KO-Theory of Complex Flag Varieties

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Zusammenfassung

Das Ziel dieser Dissertation besteht darin, die reelle topologische K-Theorie komplexer Fahnenvarietäten zu bestimmen.

Als eine der ersten nicht-gewöhnlichen Kohomologietheorien wurde *topologische K-Theorie* entdeckt. Genauer gesagt beinhaltet sie zwei Kohomologietheorien, nämlich komplexe und reelle topologische K-Theorie K^* und KO^*.

Als *komplexe Fahnenvarietäten* bezeichnen wir homogene Räume G/H, wobei G eine kompakte einfach-zusammenhängende Liegruppe und $H = C_G(S)$ der Zentralisator eines Torus $S \subset G$ ist. Jede komplexe Fahnenvarietät ist eine glatte komplexe projektive Varietät. Ihre komplexe K-Theorie K^* ist nach Arbeiten von Hodgkin und Pittie [Pit72] bekannt. Im Gegensatz hierzu ist unser Verständnis ihrer rellen K-Theorie äußerst lückenhaft und es sind nur Teilergebnisse bekannt. Zum Beispiel werden die KO-Gruppen aller Fahnenvarietäten der Form $SU(n)/H$ in [KKO14] bestimmt, aber nach unserem Kenntnisstand wurden $G = Sp(n)$ und $Spin(n)$ bisher nur in Spezialfällen behandelt.

Die klassische Herangehensweise zur Berechnung von $KO^*(G/H)$ verwendet die Atiyah-Hirzebruch-Spektralsequenz. In [Zib15] entwickelt der Autor einen anderen Ansatz: Statt die KO-Theorie direkt zu betrachten, wird der sogenannte *Wittring* eines topologischen Raumes als Kokern der Realifizierungsabbildung $r_{2*}\colon K^{2*} \to KO^{2*}$ definiert. Es wird gezeigt, wie der Wittring einer komplexen Fahnenvarietät durch die Darstellungstheorie kompakter Liegruppen verstanden werden kann und dass er den Torsionsanteil ihrer KO-Gruppen bestimmt. Der freie Anteil der KO-Gruppen wird durch die rationale Kohomologie festgelegt, deren Berechnung bereits wohlbekannt ist. In [Zib15] berechnet der Autor dann den Wittring aller *vollen Fahnenvarietäten*, das heißt aller komplexen Fahnenvarietäten G/T, wobei T ein maximaler Torus in G ist.

In dieser Dissertation benutzen wir den Ansatz in [Zib15], um den Wittring allgemeinerer komplexer Fahnenvarietäten zu berechnen. Wir berechnen die Wittringe aller komplexer Fahnenvarietäten G/H mit $G = SU(n)$, $Spin(2n + 1)$, $Sp(n)$, G_2, F_4 und berechnen zumindest die Wittgruppen für $G = Spin(2n)$. Außerdem bestimmen wir den Wittring aller *fast-vollen Fahnenvarietäten*, also derjenigen komplexen Fahnenvarietäten, die in einem gewissen Sinne den vollen Fahnenvarietäten am nächsten sind.

Abstract

The aim of this thesis is to compute the real topological K-groups of complex flag varieties.

Topological K-theory was one of the first extraordinary cohomology theories to be discovered. To be more precise, it entails two cohomology theories, namely complex and real topological K-theory K^* and KO^*.

By a *complex flag variety*, we mean a homogeneous space G/H where G is a compact simply connected Lie group and $H = C_G(S)$ is the centraliser of a torus $S \subset G$. It is a smooth complex projective variety. The complex K-theory K^* of complex flag varieties is well-understood by work of Hodgkin and Pittie [Pit72]. On the contrary, our knowledge of their real K-theory has been rather patchy and only partial results are known. For example, KO^* of all complex flag varieties of the form $SU(n)/H$ are computed in [KKO14], but to our knowledge $G = Sp(n)$ and $Spin(n)$ have been treated only in special cases.

The classical approach to compute $KO^*(G/H)$ is via the Atiyah-Hirzebruch spectral sequence. In [Zib15], the author develops a different approach: Instead of considering the KO-theory directly, the so-called *Witt ring* of a topological space is defined as the cokernel of the realification map $r_{2*}\colon K^{2*} \to KO^{2*}$. It is shown how the Witt ring of complex flag varieties can actually be understood via the representation theory of compact Lie groups, and that it determines the torsion part of their KO-groups. The free part of the KO-groups is determined by the rational cohomology groups, which are already well-understood. In [Zib15], the author then computes the Witt ring of all *full-flag varieties*, which are the complex flag varieties of the form G/T where T is a maximal torus in G.

In this thesis, we use the approach of [Zib15] to compute the Witt ring of more general complex flag varieties. We are able to compute the Witt ring of all complex flag varieties G/H where $G = SU(n)$, $Spin(2n+1)$, $Sp(n)$, G_2, F_4, and we compute at least the Witt groups for $G = Spin(2n)$. We also determine the Witt ring of all *almost full flag varieties*, which are those that are in some sense closest to the full flag varieties.

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Contents

Introduction

In more primitive forms, many ideas that are now regarded as part of K-theory can be traced back to the 19th century. However, one of the first origins of K-theory that began to establish it as a separate subject can be found in the 1950s. In 1957, Grothendieck defined the abelian group now known as $K_0(X)$ for an algebraic variety X [BS58]. In modern language, it is the Grothendieck group associated with the exact category of vector bundles over X. It was an essential ingredient in Grothendieck's reformulation of the Riemann-Roch theorem. This marked the beginning of modern algebraic K-theory.

In 1959, Atiyah and Hirzebruch gave an analogous construction in the context of complex or real topological vector bundles over topological spaces [AH59], thus establishing the subject of topological K-theory. They showed that this gives rise to extraordinary cohomology theories K^* and KO^* [AH61]. These subsequently proved to be extremely powerful and successful in applications: For example, Atiyah and Hirzebruch proved a Riemann-Roch theorem for differentiable manifolds [AH59], and Adams determined the maximal number of linearly independent vector fields on spheres [Ada62].

Our problem

In this thesis, we shall be concerned with the computation of real topological K-theory KO^* of certain homogeneous spaces, namely *complex flag varieties*. Let G be a compact connected semisimple Lie group and $H \subset G$ be a closed connected subgroup. We call G/H a *complex flag variety* if it is a smooth complex projective variety. Since any compact connected semisimple Lie group is finitely covered by a compact simply connected Lie group, we could restrict to compact simply connected Lie groups G for this definition.

There is a very satisfying classification of those subgroups $H \subset G$ such that G/H is a complex flag variety. First, any such subgroup H turns out to be the centraliser $C_G(S)$ of a torus $S \subset G$ (cf. Theorem 1.5). In turn, there is a surjective map (cf. Proposition 1.8)

$$\left\{\substack{\text{subsets of the set of nodes of}\\ \text{the Dynkin diagram of } G}\right\} \twoheadrightarrow \{C_G(S) \mid S \subset G \text{ torus}\}\,/\text{conjugation} \tag{1}$$

so that any complex flag variety G/H can be specified by a choice of a subset of the nodes of the Dynkin diagram of G, i.e. by a choice of a subset of the simple roots of G. For example, the empty subset yields a maximal torus $H = T$ and the whole set yields $H = G$.

For the compact simple Lie groups of ordinary type, one obtains the following classification (cf. Proposition 1.12).

Type	G	Subgroups H (up to conj.) s.t. G/H is a complex flag variety
A_n	$SU(n)$	$S(U(n_1) \times \ldots \times U(n_l))$ where $n_1 + \ldots + n_l = n$
B_n	$SO(2n+1)$	$SO(2m+1) \times U(n_1) \times \ldots \times U(n_l)$ where $m + n_1 + \ldots + n_l = n$
C_n	$Sp(n)$	$Sp(m) \times U(n_1) \times \ldots \times U(n_l)$ where $m + n_1 + \ldots + n_l = n$
D_n	$SO(2n)$	$SO(2m) \times U(n_1) \times \ldots \times U(n_l)$ where $m + n_1 + \ldots + n_l = n$

The main result of this thesis is the computation of the KO-groups of the above complex flag varieties and of all the flag varieties where G is of extraordinary type G_2 or F_4. To be more precise, we compute the torsion part of the KO-groups. The free part can be derived from the rational cohomology, which is already well-understood.

Known results

To put these results into perspective, we give an overview of what is currently known. The complex K-theory K^* of complex flag varieties is completely understood by the works of Hodgkin [Hod75] and Pittie [Pit72]. However, the case of real K-theory KO^* is more complicated and our knowledge has been rather patchy.

In [Fuj67], all KO-groups of complex projective space $\mathbb{CP}^n$ are computed. This is generalised to determine the KO-groups of complex Grassmannians $SU(m+n)/S(U(m) \times U(n))$ in [KH91]. Much more recently, the computation of the KO-groups of all complex flag varieties of the form $SU(n_1 + \ldots + n_l)/S(U(n_1) \times \ldots \times U(n_l))$, i.e. where G is of type A_n, has been achieved in [KKO14]. However, this paper contains some minor arithmetic mistakes so that their table of the KO-groups is flawed. This can already be seen from the fact that their expressions for the ranks that are given in their table do not always yield integers but fractions.

The computations of the KO-theory of full flag varieties G/T, where T is a maximal torus in G, are obtained for the simple groups G of ordinary type in [KKO04]. This was extended to include $G = G_2, F_4, E_6$ in [KO13].

All the mentioned results about the KO-theory of complex flag varieties were obtained in essentially the same way: The authors considered the Atiyah-Hirzebruch spectral sequence for KO^*, computed the E_2-page including the differentials, obtained the E_3-page from this and then showed in each individual case that the spectral sequence collapses on the third page. The arguments required to make this work are intricate, which is why only partial results are known.

In [Zib15], the author develops an alternative approach and computes the KO-groups of all *full flag varieties*, i.e. of all quotients G/T where G is a compact simply connected Lie group and T is a maximal torus, in an essentially type-independent way. The basic strategy is to consider the so-called *Witt groups* W^i. For a topological space X, denoting by $r_i\colon K^{2i}(X) \to KO^{2i}(X)$ the realification, they are defined as

$$W^i(X) := KO^{2i}(X)/r_i$$

Since the KO-groups are 8-periodic, the Witt groups are 4-periodic. The important observations in [Zib15] are that these Witt groups are computable via a result of Bousfield (cf. Lemma 2.3) and that they completely determine the torsion in the KO-groups of complex flag varieties. Even more is true: The *total Witt group*

$$W^*(X) := \bigoplus_{i \in \mathbb{Z}_4} W^i(X)$$

is a graded ring, and the ring structure of W^* determines part of the ring structure of KO^*. The whole computation is essentially representation-theoretic, in that our understanding of the representation theory of compact Lie groups enters crucially. The result can be concisely stated as follows:

Theorem ([Zib15, Thm. 3.3])**.** *Let G be a simply connected compact Lie group and $T \subset G$ be a maximal torus. The Witt ring of G/T is an exterior algebra on $b_{\mathbb{H}}$ generators of degree 1 and $\frac{b_{\mathbb{C}}}{2} + b_{\mathbb{R}}$ generators of degree 3, where $b_{\mathbb{C}}$, $b_{\mathbb{R}}$ and $b_{\mathbb{H}}$ denote the number of fundamental representations of G of complex, real and quaternionic type, respectively.*

Witt groups also occur in different contexts. We justify our terminology in Remark 1.7.

Our results

In this thesis, we extend the approach in [Zib15] to compute the Witt rings of more general complex flag varieties. Let us consider some of our results. Extending the above result of Zibrowius, we determine the Witt ring of *almost full flag varieties*. By an *almost full flag variety* we mean a flag variety G/H where H arises via (1) from a choice of a single node of the Dynkin diagram of G. We prove (cf. Theorem 7.9):

Theorem 0.1. *Let G/H be an almost full flag variety. Then the Witt ring of G/H is an exterior algebra with $b_{\mathbb{H}} - \epsilon$ generators of degree 1 and $\frac{b_{\mathbb{C}}}{2} + b_{\mathbb{R}} - \epsilon'$ generators of degree 3, where $\{\epsilon, \epsilon'\} = \{0, 1\}$ and $b_{\mathbb{C}}$, $b_{\mathbb{R}}$ and $b_{\mathbb{H}}$ denote the number of fundamental representations of G of complex, real and quaternionic type, respectively.*

The exact values of ϵ and ϵ' are actually determined by a criterion which can be expressed merely in terms of the root system of G (cf. Theorem 7.9 for details).

The Dynkin diagram of G_2 only has two nodes. So the only complex flag varieties of the form G_2/H are the respective full flag variety, a singleton and two almost full flag varieties. Then we immediately obtain (cf. Theorem 7.10):

Theorem 0.2. *Let H be a subgroup of G_2 obtained via (1) from a subset of the nodes* *of the Dynkin diagram of G_2 containing α nodes. The Witt ring of G/H is an exterior algebra on $2-\alpha$ generators, all of which are of degree 3.*

We will show that the Witt ring of a complex flag variety F_4/H is also always an exterior algebra with all generators in degree 3 (cf. Theorem 7.13).

However, Witt rings of complex flag varieties are in general more complicated than just exterior algebras. This can be seen from the Witt rings of complex flag varieties of ordinary type. In this thesis, we compute the Witt groups of all ordinary flag varieties. Except in some special cases for type D_n, we are able to determine W^* as a ring. As an illustration, we obtain the following result (cf. Theorem 5.11):

Theorem 0.3. *Let*

$$X := \frac{Sp(n)}{Sp(m) \times U(n_1) \times \ldots \times U(n_l)}$$

where $m + n_1 + \ldots + n_l = n$ and set

$$h := \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_l}{2} \right\rfloor \text{ and } f := \left\lfloor \frac{n}{2} \right\rfloor - h \text{ and } g := \left\lceil \frac{n-m}{2} \right\rceil.$$

Then there is a ring isomorphism

$$W^*(X) \cong \frac{\mathbb{Z}_2\left[a_1, \ldots, a_{\lfloor m/2 \rfloor}\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[b_1^{(p)}, \ldots, b_{\lfloor n_p/2 \rfloor}^{(p)}\right]}{(\mu_1, \ldots, \mu_h)} \otimes_{\mathbb{Z}_2} \bigwedge(u_1, \ldots, u_f, v_1, \ldots, v_g),$$

where

$$\mu_j = \sum_{c+c_1+\ldots+c_l=j} a_c \cdot b_{c_1}^{(1)} \ldots b_{c_l}^{(l)} + \binom{2n}{2j},$$

taking into account that we set

$$a_0 := 1 \text{ and } a_i := a_{m-i} \text{ for all } \lfloor m/2 \rfloor < i \leq m,$$
$$b_0^{(p)} := 1 \text{ for all } 1 \leq p \leq l \text{ and } b_i^{(p)} := b_{n_p-i}^{(p)} \text{ for all } 1 \leq p \leq l,\ \lfloor n_p/2 \rfloor < i \leq n_p.$$

The left factor in the above tensor product is contained in $W^0(X)$. Furthermore, $u_i \in W^3(X)$ for all $1 \leq i \leq f$ and $v_j \in W^1(X)$ for all $1 \leq j \leq g$.

Of course, our computation of the Witt rings of ordinary flag varieties includes the ordinary almost full flag varieties and Theorem 0.1 can be checked on a case-by-case basis. However, our proof of Theorem 0.1 is actually type-independent and in particular applies to the exceptional types E_6, E_7 and E_8, for which we do not give full computations here.

In summary, in this thesis we compute the Witt groups of all ordinary flag varieties and all extraordinary flag varieties of types G_2 and F_4. We can determine the ring structure except in some special cases for type D_n. We deal with each type $A_n, B_n, C_n, D_n, G_2, F_4$ individually. However, we make a general observation:

Theorem 0.4. *For all the flag varieties G/H where G is simple of type A_n, B_n, C_n, G_2 or F_4 and for all full and almost full flag varieties, we have an isomorphism*

$$W^*(G/H) \cong A \otimes_{\mathbb{Z}_2} \bigwedge(u_1, \dots, u_r)$$

where A is a $\mathbb{Z}_2$-algebra which is completely contained in $W^0(G/H)$ and each generator $u_1, \dots, u_r$ of the exterior algebra is of degree 1 or 3. Furthermore, we observe in all cases that

$$r = \frac{b_{\mathbb{C}}^G}{2} + b_{\mathbb{R}}^G + b_{\mathbb{H}}^G - \frac{b_{\mathbb{C}}^H}{2} - b_{\mathbb{R}}^H - b_{\mathbb{H}}^H$$

where $b_{\mathbb{C}}^G, b_{\mathbb{R}}^G, b_{\mathbb{H}}^G$ are the number of fundamental representations of G of complex, real, quaternionic type respectively and $b_{\mathbb{C}}^H, b_{\mathbb{R}}^H, b_{\mathbb{H}}^H$ are the number of fundamental representations of the simply connected component[1] *of H of complex, real, quaternionic type respectively.*

This theorem can be checked on a case-by-case basis from our computations, see Theorems 5.3, 5.7, 5.11, 7.9, 7.10 and 7.13. The theorem is also true for those flag varieties where G is simple of type D_n for which we were able to compute the Witt ring, see Propositions 5.14 and 5.16 and Remarks 5.15 and 5.17. For all we know, the theorem could hold in general.

We see from the above that for full or almost full flag varieties and for all exceptional flag varieties of types G_2 and F_4, we have $A = \mathbb{Z}_2$. However, Theorem 0.3 says that A can be a quite complicated ring in general. Furthermore, notice that Theorem 0.1 implies that the Witt ring of the almost full flag variety G/H is an exterior algebra with $\frac{b_{\mathbb{C}}}{2} + b_{\mathbb{R}} + b_{\mathbb{H}} - 1$ generators. This is of course the same number of exterior algebra generators as predicted by Theorem 0.4. However, the root system-theoretic determination of the grading of these generators in Theorem 7.9 does not generalise in an obvious way, as is apparent in our computations for ordinary types.

Outline of this thesis

Let us give a brief outline of this thesis. Chapter 1 serves two purposes: It introduces our main objects of study and provides some general background that we require. It starts by giving a quick review of topological K-theory, defines the Witt ring of a topological space and explains how the Witt groups of certain homogeneous spaces determine the torsion in their KO-theory. We also show how the free part of the KO-groups can be read off from the rational cohomology. Then we introduce the spaces we want to study, complex flag varieties, and give a proof of their aforementioned classification. Finally, we review some representation theory of compact Lie groups.

[1] By the classification of compact Lie groups, H is isomorphic to $(\tilde{H} \times T)/C$ for some simply connected (hence semisimple) Lie group $\tilde{H}$, a torus T and some finite central subgroup C. We call $\tilde{H}$ the *simply connected component* of H. In our case, H is the centraliser of a torus obtained from a choice of subset Θ of simple roots of G. The Dynkin diagram of $\tilde{H}$ is actually the subdiagram of the Dynkin diagram of G spanned by all the nodes corresponding to roots in Θ.

In Chapter 2, we explain Bousfield's lemma which shows that Witt rings can be computed as Tate cohomology rings. So the remainder of the chapter is spent proving some technical results about the computation of Tate cohomology that we will need later. There we will already see that the notion of regular sequences in a ring play an important role for these sorts of computations.

Therefore we dedicate Chapter 3 to first reviewing some results about regular sequences. We then prove a result about regularity of sequences of inhomogeneous elements in graded rings which will be crucial for our concrete computations.

We are then ready to give a complete outline of our computation of the Witt rings of complex flag varieties in Chapter 4. The basic approach is the same as in [Zib15] and we generalise results there to put them to work for all complex flag varieties.

The computations for simple G of ordinary type are finally conducted in Chapter 5, with the proof of some technical lemmas postponed to Chapter 6.

In Chapter 7, we deal with the extraordinary types G_2 and F_4. The basic approach is the same as for the ordinary types, but the representation-theoretic part is dealt with in a different and (at least for the extraordinary types) somewhat simpler manner, extracting the required information just from the root system of G instead of the representation ring. We also use this viewpoint to prove the result about the Witt ring of almost full flag varieties in this chapter.

Chapter 1

KO-Theory and Representations

1.1 KO-rings and Witt rings

We first want to review the construction and basic properties of topological K-theory, all of which can be found in the literature such as in [Ati94] or [Hat]. We then define our main objects of interest, namely Witt rings, and show how they relate to real topological K-theory.

For our purposes, it suffices to restrict to compact Hausdorff spaces. Therefore, we assume all spaces in this section are compact Hausdorff. To more generally define K-theory of non-compact spaces, one has to resort to a definition in terms of classifying spaces [AH61, §1.3].

So let X be a compact Hausdorff space. We denote by $\mathrm{Vect}_{\mathbb{R}}(X)$ and $\mathrm{Vect}_{\mathbb{C}}(X)$ respectively the set of isomorphism classes of real and complex vector bundles over X respectively. We allow for the fibres of a vector bundle to have different dimensions over different connected components of X. Both $\mathrm{Vect}_{\mathbb{R}}(X)$ and $\mathrm{Vect}_{\mathbb{C}}(X)$ are abelian monoids with addition induced by the direct sum of vector bundles. We define

$$\begin{aligned} KO^0(X) &:= K(\mathrm{Vect}_{\mathbb{R}}(X)) \\ K^0(X) &:= K(\mathrm{Vect}_{\mathbb{C}}(X)) \end{aligned}$$

where $K(-)$ denotes the Grothendieck group associated with an abelian monoid. Actually, $KO^0(X)$ and $K^0(X)$ are even commutative rings where the product is induced by taking tensor products of vector bundles. Moreover, KO^0 and K^0 are functorial: If $f\colon X \to Y$ is a map between compact Hausdorff spaces, then taking pullbacks of vector bundles over Y along f induces ring homomorphisms $f^*\colon KO^0(Y) \to KO^0(X)$ and $K^0(Y) \to K^0(X)$.

We can also define reduced versions of the groups $KO^0(X)$ and $K^0(X)$. If (X, x_0) is a based space, we let

$$\widetilde{KO}^0(X) := \ker(KO^0(X) \to KO^0(\{x_0\}))$$

$$\widetilde{K}^0(X) := \ker(K^0(X) \to K^0(\{x_0\}))$$

Now we want to define groups K^n and KO^n for $n \in \mathbb{Z}$ to extend K^0 and KO^0 to extraordinary cohomology theories. It turns out to be easier to first do this for reduced groups. Let (X, x_0) be a based space and SX denote the reduced suspension of (X, x_0). In order to satisfy the suspension axiom, we are forced to define the groups as follows for $n \geq 0$:

$$\begin{aligned} \widetilde{KO}^{-n}(X) &:= \widetilde{KO}^0(S^n X) \\ \widetilde{K}^{-n} &:= \widetilde{K}^0(S^n X) \end{aligned}$$

To extend this definition to positive degrees, we crucially require the celebrated Bott periodicity theorem, which shows in our context that

$$\widetilde{KO}^0(X) \cong \widetilde{KO}^0(S^8 X) \text{ and } \widetilde{K}^0(X) \cong \widetilde{K}^0(S^2 X)$$

The isomorphisms are induced by taking exterior products of elements in $\widetilde{KO}^0(X)$ and $\widetilde{K}^0(X)$ with classes in $\widetilde{KO}^0(S^8)$ and $\widetilde{K}^0(S^2)$ coming from the tautological bundles over $\mathbb{OP}^1 \cong S^8$ and $\mathbb{CP}^1 \cong S^2$ respectively. This then allows us to extend our definition of $\widetilde{KO}^n$ and $\widetilde{K}^n$ to positive degrees to obtain 8-periodic and 2-periodic reduced cohomology theories $\widetilde{KO}^*$ and $\widetilde{K}^*$ respectively. These reduced cohomology theories yield unreduced cohomology theories KO^* and K^* respectively. The product structure on KO^0 and K^0 can be extended to KO^* and K^* so that they become graded-commutative rings.

Topological K-theory was one of the first extraordinary cohomology theories to be discovered: KO^* and K^* satisfy all the Eilenberg-Steenrod axioms[1] except for the dimension axiom. We have

$$K^0(\text{pt}) \cong \mathbb{Z} \qquad K^{-1}(\text{pt}) \cong 0$$

and

n	0	-1	-2	-3	-4	-5	-6	-7
$KO^n(\text{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

As rings,

$$\begin{aligned} K^*(\text{pt}) &\cong \mathbb{Z}[t, t^{-1}] \\ KO^*(\text{pt}) &\cong \mathbb{Z}[\eta, \alpha, \lambda, \lambda^{-1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\lambda) \end{aligned}$$

Here $t \in K^{-2}(\text{pt})$ is the Bott element in complex K-theory, $\lambda \in KO^{-8}(\text{pt})$ is the Bott element in KO-theory and η and α are generators in degrees -1 and -4 respectively.

As for any generalised cohomology theory, one of the most important computational

[1] The Eilenberg-Steenrod axioms for cohomology theories can be found in [May99, §18.1].

tools is the associated spectral sequence called the *Atiyah-Hirzebruch spectral sequence*[2]. It takes the form

$$\begin{aligned} E_2^{p,q} &= H^p(X; KO^q(\mathrm{pt})) \Rightarrow KO^{p+q}(X) \\ {}'E_2^{p,q} &= H^p(X; K^q(\mathrm{pt})) \Rightarrow K^{p+q}(X) \end{aligned}$$

The differentials on the E_2-page for KO^* can be described explicitly in terms of Steenrod squares [Fuj67, §1].

There are natural maps relating real and complex K-theory. Given a real vector space, we can apply $- \otimes_{\mathbb{R}} \mathbb{C}$ to obtain a complex vector space. Conversely, given a complex vector space, it has an underlying real vector space. All this can be promoted to maps

$$\begin{aligned} \hat{c}_i &\colon KO^i(X) \to K^i(X) \\ \hat{r}_i &\colon K^i(X) \to KO^i(X) \end{aligned}$$

We will mostly be interested in even degrees, so for each i, we denote by

$$\begin{aligned} c_i &\colon KO^{2i}(X) \to K^0(X) \\ r_i &\colon K^0(X) \to KO^{2i}(X) \end{aligned} \tag{1.1}$$

the maps $\hat{c}_{2i}$ and $\hat{r}_{2i}$ composed with appropriate powers of the periodicity isomorphism for K^*. We call these maps the *complexification* and *realification* maps. The map $c_*\colon KO^{2*}(X) \to K^0(X)$ is a ring homomorphism whereas $r_*\colon K^0(X) \to KO^{2*}(X)$ is just an additive homomorphism.

We also have an involution $*\colon K^0(X) \to K^0(X)$ induced by assigning to each complex vector bundle its dual bundle.

The following relations hold [Bou90, 4.7]:

$$\begin{array}{lll} c_i r_i = \mathrm{id} + (-1)^i * & *c_i = (-1)^i c_i & r_{i-1} c_i = \eta^2 \\ r_i c_i = 2 & r_i * = (-1)^i r_i & \end{array}$$

The realification and complexification can be assembled to form a long exact sequence known as the *Bott sequence*. It takes the shape [Bot69, p. 75]

$$\ldots \to KO^{i+1}(X) \xrightarrow{\cdot\eta} KO^i(X) \xrightarrow{\hat{c}_i} K^i(X) \xrightarrow{\hat{r}_{i+2} \circ t^{-1}} KO^{i+2}(X) \to \ldots$$

We now want to think about the K-theory of homogeneous spaces. So let G be a compact connected Lie group and $H \subset G$ be a closed connected subgroup of maximal rank, i.e. one that contains a maximal torus of G.

The complex K-theory of G/H is well-understood by the works of Hodgkin and Pittie. We will explain this in greater detail in Chapter 4. For now, we just quote the following facts which are immediate from Lemma 9.2 and (9.1) in [Hod75].

[2] The Atiyah-Hirzebruch spectral sequence for topological K-theory is introduced in [AH61, §2].

Theorem 1.1. *Let G and H be as above. Then $K^1(G/H) = 0$ and $K^0(G/H)$ is a free abelian group of finite rank.*

As already indicated in the introduction, our knowledge of $KO^*(G/H)$ is rather patchy and most attempts so far have used the Atiyah-Hirzebruch spectral sequence. Following [Zib15], we use a different approach. For a space X, we define its *Witt groups* as

$$W^i(X) := KO^{2i}(X)/r_i$$

The Witt groups are clearly 4-periodic. As for any $x \in KO^{2i}(X)$, we have $x+x = r_i c_i(x) \in \operatorname{im}(r_i)$, we see that $W^i(X)$ is 2-torsion. Since $x \cdot r_i(y) = r_{i+j}(c_j(x) \cdot y)$ for any $x \in KO^{2j}(X)$ and $y \in K^0(X)$, the *total Witt group*

$$W^*(X) := \bigoplus_{i \in \mathbb{Z}_4} W^i(X)$$

inherits a product from KO^{2*} and thus becomes a ring.

In subsequent chapters, we will see how to compute the Witt ring of flag varieties. For now, let us see how $W^*(X)$ relates to $KO^*(X)$. Basically, $W^*(X)$ determines the 2-torsion in $KO^*(X)$ and the free part can be determined by different means.

Proposition 1.2 ([Zib15, Lem. 1.2])**.** *Suppose $K^1(X) = 0$ and $K^0(X)$ is free. Then for all i,*

$$\begin{aligned} KO^{2i}(X) &= W^{i+1}(X) \cdot \eta^2 \oplus \textit{free part} \\ KO^{2i+1}(X) &= W^{i+1}(X) \cdot \eta \end{aligned}$$

Remark. By Theorem 1.1, this applies in particular to $X = G/H$.

Proof. Since $K^1(X) = 0$, the Bott sequence becomes

$$0 \to KO^{2i-1}(X) \xrightarrow{\cdot\eta} KO^{2i-2}(X) \xrightarrow{c_{i-1}} K^0(X) \xrightarrow{r_i} KO^{2i}(X) \xrightarrow{\cdot\eta} KO^{2i-1}(X) \to 0$$

We see that multiplication by η induces an isomorphism between $W^i(X) \cong KO^{2i}(X)/r_i$ and $KO^{2i-1}(X)$. This proves one part of the Proposition. Moreover, multiplication by η^2 induces an isomorphism between $W^i(X)$ and $\ker(c_{i-1})$. Since $K^0(X)$ is free, $\operatorname{im}(c_{i-1})$ is also free and so the short exact sequence

$$0 \to \ker(c_{i-1}) \to KO^{2i-2}(X) \to \operatorname{im}(c_{i-1}) \to 0$$

splits. This yields

$$KO^{2i-2}(X) = \ker(c_{i-1}) \oplus \text{free part} = W^i(X) \cdot \eta^2 \oplus \text{free part} \qquad \square$$

Remark 1.3 ([Zib15, Rem. 1.3])**.** The products $KO^{\mathrm{odd}}(X) \otimes KO^{\mathrm{odd}}(X) \to KO^{\mathrm{even}}(X)$ are determined by the products in $W^*(X)$ according to the following commutative square:

$$\begin{tikzcd}
W^i(X) \otimes W^j(X) \arrow[r] \arrow[d, "\cong"', "\eta\otimes\eta"] & W^{i+j}(X) \arrow[d, "\eta^2"] \\
KO^{2i-1}(X) \otimes KO^{2j-1}(X) \arrow[r] & KO^{2i+2j-2}(X)
\end{tikzcd}$$

Here, the horizontal maps are the products.

Given that we can compute the Witt ring of G/H, we are thus left to determine the free part of $KO^{2i}(X)$.

Proposition 1.4. *Let G be a compact connected Lie group and $H \subset G$ be a closed connected subgroup of maximal rank. Then the free part of $KO^{2i}(G/H)$ has rank n_{2i} where*

$$n_0 = n_4 = dim_{\mathbb{Q}} \left(\bigoplus_{i=0}^{\infty} H^{4i}(G/H;\mathbb{Q}) \right)$$
$$n_2 = n_6 = dim_{\mathbb{Q}} \left(\bigoplus_{i=0}^{\infty} H^{4i+2}(G/H;\mathbb{Q}) \right)$$

Proof. By a result of Hopf [Hat01, Thm. 3C.4], $H^*(H;\mathbb{Q})$ is an exterior algebra over $\mathbb{Q}$ on generators of odd degree. Applying the Serre spectral sequence to the fibration $H \to EH \to BH$, we deduce that $H^*(BH;\mathbb{Q})$ is a polynomial algebra over $\mathbb{Q}$ with all generators in even degree. In particular, $H^*(BH;\mathbb{Q})$ is concentrated in even degrees. By a result of Borel [McC01, Thm. 8.3], the map

$$j^*\colon H^*(BH;\mathbb{Q}) \to H^*(G/H;\mathbb{Q})$$

is onto, where $j\colon G/H \to BH$ is the classifying map of the principal H-bundle $G \to G/H$. Hence $H^*(G/H;\mathbb{Q})$ is concentrated in even degrees. Thus we obtain that the Atiyah-Hirzebruch spectral sequence of G/H for rational KO-theory

$$E_2^{p,q} = H^p(G/H; KO^q(\mathrm{pt}) \otimes \mathbb{Q}) \Rightarrow KO^{p+q}(G/H) \otimes \mathbb{Q}$$

collapses on the E_2-page for geographical reasons. The claim follows from this. $\square$

As already indicated in the proof of the Proposition, the rational cohomology of G/H is well-understood (see [McC01, §8.1]) so that it enables us to compute the ranks of the free parts of $KO^*(G/H)$.

We will be particularly interested in computing the Witt rings and KO-theory of complex flag varieties, which we now introduce.

1.2 Complex flag varieties

In [Ser54], Serre classifies all the subgroups of a compact connected semisimple Lie group for which the associated homogeneous space is a smooth complex projective variety.

Theorem 1.5. *Let G be a compact connected semisimple Lie group. If $S \subset G$ is a torus, then $G/C_G(S)$ is a smooth complex projective variety. Conversely, if $H \subset G$ is a closed subgroup such that G/H is a smooth complex projective variety, then $H = C_G(S)$ for some torus $S \subset G$.*

Proof. See [Ser54], Théorème 1 and Théorème 2. $\square$

Remark 1.6. It is a result of Hopf that for a torus $S \subset G$, its centraliser $C_G(S)$ must be connected. For if $x \in C_G(S)$, i.e. if x commutes with all elements in S, then there must be a torus in G containing both x and S [Ada69, Prop. 4.25], and this torus is then clearly contained in $C_G(S)$.

In light of the above theorem, we define a *complex flag variety* to be the quotient of a compact connected semisimple Lie group by the centraliser of a torus in the Lie group.

Remark 1.7. Even though the results of the previous section apply to quotients of compact Lie groups by arbitrary closed connected subgroups of maximal rank, in this thesis we shall focus on complex flag varieties only. One motivation for this is that the Witt ring of a complex variety, as defined in the previous section, can be viewed from a different perspective, which we briefly explain.

The Witt ring of a *field*, classifying non-degenerate anisotropic quadratic forms, dates back to the 1930s. Subsequently, this was vastly generalised by defining 4-periodically graded Witt rings of *schemes* and *triangulated categories with duality*[3]. These algebraic Witt rings are closely related to the topological Witt rings that we defined above: Zibrowius showed [Zib11, Thm. 2.5] that for a smooth complex cellular variety X, the algebraic Witt ring of X coincides with the topological Witt ring of $X(\mathbb{C})$.

Now the projective varieties $G/C_G(S)$ are actually cellular: Letting $G_{\mathbb{C}}$ be the complexification of G, there is a parabolic subgroup $P \subset G_{\mathbb{C}}$ such that $P \cap G = C_G(S)$. So $G_{\mathbb{C}}/P \cong G/C_G(S)$ and the Bruhat decomposition for $G_{\mathbb{C}}$ shows that this is a *cellular* variety. Hence by the theorem of Zibrowius, the algebraic and topological Witt rings of $G/C_G(S)$ coincide. Thus our computations are also of interest from the viewpoint of algebraic Witt rings of schemes.

It turns out that there is a simple classification of the centralisers of tori in compact Lie groups in terms of their roots, which we shall now describe. This classification is certainly implicit in [Ser54] even though a proof is not given there. Due to a lack of reference, we

[3]See [Bal05] for an overview.

give a direct proof. We use basic facts and terminology from the representation theory of compact Lie groups and refer to [Ada69] and [BD85] as general references.

First, let us fix some notation. Let G be a compact connected Lie group (we do not require semisimple here) and $T \subset G$ be a maximal torus. We denote by LG and LT the respective tangent spaces at the identity element and by $\exp\colon LG \to G$ or $LT \to T$ the exponential map. Let $\mathrm{Ad}\colon G \to Aut(LG)$ be the adjoint representation whose restriction $\mathrm{Ad}|_T$ makes LG into a T-space. As such, LG splits into a direct sum

$$LG = LT \oplus \bigoplus_{i=1}^{m} V_i$$

of some 2-dimensional vector spaces V_i on which $t \in T$ acts via a matrix

$$R_i(t) := \begin{pmatrix} \cos(2\pi\overline{\theta}_i(t)) & -\sin(2\pi\overline{\theta}_i(t)) \\ \sin(2\pi\overline{\theta}_i(t)) & \cos(2\pi\overline{\theta}_i(t)) \end{pmatrix}$$

for some homomorphisms $\overline{\theta}_i\colon T \to \mathbb{R}/\mathbb{Z}$ induced by linear forms $\theta_i\colon LT \to \mathbb{R}$ taking integer values on the integer lattice $I := \exp^{-1}(0) \subset LT$ [Ada69, Prop. 4.12]. The linear forms $\pm\theta_i \in LT^*$ are the roots of G. If necessary, we change the signs so that the chamber

$$FWC := \{v \in LT \mid \theta_i(v) > 0 \text{ for } 1 \leq i \leq m\}$$

is non-empty and take it as our fundamental Weyl chamber. The roots $R^+ = \{\theta_i \mid 1 \leq i \leq m\}$ are then the positive roots and we denote the subset of the simple roots by Σ.

For any subset $\Theta \subset \Sigma$, we write $U_\Theta := \bigcap_{\theta\in\Theta} \ker(\theta)$, a subspace of LT. Then $\exp(U_\Theta) \subset T$ is a torus. This yields a map

$$\begin{aligned} \varphi\colon \ \{\text{subsets of } \Sigma\} &\to \{C_G(S) \mid S \subset G \text{ torus}\}\,/\text{conjugation}, \\ \Theta &\mapsto C_G(\exp(U_\Theta)) \end{aligned}$$

Proposition 1.8. *φ is surjective.*

Before we prove this, we need a lemma.

Lemma 1.9. *Let $X \in LG$ and $t \in T$. Then $X \in LC_G(t)$ if and only if $Ad(t)X = X$.*

Proof. Let $\tau_t\colon G \to G$ denote the map $x \mapsto txt^{-1}$. We have

$$\begin{aligned} X \in LC_G(t) \ &\Leftrightarrow\ sX \in LC_G(t) \quad \forall s \in \mathbb{R} \ \Leftrightarrow\ \exp(sX) \in C_G(t) \quad \forall s \in \mathbb{R} \\ &\Leftrightarrow\ \tau_t(\exp(sX)) = \exp(sX) \quad \forall s \in \mathbb{R} \end{aligned}$$

Differentiating this with respect to s at $s = 0$ yields $Ad(t)X = X$. Conversely, assuming that $Ad(t)X = X$, it follows that $Ad(t)sX = sX$ for all $s \in \mathbb{R}$. Thus from the commutativity of the square

$$\begin{array}{ccc} LG & \xrightarrow{Ad(t)} & LG \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\tau_t} & G \end{array}$$

we deduce that $\tau_t(\exp(sX)) = \exp(sX)$ for all $s \in \mathbb{R}$. □

Proof of Proposition 1.8. Suppose $S \subset G$ is a torus. Since any two maximal tori in G are conjugate, we may assume without loss of generality that $S \subset T$. Any torus is monogenic, so there exists $b \in S$ such that $S = \mathrm{cl}(\{b^n \mid n \in \mathbb{Z}\})$. Then $C_G(S) = C_G(b)$. Now pick some $X \in LS \subset LT$ with $\exp(X) = b$. Without loss of generality, we may assume that $X \in \mathrm{cl}(FWC)$ since the Weyl group acts transitively on Weyl chambers. We may also assume that $|\theta(X)| < 1$ for all roots θ (otherwise, replace X by $\frac{1}{N}X$ for sufficiently large N and b by $\exp\left(\frac{1}{N}X\right)$). We define $\Theta := \{\theta \in \Sigma \mid \theta(X) = 0\}$. We claim that $\varphi(\Theta) = C_G(b)$. Once we prove this, we are done.

Note that $X \in U_\Theta$ and so $b \in \exp(U_\Theta)$. Hence $\varphi(\Theta) = C_G(\exp(U_\Theta)) \subset C_G(b)$.

Now let $g \in C_G(b)$. Since $C_G(b) = C_G(S)$ is a compact connected Lie group (see Remark 1.6), the exponential map $LC_G(b) \to C_G(b)$ is onto. So we can pick $Y \in LC_G(b)$ with $\exp(Y) = g$. Then $\mathrm{Ad}(b)Y = Y$ by Lemma 1.9. Suppose now $Z \in U_\Theta$. We aim to show that $\mathrm{Ad}(\exp(Z))Y = Y$. If we prove this, then by Lemma 1.9, $Y \in LC_G(\exp(Z))$ and so $g = \exp(Y) \in C_G(\exp(Z))$.

We write $Y = Y_T + \sum_{i=1}^m Y_i$ where $Y_T \in LT$ and $Y_i \in V_i$ for $1 \le i \le m$. Since $\mathrm{Ad}(b)Y = Y$, we have

$$Y_T + \sum_{i=1}^m Y_i = Y_T + \sum_{i=1}^m R_i(b)Y_i$$

By definition of Θ, we have $\theta(X) = 0$ if $\theta \in \Theta$ and $0 < \theta(X) < 1$ if $\theta \in \Sigma \setminus \Theta$. Every positive root is a sum of simple roots. If $\theta \in R^+$ is not in the subspace spanned by Θ, we deduce that $\theta(X) > 0$, and we have $\theta(X) < 1$ by our choice of X. Thus for every $1 \le i \le m$, we have that $R_i(b)$ is a nontrivial rotation matrix if θ_i is not a sum of the simple roots in Θ. So from the above equation, we deduce that $Y_i = 0$ if θ_i is not a sum of roots in Θ.

Since $Z \in U_\Theta$, we have that $R_i(\exp(Z)) = \mathrm{id}$ if θ_i is a sum of simple roots in Θ. Thus we obtain that

$$\mathrm{Ad}(\exp(Z))Y = Y_T + \sum_{i=1}^m R_i(\exp(Z))Y_i = Y$$

as required. □

Remark 1.10. Let $\Theta \subset \Sigma$ and $H_\Theta := \varphi(\Theta) = C_G(\exp(U_\Theta))$. We deduce that the root system of H_Θ is precisely the sub-root system of G generated by Θ:

Suppose $X \in LG$. By Lemma 1.9, we have that $X \in LH_\Theta$ if and only if $\mathrm{Ad}(t)X = X$ for all $t \in \exp(U_\Theta)$. Hence

$$LH_\Theta = LT \oplus \bigoplus_{i \in I_\Theta} V_i$$

where

$$I_\Theta = \big\{ i \mid 1 \leq i \leq m,\ \overline{\theta}_i(t) = 0\ \forall t \in \exp(U_\Theta) \big\} = \{ i \mid 1 \leq i \leq m,\ \theta_i(X) = 0\ \forall X \in U_\Theta \}$$

We have $I_\Theta = \{ i \mid 1 \leq i \leq m,\ \theta_i \in \mathrm{span}(\Theta) \}$: The inclusion "$\supset$" is clear. For "$\subset$", note that $U_\Theta = \bigcap_{\theta \in \Theta} \ker(\theta) \not\subset \ker(\tau)$ if $\tau \in \Sigma \setminus \Theta$ since the simple roots Σ are linearly independent. This implies the claim.

Remark 1.11. There is an algebraic analogue of Proposition 1.8. It is well-known that there is a bijective correspondence between isomorphism classes of compact connected Lie groups and complex reductive groups [Lee02, Thm. 4.29]: We can assign to a compact connected Lie group its complexification, which turns out to be reductive, and conversely we can assign to a complex reductive group its maximal compact subgroup (which is unique up to conjugation).

Letting G be a complex simple algebraic group, a Zariski-closed subgroup $P \subset G$ is called *parabolic* if the quotient G/P is projective. There is a bijective correspondence between subsets of the simple roots of G and the parabolic subgroups of G up to conjugation [FH04, §23.3]. The reason why this correspondence is *bijective* in this case and not only surjective is, roughly speaking, that there are more conjugacy classes of parabolic subgroups in a complex simple algebraic group than there are conjugacy classes of centralisers of tori in compact simple Lie groups. As an example, consider $SU(3)$ whose complexification is $SL_3(\mathbb{C})$. Then the two subgroups of $SU(3)$ consisting of matrices of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \text{respectively}$$

are centralisers of tori and are actually conjugate in $SU(3)$, whereas the two parabolic subgroups of $SL_3(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \text{respectively}$$

are *not* conjugate in $SL_3(\mathbb{C})$.

We now apply Proposition 1.8 to determine all the centralisers of tori in the compact simple Lie groups of ordinary type. The compact simple simply connected Lie groups are classified by their Dynkin diagrams (cf. Figure 1.1) where each node corresponds to one of

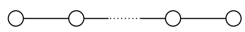
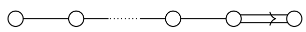
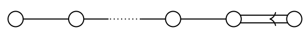
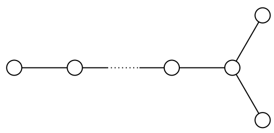
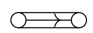
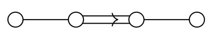
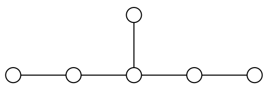
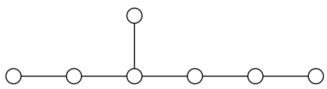
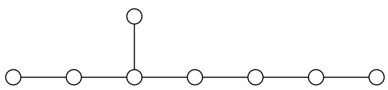
Dynkin diagram	Type	Group
	A_n	$SU(n)$
	B_n	$Spin(2n+1)$
	C_n	$Sp(n)$
	D_n	$Spin(2n)$
	G_2	G_2
	F_4	F_4
	E_6	E_6
	E_7	E_7
	E_8	E_8

Figure 1.1: Dynkin diagrams and their corresponding compact simple simply connected Lie groups

the simple roots. By Proposition 1.8, we can thus determine a centraliser of a torus H_Θ in the respective simple Lie group by marking the nodes corresponding to the simple roots in the set $\Theta \subset \Sigma$ white and marking the other nodes black. For example, the Dynkin diagram with all nodes marked black corresponds to a maximal torus.

Let us consider the symplectic group $Sp(n)$ of type C_n. There is an inclusion $U(n) \to Sp(n)$, and the maximal torus T in $U(n)$ given by all the diagonal matrices

$$\operatorname{diag}(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}) \in U(n)$$

is also a maximal torus in $Sp(n)$. A set of simple roots of $Sp(n)$ is given by $\theta_i := x_i - x_{i+1}$ for $1 \le i \le n-1$ and $\theta_n := 2x_n$. Now given $\overline{I} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and setting $I := \{1, \ldots, n\} \setminus \overline{I}$ and $\Theta := \{\theta_i \mid i \in I\}$, the torus $\exp(U_\Theta)$ in $Sp(n)$ consists of all elements in T of the form $\operatorname{diag}(y_1, \ldots, y_n)$ where:

$$y_j = y_{j+1} \text{ if } j \in I \setminus \{n\} \text{ and } y_n = y_{n-1} = \ldots = y_{1+\max(\overline{I})} = 1 \text{ if } n \in I$$

This is a subtorus of T with centraliser

$$U(i_1) \times U(i_2 - i_1) \times \ldots \times U(i_k - i_{k-1}) \times Sp(n - i_k) \subset Sp(n)$$

Hence we have shown that up to conjugation, every centraliser of a torus in $Sp(n)$ is of this form. Thus every smooth complex projective variety which is a quotient of $Sp(n)$ can be obtained as a quotient by a subgroup of this form.

A similar analysis can be done for types A_n, B_n, D_n. Before we formulate the result, we review a few facts about the group $Spin(n)$, the connected double cover of $SO(n)$. Since there is an inclusion $U(n) \hookrightarrow SO(2n)$, we have an inclusion $\tilde{U}(n) \hookrightarrow Spin(2n)$ where $\tilde{U}(n)$ denotes the connected double cover of $U(n)$. Furthermore, there are natural maps

$$\psi_{m,n} \colon Spin(m) \times Spin(n) \to Spin(m+n)$$

covering the inclusion $SO(m) \times SO(n) \hookrightarrow SO(m+n)$. Thus $\psi_{m,n}$ has the cyclic group of order 2 as kernel: If we denote, for every k, by $\epsilon \in Spin(k)$ the nontrivial element in the kernel of $Spin(k) \twoheadrightarrow SO(k)$, then $\ker(\psi_{m,n}) = \{(1,1), (\epsilon, \epsilon)\}$.

Proposition 1.12. *Up to conjugation, every centraliser of a torus in ...*

- *... $SU(n)$ is of the form $S(U(n_1) \times \ldots \times U(n_l))$ for some $n_1, \ldots, n_l$ with $\sum_{i=1}^l n_i = n$. Here, $S(U(n_1) \times \ldots \times U(n_l))$ is the closed subgroup of $U(n_1) \times \ldots \times U(n_l)$ comprising all elements $(A_1, \ldots, A_l) \in U(n_1) \times \ldots \times U(n_l)$ such that $\prod_{i=1}^l det(A_i) = 1$.*

- *... $Sp(n)$ is of the form $Sp(m) \times U(n_1) \times \ldots \times U(n_l)$ for some $m, n_1, \ldots, n_l$ with $m + \sum_{i=1}^l n_i = n$.*

- *... $Spin(2n+1)$ or $Spin(2n)$ is of the form*

$$(Spin(2m+1) \times \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l))/C$$
$$\text{or} \quad (Spin(2m) \times \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l))/C \quad \text{respectively}$$

for some $m, n_1, \ldots, n_l$ with $m + \sum_{i=1}^{l} n_i = n$, where C is the subgroup of all elements $(x_0, x_1, \ldots, x_l)$ of $Spin(2m+1) \times \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l)$ or $Spin(2m) \times \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l)$ respectively such that $x_i \in \{1, \epsilon\}$ for all $0 \leq i \leq l$ and $x_j = \epsilon$ for an even number of j.

1.3 Representation rings

It will be crucial for our computations to have a good understanding of the representation rings of compact connected Lie groups. For the whole general theory, we refer to [Ada69] and [BD85]. In this section, we briefly revisit some of the most important definitions and results that are relevant to us. We then describe the complex representation rings of the groups in Proposition 1.12, i.e. the centralisers of tori in a simply connected compact simple Lie group.

1.3.1 Preliminaries

Throughout this section, let G be a compact connected Lie group. All representations are finite-dimensional and continuous.

Theorem 1.13. *Any representation of G over $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{H}$ can be written as a direct sum of irreducible representations.*

Proof. See [Ada69, Thm. 3.20]. □

By $RO(G)$, $RSp(G)$, $R(G)$ respectively we denote the free abelian group on the isomorphism classes of irreducible F-representations of G for $F = \mathbb{R}, \mathbb{H}, \mathbb{C}$ respectively. We call elements in these groups *virtual representations* of G. By Theorem 1.13, we see that elements $\sum n_i[V_i]$, where V_i is an irreducible F-representation and $n_i \geq 0$ for all i, are in bijective correspondence with the isomorphism classes of F-representations of G. Furthermore, taking tensor products of real or complex representations of G induces products on $RO(G)$ and $R(G)$, making them into commutative rings.

We can alternatively define $R(G)$ and $RO(G)$ via equivariant K-theory. Observe that a G-equivariant real or complex vector bundle over a point is just a real or complex representation of G. Thus we can define graded representation rings extending $RO(G)$ and $R(G)$ by setting, for $i \in \mathbb{Z}$,

$$RO^i(G) := KO^i_G(\mathrm{pt})$$
$$R^i(G) := K^i_G(\mathrm{pt})$$

Similarly to the non-equivariant case, RO^* is 8-periodic and R^* is 2-periodic. The groups $RO^*(G)$ can be expressed more concretely in terms of $RO(G)$ and $RSp(G)$ [BG10, Thm. 2.2.12]. In particular, we have

$$\begin{aligned} R^0(G) &= R(G) \quad R^1(G) = 0 \\ RO^0(G) &= RO(G) \quad RO^4(G) = RSp(G) \end{aligned}$$

Analogously to the non-equivariant case, we obtain realification, complexification and a duality:

$$\begin{aligned} r_i &\colon R^0(G) \to RO^{2i}(G) \\ c_i &\colon RO^{2i} \to R^0(G) \\ * &\colon R(G) \to R(G) \end{aligned}$$

These satisfy the same relations as in the non-equivariant case. We call a virtual representation in $R(G)$ *of real type* or *of quaternionic type* if it is in the image of c_0 or c_2 respectively. Every complex self-dual irreducible representation is either of quaternionic or of real type [Ada69, Prop. 3.56].

Let us return to our ordinary $RO(G)$ and $R(G)$. Taking exterior powers of representations induces λ-operations on $RO(G)$ and $R(G)$ making them special λ-rings [AT69, Thm. 1.5]. In particular, the following identities hold:

- $\Lambda^0(x) = 1$ and $\Lambda^1(x) = x$ for all $x \in R(G)$.
- $\Lambda^n(\mu) = 0$ if μ is an actual (not just a virtual) representation and $n > \mathrm{rk}(\mu)$.
- $\Lambda^n(x + y) = \sum_{i=0}^n \Lambda^i(x)\Lambda^{n-i}(y)$ for all $x, y \in R(G)$.
- $\Lambda^n(\alpha x) = \alpha^n \Lambda^n(x)$ if α is an actual representation of rank 1 and $x \in R(G)$.

Theorem 1.14. *Let $T \subset G$ be a maximal torus. Then the inclusion induces an isomorphism $R(G) \to R(T)^W$, where $R(T)^W$ is the ring of characters invariant under the action of the Weyl group $W = N_G(T)/T$ of G.*

Proof. See [Ada69, Thm. 6.20]. □

If G is simply connected, its complex representation ring can be easily described:

Theorem 1.15. *Suppose G is simply connected. Then*

$$R(G) \cong \mathbb{Z}[\lambda_1, \lambda_1^*, \dots, \lambda_k, \lambda_k^*, \mu_1, \dots, \mu_l]$$

for some irreducible representations λ_i, μ_j of G where the μ_j are self-dual. These are called the fundamental representations *of G.*

Proof. See [BD85], Corollary VI.2.11 and Note VI.4.2. □

Example 1.16. We recall the representation rings of the compact simply connected simple Lie groups of ordinary type (cf. [BD85, Chapters VI.5 and VI.6]):

$$R(SU(n)) \cong \mathbb{Z}[\lambda_1, \dots, \lambda_{n-1}]$$

where λ_1 is the standard complex representation of rank n and $\lambda_i = \Lambda^i(\lambda_1)$ for all i. The duality is given via $\lambda_i^* = \lambda_{n-i}$ for all i. Suppose $n = 2k$, then λ_k is self-dual. It is of real type if k is even and of quaternionic type if k is odd.

$$R(Sp(n)) \cong \mathbb{Z}[\mu_1, \dots, \mu_n]$$

where μ_1 is the standard complex representation of rank $2n$ and $\mu_i = \Lambda^i(\mu_1)$ for all i. All representations are self-dual and μ_i is of real type if i is even and of quaternionic type if i is odd.

$$R(Spin(2n+1)) \cong \mathbb{Z}[\nu_1, \dots, \nu_{n-1}, \Delta]$$
$$R(Spin(2n)) \cong \mathbb{Z}[\xi_1, \dots, \xi_{n-2}, \Delta_+, \Delta_-]$$

Here, ν_1 is induced by the standard complex representation of $SO(2n+1)$ of rank $2n+1$, $\nu_i = \Lambda^i(\nu_1)$ for all i, and Δ is the *spin representation* of rank 2^n. All representations of $Spin(2n+1)$ are self-dual, λ_i is of real type for all i and Δ is of real type if $n \equiv 0, 3 \pmod 4$ and of quaternionic type if $n \equiv 1, 2 \pmod 4$.

The representation ξ_1 is induced by the standard complex representation of $SO(2n)$ of rank $2n$ and $\xi_i = \Lambda^i(\xi_1)$ for all i and Δ_+, Δ_- are the *half-spin representations* of $Spin(2n)$ of respective rank 2^{n-1}. The representations ξ_i are all self-dual and Δ_+, Δ_- are self-dual if n is even and mutually dual if n is odd. In the case that n is even, Δ_+, Δ_- are of real type if $n \equiv 0 \pmod 4$ and of quaternionic type if $n \equiv 2 \pmod 4$.

Furthermore, we can express $\nu_n = \Lambda^n(\nu_1)$ and $\xi_{n-1} = \Lambda^{n-1}(\xi_1),\ \xi_n = \Lambda^n(\xi_1)$ in terms of the polynomial generators above as follows:

$$\Delta^2 = 1 + \nu_1 + \ldots + \nu_{n-1} + \nu_n$$
$$\Delta_+\Delta_- = \xi_{n-1} + \xi_{n-3} + \xi_{n-5} + \ldots$$
$$\Delta_+^2 + \Delta_-^2 = \xi_n + 2\xi_{n-2} + 2\xi_{n-4} + \ldots$$

Theorem 1.17. *If both G and H are compact connected Lie groups, the projection maps induce an isomorphism*

$$R(G \times H) \cong R(G) \otimes_{\mathbb{Z}} R(H)$$

Proof. See [Ada69, Thm. 3.65]. □

Lemma 1.18. *Suppose $C \subset G$ is a finite central subgroup. Then $G \to G/C$ induces an injection $R(G/C) \to R(G)$, mapping $R(G/C)$ isomorphically onto the subring $R(G)^C \subset R(G)$ additively generated by all irreducible representations of G via which C acts trivially.*

Furthermore, $R(G)$ is integral over $R(G/C)$. In particular, $R(G)$ and $R(G/C)$ have the same Krull dimension.

Proof. The first part follows easily from the observation that an irreducible representation of G descends to an irreducible representation of G/C if and only if the restriction of the representation to C yields a trivial representation.

Now suppose ρ is an irreducible representation of G. By Schur's lemma, each $c \in C$ acts via ρ by multiplication by some n_cth root of unity. Let $N := \operatorname{lcm}\{n_c \mid c \in C\}$, then C acts trivially via ρ^N. This shows that all irreducible representations of G are integral over $R(G/C)$. Since they generate $R(G)$, the whole of $R(G)$ is integral over $R(G/C)$. □

Sometimes we want to consider a basis of $R(G)$ and $R(G/C)$ other than the one given by the irreducible representations:

Lemma 1.19. *Suppose $C \subset G$ is a finite central subgroup. Let $(\rho_i)_{i\in I}$ be a $\mathbb{Z}$-basis of $R(G)$ where each $\rho_i\colon G \to GL_{n_i}(\mathbb{C})$ is an actual (not just a virtual) representation of G. Suppose that each $c \in C$ acts by multiplication by a scalar via all ρ_i. Letting $J \subset I$ denote the set of $j \in I$ such that C acts trivially via ρ_j, a $\mathbb{Z}$-basis of $R(G/C)$ is given by $(\rho_j)_{j\in J}$.*

Proof. Every ρ_i can be written as a sum of irreducible representations. Since every $c \in C$ acts by multiplication by some scalar $\zeta_{c,i}$ via ρ_i, it also acts by multiplication by $\zeta_{c,i}$ via every irreducible summand of ρ_i. In particular, C acts trivially via ρ_i if and only if it acts trivially via each irreducible summand. This shows that

$$\bigoplus_{j\in J} \mathbb{Z}\rho_j \subset \bigoplus_{\substack{\sigma\in R(G/C)\\ \text{irred.}}} \mathbb{Z}\sigma \quad\text{and}\quad \bigoplus_{i\in I\setminus J} \mathbb{Z}\rho_i \subset \bigoplus_{\substack{\tau\in R(G)\setminus R(G/C)\\ \text{irred.}}} \mathbb{Z}\tau$$

Since the ρ_i form a basis of $R(G)$, we have

$$R(G) = \bigoplus_{j\in J} \mathbb{Z}\rho_j \oplus \bigoplus_{i\in I\setminus J} \mathbb{Z}\rho_i \subset \bigoplus_{\substack{\sigma\in R(G/C)\\ \text{irred.}}} \mathbb{Z}\sigma \oplus \bigoplus_{\substack{\tau\in R(G)\setminus R(G/C)\\ \text{irred.}}} \mathbb{Z}\tau = R(G)$$

So the inclusions must in fact be equalities. In particular, $\bigoplus_{j\in J} \mathbb{Z}\rho_j = \bigoplus_{\substack{\sigma\in R(G/C)\\ \text{irred.}}} \mathbb{Z}\sigma$. But this is $R(G/C)$ by Lemma 1.18. □

1.3.2 Representation rings of centralisers of tori

We will determine the representation rings of the groups that occur as centralisers of tori in simple Lie groups, which we described in Proposition 1.12.

Let us first determine the representation ring of $\tilde{U}(n)$, the connected double cover of $U(n)$, as well as the induced map $R(U(n)) \to R(\tilde{U}(n))$. We first construct $\tilde{U}(n)$ explicitly. Its representation ring is then easy to compute.

We define a surjective homomorphism

$$SU(n) \times S^1 \to U(n), \quad (A, x) \mapsto xA$$

It is easy to see that the kernel is cyclic of order n, generated by $(\zeta I_n, \zeta^{-1})$ where $\zeta := e^{\frac{2\pi i}{n}}$. Thus we have

$$U(n) \cong SU(n) \times S^1 / \langle (\zeta I_n, \zeta^{-1}) \rangle$$

Suppose n is even. Then we define

$$\tilde{U}(n) := SU(n) \times S^1 / \langle (\zeta^2 I_n, \zeta^{-2}) \rangle$$

with the obvious 2:1 projection $\pi_n \colon \tilde{U}(n) \to U(n)$ with $\ker(\pi_n) = \big\{[(I_n, 1)], [(\zeta I_n, \zeta^{-1})]\big\}$. Note that if we made the same definition for odd n, then $\tilde{U}(n) \to U(n)$ would just be an isomorphism.

If n is odd, we define

$$\tilde{U}(n) := SU(n) \times S^1 / \langle (\zeta^2 I_n, \zeta^{-1}) \rangle$$

with projection map

$$\pi_n \colon \tilde{U}(n) \to U(n), \quad [(A, x)] \mapsto [(A, x^2)]$$

It is easy to check that $\ker(\pi_n) = \{[(I_n, 1)], [(I_n, -1)]\}$. If we made this definition for even n, then $[(I_n, -1)] = [(I_n, 1)]$ in $\tilde{U}(n)$ and π_n would be an isomorphism.

We know that

$$R(SU(n) \times S^1) \cong \mathbb{Z}[\lambda_1, \dots, \lambda_{n-1}] \otimes \mathbb{Z}[x, x^{-1}]$$

where λ_1 is the standard representation of $SU(n)$ and $\lambda_i = \Lambda^i(\lambda_1)$. Now $(\zeta I_n, \zeta^{-1}) \in SU(n) \times S^1$ acts by multiplication by ζ^i via λ_i and by multiplication by ζ^{-1} via x. Hence by Lemma 1.19 (taking all the monomials as a $\mathbb{Z}$-basis of $R(SU(n) \times S^1)$), the subring of $R(SU(n) \times S^1)$ generated by representations via which $(\zeta I_n, \zeta^{-1})$ acts trivially is

$$R(U(n)) \cong \mathbb{Z}[x\lambda_1, x^2\lambda_2, \dots, x^{n-1}\lambda_{n-1}] \otimes \mathbb{Z}[x^n, x^{-n}]$$

Observe that since x is of rank 1, $\Lambda^i(x\lambda_1) = x^i\lambda_i$. We similarly determine $R(\tilde{U}(n))$ as a subring of $R(SU(n) \times S^1)$:

Suppose n is even. Since $(\zeta^2 I_n, \zeta^{-2})$ acts by multiplication by ζ^{2i} via λ_i and by multiplication by ζ^{-2} via x, we obtain

$$R(\tilde{U}(n)) \cong \mathbb{Z}[x\lambda_1, x^2\lambda_2, \ldots, x^{n-1}\lambda_{n-1}] \otimes \mathbb{Z}\left[x^{\frac{n}{2}}, x^{-\frac{n}{2}}\right]$$

Observe that since x is of rank 1, $\Lambda^i(x\lambda_1) = x^i\lambda_i$.

Suppose n is odd. Since $(\zeta^2 I_n, \zeta^{-1})$ acts by multiplication by ζ^{2i} via λ_i and by multiplication by ζ^{-1} via x, we obtain

$$R(\tilde{U}(n)) \cong \mathbb{Z}[x^2\lambda_1, x^4\lambda_2, \ldots, x^{2(n-1)}\lambda_{n-1}] \otimes \mathbb{Z}[x^n, x^{-n}]$$

Observe that since x is of rank 1, $\Lambda^i(x^2\lambda_1) = x^{2i}\lambda_i$.

In summary, we deduce:

Proposition 1.20. *The representation rings of $U(n)$ and $\tilde{U}(n)$ are*

$$\begin{aligned} R(U(n)) &\cong \ \mathbb{Z}\left[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}\right] \\ R(\tilde{U}(n)) &\cong \ \mathbb{Z}\left[x_1, \ldots, x_{n-1}, (x_n)^{\frac{1}{2}}, (x_n)^{-\frac{1}{2}}\right] \end{aligned}$$

with duality in both cases given by $x_i^* = x_n^{-1} \cdot x_{n-i}$. Here $x_1 \in R(U(n))$ is the standard representation of $U(n)$ and $x_i := \Lambda^i(x_1)$. Under this identification, the projection $\pi_n \colon \tilde{U}(n) \to U(n)$ induces the inclusion indicated by the chosen notation, and the non-trivial element in ker(π_n) acts trivially via $x_i \in R(\tilde{U}(n))$ for $1 \le i \le n$ and by multiplication by -1 via $(x_n)^{\frac{1}{2}}$.

Later, we will need to know the map on representation rings induced by the inclusion $\tilde{U}(n) \to Spin(2n)$.

Lemma 1.21. *Using the notation for the representation rings of $Spin(2n+1)$, $Spin(2n)$ and $\tilde{U}(n)$ as in Example 1.16 and Proposition 1.20, the restriction $R(Spin(2n)) \to R(\tilde{U}(n))$ maps $\xi_1 \mapsto x_1 + x_1^*$ and*

$$\Delta_+ \mapsto \begin{cases} x_n^{-\frac{1}{2}}\,(1 + x_2 + x_4 + \ldots + x_n) & \text{if } n \text{ is even} \\ x_n^{-\frac{1}{2}}\,(x_1 + x_3 + x_5 + \ldots + x_n) & \text{if } n \text{ is odd} \end{cases}$$

$$\Delta_- \mapsto \begin{cases} x_n^{-\frac{1}{2}}\,(1 + x_2 + x_4 + \ldots + x_{n-1}) & \text{if } n \text{ is odd} \\ x_n^{-\frac{1}{2}}\,(x_1 + x_3 + x_5 + \ldots + x_{n-1}) & \text{if } n \text{ is even} \end{cases}$$

The restriction $R(Spin(2n+1)) \to R(Spin(2n))$ maps $\nu_1 \mapsto \xi_1 + 1$ and $\Delta \mapsto \Delta_+ + \Delta_-$.

Proof. The claim about $R(Spin(2n)) \to R(\tilde{U}(n))$ can be proved by comparing weights. For the weights of Δ_+ and Δ_-, see [Ada96, Prop. 4.2]. The claim about $R(Spin(2n+1)) \to R(Spin(2n))$ is immediate from [Ada96, Prop. 4.4]. □

Finally, we determine the representation rings of the groups in Proposition 1.12. The following lemma will be useful:

Lemma 1.22. *Let R, S be integral domains of the same Krull dimension and $f\colon R \to S$ be a surjective ring homomorphism. Then f is an isomorphism.*

Proof. It suffices to show that f is injective. So let $P = \ker(f)$. Since f is surjective, $R/P \cong S$ and hence P is a prime ideal because S is an integral domain. We have $\dim(S) = \dim(R/P) \leq \dim(R) - \mathrm{ht}(P) \leq \dim(R)$. Since $\dim(R) = \dim(S)$ by assumption, we deduce $\mathrm{ht}(P) = 0$. But since R is an integral domain, this means that $\ker(f) = P = 0$. So f is an isomorphism. □

Type A_n

A maximal torus T in $H_A := S(U(n_1) \times \ldots \times U(n_l))$ is given by all diagonal matrices

$$\operatorname{diag}\left(z_1^{(1)}, \ldots, z_{n_1}^{(1)}, z_1^{(2)}, \ldots, z_{n_2}^{(2)}, \ldots, z_{n_l}^{(l)}\right)$$

where $z_j^{(p)} = e^{2\pi i a_j^{(p)}}$ with $\sum_{p=1}^{l} \sum_{j=1}^{n_p} a_j^{(p)} = 0$. Simple roots are given by $a_j^{(p)} - a_{j+1}^{(p)}$ for $1 \leq p \leq l$ and $1 \leq j < n_p$ and so we can deduce that the Weyl group W is $S_{n_1} \times \ldots \times S_{n_l}$, where each symmetric group S_{n_p} acts by permuting the $z_j^{(p)}$ for $1 \leq j \leq n_p$. Thus $R(T)^W$ is generated by the elementary symmetric polynomials $\sigma_k^{(p)} = \sum_{i_1 < \ldots < i_k} z_{i_1}^{(p)} \ldots z_{i_k}^{(p)}$ together with $\left(\sigma_{n_p}^{(p)}\right)^{-1} = \left(\prod_{i=1}^{n_p} z_i^{(p)}\right)^{-1}$ where $1 \leq p \leq l$ and $1 \leq k \leq n_p$. One can check that $\sigma_k^{(p)}$ is precisely the character of the representation $x_k^{(p)} := \Lambda^k\left(x_1^{(p)}\right)$, where $x_1^{(p)}$ is the representation of $S(U(n_1) \times \ldots \times U(n_l))$ where the pth block $U(n_p)$ acts via the standard representation on $\mathbb{C}^{n_p}$.

Proposition 1.23. *There is an isomorphism*

$$R(H_A) \cong \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p}^{(p)}\right] / \left(\prod_{p=1}^{l} x_{n_p}^{(p)} - 1\right)$$

The duality on $R(H_A)$ is given by $\left(x_k^{(p)}\right)^* = \left(x_{n_p}^{(p)}\right)^{-1} x_{n_p-k}^{(p)}$ under this isomorphism, using the convention that $x_0^{(p)} = 1$.

Proof. Letting $S := \bigotimes_{p=1}^{l} \mathbb{Z}\left[\left(z_1^{(p)}\right)^{\pm 1}, \ldots, \left(z_{n_p}^{(p)}\right)^{\pm 1}\right]$ be a Laurent ring, the map

$$f\colon \bigotimes_{p=1}^{l} \mathbb{Z}\left[X_1^{(p)}, \ldots, X_{n_p-1}^{(p)}\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}\left[X_{n_p}^{(p)}, \left(X_{n_p}^{(p)}\right)^{-1}\right] \to S, \quad X_i^{(p)} \mapsto \sigma_i^{(p)}$$

is injective since the elementary symmetric polynomials are algebraically independent. Now $R(T) = S/\left(\prod_{p=1}^{l} \prod_{i=1}^{n_p} z_i^{(p)} - 1\right)$. Denoting by $\pi\colon S \to R(T)$ the projection map, by the comments above, $\pi \circ f$ maps surjectively onto $R(T)^W$. We have $\ker(\pi \circ f) = f^{-1}\left(\left(\prod_{p=1}^{l}\prod_{i=1}^{n_p} z_i^{(p)} - 1\right)\right) = \left(\prod_{p=1}^{l} X_{n_p}^{(p)} - 1\right)$. The second equality here follows from the fact that f is just the embedding of the subring of S of all elements invariant under the action of $S_{n_1} \times \ldots \times S_{n_l}$. This yields the isomorphism. The fact that the duality is as described follows immediately from considering the duals of the $\sigma_i^{(p)} \in R(T)^W$. $\square$

Type B_n

Let $H_B := \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l) \times Spin(2m+1)$. By Theorem 1.17 and Proposition 1.20, we have

$$R(H_B) \cong \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm\frac{1}{2}}\right] \text{ if } m = 0$$

$$R(H_B) \cong \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm\frac{1}{2}}\right] \otimes \mathbb{Z}[y_1, \ldots, y_{m-1}, \Delta] \text{ if } m > 0$$

By ϵ we denote the nontrivial element in $\tilde{U}(n_p)$ or $Spin(2m+1)$ in the kernel of $\tilde{U}(n_p) \to U(n_p)$ or $Spin(2m+1) \to SO(2m+1)$. Let C_B be the subgroup of H_B consisting of all elements $(a_1, \ldots, a_l, b)$ where $a_i, b \in \{1, \epsilon\}$ such that ϵ occurs an even number of times.

Proposition 1.24. *If $m > 0$, there is an isomorphism*

$$\bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm 1}\right] \otimes \mathbb{Z}[y_1, \ldots, y_{m-1}] \otimes \mathbb{Z}[t] \to R(H_B/C_B)$$

where $x_i^{(p)} \mapsto x_i^{(p)}$ for all $1 \leq p \leq l$ and $1 \leq i \leq n_p$, $y_j \mapsto y_j$ for $1 \leq j \leq m-1$ and $t \mapsto \Delta \cdot \prod_{p=1}^{l}\left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}}$.

If $m = 0$, there is an isomorphism

$$\frac{\bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm 1}\right] \otimes \mathbb{Z}[t]}{\left(t^2 - \left(\prod_p x_{n_p}^{(p)}\right)^{-1}\right)} \to R(H_B/C_B)$$

where $x_i^{(p)} \mapsto x_i^{(p)}$ for all $1 \leq p \leq l$ and $1 \leq i \leq n_p$ and $t \mapsto \prod_{p=1}^{l}\left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}}$.

Under the above isomorphisms, the duality is given by

$$\begin{aligned} \left(x_i^{(p)}\right)^* &= \left(x_{n_p}^{(p)}\right)^{-1} x_{n_p-i}^{(p)} && \textit{for all } 1 \leq p \leq l,\ 1 \leq i \leq n_p \\ y_j^* &= y_j && \textit{for all } 1 \leq j \leq m-1, \\ t^* &= t \cdot \prod_p x_{n_p}^{(p)} \end{aligned}$$

Remark. By Lemma 1.18, the ring $R(H_B/C_B)$ is a subring of $R(H_B)$. This justifies our notation, for example regarding $x_i^{(p)}$ as a representation of both H_B and H_B/C_B.

Proof of Proposition 1.24. Suppose $m > 0$. Observe that $C_B \subset H_B$ acts trivially via $x_i^{(p)} \in R(H_B)$ and $y_j \in R(H_B)$ for all $1 \le p \le l$ and $1 \le i \le n_p$ and for all $1 \le j < m$. Furthermore, C_B acts trivially via $\Delta^b \cdot \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{\frac{1}{2}a_p} \in R(H_B)$ if and only if all of a_p, b are even or all of a_p, b are odd, and otherwise some elements of C_B act trivially and the others act by multiplication by -1.

This shows by Lemma 1.19 (taking all the monomials as a $\mathbb{Z}$-basis of $R(H_B)$) that the described map is surjective. The ring on the left is an integral domain of Krull dimension $1 + m + \sum_p n_p$. By Lemma 1.18, $R(H_B/C_B)$ is an integral domain with the same Krull dimension as $R(H_B)$, i.e. also $1 + m + \sum_p n_p$. So by Lemma 1.22, the map must be an isomorphism.

Similar arguments apply for $m = 0$. □

Type C_n

Let $H_C := U(n_1) \times \ldots \times U(n_l) \times Sp(m)$. From Theorem 1.17, we immediately obtain:

Proposition 1.25. *There is an isomorphism*

$$\bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm 1}\right] \otimes \mathbb{Z}[y_1, \ldots, y_m] \to R(H_C)$$

where $x_1^{(p)}$ corresponds to the standard action of the pth block $U(n_p)$ and $x_i^{(p)} = \Lambda^i\left(x_1^{(p)}\right)$, and y_1 corresponds to the standard representation of the block $Sp(m)$ on $\mathbb{C}^{2m}$ and $y_j = \Lambda^j(y_1)$. Under this isomorphism, the duality on $R(H_C)$ is given by $\left(x_i^{(p)}\right)^* = \left(x_{n_p}^{(p)}\right)^{-1} x_{n_p-i}^{(p)}$ and $y_j^* = y_j$.

Type D_n

Let $H_D := \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l) \times Spin(2m)$. By Theorem 1.17 and Proposition 1.20, we have

$$R(H_D) \cong \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm\frac{1}{2}}\right] \text{ if } m = 0$$

$$R(H_D) \cong \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm\frac{1}{2}}\right] \otimes \mathbb{Z}[y_1, \ldots, y_{m-2}, \Gamma_+, \Gamma_-] \text{ if } m \ge 2$$

The case $m = 1$ need not be considered separately. It is already contained in the case $m = 0$ since $Spin(2) \cong \tilde{U}(1)$.

As for type B_n, we let ϵ denote the nontrivial element in $\tilde{U}(n_p)$ or $Spin(2m)$ which is in the kernel of $\tilde{U}(n_p) \to U(n_p)$ or $Spin(2m) \to SO(2m)$. Let C_D be the subgroup of H_D consisting of all elements $(a_1, \ldots, a_l, b)$ where $a_i, b \in \{1, \epsilon\}$ such that ϵ occurs an even number of times.

Proposition 1.26. *If $m \geq 2$, there is an isomorphism*

$$\bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm 1}\right] \otimes \mathbb{Z}[y_1, \ldots, y_{m-2}] \otimes \mathbb{Z}[d_+, d_-] \to R(H_D/C_D)$$

where $x_i^{(p)} \mapsto x_i^{(p)}$ and $y_j \mapsto y_j$ and $d_+ \mapsto \Gamma_+ \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{\frac{1}{2}}$ and $d_- \mapsto \Gamma_- \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}}$.

If $m = 0$, there is an isomorphism

$$\frac{\bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm 1}\right] \otimes \mathbb{Z}[t]}{\left(t^2 - \prod_p x_{n_p}^{(p)}\right)} \to R(H_D/C_D)$$

where $x_i^{(p)} \mapsto x_i^{(p)}$ for all $1 \leq p \leq l$ and $1 \leq i \leq n_p$ and $t \mapsto \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{\frac{1}{2}}$.

Under these isomorphisms, the dualities are determined by

$$\left(x_i^{(p)}\right)^* = \left(x_{n_p}^{(p)}\right)^{-1} x_{n_p-i}^{(p)} \quad \textit{and} \quad y_j^* = y_j \quad \textit{and} \quad t^* = t^{-1} = t \cdot \left(\prod_p x_{n_p}^{(p)}\right)^{-1}$$

$$d_+^* = \begin{cases} d_- & \textit{if } m \textit{ is odd} \\ d_+ \cdot \prod_p \left(x_{n_p}^{(p)}\right)^{-1} & \textit{if } m \textit{ is even} \end{cases}, \quad d_-^* = \begin{cases} d_+ & \textit{if } m \textit{ is odd} \\ d_- \cdot \prod_p x_{n_p}^{(p)} & \textit{if } m \textit{ is even} \end{cases}$$

Remark. By Lemma 1.18, the ring $R(H_D/C_D)$ is a subring of $R(H_D)$. This justifies our notation, for example regarding $x_i^{(p)}$ as a representation of both H_D and H_D/C_D.

Proof. Suppose $m \geq 2$. We know that ϵ acts trivially via all $x_i^{(p)} \in R(H_D)$ and all $y_j \in R(H_D)$, but it acts by multiplication by -1 via Γ_+, Γ_- and $\left(x_{n_p}^{(p)}\right)^{\pm\frac{1}{2}}$ for $1 \leq p \leq l$. Hence C_D acts trivially via all $x_i^{(p)} \in R(H_D)$ and all $y_j \in R(H_D)$, and it acts trivially via

$$\Gamma_+^{a_-} \Gamma_-^{a_+} \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{\pm\frac{1}{2}b_p} \tag{1.2}$$

if and only if $a_+ + a_-$ and all of b_p are even or $a_+ + a_-$ and all of b_p are odd. Thus C_D acts trivially via (1.2) if and only if it can be written as a product of powers of

$$\Gamma_+ \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{\frac{1}{2}} \quad \text{and} \quad \Gamma_- \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} \quad \text{and} \quad x_{n_p}^{(p)} \text{ for } 1 \leq p \leq l$$

All this shows by Lemma 1.19 (taking the monomials as a $\mathbb{Z}$-basis of $R(H_D)$) that the described map is surjective. Now observe that both $R(H_D/C_D)$ and the domain of the map are integral domains with Krull dimension $1 + m + \sum_p n_p$ (cf. Lemma 1.18). Thus we deduce from Lemma 1.22 that the map is an isomorphism.

Similar arguments apply for $m = 0$. $\square$

Chapter 2

Tate Cohomology

In the previous chapter, we defined the Witt ring and showed that it determines the torsion in the real topological K-theory of a complex flag variety. We shall now begin to see how to compute Witt rings. The decisive result is Bousfield's lemma, which shows that Witt rings of complex flag varieties are isomorphic to Tate cohomology rings of their complex topological K-theory. This makes the Witt ring computable for us in many cases, as we shall see in subsequent chapters.

This chapter is organised as follows. In the first section, we define Tate cohomology and formulate Bousfield's lemma. We largely follow [Zib15] here. In the second section, we develop the main technical results to compute Tate cohomology. These will be used in our concrete computations. It may be advisable for the reader to skim through this section at first and refer back to it later as required. In the third section, we deal with Tate cohomology of representation rings. In particular, we compute the Tate cohomology of the representation rings of the Lie groups in Proposition 1.12, i.e. of centralisers of tori. This will be important for our computations later.

2.1 Witt rings are Tate cohomology rings

A $*$*-ring* A is a commutative unital ring with an involution $*\colon A \to A$ which is a ring isomorphism. A $*$*-ideal* of a $*$-ring is an ideal which is closed under the involution. A $*$*-module* is an abelian group with an additive involution. A $*$*-module* M *over a* $*$*-ring* A is an A-module M which is also a $*$-module, such that the involutions on A and M are compatible with the module structure, i.e. $(a \cdot m)^* = a^* \cdot m^*$ for all $a \in A$ and $m \in M$. For example, any $*$-ideal of A is a $*$-module over A. Every $*$-module is a $*$-module over $\mathbb{Z}$ with the trivial involution on $\mathbb{Z}$. A *morphism of* $*$*-modules* is a homomorphism of modules that preserves the involution and a *morphism of* $*$*-rings* is a ring homomorphism that preserves the involution. For a $*$-module M, an element $m \in M$ is *self-dual* if $m^* = m$ and *anti-self-dual* if $m^* = -m$.

Example 2.1. For any compact Hausdorff space X, its complex topological K-theory

$K^0(X)$ is a $*$-ring with the duality induced by assigning to each complex vector bundle its dual bundle.

For any compact connected Lie group G, its complex representation ring $R(G)$ is a $*$-ring with the duality induced by assigning to each complex representation its dual representation.

For a $*$-module M, we define its *Tate cohomology* to be $h^*(M) = h^+(M) \oplus h^-(M)$ where

$$h^+(M) := \frac{\ker(\mathrm{id} - *)}{\mathrm{im}(\mathrm{id} + *)}$$
$$h^-(M) := \frac{\ker(\mathrm{id} + *)}{\mathrm{im}(\mathrm{id} - *)}$$

In other words, $h^+(M)$ and $h^-(M)$ consist of the self-dual and anti-self-dual elements of M respectively modulo those elements that are self-dual and anti-self-dual respectively for trivial reasons.

If A is a $*$-ring, it is easily checked that the multiplication on A induces a ring structure on $h^*(A)$ with

$$h^+(A) \cdot h^+(A) \subset h^+(A)$$
$$h^+(A) \cdot h^-(A) \subset h^-(A)$$
$$h^-(A) \cdot h^-(A) \subset h^+(A)$$

If M is a $*$-module over the $*$-ring A, then $h^*(M)$ becomes a module over $h^*(A)$ with

$$h^+(A) \cdot h^+(M) \subset h^+(M)$$
$$h^+(A) \cdot h^-(M) \subset h^-(M)$$
$$h^-(A) \cdot h^+(M) \subset h^-(M)$$
$$h^-(A) \cdot h^-(M) \subset h^+(M)$$

Example 2.2. Let T be a torus of dimension n, then

$$R(T) \cong \mathbb{Z}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$$

with $x_i^* = x_i^{-1}$ for all i. The Tate cohomology of $R(T)$ is trivial:

$$h^*(R(T)) \cong \mathbb{Z}_2 \cdot \overline{1}$$

Let G be a compact simply connected Lie group, then

$$R(G) \cong \mathbb{Z}\left[\rho_1, \rho_1^*, \ldots, \rho_k, \rho_k^*, \mu_1, \ldots, \mu_l, \nu_1, \ldots, \nu_m\right]$$

where the μ_i are the real fundamental representations (hence self-dual) and the ν_j are the quaternionic fundamental representations (hence also self-dual). Then we have a ring isomorphism

$$\mathbb{Z}_2\left[a_1, \ldots, a_k, b_1, \ldots, b_l, c_1, \ldots, c_m\right] \to h^*(R(G))$$

where $a_i \mapsto [\rho_i \rho_i^*]$, $b_i \mapsto [\mu_i]$ and $c_i \mapsto [\nu_i]$.

We now prepare to formulate Bousfield's lemma. Let X be a compact Hausdorff space. Recall from (1.1) the complexification and realification maps $c_i \colon KO^{2i}(X) \to K^0(X)$ and $r_i \colon K^0(X) \to KO^{2i}(X)$. Since $*c_i = (-1)^i c_i$ and $c_i r_i = \mathrm{id} + (-1)^i *$, the map c_i descends to a map

$$\bar{c}_i \colon KO^{2i}(X)/r_i \to h^i(K^0(X))$$

where $KO^{2i}(X)/r_i$ denotes the cokernel of r_i. Since $c_i r_i = \mathrm{id} + (-1)^i *$ and $r_i * = (-1)^i r_i$, the map r_i descends to a map

$$\bar{r}_i \colon h^i(K^0(X)) \to c_i \backslash KO^{2i}(X)$$

where $c_i \backslash KO^{2i}(X)$ denotes the kernel of c_i.

Lemma 2.3 (Bousfield's lemma)**.** *Let X be a compact Hausdorff space with $K^1(X) = 0$. The complexification and realification maps induce isomorphisms*

$$W^0(X) \oplus W^2(X) \xrightarrow[(\bar{c}_0 \ \bar{c}_2)]{\cong} h^+(K^0(X)) \xrightarrow[\left(\begin{smallmatrix}\bar{r}_{-1}\\ \bar{r}_1\end{smallmatrix}\right)]{\cong} c\backslash KO^{-2}(X) \oplus c\backslash KO^2(X)$$
$$W^1(X) \oplus W^3(X) \xrightarrow[(\bar{c}_1 \ \bar{c}_3)]{\cong} h^-(K^0(X)) \xrightarrow[\left(\begin{smallmatrix}\bar{r}_0\\ \bar{r}_2\end{smallmatrix}\right)]{\cong} c\backslash KO^0(X) \oplus c\backslash KO^4(X)$$

The composition along each row is given by $\left(\begin{smallmatrix}\eta^2 & 0\\ 0 & \eta^2\end{smallmatrix}\right)$. Since complexification is a ring homomorphism, we get an isomorphism of rings $\bar{c} \colon W^*(X) \to h^*(K^0(X))$.

Proof. A simple proof can be found in [Zib15, §1.2]. □

Note that for any complex flag variety X, we have $K^1(X) = 0$ by Theorem 1.1. So Bousfield's lemma applies in this case.

2.2 Properties of Tate cohomology

We have now reduced the computation of Witt rings to the computation of Tate cohomology rings. So let us do the necessary groundwork by proving some basic general results about Tate cohomology. In particular, we are interested in computing the Tate cohomology of quotients of $*$-rings by $*$-ideals.

In this section, we need the notion of a *regular sequence* in a ring. It is important to deal with regular sequences in greater detail later. For now, we only require the definition, for which we refer to Definition 3.1.

The most important computational tool is the long exact sequence in Tate cohomology. It follows immediately from the snake lemma of homological algebra.

Lemma 2.4. *A short exact sequence of $*$-modules*

$$0 \to M' \to M \to M'' \to 0$$

induces a 6-periodic long exact sequence on Tate cohomology groups:

$$\begin{tikzcd} h^+(M') \arrow[r] & h^+(M) \arrow[r] & h^+(M'') \arrow[d] \\ h^-(M'') \arrow[u] & h^-(M) \arrow[l] & h^-(M') \arrow[l] \end{tikzcd}$$

We are first interested in relating the Tate cohomology of $*$-ideals to the Tate cohomology of the ambient ring.

Lemma 2.5 ([Zib15, Prop. 4.1])**.** *Let A be a $*$-ring.*

(i) If $\mu \in A$ is self-dual and not a zero divisor, then

$$h^*(A) \xrightarrow{\cdot[\mu]} h^*(\mu A)$$

is a graded isomorphism.

(ii) If λ, λ^* is a regular sequence in A, then

$$h^*(A) \xrightarrow{\cdot[\lambda\lambda^*]} h^*(\lambda, \lambda^*)$$

is a graded isomorphism.

We will also need the following lemma about the Tate cohomology of an ideal generated by two independent self-dual elements:

Lemma 2.6. *Let A be a $*$-ring with $h^-(A) = 0$ and let $\mu_1, \mu_2 \in A$ be self-dual elements that form an A-regular sequence. Suppose that $Ann_{h^+(A)}([\mu_1], [\mu_2]) = 0$. Then*

$$h^-(A\mu_1 + A\mu_2) = 0$$

and we have an isomorphism of $h^+(A)$-modules given by

$$\frac{h^+(A) \oplus h^+(A)}{h^+(A) \cdot ([\mu_2], [\mu_1])} \xrightarrow{\cong} h^+(A\mu_1 + A\mu_2), \quad \overline{(x, y)} \mapsto x[\mu_1] + y[\mu_2]$$

Proof. We define an A-module homomorphism by

$$\varphi\colon A \oplus A \to A\mu_1 + A\mu_2, \quad (a,b) \mapsto a\mu_1 + b\mu_2$$

φ is clearly onto. Let $(a,b) \in \ker(\varphi)$, i.e. $a\mu_1 + b\mu_2 = 0$. By regularity of the sequence μ_1, μ_2, we deduce that $b = b'\mu_1$ for some $b' \in A$, so $(a + b'\mu_2)\mu_1 = 0$. Again by regularity, we have $a + b'\mu_2 = 0$ and so

$$\ker(\varphi) = \{(x\mu_2, -x\mu_1) \mid x \in A\},$$

which is isomorphic to A as a $*$-module via the isomorphism $\kappa\colon A \to \ker(\varphi)$, $x \mapsto (x\mu_2, -x\mu_1)$. Now the short exact sequence of $*$-modules over A

$$0 \to A \xrightarrow{\kappa} A \oplus A \xrightarrow{\varphi} A\mu_1 + A\mu_2 \to 0$$

induces the following exact sequence on Tate cohomology, using that $h^-(A) = 0$:

$$0 \to h^-(A\mu_1 + A\mu_2) \to h^+(A) \xrightarrow{\kappa_*} h^+(A) \oplus h^+(A) \xrightarrow{\varphi_*} h^+(A\mu_1 + A\mu_2) \to 0$$

The assertions follow from this exact sequence since

$$\kappa_*\colon h^+(A) \to h^+(A) \oplus h^+(A), \quad x \mapsto (x[\mu_2], x[\mu_1])$$

is injective as $\mathrm{Ann}_{h^+(A)}([\mu_1],[\mu_2]) = 0$ by assumption. $\square$

We next consider the Tate cohomology of some quotient rings. The following will help us to reduce to considering only one self-dual element instead of two mutually dual ones.

Lemma 2.7. *Let A be a $*$-ring and $\lambda \in A$ such that λ, λ^* is a regular sequence and $\lambda\lambda^*$ is not a zero divisor. Then the natural map*

$$q\colon A/(\lambda\lambda^*) \to A/(\lambda, \lambda^*)$$

induces an isomorphism on Tate cohomology.

Proof. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & (\lambda\lambda^*) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & A/(\lambda\lambda^*) \arrow[r] \arrow[d, "q"] & 0 \\ 0 \arrow[r] & (\lambda, \lambda^*) \arrow[r] & A \arrow[r] & A/(\lambda, \lambda^*) \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

where the rows are exact. Since $\lambda\lambda^*$ is not a zero divisor and λ, λ^* is a regular sequence, Lemma 2.5 implies that the vertical maps in the following commutative diagram are isomorphisms:

$$\begin{tikzcd} h^*((\lambda\lambda^*)) \arrow[r] & h^*((\lambda,\lambda^*)) \\ h^*(A) \arrow[u, "\cdot[\lambda\lambda^*]"] \arrow[r, equal] & h^*(A) \arrow[u, "\cdot[\lambda\lambda^*]"'] \end{tikzcd}$$

Hence the homomorphism $h^*((\lambda\lambda^*)) \to h^*((\lambda,\lambda^*))$ induced by inclusion is also an isomorphism.

Now consider the long exact sequences induced by diagram (2.1). The assertion follows immediately from the five lemma. □

Remark. If λ, λ^* is a regular sequence in A, it is easily verified that $\lambda\lambda^*$ is not a zero divisor if and only if λ^*, λ is also regular.

Now we show how Tate cohomology of some quotients can be computed.

Lemma 2.8. *Let A be a $*$-ring and $\mu, \lambda \in A$ such that μ is self-dual and not a zero divisor and λ, λ^* is a regular sequence.*

(i) If $[\mu] = 0$ in $h^+(A)$, then

$$h^*(A/(\mu)) \cong h^*(A) \oplus [\overline{u}] \cdot h^*(A)$$

where $u \in A$ with $u + u^* = \mu$.

(ii) If $[\lambda\lambda^*] = 0$ in $h^+(A)$, then

$$h^*(A/(\lambda,\lambda^*)) \cong h^*(A) \oplus [\overline{u}] \cdot h^*(A)$$

where $u \in A$ with $u + u^* = \lambda\lambda^*$.

(iii) Suppose further that $h^-(A) = 0$. Then $h^-(A/\mu A) = 0$ if and only if $[\mu] \in h^+(A)$ is not a zero divisor in $h^+(A)$. Furthermore, we have $h^*(A/(\mu)) \cong h^+(A)/([\mu])$.

(iv) Suppose further that $h^-(A) = 0$. Then $h^-(A/(\lambda,\lambda^*)) = 0$ if and only if $[\lambda\lambda^*] \in h^+(A)$ is not a zero divisor in $h^+(A)$. Furthermore, we have $h^*(A/(\lambda,\lambda^*)) \cong h^+(A)/([\lambda\lambda^*])$.

Proof. Parts (i) and (ii) are proved in [Zib15, Prop. 4.1]. We prove (iii), part (iv) is similar.

The short exact sequence of $*$-modules

$$0 \to A \xrightarrow{\cdot\mu} A \to A/(\mu) \to 0$$

induces a long exact sequence on Tate cohomology. Using that $h^-(A) = 0$ by assumption, it can be written as

$$0 \to h^-(A/(\mu)) \to h^+(A) \xrightarrow{\cdot[\mu]} h^+(A) \to h^+(A/(\mu)) \to 0$$

The assertion now follows immediately. □

The following corollary follows immediately by induction from parts (iii) and (iv) of the previous lemma.

Corollary 2.9. *Let A be a $*$-ring such that $h^-(A) = 0$. Let $\mu_1, \ldots, \mu_n \in A$ be self-dual and $\lambda_1, \ldots, \lambda_m \in A$ such that $\mu_1, \ldots, \mu_n, \lambda_1, \lambda_1^*, \ldots, \lambda_m, \lambda_m^*$ is a regular sequence in A. Then the following are equivalent:*

(1) The sequence $[\mu_1], \ldots, [\mu_n], [\lambda_1\lambda_1^*], \ldots, [\lambda_m\lambda_m^*]$ is regular in $h^+(A)$.

(2) $h^-(A/(\mu_1, \ldots, \mu_k)) = 0$ for all $1 \le k \le n$ and $h^-(A/(\mu_1, \ldots, \mu_n, \lambda_1, \lambda_1^*, \ldots, \lambda_j, \lambda_j^*)) = 0$ for all $1 \le j \le m$.

If these equivalent conditions hold, then

$$h^*(A/(\mu_1, \ldots, \mu_n, \lambda_1, \lambda_1^*, \ldots, \lambda_m, \lambda_m^*)) \cong h^+(A)/([\mu_1], \ldots, [\mu_n], [\lambda_1\lambda_1^*], \ldots, [\lambda_m\lambda_m^*])$$

Let us now consider the situation where the element by which we divide is not regular in Tate cohomology, but is a zero divisor with special properties.

Lemma 2.10. *Let A be a $*$-ring with $h^-(A) = 0$ and $\mu \in A$ be a self-dual regular element. Suppose that $Ann_{h^+(A)}([\mu]) = h^+(A) \cdot [\mu]$ so that in particular $[\mu]^2 = 0$. Then*

$$h^*(A/\mu A) \cong \frac{h^+(A)}{([\mu])} \oplus \frac{h^+(A)}{([\mu])} \cdot [\overline{u}]$$

is a free $h^+(A)/([\mu])$-module of rank 2 where $u \in A$ such that $u + u^* = \mu^2$.

Proof. The short exact sequence of A-modules

$$0 \to A \xrightarrow{\cdot\mu} A \to A/\mu A \to 0$$

induces an exact sequence of $h^+(A)$-modules given by

$$0 \to h^-(A/\mu A) \xrightarrow{\partial} h^+(A) \xrightarrow{\cdot[\mu]} h^+(A) \to h^+(A/\mu A) \to 0$$

We immediately obtain $h^+(A/\mu A) \cong h^+(A)/([\mu])$. Now choose $u \in A$ such that $u + u^* = \mu^2$. Then $\overline{u} \in A/\mu A$ defines an element $[\overline{u}] \in h^-(A/\mu A)$ and $\partial([\overline{u}]) = [\mu] \in h^+(A)$ by definition of the boundary map. Now ∂ induces an isomorphism

$$h^-(A/\mu A) \underset{\cong}{\xrightarrow{\partial}} \mathrm{im}(\partial) = \mathrm{Ann}_{h^+(A)}([\mu]) = h^+(A) \cdot [\mu]$$

Since $\mathrm{Ann}_{h^+(A)}([\mu]) = h^+(A) \cdot [\mu]$, the map $h^+(A)/([\mu]) \xrightarrow{\cong} h^+(A) \cdot [\mu]$, $\overline{x} \mapsto x \cdot [\mu]$ is a well-defined isomorphism. Thus as an $h^+(A)/([\mu])$-module, $h^-(A/\mu A)$ is freely generated by $[\overline{u}] \in h^-(A/\mu A)$. □

Finally, we quotient out an ideal generated by two elements which could be zero divisors in Tate cohomology.

Lemma 2.11. *Let A be a $*$-ring with $h^-(A) = 0$ and let $\mu_1, \mu_2 \in A$ be self-dual elements that form an A-regular sequence. Suppose that $Ann_{h^+(A)}([\mu_1], [\mu_2]) = 0$. Then we have isomorphisms of $h^+(A)$-modules*

$$h^+(A/(\mu_1, \mu_2)) \xleftarrow{\cong} h^+(A)/([\mu_1], [\mu_2])$$
$$h^-(A/(\mu_1, \mu_2)) \underset{\cong}{\xrightarrow{\partial}} \frac{ker(\psi)}{h^+(A) \cdot ([\mu_2], [\mu_1])}$$

where $\psi\colon h^+(A) \oplus h^+(A) \to h^+(A),\ (a, b) \mapsto a[\mu_1] + b[\mu_2]$.

Proof. This follows immediately from Lemma 2.6 and the long exact sequence on Tate cohomology induced by $0 \to (\mu_1, \mu_2) \to A \to A/(\mu_1, \mu_2) \to 0$. □

2.3 Tate cohomology of representation rings

After a general lemma about the Tate cohomology of representation rings, we will determine the Tate cohomology of the representation rings of centralisers of tori in compact connected Lie groups. We computed these representation rings in section 1.3.2 for the ordinary types.

Lemma 2.12. *Let G be a compact connected Lie group. Then $h^-(R(G)) = 0$. Furthermore, if $C \subset G$ is a finite central subgroup, then the map $h^+(R(G/C)) \to h^+(R(G))$ induced by the quotient map $G \to G/C$ is an inclusion, and $h^+(R(G))$ is integral over its subring $h^+(R(G/C))$. In particular, they have the same Krull dimension.*

Proof. As a group, $R(G)$ is the free group on isomorphism classes of irreducible representations of G. Any irreducible representation is either self-dual or dual to another irreducible representation. This immediately implies $h^-(R(G)) = 0$.

We saw in Lemma 1.18 that $R(G/C)$ is the subgroup of $R(G)$ generated by all isomorphism classes of irreducible representations via which C acts trivially. As a $\mathbb{Z}_2$-vector space, $h^+(R(G))$ has a basis comprising all irreducible self-dual representations of G, and $h^+(R(G/C))$ has a basis comprising all irreducible self-dual representations of G via which C acts trivially. This shows that $h^+(R(G/C)) \to h^+(R(G))$ is an inclusion. If $\rho \in R(G)$ is an irreducible self-dual representation, then by Schur's Lemma, C acts trivially via ρ^N for some $N \in \mathbb{N}$. Hence $[\rho]^N \in h^+(R(G/C))$ and so $[\rho] \in h^+(R(G))$ is integral over $h^+(R(G/C))$. □

Let us now compute the Tate cohomology of the representation rings that we determined in section 1.3.2.

Type A_n

We use the notation as in Proposition 1.23.

Proposition 2.13. *If not all of $n_1, \ldots, n_l$ are even, we have an isomorphism*

$$\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] \to h^+(R(H_A)) \tag{2.2}$$

where $\beta_i^{(p)} \mapsto \left[x_i^{(p)} \left(x_i^{(p)}\right)^*\right]$.

If all of $n_1, \ldots, n_l$ are even, we have an isomorphism

$$\frac{\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_1^{(p)}, \ldots, \beta_{\frac{n_p}{2}}^{(p)}\right] \otimes \mathbb{Z}_2[\gamma]}{\left(\gamma^2 + \prod_p \beta_{\frac{n_p}{2}}\right)} \to h^+(R(H_A)) \tag{2.3}$$

where $\beta_i^{(p)} \mapsto \left[x_i^{(p)} \left(x_i^{(p)}\right)^*\right]$ and $\gamma \mapsto \left[\prod_{p=1}^{l} x_{\frac{n_p}{2}}^{(p)}\right]$.

Proof. Recall the expression for $R(H_A)$ from Proposition 1.23. The monomials in indeterminates $x_k^{(p)}$ and $\left(x_{n_q}^{(q)}\right)^{\pm 1}$ for $1 \leq p \leq l$ and $1 \leq k < n_p$ and $1 \leq q < l$ (note that we are missing out $x_{n_l}^{(l)}$) form a $\mathbb{Z}$-basis of $R(H_A)$. Since the duality takes monomials to monomials, a $\mathbb{Z}_2$-basis of $h^+(R(H_A))$ is given by the self-dual monomials in these indeterminates. Now suppose

$$z = \prod_{p=1}^{l} \prod_{i=1}^{n_p - 1} \left(x_i^{(p)}\right)^{a_i^{(p)}} \cdot \prod_{q=1}^{l-1} \left(x_{n_q}^{(q)}\right)^{b_q}$$

where $a_i^{(p)} \in \mathbb{N}_0$ and $b_q \in \mathbb{Z}$. Then

$$\begin{aligned} z^* &= \prod_{p=1}^{l-1} \prod_{i=1}^{n_p - 1} \left(x_{n_p}^{(p)}\right)^{-a_i^{(p)}} \left(x_{n_p - i}^{(p)}\right)^{a_i^{(p)}} \cdot \left(x_{n_l}^{(l)}\right)^{-a_1^{(l)} - \ldots - a_{n_l - 1}^{(l)}} \cdot \prod_{j=1}^{n_l - 1} \left(x_{n_l - j}^{(l)}\right)^{a_j^{(l)}} \cdot \prod_{q=1}^{l-1} \left(x_{n_q}^{(q)}\right)^{-b_q} \\ &= \prod_{p=1}^{l-1} \prod_{i=1}^{n_p - 1} \left(x_{n_p}^{(p)}\right)^{-a_i^{(p)}} \left(x_{n_p - i}^{(p)}\right)^{a_i^{(p)}} \cdot \prod_{j=1}^{n_l - 1} \left(x_{n_l - j}^{(l)}\right)^{a_j^{(l)}} \cdot \prod_{q=1}^{l-1} \left(x_{n_q}^{(q)}\right)^{-b_q + a_1^{(l)} + \ldots + a_{n_l - 1}^{(l)}} \end{aligned}$$

recalling that $x_{n_l}^{(l)} = \left(\prod_{p=1}^{l-1} x_{n_p}^{(p)}\right)^{-1}$ in $R(H_A)$.

So $z = z^*$ if and only if the following conditions hold:

$$a_i^{(p)} = a_{n_p - i}^{(p)} \quad \text{and} \quad 2b_q = a_1^{(l)} + \ldots + a_{n_l - 1}^{(l)} - a_1^{(q)} - \ldots - a_{n_q - 1}^{(q)}$$

for all $1 \leq p \leq l$ and $1 \leq i < n_p$ and all $1 \leq q < l$.

If $z = z^*$ and some n_p is odd, then we deduce from the above conditions that $a_{\frac{n_q}{2}}^{(q)}$ must be even for all q where n_q is even. In this case, it follows that z is a monomial in indeterminates $x_i^{(p)}\left(x_i^{(p)}\right)^*$ where $1 \leq p \leq l$ and $1 \leq i \leq \lfloor n_p/2 \rfloor$. This shows that the map (2.2) above maps a $\mathbb{Z}_2$-basis to a $\mathbb{Z}_2$-basis, so it is an isomorphism.

If all of $n_1, \ldots, n_l$ are even, then we deduce that the self-dual monomials are in bijective correspondence with the z of the form

$$z = \left(\prod_{p=1}^{l} x^{(p)}_{\frac{n_p}{2}}\right)^d \cdot \prod_{q=1}^{l}\prod_{j=1}^{n_q/2}\left(x_j^{(q)}\left(x_j^{(q)}\right)^*\right)^{a_j^{(q)}}$$

where $d \in \{0,1\}$ and $a_j^{(p)} \in \mathbb{N}_0$. Thus we see that the map (2.3) maps a $\mathbb{Z}_2$-basis to a $\mathbb{Z}_2$-basis, so it is an isomorphism. $\square$

Type B_n

We use the same notation as in Proposition 1.24. We see immediately that $h^+(R(H_B))$ is a polynomial ring over $\mathbb{Z}_2$ in $m + \sum_p \lfloor n_p/2 \rfloor$ indeterminates, so $h^+(R(H_B))$ has Krull dimension $m + \sum_p \lfloor n_p/2 \rfloor$. So $h^+(R(H_B/C_B))$ has Krull dimension $m + \sum_p \lfloor n_p/2 \rfloor$ by Lemma 2.12.

Proposition 2.14. *If not all of $n_1, \ldots, n_l$ are even, there is an isomorphism*

$$\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta^{(p)}_{\lfloor n_p/2 \rfloor}\right] \otimes \mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \to h^+(R(H_B/C_B)) \tag{2.4}$$

where $\beta_i^{(p)} \mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right]$ *and*

$$\alpha_j \mapsto \begin{cases} [y_j] \text{ if } 1 \leq j \leq m-1 \\ [\Lambda^m(y_1)] = [tt^* - 1 - y_1 - \ldots - y_{m-1}] \text{ if } j = m \end{cases}$$

If all of $n_1, \ldots, n_l$ are even, there is an isomorphism

$$\frac{\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta^{(p)}_{n_p/2}\right] \otimes \mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \mathbb{Z}_2[\delta]}{\left(\delta^2 + (1 + \alpha_1 + \ldots + \alpha_m)\beta^{(1)}_{n_1/2} \cdots \beta^{(l)}_{n_l/2}\right)} \to h^+(R(H_B/C_B)) \tag{2.5}$$

where $\beta_i^{(p)}$ and α_j are mapped as in the previous homomorphism and

$$\delta \mapsto \begin{cases} \left[\Delta \prod_p \left(x^{(p)}_{n_p}\right)^{-\frac{1}{2}} x^{(p)}_{\frac{n_p}{2}}\right] = \left[t \prod_p x^{(p)}_{\frac{n_p}{2}}\right] \text{ if } m > 0 \\ \left[\prod_p \left(x^{(p)}_{n_p}\right)^{-\frac{1}{2}} x^{(p)}_{\frac{n_p}{2}}\right] = \left[t \prod_p x^{(p)}_{\frac{n_p}{2}}\right] \text{ if } m = 0 \end{cases}$$

Proof. Suppose $m > 0$. In Proposition 1.24, we saw that $R(H_B/C_B)$ is a tensor product of a Laurent and polynomial ring. The duality takes monomials to monomials, so the self-dual monomials are a $\mathbb{Z}_2$-basis of $h^+(R(H_B/C_B))$. Suppose

$$z = t^a \prod_{p=1}^{l} \prod_{i=1}^{n_p-1} \left(x_i^{(p)}\right)^{b_i^{(p)}} \cdot \prod_{q=1}^{l} \left(x_{n_q}^{(q)}\right)^{c_q} \cdot \prod_{q=1}^{m-1} y_q^{d_q}$$

is a self-dual monomial where $a, b_i^{(p)}, d_q \in \mathbb{N}_0$ and $c_p \in \mathbb{Z}$. Then

$$z = z^* = t^a \prod_{r=1}^{l} \left(x_{n_r}^{(r)}\right)^a \prod_{p=1}^{l} \prod_{i=1}^{n_p-1} \left(x_{n_p}^{(p)}\right)^{-b_i^{(p)}} \left(x_{n_p-i}^{(p)}\right)^{b_i^{(p)}} \cdot \prod_{q=1}^{l} \left(x_{n_q}^{(q)}\right)^{-c_q} \cdot \prod_{q=1}^{m-1} y_q^{d_q}$$

This equality holds if and only if

$$2c_p = a - b_1^{(p)} - \ldots - b_{n_p-1}^{(p)} \quad \text{and} \quad b_i^{(p)} = b_{n_p-i}^{(p)}$$

for all $1 \leq p \leq l$ and $1 \leq i < n_p$.

Suppose there is some $1 \leq p \leq l$ so that n_p is odd. Then $\sum_{k=1}^{n_p-1} b_k^{(p)} = 2\sum_{k=1}^{(n_p-1)/2} b_k^{(p)}$ is even and so a is even. Then we must also have that $b_{n_r/2}^{(r)}$ is even when n_r is even. This shows that z can in fact be written as a product of powers of

$$tt^* \quad \text{and} \quad x_j^{(p)} \left(x_j^{(p)}\right)^* \quad \text{and} \quad y_q$$

for $1 \leq p \leq l$ and $1 \leq j \leq \lfloor n_p/2 \rfloor$ and $1 \leq q < m$. This proves that the above map (2.4) is surjective, noting that under the isomorphism of Proposition 1.24, tt^* corresponds to $\Delta^2 = 1 + y_1 + \ldots + y_m$.

Now suppose that $n_1, \ldots, n_l$ are all even. If we choose a odd, we must also choose $b_{n_p/2}^{(p)}$ odd for all $1 \leq p \leq l$. We deduce that z can be written as a product of powers of

$$tt^* \quad \text{and} \quad t \prod_{p=1}^{l} x_{\frac{n_p}{2}}^{(p)} \quad \text{and} \quad x_j^{(p)} \left(x_j^{(p)}\right)^* \quad \text{and} \quad y_q$$

for $1 \leq p \leq l$ and $1 \leq i < n_p$ and $1 \leq q < m$. This proves that the above map (2.5) is surjective, again noting that under the isomorphism of Proposition 1.24, tt^* corresponds to $\Delta^2 = 1 + y_1 + \ldots + y_m$.

Note that in both cases, both the domain and codomain of the surjective maps are integral domains with the same Krull dimension $m + \sum_{i=1}^{l} n_i$. Hence the maps must in fact be isomorphisms.

Similar arguments apply for $m = 0$. □

Type C_n

We use the notation of Proposition 1.25. We immediately obtain:

Proposition 2.15. *We have an isomorphism*

$$\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] \otimes \mathbb{Z}_2\left[\alpha_1, \ldots, \alpha_m\right] \to h^+(R(H_C))$$

where $\beta_i^{(p)} \mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right]$ *and* $\alpha_j \mapsto [y_j]$.

Type D_n

Recall the notation from Proposition 1.26.

Proposition 2.16. *If* $m > 2$ *is odd, we have an isomorphism*

$$\mathbb{Z}_2[\alpha_1, \ldots, \alpha_{m-1}] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] \to h^+(R(H_D/C_D))$$

where

$$\begin{aligned} \alpha_j &\mapsto \begin{cases} [y_j] \text{ if } 1 \leq j < m-1 \\ \left[\Lambda^{m-1}(y_1)\right] = [d_+d_- - y_{m-3} - y_{m-5} - \ldots] \text{ if } j = m-1 \end{cases} \\ \text{and} \quad \beta_i^{(p)} &\mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right] \end{aligned}$$

If $m \geq 2$ *is even and not all of* $n_1, \ldots, n_l$ *are even, we have an isomorphism*

$$\frac{\mathbb{Z}_2[\alpha_1, \ldots, \alpha_{m-2}, \delta_1, \delta_2, \delta]}{(\delta_1\delta_2 + \delta^2)} \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] \to h^+(R(H_D/C_D)) \tag{2.6}$$

where α_j *and* $\beta_i^{(p)}$ *are mapped as before and* $\delta_1 \mapsto \left[d_+d_+^*\right]$ *and* $\delta_2 \mapsto \left[d_-d_-^*\right]$ *and* $\delta \mapsto [d_+d_-]$.

If $m \geq 2$ *is even and all of* $n_1, \ldots, n_l$ *are even, we have an isomorphism*

$$\frac{\mathbb{Z}_2\left[\alpha_1, \ldots, \alpha_{m-2}, \delta_1, \delta_2, \delta, \zeta_1, \zeta_2\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\frac{n_p}{2}}^{(p)}\right]}{I} \to h^+(R(H_D/C_D)) \tag{2.7}$$

where

$$I = \left(\delta_1\delta_2 + \delta^2,\ \zeta_1^2 + \delta_1 Y,\ \zeta_2^2 + \delta_2 Y,\ \zeta_1\zeta_2 + \delta Y,\ \delta\zeta_1 + \zeta_2\delta_1,\ \delta\zeta_2 + \zeta_1\delta_2\right)$$

with $Y := \beta_{\frac{n_1}{2}}^{(1)} \ldots \beta_{\frac{n_l}{2}}^{(l)}$, *mapping* α_j *and* $\beta_i^{(p)}$ *as before and*

$$\delta \mapsto [\Gamma_+\Gamma_-] = [d_+d_-], \quad \delta_1 \mapsto \left[\Gamma_-^2\right] = \left[d_-^2 \prod_q x_{n_q}^{(q)}\right], \quad \delta_2 \mapsto \left[\Gamma_+^2\right] = \left[d_+^2 \prod_q \left(x_{n_q}^{(q)}\right)^{-1}\right],$$

$$\zeta_1 \mapsto \left[\Gamma_- \prod_q \left(x^{(q)}_{n_q}\right)^{-\frac{1}{2}} x^{(q)}_{\frac{n_q}{2}}\right] = \left[d_- \prod_q x^{(q)}_{\frac{n_q}{2}}\right],$$

$$\zeta_2 \mapsto \left[\Gamma_+ \prod_q \left(x^{(q)}_{n_q}\right)^{-\frac{1}{2}} x^{(q)}_{\frac{n_q}{2}}\right] = \left[d_+ \prod_q \left(x^{(q)}_{n_q}\right)^{-1} x^{(q)}_{\frac{n_q}{2}}\right].$$

If $m = 0$ and not all of the $n_1, \ldots, n_l$ are even, we have an isomorphism

$$\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta^{(p)}_1, \ldots, \beta_{\lfloor n_p/2 \rfloor}\right] \to h^+(R(H_D/C_D)) \tag{2.8}$$

where $\beta^{(p)}_i \mapsto \left[x^{(p)}_i \left(x^{(p)}_i\right)^*\right]$.

If $m = 0$ and all of the $n_1, \ldots, n_l$ are even, we have an isomorphism

$$\frac{\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta^{(p)}_1, \ldots, \beta_{\frac{n_p}{2}}\right] \otimes \mathbb{Z}_2[\gamma]}{\left(\gamma^2 + \prod_q \beta^{(q)}_{\frac{n_q}{2}}\right)} \to h^+(R(H_D/C_D)) \tag{2.9}$$

where $\beta^{(p)}_i$ is mapped as before and $\gamma \mapsto \left[\prod_q \left(x^{(q)}_{n_q}\right)^{-\frac{1}{2}} x^{(q)}_{\frac{n_q}{2}}\right] = \left[t^{-1} \prod_q x^{(q)}_{\frac{n_q}{2}}\right]$.

Proof. Suppose $m > 2$ is odd. Then $d_+^* = d_-$ and from Proposition 1.26, we see that $R(H_D/C_D)$ is a tensor product of $*$-rings, the Tate cohomology of each of which we know. Noting that $d_- d_+ = \Gamma_+ \Gamma_- = y_{m-1} + y_{m-3} + \ldots$, the result follows.

Suppose $m \geq 2$ is even. The self-dual monomials in $R(H_D/C_D)$ form a $\mathbb{Z}_2$-basis of the Tate cohomology. So suppose that

$$z = d_+^{a_+} d_-^{a_-} \cdot \prod_{i=1}^{m-2} y_i^{b_i} \cdot \prod_{p=1}^{l} \prod_{j=1}^{n_p - 1} \left(x^{(p)}_j\right)^{c^{(p)}_j} \cdot \prod_{q=1}^{l} \left(x^{(q)}_{n_q}\right)^{f_q}$$

where $a_+, a_-, b_i, c^{(p)}_j \in \mathbb{N}_0$ and $f_q \in \mathbb{Z}$. Then

$$z^* = d_+^{a_+} d_-^{a_-} \cdot \prod_{i=1}^{m-2} y_i^{b_i} \cdot \prod_{q=1}^{l} \left(x^{(q)}_{n_q}\right)^{-f_q - a_+ + a_- - c^{(q)}_1 - \ldots - c^{(q)}_{n_q - 1}} \cdot \prod_{p=1}^{l} \prod_{j=1}^{n_p - 1} \left(x^{(p)}_{n_p - j}\right)^{c^{(p)}_j}$$

So $z = z^*$ if and only if

$$c^{(p)}_j = c^{(p)}_{n_p - j} \quad \text{and} \quad 2f_q = a_- - a_+ - c^{(q)}_1 - \ldots - c^{(q)}_{n_q - 1} \tag{2.10}$$

for all $1 \leq p, q \leq l$ and $1 \leq j < n_p$.

Suppose in addition that not all of the $n_1, \ldots, n_l$ are even. Then we deduce from the above that if $z = z^*$, we must have that $a_- - a_+$ is even. Thus if q is such that n_q is even, it follows that also $c^{(q)}_{n_q/2}$ is even. All this implies that z is a product of powers of

$$x_i^{(p)}\left(x_i^{(p)}\right)^* \quad \text{and} \quad d_+^2 \cdot \prod_{q=1}^{l}\left(x_{n_q}^{(q)}\right)^{-1} \quad \text{and} \quad d_-^2 \cdot \prod_{q=1}^{l} x_{n_q}^{(q)} \quad \text{and} \quad d_+d_-$$

This shows that the map (2.6) is surjective. We know by Lemma 2.12 that $h^+(R(H_D/C_D))$ is an integral domain of the same Krull dimension as $h^+(R(H_D))$. From the description of $R(H_D)$ before Proposition 1.26, we see that if m is even, then $h^+(R(H_D))$ is a polynomial algebra over $\mathbb{Z}_2$ in $m+\sum_p \lfloor n_p/2 \rfloor$ indeterminates. So its Krull dimension is $m+\sum_p \lfloor n_p/2 \rfloor$. Observe that the ring in the domain of (2.6) is also an integral domain of the same Krull dimension. Hence the surjection must in fact be an isomorphism by Lemma 1.22.

On the other hand, still assuming $m \geq 2$ is even, suppose in addition that all of $n_1, \ldots, n_l$ are even. If $z = z^*$, we deduce from (2.10) that z is a product of powers of

$$x_i^{(p)}\left(x_i^{(p)}\right)^* \quad \text{and} \quad d_+^2 \cdot \prod_{q=1}^{l}\left(x_{n_q}^{(q)}\right)^{-1} \quad \text{and} \quad d_-^2 \cdot \prod_{q=1}^{l} x_{n_q}^{(q)} \quad \text{and} \quad d_+d_-$$
$$\text{and} \quad d_+ \prod_{q=1}^{l}\left(x_{n_q}^{(q)}\right)^{-1} x^{(q)}_{\frac{n_q}{2}} \quad \text{and} \quad d_- \prod_{q=1}^{l} x^{(q)}_{\frac{n_q}{2}}$$

This shows that the map (2.7) is surjective. The ring in the domain is an integral domain of Krull dimension at most $m+\sum_p \lfloor n_p/2 \rfloor$. We defer the proof of this to Lemma 2.17. Assuming this, the Krull dimension must be exactly $m+\sum_p \lfloor n_p/2 \rfloor$ since it surjects onto an integral domain of this Krull dimension, and the surjection must actually be an isomorphism by Lemma 1.22.

For $m = 0$, the arguments are analogous. □

Lemma 2.17. *Let*

$$S := \mathbb{Z}_2\left[\alpha_1, \ldots, \alpha_{m-2}, \delta_1, \delta_2, \delta, \zeta_1, \zeta_2\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{k_p}^{(p)}\right]$$

and consider the chain of ideals $I_1 \subset I_2 \subset I_3$ where

$$\begin{aligned} I_1 &:= \left(\delta_1\delta_2 + \delta^2\right), \\ I_2 &:= \left(\delta_1\delta_2 + \delta^2, \zeta_1^2 + \delta_1 Y\right), \\ I_3 &:= \left(\delta_1\delta_2 + \delta^2,\ \zeta_1^2 + \delta_1 Y,\ \zeta_2^2 + \delta_2 Y,\ \zeta_1\zeta_2 + \delta Y,\ \delta\zeta_1 + \zeta_2\delta_1,\ \delta\zeta_2 + \zeta_1\delta_2\right) \end{aligned}$$

with $Y := \beta_{k_1}^{(1)} \ldots \beta_{k_l}^{(l)}$. Then I_1, I_2, I_3 are prime. In particular, $ht(I_3) \geq 3$.

Proof. We consider the ring

$$T := \mathbb{Z}_2\left[\delta_1, \delta_2, \delta, \zeta_1, \zeta_2, Z\right]$$

and we define ideals of T by

$$\begin{aligned}
J_1 &:= \left(\delta_1\delta_2 + \delta^2\right), \\
J_2 &:= \left(\delta_1\delta_2 + \delta^2, \zeta_1^2 + \delta_1 Z\right), \\
J_3 &:= \left(\delta_1\delta_2 + \delta^2,\ \zeta_1^2 + \delta_1 Z,\ \zeta_2^2 + \delta_2 Z,\ \zeta_1\zeta_2 + \delta Z,\ \delta\zeta_1 + \zeta_2\delta_1,\ \delta\zeta_2 + \zeta_1\delta_2\right)
\end{aligned}$$

One can check, for example using a computer algebra system, that J_1, J_2, J_3 are prime ideals in S and that $Z \notin J_3$. For $a \in \{1,2,3\}$, we define

$$\begin{aligned}
\tilde{S}^{(a)} &:= T/J_a \otimes \mathbb{Z}_2\left[\alpha_1, \ldots, \alpha_{m-2}\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{k_p}^{(p)}\right] \\
&\cong T/J_a\left[\alpha_i, \beta_j^{(p)} \mid 1 \le i \le m-2,\ 1 \le p \le l,\ 1 \le j \le k_p\right],
\end{aligned}$$

a polynomial ring over T/J_a. We have that T/J_a is an integral domain and $\overline{Z} \neq 0$ in T/J_a. Observe that $S/I_a \cong \tilde{S}^{(a)}/\left(\overline{Z} + \beta_{k_1}^{(1)} \ldots \beta_{k_l}^{(l)}\right)\tilde{S}^{(a)}$. We want to show this is an integral domain. This now follows from:

Claim. *Let R be an integral domain and $r \in R \setminus \{0\}$. Then the ideal $(r + X_1 \ldots X_n)$ of $R[X_1, \ldots, X_n]$ is prime.*

Proof of claim. Let $A := R[X_1, \ldots, X_n]/(r + X_1 \ldots X_n)$. We want to show that A is an integral domain. We prove this by induction on n. It is clear for $n = 1$.

$\overline{X}_1$ is not a zero divisor in A: For, suppose we have $f, g \in R[X_1, \ldots, X_n]$ with $fX_1 = g \cdot (r + X_1 \ldots X_n)$. Then $(f - gX_2 \ldots X_n)X_1 = gr$, so $gr \in (X_1)$. Since R is an integral domain, $(X_1) \subset R[X_1, \ldots, X_n]$ is a prime ideal. As $r \in R \setminus \{0\}$ we deduce $g \in (X_1)$, so there is $g' \in R[X_1, \ldots, X_n]$ with $g = g'X_1$. But then $fX_1 = g'X_1(r + X_1 \ldots X_n)$ and thus $f = g' \cdot (r + X_1 \ldots X_n)$, so $f \in (r + X_1 \ldots X_n)$. This proves that $\overline{X}_1$ is not a zero divisor in A.

Now

$$A_{\overline{X}_1} \cong R\left[X_1^{\pm 1}, X_2, \ldots, X_n\right]/(r + X_1 \ldots X_n) \cong R\left[X_1^{\pm 1}, X_2, \ldots, X_n\right]/(rX_1^{-1} + X_2 \ldots X_n)$$

Setting $R' := R\left[X_1^{\pm 1}\right]$ and $r' := rX_1^{-1}$, we see by induction hypothesis that

$$A_{\overline{X}_1} \cong R'[X_2, \ldots, X_n]/(r' + X_2 \ldots X_n)$$

is an integral domain. Since $\overline{X}_1 \in A$ is not a zero divisor, this shows A is an integral domain. □

This completes the proof of our lemma. □

Chapter 3

Regular Sequences

As we already indicated in the previous chapter, regular sequences play an important role in the computation of Tate cohomology of quotient rings. In fact, as we will see in the next chapter, proving that certain polynomials form a regular sequence is at the heart of our computations of the Witt ring of complex flag varieties. We therefore dedicate a chapter to regular sequences to present the theory and results which we will need later. More specifically, the aim is to lay the necessary groundwork to prove regularity of a certain sequence of polynomials occurring in our computations.

In the first section, we recall the basic definitions and explain some results about regular sequences. Most importantly, we give equivalent criteria in terms of Krull dimension to determine if a sequence of elements of a ring is regular. We use these criteria in the second section to prove regularity of homogeneous polynomials in a particular example. It is important for our computations in the next chapter. In the third section, we consider sequences of inhomogeneous elements in a graded ring, for example a polynomial ring. We prove that we can often deduce regularity of such a sequence by just looking at the highest homogeneous components of the inhomogeneous elements. This is the central result to prove regularity of the sequence of polynomials occurring in our computation.

Everything in the first section is standard theory and can be found in the literature, for instance in [BH93] and [Mat80]. We suppose that the application in the second section should also be contained somewhere in the literature although we have not been able to find a reference. The content of the third section is our own work.

We have tried to keep this chapter reasonably self-contained and give precise references for the results we do not prove here.

Throughout this chapter, let R, S be commutative rings with a multiplicative unit. If $x_1, \dots, x_n \in R$, we denote by $(x_1, \dots, x_n)$ the ideal generated by $x_1, \dots, x_n$.

3.1 Basic facts about regular sequences

We explain some basic definitions and results. We can naturally only give a glimpse of the whole theory and have left out what we did not deem necessary for the understanding of the following two sections.

Roughly speaking, a sequence of elements in R is regular if the elements are as independent as possible.

Definition 3.1. A sequence $x_1, \dots, x_n \in R$ is *almost R-regular* (or simply *almost regular* if there is no ambiguity about the underlying ring) if for every $1 \leq i \leq n$, the map

$$R/(x_1, \dots, x_{i-1}) \xrightarrow{\cdot x_i} R/(x_1, \dots, x_{i-1})$$

is injective. If in addition, $(x_1, \dots, x_n) \neq R$ holds, it is called *R-regular* (or *regular*).

In the literature, one finds the more general notion of *M-regular sequences* for an R-module M. Our restriction to the case $M = R$ in the above definition suffices for our purposes.

Example 3.2.

(i) The standard example of a regular sequence is indeterminates in a polynomial ring: Let A be a ring and $R = A[X_1, \dots, X_n]$, then $X_1, \dots, X_n$ is clearly a regular sequence.

(ii) The order of the elements matters in regular sequences. For example, the sequence

$$X,\ Y(1-X),\ Z(1-X)$$

is regular in $\mathbb{C}[X, Y, Z]$, while

$$Y(1-X),\ Z(1-X),\ X$$

is not. We will see however that in Noetherian local rings, the order of the elements does not matter.

(iii) Any regular system of parameters in a regular local ring forms a regular sequence [Eis04, Cor. 10.15].

Almost-regularity is preserved under flat ring extensions:

Proposition 3.3. *If R and S are rings such that S is a flat R-module and $x_1, \dots, x_n \in R$ is an almost R-regular sequence, then it is almost S-regular.*

Proof. Since S is a flat R-module, $S \otimes_R -$ is an exact functor. In particular, it preserves injectivity of homomorphisms. $\square$

We will use the following consequence later:

Corollary 3.4. *Let $F \subset L$ be a field extension and R be an F-algebra. A sequence $x_1, \ldots, x_n \in R$ is almost R-regular if and only if it is almost $L \otimes_F R$-regular.*

Proof. Note that L is a free F-module since F is a field, so L is a faithfully flat F-module. Thus one implication is immediate by Proposition 3.3 and the other one follows similarly, using that L is faithfully flat. $\square$

We will now see that regularity of a sequence has strong implications for Krull dimension. First, recall that the *height of a prime ideal P*, denoted by $\mathrm{ht}(P)$, is defined as the maximal n such that there is a chain of prime ideals

$$P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n = P$$

For any ideal I, its *height* is defined as the minimal height of the prime ideals containing I.

Lemma 3.5. *Let P be a minimal prime of R and $0 \neq x \in P$. Then x is a zero divisor.*

Proof. In the localised ring R_P, we have a unique prime ideal P_P which is hence also the nilradical. Thus $x/1 \in R_P$ is nilpotent, so there are $t \in R \setminus P$ and $n \in \mathbb{N}$ such that $tx^n = 0$. So x is a zero divisor. $\square$

Proposition 3.6. *Let R be Noetherian and $x_1, \ldots, x_n \in R$ be a regular sequence. Then $ht(x_1, \ldots, x_n) = n$.*

Proof. We prove this by induction on n. Suppose $n = 1$ and let $x_1 \in R$ be a regular element. By Lemma 3.5, x_1 is not contained in any minimal prime, so $\mathrm{ht}(x_1) \geq 1$. On the other hand, $\mathrm{ht}(x_1) \leq 1$ by Krull's principal ideal theorem. Hence $\mathrm{ht}(x_1) = 1$.

Suppose now that $\mathrm{ht}(x_1, \ldots, x_{n-1}) = n - 1$. Since x_n is not a zero divisor in $R/(x_1, ..., x_{n-1})$, it cannot be contained in a minimal prime of $R/(x_1, \ldots, x_{n-1})$ by Lemma 3.5. So any minimal prime of $(x_1, \ldots, x_n)$ is not a minimal prime of $(x_1, \ldots, x_{n-1})$. Thus

$$\mathrm{ht}(x_1, \ldots, x_n) \geq \mathrm{ht}(x_1, \ldots, x_{n-1}) + 1 = n$$

By Krull's principal ideal theorem, $\mathrm{ht}(x_1, \ldots, x_n) \leq n$. Hence $\mathrm{ht}(x_1, \ldots, x_n) = n$ and we have proved the claim. $\square$

We now define one of the most important numerical invariants of a ring besides its Krull dimension:

Definition 3.7. Let R be Noetherian and I be a proper ideal. We call the maximal length of a regular sequence in I the *I-depth of R* and denote it by $\mathrm{depth}_I(R)$.

Proposition 3.6 immediately implies the following relationship between depth and Krull dimension:

Proposition 3.8. *Let R be Noetherian and I be a proper ideal. Then $depth_I(R) \leq ht(I)$.*

The rings where the notions of depth and height coincide have a rich theory and get their own terminology.

Definition 3.9. Let R be Noetherian.

(i) If R is local with maximal ideal $\mathfrak{m}$, we say that R is *Cohen-Macaulay* if $\operatorname{depth}_{\mathfrak{m}}(R) = \operatorname{ht}(\mathfrak{m})$.

(ii) In general, R is called *Cohen-Macaulay* if the localisation of R at every maximal ideal is Cohen-Macaulay.

In fact, one can show that in a Cohen-Macaulay ring R, $\operatorname{depth}_I(R) = \operatorname{ht}(I)$ for *every* proper ideal I [BH93, Cor. 2.1.4]. Cohen-Macaulay rings are useful for us precisely because the notions of depth and height coincide and so we are able to use dimension theory to show that a sequence of elements is regular. Let us first consider some (non-)examples of Cohen-Macaulay rings.

Example 3.10.

(i) Regular local rings are Cohen-Macaulay since any regular system of parameters is a regular sequence (cf. Example 3.2(iii)).

(ii) Let k be a field. Since the localisation of the polynomial ring $k[X_1, \dots, X_n]$ at any prime ideal is regular local and hence Cohen-Macaulay, $k[X_1, \dots, X_n]$ is Cohen-Macaulay.

(iii) If k is a field, $R = k[X, Y]/(X^2, XY)$ is not Cohen-Macaulay: Every element of the maximal ideal (X, Y) is annihilated by the non-trivial element X and is hence a zero divisor. So $\operatorname{depth}_{(X,Y)}(R) = 0 < \operatorname{ht}(X, Y)$.

As already indicated, we want to deduce dimension-theoretical criteria for regularity of a sequence in Cohen-Macaulay rings. We are particularly interested in polynomial rings. We will first consider regular sequences in local rings and then move on to graded rings in order to deal with polynomial rings afterwards.

We saw in Example 3.2(ii) that the order of the elements generally matters in regular sequences. However, in Noetherian local rings, it does not.

Proposition 3.11. *Suppose $(R, \mathfrak{m})$ is Noetherian local and $x_1, \dots, x_n \in R$ is a regular sequence. Then for any permutation $\sigma \in S_n$, the sequence $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ is regular.*

Proof. It suffices to prove this for $n = 2$ and $\sigma \in S_2$ the non-trivial transposition. So let $x_1, x_2 \in R$ be a regular sequence. Suppose $a \in R$ with $ax_2 = 0$. Then $a \in (x_1)$ since x_1, x_2 is regular. Say $a = a'x_1$ where $a' \in R$. Then $a'x_1x_2 = 0$. Since x_1 is not a zero divisor, we deduce $a'x_2 = 0$. Hence $a' \in (x_1)$ and so $a \in (x_1)^2$. Continuing in this way, we obtain $a \in \bigcap_{k=1}^{\infty}(x_1)^k$. So by Krull's intersection theorem, $a = 0$ as required.

Hence $x_2 \in R$ is not a zero divisor. Now let $b \in R$ with $bx_1 \in (x_2)$. Then there is $c \in R$ with $bx_1 = cx_2$, so $cx_2 \in (x_1)$, so $c \in (x_1)$ since x_1, x_2 is a regular sequence. Say $c = c'x_1$. Hence $bx_1 = c'x_2x_1$ and so $b = c'x_2 \in (x_2)$ since x_1 is not a zero divisor. □

Here is finally a dimension-theoretical criterion for regularity of sequences in local Cohen-Macaulay rings:

Theorem 3.12. *Let $(R, \mathfrak{m})$ be local Cohen-Macaulay. A sequence $x_1, \dots, x_n \in R$ is regular if and only if $ht(x_1, \dots, x_n) = n$.*

Proof. See [Mat80, Thm. 31]. □

We move on to graded rings. Many results about local rings can be carried over to analogous results about graded rings: Let R be a graded ring, i.e. there are abelian subgroups $R_i \subset R$ for $i \in \mathbb{Z}$ with $R_i = 0$ if $i < 0$ and $R = \bigoplus_{i\in\mathbb{Z}} R_i$ and $R_iR_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$. We suppose further that R is Noetherian and that $R_0 = k$ is a field.
Then R has a *unique* homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$. For this reason, as long as we restrict to considering homogeneous elements, R behaves like a local ring. We will see that with regard to the notions of depth and regular sequences, this is indeed the case. This often makes it easier to prove regularity for a sequence of homogeneous elements, for example in polynomial rings. So let us now formulate results analogous to Proposition 3.11 and Theorem 3.12.

Proposition 3.13. *Let R be graded as above and $f_1, \dots, f_n \in R$ be a regular sequence of homogeneous elements. Then any permutation of the f_i is again a regular sequence.*

Proof. It suffices to consider the case $n = 2$. So let $f_1, f_2 \in R$ be a homogeneous regular sequence. Note that f_1 and f_2 must have positive degree since $(f_1, f_2) \neq R$. Since localisation is an exact functor, $\frac{f_1}{1}, \frac{f_2}{1}$ is an $R_\mathfrak{m}$-regular sequence. By Proposition 3.11, it now suffices to show that if $f \in R$ is homogeneous such that $\frac{f}{1}$ is not a zero divisor in $R_\mathfrak{m}$, then neither is f a zero divisor in R. This is easy: Suppose $g \in R$ such that $fg = 0$. Then $\frac{f}{1} \cdot \frac{g}{1} = 0$ in $R_\mathfrak{m}$ and so $\frac{g}{1} = 0$. Hence there is $h \notin \mathfrak{m}$ such that $gh = 0$. Since $h \notin \mathfrak{m}$, it must have a non-zero homogeneous component $h_0 \in R_0 = k$ in degree 0. Suppose $g \neq 0$ and let its lowest non-zero homogeneous component be $g_k \in R_k$. Then $gh = 0$ implies $h_0g_k = 0$ and thus $g_k = 0$, but this is a contradiction. Hence $g = 0$ as required. □

We want to find equivalent criteria for the regularity of a sequence of homogeneous elements in a graded ring. From now, let R be as above and assume in addition that R is finitely generated as an $R_0 = k$-algebra. Then we can define the *Hilbert series* of R as the formal power series

$$F(R, \lambda) = \sum_{n=0}^{\infty} \dim_k(R_n)\lambda^n \in \mathbb{Z}[[\lambda]]$$

where $\dim_k(R_n)$ denotes the k-vector space dimension of R_n.

Theorem 3.14. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring such that $R_0 = k$ is a field and suppose R is finitely generated as a k-algebra. Let $f_1, \ldots, f_n \in R$ be homogeneous of positive degree. Then (1)⇔(2) if R is Cohen-Macaulay and (1)⇔(3) in general, where*

(1) $f_1, \ldots, f_n$ is an R-regular sequence.

(2) $ht(f_1, \ldots, f_n) = n$

(3) $F(R/(f_1, \ldots, f_n), \lambda) = F(R, \lambda) \cdot \prod_{i=1}^{n}(1 - \lambda^{deg(f_i)})$

Proof. We saw in Proposition 3.6 that (1) implies (2). Suppose now (2) holds. Since the f_i are homogeneous, any minimal prime over $(f_1, \ldots, f_n)$ is homogeneous [BH93, Lem. 1.5.6(a)]. By (2), we can find a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ where P_n is a minimal prime over $(f_1, \ldots, f_n)$. We may assume the P_i are homogeneous [BH93, Thm. 1.5.8], so they are contained in $\mathfrak{m}$. Thus we deduce that ht $\left(\frac{f_1}{1}, \ldots, \frac{f_n}{1}\right) = n$ in $R_{\mathfrak{m}}$. So by Theorem 3.12, it follows that $\frac{f_1}{1}, \ldots, \frac{f_n}{1}$ is $R_{\mathfrak{m}}$-regular. By the same argument as in the proof of Proposition 3.13, $f_1, \ldots, f_n$ is a R-regular sequence.

The equivalence of (1) and (3) is shown in [Sta78, Cor. 3.2]. □

3.2 Application: A regular sequence of homogeneous polynomials

We want to apply Theorem 3.14 to prove regularity of a particular sequence of homogeneous polynomials, which together with the result in the next section of this chapter will be important for our computations of Witt rings.

Let K be a field. We consider the polynomial ring

$$A = K[X_{1,1}, X_{1,2}, \ldots, X_{1,n_1}, X_{2,1}, \ldots, X_{m,1}, \ldots, X_{m,n_m}]$$

as a graded ring with grading given by $|X_{i,j}| = j$. Recall that we showed in Example 3.10(ii) that polynomial rings over a field are Cohen-Macaulay. We define certain polynomials Q_a by

$$Q_a = \sum_{i_1+\ldots+i_m=a} X_{1,i_1} X_{2,i_2} \ldots X_{m,i_m} \qquad \left(1 \leq a \leq N := \sum_{j=1}^{m} n_j\right)$$

where in this sum, we follow the convention that $X_{k,0} = 1$ for $1 \leq k \leq m$. Note that Q_a is homogeneous of degree a.

Proposition 3.15. *The Q_a for $1 \leq a \leq N$ form a regular sequence.*

Proof. Since the Q_a are homogeneous, it is sufficient by Theorem 3.14 to show that the ideal generated by the Q_a is of maximal height. So let P be a prime ideal with $Q_a \in P$ for $1 \leq a \leq N$. It is sufficient to show that $P = (X_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n_i)$.

Suppose this is false and let $\{i_1, \ldots, i_r\} = \{i \mid \exists j \colon X_{i,j} \notin P\}$. By assumption, this set is non-empty. For every $1 \le k \le r$, let j_k be the largest j with $X_{i_k,j} \notin P$.
Let $J = \sum_{k=1}^{r} j_k$. Then every monomial in Q_J either contains some $X_{i,j}$ with $i \notin \{i_1, \ldots, i_r\}$, in which case this monomial is in P, or it contains only monomials $X_{i,j}$ with $i \in \{i_1, \ldots, i_r\}$. In the latter case, there is either some $1 \le k \le r$ such that the monomial contains $X_{i_k,j}$ with $j > j_k$ (in which case this monomial is in P according to the maximality condition on j_k) or the monomial is equal to $X_{i_1,j_1} \ldots X_{i_r,j_r}$, which is not in P since P is prime. So there is only one monomial in Q_J which is not in P. Thus $Q_J \notin P$. But this is a contradiction. Hence $P = (X_{i,j})$ as required. □

Let I be the ideal in A generated by the Q_a for $1 \le a \le N$. As the Q_a form a regular sequence, by Theorem 3.14 we have $\mathrm{ht}(I) = N$ and so A/I has Krull dimension 0. Hence A/I is a finite-dimensional K-vector space. We want to use the Hilbert function of A/I to compute this dimension.

We have that

$$F(A, \lambda) = \frac{1}{\prod_{p=1}^{m} \prod_{i=1}^{n_p} (1 - \lambda^i)}$$

By Theorem 3.14, it follows that

$$\begin{aligned} F(A/I, \lambda) = F(A, \lambda) \cdot \prod_{j=1}^{N} (1 - \lambda^j) &= \frac{\prod_{j=1}^{N} (1 - \lambda^j)}{\prod_{p=1}^{m} \prod_{i=1}^{n_p} (1 - \lambda^i)} \\ &= \frac{(1-\lambda)^N \cdot \prod_{j=1}^{N} (1 + \lambda + \ldots + \lambda^{j-1})}{(1-\lambda)^N \cdot \prod_{p=1}^{m} \prod_{i=1}^{n_p} (1 + \lambda + \ldots + \lambda^{i-1})} \\ &= \frac{\prod_{j=1}^{N} (1 + \lambda + \ldots + \lambda^{j-1})}{\prod_{p=1}^{m} \prod_{i=1}^{n_p} (1 + \lambda + \ldots + \lambda^{i-1})} \end{aligned}$$

This implies:

Proposition 3.16. *$dim_K(A/I) = F(A/I, 1) = \frac{(n_1 + \ldots + n_m)!}{n_1! \cdot \ldots \cdot n_m!}$*

3.3 Regular sequences of inhomogeneous elements of graded rings

After considering homogeneous elements, we now consider sequences of inhomogeneous elements in graded rings. Our aim is to prove that if the highest homogeneous components form a regular sequence, then so do the inhomogeneous elements. This will be useful because as we have seen in the previous sections, we have dimension-theoretical tools at our disposal to prove regularity of homogeneous elements.

Suppose $R = \bigoplus_{p \in \mathbb{Z}} R_p$ is a commutative graded ring with $R_p = 0$ for $p < 0$. We define an ascending filtration of R by subgroups by

$$F^a R := \bigoplus_{p \leq a} R_p$$

This is not a filtration by ideals, but $F^a \cdot F^b \subset F^{a+b}$ for all $a, b \in \mathbb{Z}$. So $\mathrm{gr}_F R := \bigoplus_{a \in \mathbb{Z}} F^a R / F^{a-1} R$ inherits a ring structure. It is easy to see that $\mathrm{gr}_F R \cong R$. Suppose now we are given elements $y_1, \ldots, y_n \in R$ with

$$y_i = x_i + \text{lower degree terms} \quad (1 \leq i \leq n)$$

where $x_i \in R_{k_i}$ is homogeneous.

Lemma 3.17. *Suppose $x_1, \ldots, x_n$ is a regular sequence in every order in R and $r_1, \ldots, r_l \in R$ with*

$$z := r_1 y_1 + \ldots + r_l y_l \in F^a R$$

Then the term of degree a of z is in $(x_1, \ldots, x_l)$.

Proof. We write $r_i = r_i' + r_i''$ where r_i' is homogeneous of highest degree in r_i and r_i'' is a sum of terms of lower degree. We prove the claim by induction on

$$M := \max\left(\left\{\deg(r_i')\deg(x_i) \mid 1 \leq i \leq l\right\}\right)$$

If $M \leq a$, then the term in z of degree a is a sum of some of the terms $r_i' x_i$ and so is in $(x_1, \ldots, x_l)$.
Suppose $M > a$ and let $1 \leq i_1, \ldots, i_k \leq l$ be all the distinct indices such that $\deg(r_{i_p}')\deg(x_{i_p}) = M$ for $1 \leq p \leq k$. Then $r_{i_1}' x_{i_1} + \ldots + r_{i_k}' x_{i_k}$ is the term in z of highest degree M and so

$$r_{i_1}' x_{i_1} + \ldots + r_{i_k}' x_{i_k} = 0$$

since $M > a$ and $z \in F^a R$. Since the x_i are a regular sequence in any order, we deduce that $r_{i_p}' \in (x_1, \ldots, x_l)$ for $1 \leq p \leq k$. But then consider

$$z' := z - \sum_{p=1}^{k} r_{i_p}' y_{i_p} = r_1 y_1 + \ldots + r_l y_l - \sum_{p=1}^{k} r_{i_p}' y_{i_p}$$

By induction, we deduce that the term of degree a in z' is in $(x_1, \ldots, x_l)$. But since $r_{i_p}' \in (x_1, \ldots, x_n)$ for $1 \leq p \leq k$, this also holds for z. $\square$

Remark. The assertion is not true if we do not assume regularity of the sequence x_i. For example, consider $R = \mathbb{Z}_2[\alpha, \beta]$ with $\deg(\alpha) = \deg(\beta) = 1$, and set $y_1 = \alpha$ and $y_2 = \alpha^2 + \beta$ so that $x_1 = \alpha$ and $x_2 = \alpha^2$. Then $\alpha y_1 + y_2 = \beta \in F^1 R$, but $\beta \notin (x_1, x_2)$.

The filtration $F^* R$ induces a filtration $F^* R_l$ of $R_l := R/(y_1, \ldots, y_l)$ for every $1 \leq l \leq n$

via the quotient map $q_l \colon R \to R_l$.

Proposition 3.18. *We use the notation introduced in this section.*

(i) There is a surjective ring homomorphism

$$R/(x_1, \dots, x_l) \to gr_F(R_l) = \bigoplus_{a \in \mathbb{Z}} F^a R_l / F^{a-1} R_l$$

of graded rings for every $1 \leq l \leq n$.

(ii) If $x_1, \dots, x_n$ is an R-regular sequence in every order, the surjection from (i) is an isomorphism.

Remark. Since the x_i are homogeneous,

$$R/(x_1, \dots, x_l) \cong \bigoplus_{a \in \mathbb{Z}} \frac{R_a}{R_a \cap (x_1, \dots, x_l)}$$

is naturally a graded ring. Furthermore, the condition in (ii) that $x_1, \dots, x_n$ be regular in *every* order is not much stronger than that $x_1, \dots, x_n$ be regular in *some* order, as we saw in Proposition 3.13: If R is Noetherian and R_0 is a field, these conditions are equivalent.

Proof. (i) For every $a \in \mathbb{Z}$, we have a surjective additive homomorphism

$$\begin{aligned} \frac{F^a R}{F^{a-1} R} &\twoheadrightarrow \frac{\operatorname{im}(F^a R \xrightarrow{q_l} R/(y_1, \dots, y_l))}{\operatorname{im}(F^{a-1} R \xrightarrow{q_l} R/(y_1, \dots, y_l))} = \frac{F^a R_l}{F^{a-1} R_l}, \\ [x] &\mapsto [q_l(x)] \end{aligned}$$

These homomorphisms induce a surjective ring homomorphism $\mathrm{gr}_F R \to \mathrm{gr}_F R_l$ and thus we obtain a surjective ring homomorphism

$$R = \bigoplus_{a \in \mathbb{Z}} R_a \cong \mathrm{gr}_F R \to \mathrm{gr}_F R_l$$

We observe that $x_1, \dots, x_l$ are all in the kernel of this map. Hence this induces a surjective ring homomorphism

$$f \colon R/(x_1, \dots, x_l) \twoheadrightarrow \mathrm{gr}_F R_l$$

(ii) We define an inverse of the map f in (i). First, for every $a \in \mathbb{Z}$ we define a map

$$\begin{aligned} F^a R_l = \operatorname{im}(F^a R \xrightarrow{q_l} R/(y_1, \dots, y_l)) &\to R_a/(R_a \cap (x_1, \dots, x_l)), \\ q_l(x + \text{lower degree terms}) &\mapsto [x] \quad (\text{where } x \in R_a) \end{aligned}$$

This is well-defined by Lemma 3.17. It is clearly an additive homomorphism for every $a \in \mathbb{Z}$ and induces a homomorphism

$$\frac{F^a R_l}{F^{a-1} R_l} \to \frac{R_a}{R_a \cap (x_1, \dots, x_l)}$$

for every $a \in \mathbb{Z}$. All these maps induce a ring homomorphism

$$g\colon \mathrm{gr}_F R_l \to R/(x_1, \dots, x_l)$$

which is, by construction, inverse to the homomorphism f constructed in (i).

$\square$

We can now prove the main result of this section:

Corollary 3.19. *Let $R = \bigoplus_{p \in \mathbb{Z}}$ be a commutative graded ring with $R_p = 0$ for $p < 0$. Let $y_1, \dots, y_n \in R$ be such that for all $1 \leq i \leq n$,*

$$y_i = x_i + \textit{lower degree terms}$$

where $x_i \in R_{k_i}$ is homogeneous. If $x_1, \dots, x_n$ is R-regular in every order, then so is $y_1, \dots, y_n$.

Proof. Suppose $y_1, \dots, y_l$ is an R-regular sequence and suppose $r \in R$ with $r y_{l+1} \in (y_1, \dots, y_l)$. Suppose $q_l(r) \neq 0$ in $R_l = R/(y_1, \dots, y_l)$. Then there is a minimal $p \in \mathbb{Z}$ such that $q_l(r) \in F^p R_l$. Consider $[q_l(r)] \in F^p R_l / F^{p-1} R_l$ and $[q_l(y_{l+1})] = [q_l(x_{l+1})] \in F^{k_{l+1}} R_l / F^{k_{l+1}-1} R_l$. Then

$$0 = [q_l(r)] \cdot [q_l(y_{l+1})] = [q_l(r)] \cdot [q_l(x_{l+1})] \quad \text{in} \quad \frac{F^{p+k_{l+1}} R_l}{F^{p+k_{l+1}-1} R_l} \subset \mathrm{gr}_F R_l.$$

Applying the isomorphism g from the proof of Proposition 3.18 (ii) and using that $x_1, \dots, x_{l+1}$ is R-regular, we deduce that $q_l(r) = 0$ in $F^p R_l / F^{p-1} R_l$ and so $q_l(r) \in F^{p-1} R_l$. This contradicts the minimality of p. $\square$

Chapter 4

Outline of the Computation of Witt Rings

We are now ready to outline our method of computation of the Witt ring of complex flag varieties in detail. We use the approach developed in [Zib15]. There the author computes the Witt ring of all full flag varieties. We slightly generalise the approach to be able to apply it to all flag varieties.

Throughout this section, let G be a compact simply connected Lie group and H be a closed connected subgroup of maximal rank. We denote by $i\colon H \to G$ the inclusion map. We have seen that by Bousfield's lemma, $W^*(G/H) \cong h^*(K^0(G/H))$. So as a first step, we need to be able to compute $K^0(G/H)$. This is done via a theorem of Hodgkin reducing this to a computation with representation rings of G and H, which are well-understood. We then use the results of section 2.2 to compute $h^*(K^0(G/H))$ and finally see how to determine the Witt grading of $h^*(K^0(G/H)) \cong W^*(G/H)$.

4.1 K-theory of G/H

We first want to construct a map $\alpha\colon R(H) \to K^0(G/H)$.

Construction 4.1. Let $\rho\colon H \to GL_n(\mathbb{C})$ be a complex representation of H. We define

$$E_\rho := G \times_\rho \mathbb{C}^n := G \times \mathbb{C}^n/(gh, v) \sim (g, \rho(h)v) \text{ for all } g \in G,\ h \in H,\ v \in \mathbb{C}^n$$

The projection $G \times \mathbb{C}^n \to G/H$ induces a well-defined map $E_\rho \to G/H$. It is an n-dimensional complex vector bundle over G/H. This construction $\rho \mapsto E_\rho$ induces a well-defined ring homomorphism

$$\alpha\colon R(H) \to K^0(G/H)$$

It is easy to see that α is also a $*$-homomorphism with the usual dualities on $R(H)$ and $K^0(G/H)$.

We can immediately spot elements in the kernel of α:

Lemma 4.2. *Let $\sigma\colon G \to GL_n(\mathbb{C})$ be a representation of G. Then $i^*(\sigma) - n \in ker(\alpha)$.*

Proof. It is sufficient to show that $E_{i^*(\sigma)}$ is a trivial vector bundle. We do this by constructing n linearly independent sections: For $1 \leq j \leq n$, let $e_j \in \mathbb{C}^n$ be the jth standard basis vector and define

$$\begin{aligned} s_j\colon G/H &\to E_{i^*(\sigma)} = G \times \mathbb{C}^n/(gh, v) \sim (g, \sigma(h)v) \\ gH &\mapsto [(g, \sigma(g^{-1})e_j] \end{aligned}$$

It is easy to see that the s_j are well-defined continuous sections of $E_{i^*(\sigma)}$ and that they are linearly independent. □

So let $\mathfrak{a}(G) \subset R(H)$ be the ideal generated by all $i^*(\sigma) - \mathrm{rk}(\sigma) \in R(H)$ for $\sigma \in R(G)$. Lemma 4.2 shows that $\mathfrak{a}(G) \subset \ker(\alpha)$, so α induces a map

$$\overline{\alpha}\colon R(H)/\mathfrak{a}(G) \to K^0(G/H)$$

Recalling our standing assumption that G is compact, simply connected and that $H \subset G$ is of maximal rank, Hodgkin's theorem states:

Theorem 4.3 ([Pit72, Thm. 3])**.** *$\overline{\alpha}$ is an isomorphism.*

Remark 4.4. As outlined in [Zib15, §2.2 and §2.3] for example, we can apply Construction 4.1 analogously to real representations of H and real vector bundles over G/H to obtain maps

$$\alpha_O^j\colon RO^j(H) \to KO^j(G/H)$$

Let $\widetilde{RO}^j(G)$ be the kernel of the restriction map $RO^j(G) \to RO^j(\{1\})$ and denote by

$$\mathfrak{a}_O^*(G) \subset RO^*(H)$$

the ideal generated by the image of $\widetilde{RO}^*(G)$ under the restriction map i^* and by the images of $\mathfrak{a}(G)$ under the realification map r_*. Then α_O^j induces a map

$$\overline{\alpha}_O^j\colon RO^j(H)/\mathfrak{a}_O^j(G) \to KO^j(G/H)$$

See [Zib15, §2.3] for all of this. However, Proposition 2.2 and Example 2.3 in [Zib15] show that $\overline{\alpha}_O^*$ is often far from being surjective.

A crucial ingredient in the proof of Theorem 4.3 is the following result, which we shall also need later:

Theorem 4.5 ([Ste75, Thm. 1.1]). *Let K be a connected compact Lie group with $\pi_1(K)$ free and U be a closed connected subgroup of maximal rank. Then $R(U)$ is free as a module over $R(K)$ by restriction.*

Let us now apply Hodgkin's theorem 4.3 concretely. Since G is simply connected, its representation ring is a polynomial ring:

$$R(G) \cong \mathbb{Z}[\lambda_1, \lambda_1^*, \ldots, \lambda_a, \lambda_a^*, \sigma_{a+1}, \ldots, \sigma_n]$$

Here we suppose that the σ_j are self-dual. If ζ is a representation, let us write $\tilde{\zeta} := \zeta - \mathrm{rk}(\zeta)$ for the reduced virtual representation of rank 0. As $R(H)$ is a free $R(G)$-module by Theorem 4.5, it follows by Proposition 3.3 that

$$i^*(\tilde{\lambda}_1), i^*(\tilde{\lambda}_1^*), \ldots, i^*(\tilde{\lambda}_a), i^*(\tilde{\lambda}_a^*), i^*(\tilde{\sigma}_{a+1}), \ldots, i^*(\tilde{\sigma}_n)$$

is an $R(H)$-regular sequence. Letting I be the ideal generated by all these elements, we obtain by Theorem 4.3 that

$$\overline{\alpha} \colon R(H)/I \xrightarrow{\cong} K^0(G/H) \tag{4.1}$$

is an isomorphism of $*$-rings.

4.2 Tate cohomology of $K^0(G/H)$

Via the isomorphism (4.1), it is now possible to compute $h^*(K^0(G/H))$ by computing $h^*(R(H)/I)$ using the results of section 2.2. We define the following elements in $h^+(R(H))$:

$$\mu_j := \begin{cases} \left[i^*\left(\tilde{\lambda}_j \tilde{\lambda}_j^*\right)\right] & \text{if } 1 \le j \le a \\ \left[i^*\left(\tilde{\sigma}_j\right)\right] & \text{if } a < j \le n \end{cases}$$

We make the following assumptions:

(A1) There is a subset $S \subset \{1, \ldots, n\}$ such that the μ_s for $s \in S$ form an $h^+(R(H))$-regular sequence in some order.

(A2) For every $t \in \{1, \ldots, n\}$, the element μ_t is contained in the ideal $(\mu_s \mid s \in S)$.

As we will see in subsequent chapters, these assumptions are frequently satisfied in our concrete computations. In fact, showing that they hold will be the essential remaining step. However, we will see that in a few situations, these assumptions are not quite satisfied. In those cases, we will have to make slight adaptions.

Let

$$I' := \left(\tilde{\lambda}_s, \tilde{\lambda}_s^* \mid s \in S,\ 1 \le s \le a\right) + \left(\tilde{\sigma}_s \mid s \in S,\ a < s \le n\right) \subset R(H)$$

From (A1), it follows by Corollary 2.9 that as rings,

$$h^*\left(R(H)/I'\right) \cong h^+(R(H)/I') \cong \frac{h^+(R(H))}{(\mu_s \mid s \in S)}$$

Let $\overline{S} := \{1, \ldots, n\} \setminus S$. By (A2), for every $t \in \overline{S}$ there exists $u_t \in R(H)$ such that:

$$\left.\begin{array}{ll} \text{If } 1 \leq t \leq a, \text{ then } \overline{i^*\left(\tilde{\lambda}_t \tilde{\lambda}_t^*\right)} & = \overline{u}_t + \overline{u}_t^* \\ \text{If } a < t \leq n, \text{ then } \overline{i^*(\tilde{\sigma}_t)} & = \overline{u}_t + \overline{u}_t^* \end{array}\right\} \text{ in } R(H)/I'$$

Now by Lemma 2.8, we have that as modules over $h^*(R(H)/I')$,

$$h^*(R(H)/I) \cong h^*(R(H)/I') \otimes \bigwedge_{t \in \overline{S}} ([\overline{u}_t]) \tag{4.2}$$

where $[\overline{u}_t] \in h^-(R(H)/I)$ for all $t \in \overline{S}$.

To show that (4.2) is also a ring isomorphism, it suffices to show that

$$[\overline{u}_t]^2 = 0 \quad \text{for all } t \in \overline{S}$$

This is immediately implied by the following Proposition, which is also of independent interest:

Proposition 4.6. *Let X be a finite cell complex with $K^1(X) = 0$ and $x \in W^{-1}(X) \oplus W^{-3}(X)$. Then $x^2 = 0$.*

Proof. Let $x = \overline{x}_{-1} + \overline{x}_{-3}$ where $x_i \in KO^{2i}(X)$ for $i \in \{-1, -3\}$. We consider the commutative square

$$\begin{array}{ccc} W^{-1}(X) \oplus W^{-3}(X) & \xrightarrow[\cong]{\overline{c}_{\text{odd}} = (\overline{c}_{-1}, \overline{c}_{-3})} & h^-(K^0(X)) \\ \downarrow & & \downarrow \\ W^0(X) \oplus W^{-2}(X) & \xrightarrow[\cong]{\overline{c}_{\text{even}} = (\overline{c}_0, \overline{c}_{-2})} & h^+(K^0(X)) \end{array}$$

where the vertical maps are the squaring maps and the horizontal maps are isomorphisms by Bousfields lemma 2.3. We have

$$\overline{c}_{\text{even}}(x^2) = \overline{c}_{\text{odd}}(x)^2 = [c_{-1}(x_{-1})]^2 + [c_{-3}(x_{-3})]^2 \tag{4.3}$$

In $K^0(X)$, we have for $i \in \{-1, -3\}$ that $c_i(x_i) + c_i(x_i)^* = 0$. Hence in $h^-(K^0(X))$,

$$[c_i(x_i)] = [-c_i(x_i)^*] = [c_i(x_i)^*]$$

where the second equality holds since $h^-(K^0(X))$ is 2-torsion. Thus from (4.3),

$$\overline{c}_{\text{even}}(x^2) = [c_{-1}(x_{-1}) \cdot c_{-1}(x_{-1})^*] + [c_{-3}(x_{-3}) \cdot c_{-3}(x_{-3})^*] \tag{4.4}$$

Now for any $y \in K^0(X)$, it is a well-known fact that $y \cdot y^* \in \operatorname{im}(c_0)$. So (4.4) implies that $\overline{c}_{\text{even}}(x^2) \in \operatorname{im}(\overline{c}_0)$. Thus $x^2 \in W^0(X)$.

On the other hand,

$$x^2 = [x_{-1}]^2 + [x_{-3}]^2 \in W^{-2}(X)$$

Consequently, $x^2 \in W^0(X) \cap W^{-2}(X)$. So $x^2 = 0$ as asserted. □

In summary, we have proved:

Proposition 4.7. *Using the notation introduced above and assuming that (A1) and (A2) hold, we have a ring isomorphism*

$$h^*(R(H)/I) \cong \frac{h^+(R(H))}{(\mu_s \mid s \in S)} \otimes \bigwedge_{t \in \overline{S}} ([\overline{u}_t])$$

where the first factor in the tensor product is completely contained in h^+ and all the generators of the exterior algebra on the right are contained in h^-.

4.3 Witt ring of G/H

We have now computed the Tate cohomology of $R(H)/I$ and thus also the Tate cohomology of $K^0(G/H)$ via the isomorphism

$$[\overline{\alpha}] \colon h^*(R(H)/I) \xrightarrow{\cong} h^*(K^0(G/H))$$

induced by (4.1). It remains to determine the Witt grading of $h^*(K^0(G/H))$ under the isomorphism

$$W^*(G/H) \xrightarrow{\overline{c}} h^*(K^0(G/H))$$

of Bousfield's lemma. Using the notation as introduced in this chapter, we prove two lemmas to this end. They are generalisations of the assertion about the Witt grading of the Tate cohomology of full flag varieties in [Zib15, Thm. 3.3].

Lemma 4.8. *Let $x \in R(H)$ be self-dual. Then it gives rise to an element $[\alpha(x)] \in h^+(K^0(G/H))$.*
If x is a real representation, then $[\alpha(x)]$ corresponds to an element in $W^0(G/H)$ under Bousfield's isomorphism $\overline{c}$.
If x is a quaternionic representation, then $[\alpha(x)]$ corresponds to an element in $W^{-2}(G/H)$ under Bousfield's isomorphism $\overline{c}$.

Proof. Suppose x is real. Then there is $x^{\mathbb{R}} \in RO^0(H)$ such that $x = c_0\left(x^{\mathbb{R}}\right)$. From the commutative diagram

$$\begin{array}{ccc} RO^0(H) & \xrightarrow{c_0} & R(H) \\ \Big\downarrow \alpha_O^0 & & \Big\downarrow \alpha \\ KO^0(G/H) & \xrightarrow{c_0} & K^0(G/H) \end{array}$$

we deduce that

$$\alpha(x) = \alpha\left(c_0\left(x^{\mathbb{R}}\right)\right) = c_0\left(\alpha_O^0\left(x^{\mathbb{R}}\right)\right)$$

Thus $\alpha(x) \in \operatorname{im}(c_0)$. Passing to Tate cohomology, we obtain

$$[\alpha(x)] \in \operatorname{im}(\overline{c}_0\colon W^0(G/H) \to h^+(K^0(G/H)))$$

and hence the claim follows.

Similarly, if x is quaternionic, we obtain from the commutative square

$$\begin{array}{ccc} RO^4(H) & \xrightarrow{c_2} & R(H) \\ \Big\downarrow \alpha_O^4 & & \Big\downarrow \alpha \\ KO^4(G/H) & \xrightarrow{c_2} & K^0(G/H) \end{array}$$

that $\alpha(x) \in \operatorname{im}(c_2)$ and hence $[\alpha(x)] \in h^+(K^0(G/H))$ corresponds to an element in $W^{-2}(G/H)$ under Bousfield's isomorphism $\overline{c}$. $\square$

Lemma 4.9. *Let $u \in R(H)$ be such that there are $\nu_j \in R(H)$ and $\tilde{\mu}_j \in R(G)$ with $rk(\tilde{\mu}_j) = 0$ such that*

$$u + u^* = \sum_{j=1}^{n} \nu_j \cdot i^*\left(\tilde{\mu}_j\right)$$

Then u gives rise to $[\alpha(u)] \in h^-(K^0(G/H))$.

If all the $\nu_j, \tilde{\mu}_j$ are of real type, then $[\alpha(u)] \in h^-(K^0(G/H))$ corresponds to an element in $W^{-1}(G/H)$ under Bousfield's isomorphism.

If all the $\tilde{\mu}_j$ are of real type and all the ν_j are of quaternionic type, then $[\alpha(u)] \in h^-(K^0(G/H))$ corresponds to an element in $W^{-3}(G/H)$ under Bousfield's isomorphism.

Remark. Note that to apply this lemma to the $[\overline{u}_t]$ where $t \in \overline{S}$, we need to assume something stronger than (A2). We will see that in our concrete computations, this stronger condition is satisfied.

Proof. First suppose all the $\nu_j, \tilde{\mu}_j$ are of real type. Then there exist $\nu_j^{\mathbb{R}} \in RO^0(H)$ and

$\tilde{\mu}_j^{\mathbb{R}} \in RO^0(G)$ with $\nu_j = c_0\left(\nu_j^{\mathbb{R}}\right)$ and $\tilde{\mu}_j = c_0\left(\tilde{\mu}_j^{\mathbb{R}}\right)$. Thus we have

$$c_0(r_0(u)) = u + u^* = c_0\left(\sum_j \nu_j^{\mathbb{R}} \cdot i^*\left(\tilde{\mu}_j^{\mathbb{R}}\right)\right)$$

Since c_0 is injective, we have $r_0(u) = \sum_j \nu_j^{\mathbb{R}} \cdot i^*\left(\tilde{\mu}_j^{\mathbb{R}}\right) \in \mathfrak{a}_O^0(G)$. From the commutative square

$$\begin{array}{ccc} R(H)/\mathfrak{a}(G) & \xrightarrow{\overline{\alpha}} & K^0(G/H) \\ \Big\downarrow{\scriptstyle \overline{r}_0} & & \Big\downarrow{\scriptstyle r_0} \\ RO^0(H)/\mathfrak{a}_O^0(G) & \xrightarrow{\overline{\alpha}_O^0} & KO^0(G/H) \end{array}$$

we see that $\alpha(u) \in \ker(r_0)$. Hence[1] $[\alpha(u)] \in \ker\left(\overline{r}_0\colon h^-(K^0(G/H)) \to c\backslash KO^0(G/H)\right)$ and so $[\alpha(u)] \in \operatorname{im}\left(\overline{c}_3\colon W^3(G/H) \to h^-(K^0(G/H))\right)$ by Bousfield's lemma 2.3. Thus the claim follows.

Now suppose the $\tilde{\mu}_j$ are real and the ν_j are quaternionic. Then there are $\tilde{\mu}_j^{\mathbb{R}} \in RO^0(H)$ and $\nu_j^{\mathbb{H}} \in RO^4(H)$ such that $c_0\left(\tilde{\mu}_j^{\mathbb{R}}\right) = \tilde{\mu}_j$ and $c_2\left(\nu_j^{\mathbb{H}}\right) = \nu_j$. Thus we have

$$c_2(r_2(u)) = u + u^* = \sum_{j=1}^n c_2\left(\nu_j^{\mathbb{H}}\right) \cdot c_0\left(i^*\left(\tilde{\mu}_j^{\mathbb{R}}\right)\right) = c_2\left(\sum_{j=1}^n \nu_j^{\mathbb{H}} \cdot i^*\left(\tilde{\mu}_j^{\mathbb{R}}\right)\right)$$

Since c_2 is injective, we obtain

$$r_2(u) = \sum_{j=1}^n \nu_j^{\mathbb{H}} \cdot i^*\left(\tilde{\mu}_j^{\mathbb{R}}\right) \in \mathfrak{a}_O^4(G)$$

From the commutative square

$$\begin{array}{ccc} R(H)/\mathfrak{a}(G) & \xrightarrow{\overline{\alpha}} & K^0(G/H) \\ \Big\downarrow{\scriptstyle \overline{r}_2} & & \Big\downarrow{\scriptstyle r_2} \\ RO^4(H)/\mathfrak{a}_O^4(G) & \xrightarrow{\overline{\alpha}_O^4} & KO^4(G/H) \end{array}$$

we see that $\alpha(u) \in \ker(r_2)$. So by Bousfield's lemma 2.3, the claim follows. $\square$

[1] Recall that $c\backslash KO^0(G/H)$ denotes the kernel of $c_0\colon KO^0(G/H) \to K^0(G/H)$.

Chapter 5

Witt Rings of Ordinary Flag Varieties

We are finally ready to compute the Witt rings of complex flag varieties G/H where G is a compact connected simple Lie group of ordinary type and H is a centraliser of a torus in G. We follow the approach outlined in Chapter 4. We saw in Proposition 1.2 that the Witt groups determine the torsion part of the KO-groups of complex flag varieties and the free part can be read off from their rational cohomology.

In the Introduction, we already gave an overview of known topological computations of KO-groups. Complete algebraic computations of the Witt groups of flag varieties have also only been obtained in special cases: For projective spaces [Wal], split quadrics [Nen09] and Grassmannians [BC12].

5.1 Type A_n

We consider $G = SU(n)$. Up to conjugation, every centraliser of a torus in G is of the form $H(n_1, \ldots, n_l) := S(U(n_1) \times \ldots \times U(n_l))$ where $n_1 + \ldots + n_l = n$ (cf. Proposition 1.12). So we let

$$X(n_1, \ldots, n_l) := \frac{SU(n)}{S(U(n_1) \times \ldots \times U(n_l))}$$

where $n_1 + \ldots + n_l = n$. We will sometimes simply write X instead of $X(n_1, \ldots, n_l)$ and H instead of $H(n_1, \ldots, n_l)$. Suppose further that precisely k of the integers $n_1, \ldots, n_l$ are odd.

5.1.1 Representation rings and their Tate cohomology

For convenience, we first recall the representation rings of the Lie groups involved and their Tate cohomology from previous sections. We have an isomorphism

$$R(SU(n)) \cong \mathbb{Z}[\lambda_1, \ldots, \lambda_{n-1}]$$

where λ_1 is the standard n-dimensional complex representation of $SU(n)$ and $\lambda_i = \Lambda^i(\lambda_1)$ for $1 \leq i \leq n-1$. The duality on $R(SU(n))$ is given by $\lambda_i^* = \lambda_{n-i}$.

By Proposition 1.23,

$$R(S(U(n_1) \times \ldots \times U(n_l))) \cong \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p}^{(p)}\right] / \left(\prod_{p=1}^{l} x_{n_p}^{(p)} - 1\right)$$

Here, for $1 \leq p \leq l$, the element $x_1^{(p)}$ is the n_p-dimensional representation of $S(U(n_1) \times \ldots \times U(n_l))$ letting the pth block act on $\mathbb{C}^{n_p}$. Furthermore, $x_i^{(p)} = \Lambda^i\left(x_1^{(p)}\right)$ and the duality is given by $\left(x_i^{(p)}\right)^* = \left(x_{n_p}^{(p)}\right)^{-1} \cdot x_{n_p-i}^{(p)}$.

For Tate cohomology, we obtain from Proposition 2.13:

If $k > 0$, then an isomorphism is given by

$$\begin{aligned} \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] &\to h^*(R(H)), \\ \beta_i^{(p)} &\mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right] \end{aligned} \tag{5.1}$$

If $k = 0$, then an isomorphism is given by

$$\begin{aligned} \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{n_p/2}^{(p)}\right] \otimes \mathbb{Z}_2[\gamma]/\left(\gamma^2 + \beta_{n_1/2}^{(1)} \cdots \beta_{n_l/2}^{(l)}\right) &\to h^*(R(H)), \\ \beta_i^{(p)} &\mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right] \\ \gamma &\mapsto \left[x_{n_1/2}^{(1)} \cdots x_{n_l/2}^{(l)}\right] \end{aligned}$$

It will later be convenient to define the following elements in $h^*(R(H))$ (under the above identifications respectively):

$$\begin{aligned} \beta_0^{(p)} &:= 1 \text{ for } 1 \leq p \leq l, \\ \beta_i^{(p)} &:= \beta_{n_p-i}^{(p)} \text{ for } 1 \leq p \leq l \text{ and } \lfloor n_p/2 \rfloor < i \leq n_p \end{aligned} \tag{5.2}$$

Note that with these definitions, we have $\beta_i^{(p)} = \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right]$ for all $1 \leq p \leq l$ and all $0 \leq i \leq n_p$.

5.1.2 Tate cohomology of flag varieties of type A_n

Let $i\colon H \to SU(n)$ be the inclusion map. To compute $K^0(X)$ using Hodgkin's theorem, we need to determine the induced map

$$i^*\colon \mathbb{Z}\left[\lambda_1,\ldots,\lambda_{n-1}\right] \cong R(SU(n)) \longrightarrow R(H) \cong \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)},\ldots,x_{n_p}^{(p)}\right] / \left(\prod_{p=1}^{l} x_{n_p}^{(p)} - 1\right)$$

We see directly that $i^*(\lambda_1) = x_1^{(1)} + \ldots + x_1^{(l)}$. Hence we obtain for $1 \le j \le n-1$:

$$\begin{aligned} i^*\left(\lambda_j - \operatorname{rk}(\lambda_j)\right) &= \Lambda^j(i^*(\lambda_1)) - \binom{n}{j} \\ &= \Lambda^j\left(x_1^{(1)} + \ldots + x_1^{(l)}\right) - \binom{n}{j} \\ &= \sum_{a_1+\ldots+a_l=j} \Lambda^{a_1}\left(x_1^{(1)}\right)\ldots\Lambda^{a_l}\left(x_1^{(l)}\right) - \binom{n}{j} \\ &= \sum_{a_1+\ldots+a_l=j} x_{a_1}^{(1)}\ldots x_{a_l}^{(l)} - \binom{n}{j} =: P_j \end{aligned}$$

Then by Hodgkin's theorem 4.3,

$$K^0(X) \cong R(H)/(P_1,\ldots,P_{n-1})$$

We want to compute the Tate cohomology of this ring using Proposition 4.7. We have $P_j^* = P_{n-j}$ for $1 \le j \le n-1$. Thus if n is odd, we have $(n-1)/2$ mutually conjugate pairs of P_j's and if n is even, we have $n/2 - 1$ mutually conjugate pairs of P_j's and one self-conjugate one, namely $P_{n/2}$. So in order to use Proposition 4.7, we need to consider the elements $\left[P_j P_j^*\right]$ in $h^+(R(H))$. Using repeatedly that $[a + a^*] = 0$ in $h^+(R(H))$ for all $a \in R(H)$, we compute:

$$\begin{aligned} \left[P_j P_j^*\right] &= \left[\left(\sum_{a_1+\ldots+a_l=j} x_{a_1}^{(1)}\ldots x_{a_l}^{(l)} - \binom{n}{j}\right)\cdot\left(\sum_{b_1+\ldots+b_l=j} x_{b_1}^{(1)}\ldots x_{b_l}^{(l)} - \binom{n}{j}\right)^*\right] \\ &= \left[\sum_{\substack{a_1+\ldots+a_l=j \\ b_1+\ldots+b_l=j}} x_{a_1}^{(1)}\left(x_{b_1}^{(1)}\right)^*\ldots x_{a_l}^{(l)}\left(x_{b_l}^{(l)}\right)^* + \binom{n}{j}\right] \\ &= \left[\sum_{a_1+\ldots+a_l=j} x_{a_1}^{(1)}\left(x_{a_1}^{(1)}\right)^*\ldots x_{a_l}^{(l)}\left(x_{a_l}^{(l)}\right)^* + \binom{n}{j}\right] \\ &= \sum_{a_1+\ldots+a_l=j} \beta_{a_1}^{(1)}\ldots\beta_{a_l}^{(l)} + \binom{n}{j} =: \mu_j \end{aligned}$$

For the sum defining the μ_j, remember the definitions we made in (5.2).

If n is even, we also need to consider

$$[P_{n/2}] = \left[\sum_{a_1+\ldots+a_l=n/2} x_{a_1}^{(1)} \ldots x_{a_l}^{(l)} - \binom{n}{\frac{n}{2}} \right] = \begin{cases} 0 & \text{if } k > 0 \\ \gamma & \text{if } k = 0 \end{cases}$$

This equality holds for the following reason: If $k > 0$, then none of the summands $x_{a_1}^{(1)} \ldots x_{a_l}^{(l)}$ with $a_1 + \ldots + a_l = \frac{n}{2}$ is self-dual, and if $k = 0$, the only self-dual summand is $x_{n_1/2}^{(1)} \cdots x_{n_l/2}^{(l)}$.

In order to finally apply Proposition 4.7, we need to investigate the relations between the $\mu_j \in h^*(R(H))$. In other words, we need to check that (A1) and (A2) from Chapter 4 are satisfied. We state the result now, postponing the proof to Propositions 6.2 and 6.4 in the next chapter.

Proposition 5.1. *Setting $m := \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor$, the elements $\mu_1, \ldots, \mu_m$ form an $h^*(R(H))$-regular sequence and each μ_i for $m < i \leq \lfloor n/2 \rfloor$ can be written as a $\mathbb{Z}_2$-linear combination of $\mu_1, \ldots, \mu_m$.*

Remark. If $k = 0$, then $\mu_m = \gamma^2$ and so we have that $\mu_1, \ldots, \mu_{m-1}, \gamma = \left[P_{n/2}\right]$ also form an $h^*(R(H))$-regular sequence.

Let $r := \lfloor n/2 \rfloor - m$. Note that if $k = 0$, then $r = 0$. As a consequence of Proposition 5.1, we can find $u_1, \ldots, u_r \in R(H)$ such that

$$u_j + u_j^* - P_{m+j}P_{m+j}^* \in \mathbb{Z} \cdot P_1P_1^* + \ldots + \mathbb{Z} \cdot P_mP_m^* \text{ for } \begin{cases} 1 \leq j \leq r \text{ if } n \text{ is odd} \\ 1 \leq j < r \text{ if } n \text{ is even} \end{cases} \tag{5.3}$$

$$u_r + u_r^* = P_{n/2} \text{ if } n \text{ is even}$$

These give rise to $[\overline{u}_i] \in h^-(R(H)/(P_1, \ldots, P_{n-1}))$. Now from Proposition 4.7, we immediately obtain:

Proposition 5.2. *Let $m = \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor$ and $r = \lfloor n/2 \rfloor - m$. Then*

$$h^*(R(H)/(P_1, \ldots, P_{n-1})) \cong \frac{\bigotimes_{p=1}^l \mathbb{Z}_2 \left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right]}{(\mu_1, \ldots, \mu_m)} \otimes \bigwedge([\overline{u}_1], \ldots, [\overline{u}_r]) \tag{5.4}$$

where

$$\mu_j = \sum_{a_1+\ldots+a_l=j} \beta_{a_1}^{(1)} \ldots \beta_{a_l}^{(l)} + \binom{n}{j} \text{ for } 1 \leq j \leq m,$$

recalling and sticking to the definitions we made in (5.2). We have that $\beta_j^{(p)} \in h^+$ for all $1 \leq p \leq l$ and $1 \leq j \leq n_p$, and $u_i \in h^-$ for all $1 \leq i \leq r$.

5.1.3 Witt ring of flag varieties of type A_n

Recall from Bousfield's lemma 2.3 that there is an isomorphism

$$W^*(X) \xrightarrow[\cong]{\bar{c}} h^*(K^0(X)) \xleftarrow[\cong]{[\overline{\alpha}]} h^*(R(H)/(P_1, \ldots, P_{n-1}))$$

We have computed the 2-graded Tate cohomology on the right and now want to determine the 4-periodic grading of the Witt ring.

Let $b_i^{(p)} \in W^*(X)$ be the element corresponding to $\overline{\beta}_i^{(p)} \in h^*(R(H)/(P_1, \ldots, P_{n-1}))$ under the above isomorphism for $1 \leq p \leq l$ and $0 \leq i \leq n_p$. From (5.1) we see that $\overline{\beta}_i^{(p)} \in h^*(R(H)/(P_1, \ldots, P_{n-1}))$ is represented by a real representation for all p and i as $\rho\rho^*$ is always of real type for any complex representation ρ. So by Lemma 4.8,

$$b_i^{(p)} \in W^0(X) \text{ for all } 1 \leq p \leq l \text{ and } 0 \leq i \leq n_p$$

For $1 \leq i \leq r$, let $v_i \in W^*(X)$ be the element corresponding to $[\overline{u}_i] \in h^*(R(H)/(P_1, \ldots, P_{n-1}))$ under the above isomorphism. Using (5.3) and the fact that $P_j P_j^*$ is real for all j, Lemma 4.9 shows that

$$v_i \in W^{-1}(X) \text{ for } 1 \leq i \leq r-1$$

If n is even, $P_{n/2}$ is real if $n \equiv 0 \pmod 4$ and quaternionic if $n \equiv 2 \pmod 4$ (see Example 1.16). We deduce from Lemma 4.9 that

$$v_r \in \begin{cases} W^{-1}(X) \text{ if } n \not\equiv 2 \pmod 4 \\ W^{-3}(X) \text{ if } n \equiv 2 \pmod 4 \end{cases}$$

Summing up, we have proved:

Theorem 5.3. *Let $m := \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor$ and $r := \lfloor n/2 \rfloor - m$. Then as a ring,*

$$W^*(X(n_1, \ldots, n_l)) \cong \frac{\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[b_1^{(p)}, \ldots, b_{\lfloor n_p/2 \rfloor}^{(p)} \right]}{(\mu_1, \ldots, \mu_m)} \otimes \bigwedge(v_1, \ldots, v_r)$$

where

$$\mu_j = \sum_{a_1 + \ldots + a_l = j} b_{a_1}^{(1)} \ldots b_{a_l}^{(l)} + \binom{n}{j} \text{ for } 1 \leq j \leq m,$$

where we define $b_0^{(p)} := 1$ for $1 \leq p \leq l$ and $b_i^{(p)} := b_{n_p - i}^{(p)}$ for $1 \leq p \leq l$ and $\lfloor n_p/2 \rfloor < i \leq n_p$ (just as in (5.2)). We have

$$b_i^{(p)} \in W^0(X) \text{ for all } 1 \leq p \leq l \text{ and } 0 \leq i \leq n_p,$$
$$v_j \in W^{-1}(X) \text{ for all } 1 \leq j \leq r-1,$$

$$v_r \in \begin{cases} W^{-1}(X) & \text{if } n \not\equiv 2 \text{ (mod 4)} \\ W^{-3}(X) & \text{if } n \equiv 2 \text{ (mod 4)} \end{cases}$$

We want to tabulate the ranks of the Witt groups in different degrees. From Appendix A and Proposition 6.3, we immediately deduce:

Theorem 5.4. *Let*

$$m := \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor, \quad a := \frac{m!}{\lfloor n_1/2 \rfloor! \cdot \ldots \cdot \lfloor n_l/2 \rfloor!}, \quad r := \lfloor n/2 \rfloor - m.$$

We have $W^i(X(n_1, \ldots, n_l)) \cong \mathbb{Z}_2^{a \cdot z_i}$ where z_i is given as follows:

If $r = 0$ (i.e. at most one n_j is odd), then $z_0 = 1$ and $z_{-1} = z_{-2} = z_{-3} = 0$.

If $r > 0$ and $n \not\equiv 2$ (mod 4), we have:

r (4)	z_0	z_{-1}	z_{-2}	z_{-3}
0	$2^{r-2} - 2 \cdot (-4)^{\frac{r-4}{4}}$	2^{r-2}	$2^{r-2} + 2 \cdot (-4)^{\frac{r-4}{4}}$	2^{r-2}
1	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$
2	2^{r-2}	$2^{r-2} + (-4)^{\frac{r-2}{4}}$	2^{r-2}	$2^{r-2} - (-4)^{\frac{r-2}{4}}$
3	$2^{r-2} - (-4)^{\frac{r-3}{4}}$	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} - (-4)^{\frac{r-3}{4}}$

If $r > 0$ and $n \equiv 2$ (mod 4), we have:

r (4)	z_0	z_{-1}	z_{-2}	z_{-3}
0	2^{r-2}	$2^{r-2} + 2 \cdot (-4)^{\frac{r-4}{4}}$	2^{r-2}	$2^{r-2} - 2 \cdot (-4)^{\frac{r-4}{4}}$
1	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$
2	$2^{r-2} + (-4)^{\frac{r-2}{4}}$	2^{r-2}	$2^{r-2} - (-4)^{\frac{r-2}{4}}$	2^{r-2}
3	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} - (-4)^{\frac{r-3}{4}}$	$2^{r-2} - (-4)^{\frac{r-3}{4}}$

5.2 Type B_n

By Proposition 1.12, up to conjugation, all centralisers of tori in $G = Spin(2n+1)$ are of the form

$$H(m, n_1, \ldots, n_l) := Spin(2m+1) \times \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l)/C$$

where $m + n_1 + \ldots + n_l = n$ and C is the kernel of the map

$$\tilde{H}(m, n_1, \ldots, n_l) := Spin(2m+1) \times \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l) \to Spin(2n+1)$$

Note that this map factors through the map

$$\hat{H}(m, n_1, \ldots, n_l) := Spin(2m+1) \times Spin(2n_1) \times \ldots \times Spin(2n_l) \to Spin(2n+1)$$

So fix $m, n_1, \ldots, n_l \in \mathbb{N}$ with $m + n_1 + \ldots + n_l = n$ and set

$$X(m, n_1, \ldots, n_l) := Spin(2n+1)/H(m, n_1, \ldots, n_l)$$

We will sometimes simply write X and H instead of $X(m, n_1, \ldots, n_l)$ and $H(m, n_1, \ldots, n_l)$. Suppose that precisely k of the integers $n_1, \ldots, n_l$ are odd.

Remark. Note that we can write X as a quotient of $SO(2n+1)$. However, we have to work with the Spin groups here because Hodgkin's theorem 4.3, which computes $K^0(G/H)$, requires that G be simply connected.

5.2.1 Representation rings and their Tate cohomology

We have an isomorphism

$$R(Spin(2n+1)) \cong \mathbb{Z}[\lambda_1, \ldots, \lambda_{n-1}, \Delta]$$

where λ_1 is induced by the standard $(2n+1)$-dimensional complex representation of $SO(2n+1)$ via the covering map $Spin(2n+1) \twoheadrightarrow SO(2n+1)$ and Δ is the spin representation. Moreover, $\lambda_i = \Lambda^i(\lambda_1)$ for all i. All representations of $Spin(2n+1)$ are self-dual.

$$R(\hat{H}) \cong \mathbb{Z}[y_1, \ldots, y_{m-1}, \Gamma] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}\left[y_1^{(p)}, \ldots, y_{n_p-2}^{(p)}, \Gamma_+^{(p)}, \Gamma_-^{(p)}\right]$$

Here, y_1 is the standard $(2m+1)$-dimensional complex representation of the block $Spin(2m+1)$ and $y_i = \Lambda^i(y_1)$ for all i and Γ is its spin representation. Furthermore, $y_1^{(p)}$ is the standard $2n_p$-dimensional complex representation of the block $Spin(2n_p)$ and $y_i^{(p)} = \Lambda^i(y_1^{(p)})$ for all

i and $\Gamma_{\pm}^{(p)}$ are the half-spin representations.

$$R(\tilde{H}) \cong \mathbb{Z}[y_1, \ldots, y_{m-1}, \Gamma] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \ldots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm\frac{1}{2}}\right]$$

Here, y_1 is the standard $(2m+1)$-dimensional complex representation of the block $Spin(2m+1)$ and $y_i = \Lambda^i(y_1)$ for all i and Γ is its spin representation. Furthermore, $x_1^{(p)}$ is the standard n_p-dimensional representation of the block $\tilde{U}(n_p)$ and $x_i^{(p)} = \Lambda^i\left(x_1^{(p)}\right)$ for all i. The duality is given by

$$y_i^* = y_i, \quad \Gamma^* = \Gamma, \quad \left(x_i^{(p)}\right)^* = \left(x_{n_p}^{(p)}\right)^{-1} \cdot x_{n_p-i}^{(p)}$$

We computed the representation ring of H in Proposition 1.24. It is a subring of $R(\tilde{H})$. By Proposition 2.14, we have the following isomorphisms in Tate cohomology, using the notation for representations in $R(\tilde{H})$ and its subring $R(H)$ introduced above:

If $k > 0$, we have an isomorphism given by

$$\begin{aligned} \mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] &\to h^*(R(H)) \\ \alpha_i &\mapsto [\Lambda^i(y_1)] \\ \beta_i^{(p)} &\mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right] \end{aligned}$$

If $k = 0$, an isomorphism is given by

$$\begin{aligned} \frac{\mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{n_p/2}^{(p)}\right] \otimes \mathbb{Z}_2[\delta]}{\left(\delta^2 + (1 + \alpha_1 + \ldots + \alpha_m)\beta_{n_1/2}^{(1)} \cdots \beta_{n_l/2}^{(l)}\right)} &\to h^*(R(H)) \\ \alpha_i &\mapsto [\Lambda^i(y_1)] \\ \beta_i^{(p)} &\mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right] \\ \delta &\mapsto \begin{cases} \left[\Gamma \prod_p \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right] & \text{if } m > 0 \\ \left[\prod_p \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right] & \text{if } m = 0 \end{cases} \end{aligned}$$

It will be convenient to make the following definitions in $h^*(R(H))$:

$$\begin{aligned} \alpha_0 := 1 \text{ and } \alpha_{m+j} &:= \alpha_{m+1-j} \text{ for all } 1 \leq j \leq m+1 \\ \beta_0^{(p)} := 1 \text{ and } \beta_i^{(p)} &:= \beta_{n_p-i}^{(p)} \text{ for all } 1 \leq p \leq l \text{ and } \lfloor n_p/2 \rfloor < i \leq n_p \end{aligned} \tag{5.5}$$

Note that with these definitions, we have $\alpha_j = \left[\Lambda^j(y_1)\right]$ for all $0 \leq j \leq 2m+1$ and $\beta_i^{(p)} = \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right]$ for all $1 \leq p \leq l$ and $0 \leq i \leq n_p$.

5.2.2 Tate cohomology of flag varieties of type B_n

We want to compute the Tate cohomology of $K^0(X)$ following Chapter 4. Let $i\colon H \to G$ be the inclusion map. As a first step, we need to compute the restriction map $i^*\colon R(G) \to R(H)$. We see directly that

$$i^*(\lambda_1) = y_1 + \sum_{p=1}^{l} x_1^{(p)} + \left(\sum_{p=1}^{l} x_1^{(p)}\right)^*$$

Consequently, in $h^+(R(H))$ we have

$$\begin{aligned}
[i^*(\lambda_f - \mathrm{rk}(\lambda_f))] &= \left[\Lambda^f\left(i^*(\lambda_1)\right) - \mathrm{rk}(\lambda_f)\right] \\
&= \left[\Lambda^f\left(y_1 + \sum_{p=1}^{l} x_1^{(p)} + \left(\sum_{p=1}^{l} x_1^{(p)}\right)^*\right) + \binom{2n+1}{f}\right] \\
&= \sum_{j=0}^{f}\left[\Lambda^{f-j}(y_1)\right]\cdot\left[\Lambda^j\left(\sum_{p=1}^{l} x_1^{(p)} + \left(\sum_{p=1}^{l} x_1^{(p)}\right)^*\right)\right] + \binom{2n+1}{f} \\
&= \sum_{j=0}^{\lfloor f/2\rfloor}\left[\Lambda^{f-2j}(y_1)\right]\cdot\left[\Lambda^j\left(\sum_{p=1}^{l} x_1^{(p)}\right)\cdot\Lambda^j\left(\sum_{p=1}^{l} x_1^{(p)}\right)^*\right] + \binom{2n+1}{f} \\
&= \sum_{j=0}^{\lfloor f/2\rfloor}\alpha_{f-2j}\cdot\sum_{a_1+\ldots+a_l=j}\beta_{a_1}^{(1)}\ldots\beta_{a_l}^{(l)} + \binom{2n+1}{f} =: \xi_f,
\end{aligned}$$

keeping in mind the definitions we made in (5.5). We still need to compute the restriction of the spin representation Δ. So let $i_{\hat{H}}\colon \hat{H} \to G$ be the natural map. Then by [Ada96, Prop. 4.5],

$$i_{\hat{H}}^*(\Delta) = \Gamma \cdot \sum_{\epsilon_i\in\{+,-\}} \Gamma_{\epsilon_1}^{(1)}\ldots\Gamma_{\epsilon_l}^{(l)},$$

so in $h^*(R(\hat{H}))$, we have

$$\left[i_{\hat{H}}^*(\Delta)\right] = \begin{cases} 0 & \text{if } k > 0 \\ [\Gamma]\cdot\sum_{\epsilon_i\in\{+,-\}}\left[\Gamma_{\epsilon_1}^{(1)}\right]\ldots\left[\Gamma_{\epsilon_l}^{(l)}\right] & \text{if } k = 0 \end{cases}$$

This is because $\Gamma_{\pm}^{(p)}$ are self-dual if and only if n_p is even. Hence by Lemma 1.21, which describes the restriction map $R(Spin(2a)) \to R(\tilde{U}(a))$, we deduce that in $h^*(R(H))$,

$$[i^*(\Delta)] = \begin{cases} 0 & \text{if } k > 0 \\ \delta & \text{if } k = 0 \end{cases} \tag{5.6}$$

In order to apply the results of Chapter 4 to compute $h^*(K^0(X))$, we need to investigate the relations between $\xi_1, \ldots, \xi_{n-1}, [i^*(\Delta)] \in h^*(R(H))$.

Proposition 5.5. *Let*

$$S := \{i \in \mathbb{Z} \mid 1 \leq i \leq m\} \cup \left\{i \in \mathbb{Z} \mid m < i \leq 2\left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_l}{2} \right\rfloor\right) \text{ even}\right\}$$

and consider $R := \bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right]$ *as a subring of* $h^*(R(H))$.

Suppose $k > 0$**.** *Then the elements* ξ_j *for* $j \in S$ *form an* $h^*(R(H))$*-regular sequence in some order. If* $i \in \{1, 2, \ldots, n-1\} \setminus S$*, then* ξ_i *is an* R*-linear combination of the* ξ_j *for* $j \in S$*. We have* $[i^*(\Delta)] = 0$*.*

Suppose $k = 0$ **and** m **is even.** *Then the elements* ξ_j *for* $j \in S \setminus \{n\}$ *together with* $[i^*(\Delta)] = \delta$ *form an* $h^*(R(H))$*-regular sequence in some order. If* $i \in \{1, \ldots, n-1\} \setminus S$*, then* ξ_i *is an* R*-linear combination of the* ξ_j *where* $j \in S \setminus \{n\}$*.*

Suppose $k = 0$ **and** m **is odd.** *Then the elements* ξ_j *for* $j \in S$ *form an* $h^*(R(H))$*-regular sequence in some order. Let* I *be the ideal generated by these elements. Then* $\overline{\delta} \in h^+(R(H))/I$ *is a zero divisor with annihilator the ideal generated by* $\overline{\delta}$ *itself. The elements* ξ_i *for* $i \in \{1, 2, \ldots, n-1\} \setminus S$ *are* R*-linear combinations of the* ξ_j *for* $j \in S$*.*

Proof. The claims for $k > 0$ are immediate from Propositions 6.16 and 6.20, Remark 6.21 and equation (5.6).

Suppose $k = 0$ and m is even. Proposition 6.16 shows that

$$\xi_1, \xi_2, \ldots, \xi_{m-1}, \xi_m, \xi_{m+2}, \xi_{m+4}, \ldots, \xi_{n-2}, \xi_n \tag{5.7}$$

is a regular sequence in the subring

$$A := \mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] \subset h^*(R(H))$$

In $h^*(R(H))$, we have

$$\delta^2 = \sum_{i=0}^{m} \alpha_i \cdot \beta_{\frac{n_1}{2}}^{(1)} \ldots \beta_{\frac{n_l}{2}}^{(l)}$$

Remark 6.17 shows that in the sequence (5.7), we may replace ξ_n by δ^2 and still have an A-regular sequence. But since $h^*(R(H))$ is a free module of rank 2 over A, the sequence is also $h^*(R(H))$-regular by Proposition 3.3. But then replacing δ^2 by δ clearly still gives an $h^*(R(H))$-regular sequence, as required.

The statement about the linear relations is immediate from Proposition 6.20 and Remark 6.21.

Lastly, suppose $k = 0$ and m is odd. From Proposition 6.16, we deduce that the ξ_j for $j \in S$ form an $h^*(R(H))$-regular sequence. Proposition 6.20 and Remark 6.21 imply that the elements ξ_i for $i \in \{1, 2, \ldots, n-1\} \setminus S$ are R-linear combinations of the ξ_j for $j \in S$.

Now from Lemma 6.19, we have that $\alpha_i + \alpha_{i-1} \in I$ for every odd $1 \leq i \leq m$. This shows that

$$\delta^2 = (1 + \alpha_1 + \ldots + \alpha_{m-1} + \alpha_m) \cdot \beta^{(1)}_{n_1/2} \ldots \beta^{(l)}_{n_l/2} \in I$$

Let us identify the annihilator of $\overline{\delta}$ in $h^+(R(H))/I$. Let A be as defined above in the previous case, then $h^*(R(H))$ is still a free A-module of rank 2 with basis $1, \delta$. Since I is generated by elements in A, we deduce that $h^+(R(H))/I$ is a free $A/A \cap I$-module with basis 1, $\overline{\delta}$. From this and the fact that $\overline{\delta}^2 = 0$, it follows immediately that the annihilator of $\overline{\delta}$ in $h^+(R(H))/I$ is as claimed. $\square$

Now let S be as in the previous Proposition and

$$\overline{S} := \{1, \ldots, n-1\} \setminus S \text{ and } B := \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)} \left(x_1^{(p)}\right)^*, \ldots, x^{(p)}_{\lfloor n_p/2 \rfloor} \left(x^{(p)}_{\lfloor n_p/2 \rfloor}\right)^*\right],$$

where we regard B as a subring of $R(H)$. The previous Proposition implies the following:

In all cases and for all $t \in \overline{S}$, we can find $w_t \in R(H)$ such that

$$w_t + w_t^* - i^*(\lambda_t - \mathrm{rk}(\lambda_t)) \in \sum_{s \in S} B \cdot i^*(\lambda_s - \mathrm{rk}(\lambda_s)) \tag{5.8}$$

Moreover, if $k > 0$, we see from (5.6) that we can find $\eta \in R(H)$ such that

$$\eta + \eta^* = i^*(\Delta) \tag{5.9}$$

If $k = 0$ and m is odd, we can find $\kappa \in R(H)$ such that

$$\kappa + \kappa^* - i^*(\Delta)^2 \in \sum_{s \in S} B \cdot i^*(\lambda_s - \mathrm{rk}(\lambda_s)) \tag{5.10}$$

We use Proposition 5.5 to show:

Proposition 5.6. *If not all of $n_1, \ldots, n_l$ are even, then*

$$h^*(K^0(X)) \cong \frac{\mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta^{(p)}_{\lfloor \frac{n_p}{2} \rfloor}\right]}{(\xi_j \mid j \in S)} \otimes \bigwedge_{t \in \overline{S}}([\overline{w}_t]) \otimes \bigwedge([\overline{\eta}])$$

If $m, n_1, \ldots, n_l$ are even, then

$$h^*(K^0(X)) \cong \frac{\mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta^{(p)}_{\frac{n_p}{2}}\right]}{(\xi_j \mid j \in S \setminus \{n\}) + \left(\alpha_m \beta^{(1)}_{n_1/2} \ldots \beta^{(l)}_{n_l/2}\right)} \otimes \bigwedge_{t \in \overline{S}}([\overline{w}_t])$$

If $n_1, \dots, n_l$ are even and m is odd, then

$$h^*(K^0(X)) \cong \frac{\mathbb{Z}_2[\alpha_1, \dots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_1^{(p)}, \dots, \beta_{\frac{n_p}{2}}^{(p)}\right]}{(\xi_j \mid j \in S)} \otimes \bigwedge_{t \in \overline{S}} ([\overline{w}_t]) \otimes \bigwedge([\overline{\kappa}])$$

In the above,

$$\xi_j = \sum_{i=0}^{\lfloor j/2 \rfloor} \alpha_{j-2i} \cdot \sum_{a_1+\ldots+a_l=i} \beta_{a_1}^{(1)} \ldots \beta_{a_l}^{(l)} + \binom{2n+1}{j}$$

recalling the definitions we made in (5.5). In all the above expressions, the left factors of the tensor products are contained in h^+ and the generators of the exterior algebras all lie in h^-.

Proof. The first case follows immediately from Propositions 5.5 and 4.7. So does the second case, using in addition that $\alpha_{2i+1} + \alpha_{2i}$ is contained in the ideal $(\xi_j \mid j \in S,\ j \leq 2i+1)$ for all i by Lemma 6.19. The third case follows similarly from Proposition 5.5, using Lemma 2.10 in addition. $\square$

5.2.3 Witt ring of flag varieties of type B_n

Recall from Bousfield's lemma 2.3 that there is an isomorphism

$$W^*(X) \xrightarrow[\cong]{\overline{c}} h^*(K^0(X)) \xleftarrow[\cong]{[\overline{\alpha}]} h^*(R(H)/\mathfrak{a}(G))$$

We have computed the Tate cohomology on the right and want to determine the Witt grading. So under the above isomorphism, we let

- $a_i \in W^*(X)$ correspond to $\overline{\alpha}_i$ for $1 \leq i \leq m$ in all cases
- $b_j^{(p)} \in W^*(X)$ correspond to $\overline{\beta}_j^{(p)}$ for all $1 \leq p \leq l$ and $0 \leq j \leq n_p$ in all cases
- $u_t \in W^*(X)$ correspond to $[\overline{w}_t]$ for all $t \in \overline{S}$ in all cases
- $c_1 \in W^*(X)$ correspond to $[\overline{\eta}]$ if not all of $n_1, \dots, n_l$ are even
- $c_2 \in W^*(X)$ correspond to $[\overline{\kappa}]$ if $n_1, \dots, n_l$ are even and m is odd

Recall that the subring B of $R(H)$ consists entirely of real representations and $\lambda_j \in R(Spin(2n+1))$ is real for all j.

We deduce from Lemma 4.8 that $a_i \in W^0(X)$ for all $1 \leq i \leq m$ and $b_j^{(p)} \in W^0(X)$ for all $1 \leq p \leq l$ and $1 \leq j \leq \lfloor n_p/2 \rfloor$.

From equation (5.8) and Lemma 4.9, we deduce that $u_t \in W^{-1}(X)$ for all $t \in \overline{S}$.

Suppose not all of $n_1, \dots, n_l$ are even. From equation (5.9) and Lemma 4.9, recalling that $\Delta \in R(Spin(2n+1))$ is of real type if $n \equiv 0, 3 \pmod 4$ and of quaternionic type if

$n \equiv 1, 2 \pmod 4$, we deduce that

$$c_1 \in \begin{cases} W^{-1}(X) & \text{if } n \equiv 0, 3 \pmod 4 \\ W^{-3}(X) & \text{if } n \equiv 1, 2 \pmod 4 \end{cases}$$

Suppose $n_1, \dots, n_l$ are even and m is odd. Then Lemma 4.9 and equation (5.10) show that $c_2 \in W^{-1}(X)$.

In summary, we have proved:

Theorem 5.7. *Let*

$$\begin{aligned} S &:= \{i \in \mathbb{Z} \mid 1 \le i \le m\} \cup \left\{i \in \mathbb{Z} \mid m < i \le 2\left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_l}{2} \right\rfloor\right) \text{ even}\right\} \\ \overline{S} &:= \{1, \dots, n-1\} \setminus S \end{aligned}$$

If not all of $m, n_1, \dots, n_l$ are even, then

$$W^*(X) \cong \frac{\mathbb{Z}_2[a_1, \dots, a_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[b_1^{(p)}, \dots, b_{\lfloor \frac{n_p}{2} \rfloor}^{(p)}\right]}{(\xi_j \mid j \in S)} \otimes \bigwedge_{t \in \overline{S}} (u_t) \otimes \bigwedge (c)$$

If $m, n_1, \dots, n_l$ are even, then

$$W^*(X) \cong \frac{\mathbb{Z}_2[a_1, \dots, a_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[b_1^{(p)}, \dots, b_{\frac{n_p}{2}}^{(p)}\right]}{(\xi_j \mid j \in S \setminus \{n\}) + \left(a_m b_{\frac{n_1}{2}}^{(1)} \dots b_{\frac{n_l}{2}}^{(l)}\right)} \otimes \bigwedge_{t \in \overline{S}} (u_t)$$

In the above,

$$\xi_j = \sum_{i=0}^{\lfloor j/2 \rfloor} a_{j-2i} \cdot \sum_{q_1 + \ldots + q_l = i} b_{q_1}^{(1)} \dots b_{q_l}^{(l)} + \binom{2n+1}{j}$$

where we make the following definitions:

$$\begin{aligned} a_0 := 1 \text{ and } a_{m+j} := a_{m+1-j} \text{ for all } 1 \le j \le m+1 \\ b_0^{(p)} := 1 \text{ and } b_i^{(p)} := b_{n_p - i}^{(p)} \text{ for all } 1 \le p \le l \text{ and } \lfloor n_p/2 \rfloor < i \le n_p \end{aligned}$$

Furthermore, we have that all $a_i, b_j^{(p)} \in W^0(X)$ and all $u_t \in W^{-1}(X)$ and

$$c \in \begin{cases} W^{-3}(X) & \text{if not all of } n_1, \dots, n_l \text{ are even and } n \equiv 1, 2 \pmod 4 \\ W^{-1}(X) & \text{else} \end{cases}$$

It is easily checked that in all of the above cases, the number of exterior algebra generators in the above expressions is $\sum_{i=1}^{l} n_i - \lfloor n_i/2 \rfloor$. From Proposition 6.18 and

Appendix A, we can tabulate the ranks of the Witt groups in all degrees:

Theorem 5.8. *Let*

$$a := \frac{\left(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_{k+l}}{2} \rfloor\right)!}{\lfloor \frac{m}{2} \rfloor! \cdot \lfloor \frac{n_1}{2} \rfloor! \cdot \ldots \cdot \lfloor \frac{n_{k+l}}{2} \rfloor!} \quad \text{and} \quad r := \sum_{i=1}^{l} n_i - \lfloor n_i/2 \rfloor$$

We have $W^i(X) \cong \mathbb{Z}_2^{a \cdot z_i}$ where z_i is given as follows:
If $r = 0$, then $z_0 = 1$ and $z_{-1} = z_{-2} = z_{-3} = 0$.
If not all of $n_1, \ldots, n_l$ are even and $n \equiv 1$ or 2 (mod 4), then we have:

r (4)	z_0	z_{-1}	z_{-2}	z_{-3}
0	2^{r-2}	$2^{r-2} + 2 \cdot (-4)^{\frac{r-4}{4}}$	2^{r-2}	$2^{r-2} - 2 \cdot (-4)^{\frac{r-4}{4}}$
1	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$
2	$2^{r-2} + (-4)^{\frac{r-2}{4}}$	2^{r-2}	$2^{r-2} - (-4)^{\frac{r-2}{4}}$	2^{r-2}
3	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} - (-4)^{\frac{r-3}{4}}$	$2^{r-2} - (-4)^{\frac{r-3}{4}}$

Otherwise, we have:

r (4)	z_0	z_{-1}	z_{-2}	z_{-3}
0	$2^{r-2} - 2 \cdot (-4)^{\frac{r-4}{4}}$	2^{r-2}	$2^{r-2} + 2 \cdot (-4)^{\frac{r-4}{4}}$	2^{r-2}
1	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} - 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$	$2^{r-2} + 2 \cdot (-4)^{\frac{r-5}{4}}$
2	2^{r-2}	$2^{r-2} + (-4)^{\frac{r-2}{4}}$	2^{r-2}	$2^{r-2} - (-4)^{\frac{r-2}{4}}$
3	$2^{r-2} - (-4)^{\frac{r-3}{4}}$	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} + (-4)^{\frac{r-3}{4}}$	$2^{r-2} - (-4)^{\frac{r-3}{4}}$

5.3 Type C_n

Proposition 1.12 shows that up to conjugation, every centraliser of a torus in $G = Sp(n)$ is of the form

$$H(m, n_1, \dots, n_l) := Sp(m) \times U(n_1) \times \dots \times U(n_l),$$

where $n = m + n_1 + \dots + n_l$. So fix $n \in \mathbb{N}$ and $m, n_1, \dots, n_l \in \mathbb{N}$ with $n = m + n_1 + \dots + n_l$ and set

$$X(m, n_1, \dots, n_l) := \frac{Sp(n)}{Sp(m) \times U(n_1) \times \dots \times U(n_l)}$$

We will sometimes simply write X and H instead of $X(m, n_1, \dots, n_l)$ and $H(m, n_1, \dots, n_l)$.

5.3.1 Representation rings and their Tate cohomology

For convenience, we recall some representation rings and their Tate cohomology from previous sections.

$$R(Sp(n)) \cong \mathbb{Z}[\lambda_1, \dots, \lambda_n]$$

where λ_1 is the standard representation of $Sp(n)$ on $\mathbb{C}^{2n}$ and $\lambda_i = \Lambda^i(\lambda_1)$ for all i. All representations of $Sp(n)$ are self-dual. The representation ring of H is given by

$$R(H(m, n_1, \dots, n_l)) \cong \mathbb{Z}[y_1, \dots, y_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}, \dots, x_{n_p-1}^{(p)}, \left(x_{n_p}^{(p)}\right)^{\pm 1}\right]$$

where y_1 is the standard $2m$-dimensional representation of the block $Sp(m)$ and $x_1^{(p)}$ is the standard n_p-dimensional representation of the block $U(n_p)$, and

$$\begin{aligned} y_i &= \Lambda^i(y_1) \text{ for all } 0 \le i \le 2m \\ x_i^{(p)} &= \Lambda^i\left(x_1^{(p)}\right) \text{ for all } 1 \le p \le l \text{ and } 0 \le i \le n_p \end{aligned}$$

We have that $y_i = y_{2m-i}$ for all $0 \le i \le 2m$ and the duality on $R(H)$ is given by

$$\begin{aligned} y_i^* &= y_i \text{ for all } 1 \le i \le m \\ \left(x_i^{(p)}\right)^* &= \left(x_{n_p}^{(p)}\right)^{-1} \cdot x_{n_p-i}^{(p)} \text{ for all } 1 \le p \le l \text{ and } 1 \le i \le n_p \end{aligned}$$

From Proposition 2.15, we have a ring isomorphism

$$\mathbb{Z}_2[\alpha_1, \dots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \dots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right] \to h^*(R(H(m, n_1, \dots, n_l))) \tag{5.11}$$

$$\alpha_i \mapsto [y_i]$$

$$\beta_i^{(p)} \mapsto \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right]$$

In all that follows, we will identify the Tate cohomology in this way via the given isomorphism. It will be convenient to make the following definitions in $h^*(R(H))$:

$$\begin{gathered}\alpha_0 := 1 \text{ and } \alpha_{m+i} := \alpha_{m-i} \text{ for all } 1 \leq i \leq m, \\ \beta_0^{(p)} := 1 \text{ for all } 1 \leq p \leq l \text{ and } \beta_i^{(p)} := \beta_{n_p-i}^{(p)} \text{ for all } 1 \leq p \leq l,\ \lfloor n_p/2 \rfloor < i \leq n_p\end{gathered} \tag{5.12}$$

Note that with these definitions, we have $\alpha_j = \left[\Lambda^j(y_1)\right]$ for all $0 \leq j \leq 2m$ and $\beta_i^{(p)} = \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right]$ for all $1 \leq p \leq l$ and $0 \leq i \leq n_p$.

5.3.2 Tate cohomology of flag varieties of type C_n

Let $i\colon H \to Sp(n)$ be the inclusion map. We need to determine the induced map i^* on representation rings. We see directly that

$$i^*(\lambda_1) = y_1 + \sum_{p=1}^{l} x_1^{(p)} + \sum_{p=1}^{l}\left(x_1^{(p)}\right)^*$$

Hence we obtain for $1 \leq j \leq n$ that

$$\begin{aligned} i^*(\lambda_j - \mathrm{rk}(\lambda_j)) &= \Lambda^j\left(y_1 + \sum_{p=1}^{l} x_1^{(p)} + \sum_{p=1}^{l}\left(x_1^{(p)}\right)^*\right) - \binom{2n}{j} \\ &= \sum_{c+d_1+d_1'+\ldots+d_l+d_l'=j} y_c \cdot x_{d_1}^{(1)}\left(x_{d_1'}^{(1)}\right)^* \ldots x_{d_l}^{(l)}\left(x_{d_l'}^{(l)}\right)^* - \binom{2n}{j} =: P_j \end{aligned}$$

Using that $[a + a^*] = 0$ in $h^+(R(H))$ for all $a \in R(H)$, we deduce that in $h^+(R(H))$,

$$\begin{aligned} [P_j] &= \sum_{c+2d_1+\ldots+2d_l=j} \alpha_c \beta_{d_1}^{(1)} \ldots \beta_{d_l}^{(l)} + \binom{2n}{j} \\ &= \sum_{d=0}^{\lfloor j/2 \rfloor} \alpha_{j-2d} \sum_{d_1+\ldots+d_l=d} \beta_{d_1}^{(1)} \ldots \beta_{d_l}^{(l)} + \binom{2n}{j} =: \nu_j \end{aligned}$$

where we keep in mind the definitions we made in (5.12).

In order to apply Proposition 4.7 to compute $h^*(K^0(X))$, we need to determine the relations between the $\nu_j \in h^+(R(H))$, i.e. show that (A1) and (A2) from Chapter 4 are satisfied. We state the result now and postpone the proof to Propositions 6.12 and 6.14.

Proposition 5.9. *Let*

$$S := \{k \in \mathbb{N} \mid 1 \leq k \leq m\} \cup \{k \in \mathbb{N} \mid m < k \leq 2(\lfloor m/2 \rfloor + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor) \text{ even}\}$$

Then the ν_s for $s \in S$ form an $h^+(R(H))$-regular sequence in some order.
If k is even, then ν_k is a $\mathbb{Z}_2$-linear combination of the ν_s for even $s \in S$.
If k is odd, then ν_k is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2\rfloor}^{(p)}\right]$-linear combination of the ν_s for odd $s \in S$.
In particular, all ν_k are contained in the ideal $(\nu_s \mid s \in S)$.

Let

$$\overline{S} := \{1, \ldots, n\} \setminus S \text{ and } A := \bigotimes_{p=1}^{l} \mathbb{Z}\left[x_1^{(p)}\left(x_1^{(p)}\right)^*, \ldots, x_{\lfloor n_p/2\rfloor}^{(p)}\left(x_{\lfloor n_p/2\rfloor}^{(p)}\right)^*\right],$$

where A is regarded as a subring of $R(H)$. Then Proposition 5.9 shows that for all $t \in \overline{S}$, we can find $w_t \in R(H)$ such that

$$\begin{aligned} w_t + w_t^* - P_t &\in \sum_{i \in S \text{ even}} \mathbb{Z} \cdot P_i \text{ if } t \in \overline{S} \text{ is even,} \\ w_t + w_t^* - P_t &\in \sum_{i \in S \text{ odd}} A \cdot P_i \text{ if } t \in \overline{S} \text{ is odd} \end{aligned} \tag{5.13}$$

Now from Proposition 4.7, we deduce that

$$h^*(R(H)/(P_1, ..., P_n)) \cong \frac{\mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2\rfloor}^{(p)}\right]}{(\nu_s \mid s \in S)} \otimes \bigwedge_{t \in \overline{S}}([\overline{w}_t])$$

We will show in Lemma 6.11 that

$$(\nu_s \mid s \in S \text{ odd}) = (\alpha_i \mid 1 \leq i \leq m \text{ odd})$$

Hence setting $\gamma_i := \alpha_{2i}$ for $0 \leq i \leq m$, we can simplify the above expression and deduce:

Proposition 5.10. *Let $h := \lfloor m/2\rfloor + \lfloor n_1/2\rfloor + \ldots + \lfloor n_l/2\rfloor$. There is a ring isomorphism*

$$h^*(X(m, n_1, \ldots, n_l)) \cong \frac{\mathbb{Z}_2\left[\gamma_1, \ldots, \gamma_{\lfloor m/2\rfloor}\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2\rfloor}^{(p)}\right]}{(\mu_1, \ldots, \mu_h)} \otimes \bigwedge_{t \in \overline{S}}([\overline{w}_t])$$

with

$$\mu_j = \sum_{c + c_1 + \ldots + c_l = j} \gamma_c \beta_{c_1}^{(1)} \ldots \beta_{c_l}^{(l)} + \binom{2n}{2j} \quad \textit{for } 1 \leq j \leq h,$$

where we recall the definitions we made in (5.12) and that $\gamma_0 = 1$, $\gamma_i = \gamma_{m-i}$ for $\lfloor m/2\rfloor < i \leq m$.
The left factor in the tensor product is contained in h^+ and $[\overline{w}_t] \in h^-$ for all $t \in \overline{S}$.

5.3.3 Witt ring of flag varieties of type C_n

Recall from Bousfield's lemma 2.3 that there is an isomorphism

$$W^*(X) \underset{\cong}{\xrightarrow{\overline{c}}} h^*(K^0(X)) \underset{\cong}{\xleftarrow{[\overline{\alpha}]}} h^*(R(H)/(P_1,\ldots,P_n))$$

We have computed the Tate cohomology on the right and want to determine the Witt grading.

Let k be the number of odd integers among $m, n_1, \ldots, n_l$. We set

$$f := \lfloor k/2 \rfloor = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{n_1}{2} \right\rfloor - \ldots - \left\lfloor \frac{n_l}{2} \right\rfloor \text{ and } g := \left\lceil \frac{n-m}{2} \right\rceil$$

Then f is the number of even integers in $\overline{S}$ and g is the number of odd integers in $\overline{S}$.

Under the above isomorphism, let

- $b_i^{(p)} \in W^*(X)$ correspond to $\overline{\beta}_i^{(p)} \in h^*(R(H)/(P_1,\ldots,P_n))$ for all i, p.
- $a_i \in W^*(X)$ correspond to $\overline{\gamma}_i \in h^*(R(H)/(P_1,\ldots,P_n))$ for all i.
- $u_i \in W^*(X)$ for $1 \leq i \leq f$ correspond to the $[\overline{w}_t] \in h^*(R(H)/(P_1,\ldots,P_n))$ for all even $t \in \overline{S}$.
- $v_j \in W^*(X)$ for $1 \leq j \leq g$ correspond to the $[\overline{w}_t] \in h^*(R(H)/(P_1,\ldots,P_n))$ for all odd $t \in \overline{S}$.

From the isomorphism (5.11) we see that $\overline{\beta}_i^{(p)} \in h^*(R(H)/(P_1,\ldots,P_n))$ is represented by a real representation (as $\rho\rho^*$ is real for any complex representation ρ). Thus by Lemma 4.8, $b_i^{(p)} \in W^0(X)$ for all $1 \leq p \leq l$ and $1 \leq i \leq \lfloor n_p/2 \rfloor$.

As $\gamma_i = \alpha_{2i}$ is represented by a real representation (cf. (5.11), noting that $y_j \in R(H)$ is real if j is even), Lemma 4.8 implies that $a_i \in W^0(X)$ for all i.

We know that $P_i \in R(H)$ is quaternionic for odd i and real for even i. Thus we deduce from (5.13) and Lemma 4.9 that $u_i \in W^{-1}(X)$ for $1 \leq i \leq f$ and that $v_j \in W^{-3}(X)$ for $1 \leq j \leq g$.

All in all, we have shown:

Theorem 5.11. *Let*

$$h := \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_l}{2} \right\rfloor \text{ and } f := \left\lfloor \frac{n}{2} \right\rfloor - h \text{ and } g := \left\lceil \frac{n-m}{2} \right\rceil$$

There is a ring isomorphism

$$W^*(X) \cong \frac{\mathbb{Z}_2\left[a_1,\ldots,a_{\lfloor m/2 \rfloor}\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[b_1^{(p)},\ldots,b_{\lfloor n_p/2 \rfloor}^{(p)}\right]}{(\mu_1,\ldots,\mu_h)} \otimes \bigwedge(u_1,\ldots,u_f,v_1,\ldots,v_g)$$

where

$$\mu_j = \sum_{c+c_1+\ldots+c_l=j} a_c \cdot b^{(1)}_{c_1} \ldots b^{(l)}_{c_l} + \binom{2n}{2j},$$

recalling that we set

$$a_0 := 1 \text{ and } a_i := a_{m-i} \text{ for all } \lfloor m/2 \rfloor < i \leq m$$
$$b^{(p)}_0 := 1 \text{ for all } 1 \leq p \leq l \text{ and } b^{(p)}_i := b^{(p)}_{n_p-i} \text{ for all } 1 \leq p \leq l,\ \lfloor n_p/2 \rfloor < i \leq n_p$$

The left factor in the above tensor product is contained in $W^0(X)$*. Furthermore,* $u_i \in W^{-1}(X)$ *for all* $1 \leq i \leq f$ *and* $v_j \in W^{-3}(X)$ *for all* $1 \leq j \leq g$*.*

We now tabulate the ranks of the Witt groups in the different degrees. This immediately follows from Proposition 6.13 and Appendix A.

Theorem 5.12. *Let* h*,* f *and* g *be as in the previous theorem and*

$$a := \frac{h!}{\lfloor m/2 \rfloor! \lfloor n_1/2 \rfloor! \ldots \lfloor n_l/2 \rfloor!}$$

We have $W^i(X(m, n_1, \ldots, n_l)) \cong \mathbb{Z}_2^{a \cdot z_i}$ *where* z_i *is given as follows:*

If $f = g = 0$*, then* $z_0 = 1$ *and* $z_{-1} = z_{-2} = z_{-3} = 0$*.*

If $(f, g) \neq (0, 0)$*, then the* z_i *are given as in Appendix A.*

5.4 Type D_n

We consider $G = Spin(2n)$. If $m + n_1 + \ldots + n_l = n$, we define

$$\begin{aligned}\hat{H}_D(m, n_1, \ldots, n_l) &:= Spin(2m) \times Spin(2n_1) \times \ldots \times Spin(2n_l)\\ H_D(m, n_1, \ldots, n_l) &:= Spin(2m) \times \tilde{U}(n_1) \times \ldots \times \tilde{U}(n_l)\end{aligned}$$

Note that $H_D \subset \hat{H}_D$. Let $C_D(m, n_1, \ldots, n_l) := \ker(\hat{H}_D \to Spin(2n))$, then $C_D \subset H_D$ and any centraliser of a torus in $Spin(2n)$ is conjugate to a subgroup of the form $H(m, n_1, \ldots, n_l) := H_D(m, n_1, \ldots, n_l)/C_D$ by Proposition 1.12. So we fix $m, n_1, \ldots, n_l$ such that $n = m + n_1 + \ldots + n_l$ and set

$$X(m, n_1, \ldots, n_l) := Spin(2n)/H(m, n_1, \ldots, n_l)$$

Since $Spin(2) \cong \tilde{U}(1)$, we may assume that $m \neq 1$. We will sometimes simply write X instead of $X(m, n_1, \ldots, n_l)$ etc.

5.4.1 Representation rings and their Tate cohomology

We recall and fix some notation. We know that

$$R(Spin(2n)) \cong \mathbb{Z}\left[\rho_1, \ldots, \rho_{n-2}, \Delta_+, \Delta_-\right]$$

where ρ_1 is the representation induced by the complex standard representation of $SO(2n)$ of rank $2n$, $\rho_i = \Lambda^i(\rho_1)$ for all i and Δ_+, Δ_- are the half-spin representations. We have $\rho_j^* = \rho_j$ for all j and

$$\Delta_+^* = \begin{cases}\Delta_+ & \text{if } n \text{ is even}\\ \Delta_- & \text{if } n \text{ is odd}\end{cases} \qquad \Delta_-^* = \begin{cases}\Delta_- & \text{if } n \text{ is even}\\ \Delta_+ & \text{if } n \text{ is odd}\end{cases}$$

Furthermore, we write

$$R(\hat{H}_D) \cong \mathbb{Z}[\lambda_1, \ldots, \lambda_{m-2}, \Gamma_+, \Gamma_-] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}\left[\lambda_1^{(p)}, \ldots, \lambda_{n_p-2}^{(p)}, \Gamma_+^{(p)}, \Gamma_-^{(p)}\right]$$

For the representation rings of H_D and H, we use the notation as introduced in section 1.3.2 (Type D_n). We use the notation as in Proposition 2.16 to identify the Tate cohomology ring of $R(H)$. Using that notation, recall that we have

$$\begin{aligned}\alpha_{2m-j} = \alpha_j \ \forall\, 0 \leq j \leq 2m, \quad \alpha_k = 0 \text{ if } k \notin \{0, \ldots, 2m\}, \quad \alpha_0 = 1\\ \beta_{n_p-j}^{(p)} = \beta_j^{(p)} \ \forall\, 1 \leq p \leq l,\ 0 \leq j \leq n_p, \quad \beta_k^{(p)} = 0 \text{ if } k \notin \{0, \ldots, n_p\}, \quad \beta_0^{(p)} = 1\ \forall p\end{aligned} \tag{5.14}$$

Note that with these definitions, we have $\alpha_j = \left[\Lambda^j(y_1)\right]$ for all $0 \leq j \leq 2m$ and $\beta_i^{(p)} = \left[x_i^{(p)}\left(x_i^{(p)}\right)^*\right]$ for all $1 \leq p \leq l$ and $0 \leq i \leq n_p$.

5.4.2 Witt ring of flag varieties of type D_n

Let $i\colon H \to Spin(2n)$ be the inclusion map. Following Chapter 4, we need to compute the restriction $i^*\colon R(Spin(2n)) \to R(H)$. We see directly that

$$i^*(\rho_1) = y_1 + \sum_{p=1}^{l} x_1^{(p)} + \left(\sum_{p=1}^{l} x_1^{(p)}\right)^*$$

Hence we obtain for $1 \leq j \leq n$ that

$$\begin{aligned} i^*(\rho_j - \operatorname{rk}(\rho_j)) =& \Lambda^j\left(y_1 + \sum_{p=1}^{l} x_1^{(p)} + \sum_{p=1}^{l}\left(x_1^{(p)}\right)^*\right) - \binom{2n}{j} \\ =& \sum_{c+d_1+d_1'+\ldots+d_l+d_l'=j} y_c \cdot x_{d_1}^{(1)}\left(x_{d_1'}^{(1)}\right)^* \ldots x_{d_l}^{(l)}\left(x_{d_l'}^{(l)}\right)^* - \binom{2n}{j} =: P_j \end{aligned}$$

Using that $[a + a^*] = 0$ in $h^+(R(H))$ for all $a \in R(H)$, we deduce that in $h^+(R(H))$,

$$\begin{aligned} [P_j] =& \sum_{c+2d_1+\ldots+2d_l=j} \alpha_c \beta_{d_1}^{(1)} \ldots \beta_{d_l}^{(l)} + \binom{2n}{j} \\ =& \sum_{d=0}^{\lfloor j/2 \rfloor} \alpha_{j-2d} \sum_{d_1+\ldots+d_l=d} \beta_{d_1}^{(1)} \ldots \beta_{d_l}^{(l)} + \binom{2n}{j} =: \nu_j, \end{aligned}$$

keeping in mind (5.14). Note that these are very similar to the elements we obtained in the computations for type C_n. We need to investigate the relations between the ν_j.

Proposition 5.13. *Let*

$$S := \{j \in \mathbb{N} \mid 1 \leq j < m\} \cup \left\{j \in \mathbb{N} \mid m \leq j \leq 2\left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_l}{2} \right\rfloor\right) \text{ even}\right\}$$

The ν_s for $s \in S$ form an $h^+(R(H))$-regular sequence in some order.

If j is even, then ν_j is a $\mathbb{Z}_2$-linear combination of the ν_s for $s \in S$.

Regarding $h^+(R(H))$ as a $c_0(RO^0(H))$-module via the complexification $c_0\colon RO^0(H) \to R(H)$, if j is odd, then ν_j is a $c_0(RO^0(H))$-linear combination of the ν_s for $s \in S$.

In particular, all ν_j are contained in the ideal $(\nu_s \mid s \in S)$.

Proof. Suppose first that m is even. Then the subalgebra A of $h^*(R(H))$ generated by $\alpha_1, \ldots, \alpha_m$ (note that $\alpha_{m-1} = \delta + \alpha_{m-3} + \alpha_{m-5} + \ldots$ and $\alpha_m = \delta_1 + \delta_2$ if $m \geq 2$) and all $\beta_i^{(p)}$ is a polynomial ring in these indeterminates, and $h^*(R(H))$ is a free A-module. Note that $\nu_j \in A$ for all j. Now Proposition 6.12 shows that the ν_s for $s \in S$ form an A-regular sequence in this case. So by Proposition 3.3 they also form an $h^+(R(H))$-regular sequence.

Suppose now $m > 2$ is odd. Then $\alpha_m = \left[\Gamma_+^2 + \Gamma_-^2\right] = 0$ in $h^+(R(H))$ and by Proposition 2.16, $h^+(R(H))$ is a polynomial ring in indeterminates $\alpha_1, \ldots, \alpha_{m-1}$ and $\beta_i^{(p)}$ where $1 \leq p \leq l$ and $1 \leq i \leq \lfloor n_p/2 \rfloor$. Note that if j is even, then the expression for ν_j does not contain any α_i for odd i. So none of the expressions for ν_s for $s \in S$ contains α_m. Hence Proposition 6.12 shows that

$$\nu_1, \nu_2, \ldots, \nu_{m-1}, \nu_{m+1}, \nu_{m+3}, \ldots, \nu_{2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_l}{2} \rfloor)},$$

i.e. ν_s for $s \in S$, is an $h^+(R(H))$-regular sequence.

The remainder of the Proposition follows immediately from Proposition 6.14. $\square$

Define $\overline{S} := \{1, 2, \ldots, n-1\} \setminus S$. By Proposition 5.13, we can find $w_t \in R(H)$ for every $t \in \overline{S}$ such that

$$w_t + w_t^* - i^*(\rho_t - \mathrm{rk}(\rho_t)) \in \sum_{s \in S} c_0(RO^0(H)) \cdot i^*(\rho_s - \mathrm{rk}(\rho_s)) \tag{5.15}$$

Now we also need to consider the restrictions of Δ_+ and Δ_-.

If n is odd, then $\Delta_+^* = \Delta_-$ and so for the Tate cohomology computation, we need to determine the restriction of $\Delta_+\Delta_-$ (cf. Chapter 4). Since $\Delta_+\Delta_- = \rho_{n-1} + \rho_{n-3} + \rho_{n-5} + \ldots$, we have

$$\begin{aligned} h^*(K^0(G/H)) &\cong h^*\left(\frac{R(H)}{(i^*(\rho_j) - \mathrm{rk}(\rho_j) \mid 1 \leq j \leq n-2) + \left(i^*(\widetilde{\Delta}_+), i^*(\widetilde{\Delta}_-)\right)}\right) \\ &\cong h^*\left(\frac{R(H)}{(i^*(\rho_j) - \mathrm{rk}(\rho_j) \mid 1 \leq j \leq n-2) + \left(i^*(\widetilde{\Delta}_+\widetilde{\Delta}_-)\right)}\right) \\ &\cong h^*\left(\frac{R(H)}{(i^*(\rho_j) - \mathrm{rk}(\rho_j) \mid 1 \leq j \leq n-1)}\right) \end{aligned} \tag{5.16}$$

where we used Lemma 2.7 for the second isomorphism and set $\widetilde{\Delta}_\pm := \Delta_\pm - 2^{n-1}$. So it suffices in this case to consider the restriction of ρ_{n-1} which we already computed above.

Now we can compute the Tate cohomology of $K^0(X)$ if n is odd:

Proposition 5.14. *Let* $N := \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_l}{2} \rfloor$.

If n and $m > 2$ are odd or if n is odd and $m = 0$, we have a ring isomorphism

$$h^*(K^0(X)) \cong \frac{\mathbb{Z}_2\left[\gamma_1, \ldots, \gamma_{\lfloor \frac{m}{2} \rfloor}\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor \frac{n_p}{2} \rfloor}^{(p)}\right]}{(\mu_i \mid 1 \leq i \leq N)} \otimes \bigwedge_{t \in \overline{S}} ([\overline{w}_t])$$

If n is odd and $m \geq 2$ is even, we have a ring isomorphism

$$h^*(K^0(X)) \cong \frac{\mathbb{Z}_2\left[\gamma_1, \ldots, \gamma_{\frac{m}{2}}, \delta_1, \delta_2\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor \frac{n_p}{2} \rfloor}^{(p)}\right]}{(\delta_1 + \delta_2 + \gamma_{\frac{m}{2}},\ \delta_1\delta_2) + (\mu_i \mid 1 \leq i \leq N)} \otimes \bigwedge_{t \in \overline{S}}([\overline{w}_t])$$

In the above, we have

$$\mu_i := \sum_{a+a_1+\ldots+a_l=i} \gamma_a \beta_{a_1}^{(1)} \ldots \beta_{a_l}^{(l)} + \binom{2n}{2i}$$

using the conventions that

$$\gamma_0 := 1 \text{ and } \gamma_i := \gamma_{m-i} \text{ for all } \lfloor m/2 \rfloor < i \leq m,$$
$$\beta_0^{(p)} := 1 \text{ for all } 1 \leq p \leq l \text{ and } \beta_i^{(p)} := \beta_{n_p - i}^{(p)} \text{ for all } 1 \leq p \leq l,\ \lfloor n_p/2 \rfloor < i \leq n_p$$

Under the above isomorphisms, we have that the respective first factor in the tensor product is contained in h^+ and $[\overline{w}_t] \in h^-$ for all $t \in \overline{S}$.

Proof. Suppose n is odd. By equation (5.16), we deduce from Proposition 4.7 and Proposition 5.13 that we have an isomorphism

$$h^*(K^0(X)) \cong \frac{h^+(R(H))}{(\nu_s \mid s \in S)} \otimes \bigwedge_{t \in \overline{S}}([\overline{w}_t])$$

Suppose now n and $m > 2$ are odd. By Proposition 2.16,

$$h^+(R(H)) \cong \mathbb{Z}_2[\alpha_1, \ldots, \alpha_{m-1}] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor n_p/2 \rfloor}^{(p)}\right]$$

Recall from Lemma 6.11 that

$$(\nu_i \mid 1 \leq i < m \text{ odd}) = (\alpha_i \mid 1 \leq i < m \text{ odd}).$$

Setting $\gamma_i := \alpha_{2i}$, we can write

$$\nu_{2i} = \sum_{d=0}^{i} \alpha_{2i-2d} \sum_{d_1+\ldots+d_l=d} \beta_{d_1}^{(1)} \ldots \beta_{d_l}^{(l)} + \binom{2n}{2i} = \sum_{d_0+d_1+\ldots+d_l=i} \gamma_{d_0} \beta_{d_1}^{(1)} \ldots \beta_{d_l}^{(l)} + \binom{2n}{2i} = \mu_i$$

The given expression for $h^*(K^0(X))$ now follows from this.

The other case is similar. □

Remark 5.15. As in previous sections, Bousfield's lemma identifies $h^*(K^0(X))$ with $W^*(X)$. Under this identification, by Lemma 4.8 we then have in the cases of Proposition

5.14 that for all $1 \leq j \leq \lfloor m/2 \rfloor$ and all $1 \leq p \leq l$ and $1 \leq i \leq \lfloor n_p/2 \rfloor$,

$$\overline{\gamma}_j, \overline{\delta}_1, \overline{\delta}_2, \overline{\beta}_i^{(p)} \in W^0(X)$$

By (5.15), we deduce from Lemma 4.9 and the fact that the ρ_i are representations of real type that

$$[\overline{w}_t] \in W^{-1}(X) \text{ for all } t \in \overline{S}$$

The ranks of the Witt groups in different degrees can be expressed as follows: Let

$$a := \frac{N!}{\lfloor m/2 \rfloor! \lfloor n_1/2 \rfloor! \dots \lfloor n_l/2 \rfloor!}$$

It is easy to check that if not all of $m, n_1, \dots, n_l$ are even, then

$$|\overline{S}| = \left\lceil \frac{n_1}{2} \right\rceil + \dots + \left\lceil \frac{n_l}{2} \right\rceil - 1$$

Let z_i be the rank of degree i of a 4-periodically graded exterior algebra with $|\overline{S}|$ generators of degree -1. The value of z_i can be found in Appendix A (setting $f = |\overline{S}|$ and $g = 0$). Then from Proposition 6.13 and Proposition 5.14, we deduce:

If n and $m > 2$ are odd or if n is odd and $m = 0$, we have $\mathrm{rk}(W^i(X)) = a \cdot z_i$.

If n is odd and $m \geq 2$ is even, we have $\mathrm{rk}(W^i(X)) = 2a \cdot z_i$.

Now suppose n is even. Then Δ_+ and Δ_- are self-dual, so we need to consider their restrictions to H. Letting $i_{\hat{H}_D} : \hat{H}_D \to G$ be the natural map with kernel C_D, we conclude from [Ada96, Prop. 4.5] that

$$\begin{aligned} i_{\hat{H}}^*(\Delta_+) &= \sum_{\substack{\epsilon, \epsilon_i \in \{+,-\} \\ \epsilon\epsilon_1 \dots \epsilon_l = +}} \Gamma_\epsilon \Gamma_{\epsilon_1}^{(1)} \dots \Gamma_{\epsilon_l}^{(l)} \\ i_{\hat{H}}^*(\Delta_-) &= \sum_{\substack{\epsilon, \epsilon_i \in \{+,-\} \\ \epsilon\epsilon_1 \dots \epsilon_l = -}} \Gamma_\epsilon \Gamma_{\epsilon_1}^{(1)} \dots \Gamma_{\epsilon_l}^{(l)} \end{aligned} \tag{5.17}$$

Suppose that at least one of $m, n_1, \dots, n_l$ is odd, then

$$\left[i_{\hat{H}_D}^*(\Delta_+) \right] = \left[i_{\hat{H}_D}^*(\Delta_-) \right] = 0 \text{ in } h^+(R(\hat{H}_D))$$

since no summand in expressions (5.17) is self-dual. Consequently,

$$[i^*(\Delta_+)] = [i^*(\Delta_-)] = 0 \text{ in } h^+(R(H)) \tag{5.18}$$

Suppose that $m = 0$ and $n_1, \dots, n_l$ are even. Let $r := |\{j \mid n_j \equiv 2 \pmod 4\}|$. Then from

Lemma 1.21, we deduce that in $h^+(R(H))$, we have

$$[i^*(\Delta_+)] = \begin{cases} \gamma & \text{if } r \text{ is even} \\ 0 & \text{otherwise} \end{cases} \qquad [i^*(\Delta_-)] = \begin{cases} \gamma & \text{if } r \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{5.19}$$

The case that $m \geq 2, n_1, \ldots, n_l$ are all even remains. We will deal with it later. Let us now first determine the Tate cohomology in all other cases.

If n is even but not all of $m, n_1, \ldots, n_l$ are even, by (5.18) we can find $\eta_+, \eta_- \in R(H)$ such that

$$\eta_+ + \eta_+^* = i^*(\Delta_+) \quad \text{and} \quad \eta_- + \eta_-^* = i^*(\Delta_-) \tag{5.20}$$

If $m = 0$ and all of $n_1, \ldots, n_l$ are even, by (5.19) there is $\eta \in R(H)$ such that

$$\eta + \eta^* = \begin{cases} i^*(\Delta_+) & \text{if } r \text{ is odd} \\ i^*(\Delta_-) & \text{if } r \text{ is even} \end{cases} \tag{5.21}$$

We now obtain:

Proposition 5.16. *Let* $N := \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_l}{2} \rfloor$.

If n is even and $m > 2$ is odd or if n is even and $m = 0$ and not all of $n_1, \ldots, n_l$ are even, we have a ring isomorphism

$$h^*(K^0(X)) \cong \frac{\mathbb{Z}_2\left[\gamma_1, \ldots, \gamma_{\lfloor \frac{m}{2} \rfloor}\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor \frac{n_p}{2} \rfloor}^{(p)}\right]}{(\mu_i \mid 1 \leq i \leq N)} \otimes \bigwedge_{t \in \overline{S} \setminus \{n-1\}} ([\overline{w}_t], [\overline{\eta}_\pm])$$

If n is even, $m \geq 2$ is even and not all of $n_1, \ldots, n_l$ are even, we have a ring isomorphism

$$h^*(K^0(X)) \cong \frac{\mathbb{Z}_2\left[\gamma_1, \ldots, \gamma_{\frac{m}{2}}, \delta_1, \delta_2\right] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor \frac{n_p}{2} \rfloor}^{(p)}\right]}{(\delta_1 + \delta_2 + \gamma_{\frac{m}{2}},\ \delta_1\delta_2) + (\mu_i \mid 1 \leq i \leq N)} \otimes \bigwedge_{t \in \overline{S} \setminus \{n-1\}} ([\overline{w}_t], [\overline{\eta}_\pm])$$

If $m = 0$ and $n_1, \ldots, n_l$ are even, we have a ring isomorphism

$$h^*(K^0(X)) \cong \frac{\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\lfloor \frac{n_p}{2} \rfloor}^{(p)}\right]}{(\mu_i \mid 1 \leq i \leq N)} \otimes \bigwedge_{t \in \overline{S} \setminus \{n-1\}} ([\overline{w}_t]) \otimes \bigwedge([\overline{\eta}])$$

In the above, the μ_i are defined as in Proposition 5.14.

Proof. Similarly to Proposition 5.14, this follows from Proposition 4.7. □

Remark 5.17. As before, Bousfield's lemma identifies $h^*(K^0(X))$ with $W^*(X)$. Under this identification, by Lemma 4.8 we then have in the cases of Proposition 5.16 that for all $1 \leq j \leq \lfloor m/2 \rfloor$ and all $1 \leq p \leq l$ and $1 \leq i \leq \lfloor n_p/2 \rfloor$,

$$\overline{\gamma}_j, \overline{\delta}_1, \overline{\delta}_2, \overline{\beta}_i^{(p)} \in W^0(X)$$

By (5.15), we deduce from Lemma 4.9 and the fact that the ρ_i are representations of real type that

$$[\overline{w}_t] \in W^{-1}(X) \quad \text{for all } t \in \overline{S}$$

Recall that Δ_+, Δ_- are of real type if $n \equiv 0 \pmod 4$ and of quaternionic type if $n \equiv 2 \pmod 4$. Hence (5.21) and Lemma 4.9 show that if $m = 0$ and $n_1, \ldots, n_l$ are even, then

$$[\overline{\eta}] \in \begin{cases} W^{-1}(X) \text{ if } n \equiv 0 \pmod 4 \\ W^{-3}(X) \text{ if } n \equiv 2 \pmod 4 \end{cases}$$

Furthermore, if n is even but not all of $m, n_1, \ldots, n_l$ are even, then from (5.20) and Lemma 4.9, we have

$$[\overline{\eta}_+], [\overline{\eta}_-] \in \begin{cases} W^{-1}(X) \text{ if } n \equiv 0 \pmod 4 \\ W^{-3}(X) \text{ if } n \equiv 2 \pmod 4 \end{cases}$$

Let $z_i(f, g)$ be the rank of degree i of a 4-periodically graded exterior algebra with f generators of degree -1 and g generators of degree -3. Values of the $z_i(f, g)$ in terms of f and g can be found in Appendix A. Let

$$a := \frac{N!}{\lfloor m/2 \rfloor! \lfloor n_1/2 \rfloor! \ldots \lfloor n_l/2 \rfloor!}$$

It is easy to check that if n is even and $m \neq n$, then

$$|\overline{S} \setminus \{n-1\}| = \begin{cases} \left\lceil \frac{n_1}{2} \right\rceil + \ldots + \left\lceil \frac{n_l}{2} \right\rceil + m - 2\left\lfloor \frac{m}{2} \right\rfloor - 2 \text{ if not all of } m, n_1, \ldots, n_l \text{ are even} \\ \frac{n_1 + \ldots + n_l}{2} - 1 \text{ if all of } m, n_1, \ldots, n_l \text{ are even} \end{cases}$$

We deduce from Proposition 6.3 that in the cases of Proposition 5.16, we have

$$\dim_{\mathbb{Z}_2}(W^i(X)) = \begin{cases} 2a \cdot z_i(f, g) \text{ if } n \text{ is even, } m \geq 2 \text{ is even and some } n_1, \ldots, n_l \text{ is odd} \\ a \cdot z_i(f, g) \text{ otherwise} \end{cases},$$

where f and g are chosen as follows:

$$(f, g) = \begin{cases} \left(\left\lceil \frac{n_1}{2} \right\rceil + \ldots + \left\lceil \frac{n_l}{2} \right\rceil + m - 2\left\lfloor \frac{m}{2} \right\rfloor, 0\right) \text{ if } n \equiv 0 \pmod 4 \\ \left(\frac{n_1 + \ldots + n_l}{2} - 1, 1\right) \text{ if } n \equiv 2 \pmod 4, m = 0 \text{ and } n_1, \ldots, n_l \text{ are even} \\ \left(\left\lceil \frac{n_1}{2} \right\rceil + \ldots + \left\lceil \frac{n_l}{2} \right\rceil + m - 2\left\lfloor \frac{m}{2} \right\rfloor - 2, 2\right) \text{ otherwise} \end{cases}$$

Now suppose that all of $m \geq 2, n_1, \ldots, n_l$ are even. We set $r := |\{j \mid n_j \equiv 2 \pmod 4\}|$ as before. Still using the notation of Propositions 1.26 and 2.16, we have

$$\begin{aligned}
[i^*(\Delta_+)] &= \begin{cases} \left[\Gamma_+ \prod_{p=1}^l \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right] = \zeta_2 & \text{if } r \text{ is even} \\ \left[\Gamma_- \prod_{p=1}^l \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right] = \zeta_1 & \text{if } r \text{ is odd} \end{cases} \\
[i^*(\Delta_-)] &= \begin{cases} \left[\Gamma_- \prod_{p=1}^l \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right] = \zeta_1 & \text{if } r \text{ is even} \\ \left[\Gamma_+ \prod_{p=1}^l \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right] = \zeta_2 & \text{if } r \text{ is odd} \end{cases}
\end{aligned} \tag{5.22}$$

We define ideals $I_{\text{reg}} \subset I_{\text{reg}}^{\Delta} \subset I$ of $R(H)$ by

$$\begin{aligned}
I_{\text{reg}} &= (P_s \mid s \in S \setminus \{n\}) \\
I_{\text{reg}}^{\Delta} &= I_{\text{reg}} + \left(i^*(\Delta_+) - 2^{n-1},\ i^*(\Delta_-) - 2^{n-1}\right) \\
I &= (P_j \mid 1 \leq j \leq n-2) + \left(i^*(\Delta_+) - 2^{n-1},\ i^*(\Delta_-) - 2^{n-1}\right)
\end{aligned}$$

In order to compute the Tate cohomology of $K^0(X) \cong R(H)/I$, we first determine the Tate cohomology of $R(H)/I_{\text{reg}}$ and $R(H)/I_{\text{reg}}^{\Delta}$.

Let

$$A := \frac{\mathbb{Z}_2[\alpha_1, \ldots, \alpha_m] \otimes \bigotimes_{p=1}^l \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\frac{n_p}{2}}^{(p)}\right]}{(\nu_s \mid s \in S \setminus \{n\})}$$

Since $(\alpha_j \mid 1 \leq j \leq m \text{ odd}) = (\nu_j \mid 1 \leq j \leq m \text{ odd})$ by Lemma 6.11, we have $\overline{\alpha}_j = 0$ in A if j is odd, and

$$A \cong \frac{\mathbb{Z}_2[\gamma_1, \ldots, \gamma_{\frac{m}{2}}] \otimes \bigotimes_{p=1}^l \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\frac{n_p}{2}}^{(p)}\right]}{(\mu_i \mid 1 \leq i < N)}$$

where N and μ_i are defined as in Proposition 5.14. We deduce from Proposition 5.13 and Proposition 4.7 that

$$h^*\left(R(H)/I_{\text{reg}}\right) \cong \left(A \otimes \mathbb{Z}_2[\delta, \delta_1, \delta_2, \zeta_1, \zeta_2]\right)/J =: A'$$

where, setting $Y := \prod_{p=1}^l \overline{\beta}_{\frac{n_p}{2}}^{(p)}$,

$$\begin{aligned}
J = (&\delta + \overline{\alpha}_{m-1} + \overline{\alpha}_{m-3} + \ldots,\ \overline{\alpha}_m + \delta_1 + \delta_2, \\
&\delta_1\delta_2 + \delta^2,\ \zeta_1^2 + \delta_1 Y,\ \zeta_2^2 + \delta_2 Y,\ \zeta_1\zeta_2 + \delta Y,\ \delta\zeta_1 + \zeta_2\delta_1,\ \delta\zeta_2 + \zeta_1\delta_2)
\end{aligned}$$

From the above, we have $\overline{\alpha}_{m-1} = \overline{\alpha}_{m-3} = \ldots = 0$ and so $\delta \in J$. Hence

$$J = (\delta,\ \overline{\gamma}_{\frac{m}{2}} + \delta_1 + \delta_2,\ \delta_1\delta_2,\ \zeta_1^2 + \delta_1 Y,\ \zeta_2^2 + \delta_2 Y,\ \zeta_1\zeta_2,\ \zeta_2\delta_1,\ \zeta_1\delta_2)$$

We see from this that $h^*(R(H)/I_{\text{reg}})$ is a free A-module with basis $1, \overline{\delta}_1, \overline{\zeta}_1, \overline{\zeta}_2$.

Lemma 5.18. *The element $\overline{\gamma}_{\frac{m}{2}}Y$ is not a zero divisor in A. Thus neither $\overline{\gamma}_{\frac{m}{2}}$ nor Y is a zero divisor in A.*

Proof. By Proposition 5.13, $\overline{\nu}_N$ is not a zero divisor in A. But $\overline{\nu}_N = \overline{\alpha}_m \prod_p \overline{\beta}^{(p)}_{\frac{n_p}{2}}$. □

In order to deal with the fact that $\left[\overline{i^*(\Delta_\pm)}\right]$, i.e. $\overline{\zeta}_1$ and $\overline{\zeta}_2$, are zero divisors in $h^*(R(H)/I_{\text{reg}})$, we want to use Lemma 2.11 to compute $h^*\left(R(H)/I^{\Delta}_{\text{reg}}\right)$. In this direction, we first show:

Lemma 5.19. *$Ann_{A'}(\overline{\zeta}_1, \overline{\zeta}_2) = 0$*

Proof. For $i = 1, 2$, we consider the maps

$$f_i \colon A' \to A', \quad x \mapsto \overline{\zeta}_i x$$

These maps are A-linear, and taking $1, \overline{\delta}_1, \overline{\zeta}_1, \overline{\zeta}_2$ as an ordered A-basis of A', they can be represented by matrices

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Y & 0 \\ 1 & \overline{\gamma}_{\frac{m}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad F_2 = \begin{pmatrix} 0 & 0 & 0 & \overline{\gamma}_{\frac{m}{2}}Y \\ 0 & 0 & 0 & Y \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Since $\overline{\gamma}_{\frac{m}{2}}$ and Y and $\overline{\gamma}_{\frac{m}{2}}Y$ are not zero divisors in A by Lemma 5.18, we conclude

$$\text{Ann}_{A'}(\overline{\zeta}_1) = A \cdot \overline{\zeta}_2 + A \cdot (\overline{\gamma}_{\frac{m}{2}} + \overline{\delta}_1) \quad \text{and} \quad \text{Ann}_{A'}(\overline{\zeta}_2) = A \cdot \overline{\delta}_1 + A \cdot \overline{\zeta}_1$$

The claim follows immediately. □

Now let $\psi \colon A' \oplus A' \to A'$, $(a \oplus b) \mapsto a\overline{\zeta}_1 + b\overline{\zeta}_2$. To avoid clashing with the notation for an ideal generated by certain elements, we denote elements in $A' \oplus A'$ by $(a \oplus b)$ (instead of (a, b), which to us means the ideal in A' generated by a and b) where $a, b \in A'$.

Lemma 5.20. *$ker(\psi) = A \cdot ((\overline{\gamma}_{\frac{m}{2}} + \overline{\delta}_1) \oplus 0) + A \cdot (\overline{\zeta}_2 \oplus 0) + A \cdot (0 \oplus \overline{\delta}_1) + A \cdot (0 \oplus \overline{\zeta}_1)$*

Proof. We take

$$(1 \oplus 0),\ (\overline{\delta}_1 \oplus 0),\ (\overline{\zeta}_1 \oplus 0),\ (\overline{\zeta}_2 \oplus 0),\ (0 \oplus 1),\ (0 \oplus \overline{\delta}_1),\ (0 \oplus \overline{\zeta}_1),\ (0 \oplus \overline{\zeta}_2)$$

as an ordered A-basis of $A' \oplus A'$ and $1, \overline{\delta}_1, \overline{\zeta}_1, \overline{\zeta}_2$ as an ordered A-basis of A'. With respect to these ordered bases, we can represent ψ by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\gamma}_{\frac{m}{2}} Y \\ 0 & 0 & Y & 0 & 0 & 0 & 0 & Y \\ 1 & \overline{\gamma}_{\frac{m}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Since $\overline{\gamma}_{\frac{m}{2}}$ and Y are not zero divisors in A by Lemma 5.18, the claim follows. $\square$

By Lemma 2.11, the short exact sequence

$$0 \to \left(i^*(\Delta_+) - 2^{n-1},\ i^*(\Delta_-) - 2^{n-1}\right) \to \frac{R(H)}{I_{\mathrm{reg}}} \to \frac{R(H)}{I^{\Delta}_{\mathrm{reg}}} \to 0$$

induces a long exact sequence on Tate cohomology yielding isomorphisms

$$h^+\left(\frac{R(H)}{I^{\Delta}_{\mathrm{reg}}}\right) \cong \frac{A'}{(\overline{\zeta}_1, \overline{\zeta}_2)} \cong \frac{\mathbb{Z}_2[\gamma_1, \dots, \gamma_{\frac{m}{2}}] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \dots, \beta^{(p)}_{\lfloor \frac{n_p}{2} \rfloor}\right] \otimes \mathbb{Z}_2[\delta_1, \delta_2]}{(\mu_i \mid 1 \le i < N) + J'}$$

where $J' = \left(\delta_1 + \delta_2 + \overline{\gamma}_{\frac{m}{2}},\ \delta_1\delta_2,\ \delta_1 \prod_p \beta^{(p)}_{\frac{n_p}{2}},\ \delta_2 \prod_p \beta^{(p)}_{\frac{n_p}{2}}\right)$, and

$$h^-\left(R(H)/I^{\Delta}_{\mathrm{reg}}\right) \underset{\cong}{\overset{\partial}{\to}} \frac{\ker(\psi)}{A' \cdot (\overline{\zeta}_2 \oplus \overline{\zeta}_1)}$$

We have identified $\ker(\psi)$ as an A-module in Lemma 5.20. Furthermore, as A-modules,

$$\begin{aligned} A' \cdot \left(\overline{\zeta}_2 \oplus \overline{\zeta}_1\right) =& A \cdot \left(\overline{\zeta}_2 \oplus \overline{\zeta}_1\right) + A\overline{\delta}_1 \cdot \left(\overline{\zeta}_2 \oplus \overline{\zeta}_1\right) + A\overline{\zeta}_1 \cdot \left(\overline{\zeta}_2 \oplus \overline{\zeta}_1\right) + A\overline{\zeta}_2 \cdot \left(\overline{\zeta}_2 \oplus \overline{\zeta}_1\right) \\ =& A \cdot \left(\overline{\zeta}_2 \oplus \overline{\zeta}_1\right) + A \cdot \left(0 \oplus \overline{\gamma}_{\frac{m}{2}} \overline{\zeta}_1\right) + A \cdot \left(0 \oplus Y\overline{\delta}_1\right) + A \cdot \left(Y\overline{\delta}_2 \oplus 0\right) \end{aligned}$$

Hence by Lemma 5.20, we have isomorphisms of A-modules

$$h^-\left(R(H)/I^{\Delta}_{\mathrm{reg}}\right) \underset{\cong}{\overset{\partial}{\to}} \frac{\ker(\psi)}{A' \cdot (\overline{\zeta}_2 \oplus \overline{\zeta}_1)} \cong \frac{A}{(Y)} \cdot \overline{(\delta_2 \oplus 0)} \oplus \frac{A}{(Y)} \cdot \overline{(0 \oplus \delta_1)} \oplus \frac{A}{\left(\overline{\gamma}_{\frac{m}{2}}\right)} \cdot \overline{(0 \oplus \zeta_1)}$$

We want to describe the elements in $h^-\left(R(H)/I^{\pm}_{\mathrm{reg}}\right)$ corresponding to $\overline{(\delta_2 \oplus 0)}$ and $\overline{(0 \oplus \delta_1)}$ and $\overline{(0 \oplus \zeta_1)}$ under the isomorphism ∂ more concretely by giving elements in $R(H)$ representing them.

We have that (cf. Proposition 2.16)

$$\begin{aligned} &\overline{\delta}_2 \in h^+(R(H)/I_{\mathrm{reg}}) \ \text{ is represented by } \ \Gamma_+^2 \in R(H) \\ &\overline{\delta}_1 \in h^+(R(H)/I_{\mathrm{reg}}) \ \text{ is represented by } \ \Gamma_-^2 \in R(H) \\ &\overline{\zeta}_1 \in h^+(R(H)/I_{\mathrm{reg}}) \ \text{ is represented by } \ \Gamma_- \prod_{p=1}^{l} \left(x^{(p)}_{n_p}\right)^{-\frac{1}{2}} x^{(p)}_{\frac{n_p}{2}} \in R(H) \end{aligned}$$

Recall that $r = |\{j \mid n_j \equiv 2 \pmod 4\}|$. By (5.22), we can find $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3 \in R(H)$ such that

$$\tilde{u}_1 + \tilde{u}_1^* - \Gamma_+^2 \cdot i^*\left(\Delta_{(-)^{r+1}} - 2^{n-1}\right) = \Gamma_+^2 \cdot \Gamma_- \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}$$
$$\tilde{u}_2 + \tilde{u}_2^* - \Gamma_-^2 \cdot i^*\left(\Delta_{(-)^{r}} - 2^{n-1}\right) = \Gamma_-^2 \cdot \Gamma_+ \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}$$
$$\tilde{u}_3 + \tilde{u}_3^* - \Gamma_- \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)} \cdot i^*\left(\Delta_{(-)^{r}} - 2^{n-1}\right) = \Gamma_-\Gamma_+ \left(\prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right)^2$$

Let $T := \sum_{s \in S \setminus \{n\}} c_0(RO^0(H)) \cdot i^*(\rho_s - \mathrm{rk}(\rho_s))$ as a subgroup of $R(H)$. By Lemma 6.11, we can find $v \in R(H)$ and $\omega \in T$ such that

$$\Gamma_+\Gamma_- = y_{m-1} + y_{m-3} + \ldots + y_1 = v + v^* + \omega$$

Setting

$$u_1 := \tilde{u}_1 - v \cdot \Gamma_+ \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}$$
$$u_2 := \tilde{u}_2 - v \cdot \Gamma_- \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}$$
$$u_3 := \tilde{u}_3 - v \cdot \left(\prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right)^2$$

we obtain

$$\begin{aligned} u_1 + u_1^* - \Gamma_+^2 \cdot i^*\left(\Delta_{(-)^{r+1}} - 2^{n-1}\right) &\in \Gamma_+ \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)} \cdot T \\ u_2 + u_2^* - \Gamma_-^2 \cdot i^*\left(\Delta_{(-)^{r}} - 2^{n-1}\right) &\in \Gamma_- \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)} \cdot T \\ u_3 + u_3^* - \Gamma_- \prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)} \cdot i^*\left(\Delta_{(-)^{r}} - 2^{n-1}\right) &\in \left(\prod_{p=1}^{l} \left(x_{n_p}^{(p)}\right)^{-\frac{1}{2}} x_{\frac{n_p}{2}}^{(p)}\right)^2 \cdot T \end{aligned} \tag{5.23}$$

Then by definition of the boundary map ∂, the elements $u_i \in R(H)$ yield elements $[\overline{u}_i] \in h^-(R(H)/I^{\Delta}_{\mathrm{reg}})$ corresponding to $\overline{(\delta_2 \oplus 0)}$ and $\overline{(0 \oplus \delta_1)}$ and $\overline{(0 \oplus \zeta_1)}$ respectively. Thus we have an isomorphism of A-modules

$$h^-\left(R(H)/I^{\Delta}_{\mathrm{reg}}\right) \cong \frac{A}{(Y)} \cdot [\overline{u}_1] \oplus \frac{A}{(Y)} \cdot [\overline{u}_2] \oplus \frac{A}{\left(\overline{\gamma}_{\frac{m}{2}}\right)} \cdot [\overline{u}_3]$$

We have now computed $h^*(R(H)/I^{\Delta}_{\mathrm{reg}})$ as an A-module. To determine the ring structure, one would have to determine for example the products $[\overline{u}_i]\,[\overline{u}_j]$. We shall not do so here. But now using Proposition 5.13 again together with Lemma 2.8, we can immediately determine the Tate cohomology of $R(H)/I \cong K^0(X)$ as a group:

Proposition 5.21. *Suppose $m \geq 2, n_1, \ldots, n_l$ are even. Then we have a group isomorphism*

$$h^*(K^0(X)) \cong \left(h^+(R(H)/I^{\Delta}_{reg}) \oplus \frac{A}{(Y)} \cdot [\overline{u}_1] \oplus \frac{A}{(Y)} \cdot [\overline{u}_2] \oplus \frac{A}{\left(\overline{\gamma}_{\frac{m}{2}}\right)} \cdot [\overline{u}_3] \right) \otimes \bigwedge_{t \in \overline{S} \setminus \{n-1\}} ([\overline{w}_t])$$

where

$$h^+(R(H)/I^{\Delta}_{reg}) \cong \frac{\mathbb{Z}_2[\gamma_1, \ldots, \gamma_{\frac{m}{2}}] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\frac{n_p}{2}}^{(p)}\right] \otimes \mathbb{Z}_2[\delta_1, \delta_2]}{(\mu_i \mid 1 \leq i \leq N) + \left(\delta_1 + \delta_2 + \gamma_{\frac{m}{2}},\ \delta_1\delta_2,\ \delta_1 \prod_p \beta_{\frac{n_p}{2}}^{(p)}\right)}$$

$$A \cong \frac{\mathbb{Z}_2[\gamma_1, \ldots, \gamma_{\frac{m}{2}}] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_1^{(p)}, \ldots, \beta_{\frac{n_p}{2}}^{(p)}\right]}{(\mu_i \mid 1 \leq i < N)}$$

and $Y := \prod_{p=1}^{l} \overline{\beta}_{\frac{n_p}{2}}^{(p)}$ and $\mu_j = \sum_{a+a_1+\ldots+a_l=j} \gamma_a \beta_{a_1}^{(1)} \ldots \beta_{a_l}^{(l)} + \binom{2n}{2j}$.

Remark 5.22. As before, Bousfield's lemma identifies $h^*(K^0(X))$ with $W^*(X)$. Under this identification, by Lemma 4.8 we deduce that the summand $h^+\left(R(H)/I^{\Delta}_{\mathrm{reg}}\right)$ is completely contained in $W^0(X)$, and by Lemma 4.9 we have that $[\overline{w}_t] \in W^{-1}(X)$ for all $t \in \overline{S}$. Recall that Δ_+, Δ_- are real if $n \equiv 0 \pmod 4$ and quaternionic if $n \equiv 2 \pmod 4$. From (5.23) and Lemma 4.9, we deduce that the summands

$$\frac{A}{(Y)} \cdot [\overline{u}_1]\,,\ \frac{A}{(Y)} \cdot [\overline{u}_2] \subset \begin{cases} W^{-1}(X) \text{ if } n \equiv 0 \pmod 4 \\ W^{-3}(X) \text{ if } n \equiv 2 \pmod 4 \end{cases}$$

and that $\frac{A}{\left(\overline{\gamma}_{\frac{m}{2}}\right)} \cdot [\overline{u}_3] \subset W^{-1}(X)$ always.

Now from slight modifications in the computation for Proposition 3.16, we deduce:

$$\begin{aligned}
\dim_{\mathbb{Z}_2}(A/(Y)) &= \frac{(n/2-1)!}{(m/2)! \cdot (n_1/2)! \cdot \ldots \cdot (n_l/2)!} \cdot \left(\frac{n_1}{2} + \ldots + \frac{n_l}{2}\right) =: b \\
\dim_{\mathbb{Z}_2}(A/(\overline{\gamma}_{\frac{m}{2}})) &= \frac{(n/2-1)!}{(m/2)! \cdot (n_1/2)! \cdot \ldots \cdot (n_l/2)!} \cdot \frac{m}{2} =: c \\
\dim_{\mathbb{Z}_2} h^+\left(R(H)/I^{\Delta}_{\mathrm{reg}}\right) &= \frac{(n/2)!}{(m/2)! \cdot (n_1/2)! \cdot \ldots \cdot (n_l/2)!} + b =: a + b
\end{aligned}$$

Note that $a = b + c$. It is easy to check that $|\overline{S} \setminus \{n-1\}| = \frac{n_1 + \ldots + n_l}{2} - 1 =: f$. So letting z_i denote the rank of degree i of the 4-periodically graded exterior algebra with f generators of degree -1 (the values of which can be found in Appendix A), we can deduce:

$$\dim_{\mathbb{Z}_2}(W^i(X)) = \begin{cases} (a+b)(z_i + z_{i+1}) \text{ if } n \equiv 0 \pmod 4 \\ (a+b)z_i + cz_{i+1} + 2bz_{i+3} \text{ if } n \equiv 2 \pmod 4 \end{cases}$$

Chapter 6

Relations between Certain Polynomials

In this chapter, we want to give proofs for Propositions 5.1, 5.5, 5.9 and 5.13, thus filling the last gap in our computations. In the first section, we consider the polynomials μ_j occurring in our computation for type A_n and prove Proposition 5.1. These polynomials are special cases of the polynomials ν_j occurring in the computations for types C_n and D_n. We use the results of the first section to prove corresponding results for the ν_j in the second section, completing the proofs of Propositions 5.9 and 5.13. Finally, we prove Proposition 5.5 about the polynomials ξ_j from our computation for type B_n.

6.1 The polynomials μ_j

Let $n_i \in \mathbb{N}$ for each $i \in \mathbb{N}$ be such that for some $l \in \mathbb{N}_0$, the integer n_i is even if $1 \leq i \leq l$ and odd if $i > l$. For every $k \in \mathbb{N}_0$, we define

$$R_k := \frac{\bigotimes_{p=1}^{k+l} \mathbb{Z}_2\left[\beta_0^{(p)}, \ldots, \beta_{n_p}^{(p)}\right]}{\left(\beta_i^{(p)} + \beta_{n_p-i}^{(p)} \mid 1 \leq p \leq k+l,\ 0 \leq i \leq n_p\right) + \left(\beta_0^{(p)} + 1 \mid 1 \leq p \leq k+l\right)}$$

Clearly, R_k is isomorphic to a polynomial ring in $\sum_{p=1}^{k+l} \lfloor n_p/2 \rfloor$ indeterminates. The relations in R_k ensure that there is a sort of mirror symmetry among each family of generators.

The ring R_k reflects of course the Tate cohomology ring of representation rings of centralisers of tori in $SU(n)$ in our computations for type A_n. We define a mod-2 rank ring homomorphism via

$$\mathrm{rk}\colon R_k \to \mathbb{Z}_2, \quad \beta_i^{(p)} \mapsto \binom{n_p}{i}$$

This is, of course, also a reflection of the mod-2 rank function induced on Tate cohomology by the rank function on the representation ring of centralisers of tori.

Now for $k \geq 1$, we define an inclusion map

$$\kappa_{k-1} \colon R_{k-1} \to R_k, \quad \beta_i^{(p)} \mapsto \beta_i^{(p)}$$

The following polynomials are the main objects of study in this section: For $m \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, we define

$$\mu_m^{(k)} := \sum_{a_1+\ldots+a_{k+l}=m} \beta_{a_1}^{(1)} \ldots \beta_{a_{k+l}}^{(k+l)} \in R_k$$

From the interpretation of the $\mu_m^{(k)}$ as restrictions of representations of $SU(n)$ to centralisers of tori in Tate cohomology (cf. section 5.1.2) or directly from a combinatorial interpretation, it is clear that $\operatorname{rk}\left(\mu_m^{(k)}\right) = \binom{n_1+\ldots+n_{k+l}}{m}$. We define a reduced version of the above polynomials by

$$\tilde{\mu}_m^{(k)} := \mu_m^{(k)} + \operatorname{rk}\left(\mu_m^{(k)}\right)$$

These are precisely the polynomials we considered in section 5.1.2. Note that $\mu_m^{(k)} = 0$ if $m \notin \{0, 1, \ldots, n_1 + \ldots + n_{k+l}\}$ and $\mu_0^{(k)} = 1$. Furthermore,

$$\begin{aligned}
\mu_m^{(k)} &= \sum_{a_1+\ldots+a_{k+l}=m} \beta_{a_1}^{(1)} \ldots \beta_{a_{k+l}}^{(k+l)} = \sum_{a_1+\ldots+a_{k+l}=m} \beta_{n_1-a_1}^{(1)} \ldots \beta_{n_{k+l}-a_{k+l}}^{(k+l)} \\
&= \sum_{b_1+\ldots+b_{k+l}=n_1+\ldots+n_{k+l}-m} \beta_{b_1}^{(1)} \ldots \beta_{b_{k+l}}^{(k+l)} = \mu_{n_1+\ldots+n_{k+l}-m}^{(k)}
\end{aligned}$$

So we will only consider the polynomials $\mu_m^{(k)}$ for $0 \leq m \leq \lfloor (n_1 + \ldots + n_{k+l})/2 \rfloor$.

Example 6.1. Let $l = 0$ and $k = 2$ and $n_1 = 3$, $n_2 = 5$. We write the polynomials μ_j in terms of the indeterminates $\beta_i^{(p)}$ where $i \leq \lfloor n_p/2 \rfloor$:

$$\begin{aligned}
\mu_1 &= \mu_7 = \beta_1^{(1)} + \beta_1^{(2)} \\
\mu_2 &= \mu_6 = \beta_1^{(1)} + \beta_1^{(1)}\beta_1^{(2)} + \beta_2^{(2)} \\
\mu_3 &= \mu_5 = 1 + \beta_1^{(1)}\beta_1^{(2)} + \beta_1^{(1)}\beta_2^{(2)} + \beta_2^{(2)} \\
\mu_4 &= 0
\end{aligned}$$

Already in the above example, the polynomials μ_j appear to be quite complicated. We will however show that there is a quite orderly pattern of how they relate to one another.

6.1.1 Regularity and dimension

We noted above that R_k is just a polynomial ring in $\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor$ indeterminates. Hence this is also the maximal length of a regular sequence in R_k. In fact, we show:

Proposition 6.2. *The elements*

$$\mu_1^{(k)}, \mu_2^{(k)}, \ldots, \mu_{\lfloor n_1/2 \rfloor+\ldots+\lfloor n_{k+l}/2 \rfloor}^{(k)} \in R_k$$

form an R_k-regular sequence. The same holds for

$$\tilde{\mu}_1^{(k)}, \tilde{\mu}_2^{(k)}, \ldots, \tilde{\mu}_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}^{(k)} \in R_k$$

Proof. We regard R_k as a graded ring where the grading is defined by setting $|\beta_i^{(p)}| = i$ for all $1 \leq p \leq k+l$ and $0 \leq i \leq \lfloor n_p/2 \rfloor$. Note that the $\mu_m^{(k)}$ are not homogeneous with respect to this grading. But if $1 \leq m \leq \lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor$, then $\mu_m^{(k)}$ has highest homogeneous component of degree m given by

$$\sum_{\substack{a_1+\ldots+a_{k+l}=m \\ a_p \leq \lfloor n_p/2 \rfloor \text{ for } 1 \leq p \leq k+l}} \beta_{a_1}^{(1)} \ldots \beta_{a_{k+l}}^{(k+l)} =: h_m^{(k)}$$

In Proposition 3.15, we showed precisely that these $h_m^{(k)}$ form a regular sequence for $1 \leq m \leq \lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor$. Hence we deduce from Corollary 3.19 that the $\mu_m^{(k)}$ form an R_k-regular sequence for $1 \leq m \leq \lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor$.

The same proof also works for the $\tilde{\mu}_m^{(k)}$ since they have the same homogeneous components as the $\mu_m^{(k)}$ except in degree 0. □

From Proposition 3.18, we see that

$$R_k/\left(\tilde{\mu}_1^{(k)}, \ldots, \tilde{\mu}_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}^{(k)}\right) \quad \text{and} \quad R_k/\left(h_1^{(k)}, \ldots, h_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}^{(k)}\right)$$

have the same $\mathbb{Z}_2$-vector space dimension because the former ring has a filtration such that the associated graded ring is the latter ring. We computed this dimension in Proposition 3.16 and thus obtain:

Proposition 6.3.

$$dim_{\mathbb{Z}_2}\left(R_k/\left(\tilde{\mu}_1^{(k)}, \ldots, \tilde{\mu}_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}^{(k)}\right)\right) = \frac{\left(\lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_{k+l}}{2} \rfloor\right)!}{\lfloor \frac{n_1}{2} \rfloor! \cdot \ldots \cdot \lfloor \frac{n_{k+l}}{2} \rfloor!}$$

6.1.2 Linear combinations

We now want to show the following:

Proposition 6.4. *For every $\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor < m \leq \lfloor (n_1 + \ldots + n_{k+l})/2 \rfloor$, the polynomial $\mu_m^{(k)}$ (or $\tilde{\mu}_m^{(k)}$) is a $\mathbb{Z}_2$-linear combination of the polynomials*

$$\mu_0^{(k)}, \mu_1^{(k)}, \ldots, \mu_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}^{(k)} \ (or\ \tilde{\mu}_1^{(k)}, \ldots, \tilde{\mu}_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}^{(k)})$$

in R_k.

The proof will take up the remainder of this section.

Let us first see how the assertion about the $\mu_m^{(k)}$ implies the assertion about the $\tilde{\mu}_m^{(k)}$. Assuming the claim about the unreduced polynomials, it follows that for every m,

$$\tilde{\mu}_m^{(k)} \in \mathbb{Z}_2 \cdot 1 + \mathbb{Z}_2 \cdot \tilde{\mu}_1^{(k)} + \ldots + \mathbb{Z}_2 \cdot \tilde{\mu}^{(k)}_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}$$

Applying the rank function rk to both sides and using that $\operatorname{rk}\left(\tilde{\mu}_j^{(k)}\right) = 0$ for all j, we see that actually,

$$\tilde{\mu}_m^{(k)} \in \mathbb{Z}_2 \cdot \tilde{\mu}_1^{(k)} + \ldots + \mathbb{Z}_2 \cdot \tilde{\mu}^{(k)}_{\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor}$$

as required.

Thus we are left to prove the assertion about the unreduced polynomials. Let us first change the numbering of the indeterminates and the polynomials in a convenient way. In R_k, we define

$$\begin{aligned} \alpha_i^{(p)} &:= \beta^{(p)}_{i+\lfloor n_p/2 \rfloor} \text{ for all } 1 \le p \le k+l \text{ and } -\lfloor n_p/2 \rfloor \le i \le \lceil n_p/2 \rceil \\ \sigma_m^{(k)} &:= \mu^{(k)}_{m+\lfloor n_1/2 \rfloor + \ldots + \lfloor n_{k+l}/2 \rfloor} \text{ for all } m \in \mathbb{Z}. \end{aligned}$$

These definitions are chosen so that

$$\alpha_i^{(p)} = \begin{cases} \alpha_{-i}^{(p)} \text{ if } 1 \le p \le l \\ \alpha_{-i+1}^{(p)} \text{ if } l < p \le k+l \end{cases} \tag{6.1}$$

and so that the $\sigma_m^{(k)}$ form an R_k-regular sequence for all $-\lfloor n_1/2 \rfloor - \ldots - \lfloor n_{k+l}/2 \rfloor < m \le 0$ and

$$\sigma_m^{(k)} = \sigma_{-m+k}^{(k)} \text{ for all } m \tag{6.2}$$

Also note that

$$\sigma_m^{(k)} = \sum_{a_1+\ldots+a_{k+l}=m} \alpha_{a_1}^{(1)} \ldots \alpha_{a_{k+l}}^{(k+l)}$$

Now we rephrase Proposition 6.4 as follows:

Proposition 6.5. *For every $0 < m \le \lfloor k/2 \rfloor$, the polynomial $\sigma_m^{(k)}$ is a $\mathbb{Z}_2$-linear combination of the polynomials $\sigma_i^{(k)}$ for $i \le 0$.*

We prove a few first lemmas in this direction:

Lemma 6.6. *If $k \ge 2$ is even, then $\sigma_{k/2}^{(k)} = 0$.*

Proof. $\sigma_{k/2}^{(k)}$ is the sum of all monomials of the form $\alpha_{a_1}^{(1)} \ldots \alpha_{a_{k+l}}^{(k+l)}$ with $\sum_{i=1}^{k+l} a_i = k/2$. But by (6.1), we have

$$\alpha_{a_1}^{(1)} \ldots \alpha_{a_{k+l}}^{(k+l)} = \alpha_{-a_1}^{(1)} \ldots \alpha_{-a_l}^{(l)} \alpha_{-a_{l+1}+1}^{(l+1)} \ldots \alpha_{-a_{k+l}+1}^{(k+l)} \tag{6.3}$$

Since $-\sum_{i=1}^{l} a_i + \sum_{j=l+1}^{k+l}(-a_j+1) = k - \sum_{i=1}^{k+l} a_i = k/2$, the right hand side of (6.3) also gives a summand in $\sigma_{k/2}^{(k)}$, which is always distinct from the one on the left hand side since $k > 0$. So all the monomial summands in $\sigma_{k/2}^{(k)}$ cancel out each other. $\square$

Lemma 6.7. *If $k \geq 3$ is odd, then $\sum_{j\leq\lfloor k/2\rfloor} \sigma_j^{(k)} = 0$*

Proof. We have

$$\begin{aligned}
\sum_{j\leq\lfloor k/2\rfloor} \sigma_j^{(k)} &= \sum_{j\leq\lfloor k/2\rfloor}\sum_{m\in\mathbb{Z}} \left(\alpha_{-m}^{(k+l)} \cdot \kappa_{k-1}\left(\sigma_{j+m}^{(k-1)}\right)\right) \\
&= \sum_{j\leq\lfloor k/2\rfloor}\sum_{m\geq 0} \left(\alpha_{-m}^{(k+l)} \cdot \kappa_{k-1}\left(\sigma_{j+m}^{(k-1)}\right) + \alpha_{m+1}^{(k+l)} \cdot \kappa_{k-1}\left(\sigma_{j-m-1}^{(k-1)}\right)\right) \\
&= \sum_{m\geq 0} \alpha_{-m}^{(k+l)} \cdot \kappa_{k-1}\left(\sum_{j\leq\lfloor k/2\rfloor}\left(\sigma_{j+m}^{(k-1)} + \sigma_{j-m-1}^{(k-1)}\right)\right)
\end{aligned}$$

Now in R_{k-1},

$$\begin{aligned}
\sum_{j\leq\lfloor k/2\rfloor}\left(\sigma_{j+m}^{(k-1)} + \sigma_{j-m-1}^{(k-1)}\right) &= \sum_{\lfloor k/2\rfloor-m\leq j\leq\lfloor k/2\rfloor+m} \sigma_j^{(k-1)} \\
&= \sigma_{\frac{k-1}{2}}^{(k-1)} + \sum_{\frac{k-1}{2}-m\leq j<\frac{k-1}{2}} \left(\sigma_j^{(k-1)} + \sigma_{-j+k-1}^{(k-1)}\right) = 0
\end{aligned}$$

where we use (6.2) and Lemma 6.6. Hence the claim follows. $\square$

The previous lemmas already yield some of the linear relations we need. It turns out that we can deduce more linear relations from the basic ones above by induction on k. To see how to obtain these, it is useful to rephrase the problem in terms of power series. Very roughly, we shall see that obtaining a new linear relation from the basic ones above corresponds to a manipulation of the corresponding series that is easy to describe.

Let $\mathbb{Z}_2\langle t\rangle$ denote the ring of series of the form $\sum_{i\in\mathbb{Z}} a_i t^i$ with $a_i \in \mathbb{Z}_2$ for all $i \in \mathbb{Z}$ and $a_i = 0$ for $i \gg 0$. For each $k \geq 0$, we define a map

$$\psi_k\colon \mathbb{Z}_2\langle t\rangle \to R_k, \quad \sum_{j\in\mathbb{Z}} a_j t^j \mapsto \sum_{j\in\mathbb{Z}} a_j \sigma_j^{(k)}$$

This map is well-defined since $\sigma_j^{(k)} = 0$ for $j \ll 0$. Clearly, ψ_k is a group homomorphism.

Finding linear relations among the $\sigma_j^{(k)}$ is now the same as finding elements in $\ker(\psi_k)$. Let

$$P_k(t) := \sum_{j\leq\lfloor k/2\rfloor} t^j \in \mathbb{Z}_2\langle t\rangle$$

We deduce from Lemmas 6.6 and 6.7 and from (6.2):

(P1) If $k \geq 3$ is odd, then $P_k(t) \in \ker(\psi_k)$.

(P2) If $k \geq 2$ is even, then $t^{k/2} \in \ker(\psi_k)$.

(P3) For every $k \geq 0$ and every $s \in \mathbb{Z}$, we have $t^{-s} + t^{s+k} \in \ker(\psi_k)$.

We want to find more elements in the kernel of ψ_k by induction on k. So we need to be able to reduce from k to $k-1$:

Lemma 6.8. *Let $Q(t) \in \mathbb{Z}_2\langle t\rangle$ and $k \geq 1$. Then*

$$\psi_k(Q(t)) = \sum_{i=1}^{\infty} \alpha_i^{(k+l)} \cdot \kappa_{k-1}\psi_{k-1}\left((t^{-i} + t^{i-1})Q(t)\right)$$

Proof. Let $Q(t) = \sum_{j \in \mathbb{Z}} a_j t^j$. Then

$$\begin{aligned}
\psi_k(Q(t)) &= \sum_{j\in\mathbb{Z}} a_j \sigma_j^{(k)} \\
&= \sum_{j\in\mathbb{Z}} a_j \sum_{i\in\mathbb{Z}} \alpha_i^{(k+l)} \cdot \kappa_{k-1}\left(\sigma_{j-i}^{(k-1)}\right) \\
&= \sum_{i\in\mathbb{Z}} \alpha_i^{(k+l)} \cdot \kappa_{k-1}\left(\sum_{j\in\mathbb{Z}} a_j \sigma_{j-i}^{(k-1)}\right) \\
&= \sum_{i\in\mathbb{Z}} \alpha_i^{(k+l)} \cdot \kappa_{k-1}\psi_{k-1}\left(t^{-i}Q(t)\right) \\
&= \sum_{i\geq 1} \alpha_i^{(k+l)} \cdot \kappa_{k-1}\psi_{k-1}\left((t^{-i} + t^{i-1})Q(t)\right)
\end{aligned}$$

as required. □

The following result now yields the elements of $\ker(\psi_k)$ that we need:

Proposition 6.9. *If $k = 2m+1 \geq 3$ is odd, then*

$$\frac{1}{(1+t^{-1})^{2j}} \cdot P_{k-2j}(t) \in \mathit{ker}(\psi_k) \text{ for all } 0 \leq j < m$$

If $k = 2m \geq 4$ is even, then

$$\frac{1}{(1+t^{-1})^{2j+1}} \cdot P_{k-2j-1}(t) \in \mathit{ker}(\psi_k) \text{ for all } 0 \leq j < m-1$$

Before proving this, let us see how it implies Proposition 6.5.

Proof of Proposition 6.5. Suppose $k = 2m+1$ is odd. For all $0 \leq j < m$, the highest non-zero term of

$$\frac{1}{(1+t^{-1})^{2j}} \cdot P_{k-2j}(t) = (1 + t^{-1} + t^{-2} + \ldots)^{2j} \cdot \sum_{i \leq m-j} t^i$$

is of degree $m-j$. Since all these elements are in $\ker(\psi_k)$, this means that for each $0 < j \leq m = \lfloor k/2 \rfloor$, we can write $\sigma_j^{(k)}$ as a $\mathbb{Z}_2$-linear combination of the $\sigma_i^{(k)}$ for $i < j$. This implies the claim.

For $k = 2m$ even and $0 \leq j < m-1$, the highest non-zero term of

$$\frac{1}{(1+t^{-1})^{2j+1}} \cdot P_{k-2j-1}(t) = (1 + t^{-1} + t^{-2} + \dots)^{2j+1} \cdot \sum_{j \leq m-j-1} t^i$$

is of degree $m-1-j$. Thus for every $0 < j < m = k/2$, we can write $\sigma_j^{(k)}$ as a $\mathbb{Z}_2$-linear combination of the $\sigma_i^{(k)}$ for $i < j$. Thus for every $0 < j < m$, we can write $\sigma_j^{(k)}$ as a $\mathbb{Z}_2$-linear combination of the $\sigma_i^{(k)}$ for $i \leq 0$. Furthermore, from Lemma 6.6 we know that $\sigma_m^{(k)} = 0$. So we are done in this case as well. □

Proof of Proposition 6.9. We prove this by induction on k. For $k = 3$, this is (P1).

So suppose $k = 2m \geq 4$ is even and let $0 \leq j < m-1$. By Lemma 6.8, it suffices to show that

$$\frac{t^{-a} + t^{a-1}}{(1+t^{-1})^{2j+1}} \cdot P_{k-2j-1}(t) \in \ker(\psi_{k-1}) \text{ for all } a \geq 1.$$

Note that

$$\begin{aligned}
\frac{t^{-a} + t^{a-1}}{(1+t^{-1})^{2j+1}} \cdot P_{k-2j-1}(t) =&(t^{-a} + t^{a-1}) \cdot \sum_{i \leq 0} t^i \cdot \frac{P_{k-2j-1}(t)}{(1+t^{-1})^{2j}} \\
=&(t^{-a+1} + t^{-a+2} + \dots + t^{a-2} + t^{a-1}) \cdot Q_{k,j}(t) \\
=&q_{a-1}(t) \cdot Q_{k,j}(t)
\end{aligned}$$

where $Q_{k,j}(t) := P_{k-2j-1}(t) \cdot (1+t^{-1})^{-2j}$, and $q_{a-1}(t) := \sum_{|i| \leq a-1} t^i$ is a finite sum with $q_{a-1}(t) = q_{a-1}(t^{-1})$.

By induction hypothesis, we know that for all $0 \leq b < m-1$,

$$\begin{aligned}
\ker(\psi_{k-1}) \ni& \frac{1}{(1+t^{-1})^{2b}} \cdot P_{k-2b-1}(t) \\
=&(1+t^{-1})^{-2b} \cdot (1+t^{-1})^{2j} \cdot t^{j-b} \cdot \frac{P_{k-2j-1}(t)}{(1+t^{-1})^{2j}} \\
=&t^{j-b}(1+t^{-1})^{2(j-b)} \cdot Q_{k,j}(t) \\
=&r_{j-b}(t) \cdot Q_{k,j}(t)
\end{aligned}$$

where $r_{j-b}(t) := t^{j-b}(1+t^{-1})^{2(j-b)}$. For all $0 \leq b \leq j$, the element $r_{j-b}(t)$ is a finite sum with highest non-zero term of degree $j-b$ such that $r_{j-b}(t^{-1}) = r_{j-b}(t)$.

Furthermore, by (P3), for every $c \in \mathbb{Z}$, we have that $t^c + t^{-c+k-1} \in \ker(\psi_{k-1})$. We can write

$$t^c + t^{-c+k-1} = (t^c + t^{-c+2m-1}) \cdot \frac{(1+t^{-1})^{2j}}{P_{k-2j-1}(t)} \cdot \frac{P_{k-2j-1}(t)}{(1+t^{-1})^{2j}}$$

$$\begin{aligned}
&=(t^c + t^{-c+2m-1}) \cdot (1+t^{-1})^{2j} \cdot t^{-\lfloor \frac{k-2j-1}{2} \rfloor}(1+t^{-1}) \cdot Q_{k,j}(t) \\
&=(t^c + t^{-c+2m-1}) \cdot (1+t^{-1})^{2j+1} \cdot t^{-(m-j-1)} \cdot Q_{k,j}(t) \\
&=(t^{c-m} + t^{m-c-1}) \cdot t^{j+1} \cdot (1+t^{-1})^{2j+1} \cdot Q_{k,j}(t) \\
&=s_{c-m,j}(t) \cdot Q_{k,j}(t)
\end{aligned}$$

where $s_{c-m,j}(t) := (t^{c-m} + t^{m-c-1}) \cdot t^{j+1} \cdot (1+t^{-1})^{2j+1}$. Note that $s_{c-m,j}(t)$ is a finite sum, we have $s_{c-m,j}(t^{-1}) = s_{c-m,j}(t)$ and for $c \geq m$ it has highest term of degree $1+j+c-m$.

Hence we see that

$$\{r_{j-b}(t) \mid 0 \leq b \leq j\} \cup \{s_{c-m,j}(t) \mid c \geq m\}$$

is a basis of the $\mathbb{Z}_2$-vector space

$$\left\{ \sum_{i \in \mathbb{Z}} a_i t \in \mathbb{Z}_2\langle t \rangle \mid a_i = a_{-i} \text{ for all } i \in \mathbb{Z} \right\}$$

Thus for all $a \in \mathbb{Z}$, the element $q_{a-1}(t)$ can be written as a $\mathbb{Z}_2$-linear combination of these basis elements. But by the above, this shows that $q_{a-1}(t) \cdot Q_{k,j}(t)$ can be written as a $\mathbb{Z}_2$-linear combination of elements in $\ker(\psi_{k-1})$ for all $a \geq 1$. So it is itself in $\ker(\psi_{k-1})$.

The case that k is odd follows similarly. $\square$

6.2 The polynomials ν_j

We extend the results of the previous section to the more general polynomials occurring in the computations for types C_n and D_n.

Let $m, n_1, \ldots, n_l \in \mathbb{N}$ and define

$$S := \frac{\mathbb{Z}_2[\alpha_0, \alpha_1, \ldots, \alpha_{2m}] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_0^{(p)}, \beta_1^{(p)}, \ldots, \beta_{n_p}^{(p)}\right]}{I_\alpha + I_\beta}$$

where

$$\begin{aligned}
I_\alpha &= (\alpha_i + \alpha_{2m-i} \mid 0 \leq i \leq 2m) + (\alpha_0 + 1) \\
I_\beta &= \left(\beta_i^{(p)} + \beta_{n_p-i}^{(p)} \mid 1 \leq p \leq l,\ 0 \leq i \leq n_p\right) + \left(\beta_0^{(p)} + 1 \mid 1 \leq p \leq l\right)
\end{aligned}$$

We set $n := m + n_1 + \ldots + n_l$. The ring S is isomorphic to a polynomial ring in $m + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor$ indeterminates. Just as in the previous section, we define a mod-2 rank ring homomorphism

$$\mathrm{rk}\colon S \to \mathbb{Z}_2$$

via $\mathrm{rk}(\alpha_i) := \binom{2m}{i}$ and $\mathrm{rk}\left(\beta_i^{(p)}\right) := \binom{n_p}{i}$.

We define elements in S by

$$\mu_k := \sum_{a_1+\ldots+a_l=k} \beta^{(1)}_{a_1} \ldots \beta^{(l)}_{a_l}$$

$$\nu_k := \sum_{i=0}^{\lfloor k/2 \rfloor} \alpha_{k-2i} \cdot \mu_i + \binom{2n}{k}$$

From the interpretation of the ν_k as elements in the Tate cohomology of a representation ring and the above mod-2 rank function as computing the rank of representations modulo 2 (see sections 5.3 and 5.4), it is clear that $\mathrm{rk}(\nu_k) = 0$ for all k. The μ_k are basically the polynomials that we considered in the previous section. Up to the additive constant, they can be obtained as a special case of the ν_k by taking $m = 0$.

Example 6.10. Let $l = 2$ and $m = 2$, $n_1 = 3$, $n_2 = 5$. Then $n = 10$ and

$$\begin{aligned} \nu_1 &= \alpha_1 & \nu_2 &= \alpha_2 + \mu_1 \\ \nu_3 &= \alpha_1 + \alpha_1\mu_1 & \nu_4 &= 1 + \alpha_2\mu_1 + \mu_2 + 1 \\ \nu_5 &= \alpha_1\mu_1 + \alpha_1\mu_2 & \nu_6 &= \mu_1 + \alpha_2\mu_2 + \mu_3 \\ \nu_7 &= \alpha_1\mu_2 + \alpha_1\mu_3 & \nu_8 &= \mu_2 + \alpha_2\mu_3 + \mu_4 \\ \nu_9 &= \alpha_1\mu_3 + \alpha_1\mu_4 & \nu_{10} &= \alpha_2\mu_4 \end{aligned}$$

The polynomials μ_j were given in Example 6.1.

6.2.1 Regularity and dimension

Lemma 6.11. *If k is odd, then ν_k is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta^{(p)}_0, \beta^{(p)}_1, \ldots, \beta^{(p)}_{n_p}\right]$-linear combination of the α_i where i is odd.*

We have $(\nu_i \mid 1 \leq i \leq m \text{ odd}) = (\alpha_i \mid 1 \leq i \leq m \text{ odd})$. Moreover, the elements in $\{\nu_i \mid 1 \leq i \leq m \text{ odd}\}$ are S-regular in any order.

Proof. Let k be odd. Then $\binom{2n}{k} \equiv 0 \pmod 2$ and both $k - 2i$ and $2m - (k - 2i)$ are odd for every i. So from the definition, we obtain that ν_k is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta^{(p)}_0, \beta^{(p)}_1, \ldots, \beta^{(p)}_{n_p}\right]$-linear combination of the α_i where i is odd. Furthermore, note that if $0 \leq k \leq m$ is odd, then

$$\nu_k = \alpha_k + \epsilon_k$$

where each ϵ_k is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta^{(p)}_0, \beta^{(p)}_1, \ldots, \beta^{(p)}_{n_p}\right]$-linear combination of the α_i where i is odd with $1 \leq i < k$. This implies the second part of the lemma. $\square$

Proposition 6.12. *Let*

$$\begin{aligned} A_1 &:= \{\nu_i \mid 1 \leq i \leq m \text{ odd}\}, \\ A_2 &:= \{\nu_i \mid 2 \leq i \leq 2(\lfloor m/2 \rfloor + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor) \text{ even}\} \end{aligned}$$

The elements in $A := A_1 \cup A_2$ form an S-regular sequence in some order.

Proof. First note that if i is even, then the expression for ν_i does not contain any α_j for odd j. By Lemma 6.11, the elements in A_1 form an S-regular sequence. Let

$$\overline{S} := S/(A_1) = S/(\alpha_i \mid i \text{ odd})$$

In $\overline{S}$, we define $\overline{\gamma}_i := \overline{\alpha}_{2i}$. Then $\overline{\gamma}_{m-i} = \overline{\alpha}_{2(m-i)} = \overline{\alpha}_{2i} = \overline{\gamma}_i$ for all i. Note that for every i,

$$\overline{\nu}_{2i} = \sum_{j=0}^{i} \overline{\alpha}_{2i-2j} \cdot \overline{\mu}_j + \binom{2n}{i} = \sum_{j=0}^{i} \overline{\gamma}_{i-j} \cdot \overline{\mu}_j + \binom{2n}{i}$$

Up to the constant $\binom{2n}{i}$, this now is an element in $\overline{S}$ of the type that we considered in the previous section. So from Proposition 6.2, we deduce that $\overline{\nu}_2, \overline{\nu}_4, \ldots, \overline{\nu}_{2(\lfloor m/2\rfloor + \lfloor n_1/2\rfloor + \ldots + \lfloor n_l/2\rfloor)}$ is an $\overline{S}$-regular sequence. This finishes the proof. $\square$

From Propositions 3.18 and 3.16, we deduce:

Proposition 6.13.

$$dim_{\mathbb{Z}_2}\left(S/\left(A_1 \cup A_2\right)\right) = \frac{\left(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_{k+l}}{2} \rfloor\right)!}{\lfloor \frac{m}{2} \rfloor! \cdot \lfloor \frac{n_1}{2} \rfloor! \cdot \ldots \cdot \lfloor \frac{n_{k+l}}{2} \rfloor!}$$

6.2.2 Relations between the ν_i

Recall the definition of A_1 and A_2 in Proposition 6.12.

Proposition 6.14. *If i is odd, then ν_i is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_0^{(p)}, \beta_1^{(p)}, \ldots, \beta_{n_p}^{(p)}\right]$-linear combination of the ν_j where $1 \leq j \leq m$ are odd. In particular, $\nu_i \in (A_1)$. If i is even, then ν_i is a $\mathbb{Z}_2$-linear combination of elements in A_2. In particular, $\nu_i \in (A_2)$.*

Proof. The claim about odd i follows immediately from Lemma 6.11. Now consider the ν_i for even i. For each j, we define $\gamma_j := \alpha_{2j}$ in S. We have $\gamma_{m-j} = \alpha_{2m-2j} = \alpha_{2j} = \gamma_j$ for all j and then for every i,

$$\nu_{2i} = \sum_{j=0}^{i} \alpha_{2i-2j} \cdot \mu_j + \binom{2n}{i} = \sum_{j=0}^{i} \gamma_{i-j} \cdot \mu_j + \binom{2n}{i}$$

and so up to the constant $\binom{2n}{i}$, the ν_{2i} are polynomials of the form considered in the previous section in the variables γ_j and $\beta_j^{(p)}$. Thus Proposition 6.4 implies that each ν_{2i} is a $\mathbb{Z}_2$-linear combination of 1 and the $\nu_j \in A_2$. But applying the rank function, we see that it is actually a $\mathbb{Z}_2$-linear combination of just the $\nu_j \in A_2$ since $\mathrm{rk}(\nu_j) = 0$ for all j. $\square$

6.3 The polynomials ξ_j

We now consider the polynomials occurring in the computation for type B_n.

Let $m, n_1, \ldots, n_l \in \mathbb{N}$. We set $n := m + n_1 + \ldots + n_l$ and define

$$T := \frac{\mathbb{Z}_2[\alpha_0, \alpha_1, \ldots, \alpha_{2m+1}] \otimes \bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_0^{(p)}, \beta_1^{(p)}, \ldots, \beta_{n_p}^{(p)}\right]}{J_\alpha + J_\beta}$$

where

$$\begin{aligned} J_\alpha =& (\alpha_i + \alpha_{2m+1-i} \mid 0 \leq i \leq 2m+1) + (\alpha_0 + 1) \\ J_\beta =& \left(\beta_i^{(p)} + \beta_{n_p-i}^{(p)} \mid 1 \leq p \leq l,\ 0 \leq i \leq n_p\right) + \left(\beta_0^{(p)} + 1 \mid 1 \leq p \leq l\right) \end{aligned}$$

Note that T is very similar to the ring S from the previous section, but the largest index $2m+1$ occurring for the α_i is odd for T. The ring T is isomorphic to a polynomial ring in $m + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor$ indeterminates. We define a mod-2 rank ring homomorphism

$$\mathrm{rk}\colon T \to \mathbb{Z}_2$$

via $\mathrm{rk}(\alpha_i) := \binom{2m+1}{i}$ and $\mathrm{rk}\left(\beta_j^{(p)}\right) := \binom{n_p}{j}$. In T, we define elements

$$\begin{aligned} \mu_k &:= \sum_{a_1+\ldots+a_l=k} \beta_{a_1}^{(1)} \ldots \beta_{a_l}^{(l)}, \\ \xi_k &:= \sum_{i=0}^{\lfloor k/2 \rfloor} \alpha_{k-2i} \cdot \mu_i + \binom{2n+1}{k} \end{aligned}$$

From the interpretation of the ξ_k as elements in the Tate cohomology of a representation ring and the above mod-2 rank function as computing the rank of representations modulo 2 (see section 5.2), it is clear that $\mathrm{rk}(\xi_k) = 0$ for all k. These polynomials ξ_j in T formally look the same as the polynomials ν_j in S. However, due to the fact that the highest occurring index of the α_i is odd in one case and even in the other, the polynomials ξ_j and ν_j are actually qualitatively different. To illustrate this, compare the following with Example 6.10.

Example 6.15. Let $l = 2$ and $m = 2$, $n_1 = 3$, $n_2 = 5$. We then have $n = 10$ and

$$\begin{aligned} \xi_1 &= \alpha_1 + 1 & \xi_2 &= \alpha_2 + \mu_1 \\ \xi_3 &= \alpha_2 + \alpha_1\mu_1 & \xi_4 &= \alpha_1 + \alpha_2\mu_1 + \mu_2 + 1 \\ \xi_5 &= \alpha_2\mu_1 + \alpha_1\mu_2 & \xi_6 &= \alpha_1\mu_1 + \alpha_2\mu_2 + \mu_3 \\ \xi_7 &= \mu_1 + \alpha_2\mu_2 + \alpha_1\mu_3 & \xi_8 &= \alpha_1\mu_2 + \alpha_2\mu_3 + \mu_4 \\ \xi_9 &= \mu_2 + \alpha_2\mu_3 + \alpha_1\mu_4 & \xi_{10} &= \alpha_1\mu_3 + \alpha_2\mu_4 + \mu_3 \end{aligned}$$

The expressions for the μ_j were given in Example 6.1.

6.3.1 Regularity and dimension

Proposition 6.16. *The elements in*

$$\{\xi_i \mid 1 \leq i \leq m \text{ odd}\} \cup \left\{\xi_i \mid 2 \leq i \leq 2\left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_l}{2} \right\rfloor\right) \text{ even}\right\}$$

form a T-regular sequence in some order.

Proof. If $1 \leq i \leq m$ is odd, then

$$\xi_i = \alpha_i + \eta_i + \binom{2n+1}{i}$$

where $\eta_i \in (\alpha_1, \alpha_3, \ldots, \alpha_{i-2})$ for every odd $1 \leq i \leq m$. Hence it is clear that the ξ_i for odd $1 \leq i \leq m$ form a T-regular sequence.

So now let

$$\overline{T} := T/(\xi_i \mid 1 \leq i \leq m \text{ odd})$$

We see from the above that if $1 \leq i \leq m$ is odd, then $\overline{\alpha}_i \in \overline{T}$ can be expressed in terms of the $\overline{\beta}_j^{(p)}$ and a constant. Note that $\overline{T}$ is still a polynomial ring. We can define a grading on $\overline{T}$ by setting

$$\begin{aligned} \left|\overline{\beta}_i^{(p)}\right| &:= i \quad \text{for all } 1 \leq p \leq l,\ 1 \leq i \leq \lfloor n_p/2 \rfloor \\ |\overline{\alpha}_{2i}| &:= i \quad \text{for all } 1 \leq i \leq \lfloor m/2 \rfloor \end{aligned}$$

Then for all $2 \leq 2i \leq 2(\lfloor m/2 \rfloor + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor)$, we see that the highest homogeneous component of $\overline{\xi}_{2i} \in \overline{T}$ is of degree i, and Proposition 3.15 shows that they form a $\overline{T}$-regular sequence. Hence by Corollary 3.19, the elements $\overline{\xi}_{2i}$ for $2 \leq 2i \leq 2(\lfloor m/2 \rfloor + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor)$ form a $\overline{T}$-regular sequence.

This completes the proof. $\square$

Remark 6.17. The proof of the previous Proposition shows that we may replace ξ_{2i} for $1 \leq i \leq \lfloor m/2 \rfloor + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor$ by any $\delta \in T$ such that $\overline{\delta} \in \overline{T}$ has the same highest homogeneous component as $\overline{\xi}_{2i} \in \overline{T}$ and still obtain a regular sequence.

From Propositions 3.18 and 3.16, we deduce:

Proposition 6.18. *Let I be the ideal in T generated by the ξ_i where $1 \leq i \leq m$ is odd and by the ξ_i where $2 \leq i \leq 2(\lfloor m/2 \rfloor + \lfloor n_1/2 \rfloor + \ldots + \lfloor n_l/2 \rfloor)$ is even. Then*

$$dim_{\mathbb{Z}_2}(T/I) = \frac{\left(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_{k+l}}{2} \rfloor\right)!}{\lfloor \frac{m}{2} \rfloor! \cdot \lfloor \frac{n_1}{2} \rfloor! \cdot \ldots \cdot \lfloor \frac{n_{k+l}}{2} \rfloor!}$$

6.3.2 Relations between the ξ_i

Lemma 6.19. *If $1 \leq 2j+1 \leq m$, then $\alpha_{2j+1} + \alpha_{2j} \in (\xi_i \mid 1 \leq i \leq m)$. More precisely, $\alpha_{2j+1}+\alpha_{2j}$ is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_0^{(p)}, \beta_1^{(p)}, \ldots, \beta_{n_p}^{(p)}\right]$-linear combination of the ξ_i where $1 \leq i \leq m$.*

Proof. We prove this by induction on j.

For $j = 0$, the assertion holds as

$$\alpha_1 + \alpha_0 = \alpha_1 + 1 = \alpha_1 + \binom{2n+1}{1} = \xi_1 \in (\xi_i \mid 1 \leq i \leq m)$$

Suppose now $j > 0$. Then

$$\begin{aligned}
\xi_{2j+1} + \xi_{2j} =& \sum_{i=0}^{j} \alpha_{2j+1-2i} \cdot \mu_i + \binom{2n+1}{2j+1} + \sum_{i=0}^{j} \alpha_{2j-2i} \cdot \mu_i + \binom{2n+1}{2j} \\
=& \sum_{i=0}^{j} (\alpha_{2j-2i+1} + \alpha_{2j-2i})\mu_i \qquad (6.4) \\
=& \alpha_{2j+1} + \alpha_{2j} + \sum_{i=1}^{j} (\alpha_{2j-2i+1} + \alpha_{2j-2i})\mu_i
\end{aligned}$$

using that $\binom{2n+1}{2j+1} + \binom{2n+1}{2j} = \binom{2n+2}{2j+1} \equiv 0 \pmod 2$. By induction hypothesis, for all $1 \leq i \leq j$, the element $\alpha_{2j-2i+1} + \alpha_{2j-2i}$ is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2 \left[\beta_0^{(p)}, \beta_1^{(p)}, \ldots, \beta_{n_p}^{(p)}\right]$-linear combination of the ξ_p for $1 \leq p \leq m$. Hence we deduce that so is $\alpha_{2j+1} + \alpha_{2j}$. This completes the proof. $\square$

Proposition 6.20. *For all $1 \leq j \leq n$,*

$$\xi_j \in (\xi_i \mid 1 \leq i \leq m \text{ odd}) + \left(\xi_i \mid 2 \leq i \leq 2\left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n_1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_l}{2} \right\rfloor\right) \text{ even}\right) =: \hat{I}$$

Proof. By equation (6.4) and Lemma 6.19, the claim for odd j follows from the statement for even j. Let $\hat{T} := T/\hat{I}$. We need to show that the elements $\hat{\xi}_j \in \hat{T}$ represented by $\xi_j \in T$ are zero in $\hat{T}$ for even j.

Note that $(\xi_1, \ldots, \xi_m) \subset \hat{I}$. Let $\hat{\alpha}_i \in \hat{T}$ be the element represented by $\alpha_i \in T$. For $0 \leq i \leq m$, we define $\gamma_i := \hat{\alpha}_{2i} \in \hat{T}$. By Lemma 6.19, we have $\hat{\alpha}_{2i} = \hat{\alpha}_{2i+1}$ for $1 \leq 2i+1 \leq m$. Hence, for $m \geq i > \lfloor m/2 \rfloor$,

$$\gamma_i = \hat{\alpha}_{2i} = \hat{\alpha}_{2m+1-2i} = \hat{\alpha}_{2(m-i)+1} = \hat{\alpha}_{2(m-i)} = \gamma_{m-i} \text{ in } \hat{T}$$

Furthermore, for $1 \leq j \leq \lfloor n/2 \rfloor$, we can write

$$\begin{aligned}
\hat{\xi}_{2j} =& \sum_{i=0}^{j} \hat{\alpha}_{2j-2i}\hat{\mu}_i + \binom{2n+1}{2j} \\
=& \sum_{i=0}^{j} \gamma_{j-i}\hat{\mu}_i + \binom{2n+1}{2j}
\end{aligned}$$

$$= \sum_{a_0+\ldots+a_l=j} \gamma_{a_0}\hat{\beta}^{(1)}_{a_1}\ldots\hat{\beta}^{(l)}_{a_l} + \binom{2n+1}{2j}$$

Thus Proposition 6.4 implies that $\hat{\xi}_{2j}$ is a $\mathbb{Z}_2$-linear combination of

$$1,\ \hat{\xi}_2,\ \hat{\xi}_4, \ldots,\ \hat{\xi}_{2(\lfloor m/2\rfloor+\lfloor n_1/2\rfloor+\ldots+\lfloor n_l/2\rfloor)}$$

But applying the rank function and using that $\mathrm{rk}(\xi_i) = 0$ for all i, we see that $\hat{\xi}_{2j}$ is actually a $\mathbb{Z}_2$-linear combination of

$$\hat{\xi}_2,\ \hat{\xi}_4, \ldots,\ \hat{\xi}_{2(\lfloor m/2\rfloor+\lfloor n_1/2\rfloor+\ldots+\lfloor n_l/2\rfloor)}$$

But $\hat{\xi}_{2i} = 0$ in $\hat{T}$ for all $1 \leq i \leq \lfloor m/2\rfloor + \lfloor n_1/2\rfloor + \ldots + \lfloor n_l/2\rfloor$. So the assertion follows. □

Remark 6.21. We see from Lemma 6.19 and the proof of Proposition 6.20 that a slightly stronger claim holds: For all $1 \leq j \leq n$, the element ξ_j is a $\bigotimes_{p=1}^{l} \mathbb{Z}_2\left[\beta_0^{(p)}, \beta_1^{(p)}, \ldots, \beta_{n_p}^{(p)}\right]$-linear combination of the generators of $\hat{I}$ displayed above. We may even restrict to those generators $\xi_i \in \hat{I}$ for which $i < j$.

Chapter 7

Witt Rings of Extraordinary Flag Varieties

Let G be a compact connected simple Lie group of one of the extraordinary types G_2, F_4, E_6, E_7, E_8 and $H \subset G$ be a centraliser of a torus in G. To compute the Witt ring of G/H, we can, of course, use the same approach as outlined in Chapter 4. The advantage is that while we had to consider infinite families of spaces for the ordinary types, we only have finitely many flag varieties G/H derived from G of extraordinary type. However, there is a different difficulty involved: The extraordinary groups themselves are more complicated to define and so are their fundamental representations, which are cumbersome to work with. Furthermore, it is not obvious how the H lie in G (see [Yok]), and all this makes it more complicated to compute the restriction $R(G) \to R(H)$ than for the ordinary types, which is essential for our computation of $W^*(G/H)$.

To overcome this difficulty, we will explain a different approach to compute the restriction map $R(G) \to R(H)$, or at least the map $h^*(R(G)) \to h^*(R(H))$, that does not make use of explicit constructions of the fundamental representations of G or of an explicit inclusion of the subgroups H in G. The idea is, of course, to extract the required information just from the root systems, which capture enough information about the representations of G and H. We will see that for the extraordinary Lie groups, this root system approach is much easier to handle.

We could have used this approach for the Lie groups of ordinary type as well. However, there we had easy explicit descriptions of the fundamental representations of G available and it was not hard to compute the restrictions $R(G) \to R(H)$. So it would not have given us an advantage.

This chapter is organised as follows: In the first section, some notation is fixed and the general approach is explained. We recall basic definitions and facts about roots associated with a Lie group, its Weyl group and weights. We then explain the relationship between the root system of G and its representations. Finally, we consider centralisers of tori in G, whose root systems are contained in the root system of G. We shall identify their representation rings and see how to compute the duality operation on them from the root system in order to be able to compute their Tate cohomology. It will be clear that from this viewpoint, it is easy to compute the restriction $R(G) \to R(H)$, in the sense that it can easily be implemented on a computer.

In the second section, all this is applied to compute the Witt ring of almost full flag varieties. We then derive the Witt ring of G_2/H for all centralisers of a torus H. We will see that this follows more or less effortlessly from the considerations in the first section.

Finally, we compute $W^*(G/H)$ for every centraliser of a torus H in F_4. Here the root system is handled by a computer to make the computations.

We have not included computations for E_6, E_7 and E_8. The approach outlined should be just as successful for these. However, the Weyl groups are much bigger than for G_2 and F_4, making it computationally more difficult to determine the restrictions $R(G) \to R(H)$. We feel that with enough computing power and a clever and efficient implementation in a computer program, this should be feasible though.

Section 1 contains mostly standard theory as can be found in any book on representation theory of compact Lie groups. We refer to [BD85] and [Ada69] as general references. The description of the representation ring of parabolic subgroups and the duality on them can be found in [Zib14]. We have just adapted the results to our context. As far as we are aware, the computation of the Witt rings for G of type G_2 and F_4 are new.

7.1 A root system approach

Basic notation and facts

Let G be a compact connected Lie group and $T \subset G$ be a maximal torus. We denote by LG and LT the tangent spaces of G and T respectively at the identity element.

The conjugation action of G on itself induces a homomorphism $G \to Aut(G)$ and hence a homomorphism

$$\mathrm{Ad}\colon G \to Aut(LG),$$

the *adjoint representation.*

The *roots* of G are the non-zero weights of $\mathrm{Ad}|_T$. We denote them by $\pm\theta_1, \ldots, \pm\theta_m \in LT^*$. The *Weyl group* of G is given by $W = N_G(T)/T$. It is always a finite group. W acts on LT and we can choose a W-invariant inner product $\langle \cdot, \cdot \rangle$ on LT. Throughout, we thus identify LT with its dual LT^* via the isomorphism

$$LT \to LT^*, \quad v \mapsto \langle v, \cdot \rangle \tag{7.1}$$

We summarise the important facts about the action of W on LT^*. For any choice of $\epsilon_i = \pm 1$ for $1 \le i \le m$, we call

$$\{v \in LT \mid \epsilon_i \theta_i(v) > 0 \text{ for all } 1 \le i \le m\}$$

a *Weyl chamber*, provided that it is non-empty. We suppose that the signs of the roots are arranged so that the chamber corresponding to $\epsilon_i = 1$ for $1 \leq i \leq m$ is non-empty. We define the corresponding Weyl chamber to be the *fundamental Weyl chamber*, denoted by FWC. The roots $\theta_1, \ldots, \theta_m$ are then the *positive roots*. Suppose we have ordered them so that $\theta_1, \ldots, \theta_r$ are the simple roots, i.e. those that are not the sum of two positive roots. The *fundamental dual Weyl chamber* is the subset of LT^* corresponding to FWC under the isomorphism (7.1), i.e.

$$\mathrm{FDWC} = \{\alpha \in LT^* \mid \langle \alpha, \theta_i \rangle > 0 \text{ for all } 1 \leq i \leq r\}$$

We denote by $\overline{\mathrm{FWC}}$ and $\overline{\mathrm{FDWC}}$ the closure of the fundamental and fundamental dual Weyl chamber respectively. One can show that W sends roots to roots. It acts simply transitively on the set of Weyl chambers. W can be identified with the group generated by reflections $LT^* \to LT^*$ in planes orthogonal to the simple roots θ_i $(1 \leq i \leq r)$.

Moreover, let $I = \exp^{-1}(1) \subset LT$ be the *integral lattice*, where $\exp\colon LT \to T$ is the exponential map. We define the *lattice of weights* to be

$$I^* = \{\alpha \in LT^* \mid \alpha(v) \in \mathbb{Z} \text{ for all } v \in I\} \subset LT^*$$

One can see that every root is a weight. Furthermore, W sends weights to weights, so one can restrict the action of W to I^*.

We introduce a partial order on I^*. If $\omega_1, \omega_2 \in I^*$, we write $\omega_1 \leq \omega_2$ if ω_1 lies in the convex hull in LT^* of the orbit of ω_2 under W. It is clear that for any given $\omega \in I^*$, there are only finitely many $\omega' \in I^*$ such that $\omega' < \omega$. Adams [Ada69, Prop. 6.25] shows: If $\omega_1, \omega_2 \in I^* \cap \overline{\mathrm{FDWC}}$, then $\omega_1 \leq \omega_2$ if and only if $\omega_1(v) \leq \omega_2(v)$ for all $v \in \mathrm{FWC}$.

Root system and representations

The following results explain the connection between the action of W on I^* and the representation ring of G:

Theorem 7.1. *The inclusion $T \to G$ induces an isomorphism*

$$R(G) \xrightarrow{\cong} R(T)^W$$

where $R(T)^W$ is the subring of $R(T)$ of elements invariant under the action of W.

Proof. See [Ada69, Thm. 6.20]. □

If $\omega \in I^*$, we denote by $W\omega$ the orbit of ω under the action of W and define the *elementary symmetric sum* of ω by

$$S_W(\omega) := \sum_{\tau \in W\omega} e^{2\pi i \tau} \in R(T)^W$$

Lemma 7.2. *If $\omega_1, \omega_2 \in I^*$, then $S_W(\omega_1)S_W(\omega_2) = S_W(\omega_1 + \omega_2) +$ lower terms*

Proof. See [Ada69, Prop. 6.36]. $\square$

Theorem 7.3. *There is a bijective correspondence between $\overline{FDWC} \cap I^*$ and irreducible complex representations of G in which $\omega \in \overline{FDWC} \cap I^*$ corresponds to an irreducible representation χ such that*

$$\chi|_T = S_W(\omega) + \textit{lower terms}$$

Proof. See [Ada69, Thm. 6.33]. $\square$

If G is simply connected, it turns out we can say more about the structure of $\overline{\mathrm{FDWC}} \cap I^*$ and thus, by the above, about $R(G)$.

Theorem 7.4. *Suppose G is simply connected. Then there are weights $\omega_1, \ldots, \omega_r$ called* fundamental weights *determined by the conditions that*

$$\frac{2\langle \omega_i, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} = \delta_{ij} \textit{ for all } 1 \leq j \leq r$$

The map

$$\mathbb{N}_0^r \to \overline{FDWC}, \quad (n_1, \ldots, n_r) \mapsto \sum_{i=1}^r n_i \omega_i$$

is then a bijection and we have an isomorphism

$$\mathbb{Z}[\rho_1, \ldots, \rho_r] \to R(T)^W \cong R(G), \quad \rho_i \mapsto S_W(\omega_i)$$

Proof. See [Ada69, Thm. 6.41]. $\square$

Remark 7.5. In the above isomorphism of representation rings, we could as well map ρ_i to the irreducible representation with highest weight ω_i and also obtain an isomorphism. This is what Adams actually does in [Ada69, Thm. 6.41]. However, when computing the restriction map $R(G) \to R(H)$ in what follows, it will be more convenient to restrict the symmetric sums. These are, of course, generally only virtual representations of G.

We now want to describe the duality operation $*$ on $R(G)$ in terms of the weights in $\overline{\mathrm{FDWC}} \cap I^*$. Note that for any weight $\omega \in I^*$, we have

$$S_W(\omega)^* = S_W(-\omega)$$

So let w_0 be the unique element in W which takes FDWC to $-$FDWC. This is usually referred to as the *longest element of the Weyl group*[1]. Then

$$S_W(\omega)^* = S_W(-w_0.\omega)$$

where now $-w_0.\omega \in \overline{\mathrm{FDWC}}$ if $\omega \in \overline{\mathrm{FDWC}}$. So we define an involution on the weight lattice by

$$\circ\colon \overline{\mathrm{FDWC}} \cap I^* \to \overline{\mathrm{FDWC}} \cap I^*, \quad \omega \mapsto \omega^\circ := -w_0.\omega$$

Then under the correspondence of Theorem 7.3, dual irreducible representations correspond to $\circ$-dual weights. In particular, the self-dual irreducible representations correspond to fixed points of the above involution on $\overline{\mathrm{FDWC}} \cap I^*$. This often lets us easily determine the Tate cohomology of $R(G)$, for example in the following favourable situation:

Lemma 7.6. *Let $L \subset \overline{FDWC} \cap I^*$ be the subset of fixed points of the involution $\circ$. Note that L is closed under addition. If L is a free commutative monoid with fundamental system $\nu_1, \ldots, \nu_l \in L$, then*

$$\varphi\colon \mathbb{Z}_2[\alpha_1, \ldots, \alpha_l] \to h^+(R(G)), \quad \alpha_i \mapsto [S_W(\nu_i)]$$

is an isomorphism of rings.

Proof. We know from Theorem 7.1 that $h^+(R(G))$ has a $\mathbb{Z}_2$-basis given by $([S_W(\omega)])_{\omega\in L}$. For any $\omega \in L$, we may replace $[S_W(\omega)]$ by $[S_W(\omega)] + \sum_{\omega'\in\Omega} [S_W(\omega')]$ where $\Omega \subset L$ is finite and $\omega' < \omega$ for all $\omega' \in \Omega$ and still have a $\mathbb{Z}_2$-basis for $h^+(R(G))$.

Now $\mathbb{Z}_2[\alpha_1, \ldots, \alpha_l]$ has the monomials $\left(\alpha_1^{n_1} \ldots \alpha_l^{n_l}\right)_{n_1,\ldots,n_l\in\mathbb{N}_0}$ as $\mathbb{Z}_2$-basis and

$$\begin{aligned}\varphi\left(\alpha_1^{n_1} \ldots \alpha_l^{n_l}\right) &= \varphi(\alpha_1)^{n_1} \ldots \varphi(\alpha_l)^{n_l} = [S_W(\nu_1)^{n_1} \ldots S_W(\nu_l)^{n_l}] \\ &= [S_W(n_1\nu_1 + \ldots + n_l\nu_l)] + \sum_{\omega'\in\Omega} [S_W(\omega')]\end{aligned}$$

where $\Omega \subset L$ is finite and $\omega' < n_1\nu_1 + \ldots + n_l\nu_l$ for all $\omega' \in \Omega$. Since L is a free commutative monoid with fundamental system $\nu_1, \ldots, \nu_l$, we deduce from all this that φ maps a basis of $\mathbb{Z}_2[\alpha_1, \ldots, \alpha_l]$ bijectively to a basis of $h^+(R(G))$. Hence φ is an isomorphism.

$\square$

Let us consider the case that G is simply connected. The fundamental weights are precisely the indecomposable elements in $\overline{\mathrm{FDWC}} \cap I^*$. Note that the involution on the weights in $\overline{\mathrm{FDWC}}$ maps indecomposable elements to indecomposable elements, so that it maps the set of fundamental weights to itself. This implies that the fundamental

[1]We have said that W is generated by reflections in the planes orthogonal to the simple roots. For $w \in W$, we can define $l(w) \in \mathbb{N}_0$ to be the minimal length of a word that writes w as a product of reflections in the planes orthogonal to the simple roots. This defines a *length function* $l\colon W \to \mathbb{N}_0$. One can show that $l(w)$ equals the number of positive roots that are mapped to negative roots by w [Hil82, Thm. (3.4)]. From this it is clear that there is a unique element of maximal length, namely the $w_0 \in W$ which sends FDWC to $-$FDWC.

representations of G consist of self-conjugate representations and pairs of mutually conjugate representations.

Centralisers of tori

From now, we assume that G is a compact simply connected Lie group with fundamental weights $\omega_{\theta_1}, \dots, \omega_{\theta_r}$ corresponding to the simple roots $\theta_1, \dots, \theta_r$ of G. Let $\Sigma = \{\theta_1, \dots, \theta_r\}$.

We have seen in Proposition 1.8 that up to conjugation, any centraliser of a torus can be obtained from a choice of subset of the simple roots. So let $\Theta \subset \Sigma$ be such a subset and $H_\Theta \subset G$ be the corresponding centraliser of a torus. By Remark 1.10, the root system of H_Θ is the subsystem of the root system of G generated by Θ, and the Weyl group W_Θ of H_Θ is generated by reflections $LT^* \to LT^*$ in the planes orthogonal to the simple roots in Θ.

Proposition 7.7 ([Zib14, Cor. 3.5]). *We have an isomorphism of rings*

$$\mathbb{Z}[w_\theta, x_\beta^{\pm 1} \mid \theta \in \Theta, \beta \in \Sigma \setminus \Theta] \to R(T)^{W_\Theta} \cong R(H_\Theta)$$

sending w_θ to $S_{W_\Theta}(\omega_\theta)$ and x_β to $S_{W_\Theta}(\omega_\beta) = e^{2\pi i \omega_\beta}$.

Given the representation ring of H_Θ as above, it is then sufficient to understand the action of the longest element of W_Θ on I^* in order to determine the duality on $R(H_\Theta)$. So let w_0 be the longest element in W_Θ. For $\omega \in I^*$, we write $\omega^\circ := -w_0.\omega$. We have:

Proposition 7.8 ([Zib14, Cor. 3.13]). *For every $\theta \in \Theta$, there are $m_\beta^\theta \in \mathbb{Z}$ such that*

$$\omega_\theta^\circ = \omega_{\theta^\circ} + \sum_{\beta \in \Sigma \setminus \Theta} m_\beta^\theta \omega_\beta$$

Furthermore, for every $\beta \in \Sigma \setminus \Theta$, we have $\omega_\beta^\circ = -\omega_\beta$, and under the isomorphism of Proposition 7.7, the duality on $R(H_\Theta)$ corresponds to the duality $*$ given by

$$\begin{aligned} w_\theta^* &= w_{\theta^\circ} \cdot \prod_{\beta \in \Sigma \setminus \Theta} x_\beta^{m_\beta^\theta} && \text{for all } \theta \in \Theta \\ x_\beta^* &= x_\beta^{-1} && \text{for all } \beta \in \Sigma - \Theta \end{aligned}$$

Now that we can determine $R(H_\Theta)$ and the duality on it just via the root system, we can in principle compute the Tate cohomology of $K^0(G/H_\Theta)$ and thus the Witt ring of G/H_Θ: Consider the commutative square

$$\begin{array}{ccc} R(T)^W & \xleftarrow{\cong} & R(G) \\ \downarrow & & \downarrow \\ R(T)^{W_\Theta} & \xleftarrow{\cong} & R(H_\Theta) \end{array}$$

where the horizontal isomorphisms are induced by the inclusion of the maximal torus. Since $S_W(\omega_{\theta_i})$ for $1 \leq i \leq r$ are polynomial generators of $R(T)^W$ by Theorem 7.4, we deduce from Hodgkin's theorem 4.3 that

$$K^0(G/H_\Theta) \cong \frac{R(T)^{W_\Theta}}{(S_W(\omega_{\theta_i}) - s_i \mid 1 \leq i \leq r)} \tag{7.2}$$

where $s_i := |W\omega_{\theta_i}|$, the size of the W-orbit of ω_{θ_i}. Note here that for any $\omega \in I^*$, the symmetric sum $S_W(\omega) \in R(T)^W$ is also an element in the subring $R(T)^{W_\Theta} \subset R(T)^W$. The Tate cohomology of this quotient ring is now computable: We have, for every $1 \leq i \leq r$,

$$S_W(\omega_{\theta_i}) = \sum_{\omega \in W\omega_{\theta_i} \cap \overline{\mathrm{FDWC}}_\Theta} S_{W_\Theta}(\omega)$$

and we are able to determine $R(T)^{W_\Theta}$ and the duality on it from Propositions 7.7 and 7.8. Then we can, in principle, use the results of section 2.2 to compute the Tate cohomology of $K^0(G/H_\Theta)$.

7.2 Almost full flag varieties and G_2

We illustrate our approach by computing the Witt ring of almost full flag varieties. This then yields the Witt rings of all the flag varieties that are quotients of the exceptional Lie group G_2.

An *almost full flag variety* is a homogeneous space G/H where G is a compact simply connected Lie group and H is a subgroup obtained via the map φ in Proposition 1.8 from a one-element subset of the simple roots of G.

Theorem 7.9. *Let G be a compact simply connected Lie group with Weyl group W and Σ be the set of simple roots of G. We denote by $b_\mathbb{C}$, $b_\mathbb{R}$ and $b_\mathbb{H}$ the number of fundamental representations of G of complex, real and quaternionic type, respectively. Let $\Theta = \{\theta\} \subset \Sigma$ and H_Θ be the subgroup of G obtained from Θ via the map φ.*

There is a fundamental weight ω of G such that $W.(\omega + \omega^\circ) \cap \mathbb{R} \cdot \theta \neq \emptyset$. Up to the duality $\circ$ (which here denotes the duality induced by the longest element in W), this fundamental weight is unique.

The Witt ring of G/H_Θ is an exterior algebra with $b_\mathbb{H} - \epsilon$ generators of degree 1 and $\frac{b_\mathbb{C}}{2} + b_\mathbb{R} - \epsilon'$ generators of degree 3, where $(\epsilon, \epsilon') = (1, 0)$ if $\omega = \omega^\circ$ and the fundamental representation with highest weight ω is of quaternionic type, and $(\epsilon, \epsilon') = (0, 1)$ otherwise.

Proof. Let W_Θ be the Weyl group of H_Θ. By Remark 1.10, the root system of H_Θ only consists of the roots $\pm\theta$. Thus W_Θ is the group of order 2, generated by the reflection in the plane orthogonal to $\theta \in LT^*$. By Lemma 7.6, we have an isomorphism

$$\mathbb{Z}_2[\alpha] \to h^+(R(H_\Theta)), \quad \alpha \mapsto \left[S_{W_\Theta}\left(\tilde{\theta}\right)\right] = \left[e^{2\pi i\tilde{\theta}} + e^{-2\pi i\tilde{\theta}}\right]$$

where $\tilde{\theta} := s \cdot \theta$ with $s := \min\{x \in \mathbb{R}^+ \mid x \cdot \theta \in I^*\}$. Let

- ω_i and ω_i° for $1 \leq i \leq \frac{b_\mathbb{C}}{2}$ be the highest weights of the complex fundamental representations of G,
- μ_i for $1 \leq i \leq b_\mathbb{R}$ be the highest weights of the fundamental representations of real type of G,
- ν_i for $1 \leq i \leq b_\mathbb{H}$ be the highest weights of the fundamental representations of quaternionic type of G.

Using (7.2) and Lemma 2.7, we deduce

$$h^*(K^0(G/H_\Theta)) \cong h^*\left(\frac{R(H_\Theta)}{I_\omega + I_\mu + I_\nu}\right)$$

where the ideals I_ω, I_μ and I_ν are defined as

$$\begin{aligned} I_\omega =& \left(S_W(\omega_i + \omega_i^\circ) - \mathrm{rk}(S_W(\omega_i + \omega_i^\circ)) \mid 1 \leq i \leq \frac{b_\mathbb{C}}{2}\right) \\ I_\mu =& (S_W(\mu_i) - \mathrm{rk}(S_W(\mu_i)) \mid 1 \leq i \leq b_\mathbb{R}) \\ I_\nu =& (S_W(\nu_i) - \mathrm{rk}(S_W(\nu_i)) \mid 1 \leq i \leq b_\mathbb{H}) \end{aligned}$$

Suppose that for all i, j, k, we have

$$W.(\omega_i + \omega_i^\circ) \cap \mathbb{Z}\tilde{\theta} = W.\mu_j \cap \mathbb{Z}\tilde{\theta} = W.\nu_k \cap \mathbb{Z}\tilde{\theta} = \emptyset$$

Then for all i, j, k, we have that in $h^+(R(H_\Theta))$,

$$[S_W(\omega_i + \omega_i^\circ) - \mathrm{rk}(S_W(\omega_i + \omega_i^\circ))] = [S_W(\mu_j) - \mathrm{rk}(S_W(\mu_j))] = [S_W(\nu_k) - \mathrm{rk}(S_W(\nu_k))] = 0$$

So from Proposition 4.7, we deduce that $h^*(K^0(G/H_\Theta))$ is the tensor product of $h^*(R(H_\Theta))$ with an exterior algebra. But then $h^*(K^0(G/H_\Theta))$ is an infinite-dimensional $\mathbb{Z}_2$-vector space. This is a contradiction since $h^*(K^0(G/H_\Theta)) \cong W^*(G/H_\Theta)$ is finite-dimensional.

Hence there must be $c \in \mathbb{N}$ such that $c \cdot \tilde{\theta}$ is in the Weyl orbit of one of $\omega_i + \omega_i^\circ$, μ_j, ν_k. Since the $\omega_i + \omega_i^\circ$, μ_j, ν_k are linearly independent and have pairwise disjoint Weyl orbits, there is a unique such among the $\omega_i + \omega_i^\circ$, μ_j, ν_k.

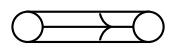
So suppose there is $w \in W$ such that $w.(\omega_i + \omega_i^\circ) = c \cdot \tilde{\theta}$ for some $c > 0$ (the other cases are completely analogous). Then $\frac{1}{c}(\omega_i + \omega_i^\circ) = w^{-1}.\tilde{\theta}$. So $\frac{1}{c}(\omega_i + \omega_i^*)$ is a weight. But since ω_i and ω_i° are part of a fundamental system of the weight lattice, we must have $c = 1$. Thus we deduce that in $h^+(R(H_\Theta))$, we have

$$[S_W(\omega_i + \omega_i^\circ)] = \left[S_{W_\Theta}\left(\tilde{\theta}\right)\right]$$

As $\left[S_{W_\Theta}\left(\tilde{\theta}\right)\right]$ generates $h^*(R(H_\Theta))$ as a polynomial ring, it follows from Proposition 4.7

that the Tate cohomology of $K^0(G/H_\Theta)$ is as claimed. The Witt grading follows from Lemma 4.9. We have that $\epsilon = 0$ and $\epsilon' = 1$ if $\tilde{\theta}$ is in the Weyl orbit of one of the $\omega_i + \omega_i^\circ$ or μ_j, and $\epsilon = 1$ and $\epsilon' = 0$ if $\tilde{\theta}$ is in the Weyl orbit of one of the ν_k. □

We want to apply the above theorem to G_2. The Dynkin diagram of G_2 is given by



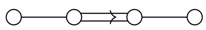
So the only flag varieties G_2/H are the full flag variety, two almost full flag varieties and the singleton. All fundamental representations of G_2 are of real type [Yok67, §6]. Thus Theorem 7.9 together with the computation of the Witt ring of full flag varieties [Zib15, Thm. 3.3] immediately implies:

Theorem 7.10. *Let Σ be the set of simple roots of G_2, let $\Theta \subset \Sigma$ and H_Θ be the centraliser of a torus in G_2 corresponding to Θ. Then $W^*(G_2/H_\Theta)$ is an exterior algebra on $2 - |\Theta|$ generators, with each generator of degree 3.*

7.3 Flag varieties of type F_4

Details on the root system of F_4 can be found in [Bou02, Plate VIII]. Since the root system of F_4 is much more complicated than for G_2 and in particular the Weyl group is much bigger, it is more convenient to make some of the computations using a computer algebra system. So we make use of the WeylGroups package[2] in Macaulay2 of Calmès and Petrov. We have included our Macaulay2 program that obtained the data provided in this section in Appendix B.

The Dynkin diagram of F_4 is given by



We denote by $\theta_1, \theta_2, \theta_3, \theta_4$ a choice of simple roots corresponding to the nodes in the Dynkin diagram from left to right. This is the same numbering as in [Bou02, Plate VIII]. Let $\Sigma = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and denote by ω_i the fundamental weight corresponding to θ_i for $1 \leq i \leq 4$. The fundamental weights constitute a $\mathbb{Z}$-basis of the weight lattice I^*. They are self-dual under the duality induced by the longest element of the Weyl group of F_4 since the only automorphism of the Dynkin diagram of F_4 is the identity.

Let $\Theta \subset \Sigma$. We are first interested in computing the Tate cohomology of $R(H_\Theta)$ using the results of section 7.1. We used Macaulay2 (cf. Appendix B) to compute the duality

$$\circ \colon I^* \to I^*, \quad \omega \mapsto -w_\Theta.\omega$$

where $w_\Theta \in W_\Theta$ is the longest element in W_Θ. The duals of the fundamental weights ω_i are given in Table 7.1, and they determine the duality on I^* since it is spanned by the

[2] The documentation can be found at `http://www2.macaulay2.com/Macaulay2/doc/Macaulay2-1.11/share/doc/Macaulay2/WeylGroups/html/`

Θ	ω_1°	ω_2°	ω_3°	ω_4°	Generators of L_Θ	L_Θ free?
$\{1,2,3,4\}$	ω_1	ω_2	ω_3	ω_4	$\omega_1, \omega_2, \omega_3, \omega_4$	yes
$\{1,2,3\}$	$\omega_1 - 2\omega_4$	$\omega_2 - 4\omega_4$	$\omega_3 - 3\omega_4$	$-\omega_4$	$\omega_1 - \omega_4$ $\omega_2 - 2\omega_4$ $2\omega_3 - 3\omega_4$	yes
$\{1,2,4\}$	$\omega_2 - 2\omega_3$	$\omega_1 - 2\omega_3$	$-\omega_3$	$\omega_4 - \omega_3$	$\omega_1 + \omega_2 - 2\omega_3$ $2\omega_4 - \omega_3$	yes
$\{1,3,4\}$	$\omega_1 - \omega_2$	$-\omega_2$	$\omega_4 - \omega_2$	$\omega_3 - \omega_2$	$2\omega_1 - \omega_2$ $\omega_3 + \omega_4 - \omega_2$	yes
$\{2,3,4\}$	$-\omega_1$	$\omega_2 - 3\omega_1$	$\omega_3 - 2\omega_1$	$\omega_4 - \omega_1$	$2\omega_2 - 3\omega_1$ $2\omega_4 - \omega_1$ $\omega_2 + \omega_4 - 2\omega_1$ $\omega_3 - \omega_1$	no
$\{1,2\}$	$\omega_2 - 2\omega_3$	$\omega_1 - 2\omega_3$	$-\omega_3$	$-\omega_4$	$\omega_1 + \omega_2 - 2\omega_3$	yes
$\{1,3\}$	$\omega_1 - \omega_2$	$-\omega_2$	$\omega_3 - \omega_2 - \omega_4$	$-\omega_4$	$2\omega_1 - \omega_2$ $2\omega_3 - \omega_2 - \omega_4$	yes
$\{1,4\}$	$\omega_1 - \omega_2$	$-\omega_2$	$-\omega_3$	$\omega_4 - \omega_3$	$2\omega_1 - \omega_2$ $2\omega_4 - \omega_3$	yes
$\{2,3\}$	$-\omega_1$	$\omega_2 - 2\omega_1 - 2\omega_4$	$\omega_3 - \omega_1 - 2\omega_4$	$-\omega_4$	$\omega_2 - \omega_1 - \omega_4$ $2\omega_3 - \omega_1 - 2\omega_4$	yes
$\{2,4\}$	$-\omega_1$	$\omega_2 - \omega_1 - 2\omega_3$	$-\omega_3$	$\omega_4 - \omega_3$	$2\omega_2 - \omega_1 - 2\omega_3$ $2\omega_4 - \omega_3$	yes
$\{3,4\}$	$-\omega_1$	$-\omega_2$	$\omega_4 - \omega_2$	$\omega_3 - \omega_2$	$\omega_3 + \omega_4 - \omega_2$	yes
$\{1\}$	$\omega_1 - \omega_2$	$-\omega_2$	$-\omega_3$	$-\omega_4$	$2\omega_1 - \omega_2$	yes
$\{2\}$	$-\omega_1$	$\omega_2 - \omega_1 - 2\omega_3$	$-\omega_3$	$-\omega_4$	$2\omega_2 - \omega_1 - 2\omega_3$	yes
$\{3\}$	$-\omega_1$	$-\omega_2$	$\omega_3 - \omega_2 - \omega_4$	$-\omega_4$	$2\omega_3 - \omega_2 - \omega_4$	yes
$\{4\}$	$-\omega_1$	$-\omega_2$	$-\omega_3$	$\omega_4 - \omega_3$	$2\omega_4 - \omega_3$	yes
$\emptyset$	$-\omega_1$	$-\omega_2$	$-\omega_3$	$-\omega_4$	0	yes

Table 7.1: In the first column, we indicate the choice of subset Θ of the simple roots by displaying the indices of the simple roots in Θ. In columns 2-5, the duals of the fundamental weights under the duality induced by the longest element in W_Θ are given. In column 6, we use the duals described in the previous columns to give generators of L_Θ, and in column 7 we determine whether L_Θ is a free commutative monoid.

fundamental weights. Given these values of w_i°, it is easy to determine generators for the abelian monoid

$$L_\Theta = \left\{\omega \in \overline{\mathrm{FDWC}}_\Theta \mid \omega^\circ = \omega\right\}$$

Such generators are also given in table 7.1. We see that except in the case $\Theta = \{\theta_2, \theta_3, \theta_4\}$, L_Θ is actually a free commutative monoid with the given generators as a fundamental system. Suppose for now that $\Theta \neq \{\theta_2, \theta_3, \theta_4\}$. We will deal with this exceptional case later. Denoting by $\tau_1^\Theta, \ldots, \tau_{a(\Theta)}^\Theta$ the basis elements of L_Θ displayed in the individual cases in Table 7.1, Lemma 7.6 yields that

$$\mathbb{Z}_2\left[\alpha_1, \ldots, \alpha_{a(\Theta)}\right] \to h^+(R(H_\Theta)), \quad \alpha_i \mapsto \left[S_{W_\Theta}\left(\tau_i^\Theta\right)\right] \tag{7.3}$$

is an isomorphism.

Now we want to determine the Tate cohomology of $K^0(F_4/H_\Theta)$, knowing by Hodgkin's theorem 4.3 that

$$K^0(F_4/H_\Theta) \cong \frac{R(H_\Theta)}{(S_W(\omega_i) - \mathrm{rk}(S_W(\omega_i)) \mid i \in \{1,2,3,4\})}$$

If $\Theta \neq \{\theta_2, \theta_3, \theta_4\}$, we can check from Table 7.2 (which was also obtained using Macaulay2) that there is always a subset $M_\Theta = \left\{n_1, \ldots, n_{a(\Theta)}\right\} \subset \{1,2,3,4\}$ such that in $h^+(R(H_\Theta))$,

$$[S_W(\omega_{n_i})] = \sum_{\gamma \in L_\Theta \cap W\omega_{n_i}} [S_{W_\Theta}(\gamma)] = \left[S_{W_\Theta}\left(\tau_i^\Theta\right)\right] \quad \text{for all } 1 \leq i \leq a(\Theta)$$

Since the $\left[S_{W_\Theta}(\tau_i^\Theta)\right]$ for $i = 1, \ldots, a(\Theta)$ are polynomial generators of $h^+(R(H_\Theta))$ by (7.3), Proposition 4.7 immediately implies that $h^*(K^0(F_4/H_\Theta))$ is an exterior algebra on $4 - a(\Theta)$ generators.

Now we consider the exceptional case.

Lemma 7.11. *Let $\Theta = \{\theta_2, \theta_3, \theta_4\}$. Then $h^*(K^0(F_4/H_\Theta)) \cong \bigwedge(u)$ with $u \in h^-$.*

Proof. By Proposition 7.7, we have an isomorphism

$$\mathbb{Z}\left[v_2, v_3, v_4, x^{\pm 1}\right] \to R(H_\Theta)$$

where $v_i \mapsto S_{W_\Theta}(\omega_i)$ for all $i \in \{2,3,4\}$ and $x \mapsto S_{W_\Theta}(\omega_1) = e^{2\pi i \omega_1}$. By Table 7.1, we have that

$$\omega_1^\circ = -\omega_1, \quad \omega_2^\circ = \omega_2 - 3\omega_1, \quad \omega_3^\circ = \omega_3 - 2\omega_1, \quad \omega_4^\circ = \omega_4 - \omega_1$$

So by Proposition 7.8, under the above isomorphism, the duality on $R(H_\Theta)$ is given by

$$x^* = x^{-1}, \quad v_2^* = v_2 x^{-3}, \quad v_3^* = v_3 x^{-2}, \quad v_4^* = v_4 x^{-1}$$

Hence it is easy to check that we have an isomorphism

$$\frac{\mathbb{Z}_2[\alpha_2, \alpha_3, \alpha_4, \beta]}{(\beta^2 + \alpha_2\alpha_4)} \to h^+(R(H_\Theta))$$

where $\alpha_i \mapsto [v_i v_i^*]$ if $i = 2, 4$ and $\alpha_3 \mapsto [v_3 x^{-1}]$ and $\beta \mapsto [v_2 v_4 x^{-2}]$.

Now from Table 7.2, we see that in $h^*(R(H_\Theta))$,

$$[S_W(\omega_1)] = [S_{W_\Theta}(2\omega_4 - \omega_1)] = [S_{W_\Theta}(2\omega_4) \cdot S_{W_\Theta}(-\omega_1)] = \left[S_{W_\Theta}(\omega_4)^2 \cdot S_{W_\Theta}(\omega_1)^{-1}\right] = \alpha_4$$

using that $S_{W_\Theta}(\omega \cdot \nu) = S_{W_\Theta}(\omega) \cdot S_{W_\Theta}(\nu)$ if $W_\Theta.\nu = \{\nu\}$ and the fact that Tate cohomology is 2-torsion. Similarly, we have

$$[S_W(\omega_2)] = [S_{W_\Theta}(2\omega_2 - 3\omega_1)] = [S_{W_\Theta}(2\omega_2) \cdot S_{W_\Theta}(-3\omega_1)] = \left[S_{W_\Theta}(\omega_2)^2 \cdot S_{W_\Theta}(\omega_1)^{-3}\right] = \alpha_2$$

and

$$[S_W(\omega_4)] = [S_{W_\Theta}(\omega_3 - \omega_1)] = [S_{W_\Theta}(\omega_3) \cdot S_{W_\Theta}(\omega_1)^{-1}] = \alpha_3$$

In a similar fashion, we want to determine the element $[S_W(\omega_3)] = [S_{W_\Theta}(\omega_2 + \omega_4 - 2\omega_1)]$ in $h^*(R(H_\Theta))$. We have

$$S_{W_\Theta}(\omega_2 + \omega_4 - 2\omega_1) = S_{W_\Theta}(\omega_2) S_{W_\Theta}(\omega_4) S_{W_\Theta}(\omega_1)^{-2} - \sum_{v \in V} b_v S_{W_\Theta}(v)$$

where $V \subset \overline{\mathrm{FDWC}}_\Theta$ such that $v < \omega_2 + \omega_4 - 2\omega_1$ for all $v \in V$ and $b_v \in \mathbb{N}$. So if $v \in V \cap L_\Theta$, then v must be a $\mathbb{N}_0$-linear combination of $2\omega_4 - \omega_1$, $2\omega_2 - 3\omega_1$, $\omega_3 - \omega_1$. By applying Lemma 7.2 inductively, we deduce that $[S_{W_\Theta}(v)]$ can be written as a polynomial over $\mathbb{Z}_2$ in $\alpha_2, \alpha_3, \alpha_4$ for all $v \in V \cap L_\Theta$. This shows that in $h^*(R(H_\Theta))$,

$$[S_{W_\Theta}(\omega_2 + \omega_4 - 2\omega_1)] = \beta + P \tag{7.4}$$

where P is a polynomial in $\alpha_2, \alpha_3, \alpha_4$.

All in all, noting that $\alpha_2, \alpha_3, \alpha_4$ is a regular sequence in $h^*(R(H_\Theta))$, we have by Corollary 2.9 that

$$h^*\left(\frac{R(H_\Theta)}{(S_W(\omega_j) - \mathrm{rk}(S_W(\omega_j)) \mid j \in \{1, 2, 4\})}\right) \cong \frac{\mathbb{Z}_2[\beta]}{(\beta + \mathrm{rk}(\beta))^2}$$

and then by (7.4), we have that

$$\left[\overline{S_W(\omega_3) - \mathrm{rk}(S_W(\omega_3))}\right] = \overline{\beta + \mathrm{rk}(\beta)}$$

Θ	$L_\Theta \cap W.\omega_1$	$L_\Theta \cap W.\omega_2$	$L_\Theta \cap W.\omega_3$	$L_\Theta \cap W.\omega_4$	Generators of L_Θ
$\{1,2,3,4\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\omega_1, \omega_2, \omega_3, \omega_4$
$\{1,2,3\}$	$\{\omega_2 - 2\omega_4\}$	$\{\omega_1 + 2\omega_3 - 4\omega_4\}$	$\{2\omega_3 - 3\omega_4\}$	$\{\omega_1 - \omega_4\}$	$\omega_1 - \omega_4$ $\omega_2 - 2\omega_4$ $2\omega_3 - 3\omega_4$
$\{1,2,4\}$	$\{\omega_1 + \omega_2 - 2\omega_3\}$	$\{\omega_1 + \omega_2 + 4\omega_4 - 4\omega_3\}$	$\{\omega_1 + \omega_2 + 2\omega_4 - 3\omega_3\}$	$\{2\omega_4 - \omega_3\}$	$\omega_1 + \omega_2 - 2\omega_3$ $2\omega_4 - \omega_3$
$\{1,3,4\}$	$\{2\omega_1 - \omega_2\}$	$\{2\omega_1 + 2\omega_3 + 2\omega_4 - 3\omega_2\}$	$\{2\omega_1 + \omega_3 + \omega_4 - 2\omega_2\}$	$\{\omega_3 + \omega_4 - \omega_2\}$	$2\omega_1 - \omega_2$ $\omega_3 + \omega_4 - \omega_2$
$\{2,3,4\}$	$\{2\omega_4 - \omega_1\}$	$\{2\omega_2 - 3\omega_1\}$	$\{\omega_2 + \omega_4 - 2\omega_1\}$	$\{\omega_3 - \omega_1\}$	$2\omega_2 - 3\omega_1$ $2\omega_4 - \omega_1$ $\omega_2 + \omega_4 - 2\omega_1$ $\omega_3 - \omega_1$
$\{1,2\}$	$\{\omega_1 + \omega_2 - 2\omega_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\omega_1 + \omega_2 - 2\omega_3$
$\{1,3\}$	$\{2\omega_1 - \omega_2\}$	$\{2\omega_1 + 4\omega_3 - 3\omega_2 - 2\omega_4\}$	$\{2\omega_1 + 2\omega_3 - 2\omega_2 - \omega_4\}$	$\{2\omega_3 - \omega_2 - \omega_4\}$	$2\omega_1 - \omega_2$ $2\omega_3 - \omega_2 - \omega_4$
$\{1,4\}$	$\{2\omega_1 - \omega_2\}$	$\{2\omega_1 + 4\omega_4 - \omega_2 - 2\omega_3\}$	$\{2\omega_1 + 2\omega_4 - \omega_2 - \omega_3\}$	$\{2\omega_4 - \omega_3\}$	$2\omega_1 - \omega_2$ $2\omega_4 - \omega_3$
$\{2,3\}$	$\{2\omega_3 - \omega_1 - 2\omega_4\}$	$\emptyset$	$\emptyset$	$\{\omega_2 - \omega_1 - \omega_4\}$	$\omega_2 - \omega_1 - \omega_4$ $2\omega_3 - \omega_1 - 2\omega_4$
$\{2,4\}$	$\{2\omega_2 - \omega_1 - 2\omega_3\}$	$\{2\omega_2 + 4\omega_4 - \omega_1 - 4\omega_3\}$	$\{2\omega_2 + 2\omega_4 - \omega_1 - 3\omega_3\}$	$\{2\omega_4 - \omega_3\}$	$2\omega_2 - \omega_1 - 2\omega_3$ $2\omega_4 - \omega_3$
$\{3,4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{\omega_3 + \omega_4 - \omega_2\}$	$\omega_3 + \omega_4 - \omega_2$
$\{1\}$	$\{2\omega_1 - \omega_2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$2\omega_1 - \omega_2$
$\{2\}$	$\{2\omega_2 - \omega_1 - 2\omega_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$2\omega_2 - \omega_1 - 2\omega_3$
$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{2\omega_3 - \omega_2 - \omega_4\}$	$2\omega_3 - \omega_2 - \omega_4$
$\{4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{2\omega_4 - \omega_3\}$	$2\omega_4 - \omega_3$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0

Table 7.2: The intersections of the Weyl orbits of the fundamental weights of F_4 with the lattice L_Θ of self-dual weights in $\overline{\mathrm{FDWC}}_\Theta$ are described. The generators of L_Θ in the last column were already described in Table 7.1.

in this Tate cohomology ring. This is a zero divisor. By Lemma 2.10, it now follows that

$$h^*(K^0(F_4/H_\Theta)) \cong h^*\left(\frac{R(H_\Theta)}{(S_W(\omega_j) - \mathrm{rk}(S_W(\omega_j)) \mid j \in \{1,2,3,4\})}\right)$$

is an exterior algebra on one generator. □

Thus we see that in all cases, including the exceptional case of Lemma 7.11, the Tate cohomology is an exterior algebra. The number of generators that arise can be described in a concise way.

Notation. If $\Theta \subset \Sigma$, the duality on the weight lattice induced by the longest element in W_Θ induces an involution of the set Θ of simple roots of H_Θ (in other words, an involution of the Dynkin diagram). Suppose Θ contains $N_\Theta^{(1)}$ self-conjugate simple roots and $N_\Theta^{(2)}$ pairs of mutually-conjugate simple roots. Then $N_\Theta^{(1)} + 2N_\Theta^{(2)} = |\Theta|$. We define $N_\Theta := N_\Theta^{(1)} + N_\Theta^{(2)}$.

Example 7.12. Suppose $\Theta = \{\theta_1, \theta_3, \theta_4\}$. Then the root system of H_Θ is $A_1 \times A_2$. The duality fixes θ_1 and exchanges θ_3 and θ_4, so $N_\Theta^{(1)} = N_\Theta^{(2)} = 1$ and $N_\Theta = 2$.

We are led to the following concise formulation of our result.

Theorem 7.13. *Let $\Theta \subset \Sigma = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. Then $W^*(F_4/H_\Theta)$ is an exterior algebra on $4 - N_\Theta$ generators, all of which are of degree 3.*

Proof. Bousfield's lemma gives an isomorphism between the Tate cohomology and the Witt ring. So all that remains to show is that the generators of the exterior algebra are in $W^{-1}(F_4/H_\Theta)$. But since all complex representations of F_4 are of real type [Yok68], this follows from Lemma 4.9. $\square$

Appendix A

Rank of Exterior Algebras

Suppose $W^* = \bigwedge(v_1, \ldots, v_f, w_1, \ldots, w_g)$ is a $\mathbb{Z}_4$-graded exterior algebra over $\mathbb{Z}_2$ with $v_i \in W^{-1}$ for all i and $w_j \in W^{-3}$ for all j. Let $u_k := \dim_{\mathbb{Z}_2} W^k$ for $0 \geq k \geq -3$.

If we regarded W^* as a $\mathbb{Z}$-graded algebra with generators in the degrees given above, its Hilbert polynomial would be

$$H(t) := (1 + t^{-1})^f \cdot (1 + t^{-3})^g \in \mathbb{Z}[t]$$

Noting that in $\mathbb{C}$, we have the identities

$$\begin{aligned} 1^0 + (-1)^0 + i^0 + (-i)^0 &= 4, \\ 1^1 + (-1)^1 + i^1 + (-i)^1 &= 0, \\ 1^2 + (-1)^2 + i^2 + (-i)^2 &= 0, \\ 1^3 + (-1)^3 + i^3 + (-i)^3 &= 0, \end{aligned}$$

we deduce that if $(f, g) \neq (0, 0)$, then

$$\begin{aligned} u_{-k} =& \frac{1}{4} \left(1^k \cdot H(1) + (-1)^k \cdot H(-1) + i^k \cdot H(i) + (-i)^k \cdot H(-i) \right) \\ =& \frac{1}{4} \left(2^{f+g} + 2 \cdot \mathrm{Re}(i^k \cdot H(i)) \right) \\ =& \frac{1}{4} \left(2^{f+g} + 2 \cdot \mathrm{Re} \left(i^k (1 - i)^f (1 + i)^g \right) \right) \end{aligned}$$

Let $\zeta := e^{\frac{\pi i}{4}}$ be a primitive eighth root of unity. Then $1 + i = \sqrt{2} \cdot \zeta$ and $1 - i = \sqrt{2} \cdot \zeta^{-1}$ and so we deduce

$$u_{-k} = 2^{f+g-2} + 2^{\frac{f+g-2}{2}} \cdot \mathrm{Re} \left(\zeta^{2k-f+g} \right)$$

Concretely, we obtain the following table for $(f, g) \neq (0, 0)$. The first two columns contain the values of f and g modulo 4. Throughout, we write $x := f + g$.

$f\ (4)$	$g\ (4)$	u_0	u_{-1}	u_{-2}	u_{-3}
0	0	$2^{x-2} - 2 \cdot (-4)^{\frac{x-4}{4}}$	2^{x-2}	$2^{x-2} + 2 \cdot (-4)^{\frac{x-4}{4}}$	2^{x-2}
1	0	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$
2	0	2^{x-2}	$2^{x-2} + (-4)^{\frac{x-2}{4}}$	2^{x-2}	$2^{x-2} - (-4)^{\frac{x-2}{4}}$
3	0	$2^{x-2} - (-4)^{\frac{x-3}{4}}$	$2^{x-2} + (-4)^{\frac{x-3}{4}}$	$2^{x-2} + (-4)^{\frac{x-3}{4}}$	$2^{x-2} - (-4)^{\frac{x-3}{4}}$
0	1	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$
1	1	$2^{x-2} + (-4)^{\frac{x-2}{4}}$	2^{x-2}	$2^{x-2} - (-4)^{\frac{x-2}{4}}$	2^{x-2}
2	1	$2^{x-2} + (-4)^{\frac{x-3}{4}}$	$2^{x-2} + (-4)^{\frac{x-3}{4}}$	$2^{x-2} - (-4)^{\frac{x-3}{4}}$	$2^{x-2} - (-4)^{\frac{x-3}{4}}$
3	1	2^{x-2}	$2^{x-2} + 2 \cdot (-4)^{\frac{x-4}{4}}$	2^{x-2}	$2^{x-2} - 2 \cdot (-4)^{\frac{x-4}{4}}$
0	2	2^{x-2}	$2^{x-2} - (-4)^{\frac{x-2}{4}}$	2^{x-2}	$2^{x-2} + (-4)^{\frac{x-2}{4}}$
1	2	$2^{x-2} + (-4)^{\frac{x-3}{4}}$	$2^{x-2} - (-4)^{\frac{x-3}{4}}$	$2^{x-2} - (-4)^{\frac{x-3}{4}}$	$2^{x-2} + (-4)^{\frac{x-3}{4}}$
2	2	$2^{x-2} + 2 \cdot (-4)^{\frac{x-4}{4}}$	2^{x-2}	$2^{x-2} - 2 \cdot (-4)^{\frac{x-4}{4}}$	2^{x-2}
3	2	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$
0	3	$2^{x-2} - (-4)^{\frac{x-3}{4}}$	$2^{x-2} - (-4)^{\frac{x-3}{4}}$	$2^{x-2} + (-4)^{\frac{x-3}{4}}$	$2^{x-2} + (-4)^{\frac{x-3}{4}}$
1	3	2^{x-2}	$2^{x-2} - 2 \cdot (-4)^{\frac{x-4}{4}}$	2^{x-2}	$2^{x-2} + 2 \cdot (-4)^{\frac{x-4}{4}}$
2	3	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} - 2 \cdot (-4)^{\frac{x-5}{4}}$	$2^{x-2} + 2 \cdot (-4)^{\frac{x-5}{4}}$
3	3	$2^{x-2} - (-4)^{\frac{x-2}{4}}$	2^{x-2}	$2^{x-2} + (-4)^{\frac{x-2}{4}}$	2^{x-2}

Appendix B

Macaulay2 Program

We include the Macaulay2 program that we used to obtain Tables 7.1 and 7.2.

```
loadPackage "WeylGroups";
R=rootSystemF4;
fundweights={weight(R,{1,0,0,0}),weight(R,{0,1,0,0}),
    weight(R,{0,0,1,0}),weight(R,{0,0,0,1})};
w1=longestWeylGroupElement(R); a=coxeterLength(w1);
L={}; for i from 0 to a do L=L|listWeylGroupElements(R,i);
#L

S={4}; P=parabolic(R,set(S)); w0=longestWeylGroupElement(R,P);
M=matrix{{},{},{},{}};
for s in S do (M=M|matrix(-w0*fundweights_(s-1)));
M

X={};
for fw in fundweights do (X=append(X,unique(delete(null,apply(L,
    i->if isMinimalRepresentative(P,i) and -w0*i*fw==i*fw then (i*fw))))));
X
```

It basically executes the following steps: After loading the root system of F_4, all the elements in the Weyl group of F_4 are saved in a list L. Then we require a subset S of the simple roots numbered as 1,2,3,4 as input and load the root system of the parabolic subgroup corresponding to S. The longest element $w0$ of the Weyl group of the parabolic subgroup is computed and then the $\circ$-duals of the fundamental weights of F_4 are output in a matrix M. This yields the first five columns of Table 7.1. Finally, by going through all the elements of the Weyl group of F_4, saved in L, the self-dual weights in the Weyl orbits of the fundamental weights are determined and saved in a list X. This yields Table 7.2.

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