Hausdorff dimension results for operator-self-similar stable random fields

Inaugural-Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

Ercan Sönmez
aus Oberndorf am Neckar

Düsseldorf, August 2016

Aus dem Mathematischen Institut
der Heinrich-Heine Universität Düsseldorf

Gedruckt mit der Genehmigung der
Mathematisch-Naturwissenschaftlichen Fakultät der
Heinrich-Heine-Universität Düsseldorf

Referent: Prof. Dr. Peter Kern
Koreferent: Prof. Dr. Hans-Peter Scheffler

Tag der mündlichen Prüfung: 17.02.2017

Abstract

The main aim of this thesis is to derive results concerning the Hausdorff dimension of random sets. The notion of Hausdorff dimension has been introduced in order to characterize sets which do possess a fractional pattern. Such sets are commonly referred to as fractals. A typical feature of fractal sets is that they exhibit reappearing patterns, i.e. many fine details of the set resemble the whole set, a phenomenon which is called self-similarity.
The sets we consider in this thesis are randomized, evolve randomly over time and are described by a random field $\{X(t) : t \in \mathbb{R}^d\}$, where t is considered to be the "time"-parameter and for any $t \in \mathbb{R}^d$ the random variable $X(t)$ is $\mathbb{R}^m$-valued. The self-similarity of the set is carried over to a statistical self-similarity, which means that a suitable time-scaling of the random field corresponds in distribution to a scaling in the state space. More precisely, if E is a suitable $d \times d$ matrix and D is a suitable $m \times m$ matrix then

$$\{X(c^E t) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^D X(t) : t \in \mathbb{R}^d\},$$

where $\stackrel{\text{f.d.}}{=}$ stands for the equality of all finite-dimensional marginal distributions. Random fields satisfying the aforementioned property are used for various applications such as the modeling of stock price evolution (see [1, 42]) and ground water modeling (see [31]).
We are interested in determining the Hausdorff dimension of the random sets given by the image

$$X([0,1]^d) = \{X(t) : t \in [0,1]^d\} \subset \mathbb{R}^m$$

and the graph

$$\operatorname{Gr} X([0,1]^d) = \{(t, X(t)) : t \in [0,1]^d\} \subset \mathbb{R}^{d+m}$$

of X over the unit cube $[0,1]^d$. As usual, the Hausdorff dimension is calculated by giving an upper and a lower bound. In our case the random field is Hölder continuous with respect to a certain quasi-metric, of which we make use in calculating an upper bound for the Hausdorff dimension and by generalizing a lemma [2, Lemma 8.2.1] which gives an upper bound for the image and the graph of Hölder continuous functions. A lower bound is calculated by relating the Hausdorff dimension to potential theoretic methods. In particular, we see that for any realization of the above random fractals one obtains the same Hausdorff dimension. Moreover, the obtained Hausdorff dimension is in general not integer.

Zusammenfassung

Das wesentliche Ziel dieser Dissertation besteht darin, die Hausdorff Dimension zufälliger Mengen zu bestimmen. Der Begriff der Hausdorff Dimension wurde eingeführt, um Mengen zu charakterisieren, die ein gebrochenes Muster aufweisen. Solche Mengen werden im Allgemeinen als Fraktale bezeichnet. Ein typisches Merkmal fraktaler Mengen ist, dass sie wiederauftretende Muster aufweisen, d.h. viele feine Details der Menge ähneln der gesamten Menge, ein Phänomen, das als Selbstähnlichkeit bezeichnet wird.

Die Mengen, die wir in dieser Dissertation betrachten, sind randomisiert, entwickeln sich zufällig im Laufe der Zeit und werden durch ein Zufallsfeld $\{X(t) : t \in \mathbb{R}^d\}$ beschrieben, wobei t als „Zeitparameter" aufgefasst wird und für jedes $t \in \mathbb{R}^d$ die Zufallsvariable $X(t)$ Werte in $\mathbb{R}^m$ hat. Die Selbstähnlichkeit der Menge wird auf eine sogenannte statistische Selbstähnlichkeit übertragen, d.h. eine geeignete Zeitskalierung des Zufallsfeldes entspricht in Verteilung einer räumlichen Skalierung. Genauer, für eine geeignete $d \times d$ Matrix E und eine geeignete $m \times m$ Matrix D gilt

$$\{X(c^E t) : t \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{c^D X(t) : t \in \mathbb{R}^d\},$$

wobei $\overset{\text{f.d.}}{=}$ für die Gleichheit aller endlich-dimensionalen Randverteilungen steht. Zufallsfelder mit der oben genannten Skalierungseigenschaft finden Anwendungen in vielen Bereichen, unter anderem in der Modellierung von Aktienpreisen (siehe [1, 42]) und in der Beschreibung von Grundwasserströmung (siehe [31]).

In dieser Arbeit interessieren wir uns für die Hausdorff Dimension der zufälligen Mengen gegeben durch das Bild

$$X([0,1]^d) = \{X(t) : t \in [0,1]^d\} \subset \mathbb{R}^m$$

und den Graphen

$$\operatorname{Gr} X([0,1]^d) = \{(t, X(t)) : t \in [0,1]^d\} \subset \mathbb{R}^{d+m}$$

von X über dem Einheitsquader $[0,1]^d$. Wie üblich wird die Hausdorff Dimension über eine obere und untere Schranke berechnet. In unserem Fall ist das Zufallsfeld Hölder-stetig bezüglich einer Pseudometrik. Davon machen wir Gebrauch, um eine obere Schranke für die Hausdorff Dimension zu berechnen, indem wir ein Lemma [2, Lemma 8.2.1] verallgemeinern, welches eine obere Schranke für das Bild und den Graphen Hölder-stetiger Funktionen liefert. Eine untere Schranke wird mithilfe von potentialtheoretischen Methoden berechnet. Insbesondere sehen wir, dass man für jede Realisierung der obigen zufälligen Fraktale die gleiche Hausdorff Dimension erhält, welche im Allgemeinen nicht ganzzahlig ist.

Acknowledgements

I would like to express my deep appreciation to Prof. Dr. Peter Kern for supervising this thesis and for giving me the opportunity to work in Düsseldorf.

This project would have been impossible without the support of the German Research Foundation (Deutsche Forschungsgemeinschaft) under grant KE1741/6-1.

My grateful thanks are extended to my colleagues at the chair for mathematical statistics and probability theory at the Heinrich-Heine-University Düsseldorf for providing a pleasant working atmosphere.

A special thanks goes to my family for supporting me.

Contents

Introduction

A real-valued self-similar field $\{X(t) : t \in \mathbb{R}^d\}$ is a random field whose finite-dimensional distributions are invariant under scaling of the "time variable" t and the corresponding $X(t)$ in the state space. More precisely, a scalar valued random field $\{X(t) : t \in \mathbb{R}^d\}$ is said to be H-self-similar for some $H \in \mathbb{R}$ if for any $c > 0$

$$\{X(ct) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^H X(t) : t \in \mathbb{R}^d\},$$

where $\stackrel{\text{f.d.}}{=}$ means equality of all finite-dimensional marginal distributions. It was first studied formally by Lamperti [28] and the parameter H is often called the self-similarity index or the Hurst index in the literature. The theoretical importance of self-similar random fields has increased significantly during the past four decades. They are useful to model various natural phenomena for instance in physics, geophysics, mathematical engineering, finance, internet traffic or ground water modeling, see, e.g., [31, 1, 2, 16, 14, 21, 22, 24, 27, 42, 43, 47, 13, 12, 8, 15, 48].

A very important class of such fields is given by Gaussian random fields and, in particular by the well-known fractional Brownian field B_H with Hurst index $H \in (0, 1)$. The random field B_H has stationary increments, i.e. it satisfies

$$\{B_H(t + h) - B_H(h) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{B_H(t) : t \in \mathbb{R}^d\}$$

for any $h \in \mathbb{R}^d$. In addition B_H is isotropic, that is

$$\{B_H(At) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{B_H(t) : t \in \mathbb{R}^d\}$$

for any orthogonal $d \times d$ matrix A. Furthermore, it is a generalization of the famous fractional Brownian motion, implicitely introduced in [23] and defined in [35].

However, Gaussian modeling is a serious drawback for applications including heavy-tailed persistent phenomena. For this purpose α-stable random fields have been introduced, where a random field $\{X(t) : t \in \mathbb{R}^d\}$ is said to be α-stable for $\alpha \in (0, 2]$ if every finite-dimensional marginal distribution $\big(X(t_1), \dots, X(t_n)\big)$ is α-stable. Self-similar α-stable fields with stationary increments have been extensively proposed as an alternative to Gaussian modeling (see, e.g., [39, 48]).

Nevertheless, certain applications [8, 12] require a random field satisfying a scaling relation with different Hurst indices in different not necessarily orthogonal directions. Such random fields are called anisotropic in the literature. In the Gaussian case a popular example of an anisotropic random field is the fractional Brownian sheet $B_{H_1,\dots,H_d}$ with Hurst indices

$H_1, \ldots, H_d > 0$. It was introduced by Kamont [23] and satisfies the property

$$\{B_{H_1,\ldots,H_d}(c_1 t_1, \ldots, c_d t_d) : t = (t_1, \ldots, t_d) \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c_1^{H_1} \ldots c_d^{H_d} B_{H_1,\ldots,H_d}(t) : t \in \mathbb{R}^d\} \quad (0.1)$$

for any $c_1, \ldots, c_d > 0$. However, $B_{H_1,\ldots,H_d}$ does not have stationary increments. We refer the reader to [5, 18, 54] and the references therein for further information on the fractional Brownian sheet.

A new class of anisotropic random fields has been recently proposed by Biermé, Meerschaert and Scheffler [9], where the anisotropic behavior is driven by a $d \times d$ matrix E. To be more precise, according to [9] a scalar valued random field $\{X(t) : t \in \mathbb{R}^d\}$ is called an operator scaling random field of order E and H, where E is a $d \times d$ matrix with positive real parts of its eigenvalues and $H > 0$ if for any $c > 0$

$$\{X(c^E t) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^H X(t) : t \in \mathbb{R}^d\}. \quad (0.2)$$

As usual, $c^E = \exp(E \log c) = \sum_{k=0}^{\infty} \frac{(\log c)^k}{k!} E^k$ is the linear operator defined through the matrix exponential. The existence of α-stable random fields satisfying (0.2) has been provided in [9] through moving-average as well as harmonizable stochastic integral representations. These fields are shown to have stationary increments. This property has been proven to be quite useful in studying their sample paths. According to [9] there exist modifications of the moving-average and harmonizable representation which almost surely satisfy a Hölder condition of certain indices in the Gaussian case $\alpha = 2$. From this, results about the Hausdorff dimension of their graphs on a compact set have been deduced. Biermé and Lacaux [10] established similar results in the stable case $\alpha \in (0, 2)$ for the harmonizable representation. In addition they showed that the moving-average stable random field does not admit any continuous modification.

Hoffmann [20] introduced the so-called operator scaling random sheets. The main idea behind such fields is to combine the property (0.1) of fractional Brownian sheets and (0.2) of operator scaling random fields in order to obtain a more general class of random fields. More precisely, according to his terminology a real-valued random field is called an operator scaling random sheet if for any $c_1, \ldots, c_n > 0$

$$\{X(c_1^{E_1} t_1, \ldots, c_n^{E_n} t_n) : t = (t_1, \ldots, t_n) \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c_1^{H_1} \ldots c_n^{H_n} X(t) : t \in \mathbb{R}^d\}, \quad (0.3)$$

where $E_1, \ldots, E_n$ are suitable matrices with positive real parts of their eigenvalues and $H_1, \ldots, H_n > 0$. By following the outline in [9] and by using the same kind of stochastic integral representations the existence of random fields satisfying (0.3) has been established in [20]. These fields have been proven to be quite flexible in modeling physical phenomena and can be applied in order to extend the well-known Cahn-Hilliard phase-field model. We refer to [3] and the references therein for further details. However, the aforementioned operator scaling random sheets do not possess stationary increments.

Another multivariate generalization of operator scaling random fields has been presented by Li and Xiao [33], i.e. to random fields with values in $\mathbb{R}^m$. The extension is to allow a scaling relation in the state space by linear operators. This concept is mainly motivated by the increasing interest in multivariate random field models in spatial statistics as well as in environmental, agricultural and ecological sciences. See [33, 47, 13] for further information. If E is a $d \times d$ real matrix and D is an $m \times m$ real matrix with positive real parts of their eigenvalues a random field $\{X(t) : t \in \mathbb{R}^d\}$ with values in $\mathbb{R}^m$ is called operator-self-similar if for any $c > 0$

$$\{X(c^E t) : t \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{c^D X(t) : t \in \mathbb{R}^d\}. \tag{0.4}$$

By defining stochastic vector integrals of deterministic matrix kernels with respect to a stable random vector measure and by using the concepts developped in [9], Li and Xiao [33] established the existence of random fields satisfying (0.4). Lastly, they mention that from both theoretical and applied point of view it would be interesting to investigate the sample path regularity and fractal properties of these fields.

In this thesis we study both operator scaling random sheets and operator-self-similar random fields and provide results about their sample path properties. The results presented in this thesis generalize various results in the literature, as will be shown in several examples. Additionally, we completely solve an open problem concerning the Hölder continuity and Hausdorff dimension of the sample paths of multivariate operator-self-similar stable random fields formulated in [33]. In particular, our results are valid for a large class of self-similar random fields.

This thesis is mainly divided into two parts. The aim of the first part is essentially to lay out the mathematical foundations required for the presented results.

Since our main focus will be on Hausdorff dimensions, we recall its definition, some basic properties and related results in the first chapter.

An essential tool in studying anisotropic random fields is the change to generalized polar coordinates with respect to scaling matrices, which was introduced in [38] and already used in [9, 10, 11, 32, 33]. Therefore, Chapter 2 is devoted to introduce these generalized polar coordinates and recall their properties.

Since the random fields we consider in this thesis are α-stable and given by stochastic integrals constructed in [42, 33], the subsequent chapter focuses on α-stable distributions and the construction of these integrals.

In the second part we present our main results. More precisely, in Chapter 4 we consider a random field X with values in $\mathbb{R}^N$, where at each time $t \in \mathbb{R}^d$ the components of the random vector $X(t)$ are N independent copies of the harmonizable operator scaling stable random sheets introduced by Hoffmann [20]. This idea is motivated by Ayache and Xiao [5]. By combining and further extending methods used in [9, 10, 11, 5] we give an upper bound on the uniform modulus of continuity of these fields. Based on this, we determine the Hausdorff dimension of the range and the graph of a trajectory of such fields over the unit cube $[0, 1]^d$. As noted above the property of stationary increments is no more true for the fields constructed in [20]. The absence of this property seems to be one of the main difficulties in determining results about their sample paths.

Finally, Chapter 5 and 6 deal with multivariate operator-self-similar stable random fields introduced by Li and Xiao [33]. As mentioned earlier they leave the open problem of investigating the sample path regularity and fractal dimensions of these fields. In particular, they conjecture that these properties such as path continuity and Hausdorff dimensions are mostly determined by the real parts of the eigenvalues of the scaling matrices in (0.4). In Chapter 5 we will solve this problem for the moving-average and harmonizable representation of such fields in the Gaussian case $\alpha = 2$ and highlight the fact that the aforementioned properties also depend on the multiplicity of the real parts of the eigenvalues of the scaling matrices. The purpose of Chapter 6 is to establish the corresponding results in the stable case $\alpha \in (0, 2)$ for the harmonizable representation.

Large parts of the last three chapters are based on the works of Sönmez [44, 45, 46].

Part I.

Foundations

1. Hausdorff dimension

In this chapter we introduce the notion of Hausdorff dimension and methods for its calculation, where the content of this chapter is strongly based on the books [17, 37] and parts of [2, 40]. For a more general introduction, information on the history and proofs of the statements below the reader is adviced to consult the aforementioned references.

1.1. Definition and properties

The basic aim is to describe the geometric structure of general Borel sets in the euclidean space $\mathbb{R}^m$ with an emphasis on fractal sets. These are typically sets having Lebesgue measure zero but being quite different from smooth curves and surfaces. Let $U \subset \mathbb{R}^m$ be a non-empty set. Recall that the diameter of U is defined as

$$\operatorname{diam}(U) = \sup\{\|x - y\| : x, y \in U\},$$

where $\|\cdot\|$ is a fixed norm on $\mathbb{R}^m$. For any $s \geq 0$ the s-dimensional Hausdorff measure of U is defined by

$$\mathcal{H}^s(U) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_{k=1}^{\infty} \operatorname{diam}(U_k)^s : U \subset \bigcup_{k=1}^{\infty} U_k,\ \operatorname{diam}(U_k) \leq \delta \right\}.$$

One can easily show that $\mathcal{H}^s(U) < \infty$ implies $\mathcal{H}^t(U) = 0$ for all $t > s$ (see [17, Chapter 2.2]). Thus, there exists a critical value, denoted by $\dim_{\mathcal{H}} U$, such that

$$\dim_{\mathcal{H}} U = \inf\{s \geq 0 : \mathcal{H}^s(U) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(U) = \infty\}.$$

$\dim_{\mathcal{H}} U$ is called the Hausdorff dimension of U.

Remark 1.1. From the definition of the Hausdorff dimension it is immediate that $\dim_{\mathcal{H}} U = m$ for any non-empty open set $U \subset \mathbb{R}^m$ and, in particular $\dim_{\mathcal{H}} \mathbb{R}^m = m$. Furthermore, as expected to hold for any reasonable definition of dimension it is monotone, that is $\dim_{\mathcal{H}} U \leq \dim_{\mathcal{H}} V$ for any two sets $U \subset V$.

In order to determine the Hausdorff dimension of U one usually gives an upper bound and a lower bound for $\dim_{\mathcal{H}} U$. In our considerations we will be interested in determining the dimension of the range and the graph of a function. More precisely, let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a function. We are interested in the Hausdorff dimension of the range

$$f([0, 1]^d) = \{f(x) : x \in [0, 1]^d\} \subset \mathbb{R}^m$$

and the graph

$$\operatorname{Gr} f([0,1]^d) = \Big\{ (x, f(x)) : x \in [0,1]^d \Big\} \subset \mathbb{R}^{d+m}$$

of f over the unit cube $[0,1]^d$. The following Lemma is well-known and gives an upper bound if the function f satisfies a Hölder condition. A proof can be found in [2, p. 193] for instance.

Lemma 1.2. *Let $f = (f_1, \ldots, f_m) : \mathbb{R}^d \to \mathbb{R}^m$ satisfy a Hölder condition of order $\alpha = (\alpha_1, \ldots, \alpha_m)$ on $[0,1]^d$, that is*

$$|f_i(x) - f_i(y)| \le c\|x - y\|^{\alpha_i}, \quad i = 1, \ldots, m$$

for some $c > 0$ and all $x, y \in [0,1]^d$. Assuming that

$$0 < \alpha_1 \le \ldots \le \alpha_m \le 1$$

we have

$$\begin{aligned} \dim_{\mathcal{H}} f([0,1]^d) &\le \min \Big\{ m, \frac{d + \sum_{i=1}^m (\alpha_m - \alpha_i)}{\alpha_m} \Big\}, \\ \dim_{\mathcal{H}} \operatorname{Gr} f([0,1]^d) &\le \min \left\{ \frac{d + \sum_{i=1}^m (\alpha_m - \alpha_i)}{\alpha_m}, d + \sum_{i=1}^m (1 - \alpha_i) \right\}. \end{aligned}$$

Corollary 1.3. *(i) Let $f : U \to \mathbb{R}^m$ be a Lipschitz transformation on some Borel set $U \subset \mathbb{R}^d$, i.e.*

$$\|f(x) - f(y)\| \le c\|x - y\| \quad \forall x, y \in U$$

with some $c > 0$. Then $\dim_{\mathcal{H}} f(U) \le \dim_{\mathcal{H}} U$.
(ii) If $f : U \to \mathbb{R}^m$ is a bi-Lipschitz transformation on the Borel set $U \subset \mathbb{R}^d$, i.e.

$$c_1\|x - y\| \le \|f(x) - f(y)\| \le c_2\|x - y\| \quad \forall x, y \in U,$$

where $0 < c_1 \le c_2 < \infty$, then $\dim_{\mathcal{H}} f(U) = \dim_{\mathcal{H}} U$. In particular, for any function $f : \mathbb{R}^d \to \mathbb{R}^m$ we have

$$\dim_{\mathcal{H}} f([0,1]^d) \le \dim_{\mathcal{H}} \operatorname{Gr} f([0,1]^d). \tag{1.1}$$

Proof. A proof of (i) and (ii) can be found in [17, Corollary 2.4]. Furthermore, (1.1) is a frequently used result. For the sake of completeness, let us prove it. Consider the projection $G : \mathbb{R}^{d+m} \to \mathbb{R}^m$ given by $G(x, y) = y$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^m$ and let $\| \cdot \|_1$ be the 1-norm. Using the fact that all norms on the euclidean space are equivalent (see, e.g., [38, Proposition 2.1.4]) for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^m$ we get

$$\begin{aligned} \|G(x_1, y_1) - G(x_2, y_2)\| &= \|y_1 - y_2\| \le \|x_1 - x_2\| + \|y_1 - y_2\| \\ &\le c\|x_1 - x_2\|_1 + c\|y_1 - y_2\|_1 = c\|(x_1, y_1) - (x_2, y_2)\|_1 \\ &\le c\|(x_1, y_1) - (x_2, y_2)\|, \end{aligned}$$

where we used the equivalence of norms in the second and last inequality and c is an unspecified positive constant. Thus, G is a Lipschitz transformation and (i) yields

$$\dim_{\mathcal{H}} f([0,1]^d) = \dim_{\mathcal{H}} G\big(\operatorname{Gr} f([0,1]^d)\big) \le \dim_{\mathcal{H}} \operatorname{Gr} f([0,1]^d).$$

□

Lemma 1.2 has been improved by Xiao [53, Lemma 2.1] to the following statement.

Lemma 1.4. *Let the assumptions of Lemma 1.2 hold. Then*

$$\dim_{\mathcal{H}} f([0,1]^d) \le \min\Big\{m; \frac{d+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j}, 1 \le j \le m\Big\},$$
$$\dim_{\mathcal{H}} \operatorname{Gr} f([0,1]^d) \le \min\left\{\frac{d+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j}, 1 \le j \le m; d+\sum_{i=1}^{m}(1-\alpha_i)\right\}.$$

In Lemma 5.5 of the second part of this thesis we further generalize Lemma 1.4.

1.2. Frostman criterion

In the underlying chapter we mainly introduced techniques for calculating the upper bound of Hausdorff dimensions. In this section we focus on the calculation of lower bounds by relating the Hausdorff dimension to potential theoretic methods. Let $U \subset \mathbb{R}^m$ be a subset and let $\mathcal{M}^1(U, \mathcal{B}(U))$ be the set of Borel-probabilty measures on U. For $\gamma \ge 0$ the γ-energy of $\mu \in \mathcal{M}^1(U, \mathcal{B}(U))$ is defined as

$$I_\gamma(\mu) = \int_U \int_U \frac{\mu(dx)\mu(dy)}{\|x-y\|^\gamma}.$$

The following Theorem is often referred to as Frostman's theorem (see, e.g., [17, Theorem 4.13]) and states that in order to find a lower bound for $\dim_{\mathcal{H}} U$ it suffices to show that there exists a probability measure $\mu \in \mathcal{M}^1(U, \mathcal{B}(U))$ with finite γ-energy.

Theorem 1.5. *Let $U \subset \mathbb{R}^m$. If there exists a probability measure $\mu \in \mathcal{M}^1(U, \mathcal{B}(U))$ with $I_\gamma(\mu) < \infty$ then $\mathcal{H}^\gamma(U) = \infty$ and, consequently* $\dim_{\mathcal{H}} U \ge \gamma$.

In this thesis we will be interested in random fields $\{X(x) : x \in \mathbb{R}^d\}$ with values in $\mathbb{R}^m$ and continuous paths. To be more precise, if we consider the image $X([0,1]^d)$ of such a random field a typical choice of a random probability measure

$$\mu \in \mathcal{M}^1\Big(X([0,1]^d), \mathcal{B}\big(X([0,1]^d)\big)\Big)$$

is the occupation measure given by

$$\mu(U) = \int_{[0,1]^d} \mathbb{1}_{\{X(x)\in U\}} dx$$

for any $U \in \mathcal{B}(X([0,1]^d))$ so that

$$\int g(x)d\mu(x) = \int_{[0,1]^d} g(X(t))dt$$

for any measurable function g and, in particular

$$\int_{X([0,1]^d)} \int_{X([0,1]^d)} \|x-y\|^{-\gamma} d\mu(x)d\mu(y) = \int_{[0,1]^d} \int_{[0,1]^d} \|X(t)-X(s)\|^{-\gamma} dtds$$

(see, e.g., [17, p. 243]). Thus, by Theorem 1.5 it suffices to show that

$$\int_{[0,1]^d} \int_{[0,1]^d} \|X(t)-X(s)\|^{-\gamma} dtds < \infty$$

in order to get $\dim_{\mathcal{H}} X([0,1]^d) \geq \gamma$, which almost surely follows from

$$\mathbb{E}\Big[\int_{[0,1]^d} \int_{[0,1]^d} \|X(t)-X(s)\|^{-\gamma} dtds\Big] < \infty.$$

Note that the latter integrand is non-negative so that Tonelli's theorem applies and yields that in order to show $\dim_{\mathcal{H}} X([0,1]^d) \geq \gamma$ almost surely one only has to prove that

$$\int_{[0,1]^d} \int_{[0,1]^d} \mathbb{E}\big[\|X(t)-X(s)\|^{-\gamma}\big] dtds < \infty.$$

Moreover, by the same arguments as above it suffices to show that

$$\int_{[0,1]^d} \int_{[0,1]^d} \mathbb{E}\Big[\big(\|t-s\|^2 + \|X(t)-X(s)\|^2\big)^{-\frac{\gamma}{2}}\Big] dtds < \infty$$

in order to obtain $\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \geq \gamma$ almost surely.

2. Generalized polar coordinates

This chapter is mainly devoted to introducing the generalized polar coordinates and can be seen as a collection of their properties which have been established in [9, 10, 32]. Throughout this chapter, we fix a matrix $E \in \mathbb{R}^{d\times d}$ with distinct positive real parts of its eigenvalues given by $0 < a_1 < \ldots < a_p$ for some $p \leq d$.

2.1. Spectral decomposition

Let f be the minimal polynomial of E, i.e. the polynomial of lowest degree such that $f(E) = 0$. Moreover, factor f into polynomials $f_1, \ldots, f_p$ such that all roots of each f_i have real part equal to a_i and define $W_i = \operatorname{Ker}\big(f_i(E)\big) = \{x \in \mathbb{R}^d : f_i(E)x = 0\}$ as the kernel of $f_i(E), 1 \leq i \leq p$. Note that $W_1, \ldots, W_p$ are vector subspaces of $\mathbb{R}^d$. Then by [38, Theorem 2.1.14]

$$\mathbb{R}^d = W_1 \oplus \ldots \oplus W_p$$

is a direct sum decomposition, that is any $x \in \mathbb{R}^d$ can uniquely be written as

$$x = x_1 + \ldots + x_p$$

for $x_i \in W_i$, $1 \leq i \leq p$. Further, one can choose an inner product on $\mathbb{R}^d$ such that the subspaces $W_1, \ldots, W_p$ are mutually orthogonal. A quite often choice in our considerations will be $\|x\|_2 = \langle x, x\rangle^{1/2}$ the associated Euclidean norm. Furthermore, we will refer to $W_1 \oplus \ldots \oplus W_p$ as the direct sum decomposition with respect to E.

2.2. Definition of polar coordinates

We now recall the definition of the generalized polar coordinates with respect to the matrix E. This definition originates from the following result taken from [38, Lemma 6.1.5]. Let us fix an arbitrary norm $\|\cdot\|$.

Lemma 2.1. *For any $x \in \mathbb{R}^d$ define*

$$\|x\|_E = \int_0^1 \|t^E x\| \frac{dt}{t}.$$

Then $\|\cdot\|_E$ is a norm on $\mathbb{R}^d$ such that $t \mapsto \|t^E x\|$ is strictly increasing for all $x \in \mathbb{R}^d \setminus \{0\}$. Moreover, if $S_E = \{x \in \mathbb{R}^d : \|x\|_E = 1\}$ denotes the unit sphere in $\mathbb{R}^d$ with respect to this norm the mapping $\psi : (0, \infty) \times S_E \to \mathbb{R}^d \setminus \{0\}$ defined by $\psi(t, \theta) = t^E\theta$ is a homeomorphism.

Since the function ψ in Lemma 2.1 is continuous and bijective, any $x \in \mathbb{R}^d \setminus \{0\}$ can uniquely be written as

$$x = \tau_E(x)^E l_E(x)$$

for some continuous functions $\tau_E(x) > 0$ and $l_E(x) \in S_E$ depending on E. $\tau_E(x)$ is called the radius with respect to E and $l_E(x)$ is called the direction with respect to E. We observe that $S_E = \{x \in \mathbb{R}^d : \tau_E(x) = 1\}$ and S_E is compact. Moreover, it is clear that $\tau_E(x) \to \infty$ as $\|x\| \to \infty$ and $\tau_E(x) \to 0$ as $\|x\| \to 0$. Hence, $\tau_E(\cdot)$ can be extended continuously by setting $\tau_E(0) = 0$. Thus, one obtains a continuous function $\tau_E : \mathbb{R}^d \to [0, \infty)$ that additionally satisfies $\tau_E(x) = \tau_E(-x)$. We will recall some more properties of this function in the next sections. Before doing so let us give the perhaps easiest example of such a function.

Example 2.2. Let I_d be the identity operator on $\mathbb{R}^d$ and $\|\cdot\|$ an arbitrary norm. Note that any $x \in \mathbb{R}^d \setminus \{0\}$ can be written as

$$x = \|x\| \cdot \frac{x}{\|x\|} = \|x\|^{I_d} \cdot \frac{x}{\|x\|} = \tau_{I_d}(x)^{I_d} l_{I_d}(x).$$

Since this representation is unique by Lemma 2.1, we obtain that $\tau_{I_d}(x) = \|x\|$ and $l_{I_d}(x) = \frac{x}{\|x\|}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. Although being quite simple, this example will be of high importance in this thesis as we shall see in the next chapters.

2.3. E-homogeneous functions

E-homogeneous functions play important roles in establishing the existence of operator-self-similar random fields and have been introduced in [9, Secion 2]. Let us briefly summarize the content of the aforementioned section.

Definition 2.3. A function $\phi : \mathbb{R}^d \to \mathbb{C}$ is called E-homogeneous if $\phi(c^E x) = c\phi(x)$ for all $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$.

Important properties of a continuous E-homogeneous function ϕ with positive values on $\mathbb{R}^d \setminus \{0\}$ are that $\phi(0) = 0$ and that ϕ attains a strictly positive maximum and minimum on the compact set S_E, that is

$$M_\phi := \max_{\theta \in S_E} \phi(\theta) > 0 \quad \text{and} \quad m_\phi := \min_{\theta \in S_E} \phi(\theta) > 0. \tag{2.1}$$

Definition 2.4. Let $\beta > 0$ and $\phi : \mathbb{R}^d \to [0, \infty)$ a continuous function. Then ϕ is called (β, E)-admissible if $\phi(x) > 0$ for all $x \neq 0$ and for any $0 < A < B$ there exists a constant $c > 0$ such that for $A \leq \|y\| \leq B$

$$|\phi(x + y) - \phi(y)| \leq c\tau_E(x)^\beta$$

holds for any $x \in \mathbb{R}^d$ with $\tau_E(x) \leq 1$.

According to [9, Remark 2.9] if ϕ is (β, E)-admissible then we necessarily have $\beta \leq a_1$.

Various examples of E-homogeneous and (β, E)-admissible functions have been given in [9, 10]. Moreover, a very important example of an E-homogeneous function is given by the radial part τ_E with respect to E. This is straightforward to see, since for any $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$

$$c^E x = c^E \tau_E(x)^E l_E(x) = \left(c\tau_E(x)\right)^E l_E(x).$$

But on the other hand

$$c^E x = \tau_E(c^E x)^E l_E(c^E x).$$

Since this representation is unique by definition, it follows that $c\tau_E(x) = \tau_E(c^E x)$ and $l_E(x) = l_E(c^E x)$.

2.4. Properties of polar coordinates

In this section we mainly recall results on how to bound the growth rate of $\tau_E(x)$ in terms of the real parts of the eigenvalues of E. Let us start with a Lemma that has been established in [9, Lemma 2.1].

Lemma 2.5. *For any small $\varepsilon > 0$ and $H > 0$ there exist constants $C_{3,1}, \ldots, C_{3,4} > 0$ such that*

$$C_{3,1}\|x\|^{\frac{H}{a_1}+\varepsilon} \leq \tau_E(x)^H \leq C_{3,2}\|x\|^{\frac{H}{a_p}-\varepsilon}$$

for all x with $\tau_E(x) \leq 1$ and

$$C_{3,3}\|x\|^{\frac{H}{a_p}-\varepsilon} \leq \tau_E(x)^H \leq C_{3,4}\|x\|^{\frac{H}{a_1}+\varepsilon}$$

for all x with $\tau_E(x) \geq 1$.

Corollary 2.6. *Let $H > 0$ and $\beta \in \mathbb{R}$. Then*

$$\lim_{\|x\|\downarrow 0} \frac{\tau_E(x)^H}{|\log[\tau_E(x)]^\beta|} = 0. \tag{2.2}$$

In particular, for any $\varepsilon > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$ with $\|x\| \leq \eta$ for some $\eta > 0$ one can find a constant $C_{3,5} > 0$ such that

$$\tau_E(x)^H |\log[\tau_E(x)]^\beta| \leq C_{3,5}\tau_E(x)^{H-\varepsilon} \tag{2.3}$$

or, analogously

$$\tau_E(x)^H |\log[1 + \tau_E(x)^{-1}]^\beta| \leq C_{3,5}\tau_E(x)^{H-\varepsilon}. \tag{2.4}$$

Proof. By Lemma 2.5 we have for $\varepsilon < \frac{H}{a_p}$

$$C_{3,1}\|x\|^{\frac{H}{a_1}+\varepsilon} \leq \tau_E(x)^H \leq C_{3,2}\|x\|^{\frac{H}{a_p}-\varepsilon}$$

for all x with $\tau_E(x) \leq 1$. Using this for an unspecified constant $c > 0$ we obtain

$$\lim_{\|x\|\downarrow 0} \frac{\tau_E(x)^H}{|\log[\tau_E(x)]^\beta|} \leq c \lim_{\|x\|\downarrow 0} \frac{\|x\|^{\frac{H}{a_p}-\varepsilon}}{|\log[\|x\|^{\frac{H}{a_1}+\varepsilon}]^\beta|} = 0,$$

where the last equality is a well-known fact. Let us now prove (2.3), which is equivalent to

$$\tau_E(x)^\varepsilon |\log[\tau_E(x)]^\beta| \leq C_{3,5}.$$

But this is obvious, since τ_E is continuous and by (2.2)

$$\lim_{\|x\|\downarrow 0} \frac{\tau_E(x)^\varepsilon}{|\log[\tau_E(x)]^{-\beta}|} = 0.$$

The proof of (2.4) is carried out analogously. □

Biermé and Lacaux [10, Corollary 3.4] proved the following improvement of the bounds in Lemma 2.5.

Lemma 2.7. *Let $\mathbb{R}^d = W_1 \oplus \ldots \oplus W_p$ be the direct sum decomposition with respect to E. For any $\eta \in (0,1)$ and $H > 0$ there exist finite constants $C_{3,6}, C_{3,7} > 0$ such that for any $x \in W_i \setminus \{0\}$ with $\|x\| \leq \eta$*

$$C_{3,6}\|x\|^{\frac{H}{a_i}} |\log[\|x\|]|^{-\frac{l_i-1}{a_i}} \leq \tau_E(x)^H \leq C_{3,7}\|x\|^{\frac{H}{a_i}} |\log[\|x\|]|^{\frac{l_i-1}{a_i}},$$

where $l_i, 1 \leq i \leq p$, are positive integers depending on W_i.

In [32, Example 6.2] it is shown that the bounds in Lemma 2.7 cannot be improved in general. However, in Example 4.6 of the second part of this thesis we will see that there is an example of a matrix E in which these bounds can be improved.

The upper bound in the following Lemma is the statement of [9, Lemma 2.2] and implies that the function $\rho : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ given by $\rho(x,y) = \tau_E(x-y)$ is a quasi-metric on $\mathbb{R}^d$ (see [41] for a definition of a quasi-metric). By using the same method with minor adjustments we can also prove the lower bound.

Lemma 2.8. *There exists a constant $C_{3,8} \geq 1$ such that for all $x, y \in \mathbb{R}^d$ we have*

$$C_{3,8}^{-1}(\tau_E(x) + \tau_E(y)) \leq \tau_E(x+y) \leq C_{3,8}(\tau_E(x) + \tau_E(y)).$$

Proof. Let us prove the lower bound. Set $G = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \tau_E(x) + \tau_E(y) = 1\}$. Note that G is bounded and, by the continuity of the function τ_E, it is closed. Hence, G is compact. Moreover, G is bounded away from zero. Thus, the continuous function $(x,y) \mapsto \tau_E(x+y)$ assumes a positive and finite minimum on G. Let us define

$$K := \min_{(x,y)\in G} \tau_E(x+y).$$

Note that $S_E \times \{0\} \subset G$. Thus, by the definition of K we have

$$K \leq \tau_E(\theta + 0) = \tau_E(\theta) = 1.$$

For $x, y \in \mathbb{R}^d \setminus \{0\}$ define $s = \big(\tau_E(x) + \tau_E(y)\big)^{-1}$. Then it follows

$$\tau_E(x+y) = s^{-1} s \tau_E(x+y) = s^{-1} \tau_E(s^E(x+y)) = s^{-1} \tau_E(s^E x + s^E y),$$

where we used that τ_E is an E-homogeneous function as shown in Chapter 2.3. Note that $(s^E x, s^E y) \in G$, since

$$\tau_E(s^E x) + \tau_E(s^E y) = s\big(\tau_E(x) + \tau_E(y)\big) = s s^{-1} = 1.$$

Therefore, we obtain

$$\tau_E(x+y) = s^{-1} \tau_E(s^E x + s^E y) \geq s^{-1} K = \big(\tau_E(x) + \tau_E(y)\big) K.$$

Hence, this proves the lower bound. □

Corollary 2.9. *For any $H > 0$ there exists a constant $C_{3,9} \geq 1$ such that for all $x, y \in \mathbb{R}^d$ we have*

$$C_{3,9}^{-1}\big(\tau_E(x)^H + \tau_E(y)^H\big) \leq \tau_E(x+y)^H \leq C_{3,9}\big(\tau_E(x)^H + \tau_E(y)^H\big).$$

Proof. Throughout this proof, let c be an unspecified positive constant which might change in each occurence. Let us first prove the left inequality. From Lemma 2.8 we get

$$\tau_E(x+y)^H \geq c\big(\tau_E(x) + \tau_E(y)\big)^H \geq c\tau_E(x)^H$$

and, analogously $\tau_E(x+y)^H \geq c\tau_E(y)^H$ so that, overall

$$\tau_E(x+y)^H \geq K_1^{-1}\big(\tau_E(x)^H + \tau_E(y)^H\big)$$

for some suitable $K_1 \geq 1$. It remains to prove the right inequality. Using the left inequality we obtain

$$\tau_E(x)^H = \tau_E(x+y-y)^H \geq c\big(\tau_E(x+y)^H + \tau_E(-y)^H\big) \geq c\tau_E(x+y)^H$$

and, analogously $\tau_E(y)^H \geq c\tau_E(x+y)^H$. Combining this, we conclude

$$2c\tau_E(x+y)^H \leq \tau_E(x)^H + \tau_E(y)^H$$

or, equivalently

$$\tau_E(x+y)^H \leq K_2\big(\tau_E(x)^H + \tau_E(y)^H\big)$$

for some suitable $K_2 \geq 1$. Now the statement follows by choosing $C_{3,9} = \max\{K_1, K_2\}$. □

Remark 2.10. As before let $\mathbb{R}^d = W_1 \oplus \ldots \oplus W_p$ denote the direct sum decomposition with respect to E. For any $1 \le k \le p$ define $O_k = W_1 \oplus \ldots \oplus W_k$. Then Corollary 2.9 along with Lemma 2.7 implies that one can find two constants $C_{3,10}, C_{3,11} > 0$ such that for any $\varepsilon > 0$, $H > 0, x_i \in W_i, k+1 \le i \le p$ and $y \in O_k$ with $x = y + \sum_{i=k+1}^{p} x_i$ and $x \le \eta$ for some $\eta \in (0,1)$ we have

$$C_{3,10}\Big(\|y\|^{\frac{H+\varepsilon}{a_1}} + \sum_{i=k+1}^{p} \|x_i\|^{\frac{H+\varepsilon}{a_i}}\Big) \le \tau_E(x)^H \le C_{3,11}\Big(\|y\|^{\frac{H-\varepsilon}{a_k}} + \sum_{i=k+1}^{p} \|x_i\|^{\frac{H-\varepsilon}{a_i}}\Big).$$

This estimate will play an important role in our considerations and we will quite often make use of it.

3. Stable distributions and integrals

This chapter serves as an introduction to α-stable distributions and stable integrals. The first section of this chapter is strongly influenced by the content of [42, Chapter 2], where a more general treatment and related results concerning stable distributions can be found. The second section can basically be seen as a short summary of [42, Chapter 6], where the theory of complex-valued stable stochastic integrals has been developed. This theory has been extended to vector-valued stochastic integrals in [33], which will be summarized in the third and last section of this chapter. From now on, throughout this thesis let us fix a probability space $(\Omega, \mathcal{A}, P)$.

3.1. α-stable random variables

We now recall the definition of α-stable random variables. Here we only focus on the case of symmetric distributions.

Definition 3.1. A random vector $X = (X_1, \dots, X_m)$ with values in $\mathbb{R}^m$ is called multivariate symmetric stable if for any $A, B > 0$ there exists $C > 0$ such that

$$AX^{(1)} + BX^{(2)} \overset{\mathrm{d}}{=} CX, \tag{3.1}$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X and $\overset{\mathrm{d}}{=}$ means equality in distribution.

Stable random variables are usually called α-stable. The term α-stable is justified by the following Theorem [42, Theorem 2.1.2].

Theorem 3.2. *Let $X = (X_1, \dots, X_m)$ be a multivariate symmetric stable random vector. Then there is a unique constant $\alpha \in (0, 2]$ such that in* (3.1)

$$C = (A^\alpha + B^\alpha)^{\frac{1}{\alpha}}.$$

Moreover, any linear combination $\sum_{k=1}^m b_k X_k$ of the components is univariate symmetric stable. X is also referred to as symmetric α-stable (SαS) with index α of stability.

Remark 3.3. If X is a Gaussian random vector with mean 0 and covariance matrix $\Sigma \in \mathbb{R}^{m \times m}$, i.e. $X \sim \mathcal{N}(0, \Sigma)$ then X is SαS with index $\alpha = 2$, since for any two independent copies $X^{(1)}$ and $X^{(2)}$ of X and any constants $A, B > 0$

$$AX^{(1)} + BX^{(2)} \sim \mathcal{N}(0, (A^2 + B^2)\Sigma) \sim (A^2 + B^2)^{\frac{1}{2}} X$$

so that (3.1) holds with $C = (A^2 + B^2)^{\frac{1}{2}}$. Moreover, any symmetric 2-stable random vector has a Gaussian distribution.

The property (3.1) of symmetric α-stable random vectors can be extended to the following result (see [42, Corollary 2.1.3]).

Corollary 3.4. *A random vector X is symmetric α-stable if and only if for any $n \geq 2$*

$$X^{(1)} + X^{(2)} + \ldots + X^{(n)} \stackrel{\mathrm{d}}{=} n^{\frac{1}{\alpha}} X,$$

where $X^{(1)}, \ldots, X^{(n)}$ are independent copies of X.

In this thesis characteristic functions of stable random vectors play an important role in calculating certain expected energy integrals as given in Chapter 1.2 and fortunately they are usually known in closed form. Let $X = (X_1, \ldots, X_m)$ be a symmetric α-stable random variable with values in $\mathbb{R}^m$ and for any $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$ let

$$\phi_\alpha(\theta) = \phi_\alpha(\theta_1, \ldots, \theta_m) = \mathbb{E}[\exp(i\langle\theta, X\rangle)] = \mathbb{E}[\exp(i \sum_{k=1}^{m} \theta_k X_k)]$$

denote its characteristic function. The following Theorem [42, Theorem 2.4.3] gives useful information on $\phi_\alpha(\theta)$.

Theorem 3.5. *X is a symmetric α-stable vector in $\mathbb{R}^m$ with $0 < \alpha < 2$ if and only if there exists a unique symmetric finite measure Γ on the unit sphere $S_m = \{x \in \mathbb{R}^m : \|x\| = 1\}$ such that*

$$\phi_\alpha(\theta) = \exp\Big(- \int_{S_m} |\langle\theta, x\rangle|^\alpha \Gamma(dx)\Big).$$

Γ is called the spectral measure of the symmetric α-stable random vector X.

Note that Theorem 3.5 only gives information on the characteristic function for $0 < \alpha < 2$. However, in the Gaussian case $\alpha = 2$ it is well-known that the characteristic function of a random vector with mean 0 and covariance matrix $\Sigma \in \mathbb{R}^{m \times m}$ is given by

$$\phi_2(\theta) = \exp(-\frac{1}{2}\theta^T \Sigma \theta).$$

Corollary 3.6. *If $m = 1$ one has $S_1 = \{-1, 1\}$ and the symmetric spectral measure of the symmetric α-stable random variable X satisfies $\Gamma(\{1\}) = \Gamma(\{-1\})$. Hence, the characteristic function of X is given by*

$$\phi_\alpha(\theta) = \exp(-\sigma^\alpha |\theta|^\alpha)$$

with

$$\sigma = (\int_{S_1} |x|^\alpha \Gamma(dx))^{\frac{1}{\alpha}} = (2\Gamma(\{1\}))^{\frac{1}{\alpha}}.$$

σ is called the scale parameter of X. Furthermore, though not being known in closed form in general except for a few special cases, the probability densities of α-stable random variables exist and are continuous, see [42, p.9].

3.2. Complex-valued SαS stochastic integrals

The aim of this section is to define stochastic integrals which have a stable distribution. Again we restrict our considerations to special cases and refer the reader to [42, Chapter 6] for a more general outline. In the following we will identify any complex-valued random variable $X = X_1 + iX_2$ with the variable (X_1, X_2) taking values in $\mathbb{R}^2$. Let us first state the following definition.

Definition 3.7. A complex-valued SαS random variable $X = X_1 + iX_2$ is called isotropic or rotationally invariant if

$$e^{i\varphi} X \overset{\mathrm{d}}{=} X$$

for any $\varphi \in [0, 2\pi)$.

The following Theorem [42, Theorem 2.6.3] characterizes the set of all isotropic SαS random variables for $0 < \alpha < 2$.

Theorem 3.8. *For $0 < \alpha < 2$ let $X = X_1 + iX_2$ be a complex-valued SαS random variable and let Γ be its spectral measure according to Theorem 3.5. Then X is isotropic if and only if Γ is uniform, that is for any $B \in \mathcal{B}(S_2)$*

$$\Gamma(B) = \Gamma(R_\varphi(B)),$$

where

$$R_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

is the matrix corresponding to the rotation by the angle $\varphi \in [0, 2\pi)$.

In the following let λ_d denote the Lebesgue measure on $\mathbb{R}^d$ and define

$$\mathcal{E}_0 = \{A \in \mathcal{B}(\mathbb{R}^d) : \lambda_d(A) < \infty\}. \tag{3.2}$$

Furthermore, let $L^0(\Omega)$ be the set of all real random variables on the underlying probability space. Because of its appearance in (3.2) λ_d is also referred to as Lebesgue control measure.

Definition 3.9. A mapping $M : \mathcal{E}_0 \to L^0(\Omega)$ is called an independently scattered random measure with Lebesgue control measure if it satisfies the following two conditions.

(i) If $A_1, \ldots, A_k \in \mathcal{E}_0$ are disjoint then the random variables $M(A_1), \ldots, M(A_k)$ are independent.

(ii) M is σ-additive, that is for any sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{E}_0$ such that $\bigcup_{n=1}^\infty A_n \in \mathcal{E}_0$ we have

$$M\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n=1}^\infty M(A_n) \quad a.s.$$

According to [42, Section 6.1] there exists a random measure, denoted M_α, such that for any $A \in \mathcal{E}_0$ the random variable $M_\alpha(A)$ is a complex-valued isotropic SαS random variable with the property that for $0 < \alpha < 2$ the spectral measure Γ_A of $M_\alpha(A)$ according to Theorem 3.5 is given by

$$\Gamma_A(B) = \lambda_d(A)\gamma(B)$$

for all $B \in \mathcal{B}(S_2)$, where γ is a finite and uniform measure on the unit circle S_2. Note that Γ_A is a uniform measure on S_2, which by Theorem 3.8 is equivalent to $M_\alpha(A)$ being isotropic. By [42, Theorem 6.3.1] for all $\theta \in \mathbb{R}^2$ the characteristic function of $M_\alpha(A)$ is given by

$$\mathbb{E}[\exp\left(i\langle\theta, M_\alpha(A)\rangle\right)] = \exp\Big(-|\theta|^\alpha c_0 \lambda_d(A)\gamma(S_2)\Big)$$

with

$$c_0 = \frac{1}{2\pi}\int_0^\pi |\cos\varphi|^\alpha d\varphi.$$

In the literature, for simplicity it is usually assumed that γ is a probability measure on S_2, which implies that $\gamma(S_2) = 1$. However, in this thesis without loss of generality we will assume that $\gamma(S_2) = \frac{1}{c_0}$ so that the characteristic function of $M_\alpha(A)$ equals

$$\mathbb{E}[\exp\left(i\langle\theta, M_\alpha(A)\rangle\right)] = \exp\Big(-|\theta|^\alpha \lambda_d(A)\Big)$$

for any $\theta \in \mathbb{R}^2$. This characteristic function remains to be true in the Gaussian case $\alpha = 2$. From now on we will call M_α a complex isotropic SαS random measure.

Having introduced a complex isotropic SαS random measure we are ready to define the stochastic integral

$$\int_{\mathbb{R}^d} f(x) M_\alpha(dx)$$

for any measurable function

$$f \in L^\alpha(\lambda) = \{f : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|^\alpha dx < \infty\},$$

where $\lambda = \lambda_d$ and $dx = \lambda_d(dx)$. Let us first assume that f is a simple function of the form $f(x) = \sum_{j=1}^n c_j \mathbb{1}_{A_j}(x)$, where $c_1, \dots, c_n \in \mathbb{C}$ and $A_1, \dots, A_n \in \mathcal{E}_0$ are disjoint. In this case we define

$$I(f) = \int_{\mathbb{R}^d} f(x) M_\alpha(dx) = \sum_{j=1}^n c_j M_\alpha(A_j).$$

Now assume that $f \in L^\alpha(\lambda)$ is arbitrary. Then according to [42, p. 277] there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ in $L^\alpha(\lambda)$ with the properties that

$$f_n(x) \to f(x)$$

as $n \to \infty$ for almost every $x \in \mathbb{R}^d$ and that the sequence $(f_n)_{n\in\mathbb{N}}$ is dominated, that is there exists a function $g \in L^\alpha(\lambda)$ such that for every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$

$$|f_n(x)| \leq g(x).$$

In addition it is shown in [42, Chapter 6] that the sequence of integrals $\big(I(f_n)\big)_{n\in\mathbb{N}}$ defined above converges in probability and one defines

$$I(f) = \operatorname*{plim}_{n\to\infty} I(f_n),$$

where $\operatorname{plim}_{n\to\infty}$ denotes the limit in probability. Furthermore, this limit does not depend on a particular choice of the approximating sequence $(f_n)_{n\in\mathbb{N}}$. The following Proposition summarizes some important properties of the integral $I(f)$.

Proposition 3.10. *(i) For any $f \in L^\alpha(\lambda)$ the integral $I(f)$ is a complex-valued isotropic $S\alpha S$ random variable.*
(ii) For any $f, g \in L^\alpha(\lambda)$ and any $a, b \in \mathbb{C}$

$$I(af + bg) = aI(f) + bI(g) \quad a.s.$$

Let $\{f_t : t \in \mathbb{R}^d\} \subset L^\alpha(\lambda)$ be a family of functions. We will usually be interested in the random field $\{I(f_t) : t \in \mathbb{R}^d\}$. The following result might be quite surprising at first glance and states that the real and imaginary parts of $\{I(f_t) : t \in \mathbb{R}^d\}$ have the same finite-dimensional distributions [42, Corollary 6.3.5].

Theorem 3.11. *Let $\{I(f_t) : t \in \mathbb{R}^d\}$ be given as above. For any $\theta_1, \dots, \theta_n \in \mathbb{R}$ and $t_1, \dots, t_n \in \mathbb{R}^d$*

$$\mathbb{E}\Big[\exp\Big(i\sum_{j=1}^n \theta_j \operatorname{Re} I(f_{t_j})\Big)\Big] = \exp\Big(-\int_{\mathbb{R}^d} |\sum_{j=1}^n \theta_j f_{t_j}(x)|^\alpha dx\Big)$$

and, in particular

$$\{\operatorname{Re} I(f_t) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{\operatorname{Im} I(f_t) : t \in \mathbb{R}^d\}.$$

Remark 3.12. Let $M = M^1 + iM^2$ be a Gaussian random measure satisfying

$$\mathbb{E}[M(A)\overline{M(B)}] = \lambda_d(A \cap B) \quad \text{and} \quad M(-A) = \overline{M(A)}$$

for all $A, B \in \mathcal{E}_0$, where $\overline{M}$ denotes the complex conjugate of M. Furthermore, let M_2 be the complex-valued isotropic SαS random measure with $\alpha = 2$ defined above. Assume that $\{f_t : t \in \mathbb{R}^d\} \subset L^2(\lambda)$ is a subset of functions satisfying

$$f_t(x) = \overline{f_t(-x)}$$

for all $t, x \in \mathbb{R}^d$. Then one obtains

$$\{\operatorname{Re}\int_{\mathbb{R}^d} f_t(x) M_2(dx) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{\int_{\mathbb{R}^d} f_t(x) M(dx) : t \in \mathbb{R}^d\}.$$

A proof of this statement and the definition of the real-valued integral

$$\int_{\mathbb{R}^d} f_t(x) M(dx)$$

can be found in [42, Section 7.2.2].

3.3. $\mathbb{R}^m$-valued SαS stochastic integrals

This section is concerned with integral representations of random vectors which have a stable distribution. These integrals have been constructed in [33] by providing both stochastic integrals of real and complex matrix-valued functions. From now on, throughout this thesis let $\|Q\| = \max_{\|x\|=1} \|Qx\|$ be the operator norm for any matrix $Q \in \mathbb{R}^{m\times m}$.

Let us first recall the definition of stochastic integrals of real matrix-valued functions. Since the details of this construction are omitted in [33], we will be more precise concerning the proofs of the statements we make. Let $\mathcal{E}_0$ be defined as in the previous section. According to [33, p. 8] there exists a random measure W_α such that $W_\alpha(A) = \big(W_\alpha^1(A), \dots, W_\alpha^m(A)\big)$ is a SαS random variable with values in $\mathbb{R}^m$ for any $A \in \mathcal{E}_0$ with the property that for $0 < \alpha < 2$ the spectral measure Γ_A of $W_\alpha(A)$ is given by

$$\Gamma_A(B) = \lambda_d(A)\gamma(B)$$

for any $B \in \mathcal{B}(S_m)$, where γ is a finite and uniform measure on S_m satisfying

$$\int_{S_m} |\langle \theta, x\rangle|^\alpha \gamma(dx) = \|\theta\|^\alpha$$

for any $\theta \in \mathbb{R}^m$. Therefore, by Theorem 3.5 the characteristic function of $W_\alpha(A)$ is given by

$$\mathbb{E}\Big[\exp\Big(i\langle \theta, W_\alpha(A)\rangle\Big)\Big] = \exp\Big(-\lambda_d(A)\|\theta\|^\alpha\Big).$$

In particular, for disjoint sets $A_1, \dots, A_n \in \mathcal{E}_0$ and $\theta_1, \dots, \theta_n \in \mathbb{R}^m$ one obtains

$$\mathbb{E}\Big[\exp\Big(i\sum_{j=1}^n \langle \theta_j, W_\alpha(A_j)\rangle\Big)\Big] = \exp\Big(-\sum_{j=1}^n \lambda_d(A_j)\|\theta_j\|^\alpha\Big).$$

This remains to be true in the Gaussian case $\alpha = 2$. Now assume that $\{Q(u) : u \in \mathbb{R}^d\}$ is a family of real $m \times m$ matrices and that the function $Q : u \mapsto Q(u)$ is measurable. In addition assume that

$$\int_{\mathbb{R}^d} \|Q(u)\|^\alpha du < \infty.$$

We want to define the integral

$$I(Q) = \int_{\mathbb{R}^d} Q(u) W_\alpha(du).$$

Suppose first that Q is a simple function of the form $Q(u) = \sum_{j=1}^n R_j \mathbb{1}_{A_j}(u)$, where $R_1, \dots R_n \in \mathbb{R}^{m\times m}$ and $A_1, \dots, A_n \in \mathcal{E}_0$ are disjoint sets. In this case we define

$$I(Q) = \sum_{j=1}^n R_j W_\alpha(A_j).$$

Thus, one obtains a SαS random vector with characteristic function given by

$$\mathbb{E}\Big[\exp\Big(i\langle\theta, I(Q)\rangle\Big)\Big] = \mathbb{E}\Big[\exp\Big(i\langle\theta, \sum_{j=1}^{n} R_j W_\alpha(A_j)\rangle\Big)\Big] = \mathbb{E}\Big[\exp\Big(i\sum_{j=1}^{n}\langle R_j^T\theta, W_\alpha(A_j)\rangle\Big)\Big]$$
$$= \exp\Big(-\sum_{j=1}^{n}\lambda_d(A_j)\|R_j^T\theta\|^\alpha\Big) = \exp\Big(-\int_{\mathbb{R}^d}\|Q^T(u)\theta\|^\alpha du\Big)$$

for all $\theta \in \mathbb{R}^m$. Now assume that Q is an arbitrary measurable matrix-valued function satisfying

$$\int_{\mathbb{R}^d}\|Q(u)\|^\alpha du < \infty.$$

Then according to [34, p. 148] there exists a sequence of simple matrix-valued functions $(Q_n)_{n\in\mathbb{N}}$ as above satisfying

$$\int_{\mathbb{R}^d}\|Q_n^T(u) - Q^T(u)\|^\alpha du \to 0 \tag{3.3}$$

as $n \to \infty$. From the definition of the integral $I(\cdot)$ for simple matrix-valued functions we further get

$$\mathbb{E}\Big[\exp\Big(i\langle\theta, I(Q_n) - I(Q_m)\rangle\Big)\Big] = \exp\Big(-\int_{\mathbb{R}^d}\|Q_n^T(u)\theta - Q_m^T(u)\theta\|^\alpha du\Big) \to 1$$

for all $\theta \in \mathbb{R}^m$ as $m, n \to \infty$ by (3.3). Thus $I(Q_n) - I(Q_m) \to 0$ in probability as $m, n \to \infty$ and the sequence $\big(I(Q_n)\big)_{n\in\mathbb{N}}$ converges in probability. We define

$$I(Q) = \operatorname*{plim}_{n\to\infty} I(Q_n).$$

This limit does not depend on the choice of the approximating sequence $(Q_n)_{n\in\mathbb{N}}$. Indeed, assume that both $(Q_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ are sequences of simple functions satisfying (3.3). Then the sequence $(T_n)_{n\in\mathbb{N}}$ defined by

$$T_n = \begin{cases} Q_n & \text{if } n \text{ even,} \\ S_n & \text{if } n \text{ odd} \end{cases}$$

satisfies (3.3) as well so that

$$\operatorname*{plim}_{n\to\infty} I(T_n)$$

exists, which in particular yields

$$\operatorname*{plim}_{n\to\infty} I(Q_n) = \operatorname*{plim}_{n\to\infty} I(S_n).$$

Thus, one obtains a SαS random vector with characteristic function given by

$$\mathbb{E}\Big[\exp\Big(i\langle\theta, I(Q)\rangle\Big)\Big] = \lim_{n\to\infty}\mathbb{E}\Big[\exp\Big(i\langle\theta, I(Q_n)\rangle\Big)\Big] = \lim_{n\to\infty}\exp\Big(-\int_{\mathbb{R}^d}\|Q_n^T(u)\theta\|^\alpha du\Big)$$
$$= \exp\Big(-\int_{\mathbb{R}^d}\|Q^T(u)\theta\|^\alpha du\Big).$$

Let us summarize the properties of the integral in the following Theorem.

Theorem 3.13. *Let $\{Q(u) : u \in \mathbb{R}^d\}$ be a family of real $m \times m$ matrices. If the function Q given by $Q(u)$ for all $u \in \mathbb{R}^d$ is measurable and satisfies*

$$\int_{\mathbb{R}^d} \|Q(u)\|^\alpha du < \infty$$

the stochastic integral

$$I(Q) = \int_{\mathbb{R}^d} Q(u) W_\alpha(du)$$

exists and is a $S\alpha S$ random vector with values in $\mathbb{R}^m$. Moreover, its characteristic function is given by

$$\mathbb{E}\Big[\exp\Big(i\langle\theta, I(Q)\rangle\Big)\Big] = \exp\Big(-\int_{\mathbb{R}^d} \|Q^T(u)\theta\|^\alpha du\Big)$$

for all $\theta \in \mathbb{R}^m$.

Let us now turn to the definition of stochastic integrals of complex matrix-valued functions by briefly summarizing the content in [33, Section 4]. Again one first has to remark that there exists a random measure $\tilde{M}_\alpha$ such that $\tilde{M}_\alpha(A)$ is a $\mathbb{C}^m$-valued SαS random variable for any $A \in \mathcal{E}_0$ with characteristic function of its real part $\operatorname{Re}\tilde{M}_\alpha(A)$ given by

$$\mathbb{E}\Big[\exp\Big(i\langle\theta, \operatorname{Re}\tilde{M}_\alpha(A)\rangle\Big)\Big] = \exp\Big(-\lambda_d(A)\|\theta\|^\alpha\Big)$$

for all $\theta \in \mathbb{R}^m$. Let $\{\tilde{Q}_1(u) : u \in \mathbb{R}^d\}$ and $\{\tilde{Q}_2(u) : u \in \mathbb{R}^d\}$ be two families of real $m \times m$ matrices and define $\tilde{Q}(u) = \tilde{Q}_1(u) + i\tilde{Q}_2(u)$ for all $u \in \mathbb{R}^d$. Assume that the function $\tilde{Q} : u \mapsto \tilde{Q}(u)$ is measurable and satisfies

$$\int_{\mathbb{R}^d} \Big(\|\tilde{Q}_1(u)\|^\alpha + \|\tilde{Q}_2(u)\|^\alpha\Big) du < \infty. \tag{3.4}$$

For notational convenience let us define the set of all complex matrix-valued measurable functions satisfying (3.4) by $\tilde{L}^\alpha(\lambda)$. We want to define the real vector-valued integral

$$\tilde{I}(\tilde{Q}) = \operatorname{Re}\int_{\mathbb{R}^d} \tilde{Q}(u)\tilde{M}_\alpha(du).$$

As usual we first assume that $\tilde{Q}$ is a simple function of the form

$$\tilde{Q}(u) = \tilde{Q}_1(u) + i\tilde{Q}_2(u) = \sum_{j=1}^n R_j \mathbb{1}_{A_j}(u) + i\sum_{j=1}^n T_j \mathbb{1}_{A_j}(u),$$

where $R_1, \ldots, R_n, T_1, \ldots, T_n \in \mathbb{R}^{m\times m}$ and $A_1, \ldots, A_n \in \mathcal{E}_0$ are disjoint. In this case we define

$$\tilde{I}(\tilde{Q}) = \sum_{j=1}^n \Big(R_j \tilde{M}_R^\alpha(A_j) - T_j \tilde{M}_I^\alpha(A_j)\Big),$$

where $\tilde{M}_R^\alpha = \operatorname{Re}\tilde{M}_\alpha$ and $\tilde{M}_I^\alpha = \operatorname{Im}\tilde{M}_\alpha$ denote the real and imaginary part of $\tilde{M}_\alpha$. If $\tilde{Q}(u) = \tilde{Q}_1(u) + i\tilde{Q}_2(u) \in \tilde{L}^\alpha(\lambda)$ is arbitrary one chooses a sequence of simple functions

$(\tilde{Q}^n = \tilde{Q}_1^n + i\tilde{Q}_2^n)_{n\in\mathbb{N}}$ satisfying

$$\int_{\mathbb{R}^d} \|\tilde{Q}_1(u)^T - \tilde{Q}_1^n(u)^T\|_M^\alpha du \to 0$$

and

$$\int_{\mathbb{R}^d} \|\tilde{Q}_2(u)^T - \tilde{Q}_2^n(u)^T\|_M^\alpha du \to 0$$

as $n \to \infty$. The sequence $\big(\tilde{I}(\tilde{Q}^n)\big)_{n\in\mathbb{N}}$ converges in probability and one defines

$$\tilde{I}(\tilde{Q}) = \operatorname*{plim}_{n\to\infty} \tilde{I}(\tilde{Q}^n),$$

where this limit does not depend on the choice of $(\tilde{Q}^n)_{n\in\mathbb{N}}$ as above.

Now let $\{\tilde{Q}_t : t \in \mathbb{R}^d\} \subset \tilde{L}^\alpha(\lambda)$ be a family of complex $m \times m$ matrices. To close this section let us summarize some important properties of the random field $\{\tilde{I}(\tilde{Q}_t) : t \in \mathbb{R}^d\}$.

Theorem 3.14. *Let $\{\tilde{I}(\tilde{Q}_t) : t \in \mathbb{R}^d\}$ be the $\mathbb{R}^m$-valued $S\alpha S$ random field defined above. Then for any $\theta_1, \ldots, \theta_n \in \mathbb{R}^m$ and $t_1, \ldots, t_n \in \mathbb{R}^d$ we have*

$$\mathbb{E}\Big[\exp\Big(i\langle\sum_{j=1}^n \theta_j, \tilde{I}(\tilde{Q}_{t_j})\rangle\Big)\Big] = \exp\Bigg(-\int_{\mathbb{R}^d}\Big(\|\sum_{j=1}^n \tilde{Q}_{t_j}^1(u)^T\theta_j\|^2 + \|\sum_{j=1}^n \tilde{Q}_{t_j}^2(u)^T\theta_j\|^2\Big)^{\frac{\alpha}{2}} du\Bigg),$$

where $\tilde{Q}_{t_j}^1 + i\tilde{Q}_{t_j}^2 = \tilde{Q}_{t_j}$.

Part II.

Main Results

4. Operator scaling stable random sheets

Having set the foundations for this thesis we are now able to present our main results. We begin this chapter with the formal definition of operator scaling stable random sheets and recall some results concerning their existence and properties established in [20]. Based on this we investigate their sample path properties in the subsequent sections.

4.1. Definition and existence

Throughout this chapter, let $d = \sum_{j=1}^{n} d_j$ for some $n \in \mathbb{N}$, $d_1, \ldots, d_n \in \mathbb{N}$ and let $E_j \in \mathbb{R}^{d_j \times d_j}$, $j = 1, \ldots, n$ be matrices with distinct real parts of their eigenvalues given by

$$0 < a_1^j < \ldots < a_{p_j}^j$$

for some $p_j \leq d_j$. Furthermore, let $q_j = \operatorname{trace}(E_j)$. We define the block diagonal matrix $E \in \mathbb{R}^{d \times d}$ as

$$E = \begin{pmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_n \end{pmatrix} = \sum_{j=1}^{n} \tilde{E}_j,$$

where the matrices $\tilde{E}_1, \ldots, \tilde{E}_n \in \mathbb{R}^{d \times d}$ are defined as

$$\tilde{E}_j = \begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & E_j & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{pmatrix}.$$

In analogy to [20, Definition 1.1.1] let us state the following definition.

Definition 4.1. A scalar valued random field $\{X(x) : x \in \mathbb{R}^d\}$ is called operator scaling random sheet if for some $H_1, \ldots, H_n > 0$ we have

$$\{X(c^{\tilde{E}_j} x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^{H_j} X(x) : x \in \mathbb{R}^d\} \tag{4.1}$$

for all $c > 0$ and $j = 1, \ldots, n$.

Let us remark that any operator scaling random sheet $\{X(x) : x \in \mathbb{R}^d\}$ according to Definition 4.1 is also an operator scaling random field of order E and $H = \sum_{j=1}^n H_j$ in the sense of (0.2), since by applying (4.1) iteratively one gets

$$\{X(c^E x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{X(c^{\tilde{E}_1} \dots c^{\tilde{E}_n} x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^{H_1} X(c^{\tilde{E}_2} \dots c^{\tilde{E}_n} x) : x \in \mathbb{R}^d\}$$
$$\dots \stackrel{\text{f.d.}}{=} \{c^{H_1} \dots c^{H_n} X(x) : x \in \mathbb{R}^d\} = \{c^H X(x) : x \in \mathbb{R}^d\}$$

for any $c > 0$. Further note that this definition is indeed a generalization of the definition of operator scaling random fields, since for $n = 1$, $d = d_1$ and $E = E_1 = \tilde{E}_1$ (4.1) coincides with the property (0.2).

Let us now turn to the existence of harmonizable operator scaling SαS random sheets constructed in [20]. Suppose that we are given functions $\psi_j : \mathbb{R}^{d_j} \to [0, \infty)$ that are continuous and E_j^T-homogeneous according to Definition 2.3. Moreover, assume that $\psi_j(x) \neq 0$ for $x \neq 0$. Let $0 < \alpha \leq 2$ and M_α be a complex-valued isotropic symmetric α-stable random measure on $\mathbb{R}^d$ with Lebesgue control measure as introduced in Chapter 3.2. The following Theorem is due to [20, Theorem 4.1.1] and provides the existence of harmonizable operator scaling SαS random sheets.

Theorem 4.2. *For any vector $x \in \mathbb{R}^d$ let $x = (x_1, \dots, x_n) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_n} = \mathbb{R}^d$. The random field*

$$X_\alpha(x) = \operatorname{Re} \int_{\mathbb{R}^d} \prod_{j=1}^n (e^{i\langle x_j, \xi_j \rangle} - 1) \psi_j(\xi_j)^{-H_j - \frac{q_j}{\alpha}} M_\alpha(d\xi), \quad x \in \mathbb{R}^d \tag{4.2}$$

exists and is stochastically continuous if and only if $H_j \in (0, a_1^j)$ for all $j = 1, \dots, n$.

Let us remark that by the definition of stable integrals given in Chapter 3.2 $X_\alpha(x)$ exists if and only if the kernel function in the integral in (4.2) satisfies

$$\int_{\mathbb{R}^d} \prod_{j=1}^n |e^{i\langle x_j, \xi_j \rangle} - 1|^\alpha \psi_j(\xi_j)^{-\alpha H_j - q_j} d\xi < \infty$$

or, equivalently

$$\prod_{j=1}^n \int_{\mathbb{R}^{d_j}} |e^{i\langle x_j, \xi_j \rangle} - 1|^\alpha \psi_j(\xi_j)^{-\alpha H_j - q_j} d\xi_j < \infty.$$

The finiteness of the above integrals has basically already been shown in the proof of [9, Theorem 4.1] and the stochastic continuity can be proven similarly as a consequence of this Theorem. Further note that from (4.2) it follows that $X_\alpha(x) = 0$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_n} = \mathbb{R}^d$ such that $x_j = 0$ for at least one $j \in \{1, \dots, n\}$.

The following result has been established in [20, Corollary 4.2.1] and shows that the random field given by (4.2) is an operator scaling random sheet. Its proof is carried out exactly as the proof of [9, Corollary 4.2 (a)] by using the characteristic function of stable integrals given in Chapter 3.2.

Corollary 4.3. *Under the conditions of Theorem 4.2 the random field $\{X_\alpha(x) : x \in \mathbb{R}^d\}$ given by* (4.2) *is operator scaling in the sense of* (4.1), *that is*

$$\{X_\alpha(c^{\tilde{E}_j}x) : x \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{c^{H_j}X_\alpha(x) : x \in \mathbb{R}^d\} \tag{4.3}$$

for all $c > 0$ and $j = 1, \dots, n$.

Thus, Theorem 4.2 and Corollary 4.3 provide a large class of operator scaling SαS random sheets. As we shall see below fractional Brownian sheets fall into the class of random fields given by (4.2). It is well-known that a fractional Brownian sheet does not have stationary increments. Thus, in general a random field given by (4.2) does not possess stationary increments. But it satisfies a slightly weaker property as the following statement shows. Let us mention that it has been proven in [20, Corollary 4.2.2] by essentially using the same arguments as in the proof of [9, Corollary 4.2 (b)].

Corollary 4.4. *Let $x = (x_1, \dots, x_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} = \mathbb{R}^d$. Under the conditions of Theorem 4.2 for any $h \in \mathbb{R}^{d_j}$, $j = 1, \dots, n$ the random field $\{X_\alpha(x) : x \in \mathbb{R}^d\}$ satisfies*

$$X_\alpha(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_n) - X_\alpha(x_1, \dots, x_{j-1}, h, x_{j+1}, \dots, x_n) \overset{\text{d}}{=} X_\alpha(x).$$

Having recalled the definition of harmonizable operator scaling SαS random sheets our main objective is to provide results related to their sample paths concerning path continuity and Hausdorff dimension. A main tool in studying sample path properties of operator scaling random sheets are the generalized polar coordinates with respect to the matrix E introduced in Chapter 2. Using (4.3), in the Gaussian case $\alpha = 2$ one can write the variance of $X_2(x), x \in \mathbb{R}^d$, as

$$\mathbb{E}[X_2^2(x)] = \mathbb{E}[X_2^2(\tau_E(x)^E l_E(x))] = \tau_E(x)^{2H}\mathbb{E}[X_2^2(l_E(x))]$$

with $H = \sum_{j=1}^n H_j$. Since in the Gaussian case many sample path properties such as path continuity can be deduced from the variance, this shows that information about the behavior of the polar coordinates $(\tau_E(x), l_E(x))$ contains information about the sample path regularity, i.e. polar coordinates are useful to characterize the sample path regularity. This property also holds in the stable case $\alpha \in (0, 2)$. Thus, before studying the sample paths of the random field $\{X_\alpha(x) : x \in \mathbb{R}^d\}$ given in Theorem 4.2 in the following section we will establish some useful properties concerning the radial part of the polar coordinates and other results which will serve as useful tools in our investigations.

4.2. Preliminaries

Throughout this chapter, let us denote by $(\tau_T(x), l_T(x))$ the generalized polar coordinates according to Chapter 2 for any matrix T with positive real parts of its eigenvalues.

Lemma 4.5. *Let E be as in Chapter 4.1. Then there exists a constant $C_{5,1} \geq 1$ such that*

$$C_{5,1}^{-1}\sum_{j=1}^n \tau_{E_j}(x_j)^H \leq \tau_E(x)^H \leq C_{5,1}\sum_{j=1}^n \tau_{E_j}(x_j)^H$$

for any $H > 0$ *and* $x = (x_1, \ldots, x_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} = \mathbb{R}^d$.

Proof. For simplicity in this proof let us assume that $n = 2$. The general case follows inductively. Furthermore, for any vector $x \in \mathbb{R}^d$ let us write $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. By Corollary 2.9 there exists a constant $c \geq 1$ such that

$$\frac{1}{c}\Big(\tau_E(x_1,0)^H + \tau_E(0,x_2)^H\Big) \leq \tau_E\big((x_1,0)+(0,x_2)\big)^H = \tau_E(x)^H \leq c\Big(\tau_E(x_1,0)^H + \tau_E(0,x_2)^H\Big)$$

for all $x = (x_1, x_2) \in \mathbb{R}^d$. Thus, it only remains to prove that $\tau_E(x_1, 0) = \tau_{E_1}(x_1)$ and $\tau_E(0, x_2) = \tau_{E_2}(x_2)$. Let us prove that $\tau_E(x_1, 0) = \tau_{E_1}(x_1)$. The assertion $\tau_E(0, x_2) = \tau_{E_2}(x_2)$ is proven exactly the same way. Note that by definition

$$\begin{aligned}(x_1,0) = \tau_E(x_1,0)^E l_E(x_1,0) &= \Big(\tau_E(x_1,0)^{E_1} l_E(x_1,0)_1, \tau_E(x_1,0)^{E_2} l_E(x_1,0)_2\Big)\\ &= \Big(\tau_E(x_1,0)^{E_1} l_E(x_1,0)_1, 0\Big),\end{aligned}$$

where we used the notation $l_E(x) = \big(l_E(x)_1, l_E(x)_2\big) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. But on the other hand one can write

$$x_1 = \tau_{E_1}(x_1)^{E_1} l_{E_1}(x_1)$$

yielding that

$$\tau_{E_1}(x_1)^{E_1} l_{E_1}(x_1) = \tau_E(x_1,0)^{E_1} l_E(x_1,0)_1.$$

Further noting that

$$l_E(x_1,0) = (l_E(x_1,0)_1, l_E(x_1,0)_2) = (l_E(x_1,0)_1, 0)$$

and taking into account the definition of the norm $\|\cdot\|_E$ given in Lemma 2.1 we obtain

$$1 = \|l_{E_1}(x_1)\|_{E_1} = \|l_E(x_1,0)_1\|_{E_1}.$$

Thus, by the uniqueness of the representation we have $\tau_{E_1}(x_1) = \tau_E(x_1, 0)$ and $l_{E_1}(x_1) = l_E(x_1, 0)_1$ as desired. This concludes the proof. □

Example 4.6. Let $n = d$ and $d_1 = \ldots = d_n = 1$. Assume that $E_1 = a_1, \ldots, E_n = a_n$ are positive and pairwise distinct. Hence, E is a diagonal matrix given by

$$E = \sum_{j=1}^{n} \tilde{E}_j = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

Note that any $x_j \in \mathbb{R}$ can uniquely be written as

$$x_j = \tau_{a_j}(x_j)^{a_j} l_{a_j}(x_j) = \pm |x_j|^{\frac{1}{a_j} \cdot a_j}$$

so that $\tau_{a_j}(x_j) = |x_j|^{\frac{1}{a_j}}$ for all $j = 1, \ldots, n$. Note that $a_1, \ldots, a_n$ correspond to the eigenvalues of E and the direct sum decomposition with respect to E is given by

$$x = x_1 e_1 + \ldots + x_d e_d =: \overline{x}_1 + \ldots + \overline{x}_n$$

for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, where $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$. By Lemma 4.5 for any $H > 0$ we further get

$$\frac{1}{c}\sum_{j=1}^{n} \|\overline{x}_j\|^{\frac{H}{a_j}} = \frac{1}{c}\sum_{j=1}^{d} |x_j|^{\frac{H}{a_j}} \leq \tau_E(x)^H \leq c\sum_{j=1}^{d} |x_j|^{\frac{H}{a_j}} = c\sum_{j=1}^{n} \|\overline{x}_j\|^{\frac{H}{a_j}}$$

for some constant $c \geq 1$. Note that this bound is an improvement of the bounds given in Remark 2.10.

We now state two technical Lemmas which will be needed in order to estimate certain expected energy integrals arising from Frostman's theorem (see Chapter 1). Lemma 4.7 below with $k = 1$ is an analogous statement to [5, Lemma 3.6] (see also [54, p. 212]). Furthermore, Lemma 4.8 with $k = 1$ is the statement of [5, Lemma 3.7]. By using the methods in [54, 5] we can establish the statements for general $k \in \mathbb{N}$.

Lemma 4.7. *Let $0 < h < 1$ be a given constant. Then for any constants $\delta > h, M > 0$, $u > 0$ and any $k \in \mathbb{N}$ there exist positive and finite constants $C_{5,2}$ and $C_{5,3}$, depending only on δ, u and M such that for all $0 < A \leq M$*

$$I(A) := \int_0^2 (A + r^h)^{-u} r^{k-1} dr \leq C_{5,2}(A^{-u+\frac{k}{\delta}} + C_{5,3}). \tag{4.4}$$

Proof. Throughout this proof, let c be an unspecified positive constant which might change in each occurence. Let us first assume that $u = \frac{k}{\delta}$. In this case we can estimate

$$I(A) = \int_0^2 (A + r^h)^{-u} r^{k-1} dr \leq \int_0^2 r^{-uh} r^{k-1} dr = \int_0^2 r^{-\frac{kh}{\delta}+k-1} dr < \infty,$$

since $\delta > h$ by assumption and, thus $-\frac{kh}{\delta} + k - 1 > -1$. Hence, $I(A)$ is bounded by a constant independent of A. So it remains to prove (4.4) for $u \neq \frac{k}{\delta}$. By using the substitution $s = A + r^h$ we obtain

$$\begin{aligned} I(A) &= \int_0^2 (A + r^h)^{-u} r^{k-1} dr = \frac{1}{h}\int_A^{A+2^h} s^{-u}(s - A)^{\frac{k-h}{h}} ds \\ &\leq \frac{1}{h}\int_A^{A+2^h} s^{-u+\frac{k}{h}-1} ds \leq \frac{1}{h}(M + 2^h)^{\frac{k}{h}-\frac{k}{\delta}} \int_A^{A+2^h} s^{-u+\frac{k}{\delta}-1} ds, \end{aligned}$$

where in the first inequality we used that $k - h > 0$. Now we have to consider two cases. First assume that $-u + \frac{k}{\delta} < 0$. Elementary integration shows that

$$I(A) \leq c\int_A^{A+2^h} s^{-u+\frac{k}{\delta}-1} ds = \frac{c}{u - \frac{k}{\delta}}\Big(A^{-u+\frac{k}{\delta}} - (A + 2^h)^{-u+\frac{k}{\delta}}\Big) \leq cA^{-u+\frac{k}{\delta}}.$$

On the other hand if $-u+\frac{k}{\delta}>0$ one gets

$$I(A) \le \frac{c}{-u+\frac{k}{\delta}}\Big((A+2^h)^{-u+\frac{k}{\delta}} - A^{-u+\frac{k}{\delta}}\Big) \le \frac{c}{-u+\frac{k}{\delta}}(M+2^h)^{-u+\frac{k}{\delta}}.$$

Overall we obtain

$$I(A) \le cA^{-u+\frac{k}{\delta}} + \frac{c}{-u+\frac{k}{\delta}}(M+2^h)^{-u+\frac{k}{\delta}} \le C_{5,2}(A^{-u+\frac{k}{\delta}} + C_{5,3})$$

for some suitable constants $C_{5,2}$ and $C_{5,3}$ independent of A. □

Lemma 4.8. *Let α, β, η be positive constants and $k \in \mathbb{N}$. For $A>0$ and $B>0$ define*

$$J := J(A,B) = \int_0^2 \frac{r^{k-1}}{(A+r^\alpha)^\beta (B+r)^\eta} dr. \tag{4.5}$$

Then there exist positive and finite constants $C_{5,4}$ and $C_{5,5}$, depending only on α, β, η such that the following holds for all reals $A, B > 0$ satisfying $A^{\frac{1}{\alpha}} \le C_{5,4}B$:

(i) if $\alpha\beta > k$ then

$$J \le C_{5,5}\frac{1}{A^{\beta-\frac{k}{\alpha}}B^\eta} \tag{4.6}$$

(ii) if $\alpha\beta = k$ then

$$J \le C_{5,5}\frac{1}{B^\eta}\log(1+B^kA^{-\frac{k}{\alpha}}) \tag{4.7}$$

(iii) if $0<\alpha\beta<k$ and $\alpha\beta+\eta \ne k$ then

$$J \le C_{5,5}\Big(\frac{1}{B^{\alpha\beta+\eta-k}}+1\Big). \tag{4.8}$$

Proof. Using the change of variables $s=\frac{r}{B}$ we get

$$\begin{aligned} J &= \int_0^{2B^{-1}} \frac{1}{(A+B^\alpha s^\alpha)^\beta}\frac{1}{(B+Bs)^\eta}B^{k-1}s^{k-1}B\,ds \\ &= \frac{1}{B^{\eta-k}}\int_0^{2B^{-1}} \frac{1}{(A+B^\alpha s^\alpha)^\beta}\frac{1}{(1+s)^\eta}s^{k-1}ds. \end{aligned}$$

Note that if $B<2$ one can split the last integral so that

$$J = \frac{1}{B^{\eta-k}}\int_0^1 \frac{1}{(A+B^\alpha s^\alpha)^\beta}\frac{s^{k-1}}{(1+s)^\eta}ds + \frac{1}{B^{\eta-k}}\int_1^{2B^{-1}} \frac{1}{(A+B^\alpha s^\alpha)^\beta}\frac{s^{k-1}}{(1+s)^\eta}ds. \tag{4.9}$$

On the other hand if $B\ge 2$ then J is bounded from above by the first term in (4.9). Hence, in the following it is sufficient to consider the case $0<B<2$. Thus, using the change of variables $t = A^{-\frac{1}{\alpha}} Bs$ we further get

$$\begin{aligned} J &\leq \frac{1}{B^{\eta-k}} \int_0^1 \frac{s^{k-1}}{(A+B^\alpha s^\alpha)^\beta} ds + \frac{1}{B^{\eta-k}} \int_1^{2B^{-1}} \frac{s^{k-1}}{(A+B^\alpha s^\alpha)^\beta s^\eta} ds \\ &= \frac{1}{B^{\eta-k}} \int_0^{A^{-\frac{1}{\alpha}}B} \frac{t^{k-1}}{(A+At^\alpha)^\beta} B^{-k+1} A^{\frac{k-1}{\alpha}} B^{-1} A^{\frac{1}{\alpha}} dt \\ &\quad + \frac{1}{B^{\eta-k}} \int_{A^{-\frac{1}{\alpha}}B}^{2A^{-\frac{1}{\alpha}}} \frac{t^{k-1}}{(A+At^\alpha)^\beta A^{\frac{\eta}{\alpha}} B^{-\eta} t^\eta} B^{-k+1} A^{\frac{k-1}{\alpha}} B^{-1} A^{\frac{1}{\alpha}} dt \\ &\leq \frac{1}{B^\eta} \frac{1}{A^{\beta-\frac{k}{\alpha}}} \int_0^{A^{-\frac{1}{\alpha}}B} \frac{1}{(1+t^\alpha)^\beta t^{1-k}} dt + \frac{1}{A^{\beta-\frac{k}{\alpha}+\frac{\eta}{\alpha}}} \int_{A^{-\frac{1}{\alpha}}B}^\infty \frac{t^{k-1}}{(1+t^\alpha)^\beta t^\eta} dt. \end{aligned} \tag{4.10}$$

From now on let c be an unspecified positive constant which might change in each occurence. Let us now first prove (i). If $\alpha\beta > k$, by using the change of variables $s = 1 + t^\alpha$ one gets

$$\int_0^{A^{-\frac{1}{\alpha}}B} \frac{1}{(1+t^\alpha)^\beta t^{1-k}} dt = c \int_1^{A^{-1}B^\alpha} \frac{(s-1)^{\frac{1-\alpha}{\alpha}}}{s^\beta (s-1)^{\frac{1-k}{\alpha}}} ds \leq c \int_1^\infty s^{-\beta} (s-1)^{\frac{k-\alpha}{\alpha}} ds < \infty,$$

since $-\beta + \frac{k-\alpha}{\alpha} < -1$. Thus, one gets an upper estimate of the last expression in (4.10) by

$$\begin{aligned} &\frac{c}{A^{\beta-\frac{k}{\alpha}} B^\eta} + \frac{1}{A^{\beta-\frac{k}{\alpha}+\frac{\eta}{\alpha}}} \int_{A^{-\frac{1}{\alpha}}B}^\infty \frac{1}{t^{\alpha\beta+\eta+1-k}} dt \\ &= \frac{c}{A^{\beta-\frac{k}{\alpha}} B^\eta} + \frac{c}{A^{\beta-\frac{k}{\alpha}+\frac{\eta}{\alpha}}} \left(A^{-\frac{1}{\alpha}} B\right)^{-\alpha\beta-\eta+k} \\ &= \frac{c}{A^{\beta-\frac{k}{\alpha}} B^\eta} + \frac{c}{B^{\alpha\beta+\eta-k}}. \end{aligned}$$

Using that $A^{\frac{1}{\alpha}} \leq C_{5,4} B$ the last expression can further be estimated by

$$\frac{c}{A^{\beta-\frac{k}{\alpha}} B^\eta} + \frac{c}{A^{\beta-\frac{k}{\alpha}} B^\eta} = C_{5,5} \frac{1}{A^{\beta-\frac{k}{\alpha}} B^\eta}$$

for some suitable $C_{5,5} > 0$. This proves (4.6). Let us now show (ii) and assume that $\alpha\beta = k$. Then the last expression in (4.10) equals

$$\frac{1}{B^\eta} \int_0^{A^{-\frac{1}{\alpha}}B} \frac{1}{(1+t^\alpha)^\beta t^{1-k}} dt + \frac{1}{A^{\frac{\eta}{\alpha}}} \int_{A^{-\frac{1}{\alpha}}B}^\infty \frac{t^{k-1}}{(1+t^\alpha)^\beta t^\eta} dt.$$

Note that for any $y \geq 0$ and $m > 0$

$$2(1+y)^m = (1+y)^m + (1+y)^m \geq 1 + y^m$$

and using this inequality with $y = t^\alpha$ and $m = \beta$

$$\frac{1}{2}(1+t^k) = \frac{1}{2}(1+t^{\alpha\beta}) \leq (1+t^\alpha)^\beta.$$

Thus, we get an upper estimate of the last integrals by

$$\begin{aligned}
&\frac{2}{B^\eta}\int_0^{A^{-\frac{1}{\alpha}}B}\frac{1}{(1+t^k)t^{1-k}}dt+\frac{1}{A^{\frac{\eta}{\alpha}}}\int_{A^{-\frac{1}{\alpha}}B}^\infty \frac{t^{k-1}}{t^{k+\eta}}dt\\
&=\frac{2}{B^\eta}\int_0^{A^{-\frac{1}{\alpha}}B}\frac{1}{t^{1-k}+t}dt+\frac{1}{A^{\frac{\eta}{\alpha}}}\int_{A^{-\frac{1}{\alpha}}B}^\infty \frac{1}{t^{1+\eta}}dt\\
&=\frac{2}{B^\eta}\int_0^{A^{-\frac{1}{\alpha}}B}\frac{t^{k-1}}{1+t^k}dt+\frac{1}{\eta}\frac{1}{A^{\frac{\eta}{\alpha}}}\frac{1}{(A^{-\frac{1}{\alpha}}B)^\eta}\\
&=\frac{2}{B^\eta}\frac{1}{k}\log(1+A^{-\frac{k}{\alpha}}B^k)+\frac{1}{\eta}\frac{1}{B^\eta}\le C_{5,5}\frac{1}{B^\eta}\log(1+A^{-\frac{k}{\alpha}}B^k)
\end{aligned}$$

for some suitable $C_{5,5}>0$. Thus, it remains to prove (4.8). So assume that $0<\alpha\beta<k$ and $\alpha\beta+\eta\neq k$. Let us first consider the case $\alpha\beta+\eta<k$. Then we can estimate J from above by

$$J\le\int_0^2 r^{-\alpha\beta-\eta+k-1}dr<\infty,$$

since $-\alpha\beta-\eta+k-1>-1$. Now assume that $\alpha\beta+\eta>k$. Let us split the last integral in (4.9) as

$$J=\frac{1}{B^{\eta-k}}\int_0^{B^{-1}A^{\frac{1}{\alpha}}}\frac{1}{(A+B^\alpha s^\alpha)^\beta}\frac{s^{k-1}}{(1+s)^\eta}ds+\frac{1}{B^{\eta-k}}\int_{B^{-1}A^{\frac{1}{\alpha}}}^{2B^{-1}}\frac{1}{(A+B^\alpha s^\alpha)^\beta}\frac{s^{k-1}}{(1+s)^\eta}ds.$$

By using the change of variables $t=A^{-\frac{1}{\alpha}}Bs$ we further get

$$\begin{aligned}
J&=\frac{1}{B^{\eta-k}}\int_0^1\frac{1}{(A+At^\alpha)^\beta}\frac{A^{\frac{1}{\alpha}}B^{-1}}{(1+A^{\frac{1}{\alpha}}B^{-1}t)^\eta}A^{\frac{k}{\alpha}-\frac{1}{\alpha}}B^{1-k}t^{k-1}dt\\
&\quad+\frac{1}{B^{\eta-k}}\int_1^{2A^{-\frac{1}{\alpha}}}\frac{A^{\frac{k-1}{\alpha}}B^{1-k}t^{k-1}A^{\frac{1}{\alpha}}B^{-1}}{(A+At^\alpha)^\beta(1+A^{\frac{1}{\alpha}}B^{-1}t)^\eta}dt\\
&\le\frac{1}{B^\eta}\frac{1}{A^{\beta-\frac{k}{\alpha}}}\int_0^1\frac{t^{k-1}}{(1+t^\alpha)^\beta}dt+\frac{1}{B^\eta}\frac{1}{A^{\beta-\frac{k}{\alpha}}}\int_1^{2A^{-\frac{1}{\alpha}}}\frac{t^{k-1}}{(1+t^\alpha)^\beta(1+A^{\frac{1}{\alpha}}B^{-1}t)^\eta}dt.
\end{aligned}$$

Since $-\alpha\beta+k-1>-1$ and $0<B<2$ by assumption, we can further get an upper estimate of the last expression by

$$\frac{c}{A^{\beta-\frac{k}{\alpha}}B^\eta}+\frac{1}{A^{\beta-\frac{k}{\alpha}}B^\eta}\int_1^{BA^{-\frac{1}{\alpha}}}\frac{t^{k-1}}{t^{\alpha\beta}}dt+\frac{1}{A^{\beta-\frac{k}{\alpha}}B^\eta}\int_{BA^{-\frac{1}{\alpha}}}^{2A^{-\frac{1}{\alpha}}}\frac{t^{k-1}}{t^{\alpha\beta+\eta}(A^{\frac{1}{\alpha}}B^{-1})^\eta}dt=:J_2.$$

Let us note that

$$\int_1^{BA^{-\frac{1}{\alpha}}}t^{-\alpha\beta+k-1}dt=c(BA^{-\frac{1}{\alpha}})^{-\alpha\beta+k}-c\le c(BA^{-\frac{1}{\alpha}})^{-\alpha\beta+k}$$

and that

$$\frac{1}{(A^{\frac{1}{\alpha}}B^{-1})^\eta}\int_{BA^{-\frac{1}{\alpha}}}^{2A^{-\frac{1}{\alpha}}}t^{-\alpha\beta-\eta+k-1}dt=\frac{c}{(A^{\frac{1}{\alpha}}B^{-1})^\eta}\Big((BA^{-\frac{1}{\alpha}})^{-\alpha\beta-\eta+k}-(A^{-\frac{1}{\alpha}})^{-\alpha\beta-\eta+k}\Big)$$

$$\leq \frac{c}{(A^{\frac{1}{\alpha}}B^{-1})^{\eta}} B^{-\alpha\beta-\eta+k} A^{\beta+\frac{\eta}{\alpha}-\frac{k}{\alpha}} = c(BA^{-\frac{1}{\alpha}})^{-\alpha\beta+k}.$$

Therefore, we can estimate J_2 from above by

$$\begin{aligned} J_2 &\leq \frac{c}{A^{\beta-\frac{k}{\alpha}}B^{\eta}} + \frac{1}{A^{\beta-\frac{k}{\alpha}}B^{\eta}} \frac{c}{(A^{-\frac{1}{\alpha}}B)^{\alpha\beta-k}} \\ &= \frac{c}{A^{\beta-\frac{k}{\alpha}}B^{\eta}} + \frac{c}{B^{\alpha\beta+\eta-k}} = \frac{c}{(A^{\frac{1}{\alpha}})^{\alpha\beta-k}B^{\eta}} + \frac{c}{B^{\alpha\beta+\eta-k}} \\ &\leq \frac{c}{B^{\alpha\beta+\eta-k}}, \end{aligned}$$

where we used that $A^{\frac{1}{\alpha}} \leq C_{5,4}B$ in the last inequality. Overall we get that

$$J \leq C_{5,5}\left(\frac{1}{B^{\alpha\beta+\eta-k}} + 1\right)$$

for some suitable constant $C_{5,5} > 0$. This completes the proof. $\square$

4.3. Uniform modulus of continuity

In this section we study the uniform modulus of continuity of the random field given in (4.2). Our approach is to apply results established in [10, 11] by using the properties stated in the preceding two sections. Throughout this section, suppose that $R \in \mathbb{R}^{d\times d}$ with $q(R) = \operatorname{trace}(R)$ and the distinct real parts of its eigenvalues are given by $0 < a_1 < \ldots < a_p$ for some $p \leq d$. Let us first state the following result which is a direct consequence of [10, Proposition 5.3].

Proposition 4.9. *Let $\{X(x) : x \in \mathbb{R}^d\}$ be a real-valued centered Gaussian field, $G_d \subset \mathbb{R}^d$ a non-empty compact set and assume that there exists a constant $C > 0$ such that for all $x, y \in G_d$*

$$\mathbb{E}[(X(x) - X(y))^2] \leq C\tau_R(x-y)^{2H} |\log \tau_R(x-y)|^{\beta} \tag{4.11}$$

for some $H \in (0, a_1)$ and $\beta \in \mathbb{R}$. Then there exists a modification X^* of X such that

$$\sup_{\substack{x,y\in G_d \\ x\neq y}} \frac{|X^*(x) - X^*(y)|}{\tau_R(x-y)^H |\log \tau_R(x-y)|^{\frac{1}{2}+\beta+\varepsilon}} < \infty \quad a.s.$$

for any $\varepsilon > 0$.

Let us remark that Proposition 4.9 is a quite general result, since it holds for any centered Gaussian random field that satisfies (4.11). A corresponding result for certain stable random fields has been proven in [11] by using series representations of stable fields as given in [29, 30, 25]. More precisely, in the following let M_α be a complex-valued isotropic SαS random measure according to Chapter 3.2. Furthermore, let Y be a scalar valued random field defined through the stochastic integral

$$Y(x) = \operatorname{Re} \int_{\mathbb{R}^d} f_\alpha(x,\xi) M_\alpha(d\xi) \tag{4.12}$$

for any $x \in \mathbb{R}^d$, where $f_\alpha(x,\cdot) \in L^\alpha(\lambda)$ is given by

$$f_\alpha(x,\xi) = \prod_{j=1}^{n} (e^{i\langle x_j, \xi_j \rangle} - 1)\psi_\alpha(\xi)$$

for all $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} = \mathbb{R}^d$ and $\psi_\alpha : \mathbb{R}^d \to [0,\infty)$ is a measurable function satisfying

$$\int_{\mathbb{R}^d} \min(1, \|\xi\|^\alpha) |\psi_\alpha(\xi)|^\alpha d\xi < \infty.$$

The following is due to [11, Proposition 5.1].

Proposition 4.10. *Assume that there exist some positive and finite constants c_ψ, K and $\beta \in (0, a_1)$ such that*

$$|\psi_\alpha(\xi)| \le c_\psi \tau_{R^T}(\xi)^{-\beta - \frac{q(R)}{\alpha}}$$

holds for almost every $\xi \in \mathbb{R}^d$ with $\|\xi\| > K$. Then there exists a modification Y^* of Y such that for any non-empty compact set $G_d \subset \mathbb{R}^d$

$$\sup_{\substack{x,y \in G_d \\ x \neq y}} \frac{|Y^*(x) - Y^*(y)|}{\tau_R(x-y)^\beta \big[\log\big(1 + \tau_R(x-y)^{-1}\big)\big]^{\delta + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \quad a.s. \tag{4.13}$$

for any $\delta > 0$.

Let us now state the main result of this section.

Proposition 4.11. *Let the assumptions of Theorem 4.2 hold and assume that $H_j = 1$ or, equivalently $a_1^j > 1$ for $j = 1, \ldots, n$. Then there exists a modification X_α^* of the random field in (4.2) such that for any $\varepsilon > 0$ and any non-empty compact set $G_d \subset \mathbb{R}^d$*

$$\sup_{\substack{x,y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\sum_{j=1}^n \tau_{E_j}(x_j - y_j) |\log \sum_{j=1}^n \tau_{E_j}(x_j - y_j)|^{\frac{1}{2} + \varepsilon}} < \infty \quad a.s.$$

if $\alpha = 2$ and

$$\sup_{\substack{x,y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\sum_{j=1}^n \tau_{E_j}(x_j - y_j) \big[\log\big(1 + \sum_{j=1}^n \tau_{E_j}(x_j - y_j)^{-1}\big)\big]^{\varepsilon + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \quad a.s.$$

if $\alpha \in (0,2)$, where we used the notation $x = (x_1, \ldots, x_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} = \mathbb{R}^d$. In particular, for any $0 < \gamma < 1$ and $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in G_d$ one can find a *positive and finite constant* $C_{5,6}$ *such that*

$$|X_\alpha^*(x) - X_\alpha^*(y)| \leq C_{5,6} \sum_{j=1}^n \tau_{E_j}(x_j - y_j)^\gamma \tag{4.14}$$

holds almost surely.

Proof. Let us first assume that $\alpha = 2$. In the following let $\|\cdot\|_p$ denote the p-norm on $\mathbb{R}^n$ for $p \geq 1$, c an unspecified positive constant, $G_d \subset \mathbb{R}^d$ an arbitrary compact set and recall that by the equivalence of norms one can find a constant c such that $\frac{1}{c}\|u\|_2 \leq \|u\|_1 \leq c\|u\|_2$ or, equivalently

$$\frac{1}{c}\sum_{i=1}^n |u_i|^2 \leq \Big(\sum_{i=1}^n |u_i|\Big)^2 \leq c\sum_{i=1}^n |u_i|^2$$

for any $u \in \mathbb{R}^n$. Further let us remark that by Theorem 3.11 the variance of the centered Gaussian random variable $X_2(x)$ in (4.2) is given by

$$\Gamma^2(x) = \mathbb{E}[X_2(x)^2] = c\int_{\mathbb{R}^d} \prod_{j=1}^n |e^{i\langle x_j, \xi_j\rangle} - 1|^2 \psi_j(\xi_j)^{-2-q_j} d\xi.$$

Note that for all $1 \leq j \leq n$ and $x = (x_1, \ldots, x_n) \in G_d$ one can find a constant $0 < M < \infty$ such that

$$\Gamma^2(x_1, \ldots, x_{j-1}, \theta, x_{j+1}, \ldots, x_n) \leq M,$$

where $\theta \in \mathbb{R}^{d_j}$ with $\tau_{E_j}(\theta) = 1$. Using all this and the elementary inequality

$$\begin{aligned}
&|X_2(x) - X_2(y)| \\
&\quad\leq \sum_{i=1}^n |X_2(x_1, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n) - X_2(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, y_n)|
\end{aligned}$$

with the convention that

$$X_2(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, y_n) = X_2(y)$$

for $i = 1$ and

$$X_2(x_1, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n) = X_2(x)$$

for $i = n$ we get for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in G_d$

$$\begin{aligned}
&\mathbb{E}[|X_2(x) - X_2(y)|^2] \\
&\leq \mathbb{E}\Big[\Big(\sum_{i=1}^n |X_2(x_1, \ldots, x_{i-1}, x_i, y_{i+1}, \ldots, y_n) - X_2(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, y_n)|\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\sum_{i=1}^n |X_2(x_1, \ldots, x_{i-1}, x_i - y_i, y_{i+1}, \ldots, y_n)|\Big)^2\Big] \\
&\leq c\mathbb{E}\Big[\sum_{i=1}^n |X_2(x_1, \ldots, x_{i-1}, x_i - y_i, y_{i+1}, \ldots, y_n)|^2\Big],
\end{aligned}$$

where we used Corollary 4.4 in the equality and the equivalence of norms in the last inequality. Using Corollary 4.3 and the generalized polar coordinates for $x_i - y_i$ we can further get an upper estimate of the last expression by

$$\begin{aligned} &c\sum_{i=1}^{n}\tau_{E_i}(x_i-y_i)^2\mathbb{E}\Big[|X_2(x_1,\dots,x_{i-1},l_{E_i}(x_i-y_i),y_{i+1},\dots,y_n)|^2\Big] \\ &\le cM\sum_{i=1}^{n}\tau_{E_i}(x_i-y_i)^2 \le cM\tau_E(x-y)^2, \end{aligned}$$

where we used Lemma 4.5 with $H=2$ in the last inequality. Therefore, X_2 satisfies (4.11) with $H=1$ and $\beta=0$ so that Proposition 4.9 yields

$$\sup_{\substack{x,y\in G_d\\ x\neq y}} \frac{|X_2^*(x)-X_2^*(y)|}{\tau_E(x-y)|\log\tau_E(x-y)|^{\frac{1}{2}+\varepsilon}} < \infty \quad a.s. \tag{4.15}$$

for any $\varepsilon > 0$ and a continuous modification X_2^* of X_2, which by Lemma 4.5 is equivalent to

$$\sup_{\substack{x,y\in G_d\\ x\neq y}} \frac{|X_2^*(x)-X_2^*(y)|}{\sum_{i=1}^n\tau_{E_i}(x_i-y_i)|\log\sum_{i=1}^n\tau_{E_i}(x_i-y_i)|^{\frac{1}{2}+\varepsilon}} < \infty \quad a.s.$$

Let us now prove (4.14). Note that (4.14) is trivially true for $x = y$, so assume $x \neq y$, $x, y \in G_d$. Then by (4.15) we obtain

$$\begin{aligned} |X_2^*(x)-X_2^*(y)| &\le c\tau_E(x-y)|\log\tau_E(x-y)|^{\frac{1}{2}+\varepsilon} \le c\tau_E(x-y)^{\gamma} \\ &\le C_{5,6}\sum_{i=1}^{n}\tau_{E_i}(x_i-y_i)^{\gamma} \end{aligned}$$

almost surely for some suitable constant $C_{5,6} > 0$ and any $0 < \gamma < 1$, where we used (2.3) in the second inequality and Lemma 4.5 in the last inequality.

Let us now assume that $\alpha \in (0,2)$. The idea is to apply Proposition 4.10 with

$$\psi_\alpha = \prod_{j=1}^{n}\psi_j(\xi_j)^{-1-\frac{q_j}{\alpha}}.$$

Let $K > 0$ be a constant. Note that $\tau_{E^T}(\xi) > 0$ is bounded away from zero for all $\xi \in \mathbb{R}^d$ with $\|\xi\| > K$. Thus, using the change to generalized polar coordinates, the fact that ψ_j is E_j^T-homogeneous and (2.1) one gets

$$\begin{aligned} \psi_\alpha(\xi) &= \psi_\alpha(\tau_{E^T}(\xi)^{E^T}l_{E^T}(\xi)) = \psi_\alpha(\tau_{E^T}(\xi)^{E_1^T}l_{E^T}(\xi)_1,\dots,\tau_{E^T}(\xi)^{E_n^T}l_{E^T}(\xi)_n) \\ &= \prod_{j=1}^{n}\psi_j(\tau_{E^T}(\xi)^{E_j^T}l_{E^T}(\xi)_j)^{-1-\frac{q_j}{\alpha}} = \prod_{j=1}^{n}\tau_{E^T}(\xi)^{-1-\frac{q_j}{\alpha}}\psi_j(l_{E^T}(\xi)_j)^{-1-\frac{q_j}{\alpha}} \\ &= \tau_{E^T}(\xi)^{-n-\frac{\sum_{j=1}^n q_j}{\alpha}}\prod_{j=1}^{n}\psi_j(l_{E^T}(\xi)_j)^{-1-\frac{q_j}{\alpha}} \\ &\le c\tau_{E^T}(\xi)^{-1-\frac{q(E)}{\alpha}}\tau_{E^T}(\xi)^{-(n-1)} \le c_\psi\tau_{E^T}(\xi)^{-1-\frac{q(E)}{\alpha}} \end{aligned}$$

for all $\xi \in \mathbb{R}^d$ with $\|\xi\| > K$ and some constant $c_\psi > 0$, where we used the elementary fact that $q(E) = \operatorname{trace}(E) = \sum_{j=1}^n q_j$ and the notation $l_{E^T}(\xi) = (l_{E^T}(\xi)_1, \ldots, l_{E^T}(\xi)_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} = \mathbb{R}^d$. Thus, since we also assumed that $a_1^j > 1$ for $j = 1, \ldots, n$, the assumptions of Proposition 4.10 are fulfilled with $\beta = 1$ and there exists a modification X_α^* of X_α such that

$$\sup_{\substack{x,y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\tau_E(x-y)\big[\log\big(1 + \tau_E(x-y)^{-1}\big)\big]^{\varepsilon + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \quad a.s.$$

for any $\varepsilon > 0$ and any non-empty compact set $G_d \subset \mathbb{R}^d$, which by Lemma 4.5 is equivalent to

$$\sup_{\substack{x,y \in G_d \\ x \neq y}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\sum_{j=1}^n \tau_{E_j}(x_j - y_j)\big[\log\big(1 + \sum_{j=1}^n \tau_{E_j}(x_j - y_j)^{-1}\big)\big]^{\varepsilon + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \quad a.s.$$

From this, (4.14) is deduced exactly as in the Gaussian case $\alpha = 2$ above by using (2.4) instead of (2.3). This completes the proof. $\square$

Proposition 4.11 immediately implies the following.

Corollary 4.12. *Let the assumptions of Theorem 4.2 hold with $H_j = 1$, $1 \leq j \leq n$, and let $G_d \subset \mathbb{R}^d$ be a non-empty compact set. Let $X_1, \ldots, X_N$ be N independent copies of X_α and define $X(x) = (X_1(x), \ldots, X_N(x))$ for any $x \in \mathbb{R}^d$. Then there exists a modification X^* of X such that for any $0 < \gamma < 1$, any norm $\|\cdot\|$ and any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in G_d$ there exists a constant $C_{5,7} > 0$ such that*

$$\|X^*(x) - X^*(y)\| \leq C_{5,7} \sum_{j=1}^n \tau_{E_j}(x_j - y_j)^\gamma \tag{4.16}$$

holds almost surely.

Proof. Without loss of generality let us assume that $\|\cdot\| = \|\cdot\|_1$ is the 1-norm on $\mathbb{R}^N$. Then by Proposition 4.11 there exist continuous modifications X_i^* of $X_i, 1 \leq i \leq N$, such that

$$|X_i^*(x) - X_i^*(y)| \leq c \sum_{j=1}^n \tau_{E_j}(x_j - y_j)^\gamma$$

for some positive constant c. Using this for $X^* = (X_1^*, \ldots, X_N^*)$ we get

$$\|X^*(x) - X^*(y)\|_1 = \sum_{i=1}^n |X_i^*(x) - X_i^*(y)| \leq nc \sum_{j=1}^n \tau_{E_j}(x_j - y_j)^\gamma$$

as desired. $\square$

4.4. Hausdorff dimension of the sample paths

Throughout this section, suppose that the assumptions of Theorem 4.2 hold and let $X_1, \ldots, X_N$ be N independent copies of the random field X_α given in (4.2). Let us define the stable random field $X = \{X(x) : x \in \mathbb{R}^d\}$ with values in $\mathbb{R}^N$ by

$$X(x) = (X_1(x), \ldots, X_N(x)), \quad x \in \mathbb{R}^d.$$

We will call X a (d, N)-harmonizable operator scaling stable random sheet. Furthermore, for $1 \le j \le n$ let

$$\mathbb{R}^{d_j} = W_1^j \oplus \ldots \oplus W_{p_j}^j$$

be the direct sum decomposition with respect to E_j according to Chapter 2.1 and let $\mu_k^j = \dim W_k^j, 1 \le k \le p_j$. Note that

$$d_j = \mu_1^j + \ldots + \mu_{p_j}^j.$$

In order to state our Theorems conveniently we will assume that

$$0 < \frac{H_1}{a_{p_1}^1} < \ldots < \frac{H_1}{a_1^1} \le \frac{H_2}{a_{p_2}^2} < \ldots < \frac{H_2}{a_1^2} \le \ldots \le \frac{H_n}{a_{p_n}^n} < \ldots < \frac{H_n}{a_1^n} < 1.$$

Furthermore, for $1 \le j \le n$ let us define $\tilde{\mu}_k^j = \mu_{p_j+1-k}^j$ and $\tilde{a}_k^j = a_{p_j+1-k}^j, 1 \le k \le p_j$. Note that we have

$$0 < \frac{H_1}{\tilde{a}_1^1} < \ldots < \frac{H_1}{\tilde{a}_{p_1}^1} \le \frac{H_2}{\tilde{a}_1^2} < \ldots < \frac{H_2}{\tilde{a}_{p_2}^2} \le \ldots \le \frac{H_n}{\tilde{a}_1^n} < \ldots < \frac{H_n}{\tilde{a}_{p_n}^n} < 1. \tag{4.17}$$

Theorem 4.13. *Let $X = \{X(x) : x \in \mathbb{R}^d\}$ be a (d, N)-harmonizable operator scaling stable random sheet satisfying (4.17). Then with probability one*

$$\dim_{\mathcal{H}} X([0,1]^d) = \min\{N; \sum_{j=1}^n \sum_{k=1}^{p_j} \frac{a_k^j}{H_j} \mu_k^j\} \tag{4.18}$$

and

$$\begin{aligned} \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) &= \min\left\{ \sum_{j=1}^n \sum_{k=1}^{p_j} \frac{a_k^j}{H_j} \mu_k^j; G(l,b), 1 \le l \le n, 1 \le b \le p_l \right\} \\ &= \begin{cases} \sum_{j=1}^n \sum_{k=1}^{p_j} \frac{a_k^j}{H_j} \mu_k^j & \text{if } \sum_{j=1}^n \sum_{k=1}^{p_j} \frac{a_k^j}{H_j} \mu_k^j \le N, \\ \min\limits_{\substack{1 \le l \le n \\ 1 \le b \le p_l}} G(l,b) & \text{else,} \end{cases} \end{aligned} \tag{4.19}$$

where

$$G(l,b) = \sum_{j=1}^{l-1} \sum_{k=1}^{p_j} \frac{\tilde{a}_k^j H_l}{\tilde{a}_b^l H_j} \tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{\tilde{a}_b^l} \tilde{\mu}_k^l + \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + (1 - \frac{H_l}{\tilde{a}_b^l})N.$$

The second equality in (4.19) is verified by the following Lemma whose proof is elementary.

Denote

$$\kappa = \min\left\{ \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{a_k^j}{H_j}\mu_k^j ; G(l,b), 1 \le l \le n, 1 \le b \le p_l \right\},$$

where $G(l,b)$ is defined as in Theorem 4.13.

Lemma 4.14. *Assume that* (4.17) *holds. The following statements are true.*

(i) If $N \ge \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{a_k^j}{H_j}\mu_k^j$ *then*

$$\kappa = \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{a_k^j}{H_j}\mu_k^j.$$

(ii) If

$$\sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j + \sum_{k=1}^{b-1} \frac{\tilde{a}_k^l}{H_l}\tilde{\mu}_k^l \le N < \sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{H_l}\tilde{\mu}_k^l \tag{4.20}$$

for some $1 \le l \le n$ *and* $1 \le b \le p_l$ *then* $\kappa = G(l,b)$ *and*

$$\kappa \in \Big(\sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + N, \sum_{j=l}^{n} d_j - \sum_{k=1}^{b-1} \tilde{\mu}_k^l + N \Big].$$

Proof. Let us first assume that $N \ge \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{a_k^j}{H_j}\mu_k^j$. Noting that $0 < \frac{H_l}{\tilde{a}_b^l} < 1$ we can estimate

$$\begin{aligned}
G(l,b) &\ge \sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j H_l}{\tilde{a}_b^l H_j}\tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{\tilde{a}_b^l}\tilde{\mu}_k^l + \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j - \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j H_l}{\tilde{a}_b^l H_j}\tilde{\mu}_k^j \\
&= -\sum_{k=b+1}^{p_l} \frac{\tilde{a}_k^l}{\tilde{a}_b^l}\tilde{\mu}_k^l - \sum_{j=l+1}^{n}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j H_l}{\tilde{a}_b^l H_j}\tilde{\mu}_k^j + \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j \\
&\ge -\sum_{k=b+1}^{p_l} \tilde{\mu}_k^l - \sum_{j=l+1}^{n}\sum_{k=1}^{p_j} \tilde{\mu}_k^j + \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j \\
&= -d_l - \sum_{j=l+1}^{n} d_j + \sum_{j=l}^{n} d_j + \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j \\
&= \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{a_k^j}{H_j}\mu_k^j,
\end{aligned}$$

where we used (4.17) in the second inequality. It remains to prove (ii). So suppose that (4.20) holds for some $1 \le l \le n$ and $1 \le b \le p_l$. Then using (4.17) we can estimate

$$\begin{aligned}
G(l,b) < &\sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j H_l}{\tilde{a}_b^l H_j}\tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{\tilde{a}_b^l}\tilde{\mu}_k^l + \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + \sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{H_l}\tilde{\mu}_k^l \\
&- \sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j H_l}{\tilde{a}_b^l H_j}\tilde{\mu}_k^j - \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{\tilde{a}_b^l}\tilde{\mu}_k^l
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=l+1}^{n} d_j + \sum_{k=b+1}^{p_l} \tilde{\mu}_k^l + \sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{H_l}\tilde{\mu}_k^l \\
&\leq \sum_{j=l+1}^{n}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j + \sum_{k=b+1}^{p_l} \frac{\tilde{a}_k^l}{H_l}\tilde{\mu}_k^l + \sum_{j=1}^{l-1}\sum_{k=1}^{p_j} \frac{\tilde{a}_k^j}{H_j}\tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{\tilde{a}_k^l}{H_l}\tilde{\mu}_k^l \\
&= \sum_{j=1}^{n}\sum_{k=1}^{p_j} \frac{a_k^j}{H_j}\mu_k^j.
\end{aligned}
$$

Now let us show that $G(l,b) \leq G(j,k)$ for all $1 \leq j \leq n$ and $1 \leq k \leq p_j$ or, equivalently $G(l,b) - G(j,k) \leq 0$. We divide the proof into the case that either $1 \leq j \leq l-1$ or $j = l$ and $1 \leq b \leq k$ and the case that either $l+1 \leq j \leq n$ or $j = l$ and $k < b \leq p_l$. Let us first consider the case that either $1 \leq j \leq l-1$ or $j = l$ and $1 \leq b \leq k$. For simplicity let us assume that $j = l$ and $1 \leq b \leq k$, since the case $1 \leq j \leq l-1$ is proven analogously. Then using (4.17) and (4.20) we can estimate

$$
\begin{aligned}
G(l,b) - G(j,k) &= \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_b^l H_i}\tilde{\mu}_q^i + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{\tilde{a}_b^l}\tilde{\mu}_q^l - \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_k^l H_i}\tilde{\mu}_q^i - \sum_{q=1}^{k} \frac{\tilde{a}_q^l}{\tilde{a}_k^l}\tilde{\mu}_q^l + \sum_{i=b+1}^{k} \tilde{\mu}_i^l \\
&\quad + \left(\frac{H_l}{\tilde{a}_k^l} - \frac{H_l}{\tilde{a}_b^l}\right) \cdot N \\
&< \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_b^l H_i}\tilde{\mu}_q^i + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{\tilde{a}_b^l}\tilde{\mu}_q^l - \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_k^l H_i}\tilde{\mu}_q^i - \sum_{q=1}^{k} \frac{\tilde{a}_q^l}{\tilde{a}_k^l}\tilde{\mu}_q^l + \sum_{i=b+1}^{k} \tilde{\mu}_i^l \\
&\quad + \left(\frac{H_l}{\tilde{a}_k^l} - \frac{H_l}{\tilde{a}_b^l}\right) \cdot \Big(\sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i}{H_i}\tilde{\mu}_q^i + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{H_l}\tilde{\mu}_q^l\Big) \\
&= -\sum_{q=b+1}^{k} \frac{\tilde{a}_q^l}{\tilde{a}_k^l}\tilde{\mu}_q^l + \sum_{i=b+1}^{k} \tilde{\mu}_i^l \leq 0.
\end{aligned}
$$

Let us now consider the case that either $l+1 \leq j \leq n$ or $j = l$ and $k < b \leq p_l$. Again let us assume for simplicity that $j = l$ and $k < b \leq p_l$, since the case $l+1 \leq j \leq n$ is proven analogously. Then (4.17) and (4.20) yield

$$
\begin{aligned}
G(l,b) - G(l,k) &= \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_b^l H_i}\tilde{\mu}_q^i + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{\tilde{a}_b^l}\tilde{\mu}_q^l - \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_k^l H_i}\tilde{\mu}_q^i - \sum_{q=1}^{k} \frac{\tilde{a}_q^l}{\tilde{a}_k^l}\tilde{\mu}_q^l - \sum_{i=k+1}^{b} \tilde{\mu}_i^l \\
&\quad + \left(\frac{H_l}{\tilde{a}_k^l} - \frac{H_l}{\tilde{a}_b^l}\right) \cdot N \\
&\leq \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_b^l H_i}\tilde{\mu}_q^i + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{\tilde{a}_b^l}\tilde{\mu}_q^l - \sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i H_l}{\tilde{a}_k^l H_i}\tilde{\mu}_q^i - \sum_{q=1}^{k} \frac{\tilde{a}_q^l}{\tilde{a}_k^l}\tilde{\mu}_q^l - \sum_{i=k+1}^{b} \tilde{\mu}_i^l \\
&\quad + \left(\frac{H_l}{\tilde{a}_k^l} - \frac{H_l}{\tilde{a}_b^l}\right)\Big(\sum_{i=1}^{l-1}\sum_{q=1}^{p_i} \frac{\tilde{a}_q^i}{H_i}\tilde{\mu}_q^i + \sum_{q=1}^{b-1} \frac{\tilde{a}_q^l}{H_l}\tilde{\mu}_q^l\Big) \\
&= \tilde{\mu}_b^l + \sum_{q=k+1}^{b-1} \tilde{\mu}_q^l - \sum_{i=k+1}^{b} \tilde{\mu}_i^l = 0
\end{aligned}
$$

as desired. Altogether if (4.20) holds we have

$$\kappa = G(l,b) \leq \tilde{\mu}_b^l + \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + N = \sum_{j=l}^{n} d_j - \sum_{k=1}^{b-1} \tilde{\mu}_k^l + N$$

and

$$\kappa = G(l,b) > \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + N.$$

This completes the proof. □

As usual the proof of Theorem 4.13 is divided into proving the upper and lower bounds separately. Let us first show that the upper bounds in (4.18) and (4.19) follow from Corollary 4.12 and a covering argument, where one has to take into account the anisotropic behavior of operator scaling random sheets. Before doing this let us state the following Remark.

Remark 4.15. Assume that the conditions of Theorem 4.2 hold. Then by Corollary 4.3 for any $c > 0$

$$\{X_\alpha(c^{\tilde{E}_j} x) : x \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{c^{H_j} X_\alpha(x) : x \in \mathbb{R}^d\}.$$

If we define $\hat{E}_j = \frac{\tilde{E}_j}{H_j}$ we have

$$\{X_\alpha(c^{\hat{E}_j} x) : x \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{(c^{\frac{1}{H_j}})^{H_j} X_\alpha(x) : x \in \mathbb{R}^d\} = \{cX_\alpha(x) : x \in \mathbb{R}^d\},$$

i.e. the random field X_α satisfies (4.1) with $\tilde{E}_j = \hat{E}_j$ and $H_j = 1, 1 \leq j \leq n$. Thus, unless stated otherwise without loss of generality we will assume that $H_j = 1, 1 \leq j \leq n$, which by Theorem 4.2 implies that $1 < a_1^j < \ldots < a_{p_j}^j$.

Proof of the upper bounds in Theorem 4.13. According to Corollary 4.12 there exists a modification X^* of X such that (4.16) holds. Note that X and X^* are indistinguishable by the continuity of X^*. Therefore, without loss of generality we will assume that X itself almost surely satisfies (4.16).

Let us first note that according to Remark 1.1

$$\dim_{\mathcal{H}} X([0,1]^d) \leq \dim_{\mathcal{H}} \mathbb{R}^N = N \quad a.s,$$

so in the proof of the upper bound in (4.18) we only need to prove the inequality

$$\dim_{\mathcal{H}} X([0,1]^d) \leq \sum_{j=1}^{n} \sum_{k=1}^{p_j} a_k^j \mu_k^j \quad a.s. \tag{4.21}$$

Throughout this proof, let c be an unspecified positive constant and let us use the notation $x = (x_1, \ldots, x_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n} = \mathbb{R}^d$ for any vector x. Furthermore, let us write

$$x_j = x_j^1 + \ldots + x_j^{p_j}$$

as the direct sum decomposition with respect to E_j for any vector $x_j \in \mathbb{R}^{d_j}, 1 \leq j \leq n$. By (4.16) combined with Remark 2.10 for any non-empty compact set $G_d \subset \mathbb{R}^d$, any $x, y \in G_d$ and any constants $0 < \gamma_{j,k} < \gamma'_{j,k} < \frac{1}{a_k^j}$, $1 \leq j \leq n, 1 \leq k \leq p_j$

$$\|X(x) - X(y)\| \leq c \sum_{j=1}^{n} \sum_{k=1}^{p_j} \|x_j^k - y_j^k\|^{\gamma'_{j,k}} \quad a.s. \tag{4.22}$$

Let us choose compact subsets $V_k^j \subset W_k^j, 1 \leq j \leq n, 1 \leq k \leq p_j$, such that

$$[0,1]^{d_j} \subset V_1^j + \ldots + V_{p_j}^j,$$

where $V_1^j + \ldots + V_{p_j}^j = \{x_j^1 + \ldots + x_j^{p_j} : x_j^i \in V_i^j, 1 \leq i \leq p_j\}$. Moreover, let $p = \max_{1 \leq j \leq n} p_j$ and if $p_j < p$ we define $V_k^j = \{0\}$, $a_k^j = \mu_k^j = 0$ for $k = p_j + 1, \ldots, p$. Then we can write

$$\begin{aligned} [0,1]^d &= [0,1]^{d_1} \times \ldots \times [0,1]^{d_n} \subset (V_1^1 + \ldots + V_{p_1}^1) \times \ldots \times (V_1^n + \ldots + V_{p_n}^n) \\ &= (V_1^1 \times \ldots \times V_1^n) + \ldots + (V_p^1 \times \ldots \times V_p^n). \end{aligned}$$

For any integer $k \geq 2$ we cover $\hat{V}_l = V_l^1 \times \ldots \times V_l^n, 1 \leq l \leq p$, by $m_{k,l}$ sub-rectangles

$$\{R_{k,l,i_{k,l}} = R_{k,l,i_{k,l},1} \times \ldots \times R_{k,l,i_{k,l},n}\}, \quad 1 \leq i_{k,l} \leq m_{k,l},$$

where each $R_{k,l,i_{k,l},j} \subset V_l^j$ has side-lengths $k^{-a_l^j}, 1 \leq j \leq n$, if and only if $V_l^j \neq \{0\}$ and $R_{k,l,i_{k,l},j} = \{0\}$ if and only if $V_l^j = \{0\}$. Thus, in case $V_l^j \neq \{0\}$ the diameter and volume of the rectangle $R_{k,l,i_{k,l},j}$ satisfy

$$\operatorname{diam}(R_{k,l,i_{k,l},j}) = c k^{-a_l^j}$$

and

$$\operatorname{vol}(R_{k,l,i_{k,l},j}) = c k^{-a_l^j \mu_l^j},$$

where the volume is taken with respect to the Lebesgue measure on $\mathbb{R}^{\mu_l^j}$. Thus, the volume of $R_{k,l,i_{k,l}}$ with respect to the Lebesgue measure on $\mathbb{R}^{d_j}$ is given by

$$\operatorname{vol}(R_{k,l,i_{k,l}}) = \prod_{j=1}^{n} \operatorname{vol}(R_{k,l,i_{k,l},j}) = c k^{-\sum_{j=1}^{n} a_l^j \mu_l^j} \tag{4.23}$$

for all $1 \leq l \leq p, 1 \leq i_{k,l} \leq m_{k,l}$ and $k \geq 2$. Note that since $\{R_{k,l,i_{k,l}}, 1 \leq i_{k,l} \leq m_{k,l}\}$ cover $\hat{V}_l$, we have

$$\sum_{i_{k,l}=1}^{m_{k,l}} \operatorname{vol}(R_{k,l,i_{k,l}}) \leq c$$

or, equivalently

$$m_{k,l} \leq c k^{\sum_{j=1}^{n} a_l^j \mu_l^j}. \tag{4.24}$$

Note that $X([0,1]^d)$ can be covered by

$$X(R_{k,1,i_{k,1}}+\ldots+R_{k,p,i_{k,p}}),\quad 1\le i_{k,1}\le m_{k,1},\ldots,1\le i_{k,p}\le m_{k,p}$$

for any integer $k\ge 2$. Furthermore, by (4.22) the diameter of the image $X(R_{k,1,i_{k,1}}+\ldots+R_{k,p,i_{k,p}})$ almost surely satisfies

$$\begin{aligned}&\operatorname{diam} X(R_{k,1,i_{k,1}}+\ldots+R_{k,p,i_{k,p}})\\ &\le c\sup_{x_j^l,y_j^l\in R_{k,l,i_{k,l}}}\sum_{j=1}^{n}\sum_{l=1}^{p_j}\|x_j^l-y_j^l\|^{\gamma'_{j,l}}\\ &\le c\sum_{j=1}^{n}\sum_{l=1}^{p_j}k^{-a_l^j\gamma'_{j,l}}\le c\max_{j,l}k^{-a_l^j\gamma'_{j,l}}=k^{-1+\delta}\end{aligned}\tag{4.25}$$

with $\delta=\max\{1-a_l^j\gamma'_{j,l},1\le j\le n,1\le l\le p_j\}$ and $1-\delta\in(0,1)$. Let us choose $\gamma'_{j,l}\in(\gamma_{j,l},\frac{1}{a_l^j})$ for all $1\le j\le n,1\le l\le p_j$ such that

$$(1-\delta)\sum_{j=1}^{n}\sum_{l=1}^{p_j}\frac{1}{\gamma_{j,l}}\mu_l^j>\sum_{j=1}^{n}\sum_{l=1}^{p_j}a_l^j\mu_l^j.$$

Then with $\beta=\sum_{j=1}^{n}\sum_{l=1}^{p_j}\frac{1}{\gamma_{j,l}}\mu_l^j$ it follows from (4.24) and (4.25) that

$$\begin{aligned}&\sum_{i_{k,1}=1}^{m_{k,1}}\cdots\sum_{i_{k,p}=1}^{m_{k,p}}\operatorname{diam} X(R_{k,1,i_{k,1}}+\ldots+R_{k,p,i_{k,p}})^{\beta}\\ &\le c m_{k,1}\ldots m_{k,p}\cdot k^{-(1-\delta)\beta}\\ &\le ck^{\sum_{l=1}^{p}\sum_{j=1}^{n}a_l^j\mu_l^j}\cdot k^{-(1-\delta)\beta}\\ &=ck^{\sum_{j=1}^{n}\sum_{l=1}^{p_j}a_l^j\mu_l^j}\cdot k^{-(1-\delta)\beta}\to 0\end{aligned}$$

as $k\to\infty$. Then by the definition of the Hausdorff dimension (see Chapter 1.1) this proves that

$$\dim_{\mathcal H} X([0,1]^d)\le\beta=\sum_{j=1}^{n}\sum_{l=1}^{p_j}\frac{1}{\gamma_{j,l}}\mu_l^j\quad a.s.$$

Since this holds for any $\gamma_{j,l}<\frac{1}{a_l^j}$ or, equivalently any $\frac{1}{\gamma_{j,l}}>a_l^j$, we derive (4.21) by letting $\frac{1}{\gamma_{j,l}}\downarrow a_l^j$.

Now we turn to the proof of the upper bound in (4.19). We will show that there are two different ways of covering $\operatorname{Gr} X([0,1]^d)$, each of which leads to an upper bound of $\dim_{\mathcal H}\operatorname{Gr} X([0,1]^d)$.

Note that for any fixed integer $k\ge 2$ the graph $\operatorname{Gr} X([0,1]^d)$ can be covered by

$$(R_{k,1,i_{k,1}}+\ldots+R_{k,p,i_{k,p}})\times X(R_{k,1,i_{k,1}}+\ldots+R_{k,p,i_{k,p}}),$$

$1\le i_{k,1}\le m_{k,1},\ldots,1\le i_{k,p}\le m_{k,p}$. Combining this with (4.25) we see that $\operatorname{Gr} X([0,1]^d)$

can be covered by $m_{k,1}\dots m_{k,p}$ cubes in $\mathbb{R}^{d+N}$ with side-lengths at most $ck^{-1+\delta}$. Then by exactly the same arguments as above we obtain

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \leq \sum_{j=1}^{n}\sum_{l=1}^{p_j} a_l^j \mu_l^j \quad a.s. \tag{4.26}$$

We fix integers $1 \leq l \leq n$ and $1 \leq b \leq p_l$. Observe that each

$$(R_{k,1,i_{k,1}} + \ldots + R_{k,p,i_{k,p}}) \times X(R_{k,1,i_{k,1}} + \ldots + R_{k,p,i_{k,p}})$$

can be covered by $\tilde{m}_{k,b,l}$ cubes in $\mathbb{R}^{d+N}$ of side-lengths $k^{-\tilde{a}_b^l}$ so that each cube has volume $ck^{-\tilde{a}_b^l(d+N)}$. From (4.23) and (4.25) it follows that

$$\operatorname{vol}\Big((R_{k,1,i_{k,1}} + \ldots + R_{k,p,i_{k,p}}) \times X(R_{k,1,i_{k,1}} + \ldots + R_{k,p,i_{k,p}})\Big) \leq ck^{-\sum_{j=1}^n \sum_{q=1}^{p_j} a_q^j \mu_q^j + (-1+\delta)N}$$

and therefore

$$\tilde{m}_{k,b,l} k^{-\tilde{a}_b^l(d+N)} \leq ck^{-\sum_{j=1}^n \sum_{q=1}^{p_j} a_q^j \mu_q^j + (-1+\delta)N} = ck^{-\sum_{j=1}^n \sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j + (-1+\delta)N}. \tag{4.27}$$

Let us further note that by (4.17)

$$\begin{aligned}
&-\sum_{j=1}^{n}\sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j + \tilde{a}_b^l \cdot d = -\sum_{j=1}^{n}\sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j + \tilde{a}_b^l \sum_{j=1}^{n}\sum_{q=1}^{p_j} \tilde{\mu}_q^j \\
&= \sum_{j=1}^{l-1}\sum_{q=1}^{p_j} (\tilde{a}_b^l - \tilde{a}_q^j)\tilde{\mu}_q^j + \sum_{j=l+1}^{n}\sum_{q=1}^{p_j} (\tilde{a}_b^l - \tilde{a}_q^j)\tilde{\mu}_q^j + \sum_{q=1}^{b-1} (\tilde{a}_b^l - \tilde{a}_q^l)\tilde{\mu}_q^l + \sum_{q=b}^{p_l} (\tilde{a}_b^l - \tilde{a}_q^l)\tilde{\mu}_q^l \\
&\leq \sum_{j=l+1}^{n}\sum_{q=1}^{p_j} (\tilde{a}_b^l - \tilde{a}_q^j)\tilde{\mu}_q^j + \sum_{q=b}^{p_l} (\tilde{a}_b^l - \tilde{a}_q^l)\tilde{\mu}_q^l.
\end{aligned}$$

Combining this with (4.27) yields

$$\tilde{m}_{k,b,l} \leq ck^{\sum_{j=l+1}^n \sum_{q=1}^{p_j} (\tilde{a}_b^l - \tilde{a}_q^j)\tilde{\mu}_q^j + \sum_{q=b}^{p_l} (\tilde{a}_b^l - \tilde{a}_q^l)\tilde{\mu}_q^l + (\tilde{a}_b^l - 1 + \delta)N}. \tag{4.28}$$

Hence, $\operatorname{Gr} X([0,1]^d)$ can be covered by $m_{k,1}\dots m_{k,p}\tilde{m}_{k,b,l}$ cubes in $\mathbb{R}^{d+N}$ with side-lenths $k^{-\tilde{a}_b^l}$. Denote

$$\eta_{b,l} = \sum_{j=1}^{l-1}\sum_{q=1}^{p_j} \frac{\tilde{a}_q^j}{\tilde{a}_b^l}\tilde{\mu}_q^j + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{\tilde{a}_b^l}\tilde{\mu}_q^l + \sum_{j=l}^{n} d_j - \sum_{q=1}^{b} \tilde{\mu}_q^l + (1 - \tilde{\gamma}_{l,b})N,$$

where $0 < \tilde{\gamma}_{l,b} < \frac{1}{\tilde{a}_b^l}$ is chosen such that $1 - \delta > \tilde{a}_b^l \tilde{\gamma}_{l,b}$. In order to obtain

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \leq \eta_{b,l} \quad a.s.$$

we want to show that

$$m_{k,1}\dots m_{k,p}\tilde{m}_{k,b,l} \cdot (k^{-\tilde{a}_b^l})^{\eta_{b,l}} \to 0$$

as $k \to \infty$. Recall from (4.24) that

$$m_{k,1} \dots m_{k,p} \le c k^{\sum_{j=1}^{n} \sum_{q=1}^{p_j} a_q^j \mu_q^j} = c k^{\sum_{j=1}^{n} \sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j}$$

and that the constants $\gamma_{j,q}, 1 \le j \le n, 1 \le q \le p_j$, are chosen such that $1 - \delta > \tilde{a}_b^l \tilde{\gamma}_{l,b}$. Combining this with (4.28) we obtain

$$\begin{aligned}
& m_{k,1} \dots m_{k,p} \tilde{m}_{k,b,l} \cdot k^{-\tilde{a}_b^l \eta_{b,l}} \\
& \le c k^{\sum_{j=1}^{n} \sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j + \sum_{j=l+1}^{n} \sum_{q=1}^{p_j} (\tilde{a}_b^l - \tilde{a}_q^j) \tilde{\mu}_q^j + \sum_{q=b}^{p_l} (\tilde{a}_b^l - \tilde{a}_q^l) \tilde{\mu}_q^l + (\tilde{a}_b^l - 1 + \delta) N - \tilde{a}_b^l \eta_{b,l}} \\
& \to 0
\end{aligned}$$

as $k \to \infty$, since the exponent in the last expression equals

$$\begin{aligned}
& \sum_{j=1}^{n} \sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j + \sum_{j=l+1}^{n} \sum_{q=1}^{p_j} (\tilde{a}_b^l - \tilde{a}_q^j) \tilde{\mu}_q^j + \sum_{q=b}^{p_l} (\tilde{a}_b^l - \tilde{a}_q^l) \tilde{\mu}_q^l + (\tilde{a}_b^l - 1 + \delta) N \\
& \quad - \tilde{a}_b^l \Bigg(\sum_{j=1}^{l-1} \sum_{q=1}^{p_j} \frac{\tilde{a}_q^j}{\tilde{a}_b^l} \tilde{\mu}_q^j + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{\tilde{a}_b^l} \tilde{\mu}_q^l + \sum_{j=l}^{n} d_j - \sum_{q=1}^{b} \tilde{\mu}_q^l + (1 - \tilde{\gamma}_{l,b}) N \Bigg) \\
& = \sum_{j=1}^{l-1} \sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j + \sum_{q=1}^{b-1} \tilde{a}_q^l \tilde{\mu}_q^l + \sum_{j=l+1}^{n} \sum_{q=1}^{p_j} \tilde{a}_b^l \tilde{\mu}_q^j + (\tilde{a}_b^l - 1 + \delta) N \\
& \quad + \sum_{q=b}^{p_l} \tilde{a}_b^l \tilde{\mu}_q^l - \sum_{j=1}^{l-1} \sum_{q=1}^{p_j} \tilde{a}_q^j \tilde{\mu}_q^j - \sum_{q=1}^{b} \tilde{a}_q^l \tilde{\mu}_q^l - \tilde{a}_b^l \Big(\sum_{j=l}^{n} d_j - \sum_{q=1}^{b} \tilde{\mu}_q^l + (1 - \tilde{\gamma}_{l,b}) N \Big) \\
& = -\tilde{a}_b^l \tilde{\mu}_b^l + \sum_{j=l+1}^{n} \sum_{q=1}^{p_j} \tilde{a}_b^l \tilde{\mu}_q^j + \sum_{q=b}^{p_l} \tilde{a}_b^l \tilde{\mu}_q^l + (\tilde{a}_b^l - 1 + \delta - \tilde{a}_b^l + \tilde{a}_b^l \tilde{\gamma}_{l,b}) N \\
& \quad - \tilde{a}_b^l \sum_{j=l}^{n} d_j + \tilde{a}_b^l \sum_{q=1}^{b} \tilde{\mu}_q^l \\
& = -\tilde{a}_b^l \tilde{\mu}_b^l + \tilde{a}_b^l \sum_{j=l+1}^{n} d_j + \sum_{q=b}^{p_l} \tilde{a}_b^l \tilde{\mu}_q^l + \big(-1 + \delta + \tilde{a}_b^l \tilde{\gamma}_{l,b} \big) N - \tilde{a}_b^l \sum_{j=l}^{n} d_j + \tilde{a}_b^l \sum_{q=1}^{b} \tilde{\mu}_q^l \\
& = \big(-1 + \delta + \tilde{a}_b^l \tilde{\gamma}_{l,b} \big) N < 0
\end{aligned}$$

by assumption. This shows that

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \le \eta_{b,l} \quad a.s.$$

Since this holds for any $0 < \tilde{\gamma}_{l,b} < \frac{1}{\tilde{a}_b^l}$, by letting $\tilde{\gamma}_{l,b} \uparrow \frac{1}{\tilde{a}_b^l}$ we derive that

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \le \sum_{j=1}^{l-1} \sum_{q=1}^{p_j} \frac{\tilde{a}_q^j}{\tilde{a}_b^l} \tilde{\mu}_q^j + \sum_{q=1}^{b} \frac{\tilde{a}_q^l}{\tilde{a}_b^l} \tilde{\mu}_q^l + \sum_{j=l}^{n} d_j - \sum_{q=1}^{b} \tilde{\mu}_q^l + (1 - \frac{1}{\tilde{a}_b^l}) N \quad a.s. \tag{4.29}$$

Combining (4.26) and (4.29) yields the upper bound in (4.19). □

Before proving the lower bounds in Theorem 4.13 we need several Lemmata. The following result is the statement of [6, Lemma 3.1 (a)].

Lemma 4.16. *Let X be a random vector with values in $\mathbb{R}^N$ having a continuous probability density. Then*

$$\mathbb{E}[\|X\|^{-\delta}] < \infty$$

for any $0 < \delta < N$.

The following Lemma is needed in order to determine a lower bound for $\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$ and it will also be of importance in the last chapter. Its proof can be seen as a generalization of the methods used in the proof of [9, Theorem 5.6].

Lemma 4.17. *Let $\{Y(x) : x \in \mathbb{R}^d\}$ be a random field with values in $\mathbb{R}^N$. If $\gamma > N$ there exists a positive and finite constant $C_{5,8} > 0$ such that*

$$\mathbb{E}\Big[(\|x-y\|^2 + \|Y(x) - Y(y)\|^2)^{-\frac{\gamma}{2}}\Big] \leq C_{5,8}\|x-y\|^{-\gamma} \int_{\mathbb{R}^N} \mathbb{E}\Big[\exp\big(i\langle\theta, \frac{Y(x) - Y(y)}{\|x-y\|}\rangle\big)\Big] d\theta$$

for any $x, y \in \mathbb{R}^d$.

Proof. Let us define a function $f_\gamma : \mathbb{R}^N \to \mathbb{R}$ given by $f_\gamma(\xi) = (\|\xi\|^2 + 1)^{-\frac{\gamma}{2}}$. Suppose first that $N \geq 2$. Then by using classical polar coordinates we obtain for some unspecified positive constant c

$$\int_{\mathbb{R}^N} f_\gamma(\xi) d\xi = \int_{\mathbb{R}^N} (\|\xi\|^2 + 1)^{-\frac{\gamma}{2}} d\xi \leq c \int_0^\infty (r^2 + 1)^{-\frac{\gamma}{2}} r^{N-1} dr.$$

Using the substitution $u = r^2 + 1$ we further calculate

$$\begin{aligned} \int_0^\infty (r^2 + 1)^{-\frac{\gamma}{2}} r^{N-1} dr &= c \int_1^\infty u^{-\frac{\gamma}{2}} (u-1)^{\frac{N-1}{2}} (u-1)^{-\frac{1}{2}} du \\ &\leq c \int_1^\infty u^{-\frac{\gamma}{2} + \frac{N-2}{2}} du < \infty, \end{aligned} \tag{4.30}$$

since $-\gamma < -N$ by assumption. If $N = 1$ (4.30) is proven analogously. Let $\hat{f}_\gamma$ be the Fourier transform of f_γ. Then using (4.30) we obtain

$$\begin{aligned} |\hat{f}_\gamma(\xi)| &= |\int_{\mathbb{R}^N} e^{i\langle\xi, y\rangle} f_\gamma(y) dy| \leq \int_{\mathbb{R}^N} |e^{i\langle\xi, y\rangle} f_\gamma(y)| dy \\ &= \int_{\mathbb{R}^N} f_\gamma(y) dy < \infty \end{aligned}$$

for any $\xi \in \mathbb{R}^N$, i.e. $\hat{f}_\gamma(\xi)$ is essentially bounded and one can find a constant c independent of ξ such that

$$|\hat{f}_\gamma(\xi)| \leq c. \tag{4.31}$$

Using Fourier inversion (see also [7, Lemma 4.1]) we can write

$$f_\gamma(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\langle \xi, y\rangle} \hat{f}_\gamma(y) dy$$

for any $\xi \in \mathbb{R}^N$ so that we can calculate

$$\begin{aligned}
&\mathbb{E}\Big[\big(\|x-y\|^2 + \|Y(x)-Y(y)\|^2\big)^{-\frac{\gamma}{2}}\Big] = \|x-y\|^{-\gamma}\mathbb{E}\Big[\Big(1+\big(\frac{\|Y(x)-Y(y)\|}{\|x-y\|}\big)^2\Big)^{-\frac{\gamma}{2}}\Big] \\
&= \|x-y\|^{-\gamma}\mathbb{E}\Big[f_\gamma\Big(\frac{\|Y(x)-Y(y)\|}{\|x-y\|}\Big)\Big] \\
&= c\|x-y\|^{-\gamma}\mathbb{E}\Big[\int_{\mathbb{R}^N} e^{i\langle \xi, \frac{\|Y(x)-Y(y)\|}{\|x-y\|}\rangle} \hat{f}_\gamma(\xi) d\xi\Big].
\end{aligned}$$

Note that the integral in the last expression is of product structure so that Fubini's theorem applies. Combining this with (4.22) the last expression becomes

$$\begin{aligned}
&c\|x-y\|^{-\gamma} \int_{\mathbb{R}^N} \mathbb{E}\Big[e^{i\langle \xi, \frac{\|Y(x)-Y(y)\|}{\|x-y\|}\rangle}\Big] \hat{f}_\gamma(\xi) d\xi \\
&\leq C_{5,8}\|x-y\|^{-\gamma} \int_{\mathbb{R}^N} \mathbb{E}\Big[e^{i\langle \xi, \frac{\|Y(x)-Y(y)\|}{\|x-y\|}\rangle}\Big] d\xi
\end{aligned}$$

for some suitable $C_{5,8} > 0$. □

In the following for any $x, y \in \mathbb{R}^d$ let

$$\begin{aligned}
\sigma(x,y) &= \|X_\alpha(x) - X_\alpha(y)\|_\alpha \\
&= \Big[\int_{\mathbb{R}^d} \Big|\prod_{j=1}^n (e^{i\langle x_j, \xi_j\rangle} - 1) - \prod_{j=1}^n (e^{i\langle y_j, \xi_j\rangle} - 1)\Big|^\alpha \prod_{j=1}^n |\psi_j(\xi_j)|^{-\alpha - q_j} d\xi\Big]^{\frac{1}{\alpha}}
\end{aligned}$$

be the scale parameter of the 1-dimensional stable random variable $X_\alpha(x) - X_\alpha(y)$ according to Corollary 3.6. The following Theorem is crucial for proving the lower bounds in Theorem 4.13. Its proof is based on [49, Theorem 1] and also on [50, 51, 52]. Let us remark that a similar method of the following proof has been applied in [50, Theorem 3.4] for certain α-stable random fields if $1 \leq \alpha \leq 2$. In the following we are able to extend this method for $0 < \alpha < 1$ and, in particular this shows that the statement of [50, Theorem 3.5] can be formulated for $0 < \alpha < 1$ as well.

Theorem 4.18. *There exists a constant $C_{5,9} > 0$, depending on $q_1, \ldots, q_n$ and d only such that for all $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots y_n) \in [\frac{1}{2}, 1)^{d_1} \times \ldots \times [\frac{1}{2}, 1)^{d_n}$ we have*

$$\sigma(x,y) \geq C_{5,9} \sum_{j=1}^n \tau_{E_j}(x_j - y_j),$$

where $\tau_{E_j}(\cdot)$ is the radial part with respect to $E_j, 1 \leq j \leq n$.

Proof. Throughout this proof, we fix $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots y_n) \in [\frac{1}{2}, 1)^{d_1} \times \ldots \times [\frac{1}{2}, 1)^{d_n}$ and an unspecified positive constant c independent of x and y. We will show that for any

$1 \leq j \leq n$

$$\sigma(x,y) \geq c_j \tau_{E_j}(x_j - y_j), \tag{4.32}$$

for some $c_j > 0$, since this implies that

$$\sigma(x,y) \geq C_{5,9} \sum_{j=1}^{n} \tau_{E_j}(x_j - y_j)$$

with $C_{5,9} = \frac{\min_{1 \leq j \leq n} c_j}{n}$. Without loss of generality assume that $j = 1$ and let $r = \tau_{E_1}(x_1 - y_1)$. Note that for $r = 0$ (4.32) always holds, thus in the following we will assume that $r > 0$. For every $1 \leq j \leq n$ we consider a so-called bump function $\delta_j \in C^\infty(\mathbb{R}^{d_j})$ with values in $[0,1]$ such that $\delta_j(0) = 1$ and δ_j vanishes outside the open ball

$$B(K_j, 0) = \{z \in \mathbb{R}^{d_j} : \tau_{E_j}(z) < K_j\}$$

for

$$\begin{aligned} K_j = \min\Big\{1, \frac{K_1^j}{K_2^j}\big(\sqrt{d_j}\frac{1}{2}\big)^{\frac{1}{a_1^j} - \frac{1}{a_{p_j}^j} + 2\varepsilon}, \frac{K_3^j}{K_4^j}\big(\sqrt{d_j}\frac{1}{2}\big)^{\frac{1}{a_{p_j}^j} - \frac{1}{a_1^j} - 2\varepsilon}, \frac{K_1^j}{K_4^j}, \frac{K_3^j}{K_2^j}, \\ K_1^j\big(\sqrt{d_j}\frac{1}{2}\big)^{\frac{1}{a_1^j} + \varepsilon}, K_3^j\big(\sqrt{d_j}\frac{1}{2}\big)^{\frac{1}{a_{p_j}^j} - \varepsilon}\Big\}, \end{aligned}$$

where $\varepsilon > 0$ is some (sufficiently) small number and $K_1^j, \ldots, K_4^j$ are the suitable constants $C_{3,1}, \ldots, C_{3,4}$ derived from Lemma 2.5 corresponding to the matrix E_j. The choice of the positive constant K_j will be clear later in this proof. Let $\hat{\delta}_j$ be the Fourier transform of δ_j. Then by the Riemann-Lebesgue Lemma (see, e.g., [26, Chapter 1 Theorem 4.1])

$$|\hat{\delta}_j(\xi_j)| \to 0$$

as $\|\xi_j\| \to \infty$ and $\hat{\delta}_j \in C(\mathbb{R}^{d_j})$. Moreover, the bump functions δ_j can be chosen such that the integral

$$c = \int_{\mathbb{R}^d} \Big(\prod_{j=1}^{n} |\psi_j(\xi_j)|^{-1-\frac{q_j}{\alpha}}\Big)^{-\frac{\beta}{k}} \prod_{j=1}^{n} |\hat{\delta}_j(\xi_j)|^{\beta} d\xi \tag{4.33}$$

is a positive and finite constant for any $\beta > 0$ and $k \in \mathbb{N}$ (see the proof of [49, Theorem 1]). By the Fourier inversion formula we can write

$$\delta_j(s_j) = \frac{1}{(2\pi)^{d_j}} \int_{\mathbb{R}^{d_j}} e^{-i\langle s_j, \lambda_j\rangle} \hat{\delta}_j(\lambda_j) d\lambda_j \tag{4.34}$$

for all $s_j \in \mathbb{R}^{d_j}$. Let $\delta_1^r(s_1) = \frac{1}{r^{q_1}}\delta_1\big((\frac{1}{r})^{E_1} s_1\big)$. Then by using the change of variables $\xi_1 =$

$(\frac{1}{r})^{E_1^T}\lambda_1$ and the fact that $d\xi_1 = \frac{1}{r^{q_1}} d\lambda_1$ we obtain from (4.34)

$$\begin{aligned}
\delta_1^r(s_1) &= \frac{1}{r^{q_1}} \frac{1}{(2\pi)^{d_1}} \int_{\mathbb{R}^{d_1}} e^{-i\langle (\frac{1}{r})^{E_1} s_1, \lambda_1\rangle} \hat{\delta}_1(\lambda_1) d\lambda_1 \\
&= \frac{1}{r^{q_1}} \frac{1}{(2\pi)^{d_1}} \int_{\mathbb{R}^{d_1}} e^{-i\langle s_1, (\frac{1}{r})^{E_1^T}\lambda_1\rangle} \hat{\delta}_1(\lambda_1) d\lambda_1 \\
&= \frac{1}{(2\pi)^{d_1}} \int_{\mathbb{R}^{d_1}} e^{-i\langle s_1, \xi_1\rangle} \hat{\delta}_1(r^{E_1^T}\xi_1) d\xi_1.
\end{aligned} \tag{4.35}$$

By Lemma 2.5 we have

$$\begin{aligned}
\tau_{E_j}(x_j) &\geq \min\Big\{K_1^j \|x_j\|^{\frac{1}{a_1^j}+\varepsilon}, K_3^j \|x_j\|^{\frac{1}{a_{p_j}^j}-\varepsilon}\Big\} \\
&\geq \min\Big\{K_1^j (\sqrt{d_j}\frac{1}{2})^{\frac{1}{a_1^j}+\varepsilon}, K_3^j (\sqrt{d_j}\frac{1}{2})^{\frac{1}{a_{p_j}^j}-\varepsilon}\Big\} \geq K_j,
\end{aligned} \tag{4.36}$$

where the second inequality follows from $x_j \in [\frac{1}{2}, 1)^{d_j}$. Furthermore, using the fact that $\tau_{E_1}(\cdot)$ is E_1-homogeneous (see Chapter 2.3) we have

$$\tau_{E_1}((\frac{1}{r})^{E_1^T}(x_1 - y_1)) = \frac{\tau_{E_1}(x_1 - y_1)}{r} = 1 \geq K_j, \tag{4.37}$$

and by Lemma 2.5 we obtain

$$\begin{aligned}
\tau_{E_1}((\frac{1}{r})^{E_1^T} x_1) &= \frac{1}{r}\tau_{E_1}(x_1) \\
&\geq \min\left\{ \frac{K_1^1}{K_4^1} \frac{\|x_1\|^{\frac{1}{a_1^1}+\varepsilon}}{\|x_1 - y_1\|^{\frac{1}{a_1^1}+\varepsilon}}, \frac{K_1^1}{K_2^1} \frac{\|x_1\|^{\frac{1}{a_1^1}+\varepsilon}}{\|x_1 - y_1\|^{\frac{1}{a_{p_1}^1}-\varepsilon}}, \frac{K_3^1}{K_4^1} \frac{\|x_1\|^{\frac{1}{a_{p_1}^1}-\varepsilon}}{\|x_1 - y_1\|^{\frac{1}{a_1^1}+\varepsilon}}, \right. \\
&\qquad \left. \frac{K_3^1}{K_2^1} \frac{\|x_1\|^{\frac{1}{a_{p_1}^1}-\varepsilon}}{\|x_1 - y_1\|^{\frac{1}{a_{p_1}^1}-\varepsilon}} \right\} \\
&\geq \min\left\{ \frac{K_1^1}{K_4^1}, \frac{K_1^1}{K_2^1}(\sqrt{d_1}\frac{1}{2})^{\frac{1}{a_1^1}-\frac{1}{a_{p_1}^1}+2\varepsilon}, \frac{K_3^1}{K_4^1}(\sqrt{d_1}\frac{1}{2})^{\frac{1}{a_{p_1}^1}-\frac{1}{a_1^1}-2\varepsilon}, \frac{K_3^1}{K_2^1} \right\} \\
&\geq K_1,
\end{aligned} \tag{4.38}$$

where we used that $x_1, y_1 \in [\frac{1}{2}, 1)^{d_1}$ in the second inequality. (4.36), (4.37) and (4.38) imply that $\delta_j(x_j) = 0, 1 \leq j \leq n, \delta_1^r(x_1 - y_1) = 0$ and $\delta_1^r(x_1) = 0$. Hence, combining this with (4.34) and (4.35) it follows that

$$\begin{aligned}
I :=& \int_{\mathbb{R}^d} \Big(\prod_{j=1}^n (e^{i\langle x_j,\lambda_j\rangle} - 1) - \prod_{j=1}^n (e^{i\langle y_j,\lambda_j\rangle} - 1)\Big) \\
&\times \prod_{j=1}^n e^{-i\langle x_j,\lambda_j\rangle} \hat{\delta}_1(r^{E_1^T}\lambda_1) \prod_{j=2}^n \hat{\delta}_j(\lambda_j) d\lambda \\
=& \int_{\mathbb{R}^{d_1}} (e^{i\langle x_1,\lambda_1\rangle} - 1) e^{-i\langle x_1,\lambda_1\rangle} \hat{\delta}_1(r^{E_1^T}\lambda_1) d\lambda_1 \\
&\times \prod_{j=2}^n \int_{\mathbb{R}^{d_j}} (e^{i\langle x_j,\lambda_j\rangle} - 1) e^{-i\langle x_j,\lambda_j\rangle} \hat{\delta}_j(\lambda_j) d\lambda_j \\
&- \Bigg(\int_{\mathbb{R}^{d_1}} (e^{i\langle y_1,\lambda_1\rangle} - 1) e^{-i\langle x_1,\lambda_1\rangle} \hat{\delta}_1(r^{E_1^T}\lambda_1) d\lambda_1 \\
&\times \prod_{j=2}^n \int_{\mathbb{R}^{d_j}} (e^{i\langle y_j,\lambda_j\rangle} - 1) e^{-i\langle x_j,\lambda_j\rangle} \hat{\delta}_j(\lambda_j) d\lambda_j \Bigg) \\
=& (2\pi)^{d_1} \ldots (2\pi)^{d_n} \Big(\delta_1^r(0) - \delta_1^r(x_1) \Big) \prod_{j=2}^n \Big(\delta_j(0) - \delta_j(x_j) \Big) \\
&- (2\pi)^{d_1} \ldots (2\pi)^{d_n} \Big(\delta_1^r(x_1 - y_1) - \delta_1^r(x_1) \Big) \prod_{j=2}^n \Big(\delta_j(x_j - y_j) - \delta_j(x_j) \Big) \\
=& (2\pi)^d \frac{1}{r^{q_1}} \delta_1(0) = (2\pi)^d \frac{1}{r^{q_1}}.
\end{aligned} \tag{4.39}$$

Let us choose $k \in \mathbb{N}$ such that $k\alpha \geq 1$. We now show that

$$\begin{aligned}
&\Bigg(\int_{\mathbb{R}^d} \Big| \prod_{j=1}^n (e^{i\langle x_j,\lambda_j\rangle} - 1) - \prod_{j=1}^n (e^{i\langle y_j,\lambda_j\rangle} - 1) \Big|^{k\alpha} \prod_{j=1}^n |\psi_j(\lambda_j)|^{-\alpha-q_j} d\lambda \Bigg)^{\frac{1}{k\alpha}} \\
&\qquad \leq 2^{n+1} \sigma(x,y)^{\frac{1}{k}}.
\end{aligned} \tag{4.40}$$

Note that for $\alpha \geq 1$ we have $k = 1$ and (4.40) is trivially true. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_n}$ let

$$z(\lambda) = \prod_{j=1}^n (e^{i\langle x_j,\lambda_j\rangle} - 1) - \prod_{j=1}^n (e^{i\langle y_j,\lambda_j\rangle} - 1)$$

and note that since $|e^{it} - 1|^2 = 2 - 2\cos t \leq 4$ for all $t \in \mathbb{R}$, it follows that

$$|z(\lambda)| \leq \prod_{j=1}^n |e^{i\langle x_j,\lambda_j\rangle} - 1| + \prod_{j=1}^n |e^{i\langle y_j,\lambda_j\rangle} - 1| \leq 2 \cdot 2^n = 2^{n+1}.$$

From this we obtain

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^d} |z(\lambda)|^{k\alpha} \prod_{j=1}^n |\psi_j(\lambda_j)|^{-\alpha H_j - q_j} d\lambda \Big)^{\frac{1}{k\alpha}} \\
&= \Big(\int_{\{\lambda\in\mathbb{R}^d : |z(\lambda)| \leq 1\}} |z(\lambda)|^{k\alpha} \prod_{j=1}^n |\psi_j(\lambda_j)|^{-\alpha-q_j} d\lambda
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\{\lambda\in\mathbb{R}^d:|z(\lambda)|>1\}}|z(\lambda)|^{k\alpha}\prod_{j=1}^n|\psi_j(\lambda_j)|^{-\alpha-q_j}d\lambda\Big)^{\frac{1}{k\alpha}}\\
&\leq\Big(\int_{\{\lambda\in\mathbb{R}^d:|z(\lambda)|\leq 1\}}|z(\lambda)|^{\alpha}\prod_{j=1}^n|\psi_j(\lambda_j)|^{-\alpha-q_j}d\lambda\\
&+\int_{\{\lambda\in\mathbb{R}^d:|z(\lambda)|>1\}}|z(\lambda)|^{k\alpha+\alpha}\prod_{j=1}^n|\psi_j(\lambda_j)|^{-\alpha-q_j}d\lambda\Big)^{\frac{1}{k\alpha}}\\
&\leq 2^{n+1}\Big(\int_{\mathbb{R}^d}|z(\lambda)|^{\alpha}\prod_{j=1}^n|\psi_j(\lambda_j)|^{-\alpha-q_j}d\lambda\Big)^{\frac{1}{k\alpha}}\\
&=2^{n+1}\sigma(x,y)^{\frac{1}{k}}.
\end{aligned}
$$

Let $\beta>1$ be the constant such that $\frac{1}{k\alpha}+\frac{1}{\beta}=1$. Recall from (4.39) that $I=|I|\in\mathbb{R}$. Then by Hölder's inequality and (4.40)

$$
\begin{aligned}
I=&\int_{\mathbb{R}^d}\Big(\prod_{j=1}^n(e^{i\langle x_j,\lambda_j\rangle}-1)-\prod_{j=1}^n(e^{i\langle y_j,\lambda_j\rangle}-1)\Big)\\
&\times\prod_{j=1}^n e^{-i\langle x_j,\lambda_j\rangle}\hat{\delta}_1(r^{E_1^T}\lambda_1)\prod_{j=2}^n\hat{\delta}_j(\lambda_j)d\lambda\\
\leq&\left(\int_{\mathbb{R}^d}\Big|\prod_{j=1}^n(e^{i\langle x_j,\lambda_j\rangle}-1)-\prod_{j=1}^n(e^{i\langle y_j,\lambda_j\rangle}-1)\Big|^{k\alpha}\prod_{j=1}^n|\psi_j(\lambda_j)|^{-\alpha-q_j}d\lambda\right)^{\frac{1}{k\alpha}}\\
&\times\Big(\int_{\mathbb{R}^d}\frac{1}{\big(\prod_{j=1}^n|\psi_j(\lambda_j)|^{-\alpha-q_j}\big)^{\frac{\beta}{k\alpha}}}|\hat{\delta}_1(r^{E_1^T}\lambda_1)\prod_{j=2}^n\hat{\delta}_j(\lambda_j)|^{\beta}d\lambda\Big)^{\frac{1}{\beta}}\\
\leq&\, 2^{n+1}\sigma(x,y)^{\frac{1}{k}}\Big(\int_{\mathbb{R}^d}\frac{1}{\big(\prod_{j=1}^n|\psi_j(\lambda_j)|^{-\alpha-q_j}\big)^{\frac{\beta}{k\alpha}}}|\hat{\delta}_1(r^{E_1^T}\lambda_1)\prod_{j=2}^n\hat{\delta}_j(\lambda_j)|^{\beta}d\lambda\Big)^{\frac{1}{\beta}}.
\end{aligned}
$$

Using the change of variables $\xi_1=r^{E_1^T}\lambda_1, d\xi_1=r^{q_1}d\lambda_1$ and that ψ_1 is E_1^T-homogeneous the last expression becomes

$$
\begin{aligned}
&2^{n+1}\sigma(x,y)^{\frac{1}{k}}\cdot r^{-\frac{1}{k}-\frac{q_1}{k\alpha}-\frac{q_1}{\beta}}\Big(\int_{\mathbb{R}^d}\big(\prod_{j=1}^n|\psi_j(\xi_j)|^{-1-\frac{q_j}{\alpha}}\big)^{-\frac{\beta}{k}}\prod_{j=1}^n|\hat{\delta}_j(\xi_j)|^{\beta}d\xi\Big)^{\frac{1}{\beta}}\\
&=c\sigma(x,y)^{\frac{1}{k}}\cdot r^{-\frac{1}{k}-q_1}=c\big(\sigma(x,y)\cdot r^{-1-kq_1}\big)^{\frac{1}{k}},
\end{aligned}
\tag{4.41}
$$

where we used that $\frac{1}{k\alpha}+\frac{1}{\beta}=1$ and (4.33) in the first equality. Overall combining (4.39) with (4.41) yields

$$(2\pi)^d r^{-q_1}\leq c\big(\sigma(x,y)\cdot r^{-1-kq_1}\big)^{\frac{1}{k}},$$

which is equivalent to (4.32) for some suitable constant c_1. This finishes the proof of the Theorem. □

Proposition 4.19. *For $\gamma>0$ let*

$$\mathcal{E}_\gamma=\int_{[\frac{1}{2},1]^{2d_1}}\cdots\int_{[\frac{1}{2},1]^{2d_n}}\Big(\sum_{j=1}^n\tau_{E_j}(x_j-y_j)\Big)^{-\gamma}dxdy.$$

Then $\mathcal{E}_\gamma$ is finite for any $0<\gamma\le\min\{N,\sum_{j=1}^n\sum_{k=1}^{p_j}\frac{a_k^j}{1+\varepsilon}\mu_k^j\}$, where $\varepsilon>0$ is a sufficiently small number.

Proof. Throughout this proof, let c and c' be two unspecified positive constants. Let us first observe that for any $0<\gamma<\min\{N,\sum_{j=1}^n\sum_{k=1}^{p_j}\frac{a_k^j}{1+\varepsilon}\mu_k^j\}$ there exist integers $1\le l\le n, 1\le b\le p_l$ such that

$$\sum_{j=l}^{n}\sum_{k=b+1}^{p_j}\frac{a_k^j}{1+\varepsilon}\mu_k^j<\gamma\le\sum_{j=l}^{n}\sum_{k=b}^{p_j}\frac{a_k^j}{1+\varepsilon}\mu_k^j, \tag{4.42}$$

where

$$\sum_{j=l}^{n}\sum_{k=p_j+1}^{p_j}\frac{a_k^j}{1+\varepsilon}\mu_k^j:=\sum_{j=l+1}^{n}\sum_{k=1}^{p_j}\frac{a_k^j}{1+\varepsilon}\mu_k^j$$

with the convention that $\sum_{j=n+1}^n\sum_{k=1}^{p_j}\frac{a_k^j}{1+\varepsilon}\mu_k^j=0$. In the following without loss of generality we will only consider the case $l=1$ and $b=1$, since the remaining cases are simpler because they require less steps of integration using Lemma 4.7. Thus, assuming (4.42) with $l=b=1$ we choose positive constants $\delta_1^1,\dots,\delta_{p_1}^1,\delta_1^2,\dots,\delta_{p_2}^2,\dots,\delta_n^1,\dots,\delta_{p_n}^n$ such that $\delta_k^j>\frac{1+\varepsilon}{a_k^j}$ for each $1\le j\le n, 1\le k\le p_j$ and

$$\sum_{k=2}^{p_1}\frac{\mu_k^1}{\delta_k^1}+\sum_{j=2}^{n}\sum_{k=1}^{p_j}\frac{\mu_k^j}{\delta_k^j}<\gamma<\frac{a_1^1}{1+\varepsilon}\mu_1^1+\sum_{k=2}^{p_1}\frac{\mu_k^1}{\delta_k^1}+\sum_{j=2}^{n}\sum_{k=1}^{p_j}\frac{\mu_k^j}{\delta_k^j}. \tag{4.43}$$

For any vector $x_j\in\mathbb{R}^{d_j}, 1\le j\le n$, let

$$x_j=x_j^1+\ldots+x_j^{p_j}$$

be its direct sum decomposition with respect to E_j. Recall that $\mathbb{R}^{d_j}=W_1^j\oplus\ldots\oplus W_{p_j}^j$. Note that

$$\mathcal{E}_\gamma\le c\int_{\|x_1\|\le 2}\cdots\int_{\|x_n\|\le 2}\Big(\sum_{j=1}^{n}\tau_{E_j}(x_j)\Big)^{-\gamma}dx.$$

Since the $W_i^j(1\le i\le p_j)$ are orthogonal in the associated euclidean norm, it follows that $\|x_j\|\le 2$ implies $\|x_j^i\|\le 2$ for $i=1,\dots,p_j$. Then Remark 2.10 yields

$$\mathcal{E}_\gamma\le c\int_{\substack{\|x_j^k\|\le 2\\ j=1,\dots,n\\ k=1,\dots,p_j}}\Big(\sum_{j=1}^{n}\sum_{k=1}^{p_j}\|x_j^k\|^{\frac{1+\varepsilon}{a_k^j}}\Big)^{-\gamma}dx_j^k.$$

By using the change to (classical) polar coordinates we further get

$$\mathcal{E}_\gamma\le c\int_{\substack{r_k^j\in(0,2)\\ j=1,\dots,n\\ k=1,\dots,p_j}}\Big(\sum_{j=1}^{n}\sum_{k=1}^{p_j}(r_k^j)^{\frac{1+\varepsilon}{a_k^j}}\Big)^{-\gamma}(r_k^j)^{\mu_k^j-1}dr_k^j.$$

Applying Lemma 4.7 with

$$A=\sum_{j=1}^{n-1}\sum_{k=1}^{p_j}(r_k^j)^{\frac{1+\varepsilon}{a_k^j}}+\sum_{k=1}^{p_n-1}(r_k^n)^{\frac{1+\varepsilon}{a_k^n}},\quad h=\frac{1+\varepsilon}{a_{p_n}^n},\quad k=\mu_{p_n}^n,\quad u=\gamma\quad\text{and}\quad\delta=\delta_{p_n}^n$$

we integrate with respect to $dr_{p_n}^n$ in the last expression and obtain that

$$\mathcal{E}_\gamma\le c'+c\int_{\substack{r_k^j\in(0,2)\\(j,k)\neq(n,p_n)}}\Big(\sum_{j=1}^{n-1}\sum_{k=1}^{p_j}(r_k^j)^{\frac{1+\varepsilon}{a_k^j}}+\sum_{k=1}^{p_n-1}(r_k^n)^{\frac{1+\varepsilon}{a_k^n}}\Big)^{-\gamma+\frac{\mu_{p_n}^n}{\delta_{p_n}^n}}\prod_{\substack{j=1,\dots,n\\k=1,\dots,p_j\\(j,k)\neq(n,p_n)}}(r_k^j)^{\mu_k^j-1}dr_k^j.$$

By repeating this procedure, i.e. by repeatedly using Lemma 4.7 to the integral in the last expression, integrating with respect to $dr_{p_n-1}^n,\dots,dr_1^n,\dots,dr_{p_2}^2,\dots,dr_1^2,dr_{p_1}^1,\dots,dr_2^1$ we derive that

$$\mathcal{E}_\gamma\le c'+c\int_0^2\Big((r_1^1)^{\frac{1+\varepsilon}{a_1^1}}\Big)^{-\gamma+(\sum_{k=2}^{p_1}\frac{\mu_k^1}{\delta_k^1}+\sum_{j=2}^{n}\sum_{k=1}^{p_j}\frac{\mu_k^j}{\delta_k^j})}\cdot(r_1^1)^{\mu_1^1-1}dr_1^1. \tag{4.44}$$

Note that from (4.43) we get

$$\frac{1+\varepsilon}{a_1^1}\cdot\Big(-\gamma+\big(\sum_{k=2}^{p_1}\frac{\mu_k^1}{\delta_k^1}+\sum_{j=2}^{n}\sum_{k=1}^{p_j}\frac{\mu_k^j}{\delta_k^j}\big)\Big)+\mu_1^1-1>-\frac{1+\varepsilon}{a_1^1}\frac{a_1^1}{1+\varepsilon}\mu_1^1+\mu_1^1-1=-1.$$

Thus, the integral on the right-hand side of (4.44) is finite. This proves the assertion. □

Proposition 4.20. *Let $1\le l\le n$ and $1\le b\le p_l$ be two integers such that*

$$\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\tilde{a}_k^j\tilde{\mu}_k^j+\sum_{k=1}^{b-1}\tilde{a}_k^l\tilde{\mu}_k^l\le N<\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\tilde{a}_k^j\tilde{\mu}_k^j+\sum_{k=1}^{b}\tilde{a}_k^l\tilde{\mu}_k^l.$$

For $\gamma>N$ define

$$\mathcal{G}_\gamma=\int_{[\frac{1}{2},1]^d\times[\frac{1}{2},1]^d}\|x-y\|^{N-\gamma}\Big(\sum_{j=1}^{n}\tau_{E_j}(x_j-y_j)\Big)^{-N}dxdy.$$

Then if $\varepsilon>0$ is sufficiently small $\mathcal{G}_\gamma$ is finite for any

$$N<\gamma\le\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{a}_k^j}{\tilde{a}_b^l}\tilde{\mu}_k^j+\sum_{k=1}^{b}\frac{\tilde{a}_k^l}{\tilde{a}_b^l}\tilde{\mu}_k^l+\sum_{j=l}^{n}d_j-\sum_{k=1}^{b}\tilde{\mu}_k^l+(1-\frac{1+\varepsilon}{\tilde{a}_b^l})N.$$

Proof. To simplify notation let $g_k^j=\frac{1+\varepsilon}{\tilde{a}_k^j},1\le j\le n,1\le k\le p_j$, and let us choose $\varepsilon>0$ small enough such that

$$\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{\mu}_k^j}{g_k^j}+\sum_{k=1}^{b-1}\frac{\tilde{\mu}_k^l}{g_k^l}<N<\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{\mu}_k^j}{g_k^j}+\sum_{k=1}^{b}\frac{\tilde{\mu}_k^l}{g_k^l}. \tag{4.45}$$

Assume that

$$N < \gamma < \sum_{j=1}^{l-1} \sum_{k=1}^{p_j} \frac{g_b^l}{g_k^j} \tilde{\mu}_k^j + \sum_{k=1}^{b} \frac{g_b^l}{g_k^l} \tilde{\mu}_k^l + \sum_{j=l}^{n} d_j - \sum_{k=1}^{b} \tilde{\mu}_k^l + (1 - g_b^l)N. \tag{4.46}$$

Let c and c' be two unspecified positive constants. Furthermore, let

$$x_j = x_j^1 + \ldots + x_j^{p_j}$$

be the direct sum decomposition with respect to E_j for any vector $x_j \in \mathbb{R}^{d_j}, 1 \leq j \leq n$. Without loss of generality let $\|\cdot\| = \|\cdot\|_2$ be the 2-norm so that by the equivalence of norms we have

$$\begin{aligned} \|x\|_2 = \|(x_1, \ldots, x_n)\|_2 &= \sqrt{\sum_{j=1}^{n} \|x_j\|_2^2} \leq c \sum_{j=1}^{n} \|x_j\|_2 \\ &= c \sum_{j=1}^{n} \sqrt{\sum_{k=1}^{p_j} \|x_j^k\|_2^2} \leq c \sum_{j=1}^{n} \sum_{k=1}^{p_j} \|x_j^k\|_2, \end{aligned}$$

where we used that the spectral components W_i^j, $1 \leq i \leq p_j$, are orthogonal in the associated euclidean norm in the last equality. Using this we obtain

$$\begin{aligned} \mathcal{G}_\gamma &\leq c \int_{\|x\| \leq 2} \|x\|^{N-\gamma} \Big(\sum_{j=1}^{n} \tau_{E_j}(x_j) \Big)^{-N} dx \\ &\leq c \int_{\substack{\|x_j^k\| \leq 2 \\ j=1,\ldots,n \\ k=1,\ldots,p_j}} \Big(\sum_{j=1}^{n} \sum_{k=1}^{p_j} \|x_j^k\| \Big)^{N-\gamma} \Big(\sum_{j=1}^{n} \sum_{k=1}^{p_j} \|x_j^k\|^{g_k^j} \Big)^{-N} dx_j^k, \end{aligned}$$

where the last inequality follows from Remark 2.10 and the fact that $\|x\| \leq 2$ implies $\|x_j^k\| \leq 2$ for $1 \leq j \leq n, 1 \leq k \leq p_j$. By using the change to (classical) polar coordinates we can further estimate

$$\mathcal{G}_\gamma \leq c \int_{\substack{r_k^j \in (0,2) \\ j=1,\ldots,n \\ k=1,\ldots,p_j}} \Big(\sum_{j=1}^{n} \sum_{k=1}^{p_j} r_k^j \Big)^{N-\gamma} \Big(\sum_{j=1}^{n} \sum_{k=1}^{p_j} (r_k^j)^{g_k^j} \Big)^{-N} \prod_{\substack{j=1,\ldots,n \\ k=1,\ldots,p_j}} (r_k^j)^{\tilde{\mu}_k^j - 1} dr_k^j. \tag{4.47}$$

In order to show that the integral in (4.47) is finite we will integrate $dr_1^1, \ldots, dr_b^l$ iteratively. Furthermore, we will assume that $l \neq 1$ and $b \neq 1$ in (4.45), since for $l = b = 1$ one can use (4.8) of Lemma 4.8 to obtain (4.49) directly. Indeed, if $l = b = 1$ (4.45) gives

$$g_1^1 N < \tilde{\mu}_1^1$$

so that by (4.8) of Lemma 4.8 with

$$B = \sum_{k=2}^{p_1} r_k^1 + \sum_{j=2}^{n} \sum_{k=1}^{p_j} r_k^j, \quad \alpha = g_1^1, \quad \beta = N, \quad k = \tilde{\mu}_1^1 \quad \text{and} \quad \eta = \gamma - N$$

we obtain

$$\mathcal{G}_\gamma \leq c \int_{\substack{r_k^j \in (0,2) \\ (j,k)=(1,2),\ldots,(1,p_1) \\ j=2,\ldots,n \\ k=1,\ldots,p_j}} \Big(\sum_{k=2}^{p_1} r_k^1 + \sum_{j=2}^{n} \sum_{k=1}^{p_j} r_k^j \Big)^{N-\gamma-g_1^1 N+\tilde{\mu}_1^1} \cdot \prod_{k=2}^{p_1} (r_k^1)^{\tilde{\mu}_k^1 - 1} \prod_{\substack{j=2,\ldots,n \\ k=1,\ldots,p_j}} (r_k^j)^{\tilde{\mu}_k^j - 1} dr_k^j,$$

which is (4.49) below with $l = b = 1$. So in the following assume that $l \neq 1$ and $b \neq 1$ in (4.45). We first integrate with respect to dr_1^1. Since $l \neq 1$ and $b \neq 1$, by (4.45) we have

$$N > \frac{\tilde{\mu}_1^1}{g_1^1},$$

which is equivalent to

$$N g_1^1 > \tilde{\mu}_1^1.$$

Thus, we can use (4.6) of Lemma 4.8 with

$$A = \sum_{k=2}^{p_1} (r_k^1)^{g_k^1} + \sum_{j=2}^{n} \sum_{k=1}^{p_j} (r_k^j)^{g_k^j}, \quad B = \sum_{k=2}^{p_1} r_k^1 + \sum_{j=2}^{n} \sum_{k=1}^{p_j} r_k^j, \quad \alpha = g_1^1, \quad \beta = N, \quad k = \tilde{\mu}_1^1$$

and $\eta = \gamma - N$ to obtain that

$$\mathcal{G}_\gamma \leq c \int_{\substack{r_k^j \in (0,2) \\ (j,k) \neq (1,1)}} \Big(\sum_{k=2}^{p_1} r_k^1 + \sum_{j=2}^{n} \sum_{k=1}^{p_j} r_k^j \Big)^{N-\gamma} \Big(\sum_{k=2}^{p_1} (r_k^1)^{g_k^1} + \sum_{j=2}^{n} \sum_{k=1}^{p_j} (r_k^j)^{g_k^j} \Big)^{-N+\frac{\tilde{\mu}_1^1}{g_1^1}} \cdot \prod_{k=2}^{p_1} (r_k^1)^{\tilde{\mu}_k^1 - 1} \prod_{\substack{j=2,\ldots,n \\ k=1,\ldots,p_j}} (r_k^j)^{\tilde{\mu}_k^j - 1} dr_k^j.$$

Note that by (4.45) we can repeat this procedure for integration with respect to $dr_2^1, \ldots, dr_{b-1}^l$ and obtain

$$\mathcal{G}_\gamma \leq c \int_{\substack{r_k^j \in (0,2) \\ (j,k)=(l,b),\ldots,(l,p_l) \\ j=l+1,\ldots,n \\ k=1,\ldots,p_j}} \Big(\sum_{k=b}^{p_l} r_k^l + \sum_{j=l+1}^{n} \sum_{k=1}^{p_j} r_k^j \Big)^{N-\gamma} \cdot \Big(\sum_{k=b}^{p_l} (r_k^l)^{g_k^l} + \sum_{j=l+1}^{n} \sum_{k=1}^{p_j} (r_k^j)^{g_k^j} \Big)^{\xi} \prod_{k=b}^{p_l} (r_k^l)^{\tilde{\mu}_k^l - 1} \prod_{\substack{j=l+1,\ldots,n \\ k=1,\ldots,p_j}} (r_k^j)^{\tilde{\mu}_k^j - 1} dr_k^j \tag{4.48}$$

with

$$\xi = -N + \sum_{j=1}^{l-1} \sum_{k=1}^{p_j} \frac{\tilde{\mu}_k^j}{g_k^j} + \sum_{k=1}^{b-1} \frac{\tilde{\mu}_k^l}{g_k^l}.$$

Note that by (4.45) we now have

$$N < \sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{\mu}_k^j}{g_k^j} + \sum_{k=1}^{b}\frac{\tilde{\mu}_k^l}{g_k^l},$$

which is equivalent to

$$\Big(N - \sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{\mu}_k^j}{g_k^j} - \sum_{k=1}^{b-1}\frac{\tilde{\mu}_k^l}{g_k^l}\Big)\cdot g_b^l < \tilde{\mu}_b^l.$$

Thus, we can integrate with respect to dr_b^l in (4.48) by using (4.8) of Lemma 4.8 with $\alpha = g_b^l$, $\beta = -\xi$, $k = \tilde{\mu}_b^l$, $\eta = \gamma - N$ and get that

$$\begin{aligned}\mathcal{G}_\gamma \le c \int_{\substack{r_k^j\in(0,2)\\(j,k)=(l,b+1),\dots,(l,p_l)\\ j=l+1,\dots,n\\ k=1,\dots,p_j}} &\Big(\sum_{k=b+1}^{p_l} r_k^l + \sum_{j=l+1}^{n}\sum_{k=1}^{p_j} r_k^j\Big)^{\xi'}\\ &\cdot \prod_{k=b+1}^{p_l}(r_k^l)^{\tilde{\mu}_k^l-1}\prod_{\substack{j=l+1,\dots,n\\k=1,\dots,p_j}}(r_k^j)^{\tilde{\mu}_k^j-1}dr_k^j\end{aligned}\tag{4.49}$$

with

$$\begin{aligned}\xi' = k - \alpha\beta - \eta &= N - \gamma - g_b^l\Big(N - \sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{\mu}_k^j}{g_k^j} - \sum_{k=1}^{b-1}\frac{\tilde{\mu}_k^l}{g_k^l}\Big) + \tilde{\mu}_b^l\\ &= (1-g_b^l)N + g_b^l\Big(\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{\mu}_k^j}{g_k^j} + \sum_{k=1}^{b-1}\frac{\tilde{\mu}_k^l}{g_k^l}\Big) + \tilde{\mu}_b^l - \gamma.\end{aligned}$$

Observe that by (4.46)

$$\xi' > -\sum_{j=l}^{n} d_j + \sum_{k=1}^{b}\tilde{\mu}_k^l = -\sum_{j=l+1}^{n} d_j - \sum_{k=b+1}^{p_l}\tilde{\mu}_k^l.$$

Note that the number of integrals in (4.49) is given by

$$p_l - b + \sum_{j=l+1}^{n} p_j$$

and the sum of the exponents in the integral in (4.49) satisfies

$$\xi' + \sum_{k=b+1}^{p_l}\tilde{\mu}_k^l - (p_l - b) + \sum_{j=l+1}^{n} d_j - \sum_{j=l+1}^{n} p_j > -(p_l - b + \sum_{j=l+1}^{n} p_j).$$

Thus, the integral in (4.49) is finite and this completes the proof. □

Proof of the lower bounds in Theorem 4.13. Throughout this proof, let c be an unspecified positive constant. Let us first prove the lower bound in (4.18). Note that by the monoticity of the Hausdorff dimension (see Remark 1.1)

$$\dim_{\mathcal{H}} X([0,1]^d) \geq \dim_{\mathcal{H}} X([\frac{1}{2},1]^d).$$

Thus, it suffices to show that

$$\dim_{\mathcal{H}} X([\frac{1}{2},1]^d) \geq \min\{N, \sum_{j=1}^{n}\sum_{k=1}^{p_j} a_k^j \mu_k^j\} \quad a.s.$$

According to Frostman's theorem (see Chapter 1.2) it suffices to show that

$$\mathcal{E}_\gamma = \mathbb{E}[\int_{[\frac{1}{2},1]^d}\int_{[\frac{1}{2},1]^d} \|X(x)-X(y)\|^{-\gamma} dxdy] < \infty$$

in order to obtain $\dim_{\mathcal{H}} X([\frac{1}{2},1]^d) \geq \gamma$ almost surely.

Let us remark that the characteristic function of the α-stable random vector $X(x)-X(y) = (X_1(x)-X_1(y),\dots,X_N(x)-X_N(y))$, $x,y \in \mathbb{R}^d$, is given by

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(i\langle\theta, X(x)-X(y)\rangle\Big)\Big] &= \mathbb{E}\Big[\exp\Big(i\sum_{j=1}^{N}\theta_j(X_j(x)-X_j(y))\Big)\Big] \\
&= \prod_{j=1}^{N}\mathbb{E}\Big[\exp\Big(i\theta_j(X_j(x)-X_j(y))\Big)\Big] \\
&= \prod_{j=1}^{N}\mathbb{E}\Big[\exp\Big(i\theta_j(X_\alpha(x)-X_\alpha(y))\Big)\Big] \\
&= \prod_{j=1}^{N}\exp\big(-|\theta_j|^\alpha\sigma(x,y)^\alpha\big) = \exp\Big(-\sum_{j=1}^{N}|\theta_j|^\alpha\sigma(x,y)^\alpha\Big)
\end{aligned}$$

for any $\theta \in \mathbb{R}^N$ with scale parameter $\sigma(x,y)$ defined as in the proof of Theorem 4.18. In particular, for $Y(x,y) = \frac{1}{\sigma(x,y)}(X(x)-X(y))$ with $x \neq y$ we obtain that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(i\langle\theta, Y(x,y)\rangle\Big)\Big] &= \mathbb{E}\Big[\exp\Big(i\langle\frac{1}{\sigma(x,y)}\theta, X(x)-X(y)\rangle\Big)\Big] \\
&= \exp\Big(-\sum_{j=1}^{N}|\theta_j|^\alpha\Big),
\end{aligned}$$

which shows that the distribution of $Y(x,y)$ is independent of x and y. Thus, since the probability densities of α-stable random variables exist and are continuous (see Corollary 3.6) by Lemma 4.16 we can find a constant c independent of x and y such that

$$\mathbb{E}[\|Y(x,y)\|^{-\gamma}] \leq c$$

for any $0 < \gamma < N$. Using this we obtain

$$\mathcal{E}_\gamma = \int_{[\frac{1}{2},1]^d}\int_{[\frac{1}{2},1]^d} \sigma(x,y)^{-\gamma}\mathbb{E}[\|Y(x,y)\|^{-\gamma}]dxdy$$

$$\begin{aligned}
&\leq c \int_{[\frac{1}{2},1]^d} \int_{[\frac{1}{2},1]^d} \sigma(x,y)^{-\gamma} dx dy \\
&\leq c \int_{[\frac{1}{2},1]^d} \int_{[\frac{1}{2},1]^d} \Big(\sum_{j=1}^{n} \tau_{E_j}(x_j - y_j) \Big)^{-\gamma} dx dy \\
&= c \int_{[\frac{1}{2},1]^{2d_1}} \cdots \int_{[\frac{1}{2},1]^{2d_n}} \Big(\sum_{j=1}^{n} \tau_{E_j}(x_j - y_j) \Big)^{-\gamma} dx dy,
\end{aligned}$$

where we used Theorem 4.18 in the last inequality. By Proposition 4.19 the integral in the last expression is finite for any

$$0 < \gamma < \min\{N, \sum_{j=1}^{n} \sum_{k=1}^{p_j} \frac{a_k^j}{1+\varepsilon} \mu_k^j\}$$

and any $\varepsilon > 0$ arbitrarily small. Thus, Frostman's criterion yields

$$\dim_{\mathcal{H}} X([\frac{1}{2},1]^d) \geq \min\{N, \sum_{j=1}^{n} \sum_{k=1}^{p_j} \frac{a_k^j}{1+\varepsilon} \mu_k^j\} \quad a.s.$$

Since this holds for any small $\varepsilon > 0$, the lower bound in (4.18) follows by letting $\varepsilon \to 0$.

Now we prove the lower bound in (4.19). First assume that

$$\sum_{j=1}^{n} \sum_{k=1}^{p_j} a_k^j \mu_k^j \leq N.$$

By Corollary 1.3 (ii) and (4.18)

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \geq \dim_{\mathcal{H}} X([0,1]^d) \geq \sum_{j=1}^{n} \sum_{k=1}^{p_j} a_k^j \mu_k^j \quad a.s.$$

Therefore, it suffices to prove that for any $1 \leq l \leq n, 1 \leq b \leq p_l$

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \geq G(l,b) \quad a.s.,$$

where $G(l,b)$ is defined as in Theorem 4.13. By Lemma 4.14 and the assumption $H_j = 1$, $1 \leq j \leq n$, (see Remark 4.15) it remains to consider the case that

$$\sum_{j=1}^{l-1} \sum_{k=1}^{p_j} \tilde{a}_k^j \tilde{\mu}_k^j + \sum_{k=1}^{b-1} \tilde{a}_k^l \tilde{\mu}_k^l \leq N < \sum_{j=1}^{l-1} \sum_{k=1}^{p_j} \tilde{a}_k^j \tilde{\mu}_k^j + \sum_{k=1}^{b} \tilde{a}_k^l \tilde{\mu}_k^l$$

for some $1 \leq l \leq n, 1 \leq b \leq p_l$. Again by Frostman's criterion (Chapter 1.2) it is sufficient to show that

$$\mathcal{G}_\gamma = \int_{[\frac{1}{2},1]^d} \int_{[\frac{1}{2},1]^d} \mathbb{E}\Big[(\|x-y\|^2 + \|X(x) - X(y)\|^2)^{-\frac{\gamma}{2}} \Big] dx dy < \infty$$

in order to obtain $\dim_{\mathcal{H}} \operatorname{Gr} X([\frac{1}{2},1]^d) \geq \gamma$ almost surely. Assume that $\gamma > N$. Applying

Lemma 4.17 with the characteristic function computed above we get

$$\begin{aligned}\mathcal{G}_\gamma &\leq c\int_{[\frac{1}{2},1]^d}\int_{[\frac{1}{2},1]^d}\|x-y\|^{-\gamma}\int_{\mathbb{R}^N}\exp\Big(-\sum_{j=1}^N|\theta_j|^\alpha\frac{\sigma(x,y)^\alpha}{\|x-y\|^\alpha}\Big)d\theta dxdy\\ &= c\int_{[\frac{1}{2},1]^d}\int_{[\frac{1}{2},1]^d}\|x-y\|^{N-\gamma}\sigma(x,y)^{-N}\int_{\mathbb{R}^N}\exp\Big(-\sum_{j=1}^N|u_j|^\alpha\Big)dudxdy,\end{aligned}$$

where we used the change of variables $u_j=\theta_j\frac{\sigma(x,y)}{\|x-y\|}$, $du_j=\frac{\sigma(x,y)}{\|x-y\|}d\theta_j$ in the last equality. Note that by using the change of variables $v_j=u_j^\alpha$ we get

$$\begin{aligned}\int_{\mathbb{R}^N}\exp\Big(-\sum_{j=1}^N|u_j|^\alpha\Big)du &= \prod_{j=1}^N\int_{\mathbb{R}}\exp(-|u_j|^\alpha)du_j=\prod_{j=1}^N 2\int_0^\infty\exp(-u_j^\alpha)du_j\\ &= c\prod_{j=1}^N\int_0^\infty v_j^{\frac{1}{\alpha}-1}\exp(-v_j)dv_j=c\Gamma(\frac{1}{\alpha})^N,\end{aligned}$$

where $\Gamma(z)=\int_0^\infty v^{z-1}e^{-v}dv$ is the gamma function. Combining this with Theorem 4.18 we can estimate

$$\mathcal{G}_\gamma\leq c\int_{[\frac{1}{2},1]^d}\int_{[\frac{1}{2},1]^d}\|x-y\|^{N-\gamma}\Big(\sum_{j=1}^n\tau_{E_j}(x_j-y_j)\Big)^{-N}dxdy.$$

By Proposition 4.20 the last expression is finite for any

$$N<\gamma<\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{a}_k^j}{\tilde{a}_b^l}\tilde{\mu}_k^j+\sum_{k=1}^b\frac{\tilde{a}_k^l}{\tilde{a}_b^l}\tilde{\mu}_k^l+\sum_{j=l}^n d_j-\sum_{k=1}^b\tilde{\mu}_k^l+(1-\frac{1+\varepsilon}{\tilde{a}_b^l})N$$

and any $\varepsilon>0$. Thus, by Frostman's theorem

$$\dim_{\mathcal{H}}\operatorname{Gr}X([\frac{1}{2},1]^d)\geq\sum_{j=1}^{l-1}\sum_{k=1}^{p_j}\frac{\tilde{a}_k^j}{\tilde{a}_b^l}\tilde{\mu}_k^j+\sum_{k=1}^b\frac{\tilde{a}_k^l}{\tilde{a}_b^l}\tilde{\mu}_k^l+\sum_{j=l}^n d_j-\sum_{k=1}^b\tilde{\mu}_k^l+(1-\frac{1+\varepsilon}{\tilde{a}_b^l})N$$

almost surely for any $\varepsilon>0$. Since this holds for any arbitrarily small $\varepsilon>0$, this proves the lower bound in (4.19) by letting $\varepsilon\to0$. This completes the proof of Theorem 4.13. □

We are now interested in properties of the 1-dimensional random field X_α given in Theorem 4.2. Let us first recall the definition of the Hölder critical exponent [9, Definition 5.1].

Definition 4.21. Let $\beta\in(0,1)$. A real-valued random field $\{X(x):x\in\mathbb{R}^d\}$ is said to have Hölder critical exponent β if there exists a modification X^* of X such that the following properties hold.

(i) For any $s\in(0,\beta)$ the sample paths of X^* almost surely satisfy a uniform Hölder condition of order s on $[0,1]^d$, i.e. there exists a positive and finite random variable A

such that

$$|X^*(x) - X^*(y)| \leq A\|x - y\|^s \tag{4.50}$$

for all $x, y \in [0,1]^d$.

(ii) For any $s \in (\beta, 1)$ (4.50) fails almost surely.

We now state the following which is an easy consequence of Theorem 4.13.

Corollary 4.22. *Assume that the conditions of Theorem 4.13 hold and let X_α be the random field given in* (4.2)*. Then with probability one*

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = d + 1 - \frac{H_1}{a^1_{p_1}}. \tag{4.51}$$

Furthermore, X_α admits $\frac{H_1}{a^1_{p_1}}$ as the Hölder critical exponent.

Proof. By Remark 4.15 without loss of generality we may assume that $H_1 = 1$ or, equivalently $a^1_1 > 1$. From Theorem 4.13 and Lemma 4.14 with $N = 1$ we get

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = G(1,1) \quad a.s.,$$

since

$$N = 1 < \tilde{a}^1_1 \tilde{\mu}^1_1$$

by assumption. By definition we have

$$G(1,1) = \tilde{\mu}^1_1 + \sum_{j=1}^n d_j - \tilde{\mu}^1_1 + (1 - \frac{1}{\tilde{a}^1_1}) \cdot 1 = d + 1 - \frac{1}{a^1_{p_1}}.$$

It remains to prove that X_α admits $\frac{1}{a^1_{p_1}}$ as the Hölder critical exponent. Let c be an unspecified positive constant. From Corollary 4.12 with $N = 1$ we get that there exists a modification X^* of X_α such that

$$|X^*(x) - X^*(y)| \leq c \sum_{j=1}^n \tau_{E_j}(x_j - y_j)^s \quad a.s.$$

for any $0 < s < 1$ and $x, y \in [0,1]^d$, which is by Lemma 4.5 equivalent to

$$|X^*(x) - X^*(y)| \leq c\tau_E(x - y)^s \quad a.s.$$

Combining this with Lemma 2.5 and the fact that, by (4.17), $a^1_{p_1}$ is the largest real part of the eigenvalues of E we get that

$$|X^*(x) - X^*(y)| \leq c\|x - y\|^s \quad a.s.$$

for any $s \in (0, \frac{1}{a^1_{p_1}})$. Thus X^* almost surely satisfies (4.50) with $s \in (0, \frac{1}{a^1_{p_1}})$. It remains to prove that (4.50) with $s \in (\frac{1}{a^1_{p_1}}, 1)$ almost surely fails. We prove this by contradiction. So assume that for some $s \in (\frac{1}{a_{p_1}^1}, 1)$ we have

$$|X^*(x) - X^*(y)| \leq c\|x-y\|^s$$

with positive probability. Then Lemma 1.2 with $m = 1$ and $\alpha_1 = s$ yields

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \leq d + 1 - s$$

with positive probability. But this contradicts (4.51). The proof of Corollary 4.22 is complete. $\square$

Remark 4.23. Corollary 4.22 shows that the components of (d, N)-harmonizable operator scaling stable random sheets almost surely satisfy a Hölder condition. However, for $N \geq 2$ the upper bounds for the Hausdorff dimension of the image and the graph one gets from Lemma 1.4 are not sharp as soon as $E \neq a_1^1 I_d$, that is as soon as the random field is anisotropic. Furthermore, comparing Theorem 4.13 for $N \geq 2$ with Corollary 4.22 is quite surprising and shows that in the 1-dimensional case the Hausdorff dimension of the graph only depends on solely one real part of the eigenvalues of the scaling matrices $E_1, \dots, E_n$, namely the largest, whereas in higher dimensions the Hausdorff dimension of the graph depends in general on all the real parts of the eigenvalues and even the multiplicity of the eigenvalues.

We close this chapter with two examples which show that Theorem 4.2 includes a very large class of random fields.

Example 4.24. Let $\alpha = 2, d_j = E_j = 1$ for all $j = 1, \dots, n$ and consider the functions $\psi(\xi_j) = |\xi_j|$ for all $\xi_j \in \mathbb{R}$. Clearly, ψ_j is 1-homogeneous and satisfies $\psi_j(\xi_j) \neq 0$ for all $\xi_j \neq 0$. Thus, by Theorem 4.2 we can define

$$X_2(x) = \operatorname{Re} \int_{\mathbb{R}^d} \prod_{j=1}^d (e^{ix_j\xi_j} - 1)|\xi_j|^{-H_j - \frac{1}{2}} M_2(d\xi), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

for all $0 < H_j < 1, 1 \leq j \leq d$. Let $\overline{M}$ be a Gaussian random measure as in Remark 3.12 and define

$$f_x(\xi) = \prod_{j=1}^d (e^{ix_j\xi_j} - 1)|\xi_j|^{-H_j - \frac{1}{2}}.$$

Note that $f_x(\xi) = \overline{f_x(-\xi)}$ for all $x, \xi \in \mathbb{R}^d$ so that by Remark 3.12

$$X_2(x) \stackrel{\mathrm{d}}{=} \int_{\mathbb{R}^d} f_x(\xi) \overline{M}(dx).$$

In [18] it is shown that up to a multiplicative constant the latter is an integral representation of the fractional Brownian sheet at time $x \in \mathbb{R}^d$ with Hurst indices $H_1, \dots, H_d$. Moreover, the statement of Theorem 4.13 becomes that with probability one

$$\dim_{\mathcal{H}} X([0,1]^d) = \min\{N, \sum_{j=1}^d \frac{1}{H_j}\}$$

and

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = \min\Big\{ \sum_{j=1}^{d} \frac{1}{H_j}; \sum_{j=1}^{k-1} \frac{H_k}{H_j} + d - k + 1 + (1 - H_k)N, 1 \leq k \leq d \Big\}.$$

Thus, Theorem 4.13 can be seen as a generalization of [5, Theorem 3.1]. In particular, from Corollary 4.22 in the 1-dimensional case we get

$$\dim_{\mathcal{H}} \operatorname{Gr} X_2([0,1]^d) = d + 1 - H_1 \quad a.s.,$$

which generalizes [4, Theorem 1.3] and X_2 admits H_1 as the Hölder critical exponent.

Example 4.25. Let $n = 1, d = d_1$ and $E = E_1$. As noted above the random field X_α given by (4.2) coincides with the operator scaling random field in [9, Theorem 4.1] and Theorem 4.13 reads as

$$\dim_{\mathcal{H}} X([0,1]^d) = \min\{N, \sum_{k=1}^{p_1} \frac{a_k^1}{H_1} \mu_k^1\}$$

and

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = \min\Big\{ \sum_{k=1}^{p_1} \frac{a_k^1}{H_1} \mu_k^1; \sum_{k=1}^{b} \frac{\tilde{a}_k^1}{\tilde{a}_b^1} \tilde{\mu}_k^1 + d - \sum_{k=1}^{b} \tilde{\mu}_k^1 + (1 - \frac{H_1}{\tilde{a}_b^1})N, 1 \leq b \leq p_l \Big\}$$

almost surely. Let us remark that in the Gaussian case $\alpha = 2$ this generalizes the Hausdorff dimension results stated in [32, Section 3]. Furthermore, in the 1-dimensional case from Corollary 4.22 we obtain

$$\dim_{\mathcal{H}} \operatorname{Gr} X_\alpha([0,1]^d) = d + 1 - \frac{H_1}{a_{p_1}^1} \quad a.s.,$$

which is the statement of [9, Theorem 5.6] for $\alpha = 2$ and [10, Proposition 5.7] for $\alpha \in (0,2)$.

5. Multivariate Gaussian operator-self-similar random fields

As noted in the Introduction in this chapter we give the solution to some open problems formulated in [33]. We first recall the definition of operator-self-similar random fields and results concerning their existence established in [33].

5.1. Definition and existence

Throughout this chapter, let $E \in \mathbb{R}^{d\times d}$ be a matrix with distinct positive real parts of its eigenvalues given by $0 < a_1 < \ldots < a_p$ for some $p \leq d, q = \operatorname{trace}(E)$ and let $D \in \mathbb{R}^{m\times m}$ be a matrix with positive real parts of its eigenvalues given by $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$. Note that $\lambda_1, \ldots, \lambda_m$ are not necessarily different.

Definition 5.1. A random field $\{X(x) : x \in \mathbb{R}^d\}$ with values in $\mathbb{R}^m$ is called multivariate operator-self-similar for E and D or (E, D)-operator-self-similar if

$$\{X(c^E x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^D X(x) : x \in \mathbb{R}^d\} \tag{5.1}$$

for all $c > 0$.

An important class of multivariate Gaussian operator-self-similar random fields is given by the so-called operator-fractional Brownian motion B_D with state space scaling exponent D introduced in [36]. The random field B_D fulfills the self-similarity relation

$$\{B_D(ct) : t \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{c^D B_D(t) : t \in \mathbb{R}^d\}$$

for any $c > 0$, i.e. it is (I_d, D)-operator-self-similar. We remark that Mason and Xiao [36] studied several sample path properties of B_D including fractal dimensions of the range and the graph of B_D. More precisely, for any arbitrary Borel set $U \subset \mathbb{R}^d$, under some additional assumptions (see [36, Theorem 4.1]), they showed that almost surely the Hausdorff dimension of the range and graph are given by

$$\dim_{\mathcal{H}} B_D(F) = \min\Big\{m, \Big(\dim_{\mathcal{H}} F + \sum_{i=1}^{j}(\lambda_j - \lambda_i)\Big)\lambda_j^{-1}, 1 \leq j \leq m\Big\}$$

and

$$\dim_{\mathcal{H}} \operatorname{Gr} B_D(F) = \begin{cases} \dim_{\mathcal{H}} B_D(F) & \text{if } \dim_{\mathcal{H}} F \leq \sum_{i=1}^{m} \lambda_i, \\ \dim_{\mathcal{H}} F + \sum_{i=1}^{m} (1-\lambda_i) & \text{if } \dim_{\mathcal{H}} F > \sum_{i=1}^{m} \lambda_i. \end{cases}$$

In particular, if $F = [0,1]^d$ they obtain that almost surely

$$\dim_{\mathcal{H}} B_D([0,1]^d) = \min \Big\{ m, \Big(d + \sum_{i=1}^{j} (\lambda_j - \lambda_i) \Big) \lambda_j^{-1}, 1 \leq j \leq m \Big\} \tag{5.2}$$

and

$$\dim_{\mathcal{H}} \operatorname{Gr} B_D([0,1]^d) = \begin{cases} \dim_{\mathcal{H}} B_D([0,1]^d) & \text{if } d \leq \sum_{i=1}^{m} \lambda_i, \\ d + \sum_{i=1}^{m} (1-\lambda_i) & \text{if } d > \sum_{i=1}^{m} \lambda_i. \end{cases} \tag{5.3}$$

In the following let $\phi : \mathbb{R}^d \to [0,\infty)$ be an E-homogeneous (β, E)-admissible function according to Definition 2.3 and Definition 2.4. Recall that $0 < \beta \leq a_1$. Moreover, let W_2 be an $\mathbb{R}^m$-valued symmetric Gaussian random measure on $\mathbb{R}^d$ according to Chapter 3.3. The following is due to [33, Theorem 2.5] and provides the existence of moving-average operator-self-similar Gaussian random fields.

Theorem 5.2. *If $\lambda_m < \beta$ the random field*

$$X_\phi(x) = \int_{\mathbb{R}^d} [\phi(x-y)^{D - \frac{q}{2} I_m} - \phi(-y)^{D - \frac{q}{2} I_m}] W_2(dy), \quad x \in \mathbb{R}^d \tag{5.4}$$

exists and is a stochastically continuous (E, D)-operator-self-similar Gaussian random field with stationary increments.

For the sake of simplicity let us denote the kernel matrix in (5.4) by

$$Q(x,y) = [\phi(x-y)^{D - \frac{q}{2} I_m} - \phi(-y)^{D - \frac{q}{2} I_m}]$$

and let us recall that according to Chapter 3.3 X_ϕ exists, since

$$\int_{\mathbb{R}^d} \|Q(x,y)\|^2 dy < \infty$$

for all $x \in \mathbb{R}^d$, as shown in the proof of [33, Theorem 2.5].

Let us now turn to the existence of harmonizable operator-self-similar Gaussian random fields constructed in [33, Theorem 2.6]. Suppose that $\psi : \mathbb{R}^d \to [0,\infty)$ is a continuous E^T-homogeneous function such that $\psi(x) \neq 0$ for $x \neq 0$. Moreover, let $\tilde{M}_2$ be a $\mathbb{C}^m$-valued symmetric Gaussian random measure on $\mathbb{R}^d$ as given in Chapter 3.3.

Theorem 5.3. *If $\lambda_m < a_1$ the random field*

$$X_\psi(x) = \mathrm{Re} \int_{\mathbb{R}^d} (e^{i\langle x,y\rangle} - 1)\psi(y)^{-D-\frac{q}{2}I_m} \tilde{M}_2(dy), \quad x \in \mathbb{R}^d \tag{5.5}$$

exists and is a stochastically continuous (E, D)-operator-self-similar Gaussian random field with stationary increments.

As in the above X_ψ is well defined, since the kernel matrix in (5.5) satisfies

$$\int_{\mathbb{R}^d} (|1 - \cos\langle x, y\rangle|^2 + |\sin\langle x, y\rangle|^2)\|\psi(y)^{-D-\frac{q}{2}I_m}\|^2 dy < \infty$$

for all $x \in \mathbb{R}^d$, which is shown in the proof of [33, Theorem 2.5].

Let us recall that an $\mathbb{R}^m$-valued random field $\{Y(x) : x \in \mathbb{R}^d\}$ is said to be proper if for every $x \in \mathbb{R}^d$ the distribution of $Y(x)$ is full, i.e. it is not supported on any proper hyperplane in $\mathbb{R}^m$, which is in the Gaussian case well-known to be equivalent to $\det \mathrm{Cov}\,(Y(x)) > 0$. In [33] it is shown that X_ψ is proper, whereas X_ϕ is proper if $\frac{q}{2}$ is not an eigenvalue of D (see [33, Remark 2.1]). For the sake of simplicity we will always assume that the latter holds in order to ensure that both X_ϕ and X_ψ are proper.

Remark 5.4. Assume that the conditions of Theorem 5.2 and Theorem 5.3 hold so that, in particular $\lambda_m < a_1$. Let X be (E, D)-operator-self-similar and define $\tilde{E} = \frac{E}{H}$ and $\tilde{D} = \frac{D}{H}$ for some $H \in (\lambda_m, a_1)$. Then X is $(\tilde{E}, \tilde{D})$-operator-self-similar as well, since for any $c > 0$

$$\{X(c^{\tilde{E}}x) : x \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{(c^{\frac{1}{H}})^D X(x) : x \in \mathbb{R}^d\} = \{c^{\tilde{D}} X(x) : x \in \mathbb{R}^d\}.$$

Note that the real parts of the eigenvalues of $\tilde{D}$ are smaller than 1, whereas the real parts of the eigenvalues of $\tilde{E}$ are larger than 1. So without loss of generality we will always assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m < 1 < a_1 < \ldots < a_p. \tag{5.6}$$

As in the previous chapter a main tool for the study of sample paths of multivariate (E, D)-operator-self-similar random fields will be the change to generalized polar coordinates with respect to the scaling matrix E. Before studying their sample paths, in the next section we state and prove a Lemma which might be of independent interest in fractal geometry.

5.2. Preliminaries

Let us adapt the notation of the previous chapters and let $(\tau_E(x), l_E(x))$ be the generalized polar coordinates with respect to E. Furthermore, let $\mathbb{R}^d = W_1 \oplus \ldots \oplus W_p$ be the direct sum decomposition with respect to E as introduced in Chapter 2.1 and define

$$\mu_k = \dim W_k, \quad \tilde{\mu}_k = \dim W_{p+1-k}, \quad \tilde{a}_k = a_{p+1-k}$$

for $1 \le k \le p$. Note that

$$\tilde{a}_1 > \tilde{a}_2 > \ldots > \tilde{a}_p. \tag{5.7}$$

The following Lemma is a generalization of Lemma 1.4.

Lemma 5.5. *Let $f = (f_1, \ldots, f_m) : [0,1]^d \to \mathbb{R}^m$ satisfy the following generalized Hölder condition with respect to E:*

$$|f_i(x) - f_i(y)| \le c\,\tau_E(x-y)^{\alpha_i}, \quad 1 \le i \le m, \tag{5.8}$$

where $c > 0$ and $0 < \alpha_i \le 1$ are constants such that

$$0 < \alpha_1 \le \alpha_2 \le \ldots \le \alpha_m \le 1. \tag{5.9}$$

Then

$$\dim_{\mathcal{H}} f([0,1]^d) \le \min\Big\{m; \frac{\sum_{k=1}^p a_k\mu_k + \sum_{i=1}^j(\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le m\Big\} \tag{5.10}$$

and

$$\begin{aligned} \dim_{\mathcal{H}} \operatorname{Gr} f([0,1]^d) \le \min\Bigg\{ & \frac{\sum_{k=1}^p a_k\mu_k + \sum_{i=1}^j(\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le m; \\ & \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=l+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m}(1 - \frac{\alpha_i}{\tilde{a}_l}), 1 \le l \le p\Bigg\}. \end{aligned} \tag{5.11}$$

Proof. Throughout this proof, let c be an unspecified positive constant which might change in each occurence. Note that we clearly have

$$\dim_{\mathcal{H}} f([0,1]^d) \le \dim_{\mathcal{H}} \mathbb{R}^m = m$$

and by Corollary 1.3 (ii)

$$\dim_{\mathcal{H}} f([0,1]^d) \le \dim_{\mathcal{H}} \operatorname{Gr} f([0,1]^d).$$

So it suffices to prove (5.11). We first show that

$$\dim_{\mathcal{H}} \operatorname{Gr} f([0,1]^d) \le \frac{\sum_{k=1}^p a_k\mu_k + \sum_{i=1}^j(\alpha_j - \alpha_i)}{\alpha_j} \tag{5.12}$$

for every fixed $1 \le j \le m$. Let us choose compact subsets $V_1 \subset W_1, \ldots, V_p \subset W_p$ such that

$$[0,1]^d \subset V_1 + \ldots + V_p,$$

where $V_1 + \ldots + V_p = \{x_1 + \ldots + x_p : x_i \in V_i, 1 \le i \le p\}$. For any integer $n \ge 2$ we cover V_l $(1 \le l \le p)$ by $k_{n,l}$ cubes $\{R_{n,l,i_l}\}$ $(1 \le i_l \le k_{n,l})$ with edge-lengths n^{-a_l} so that the diameter and volume of R_{n,l,i_l} satisfy

$$\operatorname{diam}(R_{n,l,i_l}) \leq cn^{-a_l} \tag{5.13}$$

and

$$\operatorname{vol}(R_{n,l,i_l}) \leq cn^{-a_l\mu_l},$$

where the volume is taken with respect to the μ_l-dimensional Lebesgue measure. Since the cubes $\{R_{n,l,i_l}\}$ $(1 \leq i_l \leq k_{n,l})$ cover V_l, we have

$$k_{n,l} \leq cn^{a_l\mu_l}$$

for all $1 \leq l \leq p$, which yields that

$$k_{n,1}\dots k_{n,p} \leq cn^{\sum_{l=1}^{p} a_l\mu_l}. \tag{5.14}$$

For any vector $x \in \mathbb{R}^d$ let $x = x_1+\ldots+x_p$ be the direct sum decomposition with respect to E and let $\varepsilon > 0$ denote an unspecified (small) constant which might change in each occurence. From (5.8) and Remark 2.10 we get for $1 \leq i \leq m$ and $\varepsilon > 0$

$$|f_i(x) - f_i(y)| \leq c\tau_E(x-y)^{\alpha_i} \leq c\sum_{k=1}^{p} \|x_k - y_k\|^{\frac{\alpha_i}{a_k}-\varepsilon}.$$

Thus, each $f(R_{n,1,i_1} + \ldots + R_{n,p,i_p})$ can be covered by a rectangle $T_{n,i_1,\ldots,i_p} \subset \mathbb{R}^m$ of sides $c(\frac{1}{n})^{\alpha_i-\varepsilon}$ $(1 \leq i \leq m)$. Note that

$$\operatorname{vol}(T_{n,i_1,\ldots,i_p}) \leq c(\frac{1}{n})^{\sum_{j=1}^{m}\alpha_i-\varepsilon}.$$

For each fixed $1 \leq j \leq m$ we can cover $T_{n,i_1,\ldots,i_p}$ by at most $c(\frac{1}{n})^{\sum_{i=1}^{j}(\alpha_i-\alpha_j)-\varepsilon}$ cubes $T_{n,i_1,\ldots,i_p,k}$ $(1 \leq k \leq c(\frac{1}{n})^{\sum_{i=1}^{j}(\alpha_i-\alpha_j)-\varepsilon})$ of edge-lengths $c(\frac{1}{n})^{\alpha_j}$, since

$$\begin{aligned}
c(\frac{1}{n})^{\sum_{i=1}^{j}(\alpha_i-\alpha_j)-\varepsilon}\cdot\operatorname{vol}(T_{n,i_1,\ldots,i_p,k}) &= c(\frac{1}{n})^{\sum_{i=1}^{j}(\alpha_i-\alpha_j)-\varepsilon}\cdot(\frac{1}{n})^{m\alpha_j}\\
&= c(\frac{1}{n})^{\sum_{i=1}^{j}\alpha_i+(m-j)\alpha_j-\varepsilon}\\
&\geq c(\frac{1}{n})^{\sum_{i=1}^{m}\alpha_i-\varepsilon}
\end{aligned}$$

by (5.9). Note that

$$\operatorname{Gr} f([0,1]^d) \subset \bigcup_{i_1,\ldots,i_p}\bigcup_{k}(R_{n,1,i_1}+\ldots+R_{n,p,i_p}) \times T_{n,i_1,\ldots,i_p,k}$$

and since

$$\operatorname{diam}(T_{n,i_1,\ldots,i_p,k}) \leq c(\frac{1}{n})^{\alpha_j},$$

(5.13) shows that

$$\operatorname{diam}\big((R_{n,1,i_1}+\ldots+R_{n,p,i_p})\times T_{n,i_1,\ldots,i_p,k}\big)\le c(\frac{1}{n})^{\alpha_j}. \tag{5.15}$$

Let $\gamma>\varepsilon$. Then by (5.14) and (5.15)

$$\begin{aligned}
&\sum_{i_1,\ldots,i_p}\sum_k \operatorname{diam}\big((R_{n,1,i_1}+\ldots+R_{n,p,i_p})\times T_{n,i_1,\ldots,i_p,k}\big)^{\left[\gamma+\sum_{l=1}^p a_l\mu_l+\sum_{i=1}^j(\alpha_j-\alpha_i)\right]/\alpha_j}\\
&\le c\,k_{n,1}\ldots k_{n,p}\cdot(\frac{1}{n})^{\sum_{i=1}^j(\alpha_i-\alpha_j)-\varepsilon}\cdot(\frac{1}{n})^{\gamma+\sum_{l=1}^p a_l\mu_l+\sum_{i=1}^j(\alpha_j-\alpha_i)}\\
&\le c\,(\frac{1}{n})^{\gamma-\varepsilon}\to 0
\end{aligned}$$

as $n\to\infty$. This proves

$$\dim_{\mathcal{H}}\operatorname{Gr} f([0,1]^d)\le\frac{\varepsilon+\sum_{k=1}^p a_k\mu_k+\sum_{i=1}^j(\alpha_j-\alpha_i)}{\alpha_j}$$

for all $1\le j\le m$ and $\varepsilon>0$. Hence, (5.12) follows by letting $\varepsilon\to 0$. It remains to prove

$$\dim_{\mathcal{H}}\operatorname{Gr} f([0,1]^d)\le\sum_{j=1}^k\frac{\tilde a_j}{\tilde a_k}\tilde\mu_j+\sum_{j=k+1}^p\tilde\mu_j+\sum_{i=1}^m(1-\frac{\alpha_i}{\tilde a_k}) \tag{5.16}$$

for any $1\le k\le p$. Let us fix an integer $1\le k\le p$. We observe that each

$$(R_{n,1,i_1}+\ldots+R_{n,p,i_p})\times T_{n,i_1,\ldots,i_p}$$

can be covered by $\ell_{n,k}$ cubes in $\mathbb{R}^{d+m}$ of sides $n^{-\tilde a_k}$. Further note that since

$$\operatorname{vol}\big((R_{n,1,i_1}+\ldots+R_{n,p,i_p})\times T_{n,i_1,\ldots,i_p}\big)\le cn^{-\sum_{l=1}^p\tilde a_l\tilde\mu_l-\sum_{i=1}^m(\alpha_i+\varepsilon)},$$

we can achieve that

$$\ell_{n,k}n^{-\tilde a_k(d+m)}\le cn^{-\sum_{l=1}^p\tilde a_l\tilde\mu_l-\sum_{i=1}^m(\alpha_i+\varepsilon)}$$

or, equivalently

$$\ell_{n,k}\le cn^{-\sum_{l=1}^p\tilde a_l\tilde\mu_l+d\tilde a_k-\sum_{i=1}^m(\alpha_i-\tilde a_k+\varepsilon)}$$

and the exponent in the last expression equals

$$-\sum_{l=1}^p(\tilde a_l-\tilde a_k)\tilde\mu_l-\sum_{i=1}^m(\alpha_i-\tilde a_k+\varepsilon)\le\sum_{l=k+1}^p(\tilde a_k-\tilde a_l)\tilde\mu_l-\sum_{i=1}^m(\alpha_i-\tilde a_k+\varepsilon), \tag{5.17}$$

where the last inequality follows from (5.7). Let $0<\alpha_i'<\alpha_i-\varepsilon, 1\le i\le m$, and define

$$\eta_k=\sum_{j=1}^k\frac{\tilde a_j}{\tilde a_k}\tilde\mu_j+d-\sum_{j=1}^k\tilde\mu_j+\sum_{i=1}^m(1-\frac{\alpha_i'}{\tilde a_k}).$$

Note that $\operatorname{Gr} f([0,1]^d)$ can be covered by $k_{n,1}\ldots k_{n,p}\cdot\ell_{n,k}$ cubes in $\mathbb{R}^{d+m}$ with edge-lengths

$n^{-\tilde{a}_k}$. We will now show that

$$k_{n,1}\dots k_{n,p}\cdot \ell_{n,k} n^{-\tilde{a}_k \eta_k} \to 0 \tag{5.18}$$

as $n\to\infty$ in order to obtain $\dim_{\mathcal{H}} \mathrm{Gr}\, f([0,1]^d) \le \eta_k$. Using (5.14) and (5.17) we get

$$k_{n,1}\dots k_{n,p}\cdot \ell_{n,k} n^{-\tilde{a}_k \eta_k} \le n^{\gamma}$$

with

$$\begin{aligned}
\gamma &= \sum_{j=1}^{p} \tilde{a}_j \tilde{\mu}_j + \sum_{j=k+1}^{p} (\tilde{a}_k - \tilde{a}_j)\tilde{\mu}_j - \sum_{i=1}^{m} (\alpha_i - \tilde{a}_k + \varepsilon) - \sum_{j=1}^{k} \tilde{a}_j \tilde{\mu}_j \\
&\quad - \tilde{a}_k \Big(d - \sum_{j=1}^{k} \tilde{\mu}_j + \sum_{i=1}^{m} (1 - \frac{\alpha_i'}{\tilde{a}_k}) \Big) \\
&= \tilde{a}_k \sum_{j=k+1}^{p} \tilde{\mu}_j - \sum_{i=1}^{m} (\alpha_i - \tilde{a}_k + \varepsilon) - \tilde{a}_k \sum_{j=k+1}^{p} \tilde{\mu}_j - \sum_{i=1}^{m} (\tilde{a}_k - \alpha_i') \\
&= \sum_{i=1}^{m} (\alpha_i' - \alpha_i + \varepsilon) < 0
\end{aligned}$$

by assumption so that (5.18) holds and implies $\dim_{\mathcal{H}} \mathrm{Gr}\, f([0,1]^d) \le \eta_k$. Therefore, (5.16) follows by letting $\alpha_i' \to \alpha_i - \varepsilon$ and $\varepsilon \to 0$. The proof of Lemma 5.5 is complete. □

Remark 5.6. Let $f = (f_1,\dots,f_m) : [0,1]^d \to \mathbb{R}^m$ and assume that f satisfies (5.8) with α_i replaced by β_i for every $\beta_i < \alpha_i$. Then in view of the proof of Lemma 5.5 we see that (5.10) and (5.11) are still valid.

Example 5.7. Let the assumptions of Lemma 5.5 hold with $E = I_d$ the identity operator on $\mathbb{R}^d$. By Example 2.2 we have $\tau_{I_d}(x) = \|x\|$ for all $x \in \mathbb{R}^d$. Further note that $p = 1, a_1 = 1$ and the direct sum decomposition with respect to I_d is $\mathbb{R}^d = W_1$ so that $\dim W_1 = d$. Thus, Lemma 5.5 reads as

$$\dim_{\mathcal{H}} f([0,1]^d) \le \min \Big\{ m, \frac{d + \sum_{i=1}^{j} (\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le m \Big\}$$

and

$$\dim_{\mathcal{H}} \mathrm{Gr}\, f([0,1]^d) \le \min \left\{ \frac{d + \sum_{i=1}^{j} (\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le m, d + \sum_{i=1}^{m} (1 - \alpha_i) \right\},$$

which coincides with the statements in Lemma 1.4.

5.3. Uniform modulus of continuity

From now on throughout this chapter, let the assumptions of Chapter 5.1 hold and let us write X to indicate that we consider either the random field X_ϕ in Theorem 5.2 or X_ψ in

Theorem 5.3. We will now state a result about the modulus of continuity for the components of $X = (X_1, \dots, X_m)$. Before doing this, let us recall that from the Jordan decomposition theorem (see e.g. [19, p. 129]) there exists a real invertible matrix $A \in \mathbb{R}^{m\times m}$ such that $A^{-1}DA$ is of the real canonical form, i.e.

$$A^{-1}DA = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix}$$

for some $k \le m$ and some block matrices $J_1, \dots, J_k$, where each $J_j, 1 \le j \le k$, is either a Jordan cell matrix of the form

$$J_j = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

with λ a real eigenvalue of D or J_j is of the form

$$J_j = \begin{pmatrix} \Lambda & I_2 & & & \\ & \Lambda & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & \Lambda \end{pmatrix} \quad \text{with } \Lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the complex numbers $a \pm ib, b \neq 0$, are complex conjugated eigenvalues of D.

Proposition 5.8. *If the operator D itself is of the real canonical form there exist positive and finite constants $1 \le p_j \le m$ $(j = 1, \dots, m)$, $C_{6,1}$, depending only on D, d and m, and a modification X^* of X such that for every $j = 1, \dots, m$*

$$\sup_{\substack{x,y\in[0,1]^d \\ x\neq y}} \frac{|X_j^*(x) - X_j^*(y)|}{\tau_E(x-y)^{\lambda_j} |\log \tau_E(x-y)|^{\frac{1}{2}+2(p_j-1)+\varepsilon}} \le C_{6,1} \quad a.s. \tag{5.19}$$

for every $\varepsilon > 0$. In particular, one can find a positive and finite constant $C_{6,2}$ such that for every $1 \le j \le m$ and $\varepsilon > 0$

$$|X_j^*(x) - X_j^*(y)| \le C_{6,2} \tau_E(x-y)^{\lambda_j - \varepsilon} \tag{5.20}$$

holds almost surely for any $x, y \in [0,1]^d$.

Proof. The proof of this Proposition is essentially based on the proof in [36, Proposition 4.1] and the idea is to apply Proposition 4.9. Let c be an unspecified positive constant and define

$r = \tau_E(x-y)$ for some fixed $x, y \in [0,1]^d$. By Proposition 4.9 it suffices to show that

$$\mathbb{E}\big[\big(X_j(x) - X_j(y)\big)^2\big] \leq c r^{2\lambda_j} |\log r|^{2(p_j-1)}. \tag{5.21}$$

As before let $\|Q\| = \max_{\|x\|=1} \|Qx\|$ for any matrix $Q \in \mathbb{R}^{m\times m}$. Let us recall that the operator norm is submultiplicative, i.e.

$$\|AB\| \leq \|A\|\|B\| \tag{5.22}$$

for all $A, B \in \mathbb{R}^{m\times m}$ (see, e.g., [38, Proposition 2.1.3]) and that

$$\max_{1\leq i,j\leq m} |a_{ij}| \leq \|A\| \leq \sqrt{m^3} \max_{1\leq i,j\leq m} |a_{ij}| \tag{5.23}$$

for any $A = (a_{ij}) \in \mathbb{R}^{m\times m}$ (see [36, p. 60]).

Fix $1 \leq j \leq m$ and let $J_1, \dots, J_k$ be the diagonal blocks of D for some $k \leq m$. Suppose that the block corresponding to the eigenvalue $\alpha_j = \lambda_j + i\beta_j$ is J_l. For notational simplicity we will suppress the subscript l. Denote the standard basis of $\mathbb{R}^m$ by $(e_1, \dots, e_m)$. Since X is (E, D)-operator-self-similar and has stationary increments, using the change to generalized polar coordinates with respect to E we get

$$\begin{aligned}
\mathbb{E}\big[\big(X_j(x) - X_j(y)\big)^2\big] &= \mathbb{E}\big[\langle X(x) - X(y), e_j\rangle^2\big] \\
&= \mathbb{E}\big[\langle \tau_E(x-y)^D X\big(l_E(x-y)\big), e_j\rangle^2\big] \\
&= \mathbb{E}\Big[\Big(\sum_{k=1}^m X_k\big(l_E(x-y)\big)\langle \tau_E(x-y)^D e_k, e_j\rangle\Big)^2\Big].
\end{aligned}$$

Note that $\langle r^D e_k, e_j\rangle$ is the (j,k)th entry of r^D and $a = \langle r^D e_k, e_j\rangle \neq 0$ only if $j \leq k$ and a is also an entry of r^J, since D is assumed to be of the real canonical form. Now we distinguish two cases. First we assume that J is a $p\times p$ Jordan cell matrix with eigenvalue $\lambda = \lambda_j$. Then

$$J^k = \begin{pmatrix}
\lambda^k & k\lambda^{k-1} & \dots & \dots & \binom{k}{p-1}\lambda^{k-p+1} \\
& \lambda^k & k\lambda^{k-1} & & \vdots \\
& & \ddots & \ddots & \vdots \\
& & & \ddots & k\lambda^{k-1} \\
& & & & \lambda^k
\end{pmatrix}$$

for all $k \in \mathbb{N}$ so that

$$r^J = \sum_{k=0}^\infty \frac{(\log r)^k}{k!} J^k = \begin{pmatrix}
r^\lambda & r^\lambda \log r & \dots & \dots & \frac{r^\lambda}{(p-1)!}(\log r)^{p-1} \\
& r^\lambda & r^\lambda \log r & & \vdots \\
& & \ddots & \ddots & \vdots \\
& & & \ddots & r^\lambda \log r \\
& & & & r^\lambda
\end{pmatrix}.$$

Note that for small r we have

$$\max_{1\leq k,j\leq m} |\langle r^D e_k, e_j\rangle| = \frac{r^\lambda}{(p-1)!}(\log r)^{p-1}. \tag{5.24}$$

Furthermore, since

$$\Gamma(\theta) = \mathbb{E}\big[\big(\sum_{k=1}^m X_k(\theta)\big)^2\big] = \int_{\mathbb{R}^d} \Big(\sum_{k=1}^m |e^{i\langle\theta,y\rangle} - 1| \|\psi(y)^{-D-\frac{q}{2}} e_k\|\Big)^2 dy$$

is continuous and bounded on the unit sphere S_E, there exists $0 < M < \infty$ such that $\max_{\theta\in S_E} \Gamma(\theta) \leq M$. Combining this with (5.24) we obtain

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{k=1}^m X_k(l_E(x-y))\langle r^D e_k, e_j\rangle\Big)^2\Big] &\leq c r^{2\lambda}(\log r)^{2(p-1)}\mathbb{E}\Big[\Big(\sum_{k=1}^m X_k(l_E(x-y))\Big)^2\Big]\\ &\leq cM r^{2\lambda}(\log r)^{2(p-1)},\end{aligned}$$

which proves (5.21).

Now we consider the case that J is a $2p\times 2p$ matrix of the form

$$J = \begin{pmatrix} \Lambda & I_2 & & & \\ & \Lambda & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & \Lambda \end{pmatrix} \quad \text{with } \Lambda = \begin{pmatrix} \lambda_j & -\beta_j \\ \beta_j & \lambda_j \end{pmatrix}.$$

Let us define a $2p\times 2p$ matrix $M(A_1,\dots,A_p)$ by

$$M(A_1,\dots,A_p) = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ & \ddots & \ddots & \vdots \\ & & \ddots & A_2 \\ & & & A_1 \end{pmatrix}$$

so that we can write $J = M(\Lambda, I_2, 0, \dots, 0)$ and

$$\begin{aligned} r^J &= M\Big(r^\Lambda, r^\Lambda \log r, \dots, r^\Lambda \frac{(\log r)^{p-1}}{(p-1)!}\Big)\\ &= M(r^\Lambda, 0, \dots, 0)\cdot M\Big(I_2, I_2\log r, \dots, I_2\frac{(\log r)^{p-1}}{(p-1)!}\Big).\end{aligned}$$

By using the fact that $\|r^\Lambda u\| \leq r^{\lambda_j}\|u\|$ for every $u = (u_1, u_2)\in\mathbb{R}^2$ (see [36, p. 65]) we obtain for $u \in \mathbb{R}^{2p}$ that

$$\begin{aligned} \|r^J u\| &= \|M(r^\Lambda, 0, \ldots, 0) \cdot M\Big(I_2, I_2 \log r, \ldots, I_2 \frac{(\log r)^{p-1}}{(p-1)!}\Big) u\| \\ &\leq \|M(r^{\lambda_j} I_2, 0, \ldots, 0) \cdot M\Big(I_2, I_2 \log r, \ldots, I_2 \frac{(\log r)^{p-1}}{(p-1)!}\Big) u\|. \end{aligned} \tag{5.25}$$

Assume that the component vector of $X(l_E(x-y))$ corresponding to the block r^J of r^D is given by $\tilde{X}(l_E(x-y))$. Further let $(\tilde{e}_1, \ldots, \tilde{e}_{2p})$ be the canonical basis of $\mathbb{R}^{2p}$. Then for some $1 \leq i \leq 2p$ by the Cauchy-Schwarz inequality and (5.25)

$$\begin{aligned} \mathbb{E}\big[\big(X_j(x) - X_j(y)\big)^2\big] &= \mathbb{E}[\langle r^D X(l_E(x-y)), e_j\rangle^2] \\ &= \mathbb{E}[\langle r^J \tilde{X}(l_E(x-y)), \tilde{e}_i\rangle^2] \\ &\leq \mathbb{E}[\|r^J \tilde{X}(l_E(x-y))\|^2 \|\tilde{e}_i\|^2] \\ &\leq \mathbb{E}[\|M_1(r) M_2(r) \tilde{X}(l_E(x-y))\|^2] \\ &\leq \mathbb{E}[\|M_1(r) M_2(r)\|^2 \|\tilde{X}(l_E(x-y))\|^2] \end{aligned}$$

with $M_1(r) = M(r^{\lambda_j} I_2, 0, \ldots, 0)$ and $M_2(r) = M\Big(I_2, I_2 \log r, \ldots, I_2 \frac{(\log r)^{p-1}}{(p-1)!}\Big)$. Noting that $\max_{\theta \in S_E} \mathbb{E}[\|\tilde{X}(\theta)\|^2] \leq c$ the last expression can be estimated from above by

$$c\mathbb{E}\Big[\|M_1(r) M_2(r)\|^2\Big] \leq c\|M_1(r)\|^2 \|M_2(r)\|^2 \leq c r^{2\lambda_j} (\log r)^{2(p-1)},$$

where we used (5.22) in the first inequality and (5.23) in the last inequality. Hence, this proves (5.21).

Finally, (5.20) follows from (5.19) and (2.3) exactly as in the proof of Proposition 4.11. □

5.4. Hausdorff dimension of the sample paths

In this section we state our results on the Hausdorff dimension of the range and the graph of a trajectory of X over the unit cube $[0,1]^d$. Recall that $\mathbb{R}^d = W_1 \oplus \ldots \oplus W_p$ is the direct sum decomposition with respect to E, $\mu_j = \dim W_j$, $\tilde{\mu}_j = \dim W_{p+1-j}$, $\tilde{a}_j = a_{p+1-j}$ for $1 \leq j \leq p$ so that by (5.6)

$$0 < \lambda_1 \leq \ldots \leq \lambda_m < 1 < \tilde{a}_p < \ldots < \tilde{a}_1. \tag{5.26}$$

Theorem 5.9. *With probability one*

$$\begin{aligned} \dim_{\mathcal{H}} X([0,1]^d) &= \min\Big\{m; \frac{\sum_{k=1}^p a_k \mu_k + \sum_{i=1}^j (\lambda_j - \lambda_i)}{\lambda_j}, 1 \leq j \leq m\Big\} \\ &= \begin{cases} m & \text{if } \sum_{i=1}^m \lambda_i < \sum_{k=1}^p a_k \mu_k, \\ \frac{\sum_{k=1}^p a_k \mu_k + \sum_{i=1}^l (\lambda_l - \lambda_i)}{\lambda_l} & \text{if } \sum_{i=1}^{l-1} \lambda_i < \sum_{k=1}^p a_k \mu_k \leq \sum_{i=1}^l \lambda_i \end{cases} \end{aligned} \tag{5.27}$$

and

$$\dim_{\mathcal{H}} \mathrm{Gr}\, X([0,1]^d) = \min\Big\{ \dim_{\mathcal{H}} X([0,1]^d); G(l), 1 \le l \le p\Big\}$$

$$= \begin{cases} \dim_{\mathcal{H}} X([0,1]^d) & \text{if } \sum_{k=1}^{p} a_k\mu_k \le \sum_{i=1}^{m} \lambda_i, \\ G(l) & \text{if } \sum_{k=1}^{l-1} \tilde{a}_k\tilde{\mu}_k \le \sum_{i=1}^{m} \lambda_i < \sum_{k=1}^{l} \tilde{a}_k\tilde{\mu}_k, \end{cases} \tag{5.28}$$

where

$$G(l) = \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=l+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m}(1 - \frac{\lambda_i}{\tilde{a}_l}).$$

The second equality in (5.27) and the second equality in (5.28) are verified by the following elementary Lemma whose proof is analogous to the proof of Lemma 4.14. Denote

$$\zeta = \min\Big\{m, \frac{\sum_{k=1}^{p} a_k\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}, 1 \le j \le m\Big\}$$

and

$$\kappa = \min\Big\{\sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=l+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m}(1 - \frac{\lambda_i}{\tilde{a}_l}), 1 \le l \le p\Big\}.$$

Lemma 5.10. *If* (5.26) *holds then the following statements are true.*

(i) If $\sum_{i=1}^{l-1} \lambda_i < \sum_{k=1}^{p} a_k\mu_k \le \sum_{i=1}^{l} \lambda_i$ *for some* $1 \le l \le m$ *then*

$$\zeta = \frac{\sum_{k=1}^{p} a_k\mu_k + \sum_{i=1}^{l}(\lambda_l - \lambda_i)}{\lambda_l}$$

and $\zeta \in (l-1, l]$.

(ii) If $\sum_{i=1}^{m} \lambda_i < \sum_{k=1}^{p} a_k\mu_k$ *then* $\zeta = m$.

(iii) If there is $1 \le k \le p$ *such that* $\sum_{j=1}^{k-1} \tilde{a}_j\tilde{\mu}_j \le \sum_{i=1}^{m} \lambda_i < \sum_{j=1}^{k} \tilde{a}_j\tilde{\mu}_j$ *then*

$$\kappa = \sum_{j=1}^{k} \frac{\tilde{a}_j}{\tilde{a}_k}\tilde{\mu}_j + \sum_{j=k+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m}(1 - \frac{\lambda_i}{\tilde{a}_k})$$

and $\kappa \in \Big(m + \sum_{j=k+1}^{p} \tilde{\mu}_j, m + \sum_{j=k}^{p} \tilde{\mu}_j\Big]$.

Proof. We first prove (i). So assume that for some $1 \le l \le m$

$$\sum_{i=1}^{l-1} \lambda_i < \sum_{k=1}^{p} a_k\mu_k \le \sum_{i=1}^{l} \lambda_i. \tag{5.29}$$

Define

$$\zeta_j = \frac{\sum_{k=1}^{p} a_k\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}$$

for $1 \leq j \leq m$. We first show that $\zeta = \zeta_l$. Let $l \leq b \leq m$ and note that $\lambda_b \geq \lambda_l$. Then

$$\begin{aligned}
\zeta_l - \zeta_b &= l - b + (\lambda_b - \lambda_l) \cdot \frac{\sum_{k=1}^{p} a_k\mu_k}{\lambda_l\lambda_b} - \lambda_b \cdot \frac{\sum_{i=1}^{l} \lambda_i}{\lambda_l\lambda_b} + \lambda_l \cdot \frac{\sum_{i=1}^{b} \lambda_i}{\lambda_l\lambda_b} \\
&\leq l - b + (\lambda_b - \lambda_l) \cdot \frac{\sum_{i=1}^{l} \lambda_i}{\lambda_l\lambda_b} - \lambda_b \cdot \frac{\sum_{i=1}^{l} \lambda_i}{\lambda_l\lambda_b} + \lambda_l \cdot \frac{\sum_{i=1}^{b} \lambda_i}{\lambda_l\lambda_b} \\
&= l - b + \lambda_l \cdot \frac{\sum_{i=l+1}^{b} \lambda_i}{\lambda_l\lambda_b} = l - b + \frac{\sum_{i=l+1}^{b} \lambda_i}{\lambda_b} \\
&\leq l - b + b - l = 0,
\end{aligned}$$

where we used (5.26) in the last inequality. Thus, we have $\zeta_l \leq \zeta_b$ for $l \leq b \leq m$. Similarly one shows $\zeta_l \leq \zeta_b$ for $b \leq l \leq m$ so that $\zeta = \min_{1\leq j\leq m} \zeta_j = \zeta_l$, since by (5.29)

$$\zeta_l \leq \frac{\sum_{i=1}^{l} \lambda_l}{\lambda_l} = l \leq m$$

and

$$\zeta_l > \frac{\sum_{i=1}^{l} \lambda_l - \lambda_l}{\lambda_l} = l - 1.$$

Now we prove (ii). Assume that

$$\sum_{i=1}^{m} \lambda_i < \sum_{k=1}^{p} a_k\mu_k.$$

Then

$$\begin{aligned}
\zeta_j &> \frac{\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j} = \frac{\sum_{i=j+1}^{m} \lambda_i + \sum_{i=1}^{j} \lambda_j}{\lambda_j} \\
&\geq \sum_{i=j+1}^{m} 1 + \sum_{i=1}^{j} 1 = m
\end{aligned}$$

for all $1 \leq j \leq m$, where we used (5.26) in the last inequality. This shows $\zeta = m$.

We now turn to the proof of (iii). Suppose that

$$\sum_{j=1}^{k-1} \tilde{a}_j\tilde{\mu}_j \leq \sum_{i=1}^{m} \lambda_i < \sum_{j=1}^{k} \tilde{a}_j\tilde{\mu}_j \tag{5.30}$$

for some $1 \leq k \leq p$. For $1 \leq l \leq p$ define

$$\kappa_l = \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=l+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m}(1 - \frac{\lambda_i}{\tilde{a}_l}).$$

We want to show that $\kappa = \kappa_k$. First assume that $1 \le l \le k$. Then using (5.30) and (5.26)

$$\begin{aligned}\kappa_k - \kappa_l &= \sum_{j=1}^{k} \frac{\tilde{a}_j}{\tilde{a}_k}\tilde{\mu}_j - \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j - \sum_{j=l+1}^{k} \tilde{\mu}_j + (\frac{1}{\tilde{a}_l} - \frac{1}{\tilde{a}_k})\sum_{i=1}^{m} \lambda_i \\ &\le \sum_{j=1}^{k} \frac{\tilde{a}_j}{\tilde{a}_k}\tilde{\mu}_j - \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j - \sum_{j=l+1}^{k} \tilde{\mu}_j + (\frac{1}{\tilde{a}_l} - \frac{1}{\tilde{a}_k})\sum_{j=1}^{k-1} \tilde{a}_j\tilde{\mu}_j \\ &= \tilde{\mu}_k + \sum_{j=l+1}^{k-1} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j - \sum_{j=l+1}^{k} \tilde{\mu}_j \\ &\le \tilde{\mu}_k + \sum_{j=l+1}^{k-1} \tilde{\mu}_j - \sum_{j=l+1}^{k} \tilde{\mu}_j = 0,\end{aligned}$$

i.e. $\kappa_k \le \kappa_l$. On the other hand if $k \le l \le p$ we obtain from (5.30) and (5.26) that

$$\begin{aligned}\kappa_k - \kappa_l &= \sum_{j=1}^{k} \frac{\tilde{a}_j}{\tilde{a}_k}\tilde{\mu}_j - \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=k+1}^{l} \tilde{\mu}_j + (\frac{1}{\tilde{a}_l} - \frac{1}{\tilde{a}_k})\sum_{i=1}^{m} \lambda_i \\ &< \sum_{j=1}^{k} \frac{\tilde{a}_j}{\tilde{a}_k}\tilde{\mu}_j - \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=k+1}^{l} \tilde{\mu}_j + (\frac{1}{\tilde{a}_l} - \frac{1}{\tilde{a}_k})\sum_{j=1}^{k} \tilde{a}_j\tilde{\mu}_j \\ &= -\sum_{j=k+1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=k+1}^{l} \tilde{\mu}_j \\ &\le -\sum_{j=k+1}^{l} \tilde{\mu}_j + \sum_{j=k+1}^{l} \tilde{\mu}_j = 0,\end{aligned}$$

i.e. $\kappa_k < \kappa_l$. This proves $\kappa = \min_{1 \le l \le p} \kappa_l = \kappa_k$. Finally, by (5.30) we have

$$\kappa = \kappa_k \le \tilde{\mu}_k + \sum_{j=k+1}^{p} \tilde{\mu}_j + m = \sum_{j=k}^{p} \tilde{\mu}_j + m$$

and

$$\kappa = \kappa_k > \sum_{j=k+1}^{p} \tilde{\mu}_j + m.$$

□

Proposition 5.11. *Fix $1 \le j \le m$ and let $\varepsilon > 0$ be sufficiently small. Then the integral*

$$\mathcal{E}_\gamma = \int_{[0,1]^d} \int_{[0,1]^d} \tau_E(x-y)^{-\gamma\lambda_j + \sum_{i=1}^{j}(\lambda_j - \lambda_i)} dx dy$$

is finite for any

$$0 < \gamma \le \min\Big\{m, \frac{\sum_{k=1}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}\Big\}.$$

Proof. In this proof let c and c' be two unspecified positive constants. Note that

$$\mathcal{E}_\gamma \le \int_{\|x\| \le 2} \tau_E(x)^{-\gamma\lambda_j + \sum_{i=1}^{j}(\lambda_j - \lambda_i)} dx.$$

In order to show that the integral in the last expression is finite we observe that for any

$$0 < \gamma < \min\Big\{m, \frac{\sum_{k=1}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}\Big\}$$

there exists an integer $1 \leq l \leq p$ such that

$$\frac{\sum_{k=l+1}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j} < \gamma < \frac{\sum_{k=l}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}. \tag{5.31}$$

In the following we only consider the case $l = 1$, since the remaining cases are easier because they require less steps of integration using Lemma 4.7. So assuming (5.31) with $l = 1$ we can choose positive constants $\delta_2, \ldots, \delta_p$ such that $\delta_j > \frac{1+\varepsilon}{a_j}$, $2 \leq j \leq p$ and

$$\frac{\sum_{k=2}^{p} \frac{\mu_k}{\delta_k} + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j} < \gamma < \frac{\frac{a_1}{1+\varepsilon}\mu_1 + \sum_{k=2}^{p} \frac{\mu_k}{\delta_k} + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}. \tag{5.32}$$

Let $x = x_1 + \ldots + x_p$ for $x_i \in W_i, 1 \leq i \leq p$ be the direct sum decomposition with respect to E. Since the W_i are orthogonal in the associated euclidean norm, it follows that $\|x\| \leq 2$ implies $\|x_i\| \leq 2$ for $1 \leq i \leq p$. Then by Remark 2.10

$$\mathcal{E}_\gamma \leq c \int_{\|x_1\| \leq 2} \cdots \int_{\|x_p\| \leq 2} \big(\|x_1\|^{\frac{1+\varepsilon}{a_1}} + \ldots + \|x_p\|^{\frac{1+\varepsilon}{a_p}}\big)^{-\gamma\lambda_j + \sum_{i=1}^{j}(\lambda_j - \lambda_i)} dx_1 \ldots dx_p.$$

By using the change to (classical) polar coordinates we can further estimate

$$\mathcal{E}_\gamma \leq c \int_0^2 dr_1 \ldots \int_0^2 dr_p \big(r_1^{\frac{1+\varepsilon}{a_1}} + \ldots + r_p^{\frac{1+\varepsilon}{a_p}}\big)^{-\gamma\lambda_j + \sum_{i=1}^{j}(\lambda_j - \lambda_i)} \prod_{j=1}^{p} r_j^{\mu_j - 1}. \tag{5.33}$$

Applying Lemma 4.7 to the integral in (5.33) with

$$A = \sum_{j=1}^{p-1} r_j^{\frac{1+\varepsilon}{a_j}}, \quad u = \gamma\lambda_j - \sum_{i=1}^{j}(\lambda_j - \lambda_i) \quad \text{and} \quad k = \mu_p$$

we integrate with respect to dr_p in the last expression and obtain that

$$\mathcal{E}_\gamma \leq c' + c \int_0^2 dr_1 \ldots \int_0^2 dr_{p-1} \big(r_1^{\frac{1+\varepsilon}{a_1}} + \ldots + r_{p-1}^{\frac{1+\varepsilon}{a_{p-1}}}\big)^{-\gamma\lambda_j + \sum_{i=1}^{j}(\lambda_j - \lambda_i) + \frac{\mu_p}{\delta_p}} \prod_{j=1}^{p-1} r_j^{\mu_j - 1}.$$

By repeating this procedure $(p-2)$-times, we derive

$$\mathcal{E}_\gamma \leq c' + c \int_0^2 \big(r_1^{\frac{1+\varepsilon}{a_1}}\big)^{-\gamma\lambda_j + \sum_{i=1}^{j}(\lambda_j - \lambda_i) + \sum_{k=2}^{p} \frac{\mu_k}{\delta_k}} \cdot r_1^{\mu_1 - 1} dr_1. \tag{5.34}$$

Note that from (5.32) we get

$$\frac{1+\varepsilon}{a_1} \cdot \Big(-\gamma\lambda_j + \sum_{i=1}^{j}(\lambda_j - \lambda_i) + \sum_{k=2}^{p} \frac{\mu_k}{\delta_k}\Big) + \mu_1 - 1$$

$$> \frac{1+\varepsilon}{a_1} \cdot \Big(- \frac{a_1}{1+\varepsilon}\mu_1\Big) + \mu_1 - 1 = -1.$$

Thus, the integral in (5.34) is finite and this proves the assertion. □

Proposition 5.12. *Let $1 \leq k \leq p$ be an integer such that*

$$\sum_{j=1}^{k-1} \tilde{a}_j \tilde{\mu}_j \leq \sum_{i=1}^{m} \lambda_i < \sum_{j=1}^{k} \tilde{a}_j \tilde{\mu}_j.$$

For $\gamma > m$ define

$$\mathcal{G}_\gamma = \int_{[0,1]^d} \int_{[0,1]^d} \|x-y\|^{m-\gamma} \tau_E(x-y)^{-\sum_{i=1}^m \lambda_i} dxdy.$$

Then if $\varepsilon > 0$ is sufficiently small G_γ is finite for any

$$m < \gamma \leq \sum_{j=1}^{k} \frac{\tilde{a}_j}{\tilde{a}_k} \tilde{\mu}_j + \sum_{j=k+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m} (1 - \lambda_i \frac{1+\varepsilon}{\tilde{a}_k}).$$

Proof. Let us first note that by a change of variables

$$\mathcal{G}_\gamma \leq c \int_{\|x\| \leq 2} \|x\|^{m-\gamma} \tau_E(x)^{-\sum_{i=1}^m \lambda_i} dx.$$

Let c and c' be two unspecified positive constants. To simplify notation let $g_j = \frac{\tilde{a}_j}{1+\varepsilon}$, $1 \leq j \leq p$. Then by assumption for sufficiently small $\varepsilon > 0$ we have

$$\sum_{j=1}^{k-1} g_j \tilde{\mu}_j < \sum_{i=1}^{m} \lambda_i < \sum_{j=1}^{k} g_j \tilde{\mu}_j. \tag{5.35}$$

Let us write $x = x_1 + \ldots + x_k + y$ for $x_i \in \tilde{W}_i$, $1 \leq i \leq k$ and $y \in \tilde{W}_{k+1} \oplus \ldots \oplus \tilde{W}_p$. As in the proof of Proposition 4.20 we remark that

$$\begin{aligned} \|x\| \leq c\|x\|_2 = c\sqrt{\sum_{j=1}^{k} \|x_j\|_2^2 + \|y\|_2^2} &\leq c\Big(\sum_{j=1}^{k} \|x_j\|_2 + \|y\|_2 \Big) \\ &\leq c\Big(\sum_{j=1}^{k} \|x_j\| + \|y\| \Big) \end{aligned}$$

and that $\|x\| \leq 2$ implies $\|x_i\| \leq 2$, $1 \leq i \leq k$, and $\|y\| \leq 2$. Combining this with Remark 2.10 we get

$$\begin{aligned} \mathcal{G}_\gamma \leq c \int_{\|x_1\| \leq 2} \cdots \int_{\|x_k\| \leq 2} \int_{\|y\| \leq 2} &\Big(\sum_{j=1}^{k} \|x_j\|^{\frac{1}{g_j}} + \|y\|^{\frac{1}{g_p}} \Big)^{-\sum_{i=1}^m \lambda_i} \\ &\times \Big(\sum_{j=1}^{k} \|x_j\| + \|y\| \Big)^{m-\gamma} dx_1 \ldots dx_k dy. \end{aligned}$$

By using the change to (classical) polar coordinates we further get

$$\begin{aligned}\mathcal{G}_\gamma \le c\int_0^2 dr\int_0^2 dr_k \ldots \int_0^2 dr_1 \Big(\sum_{j=1}^k r_j^{\frac{1}{g_j}} + r^{\frac{1}{g_p}}\Big)^{-\sum_{i=1}^m \lambda_i} \\ \times \Big(\sum_{j=1}^k r_j + r\Big)^{m-\gamma} \prod_{j=1}^k r_j^{\tilde{\mu}_j - 1} \cdot r^{\sum_{j=k+1}^p \tilde{\mu}_j - 1}.\end{aligned} \tag{5.36}$$

In order to show that the integral in (5.36) is finite we will integrate $dr_1, \ldots, dr_k$ iteratively. Furthermore, we will assume that $k > 1$ in (5.35), since for $k = 1$ we can use (4.8) of Lemma 4.8 to obtain (5.38) directly. Indeed, if $k = 1$ in (5.35) we have

$$\frac{1}{g_1}\sum_{i=1}^m \lambda_i < \tilde{\mu}_1$$

so that by (4.8) of Lemma 4.8 with

$$B = r, \quad \alpha = \frac{1}{g_1}, \quad \beta = \sum_{i=1}^m \lambda_i, \quad \eta = \gamma - m, \quad k = \tilde{\mu}_1$$

we obtain

$$\mathcal{G}_\gamma \le c\int_0^2 r^{m-\gamma-\frac{1}{g_1}(\sum_{i=1}^m \lambda_i)+\tilde{\mu}_1} \cdot r^{\sum_{j=2}^p \tilde{\mu}_j - 1} dr,$$

which is (5.38) below with $k = 1$. So in the following assume that $k > 1$ in (5.35). Let us first integrate with respect to dr_1. Since by (5.35)

$$\frac{1}{g_1}\sum_{i=1}^m \lambda_i > \tilde{\mu}_1$$

we can use (4.6) of Lemma 4.8 with

$$A = \sum_{j=2}^k r_j^{\frac{1}{g_j}} + r^{\frac{1}{g_p}}, \quad B = \sum_{j=2}^k r_j + r, \quad \alpha = \frac{1}{g_1}, \quad \beta = \sum_{i=1}^m \lambda_i, \quad \eta = \gamma - m, \quad k = \tilde{\mu}_1$$

to get that

$$\begin{aligned}\mathcal{G}_\gamma \le c\int_0^2 dr\int_0^2 dr_k \ldots \int_0^2 dr_2 \Big(\sum_{j=2}^k r_j^{\frac{1}{g_j}} + r^{\frac{1}{g_p}}\Big)^{-\sum_{i=1}^m \lambda_i + g_1\tilde{\mu}_1} \\ \times \Big(\sum_{j=2}^k r_j + r\Big)^{m-\gamma} \prod_{j=2}^k r_j^{\tilde{\mu}_j - 1} \cdot r^{\sum_{j=k+1}^p \tilde{\mu}_j - 1}.\end{aligned}$$

Using (5.35), we can repeat this procedure for integration with respect to $dr_2, \ldots, dr_{k-1}$ and obtain

$$\mathcal{G}_\gamma \le c \int_0^2 dr \int_0^2 dr_k \Big(r_k^{\frac{1}{g_k}} + r^{\frac{1}{g_p}} \Big)^{-\sum_{i=1}^m \lambda_i + \sum_{j=1}^{k-1} g_j \tilde{\mu}_j} \\ \times \Big(r_k + r \Big)^{m-\gamma} r_k^{\tilde{\mu}_k - 1} \cdot r^{\sum_{j=k+1}^p \tilde{\mu}_j - 1}. \tag{5.37}$$

Since by (5.35) we have

$$\Big(\sum_{i=1}^m \lambda_i - \sum_{j=1}^{k-1} g_j \tilde{\mu}_j \Big) \frac{1}{g_k} < \tilde{\mu}_k$$

we can use (4.8) of Lemma 4.8 to the integral in (5.37) with

$$B = r, \quad \alpha = \frac{1}{g_k}, \quad \beta = \sum_{i=1}^m \lambda_i + \sum_{j=1}^{k-1} g_j \tilde{\mu}_j, \quad \eta = \gamma - m, \quad k = \tilde{\mu}_k$$

and obtain

$$\mathcal{G}_\gamma \le c \int_0^2 r^{m-\gamma-\frac{1}{g_k}\left(\sum_{i=1}^m \lambda_i - \sum_{j=1}^{k-1} g_j \tilde{\mu}_j\right) + \tilde{\mu}_k} \cdot r^{\sum_{j=k+1}^p \tilde{\mu}_j - 1} dr. \tag{5.38}$$

Observe that for

$$m < \gamma < \sum_{j=1}^k \frac{g_j}{g_k} \tilde{\mu}_j + \sum_{j=k+1}^p \tilde{\mu}_j + \sum_{i=1}^m (1 - \lambda_i \frac{1}{g_k})$$

we have

$$\begin{aligned} &m - \gamma - \frac{1}{g_k} \Big(\sum_{i=1}^m \lambda_i - \sum_{j=1}^{k-1} g_j \tilde{\mu}_j \Big) + \tilde{\mu}_k + \sum_{j=k+1}^p \tilde{\mu}_j - 1 \\ &> -\tilde{\mu}_k - \sum_{j=k+1}^p \tilde{\mu}_j + \tilde{\mu}_k + \sum_{j=k+1}^p \tilde{\mu}_j - 1 = -1. \end{aligned}$$

Thus, the integral in (5.38) is finite and this completes the proof. □

We now give a proof of Theorem 5.9 which further takes into account some methods used in the proof of [36, Theorem 4.1] and [53, Theorem 2.1].

Proof of Theorem 5.9. By the Jordan decomposition Theorem (see Chapter 5.3) there exists a real invertible matrix $A \in \mathbb{R}^{m\times m}$ such that $\tilde{D} = A^{-1}DA$ is of the real canonical form. Consider the random field Y given by

$$Y(x) = A^{-1}X(x), \quad x \in \mathbb{R}^d.$$

Then using the fact that $c^{\tilde{D}} = A^{-1}c^D A$ for any $c > 0$ (see [38, Proposition 2.2.2]) and the fact that X is (E, D)-operator-self-similar we get that

$$\begin{aligned} Y(c^E x) &= A^{-1}X(c^E x) \overset{\mathrm{d}}{=} A^{-1}c^D X(x) = A^{-1}c^D A A^{-1} X(x) \\ &= c^{\tilde{D}} A^{-1} X(x) = c^{\tilde{D}} Y(x) \end{aligned}$$

for any $c > 0$ and $x \in \mathbb{R}^d$. Hence, Y is an $(E, \tilde{D})$-operator-self-similar Gaussian random field in $\mathbb{R}^m$ and has stationary increments. Moreover, the mapping $y \mapsto Ay$ is bi-Lipschitz, since by [38, Proposition 2.1.3] we have

$$\frac{1}{\|A^{-1}\|}\|y\| \leq \|Ay\| \leq \|A\|\|y\|$$

for all $y \in \mathbb{R}^m$. Therefore, from Corollary 1.3 (ii) we get

$$\dim_{\mathcal{H}} Y([0,1]^d) = \dim_{\mathcal{H}} X([0,1]^d)$$

and

$$\dim_{\mathcal{H}} \operatorname{Gr} Y([0,1]^d) = \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d).$$

Thus, without loss of generality we may and will assume that D itself is of the real canonical form. Furthermore, by Proposition 5.8 there exists a modification X^* of X such that (5.20) holds. Since X^* is continuous, X and X^* are indistinguishable and we may also assume that X itself satisfies (5.20).

Let us now first prove (5.27). Since X satisfies (5.20), the upper bound in (5.27) follows from Lemma 5.5 and Remark 5.6. It remains to prove the lower bound in (5.27). By Frostman's theorem (see Chapter 1.2) it suffices to show that

$$\mathcal{E}_\gamma = \int_{[0,1]^d}\int_{[0,1]^d} \mathbb{E}[\|X(x) - X(y)\|^{-\gamma}]dxdy < \infty$$

to obtain that $\dim_{\mathcal{H}} X([0,1]^d) \geq \gamma$ almost surely. From now on let c and c' be two unspecified positive constants and let us further recall some well-known facts taken from [53, p. 279]. For any positive definite matrix $T \in \mathbb{R}^{m\times m}$ with rank m and any vector $u \in \mathbb{R}^m$ one can estimate

$$u^T T^{-1} u \geq c u^T u. \tag{5.39}$$

Furthermore, for any $a > 0$ we have

$$\int_0^\infty (y^2 + a^2)^{-\frac{\gamma}{2}} dy = c_1(\gamma) a^{-\gamma+1} \quad \text{for} \quad \gamma > 1 \tag{5.40}$$

and

$$\int_0^\infty (y^2 + a^2)^{-\frac{\gamma}{2}} e^{-y^p} dy = c_2(\gamma) a^{-\gamma+1} + c_3(\gamma) \quad \text{for} \quad 0 < \gamma < 1, p > 0, \tag{5.41}$$

where $c_1(\gamma)$, $c_2(\gamma)$ and $c_3(\gamma)$ are positive constants depending only on γ.

Note that since X is (E, D)-operator-self-similar with stationary increments

$$\begin{aligned} X(x) - X(y) &\overset{\mathrm{d}}{=} X(x-y) = X(\tau_E(x-y)^E l_E(x-y)) \\ &\overset{\mathrm{d}}{=} \tau_E(x-y)^D X(l_E(x-y)) \end{aligned}$$

for all $x, y \in \mathbb{R}^d$. For the covariance matrix this implies that

$$\begin{aligned}\det \operatorname{Cov}\big(X(x) - X(y)\big) &= \big(\det \tau_E(x-y)^D\big)^2 \det \operatorname{Cov} X\big(l_E(x-y)\big)\\ &= \prod_{j=1}^{m} \tau_E(x-y)^{2\lambda_j} \det \operatorname{Cov} X\big(l_E(x-y)\big),\end{aligned}$$

where in the last equality we used that D is of the real canonical form and the fact that $\det \tau_E(x-y)^D = \prod_{j=1}^m \tau_E(x-y)^{\lambda_j}$. Since X is Gaussian, continuous and proper, we have

$$\det \operatorname{Cov} X\big(l_E(x-y)\big) \geq \min_{\theta \in S_E} \det \operatorname{Cov} X(\theta) > 0.$$

For $x \neq y$ let

$$Y_j(x,y) = \frac{X_j(x) - X_j(y)}{\tau_E(x-y)^{\lambda_j}}, \quad j = 1, \ldots, m.$$

Then

$$\begin{aligned}\det \operatorname{Cov}\big(Y(x,y)\big) &= \frac{1}{\prod_{j=1}^m \tau_E(x-y)^{2\lambda_j}} \det \operatorname{Cov}\big(X(x) - X(y)\big)\\ &= \det \operatorname{Cov} X\big(l_E(x-y)\big) \geq c\end{aligned}$$

for some positive constant c independent of x and y with $x \neq y$. Since $Y(x,y)$ is Gaussian, this implies that $\operatorname{Cov}\big(Y(x,y)\big)$ is a positive definite matrix with rank m. Therefore, using (5.39) with $T = \operatorname{Cov}\big(Y(x,y)\big)$ and the fact that X is a Gaussian field with stationary increments by the definition of $Y(x,y)$ we obtain for $\gamma > 0$

$$\begin{aligned}\xi_\gamma &:= \mathbb{E}[\|X(x) - X(y)\|^{-\gamma}]\\ &= \int_{\mathbb{R}^m} \frac{1}{(2\pi)^{\frac{m}{2}}} \frac{1}{\sqrt{\det \operatorname{Cov}\big(Y(x,y)\big)}} \Big[\sum_{j=1}^m \big(u_j \tau_E(x-y)^{\lambda_j}\big)^2\Big]^{-\frac{\gamma}{2}}\\ &\qquad \times \exp\Big(-\frac{1}{2} u^T \operatorname{Cov}\big(Y(x,y)\big)^{-1} u\Big) du\\ &\leq c \int_{\mathbb{R}^m} \Big[\sum_{j=1}^m \big(u_j \tau_E(x-y)^{\lambda_j}\big)^2\Big]^{-\frac{\gamma}{2}} \exp\Big(-\sum_{j=1}^m u_j^2\Big) du_1 \ldots du_m.\end{aligned} \tag{5.42}$$

We now consider two cases. First assume that there exists an integer $1 \leq l \leq m$ such that

$$\sum_{i=1}^{l-1} \lambda_i < \sum_{k=1}^{p} a_k \mu_k \leq \sum_{i=1}^{l} \lambda_i.$$

Then by Lemma 5.10 we may and will assume that $l - 1 < \gamma < l$. We first integrate with respect to du_1 in the last integral in (5.42) by using $\exp(-u_1^2) \leq 1$ and (5.40) with

$$a = \Big(\sum_{j=2}^m \big(u_j \tau_E(x-y)^{\lambda_j - \lambda_1}\big)^2\Big)^{\frac{1}{2}}$$

so that

$$\begin{aligned}\xi_\gamma &\le c\tau_E(x-y)^{-\gamma\lambda_1}\cdot\int_{\mathbb{R}^m}\Big[u_1^2+\sum_{j=2}^m\big(u_j\tau_E(x-y)^{\lambda_j-\lambda_1}\big)^2\Big]^{-\frac{\gamma}{2}}\exp\Big(-\sum_{j=2}^m u_j^2\Big)du_1\dots du_m\\ &\le c\tau_E(x-y)^{-\gamma\lambda_1}\cdot\int_{\mathbb{R}^{m-1}}\Big[\sum_{j=2}^m\big(u_j\tau_E(x-y)^{\lambda_j-\lambda_1}\big)^2\Big]^{-\frac{(\gamma-1)}{2}}\exp\Big(-\sum_{j=2}^m u_j^2\Big)du_2\dots du_m\\ &\le c\tau_E(x-y)^{-\gamma\lambda_1-(\gamma-1)(\lambda_2-\lambda_1)}\cdot\int_{\mathbb{R}^{m-1}}\Big[u_2^2+\sum_{j=3}^m\Big(u_j\tau_E(x-y)^{\lambda_j-\lambda_2}\Big)^2\Big]^{-\frac{(\gamma-1)}{2}}\\ &\qquad\qquad\times\exp\Big(-\sum_{j=3}^m u_j^2\Big)du_2\dots du_m.\end{aligned}$$

By iterating this argument for integration with respect to $du_2,\dots,du_{l-1}$ we find that

$$\begin{aligned}\xi_\gamma \le c\tau_E(x-y)^{-\gamma\lambda_l+\sum_{i=1}^l(\lambda_l-\lambda_i)}\cdot\int_{\mathbb{R}^{m-l+1}}\Big[u_l^2+\sum_{j=l+1}^m\Big(u_j\tau_E(x-y)^{\lambda_j-\lambda_l}\Big)^2\Big]^{-\frac{(\gamma-l+1)}{2}}\\ \times\exp\Big(-\sum_{j=l}^m u_j^2\Big)du_l\dots du_m.\end{aligned}$$

Note that $0<\gamma-l+1<1$ by assumption so that we can use (5.41) to integrate with respect to du_l in the last expression and obtain that

$$\begin{aligned}\xi_\gamma \le c\tau_E(x-y)^{-\gamma\lambda_l+\sum_{i=1}^l(\lambda_l-\lambda_i)}\cdot\int_{\mathbb{R}^{m-l}}\bigg(\Big[\sum_{j=l+1}^m\Big(u_j\tau_E(x-y)^{\lambda_j-\lambda_l}\Big)^2\Big]^{-\frac{(\gamma-l)}{2}}+c'\bigg)\\ \times\exp\Big(-\sum_{j=l+1}^m u_j^2\Big)du_{l+1}\dots du_m.\end{aligned}$$

Since $-(\gamma-l)>0$ by assumption, we can get an upper estimate of the integral in the last expression by a change of polar coordinates and obtain

$$\begin{aligned}&c\int_{\mathbb{R}^{m-l}}\bigg(\Big[\sum_{j=l+1}^m u_j^2\Big]^{\frac{l-\gamma}{2}}+c'\bigg)\exp\Big(-\sum_{j=l+1}^m u_j^2\Big)du_{l+1}\dots du_m\\ &\le c\int_0^\infty\Big(r^{l-\gamma}+c'\Big)e^{-r^2}r^{m-l-1}dr<\infty.\end{aligned}$$

To summarize this we have

$$\xi_\gamma\le c\tau_E(x-y)^{-\gamma\lambda_l+\sum_{i=1}^l(\lambda_l-\lambda_i)}.$$

On the other hand if

$$\sum_{i=1}^m\lambda_i<\sum_{k=1}^p a_k\mu_k$$

by Lemma 5.10 we may assume that $\gamma<m$ and the above calculations show that

$$\xi_\gamma\le c\tau_E(x-y)^{-\gamma\lambda_m+\sum_{i=1}^m(\lambda_m-\lambda_i)}.$$

Altogether we obtain that for some $1 \leq l \leq m$

$$\mathcal{E}_\gamma \leq c \int_{[0,1]^d} \int_{[0,1]^d} \tau_E(x-y)^{-\gamma\lambda_l + \sum_{i=1}^{l}(\lambda_l - \lambda_i)} dx dy.$$

By Proposition 5.11 the above integral is finite for any

$$0 < \gamma < \min\Big\{m, \frac{\sum_{k=1}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{l}(\lambda_l - \lambda_i)}{\lambda_l}\Big\},$$

where $\varepsilon > 0$ is arbitrarily small. Frostman's theorem then yields

$$\dim_{\mathcal{H}} X([0,1]^d) \geq \min\Big\{m, \frac{\sum_{k=1}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{l}(\lambda_l - \lambda_i)}{\lambda_l}\Big\}$$

almost surely. Since this holds for any arbitrarily small $\varepsilon > 0$, the lower bound in (5.27) follows by letting $\varepsilon \to 0$.

We now turn to the proof of (5.28). First assume that

$$\sum_{k=1}^{p} a_k \mu_k \leq \sum_{i=1}^{m} \lambda_i.$$

Then combining (5.27), Corollary 1.3 (ii) and Lemma 5.5 we obtain almost surely

$$\begin{aligned}\min\Big\{\frac{\sum_{k=1}^{p} a_k\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}, 1 \leq j \leq m\Big\} &= \dim_{\mathcal{H}} X([0,1]^d) \\ \leq \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) &\leq \min\Big\{\frac{\sum_{k=1}^{p} a_k\mu_k + \sum_{i=1}^{j}(\lambda_j - \lambda_i)}{\lambda_j}, 1 \leq j \leq m\Big\}\end{aligned}$$

so that $\dim_{\mathcal{H}} X([0,1]^d) = \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$ almost surely. Now we consider the case that

$$\sum_{j=1}^{l-1} \tilde{a}_j \tilde{\mu}_j \leq \sum_{i=1}^{m} \lambda_i < \sum_{j=1}^{l} \tilde{a}_j \tilde{\mu}_j$$

for some $1 \leq l \leq p$. Then the upper bound in (5.28) follows from (5.20), Lemma 5.5 and Remark 5.6. It remains to prove $\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \geq \gamma$ almost surely for all

$$0 < \gamma < \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=l+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m}(1 - \frac{\lambda_i}{\tilde{a}_l}).$$

By Lemma 5.10 we may and will assume that $\gamma > m + \sum_{j=l+1}^{p} \tilde{\mu}_j$. Again by Frostman's theorem it is sufficient to show that

$$\mathcal{G}_\gamma = \int_{[0,1]^d} \int_{[0,1]^d} \mathbb{E}\big[(\|x-y\|^2 + \|X(x) - X(y)\|^2)^{-\frac{\gamma}{2}}\big] dx dy < \infty.$$

If $Y(x,y)$ is defined as above we obtain

$$\eta_\gamma := \mathbb{E}\big[(\|x-y\|^2 + \|X(x) - X(y)\|^2)^{-\frac{\gamma}{2}}\big]$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^m} \frac{1}{(2\pi)^{\frac{m}{2}}} \frac{1}{\sqrt{\det \operatorname{Cov}\left(Y(x,y)\right)}} \Big[\|x-y\|^2 + \sum_{i=1}^m \left(u_i \tau_E(x-y)^{\lambda_i}\right)^2\Big]^{-\frac{\gamma}{2}} \\
&\qquad \times \exp\Big(-\frac{1}{2} u^T \operatorname{Cov}\left(Y(x,y)\right)^{-1} u\Big) du \\
&\leq c \int_{\mathbb{R}^m} \Big[\|x-y\|^2 + \sum_{i=1}^m \left(u_i \tau_E(x-y)^{\lambda_i}\right)^2\Big]^{-\frac{\gamma}{2}} du_1 \dots du_m \\
&= c\, \tau_E(x-y)^{-\gamma\lambda_1} \cdot \int_{\mathbb{R}^m} \Big[u_1^2 + \frac{\|x-y\|^2}{\tau_E(x-y)^{2\lambda_1}} + \sum_{i=2}^m \left(u_i \tau_E(x-y)^{\lambda_i - \lambda_1}\right)^2\Big]^{-\frac{\gamma}{2}} du_1 \dots du_m.
\end{aligned}
$$

We first integrate with respect to du_1 using (5.40) to obtain that

$$
\eta_\gamma \leq c\, \tau_E(x-y)^{-\gamma\lambda_1} \cdot \int_{\mathbb{R}^{m-1}} \Big[\frac{\|x-y\|^2}{\tau_E(x-y)^{2\lambda_1}} + \sum_{i=2}^m \left(u_i \tau_E(x-y)^{\lambda_i-\lambda_1}\right)^2\Big]^{-\frac{(\gamma-1)}{2}} du_2 \dots du_m.
$$

Since $\gamma > m$, we can repeat this procedure for integration with respect to $du_2, \dots, du_m$ and obtain that

$$
\begin{aligned}
\eta_\gamma &\leq c\, \tau_E(x-y)^{-\gamma\lambda_m + \sum_{i=1}^m (\lambda_m - \lambda_i)} \|x-y\|^{-(\gamma-m)} \tau_E(x-y)^{(\gamma-m)\lambda_m} \\
&= c\, \tau_E(x-y)^{-\sum_{i=1}^m \lambda_i} \|x-y\|^{-(\gamma-m)}
\end{aligned}
$$

so that

$$
\mathcal{G}_\gamma \leq c \int_{[0,1]^d} \int_{[0,1]^d} \tau_E(x-y)^{-\sum_{i=1}^m \lambda_i} \|x-y\|^{-(\gamma-m)} dx dy.
$$

By Proposition 5.12 the above integral is finite for all

$$
m < \gamma < \sum_{j=1}^l \frac{\tilde{a}_j}{\tilde{a}_l} \tilde{\mu}_j + \sum_{j=l+1}^p \tilde{\mu}_j + \sum_{i=1}^m (1 - \lambda_i \frac{1+\varepsilon}{\tilde{a}_l})
$$

with an arbitrarily small $\varepsilon > 0$ and we obtain

$$
\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) \geq \sum_{j=1}^l \frac{\tilde{a}_j}{\tilde{a}_l} \tilde{\mu}_j + \sum_{j=l+1}^p \tilde{\mu}_j + \sum_{i=1}^m (1 - \lambda_i \frac{1+\varepsilon}{\tilde{a}_l})
$$

almost surely. Therefore, the lower bound in (5.28) follows by letting $\varepsilon \to 0$. This completes the proof of Theorem 5.9. □

Let us close this chapter with the following two examples.

Example 5.13. As noted in Chapter 5.1 the operator fractional Brownian motion is (I_d, D)-operator-self-similar. Let us assume that $E = I_d$ in Theorem 5.2 and Theorem 5.3. Then Theorem 5.9 can be written as

$$
\dim_{\mathcal{H}} X([0,1]^d) = \min\Big\{m, \frac{d + \sum_{i=1}^j (\lambda_j - \lambda_i)}{\lambda_j}, 1 \leq j \leq m\Big\}
$$

and

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = \begin{cases} \dim_{\mathcal{H}} X([0,1]^d) & \text{if } d \leq \sum_{i=1}^{m} \lambda_i, \\ d + \sum_{i=1}^{m} (1-\lambda_i) & \text{if } d > \sum_{i=1}^{m} \lambda_i \end{cases}$$

almost surely, which coincides with (5.2) and (5.3).

Example 5.14. Assume that $D = H \cdot I_m$ for some $0 < H < 1$, i.e. D is a diagonal matrix with constant diagonal entries H. In this situation the random field X coincides with the same random field as in Example 4.25 with $\alpha = 2$ and Theorem 5.9 further becomes

$$\dim_{\mathcal{H}} X([0,1]^d) = \min\Big\{m, \sum_{k=1}^{p} \frac{a_k}{H}\mu_k\Big\}$$

and

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = \min\Big\{\sum_{k=1}^{p} \frac{a_k}{H}\mu_k; \sum_{k=1}^{l} \frac{\tilde{a}_k}{\tilde{a}_l}\tilde{\mu}_k + \sum_{k=l+1}^{p} \tilde{\mu}_k + (1 - \frac{H}{\tilde{a}_l})m, 1 \leq l \leq p\Big\}$$

almost surely so that we recover the results in Example 4.25.

6. Multivariate stable harmonizable operator-self-similar random fields

The purpose of this chapter is to establish the corresponding results of Chapter 5 for α-stable harmonizable operator-self-similar random fields. Indeed, we will see that these fields have the same kind of regularity properties as the Gaussian fields given in Chapter 5. We first recall the definition of stable harmonizable operator-self-similar random fields from [33].

6.1. Definition and existence

Throughout this chapter, let us adopt the notation of the preceding chapter and assume that (5.6) holds. Let $\psi : \mathbb{R}^d \to [0, \infty)$ be a continuous E^T-homogeneous function according to Definition 2.3 and assume that $\psi(x) \neq 0$ for $x \neq 0$. Moreover, let $\tilde{M}_\alpha$, $\alpha \in (0, 2)$, be a $\mathbb{C}^m$-valued SαS random measure on $\mathbb{R}^d$ as introduced in Chapter 3.3. Recall that $q = \text{trace}(E)$. The following is due to [33, Theorem 2.6].

Theorem 6.1. *If* (5.6) *holds the random field*

$$X_\alpha(x) = \text{Re} \int_{\mathbb{R}^d} (e^{i\langle x,y\rangle} - 1)\psi(y)^{-D-\frac{q}{\alpha}I_m} \tilde{M}_\alpha(dy), \quad x \in \mathbb{R}^d \tag{6.1}$$

is well-defined, proper, stochastically continuous and (E, D)-operator-self-similar SαS with stationary increments.

Let us recall that X_α is well-defined, since the kernel matrix in (6.1) satisfies

$$\int_{\mathbb{R}^d} |\exp^{i\langle x,y\rangle} - 1|^\alpha \|\psi(y)^{-D-\frac{q}{\alpha}I_m}\|^\alpha dy < \infty$$

for all $x \in \mathbb{R}^d$, which is shown in the proof of [33, Theorem 2.6]. Moreover, according to Theorem 3.14 the characteristic function of $X_\alpha(x)$, $x \in \mathbb{R}^d$, is given by

$$\begin{aligned}
&\mathbb{E}\big[\exp\left(i\langle\theta, X_\alpha(x)\rangle\right)\big] \\
&= \exp\left(-\int_{\mathbb{R}^d} \sqrt{(1-\cos\langle x,y\rangle)^2 + (\sin\langle x,y\rangle)^2}^{\,\alpha} \|\psi(y)^{-D^T-\frac{q}{\alpha}I_m}\theta\|^\alpha dy\right) \\
&= \exp\left(-\int_{\mathbb{R}^d} |e^{i\langle x,y\rangle} - 1|^\alpha \|\psi(y)^{-D^T-\frac{q}{\alpha}I_m}\theta\|^\alpha dy\right)
\end{aligned} \tag{6.2}$$

for any $\theta \in \mathbb{R}^m$.

Before studying the sample paths of the random field given in (6.1) we establish results about exponential powers of linear operators in the next section.

6.2. Exponential powers of linear operators

The following Proposition is due to [38, Proposition 2.2.11].

Proposition 6.2. *Let $A \in \mathbb{R}^{m \times m}$ be a matrix and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^m$. The following statements hold.*

(i) If every eigenvalue of A has real part less than β_1 then for any $t_0 > 0$ there exists a constant $C > 0$ such that $\|t^A u\| \geq C t^{\beta_1} \|u\|$ holds for all $0 < t \leq t_0$ and all $u \in \mathbb{R}^m$.

(ii) If every eigenvalue of A has real part less than β_2 then for any $s_0 > 0$ there exists a constant $C > 0$ such that $\|s^A u\| \leq C s^{\beta_2} \|u\|$ holds for all $s \geq s_0$ and all $u \in \mathbb{R}^m$.

Corollary 6.3. *Assume that D is of the real canonical form (see Chapter 5.3) and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^m$. The following statements hold.*

(i) For any $t_0 > 0$ there exists a constant $C_{7,1} > 0$ such that for any $\varepsilon > 0$

$$\|t^D \theta\| \geq C_{7,1} \sum_{j=1}^{m} t^{\lambda_j + \varepsilon} |\theta_j|$$

holds for all $0 < t \leq t_0$ and all $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$.

(ii) For any $s_0 > 0$ there exists a constant $C_{7,2} > 0$ such that for any $\varepsilon > 0$

$$\|s^{-D} \theta\| \leq C_{7,2} \sum_{j=1}^{m} s^{-\lambda_j + \varepsilon} |\theta_j|$$

holds for all $s \geq s_0$ and all $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$.

Proof. We only prove part (i). Part (ii) is proven exactly the same way. In this proof let c be an unspecified positive constant. Assume that the distinct real parts of the eigenvalues of D are given by $\overline{\lambda}_1, \ldots, \overline{\lambda}_k$ for some $k \leq m$ and let us write

$$D = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

for some block matrices J_j so that each J_j is associated with $\overline{\lambda}_j$, $1 \leq j \leq k$. Furthermore, write $\theta = (\overline{\theta}_1, \ldots, \overline{\theta}_k)$ for any $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$ and let $\|\cdot\|_1$ be the 1-norm. Then applying Proposition 6.2 and noting that every eigenvalue of each J_j has real part less than $\overline{\lambda}_j + \varepsilon$, $1 \leq j \leq k$ for all $\varepsilon > 0$, $t_0 > 0$ and all $0 < t \leq t_0$ we have

$$\|t^D \theta\| \geq c \|t^D \theta\|_1 = c \sum_{j=1}^{k} \|t^{J_j} \overline{\theta}_j\|_1$$

$$\geq c \sum_{j=1}^{k} t^{\overline{\lambda}_j+\varepsilon} \|\overline{\theta}_j\|_1 = c \sum_{j=1}^{m} t^{\lambda_j+\varepsilon} |\theta_j|,$$

where we used the equivalence of norms in the first inequality. □

6.3. Uniform modulus of continuity

As before let $(\tau_E(x), l_E(x))$ denote the generalized polar coordinates with respect to E. For notational convenience let us suppress the subscript α and simply write X instead of X_α. The following is the main result of this section.

Proposition 6.4. *Let the assumptions of Theorem 6.1 hold and suppose that D is of the real canonical form. Then there exists a modification X^* of X such that for any $\varepsilon > 0$ and any $\delta > 0$*

$$\sup_{\substack{x,y \in [0,1]^d \\ x \neq y}} \frac{|X_j^*(x) - X_j^*(y)|}{\tau_E(x-y)^{\lambda_j - \varepsilon} \big[\log\big(1 + \tau_E(x-y)^{-1}\big)\big]^{\delta + \frac{1}{2} + \frac{1}{\alpha}}} < \infty \tag{6.3}$$

holds almost surely for all $1 \leq j \leq m$. In particular, for every $\varepsilon > 0$ and $j = 1, \dots, m$ there exists a constant $C_{7,3} > 0$ such that X^* almost surely satisfies

$$|X_j^*(x) - X_j^*(y)| \leq C_{7,3} \tau_E(x-y)^{\lambda_j - \varepsilon} \tag{6.4}$$

for all $x, y \in [0,1]^d$.

The proof of Proposition 6.4 takes into account some methods used in the proof of [11, Proposition 5.1] and the key point is to remark that the components $X_j, 1 \leq j \leq m$, behave like 1-dimensional operator scaling harmonizable random fields given in [9].

Proof. Fix $1 \leq j \leq m$ and denote by $(e_1, \dots, e_m)$ the canonical basis of $\mathbb{R}^m$. The main idea is to apply Proposition 4.10 with an appropriate choice of the function ψ_α. Indeed, let Y be the random field given in (4.12), more precisely

$$Y(x) = \operatorname{Re} \int_{\mathbb{R}^d} (e^{i\langle x,\xi\rangle} - 1) \psi_\alpha(\xi) M_\alpha(d\xi)$$

for any $x \in \mathbb{R}^d$, with

$$\psi_\alpha(\xi) = \|\psi(\xi)^{-D^T - \frac{q}{\alpha} I_m} e_j\|,$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^m$. We now show that

$$\{X_j(x) : x \in \mathbb{R}^d\} \stackrel{\text{f.d.}}{=} \{Y(x) : x \in \mathbb{R}^d\}. \tag{6.5}$$

Let $t_1, \dots, t_n \in \mathbb{R}^d$ and $\theta_1, \dots, \theta_n \in \mathbb{R}$. By (6.2) and Theorem 3.11 we have

$$\mathbb{E}\Big[\exp\Big(i \sum_{k=1}^{n} \theta_k X_j(t_k)\Big)\Big] = \mathbb{E}\Big[\exp\Big(i \langle \sum_{k=1}^{n} \theta_k e_j, X(t_k)\rangle\Big)\Big]$$

$$= \exp\Big(-\int_{\mathbb{R}^d}\Big|\sum_{k=1}^n \theta_k(e^{i\langle t_k,y\rangle}-1)\Big|^\alpha \|\psi(y)^{-D^T-\frac{q}{\alpha}I_m}e_j\|^\alpha dy\Big)$$
$$= \mathbb{E}\Big[\exp\Big(i\sum_{k=1}^n \theta_k Y(t_k)\Big)\Big]$$

and this proves (6.5). Let c be an unspecified positive constant. Let us remark that Corollary 6.3 can also be applied to the block diagonal matrix $D^T + \frac{q}{\alpha}I_m$. Then using this, the fact that ψ is E^T-homogeneous and (2.1) we see that

$$\begin{aligned}\psi_\alpha(y) = \|\psi(y)^{-D^T-\frac{q}{\alpha}I_m}e_j\| &\le c\psi(y)^{-\lambda_j+\varepsilon-\frac{q}{\alpha}}\\ &= c\psi\big(\tau_{E^T}(y)^{E^T}l_{E^T}(y)\big)^{-\lambda_j+\varepsilon-\frac{q}{\alpha}}\\ &= c\tau_{E^T}(y)^{-\lambda_j+\varepsilon-\frac{q}{\alpha}}\cdot\psi\big(l_{E^T}(y)\big)^{-\lambda_j+\varepsilon-\frac{q}{\alpha}}\\ &\le c_\psi\tau_{E^T}(y)^{-(\lambda_j-\varepsilon)-\frac{q}{\alpha}}\end{aligned}$$

for all $y \in \mathbb{R}^d$ with $\|y\| > K > 0$ and some $c_\psi \in (0,\infty)$. Thus, Proposition 4.10 applies and yields that there exists a modification Y^* of Y such that Y^* satisfies (4.13) with $\beta = \lambda_j - \varepsilon$. Furthermore, since Y^* is a modification of Y, for $\theta_1,\ldots,\theta_n \in \mathbb{R}$, $t_1,\ldots,t_n \in \mathbb{R}^d$ we have

$$\sum_{i=1}^n \theta_i Y^*(t_i) = \sum_{i=1}^n \theta_i Y(t_i)$$

almost surely, which implies that

$$\{Y^*(x) : x \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{Y(x) : x \in \mathbb{R}^d\}.$$

Combining this with (6.5) yields

$$\{Y^*(x) : x \in \mathbb{R}^d\} \overset{\text{f.d.}}{=} \{X_j(x) : x \in \mathbb{R}^d\}. \tag{6.6}$$

Let us now show that there exists a (random) constant C such that

$$|X_j(x) - X_j(y)| \le C\tau_E(x-y)^{\lambda_j-\varepsilon}\Big[\log\Big(1+\tau_E(x-y)^{-1}\Big)\Big]^{\delta+\frac{1}{2}+\frac{1}{\alpha}}$$

for all $x, y \in [0,1]^d \cap \mathbb{Q}^d$, $x \neq y$, almost surely. Indeed, from (6.6) for any $x, y \in [0,1]^d$, $x \neq y$, we get

$$\begin{aligned}&P\bigg(|X_j(x)-X_j(y)| \le C\tau_E(x-y)^{\lambda_j-\varepsilon}\Big[\log\Big(1+\tau_E(x-y)^{-1}\Big)\Big]^{\delta+\frac{1}{2}+\frac{1}{\alpha}}\bigg)\\ &= P\bigg(|Y^*(x)-Y^*(y)| \le C\tau_E(x-y)^{\lambda_j-\varepsilon}\Big[\log\Big(1+\tau_E(x-y)^{-1}\Big)\Big]^{\delta+\frac{1}{2}+\frac{1}{\alpha}}\bigg)\\ &= 1,\end{aligned}$$

which yields that $P(\Omega^*) = 1$ for

$$\Omega^* := \Big\{ |X_j(x) - X_j(y)| \le C\tau_E(x-y)^{\lambda_j - \varepsilon} \Big[\log\Big(1 + \tau_E(x-y)^{-1}\Big)\Big]^{\delta + \frac{1}{2} + \frac{1}{\alpha}} \\ \forall x, y \in [0,1]^d \cap \mathbb{Q}^d, x \neq y \Big\},$$

since $[0,1]^d \cap \mathbb{Q}^d$ is countable. We now define a modification X_j^* of X_j on $[0,1]^d$ such that (6.3) holds with probability one as follows. If $\omega \notin \Omega^*$ we set

$$X_j^*(u)(\omega) = 0$$

for all $u \in [0,1]^d$. Now suppose that $\omega \in \Omega^*$. Then for any $u \in [0,1]^d \cap \mathbb{Q}^d$ we set

$$X_j^*(u)(\omega) = X_j(u)(\omega).$$

Now assume that $u \in [0,1]^d$ is arbitrary and $\omega \in \Omega^*$. Then since $[0,1]^d \cap \mathbb{Q}^d$ is dense in $[0,1]^d$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $[0,1]^d \cap \mathbb{Q}^d$ such that $\lim_{n\to\infty} u_n = u$. It follows from Corollary 2.6 that

$$\begin{aligned} &\lim_{m,n\to\infty} |X_j^*(u_n)(\omega) - X_j^*(u_m)(\omega)| \\ &\le \lim_{m,n\to\infty} C\tau_E(u_n - u_m)^{\lambda_j - \varepsilon} \Big[\log\Big(1 + \tau_E(u_n - u_m)^{-1}\Big)\Big]^{\delta + \frac{1}{2} + \frac{1}{\alpha}} \\ &= 0 \quad a.s. \end{aligned}$$

so that $\big(X_j^*(u_n)(\omega)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and, thus converges. We set

$$X_j^*(u)(\omega) = \lim_{n\to\infty} X_j^*(u_n)(\omega) = \lim_{n\to\infty} X_j(u_n)(\omega).$$

Note that this limit does not depend on the choice of $(u_n)_{n\in\mathbb{N}}$ and that X_j^* is well-defined. We now show that X_j^* is a modification of X_j. Let $u \in [0,1]^d$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]^d \cap \mathbb{Q}^d$ with $u_n \to u$ as $n \to \infty$. Since X_j is stochastically continuous, we have

$$X_j(u) = \operatorname*{plim}_{n\to\infty} X_j(u_n).$$

Moreover, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that

$$X_j(u) = \lim_{k\to\infty} X_j(u_{n_k}) \quad a.s.$$

Using this, from the definition of X_j^* we get

$$P\big(X_j^*(u) = X_j(u)\big) = P\big(\lim_{k\to\infty} X_j(u_{n_k}) = X_j(u)\big) = 1,$$

which shows that X_j^* is a modification of X_j. Let us now show that X_j^* satisfies (6.3) almost surely. Recall that τ_E is continuous. Then for any $u, v \in [0,1]^d$ we can find a constant C such that

$$\begin{aligned}|X_j^*(u)(\omega) - X_j^*(v)(\omega)| &= \lim_{n\to\infty} |X_j^*(u_n)(\omega) - X_j^*(v_n)(\omega)| \\ &\leq \lim_{n\to\infty} C\tau_E(u_n - v_n)^{\lambda_j - \varepsilon}\Big[\log\Big(1 + \tau_E(u_n - v_n)^{-1}\Big)\Big]^{\delta + \frac{1}{2} + \frac{1}{\alpha}} \\ &= C\tau_E(u - v)^{\lambda_j - \varepsilon}\Big[\log\Big(1 + \tau_E(u - v)^{-1}\Big)\Big]^{\delta + \frac{1}{2} + \frac{1}{\alpha}}\end{aligned}$$

for every $\omega \in \Omega$. This proves (6.3). Finally, (6.4) follows from (6.3) and (2.4) exactly as in the proof of Proposition 4.11. This concludes the proof. □

Proposition 6.4 compared to Proposition 5.8 shows that (E, D)-operator-self-similar stable random fields satisfy the same generalized Hölder condition with respect to the matrix E as the Gaussian ones. Therefore, it is natural to have also the same results of Theorem 5.9 for the Hausdorff dimension of their images and graphs on $[0,1]^d$, which we state in the next section.

6.4. Hausdorff dimension of the sample paths

As in the previous chapter let $\mathbb{R}^d = W_1 \oplus \ldots \oplus W_p$ be the direct sum decomposition with respect to E, define $\mu_j = \dim W_j$, $\tilde{\mu}_j = \dim W_{p+1-j}$, $\tilde{a}_j = a_{p+1-j}$ for $1 \leq j \leq p$ and assume that (5.26) holds. As before let $X = X_\alpha$ for some $\alpha \in (0,2)$ be the random field given by (6.1).

Theorem 6.5. *With probability one*

$$\begin{aligned}\dim_{\mathcal{H}} X([0,1]^d) &= \min\Big\{m; \frac{\sum_{k=1}^p a_k\mu_k + \sum_{i=1}^j (\lambda_j - \lambda_i)}{\lambda_j}, 1 \leq j \leq m\Big\} \\ &= \begin{cases} m & \text{if } \sum_{i=1}^m \lambda_i < \sum_{k=1}^p a_k\mu_k, \\ \frac{\sum_{k=1}^p a_k\mu_k + \sum_{i=1}^l (\lambda_l - \lambda_i)}{\lambda_l} & \text{if } \sum_{i=1}^{l-1} \lambda_i < \sum_{k=1}^p a_k\mu_k \leq \sum_{i=1}^l \lambda_i \end{cases}\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\dim_{\mathcal{H}} \mathrm{Gr}\, X([0,1]^d) &= \min\Big\{\dim_{\mathcal{H}} X([0,1]^d); G(l), 1 \leq l \leq p\Big\} \\ &= \begin{cases} \dim_{\mathcal{H}} X([0,1]^d) & \text{if } \sum_{k=1}^p a_k\mu_k \leq \sum_{i=1}^m \lambda_i, \\ G(l) & \text{if } \sum_{k=1}^{l-1} \tilde{a}_k\tilde{\mu}_k \leq \sum_{i=1}^m \lambda_i < \sum_{k=1}^l \tilde{a}_k\tilde{\mu}_k, \end{cases}\end{aligned} \tag{6.8}$$

where

$$G(l) = \sum_{j=1}^l \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=l+1}^p \tilde{\mu}_j + \sum_{i=1}^m (1 - \frac{\lambda_i}{\tilde{a}_l}).$$

Before proving Theorem 6.5 we first prove two Lemmata. The result stated in the following Lemma is taken from [53, p. 283]. However, since the proof is omitted in [53], for the sake of completeness let us prove it.

Lemma 6.6. *Let Y be a random vector with values in $\mathbb{R}^m$ and characteristic function ϕ. Then for each $\gamma > 0$*

$$2^{\frac{\gamma}{2}-1}\Gamma(\frac{\gamma}{2})\mathbb{E}[\|Y\|^{-\gamma}] = (2\pi)^{-\frac{m}{2}} \int_0^\infty \int_{\mathbb{R}^m} \exp\big(-\frac{\|y\|^2}{2}\big)\phi(uy)dyu^{\gamma-1}du. \tag{6.9}$$

Proof. By the definition of ϕ the right hand side of (6.9) equals

$$(2\pi)^{-\frac{m}{2}} \int_0^\infty \int_{\mathbb{R}^m} u^{\gamma-1} \exp\big(-\frac{\|y\|^2}{2}\big) \int_{\mathbb{R}^m} \exp\big(i\langle x, uy\rangle\big) P_Y(dx)dydu,$$

where P_Y denotes the distribution of Y. Using Fubini's theorem and the characteristic function of a multivariate normal distribution the last expression becomes

$$\begin{aligned}&(2\pi)^{-\frac{m}{2}} \int_0^\infty \int_{\mathbb{R}^m} u^{\gamma-1} \int_{\mathbb{R}^m} \exp\big(i\langle ux, y\rangle\big) \exp\big(-\frac{\|y\|^2}{2}\big)dyP_Y(dx)du\\ &= (2\pi)^{-\frac{m}{2}} \int_0^\infty \int_{\mathbb{R}^m} u^{\gamma-1}(2\pi)^{\frac{m}{2}} \exp\big(-\frac{1}{2}u^2\|x\|^2\big)P_Y(dx)du.\end{aligned}$$

By using the substitution $v = u\|x\|$ and Fubini's theorem again the last expression further equals

$$\begin{aligned}&\int_{\mathbb{R}^m} \int_0^\infty v^{\gamma-1}\|x\|^{1-\gamma} \exp\big(-\frac{1}{2}v^2\big)\|x\|^{-1}dvP_Y(dx)\\ &= \int_{\mathbb{R}^m} \|x\|^{-\gamma}P_Y(dx) \cdot \int_0^\infty v^{\gamma-1} \exp\big(-\frac{1}{2}v^2\big)dv\end{aligned}$$

with

$$\begin{aligned}\int_0^\infty v^{\gamma-1} \exp\big(-\frac{1}{2}v^2\big)dv &= \int_0^\infty 2^{\frac{\gamma}{2}-1}u^{\frac{\gamma}{2}-1} \exp(-u)du\\ &= 2^{\frac{\gamma}{2}-1}\Gamma(\frac{\gamma}{2}),\end{aligned}$$

where we used the change of variables $u = \frac{1}{2}v^2$ in the first equality. This proves (6.9). □

The following Lemma shows that the assumption (H_3) made in [53, Section 3], which states that there is an upper bound for the characteristic function of multivariate SαS random fields in terms of the characteristic function of their components, is superfluous in order to determine the Hausdorff dimension of the range and the graph of the sample paths of multivariate α-stable random fields. This assumption is used in the proof of [53, Theorem 3.1] in order to derive a statement which coincides with the statement of the following Lemma. However, in the following proof we will see that the aforementioned assumption is in fact superfluous.

Lemma 6.7. *Assume that D is of the real canonical form. Then for all $t \in [0,1]^d$, $\theta \in \mathbb{R}^m$*

and $\varepsilon > 0$ there exists a constant $C_{7,4} > 0$, depending only on ε, such that

$$\mathbb{E}\big[\exp\big(i\langle X(t),\theta\rangle\big)\big] \le \exp\Big(-C_{7,4}\sum_{j=1}^{m}\big|\tau_E(t)^{\lambda_j+\varepsilon}\theta_j\big|^{\alpha}\Big).$$

Let us recall that the characteristic function of the SαS random vector $X(t)$ is given by (6.2).

Proof. Let $(\tau_E(t), l_E(t))$ be the generalized polar coordinates of t with respect to E and recall that $S_E = \{t \in \mathbb{R}^d : \tau_E(t) = 1\}$. Further, let c be an unspecified positive constant. Using (6.2) and the change to generalized polar coordinates we get

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(i\langle X(t),\theta\rangle\big)\big]\\
&= \exp\Big(-\int_{\mathbb{R}^d}\big|\exp\big(i\langle \tau_E(t)^E l_E(t), y\rangle\big)-1\big|^{\alpha}\big\|\psi(y)^{-D^T-\frac{q}{\alpha}I_m}\theta\big\|^{\alpha}dy\Big)\\
&= \exp\Big(-\int_{\mathbb{R}^d}\big|\exp\big(i\langle l_E(t), \tau_E(t)^{E^T} y\rangle\big)-1\big|^{\alpha}\big\|\psi(y)^{-D^T-\frac{q}{\alpha}I_m}\theta\big\|^{\alpha}dy\Big)\\
&= \exp\Big(-\int_{\mathbb{R}^d}\big|\exp\big(i\langle l_E(t), z\rangle\big)-1\big|^{\alpha}\big\|\tau_E(t)^{D^T+\frac{q}{\alpha}I_m}\psi(z)^{-D^T-\frac{q}{\alpha}I_m}\theta\big\|^{\alpha}\tau_E(t)^{-q}dz\Big),
\end{aligned}$$

where we used the substitution $z = \tau_E(t)^{E^T}y$, $dz = \tau_E(t)^q dy$ and the E^T-homogeneity of ψ in the last equality. Let us note that the function

$$\Gamma(\xi) = \int_{\mathbb{R}^d}|e^{i\langle\xi,z\rangle-1}|^{\alpha}\psi(z)^{-(\lambda_m-\varepsilon)\alpha-q}dz$$

is positive and finite on the compact S_E and, hence due to the continuity of Γ

$$m_\alpha := \min_{\xi\in S_E}\Gamma(\xi) > 0.$$

Using this and Corollary 6.3 the above calculations show that for all $t \in [0,1]^d$ we can estimate

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(i\langle X(t),\theta\rangle\big)\big]\\
&\le \exp\Big(-c\int_{\mathbb{R}^d}\big|\exp\big(i\langle l_E(t), z\rangle\big)-1\big|^{\alpha}\psi(z)^{-(\lambda_m-\varepsilon)\alpha-q}dz\Big|\sum_{j=1}^{m}\tau_E(t)^{\lambda_j+\varepsilon+\frac{q}{\alpha}}\theta_j\Big|^{\alpha}\tau_E(t)^{-q}\Big)\\
&= \exp\Big(-c\int_{\mathbb{R}^d}\big|\exp\big(i\langle l_E(t), z\rangle\big)-1\big|^{\alpha}\psi(z)^{-(\lambda_m-\varepsilon)\alpha-q}dz\Big|\sum_{j=1}^{m}\tau_E(t)^{\lambda_j+\varepsilon}\theta_j\Big|^{\alpha}\Big)\\
&\le \exp\Big(-cm_\alpha\cdot\Big|\sum_{j=1}^{m}\tau_E(t)^{\lambda_j+\varepsilon}\theta_j\Big|^{\alpha}\Big)
\end{aligned}$$

so that, in particular

$$\mathbb{E}\big[\exp\big(i\langle X(t),\theta\rangle\big)\big] \le \exp\Big(-cm_\alpha\cdot\Big|\tau_E(t)^{\lambda_j+\varepsilon}\theta_j\Big|^{\alpha}\Big) \tag{6.10}$$

for all $1 \leq j \leq m$. Now let $X^{(1)}, \ldots, X^{(m)}$ be independent copies of $X(t)$. Since $X(t)$ is SαS, from Corollary 3.4 we get

$$m^{-\frac{1}{\alpha}}(X^{(1)} + \ldots + X^{(m)}) \stackrel{\mathrm{d}}{=} X(t).$$

Using this and (6.10) we obtain

$$\begin{aligned}
\mathbb{E}\big[\exp\big(i\langle X(t), \theta\rangle\big)\big] &= \mathbb{E}\big[\exp\big(i\langle m^{-\frac{1}{\alpha}} \sum_{j=1}^{m} X^{(j)}, \theta\rangle\big)\big] \\
&= \prod_{j=1}^{m} \mathbb{E}\big[\exp\big(i m^{-\frac{1}{\alpha}} \langle X^{(j)}, \theta\rangle\big)\big] \\
&\leq \prod_{j=1}^{m} \exp\Big(- c|\tau_E(t)^{\lambda_j+\varepsilon}|\theta_j||^{\alpha}\Big) \\
&= \exp\Big(- c\sum_{j=1}^{m} |\tau_E(t)^{\lambda_j+\varepsilon}|\theta_j||^{\alpha}\Big)
\end{aligned}$$

as desired. $\square$

Proof of Theorem 6.5. As in the proof of Theorem 5.9 without loss of generality we will assume that D is of the real canonical form and that X satisfies (6.4).

Let us first prove (6.7). Since X satisfies (6.4), the upper bound in (6.7) follows from Lemma 5.5 and Remark 5.6. Now we prove the lower bound by applying Frostman's theorem (see Chapter 1.2). Let

$$\mathcal{E}_\gamma = \int_{[0,1]^d} \int_{[0,1]^d} \mathbb{E}[\|X(t) - X(s)\|^{-\gamma}] dt ds$$

and, throughout this proof let c be an unspecified positive constant. Using the fact that X has stationary increments, Lemma 6.6 and Lemma 6.7 for all $\lambda_j' > \lambda_j$, $1 \leq j \leq m$, we obtain

$$\begin{aligned}
\zeta_\gamma &:= \mathbb{E}[\|X(t) - X(s)\|^{-\gamma}] = \mathbb{E}[\|X(t-s)\|^{-\gamma}] \\
&\leq c \int_0^\infty \int_{\mathbb{R}^m} \exp(-\frac{\|y\|^2}{2}) \mathbb{E}\Big[\exp\big(i\langle X(t-s), uy\rangle\big)\Big] dy u^{\gamma-1} du \\
&\leq c \int_0^\infty \int_{\mathbb{R}^m} \exp\Bigg(- \frac{\|y\|^2}{2} - c\sum_{j=1}^{m} |\tau_E(t-s)^{\lambda_j'} u|y_j||^{\alpha}\Bigg) dy_1 \ldots dy_m u^{\gamma-1} du \\
&= c \int_{\mathbb{R}^m} \exp\Bigg(- c\sum_{j=1}^{m} |\tau_E(t-s)^{\lambda_j'} |x_j||^{\alpha}\Bigg) \\
&\qquad \times \int_0^\infty u^{\gamma-m-1} \exp\big(- \frac{\|x\|^2}{2u^2}\big) du dx_1 \ldots dx_m,
\end{aligned} \tag{6.11}$$

where we used Fubini's theorem and the change of variables $x = uy$ in the last equality. Note that by using the substitution $v = \frac{\|x\|^2}{2u^2}$

$$\int_0^\infty u^{\gamma-m-1} \exp\big(- \frac{\|x\|^2}{2u^2}\big) du = \int_0^\infty \frac{\|x\|^{\gamma-m-1}}{\sqrt{2}^{\gamma-m-1}} v^{-\frac{\gamma-m-1}{2}} 2\|x\| \frac{1}{\sqrt{2}^3} v^{-\frac{3}{2}} \exp(-v) dv$$

$$
\begin{aligned}
&= c\|x\|^{\gamma-m}\int_0^\infty v^{\frac{m}{2}-\frac{\gamma}{2}-1}\exp(-v)dv\\
&= c\|x\|^{\gamma-m}\Gamma(\frac{m}{2}-\frac{\gamma}{2}).
\end{aligned}
$$

Combining this with (6.11) we get

$$
\begin{aligned}
\zeta_\gamma &\le c\int_{\mathbb{R}^m}\|x\|^{\gamma-m}\exp\Bigg(-c\sum_{j=1}^m|\tau_E(t-s)^{\lambda'_j}|x_j||^\alpha\Bigg)dx\\
&= c\tau_E(t-s)^{-\sum_{j=1}^m\lambda'_j}\int_{\mathbb{R}^m}\Big[\sum_{i=1}^m\Big(\tau_E(t-s)^{-\lambda'_i}|y_i|\Big)^2\Big]^{\frac{\gamma-m}{2}}\exp\Big(-\sum_{i=1}^m c|y_i|^\alpha\Big)dy,
\end{aligned}
\tag{6.12}
$$

where we used the change of variables $y_i=\tau_E(t-s)^{\lambda'_i}x_i$ in the last equality. Now let us first consider the case where there is an integer $0\le l\le m-1$ such that

$$
\sum_{i=1}^{m-l-1}\lambda_i<\sum_{k=1}^p a_k\mu_k\le\sum_{i=1}^{m-l}\lambda_i.
$$

By Lemma 5.10 we may and will assume that $m-l-1<\gamma<m-l$. By using (5.40) and (5.41) we will integrate with respect to $dy_m,\dots,dy_{m-l}$ in the integral in (6.12). We first integrate with respect to dy_m to find that

$$
\begin{aligned}
\zeta_\gamma \le c\tau_E(t-s)^{-\sum_{j=1}^m\lambda'_j+(m-\gamma)\lambda'_m}&\int_{\mathbb{R}^m}\Big[y_m^2+\sum_{i=1}^{m-1}\Big(\tau_E(t-s)^{\lambda'_m-\lambda'_i}|y_i|\Big)^2\Big]^{\frac{\gamma-m}{2}}\\
&\times\exp\Big(-\sum_{i=1}^{m-1}c|y_i|^\alpha\Big)dy_m\dots dy_1\\
\le c\tau_E(t-s)^{-\sum_{j=1}^m\lambda'_j+(m-\gamma)\lambda'_m}&\int_{\mathbb{R}^{m-1}}\Big[\sum_{i=1}^{m-1}\Big(\tau_E(t-s)^{\lambda'_m-\lambda'_i}|y_i|\Big)^2\Big]^{\frac{\gamma-m-1}{2}}\\
&\times\exp\Big(-\sum_{i=1}^{m-1}c|y_i|^\alpha\Big)dy_{m-1}\dots dy_1.
\end{aligned}
$$

By repeating this argument for $dy_{m-1}\dots dy_{m-l}$ exactly as in the proof of Theorem 5.9 we obtain

$$
\begin{aligned}
\zeta_\gamma &\le c\tau_E(t-s)^{-\sum_{j=1}^{m-l}\lambda'_j+(m-l-\gamma)\lambda'_{m-l}}\\
&= c\tau_E(t-s)^{\sum_{j=1}^{m-l}(\lambda'_{m-l}-\lambda'_j)-\gamma\lambda'_{m-l}}.
\end{aligned}
$$

On the other hand if

$$
\sum_{i=1}^m\lambda_i<\sum_{k=1}^p a_k\mu_k
$$

by Lemma 5.10 we can assume that $\gamma<m$ and the above calculations yield

$$
\zeta_\gamma\le c\tau_E(t-s)^{-\gamma\lambda'_m+\sum_{j=1}^m(\lambda'_m-\lambda'_j)}.
$$

Altogether we have shown that for some $1 \leq l \leq m$

$$\mathcal{E}_\gamma \leq c \int_{[0,1]^d} \int_{[0,1]^d} \tau_E(t-s)^{-\gamma\lambda_l' + \sum_{j=1}^{l}(\lambda_l' - \lambda_j')} dt ds.$$

By Proposition 5.11 the last integral is finite for any

$$0 < \gamma < \min\Big\{m, \frac{\sum_{k=1}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{l}(\lambda_l' - \lambda_i')}{\lambda_l'}\Big\},$$

with an arbitrarily small $\varepsilon > 0$ so that

$$\dim_{\mathcal{H}} X([0,1]^d) \geq \min\Big\{m, \frac{\sum_{k=1}^{p} \frac{a_k}{1+\varepsilon}\mu_k + \sum_{i=1}^{l}(\lambda_l' - \lambda_i')}{\lambda_l'}\Big\}$$

almost surely. Since this holds for any $\varepsilon > 0$ and $\lambda_j' > \lambda_j$, $1 \leq j \leq m$, the lower bound in (6.7) follows by letting $\varepsilon \to 0$ and $\lambda_j' \to \lambda_j$, $1 \leq j \leq m$.

Now we turn to the proof of (6.8). If $\sum_{k=1}^{p} a_k\mu_k \leq \sum_{i=1}^{m} \lambda_i$ then the almost sure equality

$$\dim_{\mathcal{H}} X([0,1]^d) = \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$$

is proven exactly by the same argument as in the proof of Theorem 5.9. So let us consider the case that

$$\sum_{j=1}^{l-1} \tilde{a}_j\tilde{\mu}_j \leq \sum_{i=1}^{m} \lambda_i < \sum_{j=1}^{l} \tilde{a}_j\tilde{\mu}_j$$

for some $1 \leq l \leq p$. Note that the upper bound in (6.8) follows from (6.4), Lemma 5.5 and Remark 5.6 as before. It remains to prove the lower bound in (6.8). Again we will do this by applying Frostman's theorem. Let

$$0 < \gamma < \sum_{j=1}^{l} \frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j + \sum_{j=l+1}^{p} \tilde{\mu}_j + \sum_{i=1}^{m}(1 - \frac{\lambda_i}{\tilde{a}_l})$$

and according to Lemma 5.10 assume that

$$m + \sum_{j=l+1}^{p} \tilde{\mu}_j < \gamma < m + \sum_{j=l}^{p} \tilde{\mu}_j.$$

For $s, t \in [0,1]^d$ define

$$\eta_\gamma = \mathbb{E}\big[(\|t-s\|^2 + \|X(t) - X(s)\|^2)^{-\frac{\gamma}{2}}\big].$$

From the fact that X has stationary increments, Lemma 4.17 and Lemma 6.7 we get for any $\lambda_j' > \lambda_j$, $1 \leq j \leq m$,

$$\eta_\gamma \leq c\|s-t\|^{-\gamma} \int_{\mathbb{R}^m} \mathbb{E}\Big[\exp\Big(i\langle y, \frac{X(t-s)}{\|t-s\|}\rangle\Big)\Big] dy$$

$$
\begin{aligned}
&\leq c\|s-t\|^{-\gamma}\int_{\mathbb{R}^m}\exp\Big(-c\sum_{j=1}^m|\tau_E(t-s)^{\lambda_j'}|\frac{y_j}{\|s-t\|}||^\alpha\Big)dy\\
&= c\|s-t\|^{m-\gamma}\int_{\mathbb{R}^m}\exp\Big(-c\sum_{j=1}^m|\tau_E(t-s)^{\lambda_j'}|x_j||^\alpha\Big)dx\\
&= c\|s-t\|^{m-\gamma}\tau_E(t-s)^{-\sum_{j=1}^m\lambda_j'},
\end{aligned}
$$

where we used that

$$
\int_{\mathbb{R}^m}\exp\Big(-\sum_{j=1}^m|u_j|^\alpha\Big)du<\infty
$$

in the last equality as shown in the proof of the lower bound in Theorem 4.13. Overall we have shown that

$$
\begin{aligned}
\mathcal{G}_\gamma &= \int_{[0,1]^d}\int_{[0,1]^d}\mathbb{E}\Big[(\|s-t\|^2+\|X(s)-X(t)\|^2)^{-\frac{\gamma}{2}}\Big]dsdt\\
&\leq c\int_{[0,1]^d}\int_{[0,1]^d}\|s-t\|^{m-\gamma}\tau_E(t-s)^{-\sum_{j=1}^m\lambda_j'}dsdt.
\end{aligned}
$$

By Proposition 5.12 the above integral is finite for any

$$
m<\gamma<\sum_{j=1}^l\frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j+\sum_{j=l+1}^p\tilde{\mu}_j+\sum_{i=1}^m(1-\lambda_i'\frac{1+\varepsilon}{\tilde{a}_l})
$$

and any $\varepsilon>0$ as soon as $\lambda_j'>\lambda_j$, $1\leq j\leq m$, such that

$$
\sum_{j=1}^{l-1}\tilde{a}_j\tilde{\mu}_j\leq\sum_{i=1}^m\lambda_i'<\sum_{j=1}^l\tilde{a}_j\tilde{\mu}_j.
$$

This proves that

$$
\dim_{\mathcal{H}}\operatorname{Gr}X([0,1]^d)\geq\sum_{j=1}^l\frac{\tilde{a}_j}{\tilde{a}_l}\tilde{\mu}_j+\sum_{j=l+1}^p\tilde{\mu}_j+\sum_{i=1}^m(1-\lambda_i'\frac{1+\varepsilon}{\tilde{a}_l})
$$

almost surely for any $\varepsilon>0$ and $\lambda_j'>\lambda_j$, $1\leq j\leq m$. Thus, the lower bound in (6.8) follows by letting $\varepsilon\to0$ and $\lambda_j'\to\lambda_j$, $1\leq j\leq m$. Finally, the second equality in (6.7) and the second equality in (6.8) follow from Lemma 5.10. □

Let us close this thesis with the following short remark about the relation between the time scaling matrix E and the state space scaling operator D.

Remark 6.8. In (5.26) we made use of the fact that the matrices E and D of (E,D)-operator-self-similar random fields are in general not unique. However, Theorem 5.9 and Theorem 6.5 enlighten the property that the quotients of the real parts of the eigenvalues of E and D are always unique, since the Hausdorff dimension of the range and the graph depends on the quotients $\frac{a_i}{\lambda_j}$ and $\frac{\lambda_j}{a_i}$, $1\leq i\leq p$, $1\leq j\leq m$. Let us also remark that any (E,D)-operator-self-similar random field is also $(\frac{E}{H}, \frac{D}{H})$-operator-self-similar for any $H \in (\lambda_m, a_1)$. However, the quotients of the real parts of E and D are the same as those of $\frac{E}{H}$ and $\frac{D}{H}$.

Index of notation

$\overset{\text{f.d.}}{=}$	equality of finite-dimensional distributions
$\operatorname{diam}(U)$	diameter of the set U
$\lVert \cdot \rVert$	arbitrary norm
$\mathcal{H}^s(U)$	s-dimensional Hausdorff measure of the set U
$\dim_{\mathcal{H}} U$	Hausdorff dimension of the set U
$f(U)$	range of the function f over the set U
$\operatorname{Gr} f(U)$	graph of the function f over the set U
$\lVert \cdot \rVert_p$	p-norm for $p \geq 1$
$\mathcal{B}(U)$	Borel σ-algebra over the set U
$\mathbb{R}^{k \times n}$	set of real matrices with k rows and n columns
$(\tau_E(x), l_E(x))$	generalized polar coordinates of the vector x with respect to the matrix E
S_E	unit sphere with respect to τ_E
I_k	identity operator on $\mathbb{R}^k$
$(\Omega, \mathcal{A}, P)$	probability space
$\overset{\text{d}}{=}$	equality in distribution
SαS	symmetric α-stable
$\mathcal{N}(\mu, \Sigma)$	multivariate normal distribution with mean vector μ and covariance matrix Σ
$X \sim \mathcal{N}(\mu, \Sigma)$	X is normally distributed with mean vector μ and covariance matrix Σ
S_m	unit sphere in $\mathbb{R}^m$ with respect to a norm $\lVert \cdot \rVert$
A^T	transpose of the matrix A
λ_d	Lebesgue measure on $\mathbb{R}^d$
$\mathcal{E}_0$	$\{A \in \mathcal{B}(\mathbb{R}^d) : \lambda_d(A) < \infty\}$
a.s.	almost surely
$L^\alpha(\lambda)$	$\{f : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} \lvert f(x)\rvert^\alpha \lambda_d(dx) < \infty\}$
plim	limit in probability
$\operatorname{trace}(E)$	trace of the matrix E
$\operatorname{vol}(U)$	volume of the set U
$C^\infty(U)$	set of smooth functions on U
$C(U)$	set of continuous functions on U
$\Gamma(z)$	gamma function, i.e. $\int_0^\infty v^{z-1} e^{-v} dv$

Bibliography

[1] P. Abry, P. Goncalves, J. Lévy, Lois d'échelle, fractales et ondelettes, volume 1. Hermes, 2002.

[2] R.J. Adler, The Geometry of Random Fields, John Wiley & Sons, 1981.

[3] D. Anders, A. Hoffmann, H.P. Scheffler, K. Weinberg, Application of operator-scaling anisotropic random fields to binary mixtures, Philosophical Magazine 91 (29) (2011) 3766–3792.

[4] A. Ayache, Hausdorff dimension of the graph of the fractional Brownian sheet, Rev. Mat. Iberoamericana 20 (2) (2004), 395–412.

[5] A. Ayache, Y. Xiao, Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets, J. Fourier Anal. Appl. 11 (4) (2005) 407–439.

[6] P. Becker-Kern, M.M. Meerschaert, H.P. Scheffler, Hausdorff dimension of operator stable sample paths, Monatshefte Math., 140 (2003) 91–101.

[7] A. Benassi, S. Cohen, J. Istas, Local self-similarity and the Hausdorff dimension, C.R. Acad. Sci. Paris Ser. I 336 (3) (2003) 267–272.

[8] D. Benson, M.M. Meerschaert, B. Baeumer, H.P. Scheffler, Aquifer operator-scaling and the effect on solute mixing and dispersion, Water Resour. Res (42) (2006) 1–18.

[9] H. Biermé, M.M. Meerschaert, H.P. Scheffler, Operator scaling stable random fields, Stoch. Proc. Appl., 117 (3) (2007) 312–332.

[10] H. Biermé, C. Lacaux, Hölder regularity for operator scaling stable random fields, Stoch. Proc. Appl., 119 (2009) 2222–2248.

[11] H. Biermé, C. Lacaux, Modulus of continuity of some conditionally sub-Gaussian fields, application to stable random fields, Bernoulli (21) (2015) 1719–1759.

[12] A. Bonami, A. Estrade, Anisotropic analysis of some Gaussian models, J. Fourier Anal. Appl. 9 (2003) 215–236.

[13] J.P. Chilés, P. Delfiner, Geostatistics: Modeling Spatial Uncertainty, John Wiley & Sons, 1999.

[14] E. Csáki, M. Csörgő, Inequalities for increments of stochastic processes and moduli of continuity, Ann. Probab. (23) (1992) 1031–1052.

[15] S. Davies, P. Hall, Fractal analysis of surface roughness by using spatial data (with discussion), J. Roy. Statist. Soc. Ser. B (61) (1999) 3–37.

[16] P. Embrechts, M. Maejima, Self-similar Processes, Princeton University Press, 2002.

[17] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, John Wiley & Sons, 1990.

[18] E. Herbin, From N parameter fractional Brownian motions to N parameter multifractional Brownian motions, Rocky Mountain J. Math. (36) (2006) 1249–1284.

[19] M.W. Hirsch, S. Smale, Differential equations, dynamical systems, and linear algebra, Pure and Applied Mathematics, Vol. 60., Academic Press New York-London, 1974.

[20] A. Hoffmann, Operator scaling stable random sheets with application to binary mixtures, Dissertation Universität Siegen, 2011.

[21] W.N. Hudson, J.D. Mason, Operator-self-similar processes in a finite-dimensional space, Trans. Amer. Math. Soc. (273) (1982) 281–297.

[22] J.P. Kahane, Some random series of functions, Cambridge University Press, 1985.

[23] A. Kamont, On the fractional anisotropic Wiener field, Probab. Math. Statist. (16) (1996) 85–98.

[24] A.N. Kolmogorov, Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum, C.R. Acad. Sci. URSS (26) (1940) 115–118.

[25] N. Kôno, M. Maejima, Self-similar stable processes with stationary increments, In: Stable processes and related topics, Birkhäuser Boston, Boston (1991) 275–295.

[26] H. Kumano-go, Pseudo-differential Operators, MIT Press, Massachusetts, 1982.

[27] T. L. Laha, V.K. Rohatgi, Operator self-similar stochastic processes in $\mathbb{R}^d$, Stoch. Proc. Appl., 12 (1982) 73–84.

[28] J. M. Lamperti, Semi-stable stochastic proceses, Trans. Amer. Math. Soc. 104 (1962), 62–78.

[29] R. LePage, Multidimensional infinitely divisible variables and processes, In: Probability in Banach spaces, Lecture Notes in Math, Springer, Berlin (1981) 279–284.

[30] R. LePage, Conditional moments for coordinates of stable vectors, In: Probability on vector spaces, Lecture Notes in Math, Springer, Berlin (1987) 279–284.

[31] J. Lévy Véhel, Fractals in engineering: From theory to industrial applications, Springer, New York, 1997.

[32] Y. Li, W. Wang, Y. Xiao, Exact moduli of continuity for operator scaling Gaussian random fields, Bernoulli (21) (2015), 930–956.

[33] Y. Li, Y. Xiao, Multivariate operator-self-similar random fields, Stoch. Proc. Appl., 121 (2011) 1178–1200.

[34] M. Maejima, J.D. Mason, Operator-self-similar stable processes, Stoch. Proc. Appl., 54 (1994) 139–163.

[35] B.B. Mandelbrot, J. Van Ness, Fractional Brownian motion, fractional noises and applications, Siam Review (10) (1968) 422–437.

[36] J.D. Mason, Y. Xiao, Sample path properties of operator-self-similar Gaussian random fields, Theory Probab. Appl. (46) (2002) 58–78.

[37] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge University Press, 1995.

[38] M.M. Meerschaert, H.P. Scheffler, Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice, Wiley-Interscience, New York, 2001.

[39] T. Mikosch, S. Resnick, H. Rootzén, A. Stegeman, Is network traffic approximated by stable Lévy motion or fractional Brownian motion?, Ann. Appl. Probab. (12) (2002) 23–68.

[40] P. Mörters, Y. Peres, Brownian Motion, Cambridge University Press, 2010.

[41] M. Paluszyński, K. Stempak, On quasi-metric and metric spaces, Proc. Amer. Math. Soc. (12) (2009) 4307–4312.

[42] G. Samorodnitsky, M.S. Taqqu, Stable non-Gaussian Random Processes, Chapman and Hall, New York, 1994.

[43] K. Sato, Self-similar processes with independent increments, Probab. Th. Rel. Fields (89) (1991) 285–300.

[44] E. Sönmez, The Hausdorff dimension of the graph of operator scaling stable random sheets, Preprint available at https://arxiv.org/pdf/1505.02010.pdf. (2015) 1–20.

[45] E. Sönmez, The Hausdorff dimension of multivariate operator-self-similar Gaussian random fields, Preprint available at https://arxiv.org/pdf/1511.09311.pdf. (2016) 1–22.

[46] E. Sönmez, Sample path properties of multivariate operator-self-similar stable random fields, Preprint available at https://arxiv.org/pdf/1602.01282.pdf. (2016) 1–14.

[47] H. Wackernagel, Multivariate Geostatistics: An Introduction with Applications, Springer-Verlag, New York, 1998.

[48] W. Willinger, V. Paxson, M.S. Taqqu, Self-similarity and heavy tails: Structural modelling of network traffic, a practical guide to heavy tails, Birkhäuser Boston, Boston, 1998.

[49] D. Wu, Y. Xiao, Geometric properties of fractional Brownian sheets, J. Fourier Anal. Appl. (13) (2007) 1–37.

[50] Y. Xiao, Properties of strong local nondeterminism and local times of stable random fields, Random Fields and Appl. (6) (2011) 279–310.

[51] Y. Xiao, Sample path properties of anisotropic Gaussian random fields, In: D. Khoshnevisan and F. Rassoul-Agha, editors, A Minicourse on Stochastic Partial Differential Equations, Springer, New York (2009) 145–212.

[52] Y. Xiao, Strong local nondeterminism and sample path properties of Gaussian random fields, Asymptotic Theory in Probab. and Statistics with Appl. (2007) 136–176.

[53] Y. Xiao, Dimension results for Gaussian vector fields and index-alpha stable fields, Ann. Probab. (23) (1995) 273–291.

[54] Y. Xiao, T. Zhang, Local times of fractional Brownian sheets, Probab. Theory Related Fields 124 (2002) 204–226.

Eidesstattliche Erklärung

Ich versichere an Eides statt, dass die Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist. Die Dissertation wurde in der vorgelegten oder ähnlicher Form noch bei keiner anderen Institution eingereicht. Ich habe bisher keine erfolglosen Promotionsversuche unternommen.

Düsseldorf, den 10.08.2016

Ercan Sönmez