm{D}}{=} - \int_{e^t}^{1} \mathrm{d}Z_u \, \overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} = Y_t.
\end{aligned}$$

Together with the independent increments Y therefore has $\log(c)$-stationary increments. Thus Y is a $\log(c)$-semi-Lévy process. □

Corollary 5.16. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a Lévy process which satisfies (3.3) and let $Z = (Z_t)_{t>0}$ be a right X-selfsimilar process with independent increments. Then the stochastic process $Y = (Y_t)_{t\in\mathbb{R}}$ defined as in (5.6) is a Lévy process.*

Proof. X is a $\log(c)$-semi-Lévy process for all $c > 1$ and Z is right $X_{\log(c)}$-semiselfsimilar for all $c > 1$. Then Y is a $\log(c)$-semi-Lévy process for all $c > 1$ by Lemma 5.15 and thus a Lévy process. □

Theorem 5.17 and its subsequent corollary show how a right MGOU process can be constructed out of a right semiselfsimilar respectively right selfsimilar process by means of the right random Lamperti transform. Conversely, Theorem 5.19 and its subsequent corollary show how a right semiselfsimilar respectively right selfsimilar process can be constructed out of a right MGOU process by means of the right random inverse Lamperti transform.

Theorem 5.17. *Let $X = (X_t)_{t\geq 0}$ be a $\log(c)$-semi-Lévy process for some $c > 1$ which satisfies (3.3) and let $Z = (Z_t)_{t\geq 1}$ be right $X_{\log(c)}$-semiselfsimilar. By Lemma 5.15 $Y = (Y_t)_{t\geq 0}$ defined as in (5.6) is a $\log(c)$-semi-Lévy process. Then the right X-random Lamperti transform $\overrightarrow{\mathsf{Lam}}(Z) = V = (V_t)_{t\geq 0}$ is a $\log(c)$-stationary right MGOU process driven by (X, Y).*

Proof. By Theorem 5.13(a) $\overrightarrow{\mathsf{Lam}}(Z)$ is $\log(c)$-stationary. We prove that $\overrightarrow{\mathsf{Lam}}(Z)$ solves the stochastic integral equation

$$V_t = V_0 + L_t + \int_0^t V_{u-}\,\mathrm{d}U_u\,,$$

with $(U, L) = (U_t, L_t)_{t\geq 0}$ defined as in (4.18), which is equivalent to the stochastic differential equation (4.19). By (3.7b) $\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overleftarrow{\mathsf{Exp}}(U_t)$ for all $t \geq 0$ and thus the right X-random Lamperti transform of Z can be written as $V_t = \overrightarrow{\mathsf{Lam}}(Z_t) = Z_{e^t}\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = Z_{e^t}\overleftarrow{\mathsf{Exp}}(U_t)$. By (3.1a) and the integration by parts formula we have for all $t \geq 0$

$$\begin{aligned}
V_0 + L_t + \int_0^t V_{u-}\,\mathrm{d}U_u &= Z_1 + L_t + \int_0^t Z_{e^u-}\overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}U_u \\
&= Z_1 + L_t + \int_0^t Z_{e^u-}\,\mathrm{d}\overleftarrow{\mathsf{Exp}}(U_u) \\
&= Z_1 + L_t + Z_{e^t}\overleftarrow{\mathsf{Exp}}(U_t) - Z_1 - \int_0^t \mathrm{d}Z_{e^u}\,\overleftarrow{\mathsf{Exp}}(U_{u-}) - [Z_{e^t}, \overleftarrow{\mathsf{Exp}}(U_t)] \\
&= L_t + Z_{e^t}\overrightarrow{\mathsf{Exp}}(X_t)^{-1} - \int_1^{e^t} \mathrm{d}Z_u\,\overrightarrow{\mathsf{Exp}}(X_{\log(u)-})^{-1} - [Z_{e^t}, \overleftarrow{\mathsf{Exp}}(U_t)] \\
&= L_t + V_t - Y_t - [Z_{e^t}, \overleftarrow{\mathsf{Exp}}(U_t)] \\
&= V_t + [Y_t, U_t] - [Z_{e^t}, \overleftarrow{\mathsf{Exp}}(U_t)] = V_t
\end{aligned}$$

since $L_t = Y_t + [Y_t, U_t]$ by (4.21a), and (2.18e) together with (3.1a) yields

$$[U_t, Y_t] = \left[\int_1^{e^t} \mathrm{d}Z_u\,\overleftarrow{\mathsf{Exp}}(U_{\log(u)-}), U_t\right] = \left[\int_0^t \mathrm{d}Z_{e^u}\,\overleftarrow{\mathsf{Exp}}(U_{u-}), U_t\right]$$

$$= \left[Z_{e^t}, \int\limits_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\,\mathrm{d}U_u \right] = \left[Z_{e^t}, \int\limits_0^t \mathrm{d}\overleftarrow{\mathsf{Exp}}(U_u) \right] = [Z_{e^t}, \overleftarrow{\mathsf{Exp}}(U_t)].$$

By Theorem 4.9 $\overrightarrow{\mathsf{Lam}}(Z)$ is a right MGOU process driven by (X,Y). □

Corollary 5.18. *Let $X = (X_t)_{t\geq 0}$ be a* Lévy *process which satisfies (3.3) and let $Z = (Z_t)_{t\geq 1}$ be right X-selfsimilar. By Corollary 5.16 $Y = (Y_t)_{t\geq 0}$ defined as in (5.6) is a* Lévy *process. Then the right X-random* Lamperti *transform $\overrightarrow{\mathsf{Lam}}(Z) = V = (V_t)_{t\geq 0}$ is a stationary right MGOU process driven by (X,Y).*

Proof. X is a $\log(c)$-semi-Lévy process for all $c > 1$ and Z is right $X_{\log(c)}$-semiselfsimilar for all $c > 1$. By Theorem 5.17 $\overrightarrow{\mathsf{Lam}}(Z)$ is a $\log(c)$-stationary right MGOU process driven by (X,Y) for all $c > 1$ and thus a stationary right MGOU process. □

Theorem 5.19. *Let $(X,Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-*Lévy *process for some $c > 1$ such that X satisfies (3.3) and let $V = (V_t)_{t\in\mathbb{R}}$ be the $\log(c)$-stationary right MGOU process driven by (X,Y) as in (4.26b). Then the right X-random inverse* Lamperti *transform $\overrightarrow{\mathsf{Lam}}^{-1}(V)$ is right $X_{\log(c)}$-semiselfsimilar.*

Proof. Because (X,Y) has $\log(c)$-stationary increments we have for $t \geq 1$

$$\begin{aligned}
\overrightarrow{\mathsf{Lam}}^{-1}(V_{ct})\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} &= V_{\log(ct)}\overrightarrow{\mathsf{Exp}}(X_{\log(ct)})\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \\
&= \int\limits_{-\infty}^{\log(ct)} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \\
&= \int\limits_{-\infty}^{\log(t)} \mathrm{d}Y_{u+\log(c)}\, \overrightarrow{\mathsf{Exp}}(X_{(u+\log(c))-})\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \\
&= \int\limits_{-\infty}^{\log(t)} \mathrm{d}(Y_{u+\log(c)} - Y_{\log(c)})\, \overrightarrow{\mathsf{Exp}}(X_{(u+\log(c))-} - X_{\log(c)}) \\
&\overset{\mathrm{D}}{=} \int\limits_{-\infty}^{\log(t)} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) = V_{\log(t)}\overrightarrow{\mathsf{Exp}}(X_{\log(t)}) = \overrightarrow{\mathsf{Lam}}^{-1}(V_t)
\end{aligned}$$

and for $0 < t \leq 1$

$$\begin{aligned}
\overrightarrow{\mathsf{Lam}}^{-1}(V_{c^{-1}t})\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} &= V_{\log(c^{-1}t)}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1}t)})\overrightarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \\
&= \int\limits_{-\infty}^{\log(c^{-1}t)} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \\
&= \int\limits_{-\infty}^{\log(t)} \mathrm{d}Y_{u-\log(c)}\, \overrightarrow{\mathsf{Exp}}(X_{(u-\log(c))-})\overrightarrow{\mathsf{Exp}}(X_{-\log(c)})^{-1} \\
&= \int\limits_{-\infty}^{\log(t)} \mathrm{d}(Y_{u-\log(c)} - Y_{-\log(c)})\, \overrightarrow{\mathsf{Exp}}(X_{(u-\log(c))-} - X_{-\log(c)})
\end{aligned}$$

$$\overset{\mathrm{D}}{=} \int_{-\infty}^{\log(t)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) = V_{\log(t)} \overrightarrow{\mathsf{Exp}}(X_{\log(t)}) = \overrightarrow{\mathsf{Lam}}^{-1}(V_t).$$

Together with the $\log(c)$-stationary increments of X this yields (5.5a) and (5.5b) and thus $\overrightarrow{\mathsf{Lam}}^{-1}(V)$ is right $X_{\log(c)}$-semiselfsimilar. □

Corollary 5.20. *Let $(X, Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ be a* Lévy *process such that X satisfies (3.3) and let $V = (V_t)_{t\in\mathbb{R}}$ be the stationary right MGOU process driven by (X, Y) as in (4.26b). Then the right X-random inverse* Lamperti *transform $\overrightarrow{\mathsf{Lam}}^{-1}(V)$ is right X-selfsimilar.*

Proof. (X, Y) is a $\log(c)$-semi-Lévy process for all $c > 1$. Then $\overrightarrow{\mathsf{Lam}}^{-1}(V)$ is right $X_{\log(c)}$-semiselfsimilar for all $c > 1$ by Theorem 5.19 and thus right X-selfsimilar. □

6 Connection between MGOU and semistable hemigroups

Chapter 6 introduces random semistable hemigroups as a generalization of semistable hemigroups that incorporates random scaling. We prove that a random semistable hemigroup admits an integral representation and show how random semistable hemigroups are connected to random semiselfsimilar processes, periodically stationary processes, and MGOU processes. These connections extend to the random stable, random selfsimilar, and stationary case.

Due to the non-commutativity of matrix multiplication there are two types of random semistable hemigroups in dimension $n \geq 2$. Chapter 6.1 discusses random left semistable hemigroups and the connection to random left semiselfsimilarity and left MGOU processes while chapter 6.2 discusses random right semistable hemigroups and the connection to random right semiselfsimilarity and right MGOU processes.

6.1 Left MGOU and left semistable hemigroups

In the left case the generalization of a stable and semistable hemigroup is defined as follows.

Definition 6.1. Let $X = (X_t)_{t\in\mathbb{R}}$ be a semimartingale with $X_0 = \mathbf{0}$ which satisfies (3.3).

(a) A family $(Z_{s,t})_{0\leq s\leq t}$ of random variables is called **random left stable hemigroup with exponent** X or simply **left** X**-stable hemigroup** if it satisfies the following four conditions.

(L1) $Z_{q,r}$ and $Z_{s,t}$ are independent for all $0 \leq q \leq r \leq s \leq t$.

(L2) $Z_{r,s} + Z_{s,t} = Z_{r,t}$ for all $0 \leq r \leq s \leq t$.

(L3) For all $c > 1$

$$\left(X_{\log(t)+\log(c)} - X_{\log(c)}, \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{cs,ct}\right)_{1\leq s\leq t} \stackrel{\mathrm{D}}{=} (X_{\log(t)}, Z_{s,t})_{1\leq s\leq t}, \tag{6.1a}$$

$$\left(X_{\log(t)-\log(c)} - X_{-\log(c)}, \overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}s,c^{-1}t}\right)_{0<s\leq t\leq 1} \stackrel{\mathrm{D}}{=} (X_{\log(t)}, Z_{s,t})_{0<s\leq t\leq 1}. \tag{6.1b}$$

(L4) The map $(s,t) \longmapsto Z_{s,t}$ is continuous with respect to convergence in distribution.

(b) A family $(Z_{s,t})_{0\leq s\leq t}$ of random variables is called **random left semistable hemigroup with exponent** $X_{\log(c)}$ or simply **left** $X_{\log(c)}$**-semistable hemigroup** if it satisfies (L1), (L2), (L4), and (L3) holds true for some $c > 1$.

In the definition of a left semistable hemigroup we can without loss of generality assume $c > 1$ because for $c = 1$ (6.1a) and (6.1b) are automatically fulfilled and for $0 < c < 1$ swapping (6.1a) and (6.1b) and rescaling the time parameters of $(Z_{s,t})_{0\leq s\leq t}$ leads to a left semistable hemigroup with $c^{-1} > 1$. In particular, if $(Z_{s,t})_{0\leq s\leq t}$ is a left $X_{\log(c)}$-semistable hemigroup for all $c > 1$ then it is a left X-stable hemigroup.

If the exponent X is a semi-Lévy process there is a one-to-one correspondence between left semistable hemigroups and left semiselfsimilar processes. If the exponent X is a Lévy process the one-to-one correspondence extends to left stable hemigroups and left selfsimilar processes.

Theorem 6.2. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* $\log(c)$*-semi-*Lévy *process for some $c > 1$ which satisfies (3.3).*

(a) *If $(Z_{s,t})_{0\le s\le t}$ is a left $X_{\log(c)}$-semistable hemigroup, then the process $Z = (Z_t)_{0<t\le 1}$ defined by $Z_t := Z_{0,t}$ has independent increments and is left $X_{\log(c)}$-semiselfsimilar.*

(b) *If $Z = (Z_t)_{t\ge 0}$ is a left $X_{\log(c)}$-semiselfsimilar process that is continuous in probability and has independent increments, then the family $(Z_{s,t})_{0\le s\le t}$ defined by $Z_{s,t} := Z_t - Z_s$ is a left $X_{\log(c)}$-semistable hemigroup.*

Proof. (a) By (L1) and (L2) Z has independent increments. By (L3) we have for $0 < t \le 1$

$$\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}t} = \overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{0,c^{-1}t} \overset{\mathrm{D}}{=} Z_{0,t}.$$

Together with the $\log(c)$-stationary increments of X this yields (5.2b) and thus Z is left $X_{\log(c)}$-semiselfsimilar.

(b) (L1) is clear from the independent increments of Z. (L2) follows by direct calculation since for all $0 \le r \le s \le t$

$$Z_{r,s} + Z_{s,t} = Z_s - Z_r + Z_t - Z_s = Z_t - Z_r = Z_{r,t}.$$

(L3) holds true because for $1 \le s \le t$

$$\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{cs,ct} = \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}(Z_{ct} - Z_{cs}) \overset{\mathrm{D}}{=} Z_t - Z_s = Z_{s,t}$$

and for $0 \le s \le t \le 1$

$$\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}s,c^{-1}t} = \overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1}(Z_{c^{-1}t} - Z_{c^{-1}s}) \overset{\mathrm{D}}{=} Z_t - Z_s = Z_{s,t}.$$

Together with the $\log(c)$-stationary increments of X this yields (6.1a) and (6.1b). (L4) is a consequence of the continuity in probability of Z and the fact that convergence in probability implies convergence in distribution.

□

Corollary 6.3. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* Lévy *process which satisfies (3.3).*

(a) *If $(Z_{s,t})_{0\le s\le t}$ is a left X-stable hemigroup, then the process $Z = (Z_t)_{0<t\le 1}$ defined by $Z_t := Z_{0,t}$ has independent increments and is left X-selfsimilar.*

(b) *If $Z = (Z_t)_{t\ge 0}$ is a left X-selfsimilar process that is continuous in probability and has independent increments, then the family $(Z_{s,t})_{0\le s\le t}$ defined by $Z_{s,t} := Z_t - Z_s$ is a left X-stable hemigroup.*

Proof. X is a $\log(c)$-semi-Lévy process for all $c > 1$ and we can apply Theorem 6.2.

(a) Since $(Z_{s,t})_{0\le s\le t}$ is a left $X_{\log(c)}$-semistable hemigroup for all $c > 1$, Z is left $X_{\log(c)}$-semiselfsimilar for all $c > 1$ by Theorem 6.2(a) and thus left X-selfsimilar. The independent increments of Z also follow from Theorem 6.2(a).

(b) Since Z is $X_{\log(c)}$-semiselfsimilar for all $c > 1$, $(Z_{s,t})_{0\le s\le t}$ is a left $X_{\log(c)}$-semistable hemigroup for all $c > 1$ by Theorem 6.2(b) and thus a left X-stable hemigroup.

□

Theorem 6.4 and its subsequent corollary show that a left semistable respectively left stable hemigroup admits the integral representation (6.2). Conversely, Theorem 6.6 and its subsequent corollary show that the stochastic integral in (6.2) defines a left semistable respectively left stable hemigroup.

Theorem 6.4. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-*LÉVY *process for some $c > 1$ which satisfies (3.3) and let $(Z_{s,t})_{0\le s\le t}$ be a left $X_{\log(c)}$-semistable hemigroup. By Theorem 6.2(a) $Z = (Z_t)_{0<t\le 1}$ defined by $Z_t := Z_{0,t}$ is a left $X_{\log(c)}$-semiselfsimilar process with independent increments and by Lemma 5.5 $Y = (Y_t)_{t\le 0}$ defined as in (5.3) is a $\log(c)$-semi-*LÉVY *process. Then for all $0 < s \le t \le 1$ the random variable $Z_{s,t}$ has the integral representation*

$$Z_{s,t} = \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\,. \tag{6.2}$$

Proof. For $0 < s \le t \le 1$ we have

$$\begin{aligned}
\int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u &= \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}\Bigg(-\int_{e^u}^{1} \overleftarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\,\mathrm{d}Z_v\Bigg) \\
&= \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}\Bigg(\int_{1}^{e^u} \overleftarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\,\mathrm{d}Z_v\Bigg) \\
&= \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\overleftarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1}\,\mathrm{d}Z_{e^u} \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u} = Z_{e^{\log(t)}} - Z_{e^{\log(s)}} = Z_t - Z_s = Z_{s,t}.
\end{aligned}$$

□

Corollary 6.5. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* LÉVY *process which satisfies (3.3) and let $(Z_{s,t})_{0\le s\le t}$ be a left X-stable hemigroup. By Corollary 6.3(a) $Z = (Z_t)_{0<t\le 1}$ defined by $Z_t := Z_{0,t}$ is a left X-selfsimilar process with independent increments and by Corollary 5.6 $Y = (Y_t)_{t\le 0}$ defined as in (5.3) is a* LÉVY *process. Then for all $0 < s \le t \le 1$ the random variable $Z_{s,t}$ has the integral representation (6.2).*

Proof. The calculation to derive (6.2) is the same as in the proof of Theorem 6.4. □

Theorem 6.6. *Let $(X, Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-*LÉVY *process for some $c > 1$ such that X satisfies (3.3). Then $(Z_{s,t})_{0<s\le t}$ defined as in (6.2) is a left $X_{\log(c)}$-semistable hemigroup.*

Proof. (L1) is clear from the independent increments of (X, Y). (L2) follows by direct calculation since for all $0 < r \le s \le t$

$$Z_{r,s} + Z_{s,t} = \int_{\log(r)}^{\log(s)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u + \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u = \int_{\log(r)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u = Z_{r,t}.$$

(L3) holds true because for $1 \le s \le t$

$$\begin{aligned}
\overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} Z_{cs,ct} &= \overleftarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \int\limits_{\log(cs)}^{\log(ct)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u \\
&= \int\limits_{\log(s)+\log(c)}^{\log(t)+\log(c)} \overleftarrow{\mathsf{Exp}}(X_{u-} - X_{\log(c)})\,\mathrm{d}(Y_u - Y_{\log(c)}) \\
&= \int\limits_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{(u+\log(c))-} - X_{\log(c)})\,\mathrm{d}(Y_{u+\log(c)} - Y_{\log(c)}) \\
&\overset{\mathrm{D}}{=} \int\limits_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u = Z_{s,t}
\end{aligned}$$

and for $0 < s \le t \le 1$

$$\begin{aligned}
\overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} Z_{c^{-1}s,c^{-1}t} &= \overleftarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} \int\limits_{\log(c^{-1}s)}^{\log(c^{-1}t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u \\
&= \int\limits_{\log(s)-\log(c)}^{\log(t)-\log(c)} \overleftarrow{\mathsf{Exp}}(X_{u-} - X_{-\log(c)})\,\mathrm{d}(Y_u - Y_{-\log(c)}) \\
&= \int\limits_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{(u-\log(c))-} - X_{-\log(c)})\,\mathrm{d}(Y_{u-\log(c)} - Y_{-\log(c)}) \\
&\overset{\mathrm{D}}{=} \int\limits_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u = Z_{s,t}.
\end{aligned}$$

Together with the $\log(c)$-stationary increments of X this yields (6.1a) and (6.1b). (L4) is a consequence of the continuity in probability of (X, Y) and the stochastic integral and of the fact that convergence in probability implies convergence in distribution. □

Corollary 6.7. *Let $(X, Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ be a Lévy process such that X satisfies (3.3). Then $(Z_{s,t})_{0<s\le t}$ defined as in (6.2) is a left X-stable hemigroup.*

Proof. Since (X, Y) is a $\log(c)$-semi-Lévy process for all $c > 1$, $(Z_{s,t})_{0\le s\le t}$ is a left $X_{\log(c)}$-semistable hemigroup for all $c > 1$ by Theorem 6.6 and thus a left X-stable hemigroup. □

Theorem 6.8 and its subsequent corollary show how a left semistable respectively left stable hemigroup can be constructed out of a periodically stationary respectively stationary process. Conversely, Theorem 6.10 and its subsequent corollary show how a periodically stationary respectively stationary process can be constructed out of a left semistable respectively left stable hemigroup.

Theorem 6.8. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-Lévy process for some $c > 1$ which satisfies (3.3) and let $V = (V_t)_{t\in\mathbb{R}}$ be $\log(c)$-stationary and independent of X. By Theorem 5.3(b) the left X-random inverse Lamperti transform $Z = (Z_t)_{t>0}$ of V is left $X_{\log(c)}$-semiselfsimilar and by Lemma 5.5 $Y = (Y_t)_{t\in\mathbb{R}}$ defined as in (5.3) is a $\log(c)$-semi-Lévy* *process. Then* $(Z_{s,t})_{0<s\le t}$ *defined by*

$$Z_{s,t} \coloneqq \overleftarrow{\mathsf{Exp}}(X_{\log(t)})V_{\log(t)} - \overleftarrow{\mathsf{Exp}}(X_{\log(s)})V_{\log(s)} \tag{6.3a}$$

is a left $X_{\log(c)}$*-semistable hemigroup and* $Z_{s,t}$ *has the integral representation*

$$Z_{s,t} = \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u\,. \tag{6.3b}$$

Proof. By Theorem 6.2(b) the family $(Z_{s,t})_{0<s\le t}$ defined by $Z_{s,t} \coloneqq Z_t - Z_s$ is a left $X_{\log(c)}$-semistable hemigroup and by definition of the left X-random inverse Lamperti transform we have

$$Z_{s,t} = Z_t - Z_s = \overleftarrow{\mathsf{Exp}}(X_{\log(t)})V_{\log(t)} - \overleftarrow{\mathsf{Exp}}(X_{\log(s)})V_{\log(s)}$$

which is (6.3a). The integral representation in (6.3b) for $0 < s \le t \le 1$ follows with the same calculation as in the proof of Theorem 6.4. For $1 \le s \le t$ we have

$$\begin{aligned}
\int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u &= \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}\Bigg(\int_1^{e^u} \overleftarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\,\mathrm{d}Z_v\Bigg)\\
&= \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\overleftarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1}\,\mathrm{d}Z_{e^u}\\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u} = Z_{e^{\log(t)}} - Z_{e^{\log(s)}} = Z_t - Z_s = Z_{s,t}
\end{aligned}$$

and for $0 < s \le 1 \le t$ we have

$$\begin{aligned}
\int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u &= \int_{\log(s)}^{0} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}\Bigg(-\int_{e^u}^{1} \overleftarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\,\mathrm{d}Z_v\Bigg)\\
&\quad + \int_{0}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}\Bigg(\int_1^{e^u} \overleftarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\,\mathrm{d}Z_v\Bigg)\\
&= \int_{\log(s)}^{0} \overleftarrow{\mathsf{Exp}}(X_{u-})\overleftarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1}\,\mathrm{d}Z_{e^u}\\
&\quad + \int_{0}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-})\overleftarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1}\,\mathrm{d}Z_{e^u}\\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u} = Z_{e^{\log(t)}} - Z_{e^{\log(s)}} = Z_t - Z_s = Z_{s,t}.
\end{aligned}$$

□

Corollary 6.9. *Let* $X = (X_t)_{t\in\mathbb{R}}$ *be a* Lévy *process which satisfies (3.3) and let* $V = (V_t)_{t\in\mathbb{R}}$ *be stationary and independent of* X*. By Corollary 5.4(b) the left* X*-random inverse*

LAMPERTI transform $Z = (Z_t)_{t>0}$ of V is left X-selfsimilar and by Corollary 5.6 $Y = (Y_t)_{t\in\mathbb{R}}$ defined as in (5.3) is a LÉVY process. Then $(Z_{s,t})_{0<s\leq t}$ defined as in (6.3a) is a left X-stable hemigroup and $Z_{s,t}$ has the integral representation (6.3b).

Proof. By Corollary 6.3(b) the family $(Z_{s,t})_{0<s\leq t}$ defined by $Z_{s,t} := Z_t - Z_s$ is a left X-stable hemigroup. (6.3a) is derived in the same way as in the proof of Theorem 6.8 and the integral representation (6.3b) follows with the same calculations as in the proofs of Theorem 6.4 and Theorem 6.8. □

Theorem 6.10. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-LÉVY process for some $c > 1$ which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a left $X_{\log(c)}$-semistable hemigroup. Then the process $V = (V_t)_{t\leq 0}$ defined by*

$$V_t := \overleftarrow{\mathsf{Exp}}(X_t)^{-1} Z_{0,e^t} \tag{6.4}$$

is $\log(c)$-stationary.

Proof. By Theorem 6.2(a) $Z = (Z_t)_{0<t\leq 1}$ defined by $Z_t := Z_{0,t}$ is left $X_{\log(c)}$-semiselfsimilar and then by Theorem 5.3(a) $V = \overleftarrow{\mathsf{Lam}}(Z)$ is $\log(c)$-stationary. □

Corollary 6.11. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a LÉVY process which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a left X-stable hemigroup. Then the process $V = (V_t)_{t\leq 0}$ defined as in (6.4) is stationary.*

Proof. By Corollary 6.3(a) $Z = (Z_t)_{0<t\leq 1}$ defined by $Z_t := Z_{0,t}$ is left X-selfsimilar and then by Corollary 5.4(a) $V = \overleftarrow{\mathsf{Lam}}(Z)$ is stationary. □

Theorem 6.12 and its subsequent corollary show how a left semistable respectively left stable hemigroup can be constructed out of the random functional $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ that defines a left MGOU process via the random recurrence equation (4.1). Conversely, Theorem 6.14 and its subsequent corollary show how a periodically stationary respectively stationary left MGOU process can be constructed out of a left semistable respectively left stable hemigroup.

Theorem 6.12. *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a $\log(c)$-semi-LÉVY process for some $c > 1$ such that X satisfies (3.3) and let $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ be defined as in (4.3). Then the family*

$$\left(A_{\log(t)}^{-1} B_{\log(t)} - A_{\log(s)}^{-1} B_{\log(s)}\right)_{1\leq s\leq t} \tag{6.5}$$

is a left $X_{\log(c)}$-semistable hemigroup.

Proof. Inserting the formulas in (4.3) we have for all $1 \leq s \leq t$

$$\begin{aligned} A_{\log(t)}^{-1} B_{\log(t)} - A_{\log(s)}^{-1} B_{\log(s)} &= \overleftarrow{\mathsf{Exp}}(X_{\log(t)}) \overleftarrow{\mathsf{Exp}}(X_{\log(t)})^{-1} \int_0^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-}) \, \mathrm{d}Y_u \\ &\quad - \overleftarrow{\mathsf{Exp}}(X_{\log(s)}) \overleftarrow{\mathsf{Exp}}(X_{\log(s)})^{-1} \int_0^{\log(s)} \overleftarrow{\mathsf{Exp}}(X_{u-}) \, \mathrm{d}Y_u \\ &= \int_{\log(s)}^{\log(t)} \overleftarrow{\mathsf{Exp}}(X_{u-}) \, \mathrm{d}Y_u \end{aligned}$$

which is (6.2) and by Theorem 6.6 this defines a left $X_{\log(c)}$-semistable hemigroup. □

Corollary 6.13. *Let $(X,Y)=(X_t,Y_t)_{t\geq 0}$ be a* LÉVY *process such that X satisfies (3.3) and let $(A_{s,t},B_{s,t})_{0\leq s\leq t}$ be defined as in (4.3). Then the family in (6.5) is a left X-stable hemigroup.*

Proof. With the same calculation as in the proof of Theorem 6.12 we get the integral representation (6.2) which defines a left X-stable hemigroup by Corollary 6.7. □

Theorem 6.14. *Let $X=(X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-*LÉVY *process for some $c>1$ which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a left $X_{\log(c)}$-semistable hemigroup. By Theorem 6.2(a) $Z=(Z_t)_{0<t\leq 1}$ defined by $Z_t\coloneqq Z_{0,t}$ is a left $X_{\log(c)}$-semiselfsimilar process with independent increments and by Lemma 5.5 $Y=(Y_t)_{t\leq 0}$ defined as in (5.3) is a $\log(c)$-semi-*LÉVY *process. If the improper stochastic integral in (4.13a) exists then $V=(V_t)_{t\leq 0}$ defined by*

$$V_t \coloneqq \overleftarrow{\mathsf{Exp}}(X_t)^{-1}Z_{0,e^t} \tag{6.6}$$

is a $\log(c)$-stationary left MGOU process driven by (X,Y).

Proof. From the integral representation of $Z_{s,t}$ in (6.3b) we obtain by (L2)

$$V_t=\overleftarrow{\mathsf{Exp}}(X_t)^{-1}Z_{0,e^t}=\overleftarrow{\mathsf{Exp}}(X_t)^{-1}\int_{-\infty}^{t}\overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}Y_u$$

which by (4.13b) shows that V is a $\log(c)$-stationary left MGOU process driven by (X,Y). □

Corollary 6.15. *Let $X=(X_t)_{t\in\mathbb{R}}$ be a* LÉVY *process which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a left X-stable hemigroup. By Corollary 6.3(a) $Z=(Z_t)_{0<t\leq 1}$ defined by $Z_t\coloneqq Z_{0,t}$ is a left X-selfsimilar process with independent increments and by Corollary 5.6 $Y=(Y_t)_{t\leq 0}$ defined as in (5.3) is a* LÉVY *process. Then $V=(V_t)_{t\leq 0}$ defined as in (6.6) is a stationary left MGOU process driven by (X,Y).*

Proof. The calculation to derive (4.13b) is the same as in the proof of Theorem 6.14. □

6.2 Right MGOU and right semistable hemigroups

In the right case the generalization of a stable and semistable hemigroup is defined as follows.

Definition 6.16. Let $X=(X_t)_{t\in\mathbb{R}}$ be a semimartingale with $X_0=\mathbf{0}$ which satisfies (3.3).

(a) A family $(Z_{s,t})_{0\leq s\leq t}$ of random variables is called **random right stable hemigroup** with **exponent** X or simply **right X-stable hemigroup** if it satisfies the following four conditions.

(R1) $Z_{q,r}$ and $Z_{s,t}$ are independent for all $0\leq q\leq r\leq s\leq t$.

(R2) $Z_{r,s}+Z_{s,t}=Z_{r,t}$ for all $0\leq r\leq s\leq t$.

(R3) For all $c>1$

$$\Big(X_{\log(t)+\log(c)}-X_{\log(c)},Z_{cs,ct}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1}\Big)_{1\leq s\leq t}\overset{\mathrm{D}}{=}(X_{\log(t)},Z_{s,t})_{1\leq s\leq t}, \tag{6.7a}$$

$$\Big(X_{\log(t)-\log(c)}-X_{-\log(c)},Z_{c^{-1}s,c^{-1}t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1}\Big)_{0<s\leq t\leq 1}\overset{\mathrm{D}}{=}(X_{\log(t)},Z_{s,t})_{0<s\leq t\leq 1}. \tag{6.7b}$$

(R4) The map $(s,t) \longmapsto Z_{s,t}$ is continuous with respect to convergence in distribution.

(b) A family $(Z_{s,t})_{0\le s\le t}$ of random variables is called **random right semistable hemigroup with exponent** $X_{\log(c)}$ or simply **right** $X_{\log(c)}$**-semistable hemigroup** if it satisfies (R1), (R2), (R4) and (R3) holds true for some $c>1$.

In the definition of a right semistable hemigroup we can without loss of generality assume $c>1$ because for $c=1$ (6.7a) and (6.7b) are automatically fulfilled and for $0<c<1$ swapping (6.7a) and (6.7b) and rescaling the time parameters of $(Z_{s,t})_{0\le s\le t}$ leads to a right semistable hemigroup with $c^{-1}>1$. In particular, if $(Z_{s,t})_{0\le s\le t}$ is a right $X_{\log(c)}$-semistable hemigroup for all $c>1$ then it is a right X-stable hemigroup.

If the exponent X is a semi-Lévy process there is a one-to-one correspondence between right semistable hemigroups and right semiselfsimilar processes. If the exponent X is a Lévy process the one-to-one correspondence extends to right stable hemigroups and right selfsimilar processes.

Theorem 6.17. *Let $X=(X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-Lévy process for some $c>1$ which satisfies (3.3).*

(a) *If $(Z_{s,t})_{0\le s\le t}$ is a right $X_{\log(c)}$-semistable hemigroup, then the process $Z=(Z_t)_{0<t\le 1}$ defined by $Z_t := Z_{0,t}$ has independent increments and is right $X_{\log(c)}$-semiselfsimilar.*

(b) *If $Z=(Z_t)_{t\ge 0}$ is a right $X_{\log(c)}$-semiselfsimilar process that is continuous in probability and has independent increments, then the family $(Z_{s,t})_{0\le s\le t}$ defined by $Z_{s,t}:=Z_t-Z_s$ is a right $X_{\log(c)}$-semistable hemigroup.*

Proof. (a) By (R1) and (R2) Z has independent increments. By (R3) we have for $0<t\le 1$

$$Z_{c^{-1}t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} = Z_{0,c^{-1}t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} \overset{\mathrm{D}}{=} Z_{0,t}.$$

Together with the $\log(c)$-stationary increments of X this yields (5.5a) and (5.5b) and thus Z is right $X_{\log(c)}$-semiselfsimilar.

(b) (R1) is clear from the independent increments of Z. (R2) follows by direct calculation since for all $0\le r\le s\le t$

$$Z_{r,s}+Z_{s,t} = Z_s - Z_r + Z_t - Z_s = Z_t - Z_r = Z_{r,t}.$$

(R3) holds true because for $1\le s\le t$

$$Z_{cs,ct}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} = (Z_{ct}-Z_{cs})\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \overset{\mathrm{D}}{=} Z_t - Z_s = Z_{s,t}$$

and for $0\le s\le t\le 1$

$$Z_{c^{-1}s,c^{-1}t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} = (Z_{c^{-1}t}-Z_{c^{-1}s})\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} \overset{\mathrm{D}}{=} Z_t - Z_s = Z_{s,t}.$$

Together with the $\log(c)$-stationary increments of X this yields (6.7a) and (6.7b). (R4) is a consequence of the continuity in probability of Z and the fact that convergence in probability implies convergence in distribution.

□

Corollary 6.18. *Let $X=(X_t)_{t\in\mathbb{R}}$ be a Lévy process which satisfies (3.3).*

(a) *If $(Z_{s,t})_{0\le s\le t}$ is a right X-stable hemigroup, then the process $Z=(Z_t)_{0<t\le 1}$ defined by $Z_t := Z_{0,t}$ has independent increments and is right X-selfsimilar.*

(b) *If $Z = (Z_t)_{t\geq 0}$ is a right X-selfsimilar process that is continuous in probability and has independent increments, then the family $(Z_{s,t})_{0\leq s\leq t}$ defined by $Z_{s,t} := Z_t - Z_s$ is a right X-stable hemigroup.*

Proof. X is a $\log(c)$-semi-LÉVY process for all $c > 1$ and we can apply Theorem 6.17.

(a) Since $(Z_{s,t})_{0\leq s\leq t}$ is a right $X_{\log(c)}$-semistable hemigroup for all $c > 1$, Z is right $X_{\log(c)}$-semiselfsimilar for all $c > 1$ by Theorem 6.17(a) and thus right X-selfsimilar. The independent increments of Z also follow from Theorem 6.17(a).

(b) Since Z is $X_{\log(c)}$-semiselfsimilar for all $c > 1$, $(Z_{s,t})_{0\leq s\leq t}$ is a right $X_{\log(c)}$-semistable hemigroup for all $c > 1$ by Theorem 6.17(b) and thus a right X-stable hemigroup.

□

Theorem 6.19 and its subsequent corollary show that a right semistable respectively right stable hemigroup admits the integral representation (6.8). Conversely, Theorem 6.21 and its subsequent corollary show that the stochastic integral in (6.8) defines a right semistable respectively right stable hemigroup.

Theorem 6.19. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-LÉVY process for some $c > 1$ which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a right $X_{\log(c)}$-semistable hemigroup. By Theorem 6.17(a) $Z = (Z_t)_{0<t\leq 1}$ defined by $Z_t := Z_{0,t}$ is a right $X_{\log(c)}$-semiselfsimilar process with independent increments and by Lemma 5.15 $Y = (Y_t)_{t\leq 0}$ defined as in (5.6) is a $\log(c)$-semi-LÉVY process. Then for all $0 < s \leq t \leq 1$ the random variable $Z_{s,t}$ has the integral representation*

$$Z_{s,t} = \int_{\log(s)}^{\log(t)} \mathrm{d}Y_u \,\overrightarrow{\mathsf{Exp}}(X_{u-}). \tag{6.8}$$

Proof. For $0 < s \leq t \leq 1$ we have

$$\begin{aligned}
\int_{\log(s)}^{\log(t)} \mathrm{d}Y_u \,\overrightarrow{\mathsf{Exp}}(X_{u-}) &= \int_{\log(s)}^{\log(t)} \mathrm{d}\left(-\int_{e^u}^{1} \mathrm{d}Z_v \,\overrightarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1} \right) \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}\left(\int_{1}^{e^u} \mathrm{d}Z_v \,\overrightarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1} \right) \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u} \,\overrightarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1}\overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u} = Z_{e^{\log(t)}} - Z_{e^{\log(s)}} = Z_t - Z_s = Z_{s,t}.
\end{aligned}$$

□

Corollary 6.20. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a LÉVY process which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a right X-stable hemigroup. By Corollary 6.18(a) $Z = (Z_t)_{0<t\leq 1}$ defined by $Z_t := Z_{0,t}$ is a right X-selfsimilar process with independent increments and by Corollary 5.16 $Y = (Y_t)_{t\leq 0}$ defined as in (5.6) is a LÉVY process. Then for all $0 < s \leq t \leq 1$ the random variable $Z_{s,t}$ has the integral representation (6.8).*

Proof. The calculation to derive (6.8) is the same as in the proof of Theorem 6.19. □

Theorem 6.21. *Let* $(X,Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ *be a* $\log(c)$*-semi-*Lévy *process for some* $c > 1$ *such that* X *satisfies (3.3). Then* $(Z_{s,t})_{0<s\le t}$ *defined as in (6.8) is a right* $X_{\log(c)}$*-semistable hemigroup.*

Proof. (R1) is clear from the independent increments of (X,Y). (R2) follows by direct calculation since for all $0 < r \le s \le t$

$$Z_{r,s} + Z_{s,t} = \int_{\log(r)}^{\log(s)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) + \int_{\log(s)}^{\log(t)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) = \int_{\log(r)}^{\log(t)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) = Z_{r,t}.$$

(R3) holds true because for $1 \le s \le t$

$$\begin{aligned}
Z_{cs,ct}\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} &= \int_{\log(cs)}^{\log(ct)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_{\log(c)})^{-1} \\
&= \int_{\log(s)+\log(c)}^{\log(t)+\log(c)} \mathrm{d}(Y_u - Y_{\log(c)}) \, \overrightarrow{\mathsf{Exp}}(X_{u-} - X_{\log(c)}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}(Y_{u+\log(c)} - Y_{\log(c)}) \, \overrightarrow{\mathsf{Exp}}(X_{(u+\log(c))-} - X_{\log(c)}) \\
&\overset{\mathrm{D}}{=} \int_{\log(s)}^{\log(t)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) = Z_{s,t}
\end{aligned}$$

and for $0 < s \le t \le 1$

$$\begin{aligned}
Z_{c^{-1}s,c^{-1}t}\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} &= \int_{\log(c^{-1}s)}^{\log(c^{-1}t)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-})\overrightarrow{\mathsf{Exp}}(X_{\log(c^{-1})})^{-1} \\
&= \int_{\log(s)-\log(c)}^{\log(t)-\log(c)} \mathrm{d}(Y_u - Y_{-\log(c)}) \, \overrightarrow{\mathsf{Exp}}(X_{u-} - X_{-\log(c)}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}(Y_{u-\log(c)} - Y_{-\log(c)}) \, \overrightarrow{\mathsf{Exp}}(X_{(u-\log(c))-} - X_{-\log(c)}) \\
&\overset{\mathrm{D}}{=} \int_{\log(s)}^{\log(t)} \mathrm{d}Y_u \, \overrightarrow{\mathsf{Exp}}(X_{u-}) = Z_{s,t}.
\end{aligned}$$

Together with the $\log(c)$-stationary increments of X this yields (6.7a) and (6.7b). (R4) is a consequence of the continuity in probability of (X,Y) and the stochastic integral and of the fact that convergence in probability implies convergence in distribution. □

Corollary 6.22. *Let* $(X,Y) = (X_t, Y_t)_{t\in\mathbb{R}}$ *be a* Lévy *process such that* X *satisfies (3.3). Then* $(Z_{s,t})_{0<s\le t}$ *defined as in (6.8) is a right* X*-stable hemigroup.*

Proof. Since (X,Y) is a $\log(c)$-semi-Lévy process for all $c>1$, $(Z_{s,t})_{0\leq s\leq t}$ is a right $X_{\log(c)}$-semistable hemigroup for all $c>1$ by Theorem 6.21 and thus a right X-stable hemigroup. □

Theorem 6.23 and its subsequent corollary show how a right semistable respectively right stable hemigroup can be constructed out of a periodically stationary respectively stationary process. Conversely, Theorem 6.25 and its subsequent corollary show how a periodically stationary respectively stationary process can be constructed out of a right semistable respectively right stable hemigroup.

Theorem 6.23. *Let $X=(X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-Lévy process for some $c>1$ which satisfies (3.3) and let $V=(V_t)_{t\in\mathbb{R}}$ be $\log(c)$-stationary and independent of X. By Theorem 5.13(b) the right X-random inverse Lamperti transform $Z=(Z_t)_{t>0}$ of V is right $X_{\log(c)}$-semiselfsimilar and by Lemma 5.15 $Y=(Y_t)_{t\in\mathbb{R}}$ defined as in (5.6) is a $\log(c)$-semi-Lévy process. Then $(Z_{s,t})_{0<s\leq t}$ defined by*

$$Z_{s,t} := V_{\log(t)}\overrightarrow{\mathsf{Exp}}(X_{\log(t)}) - V_{\log(s)}\overrightarrow{\mathsf{Exp}}(X_{\log(s)}) \tag{6.9a}$$

is a right $X_{\log(c)}$-semistable hemigroup and $Z_{s,t}$ has the integral representation

$$Z_{s,t} = \int_{\log(s)}^{\log(t)} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}). \tag{6.9b}$$

Proof. By Theorem 6.17(b) the family $(Z_{s,t})_{0<s\leq t}$ defined by $Z_{s,t}:=Z_t-Z_s$ is a right $X_{\log(c)}$-semistable hemigroup and by definition of the right X-random inverse Lamperti transform we have

$$Z_{s,t} = Z_t - Z_s = V_{\log(t)}\overrightarrow{\mathsf{Exp}}(X_{\log(t)}) - V_{\log(s)}\overrightarrow{\mathsf{Exp}}(X_{\log(s)})$$

which is (6.9a). The integral representation in (6.9b) for $0<s\leq t\leq 1$ follows with the same calculation as in the proof of Theorem 6.21. For $1\leq s\leq t$ we have

$$\begin{aligned}
\int_{\log(s)}^{\log(t)} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) &= \int_{\log(s)}^{\log(t)} \mathrm{d}\left(\int_1^{e^u} \mathrm{d}Z_v\, \overrightarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\right) \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u}\, \overrightarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1}\overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u} = Z_{e^{\log(t)}} - Z_{e^{\log(s)}} = Z_t - Z_s = Z_{s,t}
\end{aligned}$$

and for $0<s\leq 1\leq t$ we have

$$\begin{aligned}
\int_{\log(s)}^{\log(t)} \mathrm{d}Y_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) &= \int_{\log(s)}^{0} \mathrm{d}\left(-\int_{e^u}^{1} \mathrm{d}Z_v\, \overrightarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\right) \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&\quad + \int_{0}^{\log(t)} \mathrm{d}\left(\int_1^{e^u} \mathrm{d}Z_v\, \overrightarrow{\mathsf{Exp}}(X_{\log(v)-})^{-1}\right) \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&= \int_{\log(s)}^{0} \mathrm{d}Z_{e^u}\, \overrightarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1}\overrightarrow{\mathsf{Exp}}(X_{u-})
\end{aligned}$$

$$+ \int_{0}^{\log(t)} \mathrm{d}Z_{e^u} \overrightarrow{\mathsf{Exp}}(X_{\log(e^u)-})^{-1} \overrightarrow{\mathsf{Exp}}(X_{u-})$$
$$= \int_{\log(s)}^{\log(t)} \mathrm{d}Z_{e^u} = Z_{e^{\log(t)}} - Z_{e^{\log(s)}} = Z_t - Z_s = Z_{s,t}.$$

□

Corollary 6.24. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* Lévy *process which satisfies (3.3) and let $V = (V_t)_{t\in\mathbb{R}}$ be stationary and independent of X. By Corollary 5.14(b) the right X-random inverse* Lamperti *transform $Z = (Z_t)_{t>0}$ of V is right X-selfsimilar and by Corollary 5.16 $Y = (Y_t)_{t\in\mathbb{R}}$ defined as in (5.6) is a* Lévy *process. Then $(Z_{s,t})_{0<s\leq t}$ defined as in (6.9a) is a right X-stable hemigroup and $Z_{s,t}$ has the integral representation (6.9b).*

Proof. By Corollary 6.18(b) the family $(Z_{s,t})_{0<s\leq t}$ defined by $Z_{s,t} \coloneqq Z_t - Z_s$ is a right X-stable hemigroup. (6.9a) is derived in the same way as in the proof of Theorem 6.23 and the integral representation (6.9b) follows with the same calculations as in the proofs of Theorem 6.19 and Theorem 6.23. □

Theorem 6.25. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a $\log(c)$-semi-*Lévy *process for some $c > 1$ which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a right $X_{\log(c)}$-semistable hemigroup. Then the process $V = (V_t)_{t\leq 0}$ defined by*

$$V_t \coloneqq Z_{0,e^t} \overrightarrow{\mathsf{Exp}}(X_t)^{-1} \tag{6.10}$$

is $\log(c)$-stationary.

Proof. By Theorem 6.17(a) $Z = (Z_t)_{0<t\leq 1}$ defined by $Z_t \coloneqq Z_{0,t}$ is right $X_{\log(c)}$-semiselfsimilar and then by Theorem 5.13(a) $V = \overrightarrow{\mathsf{Lam}}(Z)$ is $\log(c)$-stationary. □

Corollary 6.26. *Let $X = (X_t)_{t\in\mathbb{R}}$ be a* Lévy *process which satisfies (3.3) and let $(Z_{s,t})_{0\leq s\leq t}$ be a right X-stable hemigroup. Then the process $V = (V_t)_{t\leq 0}$ defined as in (6.10) is stationary.*

Proof. By Corollary 6.18(a) $Z = (Z_t)_{0<t\leq 1}$ defined by $Z_t \coloneqq Z_{0,t}$ is right X-selfsimilar and then by Corollary 5.14(a) $V = \overrightarrow{\mathsf{Lam}}(Z)$ is stationary. □

Theorem 6.27 and its subsequent corollary show how a right semistable respectively right stable hemigroup can be constructed out of the random functional $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ that defines a right MGOU process via the random recurrence equation (4.14). Conversely, Theorem 6.29 and its subsequent corollary show how a periodically stationary respectively stationary right MGOU process can be constructed out of a right semistable respectively right stable hemigroup.

Theorem 6.27. *Let $(X, Y) = (X_t, Y_t)_{t\geq 0}$ be a $\log(c)$-semi-*Lévy *process for some $c > 1$ such that X satisfies (3.3) and let $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ be defined as in (4.16). Then the family*

$$\left(B_{\log(t)} A_{\log(t)}^{-1} - B_{\log(s)} A_{\log(s)}^{-1}\right)_{1\leq s\leq t} \tag{6.11}$$

is a right $X_{\log(c)}$-semistable hemigroup.

Proof. Inserting the formulas in (4.16) we have for all $1 \leq s \leq t$

$$
\begin{aligned}
B_{\log(t)} A_{\log(t)}^{-1} - B_{\log(s)} A_{\log(s)}^{-1} &= \int_0^{\log(t)} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(X_{u-}) \overleftarrow{\mathsf{Exp}}(X_{\log(t)})^{-1} \overleftarrow{\mathsf{Exp}}(X_{\log(t)}) \\
&\quad - \int_0^{\log(s)} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(X_{u-}) \overleftarrow{\mathsf{Exp}}(X_{\log(s)})^{-1} \overleftarrow{\mathsf{Exp}}(X_{\log(s)}) \\
&= \int_{\log(s)}^{\log(t)} \mathrm{d}Y_u \overleftarrow{\mathsf{Exp}}(X_{u-})
\end{aligned}
$$

which is (6.8) and by Theorem 6.21 this defines a right $X_{\log(c)}$-semistable hemigroup. □

Corollary 6.28. *Let $(X, Y) = (X_t, Y_t)_{t \geq 0}$ be a Lévy process such that X satisfies (3.3) and let $(A_{s,t}, B_{s,t})_{0 \leq s \leq t}$ be defined as in (4.16). Then the family in (6.11) is a right X-stable hemigroup.*

Proof. With the same calculation as in the proof of Theorem 6.27 we get the integral representation (6.8) which defines a right X-stable hemigroup by Corollary 6.22. □

Theorem 6.29. *Let $X = (X_t)_{t \in \mathbb{R}}$ be a $\log(c)$-semi-Lévy process for some $c > 1$ which satisfies (3.3) and let $(Z_{s,t})_{0 \leq s \leq t}$ be a right $X_{\log(c)}$-semistable hemigroup. By Theorem 6.17(a) $Z = (Z_t)_{0 < t \leq 1}$ defined by $Z_t := Z_{0,t}$ is a right $X_{\log(c)}$-semiselfsimilar process with independent increments and by Lemma 5.15 $Y = (Y_t)_{t \leq 0}$ defined as in (5.6) is a $\log(c)$-semi-Lévy process. If the improper stochastic integral in (4.26a) exists then $V = (V_t)_{t \leq 0}$ defined by*

$$
V_t := Z_{0,e^t} \overrightarrow{\mathsf{Exp}}(X_t)^{-1} \tag{6.12}
$$

is a $\log(c)$-stationary right MGOU process driven by (X, Y).

Proof. From the integral representation of $Z_{s,t}$ in (6.9b) we obtain by (R2)

$$
V_t = Z_{0,e^t} \overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \int_{-\infty}^{t} \mathrm{d}Y_u \overrightarrow{\mathsf{Exp}}(X_{u-}) \overrightarrow{\mathsf{Exp}}(X_t)^{-1}
$$

which by (4.26b) shows that V is a $\log(c)$-stationary right MGOU process driven by (X, Y). □

Corollary 6.30. *Let $X = (X_t)_{t \in \mathbb{R}}$ be a Lévy process which satisfies (3.3) and let $(Z_{s,t})_{0 \leq s \leq t}$ be a right X-stable hemigroup. By Corollary 6.18(a) $Z = (Z_t)_{0 < t \leq 1}$ defined by $Z_t := Z_{0,t}$ is a right X-selfsimilar process with independent increments and by Corollary 5.16 $Y = (Y_t)_{t \leq 0}$ defined as in (5.6) is a Lévy process. Then $V = (V_t)_{t \leq 0}$ defined as in (6.12) is a stationary right MGOU process driven by (X, Y).*

Proof. The calculation to derive (4.26b) is the same as in the proof of Theorem 6.29. □

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Erklärung

Ich versichere an Eides statt, dass die Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist. Die Dissertation wurde in der vorgelegten oder ähnlicher Form noch bei keiner anderen Institution eingereicht. Ich habe bisher keine erfolglosen Promotionsversuche unternommen.

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